TO:

FROM: Ecuații diferențiale - seminar - 24.10.2017

Ecuații de ordin superior care admit reducerea ordinului

Examen: ex1: th
ex 2,3: exerciții

1) $F(t, x^{(k)}, x^{(k+1)}, \dots, x^{(n-1)}) = 0, \quad k \geq 1$

Sch. var: $y = x^{(k)} \Rightarrow F(t, y, y', \dots, y^{(n-k)}) = 0$

2) $F(t, \frac{x'}{x}, \frac{x''}{x}, \dots, \frac{x^{(n)}}{x}) = 0$

Sch. var: $y = \frac{x'}{x} \Rightarrow G(t, y, y', \dots, y^{(n-1)}) = 0$

3) $F(x, x', \dots, x^{(n)}) =$

Sch. var: $x' = y(x) \Rightarrow G(x, y, y', \dots, y^{(n-1)}) = 0$

4) Ec. Euler: $F(x, tx', t^2x'', \dots, t^n x^{(n)}) = 0$

Sch. var: $|t| = e^s \Rightarrow G(y, y', \dots, y^{(n)}) = 0$

Ecuații liniare de ordinul al II-lea cu coef. constanți

$x'' + ax' + bx = 0$

Ec. caracteristica $\lambda^2 + a\lambda + b = 0$ $\left\langle \begin{matrix}\lambda_1 \\ \lambda_2\end{matrix}\right.$

Dacă $\lambda_1, \lambda_2 \in \mathbb{R}, \lambda_1 \neq \lambda_2 \Rightarrow$ sol. generală $x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}, \quad c_1, c_2 \in \mathbb{R}$

Dacă $\lambda_1 = \lambda_2, \lambda_1, \lambda_2 \in \mathbb{R}, \Rightarrow$ — " — $x(t) = c_1 e^{\lambda t} + c_2 t e^{\lambda t}$ — " —

Dacă $\lambda_1 = \alpha + i\beta$, $\lambda_2 = \alpha - i\beta$, $\beta > 0 \Rightarrow$ — " — $x(t) = c_1 e^{\alpha t} \cos \beta t + c_2 e^{\alpha t} \sin \beta t, \quad c_1, c_2 \in \mathbb{R}$

→ Să se det. sol. generală:

a) $x''' + x'' = 0$

s.v. $y = x'' \Rightarrow x''' = y'$

$y' + y = 0$

$y' = -y \Rightarrow$ ec. liniară scalară $\Rightarrow y(t) = c \cdot e^{-t}, \quad c \in \mathbb{R}$

~~$y' = -y \Rightarrow y(t) = c \cdot e^{-t}$~~

$x'' = c \cdot e^{-t} \Rightarrow x'(t) = \int c \cdot e^{-t} \, dt = -c \cdot e^{-t} + c_1, \quad c, c_1 \in \mathbb{R}$

$\Rightarrow x(t) = \int -c \cdot e^{-t} dt + c_1 = c \cdot e^{-t} + c_1 t + c_2, \quad c, c_1, c_2 \in \mathbb{R}$

! b) $tx'' + x' + t = 0$

c) $txx'' + t(x')^2 - xx' = 0$

c) Obs. $x(t) \equiv 0$ verif. ecuația

$$t\frac{x''}{x} + t\left(\frac{x'}{x}\right)^2 - \frac{x'}{x} = 0 \quad \text{ec. omogenă} \quad (2)$$

$$y = \frac{x'}{x} \Rightarrow y(t) = \frac{x'(t)}{x(t)}$$

$$x'(t) = y(t)x(t) \Rightarrow x''(t) = y'(t)x(t) + y(t)x'(t) \;\Big|\; \frac{1}{x(t)}$$

$$\Rightarrow \frac{x''(t)}{x(t)} = y'(t) + \frac{y(t)x'(t)}{x(t)} = y'(t) + y^2(t)$$

$$t(y'+y^2) + ty^2 - y = 0$$

$$ty' + ty^2 + ty^2 - y = 0, \quad t \neq 0, t > 0$$

$$y' = \frac{-2ty^2 + y}{t} = \frac{y}{t} - 2y^2 \quad \text{(ec. Bernoulli)}$$

$$\bar{y}' = \frac{\bar{y}}{t} \Rightarrow \bar{y}(t) = c \cdot e^{\int \frac{1}{t} dt} = c \cdot e^{\ln t} = c \cdot t$$

$\Rightarrow$ se caută sol. $y(t) = c(t) \cdot t$

$$(c(t)\cdot t)' = \frac{c(t)\cdot t}{t} - 2c^2(t)t^2$$

$$c'(t)\cdot t + c(t) = c(t) - 2c^2(t)t^2$$

$$c'(t) = -2c^2(t)t \quad \text{(ec. var. sep.)}$$

$$\frac{dc}{dt} = -2c^2 \cdot t \qquad c^2 = 0 \Rightarrow c = 0 \Rightarrow c(t) \equiv 0$$

$$\frac{dc}{-c^2} = 2t\,dt \Rightarrow -\int \frac{dc}{c^2} = 2\int t\,dt \Leftrightarrow \frac{1}{c} = \frac{2t^2}{2} + k, \; k \in \mathbb{R}$$

$$\Rightarrow c(t) = \frac{1}{t^2+k}, \; k \in \mathbb{R}$$

$$y_0(t) = 0$$

$$y_k(t) = \frac{t}{t^2+k}, \; k \in \mathbb{R}$$

$$y = \frac{x'}{x} \Rightarrow \frac{x'}{x} = 0 \Rightarrow x' = 0 \Rightarrow x_0(t) \equiv c_1, \; c_1 \in \mathbb{R}$$

$$\frac{x'}{x} = \frac{t}{t^2+k} \Rightarrow x' = \frac{tx}{t^2+k} \quad \text{(ec. liniară)}, \text{ sol. } x(t) = c_2 \cdot e^{\int \frac{t}{t^2+k} dt}$$

$$x(t) = c_2 \cdot e^{\frac{1}{2}\int \frac{2t}{t^2+k}} = c_2 \cdot e^{\frac{1}{2}\ln|t^2+k|} = c_2\sqrt{t^2+k}, \; c_1, c_2 \in \mathbb{R}$$

TO:

FROM:

! d) $t^2 x x'' - (x - t x')^2 = 0$

e) $x x'' + 1 = (x')^2$

Sch. var. $x' = y(x)$

Se caută o fct. y a.î. $x'(t) = y(x(t))$ $\forall\, t \in \operatorname{dom} x$

$x''(t) = y'(x(t))\, x'(t) = y'(x(t)) \cdot y(x(t))$

$x'' = y'(x) \cdot y(x)$

$x \cdot y'(x) \cdot y(x) + 1 = y^2(x)$

$y'(x) = (y^2(x) - 1) \cdot \frac{1}{x \cdot y(x)}$

$\frac{dy}{dx} = \frac{y^2-1}{x \cdot y}$ (ec. var. sep.) $\Rightarrow \frac{y^2-1}{y} = 0 \Leftrightarrow y^2 - 1 \Rightarrow y = \pm 1$

$\frac{1}{2}\int \frac{2y}{y^2-1}\, dy = \int \frac{dx}{x} \Leftrightarrow \frac{1}{2} \ln(y^2-1) = \ln|x| + c$, $c \in \mathbb{R}$

$\ln k$, $k > 0$

$\Leftrightarrow \sqrt{|y^2-1|} = k|x|$

$\Leftrightarrow |y^2 - 1| = k^2 x^2$

$\Rightarrow y^2 - 1 = C_1 \cdot x^2$, $C_1 \in \mathbb{R}^*$

$\Rightarrow y(x) = \pm\sqrt{C_1 x^2 + 1}$, $C_1 \in \mathbb{R}^*$

$x' = 1 \Rightarrow x_1(t) = t + C_2$, $C_2 \in \mathbb{R}$

$x' = -1 \Rightarrow x_2(t) = -t + C_3$, $C_3 \in \mathbb{R}$

$x' = \pm\sqrt{C_1 x^2 + 1}$ (ec. cu var. sep.)

! f) $x x'' + (x')^2 = 0$

g) $t^2 x'' - 4 t x' + 6x = 0$

Ec. Euler

Sch. var. $|t| = e^s$ — $t = e^s$, $t > 0$; $t = -e^s$, $t < 0$

Pp. $t > 0$ $t = e^s$

$t^2 x''(t) - 4 t x'(t) + 6 x(t) = 0$

$e^{2s} x''(e^s) - 4 e^s x'(e^s) + 6 x(e^s) = 0$

$x(t) \ast x(e^s) = y(s)$

$y(s) = x(e^s) \Leftrightarrow x(t) = y(\ln t)$

$\forall\, x(\cdot)$ sol. a ec. s.v. def. $y(\cdot)$ după regula $y(s) = x(e^s)$

$y'(s) = x'(e^s) \cdot e^s \Rightarrow x'(e^s) = \frac{y'(s)}{e^s} = y'(s) \cdot e^{-s}$

$x''(e^s) \cdot e^s = y''(s) \cdot e^{-s} + y'(s) \cdot e^{-s} \Rightarrow x''(e^s) = y''(s) \cdot e^{-2s} - y'(s) \cdot e^{-2s}$

$e^{2s}\left(y''(s)\cdot e^{-2s} - y'(s)\cdot e^{-2s}\right) - 4e^{s}\cdot y'(s)e^{-s} + 6y(s) = 0$

$y''(s) - y'(s) - 4y'(s) + 6y(s) = 0$

$y''(s) - 5y'(s) + 6y(s) = 0$ (ec. liniară ord. II)

$\lambda^2 - 5\lambda + 6 = 0$

$y(s) = k\cdot e^{2s} + q\cdot e^{3s}$, $k, q \in \mathbb{R}$

$x(t) = y(\ln t) = k\cdot e^{2\ln t} + q\cdot e^{3\ln t} = kt^2 + qt^3$, $k, q \in \mathbb{R}$

h) $t^2x'' + tx' + x = 0$ (Ec. Euler)

S.V. : $|t| = e^s$, pp. $t < 0$

$t = -e^s$

$\forall x(\cdot)$ sol. a ec. S.V. def. funcția $y(\cdot)$ după regula $y(s) = x(-e^s)$

$e^{2s}x''(-e^s) - e^s\cdot x'(-e^s) + x(-e^s) = 0$

$y(s) = x(-e^s)$

$y'(s) = -x'(-e^s)$

$x'(-e^s) = -y'(s)\cdot e^{-s}$

$-x''(-e^s)e^s = -y''(s)e^{-s} + y'(s)e^{-s}$

$x''(-e^s) = y''(s)e^{-2s} - y'(s)e^{-2s}$

$e^{2s}\left(y''(s)e^{-2s} - y'(s)\cdot e^{-2s}\right) + e^s y(s)e^{-s} + y(s) = 0$

$y''(s) + y(s) = 0$

$y'' + y = 0$ (ec. lin. ord. II)

$\lambda^2 + 1 = 0$

$\lambda_1 = -1$

$\lambda_2 = 1$

$y(s) = c_1\cdot e^{0\cdot s}\cos s\cdot s_1 + c_2 e^{0\cdot s}\sin(s)$, $c_1, c_2 \in \mathbb{R}$

$y(s) = x(-e^s) \Leftrightarrow x(t) = y(\ln(-t))$

$\Rightarrow x(t) = c_1\cos(\ln(-t)) + c_2\sin(\ln(-t))$, $c_1, c_2 \in \mathbb{R}$

i) $t^2x'' + 5tx' - 5x = 0$

j) $t^2x'' - tx' - 2x = 0$