TO:

FROM: Ecuații diferențiale - curs - 21.11.2017

Ecuații liniare pe $\mathbb{R}^n$ cu coeficienți constanți

Def: $A \in L(\mathbb{R}^n, \mathbb{R}^n)$ $\quad \frac{dx}{dt} = Ax$

în coordonate $\frac{dx_i}{dt} = a_{ij}x_j \quad i = \overline{1,n}$ sistem de ecuații liniare cu coef. const.

caz part.: $A(t) \equiv A \in L(\mathbb{R}^n, \mathbb{R}^n)$

Th. (E.V.G.)

$\forall (t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n \; \exists! \; \varphi(\cdot): \mathbb{R} \to \mathbb{R}^n$ sol. cu $\varphi(t_0) = x_0$.

$S_A := \{\varphi(\cdot): \mathbb{R} \to \mathbb{R}^n;\ \varphi(\cdot)$ sol. a ec. $x' = Ax\}$

$S_A \subset C^1\{\mathbb{R}, \mathbb{R}^n\}$ subspațiu vect. $\dim(S_A) = n$

Obs: $\varphi(\cdot)$ sol. $\Rightarrow \varphi'(t) \equiv A\varphi(t) \Rightarrow \varphi''(t) = A\varphi'(t) \equiv A^2\varphi(t) \Rightarrow \dots \Rightarrow \varphi^{(n)}(t) = A^n\varphi(t), \forall n \in \mathbb{N}$

Dacă $\varphi(\cdot)$ analitică $\Rightarrow \varphi(t) = \sum_{k \geq 0} \frac{t^k}{k!}\varphi^{(k)}(0) \equiv \sum_{k \geq 0} \frac{t^k}{k!} A^k \varphi(0) = \sum_{k \geq 0} \frac{(tA)^k}{k!} \cdot \varphi(0)$

Exponențiala unei aplicații liniare

Prop: $\forall A \in L(\mathbb{R}^n, \mathbb{R}^n) \; \exists \sum_{k \geq 0} \frac{A^k}{k!} = \lim_{m \to \infty} \sum_{k \geq 0}^{m} \frac{A^k}{k!} := \exp(A) \quad (= e^A)$

Dem (schiță): $L(\mathbb{R}^n, \mathbb{R}^n) \simeq M_n(\mathbb{R}) \simeq \mathbb{R}^{n^2}$ $\quad \sum_{k \geq 0} \|u_k\| \, c\nu \Rightarrow \sum_{k \geq 0} v_k$ convergentă

$\|A\| = \sup_{\|x\| \leq 1} \|Ax\| \quad \|AB\| = \|A\| \cdot \|B\| \Rightarrow \|A^k\| \leq \|A\|^k \; \forall k$

$\sum_{k \geq 0} \left\| \frac{A^k}{k!} \right\| \leq \sum_{k \geq 0} \frac{\|A\|^k}{k!} = e^{\|A\|}$

Prop: a) $\exp(0) = I_n$

b) $AB = BA \Rightarrow \exp(A+B) = \exp(A) \cdot \exp(B)$

c) $\forall A \in L(\mathbb{R}^n, \mathbb{R}^n) \; \exists (\exp(A))^{-1} = \exp(-A)$

Prop (Legătura cu ec. diferențiale):

$A \in L(\mathbb{R}^n, \mathbb{R}^n)$ $\quad t \mapsto \exp(A)$ derivabilă; $(\exp(A))' = A\exp(tA) \equiv (\exp(A))A$

Dem: $u(t) = \sum_{k \geq 0} u_k(t)$

$\sum_{k \geq 0} u'_k(t)$ uniform convergentă pe orice mulț. compactă $\Rightarrow \exists\, u'(t) = \sum_{k \geq 0} u'_k(t)$

$\cancel{\sum_{k \geq 1} \frac{k \cdot t^{k-1} A^k}{k!}}$ $\quad \exp(tA) = \sum_{k \geq 0} \frac{(tA)^k}{k!} \quad u_k(t) = \frac{t^k A^k}{k!}$

$\sum_{k \geq 1} \frac{k t^{k-1} A^k}{k!} = \sum_{k \geq 1} \frac{t^{k-1} A^k}{(k-1)!} \equiv A \sum_{k \geq 1} \frac{t^{k-1} A^{k-1}}{(k-1)!} \equiv A\exp(tA)$

Consecință/Corolar: $\varphi(\cdot) \in S_A \Leftrightarrow \exists\, x_0 \in \mathbb{R}^n$ a.î. $\varphi(t) \equiv (\exp(tA))\, x_0$

Dem: „⇐" evident din prop.

„⇒" Fie $x_0 := \varphi(0)$ și $\psi(t) = (\exp(tA))x_0$

$\left.\begin{array}{l}\varphi(\cdot) \text{ sol. } \varphi(0)=x_0 \\ \psi(\cdot) \text{ sol. } \psi(0)=x_0\end{array}\right\} \overset{UG}{\Rightarrow} \varphi(t) \equiv \psi(t)$

Valori proprii. Vectori proprii. Sol. ec. liniare pe $\mathbb{R}^n$ cu coef. const.

Def: $A \in L(\mathbb{R}^n, \mathbb{R}^n)$, $\sigma(A) = \{\lambda \in \mathbb{C};\ \det(A - \lambda I_n) = 0\}$

$\lambda \in \sigma(A)$ s.n. val. proprii a lui A

$VP_A(\lambda) = \{u \in \mathbb{C}^n \setminus \{0\};\ (A - \lambda I_n)u = 0\}$

$u \in VP_A(\lambda)$ s.n. vector propriu coresp. valorii proprii λ

PROP: 1) $A \in L(\mathbb{R}^n, \mathbb{R}^n)$

$\lambda \in \sigma(A) \cap \mathbb{R} \Rightarrow VP_A(\lambda) \cap \mathbb{R}^n \neq \emptyset$

Dem: 1) ~~$A \in L(\mathbb{R}^n, \mathbb{R}^n)$~~

$u = v + iw \in VP_A(\lambda) \Rightarrow (A - \lambda I_n)u = 0 \Rightarrow (A - \lambda I_n)(v + iw) = 0$

$u \neq 0$

$(A - \lambda I_n)v + i(A - \lambda I_n)w = 0 \Rightarrow \begin{cases}(A - \lambda I_n)v = 0 \\ (A - \lambda I_n)w = 0\end{cases}$ v, w nu pot fi simultan 0 ($\Rightarrow u = 0$ ⨳)

De ex. $v \neq 0 \Rightarrow v \in VP_A(\lambda) \cap \mathbb{R}^n$

Th. (Structura sol. în cazul val. proprii simple)

$A \in L(\mathbb{R}^n, \mathbb{R}^n)$

PROP: 2) $A \in L(\mathbb{R}^n, \mathbb{R}^n)$

$\left.\begin{array}{l}\lambda = \alpha + i\beta \quad \lambda \in \sigma(A) \\ u = v + iw \in VP_A(\lambda)\end{array}\right. \Rightarrow \begin{array}{l}\bar{\lambda} = \alpha - i\beta \in \sigma(A) \\ \bar{u} = v - iw \in VP_A(\bar{\lambda})\end{array}$

Dem: $\lambda \in \sigma(A) \Rightarrow \det(A - \lambda I_n) = 0 \Rightarrow 0 = \overline{\det(A - \lambda I_n)} = \det(\overline{A - \lambda I_n}) = \det(A - \bar{\lambda} I_n)$

$u \in VP_A(\lambda)$ $(A - \lambda I_n)u = 0$ $\quad 0 = \overline{(A - \lambda I_n)u} = \overline{(A - \lambda I_n)} \cdot \bar{u} = (A - \bar{\lambda} I_n)\bar{u}$

PROP: 3) (legătura cu ec. dif.)

1. Dacă $\lambda \in \sigma(A) \cap \mathbb{R}$ și $u_\lambda \in VP_A(\lambda) \cap \mathbb{R}^n \Rightarrow \varphi_\lambda(t) = e^{\lambda t} \cdot u_\lambda$, $\varphi_\lambda(\cdot) \in S_A$
2. Fie $\tilde{\varphi}_\lambda(t) := e^{\lambda t} u_\lambda$. La fel ca la 1), $\tilde{\varphi}_\lambda'(t) \equiv A\tilde{\varphi}_\lambda(t)$, $\tilde{\varphi}_\lambda(t) = e^{(\alpha + i\beta)}$

$\lambda = \alpha + i\beta$ $\quad (v + iw) = e^{\alpha t} \cdot e^{i\beta t}(v + iw) = e^{\alpha t}(\cos\beta t + i\sin\beta t)(v + iw) =$

$u_\lambda = u + iw$ $\quad = e^{\alpha t}(v\cos\beta t - w\sin\beta t) + i e^{\alpha t}(v\sin\beta t + w\cos\beta t)$

Fie $\varphi_\lambda(t) := Re(\tilde{\varphi}_\lambda(t)) = e^{\alpha t}(v\cos\beta t - w\sin\beta t)$

$\varphi_{\bar{\lambda}}(t) := Im(\tilde{\varphi}_\lambda(t)) = e^{\alpha t}(v\sin\beta t + w\cos\beta t)$

$\left.\begin{array}{l}\tilde{\varphi}_\lambda(t) = \varphi_\lambda(t) + i\varphi_{\bar{\lambda}}(t) \\ \tilde{\varphi}_\lambda'(t) \equiv A\tilde{\varphi}_\lambda(t)\end{array}\right\} \Rightarrow \varphi'_\lambda(t) + i\varphi'_{\bar{\lambda}}(t) + iA\varphi_\lambda(t) \Rightarrow$

$\Rightarrow \varphi'_\lambda(t) \equiv A\varphi_\lambda(t)$, $\varphi_{\bar{\lambda}}(t) = A\bar{\lambda}(t) \Rightarrow \varphi_\lambda(\cdot), \varphi_{\bar{\lambda}}(\cdot) \in S$

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PROP: 4) (Vectori proprii liniar independenți)

$A \in L(\mathbb{R}^n, \mathbb{R}^n)$

1) $\left.\begin{array}{l}\lambda_1, \dots, \lambda_k \in \sigma(A)\ \ u_j \in VP_A(\lambda_j),\ j=\overline{1,k} \\ \lambda_j \neq \lambda_p \quad \forall j \neq p\end{array}\right\} \Rightarrow \{u_1, \dots, u_k\} \in \mathbb{C}^n$ liniar independente

2) $\lambda_1, \dots, \lambda_m \in \sigma(A) \cap \mathbb{R}$ $u_j \in VP_A(\lambda_j) \cap \mathbb{R}^n$, $j = \overline{1,m}$

$\lambda_j = \alpha_j + i\beta_j \in \sigma(A)$, $j = \overline{m+1,k}$ $u_j = v_j + i w_j \in VP_A(\lambda_j) \Rightarrow \lambda_j \neq \lambda_p$, $\forall j \neq p \Rightarrow$

$\Rightarrow \{u_1, \dots, u_m, v_{m+1}, \dots, v_k, w_{m+1}, \dots, w_k\} \subset \mathbb{R}^n$ liniar indep.

Dem: 1) Inducție după k

$k=1$ $u_1 \in VP_A(\lambda_1) \Rightarrow u_1 \neq 0$ O.k.

$k \to k+1$ $A \mid c_1 u_1 + \dots + c_k u_k + c_{k+1} u_{k+1} = 0 \Rightarrow c_j = 0, \forall j = \overline{1,k+1}$

$$c_1 \underbrace{A u_1}_{\lambda_1 u_1} + \dots + c_k \underbrace{A u_k}_{\lambda_k u_k} + c_{k+1} \underbrace{A u_{k+1}}_{\lambda_{k+1} u_{k+1}} = 0$$

$$c_1 \lambda_1 u_1 + \dots + c_k \lambda_k u_k + c_{k+1} \lambda_{k+1} u_{k+1} = 0$$

$$c_1 \lambda_{k+1} u_1 + \dots + c_k \lambda_{k+1} u_k + c_{k+1} \lambda_{k+1} u_{k+1} = 0$$

$c_1(\lambda_1 - \lambda_{k+1}) u_1 + \dots + c_k(\lambda_k - \lambda_{k+1}) u_k = 0 \Rightarrow$ ip. ind. $\Rightarrow c_j(\lambda_j - \lambda_{k+1}) = 0$, $j = \overline{1,k}$

$\Rightarrow c_j = 0$, $j = \overline{1,k} \Rightarrow c_{k+1} = 0$

2) $\sum_{j=1}^{m} c_j u_j + \sum_{j=m+1}^{k} (c_j v_j + k_j V_j) = 0 \overset{?}{\Rightarrow} c_j = 0 \ \ k_j = 0$
$j = \overline{m+1,k}$

$$\begin{cases} u_j = v_j + i w_j \\ \bar{u}_j = v_j - i w_j \end{cases} \Rightarrow \begin{array}{l} v_j = \frac{1}{2}(u_j + \bar{u}_j) \\ w_j = \frac{1}{2i}(u_j - \bar{u}_j) \end{array}$$

$$\Rightarrow \sum_{j=1}^{m} c_j u_j + \sum_{j=m+1}^{k} \left[\frac{c_j}{2}(u_j + \bar{u}_j) + \frac{k_j}{2i}(u_j - \bar{u}_j) \right] = 0$$

$$\sum_{j=1}^{m} c_j u_j + \sum_{j=m+1}^{k} \left[\left(\frac{c_j}{2} + \frac{k_j}{2i} \right) u_j + \left(\frac{c_j}{2} - \frac{k_j}{2} \right) \bar{u}_j \right] = 0$$

$\overset{1)}{\Rightarrow} c_j = 0, j = \overline{1,m}$ $\quad \begin{array}{l} \frac{c_j}{2} + \frac{k_j}{2i} = 0 \\ \frac{c_j}{2} - \frac{k_j}{2i} = 0 \end{array}, j = \overline{m+1,k} \Rightarrow \begin{array}{l} c_j \equiv 0 \\ k_j \equiv 0 \end{array}, j = \overline{m+1,k}$