TO:

FROM: Ecuații diferențiale – 5.12.2017 – Seminar

Algoritm (Ecuații afine pe $\mathbb{R}^n$) [Metoda variației constantelor]

$\frac{dx}{dt} = A(t)x + b(t)$ $\quad$ $A(\cdot): I \subseteq \mathbb{R} \to L(\mathbb{R}^n, \mathbb{R}^n)$, $b(\cdot): I \to \mathbb{R}^n$ | cont.

1. Consideră ec. liniară asociată $\frac{d\bar{x}}{dt} = A(t)\bar{x}$

Determină $\{\bar{\varphi}_1(\cdot), \dots, \bar{\varphi}_n(\cdot)\}$ sistem fundamental de soluții

Obs: Dacă $A(t) = A \in L(\mathbb{R}^n, \mathbb{R}^n) \to$ vezi Algoritm

Scrie sol. generală $\bar{x}(t) = \sum_{i=1}^{n} c_i \bar{\varphi}_i(t)$

2. Caută sol. de forma $x(t) = \sum_{i=1}^{n} c_i(t) \bar{\varphi}_i(t)$

$x(\cdot)$ sol. $\Rightarrow \sum_{i=1}^{n} c_i'(t) \bar{\varphi}_i(t) = b(t) \Rightarrow c_i'(t) = \dots$ $i = \overline{1,n}$

$c_i(t) = \dots$ $i = \overline{1,n}$

$\Rightarrow x(t) = \dots$

Algoritm (Ecuații liniare de ordin superior cu coeficienți constanți)

$x^{(n)} = \sum_{j=1}^{n} a_j x^{(n-j)}$

1. Rezolvă ec. caracteristică $\lambda^n = \sum_{j=1}^{n} a_j \lambda^{n-j} \to \sigma(a_1, \dots, a_n): (\lambda, m_\lambda)$

2. Pt. $\lambda \in \sigma(a_1, \dots, a_n)$ scrie sol. $\quad \varphi_\lambda^j(t) = \begin{cases} t^{j-1} e^{\lambda t} & \lambda \in \mathbb{R},\ j = \overline{1, m_\lambda} \\ t^{j-1} e^{\alpha t} \cos \beta t & \lambda = \alpha + i\beta,\ \beta > 0,\ j = \overline{1, m_\lambda} \\ t^{j-1} e^{\alpha t} \sin \beta t & \end{cases}$

3. Renumerotează $\{\varphi_\lambda^j(\cdot)\}_{\substack{\lambda \in \sigma(a_1, \dots, a_n) \\ j = \overline{1, m_\lambda}}} = \{\varphi_1(\cdot), \dots, \varphi_n(\cdot)\}$ sist. fundamental de sol.

Scrie sol. generală $x(t) = \sum_{i=1}^{n} c_i \varphi_i(t)$, $c_i \in \mathbb{R}$, $i = \overline{1,n}$

Să se det. sol. generală

1) $\begin{cases} x' = 2x - y + e^{-t} \\ y' = 3y - 2x + e^{-t} \end{cases}$

2) $\begin{cases} x' = x - y + 2\sin t \\ y' = 2x - y \end{cases}$

$\begin{cases} \bar{x}' = \bar{x} - \bar{y} \\ \bar{y}' = 2\bar{x} - \bar{y} \end{cases} \Rightarrow \bar{y} = -\bar{x}' + \bar{x}$

$\Rightarrow (\bar{x}-\bar{x}')' = 2\bar{x} - (\bar{x}-\bar{x}')$

$\Rightarrow \bar{x}' - \bar{x}'' = 2\bar{x} - \bar{x} + \bar{x}'$

$\Rightarrow \bar{x}'' + \bar{x} = 0$

$\lambda^2 + 1 = 0 \Rightarrow \lambda_{1,2} = \pm i$

Fie $\lambda = i \Rightarrow \bar{\varphi}_1(t) = e^{0\cdot t} \cos 1t = \cos t$

$\bar{\varphi}_2(t) = e^{0t} \sin 1t = \sin t$

$\Rightarrow \bar{x}(t) = c_1 \cos t + c_2 \sin t$

$\Rightarrow \bar{y}(t) = c_1 \cos t + c_2 \sin t + c_1 \sin t - c_2 \cos t$

$\Rightarrow \bar{y}(t) = c_1(\cos t + \sin t) + c_2(\sin t - \cos t)$, $c_1, c_2 \in \mathbb{R}$

$x(t) = c_1(t) \cos t + c_2(t) \sin t$

$y(t) = c_1(t)(\cos t + \sin t) + c_2(t)(\sin t - \cos t)$

$(c_1(t) \cos t + c_2(t) \sin t)' = c_1(t) \cos t + c_2(t) \sin(t) - c_1(t)(\cos(t) + \sin t) - c_2(t)(\sin t - \cos t) +$
$+ 2 \sin t$

$(c_1(t)(\cos t + \sin t) + c_2(t)(\sin t - \cos t))' = 2(c_1(t) \cos t + c_2(t) \sin t) - c_1(t)(\cos t + \sin t) -$
$- c_2(t)(\sin t - \cos t)$

~~$c_1(t) \sin t + c_2'(t)$~~

$c_1'(t) \cos t - c_1(t) \sin t + c_2'(t) \sin t + c_1(t) \cos t = c_1(t) \cos t + c_2(t) \sin t - c_1(t) \cos t - c_1(t) \sin t$
$- c_2(t) \sin t + c_2(t) \cos t + 2 \sin t$

$c_1'(t)(\cos t - \sin t) + c_1(t)(-\sin t + \cos t) + c_2'(t)(\sin t - \cos t) + c_2(t)(\cos t - \sin t)) =$
$= 2c_2(t) \cos t + 2c_2(t) \sin t - c_2(t) \cos t - c_1(t) \sin t - c_2(t) \sin t + c_2(t) \cos t$

$$\begin{cases} c_1'(t) \cos t + c_2'(t) \sin t = 2 \sin t \\ c_1'(t)(\cos t + \sin t) + c_2'(t)(\sin t - \cos t) = 0 \end{cases}$$

$$\begin{cases} c_1'(t) \cos t + c_2'(t) \sin t = 2 \sin t & | \cos t \\ c_1'(t) \sin t - c_2'(t) \cos t = -2 \sin t & | \sin t \end{cases}$$

$\Rightarrow$ ~~$c_1'(t) \cos^2 t + c_2'(t)$~~ $c_1'(t)(\cos^2 t + \sin^2 t) = 2 \sin t \cos t - 2 \sin^2 t$

$\Rightarrow c_1'(t) = 2 \sin t \cos t - 2 \sin^2 t$

$c_2'(t) = 2 \sin^2 t + 2 \sin t \cos t$

$\Rightarrow c_1(t) = \int (2 \sin t \cos t - 2 \sin^2 t) dt = \int (\sin 2t - 1 + \cos 2t) dt = -\frac{1}{2} \cos 2t - t + \frac{1}{2} \sin 2t + k_1$

TO:

FROM:

$c_2(t) = \int (2\sin^2 t + 2\sin t \cos t)\,dt = \int (1-\cos 2t + \sin 2t)\,dt = t - \frac{1}{2}\sin 2t - \frac{1}{2}\cos 2t + k_2$

$k_2 \in \mathbb{R}$

3) $\begin{cases} x' = 2y - x + 1 \\ y' = 3y - 2x \end{cases}$ Temă

4) $x^{IV} + x'' = 0$

$\lambda^4 + \lambda^2 = 0$

$\lambda^2(\lambda^2+1) = 0 \Rightarrow \lambda_{1,2} = 0,\ \lambda_{3,4} = \pm i$

$\lambda = 0 \quad , m_\lambda = 2$

$\varphi_1(t) = e^{0\cdot t} = 1$

$\varphi_2(t) = t\cdot e^{0t} = t$

$\lambda = i$

$\varphi_3(t) = e^{0\cdot t}\cdot \cos 1\cdot t = \cos t$

$\varphi_4(t) = e^{0\cdot t}\cdot \sin 1\cdot t = \sin t$

$\Rightarrow x(t) = c_1\varphi_1(t) + c_2\varphi_2(t) + c_3\varphi_3(t) + c_4\varphi_4(t)$

$= c_1 + c_2 t + c_3\cos t + c_4 \sin t,\ c_1, c_2, c_3, c_4 \in \mathbb{R}$

5) $x''' - 3x'' + 3x' - x = 0 \quad , x(0) = 1,\ x'(0) = 0,\ x''(0) = -1$

$\lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0$

$(\lambda-1)^3 = 0 \Rightarrow \lambda = 1,\ m_\lambda = 3$

$\left.\begin{array}{l} \varphi_1(t) = e^{1t} = e^t \\ \varphi_2(t) = te^{1t} = te^t \\ \varphi_3(t) = t^2 e^{1\cdot t} = t^2 e^t \end{array}\right\}$ sol. gen. $x(t) = c_1 e^t + c_2 t e^t + c_3 t^2 e^t,\ c_1, c_2, c_3 \in \mathbb{R}$

$x(0) = c_1 = 1$

$x'(t) = c_1 e^t + c_2 e^t + c_2 t e^t + 2c_3 t e^t + c_3 t^2 e^t = (c_1+c_2)e^t + (c_2+2c_3)te^t + c_3 t^2 e^t$

$x'(0) = c_1 + c_2 = 0 \Rightarrow c_2 = -c_1 = -1$

$x''(t) = (c_1+c_2)e^t + (c_2+2c_3)e^t + (c_2+2c_3)te^t + 2c_3 te^t + c_3 t^2 e^t$

$x''(0) = c_1 + c_2 + c_2 + 2c_3 = 1 - 2 + 2c_3 = -1 \Rightarrow 2c_3 = -2 \Rightarrow c_3 = 0$

$x(t) = e^t - te^t$