TO:

FROM: Ecuații diferențiale - 31.10.2017 - Seminar

$x^{(n)} = f(t, x, x', \dots, x^{(n-1)})$ $f(\cdot,\cdot) : D \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$

T. Peano (E.L.)

$D = \mathring{D}$, $f(\cdot,\cdot)$ cont $\Rightarrow$ E.L. pe $D$ $(\forall\, t_0, (x_0, x_0', \dots, x_0^{n-1})) \in D$ $\exists\, \varphi(\cdot) : I_0 \in V(t_0) \to \mathbb{R}$ sol. cu $\varphi(t_0) = x_0, \varphi'(t_0) = x_0', \dots, \varphi^{(n-1)}(t_0) = x_0^{n-1})$

T. Cauchy-Lipschitz (E.U.L.)

$D = \mathring{D}$, $f(\cdot,\cdot)$ cont. local-Lipschitz (II) $\Rightarrow$ E.U.L. pe $D$
$(\forall\, (t_0, (x_0, x_0', \dots, x_0^{n-1})) \in D$ $\exists!\, \varphi(\cdot) : I_0 \in V(t_0) \to \mathbb{R}$ sol. cu $\varphi(t_0) = x_0$, $\varphi'(t_0) = x_0', \dots$
$\dots \varphi^{(n-1)}(t_0) = x_0^{n-1})$

1) $\forall\, n \in \mathbb{N}$ să se determine $K_n$ = nr. sol. posibile ale pb. $x^{(n)} = t + x^3$, $x(0) = 1$, $x'(0) = 0$

$n = 0 \Rightarrow x = t + x^3$, $x(0) = 1$, $x'(0) = 0$

$\Leftrightarrow x(t) = t + x^3(t)$, $\forall\, t$, $x(0) = 1$, $x'(0) = 1$

$t = 0 \Rightarrow \underset{1}{x(0)} = \underset{1}{x^3(0)}$

$x'(t) = 1 + 3x^2(t) \cdot x'(t)$

pt. $t = 0$ $x'(0) = 1 + 3x^2(0) \cdot x'(0)$

$0 = 1 + 3 \cdot 1 \cdot 0$

$0 = 1$ $\not\Delta$ $K_0 = 0$

$n = 1 \Rightarrow x' = t + x^3$

$x(0) = 1, x'(0) = 0$

$t = 0 \Rightarrow x'(0) = x^3(0)$

$0 = 1$ $\not\Delta$ $K_1 = 0$

$n = 2 \Rightarrow x'' = t + x^3$

$x(0) = 1$

$x'(0) = 0$

$f(t, (x_1, x_2)) = t + x_1^3$

$f : \underbrace{\mathbb{R} \times \mathbb{R}^2}_{\text{deschisă}} \to \mathbb{R}$

cont., local Lipschitz (II) pt. că este $C^1$ (II)

$\overset{T.C.L.}{\Longrightarrow}$ $\exists!\, \varphi(\cdot) : I_0 \in V(0) \to \mathbb{R}$ sol. a.î. $\varphi(0) = 1$, $\varphi'(0) = 0$ $\Rightarrow K_2 = 1$

$n = 3$ $x''' = t + x^3$, $x(0) = 1$, $x'(0) = 0$

$f(t, (x_1, x_2, x_3)) = t + x_1^3$

$f : \underbrace{\mathbb{R} \times \mathbb{R}^3}_{\text{deschisă}} \to \mathbb{R}$

cont; Local Lipschitz (II)

+ cls $C_1(\text{II})$

T.C-L $\Longrightarrow$ $\forall a \in \mathbb{R}$ $\exists!$ $\varphi_a(\cdot): I_a \in V(0) \to \mathbb{R}$ sol. a.î. $\varphi_a(0)=1$, $\varphi_a'(0)=0$, $\varphi_a''(0)=a$

$\Rightarrow a \in \mathbb{R}$ arbitrar $k_3 = \infty$

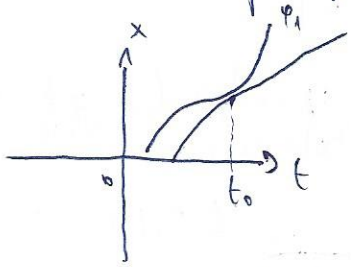
$n \geq 4$, $k_n = \infty$

2) Să se studieze posibilitatea ca graficele a două fct. distincte să fie tangente pt. fiecare din ec:

a) $x' = t^2 + x^4$

b) $x'' = t^2 + x^4$

c) $x''' = t^2 + x^4$




$\varphi_1: I_1 \to \mathbb{R}_1$, $\varphi_2: I_2 \to \mathbb{R}_2$ $\exists\, t_0 \in I_1 \cap I_2$ a.î. $\varphi_1(t_0) = \varphi_2(t_0) = x_0$

$\varphi_1'(t_0) = \varphi_2'(t_0) = x_0^1$

a) $x' = t^2 + x^4$

$f(t,x) = t^2 + x^4$

$f: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$

f cont

f local Lipschitz (II) $\overset{C.L.}{\Rightarrow}$ $\forall\, t_0, x_0 \in \mathbb{R} \times \mathbb{R}$ $\exists!$ $\varphi: I_0 \to \mathbb{R}$ sol. $\varphi(t_0) = x_0$ Nu

b) $x'' = t^2 + x^4$

$f(t,(x_1,x_2)) = t^2 + x_1^4$

$f: \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$

f cont. și l. Lipschitz (II) $\overset{C.L.}{\Rightarrow}$ $\forall (t_0,(x_0, x_0')) \in \mathbb{R} \times \mathbb{R}^2$ $\exists!$ $\varphi: I_0 \in V(t_0) \to \mathbb{R}$ sol. a.î.

$\varphi(t_0) = x_0$, $\varphi'(t_0) = x_0^1$ Nu

c) $x''' = t^2 + x^4$

$f(t,(x_1,x_2,x_3)) = t^2 + x_1^4$

$f: \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$

f cont. și l. Lipschitz (II) $\overset{C.L.}{\Longrightarrow}$ $\forall (t_0,(x_0, x_0', x_0'')) \in \mathbb{R} \times \mathbb{R}^3$ $\exists!$ $\varphi: I_0 \in V(t_0) \to \mathbb{R}$

sol. a.î. $\varphi(t_0) = x_0$, $\varphi'(t_0) = x_0^1$, $\varphi''(t_0) = x_0^2$

Dacă $\varphi_1''(t_0) \neq \varphi_2''(t_0)$ at. da

3) Să se determine $n \in \mathbb{N}$ pt. care $\exists$ o funcție $f(\cdot,\cdot): \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ continuă și local Lipschitz (II) a.î.

$\varphi_1(t) = t$

$\varphi_2(t) = t + t^3$

sunt sol. ale ec. $x^{(n)} = f(t,x)$

TO:

FROM:

$n=0 \Rightarrow x = f(t,x)$

$\varphi_1(t) = \varphi(t, \varphi_1(t))$ $\forall t, i_1, i_2$

$\Rightarrow t = f(t,t)$

$t + t^3 = f(t, t+t^3)$

$f(t,x) = x$ ok

$n=1 \Rightarrow x' = f(t,x)$

Pp. $\exists f : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ cont., local Lipschitz (II) $\xRightarrow{C.L.}$ E.U.L. pe $\mathbb{R}\times\mathbb{R}$

$\Rightarrow \forall t_0, x_0 \in \mathbb{R}\times\mathbb{R}$ $\exists!$ $\varphi : I_0 \in V(t_0)$ sol. cu $\varphi(t_0) = x_0$

? $\exists t_0$ a.î. $\varphi_1(t_0) = \varphi_2(t_0) = x_0$

$\Rightarrow t_0 = t_0 + t_0^3$

$\Rightarrow t_0 = 0 \Rightarrow x_0 = 0 \Rightarrow$ ∡ C.L. în $(0,0)$

$n=2 \Rightarrow x'' = f(t,x)$

Pp. $\exists f$ ca în ~~ipote~~ enunț ~~tt~~

Fie $g(t(x_1,x_2)) = f(t,x_1)$, $g : \mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}$ cont. și local Lipschitz (II) $\Rightarrow$ E.U.L. pe $\mathbb{R}\times\mathbb{R}^2$

$\Rightarrow \forall$ $(t_0,(x,x_0)) \in \mathbb{R}\times\mathbb{R}^2$ sol. $I_0 \in V(t_0) \in \mathbb{R}$ a.î. $\varphi(t_0) = x_0$

$\varphi'(t_0) = x_0'$

? $t_0 \in \mathbb{R}$ a.î. $\varphi_1(t_0) = \varphi_2(t_0) = x_0 \Rightarrow t_0 = 0 \Rightarrow x_0 = 0$

$\varphi_1(t_0) = \varphi_1'(t_0) = x_0'$

$1 = 1 + 3\cdot 0 \Rightarrow 1 = 1$ ok $\Rightarrow x_0' = 1 \xRightarrow{C.L.}$ în $(0,(0,1)) \Rightarrow$ ∡

$\varphi_1'(t) = 1$

$\varphi_1'(t) = 1 + 3t^2$

$n=3 \Rightarrow x''' = f(t,x)$

Pp. $\exists f$ ca în enunț

Fie $g(t,(x_1,x_2,x_3)) = f(t,x_1)$

$\Rightarrow g : \mathbb{R}\times\mathbb{R}^3 \to \mathbb{R}$ cont., local Lipschitz (II) $\xRightarrow{C.L.}$ E.U.L. pe $\mathbb{R}\times\mathbb{R}^3$

$\Rightarrow \forall (t_0(x_0, x_0', x_0^2)) \in \mathbb{R}\times\mathbb{R}^3$ $\exists!$ sol. $\varphi : I_0 \in V(t_0) \to \mathbb{R}$ a.î. $\varphi(t_0) = x_0$

$\varphi'(t_0) = x_0'$

$\varphi''(t_0) = x_0^2$

? $t_0$ a.î. $\varphi_1(t_0) = \varphi_2(t_0) = x_0$

$\varphi_1'(t_0) = \varphi_2'(t_0) = x_0'$

$\varphi_1''(t_0) = \varphi_2''(t_0) = x_0^2$

$\varphi_1''(t) = 0$

$\varphi_2''(t) = 6t$

$t_0 = 0$

$x_0 = 0$

$x_0'' = 1$

$0 = 6\cdot 0$ ok $\Rightarrow x_0^2 = 0 \Rightarrow$ ∡ C.L. în ~~$(0,(0,1,0))$~~ $(0,0,1,0)$

$n=4 \Rightarrow x^{IV} = f(t,x)$

Pp. $\exists f$ ca în enunț

Fie $g(t,(x_1,x_2,x_3,x_4)) = f(t,x_1)$

$g$ cont., local Lipschitz (II) $\xrightarrow{CL}$ EUL pe $\mathbb{R}\times\mathbb{R}^4$

$\Rightarrow \forall (t_0,(x_0,x_0^1,x_0^1,x_0^3)) \in \mathbb{R}\times\mathbb{R}^4 \ \exists!\ \varphi: I_0 \to \mathbb{R}$ sol. ~~data~~ a.î.

$\varphi(t_0) = x_0$ ~~RA~~. $\exists! t_0$ $\varphi_1(t_0) = \varphi_2(t_0) = x_0$ $t_0 = 0$

$\varphi'(t_0) = x_0'$ $\varphi_1'(t_0) = \varphi_2'(t_0) = x_0'$ $x_0 = 0$

$\varphi''(t_0) = x_0^2$ $\varphi_1''(t_0) = \varphi_2''(t_0) = x_0^2$ $x_0' = 1$

$\varphi'''(t_0) = x_0^3$ $\varphi_1'''(t_0) = \varphi_2''(t_0) = x_0^3$ $x_0'' = 0$

? $\exists$!

$\varphi_i$ sol. a ec. $x'' = f(t,x)$, $i = \overline{1,2}$

$\varphi_i^{IV}(t) = f(t,\varphi(t))$ $\forall i = \overline{1,2}$, $t \in \mathbb{R}$

$0 = f(t,t)$

$0 = f(t,t+t^3)$

Obs. $f(t,x) := 0$ verif. condițiile

pt. $n \geq 4$, $f(t,x) := 0$

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alt

$n=3$ $\varphi_1(\cdot)$ sol. ~~a ec.~~ $x''' = f(t,x)$

$0 = f(t,t)$ $\forall t$ $t=0$ $0 = f(0,0)$

$6 = f(t,t,t^3)$ $\forall t$ $6 = f(0,0)$ | $\not\exists$

$n=2$

$\varphi(\cdot)$ sol $x'' = f(t,x)$

$0 = f(t,t)$ $\forall t$

$6t = f(t,t+t^3)$

$f(\cdot,\cdot)$ local Lipschitz $\Rightarrow \exists\ L > 0$ a.î. $\|f(t,x_1) - f(t,x_2)\| \leq L\|x_1 - x_2\|$

$\forall (t,x_1),(t,x_2) \in D_0 \in V(t_0 x_0)$

$(t_0,x_0) \in \mathbb{R}\times\mathbb{R}$

$x_1 \to t$

$x_2 \to t+t^3$

$|0 - 6t| \leq L|t - t - t^3|$

$6|t| \leq L|t^3|$ $| : |t|$, $t \neq 0$

$t_0 = 0, x_0 = 0$ $D_0 \in V(0,0)$

$\frac{6}{L} \leq |t|^2$ $\forall t \in V(0)$ $t \neq 0$

$\downarrow$
$0 \Rightarrow \not\exists$