TO:

FROM: Ecuații diferențiale – curs – 7.11.2017

Existența sol. globale

Def: $f(\cdot,\cdot): I \times \mathbb{R}^n \to \mathbb{R}^n$ pp. că are prop. de:

a) DISIPATIVITATE (D) dacă $\exists\ r > 0$ $\exists\ a(\cdot): I \to \mathbb{R}_+$ continuă a.î. $|\langle x, f(t,x)\rangle| \le a(t)\|x\|^2$, $\forall t \in I$, $\forall x \in \mathbb{R}^n$, $\|x\| \ge r$

b) CREȘTERE LINIARĂ (CL) dacă $\exists\ r > 0$ $\exists\ a(\cdot): I \to \mathbb{R}_+$ continuă a.î. $\|f(t,x)\| \le a(t)\|x\|$, $\forall t \in I$, $\forall x \in \mathbb{R}^n$, $\|x\| \ge r$

c) CREȘTERE AFINĂ (CA), dacă $\exists\ r > 0$ $\exists\ a(\cdot), b(\cdot): I \to \mathbb{R}_+$ continuă a.î. $\|f(t,x)\| \le a(t)\|x\| + b(t)$ $\forall t \in I$, $\forall x \in \mathbb{R}^n$, $\|x\| \ge r$

PROP
1. C.L. $\Leftrightarrow$ CA
2. CL $\Rightarrow$ D
3. $n = 1$ CL $\Leftrightarrow$ D
4. $n > 1$ D $\not\Rightarrow$ CL

Dem: 1. CL $\Rightarrow$ CA $(b(t) \equiv 0)$

CA $\Rightarrow$ CL $\exists\ r > 0$, $\exists\ a(\cdot): I \to \mathbb{R}_+$ continuă a.î. $\|f(t,x)\| \le a(t)\|x\| + b(t)$, $\forall t \in I$, $x \in \mathbb{R}^n$, $\|x\| \ge r$

$= \|x\|\left(a(t) + \frac{b(t)}{\|x\|}\right) \le \|x\| \underbrace{\left(a(t) + \frac{b(t)}{r}\right)}_{a_1(t)}$

2. Inegalitatea C–S–B $x, y \in \mathbb{R}^n$ $|\langle x, y\rangle| \le \|x\| \cdot \|y\|$

$|\langle x, f(t,x)\rangle| \overset{C-S}{\le} \|x\| \cdot \|f(t,x)\| \le a(t)\|x\|^2$, $\forall t \in I$, $\forall x \in \mathbb{R}^n$, $\|x\| \ge r$

3. D $\Rightarrow$ CL

D $\exists\ r > 0$ $\exists\ a(\cdot): I \to \mathbb{R}_+$ a.î. $|x f(t,x)| \le a(t) \cdot \|x\|^2$, $\forall t \in I$, $\forall x \in \mathbb{R}$ $|x| \ge r$

$\Rightarrow |f(t,x)| \le a(t)|x|$, $\forall x \in \mathbb{R}^n$, $t \in I$, $|x| \ge r$

4. D $\not\Rightarrow$ CL

$f = ?$ a.î. $\langle x, f(t,x)\rangle = 0$; $\langle (x_1, x_2), (f_1(t,x), f_2(t,x))\rangle = 0$

$x_1 f_1(t,x) + x_2 f_2(t,x) = 0$ $\quad f_1(t,x) = x_2$ $\quad f_2(t,x) = -x_1$ $\quad f(t,(x_1,x_2)) = (x_2, -x_1)$

$\|f(t,x)\| = \sqrt{x_1^2 + x_2^2} = \|x\|$ ...

$\langle x, f(t,x)\rangle = 0$ $\quad \|f(t,x)\| = \|x\| \sqrt{x_1^2+x_2^2} = \|x\|^2$

Pp. că $f(\cdot,\cdot)$ are CL $\Rightarrow \exists\ r > 0$ $\exists\ a(\cdot): I \to \mathbb{R}_+$ cont. $\|x\|^2 \le a(t)\|x\|$ $\forall t \in I$, $\forall x \in \mathbb{R}^2$, $\|x\| \ge r$

$\|x\| \le a(t)$ $\forall t$, $\forall x$ Fie $t = t_0 \in I \Rightarrow \underset{\infty}{\|x\|} \le \underset{\infty}{a(t_0)}$ $\forall x \in \mathbb{R}^n$, $\|x\| \ge r$

Th (existența globală)

$f(\cdot,\cdot): I \times \mathbb{R}^n \to \mathbb{R}^n$ cont. cu (D) $\qquad \frac{dx}{dt} = f(t,x)$

Atunci $\forall (t_0, x_0) \in I \times \mathbb{R}^n$ $\exists\ \varphi(\cdot): I \to \mathbb{R}^n$ sol. cu $\varphi(t_0) = x_0$

Dem: Pp. $I = (\alpha, \beta)$ $\underset{\alpha}{+}\ \underset{a}{(}\ \underset{t_0}{+}\ \underset{b}{)}\ \underset{\beta}{+}$

Fie $(t_0, x_0) \in I \times \mathbb{R}^n$ - mulț. deschisă, $f(\cdot,\cdot)$ cont. $\Rightarrow$ T. Peano (E.L.) $\exists$ $\varphi_0(\cdot) : I_0 \in V(t_0) \to \mathbb{R}^n$ sol. cu $\varphi_0(t_0) = x_0$

T. existența sol. maximale $\Rightarrow \exists\ \varphi(\cdot) : J \to \mathbb{R}^n$ sol. maximală $\varphi(\cdot) \succ \varphi_0(\cdot) \Rightarrow \varphi(t_0) = \varphi_0(t_0) = x_0$

PROP (Intervalul de def. al sol. maximale) $J = (a, b)$

Arătăm că $J = I$. De exemplu, arătăm că $a = \alpha$ (analog $b = \beta$)

Pp. $\alpha < a$

/* Th. asupra prelungirii soluțiilor:

$f(\cdot,\cdot) : D = \mathring{D} \in \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ cont. $\frac{dx}{dt} = f(t, x)$

1. $\varphi(\cdot) : (a, b) \subset \mathbb{R} \to \mathbb{R}^n$ sol. Atunci:

2. $\varphi(\cdot)$ admite o prelungire strictă la stânga $\Rightarrow a > -\infty$ $\exists\ t_0 \in (a, b)$ $\exists\ D_0 \subset D$ compactă a.î. $(t, \varphi(t)) \in D_0,\ \forall t \in (a, b]$

$\alpha < a \Rightarrow a > \infty$, $t_0 \in (a, b)$? $D_0 \subset D$ compactă a.î. $(t, \varphi(t)) \in D_0\ \forall t \in (a, b]$

Dacă da $\Rightarrow$ T. asupra prel. sol. $\Rightarrow \varphi(\cdot)$ admite o prelungire strictă la stânga ✗ ($\varphi(\cdot)$ maximală)

$f(\cdot,\cdot)$ are (D) $\Rightarrow \exists\ r > 0$ $\exists\ a(\cdot) : I \to \mathbb{R}_+$ cont. a.î. $|\langle x, f(t,x)\rangle| \le a(t) \cdot \|x\|^2$ $\forall$ $\forall t \in I, \forall x \in \mathbb{R}^n, \|x\| > r$

$(\|\varphi(t)\|^2)' = 2\langle \varphi(t), \varphi'(t)\rangle = 2\langle\varphi(t), f(t, \varphi(t))\rangle$

Fie $A = \{t \in (a, b] ; \|\varphi(t)\| > r\}$ $B = (a, b] \setminus A$

Dacă $t \in A$ at. $(\|\varphi(t)\|^2)' = 2\langle \varphi(t), f(t, \varphi(t))\rangle \ge -2a(t)\|\varphi(t)\|^2$

Dacă $t \in B$ $(\|\varphi(t)\|^2)' = 2\langle \varphi(t), f(t, \varphi(t))\rangle \overset{C-S}{\ge} -2\|\varphi(t)\|\,\|f(t, \varphi(t))\|$

$(|\langle x, y\rangle| \le \|x\|\,\|y\|)$

$$K := \max_{t, x \in [a, b] \times \bar{B}_r(0)} \|f(t, x)\|$$

Dacă $t \in (a, b_0]$ $(\|\varphi(t)\|^2)' \ge \min(-2a(t)\|\varphi(t)\|^2, -2rk) \ge -2rk - 2a(t)\|\varphi(t)\|$

$\Big| \int_t^{t_0}$ $\Rightarrow \int_t^{t_0} (\|\varphi(s)\|^2)' ds \ge -2rk(t_0 - t) - 2\int_t^{t_0} a(s)\|\varphi(s)\|^2 ds$

$\|\varphi(t_0)\|^2 - \|\varphi(t)\|^2 \ge -2rk(t_0 - t) + 2\int_t^{t_0} a(s)\|\varphi(s)\|^2 ds$

$\|\varphi(t)\|^2 \le \|\varphi(t_0)\|^2 + 2rk(t_0 - t) + 2\int_{t_0}^{t} a(s)\|\varphi(s)\|^2 ds \le \underbrace{\|\varphi(t_0)\|^2 + 2rk(t_0 - a)}_{= M} + 2 \cdot$

$\cdot \int_{t_0}^{t} a(s)\|\varphi(s)\|^2 ds$

TO:

FROM:

$t \in [a,b] \Rightarrow \|\varphi(t)\|^2 \le M + \left|\int_{t_0}^{t} 2a(s)\|\varphi(s)\|^2 ds\right| \xrightarrow[B.G.]{Lema}$

$\left[u(t) \le M + \left|\int_{t_0}^{t} a(s) \overset{?}{v}(s) ds\right| \ \forall t \Rightarrow u(t) \le M e^{\left|\int_{t_0}^{t} 2(s) ds\right|}\right]$

$\Rightarrow \|\varphi(t)\|^2 \le M \cdot e^{2\left|\int_{t_0}^{t} a(s) ds\right|} \le M e^{2\int_a^{t_0} a(s) ds} = \rho^2$

$\Rightarrow \|\varphi(t)\| \le \rho \ \forall t \in [a, t_0] \Leftrightarrow \varphi(t) \in \bar{B}_\rho(0)$

$D_0 := [a, t_0] \times \bar{B}_\rho(0) \quad (t, \varphi(t)) \in D_0, \ \forall t \in [a,b)$

Continuitatea sol. maximale în raport cu datele inițiale și parametri

$\frac{dx}{dt} = f(t,x) \quad f(\cdot,\cdot): D = \overset{\circ}{D} \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ cont. admite UL pe D. Af. $\forall (\tau, \xi) \in D \ \exists! \ \varphi_{\tau,\xi}(\cdot)$:

$: I(\tau,\xi) = (t^-(\tau,\xi), t^+(\tau,\xi)) \to \mathbb{R}^n$ soluție maximală $\varphi_{\tau,\xi}(\tau) = \xi$

Def: S.n. curentul maximal al e.v. $f(\cdot,\cdot)$ funcția $\alpha f(\cdot,\cdot): Df \subseteq \mathbb{R} \times D \to \mathbb{R}^n$ a.î.

$\forall (\tau,\xi) \in D \ \alpha f(\cdot,\tau,\xi): I(\tau,\xi) \to \mathbb{R}^n$ sol. maximală ~~to~~ unică a pb. C. (f, τ, ξ)

$Df = \{(t,\tau,\xi) ; (\tau,\xi) \in D, \ t \in I(\tau,\xi) = (t^-(\tau,\xi), t^+(\tau,\xi))\}$

$\alpha f(\cdot,\tau,\xi) = \varphi_{\tau,\xi}(\cdot) \quad - D_1 \alpha f(t,\tau,\xi) = f(t, \alpha f(t,\tau,\xi))$

$- \alpha f(\tau,\tau,\xi) = \xi$

$- \alpha f(\cdot,\tau,\xi)$ sol. maximală

Th. asupra curentului maximal

Fie $f(\cdot,\cdot): D = \overset{\circ}{D} \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ continuă $\frac{dx}{dt} = f(t,x)$

P.p. că a) $f(\cdot,\cdot)$ admite UL pe D $\quad \Leftarrow f(\cdot,\cdot)$ local Lipschitz (II)

b) $f(\cdot,\cdot)$ – " – ~~EL~~ curent local în fiecare punct din D

Fie $\alpha f(\cdot,\cdot,\cdot): Df \to \mathbb{R}^n$ curentul maximal

Atunci 1) $Df \subset \mathbb{R} \times D$ deschisă

2) $\alpha f(\cdot,\cdot,\cdot)$ continuă

Obs: Dacă $f(\cdot,\cdot)$ cont., local-Lipschitz (II) $\Rightarrow$ T. Cauchy-Lipschitz $\Rightarrow \exists$ EUL pe D adică a)

$\Rightarrow$ Th. (Existența, unicitatea, continuitatea curentului local parametrizat) $\Rightarrow f(\cdot,\cdot)$ ~~continuă~~ admite curent local continuu în fiecare punct din D adică b)

$\frac{dx}{dt} = f(t,x,\lambda) \quad f(\cdot,\cdot,\cdot): D = \overset{\circ}{D} \subseteq \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$ continuă

$\forall \lambda \in pr_3 D \quad f(\cdot,\cdot,\lambda)$ admite UL a sol.

$\forall (t_0, x_0, \lambda_0) \in D \ \exists! \ \varphi_{t_0,x_0,\lambda_0}(\cdot): I(t_0,x_0,\lambda_0) = (t^-(t_0,x_0,\lambda_0), t^+(t_0,x_0,\lambda_0)) \to \mathbb{R}^n$ soluție maximală a pb. Cauchy $(f(\cdot,\cdot,\lambda_0), t_0, x_0)$

Def: S.n. curentul maximal parametrizat asociat e.v.p. $f(\cdot,\cdot,\cdot)$, funcția $\alpha f(\cdot,\cdot,\cdot)$:

$: Df \subseteq \mathbb{R} \times D \to \mathbb{R}^n$ a.î. $\forall (\tau,\xi,\lambda) \in D \ \alpha f(\cdot,\tau,\xi,\lambda): I(\tau,\xi,\lambda) \to \mathbb{R}^n$ sol. maximală a pb. Cauchy $(f(\cdot,\cdot,\lambda), \tau, \xi)$

$- D_1 \alpha f(t,\tau,\xi,\lambda) = f(t, \alpha f(t,\tau,\xi,\lambda), \lambda)$

$- \alpha f(\tau,\tau,\xi,\lambda) = \xi$

$-$ sol. maximală

Metodă generală de studiu: Transformarea parametrului în date inițiale

(1) $\frac{dx}{dt} = f(t, x, \lambda)$

(2) $\begin{cases} \frac{dx}{dt} = f(t, x, \lambda) \\ \frac{d\lambda}{dt} = 0_{\mathbb{R}^k} \end{cases}$ $\quad \bar{x} = (x, \lambda) \quad \bar{f}(t, (x, \lambda)) = (f(t, x, \lambda), 0_{\mathbb{R}^k})$ (2) $\frac{d\bar{x}}{dt} = \bar{f}(t, \bar{x})$

$\bar{f}(\cdot, \cdot) : D \subseteq \mathbb{R} \times (\mathbb{R}^n \times \mathbb{R}^k) \longrightarrow \mathbb{R}^n \times \mathbb{R}^k$

PROP (de echivalență):

$\varphi(\cdot)$ este sol. a ec. (1) $\Leftrightarrow$ $\bar{\varphi}(\cdot) = (\varphi(\cdot), \lambda)$ este sol. a ec. (2)

Dem: „$\Rightarrow$" $\varphi'(t) = f(t, \varphi(t), \lambda)$

$(\lambda)' \equiv 0$

„$\Leftarrow$" $\varphi(\cdot) = (\varphi_1(\cdot), \varphi_2(\cdot))$ sol. a ec. (2) $\Rightarrow \begin{cases} \varphi_1'(t) \equiv f(t, \varphi_1(t), \varphi_2(t)) \\ \varphi_2'(t) \equiv 0 \Rightarrow \exists\, \lambda \in pr_3 D \text{ a.î. } \varphi_2(t) \equiv \lambda \end{cases}$

Corolar: $\alpha_f(\cdot, \cdot, \cdot, \cdot) : D_f \subseteq \mathbb{R} \times D \to \mathbb{R}^n$ curent maximal parametrizat al ec. (1) $\Leftrightarrow$

$\alpha_{\bar{f}}(\cdot, \cdot, \cdot) : D_{\bar{f}} = D_f \to \mathbb{R}^n$ $\alpha_{\bar{f}}(t, \tau, (\xi, \lambda)) := (\alpha_f(t, \tau, \xi, \lambda), \lambda)$ este curentul maximal (neparametrizat) al ec. (2)

Th. (Continuitatea sol. maximale în raport cu parametrii)

Fie $f(\cdot, \cdot, \cdot) : D = \mathring{D} \subseteq \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^k \longrightarrow \mathbb{R}^n$ cont. local Lipschitz (II) $\frac{dx}{dt} = f(t, x, \lambda)$

Fie $\alpha_f(\cdot, \cdot, \cdot) : D_f \to \mathbb{R}^n$ curentul maximal parametrizat

Atunci 1. $D_f \subseteq \mathbb{R} \times D$ deschisă

2. $\alpha_f(\cdot, \cdot, \cdot)$ continuă

Ecuații liniare pe $\mathbb{R}^n$

Def: Fie $A(\cdot) : I \subseteq \mathbb{R} \to L(\mathbb{R}^n, \mathbb{R}^n)$ def. ec. liniara $\frac{dx}{dt} = A(t)x$

Obs: $n = 1$ $L(\mathbb{R}, \mathbb{R}) \simeq \mathbb{R}$ $x' = a(t)x$ ec. liniara scalară

$\varphi(\cdot)$ sol. a ec. $\Leftrightarrow \varphi(t) = c\, e^{\int_{t_0}^{t} a(s)\, ds}$

$B \subset \mathbb{R}^n$ bază $\quad L(\mathbb{R}^n, \mathbb{R}^n) \simeq M_n(\mathbb{R})$ $\quad A_B(t) = (a_{ij}(t))_{\substack{i = \overline{1,n} \\ j = \overline{1,n}}}$ $\quad A_B(t) = col(A(t)_{b_1}, \dots, A(t)_{b_n})$

$B = \{b_1, \dots, b_n\}$

$\frac{dx_i}{dt} = \sum_{j=1}^{n} a_{ij}(t) x_j \quad i = \overline{1,n}$

sistem de ec. liniare

TO:

FROM:

<u>Th</u> (existență & unicitate globală):

Fie $A(\cdot) : I \to L(\mathbb{R}^n, \mathbb{R}^n)$ cont. $\frac{dx}{dt} = A(t)x$

Af. $\forall (t_0, x_0) \in I \times \mathbb{R}^n \ \exists! \ \varphi_0(\cdot) : I \to \mathbb{R}^n$ sol. cu $\varphi_0(t_0) = x_0$

<u>Dem</u>: $f_{A(\cdot)}(t, x) := A(t)x$

$f_{A(\cdot)}(\cdot, \cdot)$ cont. local Lipschitz (ii) (liniară (ii)) $\overset{T}{\underset{\text{Cauchy-Lipschitz}}{\Rightarrow}}$ E.U.L $\Rightarrow$ UG

$\|f_{A(\cdot)}(t, x)\| = \|A(t, x)\| \leq \|A(t)\|$ C.L. $\Rightarrow \Delta \overset{T}{\underset{EG}{\Rightarrow}}$ EG