Game Theory Lecture Notes 5
Dynamic Games with Incomplete Information

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A Motivating Example

The chain store game with complete information

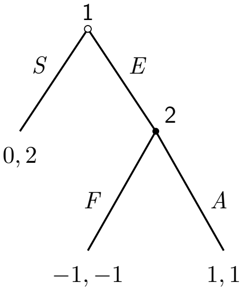


Figure 5.1: The chain-store game with complete information

- Nash equilibria: (S, F) and (E, A); only (E, A) is subgame perfect.
- (S, F) involves non-credible threat: the incumbent is not sequentially rational *off the equilibrium path*.
- Subgame perfection requires sequential rationality both *on and off* the equilibrium path; Nash does not.

A Motivating Example

The chain store game with incomplete information

- Consider an incomplete variant of the above chain store game.
- The entrant may be a competent one (C) or a weak one (W).
- The probability of a competent entrant is p.
- The competent entrant's payoffs are the same as before.
- The weak entrant's payoffs are reduced (specified in Figure 5.2).
- Only the entrant knows its own type; the incumbent does not know.
- The entrant moves first as before; the incumbent moves after the entrant enters.
- Game tree?

A Motivating Example

The chain store game with incomplete information

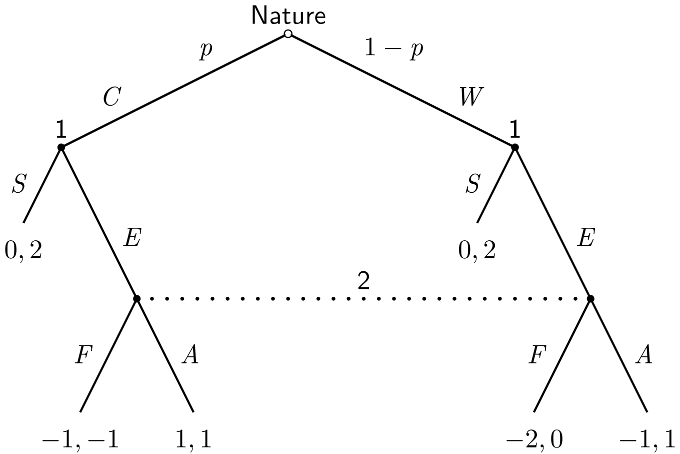


Figure 5.2: The chain-store game with incomplete information

A Motivating Example

The chain store game with incomplete information

- The entrant has two information sets.
- Every strategy of the entrant specifies an action for each of the information sets.
- Therefore, $S_1 = \{SS, SE, ES, EE\}$.
- For instance, ES is the strategy where the competent entrant enters and the weak entrant stays out.
- The incumbent has only one information set.
- Therefore, $S_2 = \{F, A\}$.

A Motivating Example

The chain store game with incomplete information

- The normal form representation:

	F	A
SS	$0, 2$	$0, 2$
SE	$-2(1-p), 2p$	$-(1-p), 2p+(1-p)$
ES	$-p, -p+2(1-p)$	$p, p+2(1-p)$
EE	$-p-2(1-p), -p$	$p-(1-p), p+(1-p)$

- Make sure you understand how it is written down.

A Motivating Example

The chain store game with incomplete information

- Consider the special case where $p = \frac{1}{2}$:

	F	A
SS	$0, 2$	$0, 2$
SE	$-1, 1$	$-\frac{1}{2}, \frac{3}{2}$
ES	$-\frac{1}{2}, \frac{1}{2}$	$\frac{1}{2}, \frac{3}{2}$
EE	$-\frac{3}{2}, -\frac{1}{2}$	$0, 1$

- Two pure strategy Nash equilibria: (SS, F) and (ES, A).
- Note that we can also formulate this extensive form game as a *Bayesian game*, and the above Nash equilibria are just the *Bayesian Nash equilibria* of this Bayesian game.
- Both equilibria are subgame perfect, but (SS, F) involves non-credible behavior: A is better than F regardless of the entrant's type.

A Motivating Example

The problem of SPE for games with incomplete information

- Subgame perfection has no bite in this game, because there are too few subgames.
- Only the whole game is a subgame. Thus, Nash is equivalent to SPE.
- This is due to the fact that although the incumbent observes the entrant's action, but it does not know the entrant's type.
- This is a common property of dynamic games with incomplete information.
- We want to find a way to extend the idea of sequential rationality to these games.

Perfect Bayesian Equilibrium

System of beliefs

- Recall the notion of *on and off* the equilibrium path.

Definition 5.1

Let $\sigma^* = (\sigma_1^*, \ldots, \sigma_n^*)$ be a Bayesian Nash equilibrium profile of strategies in a game of incomplete information. We say that an information set is **on the equilibrium path** if given σ^* and given the distribution of types, it is reached with positive probability. We say that an information set if **off the equilibrium path** if given σ^* and the distribution of types, it is reached with zero probability.

- Every information set is on the path of play under (ES, A).
- The incumbent's information set is off the path of play under (SS, A).

Perfect Bayesian Equilibrium

System of beliefs

- Recall notation:
  - H the set of all information sets;
  - $h \in H$ is one information set;
  - $x \in h$ is a node in information set h.
- We now introduce the core notion: beliefs.

Definition 5.2

A **system of beliefs** μ of an extensive-form game assigns a probability distribution over decision nodes to every information set. That is, for every information set $h \in H$ and every decision node $x \in h$, $\mu(x) \in [0, 1]$ is the probability that player i who moves in information set h assigns to his being at x, where $\sum_{x \in h} \mu(x) = 1$ for every $h \in H$.

- What is a system of beliefs for a game of perfect information?

Perfect Bayesian Equilibrium

Four requirements for a Bayesian Nash equilibrium to be perfect

Requirement 1

Every player will have a well-defined belief over where he is in each of his information sets. That is, the game will have a *system of beliefs*.

- Then, what kind of system of beliefs is reasonable?

Perfect Bayesian Equilibrium

Four requirements for a Bayesian Nash equilibrium to be perfect

Requirement 2 (Consistency)

Let $\sigma^* = (\sigma_1^*, \sigma_2^*, \ldots, \sigma_n^*)$ be a Bayesian Nash equilibrium profile of strategies. We require that in all information sets beliefs that are *on the equilibrium path* be consistent with *Bayes' rule*.

- Given σ^* and the nature's move (type distribution), let $\mathbb{P}^{\sigma^*}(x)$ be the probability that node x is reached.
- An information set h is on the equilibrium path if and only if

$$\mathbb{P}^{\sigma^*}(h) \equiv \sum_{x \in h} \mathbb{P}^{\sigma^*}(x) > 0.$$

- In this case, Requirement 2 requires

$$\mu(x) = \frac{\mathbb{P}^{\sigma^*}(x)}{\mathbb{P}^{\sigma^*}(h)}, \ \forall x \in h.$$

Perfect Bayesian Equilibrium

Four requirements for a Bayesian Nash equilibrium to be perfect

- What happens if h is off the equilibrium path?
- This is equivalent to $\mathbb{P}^{\sigma^*}(h) = 0$.
- Then, Bayes' rule does not apply. (Yes?)

Requirement 3

At information sets that are off the equilibrium path, to which Bayes' rule does not apply, any belief can be assigned.

- That is, no requirement at all is imposed on the off path beliefs.

Perfect Bayesian Equilibrium

Four requirements for a Bayesian Nash equilibrium to be perfect

- The last requirement is sequential rationality: best response to beliefs.

Requirement 4 (Sequential Rationality)

Given their beliefs, players' strategies must be sequentially rational. That is, in every information set players will play a best response to their beliefs.

- Suppose h is i's information set. Player i's strategy σ_i is sequentially rational at h given σ_{-i} and μ if

$$\mathbb{E}[v_i(\sigma_i, \sigma_{-i}, \theta)|h, \mu] \geq \mathbb{E}[v_i(s_i, \sigma_{-i}, \theta)|h, \mu], \ \forall s_i.$$

- In words, conditional on h being reached, playing σ_i is at least as good as every other strategy given σ_{-i} and μ.
- (SS, F) in the previous example is not sequentially rational given *any* belief.

Perfect Bayesian Equilibrium

Four requirements for a Bayesian Nash equilibrium to be perfect

- Imagine an information set in an extensive form game.

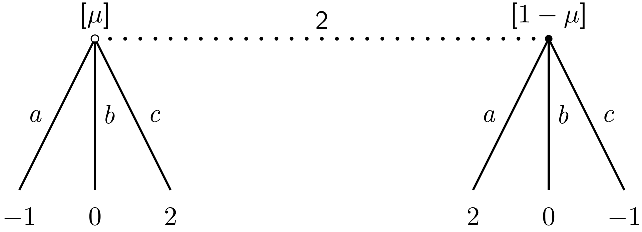


Figure 5.3: Illustration of sequential rationality; payoffs are for player 2

- Regardless of $\mu \in [0,1]$, b is not optimal.
- Thus, b at this information set is not sequentially rational to any belief.

Perfect Bayesian Equilibrium

Perfect Bayesian equilibrium

- Putting consistency and sequential rationality together leads to perfect Bayesian equilibrium.

Definition 5.3

A Bayesian Nash equilibrium profile $\sigma^* = (\sigma_1^*, \ldots, \sigma_n^*)$ together with a system of beliefs μ constitutes a **perfect Bayesian equilibrium** for an n-player game if they satisfy requirements 1 - 4.

Perfect Bayesian Equilibrium

- If all the information sets are on the path of play under a strategy profile, then this strategy profile is a perfect Bayesian equilibrium (given some belief system) if and only if it is a Bayesian Nash equilibrium.

Proposition 5.1

If a profile of strategies $\sigma^* = (\sigma_1^*, \ldots, \sigma_n^*)$ is a Bayesian Nash equilibrium of a Bayesian game Γ, and if σ^* induces all the information sets to be reached with positive probability, then σ^*, together with the belief system μ^* uniquely derived from σ^* and the distribution of types, constitutes a perfect Bayesian equilibrium for Γ.

- Make sure you understand: if all information sets are on the path, there is only one consistent belief system.

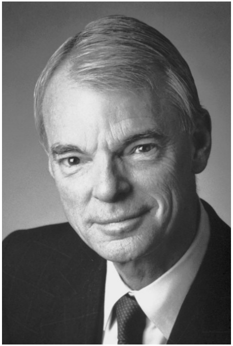
Signaling Games

General idea

- An informed player interacts with an uninformed player.
- In some instances, it may be the informed player's interest to reveal his private information to the uninformed player.
- Can the informed player *credibly signal* his type and make the uninformed player believe him?
- Examples include:
  - firm with high quality product signal its quality by providing long-term warranty;
  - the owner of a company keeps control of a significant percentage of the company when going public;
  - rich people show they are rich by buying luxury goods;
  - workers signal their intrinsic productivity to the job market by taking education.

Signaling Games

General idea



A. Michael Spence

The Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel 2001

"for their analyses of markets with asymmetric information."

Contribution: Showed how the able agents may improve the market outcome by taking costly action to signal information to poorly informed recipients. An important example is education as a signal of high individual productivity in the labor market. It is not necessary for education to have intrinsic value. Costly investment in education as such signals high ability.

Education Signaling

Setup

- Nature chooses player 1's (his) skill (productivity at work), which can be high H or low L. Only player 1 knows his own type. The probability of H is $p \in (0, 1)$, which is common knowledge.
- Player 1 can choose whether to get an MBA degree D or be content with his undergraduate degree U. The cost of getting an MBA degree is c_H for H type and c_L for L type, $c_H < c_L$. There is no cost if he chooses U.
- Player 2 (she) is an employer. She does not know player 1's type, but observes whether he owns an MBA degree or not. Then, she decides whether to assign him to be a manager M or a blue-collar worker B. At the same time, she must pay him the market wage: w_M for a manager and w_B for a blue-collar worker. Assume $w_M > w_B$.

Education Signaling

Setup

- Once employed, player 1 works and produces value to player 2. The net profit (output minus wage) to player 2 depends on player 1's skill and the job assignment:

Assignment

Skill		M	B
	H	10	5
	L	0	3

- High skilled worker is always more productive than the low skilled one.
- High skilled worker is better at managing, while low skilled one is better at blue-collar work.
- Note: we have assumed that education is completely valueless. The productivity of player 1 only depends on his own intrinsic ability and the job assignment, but is independent of whether he owns an MBA degree or not.

Education Signaling

Setup

- Player 1's payoff is the wage he obtains minus his education cost (if any).
- Player 2's payoff is the net profit.

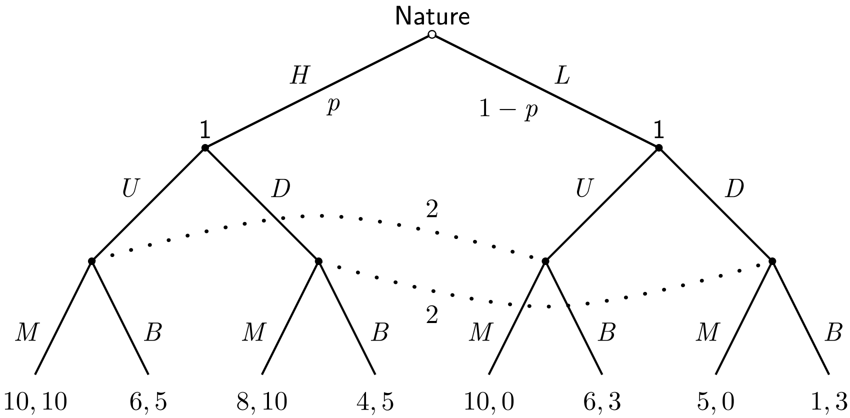


Figure 5.4: $c_H = 2$, $c_L = 5$, $w_M = 10$ and $w_B = 6$

Education Signaling

Setup

- Another, but equivalent, way to draw the game tree.

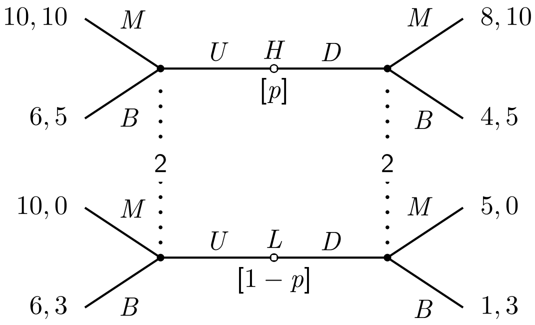


Figure 5.5: An equivalent game tree for the education signaling

Education Signaling

PBE

- Assume $p = \frac{1}{4}$. We look for PBE's.
- Pure strategies for player 1: UU, UD, DU, DD. For instance, UD means that player 1 chooses U if his type is H and chooses D if his type is L.
- Pure strategies for player 2: MM, MB, BM, BB. For instance, MB means that player 2 chooses M after observing U and chooses B after observing D.
- μ_U: player 2's belief that player 1's type is H after observing U.
- μ_D: player 2's belief that player 1's type is H after observing D.

Education Signaling

PBE

- Is there a PBE in which player 1's strategy is UD?
- If (μ_U, μ_D) is consistent, it must be $\mu_U = 1$ and $\mu_D = 0$. (Yes?)
- If player 2's strategy is sequentially rational given belief, it must that she chooses M after U and B after D. That is, player 2's strategy must be MB.
- Given player 2's strategy MB, L type wants to deviate to U.
- Therefore, there is no PBE in which player 1's strategy is UD.

Education Signaling

PBE

- Is there a PBE in which player 1's strategy is DU?
- If (μ_U, μ_D) is consistent, it must be $\mu_U = 0$ and $\mu_D = 1$.
- If player 2's strategy is sequentially rational given belief, she must play BM.
- Given player 2's strategy, no type of player 1 has an incentive to deviate.
- Therefore, (DU, BM) together with the belief system $(\mu_U = 0, \mu_D = 1)$ constitutes a PBE.
- We call this equilibrium a *separating equilibrium*: each type of player 1 chooses a different action, thus fully revealing his type to player 2.
- Although player 2 does not know player 1's type initially, but she can perfectly infer in equilibrium: after observing U, she knows it is L and after observing D, she knows it is L.

Education Signaling

PBE

- Is there a PBE in which player 1's strategy is UU?
- Consistency requires $\mu_U = p = \frac{1}{4}$. (Yes?)
- Given μ_U, player 2's best response after U is B.
- Then, for no type of player 1 to deviate, player 2 must choose B after observing D.
- Choosing B after D can be supported by the belief $\mu_D = 0$.
- Recall that, since player 2's information set after D is off the path, we have freedom in specifying μ_D. Note also that $\mu_D = 0$ is not the only belief that can support B after D. For instance, $\mu_D = 0.01$ works too.
- Therefor, (UU, BB) together with the belief $(\mu_U = \frac{1}{4}, \mu_D = 0)$ constitutes a PBE.
- This is called a *pooling equilibrium*: all types of player 1 choose the same action. Therefore, no information about player 1's type is revealed in equilibrium.

Education Signaling

PBE

- Is there a PBE in which player 1's strategy is DD?
- Consistency requires $\mu_D = \frac{1}{4}$.
- Given μ_D, player 2's best response after D is B.
- Then, L type has an incentive to deviate to U regardless of player 2's behavior after U.
- Therefore, there is no such PBE.

Education Signaling

PBE

- Other PBE?
- Observe that, in any PBE, L must play U.
- Suppose that H mixes between U and D, with probability $q \in (0, 1)$ on U.
- Consistency implies that $\mu_D = 1$ and

$$\mu_U = \frac{pq}{pq + (1-p) \times 1} < p = \frac{1}{4}.$$

- Given such μ_U, it is optimal for player 2 to play B.
- But then H is not indifferent between U and D.
- Therefore, there is no other PBE. Especially, there is no PBE in mixed strategies.

Education Signaling

PBE

- In the above separating equilibrium (DU, BM) with beliefs $\mu_U = 0$ and $\mu_D = 1$, education is used by the H type to signal that he is H, even if education itself is completely not productive.
- H can credibly do so, because L type does not want to imitate H type in equilibrium.
- The reason that L type does not want to imitate is not because L does not want to be a manager. Recall $w_M = 10 > 6 = w_B$. Rather, it is because education is too costly for him. Recall $c_L = 5 > 2 = c_H$.

Education Signaling

PBE

- As an exercise, let's consider whether there exists a PBE in mixed strategies if $p = \frac{1}{2}$?
- Following in the previous analysis, we still know that L must play U.
- Suppose again that H chooses U with probability $q \in (0, 1)$.
- Similarly as above, consistency implies that $\mu_D = 1$ and $\mu_U = \frac{pq}{pq+(1-p)} = \frac{q}{q+1}$.
- For player 1 to be indifferent between U and D, player 2 must mix between M and B with equal probability after U.
- But for she to mix, she must be indifferent between M and B after U. That is,

$$10\mu_U = 5\mu_U + 3(1 - \mu_U) \Longrightarrow \mu_U = \frac{3}{8}.$$

- Therefore, $\frac{q}{q+1} = \frac{3}{8}$ implying $q = \frac{3}{5}$.

Education Signaling

PBE

- To practice PBE in mixed strategies, consider the following signaling game.

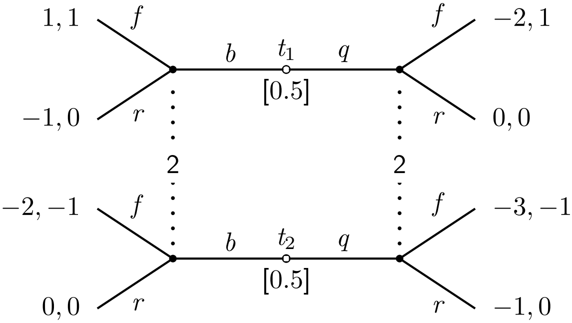


Figure 5.6: A finite signaling game.

Education Signaling

PBE

- We look for PBE in which both types mix.
- Let $\sigma(t_1) \in (0,1)$ be type t_1's probability of choosing b.
- Similarly, let $\sigma(t_2) \in (0,1)$ be type t_2's probability of choosing b.
- Let $\tau(b)$ be player 2's probability of chooosing f after b.
- Similarly, let $\tau(q)$ be player 2's probability of choosing f after q.
- For t_1 to be indifferent between b and q, we must have

$$\tau(b) - (1 - \tau(b)) = -2\tau(q).$$

- For t_2 to be indifferent between b and q, we must have

$$-2\tau(b) = -3\tau(q) - (1 - \tau(q)).$$

- These two equations together imply

$$\tau(b) = \frac{1}{2} \text{ and } \tau(q) = 0.$$

Education Signaling

PBE

- Let μ_b be player 2's belief about t_1 after b.
- Let μ_q be player 2's belief about t_1 after q.
- For player 2 to be indifferent after b, we must have

$$\mu_b - (1 - \mu_b) = 0 \Longrightarrow \mu_b = \frac{1}{2}.$$

- Consistency then requires

$$\frac{\frac{1}{2}\sigma(t_1)}{\frac{1}{2}\sigma(t_1) + \frac{1}{2}\sigma(t_2)} = \frac{1}{2} \Longrightarrow \sigma(t_1) = \sigma(t_2).$$

- This and consistency together then imply

$$\mu_q = \frac{\frac{1}{2}(1 - \sigma(t_1))}{\frac{1}{2}(1 - \sigma(t_1)) + \frac{1}{2}(1 - \sigma(t_2))} = \frac{1}{2}.$$

- Given $\mu_q = \frac{1}{2}$, player 2 is indeed optimal to play r after q.

Education Signaling

PBE

- We find a continuum of PBE:

$$\sigma(t_1) = \sigma(t_2) \in (0,1),\ \tau(b) = \frac{1}{2},\ \text{and } \tau(q) = 0,$$

with beliefs

$$\mu_b = \frac{1}{2} \text{ and } \mu_q = \frac{1}{2}.$$

Refinement

Intuitive criterion

- There are still many equilibria in signaling games even though we impose sequential rationality by PBE.
- Part of the reason is that there is no restriction at all for off path beliefs.
- Any way to further rule out some PBEs' that are less "plausible?"
- In other words, can we refine PBE further, as we refine Nash to SPE / PBE?

Refinement

Intuitive criterion

- Reconsider the education signaling.

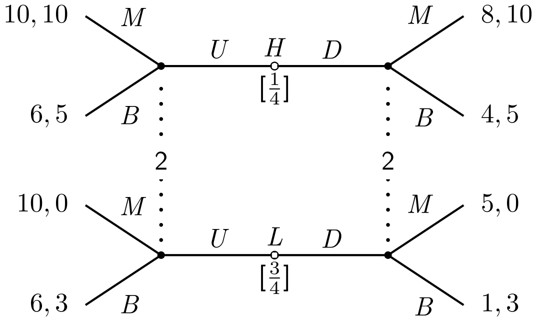


Figure 5.7: Education signaling

- A pooling equilibrium: (UU, BB). Any belief to support this behavior requires that 2 places sufficiently high probability on L type after D.

Refinement

Intuitive criterion

- Let's intuitively imagine the following hypothetical situation.
- Suppose H deviates to D and 2 observes this off path behavior.
- Before, 2 chooses, H tries to convince 2 that he is H instead of L.
- H says: "Look! If I were L type and deviated to D, then *regardless of how you would think of me*, I could at most obtain 5. But If I did not deviate, I could have obtained 6. Do you think it is reasonable for me to deviate?"
- H continues: "No, right? Therefore, now you believe that I am H. The reason I deviate to D is because I know I can convince you. Then, you would choose M instead of B, in which case this deviation is profitable for me."
- In this sense, we say (UU, BB) *fails the intuitive criterion*.

Refinement

Intuitive criterion

- Let Θ be the set of all possible types of player 1.
- For any subset $\hat{\Theta} \subset \Theta$ and $a_1 \in A_1$, let $BR_2(\hat{\Theta}, a_1) \subset A_2$ be the set of all possible player 2's best responses if player 1 has chosen a_1 and the belief μ of player 2 puts positive probability only on types in $\hat{\Theta}$. That is

$$BR_2(\hat{\Theta}, a_1) \equiv \bigcup_{\mu \in \Delta(\hat{\Theta})} \operatorname*{arg\,max}_{a_2 \in A_2} \sum_{\theta \in \hat{\Theta}} \mu(\theta) v_2(a_1, a_2, \theta).$$

- For instance, $BR_2(\{H, L\}, D) = \{M, B\}$, $BR_2(\{H\}, D) = \{M\}$ and $BR_2(\{L\}, D) = \{L\}$ in the education signaling example.

Refinement

Intuitive criterion

- Consider a PBE σ^*. Let $U(\theta)$ be type θ of player 1's equilibrium payoff.
- For any $a_1 \in A_1$, define

$$D(a_1) \equiv \left\{ \theta \in \Theta \middle| U(\theta) > \max_{a_2 \in BR_2(\Theta, a_1)} v_1(a_1, a_2, \theta) \right\}.$$

- $D(a_1)$ contains those types who, in the given equilibrium, have no incentive at all to deviate to a_1. That is, regardless of what player 2's belief is after a_1 and regardless of which best response 2 chooses, this type's payoff is strictly lower than what he would have obtained if he did not deviate.
- For instance, $D(D) = \{L\}$ in (UU, BB) in the above education signaling.
- Intuitively, player 2's belief after a_1 should rule out those types in $D(a_1)$. In other words, 2's action after a_1 should be in the set $BR_2(\Theta \backslash D(a_1), a_1)$.

Refinement

Intuitive criterion

Definition 5.4

Consider a PBE σ of a signaling game. We say that σ *fails the intuitive criterion* if there exists a type θ and an (off path) action a_1 such that

$$U(\theta) < \min_{a_2 \in BR_2(\Theta \setminus D(a_1), a_1)} v_1(a_1, a_2, \theta).$$

- Story: θ deviates to a_1 and convinces 2 that I am one of $\Theta \setminus D(a_1)$. Being convinced, 2 then chooses on action in $BR_2(\Theta \setminus D(a_1))$. Type θ finds it profitable regardless of which action in $BR_2(\Theta \setminus D(a_1))$ is actually chosen.
- If such θ and a_1 exist, we know $\theta \notin D(a_1)$ and a_1 is off-path.

Refinement

Intuitive criterion

- In the (UU, BB) equilibrium of the education signaling game,

$$U(H) = 6 < 8 = \min_{a_2 \in BR_2(\Theta \backslash D(D), D)} v_1(D, a_2, H),$$

where the second equality comes from

$$BR_2(\Theta \backslash D(D), D) = BR_2(\{H\}, D) = \{M\}.$$