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MATHS INFO PROJECT

Class of 2019

1st YEAR

Curious Oscillators

February 10 , 2017

Summary :

I. First Oscillator :

1. Theoretical study of the problem

1.1 Operating principle

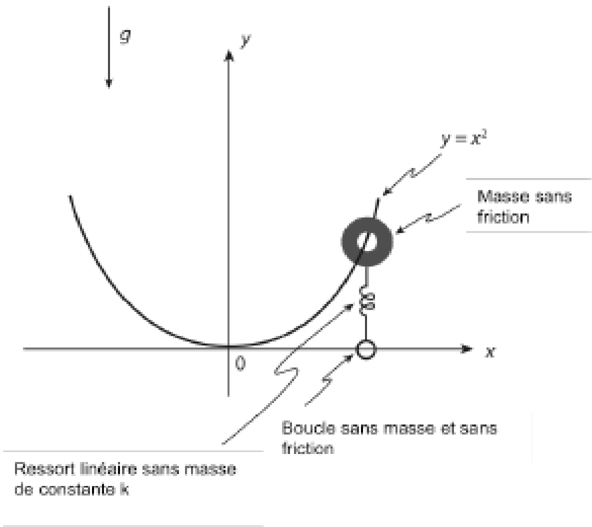

Fig 1 – System diagram

This oscillator consists in a mass **m** without friction attached to a rope shaped in the form of a parabola (as in $y = x^2$ equation). A spring without mass which rigidity is worth **k** linked to a buckle without any mass or friction is also attached to the first mass.
To launch the movement of oscillation, the mass is pulled to the right and then released at the moment t=0.

1.2 Mathematical model:

We place ourselves in a terrestrial reference frame considered as Galilean. We will study the movement of the mass on both horizontal and vertical plans.
We then set **x** as the horizontal movement and **y** as the vertical one.

To resolve this problem and calculate the movement of the mass, the equation characterizing the latter need to be solved. To write this equation we used energy theorems as no friction means no loss of energy throughout the movement and thus, leading to the conservation of the mechanical energy.
Mechanical energy being the sum of kinetic and potential energies, we calculated those two to get the equation we needed.

For the kinetic energy, we used the following formula : $Ec = \frac{1}{2}mv^2$

For the potential energy, it is the sum of the elastic potential energy of the spring and the gravitational potential energy : $Ep = Ep_g + Ep_{el} = mgy + \frac{1}{2}ky^2$

Using those two energies and the fact that the mechanical energy is the same at the beginning and the end of the movement, we deduced the following equation:

$$Em(t) = \frac{1}{2} m \left(\frac{dx}{dt}\right)^2 \left(1 + 4x^2\right) + \frac{1}{2} ky^2 + mgy \ (Eq.1)$$

According to the mechanical energy theorem, we have $\frac{dEm}{dt} = 0$ and thus by differentiating the previous equation we got the one we searched for.

1.3 Resolution

1.3.1 General case

With the current state of that one, it is impossible for us or even for a computer to solve it because it has two unknown variables. We then proceeded to set down that $y = x^2$ to suppress one of the variables:

$$\boxed{\left(1 + 4x^2\right)\frac{d^2x}{dt^2} + 4x\left(\frac{dx}{dt}\right)^2 + 2x\left(\frac{k}{m}x^2 + g\right) = 0 \ (Eq.2)}$$

Afterwards we needed to transform the equation to make it resolvable by RungeKutta method and for this purpose, we set down that $y1 = x$ and $y2 = \frac{dx}{dt}$

This ensues in a system of linear differential equations : $Y(t) = \begin{pmatrix} y1 \\ y2 \end{pmatrix} = \begin{pmatrix} x(t) \\ \frac{dx}{dt} \end{pmatrix} = \begin{pmatrix} y1 \\ \frac{dy1}{dt} \end{pmatrix} \ (Eq.3)$

1.3.2 Harmonic case

We also wanted to check if the solution could be sinusoidal so we then proceeded to set that $x = A\cos(\omega t)$. By replacing x and its differentiates by this value in(Eq .2) , we obtained the following equation: $x\left(-\omega^2 + 4A^2\omega^2 + 2g\right) + x^3\left(-8\omega^2 + 2\frac{k}{m}\right) = 0 \ (Eq.4)$

Then by writing out that each member of that equation is worth 0, we could solve it and get the initial conditions in a case of a harmonic solution.

Therefore we deduced the proper pulsation $\boxed{\omega = \frac{1}{2}\sqrt{\frac{k}{m}}}$ and the amplitude $\boxed{A = \sqrt{\frac{1}{4} - 2g\frac{m}{k}}}$

2. Simulation program

2.1 Runge-Kutta method

In order to solve the system of differential equations (Eq.3) presented in section **1,** we chose to use 4th order RungeKutta method. We also decided to work with a constant step, which means that each value is calculated at a settled interval.
This method was coded using the software Python.

2.2 Position of the problem

As we cannot enter multiple equations at once in our program, we set down a vector Y such as $Y=(y1,y2)$ and $\frac{dY}{dt}=f(Y)=(\frac{dy1}{dt},\frac{dy2}{dt})$

$\dot{Y}$ is then fully defined with the vector Y and the values of **k** and **m** chosen for example: **k**=94 $N.m^{-1}$ and **m**=1 kg

The RK4 method then allows when given a state of Y at a given time t_i to obtain the state of the same Y at the time t_{i+1}

For each step, in order to find Y_{i+1} with Y_i , RK4 calculates 4 vectors :

$$\begin{cases} k1 = f(Y) \\ k2 = f(Y + \frac{h}{2}k1) \\ k3 = f(Y + \frac{h}{2}k2) \\ k4 = f(Y + hk3) \end{cases}$$

With h being the temporal step

And then it applies the following relation: $Y_{i+1}=Y_i+\frac{h}{6}(k1+2k2+2k3+k4)$

In order to obtain every point, this program is looped as much times as needed.

3. Results and analysis

3.1 Harmonic case

We first wanted to check if the solution could be sinusoidal in the case of a harmonic solution with the initial conditions previously found in section **1**.
We ended up with the same analytical solutions. The system oscillates in a sinusoidal way which doesn't change as time passes.
As we wanted to be sure that this solution was correct, we compared the results of our RK4 program and the result of a method of resolution already included in Python called odeint.

Figures

Appendix A.1, page 10

It is clear that it is the exact same graph in both cases, which allowed us to conclude that the harmonic solution was correct .

3.2 General case

We then wanted to check if the general solution was the same as the harmonic solution, so we compared them as well for particular expressions of A and w proposed previously .

Figure

Appendix A.1, page 11

It appears that the chart is the same .

3.3 Phase portrait

One more thing that we wanted to check was the trajectory of the system so we drew up the speed of the mass in function of its position.
This resulted in an elliptical trajectory .

Figure

Appendix A.1, page 12

II. Second oscillator :

1. Theoretical study of the problem

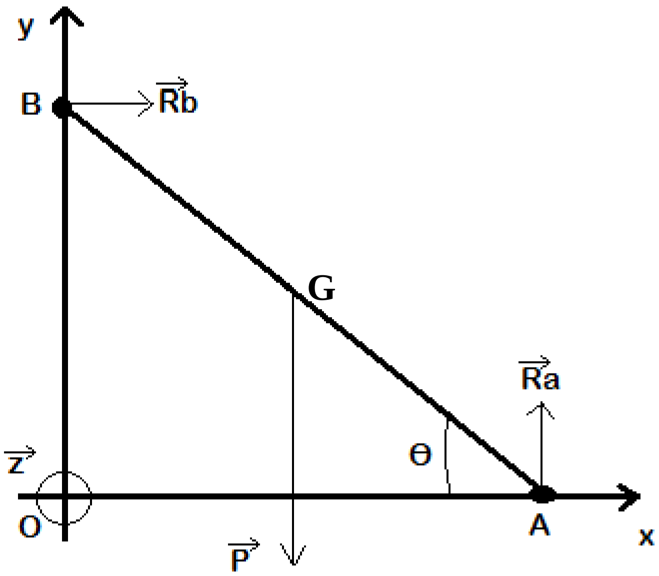

Fig 2 – System diagram

1.1 Operating principle

The second oscillator consists in a stiff shaft of length **a** without any mass linking two masses **m**. As the first oscillator, there is neither friction nor energy loss, we therefore proceeded the same way with the energy theorems excluding the fact that this oscillator posses an inertia moment due to its movement of rotation.

1.2 Mathematical model

Once more, we place ourselves in a terrestrial reference frame considered as Galilean. We will study the movement of the mass on both horizontal and vertical plans set as **x** and **y**. We also set θ as the angle between the shaft and the x axis.

Let **M** be the total mass such as $M = 2m$**.** If we consider the shaft and the two masses as a unique homogeneous solid material of mass **M**, we can calculate its inertia moment:

$$\boxed{Ig = \frac{Ma^2}{3}}$$

To get balance, there needs to be friction in A to compensate the reaction in B.

If we set G as the center of mass of the rod, then we got that : $\overrightarrow{OG} = \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB})$.

Thus: $\begin{cases} x_g &= a\cos\theta \\ y_g &= b\sin\theta \end{cases}$

This leads to xg² + yg² = a² which is the equation of a circle. The trajectory of G is therefore a circle of radius **a** which center is Θ .
As the system evolves, θ goes down, which means that it rotates in an anticlockwise direction.
Consequently, we got the following vector of rotation: $\vec{\Omega} = -\dot{\theta}\vec{ez}$

Then as the first oscillator, we need to calculate the kinetic and potential energies to apply the theorem of the mechanical energy.
For the kinetic energy, it is the sum of both rotation and translation kinetic energies:

$$Ec = Ec_r + Ec_t = \frac{1}{2}\frac{Ma^2}{3}\dot{\theta}^2 + \frac{1}{2}Mv_G{}^2 = \frac{2}{3}Ma^2\dot{\theta}^2$$

With $\vec{v_g} = \vec{\Omega} \times \overrightarrow{OG} = a\dot{\theta}(\sin\theta\,\vec{ex} - \cos\theta\,\vec{ey})$
For the potential energy, only the gravitational potential energy is present: $Ep = Mga\sin\theta$
As the mechanical energy is the same throughout the movement, we could deduce the following equation which describes the movement of the solid:

$$\boxed{\ddot{\theta} + \frac{3}{4}\frac{g}{a}\cos\theta = 0}$$

1.3 Resolution

The equation we previously got is this time directly useable in RK4 method as there is no need for setting down other variables or transforming it in a system of linear differential equations. We then proceeded to the informatics part.

2. Simulation program

2.1 Runge-Kutta method

As said above, we used the same method to solve this equation we did for the first oscillator. The method being fully explained in section **2** part **I**, we will not detail its principle and rather directly present the results as there were no particular difficulties to implement the equation in the program this time.

3. Results and analysis

3.1 General case study

As we did for the first oscillator, we wanted to check if the solution obtained with our RK4 program was the same as the one obtained with odeint.
This time we also added the Euler method of differential equation resolution and while RK4 and odeint match perfectly, Euler starts by matching with them both but then, as time passes, it tends to split apart.

Figures

Appendix A.2, page 13-14

The charts clearly show the difference between Euler and the rest. The latter is due to the step used with Euler method, since it is known that the error committed at each step is of the order of $O(h^2)$.
It is clear that one can increase the precision of this method by decreasing the size of h,

3.2 Phase portrait

Even though we settled at the beginning in the mathematical model that the trajectory of the center of mass of the system would be a circle, we still wanted to check the trajectory of the full system. Thus, we drew up the phase portrait which appears, as the first oscillator, to be elliptical.

Figure

Appendix A.2, page 15

A. Appendix

A.1 First oscillator

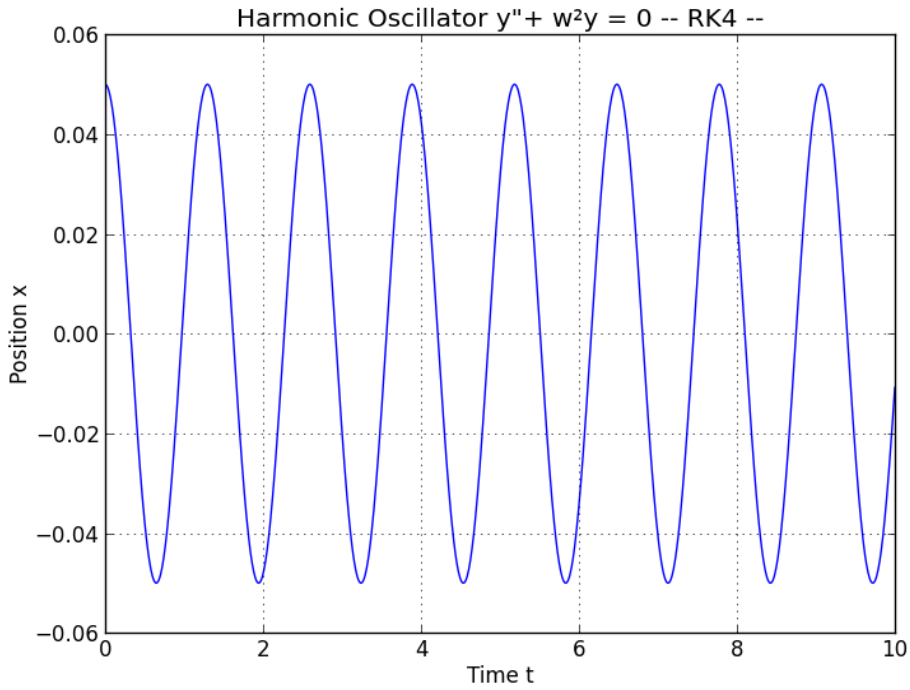

Fig A.1.1- Harmonic solution

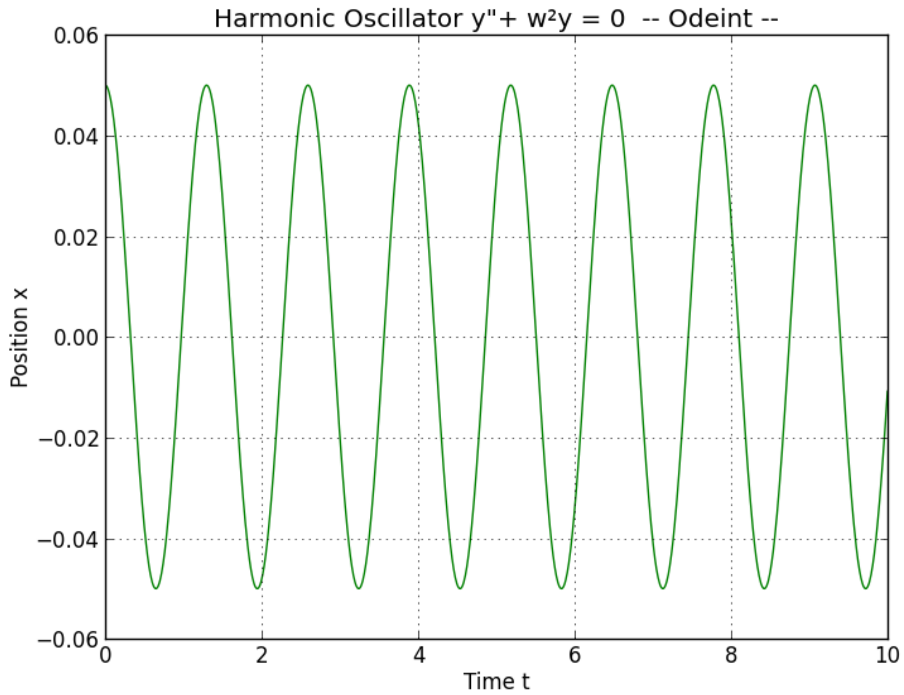

Fig A.1.2 – Odeint solution

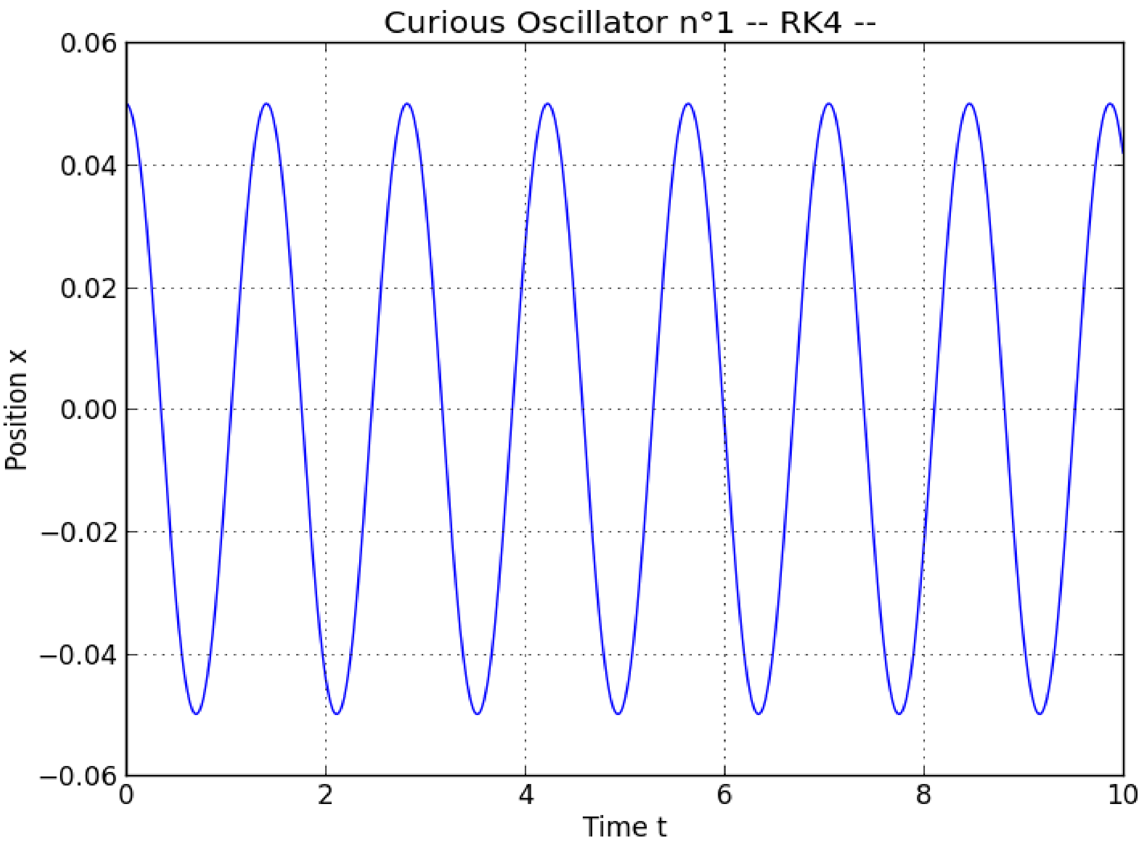

Fig A.1.3 – General solution

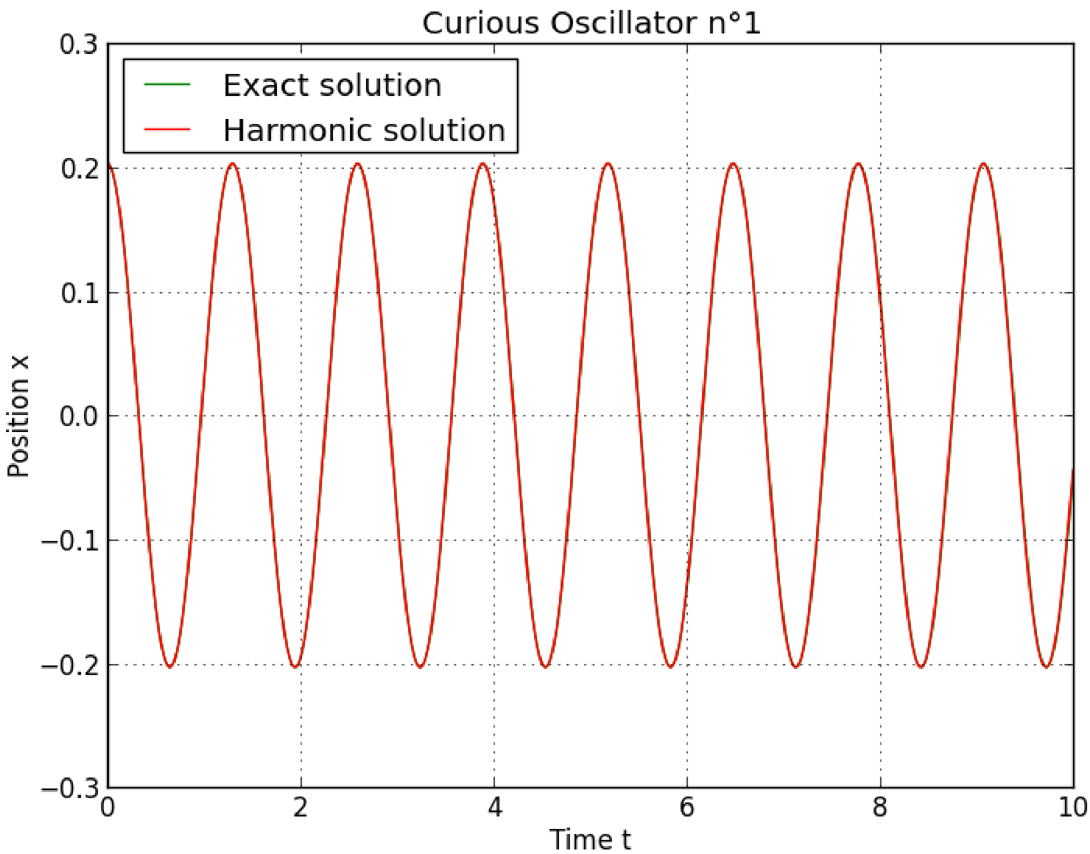

Fig A.1.4 – Comparison between general and harmonic solution

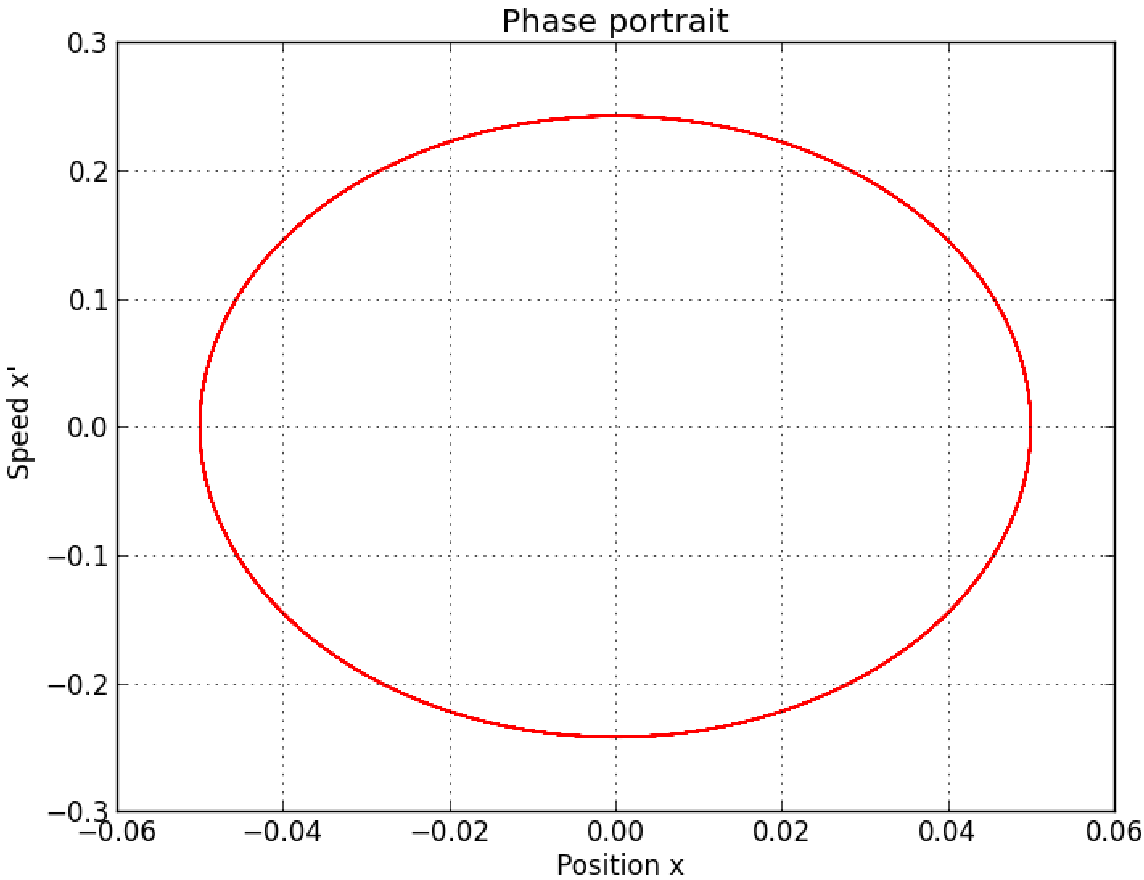

Fig A.1.5 – Phase portrait

A.2 Second oscillator

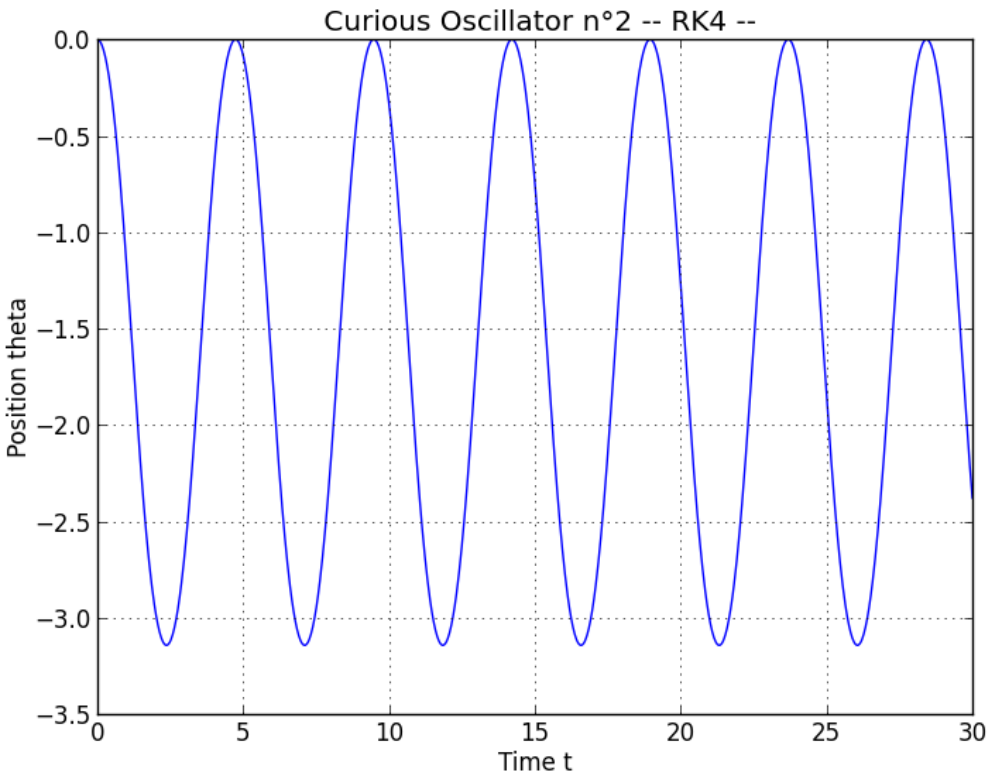

Fig A.2.1 – General solution

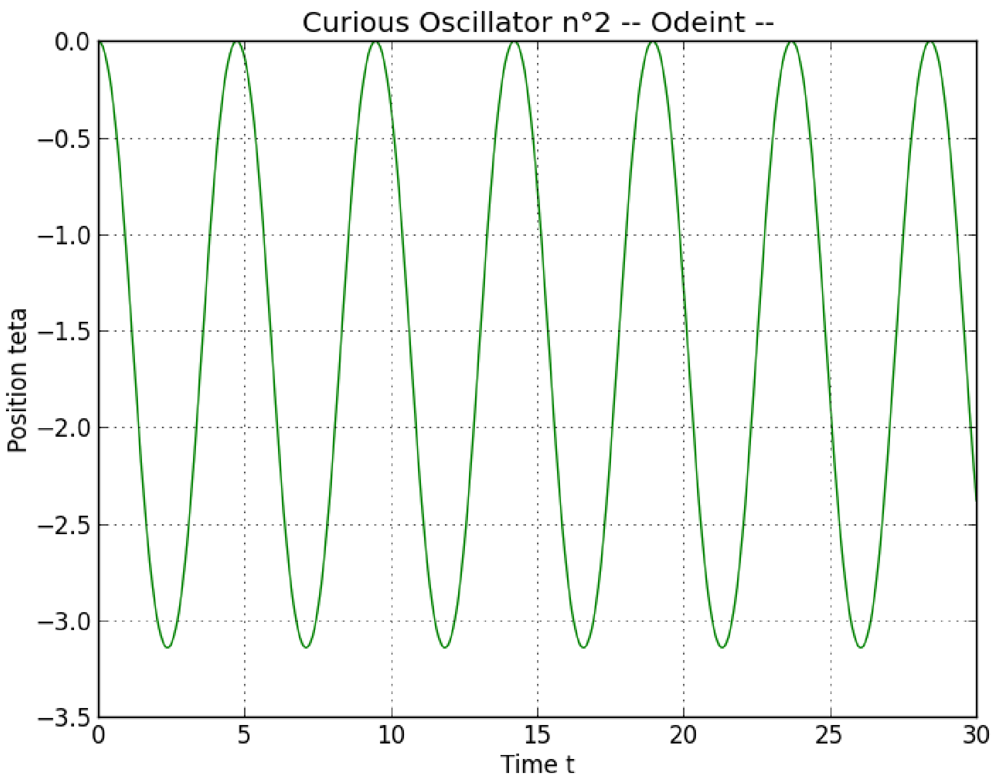

Fig A.2.2 –Odeint solution

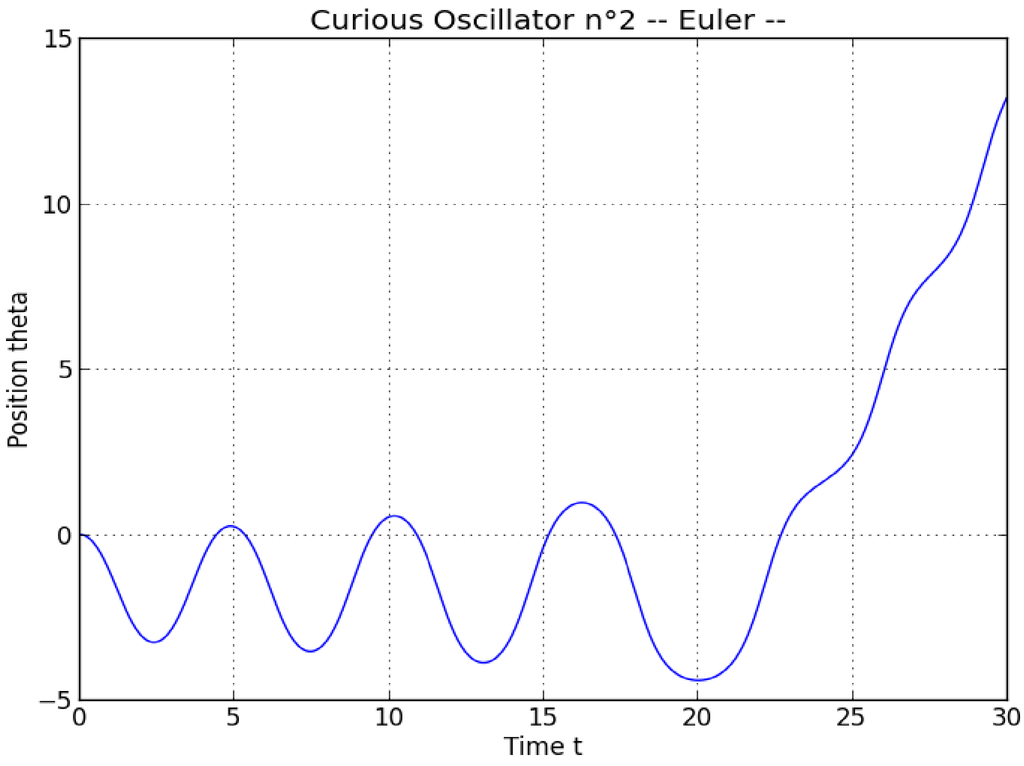

Fig A.2.3 – Euler solution

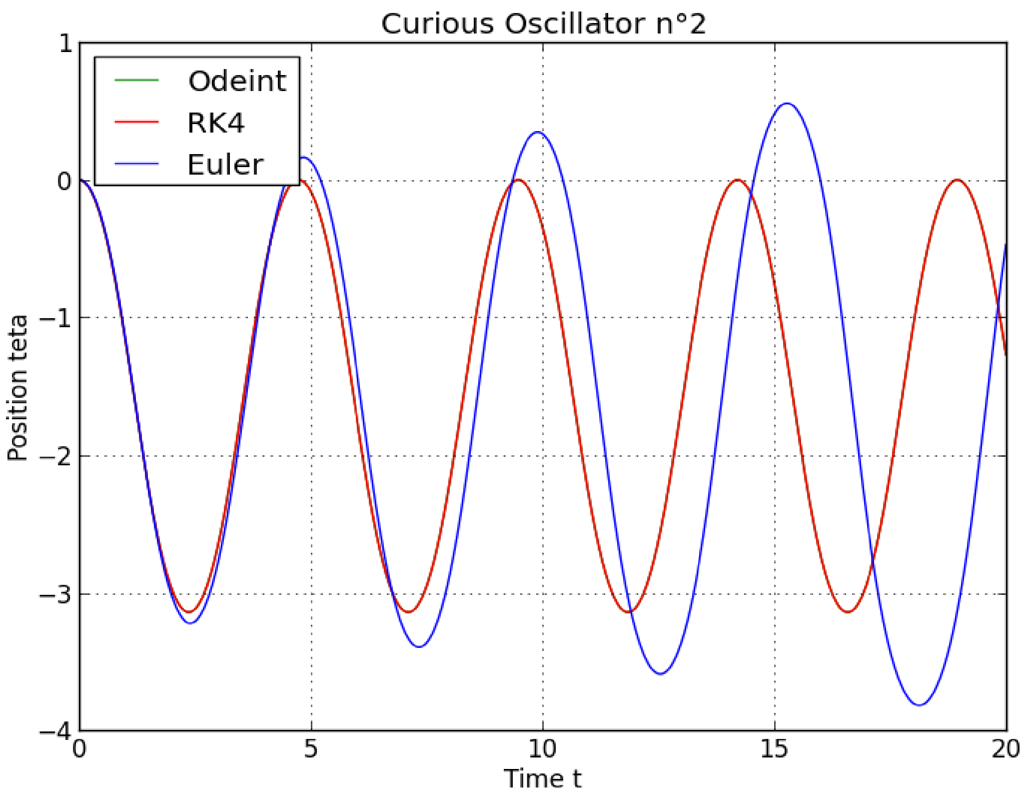

Fig A.2.4 – Comparison between all solutions

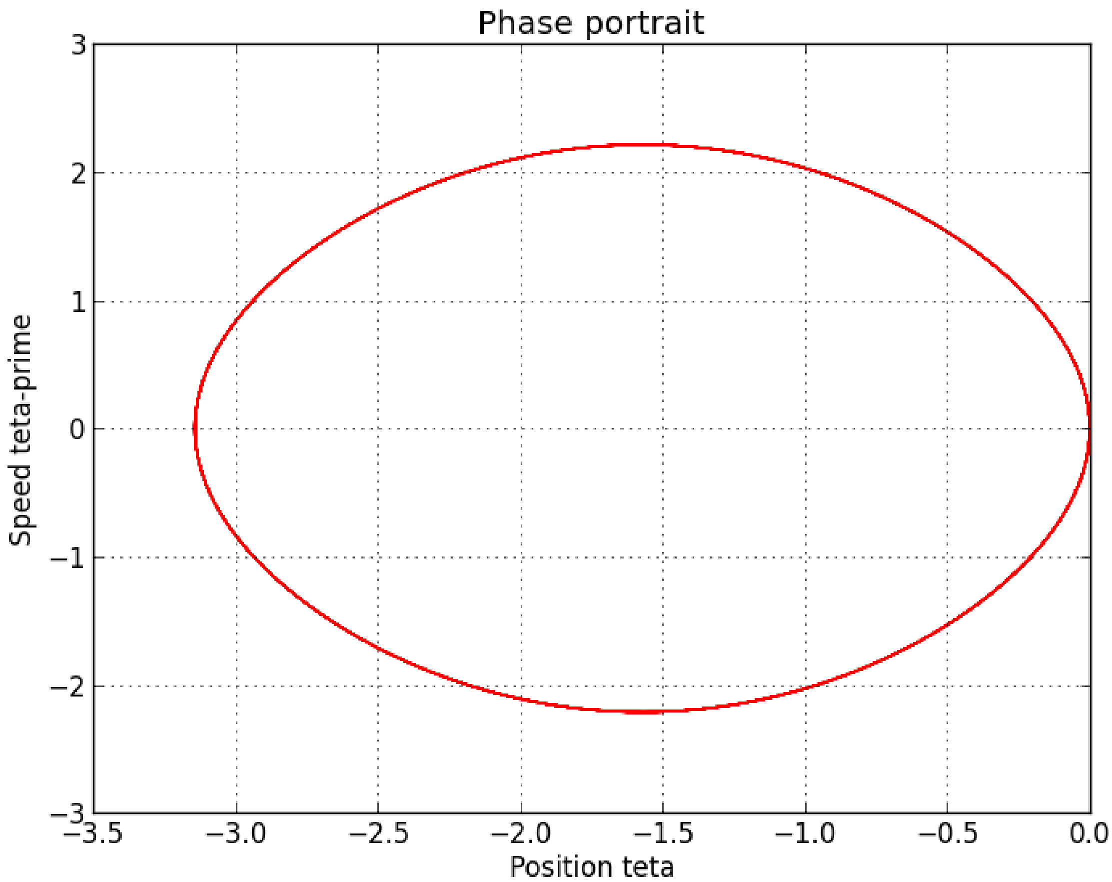

Fig A.2.5 – Phase portrait