TO:

FROM: Ecuații diferențiale - Seminar - 7.11.2017

$f(\cdot,\cdot): I \times \mathbb{R}^n \to \mathbb{R}^n$

DISIPATIVITATE (D) dacă $\exists r > 0$ $\exists a(\cdot): I \to \mathbb{R}_+$ continuă a.î. $|\langle x, f(t,x)\rangle| \le a(t)\|x\|^2$, $\forall t \in I$, $\forall x \in \mathbb{R}^n$, $\|x\| \ge r$

T(E.G.)

$f(\cdot,\cdot): I \times \mathbb{R}^n \to \mathbb{R}^n$ continuă în (D) $\frac{dx}{dt} = f(t,x)$. At. $\forall (t_0, x_0) \in I \times \mathbb{R}^n$ $\exists \varphi(\cdot): I \to \mathbb{R}^n$ sol. în $\varphi(t_0) = x_0$

1) Fie ec. $\begin{cases} x_1' = -x_2^2 \\ x_2' = x_1 x_2 \end{cases}$

a) Să se arate că $\forall \varphi$ sol. $\exists c \in \mathbb{R}$ a.î. $\|\varphi(t)\| = c$

b) — " — admite E.G.

a) Fie $\varphi(\cdot) = (\varphi_1(\cdot), \varphi_2(\cdot))$ soluție $\Rightarrow \begin{cases} \varphi_1'(t) = -\varphi_2^2(t) \\ \varphi_2'(t) = \varphi_1(t)\varphi_2(t) \end{cases}$ $\quad \|\varphi(t)\| = \sqrt{\varphi_1^2(t) + \varphi_2^2(t)}$

Fie $g(t) = \varphi_1^2(t) + \varphi_2^2(t)$

g derivabilă

$g'(t) = 2\varphi_1(t)\varphi_1'(t) + 2\varphi_2(t)\varphi_2'(t)$

$= -2\varphi_1(t)\varphi_2^2(t) + 2\varphi_1(t)\varphi_2^2(t) = 0 \Rightarrow \exists k \in \mathbb{R}$ a.î. $g(t) = k$

b) $f(t, (x_1, x_2)) = (-x_2^2, x_1x_2)$

$f: \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ continuă

ok

$|\langle x, f(t,x)\rangle| = |\langle (x_1,x_2), (-x_2^2, x_1x_2)\rangle| = |-x_1x_2^2 + x_1x_2^2| = 0 \Rightarrow (D)$

$\xRightarrow{t(EG)}$ EG pe $\mathbb{R} \times \mathbb{R}^2$

2) Fie ec. $\frac{dx_i}{dt} = \sum_{j,k=\overline{1,n}}^{n} c_{ijk} x_j x_k$, $i = \overline{1,n}$

$c_{i,j,k} = -c_{kji}$ $\forall i,j,k \in \{1,\dots,n\}$

a) $\forall \varphi(\cdot)$ sol. $\exists c \in \mathbb{R}$ a.î. $\|\varphi(t)\| = c$

b) Admite E.G.

3) Fie ec. $\begin{cases} x_1' = x_1 x_2 \\ x_2' = -2x_1^4 \end{cases}$

a) $\forall \varphi(\cdot) = (\varphi_1(\cdot), \varphi_2(\cdot))$ sol. $\exists c \in \mathbb{R}$ $\varphi_1^4(t) + \varphi_2^2(t) = c$

b) A admite EG

a) Fie $\varphi(\cdot) = (\varphi_1(\cdot), \varphi_2(\cdot))$ sol.

$\varphi_1'(t) = \varphi_1(t)\varphi_2(t)$

$\varphi_2'(t) = -2\varphi_1^4(t)$

Fie $g(t) = \varphi_1^4(t) + \varphi_2^2(t)$ continuă

$g'(t) = 4\varphi_1^3(t)\cdot\varphi_1'(t) + 2\varphi_2(t)\cdot\varphi_2'(t) = \dots = 0$

$\Rightarrow \exists c \in \mathbb{R}$ a.î. $g(t) = c$

$f(t,(x_1,x_2)) = (x_1x_2, -2x_1^4)$, $f: \underbrace{\mathbb{R}\times\mathbb{R}^2}_{\text{d}} \to \mathbb{R}^2$ cont.

$|\langle x, f(t,x)\rangle| = |(x_1,x_2),(x_1x_2,-2x^4)| = |x_1^2x_2 - 2x_1^4x_2| \leq a(t)\cdot \|x\|^2$

$= a(t)(x_1^2+x_2^2)$ $\forall t\in\mathbb{R}, x\in\mathbb{R}^2, \|x\|\geq r$

Pp. că f are (B) $\Rightarrow \exists r>0$ a.î. $\exists a(\cdot): \mathbb{R}\to\mathbb{R}_+$ continuă a.î.

Fie $t=0$

$|x_1^2x_2 - 2x_1^4x_2| \leq a(0)(x_1^2+x_2^2)$, $\forall x\in\mathbb{R}^2 ...$

$x_n = (n, n)$

$\Rightarrow |n^3 - 2n^5| \leq a(0)\,2n^2$ $\forall$, $n\in\mathbb{N}$

$\Rightarrow |n - 2n^3| \leq 2a(0)$ $\forall$ $n > n_0$ abs

b) EG: $\forall (t_0,x_0)\in\mathbb{R}\times\mathbb{R}^2$ $\exists$ $\varphi(\cdot): \mathbb{R}\to\mathbb{R}^2$ sol., $\varphi(t_0)=x_0$

Fie $(t_0,x_0)\in\underbrace{\mathbb{R}\times\mathbb{R}^2}_{\text{deschisă}}$, f continuă $\xrightarrow{\text{Th Peano}}$ $\exists$ $\varphi_0(\cdot): I_0\to\mathbb{R}^2$ sol., $\varphi_0(t_0)=x_0$

$\xrightarrow[\text{Sol. maximale}]{\text{Th existența}}$ $\exists$ $\varphi: I\to\mathbb{R}^2$ sol. maximală, φ prelungire a lui φ_0 $\Rightarrow \varphi(t_0) = \varphi_0(t_0) = x_0$

PROP (Intervalul de def. al sol. maximale) $\Rightarrow$ I deschis, $I=(a,b)$

Arătăm că $I=\mathbb{R}$, $a=-\infty$, $b=+\infty$

Arătăm că $b=+\infty$

Pp. abs. că $b<+\infty$, $t_0\in(a,b)$? $D_0\subset\mathbb{R}\times\mathbb{R}^2$ compactă a.î. $(t_0,\varphi(t))\in D_0$ $\forall t\in[t_0,b)$

Teorema asupra prelungirii soluțiilor

$f(\cdot,\cdot): D = \overset{\circ}{D} \subseteq \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ cont. $\frac{dx}{dt} = f(t,x)$

$\varphi(\cdot): (a,b)\subset\mathbb{R}\to\mathbb{R}^n$ sol.

At.: 1. $\varphi(\cdot)$ admite o prelungire strictă ($\neq\varphi(\cdot)$) la dreapta $\Leftrightarrow$ $b<+\infty$, $\exists$ $t_0\in(a,b)$ $\exists$ $D_0\subset D$ compactă a.î.

$(t,\varphi(t))\in D_0$ $\forall t\in[t_0,b)$

2. ... st.

Dacă am găsit D_0 cu ac. prop., aplicăm T asupra prelungirii sol.

$\Rightarrow \varphi(\cdot)$ admite o prelungire la dr. abs (φ maximală)

$\overset{a)}{\Rightarrow}$ $\exists$ $c\in\mathbb{R}$ a.î. $\varphi_1^4(t)+\varphi_2^2(t) = c$ $\Rightarrow \varphi_1^4(t)\leq c, \forall t$ $\Rightarrow \varphi_1^2(t)\leq\sqrt{c}, \forall t$

$\varphi_2^2(t)\leq c$, $\forall t$

$\Rightarrow \varphi_1^2(t)+\varphi_2^2(t) \leq c+\sqrt{c}, \forall t \Rightarrow \sqrt{\varphi_1^2(t)+\varphi_2^2(t)} \leq \underbrace{\sqrt{c+\sqrt{c}}}_{=k}$, $\forall t$

$\Rightarrow \|\varphi(t)\|\leq k, \forall t \Rightarrow \varphi(t)\in \overline{B}_k(0)$

$D_0 = [t_0,b]\times\overline{B}_k(0)$

D_0 compactă

TO:

FROM:

5) Fie ec. $\begin{cases} x_1' = x_2 \\ x_2' = -3x_1^5 - 2x_1^3 \end{cases}$

a) $\forall\ \varphi(\cdot) = (\varphi_1(\cdot), \varphi_2(\cdot))$ sol. $\overset{at.}{\Rightarrow} \exists\ c \in \mathbb{R}$ a.î. $\varphi_1^6(t) + \varphi_1^4(t) + \varphi_2^2(t) \equiv c$

b) Admite EG