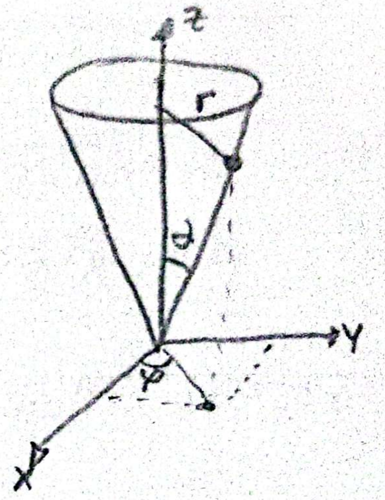
1) calcule la trayectoria que da la distancia mas corta entre dos puntos sobre la superficie de un cono invertido, con ángulo de vertice α.
Use cilindricas



$$I = \int_1^2 ds$$

$$ds^2 = dr^2 + r^2 d\theta^2 + dz^2 \quad ; \quad z = r\,ctg\,\alpha$$

$$dz = dr\,ctg\,\alpha$$

$$= dr^2 + r^2 d\theta^2 + dr^2 ctg^2\alpha$$

$$= dr^2(1 + ctg^2\alpha) + r^2 d\theta^2$$

$$= csc^2\alpha\, dr^2 + r^2 d\theta^2$$

$$= (csc^2\alpha\, \dot{r}^2 + r^2)\, d\theta^2 \quad ; \quad \dot{r} = \frac{dr}{d\theta}$$

$$I = \int_1^2 \sqrt{(csc^2\alpha\, \dot{r}^2 + r^2)}\, d\theta$$

$$\frac{d}{d\theta}\left(\frac{\partial f}{\partial \dot{r}}\right) - \frac{\partial f}{r} = 0$$

$$\frac{d}{d\theta}\left(\frac{csc^2 \dot{r}}{\sqrt{csc^2\alpha\, \dot{r}^2 + r^2}}\right) - \frac{r}{\sqrt{csc^2\alpha\, \dot{r}^2 + r^2}} = 0$$

(2) calcule el valor mínimo de la integral

$$I = \int_0^1 \left[(\dot{y})^2 + 12xy\right] dx$$

donde $y(x) \ni y(0)=0$ y $y(1)=1$

$$f(y,\dot{y},x) = \dot{y}^2 + 12xy$$

$$\frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right) - \frac{\partial f}{\partial y} = 0$$

$$\frac{\partial f}{\partial \dot{y}} = 2\dot{y}$$

$$\frac{\partial f}{\partial y} = 12x$$

$$\Rightarrow \frac{d}{dx}(2\dot{y}) - 12x = 0$$

$$\Rightarrow \frac{d}{dx}\dot{y} - 6x = 0$$

$$\ddot{y} = 6x$$

$$\Rightarrow \int d\dot{y} = \int 6x\,dx$$

$$\dot{y} = 3x^2 + C_1$$

$$\Rightarrow \int dy = \int 3x^2 + C_1\,dx$$

$$y = x^3 + C_1 x + C_2$$

como $y(0)=0 \Rightarrow$

$$y(0) = (0)^3 + C_1(0) + C_2 = 0$$

$$\Rightarrow C_2 = 0 \Rightarrow$$

$$y(x) = x^3 + C_1 x$$

y $y(1)=1 \Rightarrow$

$$y(1) = (1)^3 + C_1(1) = 1$$

$$= 1 + C_1 = 1$$

$$C_1 = 0$$

$$\Rightarrow$$

$$\underline{y(x) = x^3}$$

$$I = \int_0^1 \left[\dot{y}^2 + 12xy\right] dx \qquad \begin{aligned}\dot{y} &= 3x^2\\ y &= x^3\end{aligned}$$

$$\Rightarrow$$

$$I = \int_0^1 9x^4 + 12x^4\,dx$$

$$I = \int_0^1 21x^4\,dx$$

$$I = 21\left(\frac{x^5}{5}\right)\Big|_0^1$$

$$I = \frac{21}{5}$$

3) Encuentre la geodésica entre los puntos $P_1=(a,0,0)$ y $P_2=(-a,0,\pi)$ sobre la superficie $x^2+y^2-a^2=0$. Use cilíndricas

$$I=\int_1^2 ds \qquad ds=\sqrt{dr^2+r^2d\theta^2+dz^2}$$

$r=a \Rightarrow dr=0$

$z=z \qquad x=\cos(\theta)r$

$\cos^{-1}\left(\frac{x}{r}\right)=\theta$

$$ds=\sqrt{a^2d\theta^2+dz^2}$$

$$ds=\sqrt{a^2\dot{\theta}^2+1}\,dz \; ; \; \dot{\theta}=\frac{d\theta}{dz}$$

$\theta_2=\pi$

$\theta_1=0$

$$I=\int_1^2\sqrt{a^2\dot{\theta}^2+1}\,dz$$

$$\frac{d}{dz}\left(\frac{\partial}{\partial\dot{\theta}}\left(\sqrt{a^2\dot{\theta}^2+1}\right)\right)-\frac{\partial}{\partial\theta}\left(\sqrt{a^2\dot{\theta}^2+1}\right)=0$$

$$\frac{d}{dz}\left(\frac{a^2\dot{\theta}}{\sqrt{a^2\dot{\theta}^2+1}}\right)-0=0$$

$$\frac{a^2\dot{\theta}}{\sqrt{a^2\dot{\theta}^2+1}}=c$$

$$a^4\dot{\theta}^2=c^2(a^2\dot{\theta}^2+1)$$

$$\dot{\theta}^2=\frac{c^2}{a^2(a^2-c^2)}$$

$$\dot{\theta}=\frac{c}{a\sqrt{a^2-c^2}}$$

$$\theta=\frac{c_1}{a\sqrt{a^2-c_1^2}}z+c_2$$

$$\theta=\frac{\frac{a^2}{(1-a^2)^{1/2}}}{a\sqrt{a^2-\frac{a^4}{1-a^2}}}z$$

Para P_1 $z=0$; $\theta(0)=0$

Para P_2 $z=\pi$; $\theta(\pi)=\pi$

$$\theta(0)=\frac{c_1}{a\sqrt{a^2-c_1^2}}(0)+c_2=0$$

$$c_2=0$$

$$\theta=\frac{c_1}{a\sqrt{a^2-c_1^2}}z$$

$$\theta(\pi)=\frac{c_1}{a\sqrt{a^2-c_1^2}}\pi=\pi$$

$$=\frac{c_1}{a\sqrt{a^2-c_1^2}}=1$$

$$c_1=a\sqrt{a^2-c_1^2}$$

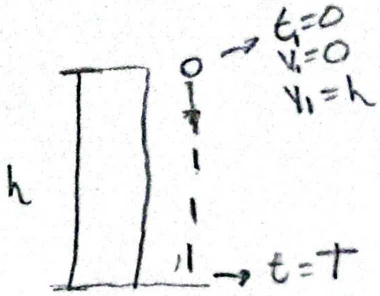
$$c_1^2=a^4-a^2c_1^2$$

$$c_1^2(1-a^2)=a^4 \qquad c_1=\frac{a^2}{(1-a^2)^{1/2}}$$

4) un cuerpo se deja ~~cae~~ caer desde una altura h y alcanza el suelo en un tiempo T. la ecuación de movimiento concebiblemente prodiá tener cualquiera de las formas

$$y = h - g_1 t \quad , \quad y = h - \frac{1}{2} g_2 t^2 \quad , \quad y = h - \frac{1}{4} g_3 t^3$$

g_1, g_2, g_3 = ctte. Demuestre que la forma correcta es aquella que produce el minimo ~~esfuerzo~~ valor de la acción



$$S = \int_0^T L\,dt \Rightarrow L = T - V$$

$$T = \frac{1}{2} m \dot{y}^2$$

$$V = mgy$$

$$L = \frac{1}{2} m \dot{y}^2 - mgy$$

$$S = \int_0^T \frac{1}{2} m \dot{y}^2 - mgy \; dt$$

aplicamos e.L

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0$$

$$\frac{\partial L}{\partial \dot{y}} = m\dot{y}$$

$$\frac{\partial L}{\partial y} = -mg$$

$$\frac{d}{dt}(m\dot{y}) + mg = 0$$

$$m\ddot{y} = -mg$$

$$\ddot{y} = -g$$

$$\dot{y} = -gt + C_1$$

$$y = -\frac{1}{2} g t^2 + C_1 t + C_2$$

como $y(0) = h$ y $\dot{y}(0) = 0 \Rightarrow$

$$\dot{y}(0) = -g(0) + C_1 = 0$$

$$C_1 = 0$$

$$y = -\frac{1}{2} g t^2 + C_2$$

$$y(0) = -\frac{1}{2} g (0)^2 + C_2 = h$$

$$C_2 = h$$

$$y(t) = h - \frac{1}{2} g t^2$$

5) El langrangiano de una particula de masa M es

$$\mathcal{L} = \frac{m^2 \dot{x}^4}{12} + M \dot{x}^2 f(x) - f^2(x)$$

donde $f(x)$ es una función diferenciable de x. Encuentre la ecuación de movimiento

aplicamos e.L

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) - \frac{\partial \mathcal{L}}{\partial x} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{m^2 \dot{x}^3}{3} + 2M \dot{x} f(x)$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = m^2 \dot{x}^2 \ddot{x} + 2m \ddot{x} f(x) + 2m \dot{x} f'(x) \dot{x} \quad ; \; f'(x) = \frac{df}{dx}$$

$$\frac{\partial \mathcal{L}}{\partial x} = m \dot{x}^2 f'(x) - 2 f(x) f'(x)$$

=>

$$m^2 \dot{x}^2 \ddot{x} + 2m \ddot{x} f(x) + 2m \dot{x}^2 f'(x) - m \dot{x}^2 f'(x) - 2f(x) f'(x) = 0$$

$$m^2 \dot{x}^2 \ddot{x} + 2m \ddot{x} f(x) + m \dot{x}^2 f'(x) - 2f(x) f'(x) = 0 \; //$$

6) $L = \frac{1}{2} g_{ab}(q_c) \dot{q}^a \dot{q}^b$

Demostrar que las ecuacione de lagrange para el sistema viene dada por

$$\ddot{q}^a + \Gamma^a_{bc} \dot{q}^b \dot{q}^c = 0$$

con $g_{ab} = g_{ba}$, $g^{ab} g_{bc} = \delta^a_c$, $\Gamma^a_{bc} = \frac{1}{2} g^{ad}\left(\frac{\partial g_{bd}}{\partial q^c} + \frac{\partial g_{cd}}{\partial q^b} - \frac{\partial g_{bc}}{\partial q^d}\right)$

E.L

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^a}\right) - \frac{\partial L}{\partial q^a} = 0$$

① $$\frac{\partial L}{\partial \dot{q}^a} = \frac{1}{2}\frac{\partial}{\partial \dot{q}^a}\left(g_{bc}(q_d)\dot{q}^b\dot{q}^c\right) = \frac{1}{2}\left(g_{bc}\delta^b_a \dot{q}^c + g_{bc}\delta^c_a \dot{q}^b\right)$$
$$= \frac{1}{2}\left(g_{ac}\dot{q}^c + g_{ba}\dot{q}^b\right) = \frac{1}{2}\left(g_{ac}\dot{q}^c + g_{ab}\dot{q}^b\right)$$

② $$\frac{d}{dt}\left(\frac{\partial L}{\dot{q}^a}\right) = \frac{1}{2}\left(\frac{\partial g_{ac}}{\partial q^b}\dot{q}^b\dot{q}^c + g_{ac}\ddot{q}^c + \frac{\partial g_{ab}}{\partial q^c}\dot{q}^c\dot{q}^b + g_{ab}\ddot{q}^b\right)$$
$$= \frac{1}{2}\left(\frac{\partial g_{ac}}{\partial q^b}\dot{q}^b\dot{q}^c + \frac{\partial g_{ab}}{\partial q^c}\dot{q}^b\dot{q}^c + g_{ab}\ddot{q}^b + g_{ab}\ddot{q}^b\right)$$
$$= \frac{1}{2}\frac{\partial g_{ac}}{\partial q^b}\dot{q}^b\dot{q}^c + \frac{1}{2}\frac{g_{ab}}{\partial q^c} + g_{ab}\ddot{q}^b$$

③ $$\frac{\partial L}{\partial q^a} = \frac{1}{2}\frac{\partial}{\partial q^a}\left(g_{bc}(q^d)\dot{q}^b\dot{q}^c\right) = \frac{1}{2}\frac{\partial g_{bc}}{\partial q^d}\delta^d_a\dot{q}^b\dot{q}^c = \frac{1}{2}\frac{\partial g_{bc}}{\partial q^a}\dot{q}^b\dot{q}^c$$

$\Rightarrow$

$$\frac{d}{dt}\left(\frac{\partial L}{\dot{q}^a}\right) - \frac{\partial L}{\partial q^a} = g_{ab}\ddot{q}^b + \frac{1}{2}\frac{\partial g_{ac}}{\partial q^b}\dot{q}^b\dot{q}^c + \frac{1}{2}\frac{g_{ab}}{\partial q^c} - \frac{1}{2}\frac{\partial g_{bc}}{\partial q^a}\dot{q}^b\dot{q}^c = 0$$
$$= g_{ab}\ddot{q}^b + \frac{1}{2}\left(\frac{\partial g_{ac}}{\partial q^b} + \frac{\partial g_{ab}}{\partial q^c} - \frac{\partial g_{bc}}{\partial q^a}\right)\dot{q}^b\dot{q}^c = 0$$

$$\Rightarrow g^{ad} g_{db}\ddot{q}^b + \frac{1}{2}g^{ad}\left(\frac{\partial g_{cd}}{\partial q^b} + \frac{\partial g_{bd}}{\partial q^c} - \frac{\partial g_{bc}}{\partial q^d}\right)\dot{q}^b\dot{q}^c = 0$$

$$\delta^a_b \ddot{q}^b + \Gamma^a_{bc} \dot{q}^b \dot{q}^c = 0$$