Here W_t always denotes the standard Wiener process.

1. [10 points] Let τ denote the first moment of time when $|W_t| = 100$.
   (a) What is the distribution of W_τ?
   (b) Are W_τ and τ independent?
   (c) Assuming that some version of Doob's theorem may be applied find $\mathrm{E}(e^{-2\tau})$.

   Hint: maybe $\exp(2W_t - 2t)$ will help?

2. [10 points] The random variables $X_1, X_2, \ldots, X_n, \ldots$ are independent uniformly distributed on $[0; 1]$. I am summing them until the first X_i greater than 0.5 is added. After this term I stop. Let's denote by S the total sum and by N — the number of terms added. Find $\mathrm{E}(S|N)$, $\mathrm{Var}(S|N)$, $E(S)$

   Hints: If U is uniform on $[a; b]$ then $\mathrm{Var}(U) = (b-a)^2/12$. If G has geometric distribution (the number of throws to get the first success) then $\mathrm{E}(G) = 1/p$ where p is the probability of success.

3. [10 points] Find $\mathrm{Var}\left(\int_0^t W_s\,ds\right)$.

   You may use the following guiding steps:
   (a) Find $d(tW_t)$ in short and full forms
   (b) Find $\mathrm{E}\left(2tW_t \int_0^t s\,dW_s\right)$
   (c) Find $\mathrm{E}\left(\left(\int_0^t s\,dW_s\right)^2\right)$
   (d) Find $\mathrm{E}\left(\int_0^t W_s\,ds\right)$
   (e) $(a-b)^2 = a^2 - 2ab + b^2$:)

4. [10 points] The risk-free interest rate is equal to 0.1. The volatility of the share is equal to $\sigma = 1$. You have an option to receive 1$ two years later if the percentage change of price during the first year is less than during the second year. Assume the framework of the Black and Scholes model. What is the fair price of this option?

5. [20 points] Solve the stochastic differential equation

$$dX_t = t\,dt + X_t\,dW_t,\ X_0 = 1$$

   You may use the following guiding steps:
   (a) Solve a less difficult stochastic differential equation, $dY_t = -Y_t\,dW_t$, $Y_0 = 1$
   (b) Find dA_t, where $A_t = X_t Y_t$
   (c) Find any deterministic function $f(t)$ such that $d(f(t)A_t)$ contains neither X_t nor A_t
   (d) Using full form of $d(f(t)A_t)$ find X_t. It may depend on some integrals :)