TO:

FROM: Ecuații diferențiale - 5.12.2017 - curs

Ecuații afine pe $\mathbb{R}^n$

$\frac{dx}{dt} = A(t)x + b(t)$ $\quad$ $\left.\begin{array}{l} A(\cdot): I \subseteq \mathbb{R} \to L(\mathbb{R}^n, \mathbb{R}^n) \\ b(\cdot): I \to \mathbb{R}^n \end{array}\right|$ continue

Th. (E.U.G)

Fie $A(\cdot): I \subseteq \mathbb{R} \to L(\mathbb{R}^n, \mathbb{R}^n)$, $b(\cdot): I \to \mathbb{R}^n$ continue bf. $\frac{dx}{dt} = A(t)x + b(t)$. At. $\forall (t_0, x_0) \in I \times \mathbb{R}^n$ $\exists!$ $\varphi(\cdot): I \to \mathbb{R}^n$ soluție cu $\varphi(t_0) = x_0$

$$S_{A(\cdot), b(\cdot)} := \{ \varphi(\cdot): I \to \mathbb{R}^n,\ \varphi(\cdot) \text{ sol. a ec. } \frac{dx}{dt} = A(t)x + b(t) \}$$

PROP (varietatea soluțiilor)

$$S_{A(\cdot), b(\cdot)} = S_{A(\cdot)} + \{ \varphi_0(\cdot) \}, \qquad \varphi_0(\cdot) \in S_{A(\cdot), b(\cdot)}$$

Dem: „$\subseteq$" Fie $\varphi_0(\cdot) \in S_{A(\cdot), b(\cdot)} \overset{?}{\Rightarrow} \varphi(\cdot) - \varphi_0(\cdot) \in S_{A(\cdot)}$

$\varphi'(t) = A(t) \cdot \varphi(t) + b(t)$

$\varphi_0'(t) = A(t)\varphi_0(t) + b(t)$

$(\varphi - \varphi_0)'(t) = A(t)(\varphi(t) - \varphi_0(t)) = A(t)(\varphi - \varphi_0)(t)$, i.e. $(\varphi - \varphi_0)(\cdot) \in S_{A(\cdot)}$

„$\supseteq$" Fie $\varphi_0(\cdot) \in S_{A(\cdot), b(\cdot)}$ și fie $\psi(\cdot) \in S_{A(\cdot)} \overset{?}{\Rightarrow} \psi(\cdot) + \varphi_0(\cdot) \in S_{A(\cdot), b(\cdot)}$

$\varphi_0'(t) = A(t)\varphi_0(t) + b(t)$

$\psi'(t) = A(t)\psi(t)$

$+$

$\varphi_0'(t) + \psi'(t) = A(t)(\varphi_0(t) + \psi_0(t)) + b(t) = A(t)(\psi + \varphi_0)(t) + b(t)$, i.e.

$(\psi + \varphi_0)(\cdot) \in S_{A(\cdot), b(\cdot)}$

$(\psi + \varphi_0)'(t)$

Corolar: Fie $\{\varphi_1(\cdot), \dots, \varphi_n(\cdot)\} \subset S_{A(\cdot)}$, sistem fundamental de soluții pt. ec. liniară asociată $\frac{d\bar{x}}{dt} = A(t)\bar{x}$ și fie $\varphi_0(\cdot) \in S_{A(\cdot), b(\cdot)}$. Atunci $\psi(\cdot) \in S_{A(\cdot), b(\cdot)} \Leftrightarrow$

$\exists c_1, \dots, c_n \in \mathbb{R}$ a.î. $\psi(t) = \sum_{i=1}^{n} c_i \varphi_i(t) + \varphi_0(t)$ sol. generală a ec. afine

Th. (Principiul variației constantelor)

Fie $X(\cdot): I \longrightarrow M_n(\mathbb{R})$ matrice fundamentală de soluții pt. ec. liniară asociată $\frac{d\bar{x}}{dt} = A(t)\bar{x}$. At. $\varphi(\cdot) \in S_{A(\cdot), b(\cdot)} \Leftrightarrow \exists c(\cdot)$ primitivă a funcției $t \longrightarrow X(t)^{-1} b(t)$

a.î. $\varphi(t) = X(t) c(t)$

Dem: „$\Rightarrow$" $\varphi(\cdot) \in S_{A(\cdot), b(\cdot)}$

Fie $c(t) := X^{-1}(t)\varphi(t) \Rightarrow \varphi(t) \equiv X(t)c(t)$

$\varphi(\cdot)$ soluție $\Rightarrow \varphi'(t) \equiv A(t)\varphi(t) + b(t)$ $\Big\} \Rightarrow$

$$\Rightarrow \underbrace{X'(t)}_{A(t)X(t)c(t)} c(t) + X(t)c'(t) \equiv A(t)\cancel{X(t)c(t)} + b(t)$$

$X(t)c'(t) \equiv b(t) \Rightarrow c'(t) \equiv X^{-1}(t)b(t)$ o.k.

„$\Leftarrow$" $\varphi(t) \equiv X(t)c(t) \Rightarrow \varphi'(t) \equiv X'(t)c(t) + X(t)c'(t) \equiv A(t)X(t)c(t) +$
$+ X^{-1}(t)b(t) \equiv A(t)\varphi(t) + b(t)$

Corolar: Fie $\{\bar{\varphi}_1(t), \dots, \bar{\varphi}_n(t)\} \subset S_n$ sistem fundamental de sol. pt. ec. liniară asociată $\frac{d\bar{x}}{dt} = A(t)\bar{x}$. At. $\varphi(\cdot) \in S_{A(\cdot), b(\cdot)} \Leftrightarrow \exists\, c(\cdot) = \begin{pmatrix} c_1(\cdot) \\ c_n(\cdot) \end{pmatrix}$ primitivă a funcției $t \longrightarrow [col(\bar{\varphi}_1(t), \dots, \bar{\varphi}_n(t))]^{-1} b(t)$ a.î.

$$\varphi(t) = \sum_{i=1}^{n} c_i(t)\bar{\varphi}_i(t)$$

Dem: $\{\bar{\varphi}_1(\cdot), \dots, \bar{\varphi}_n(\cdot)\} \subset S_{A(\cdot)}$ sist. fundamental de sol. $\Rightarrow X(t) =$
$= col(\bar{\varphi}_1(t), \dots, \bar{\varphi}_n(t)),\ t \in I$ matrice fundamentală de soluții

$X(t)c(t) = col(\bar{\varphi}_1(t), \dots, \bar{\varphi}_n(t)) \begin{pmatrix} c_1(t) \\ c_n(t) \end{pmatrix} = \sum_{i=1}^{n} c_i(t)\bar{\varphi}_i(t)$ + Th. $\Rightarrow$ ced b (teorema)

Algoritm (Metoda variației constantelor pt. ec. afine pe $\mathbb{R}^n$)

$$\frac{dx}{dt} = A(t)x + b(t)$$

1. Considerăm ec. liniară asociată $\frac{d\bar{x}}{dt} = A(t)\bar{x}$

Determinăm $\{\bar{\varphi}_1(t), \dots, \bar{\varphi}_n(t)\}$ sistem fundamental de soluții

Obs: Dacă $A(t) \equiv A \in L(\mathbb{R}^n, \mathbb{R}^n) \longrightarrow$ vezi algoritm

Scriu sol. generală $\bar{x}(t) \equiv \sum_{i=1}^{n} c_i\bar{\varphi}_i(t)$

2. (Variația constantelor propriu-zise)

Caută sol. de forma $x(t) \equiv \sum_{i=1}^{n} c_i(t)\bar{\varphi}_i(t)$

$x(\cdot)$ sol. $\Rightarrow \sum_{i=1}^{n} c_i'(t)\bar{\varphi}_i(t) \equiv b(t) \xRightarrow{\text{Kramer}} c_i'(t) = \dots\ i = \overline{1,n}$

$\Rightarrow c_i(t) = \dots,\ i = \overline{1,n}$

$\Rightarrow x(t) = \dots$

TO:

FROM:

Ecuații diferențiale liniare de ordin superior

Def: $a_1(\cdot), \dots, a_n(\cdot) : I \subseteq \mathbb{R} \to \mathbb{R}$ definesc ec. $x^{(n)} = \sum_{j=1}^{n} a_j(t) x^{(n-j)}$ (1)

Metodă generală de studiu = sistemul canonic asociat

$$(2) \begin{cases} \frac{dx_1}{dt} = x_2 \\ \frac{dx_2}{dt} = x_3 \\ \dots \\ \frac{dx_{n-1}}{dt} = x_n \\ \frac{dx_n}{dt} = \sum_{j=1}^{n} a_j(t) x_{n-j+1} \end{cases}$$

$$\tilde{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$A(t) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots \\ 0 & 0 & 0 & \dots & 1 \\ a_1(t) & & \dots & & \end{pmatrix} = \text{comp}(a_1(t), \dots, a_n(t))$$ (matrice companion)

$$(2) \quad \frac{d\tilde{x}}{dt} = A(t)\tilde{x}$$

PROP (de echivalență)

$\varphi(\cdot)$ este sol. ~~a 1.~~ a ec. (1) $\Leftrightarrow$ $\tilde{\varphi}(\cdot) = (\varphi(\cdot), \varphi'(\cdot), \dots, \varphi^{(n-1)}(\cdot))$ este sol. a ec. (2)

Th. (E.U.G.)

Fie $a_1(\cdot), \dots, a_n(\cdot) : I \subseteq \mathbb{R} \to \mathbb{R}$ cont. def. $x^{(n)} = \sum_{j=1}^{n} a_j(t) x^{(n-j)}$. Atunci $\forall (t_0, (x_0, x'_0, \dots, x_0^{(n-1)}) \in I \times \mathbb{R}^n \; \exists! \; \varphi(\cdot) : I \to \mathbb{R}$ sol. cu $\varphi(t_0) = x_0$, $\varphi'(t_0) = x_0', \dots, \varphi^{(n-1)}(t_0) = x_0^{n-1}$

Dem: Prop. de echivalență + T (E.U.G.) pt. ec. liniară pe $\mathbb{R}^n$ aplicată ec. (2)

$$S_{a_1(\cdot) \dots a_n(\cdot)} := \{ \varphi(\cdot) : I \to \mathbb{R} ; \varphi(\cdot) \text{ sol. a ec. } x^{(n)} = \sum_{i=1}^{n} a_j(t) x^{(n-j)} \}$$

Th. (Spațiul soluțiilor)

Fie $a_1(\cdot), \dots, a_n(\cdot) : I \subseteq \mathbb{R} \to \mathbb{R}$ cont. def $x^{(n)} = \sum_{j=1}^{n} a_j(t) x^{(n-j)}$

Atunci $S_{a_1(\cdot), \dots, a_n(\cdot)} \subset C^{(n)}(I, \mathbb{R})$ subsp. vectorial $\dim(S_{a_1(\cdot), \dots, a_n(\cdot)}) = n$

Dem:

$A(t) = \text{comp}(a_1(t), \dots, a_n(t))$

$T : S_{a_1(\cdot), \dots, a_n(\cdot)} \to S_{A(\cdot)}$ $\quad T(\varphi(\cdot)) = \tilde{\varphi}(\cdot) \; (= \varphi(\cdot), \varphi'(\cdot), \dots, \varphi^{(n-1)}(\cdot))$

T liniară & bijectivă $\dim(S_{A(\cdot)}) = n \Rightarrow$ q.e.d.

Def: $\{\varphi_1(\cdot), \dots, \varphi_n(\cdot)\} \subset S_{a_1(\cdot), \dots, a_n(\cdot)}$ bază s.n. sistem fundamental de sol.

Dacă $\{\varphi_1(\cdot), \dots, \varphi_n(\cdot)\} \subset S_{a_1(\cdot), \dots, a_n(\cdot)}$ sistem fundamental de soluții

$\varphi(\cdot) \in S_{a_1(\cdot), \dots, a_n(\cdot)} \iff \exists\, c_1, \dots, c_n \in \mathbb{R}$ a.î. $\underbrace{\varphi(t) = \sum_{i=1}^{n} c_i \varphi_i(t)}_{\text{sol. generală}}$

Ecuații liniare de ordin superior cu coeficienți constanți

Def: $a_1, \dots, a_n \in \mathbb{R}$ def. $x^{(n)} = \sum_{i=1}^{n} a_j x^{(n-j)}$

Caz particular: $a_j(t) = a_j \in \mathbb{R}$, $j = \overline{1,n}$

Sistemul canonic asociat

$$\begin{cases} \dfrac{dx_1}{dt} = x_2 \\ \dfrac{dx_2}{dt} = x_3 \\ - - - - - \\ \dfrac{dx_{n-1}}{dt} = x_n \\ \dfrac{dx_n}{dt} = \sum_{j=1}^{n} a_j x_{n-j+1} \end{cases} \qquad \tilde{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$A = \mathrm{comp}(a_1, \dots, a_n) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & \dots & 1 \\ a_n & a_{n-1} & \dots & \dots & a_1 \end{pmatrix}$$

$$\iff \frac{d\tilde{x}}{dt} = A\tilde{x}$$

PROP: $\underset{\text{(spectrul matricii A)}}{\sigma(\mathrm{comp}(a_1, \dots, a_n))} = \{\lambda \in \mathbb{C} ;\ \lambda^n = \sum_{i=1}^{n} g \lambda^{n-j}\} = \sigma(a_1, \dots, a_n)$

Def: $CV(\sigma(a_1, \dots, a_n)) = \{\varphi(t) = \sum_{j=1}^{l} e^{\lambda_j t} P_j(t) e^{\alpha_j t}(P_j(t) \cos \beta_j t + Q_j(t) \sin \beta_j t)$,

unde $\sigma(a_1, \dots, a_n) = \{\lambda_1, \dots, \lambda_l\} \in \mathbb{R}$, $\lambda_{l+1} = \alpha_{l-1} + i\beta_{l+1}$ $\overline{\lambda}_{l-1}, \dots, \lambda_k = \alpha_k + i\beta_k$

$\overline{\lambda}_k \to$ cu ordinele de multiplicitate $m_1, \dots, m_2$ și $P_j(t), Q_j(t)$ sunt polinoame

de grad $\leq m_j - 1\}$

Th. (Structura soluțiilor)

$$S_{a_1, \dots a_n} = CP(\sigma(a_1, \dots, a_n))$$

Dem: $\underset{\text{dim}=n}{S_{a_1, \dots, a_n}} \underset{\text{Prop. echivalență}}{\simeq} S_A \subseteq \underset{\text{dim}=n}{CP(\sigma(a_1, \dots, a_n))}$ T. structura sol.

TO:

FROM:

Algoritm: $x^{(n)} = \sum_{j=1}^{n} a_j x^{(n-j)}$ $\quad a_1, a_2, \dots, a_n \in \mathbb{R}$

1. Rezolvă ec. caracteristică $\lambda^n = \sum_{j=1}^{n} a_j \lambda^{n-j} \to \sigma(a_1, \dots, a_n) : (\lambda, m_\lambda)$

2. Pt. $\lambda \in \sigma(a_1, \dots, a_n)$ scrie soluțiile

$$\varphi_\lambda^j(t) = \begin{cases} t^{j-1} e^{\lambda t} & \lambda \in \mathbb{R},\ j = \overline{1, m_\lambda} \\ t^{j-1} e^{\alpha t} \cos\beta t & \lambda = \alpha + i\beta \\ t^{j-1} e^{\alpha t} \sin\beta t & \beta > 0 \quad j = \overline{1, m_\lambda} \end{cases}$$

3. Se renumerotează $\{\varphi_\lambda^j(\cdot)\}_{\lambda \in \sigma(a_1, \dots, a_n),\ j = \overline{1, m_\lambda}} = \{\varphi_1(\cdot), \dots, \varphi_n(\cdot)\}$ sist. fundamental de sol.

Scrie sol. generală $x(t) = \sum_{i=1}^{n} c_i \varphi_i(t)$, $c_i \in \mathbb{R}$, $i = \overline{1,n}$

Ecuații afine de ordin superior

Def: $a_1(\cdot), \dots, a_n(\cdot), b(\cdot) : I \subseteq \mathbb{R} \to \mathbb{R}$ def. $x^{(n)} = \sum_{j=1}^{n} a_j(t) x^{(n-j)} + b(t)$ (1)

Sistemul canonic asociat

$$(2) \quad \begin{cases} \frac{dx_1}{dt} = x_2 \\ \frac{dx_2}{dt} = x_3 \\ \vdots \\ \frac{dx_{n-1}}{dt} = x_n \\ \frac{dx_n}{dt} = \sum_{j=1}^{n} a_j(t) x_{n-j+1} + b(t) \end{cases}$$

$\tilde{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $A(t) = \operatorname{comp}(a_1(t), \dots, a_n(t))$, $\tilde{b}(t) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ b(t) \end{pmatrix}$

(2) $\frac{d\tilde{x}}{dt} = A(t)\tilde{x} + \tilde{b}(t)$

PROP (de echivalență)

$\varphi(\cdot)$ sol. a ec. (1) $\iff \tilde{\varphi}(\cdot) = (\varphi(\cdot), \varphi'(\cdot), \dots, \varphi^{(n-1)}(\cdot))$ e sol. a ec. (2)

Th. (E.U.G.)

Fie $a_1, a_2, \dots, a_n, b : I \to \mathbb{R}$ continue, def. $x^{(n)} = \sum_{j=1}^{n} a_j(t) x^{(n-j)} + b(t)$

At. $\forall (t_0, (x_0, x_0^1, \dots, x_0^{n-1})) \in I \times \mathbb{R}^n$ $\exists! \varphi(\cdot) : I \to \mathbb{R}$ soluție $\varphi(t_0) = x_0$, $\varphi'(t_0) = x_0^1, \dots, \varphi^{(n-1)}(t_0) = x_0^{n-1}$

Dem: Prop. de echivalență + Th. (E.U.G.) pt. ec. afine pe $\mathbb{R}^n$ aplicată ec. (2)

$S_{a_1(\cdot), \dots, a_n(\cdot), b(\cdot)} = \{\varphi(\cdot) : I \to \mathbb{R};\ \varphi(\cdot)$ sol. a ec. $x^{(n)} = \sum_{j=1}^{n} a_j(t) x^{(n-j)} + b(t)\}$

PROP (Varietatea soluțiilor)

$$S_{a_1(\cdot),\dots,a_n(\cdot),b(\cdot)} = S_{a_1(\cdot),\dots,a_n(\cdot)} + \{\varphi_0(\cdot)\},\ \varphi_0(\cdot) \in S_{a_1(\cdot),\dots,a_n(\cdot),b(\cdot)}$$

Dem: „$\subseteq$" Fie $\varphi_0(\cdot) \in S_{a_1(\cdot),\dots,a_n(\cdot),b(\cdot)}$ $\overset{?}{\Rightarrow}$ $\varphi(\cdot) - \varphi_0(\cdot) \in S_{a_1(\cdot),\dots,a_n(\cdot)}$

fie $\varphi(\cdot) \in S_{a_1(\cdot),\dots,a_n(\cdot),b(\cdot)}$

$$\varphi^{(n)}(t) = \sum_{j=1}^{n} a_j(t)\,\varphi^{(n-j)}(t) + b(t)$$

$$\varphi_0^{(n)}(t) = \sum_{j=1}^{n} a_j(t)\,\varphi_0^{(n-j)}(t) + b(t)$$

$$(\varphi-\varphi_0)^{(n)}(t) = \sum_{j=1}^{n} a_j(t)(\varphi-\varphi_0)^{(n-j)}(t) \quad \varphi-\varphi_0 \in S_{a_1(\cdot),\dots,a_n(\cdot)}$$

„$\supseteq$" Fie $\varphi(\cdot) \in S_{a_1(\cdot),\dots,a_n(\cdot)}$, $\varphi(\cdot) \in S_{a_1(\cdot),\dots,a_n(\cdot),b(\cdot)}$ $\overset{?}{\Rightarrow}$

$\overset{?}{\Rightarrow}$ $\varphi(\cdot)+\varphi_0(\cdot) \in S_{a_1(\cdot),\dots,a_n(\cdot),b(\cdot)}$

$$\varphi^{(n)}(t) = \sum_{j=1}^{n} a_j(t)\,\varphi^{(n-j)}(t)$$

$$\varphi_0^{(n)}(t) = \sum_{j=1}^{n} a_j(t)\,\varphi_0^{(n-j)}(t) + b(t)$$

$$+\ (\varphi+\varphi_0)^{(n)}(t) = \sum_{j=1}^{n} a_j(t)(\varphi+\varphi_0)^{(n-j)}(t),\ \text{i.e. } \varphi+\varphi_0 \in S_{a_1(\cdot)\dots a_n, b}$$

Corolar: Dacă $\{\bar\varphi_1(\cdot),\dots,\bar\varphi_n(\cdot)\}$ sistem fundamental de sol. pt. ec. liniară asociată $x^{(n)} = \sum_{j=1}^{n} a_j(t)\,x^{(n-j)}$ și $\varphi_0(\cdot) \in S_{a_1(\cdot),\dots,a_n(\cdot)\,b(\cdot)}$ at.

at. $\varphi(\cdot) \in S_{a_1(\cdot),\dots,a_n(\cdot),b(\cdot)}$ $\Leftrightarrow$ $\exists\, c_1,\dots,c_2 \in \mathbb{R}$ a.î. $\underbrace{\varphi(t) = \sum_{i=1}^{n} c_i\,\bar\varphi_j(t) + \varphi_0(t)}_{\text{sol. generală}}$