Curs 9

Ecuații liniare pe $\mathbb{R}^n$ cu coeficienți constanți

Structura soluțiilor în cazul general

Lemă $A \in L(\mathbb{R}^n, \mathbb{R}^n)$ $\frac{dx}{dt} = Ax$, $\lambda \in \sigma(A)$ $P_j \in \mathbb{C}^n$

$$\tilde{\varphi}(t) := e^{\lambda t} \sum_{j=0}^{n-1} P_j t^j$$

Atunci $\tilde{\varphi}'(t) \equiv A\tilde{\varphi}(t) \iff \begin{cases} (A - \lambda I_m)^m P_0 = 0 \\ P_j = \frac{1}{j!}(A - \lambda I_m)^j \cdot P_0, \ j = \overline{1, m-1} \end{cases}$

Dem $\tilde{\varphi}'(t) \equiv A \cdot \tilde{\varphi}(t)$

$$\lambda \cancel{e^{\lambda t}} \sum_{j=0}^{m-1} P_j t^j + \cancel{e^{\lambda t}} \sum_{j=1} j P_j t^{j-1} \equiv A \cancel{e^{\lambda t}} \sum_{j=0}^{m-1} P_j t^j$$

$$\lambda \sum_{j=0}^{m-1} P_j t^j + \sum_{j=0}^{m-2} (j+1) P_{j+1} t^j \equiv A \sum_{j=0}^{m-1} P_j t^j$$

$j = \overline{0, m-2}$ $\quad \lambda \cdot P_j + (j+1) P_{j+1} = A \cdot P_j$

$j = m-1$ $\quad \lambda \cdot P_{m-1} = A \cdot P_{m-1}$

$j = \overline{0, m-2}$ $\quad P_{j+1} = \frac{1}{j+1}(A - \lambda i_m) \cdot P_j = \frac{1}{(j+1)} \cdot \frac{1}{j} \cdot$

$\cdot (A - \lambda i_m)^2 P_{j-1} = \dots = \frac{1}{(j+1)!}(A - \lambda i_m)^{j+1} \cdot P_0$

$j = m-1$ $\quad \left.\begin{array}{l} (A - \lambda i_m) P_{m-1} = 0 \\ P_{m-1} = \frac{1}{(m-1)!} \cdot (A - \lambda i_m)^{m-1} \cdot P_0 \end{array}\right\} \Rightarrow \frac{1}{(m-1)!}(A - \lambda i_m)^m \cdot P_0 = 0$

Teoremă asupra forma canonică Jordan a unei matrice

$\forall A \in M_m(\mathbb{C}) \ \exists C \in M_n(\mathbb{C}) \ \det C \neq 0$ a.î.

celulă

unde $\lambda_r \in \sigma(A)$

$s_{r_i} = \dim J_r \leq$ multiplicitatea $(\lambda_r) = m_{\lambda_r}$

$$\sum_{\lambda_r = \lambda} \dim J_r = m_{\lambda_r}$$

$\boxed{\text{Forma echivalentă}}$

$\forall A \in L(\mathbb{R}^n, \mathbb{R}^n)\ \exists\ B_J = \{ b_r^s ;\ s = \overline{1, s_r},\ r = \overline{1, p} \}$

$\subset \mathbb{C}^n$ baza canonică Jordan a.î. (s_r — dim celulei Jordan)

a) $\forall r = \overline{1,p}\ \exists\ \lambda_r \in \sigma(A)$

$$\begin{cases} A \cdot b_r^1 = \lambda_r b_r^1 + b_r^2 \\ A b_r^2 = \lambda_r b_r^2 + b_r^3 \\ \ldots \\ A b_r^{s_r - 1} = \lambda_r b_r^{s_r-1} + b_r^{s_r} \\ A b_r^{s_r} = \lambda_r b_r^{s_r} \end{cases}$$

2) $s_r \leq m_{\lambda_r}$ $\quad \sum_{\lambda_r = \lambda} s_r = m_\lambda\ \forall \lambda \in \sigma(A)$

3) Dacă $\lambda_r \in \mathbb{R} \Rightarrow b_r^s \in \mathbb{R}^n$, dacă $\operatorname{Im} \lambda_r > 0$ și $\lambda_r = \overline{\lambda_t}$

atunci $b_r^s = \overline{b_t^s}$

C1) $A \in L(\mathbb{R}^n, \mathbb{R}^n)$ $B_J \subset \mathbb{C}^n$ baza canonică Jordan

Atunci $B_J^{\mathbb{R}^n} = \{ b_r^s ;\ \lambda_r \in \mathbb{R},\ s = \overline{1, s_r} \} \cup$

$\cup \{ \operatorname{Re}(b_r^s), \operatorname{Im}(b_r^s);\ \operatorname{Im} \lambda_r > 0,\ s = \overline{1, s_r} \} \subset \mathbb{R}^n$ bază

(urma pe $\mathbb{R}^n$ a bazei canonice Jordan)

$$\sum_{\substack{\lambda_r \in \mathbb{R} \\ s=\overline{1,s_r}}} c_r^s b_r^s + \sum_{\operatorname{Im}\lambda_r > 0} \left[c_r^s \operatorname{Re}(b_r^s) + k_r^s \operatorname{Im}(b_r^s) \right] = 0$$

$$\overset{?}{\Rightarrow} c_r^s = k_r^s = 0 \quad \forall s, \forall r$$

$$b_r^s = \operatorname{Re}(b_r^s) + i \cdot \operatorname{Im}(b_r^s)$$

$$\overline{b_r^s} = \operatorname{Re}(b_r^s) - i \cdot \operatorname{Im}(b_r^s)$$

$$\operatorname{Re}(b_r^s) = \frac{1}{2}(b_r^s + \overline{b_r^s})$$

$$\operatorname{Im}(b_r^s) = \frac{1}{2i} \cdot (b_r^s - \overline{b_r^s})$$

$$\sum_{\substack{\lambda_r \in \mathbb{R} \\ s=\overline{1,s_r}}} c_r^s b_r^s + \sum_{\substack{\operatorname{Im}\lambda_r > 0 \\ s=\overline{1,s_r}}} c_r^s \frac{1}{2}(b_r^s + \overline{b_r^s}) + k_r^s \frac{1}{2i}(b_r^s - \overline{b_r^s}) = 0$$

$$\sum_{\substack{\lambda_r \in \mathbb{R} \\ s=\overline{1,s_r}}} c_r^s b_r^s + \sum_{\substack{\operatorname{Im}\lambda_r > 0 \\ s=\overline{1,s_r}}} \left[\left(\frac{c_r^s}{2} + \frac{k_r^s}{2i} \right) b_r^s + \left(\frac{c_r^s}{2} - \frac{k_r^s}{2i} \right) \overline{b_r^s} \right] = 0 \xrightarrow[\text{bază}]{B_J}$$

$$\Rightarrow c_i^s = 0 \quad \lambda_r \in \mathbb{R}, \ s = \overline{1,s_r} \qquad \begin{cases} \dfrac{c_r^s}{2} + \dfrac{k_r^s}{2} = 0 \\[2ex] \dfrac{c_r^s}{2} - \dfrac{k_r^s}{2} = 0 \end{cases} \quad \operatorname{Im}\lambda_r > 0, \ s = \overline{1,s_r} \Rightarrow$$

$$\Rightarrow c_r^s = k_r^s = 0 \quad \operatorname{Im}\lambda_r > 0, \ s = \overline{1,s_r}$$

C2 $A \in L(\mathbb{R}^n, \mathbb{R}^n)$ $B_J \subset \mathbb{C}^n$ baza canonică Jordan

$\lambda \in \sigma(A)$ $B_J^\lambda := \{ b_r^s \mid \lambda_r = \lambda, s = \overline{1,s_r} \}$

Atunci: 1) $\operatorname{card}(B_J^\lambda) = m_\lambda$

2) $(A - \lambda I_n)^{m_\lambda} b_r^s = 0 \quad \forall b_r^s \in B_J^\lambda$

liniar indep.

$\boxed{\text{Dem}}$ 1) & 3) imediat din T

2) $A b_r^1 = \lambda b_r^1 + b_r^2 \Rightarrow b_r^2 = (A - \lambda i_m) b_r^1$

$A b_r^2 = \lambda b_r^2 + b_r^3 \Rightarrow b_r^3 = (A - \lambda i_m) b_r^2 = (A-\lambda i_m$

$\vdots$

$A b_r^{s_r - 1} = \lambda b_r^{s_r - 1} + b_r^{s_r}$

$b_r^{s_r} = (A - \lambda i_m) b_r^{s_r - 1} = \ldots = (A - \lambda$

$A b_r^{s_r} = \lambda b_r^{s_r}$

$\Downarrow$

$(A - \lambda i_m) b_r^{s_r} = 0.$

$\Rightarrow (A - \lambda i_m)(A - \lambda i_m)^{s_r - 1} b_r^1 = 0$

$$\left.\begin{array}{l} (A - \lambda i_m)^{s_r} b_r^1 = 0 \\ s_r \leq m_\lambda \end{array}\right\} \Rightarrow (A - \lambda i_m)^{m_\lambda} b_r^1 = 0.$$

$$(A - \lambda i_m)^{m_\lambda} b_r^s = (A - \lambda i_m)^{m_\lambda} \cdot (A - \lambda i_m)^{s-1} b_r^1 =$$

$$= (A - \lambda i_m)^{m_\lambda + s - 1} b_r^1 = 0.$$

$\boxed{\text{T (Structura sol în cazul general)}}$

$A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ $\frac{dx}{dt} = A x$

Atunci: 1. $\forall \lambda \in \sigma(A) \cap \mathbb{R}$ $m_\lambda = m \geq 1$ $\exists P_j^{\lambda l} \in \mathbb{R}^n$ a.î

$$\varphi_{\lambda l}(t) = e^{\lambda t} \cdot \sum_{j=0}^{m-1} P_j^{\lambda l} t^j, \quad l = \overline{1,m}$$

$\{\varphi_{\lambda l}(\cdot)\}_{l = \overline{1,m}} \subset S_A$ liniar independente

2. $\forall \lambda \in \sigma(A)$ $\operatorname{Im} \lambda > 0$ $m_\lambda = m \geq 1$ $\exists P_j^{\lambda l} \in \mathbb{C}^n$ a

$$\varphi_{\lambda l}(t) = \operatorname{Re}\left(e^{\lambda t} \sum_{j=0}^{m-1} P_j^{\lambda l} t^j\right), \quad l = \overline{1,m}$$

$\{\varphi_{\lambda l}(\cdot), \varphi_{\overline{\lambda l}}(\cdot)\}_{l=\overline{1,m}} \subset S_A$ liniar independente

3. $\{\varphi_{\lambda l}(\cdot)\}_{\lambda\in\sigma(A),\ l=\overline{1,m_\lambda}} \subset S_A$ sistem fundamental de soluții (bază în spațiul soluțiilor)

$\boxed{\text{Dem}}$ $A\in L(\mathbb{R}^n,\mathbb{R}^n) \longrightarrow B_J=\{b_r^s,\ s=\overline{1,s_r},\ r=\overline{1,p}\}$

$\subset \mathbb{C}^n$ bază canonică Jordan

1. Fie $\lambda\in\sigma(A)\cap\mathbb{R}$ $m_\lambda=m\geq 1$

Fie $B_J^\lambda=\{b_r^s;\ \lambda_r=\lambda,\ s=\overline{1,s_r}\} \overset{not}{=} \{p_0^{\lambda 1},\dots,p_0^{\lambda m}\}$

liniar indep.

$p_j^{\lambda l} := \frac{1}{j!}(A-\lambda I_n)^j p_0^{\lambda l},\ j=\overline{1,m-1}\quad l=\overline{1,m}$

C2 $\Longrightarrow (A-\lambda I_n)^m p_0^{\lambda l}=0,\ l=\overline{1,m}$

Fie $\varphi_{\lambda l}(t)=e^{\lambda t}\sum_{j=0}^{m-1} p_j^{\lambda l}\, t^j$ $\Big\}\xrightarrow{\text{Lema}}$

$\Rightarrow \varphi'_{\lambda l}(t)=A\cdot\varphi_{\lambda l}(t)$

$\Rightarrow \varphi_{\lambda l}(\cdot)\in S_A\ \forall\, l=\overline{1,m}$

$\{\varphi_{\lambda l}(\cdot)\}_{l=\overline{1,m}} \subset S_A$ liniar independente

$\Updownarrow$ PROP (Sol. liniar independente)

$\{\varphi_{\lambda l}(0)\}_{l=\overline{1,m}} \subset \mathbb{R}^m$ liniar indep

$\|$

$\{p_0^{\lambda l}\}_{l=\overline{1,m}}$ liniar indep

2. $\lambda\in\sigma(A)$ $\operatorname{Im}\lambda>0$ $m_\lambda=m\geq 1$

$$P^{\lambda l}_{j} := \frac{1}{j!}(A-\lambda I_n)^j P^{\lambda_n}_{0} \quad j=\overline{1,m-1}, \quad l=\overline{1,m}$$

<u>Lemă</u> $\Rightarrow \overset{\vee}{\varphi}{}'_{\lambda l}(t) \equiv A\overset{\vee}{\varphi}_{\lambda l}(t)$

$$\overset{\vee}{\varphi}_{\lambda l}(t) = \underbrace{Re(\overset{\vee}{\varphi}_{\lambda l}(t))}_{\varphi_{\lambda l}(t)} + i\cdot \underbrace{Im(\overset{\vee}{\varphi}_{\lambda l}(t))}_{\varphi_{\overline{\lambda l}}(t)} = \varphi_{\lambda l}(t) + i\cdot\varphi_{\overline{\lambda l}}$$

$$\underline{\varphi'_{\lambda l}(t)} + i\cdot \underline{\varphi'_{\overline{\lambda l}}(t)} \equiv A\,\underline{\varphi_{\lambda l}(t)} + i\cdot A\,\underline{\varphi_{\overline{\lambda} l}(t)}$$

$$\Rightarrow \begin{cases}\varphi'_{\lambda l}(t) \equiv A\cdot\varphi_{\lambda l}(t)\\ \varphi'_{\overline{\lambda l}}(t) \equiv A\cdot\varphi_{\overline{\lambda l}}(t)\end{cases} \Rightarrow \varphi_{\lambda l}(\cdot), \varphi_{\overline{\lambda l}}(\cdot) \in S_A \quad \forall\, l=\overline{1,m}$$

$$\{\varphi_{\lambda l}(\cdot), \varphi_{\overline{\lambda l}}(\cdot)\}_{l=\overline{1,m}} \subset S_A \text{ liniar indep.}$$

$\Updownarrow$ PROP (sol liniar independente)

$$\{\varphi_{\lambda l}(0), \varphi_{\overline{\lambda l}}(0)\}_{l=\overline{1,m}} \subset \mathbb{R}^n \text{ liniar indep}$$

$$\parallel$$

$$\{Re(P^{\lambda l}_0), Im(P^{\lambda l}_0)\}_{l=\overline{1,m}}$$

$$\parallel$$

$$\{Re(b^S_r), Im(b^S_r)\}_{l=\overline{1,m}} \subset \mathbb{R}^n \text{ liniar indep} \quad C1 \quad C2$$

3. n soluții $\Rightarrow$ E suficient să verificăm doar liniar independența

$\{\ldots\}$ ... liniar independente

$\{\varphi_{\lambda e}(\cdot)\}_{\substack{\lambda\in\sigma(A)\\ e=\overline{1,m_\lambda}}} \subset \mathbb{R}^n$ liniar independente

$\parallel$

$\{b^s_r,\ \lambda_r\in\mathbb{R},\ s=\overline{1,s_r}\}\cup\{Re(b_r^s),\ Im(b_r^s),\ Im\,\lambda_r>0,$
$s=\overline{1,s_r}\}\subset\mathbb{R}^n$ liniar independente $\mathbb{C}_1$

$\boxed{\text{Algoritm}}$ $\quad \frac{dx}{dt}=Ax$

1. Rezolvăm ec caracteristică $\det(A-\lambda i_m)=0 \longrightarrow \sigma(A)$
   $\sigma(A):(\lambda,\ m_\lambda)$
   $\hookrightarrow$ ordin de multiplicitate a răd. λ

2. Dacă $\lambda\in\sigma(A)\cap\mathbb{R},\ m_\lambda=1$ caută $u_\lambda\in\mathbb{R}^n\setminus\{0\}$ a.î. $(A-\lambda i_m)u_\lambda=0$
   Scrie sol $\varphi_\lambda(t)=e^{\lambda t}\cdot u_\lambda$

3. Dacă $\lambda\in\sigma(A)\cap\mathbb{R},\ m_\lambda=m>1$
   Caută $\{P_0^{\lambda 1},\dots P_0^{\lambda m}\}\subset Ker(A-\lambda\cdot i_n)^m$ liniar independentă (în $\mathbb{R}^n$)
   Scrie $P_j^{\lambda e}=\frac{1}{j!}(A-\lambda\cdot i_n)^j P_0^{\lambda e},\ j=\overline{1,m-1},\ e=\overline{1,m}$
   Scrie sol $P_{\lambda e}(t)=e^{\lambda t}\cdot\sum_{j=0}^{m-1}P_j^{\lambda e}t^j\quad e=\overline{1,m}$

4. Dacă $\lambda=\alpha+i\beta\in\sigma(A),\ \beta>0,\ m_\lambda=1.$
   Caut $u_\lambda\in\mathbb{C}^n\setminus\{0\}$ a.î. $(A-\lambda i_m)u_\lambda=0.$
   Scrie sol $\varphi_\lambda(t)=Re(e^{\lambda t},u_\lambda)$
   $\varphi_{\bar\lambda}(t)=Im(e^{\lambda t},u_\lambda)$

5. Dacă $\lambda=\alpha+i\beta\in\sigma(A),\ \beta>0,\ m_\lambda=m\geq 2$

Scrie $P_j^{\lambda e} = \frac{1}{j!}(A-\lambda I_m)^j P_0^{\lambda e}$, $j=\overline{1,m-1}$, $e=\overline{1,m}$

Scrie soluțiile:

$$\varphi_{\lambda e}(t) = Re\left(e^{\lambda t}\sum_{j=0}^{m-1} P_j^{\lambda t} t^j\right) \quad e=\overline{1,m}$$

$$\varphi_{\overline{\lambda e}}(t) = Im\left(e^{\lambda t}\sum_{j=0}^{m-1} P_j^{\lambda e} t^j\right)$$

6. Renumerotează $\{\varphi_{\lambda e}(\cdot)\}_{\substack{\lambda\in\sigma(A)\\ e=\overline{1,m_\lambda}}} =$

$= \{\varphi_1(\cdot), \varphi_2(\cdot), \dots \varphi_n(\cdot)\}$ sistem fundamental de soluții

Scrie soluția generală $\varphi(t) = \sum_{i=1}^{n} c_i\varphi_i(t)$, $c_i\in\mathbb{R}$, $i=\overline{1,n}$

$\boxed{\text{Ecuații diferențiale afine pe } \mathbb{R}^n}$

$\boxed{\text{Def}}$ $A(\cdot): I\subseteq\mathbb{R}\to L(\mathbb{R}^n,\mathbb{R}^n)$, $b(\cdot): I\to\mathbb{R}^n$ def

$$\frac{dx}{dt} = A(t)\cdot x + b(t)$$

În coordonate ($B\subset\mathbb{R}^n$ bază) $A_B(t) = (a_{ij}(t))_{i,j=\overline{1,n}}$

$$b(t) = \begin{pmatrix} b_1(t)\\ \vdots \\ b_n(t)\end{pmatrix}, \quad \frac{dx_i}{dt} = \sum_{j=1}^{n} a_{ij}(t)x_j + b_i(t), \quad i=\overline{1,n}$$

(sistem de ecuații afine)

Dacă $n=1$, $L(\mathbb{R},\mathbb{R}) \simeq \mathbb{R} \to x' = a(t)x + b(t)$,

$a(\cdot), b(\cdot): I\subseteq\mathbb{R}\to\mathbb{R}$ (ecuație afină scalară)

$\downarrow$

Metoda variației constantelor

T(E.U.G)

$$\frac{dx}{dt} = A(t)x + b(t)$$

Atunci $\forall (t_0, x_0) \in \mathring{I} \times \mathbb{R}^n \quad \exists!\ \varphi(\cdot) : \mathring{I} \to \mathbb{R}^n$ soluție cu $\varphi(t_0) = x_0$.

Dem: $f_{A(\cdot), b(\cdot)}(t, x) = A(t) \cdot x + b(t)$

$f_{A(\cdot), b(\cdot)}(\cdot, \cdot)$ cont local Lipschitz $(\Pi) \xRightarrow[\text{Cauchy-Lipschitz}]{T.}$ EUL pe $\mathring{I} \times \mathbb{R}^n$

$\Longleftrightarrow$ T.U.G

$$\| f_{A(\cdot), b(\cdot)}(t, x) \| = \| A(t)x + b(t) \| \leq \| A(t) \| \cdot \| x \| + \| b(t) \|$$

adică C.A $\Rightarrow D \xRightarrow[E.G]{T} E.G$

$$S_{A(\cdot), b(\cdot)} := \{ \varphi(\cdot) : \mathring{I} \to \mathbb{R}^n ;\ \varphi(\cdot) \text{ soluție } x' = A(t) \cdot x + b(t) \}$$

Seminar 9

Algoritm (cazul general)

$$\frac{dx}{dt} = Ax$$

1 Rezolvă ec. caracteristică

$\det(A - \lambda I_n) = 0 \rightarrow \sigma(A) : (\lambda, m_\lambda)$

2. Dacă $\lambda \in \sigma(A) \cap \mathbb{R}$, $m_\lambda = 1$ caută $u_\lambda \in \mathbb{R}^n \setminus \{0\}$

$(A - \lambda I_n) u_\lambda = 0$

Scrie sol $\varphi_\lambda(t) = e^{\lambda t} \cdot u_\lambda$

3. Dacă $\lambda \in \sigma(A) \cap \mathbb{R}$, $m_\lambda = m > 1$

Caută $\{p_0^{\lambda 1}, \dots p_0^{\lambda m}\} \in \operatorname{Ker}(A - \lambda I_n)^m$ liniar indep (în $\mathbb{R}^n$)

Scrie $p_j^{\lambda \ell} = \frac{1}{j!}(A - \lambda I_n)^j p_0^{\lambda \ell}$, $j = \overline{1, m-1}$, $\ell = \overline{1, m}$

Scrie soluțiile $\varphi_{\lambda \ell}(t) = e^{\lambda t} \sum_{j=0}^{m-1} p_j^{\lambda \ell} t^j$ $\ell = \overline{1, m}$

4. Dacă $\lambda = \alpha + i \cdot \beta \in \sigma(A)$, $\beta > 0$, $m_\lambda = 1$

Caută $u_\lambda \in \mathbb{C}^n \setminus \{0\}$ a.î. $(A - \lambda I_n) u_\lambda = 0$

Scrie sol $\varphi_\lambda(t) = \operatorname{Re}(e^{\lambda t} \cdot u_\lambda)$, $\varphi_{\bar\lambda}(t) = \operatorname{Im}(e^{\lambda t}$

5. Dacă $\lambda = \alpha + i\beta \in \sigma(A)$, $\beta > 0$, $m_\lambda = m > 1$

Caută $\{p_0^{\lambda 1}, \dots p_0^{\lambda m}\} \subset \operatorname{Ker}(A - \lambda I_n)^m$ liniar inde (în $\mathbb{C}^n$)

Scrie $p_j^{\lambda \ell} = \frac{1}{j!}(A - \lambda I_n)^j p_0^{\lambda \ell}$, $j = \overline{1, m-1}$, $\ell =$

Scrie soluțiile $\varphi_{\lambda \ell}(t) = \operatorname{Re}(e^{\lambda t} \sum_{j=0}^{m-1} p_j^{\lambda \ell} t^j)$ $\ell =$

6 Renumerotează $\{\varphi_{r_\ell}(\cdot)\}_{\substack{\lambda \in \sigma(A) \\ \ell=\overline{1,m}}} = \{\varphi_1(\cdot), \dots, \varphi_n(\cdot)\}$

sistem fundamental de soluții

Scrie sol. generală $\varphi(t) = \sum_{i=1}^{n} c_i \varphi_i(t), \ c_i \in \mathbb{R}$

<u>Ex</u>: Să se determine soluția generală:

1) $\begin{cases} x' = x - y + z \\ y' = x + y - z \\ z' = 2z - y \end{cases} \qquad A = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 0 & -1 & 2 \end{pmatrix}$

$$\det(A - \lambda I_n) = \begin{vmatrix} 1-\lambda & -1 & 1 \\ 1 & 1-\lambda & -1 \\ 0 & -1 & 2-\lambda \end{vmatrix} =$$

$$= (1-\lambda)^2 \cdot (2-\lambda) - \cancel{1} - \cancel{1} + \cancel{\lambda} + \cancel{2} - \cancel{\lambda}$$

$$= (1-\lambda)^2 \cdot (2-\lambda)$$

$\lambda_1 = 2$

$\lambda_2 = \lambda_3 = 1$

$\lambda = 2 \Rightarrow \overset{?}{u}$ a.î. $(A - 2I_3)u = 0$, unde $u = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$$\begin{pmatrix} -1 & -1 & 1 \\ 1 & -1 & -1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} -a - b + c = 0 \\ a - b - c = 0 \Rightarrow a = c \\ -b = 0 \Rightarrow b = 0 \end{cases}$$

$\Rightarrow \begin{pmatrix} a \\ 0 \\ a \end{pmatrix}, \ a = 1$

Soluția $\varphi_1(t) = e^{2t} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$\lambda = 1,\ m_\lambda = 2$

$$(A - I_3)^2 = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -1 & 2 \\ 0 & 0 & 0 \\ -1 & -1 & 2 \end{pmatrix}$$

$P_0 = ?$ a.î. $(A - I_3)^2 \cdot P_0 = 0$, $P_0 = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \Rightarrow$

$$\Rightarrow \begin{pmatrix} -1 & -1 & 2 \\ 0 & 0 & 0 \\ -1 & -1 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow a + b = 2c \Rightarrow c = \frac{a+b}{2}$$

$$\Rightarrow P_0 = \begin{pmatrix} a \\ b \\ \frac{a+b}{2} \end{pmatrix}$$

$P_{01} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \quad P_{02} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$

$P_j = \frac{1}{j!}(A - I_3)^j P_0,\ j = \underline{1}$

$P_{11} = (A - I_3) \cdot P_{01} \qquad P_{12} = (A - I_3) \cdot P_{02}$

$$P_{11} = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$P_{12} = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$$

$\varphi_2(t) = e^t (P_{01} + t \cdot P_{11})$

$\varphi_3(t) = e^t (P_{02} + t \cdot P_{12})$

$\varphi_2(t) = e^t \begin{pmatrix} 2 + t \\ t \\ 1 + t \end{pmatrix} \qquad \varphi_3(t) = e^t \begin{pmatrix} -t \\ 2 - t \\ 1 - t \end{pmatrix}$

2) $\begin{cases} x' = 4x - y \\ y' = 3x + y - z \\ z' = x + z \end{cases}$ $\quad A = \begin{pmatrix} 4 & -1 & 0 \\ 3 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix}$

$$\det(A - \lambda \cdot I_n) = \begin{vmatrix} 4-\lambda & -1 & 0 \\ 3 & 1-\lambda & -1 \\ 1 & 0 & 1-\lambda \end{vmatrix} =$$

$= (4-\lambda)(1-\lambda)^2 + 1 + 3(1-\lambda)$

$= (1-\lambda)(3 + (4-\lambda)(1-\lambda)) + 1$

$= -\lambda^3 + 6\lambda^2 - 12\lambda + 8 = (2-\lambda)^3$

$\lambda = 2 \quad m_\lambda = 3$

$(A - 2I_3)^3 = ?$

$$(A - 2I_3)^2 = \begin{pmatrix} 2 & -1 & 0 \\ 3 & -1 & -1 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 2 & -1 & 0 \\ 3 & -1 & -1 \\ 1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ 2 & -2 & 2 \\ 1 & -1 & 1 \end{pmatrix}$$

$$(A - 2I_3)^3 = \begin{pmatrix} 2 & -1 & 0 \\ 3 & -1 & -1 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 \\ 2 & -2 & 2 \\ 1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$(A - 2I_3)^3 = 0 \Rightarrow (A - 2I_3)^3 \cdot P_0 = 0, \ \forall P_0 \in \mathbb{R}^3$

$P_{01} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad P_{02} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad P_{03} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \to$ liniar indep. (baza canonică)

$P_j = \frac{1}{j!} (A - 2I_3)^j P_0, \ j = 1, 2$

$$P_{11} = (A - 2I_3)^1 \cdot P_{01} = \begin{pmatrix} 2 & -1 & 0 \\ 3 & -1 & -1 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$$

$P_{12} = \begin{pmatrix} -1 \\ -1 \\ \end{pmatrix} \quad P_{13} = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}$

$$P_{21} = \frac{1}{2}\cdot(A-\lambda I_3)^2\cdot P_{01} = \frac{1}{2}\begin{pmatrix} 1 & -1 & 1 \\ 2 & -2 & 2 \\ 1 & -1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

$$P_{22} = \frac{1}{2}\begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix} \qquad P_{23} = \frac{1}{2}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

$$\varphi_1(t) = e^{2t}(P_{01} + t\cdot P_{11} + t^2\cdot P_{21}) \quad \varphi_2(t) = e^{2t}(P_{02} + t\cdot P_{12} + t^2 \ldots$$

$$\varphi_3(t) = e^{2t}(P_{03} + t\,P_{13} + t^2 P_{23})$$

$$\varphi_1(t) = e^{2t}\begin{pmatrix} 1 + 2t + t^2\cdot\frac{1}{2} \\ 3t + t^2 \\ t + t^2\cdot\frac{1}{2} \end{pmatrix}$$

$$\varphi_2(t) = e^{2t}\begin{pmatrix} -t - \frac{1}{2}t^2 \\ 1 - t - t^2 \\ -t^2\cdot\frac{1}{2} \end{pmatrix}$$

$$\varphi_3(t) = e^{2t}\cdot\begin{pmatrix} t^2\cdot\frac{1}{2} \\ -t + t^2 \\ -1 - t + \frac{1}{2}\cdot t^2 \end{pmatrix}$$

3) $\begin{cases} x' = 2x + y \\ y' = 2y + 4z \\ z' = x - z \end{cases}$

4) $\begin{cases} x' = 2x + y \\ y' = 4y - x \end{cases}$

$$A = \begin{pmatrix} 2 & 1 \\ 2 & -1 \end{pmatrix} \quad \det(A - I_2\lambda) = \begin{vmatrix} 2-\lambda & 1 \\ 1 & -1-\lambda \end{vmatrix}$$

$$= (2-\lambda)(-1-\lambda) - 1 = -2 - 2\lambda + \lambda + \lambda^2 - 1$$

$$A=\begin{pmatrix}2 & 1\\ -1 & 4\end{pmatrix} \quad \det(A-\lambda \cdot I_2)=\begin{vmatrix}2-\lambda & 1\\ -1 & 4-\lambda\end{vmatrix}=$$

$$=(2-\lambda)(4-\lambda)+1=(\lambda-3)^2$$

$\lambda_{1,2}=3$

$\lambda=3,\ m_\lambda=2$

$$(A-3I_2)^2=\begin{pmatrix}-1 & 1\\ -1 & 1\end{pmatrix}\begin{pmatrix}-1 & 1\\ -1 & 1\end{pmatrix}=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$$

$(A-3I_2)^2\cdot P_0=0,\ \forall P_0\in\mathbb{R}^2$

$$P_{01}=\begin{pmatrix}1\\0\end{pmatrix} \quad P_{02}=\begin{pmatrix}0\\1\end{pmatrix}$$

$$P_j=\frac{1}{j}\left(A-3I_2\right)^j\cdot P_0,\ j=1$$

$$P_{11}=(A-3I_2)\cdot P_{01} \qquad P_{12}=(A-3I_3)\cdot P_{02}$$

$$P_{11}=\begin{pmatrix}-1\\-1\end{pmatrix} \quad P_{12}=\begin{pmatrix}1\\1\end{pmatrix}$$

$$\varphi_1=e^{3t}\cdot(P_{01}+t\cdot P_{11})=\begin{pmatrix}1-t\\-t\end{pmatrix}\cdot e^{3t}$$

$$\varphi_2=e^{3t}\cdot(P_{02}+t\cdot P_{12})=\begin{pmatrix}t\\1+t\end{pmatrix}\cdot e^{3t}$$

A doua metodă (recomandată pt. dimensiunea 2)

$y=x'-2x$

$(x'-2x)'=4(x'-2x)-x \Rightarrow x''-2x'=4x'-8x-x$

$\Rightarrow x''-6x'+9x=0$

$\lambda^2-6\lambda+9=0 \Rightarrow (\lambda-3)^2=0$

$y(t) = 3 \cdot c_1 e^{3t} + c_2 \cdot e^{3t} + 3 c_2 t e^{3t} - 2 c_1 e^{3t} -$
$- 2 c_2 t e^{3t}$
$= (c_1 + c_2) \cdot e^{3t} + c_2 t e^{3t}$

5) $\begin{cases} x' = 5x + 3y \\ y' = -3x - y \end{cases}$

6) $\begin{cases} x' = 2y - 3x \\ y' = y - 2x \end{cases}$

Curs 10

Ecuații afine pe $\mathbb{R}^n$

PROP (Varietatea soluțiilor)

$$S_{A(\cdot),b(\cdot)} = \underbrace{S_{A(\cdot)}}_{\text{Sol. ec. liniare}} + \underbrace{\{\varphi_0(\cdot)\}}_{\text{sol. particulară a ec. afine}}, \quad \varphi_0(\cdot) \in S_{A(\cdot),b(\cdot)}$$

Dem „$\subseteq$" Fie $\varphi_0(\cdot) \in S_{A(\cdot),b(\cdot)} \overset{?}{\Longrightarrow} \varphi(\cdot) - \varphi_0(\cdot) \in S_{A(\cdot)}$

$$\varphi'(t) \equiv A(t)\varphi(t) + b(t)$$

$$\varphi_0'(t) \equiv A(t)\varphi_0(t) + b(t)$$

$(\varphi - \varphi_0)'(t) \equiv A(t)(\varphi(t) - \varphi_0(t)) \equiv A(t)(\varphi - \varphi_0)(t)$, i.e. $(\varphi - \varphi_0)(\cdot) \in S_{A(\cdot)}$

„$\supseteq$" Fie $\varphi_0(\cdot) \in S_{A(\cdot),b(\cdot)}$, și fie $\psi(\cdot) \in S_{A(\cdot)} \overset{?}{\Longrightarrow} \psi(\cdot) + \varphi_0(\cdot) \in S_{A(\cdot),b(\cdot)}$

$$\varphi_0'(t) \equiv A(t)\varphi_0(t) + b(t)$$

$$\psi'(t) \equiv A(t)\psi(t)$$

$$\underbrace{\varphi_0'(t) + \psi'(t)}_{=(\psi+\varphi_0)'(t)} \equiv A(t)(\varphi_0(t) + \psi(t)) + b(t) \equiv A(t)(\psi + \varphi_0)(t) + b(t)$$

Corolar Fie $\{\overline{\varphi_1}(\cdot), \dots, \overline{\varphi_n}(\cdot)\} \subset S_{A(\cdot)}$ sistem fundamental de soluții pt. ec. liniară asociată $\frac{d\overline{x}}{dt} = A(t)\overline{x}$ și fie $\varphi_0(\cdot) \in S_{A(\cdot),b(\cdot)}$

Atunci $\varphi(\cdot) \in S_{A(\cdot),b(\cdot)} \Longleftrightarrow \exists\, c_1, \dots, c_n \in \mathbb{R}$ a.î.

$$\varphi(t) = \sum_{i=1}^{n} c_i \overline{\varphi_i}(t) + \varphi_0(t) \quad \text{soluție generală a ec. afine}$$

T (Principiul variației constantelor)

Fie $X(\cdot): \overset{\circ}{I} \to M_n(\mathbb{R})$ matrice fundamentală de soluții pt. ec. liniară asociată $\frac{d\overline{x}}{dt} = A(t)\overline{x}$.

Atunci $\varphi(\cdot) \in S_{A(\cdot),b(\cdot)} \Longleftrightarrow \exists\, C(\cdot)$ primitivă a funcției

Dem "$\Rightarrow$" $\varphi(\cdot) \in S_{A(\cdot), b(\cdot)}$

Fie $c(t) := X^{-1}(t)\varphi(t) \Rightarrow \varphi(t) = X(t)\cdot c(t)$

$\varphi(t)$ soluție $\Rightarrow \varphi'(t) \equiv A(t)$
$+ b($

$\Rightarrow \underbrace{X'(t)c(t)}_{\substack{\| \\ A(t)\cancel{X(t)}c(t)}} + X(t)c'(t) \equiv A(t)\cancel{X(t)}c(t) + b(t)$

$$X(t)c'(t) \equiv b(t) \Rightarrow c'(t) = X^{-1}(t)b(t)$$

"$\Leftarrow$" $\varphi(t) \equiv X(t)\cdot c(t) \Rightarrow \varphi'(t) \equiv \underbrace{X'(t)}_{\substack{\| \\ A(t)X(t)}}c(t) + X(t)c'(t)$

$A(t)X(t)c(t), + X(t)\cdot X^{-1}(t)$

$\equiv A(t)\varphi(t) + b(t)$

Corolar Fie $\{\overline{\varphi_1}(\cdot), \dots \overline{\varphi_n}(\cdot)\} \subset S_{A(\cdot)}$ sistem fundam
de soluții pt. ec. liniară asociată $\frac{d\overline{x}}{dt} = A(t)\overline{x}$.

Atunci $\varphi(\cdot) \in S_{A(\cdot), b(\cdot)} \Leftrightarrow \exists\ c(\cdot) = \begin{pmatrix} c_1(\cdot) \\ \vdots \\ c_n(\cdot) \end{pmatrix}$ primiti
funcției $t \to (col(\overline{\varphi_1}(t), \dots \varphi_n(t)))^{-1} \cdot b(t)$ a.î.

$$\varphi(t) = \sum_{i=1}^{n} c_i(t) \cdot \overline{\varphi_i}(t)$$

Dem $\{\overline{\varphi_1}(\cdot), \dots \overline{\varphi_n}(\cdot)\} \subset S_{A(\cdot)}$ sistem fundamental
soluții $\Rightarrow X(t) = col(\overline{\varphi_1}(t), \dots \overline{\varphi_n}(t)), t \in I$ matrice
fundamentală de soluții $X(t)c(t) = col(\overline{\varphi_1}(t) \dots \overline{\varphi_n}(t))$

$= \sum_{i=1}^{n} c_i(t)\overline{\varphi_i}(t) + T \Rightarrow$ q.e.d

Algoritm (metoda variației constantelor pt. ec. afine pe

$$\frac{dx}{dt} = A(t)x + b(t)$$

1 Consideră ec. liniară asociată $\frac{d\overline{x}}{dt} = A(t)\overline{x}$

Scrie soluția generală $\overline{X}(t) = \sum_{i=1}^{n} c_i \overline{\varphi_i}(t)$

2. (Variația constantelor propriu-zise)

Caută soluții de forma $X(t) = \sum_{i=1}^{n} c_i(t) \cdot \overline{\varphi_i}(t)$

$$X(\cdot) \text{ sol} \Longrightarrow \sum_{i=1}^{n} c_i'(t)\, \overline{\varphi_i}(t) = b(t) \Longrightarrow c_i'(t) = \dots,\ i=\overline{1,n}$$

$$\Longrightarrow c_i(t) = \dots\ \ i=\overline{1,n}$$

$$\Longrightarrow x(t) = \dots$$

Ecuații diferențiale liniare de ordin superior

Def: $a_1(\cdot), \dots a_n(\cdot) : I \subseteq \mathbb{R} \to \mathbb{R}$ def $x^{(n)} = \sum_{j=1}^{n} a_j(t) x^{(n-j)}$ (1)

Metoda generală de studiu (sistemul canonic asociat)

$$(2)\begin{cases} \dfrac{dx_1}{dt} = x_2 \\ \dfrac{dx_2}{dt} = x_3 \\ \dots \\ \dfrac{dx_{n-1}}{dt} = x_n \\ \dfrac{dx_n}{dt} = \sum_{j=1}^{n} a_j(t) x_{n-j+1} \end{cases} \qquad \hat{X} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \qquad A(t) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & & & & \\ 0 & 0 & 0 & \dots & 1 \\ a_n(t) & \dots & & \dots & a_1(t) \end{pmatrix}$$

Matrice companion

$$\stackrel{not}{=} \text{comp}(a_1(t), \dots a_n(t))$$

$$(2)\quad \frac{d\hat{x}}{dt} = A(t)\hat{x}$$

PROP (de echivalență)

$\varphi(\cdot)$ sol a ec(1) $\Longleftrightarrow$ $\hat{\varphi}(\cdot) = (\varphi(\cdot), \dots \varphi^{(n-1)}(\cdot))$ este soluție a ec. (2)

T (E. U. G)

Fie $a_1(\cdot), \dots a_n(\cdot) : I \subseteq \mathbb{R} \to \mathbb{R}$ cont def $x^{(n)} = \sum_{j=1}^{n} a_j(t) x^{(n-j)}$

sol. cu $\varphi(t_0) = x_0,\ \varphi'(t_0) = x_0^1, \dots \varphi^{(n-1)}(t_0) = x_0^{n-1}$.

<u>Dem</u> Prop. de echivalență + T (E U G) pt. ec. liniare pe $\mathbb{R}^n$ aplicată ec. (2)

$$S_{a_1(\cdot), \dots a_n(\cdot)} := \{\varphi(\cdot) : \dot{I} \to \mathbb{R};\ \varphi(\cdot) \text{ soluție a ec. } x^{(n)} = \sum_{j=1}^{n} a_j(t)\, x^{(n-j)}$$

T (Spațiul soluțiilor)

Fie $a_1(\cdot), \dots a_n(\cdot) : \dot{I} \subseteq \mathbb{R} \to \mathbb{R}$ cont. def. $x^{(n)} = \sum_{j=1}^{n} a_j(t)\, x^{(n-j)}$

Atunci $S_{a_1(\cdot), \dots a_n(\cdot)} \subset C^{(n)}(\dot{I}, \mathbb{R})$ subsp. vectorial dim $(S_{a_1(\cdot) \dots}$

$A(t) = \text{comp}(a_1(t), \dots a_n(t))$

$T : S_{a_1(\cdot), \dots a_n(\cdot)} \longrightarrow S_{A(\cdot)}$ $T(\varphi(\cdot)) = \overset{\vee}{\varphi}(\cdot)\ (= \varphi(\cdot), \varphi'(\cdot), \dots$

T. liniară și bijectivă dim $(S_{A(\cdot)}) = n \Rightarrow$ q.e.d

<u>Def:</u> $\{\varphi_1(\cdot), \dots \varphi_n(\cdot)\} \subset S_{a_1(\cdot), \dots a_n(\cdot)}$ bază s.n. sistem fundamental de soluții. Dacă $\{\varphi_1(\cdot), \dots \varphi_n(\cdot)\} \subset S_{a_1(\cdot), \dots a_n(\cdot)}$ sistem fundamental de soluții $\varphi(\cdot) = S_{a_1(\cdot), \dots a_n(\cdot)} \Longleftrightarrow$

$$\exists\, c_1, \dots c_n \in \mathbb{R} \text{ a.î. } \underbrace{\varphi(t) = \sum_{i=1}^{n} c_i\, \varphi_i(t)}_{\text{soluția generală}}$$

Ecuații liniare de ordin superior cu coeficienți constanți

<u>Def:</u> $a_1, \dots a_n \in \mathbb{R}$ def $x^{(n)} = \sum_{j=1}^{n} a_j\, x^{(n-j)}$

Caz particular $a_j(t) \equiv a_j \in \mathbb{R},\ j = \overline{1,n}$

Sistem algebric asociat: [redacted]

$$(2)\begin{cases} \dfrac{dx_1}{dt} = x_2 \\ \vdots \\ \dfrac{dx_{n-1}}{dt} = x_n \\ \dfrac{dx_n}{dt} = \sum_{j=1}^{n} a_j x_{n-j+1} \end{cases} \qquad \hat{X} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \qquad A = comp(a_1, \dots a_n) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ a_n & a_{n-1} & \dots & & a_1 \end{pmatrix}$$

$$(2) \quad \frac{d\hat{X}}{dt} = A\hat{X}$$

<u>PROP</u> $\underset{\text{spectrul}}{\underset{\downarrow}{\sigma}}(comp(a_1, \dots a_n)) = \{ \lambda \in \mathbb{C};\ \lambda^n = \sum_{j=1}^{n} a_j \lambda^{n-j} \} =: \sigma(a_1 \dots$

<u>Def</u> $\underset{\text{cvasi-polinoame}}{\underset{\downarrow}{CP}}(\sigma(a_1, \dots a_n)) = \{ \varphi(t) = \sum_{j=1}^{n} e^{\lambda_j t} P_j(t) + \sum_{j=l+1}^{k} e^{\alpha_j t} \cdot$

$\cdot (P_j(t) \cos \beta_j t + Q_j(t) \sin \beta_j t)$, unde

$\sigma(a_1, \dots a_n) = \{ \lambda_1, \dots \lambda_l \in \mathbb{R}, \lambda_{l+1} = \alpha_{l+1} + i \cdot \beta_{l+1}, \overline{\lambda}_{l+1}, \dots$

$\lambda_k = \alpha_k + i \cdot \beta_k, \overline{\lambda_k} \to$ cu ordinele de multiplicitate $m_1, \dots m$

și $P_j(t), Q_j(t)$ sunt polinoame de grad $\leq m_j - 1$

T(structura soluțiilor)

$$S_{a_1 \dots a_n} = CP(\sigma(a_1, \dots a_n))$$

<u>Dem</u>: $\underset{\dim = n}{\underset{\downarrow}{S_{a_1 \dots a_n}}} \underset{\substack{\uparrow \\ \text{prop. de} \\ \text{echivalență}}}{\simeq} S_A \underset{\substack{\uparrow \\ T(\text{struct. sol})}}{\subseteq} \underset{\dim = n}{\underset{\downarrow}{CP(\sigma(a_1 \dots a_n))}}$

<u>Algoritm</u> $x^{(n)} = \sum_{j=1}^{n} a_j x^{(n-j)}$ $\quad a_1, \dots a_n \in \mathbb{R}$

1. Rezolvă ec. caracteristică $\lambda^n = \sum_{j=1}^{n} a_j \lambda^{n-j} \to \sigma(a_1, \dots a_n)$

2. Pt. $\lambda \in \sigma(a_1, \dots a_n)$ scrie soluțiile

$$\varphi_\lambda^j(t) = \begin{cases} t^{j-1} e^{\lambda t}, & \lambda \in \mathbb{R}, \ j = \overline{1, m_\lambda} \\ t^{j-1} e^{\alpha t} \cos \beta t & \lambda = \alpha + i\beta \\ t^{j-1} e^{\alpha t} \sin \beta t & \beta > 0 \end{cases} \quad j = \overline{1, m_\lambda}$$

3. Renumerotarea : $\{\varphi_\lambda^j(\cdot)\}_{\substack{\lambda \in \sigma(a_1, \dots a_n) \\ j = \overline{1, m_\lambda}}} =$

$= \{\varphi_1(\cdot), \dots \varphi_n(\cdot)\}$ sistem fundamental de soluții

Scrie sol. generală $x(t) = \sum_{i=1}^{n} c_i \varphi_i(t)$, $c_i \in \mathbb{R}$ $i = \overline{1, n}$

Ecuații afine de ordin superior

Def: $a_1(\cdot), \dots a_n(\cdot), b(\cdot) : I \subseteq \mathbb{R} \to \mathbb{R}$ def $x^{(n)} = \sum_{j=1}^{n} a_j(t) x^{(n-$

Sistemul canonic asociat

$$(2) \begin{cases} \dfrac{dx_1}{dt} = x_2 \\ \dfrac{dx_2}{dt} = x_3 \\ \vdots \\ \dfrac{dx_{n-1}}{dt} = x_n \\ \dfrac{dx_n}{dt} = \sum_{j=1}^{n} a_j(t) x_{n-j+1} + b(t) \end{cases}$$

$\hat{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ $A(t) = \text{comp}(a_1(t), \dots a_n(t))$

$\hat{b}(t) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ b(t) \end{pmatrix}$

(2) $\dfrac{d\hat{x}}{dt} = A(t)\hat{x} + \hat{b}(t)$

PROP (de echivalență)

$\varphi(\cdot)$ este soluție a ec. (1) $\Longleftrightarrow \hat{\varphi}(\cdot) = (\varphi(\cdot), \dots \varphi^{(n-1)}(\cdot))$

Sol. a ec. (2)

T(EUG)

Fie $a_1(\cdot), \dots a_n(\cdot), b(\cdot): \dot{I} \to \mathbb{R}$ continue def. $x^{(n)} = \sum_{j=1}^{n} a_j(t) x^{(n-j)} + b(t)$

Atunci $\forall (t_0, (x_0, x_0^1, \dots x_0^{n-1})) \in \dot{I} \times \mathbb{R}^n \ \exists! \ \varphi(\cdot): \dot{I} \to \mathbb{R}$ soluție

$\varphi(t_0) = x_0, \ \varphi'(t_0) = x_0^1, \dots \ \varphi^{(n-1)}(t_0) = x_0^{n-1}$

Dem: Prop. de echivalență + T(EUG) pt. ec. afine pe $\mathbb{R}^n$ aplicată ec. (2)

$$S_{a_1(\cdot), \dots a_n(\cdot) b(\cdot)} = \left\{ \varphi(\cdot): \dot{I} \to \mathbb{R}, \ \varphi(\cdot) \text{ sol a ec. } x^{(n)} = \sum_{j=1}^{n} a_j(t) x^{(n-j)} + b(t) \right\}$$

PROP (varietatea soluțiilor)

$$S_{a_1(\cdot), \dots a_n(\cdot) b(\cdot)} = S_{a_1(\cdot), \dots a_n(\cdot)} + \{\varphi_0(\cdot)\}, \ \varphi_0(\cdot) \in S_{a_1(\cdot) \dots a_n(\cdot) b(\cdot)}$$

Dem „$\subseteq$" Fixăm $\varphi_0(\cdot) \in S_{a_1(\cdot) \dots a_n(\cdot) b(\cdot)}$

Fie $\varphi(\cdot) \in S_{a_1(\cdot), \dots a_n(\cdot) b(\cdot)}$ $\overset{?}{\Longrightarrow} \varphi(\cdot) - \varphi_0(\cdot) \in S_{a_1(\cdot) \dots a_n(\cdot)}$

$\varphi^{(n)}(t) = \sum_{j=1}^{n} a_j(t) \varphi^{(n-j)}(t) + b(t)$

$\varphi_0^{(n)}(t) = \sum_{j=1}^{n} a_j(t) \varphi_0^{(n-j)}(t) + b(t)$

$(\varphi - \varphi_0)^{(n)}(t) = \sum_{j=1}^{n} a_j(t) (\varphi - \varphi_0)^{(n-j)}(t)$ $\quad \varphi - \varphi_0 \in S_{a_1(\cdot) \dots a_n(\cdot)}$

„$\supseteq$" Fie $\psi(\cdot) \in S_{a_1(\cdot), \dots a_n(\cdot)}$ $\varphi_0(\cdot) \in S_{a_1, \dots a_n(\cdot) b(\cdot)}$ $\overset{?}{\Longrightarrow}$

$\psi(\cdot) + \varphi_0(\cdot) \in S_{a_1(\cdot) a_2(\cdot), \dots a_n b(\cdot)}$

$\psi^{(n)}(t) = \sum_{j=1}^{n} a_j(t) \psi^{(n-j)}(t)$

$\varphi_0^{(n)}(t) = \sum_{j=1}^{m} a_j(t) \varphi_0^{(n-j)}(t) + b(t)$

Corolar: Dacă $\{\overline{\varphi_1(\cdot)}, \dots \overline{\varphi_n(\cdot)}\}$ sistem fundamental de soluții pt. ec. lin. asociată $x^{(n)} = \sum_{j=1}^{n} a_j(t) x^{(n-j)}$ și $\varphi_0(\cdot) \in S_{a_1(\cdot), \dots a_n(\cdot) b(\cdot)}$ atunci $\varphi(\cdot) \in S_{a_1(\cdot), \dots a_n(\cdot) b(\cdot)}$ $\Longleftrightarrow$

$$\exists\, c_1 \dots c_n \in \mathbb{R} \text{ a.î. } \underbrace{\varphi(t) = \sum_{i=1}^{n} c_i \overline{\varphi_i}(t) + \varphi_0(t)}_{\text{sol. generală}}$$

Seminar 10

$\boxed{\text{Ecuații afine pe } \mathbb{R}^n - \text{Algoritm}}$

(Metoda variației constantelor)

$$\left.\begin{array}{l} A(\cdot): I \subseteq \mathbb{R} \to L(\mathbb{R}^n, \mathbb{R}^n) \\ b(\cdot): I \to \mathbb{R}^n \end{array}\right| \text{cont}$$

1. Considerăm ec. liniară asociată $\frac{d\overline{x}}{dt} = A(t) \cdot \overline{x}$

Determină $\{\overline{\varphi_1(\cdot)}, \dots \overline{\varphi_n(\cdot)}\}$ sistem fundamental de solu

Obs: Dacă $A(t) \equiv A \in L(\mathbb{R}^n, \mathbb{R}^n) \to$ vezi Algoritm

Scrie sol. generală $\overline{x}(t) = \sum_{i=1}^{n} c_i \overline{\varphi_i}(t)$

2. Caută sol. de forma $x(t) = \sum_{i=1}^{n} c_i(t) \overline{\varphi_i}(t)$

$$x(\cdot) \text{ sol} \Longrightarrow \sum_{i=1}^{n} c_i'(t) \overline{\varphi_i}(t) = b(t) \Rightarrow c_i'(t) = \dots \quad i = \overline{1,n}$$

$$\Rightarrow c_i(t) = \dots \quad i = \overline{1,n}$$

$$\Rightarrow x(t) = \dots$$

$\boxed{\text{Ecuații liniare de ordin superior cu coeficienți constanți} - \text{Algo}}$

$$x^{(n)} = \sum_{j=1}^{n} a_j x^{(n-j)} \quad a_j \in \mathbb{R},\ j = \overline{1,n}$$

1. Rezolvă ec. caracteristică: $\lambda^n = \sum_{j=1}^{n} a_j \lambda^{n-j} \to \sigma(a_1, \dots a_n)$:

2. Pt $\lambda \in \sigma(a_1, \dots a_n)$ scrie sol

$$\varphi_\lambda^j(t) = \begin{cases} t^{j-1} e^{\lambda t} & \lambda \in \mathbb{R}, j = \overline{1, m_\lambda} \\ t^{j-1} e^{\alpha t} \cos \beta t & \\ t^{j-1} e^{\alpha t} \sin \beta t & \lambda = \alpha + i\beta, \beta > 0, j = \overline{1, m_\lambda} \end{cases}$$

3. Renumerotează $\{\varphi_\lambda^j(\cdot)\}_{\substack{\lambda \in \sigma(a_1, \dots a_n) \\ j = \overline{1, m_\lambda}}} = \{\varphi_1(\cdot), \dots \varphi_n(\cdot)\}$ sistem

<u>fundamental de soluții</u>. Scrie sol generală $x(t) = \sum_{i=1}^{n} c_i \varphi_i(t)$, $c_i \in \mathbb{R}, i = \overline{1, n}$

<u>Ex</u>: Să se determine soluția generală

1) $\begin{cases} x' = 2x - y + e^{-t} \\ y' = 3y - 2x + e^{-t} \end{cases}$

$\begin{cases} \bar{x}' = 2\bar{x} - \bar{y} \\ \bar{y}' = 3\bar{y} - 2\bar{x} \end{cases}$ $\leftarrow$ $\bar{y} = 2\bar{x} - \bar{x}'$

$\bar{y}' = 3(2\bar{x} - \bar{x}') - 2\bar{x}$

$\bar{y}' = 6\bar{x} - 3\bar{x}' - 2\bar{x}$

$\bar{y}' = 4\bar{x} - 3\bar{x}'$

$(2\bar{x} - \bar{x}')' = 4\bar{x} - 3\bar{x}'$ } Ar fi trebuit să înlocuiesc direct

$2\bar{x}' - \bar{x}'' = 4\bar{x} - 3\bar{x}'$

$-\bar{x}'' = -5\bar{x}' + 4\bar{x}$

$-\lambda^2 = -5\lambda + 4 \iff \lambda^2 - 5\lambda + 4$

$\iff (\lambda - 1)(\lambda - 4) = 0$

$\lambda_1 = 1, \lambda_2 = 4$

$\Rightarrow \bar{x}(t) = c_1 e^t + c_2 e^{4t}$, $c_1, c_2 \in \mathbb{R}$

$\bar{y}(t) = -(c_1 e^t + c_2 e^{4t})' + 2(c_1 e^t + c_2 e^{4t})$

$x(t) = c_1 e^{t} + c_2 e^{4t}$

$$\begin{cases} y(t) = c_1(t) e^{t} - 2 c_2(t) e^{4t} \\ x(t) = c_1(t) e^{t} + c_2(t) e^{4t} \end{cases}$$

$(c_1(t) \cdot e^{t} + c_2(t) e^{4t})' = 2(c_1(t) e^{t} + c_2(t) e^{4t}) -$
$- c_1(t) e^{t} + 2 c_2(t) e^{4t} + e^{-t}$

$c_1'(t) e^{t} + c_1(t) e^{t} + c_2'(t) e^{4t} + 4 c_2(t) e^{4t} =$
$= 2 c_1(t) e^{t} + c_2(t) e^{4t} - c_1(t) e^{t} + 2 c_2(t) e^{4t} + e^{-t}$

$(c_1(t) e^{t} - 2 c_2(t) e^{4t})' = 3(c_1(t) e^{t} - 2 c_2(t) e^{4t})$
$-2(c_1(t) e^{t} + c_2(t) e^{4t}) + e^{-t}$

$\Leftrightarrow c_1'(t) \cdot e^{t} + c_1(t) e^{t} - 2 c_2'(t) e^{4t} - 8 c_2(t) \cdot e^{4t}$
$= 3 c_1(t) e^{t} - 6 c_2(t) e^{4t} - 2 c_1(t) e^{t} - 2 c_2(t) e^{4t}$

$\Leftrightarrow c_1'(t) e^{t} + \cancel{c_1(t) e^{t}} - 2 c_2'(t) e^{4t} - \cancel{8 c_2(t) e^{4t}}$
$= \cancel{c_1(t) e^{t}} - \cancel{8 c_2(t) e^{4t}} + e^{-t}$

$$\begin{cases} c_1'(t) e^{t} + c_2'(t) e^{4t} = e^{-t} \\ c_1'(t) e^{t} - 2 c_2'(t) e^{4t} = e^{-t} \end{cases}$$

$3 c_2'(t) = 0 \Rightarrow c_2'(t) = 0$

$c_1'(t) e^{t} = e^{-t} \Rightarrow c_1'(t) = e^{-2t}$

$c_2(t) = k_2, \; k_2 \in \mathbb{R}$

$c_1(t) = -\frac{1}{2} e^{-2t} + k_1, \; k_1 \in \mathbb{R}$

$$\begin{cases} x(t) = (-\frac{1}{2} e^{-2t} + k_1) e^{t} + k_2 e^{4t}, \; k_1, k_2 \in \mathbb{R} \\ y(t) = (-\frac{1}{2} e^{-2t} + k_1) e^{t} - 2 k_2 e^{4t}, \; k_1, k_2 \in \mathbb{R} \end{cases}$$

2) $\begin{cases} x' = x - y + 2\sin t \\ y' = 2x - y \end{cases}$

$\begin{cases} \bar{x}' = \bar{x} - \bar{y} \Rightarrow \bar{y} = \bar{x} - \bar{x}' \\ \bar{y}' = 2\bar{x} - \bar{y} \end{cases}$

$(\bar{x} - \bar{x}')' = 2\bar{x} - \bar{x} + \bar{x}'$

$\cancel{\bar{x}'} - \bar{x}'' = \bar{x} + \cancel{\bar{x}'} \Rightarrow \bar{x}'' + \bar{x} = 0.$

$\lambda^2 + 1 = 0 \Rightarrow \lambda = \pm 1$

Fie $\lambda = i$

$\Rightarrow \bar{\varphi}_1(t) = e^{0t} \cdot \cos 1t = \cos t$

$\bar{\varphi}_2(t) = e^{0t} \sin 1t = \sin t$

$\Rightarrow \bar{x}(t) = c_1 \cos t + c_2 \sin t$

$\Rightarrow \bar{y}(t) = c_1 \cos t + c_2 \sin t + c_1 \sin t - c_2 \cos t$

$\bar{y}(t) = c_1(\cos t + \sin t) + c_2(\sin t - \cos t)$

$c_1, c_2 \in \mathbb{R}$

$\begin{cases} x(t) = c_1(t) \cdot \cos t + c_2(t) \cancel{\cos t}\ \sin t \\ y(t) = c_1(t)(\cos t + \sin t) + c_2(t) \cdot (-\cos t + \sin t) \end{cases}$

$(c_1(t) \cos t + c_2(t) \sin t)' = \cancel{c_1(t) \cos t} + \cancel{c_2(t) \sin t} -$

$- \cancel{c_1(t) \cos t} - c_1(t) \sin t + c_2(t) \cos t - \cancel{c_2(t) \sin t} + 2\sin t$

$c_1'(t) \cdot \cos t - \cancel{c_1(t)} \sin t + c_2'(t) \sin t + \cancel{c_2(t)} \cos t =$

$- \cancel{c_1(t) \sin t} + \cancel{c_2(t)} \cos t + 2\sin t$

$(c_1(t)(\cos t+\sin t)+c_2(t)(\sin t-\cos t))$

$= 2(c_1(t)\cos t + c_2(t)\sin t) - c_1(t)(\cos t+\sin t) -$

$- c_2(t)(\sin t - \cos t)$

$\Leftrightarrow c_1'(t)(\cos t+\sin t) + c_1(t)(-\sin t+\cos t)$

$+ c_2'(t)(\sin t-\cos t) + c_2(t)(\cos t+\sin t) =$

$= 2c_1(t)\cos t + 2c_2(t)\sin t - c_1(t)\cos t - c_1(t)\cdot \sin$

$- c_2(t)\sin t + c_2(t)\cos t$

$\Leftrightarrow c_1'(t)(\cos t+\sin t) - \cancel{c_1(t)}\sin t + \cancel{c_1(t)}\cos t$

$+ c_2'(t)(\sin t-\cos t) + \cancel{c_2(t)}\cos t + \cancel{c_2(t)}\sin t$

$= \cancel{c_1(t)}\cos t + \cancel{c_2(t)}\sin t - \cancel{c_1(t)}\sin t + \cancel{c_2(t)}\cos$

$\Leftrightarrow c_1'(t)(\cos t+\sin t) + c_2'(t)(\sin t-\cos t) = 0.$

$$\begin{cases} c_1'(t)\cos t + c_2'(t)\sin t = 2\sin t \\ c_1'(t)(\cos t+\sin t) + c_2'(t)(\sin t-\cos t) = 0. \end{cases}$$

$$\begin{cases} c_1'(t)\cos t + c_2'(t)\sin t = 2\sin t & \mid \cos t / \sin \\ c_1'(t)\sin t - c_2'(t)\cos t = -2\sin t & \mid \sin t / \cos \end{cases}$$

$$\begin{cases} c_1'(t)\cos^2 t + c_2'(t)\sin t\cos t = 2\sin t\cos t \\ c_1'(t)\sin^2 t - c_2'(t)\cos t\cdot\sin t = -2\sin^2 t \end{cases}$$

$c_1'(t)(\sin^2 t+\cos^2 t) = 2\sin t\cos t - 2\sin^2 t$

$c_1'(t) = 2\sin t\cos t - 2\sin^2 t$

$c_1'(t) = -2\sin^2 t + 2\sin t\cos t$

$$\Rightarrow c_1(t) = \int (2\sin t\cos t - 2\sin^2 t)\, dt$$

$$= \int (\sin 2t - \cancel{1} + \cos 2t)\, dt$$

$$= -\frac{1}{2}\cos 2t - t + \frac{1}{2}\sin 2t + k_1,\ k_1 \in \mathbb{R}$$

$$\Rightarrow c_2(t) = \int (2\sin^2 t + 2\sin t\cos t)\, dt$$

$$= \int (1 - \cos 2t + \sin 2t)\, dt$$

$$= t - \frac{1}{2}\sin 2t - \frac{1}{2}\cos 2t + k_2,\ k_2 \in \mathbb{R}.$$

3) $\begin{cases} x' = 2y - x + 1 \\ y' = 3y - 2x \end{cases}$

4) $x^{IV} + x'' = 0$

$$\lambda^4 + \lambda^2 = 0 \Leftrightarrow \lambda^2(\lambda^2 + 1) = 0 \Rightarrow \begin{cases} \lambda_1 = 0,\ \lambda_2 = 0 \\ \lambda_3 = i,\ \lambda_4 = -i \end{cases}$$

1 răd reală dublă, 2 răd ~~…~~ complexe conjugate.

$\lambda = 0,\ m_\lambda = 2$

$\varphi_1(t) = e^{0t} \Rightarrow \varphi_1(t) = 1$.

$\varphi_2(t) = t \cdot e^{0t} \Rightarrow \varphi_2(t) = t$

$\lambda = i$

$\varphi_3(t) = e^{0t} \cos 1t = \cos t$

$\varphi_4(t) = e^{0t} \sin 1t = \sin t$

$$\Rightarrow x(t) = c_1\varphi_1(t) + c_2\varphi_2(t) + c_3\varphi_3(t) + c_4\varphi_4(t)$$

$$= c_1 + c_2 t + c_3\cos t + c_4 \sin t$$

$c_1, c_2, c_3, c_4 \in \mathbb{R}$

5) $x''' - 3x'' + 3x' - x = 0$.

$x(0) = 1,\ x'(0) = 0,\ x''(0) = -1$.

$\lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0$.

$(\lambda - 1)^3 = 0, \quad \lambda = 1, \ m_\lambda = 3$

$$\varphi_1(t) = e^{1t} = e^t$$
$$\varphi_2(t) = t \cdot e^t$$
$$\varphi_3(t) = t^2 \cdot e^t$$

$\Rightarrow x(t) = c_1 e^t + c_2 e^t \cdot t + c_3 e^t \cdot t^2$

$c_1, c_2, c_3 \in \mathbb{R}$.

$x(0) = c_1 \cdot e^0 = 1 \Rightarrow c_1 = 1$

$x'(t) = c_1 e^t + c_2 e^t + c_2 t e^t + 2c_3 t e^t + c_3 t$

$= e^t(c_1 + c_2) + e^t \cdot t(2c_3 + c_2) + e^t \cdot t^2$

$\overset{t=0}{\Rightarrow} e^0(1 + c_2) = 0$

$c_2 = -1$.

$x''(t) = e^t(c_1 + c_2) + e^t(c_2 + c_3) + e^t t(c_2 + c_3$

$2c_3 t e^t + e^t t^2 c^3$

$= e^t(c_1 + 2c_2 + c_3) + e^t \cdot t(c_2 + 3c_3) + e^t$

$\overset{t=0}{\Rightarrow} e^0(1 - 2 + 2c_3) = -1$

$c_3 = 0$

$x(t) = e^t - t e^t \cancel{- t^2 e^t}$