TO:

FROM: Ecuații diferențiale - curs - 24.10.2017

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Fie $M \geq 0$, $t_0 \in I \subseteq \mathbb{R}$, $u(\cdot), v(\cdot): I \to \mathbb{R}_+$ continue.
Dacă $u(t) \leq M + \left|\int_{t_0}^t u(s)v(s)\,ds\right| \ \forall t \in I$, at. $u(t) \leq M e^{\left|\int_{t_0}^t v(s)\,ds\right|}, \ \forall t \in I$

Dem:
$$u(t) \leq \underbrace{\left(M + \left|\int_{t_0}^t u(s)v(s)\,ds\right|\right) \cdot e^{-\left|\int_{t_0}^t v(s)\,ds\right|}}_{\psi(t)} e^{\left|\int_{t_0}^t v(s)\,ds\right|}$$

Arătăm că $\psi(t) \leq M \ \forall t \Rightarrow$ q.e.d.
$I_+ = \{t \in I; t \geq t_0\}$ $\quad I_- = \{t \in I; t \leq t_0\}$

Arătăm că $\psi(\cdot)$ este $\searrow$ pe I_+ și $\nearrow$ pe I_-, $\psi(t_0) = M \to \psi(t) \leq M \ \forall t$
De ex. pt. $t \in I_+$ $\quad \psi(t) = \left(M + \int_{t_0}^t u(s)v(s)\,ds\right) e^{-\int_{t_0}^t v(s)\,ds}$

$$\psi'(t) = u(t)v(t) \cdot e^{-\int_{t_0}^t v(s)\,ds} + \left(M + \int_{t_0}^t u(s)v(s)\,ds\right) e^{-\int_{t_0}^t v(s)\,ds} (-v(t)) =$$
$$= v(t) e^{-\int_{t_0}^t v(s)\,ds} \left[u(t) - \left(M + \int_{t_0}^t u(s)v(s)\,ds\right)\right] \leq 0 \ \forall t \Rightarrow \psi(\cdot) \searrow \text{ pe } I_+$$

Th. Cauchy - Lipschitz (E.U.L.)

Fie $f(\cdot,\cdot): D = \mathring{D} \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ cont, local Lipschitz (II) $\quad \frac{dx}{dt} = f(t,x)$
At. $f(\cdot,\cdot)$ admite E.U.L. pe D ($\forall (t_0, x_0) \in D \ \exists I_0 = [t_0 - a, t_0 + a] \in V(t_0)$ $\exists!$
$\varphi(\cdot): I_0 \to \mathbb{R}^n$ sol. cu $\varphi(t_0) = x_0$)

Dem: Fie $(t_0, x_0) \in D = \mathring{D}$, $f(\cdot,\cdot)$ cont. $\Rightarrow$ T. Peano $\exists I_0 = [t_0 - a, t_0 + a] \in V(t_0)$ $\exists$
$\varphi_0(\cdot): I_0 \to \mathbb{R}^n$ sol. cu $\varphi(t_0) = x_0$

Pp. că $\exists \varphi_1(\cdot): I_0 \to \mathbb{R}^n$ sol. cu $\varphi_1(t_0) = x_0$
φ_0 sol. $\Rightarrow$ $\Leftrightarrow$ Ec. integrală asociată $\varphi(t) = \varphi(t_0) + \int_{t_0}^t f(s, \varphi(s))\,ds \ \forall t \in I_0$
φ_1 sol. $\Rightarrow$ " $\quad \varphi_1(t) = \varphi_1(t_0) + \int_{t_0}^t f(s, \varphi_1(s))\,ds \ \forall t \in I_0$

$$\varphi(t) - \varphi_1(t) = \int_{t_0}^t f(s, \varphi(s))\,ds - \int_{t_0}^t f(s, \varphi_1(s))\,ds = \int_{t_0}^t \left[f(s,\varphi(s)) - f(s, \varphi_1(s))\right] ds$$

$$\|\varphi(t) - \varphi_1(t)\| = \left\|\int_{t_0}^t \left[f(s,\varphi(s)) - f(s,\varphi_1(s))\right] ds\right\| \leq \left|\int_{t_0}^t \|f(s,\varphi(s)) - f(s,\varphi_1(s))\|\,ds\right|$$

Fie $D_0 = \{(t, \varphi(t)), (t, \varphi_1(t)); t \in I_0 = [t_0 - a, t_0 + a]\} \subset D$ compactă $\left(\begin{matrix}\varphi(\cdot), \varphi_1(\cdot) \text{ cont} \\ [t_0 - a, t_0 + a] \text{ compact}\end{matrix}\right)$

$f(\cdot,\cdot)$ local Lipschitz (II), $D_0 \subset D$ compact $\Longrightarrow f(\cdot,\cdot)\big|_{D_0}$ este Lipschitz $\Rightarrow \exists L \geq 0$
a.î. $\|f(t, x_1) - f(t, x_2)\| \leq L \|x_1 - x_2\| \ \forall (t, x_1), (t, x_2) \in D$

$$u(t) \leq \left| \int_{t_0}^{t} L \| \varphi(s) - \varphi_1(s) \| ds \right| = \left| \int_{t_0}^{t} L\, u(s)\, ds \right| \; \forall t \xrightarrow[B-G]{\text{lema}} \begin{array}{l} u(t) \leq 0 \cdot e^{\cdots} = 0 \; \forall \\ u(t) \geq 0 \; \forall t \end{array}$$

$\Rightarrow u(t) = 0 \Rightarrow$ q.e.d.

Obs: $x' = x^{2/3} + a$, $a \in \mathbb{R}$

Dacă $a \neq 0$ ec. are E.U.L. deci funcția $x \to x^{\frac{2}{3}} + a$ nu este local Lipschitz (în $x = 0$)

Ecuațiile diferențiale de ordin superior. Existența & unicitatea soluțiilor

Def: a) $f(\cdot,\cdot) : D \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ def. ec. de ordin n $\quad x^{(n)} = f(t, x, x', \dots, x^{(n-1)})$

b) $\varphi(\cdot) : I \subseteq \mathbb{R} \to \mathbb{R}$ s.n. sol. a ec. dacă este de n-ori derivabilă și

$\varphi^{(n)}(t) = f(t, \varphi(t), \varphi'(t), \dots, \varphi^{(n-1)}(t))$, $t \in I$

Mai general $F(\cdot,\cdot) : D \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ $\quad F(t, x, x', \dots, x^{(n-1)})$

Metodă generală de studiu = sistemul canonic asociat

(1) $x^{(n)} = f(t, x, x', \dots, x^{(n-1)})$

$$(2) \begin{cases} \dfrac{dx_1}{dt} = x_2 \\ \dfrac{dx_2}{dt} = x_3 \\ \quad \vdots \\ \dfrac{dx_{n-1}}{dt} = x_n \\ \dfrac{dx_n}{dt} = f(t, x_1, x_2, \dots, x_n) \end{cases}$$

$$\tilde{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad \tilde{f}(t, (x_1, \dots, x_n)) = \begin{pmatrix} x_2 \\ x_3 \\ \vdots \\ x_n \\ f(t, x_1, \dots, x_n) \end{pmatrix}$$

$$(2)\; \frac{d\tilde{x}}{dt} = f(t, \tilde{x}) \qquad \tilde{f}(\cdot,\cdot) : D \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$$

PROP. (de echivalență)

$\varphi(\cdot) : I \subseteq \mathbb{R} \to \mathbb{R}$ este sol. a ec. (1) $\Leftrightarrow \tilde{\varphi}(\cdot) = (\varphi(\cdot), \varphi'(\cdot), \dots, \varphi^{(n-1)}(\cdot)) : I \to \mathbb{R}^n$ sol. a ec. (2)

Dem: "$\Rightarrow$" $\quad \varphi'(t) = \varphi'(t)$

$\varphi''(t) = \varphi''(t)$

$\vdots$

$(\varphi^{(n-2)}(t))' = \varphi^{(n-1)}(t)$

$\varphi^{(n)}(t) = (\varphi^{(n-1)}(t))' = f(t, \varphi'(t), \dots, \varphi^{(n-1)}(t))$ $\left(\begin{array}{l}\varphi(\cdot) \\ \text{sol. a ec. (1)}\end{array}\right)$

"$\Leftarrow$" Fie $\hat{\varphi}(\cdot) = (\varphi_1(\cdot), \varphi_2(\cdot), \dots, \varphi_n(\cdot))$ sol. a ec. (2)

$\varphi_1'(t) = \varphi_2(t) \Rightarrow \varphi_2(t) = \varphi_1'(t)$ $\quad \vdots$

$\varphi_2'(t) = \varphi_3(t) \Rightarrow \varphi_3(t) = \varphi_2'(t)$ $\quad \varphi_{n-1}'(t) = \varphi_n(t)$

$\vdots$ $\quad \varphi_1''(t)$ $\quad \varphi_n'(t) = f(t, \varphi_1(t), \dots, \varphi_n(t))$

TO:

FROM:

$(\varphi_1^{(n-1)}(t))' \equiv f(t, \varphi_1(t), \varphi_1'(t), \dots, \varphi_1^{(n-1)}(t))$

$(1)\ x^{(n)} = f(t, x, x', \dots, x^{(n-1)})$

<u>Problema Cauchy</u>

Problema Cauchy pt. ec. (2) se dau: $\tilde{f}(\cdot,\cdot) \to \frac{d\tilde{x}}{dt} = \tilde{f}(t, \tilde{x})$; $(t_0, \tilde{x_0}) \in D$ - condiție inițială

Se caută $\tilde{\varphi}(\cdot): I \subseteq \mathbb{R} \to \mathbb{R}^n$ sol. a ec. (2) cu $\tilde{\varphi}(t_0) = \tilde{x_0}$

$\tilde{x_0} = (x_0, x_0^1, \dots, x_0^{n-1})$

Prop. de echivalență $\tilde{\varphi}(\cdot) = (\varphi(\cdot), \varphi'(\cdot), \dots, \varphi^{(n-1)}(\cdot))$ cu $\varphi(\cdot)$ sol. ec. (1)
$\Rightarrow \varphi(\cdot)$ sol. a ec. (1) cu $\varphi(t_0) = x_0$, $\varphi'(t_0) = x_0^1, \dots, \varphi^{(n-1)}(t_0) = x_0^{n-1}$

Problema Cauchy pt. (1). Se dau: $f(\cdot,\cdot) \to x^{(n)} = f(t, x, x', \dots, x^{(n-1)})$; $(t_0, (x_0, x'_0, \dots, x_0^{n-1})) \in D$ condiție inițială

Se caută $\varphi(\cdot): I \subseteq \mathbb{R} \to \mathbb{R}^n$ sol. a ec. cu $\varphi(t_0) = x_0$, $\varphi'(t_0) = x_0', \dots, \varphi^{(n-1)}(t_0) = x_0^{n-1}$

În ac. caz, spunem că $\varphi(\cdot)$ este sol. a prob. Cauchy $(f, t_0, x_0, x_0^1, \dots, x_0^{n-1})$

Th. Peano (pt. ecuații de ordin superior)

Fie $f(\cdot,\cdot)\ D = \mathring{D} \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ cont. def. $x^{(n)} = f(t, x, x', \dots, x^{(n-1)})$

Atunci $f(\cdot,\cdot)$ admite E.L. pe D $(\forall (t_0, (x_0, x_0', \dots, x_0^{n-1})) \in D\ \exists\ \varphi(\cdot): I_0 \in V(t_0) \to \mathbb{R}$ sol. a ec. cu $\varphi(t_0) = x_0$, $\varphi'(t_0) = x_0', \dots, \varphi^{(n-1)}(t_0) = x_0^{n-1})$

<u>Dem</u>: Fie $(t_0, (x_0, x_0^1, x_0^2, \dots, x_0^{n-1})) \in D$ $\tilde{x_0} := (x_0, x_0^1, \dots, x_0^{n-1})$; $(t_0, \tilde{x_0}) \in D$

$(2)\ \frac{d\tilde{x}}{dt} = \tilde{f}(t, \tilde{x})$ $\quad \tilde{x} = (x_1, \dots, x_n)$ $\tilde{f}(t, (x_1, \dots, x_{n-1})) = (x_2, x_3, \dots, x_n, f(t, (x_1, x_2, \dots, x_n)))$

$(t_0, \tilde{x_0}) \in D = \mathring{D}$; $\tilde{f}(\cdot,\cdot)$ cont $(f(\cdot,\cdot)$ cont$)$ $\Rightarrow$ Th. Peano pt. ec. (2) $\Rightarrow \exists\ \tilde{\varphi}(\cdot): I_0 \in V(t_0) \to \mathbb{R}^n$ sol. a ec. (2) cu $\tilde{\varphi}(t_0) = \tilde{x_0}$

Prop. de echiv.: $\tilde{\varphi}(\cdot) = (\varphi(\cdot), \varphi'(\cdot), \dots, \varphi^{(n-1)}(\cdot))$ cu $\varphi(\cdot)$ sol. a ec. (1)
$\tilde{\varphi}(t_0) = \tilde{x_0} \Leftrightarrow \varphi(t_0) = x_0$, $\varphi'(t_0) = x_0', \dots, \varphi^{(n-1)}(t_0) = x_0^{n-1}$ q.e.d.

Th. Cauchy - Lipschitz (pt. ec. de ordin superior)

Fie $f(\cdot,\cdot) : D = \mathring{D} \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ cont., local Lipschitz (ii) $x^{(n)} = f(t, x, x', \dots, x^{(n-1)})$ (1)

Atunci $\forall (t_0, (x_0, x_0', \dots, x_0^{(n-1)})) \in D$ $\exists!$ $\varphi(\cdot) : I_0 \in V(t_0) \to \mathbb{R}$ sol. cu $\varphi(t_0) = x_0$,
$\varphi'(t_0) = x_0^1, \dots, \varphi^{(n-1)}(t_0) = x_0^{n-1}$

Dem: Fie $(t_0, (x_0, x_0', \dots, x_0^{(n-1)})) \in D$ $\tilde{x_0} := (x_0, x_0', \dots, x_0^{(n-1)})$

(2) $\frac{d\tilde{x}}{dt} = \tilde{f}(t, \tilde{x})$ $\tilde{x} = (x_1, \dots, x_n)$ $\tilde{f}(t, (x_1, \dots, x_n)) = (x_2, x_3, \dots x_n, f(t, x_1, \dots, x_n))$

$f(\cdot,\cdot)$ cont., local Lipschitz (ii) $\Rightarrow \tilde{f}(\cdot,\cdot)$ cont. local Lipschitz (ii), $D = \mathring{D}$ $\Big\} \Rightarrow$ T. Cauchy - Lipschitz pt. ec. (2)

$\Rightarrow$ pt. $(t_0, \tilde{x_0}) \in D$ $\exists!$ $\tilde{\varphi}(\cdot) : I_0 \in V(t_0) \to \mathbb{R}^n$ sol. a ec. (2) $\tilde{\varphi}(t_0) = \tilde{x_0}$
Prop. de echiv. $\tilde{\varphi}(\cdot) = (\varphi(\cdot), \varphi'(\cdot), \dots, \varphi^{(n-1)}(\cdot))$ cu $\varphi(\cdot)$ sol. a ec. (1)
$\tilde{\varphi}(t_0) = \tilde{x_0} \Leftrightarrow \varphi(t_0) = x_0$, $\varphi'(t_0) = x_0', \dots, \varphi^{(n-1)}(t_0) = x_0^{(n-1)}$

Continuitatea sol. locale în raport cu datele inițiale și parametrii

$\frac{dx}{dt} = f(t, x)$ $f(\cdot,\cdot) : D \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ admite E.U.L. pe D

$\forall (\tau, \xi) \in D$ $\exists!$ $\varphi_{\tau,\xi}(\cdot) : I_{\tau\xi} \in V(\tau) \to \mathbb{R}^n$ sol. a pb. Cauchy (f, τ, ξ)

Def: S.n. curentul local asociat câmpului vectorial $f(\cdot,\cdot)$ în $(t_0, x_0) \in D$ funcția
$\varphi(\cdot,\cdot) : I_1 \times I_0 \times G_0 \in V(t_0, t_0, x_0) \to \mathbb{R}^n$ cu propr. $\forall (\tau, \xi) \in I_0 \times G_0$
$\varphi(\cdot, \tau, \xi)$ sol. a pb. Cauchy (f, τ, ξ)

Def: S.n. câmp vectorial parametrizat $f(\cdot,\cdot,\cdot) : D \subseteq \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}$ a.î. $\forall \lambda$
$\in Pr_3 D$ $f(\cdot,\cdot,\lambda)$ c.v. def. $\frac{dx}{dt} = f(t, x, \lambda)$
$\to$ ~~formula~~ familie parametrizată de ec. diferențiale

Def: S.n. curent local parametrizat asociat c.v.p. $f(\cdot,\cdot,\cdot)$ în $(t_0, x_0, \lambda_0) \in D$
funcția $\varphi(\cdot,\cdot,\cdot,\cdot) : I_1 \times I_0 \times G_0 \times \Lambda_0 \in V(t_0, t_0, x_0, \lambda_0)$ a.î. $\forall$
$(\tau, \xi, \lambda) \in I_0 \times G_0 \times \Lambda_0$ $\varphi(\cdot, \tau, \xi, \lambda) : I_1 \to \mathbb{R}^n$ e sol. a Pb. Cauchy
$(f(\cdot,\cdot,\lambda), \tau, \xi)$

TO:

FROM:

Th (E.U. și continuitatea curentului local parametrizat)

Fie $f(\cdot,\cdot,\cdot): D = \mathring{D} \subseteq \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^k \to \mathbb{R}^n$ cont., local-Lipschitz (II) $\frac{dx}{dt} = f(t,x,\lambda)$
Atunci $\forall (t_0,x_0,\lambda_0) \in D\ \exists!\ \alpha(\cdot,\cdot,\cdot,\cdot): I_1 \times I_0 \times G_0 \times \Lambda_0 \in V(t_0,t_0,x_0,\lambda_0) \to \mathbb{R}^n$

Continuă curent local parametrizat

Dem: (Schiță) Fie $(t_0,x_0,\lambda_0) \in D$

1. $E := I_1 \times I_0 \times G_0 \times \Lambda_0 = ?$

$(t_0,x_0,\lambda_0) \in D = \mathring{D} \Rightarrow \exists\ \delta, r, \eta > 0$ a.î. $\underbrace{\bar{B}_\delta(t_0)\times\bar{B}_r(x_0)\times\bar{B}_\eta(\lambda_0)}_{n: D_0 \text{ compactă}} \subset D$

$K := \max_{(t,x,\lambda)\in D_0} \|f(t,x,\lambda)\|$

$K = 0 \Rightarrow \alpha(t,\tau,\xi,\lambda) = \xi$

$K > 0$ $a := \min\left\{\delta, \frac{r}{4K}\right\}$ $b := \frac{r}{2}$

$E = \bar{B}_a(t_0)\times\bar{B}_a(t_0)\times\bar{B}_b(x_0)\times\bar{B}_\eta(\lambda_0)$

$\left.\begin{array}{l} f(\cdot,\cdot,\cdot) \text{ local Lipschitz (II)} \\ D_0 \text{ compactă} \end{array}\right\} \Rightarrow \exists L > 0$ a.î. $\|f(t,x_1,\lambda) - f(t,x_2,\lambda)\| \le L\|x_1 - x_2\|$
$\forall (t,x_1,\lambda)$
$(t,x_2,\lambda) \in D_0$

2. $\alpha_m(\cdot,\cdot,\cdot,\cdot): E \to \mathbb{R}^n$ șirul aproximațiilor succesive al lui Picard

$\alpha_0(t,\tau,\xi,\lambda) = \xi$

$\alpha_m(t,\tau,\xi,\lambda) = \xi + \int_\tau^t f(s,\alpha_m(s,\tau,\xi,\lambda),\lambda)\,ds$, $m \ge 1$

a) $\alpha_m(\cdot,\cdot,\cdot,\cdot)$ - continuă $\forall m$ (inducție după m)

b) $\alpha_m(t,\tau,\xi,\lambda) \in \bar{B}_r(x_0)$ $\forall m, \forall (t,\tau,\xi,\lambda) \in E$

c) $\alpha_m(\cdot,\cdot,\cdot,\cdot)$ șir ~~este~~ uniform Cauchy $\Rightarrow$ $\alpha_m(\cdot,\cdot,\cdot,\cdot) \xrightarrow[m\to\infty]{u} \alpha(\cdot,\cdot,\cdot,\cdot)$

3. $\alpha(\cdot,\cdot,\cdot,\cdot)$ curent local parametrizat (ec. int. a sis)

4. unicitatea ($\simeq$ dem. Th. Cauchy-Lipschitz)