TO:

FROM: Ecuații diferențiale - seminar - 21.11.2017

Algoritm (cazul valorilor

$\frac{dx}{dt} = Ax \qquad A \in L(\mathbb{R}^n, \mathbb{R}^n)$

1. Rezolvă ec. caracteristică $\det(A - \lambda I_n) = 0 \Rightarrow \sigma(A) = \{\lambda_1, \dots, \lambda_n\}$ distincte
2. Dacă $\lambda \in \sigma(A) \cap \mathbb{R}$ caută $u_\lambda \in \mathbb{R}^n \setminus \{0\}$ a.î. $(A - \lambda I_n)u = 0$. Scrie sol. $\varphi_\lambda(t) = e^{\lambda t} \cdot u_\lambda$
3. Dacă $\lambda = \alpha + i\beta \in \sigma(A)$, $\beta > 0$ caută $u_\lambda \in \mathbb{C}^n \setminus \{0\}$ și $(A - \lambda I_n)u_\lambda = 0$
   Scrie soluțiile $\varphi_\lambda(t) = \mathrm{Re}(e^{\lambda t} \cdot u_\lambda)$, $\varphi_{\bar\lambda}(t) = \mathrm{Im}(e^{\lambda t} \cdot u_\lambda)$
4. Renumerotează $\{\varphi_\lambda(\cdot)\}_{\lambda \in \sigma(A)} = \{\varphi_1(\cdot), \dots, \varphi_n(\cdot)\}$ sistem fundamental de soluții
   Scrie sol. generală $\varphi(t) = \sum_{i=1}^{n} c_i \varphi_i(t)$, $c_i \in \mathbb{R}, i = \overline{1,n}$

Să se determine soluția generală: 1) $\begin{cases} x' = x + z - y \\ y' = x + y - z \\ z' = 2x - y \end{cases}$

$A = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 2 & -1 & 0 \end{pmatrix}$

$\det(A - \lambda I_3) = 0 \Leftrightarrow \begin{vmatrix} 1-\lambda & -1 & 1 \\ 1 & 1-\lambda & -1 \\ 2 & -1 & -\lambda \end{vmatrix} = -\lambda(1-\lambda)^2 - 1 + 2 - (2(1-\lambda) - 1 + \lambda + \lambda) =$

$= -\lambda(1 - 2\lambda + \lambda^2) + 1 - (2 - 2\lambda - 1 + 2\lambda) = \cancel{-\lambda + 2\lambda^2 - \lambda^3} \dots = -\lambda^3 + 2\lambda^2 + \lambda - 2$

$= -\lambda^3 + 2\lambda^2 + \lambda - 2 = -\lambda^2(\lambda - 1) + \lambda(\lambda - 1) + 2(\lambda - 1) =$

$= (\lambda - 1)(-\lambda^2 + \lambda + 2) = 0$

$\lambda_1 = 1$

$\Delta = 1 + 4 \cdot 2 = 9$

$\lambda_2 = 2$

$\lambda_3 = -1$

$\lambda = 1 \Rightarrow$ se caută $u = ?$ a.î. $(A - I_3)u = 0$, $u = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} -b + c = 0 & \Rightarrow c = b \\ a - c = 0 & \Rightarrow c = a \\ 2a - b - c = 0 \end{cases} \Rightarrow u = \begin{pmatrix} a \\ a \\ a \end{pmatrix}, a \in \mathbb{R} \setminus \{0\}$

$a = 1 \Rightarrow u = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \Rightarrow$

$\Rightarrow \varphi_1(t) = e^t \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ sol.

$\lambda = 2$, $u = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, $(A - \lambda I_3)u = 0 \Rightarrow \begin{pmatrix} -1 & -1 & 1 \\ 1 & -1 & -1 \\ 2 & -1 & -2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\Rightarrow \begin{matrix} -a - b + c = 0 \\ a - b - c = 0 \\ 2a - \cancel{4}b - 2c = 0 \end{matrix} \Rightarrow \begin{matrix} b = 0 \\ a = c \end{matrix} \Rightarrow u = \begin{pmatrix} a \\ 0 \\ a \end{pmatrix} \Rightarrow$ sol $\varphi_2(t) = e^{2t}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$\lambda = -1$, $u = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ a.î. $(A + I_3)u = 0$

$\begin{pmatrix} 2 & -1 & 1 \\ 1 & 2 & -1 \\ 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \Rightarrow \begin{cases} \cancel{-b + c = 0} \\ \cancel{a - c = 0} \\ 2a - b + c = 0 \end{cases} \Rightarrow \begin{cases} 2a - b + c = 0 \\ a + 2b - c = 0 \\ 2a - b + c = 0 \end{cases}$

$$\Rightarrow \begin{cases} b=0 \Rightarrow b=-3a \\ c=a+2b=-5a \end{cases} \Rightarrow \begin{pmatrix} a \\ -3a \\ -5a \end{pmatrix}$$

$$\varphi_3(t)=e^{-t}\begin{pmatrix} -1 \\ 3 \\ 5 \end{pmatrix} \Rightarrow \varphi(t)=c_1\varphi_1(t)+c_2\varphi_2(t)+c_3\varphi_3(t)=$$

$$=c_1\cdot e^{t}\cdot\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}+c_2\cdot e^{2t}\cdot\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}+c_3\cdot e^{-t}\begin{pmatrix} -1 \\ 3 \\ 5 \end{pmatrix}=\begin{pmatrix} c_1e^{t}+c_2e^{2t}-c_3e^{-t} \\ c_1e^{t}+3c_3e^{-t} \\ c_1e^{t}+c_2e^{2t}+5c_3e^{-t} \end{pmatrix}$$

$$\Rightarrow x(t)=c_1e^{t}+c_2e^{t}-c_3e^{-t}$$
$$y(t)=c_1e^{t}+3c_3e^{-t}$$
$$z(t)=c_1e^{t}+c_2e^{2t}+5c_3e^{-t}\quad,\ c_1,c_2,c_3\in\mathbb{R}$$

2) $\begin{cases} x'=x+y \\ y'=x+3y-z \\ z'=2y+3z-x \end{cases}$

$$A=\begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & -1 \\ -1 & 2 & 3 \end{pmatrix}$$

$$\det(A-\lambda I_3)=\begin{vmatrix} 2-\lambda & 1 & 0 \\ 1 & 3-\lambda & -1 \\ -1 & 2 & 3-\lambda \end{vmatrix}=(2-\lambda)(3-\lambda)^2+1-\left(-2(2-\lambda)+3-\lambda\right)$$

$$=(2-\lambda)(9+\lambda^2-6\lambda+2-1)$$

$$=(2-\lambda)(\lambda^2-6\lambda+10)\Rightarrow \lambda_1=2$$

$$\lambda^2-6\lambda+10=0$$

$$\Delta=\cancel{36-40}=-4 \Rightarrow \lambda_{2,3}=3\pm i$$

$\lambda_1=2 \Rightarrow u=?$ a.î. $(A-2I_3)u=0$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & -1 \\ -1 & 2 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix}=0 \Rightarrow \begin{matrix} b=0 \\ a+b-c=0 \Rightarrow a=c \end{matrix} \Rightarrow u=\begin{pmatrix} a \\ 0 \\ a \end{pmatrix}$$

$$a=1 \Rightarrow u=\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \Rightarrow \varphi_1(t)=e^{2t}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$\lambda_2=3+i$, $u=?$ a.î. $\cancel{3+i\ I_3}$ $(A-(3+i)I_3)u=0$

$$\begin{pmatrix} -1-i & 1 & 0 \\ 1 & -i & -1 \\ -1 & 2 & -i \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} (-1-i)a+b=0 \\ a-ib-c=0 \\ -a+2b-ic=0 \end{matrix} \Rightarrow \begin{matrix} b=a(1+i) \\ c=a-ib=a-i(1+i)a \\ =a(1-i+1)=a(2-i) \end{matrix}$$

$$\Rightarrow u=\begin{pmatrix} a \\ a(1+i) \\ a(2-i) \end{pmatrix} \quad a=1 \Rightarrow u=\begin{pmatrix} 1 \\ 1+i \\ 2-i \end{pmatrix}$$

$$\hat{\varphi}(t)=e^{(3+i)t}\cdot\begin{pmatrix} 1 \\ 1+i \\ 2-i \end{pmatrix}=e^{3t}\cdot e^{it}\begin{pmatrix} 1 \\ 1+i \\ 2-i \end{pmatrix}$$

TO:

FROM:

$$= e^{3t} \cdot \overbrace{(\cos t + i\sin t)}^{\cos} \cdot \begin{pmatrix} 1 \\ 1+i \\ 2-i \end{pmatrix} = e^{3t} \begin{pmatrix} \cos t + i\sin t \\ \cos t - \sin t + i(\sin t + \cos t) \\ 2\cos t + \sin t + i(2\sin t - \cos t) \end{pmatrix} =$$

$$= e^{3t} \begin{pmatrix} \cos t \\ \cos t - \sin t \\ 2\cos t - \sin t \end{pmatrix} + i e^{3t} \begin{pmatrix} \sin t \\ \sin t + \cos t \\ 2\sin t - \cos t \end{pmatrix}$$

$$\varphi_2(t) = e^{3t} \begin{pmatrix} \cos t \\ \cos t - \sin t \\ 2\cos t + \sin t \end{pmatrix} \qquad \varphi_3(t) = e^{3t} \begin{pmatrix} \sin t \\ \sin t + \cos t \\ 2\sin t - \cos t \end{pmatrix}$$

3) $\begin{cases} x' = x - 2y \\ y' = 3y + x \end{cases} \Rightarrow A = \begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix} \Rightarrow \det(A - \lambda I_2) = \begin{vmatrix} 1-\lambda & -2 \\ 1 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) + 2 =$

$= 3 - \lambda - 3\lambda + \lambda^2 + 2 = \lambda^2 - 4\lambda + 5 \Rightarrow \lambda_1 = 2+i$
$\lambda_2 = 2-i$

$\lambda_1 = 2+i$

căutăm $u = \begin{pmatrix} a \\ b \end{pmatrix}$ a.î. $(A - (2+i) I_2) u = 0 \Rightarrow \left(\begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix} - \begin{pmatrix} 2+i & 0 \\ 0 & 2+i \end{pmatrix} \right) \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Leftrightarrow \begin{pmatrix} -1-i & -2 \\ 1 & -1-i \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} -a - ai - 2b = 0 \\ a + b - bi = 0 \end{cases} \Rightarrow a = b(1-i) = 1 \quad u = \begin{pmatrix} b(1-i) \\ b \end{pmatrix}$

$b(i-1)$

$\Rightarrow$

pt $b = 1 \Rightarrow u = \begin{pmatrix} 1-i \\ 1 \end{pmatrix}$

Fie $\varphi(t) = e^{(2+i)t} \begin{pmatrix} 1-i \\ 1 \end{pmatrix} = e^{2t} \cdot e^{it} \begin{pmatrix} 1-i \\ 1 \end{pmatrix}$

$= e^{2t} (\cos t + i\sin t) \begin{pmatrix} 1-i \\ 1 \end{pmatrix}$

$= e^{2t} \begin{pmatrix} \cos t + \sin t + i(\sin t - \cos t) \\ \cos t + i\sin t \end{pmatrix}$

$= e^{2t} \begin{pmatrix} \cos t + \sin t \\ \cos t \end{pmatrix} + i e^{2t} \begin{pmatrix} \sin t - \cos t \\ \sin t \end{pmatrix}$

$\varphi_1(t) = e^{2t} \begin{pmatrix} \cos t + \sin t \\ \cos t \end{pmatrix} \qquad \varphi_2(t) = e^{2t} \begin{pmatrix} \sin t - \cos t \\ \sin t \end{pmatrix}$

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3) $\begin{cases} x' = x - y \\ y' = 3y + x \Rightarrow x = y' - 3y \end{cases} \Rightarrow \quad (y' - 3y)' = y' - 3y - 2y \Rightarrow y'' - 3y' = y' - 5y$

$\Rightarrow y'' - 2y' + 5y = 0$

ec. caracteristică : $\lambda^2 - 4\lambda + 5 = 0 \Rightarrow \lambda_{1,2} = 2 \pm i \Rightarrow y(t) =$

$\Rightarrow y(t) = c_1 e^{2t} \cos t + c_2 e^{2t} \sin t, \quad c_1, c_2 \in \mathbb{R}$

$y(t) = 2c_1 e^{2t} \cos t + c_1 e^{2t} (-\sin t) + 2c_2 e^{2t} \sin t + c_2 e^{2t} \cos t$
$- 3c_1 e^{2t} \cos t - 3c_2 e^{2t} \sin t =$
$= (-c_1 + c_2) e^{2t} \cos t + (-c_1 - c_2) e^{2t} \sin t, \quad c_1, c_2 \in \mathbb{R}$

~~4) $\begin{cases} x' = 2x + y \\ y' = x + 3y - z \\ z' = -x + 2y + 3z \end{cases} \Rightarrow A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & -1 \\ -1 & 2 & 3 \end{pmatrix}$~~

~~$\det(A - \lambda I_3) = |$~~

4) $\begin{cases} x' = x - y - z \\ y' = x + y \\ z' = 3x + z \end{cases} \Rightarrow A = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \Rightarrow \det(A - \lambda I_3) = \begin{vmatrix} 1-\lambda & -1 & -1 \\ 1 & 1-\lambda & 0 \\ 3 & 0 & 1-\lambda \end{vmatrix} =$

$= (1-\lambda)^3 - \left(-3(1-\lambda) - 1(1-\lambda)\right) = (1-\lambda)(1 - 2\lambda + \lambda^2) - (-3 + 3\lambda - 1 + \lambda) =$

$= 1 - 2\lambda + \lambda^2 - \lambda + 2\lambda^2 - \lambda^3 + 4 - 4\lambda = -\lambda^3 + 3\lambda^2 - 7\lambda + 5$

$= (\lambda - 1)(\lambda^2 - 2\lambda + 5) \Rightarrow \lambda_1 = 1$

$\lambda_{2,3} = 1 \pm 2i$

$\lambda = 1$ caut un vector $u = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ a.î. $(A - I_3)u = 0 \Rightarrow \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & 0 \\ 3 & 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} \Rightarrow \begin{cases} a = 0 \\ c = -b \end{cases}$

$\Rightarrow u = \begin{pmatrix} 0 \\ b \\ -b \end{pmatrix} \Rightarrow$ sol. $y_1(t) = e^t \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

$\lambda = 1 + 2i$, caut $u = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ a.î. $\left(A - (1 + 2i) I_3\right) u = 0$

$\Rightarrow \begin{pmatrix} 2i & -1 & -1 \\ 1 & 2i & 0 \\ 3 & 0 & 2i \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} -2ai - b - c = 0 \\ a - 2ib = 0 \Rightarrow a = 2ib \\ 3a - 2ic = 0 \Rightarrow c = \frac{3}{2i} a = \frac{3}{2i}(2ib) = 3b \end{cases}$

$u = \begin{pmatrix} 2ib \\ b \\ 3b \end{pmatrix} \quad b = 1 \Rightarrow \begin{pmatrix} 2i \\ 1 \\ 3 \end{pmatrix}$

$\varphi(t) = e^{(1+2i)t}\begin{pmatrix} 2i \\ 1 \\ 3 \end{pmatrix} = e^t e^{2it}\begin{pmatrix} 2i \\ 1 \\ 3 \end{pmatrix} = e^t(\cos 2t + i \sin 2t)\begin{pmatrix} 2i \\ 1 \\ 3 \end{pmatrix} =$

$= e^t \begin{pmatrix} -2\sin 2t + 2i \sin 2t \\ \cos 2t + i \sin 2t \\ 3\cos 2t + 3i \sin 2t \end{pmatrix} = e^t \begin{pmatrix} -2\sin t \\ \cos 2t \\ 3i \cos 2t \end{pmatrix} + i e^t \begin{pmatrix} 2\cos 2t \\ \sin 2t \\ 3\sin 2t \end{pmatrix}$

$\varphi_2(t) = \begin{pmatrix} -2\sin t \\ \cos t \\ 3\cos t \end{pmatrix} \cdot e^t \qquad \varphi_3(t) = e^t \begin{pmatrix} 2\cos 2t \\ \sin 2t \\ 3\sin 2t \end{pmatrix}$

Temă: 1) $\begin{matrix} x' = x - y \\ y' = y - 4x \end{matrix}$ 2) $\begin{matrix} x' = 2x + 2y \\ y' = 2x - y \end{matrix}$ 3) $\begin{cases} x' = x - y + z \\ y' = x + 2y - z \\ z' = x - y + z \end{cases}$