TO:

FROM: Ecuații diferențiale - 28.11.2017 - Seminar

Algoritm (cazul general) $\frac{dx}{dt} = Ax$

1. Rezolvă ec. caracteristică $\det(A - \lambda I_n) = 0 \to \sigma(A) : (\lambda, m_\lambda)$
2. Dacă $\lambda \in \sigma(A) \cap \mathbb{R}$, $m_\lambda = 1$ caută $u_\lambda \in \mathbb{R}^n \setminus \{0\}$ a.î. $(A - \lambda I_n) u_\lambda = 0$
   Scrie sol. $\varphi_\lambda(t) = e^{\lambda t} \cdot u_\lambda$
3. Dacă $\lambda \in \sigma(A) \cap \mathbb{R}$, $m_\lambda = m > 1$
   ~~Scrie~~ Caută $\{P_0^{\lambda 1}, \dots, P_0^{\lambda m}\} \subset \operatorname{Ker}(A - \lambda I_n)^m$ liniar indep. (în $\mathbb{R}^n$)
   Scrie $P_j^{\lambda l} = \frac{1}{j!}(A - \lambda I_n)^j P_0^{\lambda l}$ $j = \overline{1, m-1}$, $l = \overline{1, m}$
   Scrie sol. $\varphi_{\lambda l}(t) = e^{\lambda t} \cdot \sum_{j=0}^{m-1} P_j^{\lambda l} t^j$ $l = \overline{1, m}$
4. Dacă $\lambda = \alpha + i\beta \in \sigma(A)$, $\beta > 0$, $m_\lambda = 1$,
   Caută $u_\lambda \in \mathbb{C}^m \setminus \{0\}$ a.î. $(A - \lambda I_n) u_\lambda = 0$
5. Dacă $\lambda = \alpha + i\beta \in \sigma(A)$, $\beta > 0$, $m_\lambda = m > 1$
   Caută $\{P_0^{\lambda 1}, \dots, P_0^{\lambda m}\} \subset \operatorname{Ker}(A - \lambda I_n)^m$ liniar indep (în $\mathbb{R}^n$)
   Scrie $P_j^{\lambda l} = \frac{1}{j!}(A - \lambda I_n)^j P_0^{\lambda l}$ $j = \overline{1, m-1}$, $l = \overline{1, m}$
   Scrie sol. $\varphi_{\lambda l}(t) = \operatorname{Re}\left(e^{\lambda t} \cdot \sum_{j=0}^{m-1} P_j^{\lambda l} t^j\right)$
   $\varphi_{\bar\lambda l}(t) = \operatorname{Im}\left(e^{\lambda t} \cdot \sum_{j=0}^{m-1} P_j^{\lambda l} t^j\right)$ $l = \overline{1, m}$
6. Renumerotează $\{\varphi_{\lambda l}(\cdot)\}_{\lambda \in \sigma(A), l = \overline{1,m}} = \{\varphi_1(\cdot), \dots, \varphi_n(\cdot)\}$ sist. fund. de sol.

Scrie sol. generală $\varphi(t) = \sum_{i=1}^{n} c_i \varphi_i(t)$ $c_i \in \mathbb{R}$

Să se det. sol. generală:

1) $\begin{cases} x' = x - y + z \\ y' = x + y - z \\ z' = 2z - y \end{cases}$ $A = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 0 & -1 & 2 \end{pmatrix}$

$\det(A - \lambda I_n) = \begin{vmatrix} 1-\lambda & -1 & 1 \\ 1 & 1-\lambda & -1 \\ 0 & -1 & 2-\lambda \end{vmatrix} = (1-\lambda)^2(2-\lambda) - 1 - (1-\lambda) + 2 - \lambda$

$= (1-\lambda)^2(2-\lambda)$

$\lambda_1 = 2$

$\lambda_2 = \lambda_3 = 1$

$\lambda = 2 \Rightarrow u = ?$ a.î. $(A - 2I_3)u = 0$, $u = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$$\begin{pmatrix} -1 & -1 & 1 \\ 1 & -1 & -1 \\ 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \Rightarrow \begin{cases} -a - b + c = 0 \\ a - b - c = 0 \\ -b = 0 \end{cases} \Rightarrow a = c \Rightarrow \begin{pmatrix} a \\ 0 \\ a \end{pmatrix}$$

$a = 1$

Soluția $\varphi_1(t) = e^{2t}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$\lambda = 1$, $m_\lambda = 2$

$$(A - I_3)^m = (A - I_3)^2 = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -1 & 2 \\ 0 & 0 & 0 \\ -1 & -1 & 2 \end{pmatrix}$$

$$\Rightarrow P_0 = ? \text{ a.î. } (A - I_3)^2 P_0 = 0,\ P_0 = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \Rightarrow \begin{pmatrix} -1 & -1 & 2 \\ 0 & 0 & 0 \\ -1 & -1 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow a + b = 2c \Rightarrow c = \frac{a+b}{2} \Rightarrow P_0 = \begin{pmatrix} a \\ b \\ \frac{a+b}{2} \end{pmatrix}$$

$$P_{01} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \quad P_{02} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$$

$$P_j = \frac{1}{j!}(A - I_3)^j P_0,\ j = 1 \Rightarrow P_1 = (A - I_3)P_{01},\ P_{12} = (A - I_3)P_{02}$$

$$P_1 = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$P_{12} = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$$

$$\varphi_2(t) = e^t(P_{01} + tP_{11}) = e^t\begin{pmatrix} 2+t \\ t \\ 1+t \end{pmatrix}$$

$$\varphi_3(t) = e^t(P_{02} + tP_{12}) = e^t\begin{pmatrix} -t \\ 2-t \\ 1-t \end{pmatrix}$$

+ soluția generală

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2) $\begin{cases} x' = 4x - y \\ y' = 3x + y - z \\ z' = x + z \end{cases}$ $\quad A = \begin{pmatrix} 4 & -1 & 0 \\ 3 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix}$

$$\det(A - \lambda I_3) = \begin{vmatrix} 4-\lambda & -1 & 0 \\ 3 & 1-\lambda & -1 \\ 1 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^2(4-\lambda) + 1 + 3(1-\lambda)$$

$$= (1-\lambda)^2(4-\lambda) + 4 - 3\lambda =$$

$$= 4 - \lambda - 8\lambda + 2\lambda^2 + 4\lambda^2 - \lambda^3 + 4 - 3\lambda = -\lambda^3 + 6\lambda^2 - 12\lambda + 8 = -(\lambda - 2)^3$$

$\lambda = 2$, $m_\lambda = 3$

$(A - 2I_3)^3 = ?$

$$\cancel{(A - 2I_3)^2 = \begin{pmatrix} 4 & -1 & 0 \\ 3 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 4 & -1 & 0 \\ 3 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 13 & -5 & 1 \\ 14 & -2 & -2 \\ 5 & -1 & 1 \end{pmatrix}}$$

TO:

FROM: $= \begin{pmatrix} 1 & -1 & 1 \\ 2 & -2 & 2 \\ 1 & -1 & 1 \end{pmatrix}$

$$(A-2I_3)^3 = \begin{pmatrix} 1 & -1 & 1 \\ 2 & -2 & 2 \\ 1 & -1 & 1 \end{pmatrix}\begin{pmatrix} 2 & -1 & 0 \\ 3 & -1 & -1 \\ 1 & 0 & -1 \end{pmatrix} = O_3$$

$(A-2I_3)^3 = 0 \Rightarrow (A-2I_3)^3 P_0 = 0 \quad \forall P_0 \in \mathbb{R}^3$

$P_{01} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad P_{02} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad P_{03} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ liniar independenți

$P_j = \frac{1}{j!}(A-2I_3)^j P_0 \qquad , j = \overline{1,2}$

$$P_{11} = (A-2I_3)P_{01} = \begin{pmatrix} 2 & -1 & 0 \\ 3 & -1 & -1 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$$

$P_{12} = (A-2I_3)\cdot P_{02} = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}$

$P_{13} = (A-2I_3)P_{03} = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}$

$$P_{21} = \frac{1}{2}(A-2I_3)^2 P_{01} = \frac{1}{2}\begin{pmatrix} 1 & -1 & 1 \\ 2 & -2 & 2 \\ 1 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

$P_{22} = \frac{1}{2}(A-2I_3)^2 P_{02} = \frac{1}{2}\begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}$

$P_{23} = \frac{1}{2}(A-2I_3)^2 P_{03} = \frac{1}{2}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

$\varphi_1(t) = e^{2t}(P_{01} + tP_{11} + t^2 P_{21}) \quad = e^{2t}\begin{pmatrix} 1+2t+\frac{t^2}{2} \\ 3t + t^2 \\ t + \frac{t^2}{2} \end{pmatrix}$

$\varphi_2(t) = e^{2t}(P_{02} + tP_{12} + t^2 P_{22})$

$\varphi_3(t) = e^{2t}(P_{03} + tP_{13} + t^2 P_{23})$

$\varphi_2(t) = e^{2t}\begin{pmatrix} -t - \frac{t^2}{2} \\ 1-t-t^2 \\ -\frac{t^2}{2} \end{pmatrix}$

$\varphi_2(t) = e^{2t}\begin{pmatrix} \frac{t^2}{2} \\ -t+t^2 \\ 1-t+\frac{t^2}{2} \end{pmatrix}$

3) $\begin{cases} x' = 2x + y \\ y' = 2y + 4z \\ z' = x - z \end{cases} \qquad A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 4 \\ 1 & 0 & -1 \end{pmatrix}$

$$\det(A-\lambda I_3) = \begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 4 \\ 1 & 0 & -1-\lambda \end{vmatrix} = (2-\lambda)^2(-1-\lambda) + 4 - 0$$

$= (4-4\lambda+\lambda^2)(-1-\lambda) + 4 = -4 - \cancel{4\lambda} + \cancel{4\lambda} + 4\lambda^2 - \lambda^2 - \lambda^3 =$

$= -\lambda^3 + 3\lambda^2 \ldots$

$= \lambda^2(3-\lambda)$

$\lambda_1 = 3$

$\lambda_2 = 0,\ m_\lambda = 2$

$\lambda = 3$

$u \in \mathbb{R}^3$ a.î. $(A - 3I_3)u = 0,$ $u = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$$\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 4 \\ 1 & 0 & -4 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} -a + b = 0 \Rightarrow a = b \\ -b + 4c = 0 \Rightarrow b = 4c \Rightarrow b = a \\ a - 4c = 0 \Rightarrow a = 4c \end{cases}$$

$$\Rightarrow u = \begin{pmatrix} 4c \\ 4c \\ c \end{pmatrix}$$

$$c = 1 \Rightarrow u = \begin{pmatrix} 4 \\ 4 \\ 1 \end{pmatrix} \Rightarrow \varphi_1(t) = e^{3t}\begin{pmatrix} 4 \\ 4 \\ 1 \end{pmatrix}$$

$\lambda = 0,\ m_\lambda = 2$

$$A^2 = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 4 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 4 \\ 1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 4 & 4 & 4 \\ 4 & 4 & 4 \\ 1 & 1 & 1 \end{pmatrix}$$

$P_0 = ?$ a.î. $A^2 \cdot P_0 = 0,\ P_0 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$$\begin{pmatrix} 4 & 4 & 4 \\ 4 & 4 & 4 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \Rightarrow a + b + c = 0 \Rightarrow c = -a - b$$

$$P_0 = \begin{pmatrix} a \\ b \\ -a-b \end{pmatrix}$$

$$a = 1, b = 0 \Rightarrow P_{01} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$a = 0, b = 1 \Rightarrow P_{02} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$$P_j = \frac{1}{j!}(A^j \cdot P_0) \quad j = 1$$

$$P_{11} = A \cdot P_{01} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 4 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \\ 2 \end{pmatrix}$$

$$P_{12} = A \cdot P_{02} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 4 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$\varphi_2(t) = (P_{01} + t P_{11})$

$\varphi_3(t) = (P_{02} + t P_{12})$

$$\varphi_2(t) = \begin{pmatrix} 1 + 2t \\ -4t \\ -1 + 2t \end{pmatrix}$$

$$\varphi_3(t) = \begin{pmatrix} t \\ 1 - 2t \\ -1 + t \end{pmatrix}$$

TO:

FROM:

4) $\begin{cases} x' = 2x + y \\ y' = 4y - x \end{cases} \Rightarrow A = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix}$

$\det(A - \lambda I_3) = \begin{vmatrix} 2-\lambda & 1 \\ -1 & 4-\lambda \end{vmatrix} = (2-\lambda)(4-\lambda) + 1 = 8 - 2\lambda - 4\lambda + \lambda^2 = \lambda^2 - 6\lambda + 9$

$\cancel{\lambda_{1,2} = \frac{6 \pm \sqrt{36-32}}{2} \Rightarrow \lambda_1 = 4, \lambda_2 = 1} \Rightarrow \lambda_{1,2} = 3$

$\lambda = 3 \quad m_\lambda = 2$

$(A - 3I_2)^2 = ?$

$A - 3I_2 = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}$

$(A - 3I_2)^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow (A - 3I_2)^2 \cdot P_0 = 0 \quad \forall P_0 \in \mathbb{R}^2$ (at. se ia P_0 din baza canonică)

$P_{01} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, P_{02} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$P_j = \frac{1}{j!}(A - 3I_2)^j P_0 \quad , j = 1$

$P_{11} = (A - 3I_2)P_{01} = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$

$P_{12} = (A - 3I_2)P_{02} = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\varphi_1(t) = e^{3t}(P_{01} + tP_{11})$

$\varphi_2(t) = e^{3t}(P_{02} + tP_{12})$

$\varphi_1(t) = e^{3t}\begin{pmatrix} 1-t \\ -t \end{pmatrix}$

$\varphi_2(t) = e^{3t}\begin{pmatrix} t \\ 1+t \end{pmatrix}$

4') $\begin{cases} x' = 2x + y \\ y' = 4y - x \end{cases}$

$y = x' - 2x$

$(x' - 2x)' = 4x' - 8x - x \Rightarrow x'' - 2x' = 4x' - 7x$

$\Rightarrow x'' - 6x' + 9x = 0$ ec. liniară cu coef. const.

$\lambda^2 - 6\lambda + 9 = (\lambda - 3)^2 = 0 \Rightarrow \lambda_{1,2} = 3$

$x(t) = c_1 e^{3t} + c_2 \cdot t e^{3t}, \quad c_1, c_2 \in \mathbb{R}$

$y(t) = 3c_1 e^{3t} + c_2 e^{3t} + 3c_2 t \cdot e^{3t} - 2c_1 e^{3t} - 2c_2 t e^{3t} =$

$= (c_1 + c_2)e^{3t} + c_2 t e^{3t} + c_2 t e^{3t}$

5) $\begin{cases} x' = 5x + 3y \\ y' = -3x - y \end{cases}$

6) $\begin{cases} x' = 2y - 3x \\ y' = y - 2x \end{cases}$