TO:

FROM: Ecuații diferențiale – seminar – 14.11.2017

$\frac{dx}{dt} = A(t)x$ $\quad A(\cdot): I \subseteq \mathbb{R} \to L(\mathbb{R}^n, \mathbb{R}^n)$ continuă

$S_{A(\cdot)} := \{\varphi(\cdot): I \to \mathbb{R}^n,\ \varphi(\cdot)$ sol. $x' = A(t)x\}$

WRONSKIANUL asociat soluțiilor $\varphi_1(\cdot), \dots, \varphi_n(\cdot) \in S_{A(\cdot)}$

$W_{\varphi_1,\dots,\varphi_n}(t) := \det[\mathrm{col}(\varphi_1(t), \dots, \varphi_n(t))], t \in I$

T. Liouville: $W_{\varphi_1 \dots \varphi_n}(t) = W_{\varphi_1 \dots \varphi_n}(\tau)\, e^{\int_\tau^t Tr(A(s))\,ds}, \forall t, \tau \in I$

1) Fie ec. $\begin{cases} x' = e^t y - e^{2t} x \\ y' = (e^t - e^{3t})x + e^{2t} y \end{cases}$

$\varphi_2(\cdot) = ?$ a.î. $\{\varphi_1(\cdot), \varphi_2(\cdot)\}$ sistem fundamental de soluții

$S_{A(\cdot)} \subset C^1(I, \mathbb{R}^n)$ subsp. vect. $\dim(S_{A(\cdot)}) = n$

$\{\varphi_1(\cdot), \dots, \varphi_n(\cdot)\} \subset S_{A(\cdot)}$ bază s.n. sist. fundamental de soluții

$\varphi(\cdot) \in S_{A(\cdot)} \iff \exists\, c_i \in \mathbb{R}$ a.î. $\varphi(t) = \sum_{i=1}^{n} c_i \varphi_i(t)$ soluția generală

Fie $\varphi_2(t) = \begin{pmatrix} a(t) \\ b(t) \end{pmatrix}$

φ_2 soluție $\Rightarrow \begin{cases} a'(t) = e^t b(t) - e^{2t} a(t) \\ b'(t) = (e^t - e^{3t}) a(t) + e^{2t} b(t) \end{cases}$

$A(t) = \begin{pmatrix} -e^{2t} & e^t \\ e^t - e^{3t} & e^{2t} \end{pmatrix}$

$W(t) = \det[\mathrm{col}(\varphi_1(t), \varphi_2(t))] = \det \begin{pmatrix} 1 & a(t) \\ e^t & b(t) \end{pmatrix} = b(t) - e^t a(t)$

Th. Liouville $\Rightarrow W(t) = W(\tau) \cdot e^{\int_\tau^t Tr(A(s))\,ds} = W(\tau) \cdot e^{\int 0} = W(\tau), \forall t, \tau \in I$

$\Rightarrow \exists\, c \in \mathbb{R}$ a.î. $W(t) = c \iff b(t) - e^t \cdot a(t) = c \Rightarrow b(t) = e^t a(t) + c$

$a'(t) = e^t(c + e^t \cdot a(t)) - e^{2t} \cdot a(t) = e^t \cdot c + e^{2t} a(t) - e^{2t} \cdot a(t) = c \cdot e^t$

$a(t) = \int c \cdot e^t\, dt = c \cdot e^t + k,\ k \in \mathbb{R}$

$\Rightarrow b(t) = c + e^t(c \cdot e^t + k) = c + c \cdot e^{2t} + k \cdot e^t,\ t \in \mathbb{R}$

$\Rightarrow \begin{pmatrix} c \cdot e^t + k \\ c + c \cdot e^{2t} + k \cdot e^t \end{pmatrix}, k \in \mathbb{R}$ soluția generală

$\varphi(t) = c \cdot \underbrace{\begin{pmatrix} e^t \\ 1 + e^{2t} \end{pmatrix}}_{\varphi_2(t)} + k \cdot \underbrace{\begin{pmatrix} 1 \\ e^t \end{pmatrix}}_{\varphi_1(t)}, \quad c, k \in \mathbb{R}$

2) $\begin{cases} x' = y - tx \\ y' = (1 - t^2)x + ty \end{cases}$ $\qquad \varphi(t) = \begin{pmatrix} t \\ t^2 + 1 \end{pmatrix}$ sol

$\varphi_2(t) = ?$ a.î. $\{\varphi_1(\cdot), \varphi_2(\cdot)\}$ sistem fundamental de soluții

$\varphi_2(t) = \begin{pmatrix} a(t) \\ b(t) \end{pmatrix}$ $\qquad A(t) = \begin{pmatrix} -t & 1 \\ 1 - t^2 & t \end{pmatrix}$

φ_2 soluție $\Rightarrow a'(t) = b(t) - t a(t)$

$b'(t) = (1-t^2)a(t) + t b(t)$

$W(t) = \det(\mathrm{col}(\varphi_1(t), \varphi_2(t))) = \det\begin{pmatrix} t & a(t) \\ t^2+1 & b(t) \end{pmatrix} =$

$= t b(t) - (t^2+1)a(t) \overset{\text{Th Liouville}}{=\!=} W(t) = W(\tau) e^{\int_\tau^t Tr(A(s))ds} =$

$= W(\tau) e^0 = W(\tau) \ \forall t, \tau \Rightarrow \exists c \in \mathbb{R}$ a.î. $W(t) = c \ \forall t$

$\Rightarrow t b(t) - (t^2+1) a(t) = c \ \forall t$

$b(t) = \frac{c}{t} + \frac{t^2+1}{t} \cdot a(t)$, pp. $t > 0$

$a'(t) = \frac{c}{t} + \frac{(t^2+1)}{t} \cdot a(t) - t a(t) = \frac{c}{t} + \frac{a(t)}{t}$

$a' = \frac{a}{t} + \frac{c}{t}$ (ec. afină scalară)

$\bar{a}' = \frac{\bar{a}}{t}$

$\bar{a}(t) = k \cdot e^{\int \frac{1}{t} dt} = k e^{\ln|t|} = k \cdot t, \forall k \in \mathbb{R}$

$a(t) = k(t) t \Rightarrow (k(t) \cdot t)' = \frac{k(t) \cdot t}{t} + \frac{c}{t} \Rightarrow k'(t) \cdot t + k(t) = k(t) + \frac{c}{t}$

$\Rightarrow k'(t) = \frac{c}{t^2} \Rightarrow k(t) = \int \frac{c}{t^2} dt = -\frac{c}{t} + d$, $d \in \mathbb{R}$

$\Rightarrow a(t) = -c + d \cdot t$

$\Rightarrow b(t) = \frac{c}{t} + \left(\frac{t^2+1}{t}\right)(-c + d \cdot t) = \frac{c}{t} + \frac{-ct^2 - c + dt^3 + d \cdot t}{t} =$

$= -ct + dt^2 + d$

$\Rightarrow \varphi_2(t) = \begin{pmatrix} -c + d \cdot t \\ -ct + dt^2 + d \end{pmatrix}$, $c, d \in \mathbb{R}$

$\varphi_2(t) = -c \underbrace{\begin{pmatrix} 1 \\ t \end{pmatrix}}_{\varphi_2} + d \underbrace{\begin{pmatrix} t \\ t^2+1 \end{pmatrix}}_{\varphi_1}$

2) $\begin{cases} x' = y - tx \\ y' = (1-t^2)x + ty \end{cases}$

Să se det. sol. generală

$y = x' + tx$

$y(t) = x'(t) + tx(t)$

$(x'(t) + tx(t))' = (1-t^2)x(t) + t(x'(t) + tx(t))$

$x''(t) + tx'(t) + x(t) = x(t) - t^2 x(t) + tx'(t) + t^2 x(t)$

$x''(t) = 0 \Rightarrow x'(t) = c, c \in \mathbb{R}$

$\Rightarrow x(t) = ct + c_2$, $c, c_2 \in \mathbb{R}$

$\Rightarrow y(t) = c + ct^2 + c_2 t$, $c, c_2 \in \mathbb{R}$

TO:

FROM:

pt $\begin{cases} x' = y + tx \\ y' = (1-t^2)x + ty \end{cases}$

$x' = y + tx \Rightarrow y(t) = x'(t) - tx(t)$

$(x'(t) - tx(t))' = (1-t^2)x(t) + t(x'(t) - tx(t))$

$x''(t) - x(t) - tx'(t) = (1-t^2)x(t) + tx'(t) - t^2 x(t)$

$x'' - 2tx' + (2t^2 - 2)x = 0$

3) $I = ?$ $A(\cdot) = ?$ $A(\cdot) : I \to L(\mathbb{R}^2, \mathbb{R}^2)$ a.î. $\varphi_1(\cdot), \varphi_2(\cdot) \in S_{A(\cdot)}$, unde $\varphi_1(t) = \begin{pmatrix} 1 \\ t^2 \end{pmatrix}$, $\varphi_2(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Fie $A(\cdot) = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix}$

$\Rightarrow \begin{cases} x' = a(t)x + b(t)y \\ y' = c(t)x + d(t)y \end{cases}$

φ_1 sol. $\Rightarrow \begin{cases} 0 = a(t) + b(t)t^2 \\ 2t = c(t) + d(t)t^2 \end{cases} \Rightarrow \begin{cases} 0 = a(t) + b(t) \\ 0 = c(t) + d(t) \end{cases} \Rightarrow \begin{cases} b(t)(t^2-1) = 0 \\ 2t = d(t)t^2 - d(t) = d(t)(t^2-1) \end{cases}$

$t^2 - 1 \neq 0 \Rightarrow t \neq \pm 1$ sau $I = (-\infty, -1)$
sau $I = (-1, 1)$
sau $I = (1, +\infty)$

$\Rightarrow \begin{cases} b(t) = 0 \\ d(t) = \dfrac{2t}{t^2-1} \end{cases}$

$a(t) = 0$

$c(t) = -d(t) = \dfrac{-2t}{t^2-1}$

$A(t) = \begin{pmatrix} 0 & 0 \\ \dfrac{2t}{t^2-1} & \dfrac{-2t}{t^2-1} \end{pmatrix}$

$\dfrac{dx}{dt} = A(t)x$ $A(\cdot) : I \to L(\mathbb{R}^n, \mathbb{R}^n)$ cont.

Def: 1. S.n. matrice de soluții $X(t) = \mathrm{col}(\varphi_1(t), \dots, \varphi_m(t))$ unde $\varphi_1(\cdot), \dots, \varphi_m(\cdot) \in S_{A(\cdot)}$

2. $X(\cdot) : I \to M_{n,m}(\mathbb{R})$ s.n. soluție matriceală dacă $\exists\, B \subset \mathbb{R}^n$ bază cu $X'(t) = A_B(t)X(t)$

PROP: $X(\cdot)$ matrice de soluții $\Leftrightarrow$ $X(\cdot)$ soluție matriceală

$\varphi_1(\cdot), \varphi_2(\cdot)$ sol. $\Rightarrow X(t) = \mathrm{col}(\varphi_1(t), \varphi_2(t))$ matrice de soluții

$\Rightarrow X$ sol. matriceală $\Rightarrow X'(t) = A(t)X(t) \quad | X^{-1}(t)$

Alegem I a.î. pt. $t \in I$ $\exists\, X^{-1}(t)$

$X(t) = \begin{pmatrix} 1 & 1 \\ t^2 & 1 \end{pmatrix}$ $\det X(t) = 1 - t^2 \neq 0$ $t \neq \pm 1 \Rightarrow I = (-\infty, -1)$ sau $I = (-1, 1)$ sau $I = (1, +\infty)$

$\Rightarrow A(t) = X'(t)X^{-1}(t) = \dots = \begin{pmatrix} 0 & 0 \\ \dots & \dots \end{pmatrix}$