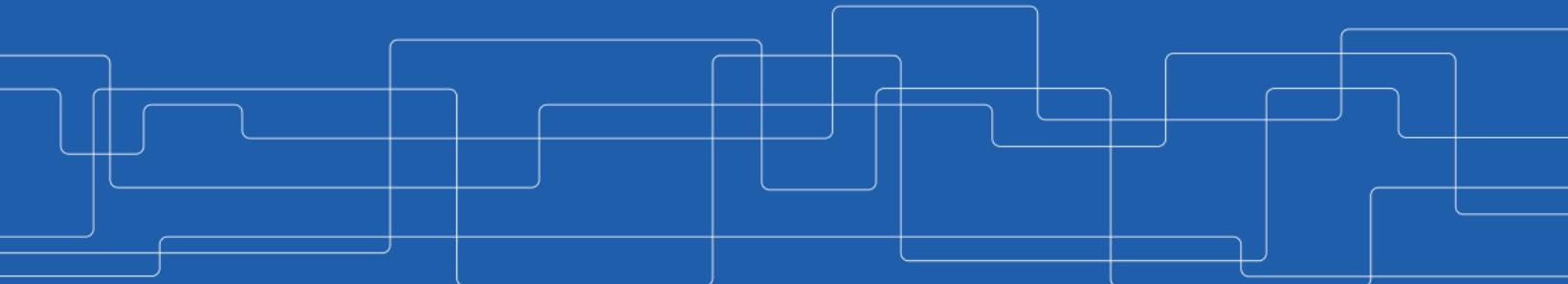




# Deep Feedforwards Networks

Amir H. Payberah  
[payberah@kth.se](mailto:payberah@kth.se)  
28/11/2018



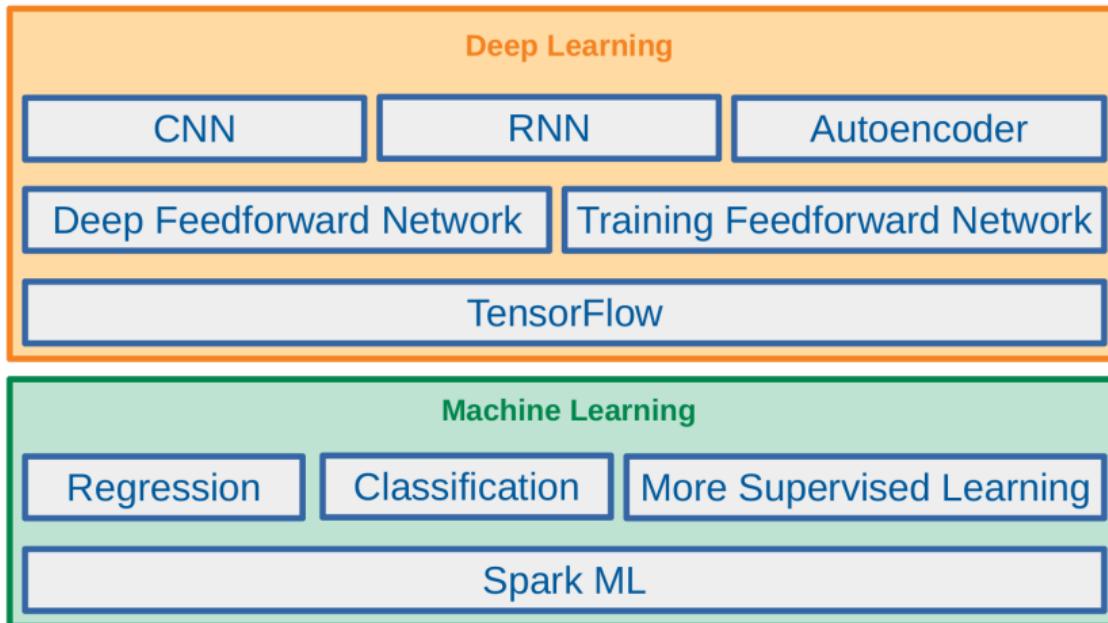


# The Course Web Page

<https://id2223kth.github.io>

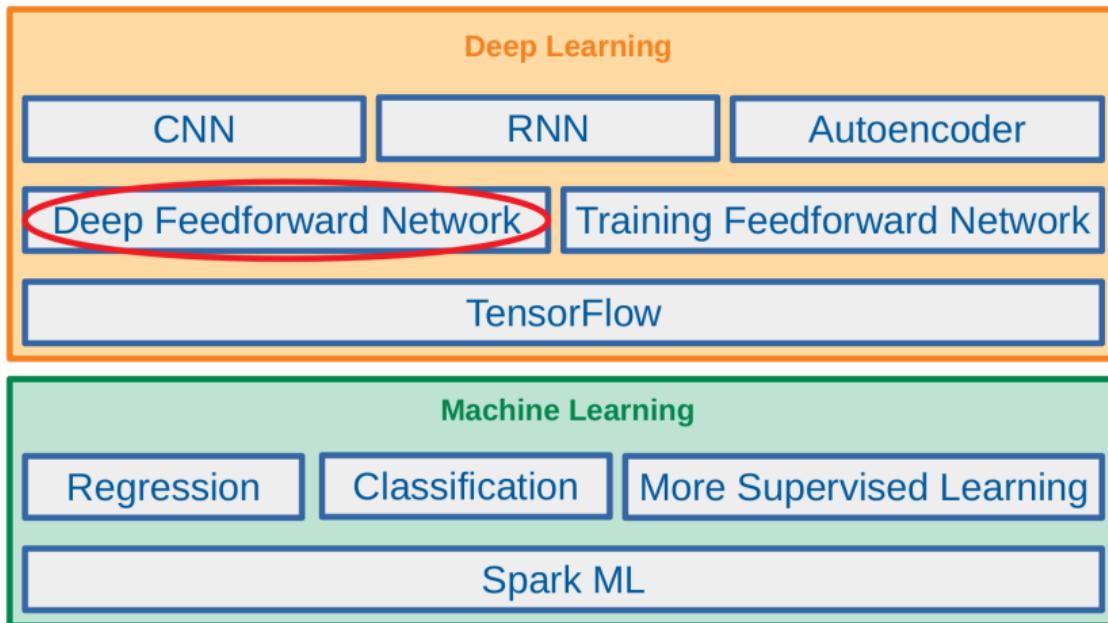


# Where Are We?





# Where Are We?



# Nature ...

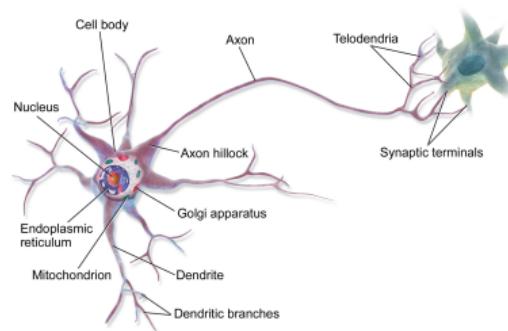
► Nature has inspired many of our inventions

- Birds inspired us to fly
- Burdock plants inspired velcro
- Etc.



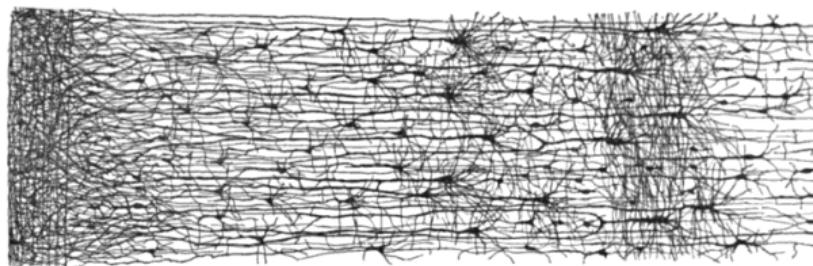
# Biological Neurons (1/2)

- ▶ Brain architecture has inspired artificial neural networks.
- ▶ A biological neuron is composed of
  - Cell body, many dendrites (branching extensions), one axon (long extension), synapses
- ▶ Biological neurons receive signals from other neurons via these synapses.
- ▶ When a neuron receives a sufficient number of signals within a few milliseconds, it fires its own signals.



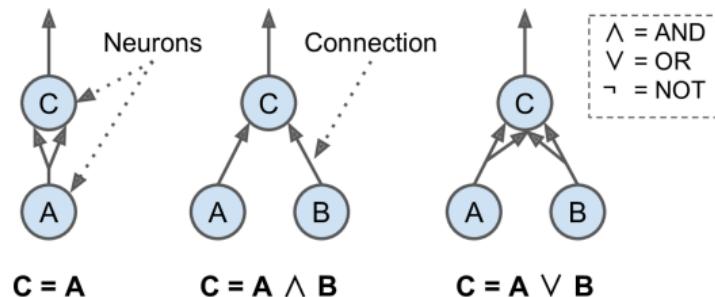
## Biological Neurons (2/2)

- ▶ Biological neurons are organized in a vast **network of billions of neurons**.
- ▶ Each neuron typically is **connected to thousands of other neurons**.



# A Simple Artificial Neural Network

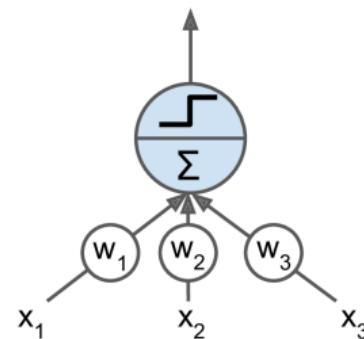
- ▶ One or more **binary inputs** and **one binary output**
- ▶ Activates its **output** when more than a **certain number of its inputs are active**.



# The Linear Threshold Unit (LTU)

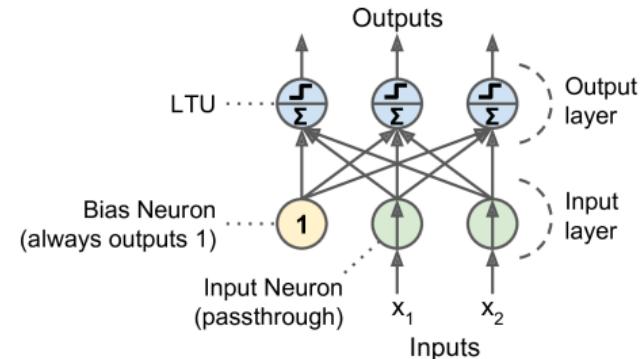
- ▶ Inputs of a LTU are **numbers** (not binary).
- ▶ Each **input connection** is associated with a **weight**.
- ▶ Computes a **weighted sum of its inputs** and applies a **step function** to that **sum**.

- ▶  $z = w_1x_1 + w_2x_2 + \dots + w_nx_n = \mathbf{w}^T\mathbf{x}$
- ▶  $\hat{y} = \text{step}(z) = \text{step}(\mathbf{w}^T\mathbf{x})$



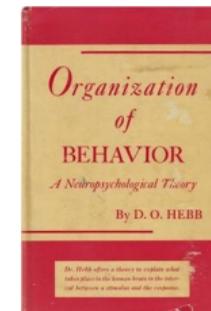
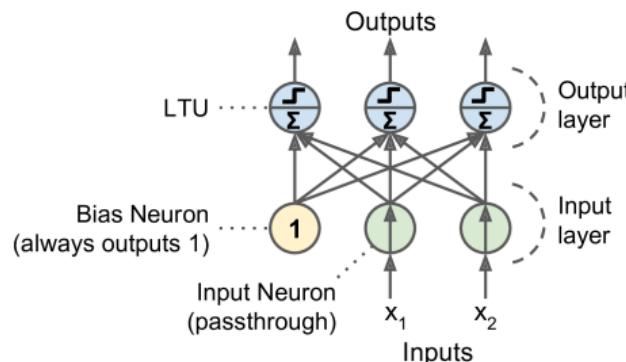
# The Perceptron

- ▶ The **perceptron** is a **single layer** of LTUs.
- ▶ The **input neurons** output whatever **input** they are fed.
- ▶ A **bias neuron**, which just **outputs 1** all the time.
- ▶ If we use **logistic function (sigmoid)** instead of a **step** function, it computes a **continuous** output.



# How is a Perceptron Trained? (1/2)

- ▶ The **Perceptron training algorithm** is inspired by **Hebb's rule**.
- ▶ When a **biological neuron** often **triggers another neuron**, the **connection** between these two neurons grows **stronger**.



# How is a Perceptron Trained? (2/2)

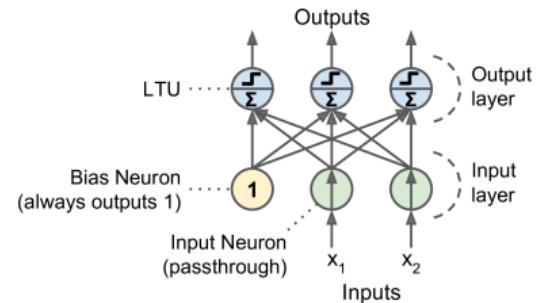
- ▶ Feed one training instance  $\mathbf{x}$  to each neuron  $j$  at a time and make its prediction  $\hat{y}_j$ .
- ▶ Update the connection weights.

$$\hat{y}_j = \sigma(\mathbf{w}_j^T \mathbf{x} + b)$$

$$J(\mathbf{w}_j) = \text{cross\_entropy}(y_j, \hat{y}_j)$$

$$\mathbf{w}_{i,j}^{(\text{next})} = \mathbf{w}_{i,j} - \eta \frac{\partial J(\mathbf{w}_j)}{\partial w_i}$$

- ▶  $w_{i,j}$ : the weight between neurons  $i$  and  $j$ .
- ▶  $x_i$ : the  $i$ th input value.
- ▶  $\hat{y}_j$ : the  $j$ th predicted output value.
- ▶  $y_j$ : the  $j$ th true output value.
- ▶  $\eta$ : the learning rate.





# Perceptron in TensorFlow



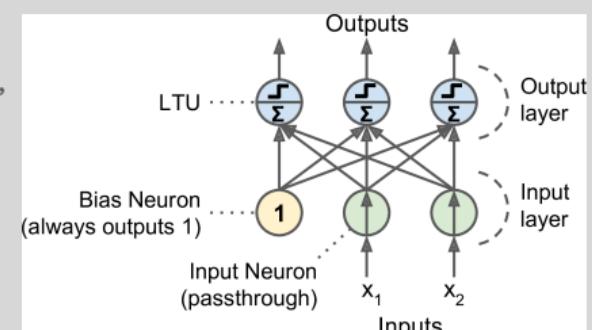
# Perceptron in TensorFlow - First Implementation (1/3)

- ▶ `n_neurons`: number of neurons in a layer.
- ▶ `n_features`: number of features.

```
n_neurons = 3
n_features = 2

# placeholder
X = tf.placeholder(tf.float32, shape=(None, n_features),
                   name="X")
y_true = tf.placeholder(tf.int64, shape=(None),
                       name="y")

# variables
W = tf.get_variable("weights", dtype=tf.float32,
                     initializer=tf.zeros((n_features, n_neurons)))
b = tf.get_variable("bias", dtype=tf.float32,
                     initializer=tf.zeros((n_neurons)))
```





# Perceptron in TensorFlow - First Implementation (2/3)

$$\hat{y}_j = \sigma(\mathbf{w}_j^T \mathbf{x} + b)$$

```
# make the network
z = tf.matmul(X, W) + b
y_hat = tf.nn.sigmoid(z)
```

$$J(\mathbf{w}_j) = \text{cross\_entropy}(y_j, \hat{y}_j) = - \sum_i^m y_j^{(i)} \log(\hat{y}_j^{(i)})$$

```
# define the cost
cross_entropy = -tf.reduce_sum(y_true * tf.log(y_hat), axis=1)
cost = tf.reduce_mean(cross_entropy)
```

$$\mathbf{w}_{i,j}^{(\text{next})} = \mathbf{w}_{i,j} - \eta \frac{\partial J(\mathbf{w}_j)}{\mathbf{w}_i}$$

```
# train the model
# 1. compute the gradient of cost with respect to W and b
# 2. update the weights and bias
learning_rate = 0.1
new_W = W.assign(W - learning_rate * tf.gradients(xs=W, ys=cost))
new_b = b.assign(b - learning_rate * tf.gradients(xs=b, ys=cost))
```



## Perceptron in TensorFlow - First Implementation (3/3)

- ▶ Execute the network.

```
# execute the model
init = tf.global_variables_initializer()

n_epochs = 100
with tf.Session() as sess:
    init.run()
    for epoch in range(n_epochs):
        sess.run([new_W, new_b, cost], feed_dict={X: training_X, y_true: training_y})
```



# Perceptron in TensorFlow - Second Implementation (1/2)

$$\hat{y}_j = \sigma(\mathbf{w}_j^T \mathbf{x} + b)$$

```
# make the network
z = tf.matmul(X, W) + b
y_hat = tf.nn.sigmoid(z)
```

$$J(\mathbf{w}_j) = \text{cross\_entropy}(y_j, \hat{y}_j) = - \sum_i^m y_j^{(i)} \log(\hat{y}_j^{(i)})$$

```
# define the cost
cross_entropy = tf.nn.sigmoid_cross_entropy_with_logits(z, y_true)
cost = tf.reduce_mean(cross_entropy)
```

$$\mathbf{w}_{i,j}^{(\text{next})} = \mathbf{w}_{i,j} - \eta \frac{\partial J(\mathbf{w}_j)}{\mathbf{w}_i}$$

```
# train the model
learning_rate = 0.1
optimizer = tf.train.GradientDescentOptimizer(learning_rate)
training_op = optimizer.minimize(cost)
```



## Perceptron in TensorFlow - Second Implementation (2/2)

- ▶ Execute the network.

```
# execute the model
init = tf.global_variables_initializer()

n_epochs = 100
with tf.Session() as sess:
    init.run()
    for epoch in range(n_epochs):
        sess.run(training_op, feed_dict={X: training_X, y_true: training_y})
```



# Perceptron in Keras

- ▶ Build and execute the network.

```
n_neurons = 10
y_hat = tf.keras.Sequential([layers.Dense(n_neurons, activation="sigmoid")])

y_hat.compile(optimizer=tf.train.GradientDescentOptimizer(0.001), loss="binary_crossentropy",
              metrics=["accuracy"])

n_epochs = 100
y_hat.fit(training_X, training_y, epochs=n_epochs)
```

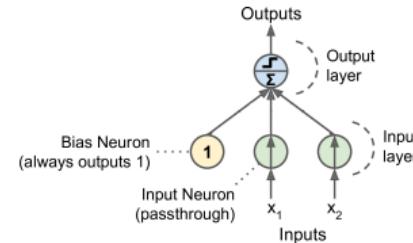


# Multi-Layer Perceptron (MLP)

# Perceptron Weakness (1/2)

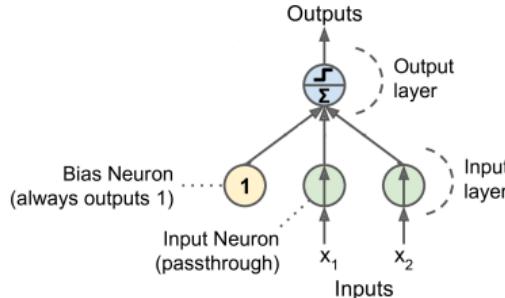
- Incapable of solving some **trivial problems**, e.g., **XOR** classification problem. **Why?**

A	B	A XOR B
0	0	0
0	1	1
1	0	1
1	1	0



$$\mathbf{x} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \quad \mathbf{y} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

## Perceptron Weakness (2/2)



$$\mathbf{x} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \quad \mathbf{y} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \quad \hat{\mathbf{y}} = \text{step}(z), z = w_1x_1 + w_2x_2 + b$$

$$J(\mathbf{w}) = \frac{1}{4} \sum_{\mathbf{x} \in \mathcal{X}} (\hat{\mathbf{y}}(\mathbf{x}) - \mathbf{y}(\mathbf{x}))$$

- ▶ If we minimize  $J(\mathbf{w})$ , we obtain  $w_1 = 0$ ,  $w_2 = 0$ , and  $b = \frac{1}{2}$ .
- ▶ But, the model outputs 0.5 everywhere.

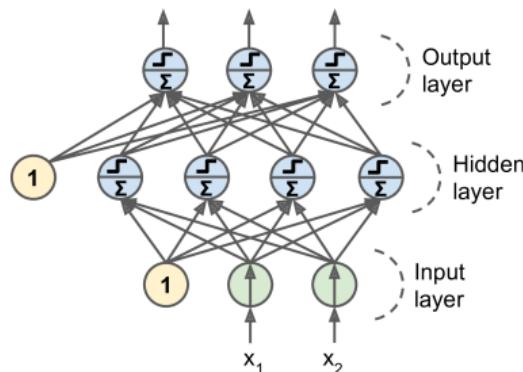


# Multi-Layer Perceptron (MLP)

- ▶ The **limitations** of Perceptrons can be eliminated by **stacking multiple Perceptrons**.
- ▶ The resulting network is called a **Multi-Layer Perceptron (MLP)** or **deep feedforward neural network**.

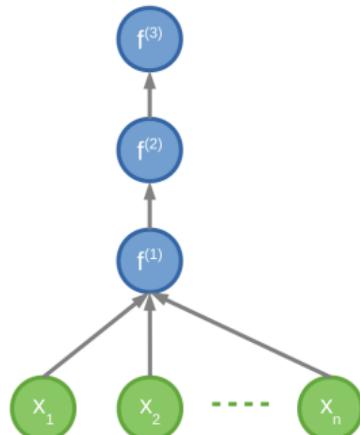
# Feedforward Neural Network Architecture

- ▶ A feedforward neural network is composed of:
  - One **input layer**
  - One or more **hidden layers**
  - One final **output layer**
- ▶ Every layer except the output layer includes a **bias neuron** and is **fully connected** to the **next layer**.

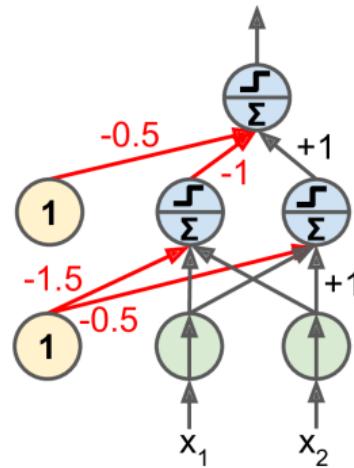


# How Does it Work?

- ▶ The model is associated with a directed acyclic graph describing how the functions are composed together.
- ▶ E.g., assume a network with just a single neuron in each layer.
- ▶ Also assume we have three functions  $f^{(1)}$ ,  $f^{(2)}$ , and  $f^{(3)}$  connected in a chain:  $\hat{y} = f(x) = f^{(3)}(f^{(2)}(f^{(1)}(x)))$
- ▶  $f^{(1)}$  is called the first layer of the network.
- ▶  $f^{(2)}$  is called the second layer, and so on.
- ▶ The length of the chain gives the depth of the model.

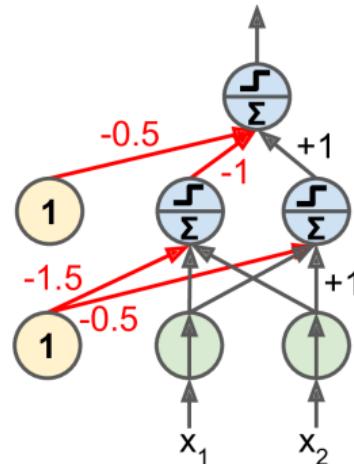


## XOR with Feedforward Neural Network (1/3)



$$\mathbf{X} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \quad \mathbf{y} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{w}_x = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \mathbf{b}_x = \begin{bmatrix} -1.5 \\ -0.5 \end{bmatrix}$$

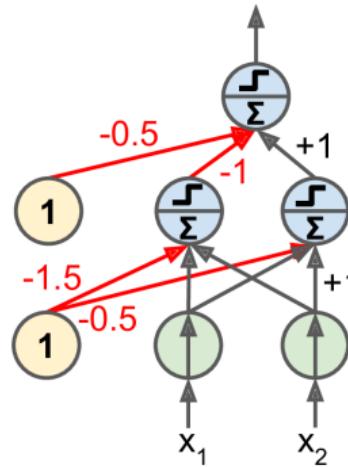
## XOR with Feedforward Neural Network (2/3)



$$\mathbf{out}_h = \mathbf{X}\mathbf{W}_x^T + \mathbf{b}_x = \begin{bmatrix} -1.5 & -0.5 \\ -0.5 & 0.5 \\ -0.5 & 0.5 \\ 0.5 & 1.5 \end{bmatrix} \quad \mathbf{h} = \text{step}(\mathbf{out}_h) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\mathbf{w}_h = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \mathbf{b}_h = -0.5$$

## XOR with Feedforward Neural Network (3/3)



$$\mathbf{out} = \mathbf{w}_h^T \mathbf{h} + \mathbf{b}_h = \begin{bmatrix} -0.5 \\ 0.5 \\ 0.5 \\ -0.5 \end{bmatrix} \quad \text{step}(\mathbf{out}) = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$



# How to Learn Model Parameters $W$ ?



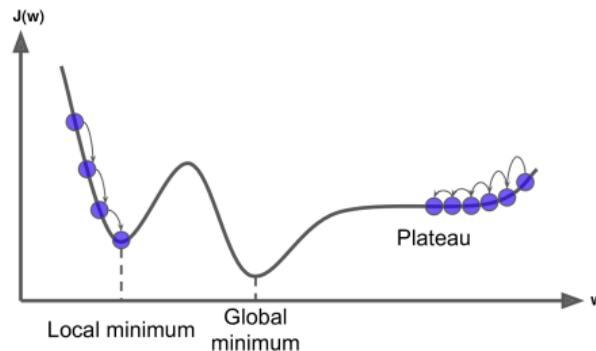
## Feedforward Neural Network - Cost Function

- ▶ We use the **cross-entropy** (minimizing the negative log-likelihood) between the training data  $y$  and the model's predictions  $\hat{y}$  as the **cost function**.

$$\text{cost}(y, \hat{y}) = - \sum_j y_j \log(\hat{y}_j)$$

# Gradient-Based Learning (1/2)

- ▶ The **most significant difference** between the **linear models** we have seen so far and **feedforward neural network**?
- ▶ The **non-linearity** of a neural network causes its **cost functions** to become **non-convex**.
- ▶ Linear models, with **convex cost function**, **guarantee** to find **global minimum**.
  - Convex optimization converges starting from **any initial parameters**.



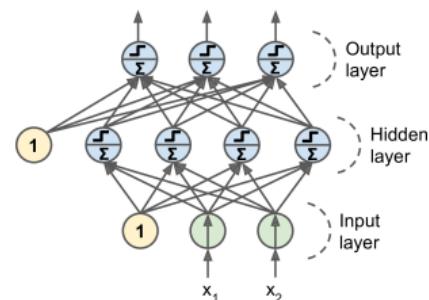


## Gradient-Based Learning (2/2)

- ▶ Stochastic gradient descent applied to **non-convex cost functions** has no such convergence guarantee.
- ▶ It is **sensitive** to the values of the **initial parameters**.
- ▶ For feedforward neural networks, it is important to **initialize** all **weights** to small random values.
- ▶ The **biases** may be **initialized** to zero or to small positive values.

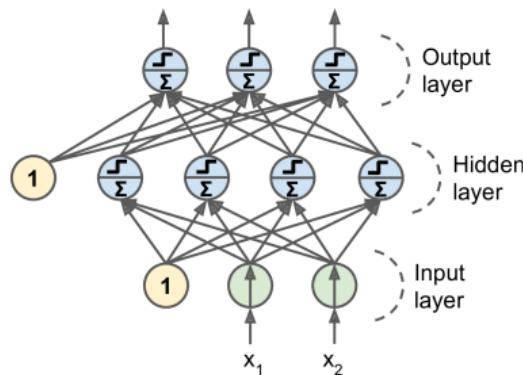
# Training Feedforward Neural Networks

- ▶ How to **train** a **feedforward neural network**?
- ▶ For each training instance  $\mathbf{x}^{(i)}$  the algorithm does the following **steps**:
  1. **Forward pass**: make a **prediction** (compute  $\hat{y}^{(i)} = f(\mathbf{x}^{(i)})$ ).
  2. Measure the **error** (compute  $\text{cost}(\hat{y}^{(i)}, y^{(i)})$ ).
  3. **Backward pass**: go through each layer in **reverse** to measure the **error contribution** from **each connection**.
  4. **Tweak the connection weights** to **reduce the error** (update  $\mathbf{W}$  and  $\mathbf{b}$ ).
- ▶ It's called the **backpropagation** training algorithm



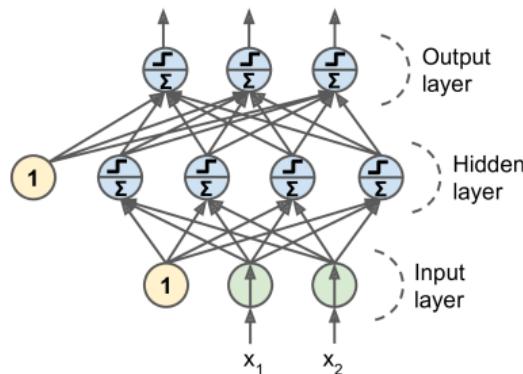
## Output Unit (1/3)

- ▶ Linear units in neurons of the output layer.
- ▶ Given  $\mathbf{h}$  as the output of neurons in the layer before the output layer.
- ▶ Each neuron  $j$  in the output layer produces  $\hat{y}_j = \mathbf{w}_j^T \mathbf{h} + b_j$ .
- ▶ Minimizing the cross-entropy is then equivalent to minimizing the mean squared error.



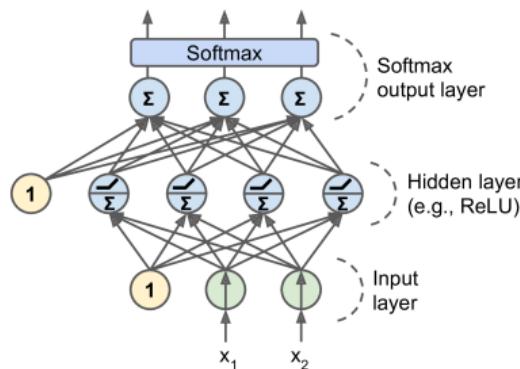
## Output Unit (2/3)

- ▶ Sigmoid units in neurons of the output layer (binomial classification).
- ▶ Given  $\mathbf{h}$  as the output of neurons in the layer before the output layer.
- ▶ Each neuron  $j$  in the output layer produces  $\hat{y}_j = \sigma(\mathbf{w}_j^T \mathbf{h} + b_j)$ .
- ▶ Minimizing the cross-entropy.



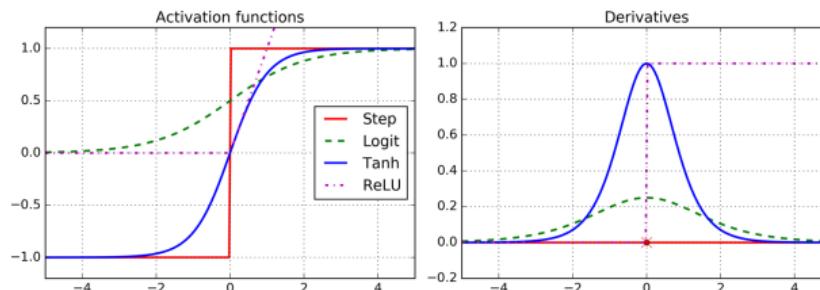
## Output Unit (3/3)

- ▶ Softmax units in neurons of the output layer (multinomial classification).
- ▶ Given  $\mathbf{h}$  as the output of neurons in the layer before the output layer.
- ▶ Each neuron  $j$  in the output layer produces  $\hat{y}_j = \text{softmax}(\mathbf{w}_j^T \mathbf{h} + b_j)$ .
- ▶ Minimizing the cross-entropy.



# Hidden Units

- ▶ In order for the **backpropagation** algorithm to work properly, we need to **replace the step function** with **other activation functions**. **Why?**
- ▶ Alternative activation functions:
  1. Logistic function (sigmoid):  $\sigma(z) = \frac{1}{1+e^{-z}}$
  2. Hyperbolic tangent function:  $\tanh(z) = 2\sigma(2z) - 1$
  3. Rectified linear units (ReLUs):  $\text{ReLU}(z) = \max(0, z)$





# Feedforward Network in TensorFlow

# Feedforward in TensorFlow - First Implementation (1/3)

- ▶ `n_neurons_h`: number of neurons in the hidden layer.
- ▶ `n_neurons_out`: number of neurons in the output layer.
- ▶ `n_features`: number of features.

```
n_neurons_h = 4
n_neurons_out = 3
n_features = 2

# placeholder
X = tf.placeholder(tf.float32, shape=[None, n_features], name="X")
y_true = tf.placeholder(tf.int64, shape=[None], name="y")

# variables
W1 = tf.get_variable("weights1", dtype=tf.float32,
    initializer=tf.zeros((n_features, n_neurons_h)))
b1 = tf.get_variable("bias1", dtype=tf.float32, initializer=tf.zero((n_neurons_h)))

W2 = tf.get_variable("weights2", dtype=tf.float32,
    initializer=tf.zeros((n_features, n_neurons_out)))
b2 = tf.get_variable("bias2", dtype=tf.float32, initializer=tf.zero((n_neurons_out)))
```

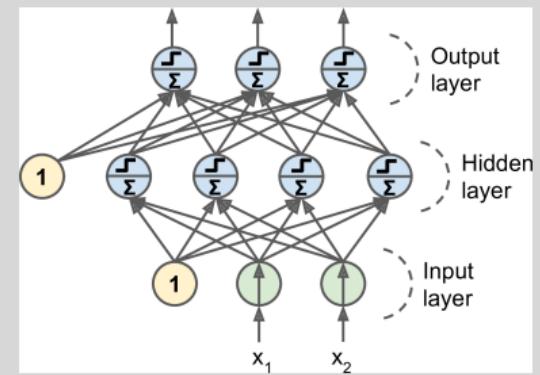
# Feedforward in TensorFlow - First Implementation (2/3)

- ▶ Build the network.

```
# make the network
h = tf.nn.sigmoid(tf.matmul(X, W1) + b1)
z = tf.matmul(h, W2) + b2
y_hat = tf.nn.sigmoid(z)

# define the cost
cross_entropy =
    tf.nn.sigmoid_cross_entropy_with_logits(z, y_true)
cost = tf.reduce_mean(cross_entropy)

# train the model
learning_rate = 0.1
optimizer = tf.train.GradientDescentOptimizer(learning_rate)
training_op = optimizer.minimize(cost)
```





## Feedforward in TensorFlow - First Implementation (3/3)

- ▶ Execute the network.

```
# execute the model
init = tf.global_variables_initializer()

n_epochs = 100
with tf.Session() as sess:
    init.run()
    for epoch in range(n_epochs):
        sess.run(training_op, feed_dict={X: training_X, y_true: training_y})
```

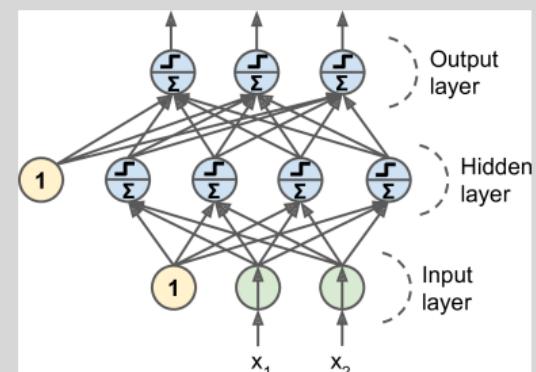
# Feedforward in TensorFlow - Second Implementation

```
n_neurons_h = 4
n_neurons_out = 3
n_features = 2

# placeholder
X = tf.placeholder(tf.float32, shape=[None, n_features],
                   name="X")
y_true = tf.placeholder(tf.int64, shape=[None],
                       name="y")

# make the network
h = tf.layers.dense(X, n_neurons_h, name="hidden",
                    activation=tf.sigmoid)
z = tf.layers.dense(h, n_neurons_out, name="output")

# the rest as before
```





# Feedforward in Keras

```
n_neurons_h = 4
n_neurons_out = 3
n_epochs = 100
learning_rate = 0.1

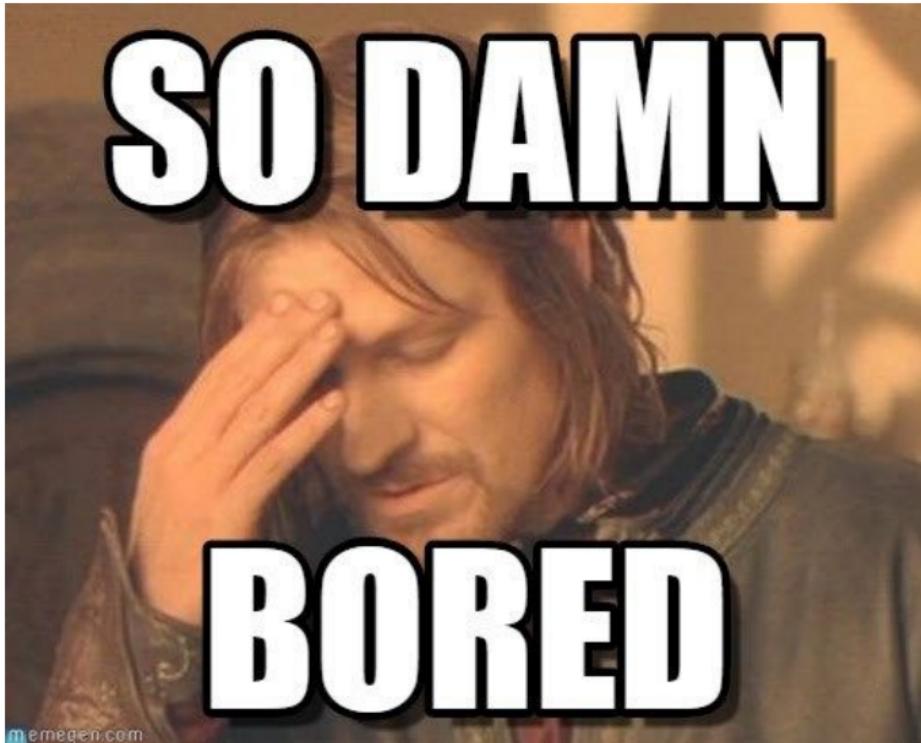
model = tf.keras.Sequential()
model.add(layers.Dense(n_neurons_h, activation="sigmoid"))
model.add(layers.Dense(n_neurons_out, activation="sigmoid"))

model.compile(optimizer=tf.train.GradientDescentOptimizer(learning_rate=.001),
              loss="binary_crossentropy", metrics=["accuracy"])

model.fit(training_X, training_y, epochs=n_epochs)
```



# Dive into Backpropagation Algorithm



[<https://i.pinimg.com/originals/82/d9/2c/82d92c2c15c580c2b2fce65a83fe0b3f.jpg>]



## Chain Rule of Calculus (1/2)

- ▶ Assume  $x \in \mathbb{R}$ , and two functions  $f$  and  $g$ , and also assume  $y = g(x)$  and  $z = f(y) = f(g(x))$ .
- ▶ The **chain rule of calculus** is used to compute the **derivatives of functions**, e.g.,  $z$ , formed by **composing other functions**, e.g.,  $g$ .
- ▶ Then the **chain rule** states that  $\frac{dz}{dx} = \frac{dz}{dy} \frac{dy}{dx}$
- ▶ Example:

$$z = f(y) = 5y^4 \text{ and } y = g(x) = x^3 + 7$$

$$\frac{dz}{dx} = \frac{dz}{dy} \frac{dy}{dx}$$

$$\frac{dz}{dy} = 20y^3 \text{ and } \frac{dy}{dx} = 3x^2$$

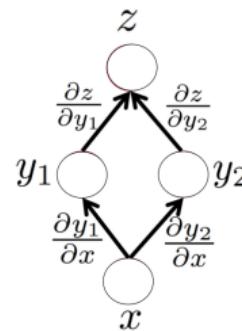
$$\frac{dz}{dx} = 20y^3 \times 3x^2 = 20(x^3 + 7) \times 3x^2$$

## Chain Rule of Calculus (2/2)

- ▶ Two paths chain rule.

$z = f(y_1, y_2)$  where  $y_1 = g(x)$  and  $y_2 = h(x)$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y_1} \frac{\partial y_1}{\partial x} + \frac{\partial z}{\partial y_2} \frac{\partial y_2}{\partial x}$$



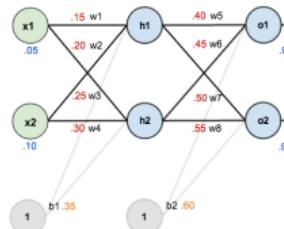


# Backpropagation

- ▶ Backpropagation training algorithm for MLPs
- ▶ The algorithm repeats the following steps:
  1. Forward pass
  2. Backward pass

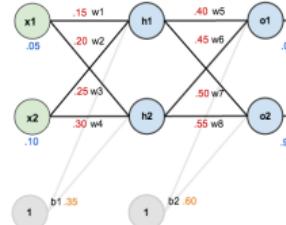
# Backpropagation - Forward Pass

- ▶ Calculates outputs given input patterns.
- ▶ For each training instance
  - Feeds it to the network and computes the output of every neuron in each consecutive layer.
  - Measures the network's output error (i.e., the difference between the true and the predicted output of the network)
  - Computes how much each neuron in the last hidden layer contributed to each output neuron's error.



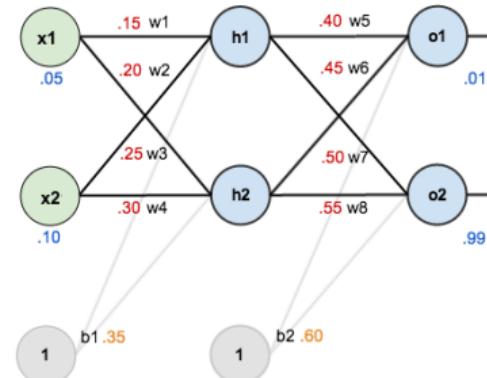
# Backpropagation - Backward Pass

- ▶ Updates weights by calculating gradients.
- ▶ Measures how much of these error contributions came from each neuron in the previous hidden layer
  - Proceeds until the algorithm reaches the input layer.
- ▶ The last step is the gradient descent step on all the connection weights in the network, using the error gradients measured earlier.



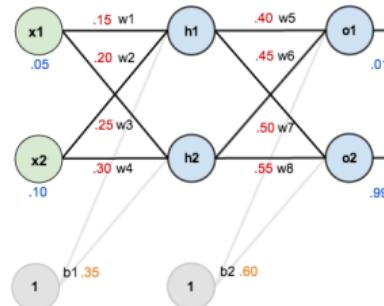
# Backpropagation Example

- ▶ Two **inputs**, two **hidden**, and two **output** neurons.
- ▶ Bias in **hidden** and **output** neurons.
- ▶ Logistic activation in all the neurons.
- ▶ Squared error function as the cost function.



## Backpropagation - Forward Pass (1/3)

- ▶ Compute the **output** of the **hidden layer**



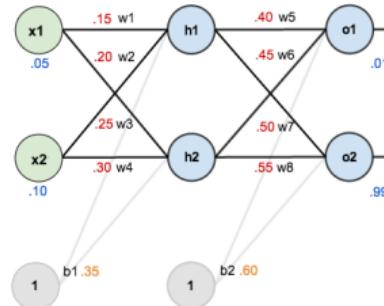
$$\text{net}_{h1} = w_1 x_1 + w_2 x_2 + b_1 = 0.15 \times 0.05 + 0.2 \times 0.1 + 0.35 = 0.3775$$

$$\text{out}_{h1} = \frac{1}{1 + e^{\text{net}_{h1}}} = \frac{1}{1 + e^{0.3775}} = 0.59327$$

$$\text{out}_{h2} = 0.59688$$

## Backpropagation - Forward Pass (2/3)

- ▶ Compute the **output** of the **output layer**



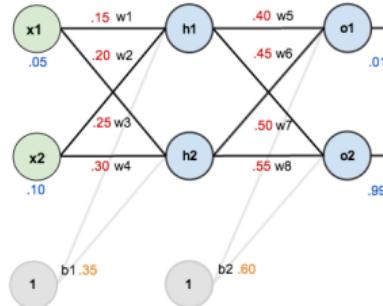
$$\text{net}_{o1} = w_5 \text{net}_{h1} + w_6 \text{net}_{h2} + b_2 = 0.4 \times 0.59327 + 0.45 \times 0.59688 + 0.6 = 1.1059$$

$$\text{out}_{o1} = \frac{1}{1 + e^{\text{net}_{o1}}} = \frac{1}{1 + e^{1.1059}} = 0.75136$$

$$\text{out}_{o2} = 0.77292$$

## Backpropagation - Forward Pass (3/3)

- ▶ Calculate the **error** for each output



$$E_{o1} = \frac{1}{2}(\text{target}_{o1} - \text{output}_{o1})^2 = \frac{1}{2}(0.01 - 0.75136)^2 = 0.27481$$

$$E_{o2} = 0.02356$$

$$E_{\text{total}} = \sum \frac{1}{2}(\text{target} - \text{output})^2 = E_{o1} + E_{o2} = 0.27481 + 0.02356 = 0.29837$$

This class is boring...

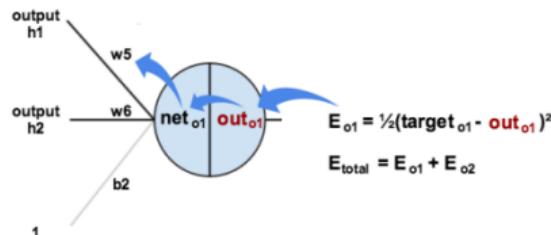


can we learn about dragons?

[<http://marimancusi.blogspot.com/2015/09/are-you-book-dragon.html>]

# Backpropagation - Backward Pass - Output Layer (1/6)

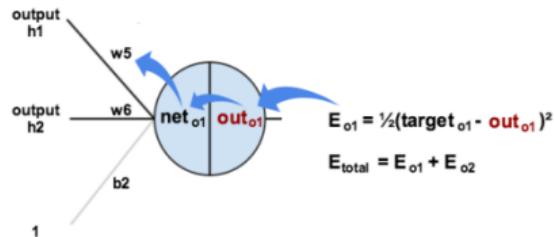
- ▶ Consider  $w_5$
- ▶ We want to know how much a **change** in  $w_5$  affects the **total error** ( $\frac{\partial E_{\text{total}}}{\partial w_5}$ )
- ▶ Applying the **chain rule**



$$\frac{\partial E_{\text{total}}}{\partial w_5} = \frac{\partial E_{\text{total}}}{\partial \text{out}_{o1}} \times \frac{\partial \text{out}_{o1}}{\partial \text{net}_{o1}} \times \frac{\partial \text{net}_{o1}}{\partial w_5}$$

## Backpropagation - Backward Pass - Output Layer (2/6)

- First, how much does the **total error** change with **respect to the output?** ( $\frac{\partial E_{\text{total}}}{\partial \text{out}_{o1}}$ )



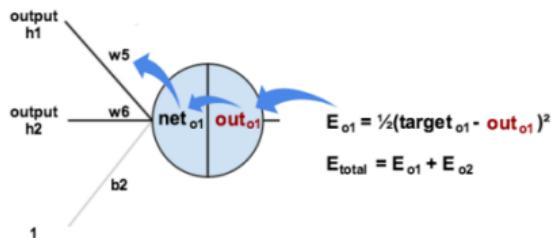
$$\frac{\partial E_{\text{total}}}{\partial w_5} = \frac{\partial E_{\text{total}}}{\partial \text{out}_{o1}} \times \frac{\partial \text{out}_{o1}}{\partial \text{net}_{o1}} \times \frac{\partial \text{net}_{o1}}{\partial w_5}$$

$$E_{\text{total}} = \frac{1}{2}(\text{target}_{o1} - \text{out}_{o1})^2 + \frac{1}{2}(\text{target}_{o2} - \text{out}_{o2})^2$$

$$\frac{\partial E_{\text{total}}}{\partial \text{out}_{o1}} = -2 \frac{1}{2}(\text{target}_{o1} - \text{out}_{o1}) = -(0.01 - 0.75136) = 0.74136$$

## Backpropagation - Backward Pass - Output Layer (3/6)

- ▶ Next, how much does the  $\text{out}_{o1}$  change with respect to its total input  $\text{net}_{o1}$ ?  
 $(\frac{\partial \text{out}_{o1}}{\partial \text{net}_{o1}})$



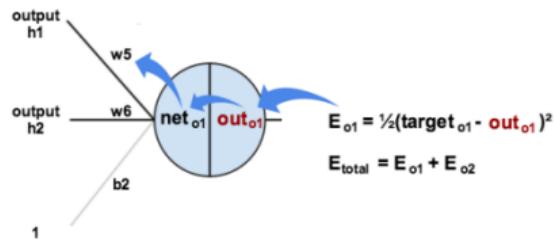
$$\frac{\partial E_{\text{total}}}{\partial w_5} = \frac{\partial E_{\text{total}}}{\partial \text{out}_{o1}} \times \frac{\partial \text{out}_{o1}}{\partial \text{net}_{o1}} \times \frac{\partial \text{net}_{o1}}{\partial w_5}$$

$$\text{out}_{o1} = \frac{1}{1 + e^{-\text{net}_{o1}}}$$

$$\frac{\partial \text{out}_{o1}}{\partial \text{net}_{o1}} = \text{out}_{o1}(1 - \text{out}_{o1}) = 0.75136(1 - 0.75136) = 0.18681$$

## Backpropagation - Backward Pass - Output Layer (4/6)

- Finally, how much does the total  $\text{net}_{o1}$  change with respect to  $w_5$ ? ( $\frac{\partial \text{net}_{o1}}{\partial w_5}$ )



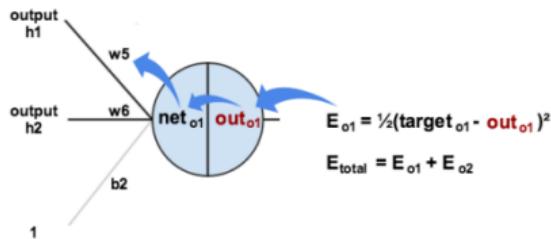
$$\frac{\partial E_{\text{total}}}{\partial w_5} = \frac{\partial E_{\text{total}}}{\partial \text{out}_{o1}} \times \frac{\partial \text{out}_{o1}}{\partial \text{net}_{o1}} \times \frac{\partial \text{net}_{o1}}{\partial w_5}$$

$$\text{net}_{o1} = w_5 \times \text{out}_{h1} + w_6 \times \text{out}_{h2} + b_2$$

$$\frac{\partial \text{net}_{o1}}{\partial w_5} = \text{out}_{h1} = 0.59327$$

## Backpropagation - Backward Pass - Output Layer (5/6)

- ▶ Putting it all together:

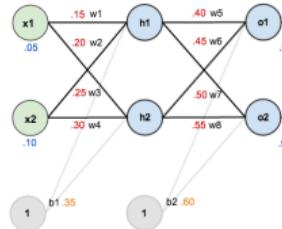


$$\frac{\partial E_{total}}{\partial w_5} = \frac{\partial E_{total}}{\partial out_{o1}} \times \frac{\partial out_{o1}}{\partial net_{o1}} \times \frac{\partial net_{o1}}{\partial w_5}$$

$$\frac{\partial E_{total}}{\partial w_5} = 0.74136 \times 0.18681 \times 0.59327 = 0.08216$$

# Backpropagation - Backward Pass - Output Layer (6/6)

- ▶ To decrease the error, we subtract this value from the **current weight**.
- ▶ We assume that the **learning rate** is  $\eta = 0.5$ .



$$w_5^{(\text{next})} = w_5 - \eta \times \frac{\partial E_{\text{total}}}{\partial w_5} = 0.4 - 0.5 \times 0.08216 = 0.35891$$

$$w_6^{(\text{next})} = 0.40866$$

$$w_7^{(\text{next})} = 0.51113$$

$$w_8^{(\text{next})} = 0.56137$$

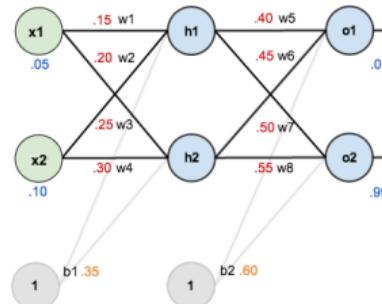


[<https://makeameme.org/meme/oooh-this>]

# Backpropagation - Backward Pass - Hidden Layer (1/8)

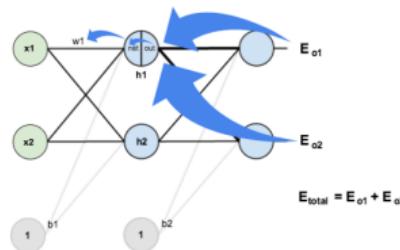
- ▶ Continue the **backwards pass** by calculating new values for  $w_1$ ,  $w_2$ ,  $w_3$ , and  $w_4$ .
- ▶ For  $w_1$  we have:

$$\frac{\partial E_{\text{total}}}{\partial w_1} = \frac{\partial E_{\text{total}}}{\partial \text{out}_{h1}} \times \frac{\partial \text{out}_{h1}}{\partial \text{net}_{h1}} \times \frac{\partial \text{net}_{h1}}{\partial w_1}$$



## Backpropagation - Backward Pass - Hidden Layer (2/8)

- Here, the **output** of each hidden layer neuron contributes to the **output** of multiple **output neurons**.
- E.g.,  $out_{h1}$  affects both  $out_{o1}$  and  $out_{o2}$ , so  $\frac{\partial E_{total}}{\partial out_{h1}}$  needs to take into consideration its effect on the both output neurons.

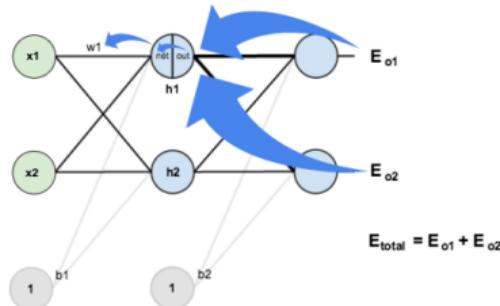


$$\frac{\partial E_{total}}{\partial w_1} = \frac{\partial E_{total}}{\partial out_{h1}} \times \frac{\partial out_{h1}}{\partial net_{h1}} \times \frac{\partial net_{h1}}{\partial w_1}$$

$$\frac{\partial E_{total}}{\partial out_{h1}} = \frac{\partial E_{o1}}{\partial out_{h1}} + \frac{\partial E_{o2}}{\partial out_{h1}}$$

## Backpropagation - Backward Pass - Hidden Layer (3/8)

- ▶ Starting with  $\frac{\partial E_{o1}}{\partial \text{out}_{h1}}$



$$\frac{\partial E_{\text{total}}}{\partial \text{out}_{h1}} = \frac{\partial E_{o1}}{\partial \text{out}_{h1}} + \frac{\partial E_{o2}}{\partial \text{out}_{h1}}$$

$$\frac{\partial E_{o1}}{\partial \text{out}_{h1}} = \frac{\partial E_{o1}}{\partial \text{out}_{o1}} \times \frac{\partial \text{out}_{o1}}{\partial \text{net}_{o1}} \times \frac{\partial \text{net}_{o1}}{\partial \text{out}_{h1}}$$

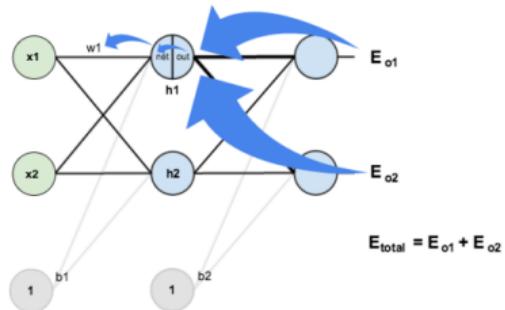
$$\frac{\partial E_{o1}}{\partial \text{out}_{o1}} = 0.74136, \frac{\partial \text{out}_{o1}}{\partial \text{net}_{o1}} = 0.18681$$

$$\text{net}_{o1} = w_5 \times \text{out}_{h1} + w_6 \times \text{out}_{h2} + b_2$$

$$\frac{\partial \text{net}_{o1}}{\partial \text{out}_{h1}} = w_5 = 0.40$$

# Backpropagation - Backward Pass - Hidden Layer (4/8)

- ▶ Plugging them together.



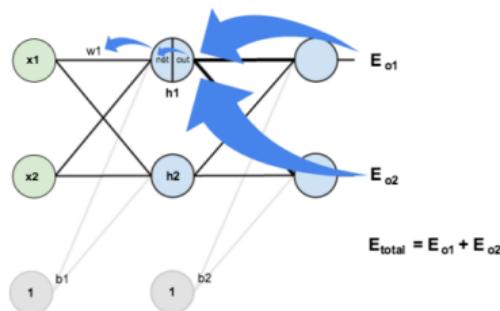
$$\frac{\partial E_{o1}}{\partial \text{out}_{h1}} = \frac{\partial E_{o1}}{\partial \text{out}_{o1}} \times \frac{\partial \text{out}_{o1}}{\partial \text{net}_{o1}} \times \frac{\partial \text{net}_{o1}}{\partial \text{out}_{h1}} = 0.74136 \times 0.18681 \times 0.40 = 0.0554$$

$$\frac{\partial E_{o2}}{\partial \text{out}_{h1}} = -0.01905$$

$$\frac{\partial E_{\text{total}}}{\partial \text{out}_{h1}} = \frac{\partial E_{o1}}{\partial \text{out}_{h1}} + \frac{\partial E_{o2}}{\partial \text{out}_{h1}} = 0.0554 + -0.01905 = 0.03635$$

## Backpropagation - Backward Pass - Hidden Layer (5/8)

- Now we need to figure out  $\frac{\partial \text{out}_{h1}}{\partial \text{net}_{h1}}$ .



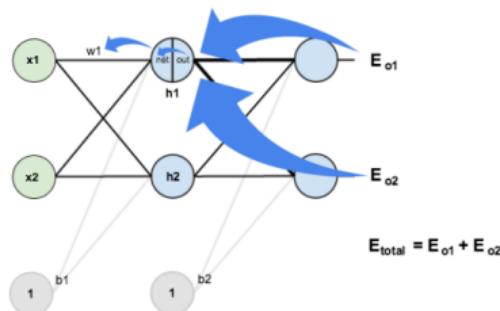
$$\frac{\partial E_{\text{total}}}{\partial w_1} = \frac{\partial E_{\text{total}}}{\partial \text{out}_{h1}} \times \frac{\partial \text{out}_{h1}}{\partial \text{net}_{h1}} \times \frac{\partial \text{net}_{h1}}{\partial w_1}$$

$$\text{out}_{h1} = \frac{1}{1 + e^{-\text{net}_{h1}}}$$

$$\frac{\partial \text{out}_{h1}}{\partial \text{net}_{h1}} = \text{out}_{h1}(1 - \text{out}_{h1}) = 0.59327(1 - 0.59327) = 0.2413$$

# Backpropagation - Backward Pass - Hidden Layer (6/8)

- ▶ And then  $\frac{\partial \text{net}_{h1}}{\partial w_1}$ .



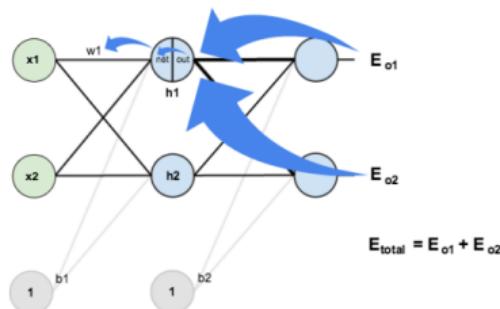
$$\frac{\partial E_{\text{total}}}{\partial w_1} = \frac{\partial E_{\text{total}}}{\partial \text{out}_{h1}} \times \frac{\partial \text{out}_{h1}}{\partial \text{net}_{h1}} \times \frac{\partial \text{net}_{h1}}{\partial w_1}$$

$$\text{net}_{h1} = w_1 x_1 + w_2 x_2 + b_1$$

$$\frac{\partial \text{net}_{h1}}{\partial w_1} = x_1 = 0.05$$

# Backpropagation - Backward Pass - Hidden Layer (7/8)

- ▶ Putting it all together.

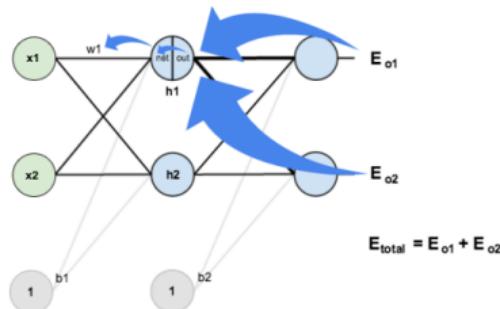


$$\frac{\partial E_{\text{total}}}{\partial w_1} = \frac{\partial E_{\text{total}}}{\partial \text{out}_{h1}} \times \frac{\partial \text{out}_{h1}}{\partial \text{net}_{h1}} \times \frac{\partial \text{net}_{h1}}{\partial w_1}$$

$$\frac{\partial E_{\text{total}}}{\partial w_1} = 0.03635 \times 0.2413 \times 0.05 = 0.00043$$

# Backpropagation - Backward Pass - Hidden Layer (8/8)

- ▶ We can now update  $w_1$ .
- ▶ Repeating this for  $w_2$ ,  $w_3$ , and  $w_4$ .



$$w_1^{(\text{next})} = w_1 - \eta \times \frac{\partial E_{\text{total}}}{\partial w_1} = 0.15 - 0.5 \times 0.00043 = 0.14978$$

$$w_2^{(\text{next})} = 0.19956$$

$$w_3^{(\text{next})} = 0.24975$$

$$w_4^{(\text{next})} = 0.2995$$



# Summary



# Summary

- ▶ LTU
- ▶ Perceptron
- ▶ Perceptron weakness
- ▶ MLP and feedforward neural network
- ▶ Gradient-based learning
- ▶ Backpropagation: forward pass and backward pass
- ▶ Output unit: linear, sigmoid, softmax
- ▶ Hidden units: sigmoid, tanh, relu



# Reference

- ▶ Ian Goodfellow et al., Deep Learning (Ch. 6)
- ▶ Aurélien Géron, Hands-On Machine Learning (Ch. 10)



# Questions?