TO:

FROM: Ecuații diferențiale - 12.12.2017 - Seminar

Algoritm (Ecuații afine de ordin superior):

$$x^{(n)} = \sum_{j=1}^{n} a_j(x)\, x^{(n-j)} + b(t) \qquad a_1(\cdot), \dots, a_n(\cdot), b(\cdot) : I \subseteq \mathbb{R} \to \mathbb{R} \text{ cont.}$$

1. Consideră ec. liniară asociată $\bar{x}^{(n)} = \sum_{j=1}^{n} a_j(x)\, \bar{x}^{(n-j)}$

Determină $\{\varphi_1(\cdot), \dots, \varphi_n(\cdot)\}$ sist. fundamental de soluții

Obs: Dacă $a_j(t) = a_j\ \forall j = \overline{1,n} \to$ vezi Algoritm

Scrie sol. generală $\bar{x}(t) = \sum_{i=1}^{n} c_i \varphi_i(t)$ $c_i \in \mathbb{R}, i = \overline{1,n}$

2. Variația constantelor

Caută soluții de forma $x(t) = \sum_{i=1}^{n} c_i(t, \varphi_i(t))$

Rezolvă sist. algebric:

$$\sum_{i=1}^{n} c_i'(t)\varphi_i(t) = 0$$

$$\sum_{i=1}^{n} c_i'(t)\varphi_i'(t) = 0 \implies c_i'(t) = \dots \quad i = \overline{1,n}$$

$$\implies c_i(t) = \dots \quad i = \overline{1,n}$$

$$\implies x(t) = \dots$$

$$\sum_{i=1}^{n} c_i'(t)\varphi_i^{(n-2)}(t) = 0$$

$$\sum_{i=1}^{n} c_i'(t)\varphi_i^{(n-1)}(t) = b(t)$$

Să se det. sol. generală:

1) $x'' - 5x' + 6x = 1$

$\bar{x}'' - 5\bar{x}' + 6\bar{x} = 0$

$\lambda^2 - 5\lambda + 6 = 0$

$\Delta = 25 - 24 = 1 \implies \lambda_{1,2} = 2, 3$

Sol. generală $\bar{x}(t) = c_1 e^{2t} + c_2 e^{3t}$, $c_1, c_2 \in \mathbb{R}$

Caut sol. de forma: $x(t) = c_1(t) e^{2t} + c_2(t) e^{3t}$

$$\begin{cases} c_1(t) \cdot e^{2t} + c_2(t) \cdot e^{3t} = 0 \\ c_1'(t)(e^{2t})' + c_2'(t) \cdot (e^{3t})' = 1 \end{cases}$$

$$\begin{cases} c_1'(t) \cdot e^{2t} + c_2'(t) \cdot e^{3t} = 0 \quad | \cdot 2 \\ 2c_1'(t) \cdot e^{2t} + 3c_2'(t) \cdot e^{3t} = 1 \end{cases} \implies \begin{cases} 2c_1'(t) \cdot e^{2t} + 2c_2'(t) \cdot e^{3t} = 0 \\ 2c_1'(t) \cdot e^{2t} + 3c_2'(t) \cdot e^{3t} = 1 \end{cases}$$

$$c_2'(t) = e^{-3t}$$

$c_1'(t)\cdot e^{2t} + e^{-3t}\cdot e^{3t} = 0$

$c_1'(t) = e^{-2t} \Rightarrow c_1(t) = -\frac{1}{2}e^{-2t} + k_1, k_1 \in \mathbb{R}$

$x(t) = \left(-\frac{1}{2}\cdot e^{-2t} + k_1\right)\cdot e^{2t} + \left(-\frac{1}{3}e^{-3t} + k_2\right)\cdot e^{3t}$

2) $x'' - 6x' + 10x = t$

$\bar{x}'' - 6\bar{x}' + 10\bar{x} = 0$

Ec. car. $\lambda^2 - 6x + 10 = 0$

$\lambda_{1,2} = 3 \pm i$

$\lambda = 3+i \Rightarrow \bar{x}(t) = c_1\cdot e^{3t}\cdot\cos 1\cdot t + c_2\cdot e^{3t}\cdot\sin 1\cdot t$

Caut sol. de forma

$x(t) = c_1(t)\, e^{3t}\cos t + c_2(t)\, e^{3t}\sin t$

$$\begin{cases} c_1'(t)\, e^{3t}\cos t + c_2'(t)\, e^{3t}\sin t = 0 \\ c_1'(t)(e^{3t}\cos t)' + c_2'(t)(e^{3t}\sin t)' = at \end{cases}$$

$$\begin{cases} c_1'(t)\, e^{3t}\cos t + c_2'(t)\sin t = 0 \\ c_1'(t)\,(3e^{3t}\cos t - e^{3t}\sin t) + c_2'(t)(3\cdot e^{3t}\sin t + e^{3t}\cos t) = t \end{cases} \quad \begin{matrix} \mid\cdot\cos t \\ \mid\cdot(-\sin t) \end{matrix}$$

$$\begin{cases} c_1'(t)\, e^{3t}\cos t + c_2'(t)\, e^{3t}\sin t = 0 & \mid\cdot\sin t \\ -c_1'(t)\, e^{3t}\sin t + c_2'(t)\, e^{3t}\cos t = t & \mid\cdot\cos t \end{cases}$$

$\cancel{+ c_2'(t)\, e^{3t}\sin t\cos t = \sin t + t\cos t}$

$\Rightarrow$

$+ c_2'(t)\, e^{3t}(\sin^2 t + \cos^2 t) = t\cos t$

$\Rightarrow$ /

$\Rightarrow c_2'(t) = e^{-3t}\, t\cos t$

$c_1'(t)e^{3t}\cos^2 t + c_1'(t)\, e^{3t}\sin^2(t) = -t\sin t$

$c_1'(t) = e^{-3t}(t\sin t)$

$\Rightarrow c_1(t) = \int -e^{-3t}\sin t\, dt$

$c_2(t) = \int e^{-3t}\, t\cos t\, dt = e^{-3t}\, t\sin t - \int (e^{-3t}t)\sin t\, dt =$

$= e^{-3t}\, t\sin t - \int(-3t\, e^{-3t} + e^{-3t})\sin t\, dt =$

$= e^{-3t}t\sin t + 3\int e^{-3t}\, t\sin t + \int e^{-3t}\sin t\, dt$

$\int e^{-3t}\sin t\, dt = -\int e^{-3t}(\cos t)'\, dt = -e^{-3t}\cos t - 3\int e^{-3t}(\cos t)\, dt$

$\int e^{-3t}\cos t\, dt = e^{-3t}\sin t + 3\int e^{-3t}\sin t\, dt$

TO:

FROM:

$$\begin{cases} A + 3B = -e^{-3t}\cos t \\ -3A + B = e^{-3t}\sin t \end{cases} \Rightarrow A\ldots, B\ldots$$

3) $x''' + 3x'' + 3x' + x = e^{-t}$

$\bar{x}''' + 3\bar{x}'' + 3\bar{x}' + \bar{x} = 0$

$\lambda^3 + 3\lambda^2 + 3\lambda + 1 = 0$

$(\lambda+1)^3 = 0 \Rightarrow \lambda_{1,2,3} = -1$

e^{-t} sol. generală: $\bar{x}(t) = c_1e^{-t} + c_2te^{-t} + c_3t^2e^{-t}$

te^{-t}

t^2e^{-t}

Căutăm sol. de forma $x(t) = c_1(t)e^{-t} + c_2(t)te^{-t} + c_3(t)t^2e^{-t}$

$$\begin{cases} c_1'(t)e^{-t} + c_2'(t)te^{-t} + c_3'(t)t^2e^{-t} = 0 \\ -c_1'(t)e^{-t} + c_2'(t)(e^{-t} - te^{-t}) + c_3'(t)(2te^{-t} - t^2e^{-t}) = 0 \\ c_1'(t)e^{-t} + c_2'(t)(-2e^{-t} + te^{-t}) + c_3'(t)(2e^{-t} - 4te^{-t} + t^2e^{-t}) = e^{-t} \end{cases} \Bigg| : \frac{1}{e^{-t}}$$

$$+\begin{cases} \Rightarrow c_2'(t) + c_3'(t)2t = 0 & (1+2) \\ -c_2'(t) + c_3'(t)(2-2t) = 1 & (2+3) \end{cases}$$

$2c_3'(t) = 1$

$c_3'(t) = \frac{1}{2} \Rightarrow c_2'(t) = -t \Rightarrow c_1'(t) = -c_2'(t)t - c_3'(t)t^2 =$

$= t^2 - \frac{1}{2}t^2 = \frac{t^2}{2}$

$\Rightarrow c_1(t) = \frac{t^3}{6} + k_1,\ k_1 \in \mathbb{R}$

$c_2(t) = -\frac{t^2}{2} + k_2,\ k_2 \in \mathbb{R}$

$c_3(t) = \frac{t}{2} + k_3,\ k_3 \in \mathbb{R}$

~~$\Rightarrow x(t) = \frac{t^3}{6} + k_1$~~

$\Rightarrow x(t) = \left(\frac{t^3}{6} + k_1\right)e^{-t} + \left(\frac{t^2}{2} + k_2\right)te^{-t} +$

$+ \left(\frac{1}{2} + k_3\right)t^2e^{t}$

4) $x^{IV} + x'' = \sin t$

$\bar{x}^{IV} + \bar{x}'' = 0$

$\lambda^4 + \lambda^2 = 0 \Rightarrow \lambda_{1,4} = 0,\ \lambda_{2,3} = \pm i$

$\lambda = 0 \Rightarrow \varphi_1(t) = e^{0\cdot t} = 1$

$\varphi_2(t) = te^{0\cdot t} = t$

$\lambda = i \Rightarrow \varphi_3(t) = e^{0\cdot t}\cos 1\cdot t = \cos t$

$\varphi_4(t) = e^{0t}\sin 1t = \sin t$

$\bar{x}(t) = c_1 + c_2t + c_3\cos t + c_4\sin t$

Căutăm soluțiile: $x(t) = c_1(t) + c_2(t)t + c_3(t)\cos t + c_4(t)\cos t$

$$\begin{cases} c_1'(t) + c_2'(t)t + c_3'(t)\cos t + c_4'(t)\sin t = 0 \\ c_1'(t)\cdot 0 + c_2'(t)\cdot t' - c_3'(t)\cdot \sin t + c_4'(t)\cdot \cos t = 0 \\ c_2'(t)\cdot 0 - c_3'(t)\cos t - c_4'(t)\sin t = 0 \\ \qquad c_3'(t)\sin t - c_4'(t)\cos t = \sin t \end{cases}$$

$$\begin{cases} -c_3'(t)\cos t - c_4'(t)\sin t = 0 & \mid \sin t \\ c_3(t)\sin t - c_4'(t)\cos t = \sin t & \mid \cos t \end{cases}$$

$$-c_4'(t)(\sin^2 t - \cos^2 t) = \sin t\cos t$$

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$$-c_4(t)(\sin^2 t + \cos^2 t) = \sin t\cdot\cos t$$

$$c_4'(t) = -\sin t\cos t$$

$$-c_3'(t)(\cos^2 t + \sin^2 t) = -\sin^2 t$$

$$c_3'(t) = \sin^2 t$$

$$c_2'(t) = \sin^3 t + \sin t\cos^2 t = \sin t(\sin^2 t + \cos^2 t) = \sin t$$

$$c_1'(t) = -t\sin t - \sin^2 t\cdot\cos t + \sin^2 t\cos t = -t\sin t$$

$$c_1(t) = \int -t\sin t\,dt = t\cos t - \int \cos t\,dt = t\cos t - \sin t + k_1$$

$$c_2(t) = \int \sin t\,dt = -\cos t + k_2$$

$$c_3(t) = \int \sin^2 t\,dt = \int \frac{1-\cos 2t}{2}\,dt = \frac{1}{2}t - \int \frac{\cos 2t}{2}\,dt = \frac{1}{2}t - \frac{1}{4}\cos 2t + k_3$$

$$c_4(t) = \int -\sin t\cos t\,dt = -\frac{1}{2}\int \sin 2t\,dt = \frac{1}{4}\cos 2t + k_4$$