3

TO:

FROM: Ecuații diferențiale – 17.10. – seminar

Ecuații Bernoulli

$\frac{dx}{dt} = a(t)x + b(t)x^{\alpha}$ $\quad a(\cdot), b(\cdot) : I \subseteq \mathbb{R} \to \mathbb{R}$ cont., $\alpha \in \mathbb{R} \setminus \{0,1\}$

Algoritm

1. Consider ec. liniară asociată $\frac{d\bar{x}}{dt} = a(t)\bar{x}$. Scriu sol. generală $\bar{x}(t) = c \cdot e^{A(t)}$
2. Variația constantelor

Caut sol. de forma $x(t) = c(t)e^{A(t)}$

$x(\cdot)$ sol. $\Rightarrow \frac{dc}{dt} = e^{(\alpha-1)A(t)} b(t) c^{\alpha}$ ec. v. sep. $\overset{\text{Algoritm}}{\Longrightarrow}$ $c(t) \ldots$

$x(t) \ldots$

- Să se det. sol. generală:

1) $x' = x^2 e^t - 2x$
2) $tx' + x = t^5 x^3 e^t$
3) $tx' = 2t^2\sqrt{x} + 4x$
4) $x' + \frac{x}{t} = \frac{1}{t^2 x^2}$

1) $\bar{x}' = -2\bar{x} \Rightarrow \bar{x}(t) = ce^{-2t}$

Caut sol. de forma $x(t) = c(t)e^{-2t}$

$(c(t) \cdot e^{-2t})' = (c(t)e^{-2t})^2 e^t - 2(c(t)e^{-2t})$

$c'(t)e^{-2t} - 2c(t) \cdot e^{-2t} = c^2(t)e^{-3t} - 2c(t) \cdot e^{-2t}$

$\Rightarrow c'(t) = c^2(t)e^{-t}$

$\frac{dc}{dt} = c^2 e^{-t}$ ec. cu var. sep.

$c^2 = 0 \Rightarrow c(t) \equiv 0$

$\frac{dc}{c^2} = e^{-t}dt$

$\int \frac{dc}{c^2} = \int e^{-t} dt$

$-\frac{1}{c} = -e^{-t} + k, k \in \mathbb{R} \Rightarrow c(t) = \frac{1}{e^{t} - k}, t \in \mathbb{R}$

$\Rightarrow x_0(t) = 0$

$x_k(t) = \frac{e^{-2t}}{e^{-t} - k}, t \in \mathbb{R}$

2) $tx' + tx = t^5 x^3 e^t$

$x' = -\frac{x}{t} + t^4 e^t x^3, \quad t \neq 0, t > 0$

$\bar{x}' = -\frac{\bar{x}}{t} \Rightarrow$ ~~sol. de forma~~ $x(t) = c \cdot e^{\int -\frac{1}{t}dt} = c \cdot e^{-\ln(t)} = c\frac{1}{t}$

Sol. de forma $x(t) = c(t) \cdot \frac{1}{t}$

$(c(t)\frac{1}{t})' = \frac{-c(t) \cdot \frac{1}{t}}{t} + t^4 e^t (c(t) \cdot \frac{1}{t})^3$

$c'(t)\frac{1}{t} + \cancel{c(t)(-\frac{1}{t^2})} = \cancel{-\frac{c(t)}{t^2}} + \frac{t^4 e^t c^3(t)}{t^3} \Rightarrow c'(t) = t^2 e^t c^3(t)$

1

$$\frac{dc}{dt} = t^2 e^t c^3$$

$c^3 = 0 \Rightarrow c = 0 \Rightarrow c(t) \equiv 0$

$$\frac{dc}{c^3} = t^2 e^t \, dt$$

$$\int \frac{1}{c^3} \, dc = \int t^2 e^t \, dt$$

$$\frac{1}{-2c^2} = t^2 e^t - \int 2t e^t \, dt = t^2 e^t - 2t e^t + \int 2 e^t \, dt = t^2 e^t - 2t e^t + 2e^t + k, \ k \in \mathbb{R}$$

$$\Rightarrow c(t) = \pm \frac{1}{\sqrt{-2(t^2 e^t - 2te^t + 2e^t + k)}}, \ k \in \mathbb{R}$$

$x_0(t) = 0$

$x_k(t) = \pm \frac{1}{t(\sqrt{\dots})}$

⋮

<u>Ecuatii Riccati</u>

$$\frac{dx}{dt} = a(t)x^2 + b(t)x + c(t) \qquad , \ a(\cdot), b(\cdot), c(\cdot) : I \subseteq \mathbb{R} \to \mathbb{R}, \text{ cont.}$$

$\exists \ \varphi_0(\cdot)$ sol.

<u>Algoritm</u>

Sch. var. $y = x - \varphi_0(t)$ [$\forall x(\cdot)$ sol. a.v. dif. $y(\cdot)$ după regula $y(t) = x(t) - \overset{\varphi_0(t)}{\cancel{x}}$]

$\Rightarrow$ ec. Bernoulli în $y \Rightarrow$ Algoritm $\Rightarrow y(t) = \dots \Rightarrow x(t) = y(t) + \varphi_0(t) = \dots$

1) $x' = x^2 - 2xe^t + e^{2t} + e^t$, $\varphi(t) = e^t$ sol

2) $x' = x^2 + 6x - 4t^2 + 11$, $\varphi_0(t) = 2t - 3$ sol.

3) $x' + x^2 - \frac{2}{t} x + \frac{2}{t^2} = 0$, $\varphi_0(t) = \frac{2}{t}$ sol

4) $x' = -x^2 \sin t + \frac{2 \sin t}{\cos^2 t}$, $\varphi_0(t) = \frac{1}{\cos t}$ sol.

1) $y = x - e^t$, $y(t) = x(t) - e^t$

$\Leftrightarrow x(t) = y(t) + e^t$

$(y(t) + e^t)' = (y(t) + e^t)^2 - 2(y(t) + e^t)e^t + e^{2t} + e^t$

$y'(t) + e^t = y^2(t) + 2y(t)e^t + e^{2t} - 2y(t)e^t - 2e^{2t} + e^{2t} + e^t$

$y'(t) = y^2(t)$

$\frac{dy}{dt} = y^2$ ec. cu var. sep.

$y^2 = 0 \Rightarrow y = 0 \Rightarrow y(t) = 0$

$\frac{dy}{y^2} = dt \Rightarrow \int \frac{dy}{y^2} = \int dt \Rightarrow -\frac{1}{y} = t + k, \ k \in \mathbb{R}$

TO:

FROM:

$\Rightarrow y_k(t) = -\frac{1}{t+k},\ k \in \mathbb{R}$

$\Rightarrow x(t) = y(t) + e^t$

$x_k(t) = e^t - \frac{1}{t+k},\ k \in \mathbb{R}$

2) $y(t) = x(t) - 2t + 3$

$x(t) = y(t) + 2t - 3$

$(y(t) + 2t - 3)' = (y(t) + 2t - 3)^2 + 6(y(t) + 2t - 3) - 4t^2 + 11$

$y'(t) + 2 = y^2(t) + 4t^2 + 9 + 4t\,y(t) - 6y(t) - 12t + 6y(t) + 12t - 18 - 4t^2 + 11$

$y'(t) = y^2(t) + 4t\,y(t)$

$y'(t) = y^2(t) + 4t\,y(t)$

$y' = 4ty + y^2$

$y' = 4ty$

$y(t) = c \cdot e^{2t^2}$

$y(t) = c(t) \cdot e^{2t^2}$

$(c(t)\,e^{2t^2})' = (c(t)\,e^{2t^2})^2 + 4t(c(t)\,e^{2t^2})$

$c'(t)\,e^{2t^2} + c(t)\,e^{2t^2} \cdot 4t = c^2(t)\,e^{4t^2} + c(t)\,e^{2t^2}\,4t$

$c'(t) = c^2(t)\,e^{2t^2}$

$\frac{dc}{dt} = c^2 \cdot e^{2t^2}$

$c^2 = 0 \Rightarrow c = 0 \Rightarrow c(t) \equiv 0$

$\frac{dc}{c^2} = e^{2t^2}dt$

$\int \frac{dc}{c^2} = \int e^{2t^2}dt$

$-\frac{1}{c} = \int_0^t e^{2s^2}ds + k,\ k \in \mathbb{R}$

$\Rightarrow c(t) = -\frac{1}{\int_0^t e^{2s^2}ds + k}$

$y_0(t) = 0$

$y_k(t) = \frac{-e^{2t^2}}{\int_0^t e^{2s^2}ds + k},\ k \in \mathbb{R}$

$x(t) = y(t) + 2t - 3$

$x_0(t) = 2t - 3$

$x_k(t) = \frac{-e^{2t^2}}{\int_0^t e^{2s^2}ds + k}$

<u>Ecuații omogene</u>

$\frac{dx}{dt} = f\left(\frac{x}{t}\right),\ f(\cdot): D \subseteq \mathbb{R} \to \mathbb{R}$

Algoritm: sch. var. $y = \frac{x}{t}$ $\forall x(\cdot)$ sol. a e. def. funcț. $y(\cdot)$ după reg. $y(t) = \frac{x(t)}{t}$

$\Rightarrow \frac{dy}{dt} = \frac{f(y) - y}{t}$ ec. var. sep. $\longrightarrow$ Algoritm $\Rightarrow \ldots$

1) $2t^2x' = t^2 + x^2$

2) $x' = \dfrac{x+\sqrt{tx}}{t}$

3) $tx' = x + \sqrt{x^2 - t^2}$

4) $x' = \dfrac{2tx}{t^2 - x^2}$

1) $2t^2x' = t^2 + x^2 \qquad t^2 \neq 0 \,|\, 2t^2$
$\qquad\qquad\qquad\qquad\qquad t > 0$

$x' = \frac{1}{2} + \frac{1}{2}\left(\frac{x}{t}\right)^2$

Sch. var. $y = \frac{x}{t} \Rightarrow y(t) = \frac{x(t)}{t}$

$\Leftrightarrow x(t) = y(t)$

$\Rightarrow (t \cdot y(t))' = \frac{1}{2} + \frac{1}{2} y^2(t)$

$\Rightarrow y(t) + t y'(t) = \frac{1}{2} + \frac{1}{2} \cdot y^2(t)$

$\Leftrightarrow y'(t) = \frac{1}{t}\left(\frac{1}{2} + \frac{1}{2} y^2(t) - y(t)\right)$

$\Leftrightarrow \frac{dy}{dt} = \frac{(y-1)^2}{2t}$

$(y-1)^2 = 0 \Rightarrow y = 1 \Rightarrow y_0(t) = 1$

$\Rightarrow \cancel{\frac{dy}{dt}}$

$\Rightarrow \frac{dy}{(y-1)^2} = \frac{1}{2t}\, dt$

$\int \frac{dy}{(y-1)^2} = \frac{1}{2}\int t\, dt \Leftrightarrow \cancel{-\frac{1}{y-1} = \frac{1}{4}t^2 + k, k \in \mathbb{R}}$

$\Rightarrow \cancel{y(t) = 1} \Leftrightarrow \frac{1}{1-y} = \frac{1}{4}t^2 + k, k \in \mathbb{R} \Rightarrow y_k(t) = -\frac{1}{\frac{1}{4}t^2 + k} - 1, k \in \mathbb{R}, y_0(t) = 1$

$\Rightarrow x_0(t) = t$

$x_k(t) = t\left(-\frac{1}{\frac{1}{4}t^2 + k} - 1\right), k \in \mathbb{R}$

2) $x' = \frac{x + \sqrt{tx}}{t}, t \neq 0, t > 0$

$x' = \frac{x}{t} + \sqrt{\frac{tx}{t^2}}$

$x' = \frac{x}{t} + \sqrt{\frac{x}{t}}$

Sch. var. $y = \frac{x}{t} \Rightarrow y(t) = \frac{x(t)}{t} \Rightarrow x(t) = y(t)t$

$(y(t)t)' = y(t) + \sqrt{y(t)} \Rightarrow y'(t)t + y(t) = y(t) + \sqrt{y(t)}$

TO:

FROM:

$$y'(t) = \frac{\sqrt{y(t)}}{t} \Rightarrow y' = \sqrt{\frac{y}{t}}$$

$$\Rightarrow \frac{dy}{dt} = \frac{\sqrt{y}}{t}$$

$$dy\,t = dt\sqrt{y}$$

$$\frac{dy}{\sqrt{x}} = \frac{dt}{t} \quad / y$$

$$\int \frac{1}{\sqrt{y}}\,dy = \int \frac{1}{t}\,dt$$

$$2\sqrt{y} = \ln t + c$$

$$\Rightarrow y(t) = \left(\frac{\ln t + c}{2}\right)^2, \quad c \in \mathbb{R}$$

$$x(t) = \left(\frac{\ln c + c}{2}\right)^2, \quad c \in \mathbb{R}$$