1. (**Theoretical**) Hacer pasos intermedios para regla de trapecio simple, Ecuación (1.74).

Partimos de:

$$f(x) \approx p_1(x) = \frac{x-b}{a-b}f(a) + \frac{x-a}{b-a}f(b), \qquad \forall x \in [a,b]. \tag{1.73}$$

$$\int \frac{x-b}{a-b} f(a) + \frac{x-a}{b-a} f(b)$$

$$\int \frac{x-b}{a-b} f(a)\, dx + \int \frac{x-a}{b-a} f(b)\, dx$$

$$\frac{f(a)}{a-b}\int x-b\, dx + \frac{f(b)}{b-a}\int x-a\, dx$$

$$\frac{f(a)}{a-b}\left(\frac{x^2}{2}-bx\right) + \frac{f(b)}{b-a}\left(\frac{x^2}{2}-ax\right)\Bigg|_a^b$$

$$\frac{f(a)}{a-b}\left(\frac{b^2}{2}-b^2\right) + \frac{f(b)}{b-a}\left(\frac{b^2}{2}-ab\right) - \frac{f(a)}{a-b}\left(\frac{a^2}{2}-ba\right) + \frac{f(b)}{b-a}\left(\frac{a^2}{2}-a^2\right)$$

$$\frac{f(a)}{a-b}\left(-\frac{b^2}{2}\right) + \frac{f(b)}{b-a}\left(\frac{b^2}{2}-ab\right) - \frac{f(a)}{a-b}\left(\frac{a^2}{2}-ba\right) - \frac{f(b)}{b-a}\left(-\frac{a^2}{2}\right)$$

$$\frac{f(a)}{a-b}\left(-\frac{b^2}{2}-\frac{a^2}{2}+ba\right) + \frac{f(b)}{b-a}\left(\frac{b^2}{2}-ab+\frac{a^2}{2}\right)$$

$$\frac{f(a)}{a-b}\left(-\frac{1}{2}(b^2-2ab+a^2)\right) + \frac{f(b)}{b-a}\left(-\frac{1}{2}(b^2-2ab+a^2)\right) = \frac{f(b)}{b-a}\frac{(b-a)^2}{2} - \frac{f(a)}{b-a}\frac{(b-a)^2}{2}$$

$$\frac{f(b)}{b-a}\frac{(b-a)^2}{2} - \frac{f(a)}{a-b}\frac{(b-a)^2}{2} = f(b)\frac{(b-a)}{2} + f(a)\frac{(b-a)}{2}$$

Se llega a:

$$I = \int_a^b f(x)dx \simeq \int_a^b p_1(x)\,dx = \frac{b-a}{2}\left(f(a)+f(b)\right)$$

3. (**Theoretical**) Hacer los pasos intermedios para encontrar la regla de Simpson simple, Ecuación (1.87).

Partimos de:

$$f(x) \approx p_2(x) = \frac{(x-b)(x-x_m)}{(a-b)(a-x_m)}f(a)+\frac{(x-a)(x-b)}{(x_m-a)(x_m-b)}f(x_m)+\frac{(x-a)(x-x_m)}{(b-a)(b-x_m)}f(b), \quad \forall x \in [a,b]. \tag{1.86}$$

Teniendo en cuenta que:

$$x_m = \frac{a+b}{2} \qquad h = \frac{b-a}{2}$$

$$\underbrace{\int_a^b \frac{(x-b)(x-x_m)}{(a-b)(a-x_m)} f(a)\,dx}_{1} + \underbrace{\int_a^b \frac{(x-a)(x-b)}{(x_m-a)(x_m-b)} f(x_m)\,dx}_{2} + \underbrace{\int_a^b \frac{(x-a)(x-x_m)}{(b-a)(b-x_m)} f(b)\,dx}_{3}$$

$$\int x^2 - bx - x x_m + b x_m \, dx$$

1

$$\frac{f(a)}{(a-b)(a-x_m)}\left[\frac{x^3}{3} - \frac{x^2}{2}(b+x_m) + x b x_m\right]\Bigg|_a^b$$

$$\frac{b^3}{3} - \frac{b^2}{2}(b+x_m) + b^2 x_m - \frac{a^3}{3} + \frac{a^2}{2}(b+x_m) - a b x_m$$

$$\frac{b^3}{3} - \frac{b^3}{2} + \frac{b^2 x_m}{2} - \frac{a^3}{3} + \frac{a^2 b}{2} + \frac{a^2 x_m}{2} - a b x_m$$

$$-\frac{b^3}{6} + \frac{b^2}{2}x_m - \frac{a^3}{3} + \frac{a^2(b+x_m)}{2} - a b x_m$$

$$\cancel{-\frac{b^3}{6}} + \cancel{\frac{b^2 a}{4}} + \cancel{\frac{b^3}{4}} - \cancel{\frac{a^3}{3}} + \frac{3a^2 b + \cancel{a^3}}{4} - \frac{a^2 b}{2} - \cancel{\frac{ab^2}{2}}$$

$$-\frac{1}{12}\left[-b^3 + 3b^2 a - 3a^2 b + a^3\right] = -\frac{(a-b)^3}{12}$$

$$-\frac{f(a)}{(a-b)(a-x_m)}\frac{(a-b)^3}{12} = -\frac{2f(a)}{\cancel{(a-b)^2}}\cdot\frac{(a-b)^{\cancel{3}}}{12} = \frac{f(a)\,(b-a)}{6} = \boxed{\frac{h\,f(a)}{3}}$$

(2) $$\int x^2 - bx - ax + ab \;dx$$

$$\frac{f(x_m)}{(x_m-a)(x_m-b)}\left[\frac{x^3}{3} - \frac{x^2}{2}(b+a) + abx\right]\Bigg|_a^b$$

$$\frac{\cancel{b}^3}{3} - \frac{\cancel{b}^2}{2}(\cancel{b}+\cancel{a}) + \cancel{a}b^2 - \frac{a^3}{3} + \frac{a^2}{2}(b+a) - a^2b$$

$$\frac{1}{6}\left[-b^3+3b^2a-3a^2b+a^3\right] = \frac{1}{6}(a-b)^3$$

$$\frac{f(x_m)}{(x_m-a)(x_m-b)}\cdot\frac{(a-b)^3}{6} = \frac{4f(x_m)}{\cancel{(b-a)^2}}\cdot\frac{(b-a)^{\cancel{3}}}{6} = \boxed{\frac{4h\,f(x_m)}{3}}$$

(3) $$\int x^2 - x x_m - xa + a x_m \;dx$$

$$\frac{f(b)}{(b-a)(b-x_m)}\left[\frac{x^3}{3} - \frac{x^2}{2}(x_m+a) + a x_m x\right]\Bigg|_a^b$$

$$\frac{\cancel{b}^3}{3} - \frac{\cancel{b}^2}{2}(\cancel{x_m}+a) + a\cancel{x_m}b - \frac{a^3}{3} + \frac{a^2}{2}(x_m+a) - a^2x_m$$

$$\frac{b^3}{12} - \frac{3b^2 a}{\cancel{2}} + \frac{a^2b}{\cancel{2}} + \frac{b^2a}{\cancel{2}} - \frac{a^3}{12} + \frac{a^2b}{\cancel{2}} - \frac{a^2b}{\cancel{2}}$$

$$\frac{b^3}{12} - \frac{b^2a}{4} + \frac{a^2b}{4} - \frac{a^3}{12} = \frac{1}{12}\left[b^3 - 3b^2a + 3a^2b - a^3\right]$$

$$\frac{f(b)}{(b-a)(b-x_m)} \cdot \frac{(b-a)^3}{12} = \frac{2f(b)}{\cancel{(b-a)^2}} \cdot \frac{(b-a)^{\cancel{3}}}{12} = \boxed{\frac{h\,f(a)}{3}}$$

Reemplazamos en la original:

$$\int_a^b P_2(x)\,dx = \frac{h\,f(a)}{3} + \frac{4h\,f(x_m)}{3} + \frac{h\,f(a)}{3}$$

$$\int_a^b f(x)\,dx \simeq \int_a^b P_2(x)\,dx = \frac{h}{3}\left[f(a) + 4f(x_m) + f(a) \right]$$

12. Evaluar:

$$\int_1^2 \frac{1}{x^2} dx, \tag{1.142}$$

por medio de la cuadratura de Gauss-Legendre con dos y tres puntos. Rpta: $I_2 = 0{,}497041$, $I_3 = 0,499874$.

$n=2$

$$f(x) = \frac{1}{x^2}$$

Para resolver usamos la siguiente formula:

$$\int_a^b f(x)dx = \frac{1}{2}(b-a)\sum_{k=0}^{n} w_k f(\frac{1}{2}[t_k(b-a)+a+b])$$

como $n=2$

$t_k = \pm\sqrt{\frac{1}{3}}$

$w_k = 1$

$$I_2 = \frac{(2-1)}{2}\sum_{k=0}^{2} w_k \frac{1}{\left(\frac{t_k(2-1)+1+2}{2}\right)^2}$$

$$= \frac{1}{2}\sum_{k=0}^{2} w_k \cdot \frac{4}{(t_k+3)^2}$$

$$= \frac{1}{2}\left[1\cdot\frac{4}{(\sqrt{\frac{1}{3}}+3)^2} + 1\cdot\frac{4}{(-\sqrt{\frac{1}{3}}+3)^2}\right]$$

$I_2 = 0{,}49704142$

Como $n = 3$

$$t_K = 0, \ \pm\sqrt{\frac{3}{5}}$$

$$w_K = 8/9, \ 5/9$$

$$I_3 = \frac{(2-1)}{2} \sum_{K=0}^{3} w_K \frac{1}{\left(\frac{t_K(2-1)+1}{2}+2\right)^2}$$

$$= \frac{1}{2} \sum_{K=0}^{3} w_K \cdot \frac{4}{(t_K+3)^2}$$

$$= \frac{1}{2}\left[\frac{8}{9}\cdot\frac{4}{(0+3)^2} + \frac{5}{9}\cdot\frac{4}{\left(\sqrt{\frac{3}{5}}+3\right)^2} + \frac{5}{9}\cdot\frac{4}{\left(-\sqrt{\frac{3}{5}}+3\right)^2}\right]$$

$$I_3 = 0,49987402$$

13. Escribir el polinomio $p(x) = 3 + 5x + x^2$ en la base de Legendre. Rpta: $p(x) = \frac{10}{3}p_0(x) + 5p_1(x) + \frac{2}{3}p_2(x)$

Los primeros 4 polinomios de Legendre son:

$$\begin{aligned} p_0(x) &= 1 \\ p_1(x) &= x \\ p_2(x) &= \frac{1}{2}(3x^2 - 1) \\ p_3(x) &= \frac{1}{2}(5x^3 - 3x) \end{aligned}$$

Tratamos de representar cada parte del polinomio con estos:

$$3 = P_0(x) \cdot 3$$

$$5x = P_1(x) \cdot 5$$

$$x^2 = \frac{2}{3} P_2(x) = \frac{2}{3} \cdot \frac{1}{2}(3x^2 - 1) = x^2 \underbrace{- \frac{1}{3}}_{\text{Este factor "sobra"}}$$

Se puede compensar el $-\frac{1}{3}$ con:

$$p_0(x) \cdot \frac{10}{3} + P_2(x) \cdot \frac{2}{3} = 3 + x^2$$

Finalmente se tiene que:

$$P(x) = 3 + 5x + x^2 = \frac{10}{3} P_0(x) + 5 P_1(x) + \frac{2}{3} P_2(x)$$