Problem 1.
$Y \sim U[0;1]$, $X = Y^3$, $A = \{Y > 0.8\}$, $Z = 1_A - Y$. Find $E(Z|X)$, $E(X|Z)$, $E(X|1_A)$
Solution:
We note that $X = Y^3$ and $Z = 1_A - Y$ are deterministic functions of Y, so:

$$E(X|Z) = X$$

(+3 points)
and

$$E(Z|X) = Z$$

(+3 points)
The most difficult:

$$E(X|1_A) = E(Y^3|A)1_A + E(Y^3|A^c)1_{A^c}$$

(+4 points)

Problem 2.
$Y_t = W_t^3 + f(t)W_t$. Find $f(t)$ if it is known that Y_t is a martingale.
Solution:
Using the Ito's lemma we find $dY_t = 3W_t^2 dW_t + f'(t)W_t dt + 3W_t dt$.
(+5 points for the use of the Ito's lemma)
Y_t is a martingale, so $f' = -3$
We find $f'(t) = -3$ and $f(t) = -3t + c$ where $c \in \mathbb{R}$
(+5 points for the rest. The most common error: missing c gives -2 points penalty)
Alternative way:
Using the definition of a martingale:

$$E(Y_t|\mathcal{F}_s) = Y_s$$

Thus we have an equation $f(t) - f(s) = -3(t-s)$ for all s and t
(+5 points)
Plugging $s = 0$ we have $f(t) = -3t + f(0)$.
(+5 points)

Problem 3.

$$Y_n = \frac{nX_1 + X_2 + ... + X_{n+1}}{2n}$$

. In what sense does the Y_n converge? What is the limit? Find the expected value and the variance of the limit.

Solution:

A little bit of algebra and we have:

$$Y_n = \frac{X_1}{2} + 0.5\frac{X_2 + ... + X_{n+1}}{n}$$

The Strong Law of Large Numbers (as):

$$\frac{X_2 + ... + X_{n+1}}{n} \to E(X_i) = 2010$$

So (as):

$$Y_n \to \frac{X_1}{2} + 1005$$

(+4 points for the correct limit)

And $E(\lim Y_n) = 2010$, $Var(\lim Y_n) = 2011/4$

(+1 point for the expected value, +2 points for the variance)

Almost surely convergence implies convergence in probability and in distribution.

(+1 point for all convergence types except L^2)

We check L^2 convergence:

$$E((Y_n - \frac{X_1 + 2010}{2})^2) = \frac{1}{4}E((\bar{X} - 2010)^2) = \frac{1}{4}Var(\bar{X}) = \frac{2011}{4n}$$

This tends to zero and we have established L^2 convergence.

L^2 convergence implies L^1 convergence.

(+2 points for the L^2 convergence)

Problem B1.

$$dY = \frac{W}{Y}dW - \frac{W^2}{2Y^3}dt$$

a) Verify whether $Z_t = Y_t^2$ is a martingale
b) Solve this equation if $Y_0 = 1$
Solution:
a) Apply the Ito's lemma for $Z_t = Y_t^2$:

$$dZ = 2YdY + \frac{1}{2}2(dY)^2$$

Where $(dY)^2 = \frac{W^2}{Y^2}dt$
(+5 points)
This simplifies to

$$dZ = WdW$$

There is no dt term and we conclude that Z_t is a martingale.
(+5 points)
b) Calculation of the stochastic integral $\int_0^t W_s dW_s = W_t^2 - t$
(+5 points)
So we obtain $Z_t = Z_0 + W_t^2 - t$. And using initial conditions:
$Y_t = \sqrt{1 + W_t^2 - t}$
(+5 points)