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TO:

FROM: Ecuații diferențiale - curs - 14.11.2017

Ecuații liniare pe $\mathbb{R}^n$

$\frac{dx}{dt} = A(t)x$ $\qquad A(\cdot): I \subseteq \mathbb{R} \to L(\mathbb{R}^n, \mathbb{R}^n)$ continuă

$B \subset \mathbb{R}^n$ $\{b_1, \dots, b_n\}$ $\quad A_B(t) = col(A(t)b_1, \dots, A(t)b_n)$

$\underbrace{\frac{dx}{dt} = \sum_{j=1}^{n} a_{ij}(t)x_j}_{\text{sist. ec. liniare}} \quad i = \overline{1,n}$

$n=1$ ec. liniară scalară $x' = a(t)x$ $\quad \varphi(\cdot)$ sol $\Leftrightarrow \varphi(x) = c e^{\int_{t_0}^{t} a(s)ds}$, $t_0 \in I$

$a(\cdot): I \subseteq \mathbb{R} \to \mathbb{R}$ cont.

Th. (E.U.G)

Fie $A: I \subseteq \mathbb{R} \to L(\mathbb{R}^n, \mathbb{R}^n)$ continuă $\frac{dx}{dt} = A(t)x$. $\forall (t_0, x_0) \in I \times \mathbb{R}^n$ $\exists! \varphi(t_0, x_0)(\cdot): I \to \mathbb{R}^n$ sol. cu $\varphi_{t_0,x_0}(t_0) = x_0$

$S_{A(\cdot)} := \{\varphi(\cdot): I \to \mathbb{R}^n ;\ \varphi(\cdot)$ soluție $x' = A(t)x\}$

$A(\cdot)$ cont. $S_{A(\cdot)} \subseteq C^1(I, \mathbb{R}^n)$

PROP (sol. banală):

Fie $A: I \subseteq \mathbb{R} \to L(\mathbb{R}^n, \mathbb{R}^n)$ cont. $\frac{dx}{dt} = A(t)x$. Dacă $\varphi(\cdot) \in S_{A(\cdot)}$ a.î. $\exists t_0 \in I$ $\varphi(t_0) = 0$ at. $\varphi(t) \equiv 0$

Dem: $\varphi(\cdot): I \to \mathbb{R}^n$ sol. $\varphi(t_0) = 0$

Fie $\psi(\cdot): I \to \mathbb{R}^n$ $\psi(t) \equiv 0$ sol. $\psi(t_0) = 0$ $\Big\} \overset{UG}{\Longrightarrow} \varphi(t) \equiv \psi(t)$ q.e.d.

Th. (Spațiul soluțiilor):

Fie $A(\cdot): I \subseteq \mathbb{R} \to L(\mathbb{R}^n, \mathbb{R}^n)$ cont. $\frac{dx}{dt} = A(t)x$. Atunci $S_{A(\cdot)} \subset C^1(I, \mathbb{R}^n)$ subsp. vectorial $\dim(S_{A(\cdot)}) = n$

Dem: $\forall c_1, c_2 \in \mathbb{R}$, $\varphi_1, \varphi_2 \in S_{A(\cdot)} \Rightarrow c_1\varphi_1 + c_2\varphi_2 \in S_{A(\cdot)}$!

$(c_1\varphi_1 + c_2\varphi_2)'(t) = c_1\varphi_1'(t) + c_2\varphi_2'(t) = c_1 A(t)\varphi_1(t) + c_2 A(t)\varphi_2(t) = \equiv$

$= A(t)(c_1\varphi_1(t) + c_2\varphi_2(t)) = A(t)(c_1\varphi_1 + c_2\varphi_2)(t)$

Arătăm că $S_{A(\cdot)} \simeq \mathbb{R}^n$. Aplicația de evaluare în punctul $t_0 \in I$: $E_{t_0}: S_{A(\cdot)} \to \mathbb{R}^n$.

$E_{t_0}(\varphi) := \varphi(t_0)$. Arătăm că E_{t_0}: a) liniară b) injectivă c) surjectivă

? a) $c_1, c_2 \in \mathbb{R}$, $\varphi_1, \varphi_2 \in S_{A(\cdot)}$, $E_{t_0}(c_1\varphi_1 + c_2\varphi_2) = (c_1\varphi_1 + c_2\varphi_2)(t_0) = c_1\varphi_1(t_0) + c_2\varphi_2(t_0) =$

$= c_1 E(\varphi) + c_2 E(\varphi)$

b) $E_{t_0}(\varphi_1) = E_{t_0}(\varphi_2)$ $\quad \varphi_1(t_0) = \varphi_2(t_0) \overset{UG}{\Longrightarrow} \varphi_1 = \varphi_2$

c) $\forall \xi \in \mathbb{R}^n\ \exists \varphi \in S_{A(\cdot)}$ a.î. $\underbrace{E_{t_0}(\varphi)}_{\varphi(t_0)} = \xi$

T.E.G. apl. în $(t_0, \xi) \Rightarrow \exists \varphi$. sol. cu $\varphi(t_0) = \xi$

Def: S.n. sistem fundamental de soluții al ec. $x' = A(t)x$ mulțimea $\{\varphi_1(\cdot), \dots \varphi_n(\cdot)\} \subset S_{A(\cdot)}$ bază

Obs. $\varphi \in S_{A(\cdot)} \Leftrightarrow \exists c_i \in \mathbb{R}, i = \overline{1,n}$ a.î. $\varphi(t) = \sum_{i=1}^{n} c_i\varphi_i(t)$ soluția generală a ec.

PROP (sol. liniar independentă)

Fie $A(\cdot): I \subseteq \mathbb{R} \to L(\mathbb{R}^n, \mathbb{R}^n)$ cont. $\frac{dx}{dt} = A(t)x$

U.A.S.E.:

a) $\{\varphi_1(\cdot), \dots, \varphi_m(\cdot)\} \subset S_{A(\cdot)}$ sunt liniar independente

b) $\exists$ ~~I $\ni t_0$ a.î.~~ $t_0 \in I$ a.î. $\{\varphi_1(t_0), \dots, \varphi_m(t_0)\} \subseteq \mathbb{R}^n$ sunt liniar indep.

c) $\{\varphi_1(t), \dots, \varphi_m(t)\} \subset \mathbb{R}^n$ sunt liniar indep. $\forall t \in I$

Dem: a) $\Rightarrow$ c)

Fie $t \in I$ $\quad c_1\varphi_1(t) + \dots + c_m\varphi_m(t) = 0$

$c_1 E_t(\varphi_1) + \dots + c_m E_t(\varphi_m) = 0$

$E_t(c_1\varphi_1 + \dots + c_m\varphi_m) = 0$

$\Rightarrow c_1\varphi_1 + \dots + c_m\varphi_m = 0 \overset{a)}{\Rightarrow} c_1 = \dots = c_m = 0$

c) $\Rightarrow$ b) evident

b) $\Rightarrow$ a) $\quad c_1\varphi_1 + \dots + c_m\varphi_m = 0$

$E_{t_0}(c_1\varphi_1 + \dots + c_m\varphi_m) = 0$

$c_1\varphi_1(t_0) + \dots + c_m\varphi_m(t_0) = 0 \overset{b)}{\Rightarrow} c_1 = \dots = c_m = 0$

Obs: a) $\{\varphi_1(\cdot) \dots \varphi_m(\cdot)\} \subset C^1(I, \mathbb{R}^n)$ sunt liniar independente

b) $\exists t_0 \in I$ a.î. $\{\varphi_1(t_0), \dots, \varphi_m(t_0)\} \subset \mathbb{R}^n$ liniar indep.

c) $\{\varphi_1(t), \dots, \varphi_m(t)\} \subset \mathbb{R}^n$ liniar indep $\forall t \in I$

Matrici de soluții. Soluții matriceale. WRONSKIAN

Def: a) $\varphi_1(\cdot), \dots, \varphi_m(\cdot) \in S_{A(\cdot)}$ $\quad X(t) = col(\varphi_1(t), \dots, \varphi_m(t))$ s.n. matrice de soluții

b) $X(\cdot): I \to M_{n,m}(\mathbb{R})$ s.n. sol. matriceală dacă $\exists\, B \subset \mathbb{R}^n$ bază a.î.

$X'(t) = A_B(t)X(t)$

PROP: $X(\cdot): I \to M_{m,n}(\mathbb{R})$ $\quad X(\cdot)$ este matrice de soluții $\Leftrightarrow X(\cdot)$ este soluție matriceală

Dem: „$\Rightarrow$" $X(t) = col(\varphi_1(t), \dots, \varphi_m(t))$ $\quad \varphi_i(\cdot) \in S_{A(\cdot)}$ $\quad i = \overline{1,m}$

$X'(t) = col(\varphi_1'(t), \dots, \varphi_m'(t)) = col(A(t)\varphi_1(t), \dots, A(t)\varphi_m(t)) =$

$= A_B(t)\, col(\varphi_1(t), \dots, \varphi_m(t)) = A_B(t)X(t)$

„$\Leftarrow$" $X'(t) = A_B(t)X(t)$. Fie $X(t) = col(\varphi_1(t), \dots, \varphi_m(t)) \Rightarrow$

$\Rightarrow col(\varphi_1'(t), \dots, \varphi_m'(t)) = A_B(t)\, col(\varphi_1(t), \dots, \varphi_m(t)) =$

$= col(A(t)\varphi_1(t), \dots, A(t)\varphi_m(t)) \Rightarrow \varphi_i'(t) = A(t)\varphi_i(t)$ $\quad i = \overline{1,m}$ OK

Def: S.n. WRONSKIANUL soluțiilor $\varphi_1(\cdot), \dots, \varphi_m(\cdot) \in S_{A(\cdot)}$ funcția

$W_{\varphi_1 \dots \varphi_m}(t) := \det[col(\varphi_1(t), \dots, \varphi_m(t))]$, $t \in I$

TO:

FROM:

Th. lui Liouville:

Fie $A: I \in \mathbb{R} \to L(\mathbb{R}^n, \mathbb{R}^n)$ cont. $\frac{dx}{dt} = A(t)x$

Fie $\varphi_1(\cdot), \dots, \varphi_n(\cdot) \in S_{A(\cdot)}$. Atunci $W_{\varphi_1 \dots \varphi_n}(t) = W_{\varphi_1 \dots \varphi_n}(t_0) \cdot e^{\int_{t_0}^{t} Tr(A(s))ds}, \forall t, t_0 \in I$

Dem:

Obs. $T \ominus t \to W_{\varphi_1 \dots \varphi_n}(t)$ este sol. ec. liniare scalare $\frac{dy}{dt} = Tr(A(t))y$

$y(t) = c \cdot e^{\int_{t_0}^{t} Tr(A(s))ds} = y(t_0) e^{\int_{t_0}^{t} Tr(A(s))ds}$

$t = t_0 \quad y(t_0) = c$

$B \subset \mathbb{R}^n$ bază $A_B(t) = (a_{ij}(t))_{\substack{i=\overline{1,n} \\ j=\overline{1,n}}}$ $\quad \varphi_i(\cdot) = (\varphi_i^j(\cdot))_{j=\overline{1,n}} \quad i = \overline{1,n}$

$W_{\varphi_1 \dots \varphi_n}(t) = \det[\text{col}(\varphi_1(t), \dots, \varphi_n(t))] = \det(\varphi_i^j(t))_{i=\overline{1,n}} = \sum_{\sigma \in S} \text{sgn}(\sigma) \varphi^1_{\sigma(1)}(t) \dots$

$\dots \varphi^j_{\sigma(j)}(t) \dots \varphi^n_{\sigma(n)}(t) =$

$= \sum_{j=1}^{n} \sum_{\sigma \in S_n} \text{sgn}\,\sigma\, \varphi^1_{\sigma(1)} \dots \varphi^{j-1}_{\sigma(j-1)} \underbrace{(\varphi^j_{\sigma(j)}(t))'}_{\varphi_i \text{ soluție}} \varphi^{j+1}_{\sigma(j+1)}(t) \dots \varphi^n_{\sigma(n)}(t)$

$= \sum_{k=1}^{n} a_{jk}(t) \varphi^k_{\sigma(j)}(t)$

$= \sum_{j,k=1}^{n} a_{jk}(t) \underbrace{\sum_{\sigma \in S_n} \text{sgn}\,\sigma\, \varphi^1_{\sigma(1)}(t) \dots \varphi^{j-1}_{\sigma(j-1)}(t) \varphi^k_{\sigma(j)}(t) \varphi^{j+1}_{\sigma(j-1)}(t) \dots \varphi^n_{\sigma(n)}(t)}_{= \Delta_{jk}(t)} =$

$= \sum_{j,k}^{n} a_{jk}(t) \Delta_{jk}(t) =$

$\Delta_{jk}(t) = \begin{cases} W_{\varphi_1 \dots \varphi_n}(t) & j = k \\ 0 & j \neq k \end{cases}$

$= \sum_{j=1}^{n} a_{jj}(t) W_{\varphi_1 \dots \varphi_n}(t) = Tr(A(t)) \cdot W_{\varphi_1 \dots \varphi_n}(t)$

PROP (curentul global al ec. liniare)

~~Fie A : I → R~~ Fie $A: I \subset \mathbb{R} \to L(\mathbb{R}^n, \mathbb{R}^n)$ cont. $\frac{dx}{dt} = A(t)x$

Fie $\alpha_{A(\cdot)}(\cdot, \cdot, \cdot): I \times I \times \mathbb{R}^n \to \mathbb{R}^n$ curentul global al ec.

Atunci $\alpha_{A(\cdot)}(t, \tau, \cdot) \in L(\mathbb{R}^n, \mathbb{R}^n) \quad \forall t, \tau \in I$

Dem: ? $c_1, c_2 \in \mathbb{R}$, $\xi_1, \xi_2 \in \mathbb{R}^n$ $\cancel{\alpha_{A(\cdot)}(t, \tau, c_1 \xi_1 (c_2 \xi_2)) =}$

$\alpha_{A(\cdot)} = (t, \tau, c_1\xi_1 + c_2\xi_2) = c_1 \alpha_{A(\cdot)}(t, \tau, \xi_1) + c_2 \alpha_A(t, \tau, \xi_2)$, $\tau \in I$

$t \to \alpha_{A(\cdot)}(t, \tau, c_1\xi_1 + c_2\xi_2)$ sol. gen. a pb. Cauchy $(f_{A(\cdot)}, \tau, c_1\xi_1 + c_2\xi_2)$ (1)

$(f_{A(\cdot)}(t, x) = A(t)x)$

$t \to \alpha_A(t, \tau, \xi_1)$ soluție globală a pb. Cauchy $(f_{A(\cdot)}, \tau, \xi_1)$ $\Big| \xrightarrow[\text{sp. vect}]{S_{A(\cdot)}}$

$t \to \alpha_{A(\cdot)}(t, \tau, \xi_2)$ $\quad$ " $\quad (f_{A(\cdot)}, \tau, \xi_2)$

$\Rightarrow t \to c_1 A(\cdot)(t, \tau, \xi_1) + c_2 A(\cdot)(t, \tau, \xi_2)$ e sol. a pb. Cauchy $(f_{A(\cdot)}, \tau, c_1\xi_1 + c_2\xi_2)$ (2)

Din (1), (2), UG $\Rightarrow$ q.e.d.

Def: S.n. rezolvanta ec. liniare $x' = A(t)x$ funcția $R_{A(\cdot)}(\cdot, \cdot): I \times I \to L(\mathbb{R}^n, \mathbb{R}^n)$

$R_{A(\cdot)}(t, \tau)\xi := \varphi_{A(\cdot)}(t, \tau, \xi) = \varphi_{\tau, \xi}(t)$

„Rezolvă ecuația" $\varphi_{\tau,\xi}(t) = R_{A(\cdot)}(t,\tau)\xi$

Def: S.n. matrice fundamentală de soluții a ec. $x' = A(t)x$ $\quad X(t) = \mathrm{col}(\varphi_1(t), \dots, \varphi_n(t))$ unde $\{\varphi_1(\cdot), \dots, \varphi_n(\cdot)\} \subset S_{A(\cdot)}$ sistem fundamental de soluții.

$\varphi(\cdot) \in S_{A(\cdot)} \Leftrightarrow \exists c \in \mathbb{R}$ a.î. $\varphi(t) = X(t)c$

$\left[c \in \mathbb{R}^n,\ c = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}\ X(\cdot) \text{ f sol. } X(t) = \mathrm{col}(\varphi_1(t), \dots \varphi_n(t))\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \sum_{i=1}^{n} c_i \varphi_i(t)\right]$

<u>Th (Proprietățile rezolvantei)</u>:

Fie $A: I \subset \mathbb{R} \to L(\mathbb{R}^n, \mathbb{R}^n)$ cont. $\frac{dx}{dt} = A(t)x$

Fie $R_{A(\cdot)}(\cdot,\cdot): I \times I \to L(\mathbb{R}^n, \mathbb{R}^n)$ rezolvanta ec.

Atunci:
1) $R_{A(\cdot)}(t,t) = I_t \quad, \forall t$
2) $R_{A(\cdot)}(t,\tau) R_{A(\cdot)}(\tau,s) = R_{A(\cdot)}(t,s) \quad \forall t,\tau,s \in I$
3) $\exists (R_{A}(t,\tau))^{-1} = R_{A(\cdot)}(\tau,t)\ , \forall t,\tau \in I$
4) $\forall B \subset \mathbb{R}^n$ bază, $\forall X(\cdot): I \to M_n(\mathbb{R})$ matrice fundamentală de soluții
$R^B_{A(\cdot)}(t,\tau) = X(t)X^{-1}(\tau) \quad \forall t,\tau \in I$
5) $\forall B = \{b_1, \dots, b_n\} \subset \mathbb{R}^n$ bază, $\forall \tau \in I \cdot t \to R^B_{A(\cdot)}(t,\tau) =$
$= \mathrm{col}(\varphi_{\tau b_1}(t) \dots, \varphi_{\tau b_2}(t))$ matrice fundamentală de soluții
6) $\det(R_{A(\cdot)}(t,\tau)) = e^{\int_{t_0}^{t} Tr(A(\cdot))\,ds}\ , \forall t,\tau \in I$

Dem: 6) $\det(R_{A(\cdot)}(t,\tau)) = \det(R^B_{A(\cdot)}(t,\tau)) \stackrel{5)}{=} \det(\mathrm{col}(\varphi_{\tau,b_1}(t) \dots \varphi_{\tau,b_n}(t))) =$

$= W_{\varphi_{\tau b_1}, \dots \varphi_{\tau b_n}}(t) \underset{\text{Liouville}}{\overset{T}{=}} W_{\varphi_{\tau b_1}, \dots \varphi_{\tau b_2}}(\tau) \cdot e^{\int_{\tau}^{t} Tr(A(s))\,ds} \stackrel{=}{} e^{\int_{\tau}^{t} Tr(A(\cdot))\,ds} =$

$= \det(\mathrm{col}(\varphi_{\tau,b_1}(\tau) \dots \varphi_{\tau,b_n}(\tau)))\, e$

$= \underbrace{\det(\mathrm{col}(b_1, \dots, b_n))}_{=1 \text{ baza canonică}} e^{\int_{\tau}^{t} Tr(A(s))\,ds}$