Teorie dem:

$f(\cdot,\cdot): D \subseteq \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ $\qquad \frac{dx}{dt} = f(t,x)$, $x' = f(t,x)$

$$\varphi: I\subseteq \mathbb{R}\to\mathbb{R}^n \text{ sol.} \Rightarrow \begin{cases} \text{Graph } \varphi(\cdot) = \{(t,\varphi(t)),\ t\in I\} \subset D \\ \varphi(\cdot) \text{ der.} \\ \varphi'(t) = f(t,\varphi(t)) \end{cases}$$

Ec. cu var. sep.

$\frac{dx}{dt} = a(t)\cdot b(x)$, ~~a, b~~ $a: I\to\mathbb{R}$ $\ b: J\to\mathbb{R}$ cont.

Ec. liniare scalare

$\frac{dx}{dt} = a(t)x$, $a: I\subseteq\mathbb{R}\to\mathbb{R}$ cont.

Ec. afine

$\frac{dx}{dt} = a(t)x + b(t)$, $a,b: I\subseteq\mathbb{R}\to\mathbb{R}$ cont.

Ec. de tip Bernoulli

$\frac{dx}{dt} = a(t)x + b(t)\cdot x^{\alpha}$, $a,b: I\subseteq\mathbb{R}\to\mathbb{R}$ cont. $\alpha\in\mathbb{R}\setminus\{0,1\}$

Ec. de tip Ricatti

$\frac{dx}{dt} = a(t)x^2 + b(t)\cdot x + c(t)$, $a,b,c: I\subseteq\mathbb{R}\to\mathbb{R}$ cont.

Ec. omogene

$\frac{dx}{dt} = f\left(\frac{x}{t}\right)$, $f: D\subset\mathbb{R}\to\mathbb{R}$ cont.

Ec. de ord. sup. ce admit red. ord.

$F(\cdot,\cdot) : D \subseteq \mathbb{R} \times \mathbb{R}^{n+1} \to \mathbb{R}$, $F(t, x, x', \dots x^{(n)}) = 0$

* $F(t, x^{(k)}, x^{(k+1)}, \dots x^{(n)}) = 0$

$y = x^{(k)}$

* $F(t. \frac{x'}{x}, \frac{x''}{x}, \dots \frac{x^{(n)}}{x}) = 0$

$y = \frac{x'}{x}$

* Autonome $F(x, x', \dots x^{(n)}) = 0$

$x' = y(x)$

* Ec. de tip Euler

$F(x, tx', t^2x'', \dots t^n(x^{(n)})) = 0$

$|t| = e^s$ $y(s) = x(e^s)$ $x(t) = y(\ln(t))$

Pb. Cauchy

$f(\cdot,\cdot)$ $x' = f(t, x)$ $(t_0, x_0) \in D$ φ sol. cu $\varphi(t_0) = x_0$

φ e sol. a pb. Cauchy (φ, t_0, x_0)

T. Peano

$f : D = \mathring{D} \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ cont. $x' = f(t, x) \Rightarrow$ f admite EL

pe D ($\forall (t_0, x_0) \in D$ $\exists I_0 \in \mathcal{V}(t_0)$ $\exists \varphi : I_0 \to \mathbb{R}^n$ sol. cu $\varphi(t_0) = x_0$)

Fc local Lipschitz

$g(\cdot) : G \subseteq \mathbb{R}^n \to \mathbb{R}^n$ s.n. local Lipschitz în $x_0 \in G$ dacă

$\exists r > 0, \exists L > 0$ a.î. $\| g(x_1) - g(x_2) \| \le L \| x_1 - x_2 \|$ $\forall x_1, x_2 \in B_r(x_0) \cap G$

Fc. local Lipschitz în rap. cu var. a II-a

—„— $\forall D_0 \subset D$ compact $\exists L>0$ a.î. $\|f(t,x_1)-f(t,x_2)\| \le L \cdot \|x_1-x_2\|$, $\forall (t,x_1),(t,x_2) \in D_0$

Lema Bellman - Gronwall

$M \ge 0$, $u, v: I \subseteq \mathbb{R} \to \mathbb{R}_+$ cont., $t_0 \in I$. Dacă:

$u(t) \le M + \left| \int_{t_0}^{t} \mu(s) v(s) ds \right|$, $\forall t \in I$ at.

$u(t) \le M \cdot e^{\left| \int_{t_0}^{t} v(s) ds \right|}$, $\forall t \in I$

Teorema Cauchy - Lipschitz

$f: D = \mathring{D} \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ cont, local Lipschitz (II)

$\Rightarrow f(\cdot,\cdot)$ admite EUL pe D ($\forall (t_0,x_0) \in D$ $\exists I_0 = [t_0-a, t_0+a] \in V(t_0)$ $\exists! \varphi: I_0 \to \mathbb{R}^n$ sol. cu $\varphi(t_0) = x_0$)

Ec. dif. de ord. sup.

$f: D \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, $x^{(n)} = f(t, x, x', \dots x^{(n-1)})$

$\varphi: I \subseteq \mathbb{R} \to \mathbb{R}$ s.n. sol. dacă e de n ori der. și

$\varphi^{(n)}(t) = f(t, \varphi(t), \varphi'(t), \dots \varphi^{(n-1)}(t))$ $\forall t \in I$

Teoremă (Unicitate Globală)

Fie $f(\cdot,\cdot): D \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ cont. $\frac{dx}{dt} = f(t,x)$

$f(\cdot,\cdot)$ admite U.G. a sol. pe $D \Leftrightarrow f(\cdot,\cdot)$ admite U.L. a sol. pe D

Teorema asupra prel. sol.

$f: D = \mathring{D} \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ cont. $\varphi: (a,b) \subset \mathbb{R} \to \mathbb{R}^n$ sol.

1. φ adm. prel. str. la dr. $\Leftrightarrow b < +\infty$, $\exists t_0 \in (a,b)$, $\exists D_0 \subset D$ compact a.î. $(t, \varphi(t)) \in D_0$ $\forall t \in [t_0, b)$

2. φ adm. prel. str. la st. $\Leftrightarrow a > -\infty$, $\exists t_0 \in (a,b)$, $\exists D_0 \subset D$ compact a.î. $(t, \varphi(t)) \in D_0$ $\forall t \in (a, t_0]$

Teoremă (Existența sol. maximale)

$f(\cdot,\cdot) : D = \mathring{D} \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ cont. $\frac{dx}{dt} = f(t,x)$

$\forall\, \varphi \in S_f\ \exists\, \varphi_1 \in S_f$ sol. maximală $\varphi_1 \succ \varphi$

Teoremă (Existența și unicitatea sol. maximale)

—||— (ca sus)

~~$\forall (t_0, x_0$~~ $\forall\, \varphi(\cdot) \in S_f\ \exists!\, \varphi_1 \in S_f$ maximală $\varphi_1 \succ \varphi$

$\forall (t_0,x_0) \in D\ \ \exists!\ \varphi_{t_0,x_0} : I(t_0,x_0) = (t-(t_0,x_0), t+(t_0,x_0)) \to \mathbb{R}^n$ sol. maximală a pb. Cauchy (f, t_0, x_0)

Existența globală a sol.

$f : I \times \mathbb{R}^n \to \mathbb{R}^n$ cont. cu prop. de disipativitate $\frac{dx}{dt} = f(t,x)$

$\Rightarrow f$ admite $\exists G$ a sol. pe $I \times \mathbb{R}^n$ ($\forall (t_0, x_0) \in I \times \mathbb{R}^n\ \exists\ \varphi : I \to \mathbb{R}^n$ sol. cu $\varphi(t_0) = x_0$)

Ec. liniare pe $\mathbb{R}^n$

$A : I \subseteq \mathbb{R} \to L(\mathbb{R}^n, \mathbb{R}^n)$ def. ec. liniară $\frac{dx}{dt} = A(t) \cdot x$

$S_{A(\cdot)} \subset C^1(I, \mathbb{R}^n)$ e subspațiu vect. cu $\dim(S_{A(\cdot)}) = n$

↑ spațiul soluțiilor

WRONSKIANUL sol. $\varphi_1, \varphi_2, \dots \varphi_n \in S_{A(\cdot)}$ e fc.

$W_{\varphi_1 \dots \varphi_n}(t) = \det[\mathrm{col}(\varphi_1(t), \dots \varphi_n(t))],\ t \in I$

Teorema lui Liouville

A cont. $\Rightarrow$ $W_{\varphi_1, \dots \varphi_n}(t) = W_{\varphi_1 \dots \varphi_n}(t_0) \cdot e^{\int_{t_0}^{t} Tr(A(s))\, ds}$

Rezolvanta ec. liniare $x' = A(t) \cdot x$ e funcția

$$R_{A(\cdot)}(\cdot,\cdot) : I \times I \to L(\mathbb{R}^n, \mathbb{R}^n)$$

$$R_{A(\cdot)}(t,\tau)\xi := \alpha_{A(\cdot)}(t, \tau, \xi) = \varphi_{\tau,\xi}(t)$$

„Rezolvă ecuația" $\varphi_{\tau,\xi}(t) = R_{A(\cdot)}(t,\tau)\xi$

Exp. unei apl. liniare

$$\forall A \in L(\mathbb{R}^n, \mathbb{R}^n) \ \exists \sum_{k \geq 0} \frac{A^k}{k!} = \lim_{m \to \infty} \sum_{k=0}^{m} \frac{A^k}{k!} =: \exp(A) \ (= e^A)$$

Ec. liniare pe $\mathbb{R}^n$ cu coef. constanți

Sol. în caz general

$$A \in L(\mathbb{R}^n, \mathbb{R}^n) \quad \frac{dx}{dt} = Ax, \ \lambda \in \sigma(A) \quad P_j \in \mathbb{C}^n$$

$$\tilde{\varphi}(t) := e^{\lambda t} \sum_{j=0}^{m-1} P_j \cdot t^j \quad \text{unde} \begin{cases} (A - \lambda I_n)^m P_0 = 0 \\ P_j = \frac{1}{j!}(A - \lambda I_n)^j \cdot P_0, \ j = \overline{1, m-1} \end{cases}$$

Ec. afine pe $\mathbb{R}^n$

$$\frac{dx}{dt} = A(t) \cdot x + b(t)$$

$$b(t) = \begin{pmatrix} b_1(t) \\ \vdots \\ b_n(t) \end{pmatrix} \qquad \frac{dx_i}{dt} = \sum_{j=1}^{n} a_{ij}(t)\, x_j + b_i(t), \ i = \overline{1,n}$$

Principiul var. const.

Fie rez. ec. lin. $\frac{d\bar{x}}{dt} = A(t)\,\bar{x}$

$$\varphi(\cdot) \in S_{A,b} \iff \exists\, C \text{ primitivă a fc. } X^{-1}(t)\, b(t)$$

Ec. dif. lin. de ord. sup. $a_1, a_2 \ldots a_n : I \subseteq \mathbb{R} \to \mathbb{R}$

$$\tilde{X} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad A(t) = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \ldots & & & & \\ 0 & 0 & 0 & \ldots & 1 \\ a_n(t) & \ldots & & & a_1(t) \end{pmatrix} \qquad \frac{d\tilde{x}}{dt} = A(t)\,\tilde{x}$$

$x_i' = x_{i+1}$

Ec. afine de ord. sup.

$-\|-\quad \tilde{b}(t) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ b(t) \end{pmatrix} \qquad \frac{d\tilde{x}}{dt} = A(t)\tilde{x} + \tilde{b}(t)$

Int. primă

$F: D \to \mathbb{R}$ e int. pr. a lui $f: D \subset \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ dacă pt. orice sol. $x = x(t)$ $\exists$ o const. $C \in \mathbb{R}$ a.î. $F(t, x(t)) = C$

$x(t) = (x_1(t), \dots x_n(t))$

Criteriu int. prime

$$F \text{ int. primă} \iff \frac{d}{dt}\left[F(t, x(t))\right] = 0 \iff \frac{\partial F}{\partial t}(t, x(t)) + \frac{\partial F}{\partial x_1} x_1'(t) + \dots + \frac{\partial F}{\partial x_n} x_n'(t) = 0 \iff \frac{\partial F}{\partial t}(t, x(t)) + \frac{\partial F}{\partial x_1} f_1(t, x(t)) + \dots + \frac{\partial F}{\partial x_n} f_n(t, x(t)) = 0$$