Problem Set 6

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2024-09-25

Question 1A

- Polynomial for $x \leq \xi$ $f_1(x)$

For $x \leq \xi$, the term $(x - \xi)^3 = 0$, so the function simplifies to:

$f_1(x) = \beta_0 + \beta_1 x + \beta_2 x^2 + \beta_3 x^3$

Thus, $f_1(x)$ is a cubic polynomial for $x \leq \xi$ with coefficients:

$a_1 = \beta_0$

$b_1 = \beta_1$

$c_1 = \beta_2$

$d_1 = \beta_3$

- Polynomial for $x > \xi$ is $f_2(x)$:

$f_2(x) = \beta_0 + \beta_1 x + \beta_2 x^2 + \beta_3 x^3 + \beta_4 (x - \xi)^3$

The term $\beta_4 (x - \xi)^3$ is:

$\beta_4 (x - \xi)^3 = \beta_4 (x^3 - 3\xi x^2 + 3\xi^2 x - \xi^3)$

The function $f_2(x)$ becomes:

$$f_2(x) = \beta_0 + \beta_1 x + \beta_2 x^2 + \beta_3 x^3 + \beta_4 (x^3 - 3\xi x^2 + 3\xi^2 x - \xi^3)$$

$$f_2(x) = \beta_0 + \beta_1 x + \beta_2 x^2 + (\beta_3 + \beta_4) x^3 - 3\beta_4 \xi x^2 + 3\beta_4 \xi^2 x - \beta_4 \xi^3$$

This will be written as:

$$f_2(x) = (\beta_0 - \beta_4 \xi^3) + (\beta_1 + 3\beta_4 \xi^2) x + (\beta_2 - 3\beta_4 \xi) x^2 + (\beta_3 + \beta_4) x^3$$

So the coefficients of $f_2(x)$ are:

$$a_2 = \beta_0 - \beta_4 \xi^3$$

$$b_2 = \beta_1 + 3\beta_4 \xi^2$$

$$c_2 = \beta_2 - 3\beta_4\xi$$

$$d_2 = \beta_3 + \beta_4$$

The cubic polynomial for $x > \xi$ is:

$$f_2(x) = (\beta_0 - \beta_4\xi^3) + (\beta_1 + 3\beta_4\xi^2)x + (\beta_2 - 3\beta_4\xi)x^2 + (\beta_3 + \beta_4)x^3$$

Question 1B

For continuity at $x = \xi$, the values of $f_1(x)$ and $f_2(x)$ must be equal at $x = \xi$. That is:

$$f_1(\xi) = f_2(\xi)$$

Evaluate $f_1(\xi)$

$f_1(\xi) = \beta_0 + \beta_1\xi + \beta_2\xi^2 + \beta_3\xi^3$

Evaluate $f_2(\xi)$

Evaluate $f_2(\xi) = (\beta_0 - \beta_4\xi^3) + (\beta_1 + 3\beta_4\xi^2)\xi + (\beta_2 - 3\beta_4\xi)\xi^2 + (\beta_3 + \beta_4)\xi^3$

$$f_2(\xi) = \beta_0 + \beta_1\xi + \beta_2\xi^2 + \beta_3\xi^3$$

$$f_2(\xi) = \beta_0 + \beta_1\xi + \beta_2\xi^2 + (\beta_3 + \beta_4)\xi^3$$

The following terms cancel out: β_4 and $\xi_3\xi^3\xi_3$

$$f_2(\xi) = \beta_0 + \beta_1\xi + \beta_2\xi^2 + \beta_3\xi^3$$

$$f_1(\xi) = f_2(\xi)$$

Question 1C

First, compute the first derivative of $f_1(x)$:

$$f_1'(x) = \beta_1 + 2\beta_2 x + 3\beta_3 x^2$$

At $x = \xi$:

$$f_1'(\xi) = \beta_1 + 2\beta_2\xi + 3\beta_3\xi^2$$

The first derivative of $f_2(x)$ is:

$$f_2'(x) = \beta_1 + 2\beta_2 x + 3\beta_3 x^2$$

$$f_2'(x) = \beta_1 + 3\beta_4\xi^2 + 2(\beta_2 - 3\beta_4\xi)x + 3(\beta_3 + \beta_4)x^2$$

At $x = \xi$:

$$f_2'(\xi) = \beta_1 + 3\beta_4\xi^2 + 2(\beta_2 - 3\beta_4\xi)\xi + 3(\beta_3 + \beta_4)\xi^2$$

Simplify:

$$f_2'(\xi) = \beta_1 + 3\beta_4\xi^2 + 2\beta_2\xi - 6\beta_4\xi^2 + 3\beta_3\xi^2 + 3\beta_4\xi^2$$

After combining like terms:

$$f_2'(\xi) = \beta_1 + 2\beta_2\xi + 3\beta_3\xi^2$$

Thus:

$$f_1'(\xi) = f_2'(\xi)$$

This confirms that the first derivative is continuous at $x = \xi$.

Question 1D

The second derivative of $f_1(x)$ is:

$$f_1''(x) = 2\beta_2 + 6\beta_3 x$$

At $x = \xi$:

$$f_1''(\xi) = 2\beta_2 + 6\beta_3\xi$$

The second derivative of $f_2(x)$ is:

$$f_2''(x) = 2(\beta_2 - 3\beta_4\xi) + 6(\beta_3 + \beta_4)x$$

At $x = \xi$:

$$f_2''(\xi) = 2(\beta_2 - 3\beta_4\xi) + 6(\beta_3 + \beta_4)\xi$$

Simplifying:

$$f_2''(\xi) = 2\beta_2 - 6\beta_4\xi + 6\beta_3\xi + 6\beta_4\xi$$

The β_4 terms cancel out, so:

$$f_2''(\xi) = 2\beta_2 + 6\beta_3\xi$$

Thus:

$$f_1''(\xi) = f_2''(\xi)$$

This confirms that the second derivative is continuous at $x = \xi$.

Question 2A

$$\lambda = \infty, m = 0$$

- When $\lambda = \infty$, the penalty term dominates. In this case, since $m = 0$, the penalty applies directly to $g(x)$ itself, implying that $g_b(x)$ must be constant. This is because the only way to minimize the penalty term $\int [g(x)]^2 dx$ when λ is very large is for $g(x)$ to be a constant function.
- **Result:** $g_b(x)$ is a constant function, likely the mean of the y_i's.

Question 2B

$$\lambda = \infty, m = 1$$

- With $\lambda = \infty$ and $m = 1$, the penalty applies to the first derivative $g'(x)$. To minimize the penalty, $g'(x) = 0$, meaning $g(x)$ must be a constant function again.
- **Result:** $g_b(x)$ is constant, same as in part (a).

Question 2C

$$\lambda = \infty, m = 2$$

- For $\lambda = \infty$ and $m = 2$, the penalty applies to the second derivative $g''(x)$. Minimizing the penalty forces $g''(x) = 0$, meaning $g_b(x)$ must be a linear function (since a function with zero second derivative is a straight line).
- **Result:** $g_b(x)$ is a linear function, $g_b(x) = ax + b$, where a and b are constants.

Question 2D

$$\lambda = \infty, m = 3$$

- For $\lambda = \infty$ and $m = 3$, the penalty applies to the third derivative $g^{(3)}(x)$. Minimizing the penalty forces $g^{(3)}(x) = 0$, meaning $g_b(x)$ must be a quadratic function (since a function with zero third derivative is a quadratic polynomial).
- **Result:** $g_b(x)$ is a quadratic function, $g_b(x) = ax^2 + bx + c$, where a, b, and c are constants.

Question 2E

$$\lambda = \infty, m = 2$$

- When $\lambda = 0$, there is no penalty for roughness, so $g_b(x)$ will minimize the first term in the loss function alone. This would make $g_b(x)$ exactly interpolate the data points, meaning it passes through every (x_i, y_i) pair. Since $m = 2$, we would expect a smooth curve, but the penalty term doesn't influence the result when $\lambda = 0$.
- **Result:** $g_b(x)$ will interpolate the data points and can be any flexible, non-penalized smooth curve.

Question 3A

```
library(ISLR)
```

```
## Warning: package 'ISLR' was built under R version 4.4.1
```

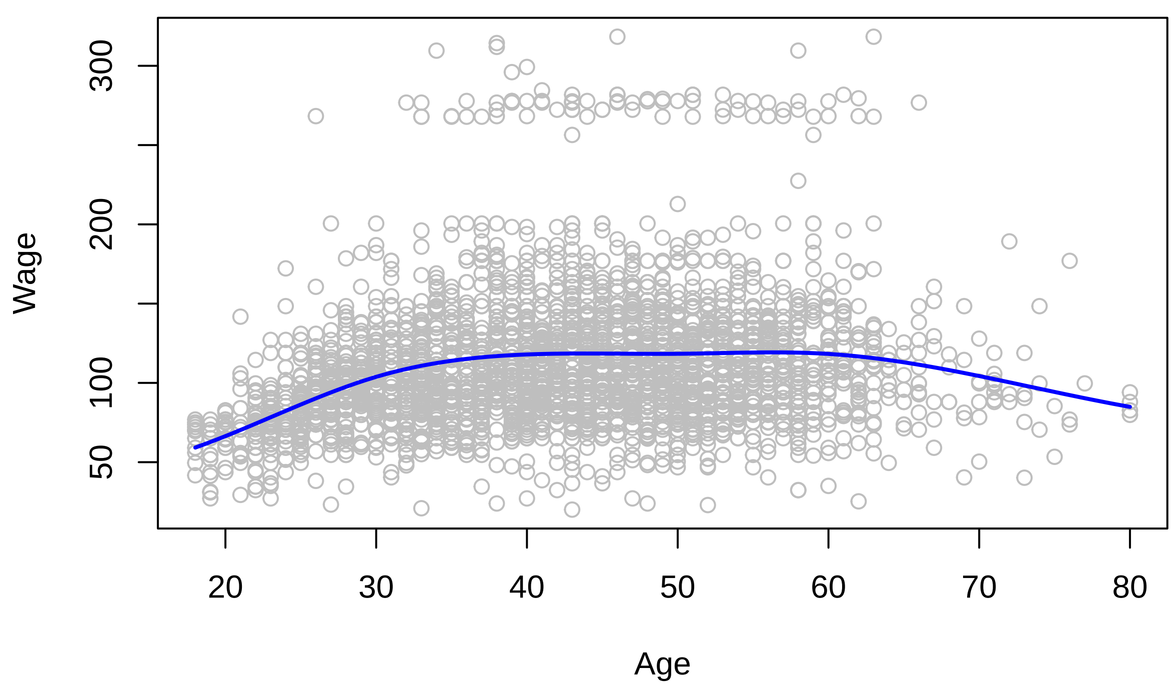
```
library(splines)

# Load the Wage dataset
data(Wage)

# Fit the model using cubic splines with knots at 30, 50, and 60
fit_cubic_spline <- lm(wage ~ bs(age, knots = c(30, 50, 60), degree = 3), data = Wage)

# Plot the data
plot(Wage$age, Wage$wage, col = "gray", xlab = "Age", ylab = "Wage")

# Add the fitted spline curve to the plot
age_grid <- seq(min(Wage$age), max(Wage$age), length.out = 100)
wage_preds <- predict(fit_cubic_spline, newdata = list(age = age_grid))
lines(age_grid, wage_preds, col = "blue", lwd = 2)
```



```r
predict(fit_cubic_spline, newdata = list(age = 30))
```

```
##        1
## 103.8223
```

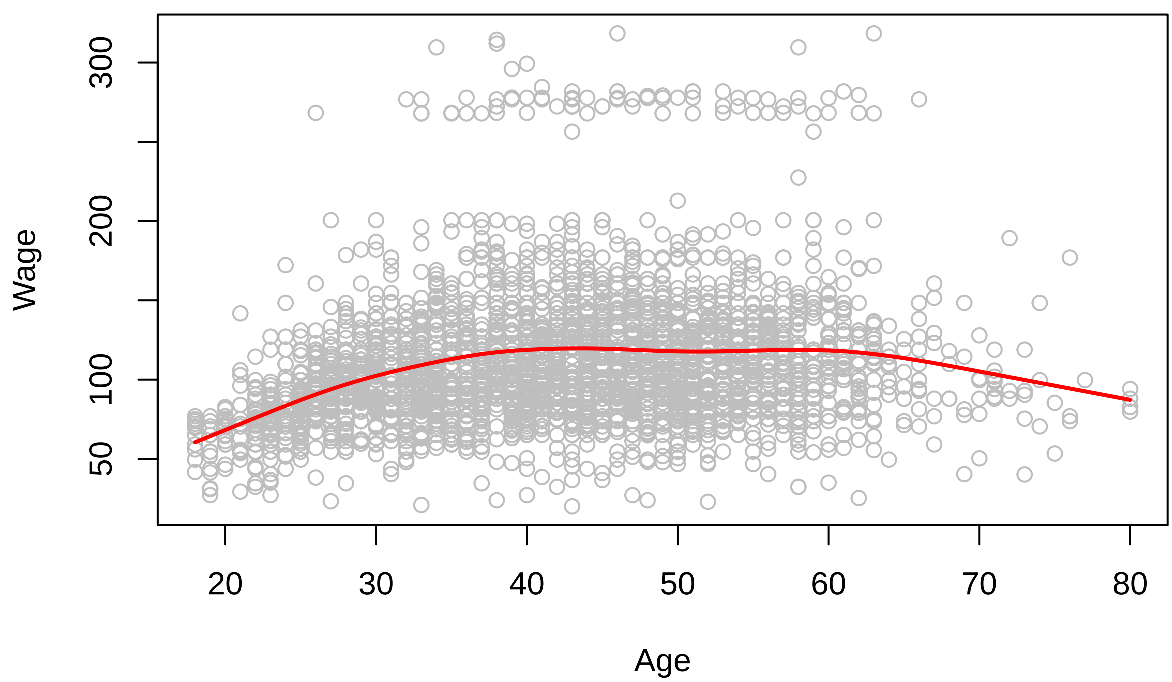
Question 3B

```r
# Fit the smoothing spline with cross-validation
fit_smoothing_spline <- smooth.spline(Wage$age, Wage$wage, cv = TRUE)
```

```
## Warning in smooth.spline(Wage$age, Wage$wage, cv = TRUE): cross-validation with
## non-unique 'x' values seems doubtful
```

```r
# Plot the data
plot(Wage$age, Wage$wage, col = "gray", xlab = "Age", ylab = "Wage")

# Add the fitted smoothing spline curve
lines(fit_smoothing_spline, col = "red", lwd = 2)
```




```r
predict(fit_smoothing_spline, x = 30)$y
```

```
## [1] 102.3999
```

Question 3C

```r
library(gam)
```

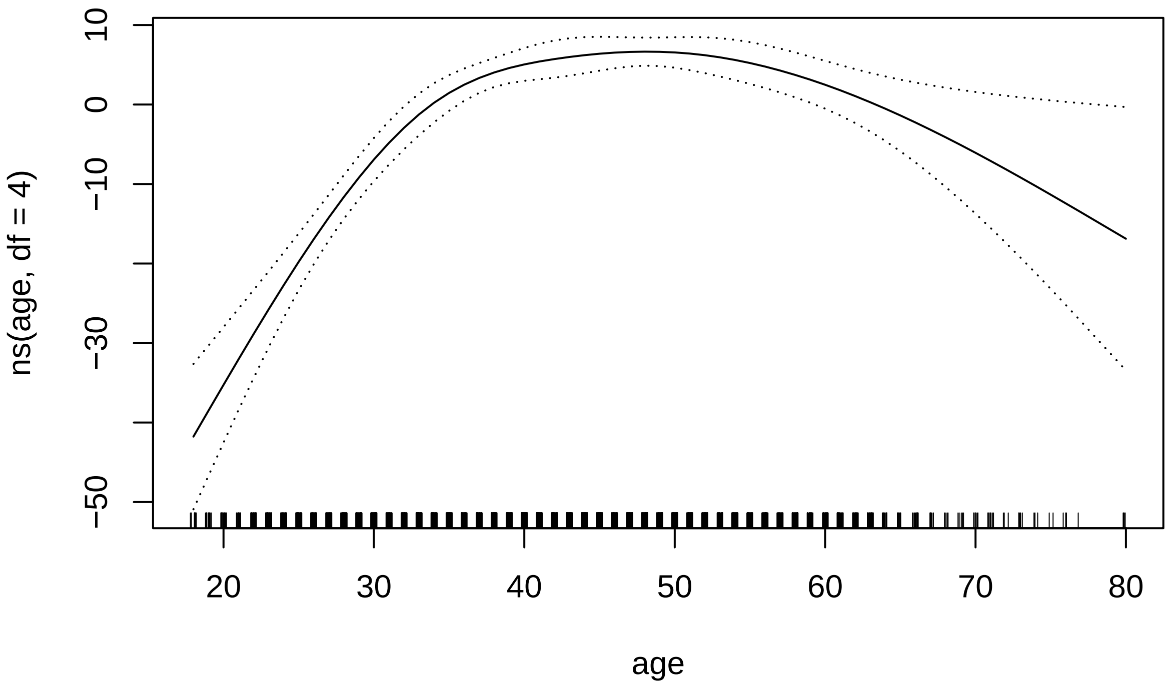
```
## Warning: package 'gam' was built under R version 4.4.1
```

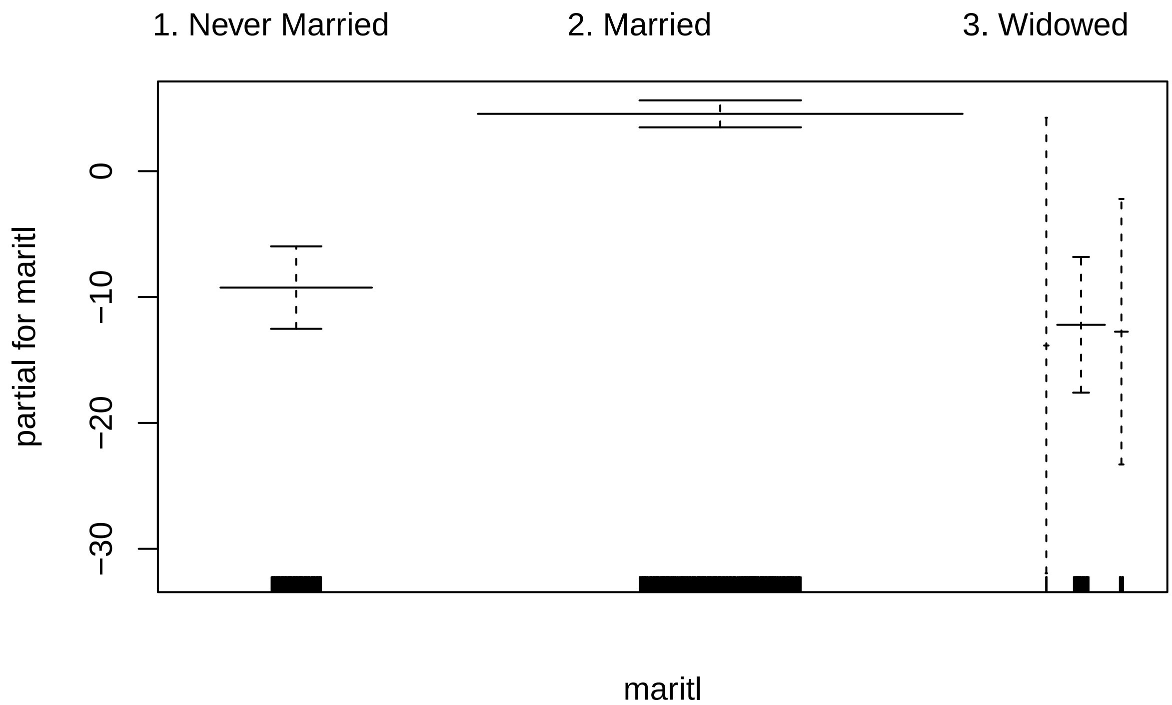
```
## Loading required package: foreach
```

```
## Loaded gam 1.22-5
```

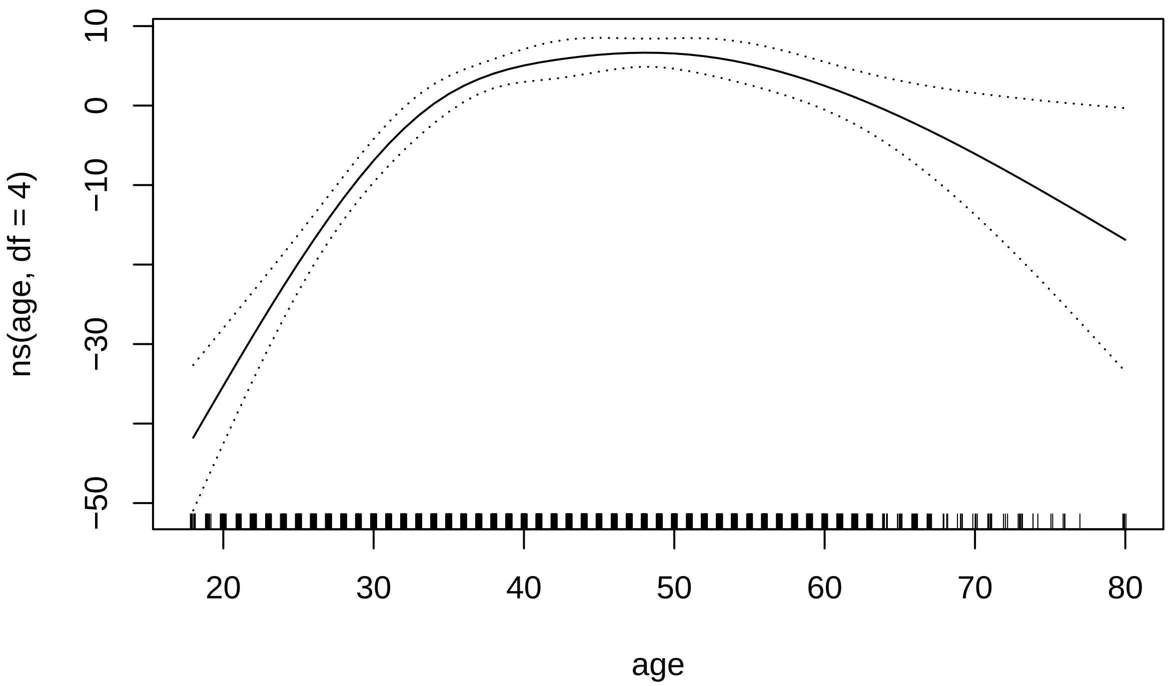
```r
# Fit the GAM model
fit_gam <- gam(wage ~ ns(age, df = 4) + maritl, data = Wage)

# Plot the effects of the GAM model
plot(fit_gam, se = TRUE)
```

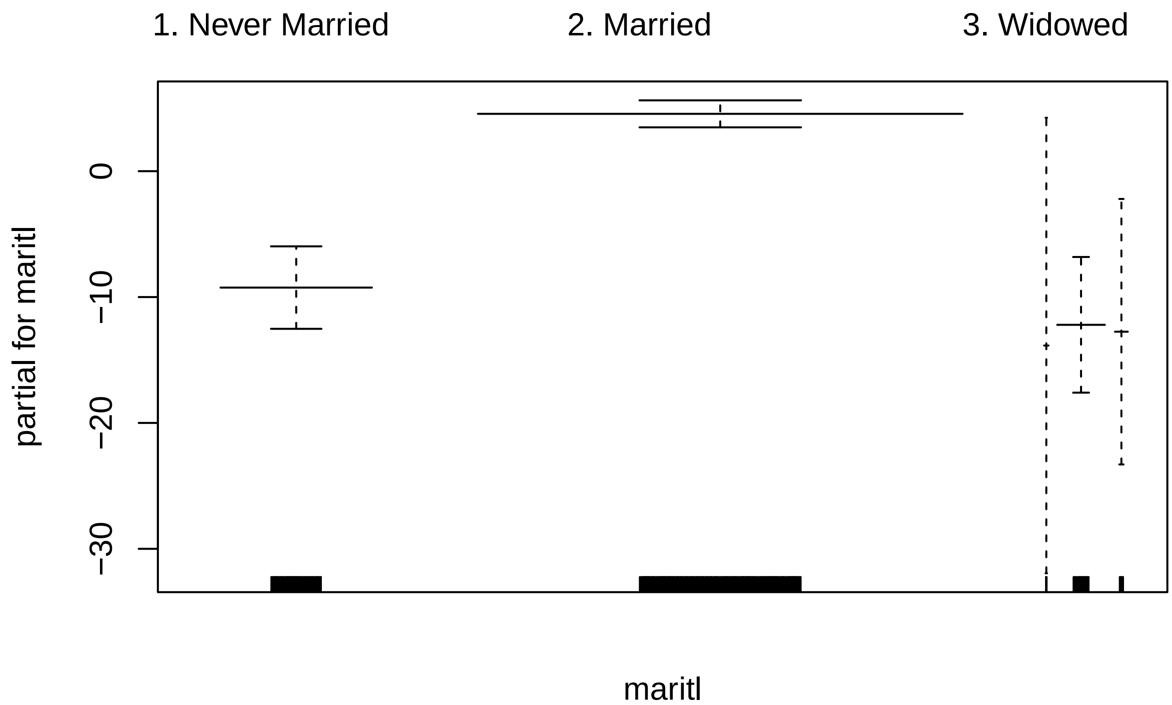





```r
# Plot the effects of the GAM model
plot(fit_gam, se = TRUE)
```



10
0
−10
−30
−50
ns(age, df = 4)
20
30
40
50
60
70
80
age




```r
predict(fit_gam, newdata = data.frame(age = 30, maritl = "2. Married"))
```

```
##       1
## 109.339
```