Stochastic calculus part, 12.01.2016

Here W_t always denotes the standard Wiener process.

1. [10 points] You throw a fair coin until «head» appears. Let's denote the result of the first toss by Y_1 (0 for tail and 1 for head) and the total number of throws by N. Find $\mathrm{E}(Y_1|N)$, $\mathrm{Var}(Y_1|N)$ and $\mathrm{E}(N|Y_1)$

2. [10 points] Consider τ, the first moment of time when the standard Wiener process will touch the barrier $4y^2 = x + 1$, or formally, $\tau = \inf\{t \mid t \geq 0, \ |W_t| = 0.5\sqrt{t+1}\}$. Find $\mathrm{E}(\tau)$.

   Hint: you may find the process $M_t = W_t^2 - t$ useful, you may also suppose that technical conditions of Doob's theorem are satisfied

3. [10 points] Let

$$Y_t = \exp\left(-6t^3 + \int_0^t f(s)\,dW_s\right),$$

   where f is some deterministic function.

   (a) Using Ito's lemma find dY_t

   (b) Find at least one function f such that Y_t is a martingale

4. [10 points] The risk-free interest rate is equal to 0.1. The volatility of the share is equal to $\sigma = 1$. You have an option to receive 1$ two years later if the price growth during the second year is higher than 5%. Assume the framework of the Black and Scholes model. What is the fair price of this option?

5. [20 points] Consider the stochastic differential equation

$$dX_t = X_t^3\,dt + X_t^2\,dW_t,$$

   (a) Apply Ito's lemma to $Y_t = f(X_t)$

   (b) Find all the functions f that makes Y_t a martingale

   (c) Find all the solutions of the stochastic differential equation

   (d) Find the solution such that $X_0 = 2$.

   (e) Find the probability that the sample path of X_t will be continuous for $t \in [0; \infty)$

Optimal control part

6. (10 points) Given the system of differential equations

$$\begin{cases} \dot{x} = \sin(x+y) \\ \dot{y} = \sin(x-y) \end{cases}$$

explore the behavior of its solutions in the neighborhood of the 2 points: $A(\pi, \pi)$ and $B(3\pi/2, 3\pi/2)$. Draw the phase portraits near A and B based on the knowledge of their eigenvalues, eigenvectors where possible.

7. (10 points) Solve the bounded control problem: maximize

$$\int_0^1 (2x - u^2/2)\, dt$$

subject to constraints $\dot{x} = u - x + t^2$, $x(0) = 0$, $-1 \leq u \leq 0$. Verify that the maximizer has been found by applying one of the sufficient conditions.

8. (20 points) Solve so-called "limit pricing problem". A homogeneous product is produced by a dominant firm along with the "competitive fringe" consisting of the $x(t)$ identical firms (x is a continuous variable). Demand on good is given by $f(p) \in C^2$, where $p(t)$ is the price, $f'(p) < 0$ for $p > 0$. Let the dominant firm has a constant returns to scale technology with the marginal costs $c = const > 0$. Then $(p-c)f(p)$ is a strictly concave function. The problem of the firm is to maximize the discounted stream of profits

$$\int_0^\infty e^{-rt}(p-c)(f(p)-x)\, dt$$

(each fringe firm produces only one unit of good) subject to constraint $\dot{x} = k(p - \bar{p})$ where k is some number, $\bar{p}$ is the equilibrium price and $r > 0$. Also $x(0) = x_0$.

   (a) Application of the current value Hamiltonian is required here. Derive the system of the first-order conditions.

   (b) By eliminating the Lagrange multiplier reduce the system to 2 equations with respect to (x, p). Check that $H_{pp} < 0$.

   (c) Show that if the dominant firm operates at the price level $\bar{p} = c$ then the fringe firms supply the entire market in the equilibrium.

   (d) Let the equilibrium price $\bar{p}$ be slightly above c. By evaluating the derivative $\frac{\partial x^s}{\partial \bar{p}}$ at $\bar{p} = c$ show that the dominant firm's market share becomes positive if $\bar{p}$ is slightly above c (x^s is the equilibrium number of the fringe firms).