Stochastic calculus part, Retake, 15.02.2016

Here W_t always denotes the standard Wiener process.

1. [10 points] You throw a fair coin infinite number of times. Let's denote the result of the second toss by Y_2 (0 for tail and 1 for head) and the number of throw of the first «head» by N. Find $\mathrm{E}(Y_2|N)$, $\mathrm{Var}(Y_2|N)$ and $\mathrm{E}(N|Y_2)$

2. [10 points] The process Y_t is given by $Y_t = 2W_t + 5t$. The stopping time τ is given by $\tau = \min\{t|Y_t^2 = 100\}$. Find the distribution of the random variable Y_τ and the expected value $\mathrm{E}(\tau)$.

   Hint: you may find the martingales a^{Y_t} and $Y_t - f(t)$ useful

3. [10 points] Let $X_0 = 2016$ and $dX_t = 2t\,dt + t^2\,dW_t$. Find $\mathrm{E}(X_t)$ and $\mathrm{Var}(X_t)$.

4. [10 points] The risk-free interest rate is equal to 0.1. The volatility of the share is equal to $\sigma = 1$. The price of a share at $t = 0$ is $S_0 = 100$. You have an option to receive 1$ **two** years later if the price of the share after **one** year is more than 105. Assume the framework of the Black and Scholes model. What is the fair price of this option?

5. [20 points] Consider the stochastic differential equation

$$dX_t = 8W_t^2 X_t\,dt + 4W_t X_t\,dW_t, \text{ where } X_0 = 1$$

   (a) Apply Ito's lemma to $Y_t = \ln X_t$

   (b) Find the solution of the initial stochastic differential equation

Optimal control part

6. Find extremals that provide the highest or lowest values of the following integral

$$J(y) = \int_1^e \left[\frac{1}{2}x(y')^2 + \frac{2yy'}{x} - \frac{y^2}{x^2}\right] dx$$

with the boundary values $y(1) = 1$, $y(e) = 2$.

(a) (10 points) Find the extremal(s).

(b) (10 points) Let $\tilde{y}(x)$ be the extremal you have found. Let $h(x) \in C^1$ and $h(1) = h(e) = 0$ ($h(x)$ is not identically zero). Prove that $J(\tilde{y} + h) - J(\tilde{y}) > 0$.

7. Consider Ramsey's model. Maximize the integral $I = \int_0^\infty [u(c) - B]dt$ subject to $\dot{k} = f(k) - c - \delta k$, $k(0) = k_0$. Function $u(c)$ monotonically increases and tends to B at the infinity, moreover I converges.

(a) (5 points) Derive Ramsey's Law $\frac{d}{dt}u'(c) = u'(c)[\delta - f'(k)]$.

(b) (5 points) Let

$$u(c) = \begin{cases} 2c - \frac{c^2}{B}, & \text{for } c \leq B \\ B, & \text{otherwise} \end{cases}$$

Production function $f(k) = \alpha k$ and $\alpha > \delta$. Find the optimal solutions c^*, k^*. Do these solutions necessarily have an economic sense?

8. (10 points) Solve the problem on the bounded optimal control $\int_0^T (1 - \beta - u)x dt \to \max$, subject to $\dot{x} = \alpha x u$, $x(0) = x_0$, $0 \leq u \leq 1$. In this problem $\alpha > 0$ and $0 < \beta < 1$.