TO:

FROM: Ecuații diferențiale – curs – 10.10.2017

Teoria elementară a ec. dif.

I Ec. dif. scalare
 1. Ec. cu variabile scalare
 2. Ec. liniare scalare
 3. Ec. afine scalare

II Ecuații Bernoulli

$\frac{dx}{dt} = a(t)x + b(t)x^{\alpha}$ $\quad a(t), b(t): I \subseteq \mathbb{R} \to \mathbb{R}$ cont, $\alpha \in \mathbb{R} \setminus \{0,1\}$ $\quad \alpha = 0 \;\; x' = a(t)x + b(t)$

$\alpha = 1 \;\; x' = a(t)x + b(t)x = \underbrace{(a(t)+b(t))}_{c(t)}x$

Prop. (Principiul variației constantelor)

Fie $A(\cdot)$ primitivă a lui $a(\cdot)$

At. $\varphi(\cdot): I \to \mathbb{R}$ e sol. a ec. $\iff \exists\, c(\cdot)$ sol. a ec. cu var. sep. $\frac{dc}{dt} = e^{(\alpha-1)A(t)} b(t) c^{\alpha}$ a.î. $\varphi(t) = c(t) \cdot e^{A(t)}$

Dem.: „$\Rightarrow$" $\cancel{\text{Fie } c(t) := \varphi(t) e^{-A(t)} + b(t)(c(t) + e^{A(t)})^{\alpha}}$

$\cancel{c(t)e^{A(t)} + c(t)e^{A(t)}}$

$\cancel{\varphi(\cdot) \text{ sol.} \Rightarrow (c(t)e^{A(t)})' = a(t)c(t)e^{A(t)} + b(t)(}$

Fie $c(t) := \varphi(t) e^{-A(t)} \Rightarrow \varphi(t) = c(t) e^{A(t)}$

$\varphi(\cdot)$ sol. $\Rightarrow (c(t)e^{A(t)})' = a(t)c(t)e^{A(t)} + b(t)(c(t)e^{A(t)})^{\alpha}$

...

$\Rightarrow c'(t) = b(t)c^{\alpha}(t)e^{(\alpha-1)A(t)}$ OK

„$\Leftarrow$" $\varphi(t) = c(t)e^{A(t)} \Rightarrow \varphi'(t) = c'(t)e^{A(t)} + c(t)e^{A(t)}a(t) =$

$= e^{(\alpha-1)A(t)} b(t) c^{\alpha}(t) \cancel{e^{A}} \cdot e^{A(t)} + \varphi(t) \cdot a(t) =$

$= e^{\alpha A(t)} c^{\alpha}(t) b(t) + \varphi(t) a(t) = (\varphi(t))^{\alpha} b(t) + \varphi(t) a(t)$

Algoritm: $\frac{dx}{dt} = a(t)x + b(t)x^{\alpha}$

1. Considerăm ec. liniară asociată $\frac{d\bar{x}}{dt} = a(t)\bar{x}$
 Scriem sol. generală $\bar{x}(t) = ce^{A(t)}$
2. Variația constantelor
 Căutăm sol. de forma $x(t) = c(t)e^{A(t)}$
 $x(\cdot)$ sol. $\Rightarrow c'(t) = e^{(\alpha-1)A(t)} b(t) c^{\alpha}(t)$ ec. cu var. sep. $\to$ Vezi Algoritm

5. Ecuații Ricatti

$\frac{dx}{dt} = a(t)x^2 + b(t)x + c(t)$ $\quad \cancel{a, b, c}\;\; a(\cdot), b(\cdot), c(\cdot): I \subseteq \mathbb{R} \to \mathbb{R}$ cont.

Liouville $x' = x^2 + t^2$ nu e integrabilă prin cuadraturi

Caz particular: se cunoaște o sol. particulară $\varphi_0(\cdot)$

Prop: Fie $\varphi_0 : I \subseteq \mathbb{R} \to \mathbb{R}$ sol. a ec.

At. $\varphi : I \to \mathbb{R}$ e sol. a ec. „$\Leftrightarrow$" $\psi(t) = \varphi(t) - \varphi_0(t)$ e sol. unei ec. Bernoulli:

$y' = 2a(t)\varphi_0(t) + b(t)y + a(t)y^2$

Dem. „$\Rightarrow$" $\varphi(\cdot), \varphi_0(\cdot)$ sol. $\Rightarrow$ $\varphi'(t) \equiv a(t)\varphi^2(t) + b(t)\varphi(t) + c(t)$

$\varphi_0'(t) \equiv a(t)\varphi_0^2(t) + b(t)\varphi_0(t) + c(t)$

$\psi(t) = \varphi(t) - \varphi_0(t) \Rightarrow \varphi(t) = \psi(t) + \varphi_0(t)$

$(\psi(t) + \varphi_0(t))' \equiv a(t)(\psi(t) + \varphi_0(t))^2 + b(t)(\psi(t) + \varphi_0(t)) + c(t)$

$\psi'(t) + \varphi_0'(t) \equiv a(t)\psi^2(t) + 2a(t)\psi(t)\varphi_0(t) + a(t)\varphi_0^2(t) + b(t)\psi(t) + b(t)\varphi_0(t) + c(t)$

$\psi'(t) = (2a(t)\varphi_0(t) + b(t))\psi(t) + a(t)\psi^2(t)$

„$\Leftarrow$" $\psi(t) = \varphi(t) - \varphi_0(t)$ $\qquad$ $\varphi_0(\cdot)$ sol $\Rightarrow$ $\varphi_0'(t) \equiv a(t)\varphi_0^2(t) + b(t)\varphi_0(t) + c(t)$

$\psi(\cdot)$ sol $\Rightarrow$ $\psi'(t) = (2a(t)\varphi_0(t) + b(t))\psi(t) + a(t)\psi^2(t)$

$(\varphi(t) - \varphi_0(t))' = (2a(t)\varphi_0(t) + b(t))(\varphi(t) - \varphi_0(t)) + a(t)(\varphi(t) - \varphi_0(t))^2$

$\varphi'(t) - \varphi_0'(t) \equiv (2a(t)\varphi_0(t) + b(t))\varphi(t) - 2a(t)\varphi_0^2(t) - b(t)\varphi_0(t) + a(t)\varphi^2(t) - 2a(t) \cdot$

$\cdot \varphi(t)\cdot\varphi_0(t) + a(t)\varphi_0^2(t) - a(t)\varphi_0^2(t) - b(t)\varphi_0(t) - c(t)$

$\Rightarrow \varphi'(t) \equiv a(t)\varphi^2(t) + b(t)\varphi(t) + c(t)$ ok

Algoritm: $x' = a(t)x^2 + b(t)x + c(t)$, $\varphi_0(\cdot)$ sol.

S.V.: $y = x - \varphi_0(t)$ $[y(t) = x(t) - \varphi_0(t) \Leftrightarrow x(t) = y(t) + \varphi_0(t)]$

$x(\cdot)$ sol. $\Rightarrow y' = (2a(t)\varphi_0(t) + b(t))y + a(t)y^2 \to$ ec. Bernoulli, vezi algoritm

$\Rightarrow y(t) = \ldots \Rightarrow x(t) = y(t) + \varphi_0(t)$

Ec. omogene

$$\frac{dx}{dt} = f\left(\frac{x}{t}\right) \qquad f : D \subseteq \mathbb{R} \to \mathbb{R} \text{ cont.}$$

Prop: $\varphi(\cdot) : I \subseteq \mathbb{R} \to \mathbb{R}$ este sol. a ec. omogene „$\Leftrightarrow$" $\psi(\cdot) : I \to \mathbb{R}$, $\psi(t) = \frac{\varphi(t)}{t}$ e sol. a ec. cu var. sep.

$$\frac{dy}{dt} = \frac{f(y) - y}{t}$$

Dem: „$\Rightarrow$" $\psi(t) = \frac{\varphi(t)}{t} \Leftrightarrow \varphi(t) = t\cdot\psi(t)$. $\varphi(\cdot)$ sol. $\Rightarrow (t\cdot\psi(t))' \equiv f\left(\frac{t\cdot\psi(t)}{t}\right)$

$t\psi'(t) + \psi(t) \equiv f(\psi(t))$ $\qquad \psi'(t) \equiv \frac{f(\psi(t)) - \psi(t)}{t}$ ok

TO:

FROM:

"$\Leftarrow$" $\varphi(t)=\frac{b(t)}{t}$ $\varphi(\cdot)$ sol. $\left(\frac{b(t)}{t}\right)'=\frac{f\left(\frac{b(t)}{t}\right)-\frac{b(t)}{t}}{t}$

$\frac{\varphi'(t)t-\varphi(t)}{t^2}=\frac{tf\left(\frac{b(t)}{t}\right)-\varphi(t)}{t^2} \Rightarrow \varphi'(t)=f\left(\frac{b(t)}{t}\right)$ OK

Algoritm: $\frac{dx}{dt}=f\left(\frac{x}{t}\right)$

S.V.: $y=\frac{x}{t}$ $\left(y(t)=\frac{x(t)}{t} \Leftrightarrow x(t)=t\cdot y(t)\right)$

$x(\cdot)$ sol $\Rightarrow y'=\frac{f(y)-y}{t}$ ec. cu var. sep. $\rightarrow$ vezi Algoritm $\Rightarrow y(t)=\dots$

$\Rightarrow x(t)=ty(t)=\dots$

II Ec. de ordin superior care admit reducerea ordinului

Def: a) $F(\cdot,\cdot): D\subseteq \mathbb{R}\times\mathbb{R}^{n+1}\to\mathbb{R}$ def. ec. dif. de ordin n $F(t,x,x',\dots,x^{(n)})=0$

b) $\varphi: I\subseteq\mathbb{R}\to\mathbb{R}$ este sol. a ec. dacă este de n ori derivabilă și verifică $F(t,\varphi(t),\varphi'(t),\dots,\varphi^{(n-1)}(t))=0$

1) $F(t,x^{(k)},x^{(k+1)},\dots,x^{(n)})=0$, $k\geq 1$

S.V. $y=x^{(k)} \Rightarrow F(t,y,y',\dots,y^{(n-k)})=0 \Rightarrow y(t)=\dots$

$\Rightarrow x(t)=\dots$

2) $F\left(t,\frac{x'}{x},\frac{x''}{x},\dots,\frac{x^{(n)}}{x}\right)=0$

S.V.: $y=\frac{x'}{x}$ ($\forall x(\cdot)$ sol., s.v. def. $y(\cdot)$ după regula $y(t)=\frac{x'(t)}{x(t)}$)

$\Rightarrow G(t,y,y',\dots,y^{(n-1)})=0 \Rightarrow y(t)=\dots \Rightarrow x'=y(t)x$ ec. liniară $\Rightarrow x(t)=\dots$

3) Ec. autonome $F(x,x',\dots,x^{(n)})=0$

S.V. $x'=y(x)$

Se caută o funcție $y(\cdot)$ a.î. $x'(t)=y(x(t))$

$\Rightarrow G(x,y,y',\dots,y^{(n-1)})=0$

4) Ec. de tip Euler

$F(x,tx',t^2x'',\dots,t^{n}x^{(n)})=0$

S.V. $|t|=e^s$ $\to t=e^s$ $t>0$

$\to t=-e^s$ $t<0$

$t=e^s$ $\forall x(\cdot)$ sol. a ec. s.v. def. $y(\cdot)$ după regula $y(s)=x(e^s) \Leftrightarrow x(t)=y(\ln t)$

Noțiuni fundamentale

$\frac{dx}{dt}=f(t,x)$ $f(\cdot): D\subseteq\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^n$

Problema Cauchy: Se știe $f(\cdot)$ care def. ecuația $x'=f(t,x)$ și $(t_0,x_0)\in D$ condiție inițială

Se caută $\varphi(\cdot): I\subseteq\mathbb{R}\to\mathbb{R}^n$ soluție a ec. cu $\varphi(t_0)=x_0$

În ac. caz, spunem că φ este sol. a problemei Cauchy (f,t_0,x_0)

Def: Spunem că $f(\cdot)$ (sau ec. $x'=f(t,x)$) admite proprietatea de:

a) existență locală (E.L.) în $(t_0, x_0) \in D$ dacă $\exists\, I_0 \in V(t_0)$ $\exists\, \varphi(\cdot): I_0 \to \mathbb{R}^n$ sol. a pb. Cauchy dată de (f, t_0, x_0)

b) unicitate locală (U.L.) în $(t_0, x_0) \in D$ dacă $\forall\, \varphi_i : I_i \to \mathbb{R}^n$, $i = 1,2$ soluții ale ac. pb. Cauchy ~~f(t~~ (f, t_0, x_0) $\exists\, I_0 \in V(t_0)$ a.î. $\varphi_1\big|_{I_1 \cap I_2 \cap I_0} = \varphi_2\big|_{I_1 \cap I_2 \cap I_0}$

c) existența globală (E.G.) în $(t_0, x_0) \in D$ dacă $D = I \times G$ ($I \subseteq \mathbb{R}$, $G \subseteq \mathbb{R}^n$) și $\exists\, \varphi(\cdot): I \to \mathbb{R}^n$ soluție a pb. Cauchy (f, t_0, x_0)

d) unicitate globală (U.G.) în $(t_0, x_0) \in D$ dacă $\forall\, \varphi_i : I_i \to \mathbb{R}^n$, $i = 1,2$ sol. a prob. Cauchy (f, t_0, x_0) $\varphi_1\big|_{I_1 \cap I_2} = \varphi_2\big|_{I_1 \cap I_2}$

__Prop:__ „Ecuația integrală asociată unei ecuații diferențiale"

Fie $f(\cdot): D \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ cont., $\frac{dx}{dt} = f(t, x)$

At. $\varphi(\cdot): I \subseteq \mathbb{R} \to \mathbb{R}^n$ sol. „$\Leftrightarrow$" 1. $\varphi(\cdot)$ continuă
2. $\varphi(t) = \varphi(t_0) + \int_{t_0}^{t} f(s, \varphi(s))\, ds$ $\forall\, t, t_0 \in I$

__Dem:__ „$\Rightarrow$" $\varphi(\cdot)$ sol. $\Rightarrow$ $\varphi(\cdot)$ derivabilă $\Rightarrow$ $\varphi(\cdot)$ continuă

$\Rightarrow \varphi'(s) = f(s, \varphi(s))$ $\forall\, s \in I$ $\Big| \int_{t_0}^{t}$

$\varphi(t) - \varphi(t_0) \overset{L.N.}{=} \int_{t_0}^{t} \varphi'(s)\, ds = \int_{t_0}^{t} f(s, \varphi(s))\, ds$

„$\Leftarrow$" $\varphi(\cdot)$ cont., $f(\cdot,\cdot)$ cont. $\Rightarrow$ $s \longrightarrow f(s, \varphi(s))$ cont. $\Rightarrow$ $t \mapsto \int_{t_0}^{t} f(s, \varphi(s))$ derivabilă $\overset{2}{\Rightarrow}$

$\overset{2}{\Rightarrow}$ $\varphi(\cdot)$ derivabilă $\overset{(\,)'}{\Rightarrow}$ $\varphi'(t) = f(t, \varphi(t))$ OK

__T. Peano__ (Existența locală a sol.)

Fie ~~$f(\cdot,\cdot): D \subseteq \mathbb{R} \times \mathbb{R} \to \mathbb{R}$~~ $f(\cdot,\cdot): D = \mathring{D} \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ continuă $\frac{dx}{dt} = f(t, x)$

At. $f(\cdot,\cdot)$ admite propr. de E.L. pe D ($\forall\, (t_0, x_0)$ $\exists\, I_0 \in V(t_0)$)

$\exists\, \varphi(\cdot): I_0 \to \mathbb{R}^n$ sol. cu $\varphi(t_0) = x_0$