Exam 12.01.2011 on stochastic calculus.

Notation: W_t is the standard Wiener process.

Part A (10 points each problem). Time allowed: 120 minutes.

1. The joint probability density function of X and Y is given by

$$f(x,y) = \begin{cases} x+y, & x \in [0;1],\ y \in [0;1] \\ 0, & \text{otherwise} \end{cases}$$

   Find $\mathbb{E}(Y|X)$ in terms of X, find the probability density function of $\mathbb{E}(Y|X)$

2. Consider the process $X_t = \int_0^t sW_s\, dW_s$. Find $\mathbb{E}(X_t)$, $Var(X_t)$, $Cov(X_t, W_t)$

3. The process Y_t is given by $Y_t = 2W_t + 5t$. The stopping time τ is given by $\tau = \min\{t | Y_t^2 = 100\}$. Find the distribution of the random variable Y_τ and the expected value $\mathbb{E}(\tau)$.

   Hint: you may find the martingales a^{Y_t} and $Y_t - f(t)$ useful

4. Find $\mathbb{P}(W_2 - W_1 > 2)$

5. In the framework of Black and Scholes model find the price of an asset which gives you the payoff of 1 rubble only if the final price S_t is at least two times bigger than the initial price S_0 of the asset.

6. Consider the free end problem, where $T > 0$ is not given

$$\int_0^T (\dot{x}^2 - x + 1)dt \to extr$$

   At the left end $x(0) = 0$ Find the optimal T value and the extremal. Check that you solved the optimality problem or show the opposite.

Part B (20 points each prolem). Time allowed: 60 minutes.

1. Consider the stochastic differential equation

$$dX_t = (\sqrt{1 + X_t^2} + 0.5X_t)dt + \sqrt{1 + X_t^2}dW_t, \quad X_0 = 0$$

   (a) Suppose that Y_t is another process that depens only on X_t, i.e. $Y_t = f(X_t)$. Find dY using the Ito's lemma.
   (b) Find such function f that the term before dW in dY is constant.
   (c) Find X_t
   (d) Sketch $\mathbb{P}(X_t > 0)$ as the function of t.

2. Consider the neoclassical optimal growth model

$$\int_0^{\infty} e^{-rt} \left(\bar{U} - \frac{1}{c(t)} \right) dt \to \max$$

subject to $\dot{k} = A \ln(1 + k) - c - \delta k$, where $A > r > \delta > 0$, $k(0) = k_0$, $\bar{U} > 0$.

   (a) Derive necessary conditions, using the current value Hamiltonian
   (b) Sketch the phase diagram for this problem with the capital intensity and consumption labeled on the horizontal and vertival axes, respectively
   (c) Check that the steady state solution exists. Provide explanation.
   (d) Explore the stability of the steady state, using the Jacobian
   (e) Why are you sure the found growth path maximizes the discounted stream of utility?