Week4

IN0003

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What about the last week?

- The video of central Tutorial is linked in this exercise sheet.
- This time we will do more about the **Prove the termination**
- The basic **WP-calulation** and **concept to find a sufficient loop invariant**
- I hope you also have attended the tutorial from last week.
- My slide would almost same as the Solution. I do have graph this time! LaTeX Yay!
- Any questions?

Ocaml Setup

We are going to start OCaml programming in next week's exercises. Please prepare your machines accordingly and bring them to the sessions. A detailed installation guide will be published soon. Please check the website and Moodle.
Maybe that will spend you a while. Feel free to ask questions about that in Piazza.
Refer to my first slide in github, where I provide the installation guide form last year.

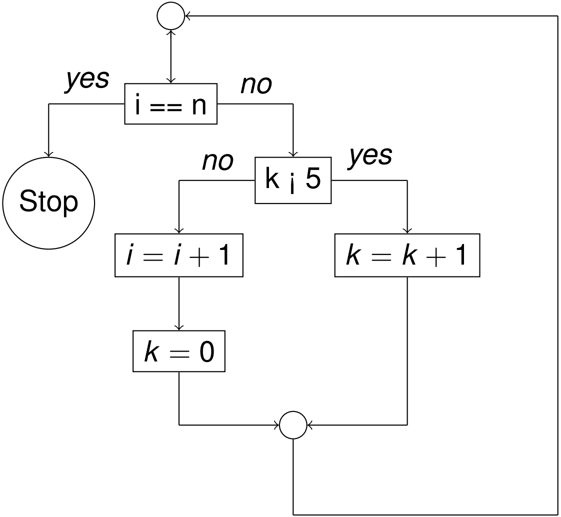
Assignment 4.1 Termination

- This Assignment requires us to understand the concepts. (That is like a Summary)
- My suggestion: Now you just go through that very quickly. Remember to read it carefully after this tutorial.
- The you can understand completely all the things!

What can be applied later?

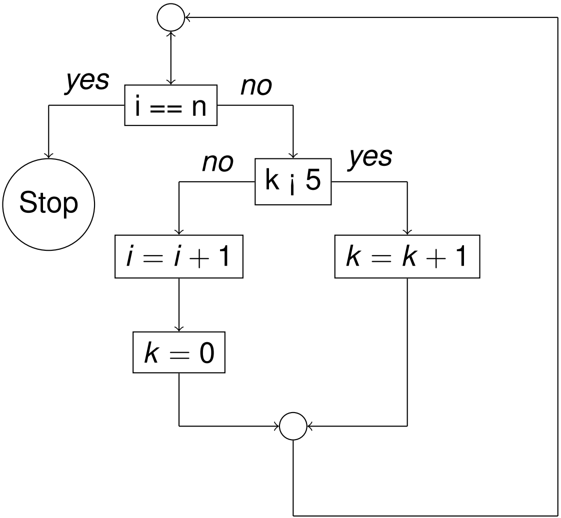
- Choose a variable of the program that is strictly increasing (or decreasing) in every single iteration. Then, if we can prove that there is an upper (or lower) bound the variable cannot reach, the loop must run only a finite number of iterations, because otherwise it would eventually reach the bound. The common strategy is to choose a variable that is decreasing by exactly 1 in every iteration and a lower bound of 0.
- If the program does not provide a variable that satisfies these requirements, we introduce a new variable (typically named r).

Assignment 4.2 Counter Time



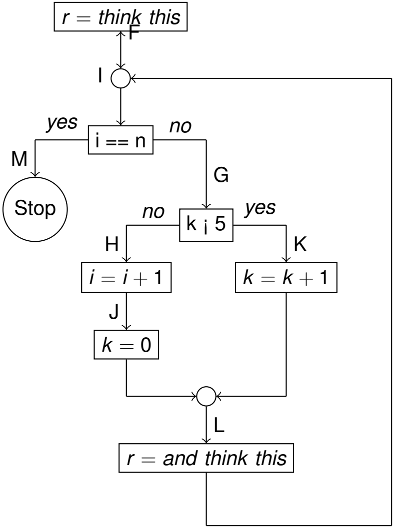
Do we have a r, which is monotonously rising or falling?

Assignment 4.2 Counter Time



Do we have a r, which is monotonously rising or falling?
NO. i, k don't monotonously change every time!

Introduce r variable!



Given the same Annotation Points from I to M.
Hint: To use a table to find the expression of r!

Find r expression

- r should be introduced, so this new variable needs to be initialized.
- The two place that r appeares, must have a same expression. We assume to reduce the r.
- r should in each loop reduced and always greater than 0 in loop.(Leaving the loop with $r = 0$, we prove that later)
- What should this expression consists of?

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- The two place that r appeares, must have a same expression. We assume to reduce the r.
- r should in each loop reduced and always greater than 0 in loop.(Leaving the loop with $r = 0$, we prove that later)
- What should this expression consists of?
- n, i, k. Here we just initial r with a positive number, then try to decrease it until 0!
- For n is a already a postive number.

Var	Time0	Time1	Time2	Time3	Time4	Time5	Time6
i	0	0	0	0	0	0	1
k	0	1	2	3	4	5	0

Var	Time7	Time8	Time9	Time10	Time11	Time12
i	1	1	1	1	1	2
k	1	2	3	4	5	0

Find *r* expression

Var	Time0	Time1	Time2	Time3	Time4	Time5	Time6
i	0	0	0	0	0	0	1
k	0	1	2	3	4	5	0

- We found that *i* every 6 times with increment 1. Also every 6 times the k from 1 to 5 then 0.
- The period is 6!
- How to design a r expression to guarantee that has decreases *r* every time and stay positive in the loop? Any ideas?

Find *r* expression

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i	0	0	0	0	0	0	1
k	0	1	2	3	4	5	0

- We found that i every 6 times with increment 1. Also every 6 times the k from 1 to 5 then 0.
- The period is 6!
- How to design a r expression to guarantee that has decreases r every time and stay positive in the loop? Any ideas?
- Very intuitively. We initialize the r with the n, the only positive value we have.
- If we want to have a decreasing r, we should multiple the increasing variables with -1. (Here are i and k).
- So Frist step: $r = c_1 * n - c_2 * i - c_3 * k - c_4$. Now only to solve these 4 constants.

Find r expression

Var	Time0	Time1	Time2	Time3	Time4	Time5	Time6
i	0	0	0	0	0	0	1
k	0	1	2	3	4	5	0

- So Frist step: $r = c_1 * n - c_2 * i - c_3 * k - c_4$. Now only to solve these 4 constants.(In this case they are non-negative)
- When we look at time 1 to time 5. The r is reducing.
- But consider about the time6. Which c_2 and c_3 could make the r to keep reducing?

Find r expression

Var	Time0	Time1	Time2	Time3	Time4	Time5	Time6
i	0	0	0	0	0	0	1
k	0	1	2	3	4	5	0

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- When we look at time 1 to time 5. The r is reducing.
- But consider about the time6. Which c_2 and c_3 could make the r to keep reducing?
- Second Step: $r = c_1 * n - 6 * i - 1 * k - c_4$. When i increases itself by 1, and the k decreases itself by 5. So $-(6 * 1) - (-5) = -1$ and r is still decreasing.
- Do you figure out that? What is the period of the i and k?

Find r expression

Try to figure out the table of the last 6 rounds(Think about when to leave the loop and how many times do we have until to leave):

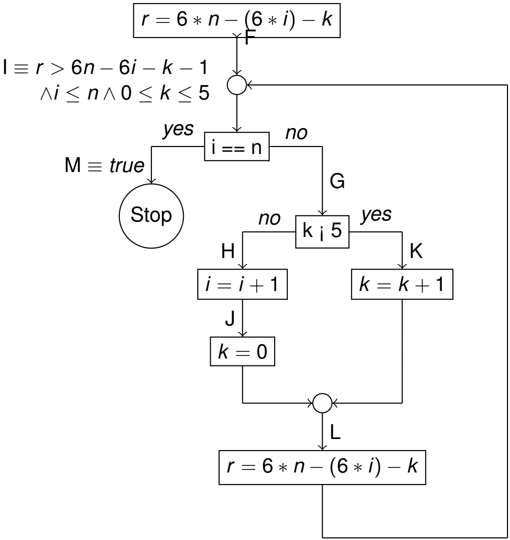
Find r expression

Try to figure out the table of the last 6 rounds(Think about when to leave the loop and how many times do we have until to leave):

Times	$6*n-6$	$6*n-5$	$6*n-4$	$6*n-3$	$6*n-2$	$6*n-1$	$6*n$
i	n-1	n-1	n-1	n-1	n-1	n-1	n
k	0	1	2	3	4	5	0
r	6	5	4	3	2	1	0

- Third Step: The time to leave the loop when $i == n$. So $c_1 = c_2 = 6, c_4 = 0$ to make $r = 0$, when leaving the loop.
- Until now we have the loop invariant on the modified graph.
- Pay attention to informations in the table. $i \leq n \wedge 0 \leq k \leq 5$
- $\mathrm{I} \equiv r = 6n - 6i - k \wedge i \leq n \wedge 0 \leq k \leq 5$

Prove the locally consistent with WP



Prove the locally consistent with WP

Now we have to find locally consistent annotations such that

- $G \implies r > 0$ and
- $L \implies r > 6n - 6i - k$

The first one to make sure that the introduced variable stay always positive in loop. The second one make sure that r is always decreasing.

So that the termination will be proved, if you can prove this two locally consistent annotations G and I.

You can have a negative variable and let it increase. Also works.

Prove the locally consistent with WP

- Now it time to practice the WP, which is exactly the same as the last exercise sheet.
- Recap from last time: If you are 100% sure that your loop invariant is correct, consider to make some assertion **stronger** to come to the locally consistent annotations G and I.
- (This time we do the loop invariant together, so it is already 100% correct. So you get the point!)

Prove the locally consistent with WP

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- Recap from last time: If you are 100% sure that your loop invariant is correct, consider to make some assertion **stronger** to come to the locally consistent annotations G and I.
- (This time we do the loop invariant together, so it is already 100% correct. So you get the point!)
- (Switch to the orignal solution. Because this WP part is the same thing as the last exercise.)

Recap about the 4.2

- If we don't have a **r variable**, then introduce one.
- Try to figure out the expression of r using the table. (When to exit the loop?)
- Using the expression of r and information from table to summary a loop invariant.
- Try to prove the locally consistent. $G \implies r > 0$ and $L \implies r > \textit{the expression of } r$
- Then using the WP like we did in the exercise 1 and 2 and 3. Nothing new.

Assignment 4.3 A Wavy Approach

- Try to solve this on your own using the recap about the 4.2.
- Ask keep asking yourself with these questions.
- Can I read out a formular and solve it? Can I have a table for a overview of all variables?
- Which expression should the r have?
- Which is the sufficient loop invariant?
- How to prove the locally consistent?
- Then you have the prove of termination!

This is much easier and so I simply the instruction. Here I just list the situation with postive a. (The negative situation can be somehow very same!)
I assume a_0 to be the readed value to avoid the mis-understanding because of the $a = \mp a \pm 1$ assignment.

Var	Time0	Time1	Time2	Time3	Time4	Time5	Time6	
a	$-a_0$	$-a_0$+1	a_0-2	$-a_0$+3	a_0-4	$-a_0$+5	a_0-6	

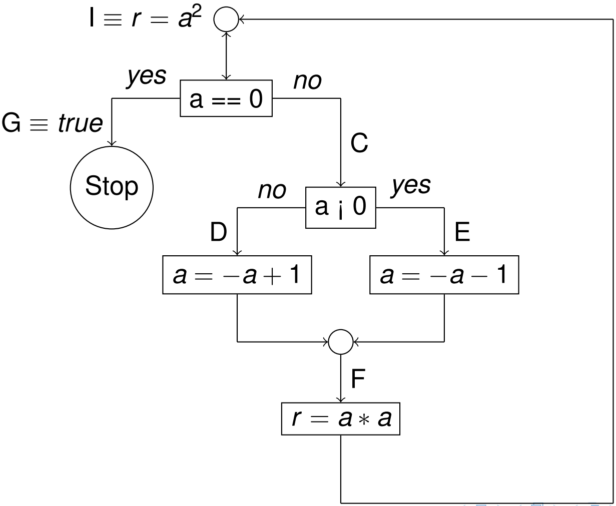
We can easliy find that the absolute value $|a|$ is always decreasing.
$r = a^2$ for this time. Because we only have a single value a. So r can only have relation with a. (Tricky!!!)

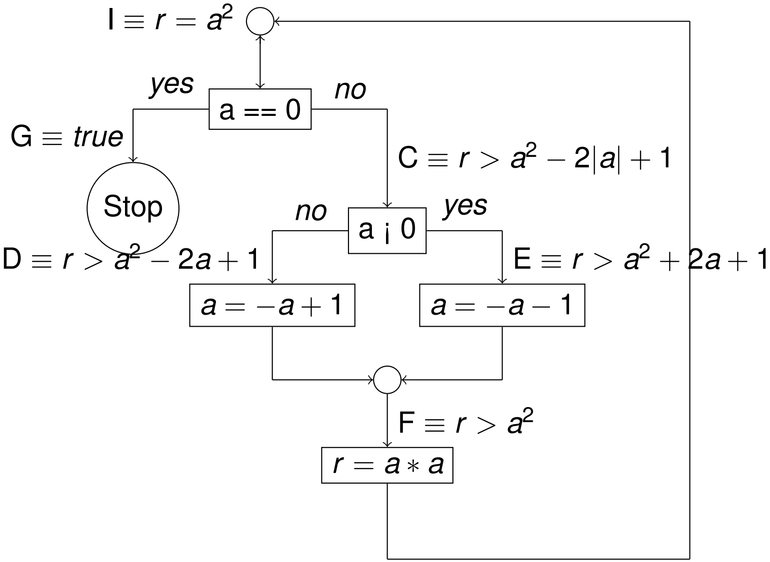
Then, we start with a loop invariant of $r = a^2$ and try to find locally consistent annotations such that

(1) $C \implies r > 0$ and

(2) $F \implies r > a^2$

are satisfied:





$I \equiv r = a^2$
a == 0
yes
no
$G \equiv true$
Stop
$C \equiv r > a^2 - 2|a| + 1$
a ¡ 0
no
yes
$D \equiv r > a^2 - 2a + 1$
$E \equiv r > a^2 + 2a + 1$
$a = -a + 1$
$a = -a - 1$
$F \equiv r > a^2$
$r = a * a$

Summary

- Remember to re-read the *Assignment*4.1. Try to match each point in into the termination exercises.
- Recommand the recording of last year's exercise.

How to prove the termination?

- If we don't have a **r variable**, then introduce one.
- Try to figure out the expression of r using the table. (When to exit the loop?)
- Using the expression of r and information from table to summary a loop invariant.
- Try to prove the locally consistent. $G \implies r > 0$ and $L \implies r > \textit{the expression of } r$
- Then using the WP like we did in the exercise 1 and 2 and 3. Nothing new.

My opinion: the assignment 4.2 worth the time, you spend for reading times of it. Assignment 4.3 is somehow a tricky exercise. It can test you, if you understand this concept.