TO:

FROM: Ecuații diferențiale – seminar – 10.10.2017

1) Ec. cu var. separabile

$\frac{dx}{dt} = a(t)\cdot b(x)$ ~~a(·), b(·)~~ $a(\cdot): I \subseteq \mathbb{R} \to \mathbb{R}$ cont.
$b(\cdot): J \subseteq \mathbb{R} \to \mathbb{R}$ cont.

Algoritm:
1. Rezolvă ec. algebrică $b(x)=0 \Rightarrow x_1, x_2, \dots, x_n$
Scrie $\varphi_1(t) \equiv x_1, \varphi_2(t) \equiv x_2, \dots, \varphi_n(t) \equiv x_n$ soluții staționare
2. Pe J_0 - se „separă" variabilele $\frac{dx}{dt} = a(t)\,dt$

- se integrează $\int \frac{dx}{b(x)} = \int a(t)\,dt$
$B(x) = A(t) + c,\ c \in \mathbb{R}$
Sol. generală sub formă implicită
- se inversează ~~x = φ(t)~~ $x = \varphi(t,c)\ (= B^{-1}(A(t)+c)),\ c \in \mathbb{R}$
Sol. generală sub forma explicită

Să se determine sol. generală
1) $x'(t^2-1) = x+1$
2) $tx' - x = x^2$
3) $x - tx' = 1 + t^2 x'$
4) $\sqrt{t^2+1}\,x' - x = 0$

1) $x' = \frac{x+1}{t^2-1} \Leftrightarrow \frac{dx}{dt} = \frac{x+1}{t^2-1},\ t \neq \pm 1$ $\int \frac{dt}{(t+1)(t-1)} = \frac{1}{2}\int\left(\frac{1}{t+1} + \frac{1}{t-1}\right) = \frac{1}{2}\left(\ln|t-1| - \ln|t+1|\right)$

$x+1=0 \Rightarrow x=-1 \Rightarrow \varphi_1(1) = -1$

$\frac{dx}{x+1} = \frac{dt}{t^2-1} \,\Big/ \int \Rightarrow \int \frac{dx}{x+1} = \int \frac{dt}{t^2-1} \Rightarrow \ln(|x+1|) = \frac{1}{2}\left(\ln|t-1| - \ln|t+1|\right) + c$

$\ln|t+1| = \frac{1}{2}\ln\sqrt{\frac{t-1}{t+1}} + \ln k,\ k>0$
$\Rightarrow |x+1| = k\sqrt{\frac{t-1}{t+1}}$

$x+1 = k\sqrt{\frac{t-1}{t+1}},\ k \in \mathbb{R}\setminus\{0\} \Rightarrow x(t) = -1 + k\sqrt{\frac{t-1}{t+1}},\ k \in \mathbb{R}\setminus\{0\}$

2) $tx' - x = x^2$
$x' = \frac{x^2+x}{t},\ t \neq 0$
$\frac{dx}{dt} = \frac{1}{t}(x^2+x)$
$x^2 + x = 0 \Rightarrow x(x+1) = 0 \Rightarrow x=0,\ x=-1$
$\Rightarrow \varphi_1(t) \equiv 0,\ \varphi_2(t) \equiv -1$
$\frac{dx}{x^2+x} = \frac{dt}{t} \,\Big/ \int \Rightarrow \int \frac{dx}{x^2+x} = \int \frac{dt}{t} \Rightarrow \ln(|x|) - \ln(|x+1|) = \ln(|t|) + c$

$\ln(|x|) - \ln(|x+1|) = \ln(|t|) + \ln(k),\ k>0 \Rightarrow \left|\frac{x}{x+1}\right| = k|t|,\ k>0$

$$\frac{x}{x+1}=k|t|,\ k\in\mathbb{R}\setminus\{0\} \Rightarrow x(t)=\frac{k|t|}{1-k|t|},\ k\in\mathbb{R}\setminus\{0\}$$

4) $\sqrt{t^2+1}\,x'-x=0$

$$x'=\frac{x}{\sqrt{t^2+1}}$$

$$\frac{dx}{dt}=x\cdot\frac{1}{\sqrt{t^2+1}}$$

$$x=0 \Rightarrow \varphi(t)=0$$

$$\frac{dx}{x}=\frac{dt}{\sqrt{t^2-1}}\ \Big|\int$$

$$\Rightarrow \int\frac{dx}{x}=\int\frac{dt}{\sqrt{t^2+1}} \Rightarrow \ln|x|=\ln\left(t+\sqrt{t^2+1}\right)+c,\ c\in\mathbb{R} \Rightarrow c=\ln k$$

$$\Rightarrow x(t)=k\left(t+\sqrt{t^2+1}\right),\ k\in\mathbb{R}\setminus\{0\}$$

<u>Ecuațiile liniare scalare</u>

$$\frac{dx}{dt}=a(t)x \qquad a(\cdot):I\subseteq\mathbb{R}\to\mathbb{R} \text{ cont.}$$

Soluția generală $x(t)=c\cdot e^{A(t)}$ $A(\cdot)$ primitivă a lui $a(\cdot)$

<u>Ecuații afine scalare</u>

$$\frac{dx}{dt}=a(t)x+b(t) \qquad a(\cdot),b(\cdot):I\subseteq\mathbb{R}\to\mathbb{R} \text{ cont.}$$

<u>Algoritm</u> (Metoda variației constantelor)

1. Considerăm ec. liniară asociată $\frac{d\bar{x}}{dt}=a(t)\bar{x}$

Scrie sol. generală $\bar{x}(t)=c\cdot e^{A(t)}$

2. „Variația constantelor"

Se caută sol. de forma $x(t)=c(t)e^{A(t)}$

Condiția: $x(\cdot)$ sol.

$$\Rightarrow (c(t)\cdot e^{A(t)})'=a(t)c(t)e^{A(t)}+b(t)$$

$$c'(t)e^{A(t)}+c(t)e^{A(t)}a(t)=a(t)c(t)e^{A(t)}+b(t)$$

$$c'(t)=b(t)e^{-A(t)}$$

$$\Rightarrow c(t)=\int b(t)e^{-A(t)}+k,\ k\in\mathbb{R} \Rightarrow x(t)=\left(\int b(t)e^{-A(t)}+k\right)e^{A(t)},\ k\in\mathbb{R}$$

TO:

FROM:

Să se det. sol. generală

1) $x' + x\, tg\, t = \frac{1}{\cos t}$

2) $t x' - x = t^2 e^t$

3) $x' = \frac{2}{t}x + t^2 \cos t$

4) $x' = \frac{2x + \ln t}{t \ln t}$

5) $x = t(x' - t\cos t)$

1) $x' = -tg(t)\,x + \frac{1}{\cos t}$

$\frac{dx}{dt} = -tg(t)\,x + \frac{1}{\cos t}$

$\frac{d\bar{x}}{dt} = -tg(t)\,\bar{x}$

$\bar{x}(t) = \cancel{c \cdot e}\; c \cdot e^{\int -tg(t)\,dt} = c \cdot e^{-\int \frac{\sin(t)}{\cos t}\,dt} = c \cdot e^{\int \frac{\cos'(t)}{\cos(t)}\,dt} = c \cdot e^{\ln|\cos t|} = c \cdot \cos(t), c \in \mathbb{R}$

$x(t) = c(t)\cos(t)$

$(c(t) \cdot \cos t)' = -tg\, t \cdot c(t)\cos t + \frac{1}{\cos t}$

$c'(t) \cdot \cos t - \cancel{c(t)\sin t} = -\cancel{\sin(t)\, c(t)} + \frac{1}{\cos t}$

$c'(t) = \frac{1}{\cos^2 t} \Rightarrow c(t) = \int \frac{1}{\cos^2 t}\,dt$

$\Rightarrow x(t) = \left(\int \frac{1}{\cos^2 t}\,dt\right)\cos t$

2) $x' = \frac{t^2 e^t + x}{t}$

$\frac{dx}{dt} = \frac{x}{t} + t \cdot e^t, t > 0$

$\frac{d\bar{x}}{dt} = \frac{\bar{x}}{t}$

$\bar{x}(t) = c \cdot e^{\int \frac{1}{t}\,dt} = c \cdot e^{\ln|t|} = c \cdot t$

Căutăm sol. de forma $x(t) = c(t) \cdot t$

$\Rightarrow (c(t) \cdot t)' = \frac{c(t) \cdot t}{t} + t \cdot e^t$

$\Leftrightarrow c'(t)t + \cancel{c(t)} = \cancel{c(t)} + t e^t \Rightarrow c'(t) = e^t \Rightarrow c(t) = \int e^t\,dt = e^t + k, k \in \mathbb{R}$

$\Rightarrow x(t) = c(t) \cdot t \Rightarrow x(t) = (e^t + k)t, k \in \mathbb{R}$

4) $x' = \dfrac{2x + \ln t}{t \ln t}$

$\dfrac{dx}{dt} = x \underbrace{\dfrac{2}{t\ln t}}_{a(t)} + \underbrace{\dfrac{1}{t}}_{b(t)}$, $t \neq 0$, $t > 0$

$\dfrac{d\bar{x}}{dt} = \dfrac{2}{t\ln t} \cdot \bar{x}$

$\bar{x} = c \cdot e^{\int \frac{2}{t\ln t} dt} = c \cdot e^{2\int (\ln t)' \frac{1}{\ln t} dt}$

$= 2c \cdot e^{2\ln(\ln t))} = c \cdot e^{\ln(\ln t)^2} = c^{\ln^2 t}$

$x(t) = c(t) \ln^2 t$

$(c(t)\ln^2 t)' = c(t) \ln^2 t \cdot \dfrac{2}{t \cdot \ln t} + \dfrac{1}{t}$

$\Rightarrow c'(t)\ln^2 t + c(t) \ln t \cdot \dfrac{1}{t} = c(t) \ln^2 t \dfrac{2}{t \ln t} + \dfrac{1}{t}$

$\Rightarrow c'(t) = \dfrac{1}{t\ln^2 t} \Rightarrow c(t) = \int \dfrac{1}{t \ln^2 t} dt$

$\Rightarrow \int (\ln t)' \dfrac{1}{\ln^2 t} dt = -\dfrac{1}{\ln t} + k$, $k \in \mathbb{R}$

$\Rightarrow x(t) = -\ln t + k\ln^2 t$, $k \in \mathbb{R}$

5) $x = t(x' - t\cos t)$

$x' = \dfrac{x}{t} + t\cos t$, $t \neq 0$, $t > 0$

$\dfrac{d\bar{x}}{dt} = \dfrac{1}{t} \cdot \bar{x}$

$\bar{x}(t) = c \cdot e^{\int \frac{1}{t} dt} = c \cdot e^{\ln t} = c \cdot t$

Sol. de forma: $x(t) = c(t) \cdot t$

$(c(t) \cdot t)' = \dfrac{c(t) \cdot \cancel{t}}{\cancel{t}} + t\cos t$

$c'(t) \cdot t + c(t) = c(t) + t \cdot \cos t$

$c'(t) = \dfrac{t \cdot \cos t}{t}$

$c'(t) = \cos t \Rightarrow c(t) = \int \cos t \, dt = \sin t + k$, $k \in \mathbb{R}$

$x(t) = t(\sin t + k)$, $k \in \mathbb{R}$

TO:

FROM:

3) ~~$x' = x + \ln t$~~ $x' = \frac{2}{t}x + t^2\cos t$, $t \neq 0$, $t > 0$

$\frac{d\bar{x}}{dt} = \frac{2}{t}\bar{x}$

$\bar{x}(t) = c \cdot e^{\int \frac{2}{t} dt} = c \cdot e^{2\ln t} = c \cdot t^2$

~~$\Rightarrow$~~ căutăm sol. de forma $x(t) = c(t) \cdot t^2$

$(c(t) \cdot t^2)' = \frac{2}{t} c(t) t^2 + t^2 \cos t$

$c'(t) \cdot t^2 + c(t) \cdot 2t = \frac{2}{t} c(t) t^2 + t^2 \cos t$

$c'(t) = \cos t \Rightarrow c(t) = \sin(t) + k, k \in \mathbb{R}$

$x(t) = t^2(\sin(t) + k), k \in \mathbb{R}$