TO:

FROM: Ecuații diferențiale – 17.10. – curs

$\frac{dx}{dt} = f(t,x) \qquad f(\cdot,\cdot): D \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$

PROP (ec. integrală asociată unei ec. diferențiale)

$f(\cdot,\cdot): D \subseteq \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ cnt. $\frac{dx}{dt} = f(t,x)$

At. $\varphi(\cdot): I \subseteq \mathbb{R} \to \mathbb{R}^n$ graph $\varphi(\cdot) \subset D$ este sol. $\iff$
1. $\varphi(\cdot)$ cnt.
2. $\varphi(t) = \varphi(t_0) + \int_{t_0}^{t} f(s, \varphi(s))\,ds \quad t, t_0 \in I$

Th. Peano (Existența locală a unei sol.)

Fie $f(\cdot,\cdot): D = \overset{\circ}{D} \subseteq \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ cnt. $\frac{dx}{dt} = f(t,x)$

At. $f(\cdot,\cdot)$ admite E.L. pe D ($\forall (t_0,x_0) \in D \ \exists\ \varphi(\cdot): I_0 \in V_0(t_0) \to \mathbb{R}^n$ sol. cu $\varphi(t_0) = x_0$)

Dem: Fie $(t_0, x_0) \in D$

Planul dem.
1. $I_0 = [t_0 - a, t_0 + a]$ $a = ?$
2. $\varphi_m(\cdot): I_0 \to \mathbb{R}^n$ și ← egal mărginit, egal uniform continuu
   Arzela-Ascoli
3. Th. ~~Arzela - Ascoli~~ $\exists\ \varphi_{m_p}(\cdot) \xrightarrow{u} \varphi(\cdot)$, $\varphi(\cdot): I_0 \to \mathbb{R}^n$ continuă

1. $(t_0, x_0) \in D = \overset{\circ}{D} \Rightarrow \exists\ \delta, \gamma > 0$ a.î. $\bar{B}_\delta(t_0) \times \bar{B}_\gamma(x_0) \subset D$

$K := \max\limits_{(t,x) \in \bar{B}_\delta(t_0) \times \bar{B}_\gamma(x_0)} \| f(t,x) \|$

$K = 0 \Rightarrow f(t,x) \equiv 0 \Rightarrow \varphi(t) \equiv x_0$

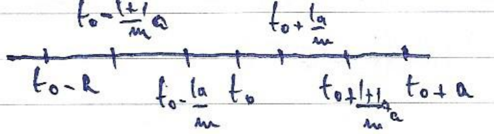
$K > 0$: $a := \min\{\delta, \frac{\gamma}{K}\} \qquad I_0 = [t_0 - a, t_0 + a]$

2. Șirul lui Tonelli

$m \geq 1 \quad \varphi_m(\cdot): I_0 \to \mathbb{R}^n$

$$\varphi_m(t) = \begin{cases} x_0, & t \in [t_0 - \frac{a}{m}, t_0 + \frac{a}{m}] \\ x_0 + \int_{t_0}^{t - \frac{a}{m}} f(s, \varphi_m(s))\,ds, & t \in [t_0 + \frac{a}{m}, t_0 + a] \\ x_0 + \int_{t_0}^{t + \frac{a}{m}} f(s, \varphi_m(s))\,ds, & t \in [t_0 - a, t_0 - \frac{a}{m}] \end{cases}$$

Arătăm că
a) $\varphi_m(\cdot)$ este bine definit
b) $\varphi_m(t) \in \bar{B}_\gamma(x_0)$, $\forall\, m \geq 1, \forall\, t \in I_0$
c) $\varphi_m(\cdot)$ este egal Lipschitz $\|\varphi_m(t') - \varphi_m(t'')\| \leq K|t' - t''|\ \forall\, t', t'' \in I_0$

a)


$l=1 \quad \varphi_m(t) \equiv x_0$ ok

$l \mapsto l+1$ Pp. de ex. că $t\in\left[t_0+\frac{l}{m}, t_0+\frac{l+1}{m}a\right] \Rightarrow t-\frac{a}{m}\in\left[t_0+\frac{l-1}{m}a, t_0+\frac{l}{m}\cdot a\right]$

$\Rightarrow \varphi_m(t)=x_0+\int_{t_0}^{t_0-\frac{a}{m}} f(s,\varphi_m(s))\,ds$ o.k.

$l=1 \quad \varphi_m(t)=x_0\in\bar{B}_r(x_0)$

$l\mapsto l+1$ Pp. că $t\in\left[t_0+\frac{l}{m}\cdot a, t_0+\frac{l+1}{m}\cdot a\right]$

$$\|\varphi_m(t)-x_0\| = \left\|x_0+\int_{t_0}^{t-\frac{a}{m}} f(s,\varphi_m(s))\,ds - x_0\right\| \le \left|\int_{t_0}^{t-\frac{a}{m}} \underbrace{\|f(s,\underbrace{\varphi_m(s)}_{\in\bar{B}_r(x_0)})\|}_{\le K}\,ds\right| \le K\left|t-\frac{a}{m}-t_0\right| \le$$

$$\le K\frac{l}{m}\cdot a \le K\cdot a \le K\cdot\frac{r}{K}=r$$

$\varphi_m(t)\in\bar{B}_r(x_0)\ \forall m\ \forall t \qquad \|\varphi_m(t)\|\le\|\varphi_m(t)-x_0\|+\|x_0\|\le r+\|x_0\|\ \forall m\ \forall t$

c) $t', t''\in I_0$ De ex. pp. $t',t''\in\left[t_0+\frac{a}{m}, t_0+a\right]$

$$\|\varphi_m(t')-\varphi_m(t'')\| = \left\|\cancel{x_0}+\int_{t_0}^{t'} f(s,\varphi_m(s))\,ds-\cancel{x_0}-\int_{t_0}^{t''} f(s,\varphi_m(s))\,ds\right\| =$$

$$=\left\|\int_{t''-\frac{a}{m}}^{t'-\frac{a}{m}} f(s,\varphi_m(s))\,ds\right\| \le \left|\int_{t''-\frac{a}{m}}^{t'-\frac{a}{m}} \overbrace{\|f(s,\underbrace{\varphi_m(s)}_{\in\bar{B}_r(x_0)})\|}^{\le K}\,ds\right| \quad \text{o.k.}$$

Analog celelalte cazuri.

3. Th. Arzela – Ascoli $\Rightarrow \exists\ \varphi_{m_p}(\cdot)\xrightarrow[p\to\infty]{u}\varphi(\cdot) \quad \varphi(\cdot): I_0\to\bar{B}_r(x_0)$ continuă

4. $$\varphi_{m_p}(t)=\begin{cases} x_0, & t\in\left[t_0-\frac{a}{m_p}, t_0+\frac{a}{m_p}\right] \\ x_0+\int_{t_0}^{t} f(s,\varphi_{m_p}(s))\,ds+\int_{t}^{t-\frac{a}{m_p}} f(s,\varphi_{m_p}(s))\,ds, & t\in\left[t_0+\frac{a}{m_p}, t_0+a\right] \\ x_0+\int_{t_0}^{t} f(s,\varphi_{m_p}(s))\,ds+\int_{t}^{t+\frac{a}{m_p}} f(s,\varphi_{m_p}(s))\,ds, & t\in\left[t_0-a, t_0-\frac{a}{m_p}\right] \end{cases}$$

$$\left.\begin{matrix}\varphi_{m_p}(\cdot)\xrightarrow{u}\varphi(\cdot)\\ f(\cdot,\cdot)\ \text{cont}\end{matrix}\right\}\Rightarrow \int_{t_0}^{t} f(s,\varphi_{m_p}(s))\,ds \longrightarrow \int_{t_0}^{t} f(s,\varphi(s))\,ds$$

$$\left\|\int_{t}^{t-\frac{a}{m_p}} f(s,\varphi_{m_p}(s))\,ds\right\| \le \left|\int_{t}^{t-\frac{a}{m_p}} \underbrace{\|f(s,\varphi_{m_p}(s))\|}_{\le K}\,ds\right| \le K\cdot\frac{a}{m_p}\xrightarrow[m_p\to\infty]{} 0$$

$m_p\to\infty \quad \varphi(t)=x_0+\int_{t_0}^{t} f(s,\varphi(s))\,ds \quad \forall t\in I_0$; $\left.\begin{matrix}\varphi(t)=\varphi(t_0)+\int_{t_0}^{t} f(s,\varphi(s))\,ds \quad \forall t\in I\\ \varphi(\cdot)\ \text{continuă}\end{matrix}\right| \Rightarrow$

$\Rightarrow$

PROP (ec. integrală asociată) $\Rightarrow \varphi(\cdot)$ soluția $\varphi(t_0)=x_0$, q.e.d.

TO:

FROM:

Obs $D=\mathring{D}$ $f(\cdot,\cdot)$ cont.

1) $x' = \mathcal{D}(t)$, $\mathcal{D}(t)=\begin{cases}1, & t\in \mathbb{Q}\\ 0, & t\notin \mathbb{Q}\end{cases}$ nu are E.L. în niciun punct

2) $x' = \operatorname{sgn} x$ $\quad \operatorname{sgn} x = \begin{cases}-1, & x<0\\ 0, & x=0\\ 1, & x>0\end{cases}$ are $\bar{E}$.U.L. în fiecare punct

Funcții local Lipschitz

Def: 1) $g(\cdot): \mathring{G} \subseteq \mathbb{R}^n \to \mathbb{R}^n$ s.n. Lipschitz (global) ~~dacă~~ pe G dacă $\exists\, L>0$ a.î. $\|g(x_1)-g(x_2)\| \le L\|x_1-x_2\|$

2) $g(\cdot): G \subseteq \mathbb{R}^n \to \mathbb{R}^n$ s.n. local Lipschitz în $x_0\in G$ dacă $\exists\, r>0$ și $L>0$ a.î. $\|g(x_1)-g(x_2)\| \le L\|x_1-x_2\|$ $\forall x_1,x_2 \in B_r(x_0)\cap G$

Obs: $g(\cdot)$ Lipschitz $\Rightarrow$ $g(\cdot)$ local Lipschitz $\Rightarrow$ $g(\cdot)$ continuă

Ex: $g(x)=x^2$ este local Lipschitz, dar nu Lipschitz (global)

$g(x)=x^{\frac{1}{3}}$ este continuă, dar nu local Lipschitz (în $x=0$)

PROP: $g: G=\mathring{G}\subseteq \mathbb{R}^n \to \mathbb{R}^n$. g local Lipschitz "$\Leftrightarrow$" $g(\cdot)|_{G_0}$ este Lipschitz (global) $\forall\, G_0$ compact $\subset G$

Dem: "$\Leftarrow$" Evident

"$\Rightarrow$" Pp. că $\exists\, G_0$ compact $\subset G$ a.î. $g(\cdot)|_{G_0}$ nu este Lipschitz

$\forall m\in\mathbb{N}$ $\exists\, x^1_m, x^2_m \in G_0$ a.î. $\|g(x^1_m)-g(x^2_m)\| > m\|x^1_m - x^2_m\|$

$\Rightarrow \|x^1_m - x^2_m\| < \frac{1}{m}\|g(x^1_m)-g(x^2_m)\| \le \frac{1}{m}\left(\|g(x^1_m)\|+\|g(x^2_m)\|\right) \le \frac{2k}{m}$ (*)

$g(\cdot)$ local Lipschitz $\Rightarrow$ $g(\cdot)$ continuă

G_0 compactă $\quad k := \max\limits_{x\in G_0}\|g(x)\|$

$x^1_m, x^2_m \in G_0$ compactă $\Rightarrow \exists\, x^1_{m_l} \to x^1 \in G_0$

$x^2_{m_l} \to x^2 \in G_0$

(*) $\|x^1_{m_l} - x^2_{m_l}\| \le \frac{2k}{m_l}$, $m_l \to \infty \Rightarrow \|x^1 - x^2\| \le 0 \Rightarrow x^1 = x^2 =: x_0$

g local Lipschitz în $x_0\in G$

$\Rightarrow \exists\, r>0\ \exists\, L>0$ a.î. $\|g(x_1)-g(x_2)\| \le L\|x_1-x_2\|$ $\forall x_1,x_2 \in B_r(x_0)$

$x^1_{m_l}, x^2_{m_l} \to x_0 \Rightarrow \exists\, l_0\in\mathbb{N}$, $l\ge l_0$, $x^1_{m_l}, x^2_{m_l} \in B_r(x_0)$

$\|g(x^1_{m_l}) - g(x^2_{m_l})\| \le L\|x^1_{m_l} - x^2_{m_l}\|$

(1) & (2) $\Longrightarrow m_l\|x^1_{m_l} - x^2_{m_l}\| \le L\|x^1_{m_l} - x^2_{m_l}\|$ $\quad m_l \to \infty$ ab

PROP: $g: G=\mathring{G}\subseteq \mathbb{R}^n \to \mathbb{R}^n$ derivabilă cu $Dg(\cdot)$ mărginită. At. $g(\cdot)$ este local Lipschitz.

Dem: Th. medii $\quad \|g(x)-g(y)\| \le \sup\limits_{z\in[x,y]}\|Dg(z)\|\cdot\|x-y\|$. $\forall x,y\in G$, $[x,y]\subset G$

$[x,y] = \{x+s(y-x);\ s\in[0,1]\}$

Fie $x_0 \in G = \overset{\circ}{G} \Rightarrow \exists$ ~~V~~ $\cancel{x_0}$ a.î. $\underbrace{\bar{B}_R(x_0)}_{m.\ convexă} \subset G$

$L := \sup\limits_{z \in B_R(x_0)} \|Dg(z)\|$

Corolar $g(\cdot): G = \overset{\circ}{G} \subseteq \mathbb{R}^n \to \mathbb{R}^n$ de clasă C^1. At. $g(\cdot)$ este local Lipschitz.

Funcții local Lipschitz în raport cu variabila a doua

Def: $f(\cdot,\cdot): D \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ s.n. local Lipschitz în rap. cu var. a doua (II) în $(t_0, x_0) \in D$ dacă $\exists D_0 \in V(t_0, x_0)$, $\exists L > 0$ a.î. $\|f(t, x_1) - f(t, x_2)\| \leq L\|x_1 - x_2\|$, $\forall (t, x_1), (t, x_2) \in D_0$

PROP: Fie $f(\cdot,\cdot): D = \overset{\circ}{D} \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ continuă. At. $f(\cdot,\cdot)$ local Lipschitz (II) (pe D)
"$\Leftrightarrow$" $\forall D_0 \subset D$ e compactă $\exists L > 0$ a.î. $\|f(t, x_1) - f(t, x_2)\| \leq L\|x_1 - x_2\|$ $\forall$ $(t, x_1), (t, x_2) \in D_0$

PROP: Fie $f(\cdot,\cdot): D = \overset{\circ}{D} \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ C^1(II) (de clasă C^1 în rap. cu var. a II-a)
$\left[\exists D_2 f(t,x) \left(\frac{\partial}{\partial x} f(t,x)\right)\right.$ și $\left.(t,x) \to D_2 f(t,x) \text{ continuă}\right]$. Atunci $f(\cdot,\cdot)$ local Lipschitz (II).

Lema Bellmann – Gronwall

$M \geq 0$, $u(\cdot), v(\cdot): I \subseteq \mathbb{R} \to \mathbb{R}_+$ continue, $t_0 \in I$. Dacă ~~u(t) ≤ M +~~
$u(t) \leq M + \left|\int_{t_0}^{t} u(s)v(s)\,ds\right|$ $\forall t \in I$. Atunci $u(t) \leq M \cdot e^{\left|\int_{t_0}^{t} v(s)\,ds\right|}$, $\forall t \in I$