EMP

~~EDO~~ Def

(*) $x'(t) = f(t, x(t))$, $x(t) = (x_1(t), \dots, x_n(t))$

$\uparrow$ sistem ec. dif. $\quad f(\cdot,\cdot) = (f_1(\cdot,\cdot), \dots, f_n(\cdot,\cdot))$

$f : D \subseteq R \times R^n \to R^n$

integrală primă pt. (*)

Def O fc. $F : D \to R$ integrală primă dacă pt orice sol. $x(t)$ $\exists$ o constantă $c \in R$ a.î. $F(t, x(t))$ const. $= c$ este o ecuație implicită a soluției

$(x(t), y(t)) \to$ sist. 2 ecuații , $F(t, x(t), y(t)) = ct$ (ex. $x(t) + y(t)$)

Criteriu pt. integrale prime

$$\boxed{F \text{ integ primă}} \Leftrightarrow \frac{d}{dt}\left[F(t, x(t))\right] = 0$$

$$\Leftrightarrow \frac{\partial F}{\partial t}(t, x(t)) + \frac{\partial F}{\partial x_1} \cdot x_1'(t) + \frac{\partial F}{\partial x_2} x_2'(t) + \dots + \frac{\partial F}{\partial x_n} x_n'(t) = 0$$

$$\Leftrightarrow \boxed{\frac{\partial F}{\partial t}(t, x(t)) + \frac{\partial F}{\partial x_1} f_1(t, x(t)) + \dots + f_n \frac{\partial F}{\partial x_n} f_n(t, x(t)) = 0}$$

Ex 1 Fie sis. $\begin{cases} x'(t) = \dfrac{x^2(t) - 2t}{y(t)} & (1) \\ y'(t) = -x(t) & (2) \end{cases}$

a) Ar. că $F(t, x, y) = t^2 + xy$ este integr. primă

b) Sol. gen. a sistemului

Dem

a) F_1 integr. primă $\Leftrightarrow$ $\frac{\partial F}{\partial t} + \frac{\partial F}{\partial x}\underbrace{\left(\frac{x^2-2t}{y}\right)}_{f_1(t,x,y)} + \frac{\partial F}{\partial y}\underbrace{(-x)}_{f_2(t,x,y)}$

$\Leftrightarrow 2t + y\left(\frac{x^2-2t}{y}\right) + x\cdot(-x) = 0$ adev.

---

b) $\underbrace{t^2+xy}_{F(t,x,y)} = c \Rightarrow xy = c - t^2$

$x(t) = \frac{c-t^2}{y(t)}$

Dem 2 $\Rightarrow y'(t) = \frac{t^2-c}{y(t)} \Rightarrow y'(t)\cdot y(t) = t^2 - c$

$\int \underbrace{y(t)\,y'(t)\,dt} = \int t^2 - c \; dt$

$\frac{1}{2}y^2(t) = \frac{t^3}{3} - c_1 t + C_2$

$y(t) = \pm\sqrt{\frac{2t^3}{3} - 2c_1 t + 2C_2}$

$x(t) = \frac{c - t^2}{\sqrt{\ldots}}$

Ex 2

$$\begin{cases} x'(t) = \dfrac{x^3(t)}{y(t)} & (1) \\ y'(t) = y(t) - x^2(t) & (2) \end{cases} \qquad f_1(t,x,y) = \frac{x^3}{y} \qquad f_2(t,x,y) = y - x^2$$

a) Ar. că $F_1(t,x,y) = xy\,e^{-t}$ integrală primă

b) Det. sol. generală

Dem

a) $$\frac{\partial F}{\partial t} + \frac{\partial F}{\partial x}\left(\frac{x^3(t)}{y(t)}\right) + \frac{\partial F}{\partial y}\left(\left(y(t) - x^2(t)\right)\right)$$

$$= -xy\,e^{-t} + y e^{-t}\frac{x^3}{y} + x e^{-t}(y - x^2) = 0 \text{ int. primă}$$

b) $F_1(t,x,y) = c \in \mathbb{R} \Rightarrow xy\,e^{-t} = c \Leftrightarrow$

$$y = \frac{c}{x e^{-t}} \qquad x = \frac{c}{y e^{-t}} \qquad (2)\ x = 0 \Rightarrow y' = y \Rightarrow y = C \cdot e^{t}$$

$$(1) \Rightarrow x'(t) = \frac{x^3}{c} \cdot x e^{-t} = \frac{x^4 e^{-t}}{c}$$

$$\frac{x'}{x^4} = \frac{e^{-t}}{c} \Rightarrow \int \frac{x'}{x^4}\,dt = \frac{1}{c}\int e^{-t}\,dt$$

$$\Rightarrow -\frac{1}{3}x^{-3} = \frac{1}{c}\left(-e^{-t}\right) + C_2 \quad \Big|\cdot(-3)$$

$$x^{-3} = \frac{3}{c}\cdot e^{-t} + C_2$$

$$x = \left(\frac{3}{c}\,e^{-t} + C_2\right)^{\frac{1}{3}}$$

$$(2) \Rightarrow y' = \frac{C}{\left(\frac{3}{c}\cdot e^{-2t} + C_2 e^{-t}\right)} - \left(\frac{3}{c}\cdot e^{-t} + C_2\right)^{\frac{2}{3}}$$

Ex 3

$x' = Ax$
$\in M_{n \times n}(\mathbb{R})$

De fapt $n = 1, 2, 3$

$x(t) = e^{tA} \cdot c$

Caz. $n = 1$ ✓

Caz. $n = 2$ $\begin{cases} x'(t) = 5x(t) + 3y(t) \\ y'(t) = -3x(t) - y(t) \end{cases}$

Metodă directă

$x''(t) \overset{(1)}{=} 5x'(t) + 3y'(t)$

$\overset{(2)}{=} 5x'(t) + 3(-3x(t) - y(t))$

$= 5x'(t) - 9x(t) - 3y(t)$

$\overset{(1)}{=} 5x'(t) - 9x(t) - \cancel{3} \frac{x'(t) - 5x(t)}{\cancel{3}}$

$= 4x'(t) - 4x(t)$

$x''(t) - 4x'(t) + 4x(t) = 0$

ec. carac.

$\lambda^2 - 4\lambda + 4 = 0 \qquad (\lambda - 2)^2 = 0 \qquad \lambda_{1,2} = 2$

$x(t) = c_1 e^{2t} + c_2 t e^{2t}$

$\Rightarrow y(t) = \frac{1}{3}(x'(t) - 5x(t))$

$= \frac{1}{3}\left(2c_1 e^{2t} + c_2 e^{2t}(2t+1) - 5c_1 e^{2t} - 5c_2 t e^{2t}\right)$

$= \frac{1}{3}(c_2 - 3c_1)e^{2t} + \frac{1}{3} t e^{2t}(2 - 5c_2)$

Sol 2

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \underbrace{\begin{pmatrix} 5 & 3 \\ -3 & -1 \end{pmatrix}}_{A} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

Det val. proprii ale lui A

$$\det(A - \lambda I_2) = 0 \iff \begin{vmatrix} 5-\lambda & 3 \\ -3 & -1-\lambda \end{vmatrix} = 0$$

$$\iff (5-\lambda)(-1-\lambda) + 9 = 0$$

$$\lambda^2 - 4\lambda + 4 = 0$$

$$\Rightarrow \lambda_1 = \lambda_2 = 2 = m_a$$

Vectorii pr. asociați pt $\lambda_1 = 2$

$$(A - 2I_2) v = 0$$

$$\iff \begin{pmatrix} 3 & 3 \\ -3 & -3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$3v_1 + 3v_2 = 0 \qquad \Rightarrow v = \begin{pmatrix} \alpha \\ -\alpha \end{pmatrix}$

$v_2 = \alpha$

$v_1 = \alpha \qquad S(\lambda_1 = 2) = \{\alpha(-1), \alpha \in \mathbb{R}\}$

$$m_g = \dim(S(\lambda_1 = 2)) = 1 < 2 = m_a$$

O soluție este $x(t) = c_1 \cdot e^{2t} \cdot v_1 = \boxed{c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t}}$

O variantă: Caut sol. de forma $x(t) = P(t) e^{2t}$, unde $P(t)$ are grad $m_a - m_g = 1$

$$x(t)=\begin{pmatrix} a_1+b_1t \\ a_2+b_2t \end{pmatrix} e^{2t}$$

$$x' = Ax$$

$$\Leftrightarrow e^{2t}\left[\begin{pmatrix} (b_1+2a_1)+2b_1t \\ (b_2+2a_2)+2b_2t \end{pmatrix}\right] = \begin{pmatrix} 5 & 3 \\ -3 & -1 \end{pmatrix}\begin{pmatrix} a_1+b_1t \\ a_2+b_2t \end{pmatrix}$$

$$\Leftrightarrow \begin{cases} b_1+2a_1 = 5a_1+3a_2 \\ 2b_1 = 5b_1+3b_2 \\ b_2+2a_2 = -3a_1-a_2 \\ 2b_2 = -3b_1-b_2 \end{cases} \Leftrightarrow \begin{cases} b_1-3a_1-3a_2=0 \\ -3b_1-3b_2=0 \\ b_2+3a_2+3a_1=0 \\ 3b_2+3b_1=0 \end{cases}$$

$$\Rightarrow \begin{cases} b_1+b_2=0 \\ b_1-3a_1-3a_2=0 \end{cases} \Leftrightarrow \begin{pmatrix} 0 & 0 & 1 & 1 \\ -3 & -3 & 1 & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{pmatrix} \Rightarrow$$

$\Rightarrow$ b_1, b_2 principale
a_1, a_2 secundare

$$\begin{cases} b_1+b_2=0 \\ b_1=3a_1+3a_2 \end{cases}$$

$$a_1=\alpha, \; a_2=\beta \Rightarrow \begin{cases} b_1 = 3\alpha+3\beta \\ b_2 = -3\alpha-3\beta \end{cases}$$

$$x(t)=\begin{pmatrix} \alpha+(3\alpha+3\beta)t \\ \beta-(3\alpha+3\beta)t \end{pmatrix} e^{2t} = \alpha\begin{pmatrix} 1+3t \\ -3t \end{pmatrix} e^{2t} + \beta e^{2t}\begin{pmatrix} 3t \\ 1-3t \end{pmatrix}$$

$$\alpha=\beta \Rightarrow x(t)=\begin{pmatrix} 1+3t \\ 1-3t \end{pmatrix} e^{2t}$$

$$\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t}, \begin{pmatrix} 1+3t \\ 1-3t \end{pmatrix} e^{2t}\right\}$$

Ex. Ce dracului mai e și asta?

$e^{tA} = ?$

$\lambda_1 = \lambda_2 = 2$

A_1, A_2

$$f(A) = f(\lambda_1)A + f'(\lambda_1)A_2$$

$$= f(2)A_1 + f'(2)A_2$$

$f(x) = 1$

$f(x) = x$

$f(x) = x^2$

$$\Rightarrow \begin{cases} I_2 = A_1 \\ f(A) = 2A_1 + A_2 \end{cases}$$

$f(x) = e^{tx}$

$$e^{tA} = e^{2t}A_1 + te^{2t}A_2$$

$\lambda_1 = \lambda_2 = 1$

$\downarrow$

$c_1 v_1 e^t$

$$e^{tA} = \sum_{n=0}^{\infty} \frac{(tA)^n}{n!}$$

$\lambda_3 = 4$

$\Downarrow$

$c_3 v_3 e^{4t}$

$m_a = m_g = 2$

$$S(\lambda_1 = 1) = \langle v_1, v_2 \rangle$$