Test 1.

Problem 1. Compare the powers of the set $\mathbb{Q}$ (the set of all rational numbers) and the set $\mathbb{Q}^2$ (the set of all pairs of rational numbers).

Problem 2. Let $S_n$ be a symmetric random walk, i.e. $S_n = X_1 + X_2 + ... + X_n$ where $X_n$ are independent and identically distributed with $P(X_n = 1) = P(X_n = -1) = 0.5$. Let $T$ be the time of the second local maximum plus one. For example: if the sequence of $X_n$ is given by $+1, +1, -1, -1, -1, +1, -1, -1, +1, ...$ then $T = 7$. We define the following sigma-algebras: $\mathcal{F}_n := \sigma(X_1, X_2, ..., X_n)$ and $\mathcal{F}_T$ - all events which may be distninguished by a rational agent who knows all $X_i$ up to time $T$.

a. Give an example of event $A$ such that $A \notin \mathcal{F}_T$ but $A \in \mathcal{F}_{2010}$

b. Give an example of event $B$ such that $B \in \mathcal{F}_T$ but $B \notin \mathcal{F}_{2010}$

c. Prove that $\mathcal{F}_T$ is different from each $\mathcal{F}_n$

Problem 3. Let $X$ be uniform on $[0; 1]$. We define $A := \{X > 0.1\}$ and $Y := X^2$. Find $E(Y|\sigma(1_A))$, $E(1_A|\sigma(Y))$ and $E(1_A + Y|Y - 1_A)$.