Final exam retake on mathematics for the Master degree students, ICEF February, 3rd, 2012
Section A. 10 points for each problem.

1. Solve the bounded control problem with the free end value where $T > 1$ is specified in advance but $x(T)$ is not.
$$\int_0^T (x-u)\,dt \to \max,\ x(0) = 1$$
$x' = u$, $0 \leqslant u \leqslant x$. What guarantees that a maximizer would be found?

2. Two stochastic processes are defined by the system of SDE with the Brownian motions W_{1t}, W_{2t} independent of each other
$$\begin{cases} dX_t = (2 + 5t + X_t)dt + 3dW_{1t} \\ dY_t = 4Y_t dt + 8Y_t dW_{1t} + 6dW_{2t} \end{cases}$$
Calculate $d(X_tY_t)$

3. Find the value of the constant a such that the process $X_t = W_t^3 - a\int_0^t W_s\,ds$ is a martingale.

4. The process Y_t is given by $Y_t = 2W_t + 5t$. The stopping time τ is given by $\tau = \min\{t|Y_t^2 = 100\}$. Find the distribution of the random variable Y_τ and the expected value $\mathbb{E}(\tau)$.
Hint: you may find the martingales a^{Y_t} and $Y_t - f(t)$ useful

5. What is the expected value and variance of W_t^2 for $t > s$ given that $W_s = x$?

6. In the framework of Black and Scholes model find the price at the time 0 of an asset which gives you the payoff $\max\{\ln(S_t), 0\}$ at the time t. Here S_t is the price of the underlying asset.

Section B. 20 points for each problem.

1. The goal of this exercise is to solve the SDE
$$dX_t = \frac{1}{X_t}dt + X_t dW_t$$
with initial condition $X_0 = 1$.
   (a) Apply the Ito's lemma to the process $Y_t = \exp(f(t) + g(W_t))X_t^2$
   (b) Find non-constant functions f and g such that the coefficient before dW_t in the expression for dY_t is zero.
   (c) Find X_t. The final expression may contain a Riemann integral of some stochastic process.

2. A typical firm with the production function $x = f(l)$ in an economy employs 1 unit of the capital paying for it $\bar{r}$. There are n identical firms in this economy where $n(t)$ is a function of time. Let the output of a firm be $x(t)$. Then the aggregate supply equals nx. Assuming that the market is in the equilibrium we use the inverse demand function $p(nx)$, where $p'(y) < 0$. Equation defining the dynamics of labor is given by $\frac{dl}{dt} = a(w - \bar{w})$, where $a > 0$ and $\bar{w}$ is some equilibrium wage rate. The growth (or fall) in number of firms is governed by the equation $\frac{dn}{dt} = b(px - \bar{w}l - \bar{r})$, where $b > 0$. Write down the system of ODE for the unknowns $l(t)$, $n(t)$, using the first-order condition for the profit-maximizing firm, find and classify the steady-state solutions (if any exist). Production function is twice differentiable and concave everywhere.