TO:

FROM: Ecuații diferențiale - 31.10.2017 - curs

Existența și unicitatea globală a soluțiilor

Th. (U.G.)

$f(\cdot,\cdot) : D \subseteq \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ cont. $\frac{dx}{dt} = f(t,x)$

At. $f(\cdot,\cdot)$ admite U.G. a sol. pe $D \iff f(\cdot,\cdot)$ admite U.L. a sol. pe D

Dem: „$\Rightarrow$" Evident

„$\Leftarrow$" Fie $\varphi_1 : I_1 \to \mathbb{R}^n$, $\varphi_2 : I_2 \to \mathbb{R}^n$ soluții a.î. $\exists\, t_0 \in I_1 \cap I_2$ a.î. $\varphi_1(t_0) = \varphi_2(t_0)$

$\overset{?}{\Rightarrow} \varphi_1|_{I_1\cap I_2} = \varphi_2|_{I_1\cap I_2}$

Tehnică generală de globalizare

$I^* = \{t\in I_1\cap I_2 ;\ \varphi_1(t) = \varphi_2(t)\} \subset I_1\cap I_2$ - interval $\Rightarrow$ mulțime conexă

mult. conexă = deschisă, închisă, nu admite submult. cu ac. proprietăți

Arăt că
a) $I^* \neq \emptyset$
b) I^* închisă
c) I^* deschisă
$\Rightarrow I^* = I_1\cap I_2$

a) $I^*\neq\emptyset$ pt. că $t_0 \in I_1\cap I_2$

b) $I^* = \{t\in I_1\cap I_2 ;\ \varphi_1(t) = \varphi_2(t)\} = \{t\in I_1\cap I_2 ;\ (\varphi_1-\varphi_2)(t) = 0\} =$

$= \underbrace{(\varphi_1-\varphi_2)^{-1}}_{\text{fct. cont.}}\underbrace{(\{0\})}_{\text{mult. închisă}} \Rightarrow$ închisă

c) I^* deschisă

$t_1\in I^* \Rightarrow \varphi_1(t_1) = \varphi_2(t_1) \overset{U.L.}{\Rightarrow} \exists\, I_0 \in \mathcal{V}(t_1)$ a.î. $\varphi_1|_{I_0\cap(I_1\cap I_2)} = \varphi_2|_{I_0\cap(I_1\cap I_2)}$

$\Rightarrow I_0\cap(I_1\cap I_2) \subset I^*$ deci I^* deschisă

Prelungirea soluțiilor. Soluții maximale

Def: a) $\varphi_1 : I_1 \subseteq \mathbb{R} \to \mathbb{R}^n$ s.n. prelungire a funcției $\varphi_2 : I_2 \subseteq \mathbb{R} \to \mathbb{R}^n$ dacă $I_1 \supset I_2$ și $\varphi_1|_{I_2} = \varphi_2(\cdot)$; not. $\varphi_1(\cdot) \succ \varphi_2(\cdot)$

b) $\varphi_1(\cdot) : (a, b_1) \subseteq \mathbb{R}\to\mathbb{R}^n$ s.n. prelungire la dreapta a lui $\varphi_2(\cdot) : (a, b_2) \subseteq \mathbb{R}\to\mathbb{R}^n$ dacă $b_2 < b_1$ și $\varphi_1(\cdot)|_{(a,b_2)} = \varphi_2(\cdot)$ not. $\varphi_1(\cdot) \succ_d \varphi_2(\cdot)$

c) $\varphi_1(\cdot) : (a_1, b) \subseteq \mathbb{R}\to\mathbb{R}^n$ s.n. prelungire la st. a lui $\varphi_2(\cdot) : (a_2, b) \subseteq \mathbb{R}\to\mathbb{R}^n$ dacă $a_1 < a_2$ și $\varphi_1(\cdot)|_{(a_2,b)} = \varphi_2(\cdot)$. not. $\varphi_1(\cdot) \succ_s \varphi_2(\cdot)$

Prop: Fie $f(\cdot,\cdot) : D \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ $\quad \frac{dx}{dt} = f(t,x)$

$S_f := \{\varphi(\cdot) \mid \varphi(\cdot) : I \subseteq \mathbb{R} \to \mathbb{R}^n \ \varphi(\cdot) \text{ sol. a ec. } \frac{dx}{dt} = f(t,x)\}$

Atunci $(S_f, \preceq)$, $(S_f, \preceq_d)$, $(S_f, \preceq_s)$ sunt mulț. ordonate

Dem: Ex!

Def: $\varphi(\cdot) \in S_f$ s.n. sol. maximală dacă este un element maximal $(S_f, \preceq)$ (= sol. saturată)

Teorema asupra prelungirii soluțiilor

Fie $f(\cdot,\cdot) : D = \overset{\circ}{D} \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ cont. $\quad \frac{dx}{dt} = f(t,x)$

Fie $\varphi(\cdot) : (a,b) \subseteq \mathbb{R} \to \mathbb{R}^n$ soluție

Atunci:

1. φ admite o prelungire strictă la dr. ~~dacă~~ $\Leftrightarrow$ $b < +\infty$, $\exists\, t_0 \in (a,b)$ $\exists\, D_0 \subset D$ compactă a.î. $(t, \varphi(t)) \in D_0 \ \forall t \in [t_0, b)$

2. $\varphi(\cdot)$ admite o prelungire strictă la st. $\Leftrightarrow$ $a > -\infty$, $\exists\, t_0 \in (a,b)$, $\exists\, D_0 \subset D$ compactă a.î. $(t, \varphi(t)) \in D_0 \ \forall t \in (a, t_0]$

Dem: 1. (2) analog)

"$\Rightarrow$" $\varphi(\cdot)$ admite o prelungire str. la dreapta $\Rightarrow \exists\, \varphi_1(\cdot) : (a_1, b_1) \to \mathbb{R}^n$ sol.

$b < b_1$ $\varphi_1(\cdot) \succ \varphi(\cdot)$ ...

$D_0 := \{(t, \varphi_1(t)) ; t \in \underbrace{[t_0, b]}_{\text{compact}}\} \underset{\text{compactă}}{\subset} D$ $\quad (\varphi_1(\cdot) \text{ cont.})$

$t \in [t_0, b)$ $(t, \varphi(t)) = (t, \varphi_1(t)) \in D_0$ ok

"$\Leftarrow$" Planul: 1. $\exists \lim\limits_{\substack{t \to b \\ t < b}} \varphi(t) =: \xi$

2. Aplicăm Th. Peano în (b, ξ) $\Rightarrow \exists\, \varphi_0(\cdot) : [b - \alpha, b + \alpha] \to \mathbb{R}^n$ sol. a pb. $c(f, b, \xi)$

3. $\varphi_1(t) := \begin{cases} \varphi(t) & t \in (a,b) \\ \varphi_0(t) & t \in [b, b+\alpha] \end{cases}$

$\varphi_1(\cdot) \underset{\neq}{\succ} \varphi(\cdot)$

TO:

FROM:

1. Criteriul lui Cauchy

$\left[\forall \varepsilon > 0, \exists \delta_\varepsilon > 0 \text{ a.î. } \forall t', t'' \in (b-\delta_\varepsilon, b) \; \|\varphi(t') - \varphi(t'')\| < \varepsilon \Rightarrow \exists \lim_{\substack{t\to b\\ t<b}} \varphi(t)\right]$

$\varphi(\cdot)$ sol. $\Rightarrow$ ecuația integrală asociată $\varphi(t) = \varphi(t_0) + \int_{t_0}^{t} f(s, \varphi(s))\,ds \; \forall t, t_0 \in (a,b)$

$$\varphi(t') = \varphi(t_0) + \int_{t_0}^{t'} f(s, \varphi(s))\,ds$$

$$\varphi(t'') = \varphi(t_0) + \int_{t_0}^{t''} f(s, \varphi(s))\,ds$$

$$\|\varphi(t') - \varphi(t'')\| = \left\|\int_{t''}^{t'} f(s,\varphi(s))\,ds\right\| < \left|\int_{t''}^{t'} \underbrace{\|f(s,\varphi(s))\|}_{\leq K}\,ds\right| \leq K|t'-t''|$$

$$K := \max_{(t,x)\in D_0} \|f(t,x)\| \qquad \left(\delta_\varepsilon = \frac{\varepsilon}{K}\right)$$

3. $\varphi_1(\cdot)$ soluție $\quad \varphi_1(t) := \begin{cases} \varphi(t) & t \in (a,b) \\ \varphi_0(t) & t \in [b, b+\alpha] \end{cases}$

$$\varphi'_{1,s}(b) = \lim_{\substack{t\to b\\ t<b}} \frac{\varphi_1(t) - \varphi_1(b)}{t-b} = \lim_{\substack{t\to b\\ t<b}} \frac{\varphi(t) - \xi}{t-b} \overset{L'H}{=} \lim_{\substack{t\to b\\ t<b}} \frac{\varphi'(t)}{1} = \lim_{\substack{t\to b\\ t<b}} f(t, \varphi(t)) =$$

$$= f(b, \xi) = f(b, \varphi_1(b)) = \varphi'_{0,d}(b) = \varphi'_{1,d}(b)$$

<u>Th.</u> (Existența soluțiilor maximale)

Fie $f(\cdot,\cdot) : D = \overset{\circ}{D} \subseteq \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ cont. $\frac{dx}{dt} = f(t,x)$

At $\forall \varphi(\cdot) \in S_f \; \exists \varphi_1(\cdot) \in S_f$ sol. maximală $\varphi_1(\cdot) \succ \varphi(\cdot)$

<u>Dem</u>: Fie $\varphi(\cdot) \in S_f$

$S_{f,\varphi} = \{\psi(\cdot) \mid \psi(\cdot) \in S_f, \psi(\cdot) \succ \varphi(\cdot)\} \neq \emptyset \; (\varphi(\cdot) \in S_{f,\varphi})$

<u>Lema lui Zorn</u>

$\forall \mathcal{P} \in S_{f,\varphi}$ total ordonată admite un majorant $\Rightarrow \exists \varphi_1(\cdot) \in S_{f,\varphi}$ element maximal

Fie $\mathcal{P} = \{\varphi_j\}_{j\in J} \subset S_{f,\varphi}$ total ordonată

$\forall i, j \in J$ sau $\varphi_i(\cdot) \succ \varphi_j(\cdot)$

sau $\varphi_j(\cdot) \succ \varphi_i(\cdot)$

Fie $I_j = \text{dom. } \varphi_j(\cdot), j \in J$

$I^* := \bigcup_{j\in J} I_j \quad \varphi_*(\cdot) : I^* \to \mathbb{R}^n \quad \varphi_*(t) = \varphi_j(t)$ dacă $t \in I_j$

Arătăm a) I^* interval

b) $\varphi_*(\cdot)$ soluție

a) I^* interval

$t_1, t_2 \in I^* \Rightarrow [t_1, t_2] \subset I^*$

$t_i \in I^*, i \in \{1,2\} \Rightarrow \exists j_1, j_2 \in J$ a.î. $t_1 \in I_{j_1} = \text{dom } \varphi_{j_1}(\cdot)$

$t_2 \in I_{j_2} = \text{dom } \varphi_{j_2}(\cdot)$

sau $\varphi_{j_1}(\cdot) \succ \varphi_{j_2}(\cdot)$ $\quad I_{j_1} \supset I_{j_2} \Rightarrow t_1, t_2 \in I_{j_1}$ interval $\Rightarrow [t_1, t_2] \subset I_{j_1} \Rightarrow [t_1, t_2] \subset I^*$

sau $\varphi_{j_2}(\cdot) \succ \varphi_{j_2}(\cdot)$ $\quad I_{j_2} \supset I_{j_1} \Rightarrow t_1, t_2 \in I_{j_2}$ interval $\Rightarrow [t_1, t_2] \subset I_{j_2} \Rightarrow [t_1, t_2] \subset I^*$

b) $\varphi_*(\cdot)$ soluție

Fie $t \in I^* \Rightarrow \exists j \in J$ a.î. $t \in I_j \Rightarrow \varphi_*(\cdot)\big|_{I_j} = \varphi_j(\cdot)$

$\varphi_*'(t) = \varphi'(\ldots$

Prop (intervalul de definiție a soluțiilor maximale)

Fie $f(\cdot,\cdot): D = \overset{\circ}{D} \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ cont. $\quad \frac{dx}{dt} = f(t, x)$

Fie $\varphi(\cdot): I \subseteq \mathbb{R} \to \mathbb{R}^n$ $\varphi(\cdot) \in S_f$ soluție maximală

At. $I \subseteq \mathbb{R}$ interval deschis

Dem: Fie $\varphi(\cdot): I \subseteq \mathbb{R} \to \mathbb{R}^n$ sol. maximală

Pp. că I nu este deschis $\Rightarrow I = [a, b]$ sau $I = [a, b)$ sau $I = (a, b]$

Pp. de exemplu $I = (a, b]$

Apl. Th. Peano în $(b, \varphi(b)) \Rightarrow \exists\, \varphi_1(\cdot): [b-\alpha, b+\alpha] \to \mathbb{R}^n$ sol. a probl. Cauchy $(f, b, \varphi(b))$

Fie $\varphi_2(t) = \begin{cases} \varphi(t), & t \in (a, b] \\ \varphi_1(t), & t \in [b, b+\alpha]. \end{cases}$

$\varphi_2(\cdot) \underset{\neq}{\succ} \varphi(\cdot)$ $\quad \varphi_2(\cdot)$ sol. ⨳ ($\varphi(\cdot)$ maximală)

Th. (Existența & unicitatea soluțiilor maximale)

Fie $f(\cdot,\cdot): D = \overset{\circ}{D} \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ cont. $\frac{dx}{dt} = f(t, x)$ admite prop. UL pe D

Atunci: 1. $\forall\, \varphi(\cdot)$ sol. $\in S_f$, $\exists!\, \varphi_1(\cdot) \in S_f$ maximală $\varphi_1(\cdot) \succ \varphi(\cdot)$

2. $\forall (t_0, x_0) \in D$ $\exists!\, \varphi_{t_0, x_0}(\cdot): I(t_0, x_0) = (t^-(t_0, x_0), t^+(t_0, x_0)) \to \mathbb{R}^n$ sol. maximală a pb. Cauchy (f, t_0, x_0)

Dem: 1. Fie $\varphi(\cdot) \in S_f$ T. existența sol. maximale $\Rightarrow \exists\, \varphi_1(\cdot): I_1 \subseteq \mathbb{R} \to \mathbb{R}^n$ maximală $\varphi_1(\cdot) \succ \varphi(\cdot)$

Pp. $\exists\, \varphi_2(\cdot): I_2 \subseteq \mathbb{R} \to \mathbb{R}^n$ maximală $\varphi_2 \succ \varphi(\cdot)$, $\varphi_2(\cdot) \neq \varphi_1(\cdot)$

$I_1 \neq I_2$ ($\varphi_1(\cdot), \varphi_2(\cdot) \succ \varphi(\cdot) \overset{U.L.}{\underset{U.G.}{\Rightarrow}} \varphi_1(\cdot)\big|_{I_1 \cap I_2} = \varphi_2\big|_{I_1 \cap I_2}$ $\quad I_1 = I_2 \Rightarrow \varphi_1 = \varphi_2$ ⨳)

TO:

FROM:

Fie $\varphi_3(t) = \begin{cases} \varphi_1(t), & t \in I_1 \\ \varphi_2(t), & t \in (I_1 \cup I_2) \setminus I_1 \end{cases}$

$\varphi_3(\cdot) \underset{\neq}{\succ} \varphi_1(\cdot)$ (prelungire strictă), & ($\varphi_1(\cdot)$ maximală)

2. Fie $(t_0, x_0) \in D = \mathring{D}$, $f(\cdot,\cdot)$ continuă $\Rightarrow$ T. Peano $\Rightarrow$ $\exists\ \varphi_0(\cdot) : [t_0 - a, t_0 + a] \to \mathbb{R}^n$ sol. care $\varphi_0(t_0) = x_0$. T. existența sol. maximale $\Rightarrow$ $\exists\ \varphi_{t_0, x_0} : I(t_0, x_0) \to \mathbb{R}^n$ sol. maximală
$\varphi_{t_0, x_0} \succ \varphi_0(\cdot)$ $\quad \varphi_{t_0, x_0}(t_0) = \varphi_0(t_0) = x_0$

<u>Prop</u> (Intervalul de definiție al sol. maximale) $\quad I(t_0, x_0) := (t^-(t_0, x_0), t^+(t_0, x_0))$
Unicitatea: Analog cu 1.

<u>Existența globală a soluțiilor</u>

<u>Def</u>: $f(\cdot,\cdot) : I \times \mathbb{R}^n \to \mathbb{R}^n$, $I \subseteq \mathbb{R}$ interval are prop. de DISIPATIVITATE (D) dacă $\exists\ r > 0$ și $\exists\ a(\cdot) : I \to \mathbb{R}_+$ continuă a.î.:

$$|\langle x, f(t,x) \rangle| \leq a(t) \|x\|^2 \quad \forall t \in I,\ \forall x \in \mathbb{R}^n\ \|x\| > r$$

<u>T (E.G.)</u>

Fie $f(\cdot,\cdot) : \underline{\underline{I \times \mathbb{R}^n}} \to \mathbb{R}^n$, continuă cu (D) $\therefore \frac{dx}{dt} = f(t,x)$

At. $f(\cdot,\cdot)$ admite E.G. a sol. pe $I \times \mathbb{R}^n$ ($\forall (t_0, x_0) \in I \times \mathbb{R}^n$ $\exists\ \varphi(\cdot) : I \to \mathbb{R}^n$ sol. cu $\varphi(t_0) = x_0$ a)