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FROM: Ecuații diferențiale - 19.12.2017 - Seminar

## Integrale prime

$\frac{dx}{dt} = f(t,x)$ $\quad f(\cdot,\cdot): D \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, $f(\cdot,\cdot) = (f_1(\cdot,\cdot), \dots, f_n(\cdot,\cdot))$

Def: $F(\cdot,\cdot): D_0 \subset D \to \mathbb{R}$ s.n. integrală primă dacă $\forall\ \varphi(\cdot,\cdot)$ sol. cu $\mathrm{graph}\,\varphi(\cdot) \subset D_0$ $\exists\ c_\varphi \in \mathbb{R}$ a.î. $F(t,\varphi(t)) \equiv c$

Criteriu: $D = \overset{\circ}{D}$, $D_0 = \overset{\circ}{D_0}$, $f(\cdot,\cdot)$ cont., $F(\cdot,\cdot)$ derivabilă, integrală primă
$\iff \frac{\partial F}{\partial t}(t,x) + \sum_{i=1}^{n} \frac{\partial F}{\partial x_i}(t,x)\, f_i(t,x) \equiv 0$ pe $D_0$

Def: $F_1(\cdot,\cdot), \dots, F_k(\cdot,\cdot): D_0 \to \mathbb{R}$ integrale prime s.n. funcțional independente

$$\mathrm{rang}\left(\frac{\partial F_i}{\partial x_j}(t,x)\right)_{\substack{i=\overline{1,k}\\ j=\overline{1,n}}} = k \text{ (maxim)} \leq n \quad \forall (t,x) \in D_0$$

$F_1(\cdot,\cdot), \dots, F_n(\cdot,\cdot): D_0 \to \mathbb{R}$ integrale prime funcțional independente
$F_i(t,x) = c_i$ $\quad i=\overline{1,n}$, $c_i \in \mathbb{R}$, $i=\overline{1,n}$
$\to$ sol. generală sub forma implicită

Algoritm: (Reducerea ordinului cu ajutorul integralelor prime)

$F_1(\cdot,\cdot), \dots, F_k(\cdot,\cdot)$ integrale ~~p~~ funcțional independente
Ip. $\det\left(\frac{\partial F_i}{\partial x_j}(t,x)\right)_{\substack{i=\overline{1,k}\\ j=\overline{1,n}}} \neq 0 \quad \forall (t,x) \in D_0$

1. $$\begin{cases} F_1(t,x_1,x_2,\dots,x_k,x_{k+1},\dots,x_n) = c_1 \\ \dots \\ F_k(t,x_1,x_2,\dots,x_k,x_{k+1},\dots,x_n) = c_k \end{cases} \Rightarrow x_j = \varphi_j(t,x_{k+1},\dots,x_n,c_1,\dots,c_k),\ j=\overline{1,k}$$

Rezolvă sistemul în necunoscutele $x_1, \dots, x_k$

2. Integrează sist. de ec. urm.:

$$\begin{cases} \frac{dx_{k+1}}{dt} = f_{k+1}(t, \varphi_1(t,x_{k+1},\dots,x_n,c_1,\dots,c_k), \dots, \varphi_k(t,x_{k+1},c_1,\dots,c_k), x_{k+1},\dots,x_n) \\ \dots \\ \frac{dx_n}{dt} = f_n(t, \varphi_1(t,x_{k+1},\dots,x_n,c_1,\dots,c_k), \dots, \varphi_k(t,x_{k+1},\dots,x_n,c_1,c_k), x_{k+1},\dots,x_n) \end{cases}$$

$n-k$ ecuații

1) Fie ec. $\begin{cases} x' = \dfrac{x^2-2t}{y} \\ y' = -x \end{cases}$

a) $F_1(t,(x,y)) = t^2 + xy$ este integrală primă

b) Să se det. sol. generală

a) Apl. criteriul

$$\frac{\partial F_1}{\partial t}(t,(x,y)) + \frac{\partial F_1}{\partial x}(t,(x,y)) \cdot \frac{x^2-2t}{y} + \frac{\partial F_1}{\partial y}(t,(x,y)) \cdot (-x) =$$

$$= 2t + y \cdot \frac{(x^2-2t)}{y} - x^2 = 0 \Rightarrow F_1 \text{ integrală primă}$$

b) $F_1(t,(x,y)) = c$

$\Rightarrow t^2 + xy = c \Rightarrow \cancel{x} = \dfrac{c-t^2}{y}$

$\Rightarrow y' = -\dfrac{(c-t^2)}{y}$

$\dfrac{dy}{dt} = \dfrac{-(c-t^2)}{y} \Rightarrow y\,dy = (t^2-c)\,dt$

$\int y\,dy = \int (t^2-c)\,dt \Rightarrow \frac{1}{2}y^2 = \frac{1}{3}t^3 - ct + k,\ k \in \mathbb{R}$

$\Rightarrow y(t) = \pm\sqrt{\frac{2}{3}t^3 - 2ct + 2k}$

$\Rightarrow x(t) = \dfrac{c-t^2}{\pm\sqrt{\frac{2}{3}t^3 - 2ct + 2k}}$

$\begin{cases} x' = \dfrac{x^2-2t}{y} & \mid y \\ y' = -x & \mid x \end{cases}$

$x'y + xy' = -2t$

$(xy)' + (t^2)' = 0$

$(xy + t^2)' = 0$

$\Rightarrow \exists\, c \in \mathbb{R}$ a.î. $x(t)y(t) + t^2 \equiv c$

$\Rightarrow F_1(t,(x,y)) = xy + t^2$

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2. Fie ec. $\begin{cases} x' = \dfrac{x^3}{y} \\ y' = y - x^2 \end{cases}$

a) $F_1(t,(x,y)) = xye^{-t}$ este integrală primă

b) Să se det. sol. generală

a) Criteriu: $\dfrac{\partial F_1}{\partial t}(t,(x,y)) + \dfrac{\partial F_1}{\partial x}(t,(x,y)) \cdot \dfrac{x^3}{y} + \dfrac{\partial F_1}{\partial y}(t,(x,y)) \cdot (y - x^2) =$

$= -xye^{-t} + ye^{-t} \cdot \dfrac{x^3}{\cancel{y}} + xe^{-t}(y - x^2) = 0 \Rightarrow$ integrală primă

b) $F_1(t,(x,y)) = c \Rightarrow xye^{-t} = c \Rightarrow y = \dfrac{ce^t}{x} \Rightarrow x' = \dfrac{x^4}{ce^t} = \dfrac{1}{c}e^{-t}x^4$ (ec. var. sep.)

$\dfrac{dx}{dt} = \dfrac{1}{c}e^{-t}x^4$

$x^4 = 0 \Rightarrow x = 0 \Rightarrow x(t) \equiv 0$

$\Rightarrow y' = y$ liniară scalară cu sol. $y(t) = k \cdot e^t$

$\dfrac{dx}{x^4} = \dfrac{1}{c}e^{-t}\,dt \Rightarrow \int \dfrac{dx}{x^4} = \dfrac{1}{c}\int e^{-t}\,dt \Leftrightarrow -\dfrac{1}{3}x^{-3} = -\dfrac{1}{c}e^{-t} + p,\ p \in \mathbb{R}$

$\Rightarrow \dfrac{1}{x^3} = \dfrac{3}{c}e^{-t} + p \Rightarrow x(t) = \sqrt[3]{\dfrac{1}{\frac{3}{c}e^{-t} + p}}$

$y(t) = \dfrac{ce^t}{\sqrt[3]{\dfrac{1}{\frac{3}{c}e^{-t} + p}}}$, $c, p \in \mathbb{R}$

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$x' = \dfrac{x^3}{y} \quad | \cdot y$

$y' = y - x^2 \quad | \cdot x$

$x'y + xy' = xy$

$(xy)' = xy$

$z = xy$

$z' = z$

$z(t) \equiv ce^t$

$x(t)y(t) \equiv ce^t$

$x(t)y(t)e^{-t} \equiv c$

$F_1(t,(x,y)) = xye^{-t}$

3) Fie ec. $\begin{cases} x' = \dfrac{t-x}{t+x+y} \\ y' = \dfrac{x-t}{t+x+y} \end{cases}$

a) $F_1(t,(x,y)) = x + y$ integrală primă

b) Să se det. sol. generală

a) $\dfrac{\partial F_1}{\partial t}(t,(x,y)) + \dfrac{\partial F_1}{\partial x}(t,(x,y))\dfrac{t-x}{t+x+y} + \dfrac{\partial F_1}{\partial y}(t,(x,y))\dfrac{x-t}{t+x+y} =$

$= 0 + 1 \cdot \dfrac{t-x}{t+x+y} + 1 \cdot \dfrac{x-t}{t+x+y} = 0 \Rightarrow$ integrală primă

b) $F_1(t,(x,y)) = c \Rightarrow x+y=c \Rightarrow y = c-x \Rightarrow x' = \dfrac{t-x}{t+x+(t-x)} = \dfrac{t-x}{t+c}$

$x' = -\dfrac{x}{t+c} + \dfrac{t}{t+c}$ (ec. afină scalară → met. var. constantelor)

$\bar{x}' = -\dfrac{\bar{x}}{t+c} \Rightarrow \bar{x}(t) = k e^{\int -\frac{1}{t+c} dt} = k e^{-\ln|t+c|} = \dfrac{k}{t+c}$

$x(t) = \dfrac{k(t)}{t+c} \Rightarrow \left(\dfrac{k(t)}{t+c}\right)' = \dfrac{-k(t)}{(t+c)^2} + \dfrac{t}{t+c}$

$\dfrac{k'(t)}{t+c} + \dfrac{k(t)\cdot}{(t+c)^2} = \dfrac{-k(t)}{(t+c)^2} - \dfrac{t}{t+c}$

$k'(t) = t \Rightarrow k(t) = \dfrac{t^2}{2} + a,\ a \in \mathbb{R} \Rightarrow x(t) = \dfrac{\frac{t^2}{2}+a}{t+c}$

$$\left.\begin{array}{l} x(t) = \dfrac{\frac{t^2}{2}+a}{t+c} \\ y(t) = c - \dfrac{\frac{t^2}{2}+a}{t+c} \end{array}\right\} ,\ a, c \in \mathbb{R} \text{ Sol. gen.}$$

4) $\begin{cases} x' = \dfrac{x^2}{y} \\ y' = -\dfrac{y^2}{x} \end{cases}$

a) $F_1(t,(x,y)) = \operatorname{arctg} \dfrac{x}{y} - t$ integrală primă

b) Sol. generală

a) $\frac{\partial F_1}{\partial t}(t,(x,y)) + \frac{\partial F_1}{\partial x}(t,(x,y))\frac{x^2}{y} + \frac{\partial F_1}{\partial y}(t,(x,y))\left(-\frac{y^2}{x}\right) =$

$= -1 + \dfrac{1}{1+\left(\frac{x}{y}\right)^2} \cdot \dfrac{1}{y} \cdot \dfrac{x^2}{y} + \dfrac{1}{1+\left(\frac{x}{y}\right)^2} \cdot \left(-\dfrac{x}{y^2}\right) \cdot \left(-\dfrac{y^2}{x}\right) =$

$= -1 + \dfrac{x^2}{1+\left(\frac{x}{y}\right)^2} + \dfrac{xy^2}{\left(1+\frac{x^2}{y^2}\right)xy^2} = -1 + \dfrac{x^2y^2}{x^2y^2} = 0$

b) $F_1(t,(x,y)) = c$

$\operatorname{arctg} \dfrac{x}{y} - t = c \Rightarrow \dfrac{x}{y} = \operatorname{tg}(t+c)$

$y = \dfrac{x}{\operatorname{tg}(t+c)} \Rightarrow x' = \dfrac{x^2}{\frac{x}{\operatorname{tg}(t+c)}} = x \operatorname{tg}(t+c)$ (ec. liniară)

$\Rightarrow x(t) = k \cdot e^{\int \operatorname{tg}(t+c)\,dt} = k \cdot e^{\int \frac{\sin(t+c)}{\cos(t+c)} dt} = k \cdot e^{\int -\frac{\cos'(t+c)}{\cos(t+c)} dt} =$

$= k \cdot e^{-\ln(\cos(t+c))} = k \cdot \dfrac{1}{\cos(t+c)}$

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$$y(t) = \frac{k \cdot \frac{1}{\cos(t+c)}}{tg(t+c)} = \frac{k}{\sin(t+c)}, \quad c, k \in \mathbb{R}$$

5) $\begin{cases} x' = \dfrac{x^2}{y-t} \\ y' = x+1 \end{cases}$

a) $F_1(t,(x,y)) = \frac{x}{y-t}$ integrală primă

b) Să se det. sol. generală

a) $\frac{\partial F_1}{\partial t}(t(x,y)) + \frac{\partial F_1}{\partial x}(t(x,y)) \cdot \frac{x^2}{y-t} + \frac{\partial F_1}{\partial y}(t,(x,y)) \cdot (x+1) =$

$= \frac{x}{(y-t)^2} + \frac{1}{y-t} \cdot \frac{x^2}{y-t} + \frac{-x}{(y-t)^2} \cdot x+1 =$

$= \frac{x+x^2}{(y-t)^2} + \frac{-x(x+1)}{(y-t)^2} = 0$

b) $\frac{x}{y-t} = c \Rightarrow x' = cx$ (liniară scalară) $\Rightarrow x(t) = k e^{ct}$

$\Rightarrow y(t) - t = \frac{x(t)}{c} = \frac{k}{c} e^{ct} \Rightarrow y(t) = t + \frac{k}{c} e^{ct}$