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TIC Santé 2A, M1 IRIV ASI
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An introduction to optimization

Exercises

Part 1.

1.1 Consider the problem

$$\begin{aligned} \min_{\mathbf{x}} \quad & 3{x_1}^2 + 0.5{x_2}^2 + 2x_2 + 2 \\ \text{s.t.} \quad & x_1, x_2 \geq 0. \end{aligned}$$

Verify whether the FONC are satisfied at the following points : $[1 \;\; 2]^T$, $[0 \;\; 3]^T$, $[1 \;\; 0]^T$ and $[0 \;\; 0]^T$.

1.2 Find the saddle points, local minimizers and local maximizers (if they exist) of the following functions :

$$\begin{aligned} f_1(\mathbf{x}) &= 4.5 - x_1 + 2x_2 - 0.5{x_1}^2 - 2{x_2}^2 \\ f_2(\mathbf{x}) &= 2{x_1}^3 + x_1{x_2}^2 + 5{x_1}^2 + {x_2}^2 \end{aligned}$$

1.3 Investigate (using two different methods) whether $\mathbf{d} = [2 \;\; -1]^T$ is a descent or an increase direction at $\mathbf{x}_0 = [1 \;\; 1]^T$ with respect to the function

$$f(\mathbf{x}) = {x_1}^2 + x_1x_2 - 4{x_2}^2 + 5$$

1.4 Solve the following unconstrained quadratic programming problem :

$$\begin{aligned} \min_{\mathbf{x}} \quad & f(\mathbf{x}) = (x_2 + x_1 - 3)^2 + 2(x_2 - x_1 + 1)^2 \\ \text{s.t.} \quad & \mathbf{x} \in \mathbb{R}^2 \end{aligned}$$

In order to verify your answer (graphically), plot using Matlab the level sets or the graph of the cost function in the domain :

$$\begin{aligned} -5 \; \leq & \; x_1 \leq \; 5 \\ -5 \; \leq & \; x_2 \leq \; 5 \end{aligned}$$

Part 2.

2.1 Write a Matlab routine to solve the problem :

$$\begin{aligned} \min_{x} \quad & f(x) = x^4 + 4x^3 + 9x^2 + 6x + 6 \\ \text{s.t.} \quad & x \in [-2, 2] \end{aligned}$$

using the Golden section with a tolerance $\epsilon = 10^{-2}$.

2.2 Write a Matlab routine to solve the problem :

$$\begin{aligned} \min_{x} \quad & f(x) = 2x^4 - 5x^3 + 100x^2 + 30x - 75 \\ \text{s.t.} \quad & x \in \mathbb{R} \end{aligned}$$

with a stopping criterion $|\frac{df}{dx}(x_k)| \leq 10^{-4}$ using :

1. Newton's method, with an initial condition $x_0 = 2$
2. Secant method, with initial conditions $x_0 = 2.1$ and $x_1 = 2$

Part 3.

3.1 Consider the following problem

$$\begin{aligned} \min_{\mathbf{x}} \quad & f(\mathbf{x}) = 1 + 2x_1 e^{-{x_1}^2 - {x_2}^2} \\ \text{s.t.} \quad & \mathbf{x} \in \mathbb{R}^2 \end{aligned}$$

Write a Matlab routine to solve the problem using the steepest descent method. Consider the stopping criterion $\|\nabla f(\mathbf{x}_k)\| < \epsilon = 10^{-3}$. For the initial condition $\mathbf{x}_0$ consider the cases $[-0.5 \;\; 0.5]^T$, $[0.5 \;\; -0.5]^T$ and $[1 \;\; 1]^T$.
Solve the line search problem analytically.

3.2 Consider the problem of finding the minimizer of Rosenbrock's function :

$$\begin{aligned} \min_{\mathbf{x}} \quad & f(\mathbf{x}) = 100(x_2 - {x_1}^2)^2 + (1 - x_1)^2 \\ \text{s.t.} \quad & \mathbf{x} \in \mathbb{R}^2 \end{aligned}$$

1. Show analytically that $[1 \;\; 1]^T$ is the unique global minimizer.
2. Implement the steepest descent method with a stopping criterion $\|\mathbf{x}_{k+1} - \mathbf{x}_k\| / \|\mathbf{x}_k\| < \epsilon_1 = 10^{-3}$, starting from $\mathbf{x}_0 = [-1 \;\; 2]^T$.
   At each iteration, use Newton's method for the line search, with initial value $\alpha_0 = 0.1$, and a stopping criterion $\frac{|\alpha_{k+1} - \alpha_k|}{\alpha k} < \epsilon_2 = 10^{-3}$.

3.3 Consider the problem

$$
\begin{aligned}
\min_{\mathbf{x}} \quad & f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T Q \mathbf{x} + \mathbf{q}^T \mathbf{x} \\
\text{s.t.} \quad & \mathbf{x} \in \mathbb{R}^2
\end{aligned}
$$

for the cases :

$$Q = \lambda I,\ \lambda > 0, \qquad \forall \mathbf{q} \in \mathbb{R}^2 \qquad \forall \mathbf{x}_0 \in \mathbb{R}^2$$

$$Q = \begin{bmatrix} 10 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \mathbf{q} = \begin{bmatrix} -3 \\ -3 \end{bmatrix}, \qquad \mathbf{x}_0 = \begin{bmatrix} -2 \\ -7 \end{bmatrix}$$

$$Q = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \mathbf{q} = \begin{bmatrix} -3 \\ -3 \end{bmatrix}, \qquad \mathbf{x}_0 = \begin{bmatrix} -2 \\ -7 \end{bmatrix}$$

1. Rank the three cases in term of speed of convergence when applying the steepest descend method.
2. Write a Matlab routine to implement the steepest descend method and the conjugate gradient method, and compare their performance.

3.4 Consider the function

$$f(\mathbf{x}) = \frac{5}{2}{x_1}^2 + \frac{1}{2}{x_2}^2 + 2x_1x_2 - 3x_1 - x_2$$

1. Express the function in a standard quadratic form.
2. Write the steps of the conjugate gradient algorithm to find the minimizer of $f(\cdot)$ starting from $\mathbf{x}_0 = [0\ \ 0]^T$.

3.5 Using the DFP algorithm, we want to find the solution to the following problem

$$
\begin{aligned}
\min_{\mathbf{x}} \quad & f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T \underbrace{\begin{bmatrix} 2 & 2 \\ 2 & 10 \end{bmatrix}}_{Q} \mathbf{x} + \underbrace{[2\ \ 0]}_{\mathbf{q}^T} \mathbf{x} \\
\text{s.t.} \quad & \mathbf{x} \in \mathbb{R}^2
\end{aligned}
$$

1. Find the formula for α_k in terms of Q, $\mathbf{g}_k = \nabla f(\mathbf{x}_k)$ and $\mathbf{d}_k$.
2. Implement the algorithm using Matlab starting from $\mathbf{x}_0 = [0\ \ 0]^T$.

Part 4.

4.1 Let $A \in \mathbb{R}^{m\times n}$, $\mathbf{b} \in \mathbb{R}^m$ with $m \leq n$, rank $A = m$ and $\mathbf{x}_0 \in \mathbb{R}^n$. Consider the problem

$$\begin{array}{ll} \arg\min\limits_{\mathbf{x}} & \|\mathbf{x} - \mathbf{x}_0\|^2 \\ \text{s.t.} & A\mathbf{x} = \mathbf{b} \end{array}$$

Show that the problem has the following unique solution

$$\mathbf{x}^\star = A^T(AA^T)^{-1}\mathbf{b} + (\mathbf{I} - A^T(AA^T)^{-1}A)\mathbf{x}_0$$

4.2 Plot the curves corresponding to the following equations

$$\begin{aligned} 4{x_2}^2 &= 20 - {x_1}^2 \\ x_2 &= {x_1}^4 - 10 \end{aligned}$$

1. Formalize a least square optimization problem which permits to find the intersections of the previous two curves.
2. Implement a Gauss-Newton method to solve the problem and run it for different initial conditions.

4.3 We want to find the parameters of a process that has an output which is linear in time. We conduct $m \geq 2$ measurements $\{y_1, \dots, y_m\}$ at different instants $\{t_1, \dots, t_m\}$, and we want to find the line $y = a^\star t + b^\star$ which has the least squared error with respect to the measurements. That is, we want to find $a^\star$ and $b^\star$ such that

$$\begin{array}{lll} [a^\star \;\; b^\star]^T & = \arg\min\limits_{\mathbf{x}} & F(\mathbf{x}) = \sum_{i=1}^{m}(y_i - at_i - b)^2 \\ & \text{s.t.} & \mathbf{x} \in \mathbb{R}^2 \end{array}$$

Find analytically the solution $[a^\star \;\; b^\star]^T$. Write a Matlab routine to implement the solution for the data :

t	y
1.0779	6.9959
1.4268	9.4782
2.1801	12.0585
2.3138	14.5837
2.9755	17.1019
3.4526	19.4104
4.1062	22.0719
4.6181	24.5620
4.8747	26.8650

(1)

4.4 Consider the set of m perturbed measurements of a sinusoidal signal $\{y_1, \ldots, y_m\}$ at different instants $\{t_1, \ldots, t_m\}$ in (2). We want to fit a sinusoidal signal to the measured data :

$$y = a \sin(\omega t + \phi)$$

by solving a nonlinear least-squares problem

$$\begin{aligned} \arg\min_{\mathbf{x}} \quad & F(\mathbf{x}) = \sum_{i=1}^{m} f_i(\mathbf{x})^2 \\ \text{s.t.} \quad & \mathbf{x} = [a \;\; \omega \;\; \phi]^T \in \mathbb{R}^3 \end{aligned}$$

1. Find the expression of $f_i(\mathbf{x})$ in the objective function.
2. Using Matlab, find a solution to the fitting problem using Gauss-Newton method. Consider $\mathbf{x}_0 = [0.5 \; 1.25 \; 0.1]^T$.

t	y
0	0.1128
0.2300	0.0876
0.4800	0.3971
0.7300	0.5766
0.9800	0.8538
1.2300	0.7856
1.4800	0.9596
1.7300	0.8259
1.9800	0.7542
2.2300	0.8918
2.4800	0.5151
2.7300	0.4231
2.9800	0.3269
3.2300	-0.1352
3.4800	-0.2729
3.7300	-0.4683
3.9800	-0.7837
4.2300	-0.8328
4.4800	-1.1336
4.7300	-0.8258
4.9800	-1.0839
5.2300	-0.9525
5.4800	-0.5864
5.7300	-0.5132
5.9800	-0.1718
6.2300	-0.0748

(2)

Part 5.

5.1 We want to find the maximum of the function

$$f(\mathbf{x}) = x_1 + 2x_2 + x_3$$

under the constraints

$$\begin{aligned} -3x_1 + x_2 + 2x_3 &\leq 2 \\ -5x_1 + x_2 + x_3 &\geq 1 \\ x_1 + x_2 + 3x_3 &\leq 5 \\ x_i &\geq 0 \qquad i = 1, 2, 3 \end{aligned}$$

Solve the problem using the simplex method. Give the maximizer $\mathbf{x}^\star$, and the maximum value $f(\mathbf{x}^\star)$.

5.2 Consider the following problem : A farmer has an area of 120 hectares. He cultivates beets, corn and wheat. Beets cultivation yields 474 Euros per hectare and requires 10 hours of use of a tractor per hectare. Corn cultivation yields 774 Euros per hectare, but requires 30 hours of tractor per hectare. Finally, wheat cultivation yields 645 Euros per hectare and requires 30 hours of tractor per hectare.

The tractor is available for a maximum of 2500 hours a year and for storage reasons, the corn surface may not exceed 1/4 of the total surface, we seek the optimal distribution of the surfaces dedicated to the three kinds of crops that maximizes the farmer's profit.

1. Formulate the problem as a linear programming problem.
2. Solve the problem using the simplex method.