ICEF master program. Maths for economists. Exam collection.

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Содержание

1 2008-2009

1.1 Exam, 12.01.2009

Final exam consists of the two parts: A and B. Part A lasts for 110 minutes. Upon completion of that part the papers will be collected and the students will have 10 minutes break. Part B lasts for 70 minutes. Students should answer eight of the following nine questions: six from Part A and two from Part B. Points will be deducted for the insufficient explanation within your answers.

Part A. Answer all SIX questions of this section. Each question is worth 10 points.

1. The joint distribution of vector (X, Y) is given by $\mathbb{P}(X = i, Y = j) = 0.1$ for $1 \leqslant i \leqslant j \leqslant 4$. Find $\mathbb{E}(Y \mid X)$.

2. The random variable X is exponentially distributed with parameter λ. The random variable Y is exponentially distributed with parameter $1/X$. Find $\mathbb{E}(Y|X)$, $\mathbb{E}(Y)$ and $\mathrm{Var}(Y)$.
3. Let $Y_t = W_t^3 - 3tW_t$.
   (a) Using Ito's lemma find dY_t
   (b) Using your previous result find $\mathbb{E}(Y_t)$ and $\mathrm{Var}(Y_t)$
4. Using a current value Hamiltonian maximize the integral

$$\int_0^2 e^{-t}\left(x - \frac{5}{2}x^2 - 2y^2\right) dt$$

subject to the conditions $\dot{x} = y - x/2$, $x(0) = 0$, $x(2)$ is free. Find x, y, μ.
5. Solve the bounded control problem

$$\int_0^T e^{-rt}(1-u)x\, dt \to \max$$

, subject to $\dot{x} = xu$, $x(0) = 1$, $0 \leqslant u \leqslant 1$, where $0 < r < 1$.
6. Consider the following optimization problem

$$\sum_0^\infty u(a_t) \to \max$$

, subject to $\sum_0^\infty a_t \leqslant s$, $s > 0$, $a_t \geqslant 0$.
   (a) Show that if $u(a)$ is increasing and strictly concave this problem has no solution
   (b) What happens if the sum in the maximization problem is changed to $\sum_0^\infty \delta^t u(a_t)$, where $0 < \delta < 1$?

Part B. Answer two questions out of the three from this section. Each question is worth 20 points. Part B lasts for 70 minutes.

1. In the framework of the Black and Scholes model find the price at $t = 0$ of an asset that pays $\max\{0, \ln S_T\}$ at time T, where S_T denotes the price of one share at time T.
2. Let's consider the following system of stochastic differential equations

$$\begin{cases} dX_t = aX_t dt - Y_t dW_t \\ dY_t = aY_t dt + X_t dW_t \end{cases}$$

with initial conditions $X_0 = x_0$ and $Y_0 = y_0$
   (a) Find the solution of the form $X_t = f(t)\cos W_t$ and $Y_t = g(t)\sin W_t$
   (b) Prove that for any solution $D_t = X_t^2 + Y_t^2$ is nonstochastic
3. Consider the profit-maximizing problem for a representative competitive firm

$$\int_0^\infty (p - wn(t))q(t)e^{-rt}\, dt \to \max$$

, subject to (*) $\dot{x} = x(1-x) - q$, where the state variable $x(t)$ represents a renewable stock resource (fish) that evolves according to the equation (*) and $q(t) = 2\sqrt{x(t)n(t)}$ is the extraction rate. Here $n(y)$ is a labor effort with a constant wage rate w. The price of fish is assumed to be constant and equal p. The optimization problem is to choose $n(t)$ to maximize the discounted profits, $0 < r < 1$.

(a) Derive necessary conditions

(b) Draw the phase diagram for this problem with the fish stock and the multiplier labeled on the axes

(c) Show that if p/w is sufficiently large the fish stock will be driven to zero, while p/w is low there is a steady-state with a positive stock

1.2 Retake, ??.??.2009

Final exam consists of the two parts: A and B. Part A lasts for 110 minutes. Upon completion of that part the papers will be collected and the students will have 10 minutes break. Part B lasts for 70 minutes. Students should answer eight of the following nine questions: six from Part A and two from Part B. Points will be deducted for the insufficient explanation within your answers.

Part A. Answer all SIX questions of this section. Each question is worth 10 points.

1. In the first bag there balls numbered from 0 to 9, in the second bag there are balls numbered from 1 to 10. Two balls were selected. You know that one ball was selected from the first bag and one from the second one. You will select at random one ball from these two and you will know only its number. Let's denote its number by X and the number of the other of the two balls by Y.
   Find $\mathbb{E}(Y \mid X)$
2. The random variable X is uniformly distributed on $[0; a]$. The random variable Y is uniformly distributed on $[0; X]$.
   Find $\mathbb{E}(Y \mid X)$, $\mathbb{E}(Y)$ and $\mathrm{Var}(Y)$.
3. Let $Y_t = \exp\left(-aW_t - \frac{a^2}{2}t\right)$.

   (a) Using Ito's lemma find dY_t

   (b) Using your previous result find $\mathbb{E}(Y_t)$ and $\mathrm{Var}(Y_t)$

4. Find extremals for the integral $\int_0^1 \left(\frac{1}{2}\dot{x}^2 + x\dot{x} + x\right) dt$ when $x(0) = 0$ and $x(1)$ is chosen freely.
5. Solve the bounded control problem $\int_0^2 (2x - 3u)\, dt \to \max$, subject to $\dot{x} = x + u$, $x(0) = 5$, $0 \leqslant u \leqslant 2$, where $x(2)$ is free.
6. Consider the following optimization problem: maximize $\sum_0^\infty \beta^t u(c_t)$, subject to $c_t + k_{t+1} = f(k_t)$, $0 < \beta < 1$, where both the utility function $u(t)$ and the production function $f(k)$ have the standard properties of monotonicity and strict concavity. Let the state variable be k and denote the next period value of k as k'. Substitute $c = f(k) - k'$ into utility function and write down the Bellman equation.
   Using the formal differentiation of the Bellman equation with respect to k under the sign of max, and drawing FOC from Bellman equation, exclude the value function and find the equation which combines the values of u' and f' at the adjacent time periods.

Part B. Answer two questions out of the three from this section. Each question is worth 20 points. Part B lasts for 70 minutes.

1. In the framework of the Black and Scholes model find the price at $t = 0$ of an asset that pays $\min\{M, \ln S_t\}$ at time T, where S_T denotes the price of one share at time T, M — arbitrary constant, specified at the moment of the issue.
2. The price of a share in euros is driven by the equation $dS = \sigma S dW + \alpha S dt$, the dollar/euro exchange rate is driven by the equation $dU = bU dW + cU dt$. Find the current price in dollars of a European call option with maturity date T, strike price K.

3. Consider the profit-maximizing problem for a representative competitive firm $\int_0^\infty (p - c(x(t)))q(t)e^{-rt}\, dt \to \max$, subject to (*) $\dot{x} = 1 - x - q$, where the state variable $x(t) < 1$ represents a nonrenewable stock resource (oil) that depletes according to the equation (*) and $q(t)$ is the extraction rate. Here $c(x)$ is a cost function of the extraction that is defined by $c(x) = e^{-x}$. The price of oil is assumed to be constant and equal p where $1/e < p < 1$. The optimization problem is to choose $q(t)$ to maximize the discounted profits, $0 < r < 1$.

   (a) Derive necessary conditions.

   (b) Prove that the steady-state exists and is unique.

2 2009-2010

2.1 Exam, 14.01.2010

Final exam consists of the two parts: A and B. Part A lasts for 120 minutes. Upon completion of that part the papers will be collected and the students will have 10 minutes break. Part B lasts for 60 minutes. Students should answer eight of the following eight questions: six from Part A and two from Part B. Points will be deducted for the insufficient explanation within your answers.

Part A. Answer all SIX questions of this section. Each question is worth 10 points.

1. The joint distribution of the random vector (X, Y) is given by its p.d.f

$$f(x, y) = \begin{cases} ce^{x-y}, & \text{for } 0 \leqslant x, y \leqslant 1 \\ 0, & \text{otherwise} \end{cases}$$

   where c is a normalization constant. Find $\mathbb{E}(X \mid Y)$.
2. Let $Y_t = W_t^3 - tW_t^4$. Find $\mathbb{E}(Y_t)$ and $\mathrm{Var}(Y_t)$. You don't have to use Ito Lemma here.
3. Let X_n be a discrete time stochastic process that converges in probability to a random number X as $n \to \infty$. Does this condition imply that X_n converges to X in mean? Almost surely? In distribution? Support at least one of your answers with a proof or counterexample. Give a definition for each type of convergence.
4. Seek to optimize the integral $\int_0^{\frac{\ln 2}{2}} (4u - u^2 - x - 3x^2)\, dt$ subject to the conditions $\dot{x} = u + x$, $x(0) = 5/8$, $x\left(\frac{\ln 2}{2}\right)$ is free. Find x, u, λ. What kind of optimum did you find?
5. Solve the bounded control problem $\int_0^1 (2 - 5t)u\, dt$, subject to $\dot{x} = 2x + 4te^{2t}u$, $x(0) = 0$, $x(1) = e^2$, $-1 \leqslant u \leqslant 1$.
6. Consider the following optimization problem: maximize $\sum_0^\infty \left(\frac{3}{4}\right)^t \ln c_t$, subject to $c_t + k_{t+1} = \sqrt{k_t}$, $k_0 > 0$. Let the state variable be k and denote the next period value of k as k'.

   (a) Write down the Bellman equation for the value function $V(k)$

   (b) Using method of undetermined coefficients find $V(k)$

   (c) Find the optimal policy function $k' = h(k)$

Part B. Answer both questions from this section. Each question is worth 20 points. Part B lasts for 60 minutes.

1. Let X_t be a stochastic process such that $dX_t = \frac{X_\infty - X_t}{\tau} dt + \sigma dW_t$, where X_∞ and τ are non-random constants, W_t is a Wiener process, and let $Y_t = X_t e^{t/\tau}$.

(a) Use Ito Lemma to find both differential and integral expressions for Y_t and use them to express X_t (a.s.) in terms of X_∞, X_0, τ, σ and t. Here X_0 is the value of X_t at time $t = 0$.

(b) Find $\mathbb{E}(X_t)$ and $\mathrm{Var}(X_t)$. Sketch the graph of $\mathbb{E}(X_t)$ as a function of t for $X_\infty = 1$, $\tau = 1$, and $X_0 = 0$, 1, and 2. Plot a possible trajectory of X_t in each case. Is there any name or names associated with X_t?

2. Utility $U(C,P) = U_1(C) + U_2(P)$ increases with the consumption C and decreases with the level of pollution P. For $C > 0$, $P > 0$ it is known that $U_1' > 0$, $U_1'' < 0$, $U_2' < 0$ and $U_2'' < 0$. It is known that $\lim_{C\to 0} U_1'(C) = \infty$ and $\lim_{P\to 0} U_2'(C) = 0$.
Consumption lies within the range $0 \leqslant C \leqslant \bar{C}$. Consumption contributes to pollution, while pollution control reduces it; moreover environment absorbs pollution at a constant rate $b > 0$. Pollution dynamics is governed by equation $\dot{P} = C^2 - (C^*)^2 - bP$ in which the first two terms represent the net contribution to the pollution flow, where $0 < C^* < \bar{C}$.
Consider the problem of maximizing the discounted ($r > 0$) utility stream $\int_0^\infty e^{-rt} U(C,P)\,dt \to \max$ subject to $\dot{P} = C^2 - (C^*)^2 - bP$, $P(0) = P_0 > 0$, $P \geqslant 0$, $0 \leqslant C \leqslant \bar{C}$.

(a) Derive necessary conditions, using the current value Hamiltonian.

(b) Sketch the phase diagram for this problem with the pollution and the consumption labeled on the axes.

(c) Find the condition under which the steady state solution (P_s, C_s) exists and $0 < C_s < \bar{C}$.

(d) Explore the stability of the steady state. Hint. You may find the return to the variables (P, m) easier for solving that part.

3 2010-2011

3.1 Exam, 12.01.2011

Notation: W_t is the standard Wiener process.

Part A (10 points each problem). Time allowed: 120 minutes.

1. The joint probability density function of X and Y is given by

$$f(x,y) = \begin{cases} x + y, & x \in [0;1],\ y \in [0;1] \\ 0, & \text{otherwise} \end{cases}$$

Find $\mathbb{E}(Y|X)$ in terms of X, find the probability density function of $\mathbb{E}(Y|X)$

2. Consider the process $X_t = \int_0^t sW_s\,dW_s$. Find $\mathbb{E}(X_t)$, $\mathrm{Var}(X_t)$, $\mathrm{Cov}(X_t, W_t)$
3. The process Y_t is given by $Y_t = 2W_t + 5t$. The stopping time τ is given by $\tau = \min\{t|Y_t^2 = 100\}$. Find the distribution of the random variable Y_τ and the expected value $\mathbb{E}(\tau)$.
Hint: you may find the martingales a^{Y_t} and $Y_t - f(t)$ useful
4. Find $\mathbb{P}(W_2 - W_1 > 2)$
5. In the framework of Black and Scholes model find the price of an asset which gives you the payoff of 1 rubble only if the final price S_t is at least two times bigger than the initial price S_0 of the asset.

6. Consider the free end problem, where $T > 0$ is not given

$$\int_0^T (\dot{x}^2 - x + 1)dt \to extr$$

At the left end $x(0) = 0$ Find the optimal T value and the extremal. Check that you solved the optimality problem or show the opposite.

Part B (20 points each prolem). Time allowed: 60 minutes.

1. Consider the stochastic differential equation

$$dX_t = (\sqrt{1 + X_t^2} + 0.5X_t)dt + \sqrt{1 + X_t^2}dW_t, \quad X_0 = 0$$

   (a) Suppose that Y_t is another process that depens only on X_t, i.e. $Y_t = f(X_t)$. Find dY using the Ito's lemma.

   (b) Find such function f that the term before dW in dY is constant.

   (c) Find X_t

   (d) Sketch $\mathbb{P}(X_t > 0)$ as the function of t.

2. Consider the neoclassical optimal growth model

$$\int_0^\infty e^{-rt}\left(\bar{U} - \frac{1}{c(t)}\right) dt \to \max$$

subject to $\dot{k} = A\ln(1 + k) - c - \delta k$, where $A > r > \delta > 0$, $k(0) = k_0$, $\bar{U} > 0$.

   (a) Derive necessary conditions, using the current value Hamiltonian

   (b) Sketch the phase diagram for this problem with the capital intensity and consumption labeled on the horizontal and vertival axes, respectively

   (c) Check that the steady state solution exists. Provide explanation.

   (d) Explore the stability of the steady state, using the Jacobian

   (e) Why are you sure the found growth path maximizes the discounted stream of utility?

3.2 Retake, ??.01.2011

Answer all SIX questions of this section. Each question is worth 10 points.

1. The joint distribution of the random vector (X, Y) is given by its p.d.f

$$f(x, y) = \begin{cases} ce^{x-y}, \text{ for } 0 \leqslant x, y \leqslant 1 \\ 0, \text{ otherwise} \end{cases}$$

where c is a normalization constant. Find $\mathbb{E}(X \mid Y)$.

2. Let $Y_t = W_t^3 - tW_t^4$. Find $\mathbb{E}(Y_t)$ and $\text{Var}(Y_t)$. You don't have to use Ito Lemma here.
3. Let X_n be a discrete time stochastic process that converges in probability to a random number X as $n \to \infty$. Does this condition imply that X_n converges to X in mean? Almost surely? In distribution? Support at least one of your answers with a proof or counterexample. Give a definition for each type of convergence.

4. Solve the calculus of variations problem to optimize the integral $\int_{1/2}^{1} \sqrt{1+\dot{x}^2}/x \, dt \to \max$ subject to the conditions $x(1/2) = \sqrt{3}/2$, $x(1) = 1$. Justify your answer referring to the sufficiency conditions. What kind of extremum did you find?
5. Let $x(t)$ represent the revenue of a firm. The fraction of it, namely $xu(t)$, where $0 \leqslant u \leqslant 1$, is spent on investments allowing the revenue to grow according to the rule $\dot{x} = \alpha x u$ with $\alpha = const > 0$.
   Another fraction of the revenue βx, $\beta = const > 0$ serves to reimburse the costs. Maximize the profit of the firm over the finite time horizon
   $$\int_0^T (1 - \beta - u)x \, dt \to \max$$
   subject to $\dot{x} = \alpha x u$, $x(0) = x_0$, $0 \leqslant u \leqslant 1$.
6. Consider the following «cake-eating» problem: maximize
   $$\sum_0^\infty \beta^t \ln c_t$$
   subject to $W_{t+1} = W_t - c_t$, $W_t \leqslant W$, $W > 0$, $0 < \beta < 1$. Let the state variable be W and denote the next period value of W as W'.
   (a) Write down the Bellman equation for the value function $V(W)$
   (b) Using method of undetermined coefficients find $V(W)$
   (c) Find the optimal policy function $c = h(W)$

Part B. Answer both questions from this section. Each question is worth 20 points. Part B lasts for 60 minutes.

1. Let X_t be a stochastic process such that $dX_t = \frac{X_\infty - X_t}{\tau} dt + \sigma dW_t$, where X_∞ and τ are non-random constants, W_t is a Wiener process, and let $Y_t = X_t e^{t/\tau}$.
   (a) Use Ito Lemma to find both differential and integral expressions for Y_t and use them to express X_t (a.s.) in terms of X_∞, X_0, τ, σ and t. Here X_0 is the value of X_t at time $t = 0$.
   (b) Find $\mathbb{E}(X_t)$ and $\mathrm{Var}(X_t)$. Sketch the graph of $\mathbb{E}(X_t)$ as a function of t for $X_\infty = 1$, $\tau = 1$, and $X_0 = 0$, 1, and 2. Plot a possible trajectory of X_t in each case. Is there any name or names associated with X_t?
2. Consider the profit maximizing problem over the infinite time horizon
   $$\int_0^\infty e^{-rt}(2\sqrt{K} - cI) \, dt \to \max$$
   where K is capital, I is investment, $c = const$ is the unit cost of investment, r is discount rate. Let $K(0) = K_0 > 0$. The capital changes under the investment equation $\dot{K} = I - bK$, where $b > 0$ is the depreciation rate. Investment is bounded $0 \leqslant I \leqslant \bar{I}$. Suppose the parameters of the problem satisfy conditions
   $$K_0 < \frac{1}{c^2(r+b)^2} < \frac{\bar{I}}{b}$$
   (a) Derive necessary conditions, using the current value Hamiltonian
   (b) Does the steady-state solution(s) (K_s, I_s) exist? Explain. If you answer is positive explore its stability.

4 2011-2012

4.1 Retake, 03.02.2012

Section A. 10 points for each problem.

1. Solve the bounded control problem with the free end value where $T > 1$ is specified in advance but $x(T)$ is not.

$$\int_0^T (x - u)\, dt \to \max,\ x(0) = 1$$

   $x' = u$, $0 \leqslant u \leqslant x$. What guarantees that a maximizer would be found?
2. Two stochastic processes are defined by the system of SDE with the Brownian motions W_{1t}, W_{2t} independent of each other

$$\begin{cases} dX_t = (2 + 5t + X_t)dt + 3dW_{1t} \\ dY_t = 4Y_t dt + 8Y_t dW_{1t} + 6dW_{2t} \end{cases}$$

   Calculate $d(X_tY_t)$
3. Find the value of the constant a such that the process $X_t = W_t^3 - a\int_0^t W_s\, ds$ is a martingale.
4. The process Y_t is given by $Y_t = 2W_t + 5t$. The stopping time τ is given by $\tau = \min\{t | Y_t^2 = 100\}$. Find the distribution of the random variable Y_τ and the expected value $\mathbb{E}(\tau)$.
   Hint: you may find the martingales a^{Y_t} and $Y_t - f(t)$ useful
5. What is the expected value and variance of W_t^2 for $t > s$ given that $W_s = x$?
6. In the framework of Black and Scholes model find the price at the time 0 of an asset which gives you the payoff $\max\{\ln(S_t), 0\}$ at the time t. Here S_t is the price of the underlying asset.

Section B. 20 points for each problem.

1. The goal of this exercise is to solve the SDE

$$dX_t = \frac{1}{X_t}dt + X_t dW_t$$

   with initial condition $X_0 = 1$.

   (a) Apply the Ito's lemma to the process $Y_t = \exp(f(t) + g(W_t))X_t^2$

   (b) Find non-constant functions f and g such that the coefficient before dW_t in the expression for dY_t is zero.

   (c) Find X_t. The final expression may contain a Riemann integral of some stochastic process.

2. A typical firm with the production function $x = f(l)$ in an economy employs 1 unit of the capital paying for it $\bar{r}$. There are n identical firms in this economy where $n(t)$ is a function of time. Let the output of a firm be $x(t)$. Then the aggregate supply equals nx. Assuming that the market is in the equilibrium we use the inverse demand function $p(nx)$, where $p'(y) < 0$. Equation defining the dynamics of labor is given by $\frac{dl}{dt} = a(w - \bar{w})$, where $a > 0$ and $\bar{w}$ is some equilibrium wage rate. The growth (or fall) in number of firms is governed by the equation $\frac{dn}{dt} = b(px - \bar{w}l - \bar{r})$, where $b > 0$. Write down the system of ODE for the unknowns $l(t)$, $n(t)$, using the first-order condition for

the profit-maximizing firm, find and classify the steady-state solutions (if any exist). Production function is twice differentiable and concave everywhere.

5 2012-2013

5.1 Exam, 10.01.2013

Optimal control part.

1. [10 points] Solve the optimal control problem $\int_0^1 (x+u)\,dt \to \max$ subject to $\dot{x} = -x + u + t$, $x(0) = 0$, $x(1)$ is free, and $0 \leqslant u \leqslant 1$.
2. [10 points] Consider an intertemporal utility maximization problem over the finite horizon in discrete time $t = 1, \ldots, T$. The utility is $u(c) = \ln c$, the initial wealth is w and if the remaining wealth at time t was w_t, then by the beginning of the next period it becomes $w_{t+1} = (1+r)(w_t - c_t)$, where r is the discount rate and c_t is the consumption. To find the maximum value of the utility stream one can use the finite version of the Bellman equation written in the form $V_t(w) = \max\{u(c) + V_{t+1}((1+r)(w-c))\}$ where the function within the braces is maximized over the values of $0 \leqslant c \leqslant w$. It is important to note that the value function V_t is the maximum value of the utility stream from time t onwards.
   (a) [3 points] Let $V_T = \ln w$. Find V_{T-1}.
   (b) [7 points] Try to verify the conjecture that $V_t = \gamma_t \ln(1+r) + (T-t+1)\ln \frac{w}{T-t+1}$ where γ_t is some (unknown) function of time.
3. [20 points] A spill of toxic substance should be cleaned up by a company. If a cleanup rate is $u(t)$, then the area of the spill shrinks by the law $x(t) = x_0 - \int_0^t u(s)\,ds$, where x_0 is the initial area. According to the government contract by the time T (and not earlier) the area of the spill should be reduced to x_T, where $x_T < x_0$. Clearly $u(t) \geqslant 0$, and by technology requirement $u(t) \leqslant \bar{u}$, where $\bar{u} > (x_0 - x_T)/T$. Costs of cleaning are directly proportional to $u(t)$. Let the unit costs equal $c > 0$ and the costs are discounted with the discount rate $r > 0$.
   (a) [5 points] Formulate the minimization problem with the bounded control. Hint: this is a fixed endpoints problem.
   (b) [15 points] By using the current value Hamiltonian set the system of equations and solve it.

Stochastic calculus part. Here W_t always denotes the standard Wiener process.

4. [10 points] Waves are arriving on the seashore. The sizes of the waves are independent uniform on $[0;1]$ random variables X_i. If the size of i-th wave is greater than the size of its neighboring waves, $X_i > \max\{X_{i-1}, X_{i+1}\}$, then we call it a «high» wave. Let H_i be the indicator of the «high» waves: $H_i = 1$ if $X_i > \max\{X_{i-1}, X_{i+1}\}$ and $H_i = 0$ otherwise.
   Find $\mathbb{E}(X_i \mid H_i)$ and $\mathbb{E}(H_i \mid X_i)$
5. [10 points] Find $\mathbb{E}(W_s \cdot W_t)$ and $\mathbb{E}(W_r \cdot W_s \cdot W_t)$ where $r < s < t$.
6. [10 points] Consider the process $X_t = \int_0^t s^2 W_s\,dW_s$. Find $\mathbb{E}(X_t)$, $\mathrm{Var}(X_t)$, $\mathrm{Cov}(X_t, W_t)$

7. [10 points] In the framework of Black and Scholes model find the price at the time 0 of an asset which gives you the payoff $\ln(S_T)$ at the time T. Here S_t is the price of the underlying asset at time t.
8. [20 points] The goal of this exercise is to solve the SDE

$$dX_t = -9X_t^2(1-X_t)dt + 3X_t(1-X_t)dW_t, \quad X_0 = 1/2$$

   (a) Using Ito's lemma find dY_t for the process $Y_t = f(X_t)$

   (b) Find such a function $f()$ that the term before dt in dY_t is zero

   (c) Obtain a simple differential equation for Y_t. It should not contain X_t.

   (d) Solve the stochastic differential equation for Y_t

   (e) Express X_t as a function of W_t and t.

Ответы и подсказки:

1. $\mathbb{E}(H_i \mid X_i) = X_i^2$, $\mathbb{E}(X_i \mid H_i = 1) = 3/4$, $\mathbb{E}(X_i \mid H_i = 0) = 3/8$,
2. $\mathbb{E}(W_s \cdot W_t) = s$ and $\mathbb{E}(W_r \cdot W_s \cdot W_t) = 0$
3. $\mathbb{E}(X_t) = 0$, $\mathrm{Var}(X_t) = t^6/6$, $\mathrm{Cov}(X_t, W_t) = 0$
4. $C_0 = e^{-rT}(\ln S_0 + (r - \sigma^2/2)T$
5. Уравнение на f имеет вид $f''(x)(1-x) + 2f'(x) = 0$. Решается заменой $f'(x) = h(x)$.

5.2 Retake, 08.02.2013

Stochastic calculus part. Here W_t denotes standard Wiener process

1. (10 points) Random variables X and Y are jointly normal with zero expected values, unit variances and correlation ρ. Find $\mathbb{E}(Y \mid X)$ and $\mathbb{E}(Y^2 \mid X)$
2. (10 points) Consider the processes $X_t = \int_0^t s^3 W_s \, dW_s$ and $Y_t = \int_0^t W_s \, dW_s$. Find $\mathbb{E}(X_t)$, $\mathrm{Var}(X_t)$, $\mathrm{Cov}(X_t, Y_t)$
3. (10 points) The stochastic process Y_t is given by equation $Y_t = W_t^4 - 6tW_t^2 + 3t^2$. Find dY_t and $\mathbb{E}(Y_7 \mid Y_2)$
4. (20 points) Consider the stochastic differential equation

$$dX_t = X_t dt + X_t dW_t, \quad X_0 = 1$$

   (a) Solve this stochastic differential equation

   (b) Let $\tau = \inf_{t>0}\{t : X_t \geqslant R\}$. Find $\mathbb{E}(\tau)$ applying the optional stopping theorem to the process W_t.

Optimal control part.

5. (10 points) Solve the optimal control problem problem: $\int_0^1 u^2 \, dt \to \min$, subject to $\dot{x} = x + u$, $x(0) = 1$, $x(1) = 0$.
6. (10 points) Consider a maximization problem over the finite horizon in discrete time $t = 1, \ldots, T$:

$$\sum_{t=1}^{T} \left(\frac{t^3}{3} + \frac{t^2}{2} \right) a_t^2 \to \max$$

   subject to constraint $\sum_{t=1}^{T} a_t = c$, $a_t \geqslant 0$.

By introducing $w_t = \sum_{i=1}^{t} a_i$ reduce that problem to dynamic programming (3 points) and solve it (7 points).

7. (20 points) Consider the profit-maximizing problem for a representative competitive firm
$$\int_0^\infty (p - c(x(t)))q(t)e^{-rt}\,dt \to \max$$
subject to (*) $\dot{x} = 1 - x - q$, where the state variable $x(t) < 1$ represents a nonrenewable stock resource (oil) that depletes according to the equation (*) and $q(t)$ is the extraction rate. Here $c(x)$ is a cost function of the extraction that is defined by $c(x) = e^{-x}$. The price of oil is assumed to be constant and equal p where $1/e < p < 1$. The optimization problem is to choose $q(t)$ to maximize the discounted profits, $0 < r < 1$.
   (a) (10 points) Derive necessary conditions.
   (b) (10 points) Prove that the steady-state exists.

Answers and hints

1. Write the joint pdf $f(x, y)$, find $f(y \mid x)$ and then the conditional expected values by integration.
2. $\mathbb{E}(X_t) = \mathbb{E}(Y_t) = 0$, use isometry property
3. Calculate dY_t. We observe that $dY_t = f(t, W_t)dW_t$, so Y_t is a martingale and $\mathbb{E}(Y_7 \mid Y_2) = Y_2$.
4. Use the substitution $Y_t = \ln X_t$. The solution is
$$X_t = X_0 \exp(t/2 + W_t)$$

6 2013-2014

6.1 Stochastic calculus hometask

1. Let X and Y be independent exponentially distributed with parameter λ. Find $\mathbb{E}(X+Y \mid X - Y)$.
2. Let S_n be symmetric random walk with $S_0 = 2013$. The stopping time τ is the first moment when $|S_n|$ reaches the value 2014 or the value 2000. Find $\mathbb{E}(\tau S_\tau)$.
   Hint: You may construct a martingale of the form $M_n = S_n^3 - f(n)S_n$ for some function f.
3. Conditional variance is defined as $\mathrm{Var}(Y \mid X) = \mathbb{E}(Y^2 \mid X) - (\mathbb{E}(Y \mid X))^2$. Find $\mathrm{Var}(W_s \mid W_t)$.
   Hint: do not forget two cases, $t > s$ and $s < t$. You may find inversion property of the Wiener process useful.
4. Let f be a real function such that f'' is continuous. Find all such functions f that $X_t = \exp(\alpha t) f(W_t)$ is a martingale.
5. Solve the stochastic differential equation
$$dX_t = (X_t/t + t)dt + 2\sqrt{tX_t}dW_t$$
   Hint: You may suppose without a proof that the solution has the form $X_t = f(t)g(W_t)$.
6. In the framework of Black and Scholes model find the price of the asset, which pays you S_T at the fixed moment of time $T > 1$ if $S_{T-1} > 1$ and 0 otherwise.

6.2 Solution for stochastic calculus hometask

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Problem 1.

Denote $\xi = X + Y$ and $\eta = X - Y$.

Then we are gonna find $E(\xi|\eta)$.

Since $X \sim \text{Exp}(\lambda)$, it has a nice density: $p_X(u) = \lambda e^{-\lambda u} \cdot I_{\{u \geqslant 0\}}$.

Similarly, $p_Y(v) = \lambda e^{-\lambda v} \cdot I_{\{v \geqslant 0\}}$.

Since X and Y are independent, the vector (X, Y) has a two-dimensional density, which equals to the product of densities of X and Y, i.e.

$$p_{(X,Y)}(u, v) = I_{\{u \geqslant 0, v \geqslant 0\}} \lambda^2 e^{-\lambda(u+v)}$$

We have $X = X(\xi, \eta) = \dfrac{\xi + \eta}{2}$ and $Y = Y(\xi, \eta) = \dfrac{\xi - \eta}{2}$.

Then $J = \begin{vmatrix} \dfrac{\partial X}{\partial \xi} & \dfrac{\partial X}{\partial \eta} \\ \dfrac{\partial Y}{\partial \xi} & \dfrac{\partial Y}{\partial \eta} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}$.

Therefore

$$p_{(\xi,\eta)}(s, t) = p_{(X,Y)}\left(\frac{s+t}{2}, \frac{s-t}{2}\right) \cdot |J| = \frac{\lambda^2}{2} I_{\{s+t \geqslant 0\}} I_{\{s-t \geqslant 0\}} e^{-\lambda s} = \frac{\lambda^2}{2} I_{\{s \geqslant |t|\}} e^{-\lambda s}$$

is a density of a vector (ξ, η).

Further,

$$p_\eta(t) = \int\limits_{\mathbb{R}} p_{(\xi,\eta)}(s, t) \mathrm{d}s = \frac{\lambda e^{-\lambda|t|}}{2}$$

Then the conditional density is

$$p_{(\xi|\eta)}(s|t) = \frac{p_{(\xi,\eta)}(s, t)}{p_\eta(t)} = \lambda e^{-\lambda(s-|t|)} \cdot I_{\{s-|t| \geqslant 0\}}$$

And finally, denoting $z = s - |t|$, we get

$$E(\xi|\eta) = \int\limits_{\mathbb{R}} s \cdot p_{(\xi|\eta)}(s|t) \mathrm{d}s \Bigg|_{t=\eta} = \int\limits_{\mathbb{R}} (z + |t|) \cdot \lambda e^{-\lambda z} \cdot I_{\{z \geqslant 0\}} \mathrm{d}z \Bigg|_{t=\eta} =$$

$$= \int\limits_{\mathbb{R}} z \cdot \lambda e^{-\lambda z} \cdot I_{\{z \geqslant 0\}} \mathrm{d}z \Bigg|_{t=\eta} + |t| \int\limits_{\mathbb{R}} \cdot \lambda e^{-\lambda z} \cdot I_{\{z \geqslant 0\}} \mathrm{d}z \Bigg|_{t=\eta} = \left(\frac{1}{\lambda} + |t|\right)\Bigg|_{t=\eta} = \frac{1}{\lambda} + |\eta|$$

(the first integral is just the expectation of exponential distribution with parameter λ, and the second is $|t|$ times the integral of the density, which equals 1)

Answer: $E(X + Y|X - Y) = \dfrac{1}{\lambda} + |X - Y|$

Problem 2 — part 1 of 2.

At first, $S_n = S_0 + \sum\limits_{i=1}^{n} x_i$, where $\{x_i\}$ is the sequence of independent random variables with

$P(x_i = -1) = P(x_i = 1) = \frac{1}{2}$. Also denote $_n = \sigma(x_1, \ldots, x_n)$.
Observe that τ is really a stopping time, because at each moment of time we precisely know, are we in one of the points $\{2000, 2014\}$ or not.
Let's prove that $E(\tau) < \infty$ (which will also imply that $P(\tau = +\infty) = 0$).
We shall obviously reach τ after any 14 consecutive steps in one direction. Then

$$P(\tau \geqslant 14n) \leqslant P(\text{At least one of } x_1, ..., x_{14} \text{ equals } -1) \times ... \times P(\text{At least one of } x_{14(n-1)+1}, ..., x_{14n} \text{ eq}$$

$$= \left(1 - 2^{-14}\right)^n = q^n$$

where $q < 1$.
Then

$$E(\tau) = \sum_{n=1}^{+\infty} P(\tau \geqslant n) \leqslant 14 \sum_{n=0}^{+\infty} P(\tau \geqslant 14n + 1) \leqslant 14 \sum_{n=0}^{\infty} q^n = 14 \cdot 2^{14} < \infty$$

as required.
$\boxed{\text{The sequence } \{S_t\} \text{ is a martingale}}$ w.r.t. filtration $\{_n\}$, because it is obviously $\{_t\}$-adapted, has a finite (zero) expectation and

$$E(S_{t+1}|_t) = E(S_t + x_{t+1}|_t) = S_t + E(x_{t+1}|_t) = S_t$$

Further, $S_{\min\{t,\tau\}}$ is obviously bounded in t (it always lies between 2000 and 2014), so the Doob's theorem conditions are satisfied, which means that

$$E(S_\tau) = S_0 = 2013$$

But $E(S_\tau) = 2000 \cdot P(S_\tau = 2000) + 2014 \cdot P(S_\tau = 2014)$, which after the substitution $E(S_\tau) = 2013$ gives us
$\boxed{P(S_\tau = 2000) = \frac{1}{14}}$ and $\boxed{P(S_\tau = 2014) = \frac{13}{14}}$
Further, let's prove that $\boxed{\text{the sequence } \{S_t^3 - 3tS_t\} \text{ is a } \{_t\}\text{-martingale}}$. At first, it is obviously adapted and $|E(S_t^3 - 3tS_t)| < \infty$
Then

$$E(S_{t+1}^3 - 3(t+1)S_{t+1} \bigm| _t) = E\left((S_t + x_{t+1})^3 - 3(t+1)(S_t + x_{t+1}) \bigm| _t\right) =$$

$$= E\left(S_t^3 + 3S_t^2 \underline{x_{t+1}} + 3\underline{(x_{t+1}^2 - 1)}S_t - 3tS_t + \underline{x_{t+1}^3} - 3(t+1)\underline{x_{t+1}} \Big| _t \right) = S_t^3 - 3tS_t$$

(the underlined terms are going out after taking conditional expectation; $(x_{t+1}^2 - 1)$ goes out even without it)
Denote $X_t = S_t^3 - 3tS_t$.
For $t \geqslant \tau$ we have $|X_{\min\{t+1,\tau\}} - X_{\min\{t,\tau\}}| = 0$, and for $t < \tau$ we have
$|X_{\min\{t+1,\tau\}} - X_{\min\{t,\tau\}}| = |X_{t+1} - X_t| = |3S_t^2 x_{t+1} - (3t+2)x_{t+1}| \leqslant 3 \times 2014^2 + 3\tau + 2$, which expectation is bounded by the finite number $3 \times 2014^2 + 2 + 3E(\tau)$.
Then, the overall expectation $E\left(|X_{\min\{t+1,\tau\}} - X_{\min\{t,\tau\}}|\right) \leqslant 3 \times 2014^2 + 2 + 3E(\tau)$ for every t, i.e. is bounded in t.
Problem 2 — part 2 of 2.
Then again, the conditions of the Doob's theorem are satisfied, i.e.

$$E(X_\tau) = X_0 = 2013^3$$

But also $E(X_\tau) = E(S_\tau^3) - 3E(\tau S_\tau)$, i.e.

$$E(\tau S_\tau) = \frac{1}{3}\left(E(S_\tau^3) - 2013^3\right) = \frac{1}{3}\left(\frac{1}{14}2000^3 + \frac{13}{14}2014^3 - 2013^3\right) = 13 \times 2009 = 26117$$

Answer: $E(\tau S_\tau) = 26117$.

Problem 3.

Case 1. Let $s \geqslant t$.

Then

$$\begin{aligned} E\left(W_s^2 \middle| W_t\right) &= E\left(W_t^2 + 2W_t(W_s - W_t) + (W_s - W_t)^2 \middle| W_t\right) = \\ &= W_t^2 + 2W_t E(W_s - W_t) + E\left((W_s - W_t)^2\right) = W_t^2 + (s - t) \end{aligned}$$

Further,

$$E(W_s|W_t) = W_t + E(W_s - W_t|W_t) = W_t$$

Then we have $Var(W_s|W_t) = E(W_s^2|W_t) - E(W_s|W_t)^2 = (s - t)$

Case 2. Let $s < t$.

By the inverse property, $V_t = tW_{\frac{1}{t}}$ is also a standard Wiener process.

From the case 1 we have

$$\frac{1}{s} - \frac{1}{t} = Var\left(V_{\frac{1}{s}} \middle| V_{\frac{1}{t}}\right) = Var\left(\frac{1}{s}W_s \middle| \frac{1}{t}W_t\right) = \frac{1}{s^2}Var\left(W_s \middle| \frac{1}{t}\ W_t\right) = \frac{1}{s^2}Var(W_s|W_t)$$

the last equation is because obviously $\sigma(W_t) = \sigma\left(\frac{1}{t}W_t\right)$.

Then $Var(W_s|W_t) = s - \frac{s^2}{t}$.

Summing up these two cases, we get the

Answer: $Var(W_s|W_t) = \max\left\{s - t, s - \frac{s^2}{t}\right\}$

Problem 4.

Let $g(t, W_t) = X_t = e^{\alpha t} f(W_t)$, then $g_t = \alpha g$, $g_{W_t} = e^{\alpha t} f'(W_t)$ and $g_{W_t W_t} = e^{\alpha t} f''(W_t)$.

By the Itô's lemma we have

$$\begin{aligned} \mathrm{d}X_t &= \alpha X_t \delta + e^{\alpha t} f'(W_t)\mathrm{d}W_t + \frac{1}{2}e^{\alpha t} f''(W_t)\delta = \\ &= e^{\alpha t}\left(\alpha f(W_t) + \frac{1}{2}f''(W_t)\right)\delta + e^{\alpha t} f'(W_t)\mathrm{d}W_t \end{aligned}$$

Therefore

$$X_t = f(0) + \int_0^t e^{\alpha u} \left(\alpha f(W_u) + \frac{1}{2} f''(W_u) \right) \mathrm{d}u + \int_0^t e^{\alpha u} f'(W_u) \mathrm{d}W_u$$

The last term in the right-hand side is a martingale, so X_t is a martingale if and only if the coefficient near δ equals zero for all t, i.e.

$$\alpha f(W_t) + \frac{1}{2} f''(W_t) = 0 \qquad \text{a.s.}$$

which (together with continuity of f'') implies

$$\alpha f(x) + \frac{1}{2} f''(x) = 0 \qquad \forall x$$

We know how to solve the linear differential equations, so I will not go into details and I'll just write the answer.

Answer: If $\alpha < 0$, then $f(x) = C_1 e^{t\sqrt{-2\alpha}} + C_2 e^{-t\sqrt{-2\alpha}}$, where C_1, C_2 are any real constants;
if $\alpha = 0$, then $f(x) = C_1 x + C_2$;
if $\alpha > 0$, then $f(x) = C_1 \cos(t\sqrt{2\alpha}) + C_2 \sin(t\sqrt{2\alpha})$

Problem 5 — part 1 of 2.

Assume for simplicity that $X_t = f(t)g(W_t)$, where f and g are kinda nice functions (defined on $[0, +\infty)$ and continuously differentiable the required number of times on this interval; right-differentiable at zero, of course).
By the Itô's lemma we have

$$\mathrm{d}X_t = \left(f'(t)g(W_t) + \frac{1}{2} f(t)g''(W_t) \right) \delta + f(t)g'(W_t)\mathrm{d}W_t$$

Substituting this into the condition of the problem and separating the parts with δ and $\mathrm{d}W_t$ we get

$$\begin{cases} f'(t)g(W_t) + \frac{1}{2} f(t)g''(W_t) = \frac{f(t)}{t} g(W_t) + t & (1) \\ f(t)g'(W_t) = 2\sqrt{tf(t)g(W_t)} & (2) \end{cases}$$

Both equations are satisfied for every $t > 0$ almost surely.

Step 1. Let's prove that $f(t) \neq 0$ if $t > 0$. Indeed, if $f(t_0) = 0$, (1) gives us

$$f'(t_0)g(W_{t_0}) = t_0 \qquad \text{a.s.}$$

so

$$g(W_{t_0}) = \frac{t_0}{f'(t_0)} \qquad \text{a.s.}$$

But g is continuous, so this gives us

$$g(x) \equiv C = Const$$

But then (2) gives

$$X_t \equiv 0 \qquad \text{a.s.}$$

which does not satisfy the condition of the problem.
Therefore, $f(t) \neq 0$, when $t \neq 0$.

Now (2) can be transformed into

$$\frac{f(t)}{4t} g'(W_t)^2 = g(W_t) \tag{3}$$

Step 2. $g'(x)$ cannot be zero in any interval.
Suppose that $g'(x) = 0$ for all $x \in (a, b)$. From (3) we have $g(x) = 0$ almost everywhere at (a, b) (because $P(W_t \in (a, b)) \neq 0$ for $t > 0$), and also $g'(x) = 0$ at (a, b). But substituting these values to (1) provides us contradiction.

Let $A = \{x \in \mathbb{R} | g'(x) \neq 0\}$. Since g' is continuous and is not identical zero (from the step 2), the set A has a positive Lebesgue measure; therefore $P(W_t \in A) \neq 0$ for all $t > 0$.

Problem 5 — pat 2 of 2.

Now fix some $t_0 > 0$. From (3) we have that almost everywhere in the event $\{W_{t_0} \in A\}$ the equality

$$\frac{g(W_{t_0})}{g'(W_{t_0})^2} = \frac{f(t_0)}{4t_0}$$

holds. Then for almost all $x \in A$ we have $\dfrac{g(x)}{g'(x)^2} = \dfrac{f(t_0)}{4t_0}$. Now fix any other $t_1 > 0$.

We have that almost everywhere in A the equality $\dfrac{g(x)}{g'(x)^2} = \dfrac{f(t_1)}{4t_1}$ holds. Since A has a positive measure, we have

$$\frac{f(t_0)}{4t_0} = \frac{f(t_1)}{4t_1}$$

i.e. $f(t) = Ct$ for all $t > 0$ and some $C = Const$. From the continuity $\boxed{f(t) = Ct \text{ for all } t \geqslant 0}$

Substitute this to (1):

$$Cg(W_t) + \frac{1}{2} Ctg''(W_t) = Cg(W_t) + t$$

$$g''(W_t) = \frac{2}{C} \tag{4}$$

(4) holds almost everywhere, but since g'' is continuous, we have $\boxed{g''(x) \equiv \dfrac{2}{C}}$

Then $g'(x) = \frac{2x}{C} + D$ and $g(x) = \frac{x^2}{C} + Dx + E$ for some constants D, E.
From (3) we have

$$g(x) = \frac{f(t)}{4t} g'(x)^2 = \frac{C}{4} g'(x)^2$$

Substitute the values of $g'(x)$ and $g(x)$:

$$\frac{x^2}{C} + Dx + E = \frac{C}{4}\left(\frac{2x}{C} + D\right)^2$$

$$\frac{x^2}{C} + Dx + E = \frac{x^2}{C} + Dx + \frac{CD^2}{4}$$

which gives us $E = \dfrac{CD^2}{4}$ and $g(x) = (Ax+B)^2$, where $A^2 = \dfrac{1}{C}$ and $2AB = D$.

Answer: $X_t = t(W_t + C)^2$ for some constant C.
(in our notation this constant equals $\dfrac{B}{A}$)
REMARK. Precisely speaking, this is not a correct answer, because $\sqrt{tX_t}$ is always positive , so (2) does not always hold, because the left side of (2), which is $2t(W_t + C)$, can be negative with nonzero probability. In other words, we solved similar, but different equation

$$\mathrm{d}X_t = \left(\frac{X_t}{t} + t\right)\delta \;\pm\; 2\sqrt{tX_t}\mathrm{d}W_t$$

and the original problem has no solutions of the type $f(t)g(W_t)$.
Problem 6 — part 1 of 4.
The risky asset S_t satisfies the stochastic differential equation

$$\mathrm{d}S_t = \mu S_t\delta + \sigma S_t \mathrm{d}W_t$$

which implies

$$S_t = S_0 \exp\left(t\left(\mu - \frac{\sigma^2}{2}\right) + \sigma W_t\right)$$

The risk-free asset satisfies the equation

$$\mathrm{d}B_t = rB_t\delta$$

which implies

$$B_t = B_0 e^{rt}$$

Let X_t be our self-financing portfolio at time t, which contains Δ_t shares of the risky asset. The total amount of money in the risky asset is Δ_t, in the risk-free asset — $(X_t - \Delta_t S_t)$. Therefore

$$\mathrm{d}X_t = \Delta_t \mathrm{d}S_t + r(X_t - \Delta_t S_t)\delta$$

Let $\tilde{W}_t = \dfrac{\mu - r}{\sigma}t + W_t$, and $\tilde{P}$ is the probability measure, w.r.t. which $\tilde{W}_t$ is a standard Wiener process (the existence of such measure is the statement of Girsanov's theorem).
In class we explicitly derived that

$$e^{-rt}X_t = X_0 + \int_0^t \Delta_u e^{-ru}\sigma S_u \mathrm{d}\tilde{W}_u$$

The last term is a $\tilde{P}$-martingale, which means that

$$E_{\tilde{P}}\left(e^{-rT}X_T\big|\,_0\right) = X_0 \qquad (1)$$

Now remember that

$$X_T = S_T I_{\{S_{T-1}>1\}}$$

so we need to calculate

$$E_{\tilde{P}}(S_T I_{\{S_{T-1}>1\}}) = \iint\limits_{u>0,v>1} u \cdot \tilde{p}_{(S_T,S_{T-1})}(u,v)\mathrm{d}u\mathrm{d}v$$

where $\tilde{p}_{(S_T,S_{T-1})}$ is a two-dimensional density of a corresponding vector w.r.t. measure $\tilde{P}$.
Problem 6 — part 2 of 4.
But

$$\iint\limits_{u>0,v>1} u\cdot\tilde{p}_{(S_T,S_{T-1})}(u,v)\mathrm{d}u\mathrm{d}v = \int\limits_1^{+\infty}\left(\int\limits_0^{+\infty} u\cdot\tilde{p}_{(S_T,S_{T-1})}(u,v)\mathrm{d}u\right)\mathrm{d}v = \int\limits_1^{+\infty}\tilde{p}_{S_{T-1}}(v)\cdot E_{\tilde{P}}\left(S_T|S_{T-1}=v\right)\mathrm{d}v$$

Further,

$$S_{T-1} = S_0\exp\left((T-1)\left(\mu-\frac{\sigma^2}{2}\right)+\sigma W_{T-1}\right) = S_0\exp\left((T-1)\left(r-\frac{\sigma^2}{2}\right)+\sigma\tilde{W}_{T-1}\right)$$

which means that with notation $A(x) = S_0\exp\left((T-1)\left(r-\frac{\sigma^2}{2}\right)+\sigma x\right)$ we have

$$\tilde{p}_{\tilde{W}_{T-1}}(x) = \tilde{p}_{S_{T-1}}\left(A(x)\right)\cdot\sigma A(x)$$

We have $x = \sigma^{-1}\left(\ln\left(\frac{A(x)}{S_0}\right)-(T-1)\left(r-\frac{\sigma^2}{2}\right)\right)$. Therefore

$$\tilde{p}_{S_{T-1}}\left(v\right) = \frac{1}{\sigma v}\cdot\frac{1}{\sqrt{2\pi(T-1)}}\exp\left(-\frac{1}{2\sigma^2(T-1)}\left(\ln(v)-\left(\ln(S_0)+(T-1)\left(r-\frac{\sigma^2}{2}\right)\right)\right)^2\right)$$

Very ugly expression! Let's believe most of those terms will cancel out.

Now

$$S_T = S_0\exp\left(T\left(r-\frac{\sigma^2}{2}\right)+\sigma\tilde{W}_T\right) = S_{T-1}\cdot\exp\left(r-\frac{\sigma^2}{2}+\sigma\left(\tilde{W}_T-\tilde{W}_{T-1}\right)\right)$$

and $\left(\tilde{W}_T-\tilde{W}_{T-1}\right)$ is independent of the σ-algebra $\sigma(S_{T-1})\subseteq\,_{T-1}$.
Then

$$E_{\tilde{P}}\left(S_T|S_{T-1}=v\right) = E_{\tilde{P}}\left(S_{T-1}\cdot\exp\left(r-\frac{\sigma^2}{2}+\sigma\left(\tilde{W}_T-\tilde{W}_{T-1}\right)\right)\middle|\,S_{T-1}=v\right) =$$

$$= v \cdot \exp\left(r - \frac{\sigma^2}{2}\right) E_{\tilde{P}}\left(\exp\left(\sigma\left(\tilde{W}_T - \tilde{W}_{T-1}\right)\right)\right) \tag{3}$$

Further, if $\xi \sim \mathcal{N}(0,1)$, then

$$E(e^{\sigma\xi}) = \frac{1}{\sqrt{2\pi}} \int\limits_{\mathbb{R}} e^{-\frac{x^2}{2}+\sigma x} \mathrm{d}x = e^{\frac{\sigma^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} \int\limits_{\mathbb{R}} e^{-\frac{(x-\sigma)^2}{2}} \mathrm{d}x = e^{\frac{\sigma^2}{2}}$$

which gives us

$$E_{\tilde{P}}\left(S_T | S_{T-1} = v\right) = v \cdot \exp\left(r - \frac{\sigma^2}{2}\right) \exp\left(\frac{\sigma^2}{2}\right) = ve^r \tag{4}$$

which, I think, is a very nice formula (especially if we compare it with (2)).

Problem 6 — part 3 of 4.

Let's continue.

Taking (2) and (4), we have

$$\int\limits_1^{+\infty} \tilde{p}_{S_{T-1}}(v) \cdot E_{\tilde{P}}\left(S_T | S_{T-1} = v\right) \mathrm{d}v =$$

$$= e^r \int\limits_1^{+\infty} \frac{1}{\sigma} \cdot \frac{1}{\sqrt{2\pi(T-1)}} \exp\left(-\frac{1}{2\sigma^2(T-1)}\left(\ln(v) - \left(\ln(S_0) + (T-1)\left(r - \frac{\sigma^2}{2}\right)\right)\right)^2\right) \mathrm{d}v =$$

$$= e^r \int\limits_0^{+\infty} e^s \cdot \frac{1}{\sqrt{2\pi\sigma^2(T-1)}} \exp\left(-\frac{1}{2\sigma^2(T-1)}\left(s - \left(\ln(S_0) + (T-1)\left(r - \frac{\sigma^2}{2}\right)\right)\right)^2\right) \mathrm{d}s$$

We substituted $s = \ln(v)$. Of course, $\mathrm{d}v = e^s \mathrm{d}s$.

Denote also $B = \ln(S_0) + (T-1)\left(r - \frac{\sigma^2}{2}\right)$.

Then our integral equals

$$e^r \int\limits_0^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2(T-1)}} \exp\left(-\frac{s^2 - 2Bs - 2\sigma^2(T-1)s + B^2}{2\sigma^2(T-1)}\right) \mathrm{d}s =$$

$$= \exp\left(r - \frac{B^2}{2\sigma^2(T-1)} + \frac{\left(B + \sigma^2(T-1)\right)^2}{2\sigma^2(T-1)}\right) \int\limits_0^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2(T-1)}} \exp\left(-\frac{\left(s - B - \sigma^2(T-1)\right)^2}{2\sigma^2(T-1)}\right) \mathrm{d}s$$

$$= \exp\left(r - \frac{B^2}{2\sigma^2(T-1)} + \frac{\left(B + \sigma^2(T-1)\right)^2}{2\sigma^2(T-1)}\right) \cdot \tilde{P}\left(\eta > -\left(B + \sigma^2(T-1)\right)\right)$$

where $\eta \sim \mathcal{N}(0, \sigma^2(T-1))$. The integral magically turns into the probability, because under the integral there is a well-known density.

Therefore our integral finally equals

$$\exp\left(r+\frac{\left(B+\sigma^2(T-1)\right)^2-B^2}{2\sigma^2(T-1)}\right)\cdot\tilde{P}\left(\frac{\eta}{\sigma\sqrt{T-1}}>-\frac{B+\sigma^2(T-1)}{\sigma\sqrt{T-1}}\right)=$$

$$=\exp\left(r+B+\frac{\sigma^2(T-1)}{2}\right)\Phi\left(\frac{B+\sigma^2(T-1)}{\sigma\sqrt{T-1}}\right)=e^{\ln(S_0)+rT}\cdot\Phi\left(\frac{\ln(S_0)+(T-1)\left(r+\frac{\sigma^2}{2}\right)}{\sigma\sqrt{T-1}}\right)$$

where $\Phi(x)$ is a distribution function of a standard normal random value, $\Phi(x) = \int\limits_{-\infty}^{x}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{t^2}{2}\right)\delta$.

Problem 6 — part 4 of 4.

And now I'll finish this huge problem.

We had

$$X_0=e^{-rT}E_{\tilde{P}}\left(S_T I_{\{S_{T-1}>1\}}\right)=e^{-rT}\cdot e^{\ln(S_0)+rT}\Phi\left(\frac{\ln(S_0)+(T-1)\left(r+\frac{\sigma^2}{2}\right)}{\sigma\sqrt{T-1}}\right)=$$

$$=S_0\Phi\left(\frac{\ln(S_0)+(T-1)\left(r+\frac{\sigma^2}{2}\right)}{\sigma\sqrt{T-1}}\right)$$

This is our answer.

Answer: The price of our asset equals

$$S_0\cdot\Phi\left(\frac{\ln(S_0)+(T-1)\left(r+\frac{\sigma^2}{2}\right)}{\sigma\sqrt{T-1}}\right)$$

where $\Phi(x)$ is a distribution function of a standard normal random value, $\Phi(x) = \int\limits_{-\infty}^{x}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{t^2}{2}\right)\delta$

6.3 Exam, 17.01.2014

Optimal control part

1. [10 points] A collector of wine is told by a doctor that he is going to die in T years and he develops a consumption plan that maximizes the utility of consumption of wine over remaining lifetime. At present the stock of wine equals $w(0)=w_0>0$. It will be consumed fully by time T, that is $w(T)=0$. The utility function is $U(c)=c$, where c is consumption of wine. The stock of wine follows the rule $w(t)=w_0-\int_0^t c(s)\,ds$. Consumption is bounded $0\leqslant c\leqslant\bar{c}$, where $\bar{c}>w_0/T$.

State the dynamic optimization problem, if the discount rate is $r > 0$. Introduce the current value Hamiltonian for that problem. Write down the first-order conditions. Solve the system of equations.

2. [10 points] Consider the following optimization problem: maximize $\sum_{t=0}^{\infty} 0.75^t \ln c_t$, subject to $c_t + k_t = \sqrt{k_t}$, where $k_0 > 0$. Let the state variable be k and denote the next period value of k as k'.
   (a) Write down the Bellman equation for the value function $V(k)$
   (b) Using the method of undetermined coefficients find $V(k)$
   (c) Find the optimal policy function $k' = h(k)$
3. [20 points] Consider the nonlinear system of differential equations

$$\begin{cases} \dot{x} = -y + ax(x^2 + y^2) \\ \dot{y} = x + ay(x^2 + y^2) \end{cases}$$

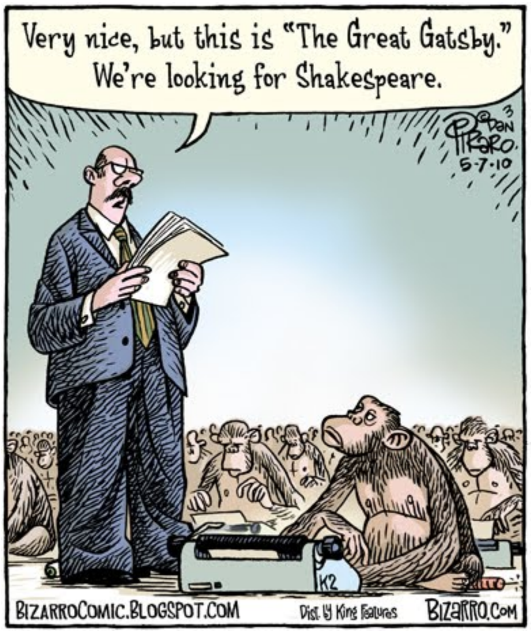
   where a is a parameter.
   (a) [5 points] Linearize the system and classify the steady-state solution
   (b) [10 points] Restore the nonlinear system and by changing the Carthesian (?) into the polar coordinates show that the system becomes

$$\begin{cases} \dot{r} = ar^3 \\ \dot{\theta} = 1 \end{cases}$$

   (c) [5 points] Solve this system and draw the phase diagram for $a < 0$, $a = 0$, $a > 0$.

Stochastic calculus part

Here W_t always denotes the standard Wiener process.



1. [10 points] It is known that $\mathrm{Var}(X) = 16$, $\mathrm{Var}(Y) = 9$, $\mathrm{Cov}(X, Y) = -1$, $\mathbb{E}(X) = \mathbb{E}(Y) = 0$. Let's denote $\hat{Y} = \mathbb{E}(Y|X)$ and $\hat{X} = \mathbb{E}(X|Y)$. It is also known that $\mathbb{E}(\hat{X}^2) = 4$ and $\mathbb{E}(\hat{Y}^2) = 1$. Find $\mathrm{Var}(\hat{Y})$, $\mathrm{Cov}(\hat{X}, Y)$, $\mathrm{Cov}(\hat{Y}, Y)$.

2. [10 points] Let's consider a Wiener process with drift, $X_t = W_t + \mu t$.
   (a) Find a non-trivial martingale of the form $M_t = e^{\beta X_t}$
   (b) Let τ be a stopping time, the first moment when X_t hits 2 or -1. Find the probability $\mathbb{P}(X_\tau = 2)$

   You may assume without proof that some version of Doob's theorem may be applied here.
3. [10 points] Is $X_t = \cos W_t$ a martingale? If not, find any non-zero function $f(t)$ such that $Y_t = f(t)X_t$ is a martingale. Find the variance of Y_t.
   Hint: You may use the fact that $\mathbb{E}(\cos W_t) = e^{-t/2}$.
4. [10 points] In the framework of Black and Scholes model find the price at the time 0 of an asset which gives you the payoff S_T^2 at the time T. Here S_t is the price of the underlying asset at time t.
5. [20 points] Consider the following stochastic differential equation

$$dX_t = -0.5e^{-2X_t}dt + e^{-X_t}dW_t,$$

   with deterministic initial value X_0.
   (a) Is X_t a martingale?
   (b) Using the substitution $Y_t = f(X_t)$ solve this differential equation. Hint: try to find a function f such that the term before dt cancels out.
   (c) Sketch the possible path of the process X_t
   (d) Using basic facts about Wiener process find the limit $\lim_{t\to\infty} \mathbb{P}(X_t \geqslant a)$ for all values of the parameter a.

marking scheme

1. 3 pts:

$$\mathrm{Var}(\hat{Y}) = \mathbb{E}(\hat{Y}^2) - (\mathbb{E}(\hat{Y}))^2 = 1 - 0^2 = 1$$

   3 pts:

$$\mathrm{Cov}(\hat{X}, Y) = \mathbb{E}(\mathbb{E}(X|Y)Y) - \mathbb{E}(\hat{X})\mathbb{E}(Y) = \mathbb{E}(\mathbb{E}(YX|Y)) - \mathbb{E}(X)\mathbb{E}(Y) = \mathrm{Cov}(X, Y)$$

   4 pts:

$$\mathrm{Cov}(\hat{Y}, Y) = \ldots = \mathrm{Cov}(\hat{Y}, \hat{Y})$$

2. $M_t = e^{-2\mu X_t}$ 5 pts, probability — 5 pts, $p = \frac{1-\exp(2\mu)}{\exp(-4\mu)-\exp(2\mu)}$
3. X_t is not a martingale — 1 pt, $Y_t = e^{t/2}X_t$ or proportional — 4 pts, variance — 5 pts $\mathrm{Var}(Y_t) = (e^t + e^{-t} - 2)/2$
4. Pricing formula — 2 pts, risk-neutral substitution — 2 pts, calculations — 6 pts, $S_0^2 e^{(r+\sigma^2)T}$.
5. (a) X_t is not a martingale — 1 pt
   (b) use of Ito's formula — 5 pts, equation $f' = f$ — 5 pts, solution $X_t = \ln(W_t + e^{X_0})$. — 5 pts
   (c) Key features: "jiggly" — 1 pt, blows down to minus infinity in finite time — 2 pts
   (d) The limit is zero.

6.4 Retake, 08.02.2014

Stochastic calculus part

1. [10 points] Is $X_t = \sin W_t$ a martingale? If not, find any non-zero function $f(t)$ such that $Y_t = f(t)X_t$ is a martingale. Find the expected value $\mathbb{E}(X_t)$
2. [10 points] In the framework of Black and Scholes model find the price of the asset, which pays you S_2/S_1 at the fixed moment of time $T = 2$
3. [10 points] Let τ be a stopping time, the first moment when W_t hits 2 or -1.
   (a) Find the probability $\mathbb{P}(W_\tau = 2)$
   (b) Find a martingale of the form $X_t = W_t^2 + f(t)$.
   (c) Find $\mathbb{E}(\tau)$
4. [10 points] The joint distribution of the random vector (X, Y) is given by its probability density function
$$f(x, y) = \begin{cases} ce^{x-y}, \text{ for } 0 \leqslant x, y \leqslant 1 \\ 0, \text{ otherwise} \end{cases}$$
where c is a normalization constant. Find $\mathbb{E}(X \mid Y)$.
5. [20 points] Let's consider the following system of stochastic differential equations
$$\begin{cases} dX_t = aX_t dt - Y_t dW_t \\ dY_t = aY_t dt + X_t dW_t \end{cases}$$
with initial conditions $X_0 = x_0$ and $Y_0 = y_0$
   (a) Find the solution of the form $X_t = f(t)\cos W_t$ and $Y_t = g(t)\sin W_t$
   (b) Prove that for any solution $D_t = X_t^2 + Y_t^2$ is nonstochastic

Answers and hints

1. $f(t) = e^{t/2}$ or proportional, $\mathbb{E}(X_t) = 0$.
2. $X_0 = e^{-r}$
3. Using Doob's theorem, $\mathbb{E}(W_\tau) = 0$, so $\mathbb{P}(W_\tau = 2) = 1/3$. Using Ito's lemma or otherwise $X_t = W_t^2 - t$ is a martingale, so $\mathbb{E}(X_\tau) = 0$ and $\mathbb{E}(\tau) = 2$.
4. $\mathbb{E}(X \mid Y) = \int_0^1 xf(x \mid Y)\, dx$. Here $f(x, y)$ maybe factored, so X and Y are independent, and $\mathbb{E}(X \mid Y) = const$.
5. Just use Ito's lemma for proposed solution form and obtain 4 equations. For point b just take dD_t and observe that it has no term before dW.