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FROM: Ecuații diferențiale - 28.11.2017 - curs

Ecuații liniare pe $\mathbb{R}^n$ cu coeficienți constanți
Structura soluțiilor în cazul general

Lemă: $A \in L(\mathbb{R}^n, \mathbb{R}^n)$ $\frac{dx}{dt} = Ax$ $\lambda \in \sigma(A)$ $P_j \in \mathbb{R}^n$ $\tilde{\varphi}(t) := e^{\lambda t} \sum_{j=0}^{m-1} P_j t^j$

Atunci $\tilde{\varphi}'(t) \equiv A\tilde{\varphi}(t) \Leftrightarrow \begin{cases} (A - \lambda I_n)^m P_0 = 0 \\ P_j = \frac{1}{j!}(A - \lambda I_n)^j P_0 \quad j = \overline{1, m-1} \end{cases}$

Dem: $\tilde{\varphi}'(t) \equiv A\tilde{\varphi}(t)$

$$\lambda \cancel{e^{\lambda t}} \sum_{j=0}^{m-1} P_j t^j + \cancel{e^{\lambda t}} \sum_{j=1}^{m-1} j P_j t^{j-1} \equiv A \cancel{e^{\lambda t}} \sum_{j=1}^{m-1} P_j t^j$$

$$\lambda \sum_{j=0}^{m-1} P_j t^j + \sum_{j=0}^{m-2} (j+1) P_{j+1} t^j \equiv A \sum_{j=0}^{m-1} P_j t^j$$

$j = \overline{0, m-2}$ $\lambda P_j + (j+1) P_{j+1} = AP_j$

$j = m-1$ $\lambda P_{m-1} = AP_{m-1}$

$j = \overline{0, m-2}$ $P_{j+1} = \frac{1}{j+1}(A - \lambda I_n)P_j = \frac{1}{(j+1)} \cdot \frac{1}{j!}(A - \lambda I_n)^2 P_{j-1} - \ldots = \frac{1}{(j+1)!}(A - \lambda I_n)^{j+1} P_0$

~~$j = m-1$~~

$j = m-1$ $\left.\begin{matrix} (A - \lambda I_n) P_{m-1} = 0 \\ P_{m-1} = \frac{1}{(m-1)!}(A - \lambda I_n)^{m-1} P_0 \end{matrix}\right\} \Rightarrow \frac{1}{(m-1)!}(A - \lambda I_n)^m P_0 = 0$

Teorema asupra

$\forall A \in M_n(\mathbb{R})$ $\exists C \in M_n(\mathbb{C})$ $\det C \neq 0$ a.î. $C^{-1}AC = \begin{pmatrix} J_1 & 0 & \ldots & 0 \\ 0 & J_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & \ldots & J_p \end{pmatrix}$ unde

$J_r = \begin{pmatrix} \lambda_r & 1 & 0 & \ldots & 0 \\ 0 & \lambda_r & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \ldots & \lambda_r \end{pmatrix}$ - celula Jordan $\lambda_r \in \sigma(A)$

$S_{ri} = \dim J_r \leq$ multiplicitatea$(\lambda_r) = m$

$\sum_{\lambda_r = \lambda} \dim J_r = m_\lambda$

Forma echivalentă

$\forall A \in L(\mathbb{R}^n, \mathbb{R}^n)$ $\exists B_j = \{b^s_r \,;\, s = \overline{1, s_r},\ r = \overline{1, p}\} \subset \mathbb{R}^n$ bază canonică Jordan a.î.:

a) $\forall r = \{1, \ldots, p\}$ $\exists \lambda_r \in \sigma(A)$

$$\begin{cases} Ab^1_r = \lambda_r b^1_r + b^2_r \\ Ab^2_r = \lambda_r b^2_r + b^3_r \\ \ldots \\ Ab^{s_r-1}_r = \lambda_r b^{s_r-1}_r + b^{s_r}_r \\ Ab^{s_r}_r = \lambda_r b^{s_r}_r \end{cases}$$

b) $s_r \leq m_{\lambda_r}$ $\sum_{\lambda_r = \lambda} = m_\lambda$ $\forall \lambda \in \sigma(A)$

c) Dacă $\lambda_r \in \mathbb{R} \Rightarrow b^s_r \in \mathbb{R}^n$, dacă $\operatorname{Im} \lambda_r > 0$ și $\lambda_r = \bar{\lambda}_t$ at. $\bar{b}^s_r = b^{\bar{s}}_t$

(C1) $A \in L(\mathbb{R}^n, \mathbb{R}^n)$ $B_J \subset \mathbb{C}^n$ baza canonică Jordan

Atunci $B_J^{\mathbb{R}^n} = \{ b^s_r ; \lambda_r \in \mathbb{R}^n, s = \overline{1,s_r} \} \cup \{ \operatorname{Re}(b^s_r), \operatorname{Im}(b^s_r) ; \operatorname{Im} \lambda_r > 0, s = \overline{1,s_r} \}$

$\subset \mathbb{R}^n$ bază (urma pe $\mathbb{R}^n$ a bazei canonice Jordan)

Dem: n vectori — să dem. doar liniar independența

$$\sum_{\substack{\lambda_r \in \mathbb{R}^n \\ s = \overline{1,s_r}}} c^s_r b^s_r + \sum_{\operatorname{Im} \lambda_r > 0} \left[c^s_r \operatorname{Re}(b^s_r) + k^s_r \operatorname{Im}(b^s_r) \right] = 0 \overset{?}{\Longrightarrow} c^s_r = k^s_r = 0 \ \forall s, r$$

$$b^s_r = \operatorname{Re}(b^s_r) + i \operatorname{Im}(b^s_r) \qquad \operatorname{Re}(b^s_r) = \frac{1}{2}(b^s_r + \bar{b}^s_r)$$

$$\bar{b}^s_r = \operatorname{Re}(b^s_r) - i \operatorname{Im}(b^s_r) \qquad \operatorname{Im}(b^s_r) = \frac{1}{2i}(b^s_r - \bar{b}^s_r)$$

$$\sum_{\substack{\lambda_r \in \mathbb{R} \\ s = \overline{1,s_r}}} c^s_r b^s_r + \sum_{\substack{\operatorname{Im} \lambda_r > 0 \\ s = \overline{1,s_r}}} \left[c^s_r \frac{1}{2} \left[b^s_r + \bar{b}^s_r \right] + k^s_r \frac{1}{2i} (b^s_r - \bar{b}^s_r) \right] = 0$$

$$\sum_{\substack{\lambda_r \in \mathbb{R} \\ s = \overline{1,s_r}}} c^s_r b^s_r + \sum_{\substack{\operatorname{Im} \lambda > 0 \\ s = \overline{1,s_r}}} \left[\left(\frac{c^s_r}{2} + \frac{k^s_r}{2i} \right) b^s_r \right] = 0 \underset{\text{bază}}{\overset{B_J}{\Longrightarrow}} \begin{cases} c^s_r = 0 & \lambda_r \in \mathbb{R} \\ \dfrac{c^s_r}{2} + \dfrac{k^s_r}{2i} = 0 & \operatorname{Im} \end{cases} \quad \ldots$$

(C2) $A \in L(\mathbb{R}^n, \mathbb{R}^n)$ $B_J \subset \mathbb{C}^n$ baza canonică Jordan $\lambda \in \sigma(A)$

$B^\lambda_J := \{ b^s_r ; \lambda_r = \lambda, s = \overline{1,s_r} \}$

Atunci 1) $\operatorname{card}(B^\lambda_J) = m_\lambda$

2) $(A - \lambda I_n)^{m_\lambda} b^s_r = 0 \ \forall \, b^s_r \in B^\lambda_J$

3) $\operatorname{Im} \lambda > 0 \Rightarrow \{ \operatorname{Re}(b^s_r), \operatorname{Im}(b^s_r) ; s = \overline{1,s_r} \} \subset \mathbb{R}^n$ liniar indep.

Dem. 1) & 3) imediat din T

$$A b^1_r = \lambda b^1_r + b^2_r \qquad \Rightarrow b^2_r = (A - \lambda I_n) b^1_r$$

$$A b^2_r = \lambda b^2_r + b^3_r \qquad \Rightarrow b^3_r = (A - \lambda I_n) b^2_r = (A - \lambda I_n)^2 b^1_r$$

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$$\begin{cases} A b^{s_r - 1}_r = \lambda b^{s_r - 1}_r + b^{s_r}_r & \Rightarrow b^{s_r}_r = (A - \lambda I_n) b^{s_r - 1}_r = \ldots = (A - \lambda I_n)^{s_r - 1} b^1_r \\ A b^{s_r}_r = \lambda b^{s_r}_r & \Rightarrow (A - \lambda I_n) b^{s_r}_r = 0 \end{cases}$$

$$(A - \lambda I_n)(A - \lambda I_n)^{s_r - 1} b^1_r = 0$$

$$(A - \lambda I_n)^{s_r} b^1_r = 0 \qquad s_r \leq m_\lambda$$

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$\Rightarrow (A-\lambda I_n)^{m_\lambda} b_r^1 = 0$

$(A-\lambda I_n)^{m_\lambda} b_r^s = (A-\lambda I_n)^{m_\lambda}(A-\lambda I_n)^{s-1} b_r^1 = (A-\lambda I_n)^{m_\lambda + s-1} b_r^1 = 0$

<u>Th</u> (Structura sol. în cazul general)

$A \in L(\mathbb{R}^n, \mathbb{R}^n)$ $\quad \frac{dx}{dt} = Ax$

Atunci: $\quad 1.\ \forall \lambda \in \sigma(A) \cap \mathbb{R}$ $\quad m_\lambda = m \geq 1$ $\ \exists\, p_j^{\lambda l} \in \mathbb{R}^n$ $\ \varphi_{\lambda l} = e^{\lambda t} \sum_{j=0}^{m-1} p_j^{\lambda l} t^j, \ l = \overline{1,m}$

$\{\varphi_{\lambda l}(\cdot)\}_{l=\overline{1,m}}$ liniar independente $\subset S_A$

$2.\ \forall \lambda \in \sigma(A)$ $\ \operatorname{Im}\lambda > 0$ $\ m_\lambda = m \geq 1$, $\ \exists\, p_j^{\lambda l} \in \mathbb{C}^n$ a.î.

$\varphi_{\lambda l}(t) = \operatorname{Re}\left(e^{\lambda t} \sum_{j=0}^{m-1} p_j^{\lambda l} t^j\right)$

$\varphi_{\bar\lambda l} = \operatorname{Im}\left(e^{\lambda t} \sum_{j=0}^{m-1} p_j^{\lambda l} t^j\right)$ $\quad l = \overline{1,m}$

$3.\ \{\varphi_{\lambda l}(\cdot)\}_{\substack{\lambda \in \sigma(A) \\ l = \overline{1,m_\lambda}}} \subset S_A$ sistem fundamental de soluții

<u>Dem</u>: $A \in L(\mathbb{R}^n, \mathbb{R}^n) \longrightarrow B_J = \{b_r^s,\ s = \overline{1,s_r},\ r = \overline{1,p}\} \subset \mathbb{R}^n$ bază canonică Jordan

Fie $B_J^\lambda = \{b_r^s,\ \lambda_r = \lambda,\ s = \overline{1,s_r}\} \overset{\text{not.}}{=} \{p_0^{\lambda 1}, \dots, p_0^{\lambda m}\}$ liniar indep.

$p_j^{\lambda l} := \frac{1}{j!}(A-\lambda I_n)^j p_0^{\lambda l}$ $\quad j = \overline{1,m-1}$ $\quad l = \overline{1,m}$

(C2) $(A-\lambda I_n)^m p_0^{\lambda l} = 0$ $\quad l = \overline{1,m}$

Fie $\varphi_{\lambda l}(t) = e^{\lambda t} \sum_{j=0}^{m-1} p_j^{\lambda l} t^j$

$\overset{\text{Lema}}{\Longrightarrow} \varphi_{\lambda l}'(t) = A\varphi_{\lambda l}(t) \Rightarrow \varphi_{\lambda l}(\cdot) \in S_A \ \forall\, l = \overline{1,m}$

$\{\varphi_{l\lambda}(\cdot)\}\ l = \overline{1,m} \subset S_A$ liniar independente

$\Updownarrow$ PROP (Sol. liniar independente)

$\{\varphi_{\lambda l}(0)\}_{l=\overline{1,m}} \subset \mathbb{R}^n$ liniar indep.

$\{p_0^{\lambda l}\}\ l = \overline{1,m}$ liniar indep.

$2.\ \lambda \in \sigma(A)$ $\ \operatorname{Im}\lambda > 0$ $\quad m_\lambda = m \geq 1$

Fie $B_J^\lambda := \{b_r^s;\ \lambda_r = \lambda,\ s = \overline{1,s_r}\} = \{p_0^{\lambda l}\}_{l=\overline{1,m}}$

$p_j^{\lambda l} := \frac{1}{j!}(A-\lambda I_n)^j p_0^{\lambda l}$ $\quad j = \overline{1,m-1}$ $\quad l = \overline{1,m}$ $\xrightarrow[\text{Lema}]{+(C2)} \tilde\varphi_{\lambda l}'(t) = A\tilde\varphi_{\lambda l}(t)$

$\tilde\varphi_{\lambda l}(t) = \underbrace{\operatorname{Re}(\tilde\varphi_{\lambda l}(t))}_{\varphi_{\lambda l}(t)} + i\,\underbrace{\operatorname{Im}(\tilde\varphi_{\lambda l}(t))}_{\varphi_{\bar\lambda l}(t)} = \varphi_{\lambda l}(t) + i\,\varphi_{\bar\lambda l}(t)$

$\xrightarrow{+(C2)} \varphi'_{\lambda l}(t) + i\,\varphi'_{\bar\lambda l}(t) = A\,\varphi_{\lambda l}(t) + i\,A\,\varphi_{\bar\lambda l}(t)$

$\Rightarrow \begin{array}{l} \varphi'_{\lambda l}(t) \equiv A\,\varphi_{\lambda l}(t) \\ \varphi'_{\bar\lambda l}(t) \equiv A\,\varphi_{\bar\lambda l}(t) \end{array} \Rightarrow \varphi_{\lambda l}(\cdot),\ \varphi_{\bar\lambda l}(\cdot) \in S_A \quad \forall\, l = \overline{1,m}$

$\{\varphi_{\lambda l}(\cdot), \varphi_{\bar\lambda l}(\cdot)\}_{l=\overline{1,m}} \subset S_A$ liniar indep.

$\Updownarrow$ PROP (sol. liniar indep.)

$\{\varphi_{\lambda l}(0), \varphi_{\bar\lambda l}(0)\}_{l=\overline{1,m}} \subset \mathbb{R}^n$ liniar indep.

$\{Re(p_0^{\lambda l}), Im(p_0^{\lambda l})\}_{l=\overline{1,m}}$

$\{Re(b_\lambda^l), Im(b_\lambda^l)\}_{l=\overline{1,m}} \subset \mathbb{R}^n$ liniar indep. (C1), (C2)

3. n soluții $\Rightarrow$ e suficient să verif. doar liniar independența

$\{\varphi_{\lambda l}(\cdot)\}_{\lambda\in\sigma(A),\ l=\overline{1,m_\lambda}} \subset S_A$ liniar independente

$\Updownarrow$ PROP (sol. independente)

$\{\varphi_{\lambda l}(0)\}_{\lambda\in\sigma(A),\ l=\overline{1,m_\lambda}} \subset \mathbb{R}^n$ liniar independenți

$\{b_r^s;\ \lambda_r \in \mathbb{R},\ s=\overline{1,s_r}\} \cup \{Re(b_r^s), Im(b_r^s),\ Im\,\lambda_r > 0,\ s=\overline{1,s_r}\} \subset \mathbb{R}^n$ liniar indep. (C1)

<u>Algoritm</u> $\quad \frac{dx}{dt} = Ax$

1. Se rezolvă ec. caracteristică $\det(A - \lambda I_n) = 0 \longrightarrow \sigma(A) : (\lambda, m_\lambda)$
2. Dacă $\lambda \in \sigma(A) \cap \mathbb{R}$, $m_\lambda = 1$ se caută $u_\lambda \in \mathbb{R}^n \setminus \{0\}$ a.î. $(A - \lambda I_n)u_\lambda = 0$
3. Dacă $\lambda \in \sigma(A) \cap \mathbb{R}$, $m_\lambda = m > 1$ căutăm $\{p_0^{\lambda 1}, \dots, p_0^{\lambda m}\} \subset \ker(A - \lambda I_n)^m$ liniar indep. (în $\mathbb{R}^n$)

   Scrie $p_j^{\lambda l} = \frac{1}{j!}(A - \lambda I_n)^j p_0^{\lambda l} \quad j = \overline{1, m-1},\ l = \overline{1,m}$

   Scrie sol. $\quad \varphi_{\lambda l}(t) = e^{\lambda t} \sum_{j=0}^{m-1} p_j^{\lambda l} t^j \quad l = \overline{1,m}$

4. Dacă $\lambda = \alpha + i\beta \in \sigma(A)$, $\beta > 0$, $m_\lambda = 1$

   Caută $u_\lambda \in \mathbb{C}^n \setminus \{0\}$ a.î. $(A - \lambda I_n)u_\lambda = 0$

   Scrie 2 sol. $\quad \varphi_\lambda(t) = Re(e^{\lambda t}, u_\lambda)$

   $\varphi_{\bar\lambda}(t) = Im(e^{\lambda t}, u_\lambda)$

TO:

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5. Dacă $\lambda = \alpha + i\beta \in \sigma(A)$, $\beta > 0$, $m_\lambda = m \geq 1$
Caută $\{p_0^{\lambda 1}, \dots, p_0^{\lambda m}\} \subset \ker(A - \lambda I_n)^m$ liniar indep. (în $\mathbb{C}^n$)
Scrie $p_j^{\lambda l} = \frac{1}{j!}(A - \lambda I_n)^j p_0^{\lambda l}$ $j = \overline{1, m-1}$, $l = \overline{1, m}$

Scrie sol. $\varphi_{\lambda l}(t) = \mathrm{Re}\left(e^{\lambda t} \cdot \sum_{j=0}^{m-1} p_j^{\lambda l} t^j\right)$ $l = 1, m$

$$\varphi_{\bar{\lambda} l}(t) = \mathrm{Im}\left(e^{\lambda t} \cdot \sum_{j=0}^{m-1} p_j^{\lambda l} t^j\right)$$

6. Renumerotează $\{\varphi_{\lambda l}(\cdot)\}_{\substack{\lambda \in \sigma(A) \\ l = \overline{1, m_\lambda}}} = \{\varphi_1(\cdot), \dots, \varphi_n(\cdot)\}$ sistem fundamental de soluții

Scrie soluția generală $\varphi(t) = \sum_{i=1}^{n} c_i \varphi_i(t)$ $c_i \in \mathbb{R}, i = \overline{1, n}$

Ecuații afine pe $\mathbb{R}^n$

Def: $A(\cdot) : I \subseteq \mathbb{R} \to L(\mathbb{R}^n, \mathbb{R}^n)$ $b(\cdot) : I \to \mathbb{R}^n$ def. ec. afină $\frac{dx}{dt} = A(t)x + b(t)$
În coordonate $(0 \subset \mathbb{R}^n)$ bază $A_0(t) = (a_{ij}(t))_{i,j = \overline{1,n}}$ $b(t) = \begin{pmatrix} b_1(t) \\ \vdots \\ b_n(t) \end{pmatrix}$

$\frac{dx_i}{dt} = \sum_{j=1}^{n} a_{ij}(t) x_j + b_i(t), i = \overline{1, n}$ Sistem ec. afine

Dacă $n = 1$ $L(\mathbb{R}, \mathbb{R}) \simeq \mathbb{R} \to \underbrace{x' = a(t)x + b(t)}_{\text{ec. afină scalară}}$ $a(\cdot), b(\cdot) : I \subseteq \mathbb{R} \to \mathbb{R}$

Principiul (Metoda) variației constantelor

Th. (E.U.G.): Fie $A(\cdot) : I \to L(\mathbb{R}^n, \mathbb{R}^n)$, $b(\cdot) : I \to \mathbb{R}^n$ continue $\frac{dx}{dt} = A(t)x + b(t)$
Atunci $\forall (t_0, x_0) \in I \times \mathbb{R}^n$ $\exists!$ $\varphi(\cdot) : I \to \mathbb{R}^n$ soluție cu $\varphi(t_0) = x_0$

Dem: $f_{A(\cdot), b(\cdot)}(t, x) = A(t)x + b(t)$

$f_{A(\cdot), b(\cdot)}$ continuă, local Lipschitz (II) $\xrightarrow[\text{C-L}]{\text{Th}}$ $\underbrace{\text{E.U.L.}}_{\substack{\Downarrow T \\ \text{U.G.}}}$ pe $I \times \mathbb{R}^n$

$\|f_{A(\cdot), b(\cdot)}(t, x)\| =$

$= \|A(t)x + b(t)\| \leq \|A(t)\| \cdot \|x\| + \|b(t)\|$ adică C.A. $\Rightarrow \mathcal{D} \xrightarrow[\text{E.G.}]{T}$ E.G.

$S_{A(\cdot), b(\cdot)} := \{\varphi(\cdot) : I \to \mathbb{R}^n ; \varphi(\cdot) \text{ soluție } x' = A(t)x + b(t)\}$