Practice problems for stochastic calculus. 2014

1. An enemy submarine is somewhere on the number line. The initial coordinate of the submarine is some unknown integer number. It is moving at some constant integer speed (units per minute).

   You can launch a torpedo each minute at any integer on the number line. If the the submarine is there, you hit it and it sinks. You have infinite number of torpedoes. You must sink this enemy sub.

   (a) Draw a picture of number line and the submarine. Just for fun!

   (b) Devise a strategy that is guaranteed to eventually hit the enemy submarine

2. Compare the power of two sets: $A = \mathbb{Q}$ — the set of all rational numbers, $B = \mathbb{Q}^2$ — the set of all possible pairs of rational numbers.

3. Using the fact that the Borel σ-algebra $\mathcal{B}$ is the smallest σ-algebra containing all subsets of the form $(-\infty; t]$ show that $\mathbb{N} \in \mathcal{B}$.

4. Let $\Omega = \mathbb{R}$. Find explicitely the smallest σ-algebra which contains the sets $A = [0; 1]$ and $B = [10; 100]$.

5. Let X be uniform on $[0; 1]$ and $Y = 1_{X<0.7} + 1_{X>0.1}$. Describe the σ-algebra $\sigma(Y)$. How many events $\sigma(X)$ contains?

6. How many different σ-algebras one may construct if Ω contains three elements? Four?

7. We throw a coin infinite number of times. Let's define the sequence of random variables X_n such that X_n is equal to 1, if the result of n-th throw is tail, and 0 otherwise. We also define a bunch of σ-algebras: $\mathcal{F}_n := \sigma(X_1, X_2, ..., X_n)$, $\mathcal{H}_n := \sigma(X_n, X_{n+1}, X_{n+2}, \ldots)$.

   Give two non-trivial (other than Ω and $\emptyset$) examples of event A, such that:

   - $A \in \mathcal{F}_{2014}$
   - $A \notin \mathcal{F}_{2014}$
   - A belongs to every $\mathcal{H}_n$

   Which σ-algebras contains the events:

   - $A = \{X_{37} > 0\}$
   - $B = \{X_{37} > X_{2014}\}$
   - $C = \{X_{37} > X_{2014} > X_{12}\}$

   Simplify where possible: $\mathcal{F}_{11} \cap \mathcal{F}_{25}$, $\mathcal{F}_{11} \cup \mathcal{F}_{25}$, $\mathcal{H}_{11} \cap \mathcal{H}_{25}$, $\mathcal{H}_{11} \cup \mathcal{H}_{25}$

8. Veniamin throws a coin until three consecutive tails appear. Let the random variable T be the number of throws. Let $\mathcal{F}_T$ be the σ-algebra of all events distinguishable by Veniamin.

   (a) Provide two non-trivial (not equal to Ω or $\emptyset$) examples of event A such that $A \in \mathcal{F}_T$ but $A \notin \sigma(T)$

   (b) Provide two non-trivial (not equal to Ω or $\emptyset$) examples of event A such that $A \in \sigma(T)$

   (c) Is it true, that $\sigma(T) \subset \mathcal{F}_T$?