31.10.2017.

CURS 5 EC. DIFERENȚIALE

Existența și unicitatea globală a soluțiilor

Teoremă (Unicitatea Globală)

Fie $f(\cdot,\cdot): D \subseteq \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ cont $\frac{dx}{dt} = f(t,x)$

Atunci $f(\cdot,\cdot)$ admite UG a sol. pe $D \Leftrightarrow f(\cdot,\cdot)$ admite UL a sol pe D.

Dem: „$\Rightarrow$" Evident.

„$\Leftarrow$" $\varphi_1: I_1 \to \mathbb{R}^n$
$\varphi_2: I_2 \to \mathbb{R}^n$ sol pt care $\exists\, t_0 \in I_1 \cap I_2$ a.î. $\varphi_1(t_0) = \varphi_2(t_0)$

$$\overset{?}{\Rightarrow} \varphi_1|_{I_1\cap I_2} = \varphi_2|_{I_1\cap I_2}$$

Tehnică generală de globalizare

$I^* = \{t \in I_1\cap I_2;\ \varphi_1(t) = \varphi_2(t)\} \subset I_1 \cap I_2$ - interval $\to$ m. conexă

[M conexă $\Leftarrow$ deschisă, închisă, nu are submulțimi cu aceste propr.]

Arăt că: a) $I^* \neq \emptyset$
b) I^* închisă
c) I^* deschisă
$\Bigg\} \Rightarrow I^* = I_1 \cap I_2$.

a) $I^* \neq \emptyset$ pt. că $t_0 \in I_1 \cap I_2$

b) $I^* = \{t\in I_1\cap I_2;\ \varphi_1(t) = \varphi_2(t)\} = \{t \in I_1\cap I_2;\ (\varphi_1-\varphi_2)(t) =$
$= \underbrace{(\varphi_1-\varphi_2)}_{\text{cont.}}{}^{-1}(\underbrace{\{0\}}_{\text{închisă}})$ - închisă

c) I^* deschisă

$t_1 \in I^* \Rightarrow \varphi_1(t_1) = \varphi_2(t_1) \overset{U.L.}{\Longrightarrow} \exists\, I_0 \in \mathcal{V}(t_1)$ a.î.
$$\varphi_1|_{I_0\cap(I_1\cap I_2)} = \varphi_2|_{I_0\cap(I_1\cap I_2)}$$
$\Rightarrow I_0 \cap (I_1\cap I_2) \subset I^*$ deci I^* deschisă.

Prelungirea soluțiilor. Soluții maximale.

Def: a) $\varphi_1(\cdot): I_1 \subseteq \mathbb{R} \to \mathbb{R}^n$ s.n. prelungire a lui $\varphi_2: I_2 \subseteq \mathbb{R} \to \mathbb{R}^n$

b) $\varphi_1(\cdot) : (a, b_1) \subseteq \mathbb{R} \to \mathbb{R}^n$ s.n. prelungire la dreapta a lui $\varphi_2(\cdot) : (a, b_2) \subseteq \mathbb{R} \to \mathbb{R}^n$ dacă $b_2 < b_1$ și $\varphi_1(\cdot)\big|_{(a,b_2)} = \varphi_2(\cdot)$.

$\varphi_1(\cdot) \succ_d \varphi_2$

c) $\varphi_1(\cdot) : (a_1, b) \subseteq \mathbb{R} \to \mathbb{R}^n$ s.n. prelungire la stânga a lui $\varphi_2(\cdot) : (a_2, b) \subseteq \mathbb{R} \to \mathbb{R}^n$ dacă $a_1 < a_2$ și $\varphi_1(\cdot)\big|_{(a_2,b)} = \varphi_2(\cdot)$.

$\varphi_1(\cdot) \succ_s \varphi_2(\cdot)$

PROP: Fie $f(\cdot,\cdot) : D \subseteq \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ $\quad \frac{dx}{dt} = f(t,x)$

$$S_f := \{ \varphi(\cdot) ;\ \varphi(\cdot) : I \subseteq \mathbb{R} \to \mathbb{R}^n \ \varphi(\cdot) \text{ sol a ec } \frac{dx}{dt} = f(t,x) \}$$

Atunci $(S_f, \succ)$, $(S_f, \succ_d)$, $(S_f, \succ_s)$ sunt mulțimi ordonate

Dem: Ex!

Def: $\varphi(\cdot) \in S_f$ s.n. soluție maximală dacă este un element maximal $(S_f, \ )$

Teorema asupra prelungirii soluțiilor:

Fie $f(\cdot,\cdot) : D = \overset{\circ}{D} \subseteq \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ cont $\quad \frac{dx}{dt} = f(t,x)$

Fie $\varphi(\cdot) : (a, b) \subset \mathbb{R} \to \mathbb{R}^n$ soluție.

Atunci

1. $\varphi(\cdot)$ admite o prelungire strictă la dreapta $\Leftrightarrow b < +\infty$ $\to$ + se poate fi prelungită cu ea însăși
$\exists\, t_0 \in (a, b)$, $\exists\, D_0 \subset D$ compactă a.î. $(t, \varphi(t)) \in D_0 \ \forall t \in [t_0, b)$
graficul fc.

2. $\varphi(\cdot)$ admite o prelungire strictă la stânga $\Leftrightarrow a > -\infty$,
$\exists\, t_0 \in (a, b)$, $\exists\, D_0 \subset D$ compactă a.î. $(t, \varphi(t)) \in D_0 \ \forall t \in (a, t_0]$

Dem: 1) (2) analog)

"$\Rightarrow$" $\varphi(\cdot)$ admite o prelungire strictă la dreapta

$\Rightarrow \exists\, \varphi_1(\cdot) : (a, b_1) \to \mathbb{R}^n$ sol ec b_1 $\varphi_1(\cdot) \succ_d \varphi(\cdot)$

$b < b_1 \Rightarrow b < +\infty$ $\quad$ Fie $t_0 \in (a, b)$

$D_0 := \{ t, \varphi_1(t) ;\ t \in \underbrace{[t_0, b]}_{\text{compact}} \} \subset D$ $\quad (\varphi_1(\cdot)$ cont$)$
compactă

$t \in [t_0, b)$ $(t, \varphi(t)) = (t, \varphi_1(t)) \in D_0$ OK.

2. Aplicarea T Peano în $(b, \xi) \Rightarrow \exists \varphi_0(\cdot) : [b-\alpha, b+\alpha] \to \mathbb{R}^n$ sol a pb. $C(f, b, \xi)$.

3. $\varphi_1(t) := \begin{cases} \varphi(t), & t \in (a, b) \\ \varphi_0(t) & t \in [b, b+\alpha] \end{cases}$.

$\varphi_1(\cdot) \underset{\neq}{\succ} \varphi(\cdot)$

1. Criteriul lui Cauchy

$[\forall \varepsilon > 0 \ \exists \delta_\varepsilon > 0$ a.î. $\forall t', t'' \in (b - \delta_\varepsilon, b)$ $\quad \|\varphi(t') - \varphi(t'')\| < \varepsilon \Rightarrow \exists \lim\limits_{\substack{t \to b \\ t < b}} \varphi(t)]$

$\varphi(\cdot)$ sol $\Rightarrow$ ec integrala asoc $\varphi(t) = \varphi(t_0) + \int_{t_0}^{t} f(s, \varphi(s)) ds$

$\forall t, t_0 \in (a, b)$

$\varphi(t') = \varphi(t_0) + \int_{t_0}^{t'} f(s, \varphi(s)) ds$

$\varphi(t'') = \varphi(t_0) + \int_{t_0}^{t''} f(s, \varphi(s)) ds$

$\| \varphi(t') - \varphi(t'') \| = \| \int_{t''}^{t'} f(s, \varphi(s)) ds \| \le | \int_{t''}^{t'} \underbrace{\|f(s, \varphi(s))\|}_{\le K} ds |$

$K := \max\limits_{(t,x) \in \Delta_0} \| f(t, x) \|$ $\quad \le \underline{K |t' - t''|}$ $\quad (\delta_\varepsilon = \frac{\varepsilon}{K})$

2. $\varphi_1(\cdot)$ soluție

$\underline{\varphi_{1,s}'(b)} = \lim\limits_{\substack{t \to b \\ t < b}} \frac{\varphi_1(t) - \varphi_1(b)}{t - b} = \lim\limits_{\substack{t \to b \\ t < b}} \frac{\varphi(t) - \xi}{t - b} \overset{L'H}{=} \lim\limits_{\substack{t \to b \\ t < b}} \varphi'(t)$

$= \lim\limits_{\substack{t \to b \\ t < b}} f(t, \varphi(t)) = \underline{f(b, \xi)} = \cancel{f(b, \varphi_1(b))} = \varphi_{0,d}'(b) = \underline{\varphi_{1,d}'(b)}$

Teorema (Existența sol. maximale)

Fie $f(\cdot, \cdot) : D = \mathring{D} \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ cont $\frac{dx}{dt} = f(t, x)$

Atunci $(\forall)$ $\varphi(\cdot) \in S_f$ $\exists$ $\varphi_1(\cdot) \in S_f$ sol maximală $\varphi_1(\cdot) \succ \varphi(\cdot)$

Dem: Fie $\varphi(\cdot) \in S_f$

$S_{f,\varphi} = \{\psi(\cdot); \psi(\cdot) \in S_f \ \psi(\cdot) \succ \varphi(\cdot)\} \neq \emptyset \ (\varphi(\cdot) \in S_{f,\varphi})$

Lema lui Zorn

$\forall$ $P \subset S_{f,\varphi}$ total ordonată admite un majorant $\Rightarrow \exists \varphi_1(\cdot) \in S_{f,\varphi}$ elem. maximal.

Fie $P = \{\varphi_j\}_{j \in J} \subset S_{f,\varphi}$ total ordonată

$\forall i, j \in J$ sau $\varphi_i(\cdot) \succ \varphi_j(\cdot)$

sau $\varphi_j(\cdot) \succ \varphi_i(\cdot)$

Fie $I_j := \text{dom } \varphi_j(\cdot), j \in J$

$I^* := \bigcup_{j \in J} I_j \quad \varphi_*(\cdot) : I^* \to \mathbb{R}^n \quad \varphi_*(t) = \varphi_j(t)$ dacă $t \in I_j$

Arătăm a) I^* interval

b) $\varphi_*(\cdot)$ sol

a) I^* interval

$t_1, t_2 \in I^* \Rightarrow [t_1, t_2] \subset I^*$

$t_i \in I^*, i = \overline{1,2} \Rightarrow \exists j_1, j_2 \in J$ a.î. $t_1 \in I_{j_1} = \text{dom } \varphi_{j_1}(\cdot)$

$t_2 \in I_{j_2} = \text{dom } \varphi_{j_2}(\cdot)$

sau $\varphi_{j_1}(\cdot) \succ \varphi_{j_2}(\cdot)$ $I_{j_1} \supset I_{j_2} \Rightarrow t_1, t_2 \in I_{j_1}$ interval

$\Rightarrow [t_1, t_2] \subset I_{j_1} \Rightarrow [t_1, t_2] \subset I^*$

sau $\varphi_{j_2}(\cdot) \succ \varphi_{j_1}(\cdot)$ $I_{j_2} \supset I_{j_1} \Rightarrow t_1, t_2 \in I_{j_2}$ interval

$\Rightarrow [t_1, t_2] \subset I_{j_2} \Rightarrow [t_1, t_2] \subset I^*$

b) $\varphi_*(\cdot)$ sol

$t \in I^* \Rightarrow \exists j \in J$ a.î. $t \in I_j \Rightarrow \varphi_*(\cdot)|I_j = \varphi_j(\cdot)$

$\underline{\varphi_*'(t)} = \varphi_j'(t) = f(t, \varphi_j(t)) = \underline{f(t, \varphi_*(t))}$

PROP (intervalul de def al sol maximale)

Fie $f(\cdot,\cdot) : D = \overset{\circ}{D} \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ cont $\frac{dx}{dt} = f(t,x)$

Fie $\varphi(\cdot) : I \subseteq \mathbb{R} \to \mathbb{R}^n$ ...

Dem: Fie $\varphi(\cdot): I \to \mathbb{R}^n$ sol maximală
Pp că I nu este deschis $\Rightarrow I = [a, b]$ sau $I = [a, b)$
sau $I = (a, b]$

Pp de ex $I = (a, b]$
Aplic T Peano în $(b, \varphi(b)) \Rightarrow \exists\ \varphi_1(\cdot): [b-\alpha, b+\alpha] \subseteq \mathbb{R}^n$
sol a pb Cauchy $(f, b, \varphi(b))$
Fie $\varphi_2(t) = \begin{cases} \varphi(t), & t \in (a, b] \\ \varphi_1(t), & t \in [b, b+\alpha] \end{cases}$
$\varphi_2(\cdot) \underset{\neq}{\succ} \varphi(\cdot)$ $\varphi_2(\cdot)$ sol $\not\propto$ ($\varphi(\cdot)$ maximală)

Teorema (Existența și unicitatea sol. maximale)
Fie $f(\cdot,\cdot): D = \overset{\circ}{D} \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ cont $\frac{dx}{dt} = f(t, x)$ admite UL pe D.
Atunci:
1) $\forall\ \varphi(\cdot) \in S_f\ \exists!\ \varphi_1(\cdot) \in S_f$ maximală $\varphi_1(\cdot) \succ \varphi(\cdot)$
2) $\forall\ (t_0, x_0) \in D\ \exists!\ \varphi_{t_0, x_0}(\cdot): I(t_0, x_0) = (t^-(t_0, x_0), t^+(t_0, x_0))$
$\to \mathbb{R}^n$ sol maximală a pb Cauchy (f, t_0, x_0)

Dem: 1. Fie $\varphi(\cdot) \in S_f$ T. existența sol. maximale $\Rightarrow$
$\exists\ \varphi_1(\cdot): I_1 \subset \mathbb{R} \to \mathbb{R}^n$ maximală $\varphi_1(\cdot) \succ \varphi(\cdot)$
Pp. că $\exists\ \varphi_2(\cdot): I_2 \subset \mathbb{R} \to \mathbb{R}^n$ maximală $\varphi_2(\cdot) \succ \varphi(\cdot)$
$\varphi_1(\cdot) \neq \varphi_2(\cdot)$
$I_1 \neq I_2$ ($\varphi_1(\cdot), \varphi_2(\cdot) \succ \varphi(\cdot) \underset{U.G}{\overset{U.L}{\Rightarrow}} \varphi_1(\cdot)|_{I_1 \cap I_2} = \varphi_2|_{I_1 \cap I_2}$
$I_1 = I_2 \Rightarrow \varphi_1 = \varphi_2$ $\not\propto$)
Fie $\varphi_3(t) = \begin{cases} \varphi_1(t), & t \in I_1 \\ \varphi_2(t), & t \in (I_1 \cup I_2) \setminus I_1 \end{cases}$
$\varphi_3(\cdot) \underset{\neq}{\succ} \varphi_1(\cdot)$ $\not\propto$ ($\varphi_1(\cdot)$ maximală)

2. Fie $(t_0, x_0) \in D = \overset{\circ}{D}$, $f(\cdot,\cdot)$ continuă $\Rightarrow$ T. Peano
$\Rightarrow \exists\ \varphi_0(\cdot): [t_0 - a, t_0 + a] \to \mathbb{R}^n$ sol cu $\varphi_0(t_0) = x_0$
T. existența sol. maximale $\Rightarrow \exists\ \varphi_{t_0, x_0}(\cdot): I(t_0, x_0) \to \mathbb{R}^n$
sol maximală ...

PROP(intervalul de def al sol max) $I(t_0, x_0) := (t^-(t_0, x_0), t^+(t_0, x_0))$

Unicitatea: Analog cu 1.

Existența globală a soluțiilor

<u>Def</u>: $f(\cdot,\cdot): I \times \mathbb{R}^n \to \mathbb{R}^n$, $I \subseteq \mathbb{R}$ interval are prop. de disipativitate (Δ) dacă $\exists\, r > 0$ $\exists\, a(\cdot): I \to \mathbb{R}_+$ continuă a.î.: $|\langle x, f(t,x)\rangle| \leq a(t) \cdot \|x\|^2 \quad \forall\, t \in I, \forall\, x \in \mathbb{R}^n \; \|x\| > r$

T(E.G)

Fie $f(\cdot,\cdot)$: <u>$I \times \mathbb{R}^n \to \mathbb{R}^n$</u>, <u>continuă</u> cu <u>(Δ)</u> $\frac{dx}{dt} = f(t,x)$

Atunci $f(\cdot,\cdot)$ admite E.G. a sol pe $I \times \mathbb{R}^n$ ($\forall\, (t_0, x_0) \in I \times \mathbb{R}^n$ $\exists\, \varphi(\cdot): I \to \mathbb{R}^n$ sol cu $\varphi(t_0) = x_0$)

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Ec. Diferențiale
Seminar 5

$x^{(n)} = f(t, x, x', \dots, x^{(n-1)})$ $\quad f(\cdot,\cdot) : D \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$

T. Peano: $D = \overset{\circ}{D}$, $f(\cdot,\cdot)$ cont $\Rightarrow$ E.L. pe D ($\forall (t_0, (x_0, x_0^1, \dots, x_0^{n-1})) \in D$
$\exists$ $\varphi(\cdot) : I_0 \in \mathcal{V}(t_0) \to \mathbb{R}$ sol cu $\varphi(t_0) = x_0$, $\varphi'(t_0) = x_0^1 \dots \varphi^{(n-1)}(t_0)$
$= x_0^{n-1}$)

T. Cauchy-Lipschitz (EUL)

$D = D^\circ$, $f(\cdot,\cdot)$ cont, local-Lipschitz (II) $\Rightarrow$ EUL pe D

$\forall (t_0, (x_0, x_0^1, \dots x_0^{n-1})) \in D$ $\exists!$ $\varphi(\cdot) : I_0 \in \mathcal{V}(t_0) \to \mathbb{R}$ sol cu $\varphi(t_0) = x_0$,
$\varphi'(t_0) = x_0^1, \dots, \varphi^{(n-1)}(t_0) = x_0^{n-1}$).

1) $\forall n \in \mathbb{N}$ Să se det K_n = nr sol posibile ale pb:
$$x^{(n)} = t + x^3, \; x(0) = 1, \; x'(0) = 0.$$

$n = 0$: $x = t + x^3$, $x(0) = 1$, $x'(0) = 0$.

$x(t) = t + x^3(t)$, $\forall t$, $x(0) = 1$, $x'(0) = 0$.

$t = 0 \Rightarrow x(0) = x^3(0)$

$1 = 1$

$x'(t) = 1 + 3x^2(t) \cdot x'(t)$

$t = 0$ $\quad x'(0) = 1 + 3x^2(0)x'(0)$

$0 = 1$ F

$\Rightarrow K_0 = 0$.

$n = 1$: $\quad x' = t + x^3$

$x(0) = 1$

$x'(0) = 0$

$t = 0 \Rightarrow x'(0) = x^3(0)$

$0 = 1$ F

$\Rightarrow K_1 = 0$

$n = 2$: $x'' = t + x^3$, $x(0) = 1$, $x'(0) = 0$ Pb. Cauchy $\Rightarrow$ Aplic T. C. Lipschitz

$f(t, (x_1, x_2)) = t + x_1^3$

T. C. Lipschitz $\exists ! \, \varphi(\cdot): J_0 \in \mathcal{V}(0) \to \mathbb{R}$ sol. a.î. $\varphi(0)=1$, $\varphi'(0)=0$

$\Rightarrow k_2 = 1$.

Pt. $n=3$: $x''' = t + x^3$, $x(0)=1$, $x'(0)=0$

Definim $f(t,(x_1,x_2,x_3)) = t + x_1^3$

$f: \underbrace{\mathbb{R}\times\mathbb{R}^3}_{\text{deschisă}} \to \mathbb{R}$, $f(\cdot)$ continuă și local Lipschitz (II) și

de cls $C^1(\mathcal{V}) \xRightarrow{T.C.L} \forall a \in \mathbb{R}, \exists ! \, \varphi_a(\cdot): J_a \in \mathcal{V}(0) \to \mathbb{R}$.

sol. a.î. $\varphi_a(0)=1$, $\varphi'_a(0)=0$, $\varphi''_a(0)=0 \Rightarrow a \in \mathbb{R}$ arbitrar,

$k_3 = \infty$.

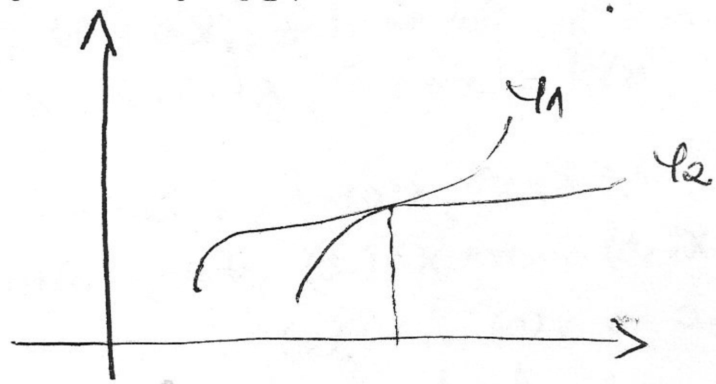
Pentru $n \geq 4 \Rightarrow k_n = \infty$.

2.) Să se studieze posibilitatea ca graficele a 2 sol. distincte să fie tangente pt fiecare din ec:

a) $x' = t^2 + x^4$

b) $x'' = t^2 + x^4$

c) $x''' = t^2 + x^4$



$\varphi_1: I_1 \to \mathbb{R}$, $\varphi_2: I_2 \to \mathbb{R}$

$\exists \, t_0 \in I_1 \cap I_2$ a.î. $\varphi_1(t_0) = \varphi_2(t_0) = x_0$

$\varphi_1'(t_0) = \varphi_2'(t_0) = x_0^1$

a) $x' = t^2 + x^4$

$f(t,x) = t^2 + x^4$

$f: \mathbb{R}\times\mathbb{R} \to \mathbb{R}$

cont., local-Lipschitz (II)

Cauchy-Lipschitz $\forall t_0, x_0 \in \mathbb{R} \; \exists ! \, \varphi(\cdot): I_0 \to \mathbb{R}$ sol $\varphi(t_0) = x_0 \, \forall t_0$

b) $x'' = t^2 + x^4$

$f(t,(x_1,x_2)) = t^2 + x_1^4$

$f: \mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}$

cont., local Lipschitz (II)

$\varphi_1'(t)=1$

$\varphi_2'(t)=1+3t^2$

$1=1+3\cdot 0$ O.K.

$\Rightarrow x_0^1=1. \xrightarrow{CL}$ în $(0,(0,1)) \Rightarrow$ ctd

- Pt $n=3$

$x'''=f(t,x)$

Pp $\exists$ f ca în enunț.

Fie $g(t,(x_1,x_2,x_3))=f(t,x_1)$

$g: \mathbb{R}\times\mathbb{R}^3\to\mathbb{R}$ cont și local-Lipschitz (II)

$\xrightarrow{C.L.}$ EUL pe $\mathbb{R}\times\mathbb{R}^3 \Rightarrow \forall (t_0,(x_0,x_0^1,x_0^2))\in\mathbb{R}\times\mathbb{R}^3$

$\exists!$ sol $\varphi: \mathcal{Y}_0\in\mathcal{V}(t_0)\to\mathbb{R}$ a.î. $\varphi(t_0)=x_0$

$\varphi'(t_0)=x_0^1$

$\varphi''(t_0)=x_0^2$

? t_0 a.î. $\left.\begin{array}{l}\varphi_1(t_0)=\varphi_2(t_0)=x_0\\ \varphi_1'(t_0)=\varphi_2'(t_0)=x_0^1\\ \varphi_1''(t_0)=\varphi_2''(t_0)=x_0^2\end{array}\right\}$ $t_0=0$, $x_0=0$, $x_0^1=1$

$\varphi_1''(t)=0$

$\varphi_2''(t)=6t$

$0=6\cdot 0 \Rightarrow x_0^2=0.$ ctd CL. în $(0,(0,1,0))$

- Pt $n=4$

$x^{(4)}=f(t,x)$

Pp. $\exists$ f ca în enunț.

Fie $g(t,(x_1,x_2,x_3,x_4))=f(t,x_1)$

$g: \mathbb{R}\times\mathbb{R}^4\to\mathbb{R}$ cont și local Lipschitz (II)

$\xrightarrow{TCL}$ EUL pe $\mathbb{R}\times\mathbb{R}^4 \Rightarrow \forall (t_0,(x_0,x_0^1,x_0^2,x_0^3))\in\mathbb{R}\times\mathbb{R}^4$

$\exists!$ sol $\varphi: \mathcal{Y}_0\in\mathcal{V}(t_0)\to\mathbb{R}$ a.î. $\left.\begin{array}{l}\varphi(t_0)=x_0\\ \varphi'(t_0)=x_0^1\\ \varphi''(t_0)=x_0^2\end{array}\right\}$ $\begin{array}{l}t_0=0\\ x_0=0\\ x_0^1=1\\ x_0^2=0\end{array}$

$\varphi'''(t_0)=x_0^3$

$\varphi_1'''(t)=0$

$\varphi_2'''(t)=6$

$\varphi_1'''(0)=\varphi_2'''(0)$

$0=6$ Fals

c) $x''' = t^2 + x^4$

$f(t,(x_1,x_2,x_3)) = t^2 + x_1^4$

$f: \mathbb{R}\times\mathbb{R}^3 \to \mathbb{R}$

cont, Local Lipschitz (II)

$\xrightarrow{TCL} \forall (t_0,(x_0,x_0^1,x_0^2)) \in \mathbb{R}\times\mathbb{R}^3\ \exists!\ \varphi: I_0 \in \mathcal{V}(t_0) \to \mathbb{R}$

sol. a.î. $\varphi(t_0) = x_0;\ \varphi'(t_0) = x_0^1,\ \varphi''(t_0) = x_0^2$

Dacă $\varphi_1''(t_0) \neq \varphi_2''(t_0) \Rightarrow$ DA!

3) Să se det $n \in \mathbb{N}$ pt care $\exists\ f(\cdot,\cdot): \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ cont și local Lipschitz (II) a.î. $\begin{cases}\varphi_1(t) = t\\ \varphi_2(t) = t + t^3\end{cases}$

sunt sol ale ec $x^{(n)} = f(t,x)$.

- Pt $n=0$ $\quad x = f(t,x)$

$\varphi_i(t) = f(t, \varphi_i(t))\ \forall t, \forall i=\overline{1,2}$.

$t = f(t,t)$

$t + t^3 = f(t, t^3+t)$

$\Rightarrow f(t,x) = x$.

- Pt $n=1$ $\quad x' = f(t,x)$

Pp. $\exists f: \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ cont și Local Lipschitz (II) $\xrightarrow{CL}$ EUL pe $\mathbb{R}\times\mathbb{R}$

$\Rightarrow \forall (t_0,x_0) \in \mathbb{R}\times\mathbb{R}\ \exists!\ \varphi: \mathcal{Y}_0 \in \mathcal{V}(t_0) \to \mathbb{R}$ sol cu $\varphi(t_0) = x_0$

? $\exists t_0$ a.î. $\varphi_1(t_0) = \varphi_2(t_0) = x_0$

$t_0 = t_0 + t_0^3$

$t_0 = 0 \Rightarrow x_0 = 0$.

$\Rightarrow \not\exists$ (C.L. în $(0,0)$).

- Pt $n=2$ $\quad x'' = f(t,x)$

Pp. că $\exists f$ ca în enunț

$g(t,(x_1,x_2)) = f(t,x_1)$

$g: \mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}$ cont, local Lipschitz (II) $\Rightarrow$ EUL pe $\mathbb{R}\times\mathbb{R}^2$

$\forall (t_0,(x_0,x_0^1)) \in \mathbb{R}\times\mathbb{R}^2\ \exists!\ \varphi: \mathcal{Y}_0 \in \mathcal{V}(t_0) \to \mathbb{R}$ sol cu

$\varphi(t_0) = x_0$

$\varphi_1^{(4)} = 0 = f(t,t)$

$\varphi_2^{(4)} = 0 = f(t, t+t^3)$

$f(t,x) \equiv 0$ verifică condițiile

- Pt $n \geq 4$: $f(t,x) \equiv 0$.

$n=3$ $\quad \varphi_i(\cdot)$ sunt sol. $\quad x''' = f(t,x)$

$0 = f(t,t) \ \forall t \qquad t=0 \quad 0 = f(0,0)$

$6 = f(t, t+t^3) \ \forall t \qquad 6 = f(0,0)$?!

$n=2$ $\quad \varphi_i(\cdot)$ sol pt $x'' = f(t,x)$

$0 = f(t,t)$

$6t = f(t, t+t^3) \ \forall t$.

f Local-Lipschitz (II) $\Rightarrow \exists L > 0$ a.î. $\|f(t,x_1) - f(t,x_2)\| \leq L\|x_1 - x_2\|$
în $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}$
$\forall (t,x_1), (t,x_2) \in \Delta_0 \in \mathcal{U}(t_0,x_0)$

$x_1 \to t$

$x_2 \to t+t^3$

$\|0 - 6t| \leq L|t - t - t^3|$

$6|t| \leq L|t^3| \quad |: |t| \quad t \neq 0$.

$t_0 = 0, x_0 = 0 \quad \Delta_0 \in \mathcal{U}(0,0)$

$\frac{6}{L} \leq \underbrace{|t^2|}_{\downarrow\, 0} \ \forall t \in \mathcal{U}(0) \ t \neq 0$?!

Curs 6 07.11.2017

Existența sol. algebrice

Def: $f(\cdot,\cdot): \mathring{I} \times \mathbb{R}^n \to \mathbb{R}^n$. Spunem că are prop. de:

a) Disipativitate (D) dacă $\exists r > 0$ $\exists a(\cdot): \mathring{I} \to \mathbb{R}_+$ cont. a.î.

$|\langle x, f(t,x)\rangle| \leq a(t)\cdot\|x\|^2 \quad \forall t \in \mathring{I}, \forall x \in \mathbb{R}^n, \|x\| > r$

b) Creștere liniară (CL) dacă $\exists r > 0$ $\exists a(\cdot): \mathring{I} \to \mathbb{R}_+$ continuă a.î. $\|f(t,x)\| \leq a(t)\cdot\|x\|, \forall t \in \mathring{I}, \forall x \in \mathbb{R}^n, \|x\| > r$

c) Creștere afină (CA) dacă $\exists r > 0$ $\exists a(\cdot), b(\cdot): \mathring{I} \to \mathbb{R}_+$ continuă a.î. $\|f(t,x)\| \leq a(t)\|x\| + b(t), \forall t \in \mathring{I}, \forall x \in \mathbb{R}^n,$ $\|x\| > r$

PROP 1. $CL \Leftrightarrow CA$

2. $CL \Rightarrow D$

3. $n = 1$ $CL \Leftrightarrow CD$

4. $n > 1$ $D \not\Rightarrow CL$

Dem: 1. $CL \Rightarrow CA$ $(b(t) \equiv 0)$

$CA \Rightarrow CL$ $\exists r > 0$ $\exists a(\cdot): \mathring{I} \to \mathbb{R}_+$ cont a.î.

$$\|f(t,x)\| \leq a(t)\|x\| + b(t) = \|x\|\left(a(t) + \frac{b(t)}{\|x\|}\right) \leq$$

$$\leq \|x\| \underbrace{\left(a(t) + \frac{b(t)}{r}\right)}_{\| \atop a_1(t)} \quad \forall t \in \mathring{I}, x \in \mathbb{R}^n, \|x\| > r$$

CL

2. inegalitatea C.B.S

$x, y \in \mathbb{R}^n, |\langle x, y\rangle| \leq \|x\|\cdot\|y\|$

$$|\langle x, f(t,x)\rangle| \overset{CBS}{\leq} \|x\|\cdot\|f(t,x)\| \leq a(t)\cdot\|x\|^2,$$

$\forall t \in \mathring{I}, x \in \mathbb{R}^n, \|x\| > r$ (D)

3. $n=1$ CL $\Rightarrow$ CD

D: $\exists\, r>0\ \exists\, a(\cdot): I \to \mathbb{R}_+$ a.î $|x \cdot f(t,x)| \le a(t)\cdot|x^2|$, $\forall t\in I$, $\forall x\in\mathbb{R}$, $|x|>r$

$|f(t,x)| \le a(t)\cdot |x|$, $\forall t\in I$, $\forall x\in\mathbb{R}$, $|x|>r$ adică CL

4. D $\not\Rightarrow$ CL

$n=2$ $f=?$ a.î $\langle x, f(t,x)\rangle = 0$

$\langle (x_1,x_2), (f_1(t,x), f_2(t,x))\rangle = 0$

$x_1 f_1(t,x) + x_2 f_2(t,x) = 0$

$f_1(t,x) = x_2$

$f_2(t,x) = -x_1$

$f(t(x_1,x_2)) = (x_2, -x_1)$

$\| f(t,x)\| = \sqrt{x_2^2 + x_1^2} = \|x\|$ are CL

Alegem $f(t(x_1,x_2)) = (x_2, -x_1)\cdot\|x\|$

$\langle x, f(t,x)\rangle = 0$, $\|f(\alpha,x)\| = \|x\|\cdot\sqrt{x_1^2+x_2^2} = \|x\|^2$

Pp. că $f(\cdot,\cdot)$ are C.L $\Rightarrow \exists\, r>0$, $\exists\, a(\cdot): \mathring{I} \to \mathbb{R}_+$ cont

$\|x\|^2 \le a(t)\cdot\|x\|$, $\forall t\in\mathring{I}$, $\forall x\in\mathbb{R}^2$, $\|x\|>r$

$\|x\| \le a(t)$ $\forall t, \forall x$

Fie $t=t_0\in\mathring{I} \Rightarrow \|x\| \le a(t_0)$, $\forall x\in\mathbb{R}^n$, $\|x\|>0$

∞ ∞ (Nu poate fi mărginită)

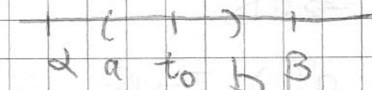
$\boxed{\text{T (Existență Globală) E.G}}$

$f(\cdot,\cdot): \mathring{I} \times \mathbb{R}^n \to \mathbb{R}^n$ cont cu (D)

$$\frac{dx}{dt} = f(t,x)$$

Atunci $\forall (t_0,x_0)\in\mathring{I}\times\mathbb{R}^n$ $\exists\, \varphi(\cdot): \mathring{I} \to \mathbb{R}^n$ sol cu $\varphi(t_0) = x_0$

Dem: Pp $\dot{I} = (\alpha, \beta)$



Fie $(t_0, x_0) \in \dot{I} \times \mathbb{R}^n$ – mult. deschisă, $f(\cdot,\cdot)$ continuă $\Longrightarrow$

T. Peano (E.L) $\exists\, \varphi_0(\cdot): \dot{I}_0 \in V(t_0) \to \mathbb{R}^n$ sol cu $\varphi_0(t_0) = x_0$.

T. existența sol maximale $\Rightarrow \exists\, \varphi(\cdot): \dot{I} \to \mathbb{R}^n$ sol maximală $\varphi(\cdot)$, $\varphi \;\; \varphi_0(\cdot) \Rightarrow \varphi(t_0) = \varphi_0(t_0) = x_0$

$\downarrow$

φ_0 extindere a lui φ

PROP (intervalul de def. al sol maximale)

Arătăm că $J = \dot{I} \iff a = \alpha,\ b = \beta$

De exemplu, arătăm că $a = \alpha$ (analog $b = \beta$).

Pp. $\alpha < a$

T. (asupra prelungirii sol)

$f(\cdot,\cdot): D = \mathring{D} \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ cont. $\frac{dx}{dt} = f(t,x)$

$\varphi(\cdot): (a,b) \subset \mathbb{R} \to \mathbb{R}^n$ sol. Atunci:

1. ...

2. $\varphi(\cdot)$ admite o prelungire strictă la stânga $\iff a > -\infty$, $\exists\, t_0 \in (a,b)\ \exists\, D_0 \subset D$ compactă a.î $(t, \varphi(t)) \in D_0$, $\forall t\ (a,b)$

$\alpha < a \Rightarrow a > -\infty$, $t_0 \in (a,b)$? $D_0 \subset D$ compactă a.î $(t, \varphi(t)) \in D_0$, $\forall t \in [a,b]$.

Dacă da $\Longrightarrow$ T. asupra prelungirii sol $\Longrightarrow$

$\varphi(\cdot)$ admite o prelungire strictă la stânga (ab $\varphi(\cdot)$ maximală)

$\varphi(\cdot,\cdot)$ are $(\Delta) \Longrightarrow \exists\, r > 0\ \exists\, a(\cdot): \dot{I} \to \mathbb{R}_+$ continuă a.î

$|\langle x, f(t,x)\rangle| \le a(t)\|x\|^2,\ \forall t \in I,\ \forall x \in \mathbb{R}^n\ \|x\| > r$.

$(\|\varphi(t)\|^2)' = 2\langle \varphi(t), \varphi'(t)\rangle = 2\langle \varphi(t), f(t, \varphi(t))\rangle$

Fie $A = \{t \in (a, t_0] ;\ \|\varphi(t)\| > r\}$

Dacă $t \in A$ $(\|\varphi(t)\|^2)' = 2 < \varphi(t), f(t, \varphi(t)) > \geq$
$-2a(t) \cdot \|\varphi(t)\|^2$

C.B.S

Dacă $t \in B$ $(\|\varphi(t)\|^2)' = 2 < \varphi(t), f(t, \varphi(t)) > \geq$

$(|<x,y>| \leq \|x\| \cdot \|y\|)$

$\geq -2 \cdot \underbrace{\|\varphi(t)\|}_{\leq r} \cdot \underbrace{\|f(t, \varphi(t))\|}_{\leq k} \geq -2rk$

$$k := \max_{(t,x) \in [a, t_0] \times \overline{B_r}(0)} \|f(t,x)\|$$

Dacă $t \in (a, t_0]$ $(\|\varphi(t)\|^2)' \geq \min(-2a(t)\|\varphi(t)\|^2, -2rk) \geq -2rk - 2a(t) \cdot \|\varphi(t)\|^2 \Big| \int_t^{t_0}$

$$\int_t^{t_0} (\|\varphi(s)\|^2)' ds \geq -2rk(t - t_0) - 2\int_t^{t_0} a(s) \cdot \|\varphi(s)\|^2 ds$$

$$\|\varphi(t_0)\|^2 - \|\varphi(t)\|^2 \geq -2rk(t - t_0) + 2\int_{t_0}^{t} a(s) \cdot \|\varphi(s)\|^2 ds$$

$$\|\varphi(t)\|^2 \leq \|\varphi(t_0)\|^2 + 2rk(t_0 - t) + 2\int_{t_0}^{t} a(s)\|\varphi(s)\|^2 ds \leq$$

$$\underbrace{\|\varphi(t_0)\|^2 + 2rk(t_0 - a)}_{= M} + 2\int_{t_0}^{t} a(s) \cdot \|\varphi(s)\|^2 ds$$

$$t \in (a, t_0] \Rightarrow \|\varphi(t)\|^2 \leq M + \left|\int_{t_0}^{t} 2a(s)\|\varphi(s)\|^2 ds\right| \xrightarrow[B6]{\text{Lema}}$$

$$\|\varphi(t)\| \leq M \cdot e^{2\left|\int_{t_0}^{t} a(s) ds\right|} \leq M \cdot e^{2\int_a^{t_0} a(s) ds} = \rho^2$$

$\Rightarrow \|\varphi(t)\| \leq \rho, \forall t \in (a, t_0] \Leftrightarrow \varphi(t) \in \overline{B_\rho}(0)$

$D_0 := [a, t_0] \times \overline{B_\rho}(0)$ $(t, \varphi(t)) \in D_0, \forall t \in [a, t_0)$

Continuitatea sol maximale în raport cu datele inițiale și parametrii

$\frac{dx}{dt} = f(t, x)$ $f(\cdot,\cdot) : D = \mathring{D} \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ cont admit UL pe D.

$\forall (\tau, \xi) \in D \ \exists! \varphi_{\tau, \xi}(\cdot) : I(\tau, \xi) = (t^-(\tau, \xi), t^+(\tau, \xi)) \to \mathbb{R}^n$ soluție maximală cu $\varphi_{\tau,\xi}(\tau) = \xi$

<u>Def</u>: s.n <u>curent maximal</u> al câmpului vectorial $f(\cdot,\cdot)$ funcția $\alpha_f(\cdot,\cdot,\cdot) : D_f \subseteq \mathbb{R} \times D \to \mathbb{R}^n$ a.î. $\forall (\tau, \xi) \in D$

$\alpha_f(\cdot, \tau, \xi) : \mathring{I}(\tau, \xi) \to \mathbb{R}$ sol. maximală unică a problemei Cauchy (f, τ, ξ).

$D_f = \{(t, \tau, \xi), (\tau, \xi) \in D, t \in \mathring{I}(\tau, \xi) = (t^-(\tau, \xi), t^+(\tau, \xi))\}$

$\alpha_f(\cdot, \tau, \xi) = \varphi_{\tau, \xi}(\cdot)$

$\to D_1 \alpha_f(t, \tau, \xi) \equiv f(t, \alpha_f(t, \tau, \xi))$

$\to \alpha_f(\tau, \tau, \xi) = \xi$

$\to \alpha_f(\cdot, \tau, \xi)$ sol. maximală

Teorema asupra curentului maximal

Fie $f(\cdot,\cdot) : D = \mathring{D} \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ continuă, $\frac{dx}{dt} = f(t, x)$

Pp. că a) $f(\cdot,\cdot)$ admite U.L pe D

b) $f(\cdot,\cdot)$ admite curent local continuu în fiecare punct din D

Fie $\alpha_f(\cdot,\cdot,\cdot): D_f \to \mathbb{R}^n$ curentul maximal.

Atunci:

1) $D_f \subseteq \mathbb{R}\times D$ deschisă

2) $\alpha_f(\cdot,\cdot,\cdot)$ continuă

<u>Obs</u>: Dacă f continuă, local-Lipschitz (II) $\Rightarrow$ T. Cauchy-Lipschitz $\Rightarrow$ EUL pe D adică a)

$\Rightarrow$ Teorema (Existența, unicitatea și continuitatea curentului local parametrizat) $\to f(\cdot,\cdot)$ admite curent local continuu în fiecare punct din D (adică b))

$$\frac{dx}{dt} = f(t,x,\lambda), \quad f(\cdot,\cdot,\cdot): D = \mathring{D} \subseteq \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^k \to \mathbb{R}^n \text{ continuă}$$

$\forall \lambda \in pr_3 D$, $f(\cdot,\cdot,\lambda)$ admite UL a sol.

$\forall (t_0,x_0,\lambda_0)\in D \ \exists! \ \varphi_{t_0,x_0,\lambda_0}(\cdot): I(t_0,x_0,\lambda_0) = (t^-(t_0,x_0,\lambda_0),$

$t^+(t_0,x_0,\lambda_0)) \to \mathbb{R}^n$ soluție maximală a pb. Cauchy

$(f(\cdot,\cdot,\lambda_0), t_0, \lambda_0)$

<u>Def</u>: Sn <u>curent maximal parametrizat</u> asociat cu $f(\cdot,\cdot,\cdot)$ funcția $\alpha_f(\cdot,\cdot,\cdot,\cdot): D_f \subseteq \mathbb{R}\times D \to \mathbb{R}^n$ a.î. $\forall(\tau,\xi,\lambda)\in D$,

$\alpha_f(\cdot,\tau,\xi,\lambda): I(\tau,\xi,\lambda) \to \mathbb{R}^n$ sol. maximală unică a pb. Cauchy $(f(\cdot,\cdot,\lambda),\tau,\xi)$

$\to D_1\alpha_f(t,\tau,\xi,\lambda) = f(t,\alpha_f(t,\tau,\xi,\lambda),\lambda)$

$\to \alpha_f(\tau,\tau,\xi,\lambda) = \xi$

$\to$ sol. maximală

Metodă generală de studiu (transf. parametrilor în date inițiale)

$$(2)\ \begin{cases} \dfrac{dx}{dt} = f(t,x,\lambda) \\ \dfrac{d\lambda}{dt} = 0_{\mathbb{R}^k} \end{cases} \qquad \overline{x} = (x,\lambda) \qquad \overline{f}(t,(x,\lambda)) = (f(t,x,\lambda), 0_{\mathbb{R}^k}).$$

$$(2)\ \frac{d\overline{x}}{dt} = \overline{f}(t,\overline{x})$$

$$\overline{f}(\cdot,\cdot) : D \subseteq \mathbb{R}\times(\mathbb{R}^n\times\mathbb{R}^k) \to \mathbb{R}^n\times\mathbb{R}^k$$

PROP (de echivalență)

$\varphi(\cdot)$ sol a ec (1) $\iff \overline{\varphi}(\cdot) = (\varphi(\cdot), \lambda)$ este sol pt. (2)

<u>Dem</u>: "$\Longrightarrow$" $\varphi'(t) \equiv f(t,\varphi(t),\lambda)$

$(\lambda)' \equiv 0$

"$\Longleftarrow$" Fie $\varphi(\cdot) = (\varphi_1(\cdot), \varphi_2(\cdot))$ sol. a ec (2) $\Longrightarrow$

$$\Longrightarrow \begin{cases} \varphi_1'(t) \equiv f(t, \varphi_1(t), \varphi_2(t)) \\ \varphi_2'(t) \equiv 0 \Longrightarrow \exists\, \lambda \in pr_3 D \text{ a.î. } \varphi_2(t) \equiv \lambda \end{cases} \Longrightarrow$$

$\Longrightarrow \varphi_1'(t) = f(t, \varphi_1(t), \lambda)$ ok

Corolar: $\alpha_f(\cdot,\cdot,\cdot,\cdot) : \Delta_f \subseteq \mathbb{R}\times D \to \mathbb{R}^n$ curent maximal parametrizat al ec. (1) $\iff \alpha_{\overline{f}}(\cdot,\cdot,\cdot,\cdot) : \Delta_{\overline{f}} = \Delta_f \to \mathbb{R}^n$,

$\alpha_{\overline{f}}(t, \tau, (\xi, \lambda)) := (\alpha_f(t,\tau,\xi,\lambda), \lambda)$ este curent maximal

(neparametrizat) al ec (2)

T (continuitatea sol. maximale în raport cu parametrii)

Fie $f(\cdot,\cdot,\cdot) : D = \mathring{D} \subseteq \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^k \to \mathbb{R}^n$ cont local Lipschit

(I) $\dfrac{dx}{dt} = f(t,x,\lambda)$.

Atunci:

1. $D_f \subseteq \mathbb{R} \times D$ deschisă
2. $\alpha_f(\cdot,\cdot,\cdot,\cdot)$ continuă

$\boxed{\text{Ecuații liniare pe } \mathbb{R}^n}$

Def: Fie $A(\cdot): \overset{\circ}{I} \subseteq \mathbb{R} \to L(\mathbb{R}^n, \mathbb{R}^n)$ def. ec. liniară $\frac{dx}{dt} = A(t) \cdot x$

Obs: $n=1$ $\quad L(\mathbb{R}, \mathbb{R}) \simeq \mathbb{R}$

$x' = a(t) \cdot x$ ec. liniară scalară

$\varphi(\cdot)$ sol a ec $\iff \varphi(t) \equiv c \cdot e^{\int_{t_0}^{t} a(s)ds}$

$B \subset \mathbb{R}^n$ bază

$B = \{b_1, \dots b_n\}$ $\quad L(\mathbb{R}^n, \mathbb{R}^n) \simeq M_n(\mathbb{R})$

$A_B(t) = (a_{ij}(t))_{\substack{i=\overline{1,n} \\ j=\overline{1,n}}}$ $\quad A_B(t) = col(A(t)b_1 \dots A(t)b_n)$

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} a_{ij}(t) x_j, \quad i = \overline{1,n} \text{ (sistem de ecuații liniare)}$$

$\boxed{T(EUG)}$

Fie $A(\cdot): \overset{\circ}{I} \to L(\mathbb{R}^n, \mathbb{R}^n)$ cont, $\frac{dx}{dt} = A(t) x$

Atunci $\forall (t_0, x_0) \in \overset{\circ}{I} \times \mathbb{R}^n$, $\exists!\ \varphi_0(\cdot): \overset{\circ}{I} \to \mathbb{R}^n$ sol cu $\varphi_0(t_0) = x_0$

Dem: $f_{A(\cdot)}(\cdot,\cdot)$ cont. local Lipschitz ($\overline{\text{II}}$) (liniară ($\overline{\text{II}}$)) $\Rightarrow$

T. Cauchy-Lipschitz $\quad EUL \overset{T}{\iff} UG$

$\|f_{A(\cdot)}(t, x)\| = \|A(t, x)\| \leq \|A(t)\| \cdot \|x\|$ $\quad C.L \Rightarrow D \underset{EG}{\overset{T}{\Longrightarrow}} EG$

07.11.2017

Seminar 6

$f(\cdot,\cdot): \mathring{I} \times \mathbb{R}^n \to \mathbb{R}^n$

DISIPATIVITATE (D)

$\exists r>0\ \exists a(\cdot): \mathring{I} \to \mathbb{R}_+$ continuă a.î. $|\langle x, f(t,x)\rangle| \leq a(t)\cdot\|x\|^2 \quad \forall t\in \mathring{I}, \forall x \in \mathbb{R}^n, \|x\| > r$

T(EG)

$f(\cdot,\cdot): \mathring{I}\times\mathbb{R}^n \to \mathbb{R}^n$ continuă cu (D) $\frac{dx}{dt} = f(t,x)$

Atunci $\forall (t_0, x_0) \in \mathring{I}\times\mathbb{R}^n\ \exists\ \varphi(\cdot): \mathring{I} \to \mathbb{R}^n$ sol cu $\varphi(t_0) = x_0$

---

1) Fie ec. $\begin{cases} x_1' = -x_2^2 \\ x_2' = x_1 x_2 \end{cases}$

a) Să se arate că $\forall\ \varphi(\cdot)$ sol. $\exists\ c \in \mathbb{R}$ a.î. $\|\varphi(t)\| \equiv c$

b) Să se arate că admite EG

a) În general, dem că o fct. e ct dacă e derivabilă, și derivata sa este nulă.

Fie $\varphi(\cdot)$ soluție, $\varphi(\cdot) = (\varphi_1(\cdot), \varphi_2(\cdot))$ (derivabilă și verifică ecuația). Deci:

$$\begin{cases} \varphi_1'(t) \equiv -\varphi_2^2(t) \\ \varphi_2'(t) \equiv \varphi_1(t)\cdot\varphi_2(t) \end{cases}$$

$\|\varphi(t)\|$ este ct.

$\|\varphi(t)\| = \sqrt{\varphi_1^2(t) + \varphi_2^2(t)}$

Fie $g(t) = \varphi_1^2(t) + \varphi_2^2(t)$

$g'(t) = 2\varphi_1(t)\cdot\varphi_1'(t) + 2\varphi_2(t)\cdot\varphi_2'(t)$

$= -2\varphi_1(t)\cdot\varphi_2^2(t) + 2\varphi_2^2(t)\cdot\varphi_1(t)$

$= 0$

$\Rightarrow \exists\ k \in \mathbb{R}$ ct. a.î. $g(t) = k$

b)

$f(t,(x_1,x_2)) = (-x_2^2, x_1 x_2)$

$f : \mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}^2$ ok $(I=\mathbb{R}, n=2)$

f continuă

Verificăm prop. de disipativitate:

$|\langle x, f(t,x)\rangle| = |\langle (x_1,x_2), f(t,(x_1,x_2))\rangle| =$
$= |\langle (x_1,x_2), (-x_2^2, x_1x_2)\rangle$

$= -x_2^2\cdot x_1 + x_1x_2^2 = 0 \to$ satisface (D)

$\xRightarrow{T.E.G}$ E.G pe $\mathbb{R}\times\mathbb{R}^2$

2) Fie ec. $\dfrac{dx_i}{dt} = \sum_{j,k=1}^{n} c_{ijk}\, x_j x_k,\ i=\overline{1,n}$

$c_{ijk} = -c_{kji}\ \ \forall i,j,k \in \{1,\dots n\}$

a) $\forall\ \varphi(\cdot)$ sol $\exists\ c\in\mathbb{R}$ a.î $\|\varphi(t)\| \equiv c$

b) Admite E.G

Fie $\varphi(\cdot)$ soluție, $\varphi(\cdot) = (\varphi_1(\cdot), \varphi_2(\cdot), \dots \varphi_n(\cdot))$, $\varphi(\cdot)$ derivabilă și verifică ecuația. Deci:

$\varphi_i'(t) = \sum_{j,k=1}^{n} c_{ijk}\, \varphi_j(t)\cdot\varphi_k(t),\ \forall\ i=\overline{1,n}$

$\|\varphi(t)\| = \sqrt{\varphi_1^2(t) + \varphi_2^2(t) + \ldots + \varphi_n^2(t)}$

Fie $g(t) = \varphi_1^2(t) + \ldots + \varphi_n^2(t)$

g deriv.

$$g'(t) = 2 \cdot \sum_{i=1}^{n} \varphi_i(t) \cdot \varphi_i'(t)$$

$$= 2 \cdot \sum_{i=1}^{n} \varphi_i(t) \cdot \sum_{j,k=1}^{n} c_{ijk} \cdot \varphi_j(t) \cdot \varphi_k(t)$$

$$= 2 \cdot \sum_{i,j,k=1}^{n} c_{ijk} \cdot \varphi_i(t) \cdot \varphi_j(t) \cdot \varphi_k(t) \underset{c_{ijk} = c_{kij}}{=} 0$$

$\Rightarrow \exists\, k \in \mathbb{R}$ a.î. $g(t) \equiv k$

b) $f(t, (x_1, \ldots x_n)) = \left(\sum_{j,k=1}^{n} c_{1jk} x_j x_k, \ldots, \sum_{j,k=1}^{n} c_{njk} x_j x_k \right)$

$f: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$

f continuă

Verificăm prop. de disipativitate.

$$|\langle x, f(t,x) \rangle| = \left|\left\langle (x_1, \ldots x_n), \left(\sum_{j,k=1}^{n} c_{1jk} x_j x_k, \ldots \sum_{j,k=1}^{n} c_{njk} x_j x_k \right) \right\rangle\right|$$

$$= x_1 \sum_{j,k=1}^{n} c_{1jk} x_{jk} + \ldots + x_n \sum_{j,k=1}^{n} c_{njk} x_j x_k$$

$$= \left| \sum_{i=1}^{n} x_i \sum_{j,k}^{n} c_{ijk} x_j x_k \right| = \left| \sum_{i,j,k=1}^{n} c_{ijk} x_i x_j x_k \right| = 0 \Rightarrow (D)$$

$\xrightarrow{TEG}$ EG pe $\mathbb{R} \times \mathbb{R}^n$

3) Fie ec. $\begin{cases} x_1' = x_1 x_2 \\ x_2' = -2 x_1^4 \end{cases}$

b) Admite EG.

a) $\varphi(\cdot)=(\varphi_1(\cdot),\varphi_2(\cdot))$ soluție $\Rightarrow \varphi(\cdot)$ deriv. și verifică ecuația.

Deci:
$$\begin{cases} \varphi_1'(t)=\varphi_1(t)\cdot\varphi_2(t) \\ \varphi_2'(t)=-2\varphi_1^4(t) \end{cases}$$

$\varphi_1^4(t)+\varphi_2^2(t)\equiv C.$

$g(t)=\varphi_1^4(t)+\varphi_2^2(t).$

$g(t)$ derivb

$$g'(t)=4\varphi_1^3(t)\cdot\varphi_1'(t)+2\varphi_2(t)\cdot\varphi_2'(t)$$
$$=4\varphi_1^4(t)\cdot\varphi_1(t)\varphi_2(t)+2\cdot\varphi_2(t)\cdot(-2\varphi_1^4(t))$$
$$=0.$$

$\Rightarrow \exists\, k\in\mathbb{R}$ a.î. $g(t)=k$.

b) $f(t,(x_1,x_2))=(x_1x_2,-2x_1^4)$

$f:\mathbb{R}\times\mathbb{R}^2\to\mathbb{R}^2$ OK

f cont

Verificăm prop. de disipativitate:

$$|\langle x, f(t,x)\rangle| = |\langle(x_1,x_2),(x_1x_2,-2x_1^4)\rangle|$$
$$=|x_1^2\cdot x_2-2x_2\cdot x_1^4|\le a(t)\|x\|^2=a(t)\cdot(x_1^2+x_2^2)$$

Pol. de grad 5 $\to$ Pol. de grad 2

Pp. că f are (D) $\Rightarrow \exists\, r>0\ \exists\, a(\cdot):\mathbb{R}\to\mathbb{R}_+$ cont a.î.

$$|x_1^2\cdot x_2-2x_1^4\cdot x_2|\le a(t)\cdot(x_1^2+x_2^2),\ \forall t\in\mathbb{R}, \forall x\in\mathbb{R}^2$$
$\|x\|>r.$

Fie $t=0$.

$$|x_1^2x_2-2x_1^4x_2|\le a(0)\cdot(x_1^2+x_2^2)\ \forall x\in\mathbb{R}^2, \|x\|>r$$

$x_n=(n,n)$

$\Rightarrow |n - 2n^3| \leq 2 \cdot a(0) \;\; \forall n > n_0$

$\Downarrow$

$\infty \quad\quad \infty$

$\forall (t_0, x_0) \in \mathbb{R} \times \mathbb{R}^2 \;\; \exists\, \varphi(\cdot) : \mathbb{R} \to \mathbb{R}^2$ sol, $\varphi(t_0) = x_0$.

Fie $(t_0, x_0) \in \underbrace{\mathbb{R} \times \mathbb{R}^2}_{\text{deschisă}}$), f cont $\Rightarrow$ T. Peano: $\exists\, \varphi_0(\cdot) : I_0 \to \mathbb{R}^2$ sol

$\varphi_0(t_0) = x_0$

T. (existența sol maximale) $\Longrightarrow \exists\, \varphi : I \to \mathbb{R}^2$ sol. maximală, φ prelungire a lui $\varphi_0 \Rightarrow \varphi(t_0) \equiv \varphi_0(t_0) = x_0$.

PROP (int. def al sol. max) $\Rightarrow I$ deschis, $I = (a, b)$

Arătăm că $I = \mathbb{R}$, $a = -\infty$, $b = +\infty$

Dem $b = +\infty$

Pp. că $b < +\infty$ $t_0 \in (a, b)$? $\exists\, D_0 \subset \mathbb{R} \times \mathbb{R}^2$ compactă a.î. $(t, \varphi(t)) \in D_0 \;\; \forall t \in [t_0, b)$

Dacă am găsit D_0 cu această prop aplicăm T asupra prelungirii sol $\Rightarrow$ $\left.\begin{array}{l}\varphi(\cdot) \text{ admite o prelungire strictă la dreapta} \\ \varphi(\cdot) \text{ sol maximală}\end{array}\right\} \infty$

$\overset{a)}{\Longrightarrow} \exists\, c \in \mathbb{R}$ a.î. $\varphi_1^4(t) + \varphi_2^2(t) \leq c \Rightarrow \varphi_1^4(t) \leq c \;\; \forall t \Rightarrow$

$\varphi_2^2(t) \leq c \;\; \forall (t)$

$\Rightarrow \varphi_1^2(t) \leq \sqrt{c}, \forall (t)$

$\Rightarrow \varphi_1^2(t) + \varphi_2^2(t) \leq c + \sqrt{c}$

$\|\varphi(t)\| \leq k, \; \forall (t) \Leftrightarrow \varphi(t) \in \overline{B_k}(0)$

$D_0 = [t_0, b] \times \overline{B}_k(0)$