Generating function #1

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Enumeration problems

- Trying to determine the number of objects of size n satisfying a certain definition.

- Examples
  - what is the number of permutations of $\{1, 2, \ldots, n\}$?
  - what is the number of binary sequences of length n?

- Generating function is a different way of writing a sequence of numbers.

Why generating function?

- We want to know solutions of various recursion relations.

- There are very hard cases to find out solution without using generating function method.

Basic use of generating function

- $A(x) = \sum_{n\geq 0} a_n x^n = \sum_{n=0}^{\infty} a_n x^n$

: $A(x)$ is a generating function of sequences of numbers a_n.

- If a generating function is well-known, we can infer the sequences of numbers.

- Q. A generating function is $A(x) = (1+x)^k$. Find the sequences of numbers a_n.

Additional use of generating function

- Cauchy product formula
  - If $\left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{m=0}^{\infty} b_m\right) = \sum_{k=0}^{\infty} c_k, \quad c_k = \sum_{l=0}^{k} a_l b_{k-l}.$
  - For generating function case,
    - $\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{m=0}^{\infty} b_m x^m\right) = \sum_{k=0}^{\infty} c_k\, x^k, \quad c_k = \sum_{l=0}^{k} a_l b_{k-l}$
- If $A(x) = \sum_{n\geq 0} a_n x^n$, $C(x) = \sum_{k\geq 0} A(x)^k$. $\;C(x) = \frac{1}{1-A(x)}$.
- Q. Two generating functions are $A(x) = (1+x)^k$, $B(x) = (1+ax)^k$. If $C(x) = A(x)B(x)$, find the sequences of numbers c_n.

Additional use of generating function

- Summation rule (by my own job)

- $f_n = \sum_{m=1}^{n} g_m, \quad F(x) = \sum_{n=0}^{\infty} f_n x^n, \quad G(x) = \sum_{n=0}^{\infty} g_n x^n.$

- $g_n = f_n - f_{n-1}, \quad G(x) = \sum_{n=0}^{\infty} g_n x^n = \sum_{n=0}^{\infty} (f_n - f_{n-1}) x^n. \quad f_{-1} = 0.$

- $G(x) = F(x) - xF(x).$

- $F(x) = \frac{1}{1-x} G(x).$

Additional use of generating function

- Derivative of generating function
- $F(x) = \sum_{n=0}^{\infty} f_n x^n$
- $\frac{\partial}{\partial x} F(x) = \sum_{n=0}^{\infty} n f_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) f_{n+1} x^n$
- $\left(\frac{\partial}{\partial x}\right)^{(m)} F(x) = \sum_{n=0}^{\infty} \frac{(n+m)!}{n!} f_{n+m} x^n$

Additional use of generating function

- Integration of generating function
- $F(x) = \sum_{n=0}^{\infty} f_n x^n$
- $\int F(x)dx = \int \sum_{n=0}^{\infty} f_n x^n \, dx = \sum_{n=0}^{\infty} \frac{1}{n+1} f_n x^{n+1}$
- $\int F(x)(dx)^{(m)} = \sum_{n=0}^{\infty} \frac{n!}{(n+m)!} f_n x^{n+m}$

Basic Problems

1. Fibonacci numbers
2. Random walk

Fibonacci numbers

- The recursion relation : $a_n = a_{n-1} + a_{n-2}$, $a_0 = 1$, $a_1 = 1$
- Generating function : $A(x) = \sum_{n=0}^{\infty} a_n x^n$
- $A(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} (a_{n-1} + a_{n-2}) x^n = \sum_{n=0}^{\infty} a_{n-1} x^n + \sum_{n=0}^{\infty} a_{n-2} x^n$
- $A(x) = \sum_{n=0}^{\infty} a_n x^{n+1} + \sum_{n=0}^{\infty} a_n x^{n+2} = xA(x) + x^2 A(x) + 1$
- $A(x) = \frac{1}{1-x-x^2} = \frac{1}{(1-\phi_+ x)(1-\phi_- x)} = \frac{a_+}{1-\phi_+ x} + \frac{a_-}{1-\phi_- x}$. $\phi_\pm = \frac{1 \pm \sqrt{5}}{2}$, $a_\pm = \frac{1}{2\sqrt{5}}(\pm 1 + \sqrt{5})$
- If $A(x) = \sum_{n \geq 0} a_n x^n$, $C(x) = \sum_{k \geq 0} A(x)^k$. $C(x) = \frac{1}{1-A(x)}$.

Fibonacci numbers

- $A(x) = \frac{1}{1-x-x^2} = \frac{1}{(1-\phi_+ x)(1-\phi_- x)} = \frac{a_+}{1-\phi_+ x} + \frac{a_-}{1-\phi_- x}$. $\quad \phi_\pm = \frac{1 \pm \sqrt{5}}{2}$, $a_\pm = \frac{1}{2\sqrt{5}}\left(\pm 1 + \sqrt{5}\right)$
- $\frac{a_\pm}{1-\phi_\pm x} = a_\pm \sum_{k=0}^{\infty} (\phi_\pm x)^k$
- $A(x) = \sum_{n=0}^{\infty} a_n x^n = a_+ \sum_{k=0}^{\infty} \phi_+^k x^k + a_- \sum_{k=0}^{\infty} \phi_-^k x^k = \sum_{n=0}^{\infty} (a_+ \phi_+^n + a_- \phi_-^n) x^n$
- $a_n = a_+ \phi_+^n + a_- \phi_-^n$.

Random walk

- The recursion relation : $a_{n+1} = a_n + \eta_n,$
- $a_0 = 0$, η : random variables from symmetric gaussian distribution $\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{\eta^2}{2\sigma^2}}$
- Generating function : $A(x) = \sum_{n=0}^{\infty} a_n x^n$
- $A(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} (a_{n-1} + \eta_{n-1}) x^n = xA(x) + x \sum_{n=0}^{\infty} \eta_n x^n$
- $A(x) = \frac{1}{1-x} (x \sum_{n=0}^{\infty} \eta_n x^n)$

Random walk

- $A(x) = \frac{1}{1-x}(x\sum_{n=0}^{\infty}\eta_n x^n)$

- A relation of generating function (one of my useful relations..)
  - $f_n = \sum_{m=1}^{n} g_m,\quad F(x) = \sum_{n=0}^{\infty} f_n x^n,\quad G(x) = \sum_{n=0}^{\infty} g_n x^n.$
  - $g_n = f_n - f_{n-1},\quad G(x) = \sum_{n=0}^{\infty} g_n x^n = \sum_{n=0}^{\infty}(f_n - f_{n-1})x^n.\quad f_{-1} = 0.$
  - $G(x) = F(x) - xF(x).$
  - $F(x) = \frac{1}{1-x}G(x).$

- $B(x) = \sum_{n=0}^{\infty}\eta_{n-1}x^n.\quad \eta_{-1} = 0.$
- $a_n = \sum_{m=1}^{n}\eta_{m-1} = \sum_{m=0}^{n}\eta_m.$ (Trivial relation)

Generating function for study

1. Define your model.

2. From your model, find the recursion relation.

3. Use generating function method.

    1. Express the generating function by using the recursion relation.

    2. Find the sequence of numbers from generating function by using generating function table or power-series expansion or inverse-Laplce transform or inverse-Mellin transform…

    ## If there is another condition of generating function, just use it.
    ## Your own generating functions' library might be good for your future job.

References

- Generating Functions (lecture notes), Michel Goemans
- A short table of generating functions and related formulas, Robert Ziff
- Cauchy product - Wikipedia