Stochastic calculus and dynamic optimization retake.
In all problems W_t denotes the standard Wiener process.

1. 15 points. The random variables X_1, X_2, ... are independent uniformly distributed on $[0;1]$. I am summing them until the first X_i greater than 0.5 is added. After this term I stop. Let's denote by S the total sum and by N — the number of terms added. Find $\mathbb{E}(S|N)$, $Var(S|N)$.

   Hint: to speed up the computations you may use the fact that $Var(X_i) = 1/12$ without proof.

2. 15 points. Consider the process $X_t = \int_0^t s^2 W_s \, dW_s$. Find $\mathbb{E}(X_t)$, $Var(X_t)$, $Cov(X_t, W_t)$

3. 20 points. For $0 < s < t$ calculate the probability $\mathbb{P}(W_t > 0 \mid W_s > 0)$.

4. 25 points. Consider the following stochastic differential equation

$$dX_t = (\sqrt{1 - X_t^2} - 0.5X_t)dt + \sqrt{1 - X_t^2} dW_t, \quad X_0 = 0 \tag{1}$$

   (a) Consider a substitution $X_t = \sin Y_t$, where Y_t is some Ito process $dY_t = A_t dt + B_t dW_t$. Using Ito's lemma find dX_t.

   (b) State conditions on A_t and B_t such that the two dX_t coincides

   (c) Find Y_t and X_t

5. 25 points. Consider the profit-maximizing problem for a representative competitive firm

$$\int_0^\infty (p - c(x(t)))q(t)e^{-rt} \, dt \to \max \tag{2}$$

   where the state variable $x(t) < 1$ represents a nonrenewable stock resource (oil) that depletes according to the equation $\dot{x} = 1 - x - q$ and $q(t)$ is the extraction rate. Here $c(x)$ is a cost function of the extraction that is defined by $c(x) = e^{-x}$. The price of oil is assumed to be constant and equal p where $e^{-1} < p < 1$. The optimization problem is to choose $q(t)$ to maximize the discounted profits, $0 < r < 1$.

   (a) Derive necessary conditions.

   (b) Prove that the steady-state exists and is unique.