Ecuații diferențiale

curs 7: 14. 11. 2017

Ecuații liniare în $\mathbb{R}^n$

$$\frac{dx}{dt} = A(t)x \quad , \quad A(\cdot): J \subseteq \mathbb{R} \to L(\mathbb{R}^n, \mathbb{R}^n) \text{ continuă}$$

Fie baza $B \subseteq \mathbb{R}^n$, $A_B = col(A(t)b_1, \dots, A(t)b_n) = (a_{ij}(t))_{\substack{i=\overline{1,n}\\ j=\overline{1,n}}}$
$B = \{(b_1, \dots, b_n)\}$

$$\underbrace{\frac{dx_i}{dt} = \sum_{j=1}^{n} a_{ij}(t)x_j, \quad i=\overline{1,n}}_{\text{sistem de ecuații liniare}}$$

- $n=1$; ecuații liniare scalare: $x' = a(t)x$

$a(\cdot): J \subseteq \mathbb{R} \to \mathbb{R}$ continuă

$\varphi(\cdot)$ soluție $\iff \varphi(t) = c \cdot e^{\int_{t_0}^{t} a(s)\,ds}$, $t_0 \in J$

Teoremă (E.U.G): Fie $A(\cdot): J \subseteq \mathbb{R} \to L(\mathbb{R}^n, \mathbb{R}^n)$ continuă, $\frac{dx}{dt} = A(t)x$.
$\forall (t_0, x_0) \in J \times \mathbb{R}^n$, $\exists!\, \varphi_{t_0,x_0}(\cdot): J \to \mathbb{R}^n$ soluție cu $\varphi_{t_0,x_0}(t_0) = x_0$.

$$S_{A(\cdot)} := \{\varphi(\cdot): J \to \mathbb{R}^n;\ \varphi(\cdot) \text{ soluție},\ x' = A(t)x\}$$

$A(\cdot)$ continuă $\Rightarrow$ $S_{A(\cdot)} \subset C^1(J, \mathbb{R}^n)$ ← clasă

Prop (Soluția banală): Fie $A(\cdot): J \subset \mathbb{R} \to L(\mathbb{R}^n, \mathbb{R}^n)$ continuă, $\frac{dx}{dt} = A(t)x$. Dacă $\varphi(\cdot) \in S_{A(\cdot)}$ astfel încât $\exists\, t_0 \in J$, $\varphi(t_0) = 0$ atunci $\varphi(t) \equiv 0$.

Dem: $\varphi(\cdot): J \to \mathbb{R}^n$ soluție, $\varphi(t_0) = 0$
Fie $\psi(\cdot): J \to \mathbb{R}^n$, $\psi(t) = 0$ soluție, $\psi(t_0) = 0$
$\overset{U.G.}{\Longrightarrow} \varphi(t) \equiv \psi(t)$ OK.

Teoremă (Spațiul soluțiilor): Fie $A(\cdot): J \subseteq \mathbb{R} \to L(\mathbb{R}^n, \mathbb{R}^n)$ continuă, $\frac{dx}{dt} = A(t)x$.
Atunci $S_{A(\cdot)} \subset C^1(J, \mathbb{R}^n)$ este subspațiu vectorial cu $\dim(S_{A(\cdot)}) = n$.

Dem: $\forall c_1, c_2 \in \mathbb{R}$, $\varphi_1(\cdot), \varphi_2(\cdot) \in S_{A(\cdot)} \Rightarrow c_1\varphi_1 + c_2\varphi_2 \in S_{A(\cdot)}$

$$(c_1\varphi_1 + c_2\varphi_2)'(t) = c_1\varphi_1'(t) + c_2\varphi_2'(t) \equiv c_1 A(t)\varphi_1(t) + c_2 A(t)\varphi_2(t) \equiv$$
$$\equiv A(t)(c_1\varphi_1(t) + c_2\varphi_2(t)) \equiv A(t)(c_1\varphi_1 + c_2\varphi_2)(t)$$

Arătăm că $S_{A(\cdot)} \simeq \mathbb{R}^n$. $\xrightarrow{}$ izomorf

Aplicația de evaluare în punctul $t_0 \in J$: $E_{t_0} : S_{A(\cdot)} \to \mathbb{R}^n$, $E_{t_0}(\varphi) := \varphi(t_0)$.

Arătăm că E_{t_0} izomorfism: - (a) liniară

- (b) injectivă

- (c) surjectivă

(a) $c_1, c_2 \in \mathbb{R}$, $\varphi_1, \varphi_2 \in S_{A(\cdot)}$:

$$E_{t_0}(c_1\varphi_1 + c_2\varphi_2) = (c_1\varphi_1 + c_2\varphi_2)(t_0) = c_1\varphi_1(t_0) + c_2\varphi_2(t_0) = c_1 E_{t_0}(\varphi_1) + c_2 E_{t_0}(\varphi_2)$$

(b) $E_{t_0}(\varphi_1) = E_{t_0}(\varphi_2)$, $\varphi_1(t_0) = \varphi_2(t_0) \overset{U.G.}{\Longrightarrow} \varphi_1 = \varphi_2$

(c) $\forall \xi \in \mathbb{R}^n$, $\exists \varphi \in S_{A(\cdot)}$ astfel încât $E_{t_0}(\varphi) = \xi$

$$\underset{\varphi(t_0)}{\overset{\parallel}{E_{t_0}(\varphi)}}$$

Dar T.E.G. aplicată în $(t_0, \xi) \Rightarrow \exists\, \varphi(\cdot) : \to \mathbb{R}^n$ soluție cu $\varphi(t_0) = \xi$

Def: Se numește sistem fundamental de soluții al ecuației $x' = A(t)x$,

$\{\varphi_1(\cdot), \dots, \varphi_n(\cdot)\} \subset S_{A(\cdot)}$ bază.

Obs: $\varphi(\cdot) \in S_{A(\cdot)} \iff \exists\, c_i \in \mathbb{R}$, $i = \overline{1,n}$ astfel încât: $\varphi(t) = \sum_{i=1}^{n} c_i \varphi_i(t)$

(soluția generală a ecuației)

Prop (Soluții liniar independente): Fie $A(\cdot) : J \subseteq \mathbb{R} \to L(\mathbb{R}^n, \mathbb{R}^n)$ continuă, $\frac{dx}{dt} = A$

Următoarele afirmații sunt echivalente:

(a) $\{\varphi_1(\cdot), \dots, \varphi_m(\cdot)\} \subset S_{A(\cdot)}$ sunt liniar independente

(b) $\exists\, t_0 \in J$ astfel încât $\{\varphi_1(t_0), \dots, \varphi_m(t_0)\} \subset \mathbb{R}^n$ sunt liniari independente (ca vect. în $\mathbb{R}^n$)

(c) $\{\varphi_1(t), \dots, \varphi_m(t)\} \subset \mathbb{R}^n$ sunt liniar independenți, $\forall\, t \in J$

Dem: (a) $\Rightarrow$ (c): Fie $c_1\varphi_1(t) + \dots + c_m\varphi_m(t) = 0$

$c_1 E_t(\varphi_1) + \dots + c_m E_t(\varphi_m) = 0$

$E_t(c_1\varphi_1 + \dots + c_m\varphi_m) = 0 \Rightarrow c_1\varphi_1 + \dots + c_m\varphi_m = 0 \overset{a)}{\Rightarrow} c_1 = \dots = c_m = 0$

(c) $\Rightarrow$ (b) evident

(b) $\Rightarrow$ (a): $c_1\varphi_1 + \dots + c_m\varphi_m = 0$

$$E_{t_0}(c_1\varphi_1+\dots+c_m\varphi_m)=0$$

$$c_1\varphi_1(t_0)+\dots+c_m\varphi_m(t_0)=0 \overset{b)}{\Longrightarrow} c_1=\dots=c_m=0$$

Obs: a) $\{\varphi_1(\cdot),\dots,\varphi_m(\cdot)\}\subset C^1(J,\mathbb{R}^n)$ sunt liniar independente

$\not\Downarrow \Uparrow$ b) $\exists\, t_0\in J$ astfel încât $\{\varphi_1(t_0),\dots,\varphi_m(t_0)\}\subset\mathbb{R}^n$ sunt liniar indep.

$\not\Downarrow \Uparrow$ c) $\{\varphi_1(t),\dots,\varphi_m(t)\}\subset\mathbb{R}^n$ sunt liniar indep., $\forall t\in J$

Matrici de soluții. Soluții matriceale. WRONSKIAN

Def: a) $\varphi_1(\cdot),\dots,\varphi_m(\cdot)\in S_{A(\cdot)}$, $X(t)=\mathrm{col}(\varphi_1(t),\dots,\varphi_m(t))$ se numește matrice de soluții

b) $X(\cdot): J\to M_{n,m}(\mathbb{R})$ se numește soluție matriceală dacă $\exists\, B\subset\mathbb{R}^n$ bază a.î. $X'(t)\equiv A_B(t)X(t)$.

Prop: $X(\cdot): J\to M_{n,m}(\mathbb{R})$, $X(\cdot)$ este matrice de soluții $\Leftrightarrow$ $X(\cdot)$ este soluție matriceală.

Dem: "$\Rightarrow$" $X(t)=\mathrm{col}(\varphi_1(t),\dots,\varphi_m(t))$, $\varphi_i(\cdot)\in S_{A(\cdot)}$, $i=\overline{1,m}$

$X'(t)=\mathrm{col}(\varphi_1'(t),\dots,\varphi_m'(t))\equiv\mathrm{col}(A(t)\varphi_1(t),\dots,A(t)\varphi_m(t))\equiv A_B(t)\cdot\mathrm{col}(\varphi_1(t),\dots,\varphi_m(t))\equiv$

$\equiv A_B(t)X(t)$

"$\Leftarrow$" $X'(t)\equiv A_B(t)\cdot X(t)$

Fie $X(t)=\mathrm{col}(\varphi_1(t),\dots,\varphi_m(t))$ $\Big\}\Rightarrow \mathrm{col}(\varphi_1'(t),\dots,\varphi_m'(t))\equiv$

$\equiv A_B(t)\,\mathrm{col}(\varphi_1(t),\dots,\varphi_m(t))\equiv$

$\equiv\mathrm{col}(A(t)\varphi_1(t),\dots,A(t)\varphi_m(t))\Rightarrow$

$\Rightarrow\varphi_i'(t)\equiv A(t)\varphi_i(t)$, $i=\overline{1,m}$ OK.

Def: Se numește WRONSKIANUL soluțiilor $\varphi_1(\cdot),\dots,\varphi_n(\cdot)\in S_{A(\cdot)}$ funcția

$$W_{\varphi_1,\dots,\varphi_n}(t):=\det[\mathrm{col}(\varphi_1(t),\dots,\varphi_n(t))],\ t\in J$$

Teorema lui Liouville: Fie $A(\cdot): J\subseteq\mathbb{R}\to L(\mathbb{R}^n,\mathbb{R}^n)$ continuă, $\frac{dx}{dt}=A(t)x$. Fie $\varphi_1(\cdot),\dots,\varphi_n(\cdot)\in S_{A(\cdot)}$. Atunci $W_{\varphi_1,\dots,\varphi_n}(t)\equiv W_{\varphi_1,\dots,\varphi_n}(t_0)\cdot e^{\int_{t_0}^{t}\underbrace{Tr}_{\text{urma}}(A(s))\,ds}$.

Dem:

Obs: A demonstra teorema revine la a demonstra că $t\to W_{\varphi_1,\dots,\varphi_n}(t)$ este soluția ecuației scalare $\frac{dy}{dt}=Tr(A(t))y$.

$$y(t) = c \cdot e^{\int_{t_0}^{t} Tr(A(s))\cdot ds}$$

$$= y(t_0)\cdot e^{\int_{t_0}^{t} Tr(A(s))\cdot ds}$$

$t = t_0 \Rightarrow y(t_0) = c$

$B \subset \mathbb{R}^n$ bază $\Rightarrow A_B(t) = \left(a_{ij}(t)\right)_{\substack{i=\overline{1,n}\\ j=\overline{1,n}}}$; $\varphi_i(\cdot) = \left(\varphi_i^j(\cdot)\right)_{j=\overline{1,n}}$, $i=\overline{1,n}$

$$W_{\varphi_1,\dots,\varphi_n}(t) = \det\left[col(\varphi_1(t),\dots,\varphi_n(t))\right] = \det\left(\left(\varphi_i^j(t)\right)_{i,j=\overline{1,n}}\right) =$$

$$= \sum_{\tau\in S_n} sgn(\tau)\,\varphi^1_{\tau(1)}(t)\dots\varphi^j_{\tau(j)}(t)\dots\varphi^n_{\tau(n)}(t)$$

$$= \sum_{j=1}^{n}\sum_{\tau\in S_n} sgn\,\tau\,\varphi^1_{\tau(1)}(t)\dots\varphi^{j-1}_{\tau(j-1)}(t)\underbrace{\left(\varphi^j_{\tau(j)}(t)\right)'}_{=\sum_{k=1}^{n} a_{jk}(t)\varphi^j_{\tau(j)}(t)}\varphi^{j+1}_{\tau(j+1)}(t)\dots\varphi^n_{\tau(n)}(t) =$$

$\varphi_i(\cdot)$ soluție

$$= \sum_{j,k=1}^{n} a_{jk}(t)\underbrace{\sum_{\tau\in S_n} sgn\,\tau\,\varphi^1_{\tau(1)}(t)\dots\varphi^{j-1}_{\tau(j-1)}(t)\,\varphi^k_{\tau(j)}(t)\,\varphi^{j+1}_{\tau(j+1)}(t)}_{\Delta}$$

$$= \sum_{j,k=1}^{n} a_{jk}(t)\Delta_{jk}(t) = \sum_{j=1}^{n} a_{jj}(t)W_{\varphi_1,\dots,\varphi_n}(t) = Tr(A(t))\cdot W_{\varphi_1,\dots,\varphi_n}(t)$$

$$\Delta_{jk}(t) = \begin{cases} W_{\varphi_1,\dots,\varphi_n}(t), & j=k\\ 0, & j\neq k\end{cases}$$

Prop (curent global al ecuațiilor liniare): Fie $A(\cdot): I\subset\mathbb{R}\to L(\mathbb{R}^n,\mathbb{R}^n)$ continuă, $\frac{dx}{dt} = A(t)x$. Considerăm $\alpha_{A(\cdot)}(\cdot,\cdot,\cdot): I\times I\times\mathbb{R}^n\to\mathbb{R}^n$ curentul global al ecuației. Atunci $\alpha_{A(\cdot)}(t,\tau,\cdot)\in L(\mathbb{R}^n,\mathbb{R}^n)$, $\forall\, t,\tau\in I$.

Dem: Fie $c_1,c_2\in\mathbb{R}$, $\xi_1,\xi_2\in\mathbb{R}^n$, $\alpha_{A(\cdot)}(t,\tau,c_1\xi_1+c_2\xi_2) = c_1\alpha_{A(\cdot)}(t,\tau,\xi_1) + c_2\alpha_A(t,\tau,\xi_2)$

$\tau\in I$

$t\to\alpha_{A(\cdot)}(t,\tau,c_1\xi_1+c_2\xi_2)$ soluție globală a problemei Cauchy$(f_{A(\cdot)},\tau,$

$t \to \alpha_{A(\cdot)}(t, \tau, \xi_1)$ soluție globală a problemei Cauchy $(f_{A(\cdot)}, \tau, \xi_1)$
$t \to \alpha_{A(\cdot)}(t, \tau, \xi_2)$ soluție globală a problemei Cauchy $(f_{A(\cdot)}, \tau, \xi_2)$ $\Big\} \; S_{A(\cdot)} \xrightarrow{} $ sp. vectorial

$\Rightarrow t \to c_1 \alpha_{A(\cdot)}(t, \tau, \xi_1) + c_2 \cdot \alpha_{A(\cdot)}(t, \tau, \xi_2)$ soluție a problemei Cauchy $(f_{A(\cdot)}, \tau, c_1\xi_1 + c_2\xi_2)$ (2)

(1), (2) + U.G $\Rightarrow$ q.e.d

Def: Se numește rezolvanta ecuației liniare $x' = A(t)x$, funcția $R_{A(\cdot)}(\cdot,\cdot) : J \times J \to L(\mathbb{R}^n, \mathbb{R}^n)$

$R_{A(\cdot)}(t, \tau)\xi := \alpha_{A(\cdot)}(t, \tau, \xi) = \varphi_{\tau,\xi}(t)$

„Rezolvă ecuația" $\varphi_{\tau,\xi}(t) = R_{A(\cdot)}(t, \tau)\xi$

Def: Se numește matrice fundamentală de soluții a ecuației $x' = A(t)x$, matricea $X(t) = col(\varphi_1(t), \dots, \varphi_n(t))$ unde $\{\varphi_1(\cdot), \dots, \varphi_n(\cdot)\} \subset S_{A(\cdot)}$ sistem fundamental de soluții.

$\varphi(\cdot) \in S_{A(\cdot)} \Leftrightarrow \exists\, c \in \mathbb{R}^n$ a.î. $\varphi(t) \equiv X(t) \cdot c$

$\left[c \in \mathbb{R}^n, \; c = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} ; \; X(\cdot) \text{ matrice fundamentală de soluții } X(t)c = \right.$
$\left. = col(\varphi_1(t), \dots, \varphi_n(t)) \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \sum_{i=1}^{n} c_i \varphi_i(t) \right]$

Teoremă (Proprietățile rezolvantei): Fie $A(\cdot) : J \subseteq \mathbb{R} \to L(\mathbb{R}^n, \mathbb{R}^n)$ continuă, $\frac{dx}{dt} = A(t)x$, fie $R_{A(\cdot)}(\cdot,\cdot) : J \times J \to L(\mathbb{R}^n, \mathbb{R}^n)$ rezolvanta ecuației. Atunci:

① $R_{A(\cdot)}(t, t) = I_n, \; \forall t$

② $R_{A(\cdot)}(t, \tau)\, R_{A(\cdot)}(\tau, s) = R_{A(\cdot)}(t, s), \; \forall t, \tau, s \in J$

③ $\exists\, (R_{A(\cdot)}(t, \tau))^{-1} = R_{A(\cdot)}(\tau, t), \; \forall t, \tau \in J$

④ $\forall B \subset \mathbb{R}^n$ bază și $\forall X(\cdot) : J_1 \to M_n(\mathbb{R})$ matrice fundamentală de soluții

$$R^{B}_{A(\cdot)}(t, \tau) = X(t)\, X^{-1}(\tau), \; \forall t, \tau \in J$$

⑤ $\forall B = \{b_1, \dots, b_n\} \subset \mathbb{R}^n$ bază, $\forall \tau \in J$, $t \to R^{B}_{A(\cdot)}(t, \tau) = col(\varphi_{\tau,b_1}(t), \dots, \varphi_{\tau,b_n}(t))$ matrice fundamentală de soluții

⑥
$$\det(R_{A(\cdot)}(t, \tau)) = e^{\int_\tau^t Tr(A(s))\,ds}, \; \forall \tau, t \in J$$

Dem: (6)

$$\det(R_{A(\cdot)}(t,\tau)) = \det(R^{B}_{A(\cdot)}(t,\tau)) \overset{5)}{=} \det(\mathrm{col}(\varphi_{\tau,b_1}(t),\dots,\varphi_{\tau,b_n}(t))) =$$

$$= W_{\varphi_{\tau,b_1},\dots,\varphi_{\tau,b_n}}(t) \underset{\text{Liouville}}{\overset{T}{=}} W_{\varphi_{\tau,b_1},\dots,\varphi_{\tau,b_n}}(\tau)\cdot e^{\int_\tau^t Tr(A(s))\cdot ds} =$$

$$= \det(\mathrm{col}(\varphi_{\tau,b_1}(\ ),\dots,\varphi_{\tau,b_n}(\ )))\cdot e^{\int_\tau^t Tr(A(s))\cdot ds} =$$

$$= \underbrace{\det(\mathrm{col}(b_1,\dots,b_n))}_{=1}\cdot e^{\int_\tau^t Tr(A(s))\cdot ds}$$

baza canonică

Ecuații diferențiale
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$\frac{dx}{dt} = A(t)x$, $A(\cdot) : J \subset \mathbb{R} \to L(\mathbb{R}^n, \mathbb{R}^n)$ continuă

$S_{A(\cdot)} = \{ \varphi(\cdot) : J \to \mathbb{R}^n ;\ \varphi(\cdot)$ soluție $x' = A(t)x \}$

WRONSKIANUL asociat soluțiilor $\varphi_1(\cdot), \dots, \varphi_n(\cdot) \in S_{A(\cdot)}$

$$W_{\varphi_1, \dots, \varphi_n}(t) := \det[\,\mathrm{col}(\varphi_1(t), \dots, \varphi_n(t))] , \ t \in J$$

Teorema lui Liouville: $W_{\varphi_1, \dots, \varphi_n}(t) = W_{\varphi_1, \dots, \varphi_n}(\tau)\, e^{\int_\tau^t Tr(A(s))\cdot ds}$, $\forall\, t, \tau \in J$

Exercițiu: Fie ecuațiile:
$$\begin{cases} x' = e^t y - e^{2t} x \\ y' = (e^t - e^{3t})x + e^{2t} y \end{cases} , \quad \varphi_1(t) = \begin{pmatrix} 1 \\ e^t \end{pmatrix} \text{ soluție}$$

$\varphi_2(\cdot) = ?$ astfel încât $\{\varphi_1(\cdot), \varphi_2(\cdot)\}$ sistem fundamental de soluții.

[$S_{A(\cdot)} \subset C^1(J, \mathbb{R}^n)$ subspațiu vectorial $\dim(S_{A(\cdot)}) = n$.
$\{\varphi_1(\cdot), \dots, \varphi_n(\cdot)\} \subset S_{A(\cdot)}$ bază se numește sistem fundamental de soluții
$\varphi(\cdot) \in S_{A(\cdot)} \iff \exists\, c_i \in \mathbb{R}$ a.î. $\varphi(t) = \sum_{i=1}^{n} c_i \varphi_i(t)$ soluția generală

Fie $\varphi_2(t) = \begin{pmatrix} a(t) \\ b(t) \end{pmatrix}$; $A(t) = \begin{pmatrix} -e^{2t} & e^t \\ e^t - e^{3t} & e^{2t} \end{pmatrix}$

φ_2 soluție $\Rightarrow \begin{cases} a'(t) = e^t b(t) - e^{2t} a(t) \\ b'(t) = (e^t - e^{3t})\cdot a(t) + e^{2t}\cdot b(t) \end{cases}$

$W(t) = \det[\,\mathrm{col}(\varphi_1(t), \varphi_2(t))] = \det\begin{pmatrix} 1 & a(t) \\ e^t & b(t) \end{pmatrix} = b(t) - e^t a(t)$

Din T. Liouville $\Rightarrow W(t) = W(\tau)\cdot e^{\int_\tau^t Tr(A(s))\cdot ds} = W(\tau)\cdot e^0 = W(\tau)$, $\forall\, t, \tau \in J$ $\Rightarrow$

$\Rightarrow \exists\, c \in \mathbb{R}$ a.î. $W(t) = c \iff b(t) - e^t a(t) = c \Rightarrow b(t) = c + e^t a(t)$

$a'(t) = e^t \cdot c + \cancel{e^{2t} a(t)} - \cancel{e^{2t} a(t)} = e^t \cdot c$

$a(t) = \int e^t \cdot c\, dt = c\cdot e^t + k$, $k \in \mathbb{R}$. , $k \in \mathbb{R}$

$b(t) = c + e^t(c\cdot e^t + k) = c + c\cdot e^{2t} + k\cdot e^t$, $k \in \mathbb{R}$

$\varphi_2(t) = \begin{pmatrix} ce^t + k \\ c + c\cdot e^{2t} + k\cdot e^t \end{pmatrix}$, $k \in \mathbb{R}$ $\to$ soluția generală

Exercițiu: Fie ecuațiile $\begin{cases} x' = y - tx \\ y' = (1-t^2)x + ty \end{cases}$. Să se determine soluția generală.

$y = x' + tx$

$y(t) = x'(t) + t \cdot x(t)$

$\left(x'(t) + t \cdot x(t)\right)' = (1-t^2)\, x(t) + t \cdot \left(x'(t) + t \cdot x(t)\right)$

$x''(t) + \cancel{x(t)} + \cancel{x'(t)\, t} = \cancel{x(t)} - \cancel{t^2 x(t)} + \cancel{t\, x'(t)} + \cancel{t^2 \cdot x(t)}$

$x''(t) = 0 \Rightarrow \exists\, c \in \mathbb{R}$ a.î. $x'(t) = c$, $c \in \mathbb{R}$

$x(t) = ct + c_2$, $c, c_2 \in \mathbb{R}$

$y(t) = c + ct^2 + c_2 t$, $c, c_2 \in \mathbb{R}$

deoarece urma matricei $= 0$

Exercițiu: Fie ecuațiile $\begin{cases} x' = y + tx \\ y' = (1-t^2)x + ty \end{cases}$

$x' = y + tx \Rightarrow y(t) = x'(t) - t\, x(t)$

$\left(x'(t) - t\, x(t)\right)' = (1-t^2)\, x + t\left(x'(t) - t\, x(t)\right)$

$x''(t) - x(t) - t\, x'(t) = x - t^2 x + t\, x'(t) - t^2 x(t)$

$x''(t) - 2t\, x'(t) + (2t^2 - 2)\, x(t) = 0$

$x'' - 2t\, x' + (2t^2 - 2)\, x = 0$

Exercițiu: Fie ecuațiile $\begin{cases} x' = y - tx \\ y' = (1-t^2)x + ty \end{cases}$, $\varphi_1(t) = \begin{pmatrix} 1 \\ t^2+1 \end{pmatrix}$ soluție. Să se găsească $\varphi_2(\cdot) = ?$ astfel încât $\{\varphi_1(\cdot), \varphi_2(\cdot)\}$ sistem fundamental de soluții.

$\varphi_2(t) = \begin{pmatrix} a(t) \\ b(t) \end{pmatrix}$, $A(t) = \begin{pmatrix} -t & 1 \\ 1-t^2 & t \end{pmatrix}$

$\varphi_2(t)$ soluție $\Rightarrow \begin{cases} a'(t) = b(t) - t\, a(t) \\ b'(t) = (1-t^2)\, a(t) + t \cdot b(t) \end{cases}$

T. Liouville $\Rightarrow W(t) = W(\tau) \cdot e^{\int_\tau^t Tr(A(s))\, ds} = W(\tau)\, e^0 = W(\tau),\ \forall t, \tau \Rightarrow$

$\Rightarrow \exists\, c \in \mathbb{R}$ astfel încât $W(t) = c \Leftrightarrow t\, b(t) - (t^2+1)\cdot a(t) = c \Rightarrow$

$\Rightarrow b(t) = \dfrac{c + (t^2+1)\cdot a(t)}{t}, \quad t > 0$

$a'(t) = \dfrac{c + (t^2+1)\cdot a(t)}{t} - t \cdot a(t)$

$a'(t) = \dfrac{c + (t^2+1)\cdot a(t) - t^2 \cdot a(t)}{t} \Leftrightarrow a'(t) = \dfrac{c}{t} + \dfrac{a(t)}{t}$

$a' = \dfrac{a}{t} + \dfrac{c}{t} \rightarrow$ ecuație afină scalară (rez. prin metoda variației constantelor)

$\bar{a}' = \dfrac{\bar{a}}{t}$

$\bar{a}(t) = k \cdot e^{\int \frac{1}{t}\, dt} = k \cdot e^{\ln|t|} = k \cdot t, \quad k \in \mathbb{R}$

$a(t) = k(t)\cdot t \Rightarrow \left(k(t) \cdot t\right)' = \dfrac{k(t)\cdot t}{t} + \dfrac{c}{t}$

$k'(t)\cdot t + k(t)\cdot t' = k(t) + \dfrac{c}{t}$

$k'(t)\cdot t + \cancel{k(t)} = \cancel{k(t)} + \dfrac{c}{t}$

$k'(t) = \dfrac{c}{t^2} \Rightarrow k(t) = \int \dfrac{c}{t^2}\, dt = c \int \dfrac{1}{t^2}\, dt =$

$= -\dfrac{c}{t} + d, \quad d \in \mathbb{R} \Rightarrow \boxed{a(t) = -c + dt}$

$b(t) = \dfrac{c}{d} + \dfrac{t^2+1}{t}(-c + dt) = \dfrac{c}{d} - \dfrac{c(t^2+1)}{t} + d(t^2+1) = -ct + dt^2 + d \Rightarrow$

$\Rightarrow \varphi_2(t) = \begin{pmatrix} -c + dt \\ -ct + dt^2 + d \end{pmatrix}, \quad c, d \in \mathbb{R}$

$\varphi_2(t) = c\underbrace{\begin{pmatrix} -1 \\ -t \end{pmatrix}}_{\varphi_2(t)} + d\underbrace{\begin{pmatrix} t \\ 1 + t^2 \end{pmatrix}}_{\varphi_1(t)}$

Exercițiu: $J = ?$, $A(\cdot) = ?$, $A(\cdot) : J \to L(\mathbb{R}^2, \mathbb{R}^2)$ astfel încât $\varphi_1(\cdot), \varphi_2(\cdot) \in S_{A(\cdot)}$, unde $\varphi_1(t) = \begin{pmatrix} 1 \\ t^2 \end{pmatrix}$ și $\varphi_2(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Metoda 1:

Fie $A(t) = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix} \rightarrow$ sistemul asociat este $\begin{cases} x' = a(t)x + b(t)y \\ y' = c(t)x + d(t)\cdot y \end{cases}$

φ_1 soluție $\Rightarrow \begin{cases} 1' = a(t) + t^2 \cdot b(t) \end{cases}$ $\Leftrightarrow \begin{cases} 0 = a(t) + t^2\, b(t) \end{cases}$

T. Liouville $\Rightarrow W(t) = W(\tau)\cdot e^{\int_\tau^t Tr(A(s))\,ds} = W(\tau)\, e^{0} = W(\tau)$, $\forall\, t,\tau \Rightarrow$

$\Rightarrow \exists\, c\in\mathbb{R}$ astfel încât $W(t) = c \Leftrightarrow t\,b(t) - (t^2+1)\cdot a(t) = c \Rightarrow$

$$\Rightarrow b(t) = \frac{c + (t^2+1)\cdot a(t)}{t}, \quad t>0$$

$$a'(t) = \frac{c + (t^2+1)\cdot a(t)}{t} - t\cdot a(t)$$

$$a'(t) = \frac{c + (t^2+1)\cdot a(t) - t^2\cdot a(t)}{t} \Leftrightarrow a'(t) = \frac{c}{t} + \frac{a(t)}{t}$$

$a' = \frac{a}{t} + \frac{c}{t} \rightarrow$ ecuație afină scalară (rez. prin metoda variației constantelor)

$$\bar{a}' = \frac{\bar{a}}{t}$$

$$\bar{a}(t) = k\cdot e^{\int \frac{1}{t}\,dt} = k\cdot e^{\ln|t|} = k\cdot t, \quad k\in\mathbb{R}$$

$$a(t) = k(t)\cdot t \Rightarrow \big(k(t)\cdot t\big)' = \frac{k(t)\cdot t}{t} + \frac{c}{t}$$

$$k'(t)\cdot t + k(t)\cdot t' = k(t) + \frac{c}{t}$$

$$k'(t)\cdot t + \cancel{k(t)} = \cancel{k(t)} + \frac{c}{t}$$

$$k'(t) = \frac{c}{t^2} \Rightarrow k(t) = \int \frac{c}{t^2}\,dt = c\int\frac{1}{t^2}\,dt =$$

$$= -\frac{c}{t} + d, \quad d\in\mathbb{R} \Rightarrow \boxed{a(t) = -c + dt}$$

$$b(t) = \frac{c}{d} + \frac{t^2+1}{t}\left(-c + dt\right) = \frac{c}{d} - \frac{c(t^2+1)}{t} + d(t^2+1) = -ct + dt^2 + d \Rightarrow$$

$$\Rightarrow \varphi_2(t) = \begin{pmatrix} -c + dt \\ -ct + dt^2 + d \end{pmatrix}, \quad c, d\in\mathbb{R}$$

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