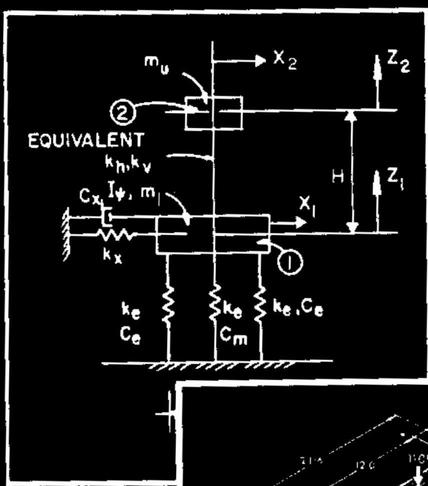
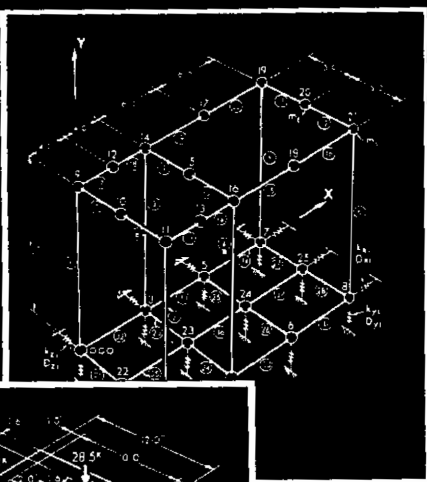
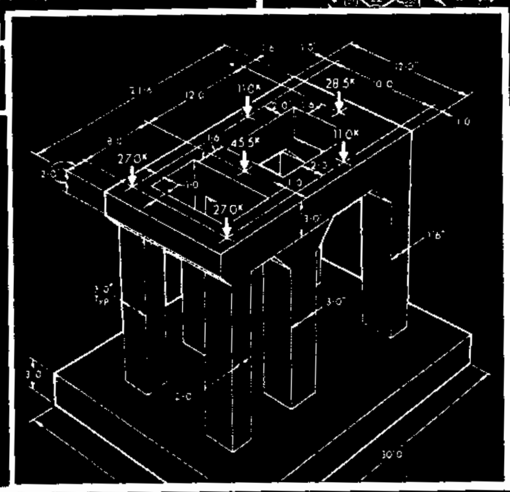
Design of Structures and Foundations for Vibrating Machines

Suresh Arya Michael O'Neill
George Pincus



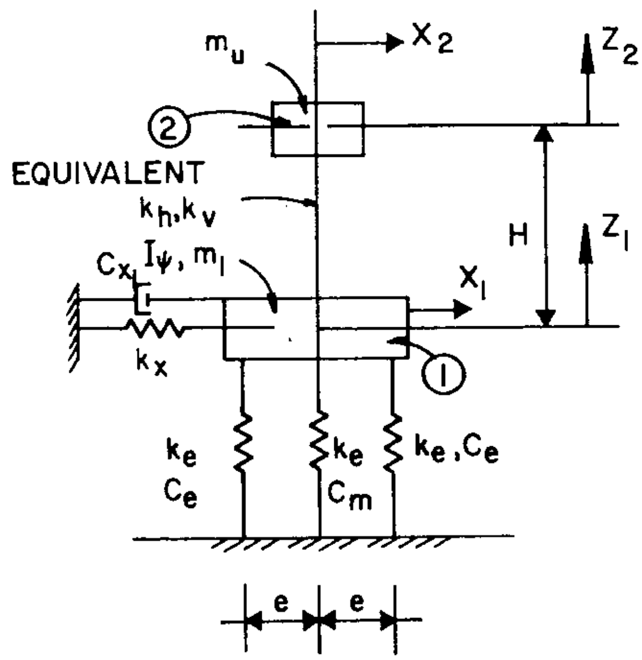




Theory and practical design applications for dynamically loaded structures.

Design of Structures and Foundations for Vibrating Machines

Design of Structures



Design of Structures and Foundations for Vibrating Machines

and Foundations for Vibrating Machines

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Contents

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Preface

The design of structures and foundations supporting dynamic loads has gradually evolved from an approximate rule-of-thumb procedure to a scientifically sound engineering procedure. Current state of the art allows engineers to reliably design structures which support increasingly heavier and larger machines. Recent advances in a number of engineering disciplines, when merged with a traditional well-established body of theoretical knowledge, have resulted in definite procedures for the analysis and design of dynamically loaded structures. However, most concepts and procedures used in the design of structures carrying dynamic machines and ultimately supported by the soil have heretofore been dispersed in texts dealing with a single aspect or a limited portion of the problem. This text brings together all those concepts and procedures for design of dynamically loaded structures. Disciplines that are involved in modern design procedures include: theory of vibrations, geotechnical engineering including soil dynamics and half-space theory, computer coding and applications, and structural analysis and design. It is assumed that the reader is an engineer or designer who is familiar with these areas. However, a basic introduction in each area is also included in the text to enhance the background of some readers.

The book includes an introductory chapter which reviews basic fundamentals. Chapter 2 describes alternatives of modeling dynamically loaded systems while Chapter 3 considers and lists the information necessary for design. Chapters 4 and 5 describe the geotechnical aspects of the problem and Chapter 5 specifically considers flexible mats and deep foundations. Finally, Chapters 6 and 7 include actual examples of different types of structures supporting dynamic machines.

This book is written by practicing engineers and engineering teachers. Practitioners and students will find the information contained here useful in their work. Also, the book will provide additional opportunities to merge the real world of design with senior- and graduate-level engineering classroom instruction. Finally, this book will serve as a model for integration of knowledge which cuts across several traditional, but previously loosely connected areas.

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March, 1979

1 Introduction-Fundamentals

The analysis and design of foundations and structures subjected to vibratory loads is considered a very complex problem because of the interaction of structural engineering, geotechnical engineering, and the theory of vibration. These foundations may be designed as a simple block, either of plain concrete or of reinforced concrete, not different in resemblance from a footing designed for static loads. The practicing engineer who is generally not theoretically motivated ordinarily shuns theoretical investigations partly because these investigations on a massive concrete block do not result in any additional reinforcement other than ordinary minimum percentage of reinforcement required by the governing codes. Even when engineering talent is available for a theoretically exact analysis, other factors such as economy, lack of high-speed computers, or design tradition result in an approximate nondynamic design. Thus, it has become imperative to devise practical design procedures which include the various aspects of design and analysis of these foundations in a way that the least effort is involved in the theoretical investigation. The design engineer should recognize that the theoretical dynamic investigation is an integral part of the design effort.

In this book, an effort has been made to use and simplify the latest theoretical knowledge available in this field (ref. 1). An easy-to-follow step-by-step routine is developed for actual design problems.

In addition, at every step of investigation, a brief description is presented explaining the physical meaning of the parameters used and role they play in the design process.

Structural System of Foundations

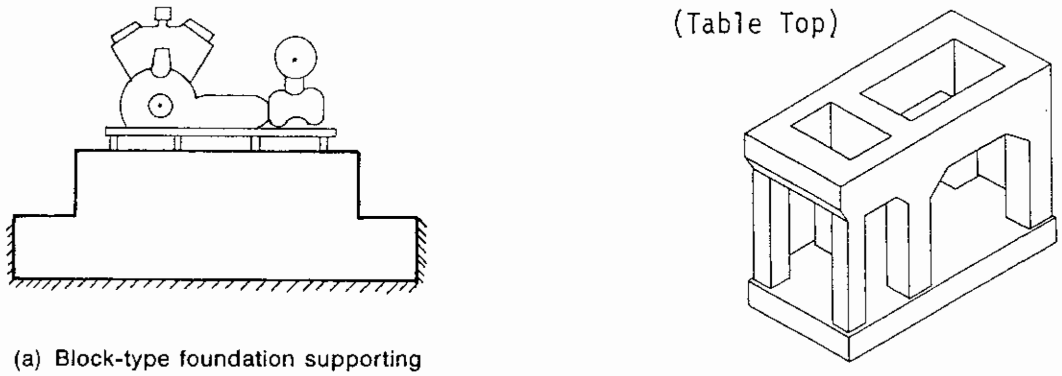
The structural form of machine foundations is generally determined by the information provided by the geotechnical consultant and the machine manufacturer. However, during the design phase, it may become necessary to adjust the dimensions or shape of the foundation, partly to meet the design criteria or to avoid interference with other fixed objects such as pipelines and building foundations. The broad categories of foundations are (a) shallow foundation (resting on soil) and (b) deep foundation (supported by piles or piers). A further classification involves the structural configuration of the foundation:

1. Block-type foundation, consisting of a thick slab of concrete directly supporting the machine and other fixed auxiliary equipment.
2. Elevated pedestal foundation (table top), consisting of a base-slab and vertical columns supporting a grid of beams at the top on which rests skid-mounted machinery. These types of foundations are illustrated in Figure 1-1.

Theoretical Approach

Vibrations developed by operating machinery produce several effects which must be considered in the design of their foundations in addition to the usual design static loads. In some cases, if the size of the machine involved is small, it may be appropriate to design the foundation for equivalent static loads instead of strictly applying the vibration design criteria. However, once the design engineer has recognized the need for a vibration analysis, it is necessary that the designer possess a clear understanding of the fundamentals of the theory of vibration (refs. 3, 5, 6 and 7), modeling techniques (refs. 2 and 8), soil dynamics (ref. 1), and in some cases, the application of computer programs (refs. 2 and 3 of Chapter 7).

In the step following the selection of the foundation gross geometry, the design engineer is faced with the



(a) Block-type foundation supporting reciprocating compressor

(b) Typical pedestal foundation

Figure 1-1. Types of foundations for vibrating machines.

vibration analysis. The usual procedure is to establish a mathematical model of the real structure which is a necessary prerequisite in order to apply the theory of vibrations. The mathematical representation of a structural system is usually defined in terms of a lumped mass, an elastic spring, and dashpot for each degree of freedom. The terms which are used in the development of the theory of vibrations are described in the Terminology section provided at the end of this chapter.

Fundamentals of Theory of Vibrations

The subject of vibration deals with the oscillatory behavior of physical systems. All physical systems built of material possessing mass and elasticity are capable of vibration at their own natural frequency which is known as a dynamic characteristic. Engineering structures subjected to vibratory forces experience vibration in differing degrees, and their design generally requires determination of their oscillatory behavior. The present "design office" state-of-the-art considers only their linear behavior because of the convenience afforded by applying the principle of superposition, and also because the mathematical techniques available for their treatment are well developed. In contrast, nonlinear behavior of systems is less well known, and the mathematical treatment is difficult to apply. However, all structures tend to become nonlinear at high amplitude of oscillation, and a nonlinear analysis is required under those conditions.

Single-Degree-of-Freedom System

An engineering structure (a fixed beam) is illustrated in Figure 1-2a. The beam is supporting a machine generating a harmonic centrifugal force. A step-by-step procedure will be described for modeling the actual structure.

Calculation of Parameters for Mathematical Model

Equivalent Mass, m_e

The beam has distributed mass along its length, and its ends are fixed against rotation. In calculating the mass for the mathematical model, it is necessary to lump the mass only at points where the dynamic force is acting, and also at those points where the dynamic response is required. In this example, the dynamic force is acting at the middle and the response is also required at the middle. The technique for obtaining the lumped mass is to equate the kinetic energies of the real and the equivalent systems (refs. 2 and 8). First, a deflected shape of the real system is assumed, Figure 1-2b which corresponds to the predominant mode. In this example, the beam can have predominant translational modes in the x-y plane, the x-z plane, and a rotational mode about the x-axis. Thus, the model has three single-degree-of-freedom systems independent of each other. Considering only the deflected shape in the x-y plane, and assuming the shape is the same as that which would be caused by a concentrated load P applied statically in the middle,

$$y_u = (Px^2/48\ EI_z)\ (3l - 4x) \qquad (1\text{-}1)$$

$$y_{umax} = Pl^3/192\ EI_z \qquad (1\text{-}2)$$

Assuming the beam's behavior stays within the elastic range and the maximum velocity at any point along the

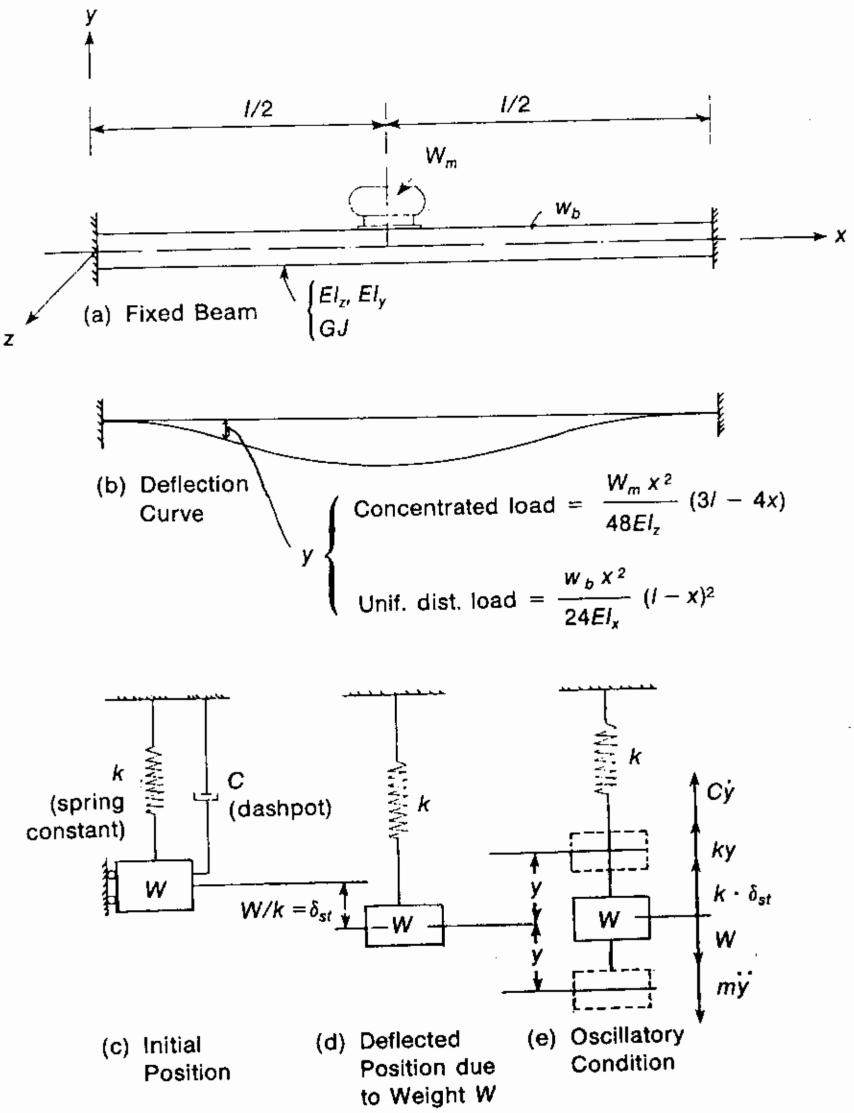


Figure 1-2. SDOF representation of a fixed beam supporting machinery in the middle.

beam is proportional to the ordinate of the deflection curve at that point, then the velocities of the beam are

$V_u = (CPx^2/48\ EI_z)\ (3l - 4x)$ and

$V_{max} = C(Pl^3/192\ EI_z)$,

where C = constant relating velocity to deflection.

Thus,

$$V_u = (4x^2/l^3)\ (3l - 4x)\ V_{max} \qquad (1\text{-}3)$$

The total kinetic energy of the beam is given by

$$KE_b = \int_0^l \tfrac{1}{2}\, m_b\, V_u^2\, dx$$
$$= 8\, m_b/l^6\, V^2_{max}(2) \int_0^{l/2} (9l^2x^4 + 16x^6 - 24lx^5)\, dx, \qquad (1\text{-}4)$$

where m_b mass per unit length.

Then,

$$\int_0^{l/2} (9l^2x^4 + 16x^6 - 24lx^5) dx = 13/1120\, l^7$$

Equation (1-4) after integration reduces to

$$KE_b = (13/70)\ V^2_{max}\, m_t\ ,$$

where $m_t = m_b\, l$, the total mass of the beam. The kinetic energy of the equivalent system is given by

$$KE_e = \tfrac{1}{2}\, m_e\, V_e^2$$

Since the kinetic energies must be equal in both systems, and since V_{max} must equal V_e,

$$KE_b = KE_e$$

$$(13/70)\ m_t\ V_{max}^2 = \tfrac{1}{2}\ m_e\ V_e^2$$

or

$$0.371\ m_t = m_e \qquad (1\text{-}5)$$

Therefore, 37.1% of the total distribution mass should be included as the corresponding mass of the mathematical model. However, the mass of the machine is located at the middle; hence, that entire mass should be considered part of the equivalent mass.

$$m_e = 0.371\ m_t + m_m \qquad (1\text{-}6)$$

Similarly, equivalent mass can be calculated in the x-z plane which will also have the same magnitude since the deflection curve of the beam remains the same. Table 1-1 lists equivalent mass factors for beams and slabs with different types of loads and support conditions.

Equivalent Spring Constant, k_e

The spring constant of an equivalent system is obtained by equating the resistances to deformation of the prototype and the mathematical model, in this case, the uniformly distributed loaded beam vs. the modeled middle loaded beam. The resistance offered by the beam per unit load is given by the reciprocal of the deflection produced by the same unit static load applied at that point. Therefore, in the example under consideration, the resistance offered due to a unit concentrated load at midspan (ref. 9) is

$$R_b = 1/(l^3/192\ EI_z)$$
$$= 192\ EI_z/l^3, \qquad (1\text{-}7)$$

which when equated to the resistance of the equivalent system gives

$$k_e = 192\ EI_z/l^3 \qquad (1\text{-}8)$$

In Table 1-1, values of spring constants for equivalent systems are presented for different types of loads and support conditions.

Equivalent Forcing Function, F(t)

The dynamic force may be distributed over a certain length of the element, and in order to obtain its equivalent, concentrated load value for application in the single-degree-of-freedom system, the work done by the actual system is equated to that done on the equivalent system. The load factor, k_l, with which the distributed dynamic force should be modified to determine the equivalent, concentrated dynamic force is given in Table 1-1. For the model shown in Figure 1-2a, the dynamic force acts at the middle of the beam, thus, a force modification factor is not required.

Formulation of Mathematical Model

A procedure for obtaining the values of various parameters in a mathematical model which equal those in an actual system is given in the preceding section. An equivalent mathematical model is shown in Figure 1-2c. The mass and the spring constants are the equivalent parameters corresponding to an actual system. An equivalent damping coefficient is not required in this particular model since it is associated with the velocity of the system only, and its effect is implicitly included when the equivalent values of mass and spring constant parameters are calculated. The chosen model has three independent degrees of displacement and/or rotation, and therefore, there are three individual equivalent models having a single degree of freedom each. The technique of mathematical formulation for each of the three models is the same, and therefore, only one single-degree-of-freedom model will be examined in detail.

A model, shown in Figure 1-2d with a weight W, is attached to a weightless spring k, and the spring stretches by an amount $\delta_{st} = W/k$. The system is initially in a state of static equilibrium with the dead weight W balanced by the restoring pull of the spring $k\delta_{st}$. Subsequently, the weight W is set into oscillation by the application of some disturbance.

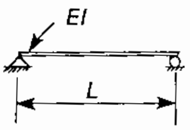
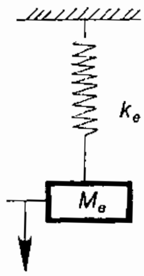
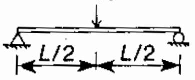
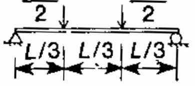
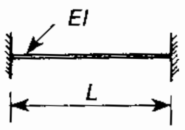
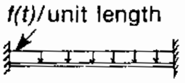
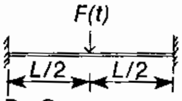
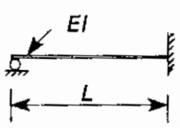
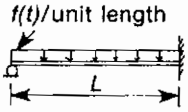
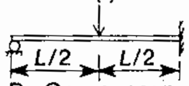
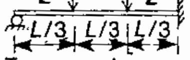
Starting at time equal to zero, the system vibrates freely with an amplitude of $\pm y$ displacement (Figure 1-2e). The forces acting on the body are applied against the direction of motion and include: the resistance offered by the spring $k\ (y + \delta_{st})$, the resistance $C\dot{y}$ offered by the viscous damping C, and inertia force $m\ \ddot{y}$, given by Newton's second law of motion. The latter force is equal to the mass of the system multiplied by its negative acceleration (the state of motion of a mass at any instant may be considered as in a state of static equilibrium upon introduction of the inertia force). The force acting in the direction of motion is the weight W. Summing up the forces, the resulting equation of motion is

$$m\ddot{y} + C\dot{y} + k\,(y + \delta_{st}) = W\,,$$

which reduces to

$$m\ddot{y} + C\dot{y} + ky = 0 \qquad (1\text{-}9)$$

Table 1-1
Dynamic Design Factors for Beams and One-Way Slabs

Type of Structural Member	Type of Dynamic Loading	Equivalent Parameters of a Single-Degree-of-Freedom System: Dynamic Load Factor k_l	Mass factor k_m: Conc. Ld.*	Mass factor k_m: Unif. Ld.	Spring Constant k_e	Equivalent Mathematical Model
 1. Simply supported beam	$f(t)$/unit length  A. Uniform load	0.64	—	0.50	$\frac{49.2\,EI}{L^3}$	 k_e, M_e
	$F(t)$  B. One concentrated load	1.0	1.0	0.49	$\frac{48.0\,EI}{L^3}$	
	$\frac{F(t)}{2}$ $\frac{F(t)}{2}$  C. Two equal concentrated loads	0.87	0.76	0.52	$\frac{49.1\,EI}{L^3}$	$F_e(t) = k_l \times$ (total dynamic load on the span)
 2. Fixed ends beam	$f(t)$/unit length  A. Uniform load	0.53	—	0.41	$\frac{203.5\,EI}{L^3}$	$M_e = k_m \times$ (total mass on the span) $k_e = k_l \times k$ where k = spring constant of a real system.
	$F(t)$  B. One concentrated load	1.0	1.0	0.37	$\frac{192.0\,EI}{L^3}$	*Equal parts of the concentrated masses are lumped at each concentrated dynamic load.
 3. Simply supported and fixed end	$f(t)$/unit length  A. Uniform load	0.58	—	0.45	$\frac{107.3\,EI}{L^3}$	
	$F(t)$  B. One concentrated load	1.0	1.0	0.43	$\frac{107.0\,EI}{L^3}$	
	$\frac{F(t)}{2}$ $\frac{F(t)}{2}$  C. Two equal concentrated loads	0.81	0.67	0.45	$\frac{106.9\,EI}{L^3}$	

This equation is an ordinary second-order linear differential equation with constant coefficients, also called a homogenous equation. For the case when a forcing function $F(t)$ is also acting, the resulting equation is classified as nonhomogenous (ref. 10) and is written as

$$m\ddot{y} + C\dot{y} + ky = F(t) \tag{1-10}$$

The solution of Equation (1-9) yields the dynamic characteristic of the system such as the natural frequency, the damped natural frequency, the critical damping coefficient, or the transient motion of the system. Each of these terms has a special significance depending upon the particular problem at hand.

Transient or Free Vibrations

A solution of the form $y = e^{st}$ is assumed for Equation (1-9) where s is a constant to be determined, and t is the independent time variable; then,

$$\dot{y} = se^{st}, \ddot{y} = s^2e^{st} \tag{1-11}$$

Upon substitution of y, $\dot{y}$, $\ddot{y}$ into Equation (1-9), the following expression is obtained:

$$(s^2 + [C/m]s + [k/m])\, e^{st} = 0 \tag{1-12}$$

Since e^{st} must be greater than zero for all values of t,

$$s^2 + (C/m)s + (k/m) = 0 \tag{1-13}$$

Equation (1-13) is a quadratic equation having two roots:

$$\begin{aligned} s_1 &= (1/2m)\,[-C + \sqrt{C^2 - 4km}] \\ s_2 &= (1/2m)\,[-C - \sqrt{C^2 - 4km}] \end{aligned} \tag{1-14}$$

Several terms, relating various parameters of Equation (1-9), are defined as:

$\omega_n = \sqrt{k/m}$ is called the circular natural frequency of the system in radians/sec;

$C_C = 2\sqrt{km}$ is the critical damping of the system in units of force/ velocity;

$D = C/C_C = C/2\sqrt{km}$ is called the damping ratio; and $\omega_d = \omega_n \sqrt{1 - D^2}$ is named the frequency of oscillation of the system with damping included.

The complete solution of Equation (1-9) is

$$y = Ae^{s_1t} + Be^{s_2t}, \tag{1-15}$$

where A and B are arbitrary constants which depend upon the initial problem conditions. The motion described by Equation (1-15) is called transient motion of the system, and the oscillations die out in a short interval of time when significant damping is present.

Equations (1-14) and (1-15) show that the nature of oscillation depends upon the value of C. Four possible values of C will be considered here (ref. 6) to illustrate the physical significance of Equation (1-15).

Case 1: C = 0 (no damping). This case reduces the problem to an undamped system, and the roots obtained from Equation (1-14) are $s_{1,2} = \pm i\omega_n$. Equation (1-15) can be written as

$$y = Ae^{i\omega_nt} + Be^{-i\omega_nt} \tag{1-16}$$

Equation (1-16) can be written in three alternate forms by the use of trigonometric identities and complex numbers:

$$y = C_1e^{i(\omega_nt-\phi)} \tag{1-17}$$

$$y = B_1 \cos \omega_nt + B_2 \sin \omega_nt \tag{1-18}$$

$$y = C_1 \cos (\omega_nt - \phi) \tag{1-19}$$

Equation (1-17) is in terms of phasors, while C_1 and ϕ are components of a complex number. B_1 and B_2 are arbitrary constants in Equation (1-18) representing the real part of the solution and can be evaluated from the initial boundary conditions. For example, at time $t = 0$, the system has a given initial displacement $y(0) = y_0$ and an initial velocity $y(0) = v_0$. Equation (1-18) then becomes

$$y = y_0 \cos \omega_nt + (v_0/\omega_n) \sin \omega_nt \tag{1-20}$$

and the velocity function,

$$\dot{y}/\omega_n = -y_0 \sin \omega_nt + (v_0/\omega_n) \cos \omega_nt \tag{1-21}$$

Equation (1-19) can be obtained from Equation (1-18) if the following substitutions are made: $B_1 = C_1 \cos \phi$, and $B_2 = C_1 \sin \phi$. Then $\tan \phi = B_2/B_1$, and $C_1^2 = B_1^2 + B_2^2$. By using the trigonometric identities,

$$y = C_1 \cos (\omega_nt - \phi) \tag{1-22}$$

$$\dot{y}/\omega_n = -C_1 \sin (\omega_nt - \phi) \tag{1-23}$$

In Equation (1-22), $C_1 = \sqrt{y_0^2 + (v_0/\omega_n)^2}$ is called amplitude of vibration, and $\phi = \tan^{-1}(v_0/\omega_ny_0)$ is called the phase angle. A graphical representation of Equation (1-22) is given in Figure 1-3a by the projections of a vector C_1 rotating about a fixed point O, with a constant velocity ω_n. The projection upon the ordinate axis represents the instantaneous displacement y, while the projection on the abscissa gives the velocity function $\dot{y}/\omega_n$ according to Equations (1-22) and (1-23), respectively. A displacement time curve based on Equation (1-20) can be obtained fom Figure 1-3a by project-

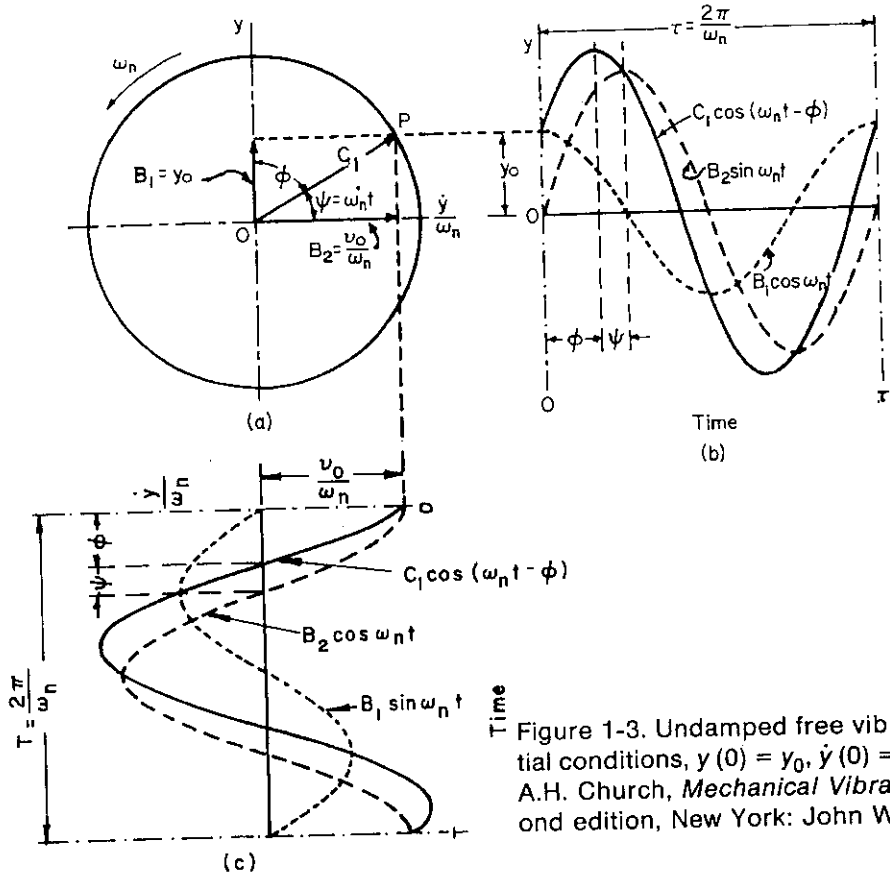


Figure 1-3. Undamped free vibrations initial conditions, $y(0) = y_0$, $\dot{y}(0) = v_0$ Source: A.H. Church, *Mechanical Vibrations*, second edition, New York: John Wiley, 1963.

ing instantaneous values of point P to the right, as shown in Figure 1-3b. A velocity versus time plot based on Equation (1-21) can be obtained by projecting point P vertically, as shown in Figure 1-3c. The variation of these terms with time is shown for a complete cycle. The time required for one complete cycle is called the period T and equals $2\pi/\omega_n$ sec. The corresponding cycle frequency is $f_n = 1/T = \omega_n/2\pi$ cps. These cycles are identically repeated since this system is undamped.

Case 2: $C^2 < 4$ km, but > 0 (underdamped). In this case, the roots of Equation (1-14) are complex conjugates, and s_1 and s_2 become

$$\left. \begin{aligned} s_1 &= \omega_n \left(-D + i\sqrt{1-D^2} \right) \\ s_2 &= \omega_n \left(-D - i\sqrt{1-D^2} \right) \end{aligned} \right\} \qquad (1\text{-}24)$$

when the damping ratio $D = C/C_C = C/2\sqrt{km} = C/2\omega_n m$ is introduced. Further substitution of Equation (1-24) into Equation (1-15) and conversion to a trigonometric form with the aid of Euler's formula $e^{\pm i\theta} = \cos\theta \pm i \sin\theta$, gives

$$y = e^{-D\omega_n t} \left(B_1 \sin \omega_n \sqrt{1-D^2}\, t + B_2 \cos \omega_n \sqrt{1-D^2}\, t \right) \qquad (1\text{-}25)$$

or

$$y = e^{-D\omega_n t}\, Y \sin(\omega_d t + \phi) \qquad (1\text{-}26)$$

Where $B_1 = Y\cos\phi$ and $B_2 = Y \sin\phi$ and $\omega_d = \omega_n\sqrt{1\text{-}D^2}$. The term ω_d is called the damped natural frequency, and Y and ϕ are arbitrary constants to be determined from the initial boundary conditions in a similar way to the procedure used in Case 1. The type of motion described by Equation (1-26) is oscillatory with frequency of ω_d and is shown in Figure 1-4a. The amplitude of oscillation Y will diminish with time and is proportional to $e^{-D\omega_n t}$ as shown by the dotted lines.

Case 3: $C^2 = 4$ km (critical damping). The damping corresponding to this case ($C = 2\sqrt{km}$, $C = C_C$ or $C/C_C = D = 1.0$) is referred to as critical damping. For this value of $C^2 = 4\,km$, Equation (1-13) has two equal roots, $s_{1,2} = -C/2m$. In this case, the general solution of the second-order differential equation is

$$y = Ae^{-(c/2m)t} + Bte^{-(c/2m)t} \qquad (1\text{-}27)$$

Substituting the value of $C/2m = 2\sqrt{km}/2m = \omega_n$ and applying the initial boundary condition, $y\,(t = 0) = y_0$, and $\dot{y}\,(t = 0) = v_0$ in Equation (1-27) gives

$$y = \{y_0 + [(v_0/\omega_n) + y_0]\,\omega_n t\}\, e^{-\omega_n t} \qquad (1\text{-}28)$$

A graphical representation of Equation (1-28) is shown in Figure 1-4b. The motion described by Equation (1-28) is aperiodic. Since critical damping represents the limit of aperiodic damping, motion is reduced to rest in the shortest possible time with no oscillation.

Case 4: $C^2 > 4\ km$ (overdamped). Referring to Equation (1-14), the roots of Equation (1-13) are real and unequal. The value of the roots, after substituting the relationship $C/m = 2\omega_n D$, is given by

$$s_{1,2} = -\omega_n D \pm \omega_n \sqrt{D^2 - 1} \qquad (1\text{-}29)$$

and the resulting solution by using Equation (1-15) is given by

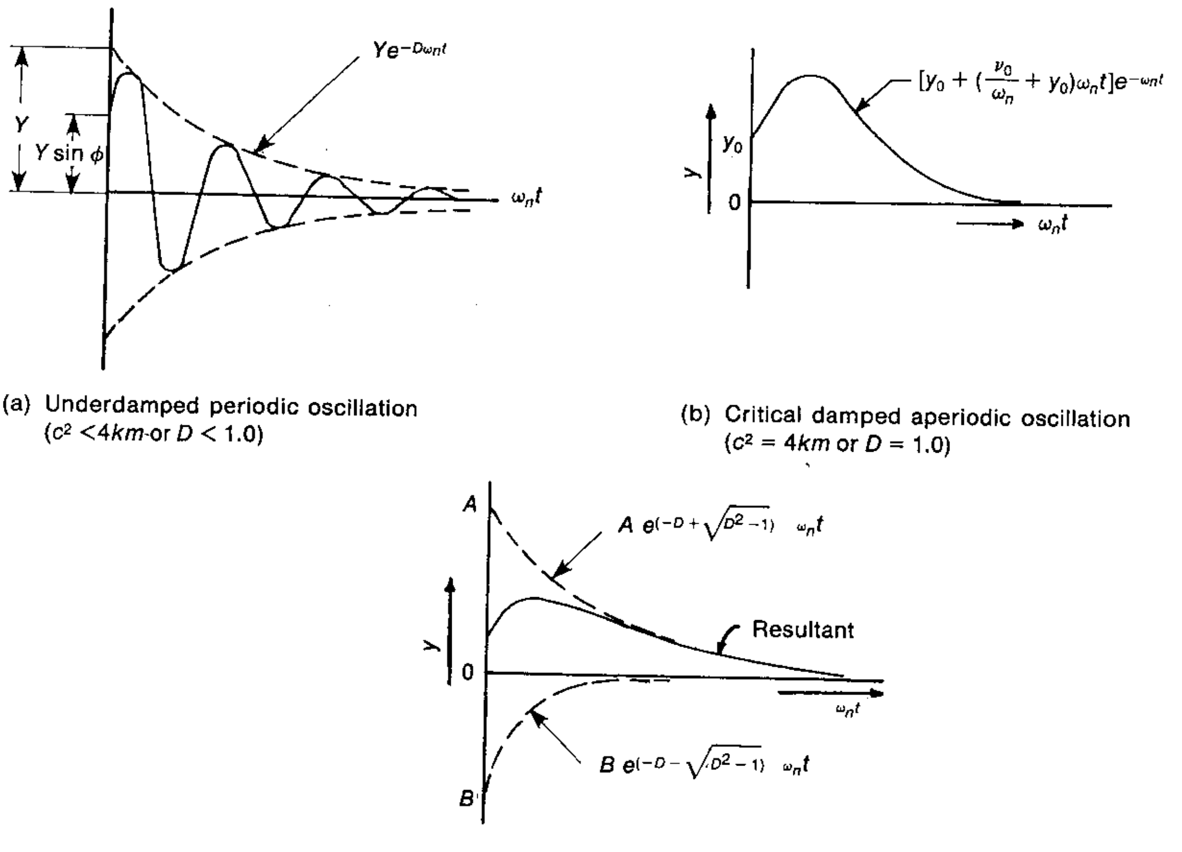
$$y = Ae^{(-D+\sqrt{D^2-1})\omega_n t} + Be^{(-D-\sqrt{D^2-1})\omega_n t} \qquad (1\text{-}30)$$

Since the roots in Equation (1-29) are real and negative for all values of $D > 1.0$, the value of y in Equation (1-30) will decrease exponentially without a change in sign. A graphical representation of Equation (1-30) is shown in Figure 1-4c, which indicates that there are no oscillations, and the system is said to be overdamped.

Steady-State Solution of Forced Vibrations

The solution of Equation (1-10) includes two parts (ref. 3): (a) transient or free vibrations and (b) steady-state or forced vibrations. Transient motion, which in mathematical terms is called the complementary function, is a solution of the homogenous equation, as previously noted.

The particular integral of Equation (1-10) gives the steady-state or forced-vibrations solution. This solution includes the influence of the forcing function. Structures and machines which are subjected to excitation forces which vary with time are susceptible to vibrations. The excitation can be in the form of a pure, simple, harmonic



(a) Underdamped periodic oscillation ($c^2 < 4km$ or $D < 1.0$)

(b) Critical damped aperiodic oscillation ($c^2 = 4km$ or $D = 1.0$)

(c) Overdamped aperiodic oscillation ($c^2 > 4km$ or $D > 1.0$)

Figure 1-4. Damped free-vibration response of SDOF system. Source: William T. Thompson, *Vibration Theory and Applications*, © 1965, pp. 39-40. Reprinted by permission of Prentice-Hall, Inc., Englewood Cliffs, N.J.

force or displacement, or it may have some other periodic form. These other periodic disturbances can be resolved into a number of harmonic components in the form of Fourier series as illustrated in Table 1-2. A third type of time-dependent excitation is in the form of a series of repeated shocks and impulses, pulse waves, or step functions or force or displacement applied to the mass or to its support.

The most common source of excitation in structures supporting machines is the internal excitation caused by an unbalanced condition in the machines or the external excitation produced by a nearby dynamic system. These excitations are generally in the form of harmonics under steady-state conditions and will be further considered here.

Equation (1-10) with a harmonic force is

$$m\ddot{y} + C\dot{y} + ky = F_0 \sin \omega t \,, \tag{1-31}$$

where ω is the frequency of the harmonic excitation. The particular integral solution for this equation is

$$y_p = A_1 \sin \omega t + A_2 \cos \omega t \tag{1-32}$$

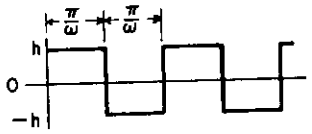
$$\text{with } \dot{y}_p = \omega A_1 \cos \omega t - \omega A_2 \sin \omega t \tag{1-33a}$$

$$\text{and } \ddot{y}_p = -\omega^2 A_1 \sin \omega t - \omega^2 A_2 \cos \omega t \tag{1-33b}$$

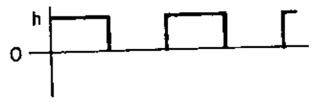
Substitution of Equations (1-32), (1-33a), and (1-33b) in Equation (1-31) and collections of the coefficients multiplying the sine and cosine terms yield

Table 1-2

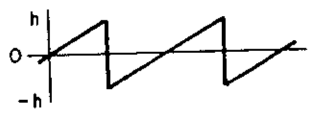
Harmonic Components of Periodic Disturbances (Ref. 7)

(1) 

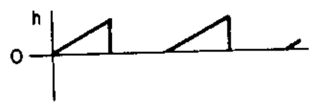
$$f(\omega t) = \frac{4h}{\pi} \sum_{k=1,3,5,\cdots}^{\infty} \frac{1}{k} \text{SIN } k\omega t$$

(2) 

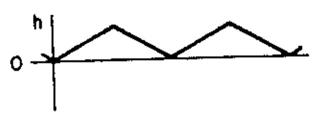
$$f(\omega t) = \frac{h}{2} + \frac{2h}{\pi} \sum_{k=1,3,5,\cdots}^{\infty} \frac{1}{k} \text{SIN } k\omega t$$

(3) 

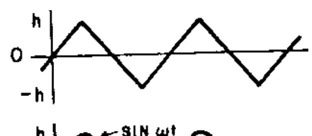
$$f(\omega t) = \frac{2h}{\pi} \sum_{k=1,2,3,\cdots}^{\infty} \frac{(-1)^{k+1}}{k} \text{SIN } k\omega t$$

(4) 

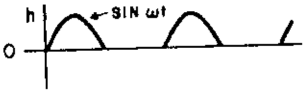
$$f(\omega t) = \frac{h}{4} - \frac{2h}{\pi^2} \sum_{k=1,3,5,\cdots}^{\infty} \frac{1}{k^2} \text{COS } k\omega t + \frac{h}{\pi} \sum_{k=1,2,3,\cdots}^{\infty} \frac{(-1)^{k+1}}{k} \text{SIN } k\omega t$$

(5) 

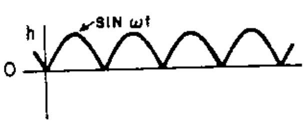
$$f(\omega t) = \frac{h}{2} - \frac{4h}{\pi^2} \sum_{k=1,3,5,\cdots}^{\infty} \frac{1}{k^2} \text{COS } k\omega t$$

(6) 

$$f(\omega t) = \frac{8h}{\pi^2} \sum_{k=1,3,5,\cdots}^{\infty} \frac{(-1)^{(k-1)/2}}{k^2} \text{SIN } k\omega t$$

(7) 

$$f(\omega t) = \frac{h}{\pi} + \frac{h}{2} \text{SIN } \omega t - \frac{2h}{\pi} \sum_{k=2,4,6,\cdots}^{\infty} \frac{1}{k^2 - 1} \text{COS } k\omega t$$

(8) 

$$f(\omega t) = \frac{2h}{\pi} - \frac{4h}{\pi} \sum_{k=2,4,6,\cdots}^{\infty} \frac{1}{k^2 - 1} \text{COS } k\omega t$$

$$[(k - m\omega^2)A_1 - C\omega A_2] \sin \omega t$$
$$+ [C\omega A_1 + (k - m\omega^2) A_2] \cos \omega t = F_0 \sin \omega t \quad (1\text{-}34)$$

and equating the sine and cosine terms on the two sides of the equation yields

$$\left.\begin{aligned} &[(k - m\omega^2) A_1 - C\omega A_2] \sin \omega t = F_0 \sin \omega t \\ &[C\omega A_1 + (k - m\omega^2) A_2] \cos \omega t = 0 \end{aligned}\right\} \quad (1\text{-}35)$$

Solving of these two simultaneous equations for the two constants A_1 and A_2 and substituting in Equation (1-32) yields

$$y_p = \frac{F_0 (k - m\omega^2) \sin \omega t - F_0 C\omega \cos \omega t}{(k - m\omega^2)^2 + (C\omega)^2} \quad (1\text{-}36)$$

The particular integrals for various forms of the forcing function are presented (in Table 1-3) to illustrate the physical feel of resulting oscillation. An alternate form for Equation (1-36) is

$$\begin{aligned} y_p &= [F_0/\sqrt{(k - m\omega^2)^2 + (C\omega)^2}] \sin (\omega t - \phi_1) \\ &= Y \sin (\omega t - \phi_1), \end{aligned} \quad (1\text{-}37)$$

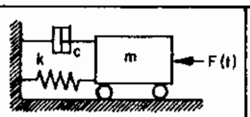
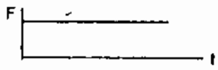
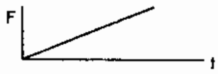
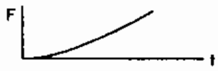
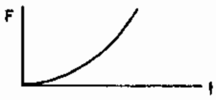
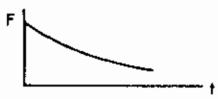
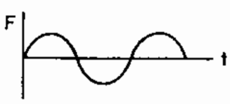
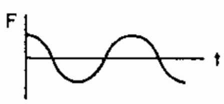
where $Y = F_0/\sqrt{(k - m\omega^2)^2 + (C\omega)^2}$

and $\phi_1 = \tan^{-1} [C\omega/(k - m\omega^2)]$

Y is the amplitude of the steady-state response, and ϕ_1 is the "phase lag" of $y_p\ (t)$ with respect to the forcing function $F_0 \sin\omega t$. Substitution of the expression for D and ω_n in the expression for Y and ϕ_1 and replacing ω/ω_n by the frequency ratio r in Equation (1-37) and rearranging Equation (1-37) in nondimensional form gives

$$\left.\begin{aligned} M &= \frac{Y}{(F_0/k)} = \frac{1}{\sqrt{[1 - r^2]^2 + [2Dr]^2}} \\ \phi_1 &= \tan^{-1} 2Dr/(1 - r^2) \end{aligned}\right\} \quad (1\text{-}38)$$

Table 1-3
Particular Integrals (Ref. 7)

	F(t)		y_p
1	F_o		$\frac{F_o}{k}$
2	$F_o t$		$\frac{F_o}{k}(t - \frac{c}{k})$
3	$F_o t^2$		$\frac{2F_o}{k}(\frac{t^2}{2} - \frac{ct}{k} - \frac{m}{k} + \frac{c^2}{k^2})$
4	$F_o t^3$		$\frac{3F_o}{k}(\frac{t^3}{3} - \frac{ct^2}{k} - \frac{2mt}{k} + \frac{2c^2t}{k^2} + \frac{4mc}{k^2} - \frac{2c^3}{k^3})$
5	$F_o e^{-st}$		$\frac{F_o}{ms^2 - cs + k} e^{-st}$
6	$F_o e^{st}$		$\frac{F_o}{ms^2 + cs + k} e^{st}$
7	$F_o \sin \omega t$		$\frac{F_o(k - m\omega^2) \sin \omega t - F_o c\omega \cos \omega t}{(k - m\omega^2)^2 + (c\omega)^2}$
8	$F_o \cos \omega t$		$\frac{F_o c\omega \sin \omega t + F_o (k - m\omega^2) \cos \omega t}{(k - m\omega^2)^2 + (c\omega)^2}$

where M is called the dynamic magnification factor, ϕ_1 has been defined earlier. Equation (1-38) shows that the M and ϕ_1 factors are functions of the frequency ratio r and the damping ratio D. These functions are shown in Figure 1-5. These curves indicate that the damping ratio D is effective in reducing the amplitude and phase angle in the region of resonance, that is when r approaches unity, the particular values of M and ϕ_1 depend on the damping ratio D.

Dynamic System Subjected to Rotating-Mass-Type Excitation

In some dynamic systems, the excitation force present arises out of unbalances in the rotating masses. Examples of such systems are reciprocating and centrifugal machines. The forces generated by a reciprocating machine are of the form (ref. 1)

$$F_x = (m_{rec} + m_{rot})\, r'\omega^2 \cos \omega t$$

$$+ m_{rec}\,[(r')^2/L]\,\omega^2 \cos 2\omega t \qquad (1\text{-}39a)$$

$$F_y = (m_{rot})\, r'\,\omega^2 \sin \omega t \qquad (1\text{-}39b)$$

where F_x and F_y are horizontal and vertical inertia forces, respectively. There are two masses: one moving with the piston at point P in Figure 1-6a called M_{rec} (reciprocating); and one moving with the crank pin at point C called M_{rot} (rotating). The crank mechanism for this type of machine is illustrated in Figure 1-6a.

The forces generated by the unbalanced rotating mass of the centrifugal machine shown in Figure 1-6b are given by

$$F_x = m_i e\omega^2 \cos \omega t \qquad (1\text{-}40a)$$

$$F_y = m_i e\omega^2 \sin \omega t \qquad (1\text{-}40b)$$

Equations (1-39) and (1-40) indicate that the magnitude of the forcing function is proportional to the rotating mass m_i, its eccentricity to the true axis e, and the speed ω. The rotating mass and its eccentricity remain constant, but the value of ω varies from start-up of the machine to its stable steady-state condition. Therefore, during that period, the maximum amplitude of the forcing function given by Equations (1-39) or (1-40) is directly proportional to the square of the operating speed.

The equation of motion for the forcing function in the centrifugal machine is given by a damped single-degree-of-freedom system in the y-direction:

$$m\ddot{y} + C\dot{y} + ky = (m_i e\omega^2)\sin \omega t \qquad (1\text{-}41)$$

By comparing Equation (1-41) with a constant-force-amplitude expression, Equation (1-31), and substituting $m_i e\omega^2 = F_0$ in Equation (1-37), the following expression is obtained:

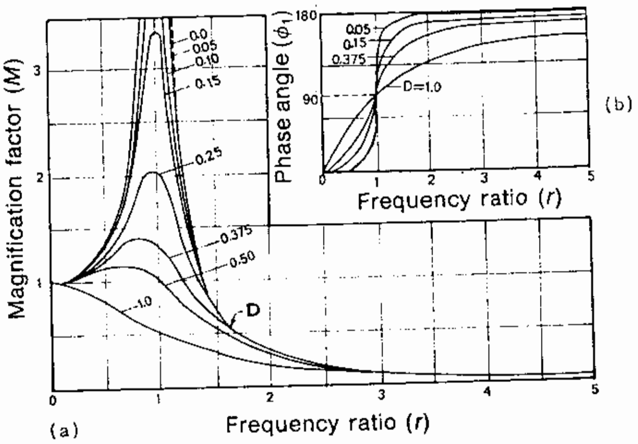


Figure 1-5. Magnification factor (M) versus frequency ratio (r); (a) and phase angle (ϕ_1) versus frequency ratio (r); (b) for a single-degree-of-freedom system subjected to a constant force amplitude force, $F = F_0 \sin \omega t$. Source: William T. Thompson, *Vibration Theory and Applications*, © 1965, p. 54. Reprinted by permission of Prentice-Hall, Inc., Englewood Cliffs, NJ.

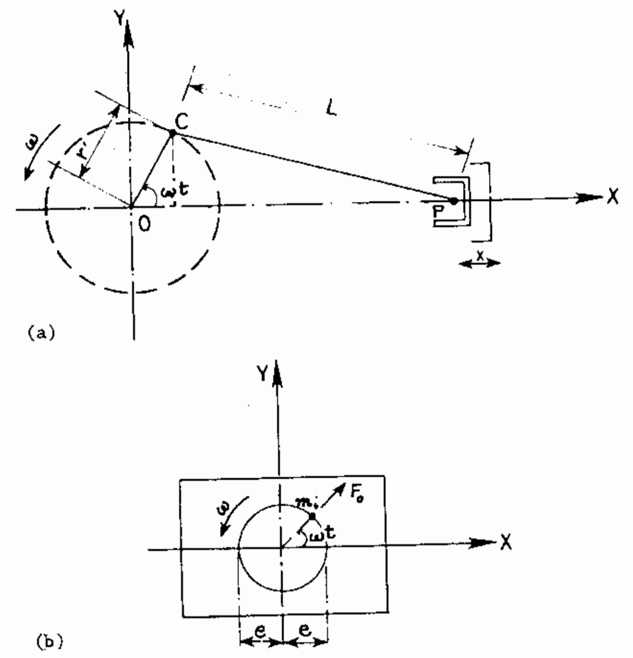


Figure 1-6. (a) Crank mechanism of a reciprocating machine; (b) Forces from a centrifugal machine (rotating mass excitation).

$$y_p = m_i e\omega^2 \sin(\omega t - \phi_1)/\sqrt{(k - m\omega^2)^2 + (C\omega)^2} \qquad (1\text{-}42a)$$

$$Y = m_i e\omega^2/\sqrt{(k - m\omega^2)^2 + (C\omega)^2} \text{ and}$$

$$\phi_1 = \tan^{-1}\frac{C\omega}{(k - m\omega^2)} \qquad (1\text{-}42b)$$

Manipulating and rearranging Equation (1-42a) in a similar manner as was done with Equations (1-37) to (1-38) gives

$$Y/[(m_i e\omega^2)/k] = 1/\sqrt{[1-r^2]^2 + [2Dr]^2} = M \qquad (1\text{-}43)$$

$$\text{or } Y/(m_i e/m) = (r)^2 M = M_r ,$$

where M_r is the dynamic magnification factor for the rotating-mass-type excitation case. Figure 1-7 shows the plot of M_r in relation to the frequency ratio r for the various values of damping ratio D.

Substituting $\mu = m_i/m$ in Equation (1-43) yields

$$M_r = Y/\mu e = (r)^2 M \qquad (1\text{-}44)$$

The term μe is called the free amplitude.

For a given system, the values of μ, e, D, and ω_n are constant, so that Figure 1-7 is, in effect, a plot of the amplitude of the mass against the rotating speed of the unbalanced force for various amounts of damping. For a small value of r, or at low rotating speed, the total mass m moves very little; at a speed approaching the natural frequency of the system, $r = 1$, the amplitude builds up to large values for small amount of damping. Further on, at higher rotating speeds, $r \rightarrow \infty$, the curves approach the value M_r equal to unity since the inertial force of the total mass is then approximately 180° out of phase with the unbalanced force.

Comparing the curves of Figures 1-5 and 1-7, it may be observed that resonant peaks occur at $r < 1$ for the case of a constant force excitation, and $r > 1$ for the case of rotating-mass-type excitation.

The various expressions which may be derived from the equations of motion of these two cases are listed in Table 1-4.

The combination of transient or free vibration (complementary function) and steady-state vibration (particular integral) gives the complete solution for Equation (1-10).

Terminology

A single-degree-of-freedom lumped mass system is presented above including the derivation of the differential equation describing the behavior of the model leading up to the development of formulae for the calculation of the dynamic response. In the area of dynamics of foundations and structures, the investigation may extend to a variety of systems, some having several degrees of freedom, and as a result, the modeling techniques and derived formulae are more complex. The fundamental principles of single-degree-of-freedom systems are also applicable to the multidegree-of-freedom systems; however, additional information from theory of vibrations is required for consideration of the more complex multidegree-of-freedom systems. A complete introduction to the theory of vibration is not presented here; however, Chapter 1 lists a number of references on the subject. The following list of terminologies summarizes the most commonly used terms (refs. 4 and 5) in the field of vibrations. An example of each defined term is also included to provide further information on its application.

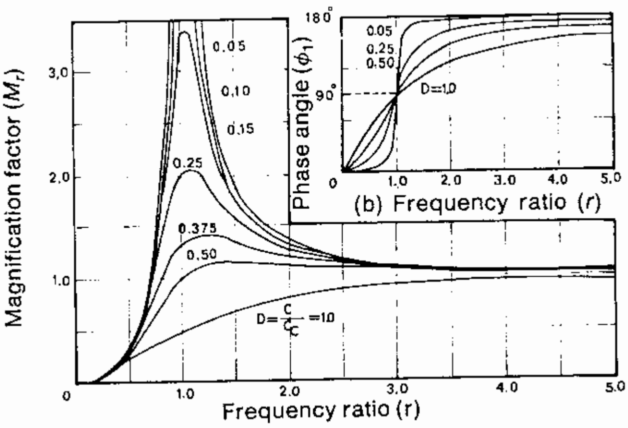


Figure 1-7. Magnification factor (M_r) vs. frequency ratio (r): (a) and phase angle (ϕ_1) versus frequency ratio (r): (b) for a single-degree-of-freedom system subjected to a rotating-type excitation, $F = m_i e\omega^2 \sin \omega t$. Source: William T. Thompson, *Vibration Theory and Applications*, © 1965, p. 60. Reprinted by permission of Prentice-Hall, Inc., Englewood Cliffs, NJ.

1. Accelerating Bodies

Acceleration

Definition: Newton's Law of Motion—a vector quantity when applied to the mass, produces a force in the direction of application.

Example: the rate of change of velocity with time. For the x-coordinate it is denoted by d^2x/dt^2 or $\ddot{x}$. See Figures 1-8 and 1-9.

Velocity

Definition: a vector quantity which represents time rate change of position for a particle or body.

Example: the rate of change of displacement with time. For the x-coordinate it is denoted by dx/dt or $\dot{x}$. See Figures 1-8 and 1-9.

Displacement

Definition: a vector quantity that represents the change of position of a particle or body from a state of equilibrium.

Example: a displacement which is a function of time. For the x-coordinate it is denoted by x. See Figures 1-8 and 1-9.

Table 1–4
Summary of Derived Expressions for a Single-Degree-of-Freedom System

Expression	Constant Force Excitation F_0 Constant	Rotating Mass-type Excitation, $F_0 = m_i e\,\omega^2$
Magnification factor	$M = \dfrac{1}{\sqrt{(1-r^2)^2 + (2Dr)^2}}$	$M_r = \dfrac{r^2}{\sqrt{(1-r^2)^2 + (2Dr)^2}}$
Amplitude at frequency f	$Y = M\,(F_0/k)$	$Y = M_r\,(m_i e/m)$
Resonant frequency	$f_{mr} = f_n \sqrt{1-2D^2}$	$f_{mr} = \dfrac{f_n}{\sqrt{1-2D^2}}$
Amplitude at resonant frequency f_r	$Y_{max} = \dfrac{(F_0/k)}{2D\sqrt{1-D^2}}$	$Y_{max} = \dfrac{(m_i e/m)}{2D\sqrt{1-D^2}}$
Transmissibility factor	$T_r = \dfrac{\sqrt{1+(2Dr)^2}}{\sqrt{(1-r^2)^2 + (2Dr)^2}}$	$\bar{T}_r = \dfrac{r^2\sqrt{1+(2Dr)^2}}{\sqrt{(1-r^2)^2 + (2Dr)^2}}$

where $r = \omega/\omega_n$
ω_n (Undamped natural circular frequency) $= \sqrt{(k/m)}$
D (Damping ratio) $= C/C_c$
C_c (Critical Damping) $= 2\sqrt{km}$
T_r = Force transmitted/F_0
$\bar{T}_r$ = Force transmitted/$m_i\, e\omega^2{}_n$

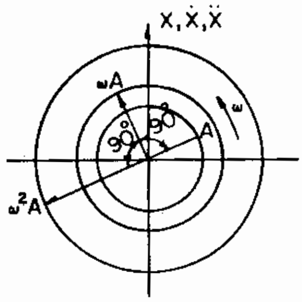


Figure 1-8. Rotating vector representation of a harmonic function $x = A \sin \omega t$.

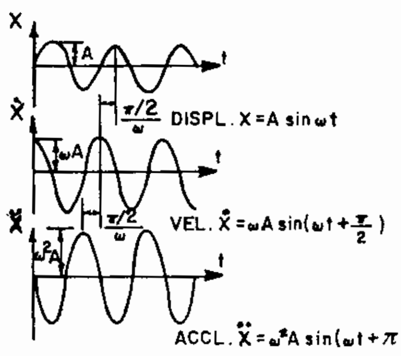


Figure 1-9. Harmonic motion representation of displacement velocity and acceleration.

2. Amplitude

Displacement

Definition: the maximum change of position of a body or some part of the system from a reference point (generally equilibrium position) at any given time.

Example: the maximum displacement of a sinusoidal quantity $x = A \sin \omega t$, which is A in this function.

Vibration

Definition: the time-varying magnitude of peak displacement (of a physical body) from a reference point.

Example: a time-varying displacement of a rotor shaft of a machine, or a foundation structure from the static equilibrium condition.

3. Analysis

Computer

Definition: resolution of complex mathematical problems into simple elements with digital (discrete number operation) or analog (continuous chart operation) computers.

Example: solution of indeterminate structures or determination of vibration in a dynamic system using computer programs.

Dynamic (Vibration)

Definition: a study of motion of a physical system at a particular time.

Example: the calculation of the amplitude of vibrations in a machine or in a foundation structure.

Matrix Method

Definition: the study of motion of masses in multi-degree-of-freedom systems.

Example: the solutions of simultaneous equations using the techniques of matrix algebra.

Modal

Definition: the dynamic analysis of a multidegree-of-freedom system, where the responses in the normal modes (each treated as independent one-degree systems) are determined separately, and then superimposed to provide the total response.

Example: vibration analysis of a "Table top", when a computer program is employed.

Static

Definition: the investigation of a physical system in equilibrium under the action of a system of stationary forces.

Example: dead-load analysis of a structural system.

4. Balancing

Static

Definition: adjustment of mass distribution of a rotating body such that statically the system is at neutral equilibrium.

Example: see Figure 1-10.

Dynamic

Definition: the adjustment of mass distribution in a rotating body such that the vibrations are controlled.

Example: see Figure 1-11.

5. Beat

Definition: the maximum resulting amplitude of two simple harmonic wave forms of slightly different frequencies which are superimposed.

Example: see Figure 1-12 in which beat frequency $(f_b) = \text{Abs.}\ [\omega_1 - \omega_2]/2\pi$. Frequency of combined oscillation $(f) = (\omega_1 + \omega_2)/4\pi$. Beat period $(T_b) = 1/f_b$. Period of resulting oscillation $(T) = 1/f$; $x_{max} = A_1 + A_2$; $x_{min} = Abs.\ (A_1 - A_2)$.

6. Conditions

Boundary

Definition: the known physical relationships at specific points of a structural body, usually at the supports.

Example: see Figure 1-13 in which (Boundary Conditions) Deflection: $y\ (x = 0, L) = 0$; Slope: $EI\ dy/dx\ (x = L/2) = 0$; Moment: EI $d^2y/dx^2\ (x = 0, L) = 0$; Shear: $EI\ d^3y/dx^3\ (x = L/2) = 0$.

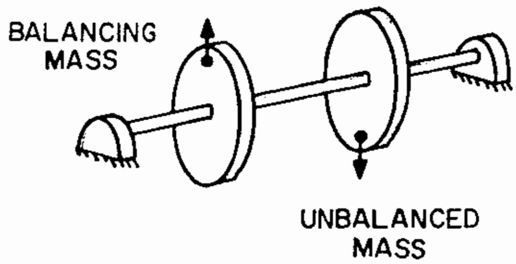


Figure 1-10. Static balancing. This system under rotation produces equal centrifugal forces, but produces unbalanced moments in shaft and pressure on the bearings.

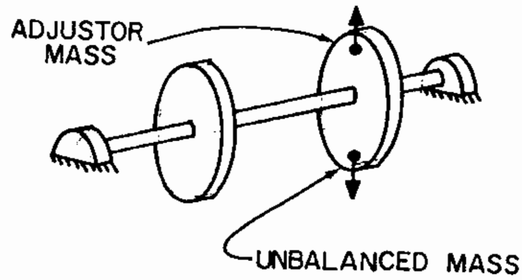


Figure 1-11. Dynamic balancing. For this system during rotation, not only are centrifugal forces balanced but the forces and moments in the shaft (in one revolution) are also balanced.

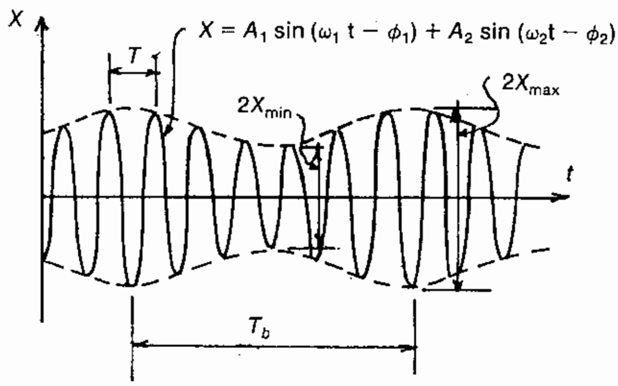


Figure 1-12. Resulting motion of two simple harmonic wave forms containing a beat.

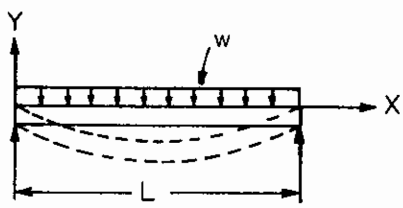


Figure 1-13. Simple beam loaded with uniform load w.

Constraint

Definition: the imposition of limitations on the behavior of a physical body.

Example: see Figure 1-14.

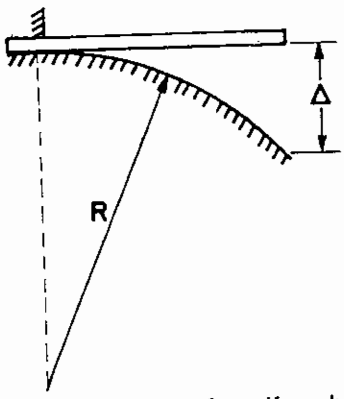


Figure 1-14. Cantilever of uniformly distributed mass. The cantilever is constrained to deform in a circular profile.

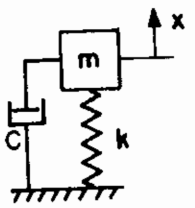


Figure 1-15. Single lumped-mass dashpot system.

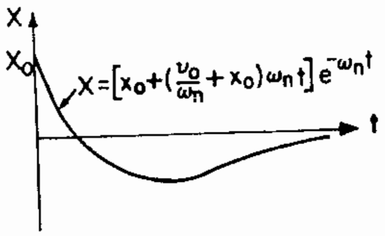


Figure 1-16. Critical damped oscillation of a single lumped-mass system.

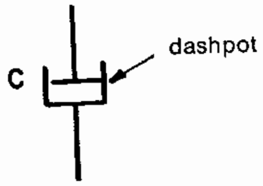


Figure 1-17. Symbol used in a lumped-mass system.

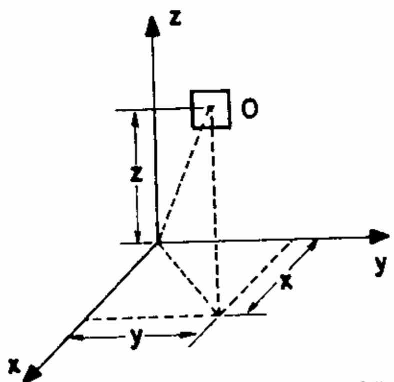


Figure 1-18. Coordinates of element "O" in space.

Initial

Definition: the known motion of a physical body at the reference time, often taken as zero.

Example: the application of brakes on a steadily moving vehicle. At the time of application of brakes, velocity = V and deceleration = 0.

7. Damping

Coefficient or Constant

Definition: a factor used in a dynamic system to account for dissipation of energy.

Example: see Figure 1-15, where equation of motion is $m\ddot{x} + C\dot{x} + kx = 0$. In this equation, the constant C accounts for viscous damping in the system.

Critical

Definition: a mathematical equality derived in viscously damped system, such that the free displacement comes to rest without oscillation.

Example: see Figure 1-16 in which critical damping $(C_c) = 2\sqrt{km}$.

Dashpot

Definition: a schematic representation of a viscous damper.

Example: see Figure 1-17.

Factor or Ratio

Definition: the ratio of actual resistance in damped harmonic motion to that necessary to produce critical damping.

Example: $D = C/C_c = C/2\sqrt{km}$.

Viscous

Definition: a type of damping assumed in a dynamics model such that the dissipation of energy during oscillation is linearly proportional to the velocity of the mass.

Example: damping force = $C\dot{x}$.

8. Coordinates

Cartesian

Definition: linear quantities that describe the location of a point in space with respect to a system of three-dimensional orthogonal axes.

Example: see Figure 1-18.

Generalized

Definition: a specification of a configuration by a set of independent geometric quantities, which may be lengths, angles, or their combinations.

Example: a set of n independent geometric coordinates which specify the configuration of an n-degree-of-freedom system.

Normal or Principal

Definition: a particular set of generalized coordinates which describes equations of motion such that there is neither static nor dynamic coupling among them.

Example: a procedure followed in modal analysis in which the general motions of the masses of a multi-degree-of-freedom system can be expressed as a superposition of its principal modes of vibration.

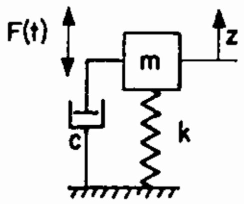


Figure 1-19. Single degree spring-lumped-mass-dashpot system.

9. Differential Equations (Equations of Motion)

Linear

Definition: an equation relating to two or more variables in terms of derivatives or differentials such that no terms involving the unknown function or its derivatives appear as products or are raised to a power different from unity. The order of a differential equation is equal to the order of the highest derivative in the equation. When the independent variable is a time function, then it is called an Equation of Motion.

Example: see Figure 1-19 for which the Equation of Motion is $m\ddot{z} + C\dot{z} + kz = F(t)$ or $m(d^2z/dt^2) + C(dz/dt) + kz = F(t)$. This is a nonhomogenous ordinary second-order linear differential equation with constant coefficients. In this equation, z and t are variables, where z is the dependent variable. In case z is dependent on more than one independent variable, then the equation will change from an ordinary to a partial differential equation. If m, C, and k are not constant and are independent of z or its derivatives, but are dependent of the variable t, then the equation is called a differential equation with variable coefficients. If the right-hand term of the equation $F(t)$ is zero, then the equation is called homogenous. The solution of a homogenous equation is called its complementary function and is given by

$$z_c(t) = Ae^{-D\omega_n t} \sin(\omega_d t + \psi)$$

where A and ψ are constants to be specified by the initial conditions. This solution gives the transient motion of the system. The solution which satisfies the nonhomogenous equation is called the particular integral. For $F(t) = F_0 \sin \omega t$ it is given by:

$$z_p(t) = F_0 \sin(\omega t - \phi)/\sqrt{(k - m\omega^2)^2 + (C\omega)^2}$$

This solution gives the steady-state response or steady-state vibration. The complete solution of the equation is the sum of the complementary function $z_c(t)$ and the particular integral $z_p(t)$.

Simultaneous

Definition: linear differential equation which contains more than one dependent variable related to a single independent variable t.

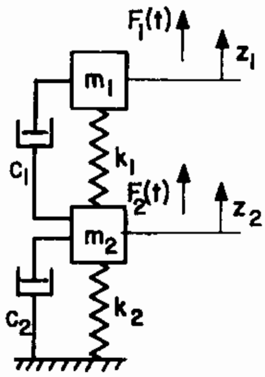


Figure 1-20. Two-degree-of-freedom system (multi-degree).

Example: see Figure 1-20 in which the Equations of Motion are $m_1\ddot{z}_1 + C_1(\dot{z}_1 - \dot{z}_2) + k_1(z_1 - z_2) = F_1(t)$ and $m_2\ddot{z}_2 + C_2\dot{z}_2 - C_1(\dot{z}_1 - \dot{z}_2) + k_2z_2 - k_1(z_1 - z_2) = F_2(t)$. This two-degree-of-freedom system contains two dependent variables, z_1, z_2, and an independent variable, t. The general solution of these two simultaneous differential equations will consist of a complementary function and a particular integral.

10. Dynamic

Eigenvalues (characteristic values or natural frequencies)

Definition: the roots of the characteristic equation which results from the expansion of the determinant of the simultaneous differential equations. (See also definition of normal modes.)

Example: when the simultaneous equations are equations of motion of the free undamped multidegree-of-freedom system, then their roots are called eigenvalues which are equal to the squares of the natural frequencies of the modes.

Eigenvectors (characteristic vectors or natural modes)

Definition: these are the characteristic vectors which are obtained by substituting the characteristic values or eigenvalues in a set of simultaneous differential equations of a multidegree-of-freedom system. Alternately, eigenvectors are the independent vibrating modes of a multidegree-of-freedom system such that during vibration the ratio of the displacements of any of two masses is constant with time.

Example: a multidegree system has exactly the same number of natural modes as degrees of freedom. Associated with each mode is a natural frequency and a characteristic shape.

Force, load

Definition: a force whose duration and amplitude is a function of time.

Example: centrifugal force generated by an unbalanced rotating mass is given by $F = m_1 e\omega^2 \sin \omega t$.

Load Factor

Definition: the ratio of the dynamic deflection at any time to the deflection which would have resulted from the static application of the dynamic load.

Example: the dynamic load factor caused by the constant centrifugal force of rotating mass on undamped one-degree system is given by $DLF = 1/[1 - (\omega/\omega_n)^2]$.

System

Definition: an elastic system which possesses mass and whose parts are capable of relative motion.

Example: an engineering structure, machine, or its components, and most physical bodies consisting of matter.

11. Excitation

Impulse

Definition: the product of force and time while force is acting on the mass.

Example: see Figure 1-21.

Inertial

Definition: excitation generated by the mass in motion.

Example: see Figure 1-22.

Harmonic, Sinusoidal

Definition: a pulsating force of the form: $F_o \sin \omega t$ or $F_o \cos \omega t$.

Example: see Figure 1-23.

Periodic

Definition: a time-function excitation which repeats itself identically at regular intervals of time.

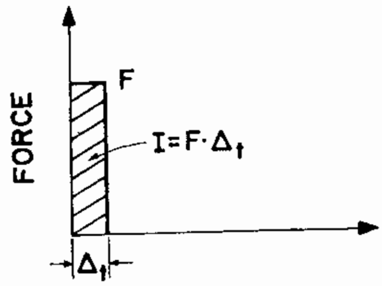


Figure 1-21. Rectangular pulse.

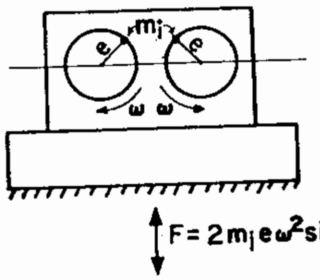


Figure 1-22. Rotating mass oscillator generated excitation: $F(t) = 2m_i e\omega^2 \sin \omega t$.

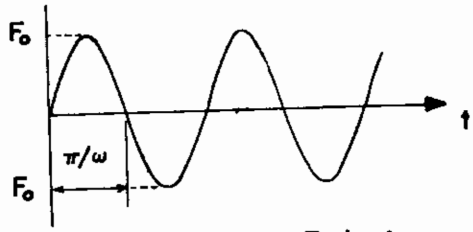


Figure 1-23. Harmonic force. The figure shows a centrifugal force of amplitude F_0 generated by a rotating machine.

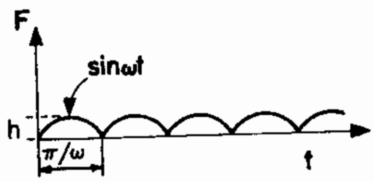


Figure 1-24. Forcing function generated by the cam of a machine.

Example: see Figure 1-24 for which the following equation holds:

$$F(t) = \frac{2h}{\pi} - \frac{4h}{\pi} \sum_{k=2,4,6\ldots}^{\infty} \frac{1}{k^2 - 1} \cos k\omega t$$

Transient

Definition: a temporary arbitrary excitation which disappears with time.

Example: see Figure 1-25.

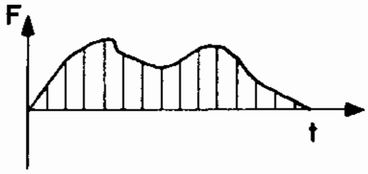


Figure 1-25. An arbitrary transient forcing function.

12. Foundation Structure (for Machines)

Block-type

Definition: a small area concrete foundation of such thickness so that the structural deformation caused by the superimposed load is negligible.

Example: see Figure 1-26.

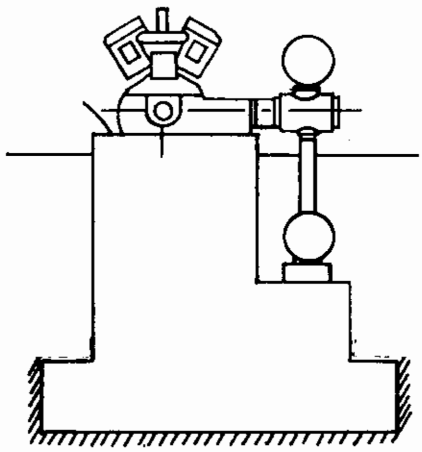


Figure 1-26. Block-type foundation for a reciprocating machine.

Elevated Frame (Table Top)

Definition: a three-dimensional elevated reinforced concrete structure consisting of beams framing into columns and supported by a heavy foundation slab. The tops of the columns are connected by a top slab or heavy longitudinal and transverse beams forming a rigid table on which the machinery rests. The foundation structure may be supported by piles or directly on the soil.

Example: see Figure 1-27.

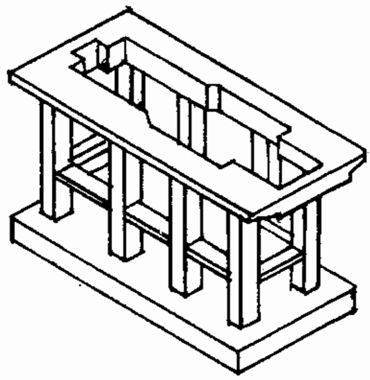


Figure 1-27. Table-top structure.

Mat Slab

Definition: a flexible concrete slab which is resting on soil and supports a machine or battery of similar machinery.

Example: see Figure 1-28.

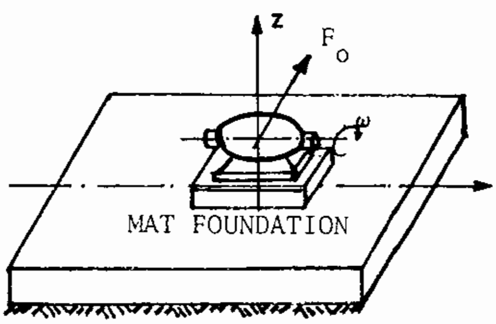


Figure 1-28. Vibrating machine supported by a mat-type foundation.

Overtuned and Undertuned

Definition: a machine foundation is said to be overtuned when the ratio of the speed of mounted machine to the natural frequency of the foundation is less than 1.0 and is called undertuned when that ratio is greater than 1.0.

Example: see Figure 1-29.

13. Frequency

Angular or Circular

Definition: the time rate of change of angular displacement given in units of radians per second. For an oscillating system, it is the number of vibrations in units of radians per second.

Example: see Figure 1-30.

Damped Natural or Harmonic

Definition: the natural frequency of a linear system which includes viscous damping C.

Example: see Figures 1-31 and 1-32 for which the following equation holds:

Damped Frequency, $\omega_d = \omega_n \sqrt{1 - D^2}$

Damping Ratio, $D = C/2\sqrt{km} < 1.0$

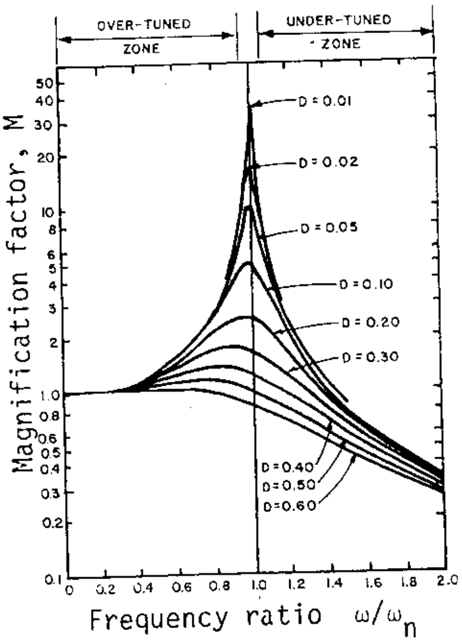


Figure 1-29. Magnification factor (*M*) versus frequency ratio for various amounts of damping ratio (*D*).

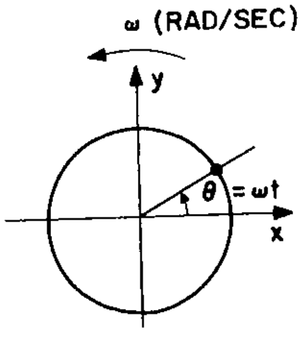


Figure 1-30. Angular or circular frequency ω.

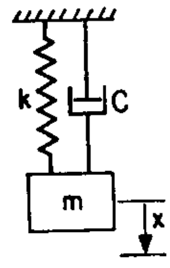


Figure 1-31. Damped-free linear system.

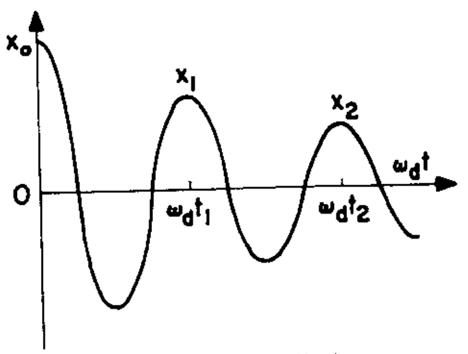


Figure 1-32. Damped-free oscillation.

Excitation, Forcing or Operating

Definition: the number of times a dynamic force achieves an identical amplitude in a time period of one second, and is given in cycles per second (Hertz).

Example: see Figure 1-33.

Fundamental

Definition: the lowest value of all natural frequencies of an oscillating system.

Example: the frequency associated with the first mode of vibration.

Natural

Definition: the dynamic property of an elastic body or system by which it oscillates repeatedly back and forth from a fixed reference point when the external force application is removed.

Example: see Figure 1-34 for which the following equation of motion holds: $m\ddot{x} + kx = 0$; Natural frequency in Hertz $(f_n) = (1/2\pi)\sqrt{k/m}$

Rayleigh's

Definition: natural frequency of a system computed by an arbitrary selection of a deflected shape which satisfies the system boundary condition so that it gives the values of maximum kinetic energy to make the lowest natural frequency a minimum. In a multidegree system, the displacement δ_1, δ_2 of the masses, caused by the masses acting as static loads,

$$\text{P.E.} = \tfrac{1}{2} W_1\delta_1 + \tfrac{1}{2} W_2\delta_2 + \ldots$$

$$\text{K.E.} = \tfrac{1}{2}\frac{W_1}{g}\delta_1^{\,2}\omega^2 + \tfrac{1}{2}\frac{W_2}{g}\delta_2^{\,2}\omega^2 + \ldots$$

$$\text{P.E.} = \text{K.E.}$$

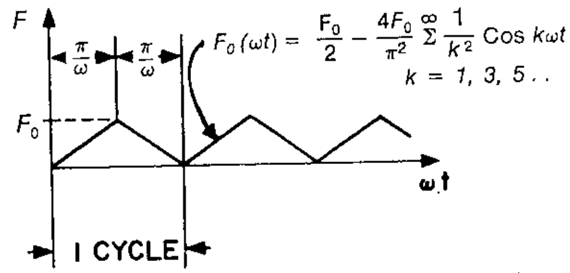


Figure 1-33. Frequency of a cam in a machine.

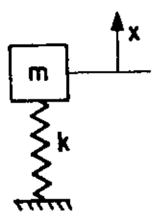


Figure 1-34. Undamped free single-degree-of-freedom system.

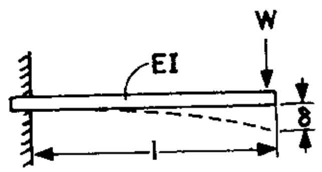


Figure 1-35. Weightless cantilever supporting load W at its end.

$$\omega^2 = \frac{\frac{1}{2}W_1\delta_1 + \frac{1}{2}W_2\delta_2 + \ldots}{\frac{\frac{1}{2}W_1\delta_1^2}{g} + \frac{\frac{1}{2}W_2\delta_2^2}{g} + \ldots}$$

$$= \frac{g\,\Sigma W_i\delta_i}{\Sigma W_i\delta_i^2}$$

Example: see Figure 1-35 for which the following holds:

$$\text{Rayleigh's Frequency } (f_n) = \frac{1}{2\pi}\sqrt{gW\delta/W\delta^2}$$

$$= \frac{1}{2\pi}\sqrt{g/\delta}$$

where $\delta = \frac{WL^3}{3EI}$

g = acceleration of gravity

14. Magnification or Amplification Factor

Definition: in a dynamic system, it is the ratio of a steady-state displacement response caused by a dynamic force to the displacement caused by an equivalent static force of a magnitude equal to the amplitude of the dynamic force.

Example: see Figure 1-36. The figure gives the response curves for a damped system subjected to a forcing function, $F(t) = F_o \sin \omega t$. Steady-state Displacement Response Amplitude,

$$x = F_0/\sqrt{(k-m\omega^2)^2 + (C\omega)^2}$$

Static Displacement $x_o = F_o/k$

Therefore, Magnification Factor,

$$M = x/x_o = 1/\sqrt{[1-(\frac{\omega}{\omega_n})^2]^2 + [2D\frac{\omega}{\omega_n}]^2}$$

15. Mass

Consistent or Continuous

Definition: a mass function which is distributed at each point of its domain and has infinite possible number of independent degrees of freedom.

Example: see Figure 1-37.

Equivalent Lumped or Lumped

Definition: a concentrated rigid mass in an idealized system which is obtained by equating the total kinetic energy of the actual system to that of the equivalent system.

Example: see Figure 1-38.

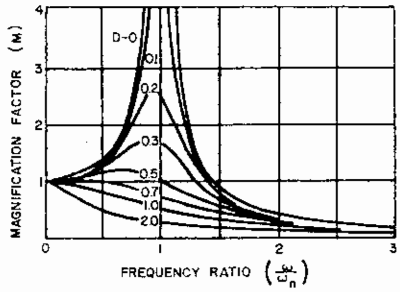


Figure 1-36. Magnification factor (M) versus frequency ratio (ω/ω_n) for various amounts of damping ratio (D).

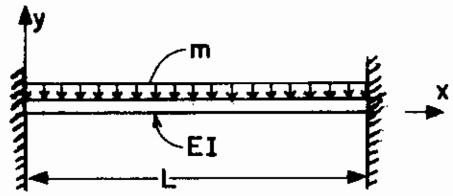


Figure 1-37. A fixed ended beam with distributed mass over the span.

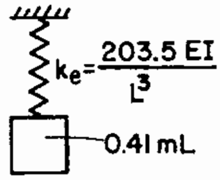


Figure 1-38. Idealized SDOF system for a fixed beam of Figure 1-37 (see also Table 1-1).

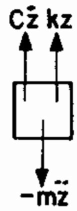


Figure 1-39. Free-body diagram of SDOF system of Figure 1-19.

16. Motion

Equation of Motion

Definition: a differential equation describing the relationship among acceleration, velocity, and displacement of a mass in a dynamic system.

Example: see Figure 1-39 for which the dynamic equilibrium condition = equation of motion, $m\ddot{z} + C\dot{z} + kz = 0$.

Periodic, Aperiodic

Definition: motion of mass which repeats itself at equal intervals of time and can be resolved into harmonics. These harmonics may be of different amplitudes and frequencies. Conversely, when the mass slowly moves back to the equilibrium position, rather than vibrating about it, the motion is said to be aperiodic.

Example: see Figure 1-40 and 1-41.

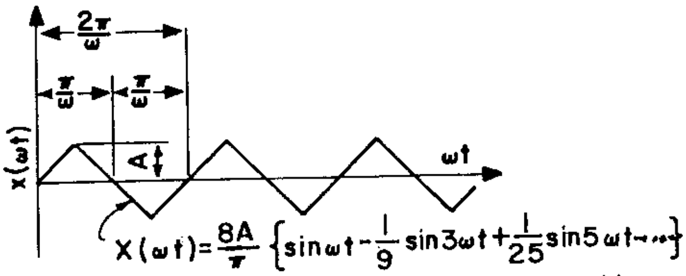


Figure 1-40. Periodic motion of a cam in a machine.

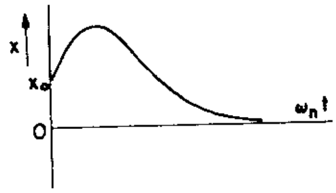


Figure 1-41. Aperiodic motion of damped-free SDOF system.

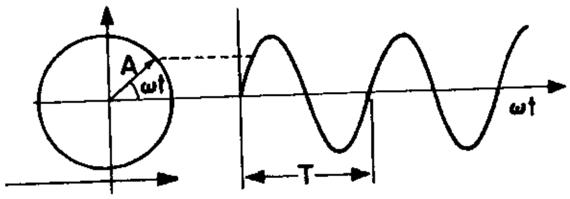


Figure 1-42. Harmonic motion A sin ωt and its vector representation.

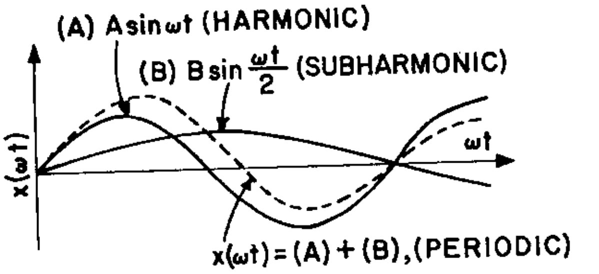


Figure 1-43. Subharmonic, harmonic, and periodic motions.

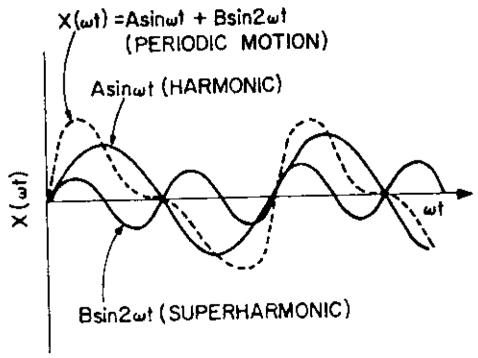


Figure 1-44. Superharmonic, harmonic, and periodic motions.

Simple Harmonic or Sinusoidal

Definition: motion of a body or parts of a system described by a trigonometric function, a sine or a cosine which repeats itself in any equal interval of time.

Example: see Figure 1-42.

Subharmonic

Definition: a sinusoidal quantity having frequencies that are fractional (1/2, 1/3, $1/n$) or a submultiple of the exciting frequency of a periodic function to which it is related.

Example: see Figure 1-43.

Superharmonic

Definition: a sinusoidal quantity having frequencies that are multiple (2, 3, n) of the exciting frequency of a periodic function to which it is related.

Example: see Figure 1-44.

17. Modes

Coupled

Definition: modes of vibration of a multidegree system where the motions are not independent but influence each other because of energy transfer from one mode to the other.

Example: see Figure 1-45 with two degrees of freedom, x and θ, for vertical and pitching oscillations, respectively.

Case I: Coupling due to mass (Center of Gravity of mass eccentric but equal strength supporting springs), also called dynamic coupling. Equations of motion:

$$m\ddot{x} + 2k_1x - k_1(L_1 - L_2)\theta = 0 \quad \text{(a)}$$

$$J_o\ddot{\theta} + k_1(L_1^2 + L_2^2)\theta - k_1(L_1 - L_2)x = 0 \quad \text{(b)}$$

In these equations, coupling is due to a mass which does not have its center of gravity at the midpoint of the system. If $L_1 = L_2$, then Equations (a) and (b) are independent.

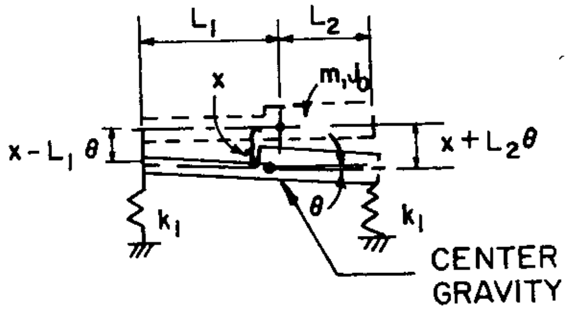


Figure 1-45. Simplified two-degree-of-freedom model of an automobile.

Case II: Coupling due to spring (center of gravity of mass centric, but unequal strength supporting springs), also called static coupling. See Figure 1-46.
Equations of motion:

$$m\ddot{x} + (k_1 + k_2)\,x - L\,(k_1 - k_2)\,\theta = 0 \qquad \text{(a)}$$

$$J_o\,\ddot{\theta} + L^2(k_1 + k_2)\,\theta - L\,(k_1 - k_2)\,x = 0 \qquad \text{(b)}$$

In these equations, coupling is due to the unequal strength springs k_1, k_2. If these springs are equal, then both Equations (a) and (b) are independent and thus, represent uncoupled (independent) modes.

Uncoupled

Definition: the modes of vibration of a multidegree system where each mode describes the complete motion of a particular type by a single independent coordinate.

Example: see Figure 1-47 with two degrees of freedom, y and θ, for vertical and pitching oscillation. Equation of motion:

$$m\ddot{y} + 2ky = F\,(t)$$

$$J_o\,\ddot{\theta} + (2kd^2)\,\theta = M_t\,(t)$$

Because of symmetry of mass center of gravity and equal values of supporting springs k, the vertical oscillation described by y and pitching oscillation described by θ are independent of each other.

First, Lowest, Fundamental

Definition: in a multidegree-of-freedom system, a mode shape which corresponds to the lowest frequency is called fundamental or first mode. The mode shapes are determined from characteristic equations.

Example: see Figure 1-48 where the beam with continuously distributed mass has infinite degrees of freedom. The frequencies ω_n and mode shapes ϕ_n are given by:

$$\omega_n = n^2\pi^2\sqrt{EIg/A\gamma}\,/L^2\,,\ \phi_n = \sin n\pi x/L\,,$$

where $n = 1, 2, 3\ldots$

E = modulus of elasticity
I = moment of inertia
A = cross-sectional area
γ = material density

Also, see Figure 1-49 for the various mode shapes.

Normal, Principal (Eigenvector)

Definition: the independent natural modes which satisfy the solution of a multidegree-of-freedom system. They have the following characteristics:

a. They represent undamped free vibration.
b. They are harmonic.

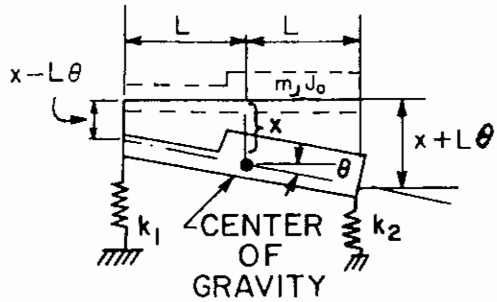


Figure 1-46. Two-degree-of-freedom system.

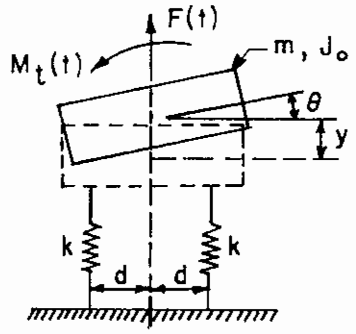


Figure 1-47. Mass with two independent degrees of freedom.

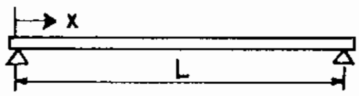


Figure 1-48. Hinged-hinged beam.

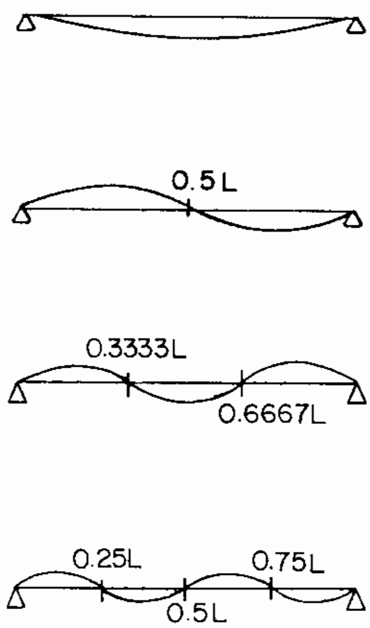


Figure 1-49. Various mode shapes of a hinged-hinged beam.

c. During vibration, at any two instants, the ratio of displacement of any two masses is constant with time. These modes are called principal modes (eigenvectors) when their amplitude is arbitrary. Where the amplitude of the principal modes is normalized to unity, then they are called normal modes (normalized eigenvectors). A corresponding normal mode shape is associated with each degree of freedom.

Example: see Figure 1-50 where the system is constrained to oscillate in the vertical direction only, and described by two independent coordinates x_1 and x_2. Equations of motion:

$$\left.\begin{aligned} m_1\ddot{x}_1 + (k_1 + k_c)x_1 - k_c x_2 = F_1(t) \\ m_2\ddot{x}_2 + (k_2 + k_c)x_2 - k_c x_1 = F_2(t) \end{aligned}\right\} \quad (1\text{-}45)$$

These are linear second-order differential equations and coupling between coordinates is due to spring k_c.

To solve for free vibration of the system, the initial conditions are:

$$F_1(t) = F_2(t) = 0$$

It is assumed that motion of every point in the system is harmonic:

$$\left.\begin{aligned} x_1 = A_1 \sin(\omega t + \psi) \\ x_2 = A_2 \sin(\omega t + \psi) \end{aligned}\right\} \quad (1\text{-}46)$$

Substituting Equation (1-46) in the homogenous part in the equations of motion the following are obtained:

$$\left.\begin{aligned} (k_1 + k_c - m_1\omega^2)A_1 - k_c A_2 = 0 \\ -k_c A_1 + (k_2 + k_c - m_2\omega^2)A_2 = 0 \end{aligned}\right\} \quad (1\text{-}47)$$

These equations are satisfied for any value of A_1 *and* A_2 if the following determinant is zero:

$$\begin{vmatrix} (k_1 + k_c - m_1\omega^2) & (-k_c) \\ (-k_c) & (k_2 + k_c - m_2\omega^2) \end{vmatrix} = 0$$

Expanding this determinant,

$$\omega^4 - \left[\frac{k_1 + k_c}{m_1} + \frac{k_2 + k_c}{m_2}\right]\omega^2 + \frac{k_1k_2 + (k_1 + k_2)k_c}{m_1m_2} = 0 \quad (1\text{-}48)$$

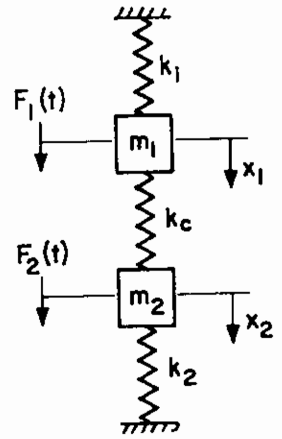


Figure 1-50. Normal mode vibration of a two-degree-of-freedom system.

This characteristic equation in quadratic form in ω^2 leads to two roots ω_1^2 and ω_2^2, which give the natural frequencies ω_1 and ω_2.

Ratio of amplitudes: From Equations (1-46), first mode $(\omega^2 = \omega_1^2)$:

$$(A_1/A_2)^{(1)} = k_c/(k_1 + k_2 - m_1\omega_1^2)$$

Second mode $(\omega^2 = \omega_2^2)$:

$$\left(\frac{A_1}{A_2}\right)^{(2)} = \frac{k_2 + k_c - m_2\,\omega_2^2}{k_c} \quad (1\text{-}49)$$

Symmetrical Case:

$k_1 = k_c = k$

$m_1 = m_2 = m$

Substituting in the characteristic Equation (1-48), $\omega^4 - (4k/m)\omega^2 + (3k^2/m^2) = 0$ which results in two roots

$\omega_{1,2}^2 = (2k/m) \pm \sqrt{4(k/m)^2 - 3(k/m)^2}$
$= k/m\,[2 \pm 1]$
$\omega_1 = \sqrt{k/m}$
$\omega_2 = \sqrt{3k/m}$

and a corresponding amplitude ratio in Equation (1-49).

First Mode $\omega_1 = \sqrt{k/m}$
$(A_1/A_2)^{(1)} = k/[2k - m(k/m)] = 1$

The masses appear to move as a single mass in either direction without deflecting the central spring.

Second Mode $\omega_2 = \sqrt{3k/m}$
$(A_1/A_2)^{(2)} = [2k - m(3k/m)]/k = -1$

The two masses move in opposite direction and there is a node at the center of the middle spring. Each half then behaves as a single-degree-of-freedom system.

See also Figure 1-51.

18. Modes of Vibrations

Definition: a dynamic system which is undergoing free vibration, where the characteristic shape is such that the motion of every particle is a simple harmonic with common frequency.

Example: see Figure 1-52 with motion described by harmonic displacement x_1 and x_2.

19. Node

Points

Definition: fictitious points used in a computer mathematical model for the purpose of determining response values usually located where the masses are lumped and/or response is to be determined.

Example: see Figure 1-53.

Vibrating Systems

Definition: a stationary point in a particular mode shape which has a constant zero amplitude from equilibrium position.

Example: see Figure 1-54.

20. Oscillation

Definition: in dynamics, it is a displacement of a mass which moves back and forth with respect to time from a reference point.

Example: see Figure 1-55.

21. Peak-to-Peak (Double Amplitude of Vibration)

Definition: an algebraic difference between opposite extremes of vibration displacement measured in a rotating mass.

Example: see Figure 1-56.

22. Period

Definition: the time duration for a single repetition of a periodic motion.

Example: see Figure 1-57, where period $(T) = 1/f$, and f = number of cycles/sec.

23. Phase

Angle

Definition A: in a dynamic system it is a measure of the time difference between a periodic excitation and

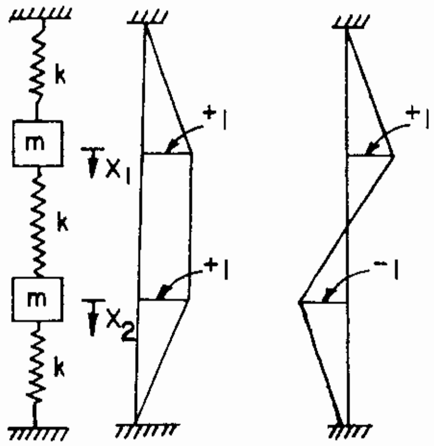


Figure 1-51. Normal modes of vibration.

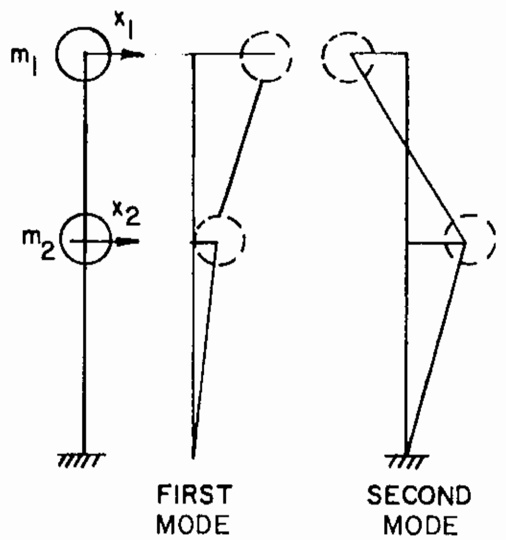


Figure 1-52. Modes of vibration of a two-degree-of-freedom system.

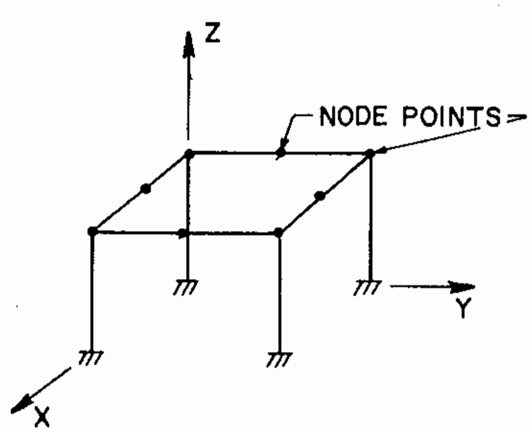


Figure 1-53. Node points in a space frame model (usually located where masses are lumped and response is determined).

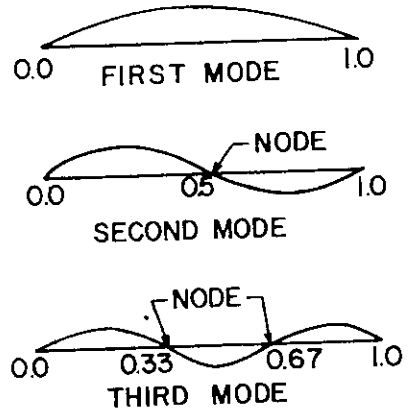


Figure 1-54. Node points in vibrating strings.

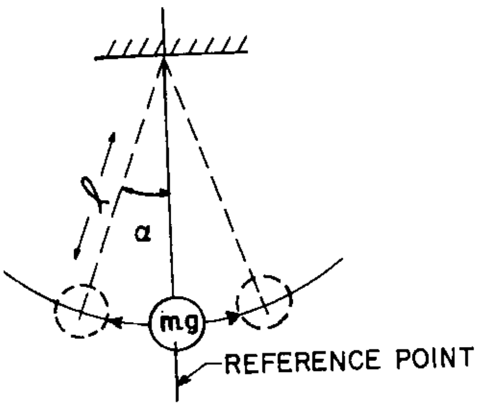


Figure 1-55. Oscillation of a simple pendulum.

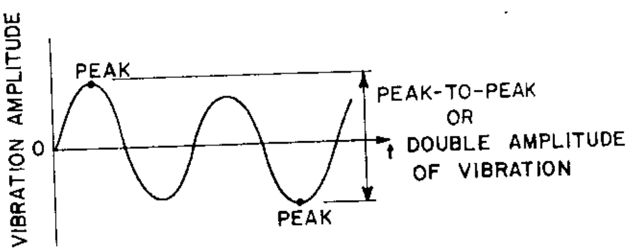


Figure 1-56. Peak-to-peak (double amplitude) of vibration.

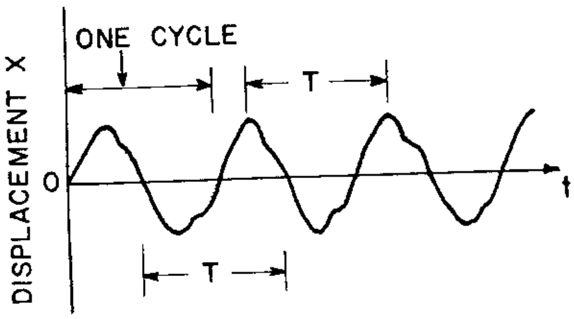


Figure 1-57. Period of periodic motion.

the resulting displacement response oscillating at the same frequency.

Example: see Figures 1-58 and 1-59.

Definition B: alternately, in rotating vector form, it is the angle lag by which the response vectors stays behind the excitation vector.

Example: see Figure 1-60.

Definition C: phase angle in a damped SDOF is given by $\phi = \tan^{-1} [C\omega / (k - m\omega^2)]$

$= \tan^{-1} [2D(\omega/\omega_n) / (1 - (\omega/\omega_n)^2)]$

Example: see Figure 1-61.

24. Resonance

Condition

Definition: a phenomenon of uncontrolled increase in vibration amplitude exhibited by a physical system when it is subjected to an external vibration force of a frequency (ω) that approaches the natural free oscillation frequency (ω_n), i.e., $(\omega/\omega_n) = 1.0$. In a damped system, a resonance condition occurs when the displacement becomes maximized as ω goes from 0 to ω_n.

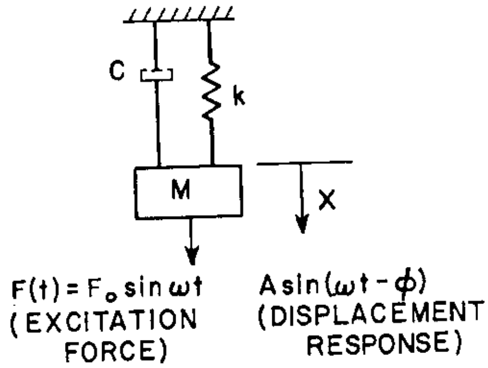


Figure 1-58. Damped SDOF subject to harmonic force, $F(t) = F_0 \sin \omega t$.

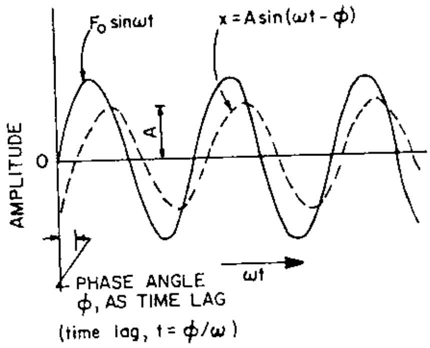


Figure 1-59. Response motion lags harmonic loading by phase angle ϕ.

Example: see Figures 1-62 and 1-63. Due to the presence of damping in every system, a resonance condition at which the vibration amplitude A will approach infinity is in fact seldom achieved.

Frequency

Definition: when the frequency of a dynamic system (related to the undamped natural frequency) equals the frequency of the applied force, a resonance condition occurs. In this condition, the response is maximized.

Example (formulae for resonance frequency):

1. For maximum amplitude magnification.
   A. Damped resonance frequency (constant force oscillator, $F = F_0 \sin \omega t$). See Figure 1-64 for which the following holds:

      Resonance frequency $(f_{mr}) = f_n \sqrt{1-2D^2}$ $(D \leq 1/\sqrt{2})$

      Magnification factor $(M) = 1/(2D\sqrt{1-D^2}) \doteq 1/2D$

   B. Damped resonance frequency (rotating mass oscillator), $F = m_i e\omega^2 \sin \omega t$. See Figure 1-65 for which the following holds:

      Resonance frequency $(f_{mr}) = f_n/\sqrt{1-2D^2}$ $(D \leq 1/\sqrt{2})$

      Magnification factor $(M_r) = A/m_i e/m = 1/(2D\sqrt{1-D^2}) \doteq 1/2D$

2. For maximum transmissibility factor, T_r. Damped resonance frequency (constant force oscillator, $F = F_o \sin \omega t$).

   Resonance frequency $(f_{mr}) = f_n \sqrt{U-1}/2D$

$$T_r = F_T/F_o = \sqrt{\frac{U}{U + [1-(U-1)/4D^2]^2 - 1}}$$

where $U = \sqrt{8D^2+1}$ and F_T = transmitted force.

25. Response

Dynamic

Definition: the time-varying displacement and/or stresses which result when a dynamic force is applied to a physical system.

Example: see Figure 1-66 for which the equation of motion is $m\ddot{x} + C\dot{x} + kx = F_o \sin \omega t$. The complete general solution $x(t)$ of this equation of motion is called dynamic response.

Steady State (forced part)

Definition: the sustained periodic motion of a physical system which has the same frequency and duration as the dynamic force.

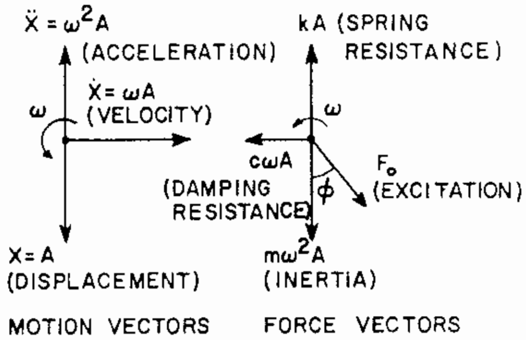


Figure 1-60. Response vector lags excitation vector by phase angle ϕ.

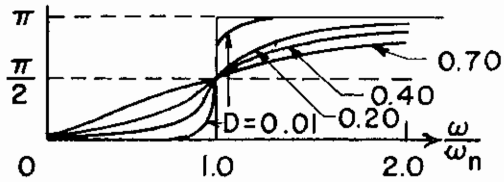


Figure 1-61. Phase angle in damped SDOF system.

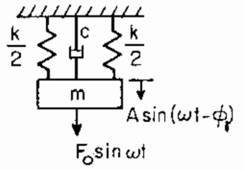


Figure 1-62. Damped SDOF system subjected to $F(t) = F_0 \sin\omega t$.

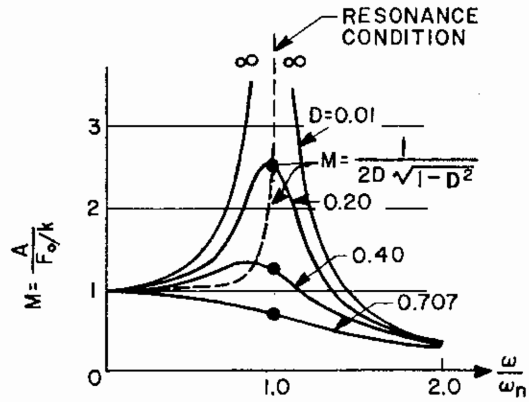


Figure 1-63. Response curve for damped SDOF system (Figure 1-62).

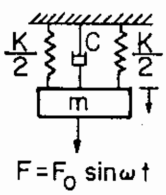


Figure 1-64. SDOF system subjected to $F = F_0 \sin \omega t$.

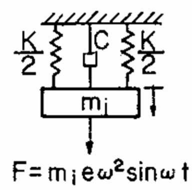


Figure 1-65. SDOF system subjected to $F = m_i e\omega^2 \sin \omega t$.

Example: see Figure 1-67 for which the following holds:
General Solution $x(t) = x_p(t) + x_c(t)$,
where $x_p(t)$ = particular integral or steady-state response
$x_c(t)$ = complementary function or transient solution

Transient

Definition: a form of free vibration, which quickly vanishes due to the presence of damping.

Example: see Figure 1-67.

26. Shaft

Critical Speed

Definition: the angular speed at which a rotating shaft exhibits dynamic instability with rapid increase in lateral amplitude. This develops when the angular speed is in resonance with the natural frequencies of lateral vibration of the shaft.

Example: see Figure 1-68.

Flexible

Definition: a rotating shaft of a machine which has a first lateral natural frequency which is lower than the rotating speed.

Example: according to an industry standard for gas turbines, the first lateral frequency of a shaft shall be at least 15% below any operating speed; the second lateral speed must be 20% above the maximum continuous speed.

Rigid (Stiff)

Definition: a rotating shaft of a machine which has a first lateral natural frequency which is greater than the rotating speed.

Example: according to industry standards for rigid-shaft compressors, the first lateral frequency of the shaft shall be at least 20% higher than the forcing frequency which may be the rotor speed or some multiple thereof.

27. Spring Stiffness

Constant

Definition: a constant of proportionality between the force and the relative deformation it produces in the direction of application in a massless structural element. An elastic spring observes Hooke's Law, that is, the spring force is linearly proportional to the spring deformation.

Example: see Figures 1-69 and 1-70.

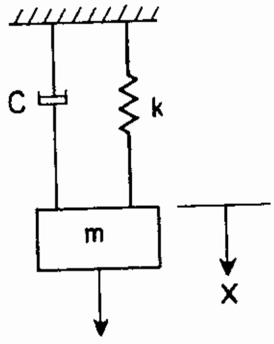


Figure 1-66. Damped SDOF system subjected to dynamic force, $F(t) = F_0 \sin \omega t$.

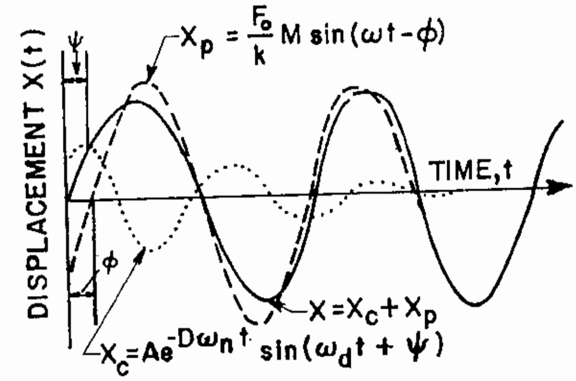


Figure 1-67. General solution of the equation of motion of Figure 1-66. From *Introduction to Structural Dynamics* by John M. Biggs, Copyright 1964, McGraw-Hill Book Co.

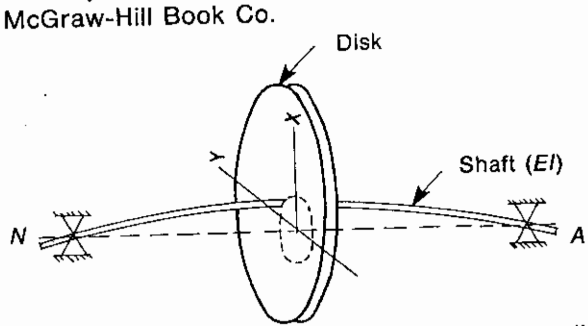


Figure 1-68. Rotating shaft with lateral amplitude in x and y directions.

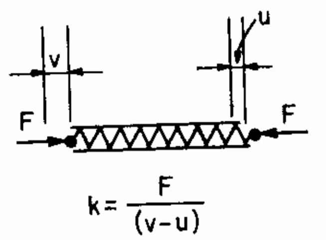


Figure 1-69. Linear spring constant.

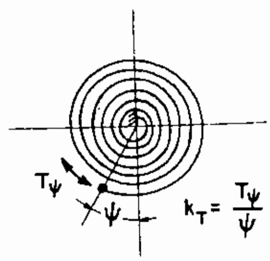


Figure 1-70. Torsional spring constant.

Equivalent

Definition: an assumed theoretical representation of an actual physical system such that force-displacement relationship in the former is equivalent to the latter.

Example: see Figures 1-71 and 1-72. For Figure 1-71 assume the following:

1. Frame weight is negligible.
2. Girder is sufficiently rigid to prevent rotation at top of columns.

Equivalent spring stiffness:

$$k_e = \frac{12(2EI)}{H^3} = \frac{12 \times 2 \times 30 \times 10^6 \times 56.4}{(20)^3 \times 144}$$
$$= 35,250 \text{ lbs./ft}$$

Linear (Elastic)

Definition: an elastic spring observes Hooke's Law, that is, the spring force is linearly proportional to the spring deformation.

Example: see Figure 1-73.

Nonlinear

Definition: in a nonlinear spring, the load in the spring is not linearly proportional to the displacement.

Example: see Figure 1-74, for which the following holds: $k(t) = \Delta p(t)/\Delta\delta(t)$.

Soil

Definition: in a soil dynamics system, a schematic representation of a linear load-deformation relationship of the soil using a linear force displacement spring.

Example: see Figures 1-75 and 1-76.

28. System

Continuous

Definition: a body which has continuously distributed mass density (ρ) and elasticity (E) in its domain. In a vibration analysis, this body has an infinite number of degrees of freedom.

Example: see Figure 1-77.

Dynamic

Definition: a structural body which has mass and elasticity and whose parts are capable of relative motion.

Example: see Figures 1-78 and 1-79.

Free

Definition: if a dynamic system is set into motion by some disturbance at initial time equal to zero and thereafter no force is applied, the resulting oscillations caused in the system are called free vibrations and the system is called a free system.

Example: see Figure 1-80.

Idealized or Equivalent

Definition: an idealized system is a convenient representation of an actual structure such that a mathematical investigation can be performed. The parameters of an idealized system are usually selected so that the deflection of the concentrated mass is the same as that for some significant point on the prototype structure. The idealized system with the equivalent parameters is called an equivalent system.

Example: see Figures 1-81 and 1-82. From Table 1-1, Case 2, equivalent parameter values are

$$k_e = (k_l)\ 384\,EI/L^3 = 0.53 \times 384\,EI/L^3$$
$$= 203.5\,EI/L^3$$
$$m_e = 0.41\,m\,L$$
$$F_e = 0.53\,F_o\,L$$

Linear

Definition: system where the principle of superposition is applicable and where cause and effect are linearly related.

Example: the influence of various forces acting on a mass is algebraically additive, as in the case of static analysis.

Nonlinear

Definition: in dynamics, the vibration whose amplitude is large such as when $\sin\theta$ cannot be represented by only the first term in its expansion but must include several terms ($\sin\theta = \theta - \theta^3/3 + \theta^5/5 - \ldots$); or when the spring-restoring force on the vibrating mass is not proportional to its displacement.

Example: see Figures 1-83 and 1-84.

Lumped-Mass Spring-Dashpot

Definition: an idealized system in which the parameters of a real elastic system have been lumped and where the translational displacements are defined.

Example: see Figure 1-85.

Single-Degree-of-Freedom (SDOF)

Definition: rectilinear or rotational motion described by a single coordinate associated with a mass.

Example: see Figures 1-86 and 1-87.

Multiple-Degree-of-Freedom (MDOF)

Definition: a rigid body in space has six degrees of freedom, namely, three coordinates to define rectilinear positions and three to define the angular positions. If

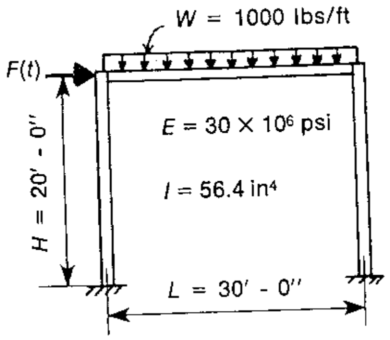


Figure 1-71. Uniformly loaded portal frame.

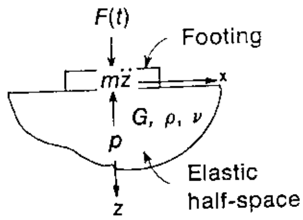


Figure 1-75. A circular footing subjected to dynamic force $F = F_0 \sin\omega t$ and resting on semi-infinite soil medium (elastic half-space).

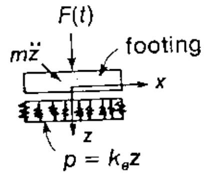


Figure 1-76. Mathematical model of the footing with an equivalent soil spring stiffness, k_e.

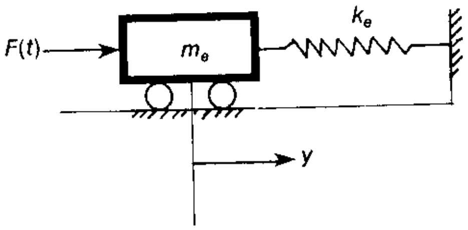


Figure 1-72. Mathematical model of portal frame (Figure 1-71) with equivalent spring stiffness, k_e.

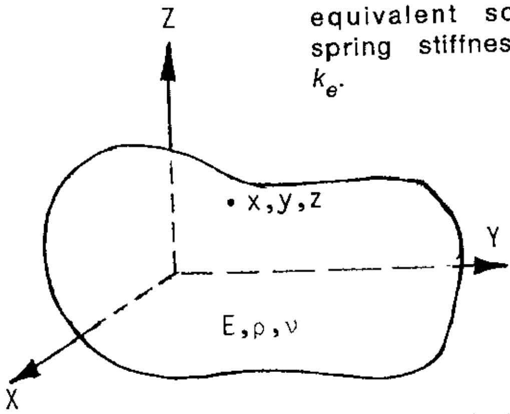


Figure 1-77. A body of continuous mass in three-dimensional space.

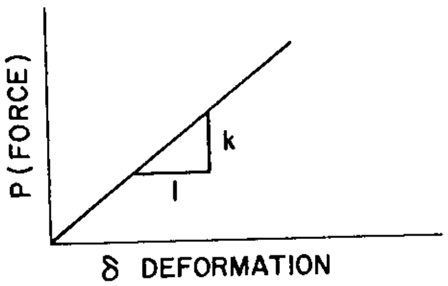


Figure 1-73. Characteristic of a linear (elastic) spring constant "k."

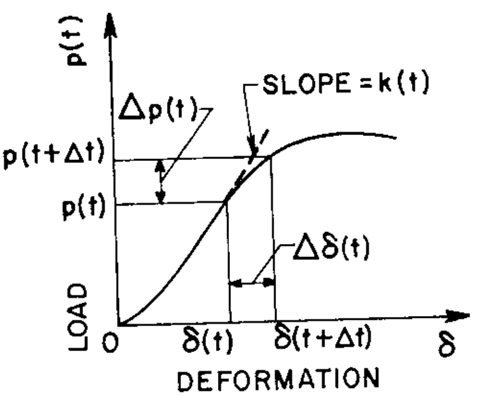


Figure 1-74. Characteristic of a nonlinear spring constant "k."

Figure 1-78. The human body—a typical dynamic system.

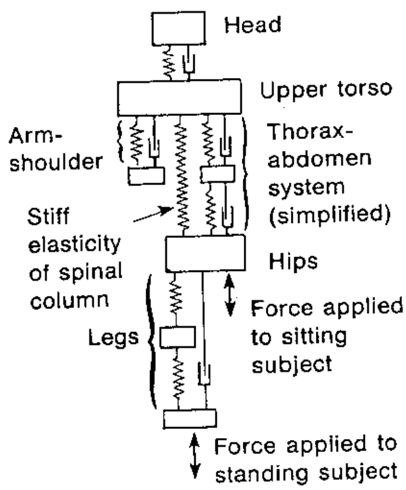


Figure 1-79. Rheological model of a human body.

Figures 1-78 and 1-79 are from *Shock and Vibration Handbook* by C.M. Harris and C.E. Crede, © 1976. Used with permission of McGraw-Hill Book Co.

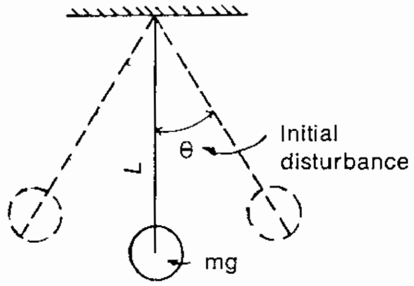


Figure 1-80. Free oscillation of a simple pendulum system.

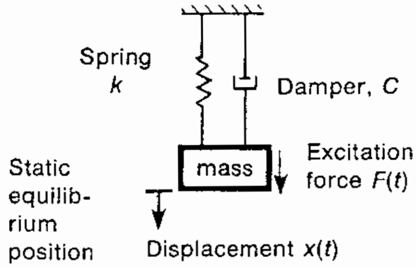


Figure 1-85. Lumped mass, spring and dashpot.

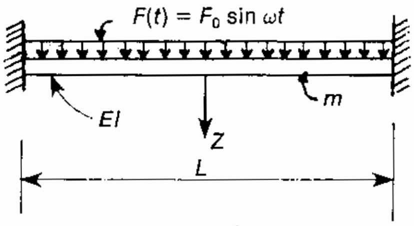


Figure 1-81. An actual physical structure of a fixed beam of a uniform mass and subjected to a uniform dynamic force. $F(t) = F_0 \sin\omega t$.

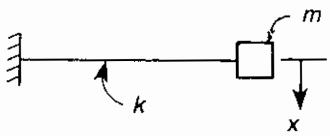


Figure 1-86. Single rectilinear motion in x-direction in a cantilever.

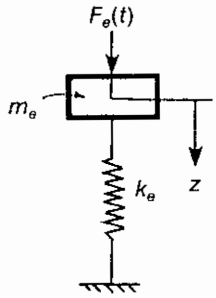


Figure 1-82. Equivalent (idealized) SDOF system.

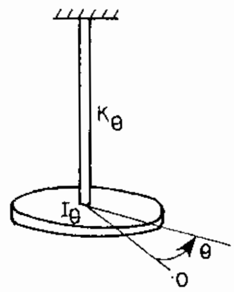


Figure 1-87. Single rotational motion in a θ direction in a torsional pendulum.

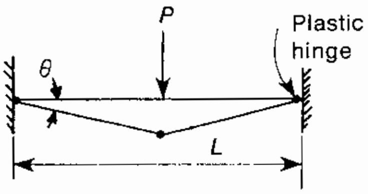


Figure 1-83. Elasto-plastic behavior in a fixed steel beam.

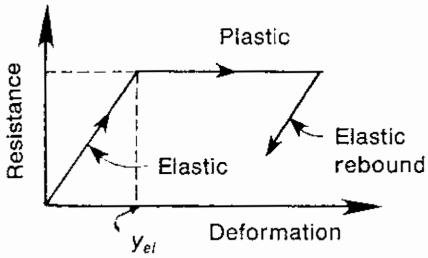


Figure 1-84. Bi-linear spring representation of the elasto-plastic system of Figure 1-83.

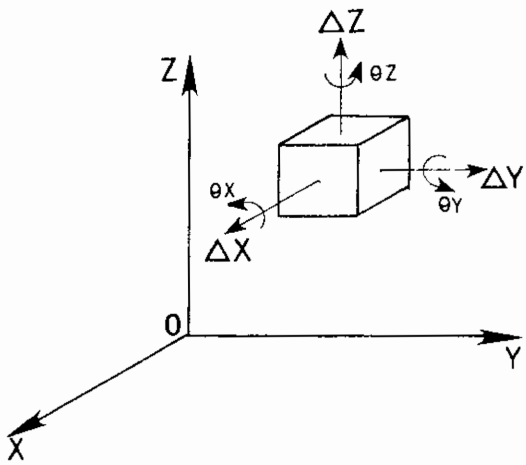


Figure 1-88. A mass element with a six-degree-of-freedom system.

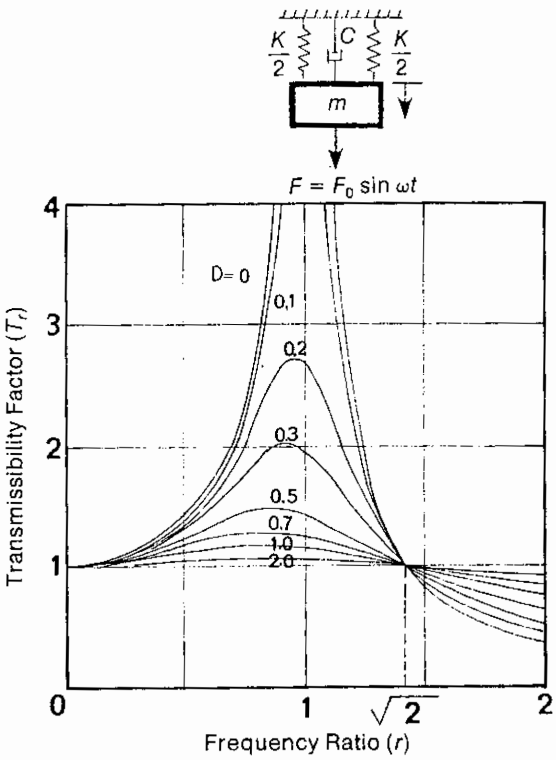


Figure 1-89. Transmissibility factor vs. frequency ratio for various damping factors.

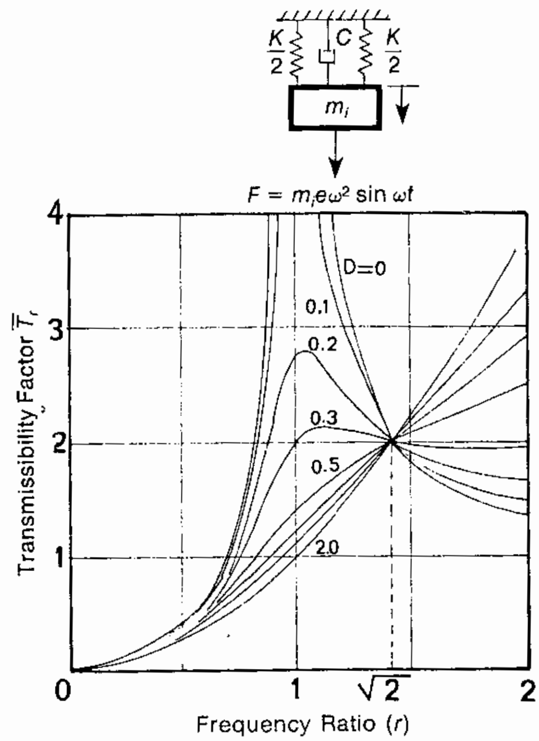


Figure 1-90. Transmissibility factor vs. frequency ratio for various damping factors.

there are n masses in a system with no constraints, then the total degrees of freedom for that system will be $6 \times n$.

Example: see Figure 1-88.

29. Transmissibility Factor

Definition: the ratio of the magnitude of the force transmitted to that of the impressed force.

Example:

1. See Figure 1-89. Constant Force Amplitude Excitation $F = F_0 \sin \omega t$.
   Transmissibility $(T_r) = F_T/F_0 = \sqrt{1 + (2Dr)^2}/\sqrt{(1-r^2)^2 + (2Dr)^2}$, where F_T is the force transmitted.
2. See Figure 1-90. Rotating Mass-Type Excitation, $F = m_i e\omega^2 \sin \omega t$.
   Transmissibility $(\overline{T}_r) = F_T/m_i\omega_n^2$
   $= r^2\sqrt{1 + (2Dr)^2}/\sqrt{(1-r^2)^2 + (2Dr)^2}$
   $= r^2 T_r$

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2 Development of Analytical Models for Dynamic Systems

A detailed dynamic analysis of a structural system as it physically appears in real life is rarely attempted. The usual practice is to choose an idealized model consisting of springs and lumped masses which will closely perform in the same way as the actual structure. It is only necessary that a proper selection of the system parameters be made such that equivalence of the idealized spring, damping element, and lumped mass in the model results in equivalent displacements at analogous points of significance in the prototype structure. In addition, the idealized model should behave, time-wise, in exactly the same manner as the actual prototype structure.

Modeling Techniques

The techniques adopted in the modeling of structures subjected to dynamic loads are still in the developing stage. The approach used in the modeling of simple systems, such as a beam supporting a vibrating load or a rigid block-type foundation supporting a machine, is straightforward. However, when the structures involved are of an indeterminate type which rest on soils and are subjected to complex dynamic forces, the modeling approach differs depending on the analyst (ref. 2). These differences do not necessarily mean disagreement in the basic fundamentals, but rather relate mostly to the accuracy and efficiency achieved in the solution. During the 1960s, the investigation of structural systems used in space exploration and more recently in structures used in nuclear power plants and offshore structures has resulted in an established state-of-the-art in the field of structural dynamics (refs. 3 and 5). The rigorous use of digital computers and finite-element analysis techniques have been the principal agents in the development of the state-of-the-art (ref. 6). Therefore, it is imperative that designers who wish to solve structural problems should have adequate exposure to these analytical tools. Modeling of any structual system is dictated by the requirements imposed on the solution. The desired solution may be for one fundamental frequency or for a spectrum of frequencies of all possible modes. In some cases it may be necessary to find the vibration response at various points of interest. There are a few practical considerations which are commonly used in all model representations. These include the following:

1. The Lumping of Mass. The logical location of equivalent lumped mass in a model should be at: (a) the point where dynamic force or load is acting; (b) a point where vibration response is desired; (c) a point where maximum static deflection will occur, e.g., at the free end of a cantilever or at the midspan of a beam; (d) the intersection point of a beam and a column; (e) the node point of finite elements in a continuous system; (f) the center of gravity of all masses, when a single-degree-of-freedom system is employed.

2. Elastic Spring Constant. The spring constant represents a linear relationship between the applied load and the displacement of the mass. A value for the spring constant is derived by determining the structural stiffness of the elastic medium existing between oscillating masses or between a mass and another infinite stiff support. Specifically, the elastic properties of: (a) a prismatic member can be represented by three linear springs and three rotational springs; (b) a thin plate can be represented by two linear springs and two rotational springs which are equivalent to stretching and bending occurring in the plane of the plate; (c) a massive concrete block approaches infinite stiffness; (d) soil reactions to the foundation loads can be represented by elastic springs capable of acting in tension and compression.

3. Damping Ratio. The dashpot of the lumped system (Figure 1-2) represents the damping in a dynamic system. Damping may occur due to several factors present in the system, for example, the frictional resistance and slippage occurring at the interface of surfaces at the contact joints or the sliding phenomenon in the molecular structure of the elastic spring.

The damping assumed in the structural system is of the viscous type and includes the following additional assumptions: (a) the internal damping present in concrete and steel structures is nominal, i.e., the damping ratio varies from 2.0 to 5.0 percent for concrete structures and 1.0 to 7.0 percent for steel structures and normally is neglected; (b) the damping agent associated with the soil is in the form of internal damping and geometric damping. The internal damping value is of small magnitude in all modes of oscillation except in the rocking mode. Geometric damping has considerable effect on the dynamic response of the system and is generally included in the model representation. Chapter 4 further describes the nature and evaluation of soil damping.

4. Forcing Function. The forcing function is normally treated as an equivalent concentrated force applied at points where masses are lumped. Torques are applied at mass points either in concentrated form or are converted into an equivalent force-couple. The effect of earthquake forces may be obtained by the application of a time-history acceleration at the mass points. A time-history displacement applied at the supports may also be used for earthquake loads as a type of forcing function (refs. 3, 4 and 5 of this chapter and ref. 4, 5 and 6 of chapter 3).

Models

Civil engineering structures of various kinds use different modeling techniques depending on the type of solution desired (ref. 1). Modeling types are given in Figures 2-1 through 2-11 (page 39) for typical structures having different constraint conditions. Also shown is the mathematical model used to represent each physical system and a short description of the model, as well as listing the applicable equations of motion. Each model is further described below.

Development of Equations of Motion

The equations of motion which describe the behavior of the mathematical model are developed using one of the following two methods (ref. 3).

(a). Dynamic equilibrium equation. In this method, the condition of equilibrium of a mass at any instant of time under the influence of forces and reactions is considered. In order to account for dynamic equilibrium, the mass inertia force is included. Consider, for example, the vertical excitation for the "Machine supported on inertia-block and vibration isolated from the foundation" shown in Figure 2-3(a). The dynamic equilibrium equations are derived as follows:

Assuming that at any instant of time, the masses m_1 and m_2 have moved up through displacements Z_1 and Z_2, respectively, from their reference position, then, for mass m_1:

1. Resistance of spring $k_{z1} = k_{z1}\,(Z_1 - Z_2)$ (acting downward)
2. Inertia force of mass $m_1 = m_1(d^2Z_1/dt^2) = m_1\ddot{Z}_1$ (acting downward—opposite to the direction of displacement).
3. Excitation force $= F_z\,(t)$ (acting upward)

Since summation of downward forces = upward forces, $\therefore\ m_1\ddot{Z}_1 + k_{z1}\,(Z_1 - Z_2) = F_z(t)$ (2-1)

Similarly for mass m_2:

1. Resistance of spring $k_{z1} = k_{z1}\,(Z_1 - Z_2)$ (acting upward)
2. Resistance of spring $k_{z2} = k_{z2}\,Z_2$ (acting downward)
3. Resistance of damping $C_{z2} = C_{z2}\,\dot{Z}_2$ (acting opposite to the direction of movement Z_2, thus acting downward)
4. Resistance of inertia force of mass $m_2 = m_2\ddot{Z}_2$ (acting opposite to the direction of movement Z_2, thus acting downward)

Equating the downward resistance to the upward resistance,

$$m_2\ddot{Z}_2 + C_{z2}\,\dot{Z}_2 + k_{z2}\,Z_2 = k_{z1}\,(Z_1 - Z_2)$$

or

$$m_2\ddot{Z}_2 + C_{z2}\,\dot{Z}_2 + k_{z1}\,(Z_2 - Z_1) + k_{z2}\,Z_2 = 0 \qquad (2\text{-}2)$$

Equations (2-1) and (2-2) are the same set of Equations (a) shown in Figure 2-3 of model. It should be noted that $\dot{Z}$ and $\ddot{Z}$ stand for the first and second derivatives of the displacement Z with respect to time t, i.e., $\dot{Z} = dz/dt$ and $\ddot{Z} = d^2z/dt^2$.

(b) Lagrange's Equation. Lagrange's equation, in its fundamental form for a conservative system in generalized coordinates q_i is given by

$$\frac{d}{dt}\left(\frac{\partial \text{K.E.}}{\partial \dot{q}_i}\right) - \frac{\partial(\text{K.E.})}{\partial q_i} + \frac{\partial(\text{P.E.})}{\partial q_i} - \frac{\partial(\text{D.E.})}{\partial q_i} = \frac{\partial(W_e)}{\partial q_i} \qquad (2\text{-}3)$$

where K.E. = kinetic energy of the system,
P.E. = potential energy of the system,
D.E. = dissipation energy of the system,
W_e = work done by the real external forces on the system.

The use of Lagrange's equation will directly yield as many equations of motion as the number of degrees of freedom of the system, given that basic energy expressions of the system are known.

This method is applied to the model discussed earlier where the dynamic equilibrium equation method was used in section (a). In this example, there are two coordinates, that is, $q_1 = Z_1$, $q_2 = Z_2$.

The energy expressions in terms of Z are as follows:

Kinetic Energy = K.E. = $\frac{1}{2} m_1 (\dot{Z}_1)^2 + \frac{1}{2} m_2 (\dot{Z}_2)^2$

Potential Energy = P.E. = $\frac{1}{2} k_{z2} (Z_2)^2 + \frac{1}{2} k_{z1} (Z_1 - Z_2)^2$

Dissipation Energy = D.E. = $(-C_{z2} \dot{Z}_2)(Z_2)$

Work by external force = $W_e = F_z (Z_1)$

The dissipation energy due to damping force must be taken as negative, since a positive damping force is always in a direction opposite to the positive displacement. The derivatives with respect to Z_1 are

$$\frac{\partial (K.E.)}{\partial \dot{q}_1} = \frac{\partial (K.E.)}{\partial \dot{Z}_1} = m_1\dot{Z}_1, \frac{d}{dt}\left(\frac{\partial (K.E.)}{\partial \dot{Z}_1}\right) = m_1\ddot{Z}_1$$

$$\frac{\partial (K.E.)}{\partial q_1} = 0$$

$$\frac{\partial (P.E.)}{\partial q_1} = \frac{\partial (P.E.)}{\partial Z_1} = k_{z1} (Z_1 - Z_2)$$

$$\frac{\partial (D.E.)}{\partial q_1} = \frac{\partial (D.E.)}{\partial Z_1} = 0$$

$$\frac{\partial (W_e)}{\partial q_1} = \frac{\partial (W_e)}{\partial Z_1} = F_Z (t)$$

Substitution of the above in Equation (2-3) leads to

$$m_1 \ddot{Z}_1 + k_{z1} (Z_1 - Z_2) = F_Z(t) \tag{2-4}$$

The derivatives with respect to Z_2 are

$$\frac{\partial (K.E.)}{\partial \dot{q}_2} = \frac{\partial (K.E.)}{\partial \dot{Z}_2} = m_2\dot{Z}_2, \frac{d}{dt}\left(\frac{\partial (K.E.)}{\partial \dot{Z}_2}\right) = m_2\ddot{Z}_2$$

$$\frac{\partial (K.E.)}{\partial q_2} = \frac{\partial (K.E.)}{\partial Z_2} = 0$$

$$\frac{\partial (P.E.)}{\partial q_2} = \frac{\partial (P.E.)}{\partial Z_2} = k_{z2}Z_2 - k_{z1} (Z_1 - Z_2)$$

$$\frac{\partial (D.E.)}{\partial q_2} = \frac{\partial (D.E.)}{\partial Z_2} = - C_{z2} \dot{Z}_2$$

$$\frac{\partial (W_e)}{\partial q_2} = \frac{\partial (W_e)}{\partial Z_2} = 0$$

Substitution of the above in Equation (2-3) leads to the equation of motion:

$$m_2\ddot{Z}_2 + C_{z2}\dot{Z}_2 + k_{z1} (Z_2 - Z_1) + k_{z2}Z_2 = 0 \tag{2-5}$$

These equations are readily verified by consideration of dynamic equilibrium given by Equations (2-1) and (2-2). The above method is generally an inefficient way of obtaining the equation of motion. Furthermore, it should be recognized that the Lagrange equation is merely a device for writing the equation of motion and is not an *independent* method of solution.

Model 1—Vibrating Machine Supported by Block-type Foundation (Figure 2-1)

This type of foundation is a very common form of physical system and is usually considered by design engineers in petrochemical and industrial plants. Three forms of dynamic mode shapes are possible and should be investigated (ref. 2). Vertical and horizontal modes are described by linear differential equations, and the solution for the natural frequencies and vibration response are easy to obtain. In the rocking mode, the coupling effect of the horizontal mode may be ignored for very shallow foundations. In that case, h_0 is zero, therefore, no coupling effect is present and thus Equation (c) of Figure 2-1 reduces to Equation (b). Similarly, Equation (d) of Figure 2-1 is also "reduced" (uncoupled) and describes the motion in coordinate ψ and is as follows:

$$I_\psi \ddot{\psi} + C_\psi \dot{\psi} + k_\psi \psi = F_y(t) H = T_\psi \cos \omega t \tag{2-6}$$

This equation along with Equations (a) and (b) of Figure 2-1 can be solved according to the procedures given in Chapter 1. An example is solved in Chapter 6 which describes the required steps in the calculation procedure. When the vibration response of the coupled modes is desired for Equations (c) and (d) of Figure 2-1, then the solution can be found by substituting

$$y = A_{y1} \sin \omega t + A_{y2} \cos \omega t \tag{2-7a}$$

$$\psi = A_{\psi1} \sin \omega t + A_{\psi2} \cos \omega t \tag{2-7b}$$

in Equations (c) and (d) and then separating the equations containing either sines or cosines. This procedure will result in four simultaneous equations with four unknowns. A complete solution of this type of equation is given in Appendix A.

Model 2—Vibrating Machine Supported by Mat-type Foundation (Figure 2-2)

This type of foundation system may be used for the situation where several small units are placed side by side or where a firm soil with a high water table is encountered at plant grade level. Due to the flexibility of the foundation mat, a high magnitude of damping will be encountered in the rocking and horizontal modes. Therefore, only the frequency and vibration response calculations in the vertical mode are required.

A single lumped mass model may be used when one set of machines is supported by a relatively rigid mat foundation. However, the model is divided into discrete lumped masses when several sets of machines are located on a flexible mat foundation. In this case, constraint conditions are applied to the boundaries in the directions of translation for the sake of stability. The spring constants for each element depend on the mat rigidity as described in Chapter 5.

Model 3—Machine Supported on an Inertia Block and Vibration Isolated from the Foundation (Figure 2-3)

In special cases and due to environmental conditions, it may be necessary to limit the vibration amplitude at the foundation base to much lower values than those usually allowed. This requirement may not be practical to achieve even by proper selection of mass or base area of the foundation. In such cases, use of an inertia block and spring absorbers is recommended.

In normal behavior, three forms of excitation are possible. Excitation in the vertical direction is independent of the other forms of oscillation. Excitation in the horizontal direction is generally coupled with the rocking mode; however, for a machine which is located at relatively low height (h is $< \frac{1}{2} b$) then investigation of the horizontal and rocking excitation independent modes is sufficient.

The parameters k_{x1} and k_{z1} are properties of the spring absorbers. Parameter m_1 is the combined mass of the machine and the inertia block together. The parameters k_{x2}, k_{z2}, $k_{\psi 2}$ and C_{x2}, C_{z2}, $C_{\psi 2}$ are spring constants and damping coefficients, respectively, of the soil in the three modes considered and should be determined using the elastic half-space theory as described in Chapter 4. Parameters m_2 and I_2 are the mass and mass moment of inertia, respectively, of the foundation.

The solution of the differential Equations (a) and (b) of Figure 2-3 can readily be found for the natural frequencies, mode shapes, transmissibility factors, and the vibration response. Often, the fundamental frequency and the transmissibility factor are the principal results of the analysis. The set of differential equations (c) of Figure 2-3 is in simultaneous form, and a manual solution is tedious to perform. This system of simultaneous equations is rarely solved by hand unless a thorough investigation of the system is required, and then the solution is obtained with the help of a computer program. However, solution for a similar type of equation of motion has been performed in Appendix A.

Model 4—Vibrating Machine Supported by a Cantilever (Figure 2-4)

It is sometimes required that a vibrating machine be supported on a cantilever. In such instances, a vibration analysis is considered necessary. Two modes (vertical and rocking) are possible (ref. 3). The calculation of the rocking mode may be ignored if the distance h is found to be small, and the cantilever arm is rigidly secured. The calculation of the vertical mode is generally performed because this provides the fundamental frequency and the largest vibration response. The mass parameter m_e is considered lumped at point O and consists of the mass of the machine plus an equivalent mass for a portion of the cantilever calculated according to the procedure explained in Chapter 1. The spring stiffness parameter k_z is the flexural stiffness of the cantilever at point O. Damping in the system varies from 0.005 to 0.05 of critical, depending on the material.

The investigation for the rocking mode is performed on a similar basis as for the vertical mode. The mass moment of inertia parameter I_ψ is calculated for the equivalent mass m_e about the point. O. The rotational spring constant k_ψ is calculated by applying a moment at point O about the x-axis (the x-axis is perpendicular to the figure).

The maximum vibration response calculated for each mode may occur at different times. Therefore, in obtaining the total response, the maximum of the sum may occur at some specific time within the interval of interest. However, obtaining this maximum value may be difficult. Therefore, a simple summation of the individual maxima is generally performed, which results in a conservative estimate of total displacement. The solution of the equations of motion has previously been described in Chapter 1.

Model 5—Vibrating Machine Supported by a Fixed Beam (Figure 2-5)

Mathematical modeling technique for this physical system is similar to the cantilever system above except that the parameter determination differs.

The mass parameter m_e is the combined mass of machine and a certain length of beam and is lumped at the intersecting axes of the machine and the beam. The spring constant k_z is a function of the flexural stiffness of the beam. Both parameters m_e and k_z can be evaluated by using the expression given for Case II in Table 1-1. The parameters I_ψ and k_ψ can be determined by following the procedure described under Model 4 above.

All modes of oscillation which may occur due to the action of the forcing function generally need investigation. In this case, the vertical mode and the rocking mode (about the x-axis) fall in this category. The vertical mode investigation is necessary because it gives the lower value of natural frequency and the higher level of vibration response. On the other hand, the rocking mode investigation provides higher value of natural frequency and a lower level of vibration response. Therefore, if the machine operating frequency is found to be very close to the rocking mode natural frequency, then the model parameters may need modification in order to avoid possible resonance conditions. Thus, both modes may need to be considered in some machine supports.

Model 6—Typical Elevated Pedestal Foundation (Table Top) (Figure 2-6)

In this physical model, there is some variation in the use of the modeling technique (ref. 1). In this example, four models are considered, and the merits of each and the effort involved in their solution is discussed. In the modeling procedure the following factors may be used in determining the type of model to be used.

Model A: Single-Lumped Mass (Uncoupled Superstructure and Foundation) (Figure 2-7)

—A preliminary investigation is required.
—Reliable information on the parameters is lacking.
—The structural framing system is not well defined and preliminary dynamic characteristics are desired.
—The beams have much higher stiffness than the columns.
—Information on vibration response is not required.

Because of the lack of interaction between superstructure and substructure, this model gives the dynamic solution of individual subsystems and generally results in a calculated higher fundamental frequency. Therefore, the engineer must use conservative criteria to avoid the resonance condition by first making certain whether the system is to be low or high tuned. This condition is generally achieved by varying the mass or stiffness of the system components. The calculation of parameters for the superstructure representation would require a familiarity with structural frame analysis for the calculation of the spring stiffness constants k_h and k_v. The mass parameters m_u is the mass of machine plus the mass of top framing and upper half-length of columns. The substructure investigation during the initial investigation phase of the total foundation is not conducted because its influence on the solution of the superstructure portion is small. However, if the substructure base slab configuration is necessary, then information on dynamic soil properties will be required, and the slab should be rigid enough such that it can be represented by a single lumped mass. The model parameters for spring stiffness k_x, k_z, k_ψ and damping constant C_x, C_z, C_ψ in the three modes of excitation can be determined from the soil properties by the elastic half-space theory as described in Chapter 4. The mass parameter m_x is the total mass of machine and of the entire foundation structure. The parameter I_ψ is the mass moment of inertia of the machine and the entire foundation at the center line and at the base of footing. The equations of motion involved in both the subsystems are linear second-order differential equations which can be solved for the natural frequency and vibration response by the procedures described in Chapter 1.

Model B: Multi-Lumped Mass (Uncoupled Superstructure and Foundation) (Figure 2-8)

This model may be used when the following types of results are desired according to the listed conditions:

—Superstructure is well defined, and the fundamental frequency is to be lower than the machine operating frequency (structure to be undertuned).
—Foundation structure is supported either on highly firm soil or rock formation or a rigid deep foundation.
—Accurate determination of vibration response is not a requirement.
—Foundation structure height is low (less than 20 ft) and foundation is not supporting more than two machines.

The dynamic characteristic of the superstructure may be calculated in either of two ways:

1. Rayleigh's Frequency (see definition in Chapter 1). In this method, the weights of structure and of the machines are applied as static forces at a discrete number of points acting in the direction of the deflected shape of the structure (assumed as the fundamental mode). The following formula is used:

$$\omega_n = \sqrt{\frac{g \sum_i F_i \delta_i}{\sum_i F_i \delta_i^2}} \tag{2-8}$$

where F_i are the equivalent lumped forces of distributed weight of structure and machine acting at point i; δ_i are displacements of the structure at points i produced by the forces F_i; g is the gravity constants; ω_n is circular natural frequency in the mode corresponding to the direction of the acting forces.

The accuracy of the natural frequency calculation obtained by this method depends entirely on how close to reality are the assumptions made in assuming the deflected shape of the structure. However, note that for the approximate results obtained by this method, the lowest frequency always gives the best approximation.

2. Modal Multidegree Lumped Mass Analysis. The normal modes (modal analysis and normal modes are defined in Chapter 1) are determined separately and then superimposed to provide the total response. A normal mode (or natural mode) of vibration is associated with each degree of freedom of lumped mass in the system. The property of a normal mode is that the system could, under certain circumstances, vibrate freely in that mode alone, and during such vibration the ratio of the displacements of any two masses is constant with time. The ratio defines the characteristic shape of the mode.

The equations of motion in matrix form for a multidegree system, but having no external acting force and no damping, have the following form:

$$m_i \ddot{y}_i + k_{ij} y_j = 0 \tag{2-9}$$

The natural frequencies of all modes are found by assuming a harmonic motion for each mode. During vibration in any single mode, the displacements of several masses is always in the same proportion, i.e., all possible positions are geometrically similar. Substitution of the assumed mode function in the equations of motion and rearrangment would result in a characteristic value problem. Briefly, solution of these equations consists of expansion of their determinant and subsequent solving for the characteristic values (or eigenvalues, natural frequencies).

The vibration response of the multidegree system due to applied forces or initial conditions is obtained by treating each normal mode as an independent one-degree system. The formulation of the problem by this method of analysis requires a thorough background in dynamics analysis and is generally attempted using a computer program. The results obtained from this method will be the natural frequencies for each degree of freedom of the masses, mode shapes for each frequency, and response at the joints in the form of vibration amplitudes, internal forces, and moments. An example problem for three degrees of freedom based on this method of analysis has been presented in Appendix A.

The investigation of the substructure is performed according to the procedures stated for Model A. The vibration response results obtained for both the subsystems are combined by some rational procedure such that maximum values are achieved at a particular moment in time.

Model C: Two-Lumped Mass with Coupled Soil-Structure Interaction (Figure 2-9)

Model C includes soil-structure interaction and also includes the true dynamic characteristics of the foundation system. This model representation may be employed when the following conditions are satisfied:

—Foundation structure is supporting not more than two machines, i.e., the length of the structure should be small and its height is not greater than 20 ft.
—Foundation structure is well defined, and reliable information on the soil is available.
—Natural frequencies of all subsystems of the model do not fall in the resonance zone (0.5 to 1.5 of the acting frequency of the forcing function).

The types of results obtained from this model are:

—Natural frequencies for all important modes which are excited by the forcing function.
—Vibration response at the axis of rotation of the machines and also at the base of the foundation.

The dynamic behavior of the model has been separated into parts a and b. Part a represents the coupled modes of horizontal and rocking oscillations while part b represents the behavior for vertical oscillation. The equations of motion of these two parts are described in Model C of Figure 2-9.

The natural frequencies of these models can be determined from the equations of motion by initially ignoring the terms containing damping and forcing functions. An exponential type of displacement function is substituted for each degree of freedom (mode shape) in these equations. A determinant is formed from the resulting equations which, when expanded, results in a characteristic or frequency equation. The solution of the frequency equation will give the expression for the natural frequencies. A general solution can be formulated with the use of natural frequency expressions to obtain the mode shapes. The constant of integration in the equations is evaluated by equating the general solution to the initial conditions. The manual calculation of vibration response of coupled mode part a is tedious and time-consuming and normally is accomplished with the use of computer programs; however, a manual solution is presented in Appendix A. The techniques used in vibration response analysis follow a modal analysis of the lumped mass multidegree system which is briefly described above in discussing Model B. The response analysis of part b is relatively simple and can be obtained manually in a manner similar to part a. However, if a computer program is used for the solution of part a, then parameters of part b can also be easily combined in order to provide the total results (ref. 2). In that case, the rocking spring k_ψ of part a should be represented by three equivalent vertical springs of equal stiffness (see Figure 2-10) with the following conditions:

$$k_\psi = 2\, k_e e^2$$
$$k_z = 3\, k_e$$

or

$$e = \sqrt{3\, k_\psi / 2\, k_z}$$

In this case k_ψ = rocking spring constant, k_z is the vertical spring constant of m_l, and e is the distance between two equivalent springs of stiffness k_e. The distribution of damping coefficient C_ψ related to C_Z is rather complex; however, a similar form of logic may be followed as has been done for the spring constant. The damping coefficients associated with the equivalent vertical spring k_e would not be all equal if the same value of e is used. Therefore, an equal value of the damping coefficient C_e has been used for the exterior springs, and a different value of damping coefficient C_m has been used for the middle spring. The member which connects the three equivalent springs should possess an infinite flexural stiffness but should also maintain the equivalent values of m_l and I_ψ.

Model D: Multi-Lumped Mass with Coupled Soil-Structure Interaction (Figure 2-11)

This model provides the design engineer with a complete insight of not only the dynamic behavior of the superstructure but also identifies the critical modes in the soil-structure system. In cases where access to a computer program capable of solving dynamics problem is available, then it is very convenient to resort to this modeling technique. In this investigation, several kinds of results can be obtained provided that the following parameters are available:

—Member sizes of the structure and geometry are available and have been proportioned such that: (a) the rigidity center of gravity of the structure in plan coincides with the center of gravity of the masses of the structure; (b) the flexural displacement of the top of the structure in either direction is uniform across the length when the top mass of the structure and of the machines are made to act as horizontal loads; (c) all columns deflect equally under static loads; (d) the center of resistance of the supporting soil coincides with the centroid of all statically imposed loads.

—Information on founding depth of the structure and on soil, such as shear modulus (G), Poisson's ratio (ν), and bearing capacity of the soil is readily available.

The results that can be obtained from the analysis of this model include:

—The natural frequencies (eigenvalues); the number of frequencies will depend on the number of active joints with lumped masses and the degrees of freedom of each joint considered in the analysis.
—The mode shapes (eigenvectors) for each of the natural frequencies.
—The vibration response (in terms of displacements, velocities and accelerations) for each mass having six degrees of freedom.
—The moments and forces at each mass joint and the reactions at the supports.

A dynamic solution of this model using the computer program STRUDL is presented in Chapter 7 (page 114).

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2. Richart, R. E., Jr., Hall, J. R., Jr., and Woods, R. D., *Vibrations of Soils and Foundations,* Englewood Cliffs, NJ: Prentice-Hall, 1970.
3. Biggs, John M., *Introduction to Structural Dynamics,* New York: McGraw-Hill, 1964.
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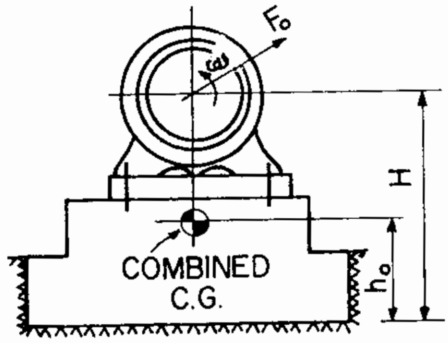
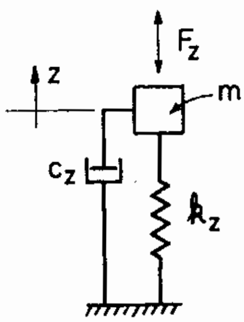


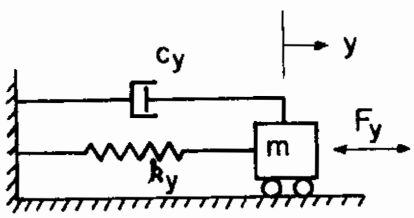
Vibrating Machine
Supported by
Block-Type Foundation
(Model 1)



Vertical Mode

Vertical mode normally behaves independent of other modes. The mass (m) of machine and foundation is assumed to be concentrated on the vertical axis. Spring constant of soil (k_z), damping in soil (C_z), inertia of mass (m_z) and the forcing function (F_z) of the machine have their line of action coinciding with the vertical axis. Equation of motion:

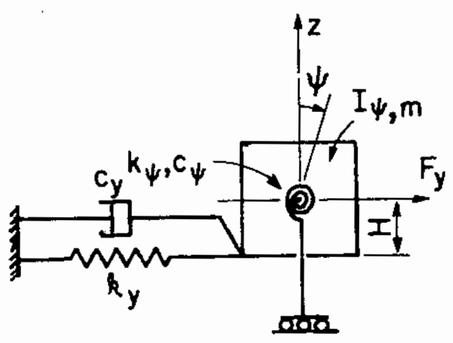
$$m\ddot{z} + C_z\dot{z} + k_z z = F_z(t) \tag{a}$$



Horizontal Mode

The representation for the horizontal mode involves an approximation. In this mode, contrary to the vertical mode, the masses do not lie on the same horizontal axis, nor the line of action of the forces coincides. Due to these reasons, this mode is normally coupled with the rocking mode. Equation of motion:

$$m\ddot{y} + C_y\dot{y} + k_y y = F_y(t) \tag{b}$$



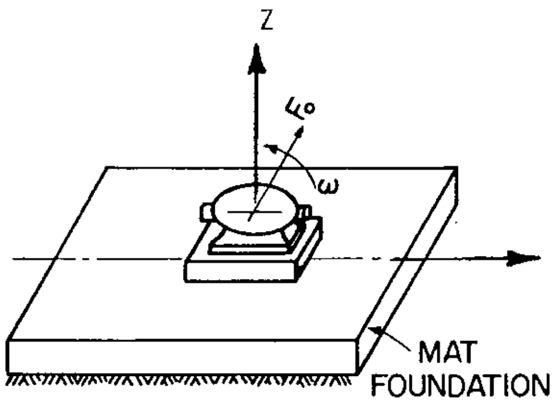
Rocking Mode

This model is a better representation of the true dynamic behavior of the structure. However, the analytical solution is difficult to attempt due to coupling of horizontal and rocking motions. This coupling effect should be investigated for the case when the machine is located high above the founding level. Equations of motion:

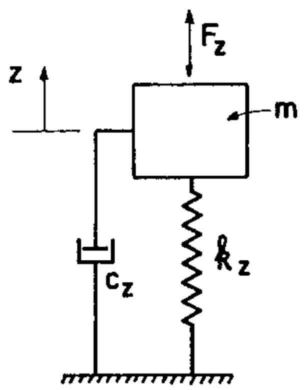
$$m\ddot{y} + C_y\dot{y} + k_y(y - \psi h_0) - h_0C_y\dot{\psi} = F_0\cos\omega t \tag{c}$$

$$I_\psi\ddot{\psi} + (C_\psi + h_0^2C_y)\dot{\psi} + (k_\psi + h_0^2k_y)\psi \tag{d}$$
$$-h_0C_y\dot{y} - h_0k_y y = F_y(t)H = T_\psi(t) = F_0H\cos\omega t$$

Figure 2-1. Model 1. Vibrating machine supported by block-type foundation.



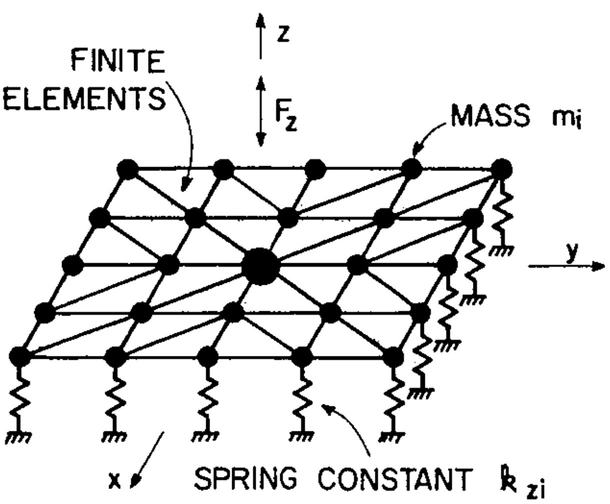
Vibrating Machine Supported by Mat-Type Foundation
(Model 2)



Single Lumped Mass Vertical Mode

For a mat foundation, which is relatively rigid (corner displacement is .85 x maximum under machine load at the center), machine and mat slab can be lumped into a single mass. Only the vertical mode needs investigation. Classification of a mat as "rigid" is discussed in Chapter 5. The spring constant k_z and the damping constant C_z are to be obtained from elastic half-space theory or from soil data. Equation of motion:

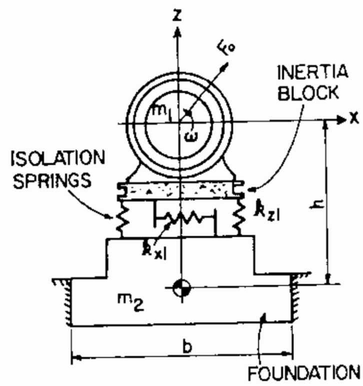
$$m\ddot{z} + C_z\dot{z} + k_z z = F_z(t)$$



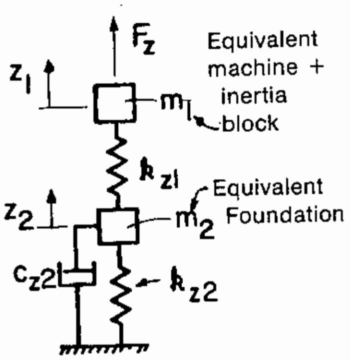
Discrete Number of Lumped Masses Vertical Mode

When the mat foundation is a thin concrete slab, and dynamic analysis is required, a discrete lumped-mass model is more appropriate for analysis. The mat may be divided into a symmetrical arrangement of finite elements (either triangular or rectangular shape) with at least bending capability. The masses at the joints can be lumped by a computer program command. The mass of the machine is to be added through the computer program input at the appropriate joints. The soil spring constant calculated for the total mat should be proportioned to the joint in relation to the peripheral areas for a rigid mat. For flexible mats, the procedures described in Chapter 5 should be used. A computer program is necessary for frequencies calculations. If vibration response calculations are desired, an equivalent damping constant of about 0.1 may be used.

Figure 2-2. Model 2. Vibrating machine supported by mat-type foundation.



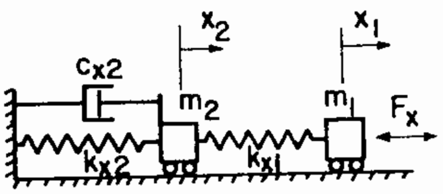
Machine Supported on Inertia-Block and Vibration Isolated from the Foundation
(Model 3)



Vertical Excitation

This model representation yields the natural frequencies and vibration response in the vertical direction. Since there are two masses in the model, there are two degrees of freedom and two natural frequencies, and mode shapes will be obtained. Damping in isolation spring is generally neglected. However, soil damping is significant and is, therefore, included. Equations of motion:

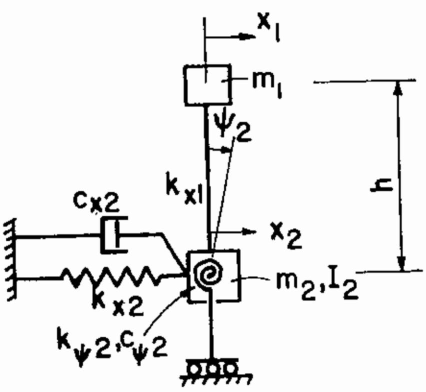
$$
\begin{aligned}
m_1\ddot{z}_1 + k_{z1}(z_1 - z_2) &= F_z(t) \\
m_2\ddot{z}_2 + C_{z2}\dot{z}_2 + k_{z1}(z_2 - z_1) + k_{z2}z_2 &= 0
\end{aligned}
\qquad (a)
$$



Horizontal Excitation

The characteristics of this model are similar to the above model except that all the parameters are related to the horizontal axis. Equations of motion:

$$
\begin{aligned}
m_1\ddot{x}_1 + k_{x1}(x_1 - x_2) &= F_x(t) \\
m_2\ddot{x}_2 + C_{x2}\dot{x}_2 + k_{x1}(x_2 - x_1) + k_{x2}x_2 &= 0
\end{aligned}
\qquad (b)
$$

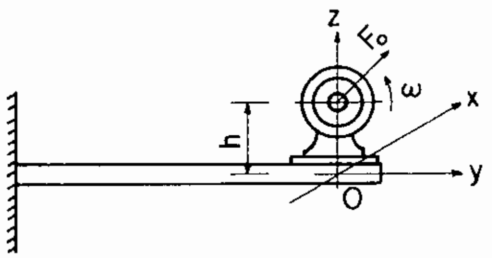


Coupled Horizontal and Rocking Modes

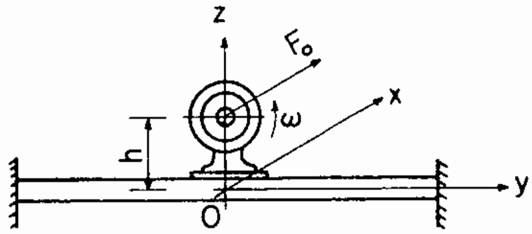
This model incorporates the coupled dynamic properties of the horizontal and rocking modes. This model has three degrees of freedom associated with the three coordinates x_1, x_2 and ψ_2. Therefore, three mode shapes are possible, each with its own natural frequency. Investigation of this model, if carried out, would make the solution of model for the horizontal excitation above unnecessary. Equations of motion:

$$
\begin{aligned}
m_1\ddot{x}_1 + k_{x1}(x_1 - x_2 - \psi_2 h) &= F_x(t) \\
m_2\ddot{x}_2 + C_{x2}\dot{x}_2 + k_{x2}x_2 - k_{x1}(x_1 - x_2 - \psi_2 h) &= 0 \\
I_2\ddot{\psi}_2 + C_{\psi 2}\dot{\psi}_2 + m_1\ddot{x}_1 h + k_{\psi 2}\psi_2 &= F_x(t)\,h
\end{aligned}
\qquad (c)
$$

Figure 2-3. Model 3. Machine supported on inertia-block and vibration isolated from the foundation.

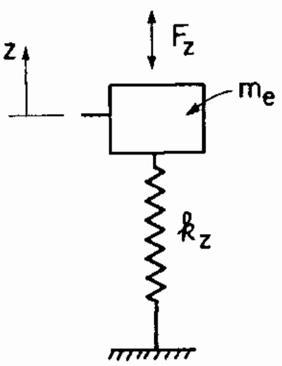


Vibrating Machine Supported by a Cantilever
(Model 4)

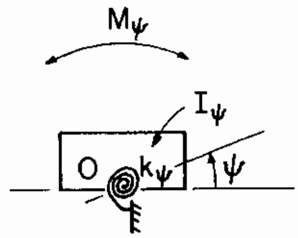


Vibrating Machine Supported by a Fixed Beam
(Model 5)

Both physical systems can be represented by a similar form of mathematical model. However, because of different physical dimensions and boundary conditions, dynamic model parameters will change. Two form of modes in the direction of dynamic forces require investigation:



Vertical Mode



Rocking Mode
(Rotation about x-Axis)

1. **Vertical Mode**—Mass (m_e) is the mass of machine + equivalent mass of support and is lumped at point *O*. Spring stiffness (k_z) is the deflection stiffness at point *O* for a flexural member. Damping in such a system is small and is neglected. Equation of motion:

$$m_e\ddot{z} + k_z z = F_0 \sin \omega t$$

2. **Rocking Mode**—Mass (I_ψ) is the moment of inertia of mass of the machine + that portion of support about point *O*. Torsional spring stiffness (k_ψ) is the rotational stiffness at point *O* for a flexural member. Damping is small and thus is not considered. Equation of motion:

$$I_\psi\ddot{\psi} + k_\psi\psi = F_0 h \cos \omega t = M_\psi \cos \omega t$$

Note: Mathematical form and description are common to both these systems.

Figures 2-4 and 2-5. Model 4 is a vibrating machine supported by a cantilever. Model 5 is a vibrating machine supported by a fixed beam.

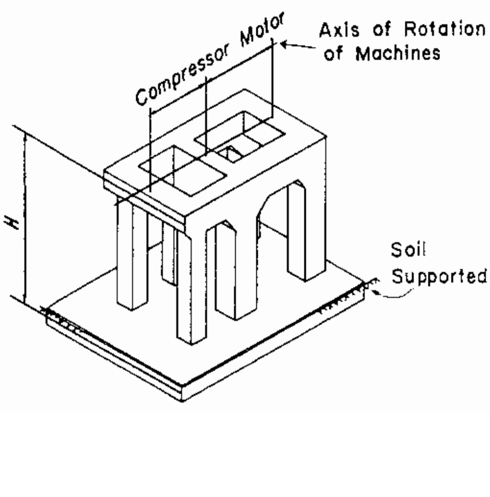
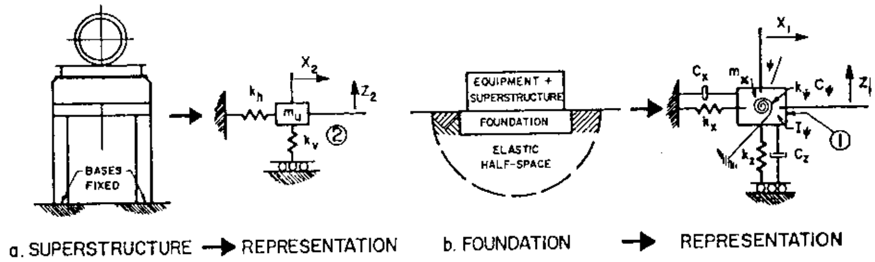


Figure 2-6. Model 6 is a typical elevated pedestal foundation "table top."



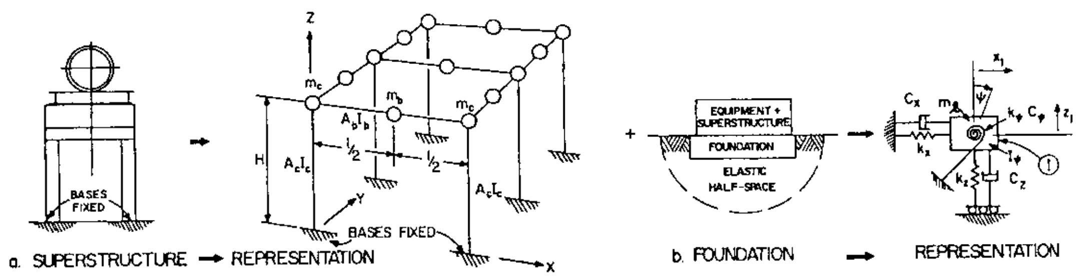
Model A

This model is composed of two parts: Figure a representing the top floor plus the supporting columns, and Figure b representing the total structure and equipment plus the bearing soil. These two subsystems are assumed to possess independent dynamic characteristics, and there is a lack of interaction behavior between the two. The basic assumption in this model is to consider the column bases as perfectly fixed. The structure is so modeled that only its three predominant motions (lateral, vertical, and rotational) are predictable at the C.G. of masses. Equations of motion:

For (a): $m_u\ddot{x}_2 + k_h x_2 = F_x(t)$
$m_u\ddot{z}_2 + k_v z_2 = F_z(t)$

For (b): $m_x\ddot{x}_1 + C_x\dot{x}_1 + k_x x_1 = F_x(t)$
$m_x\ddot{z}_1 + C_z\dot{z}_1 + k_z z_1 = F_z(t)$
$I_\psi\ddot{\psi} + C_\psi\dot{\psi} + k_\psi\psi = F_x(t)(H)$

Figure 2-7. Model A. Single-lumped mass model of table top (uncoupled superstructure and foundation).



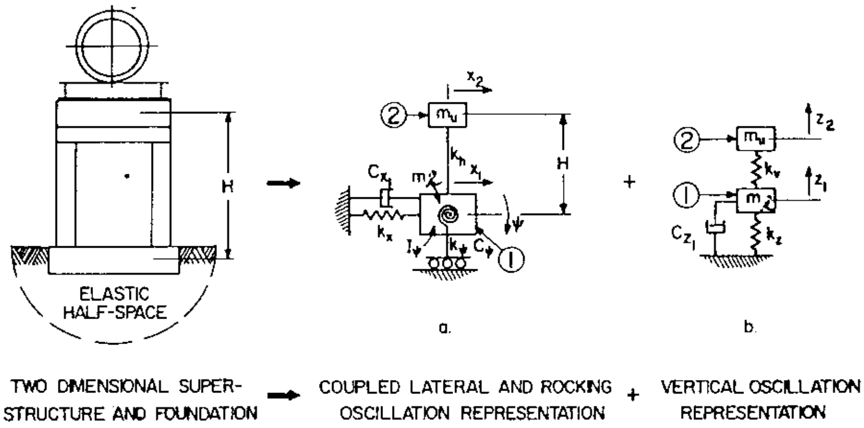
Model B

This model representation is similar to Model A, but some refinement is added by lumping the masses in the superstructure at points where dynamic response is important (see Figure a). The basic assumptions in this model are still the same, but each individual element acts independently of all others. This is generally permissible when the natural periods of the elements differ by at least a factor of two in any one direction of motion. In Figure b, the representation is similar to Model A. This model is assumed to include the complete dynamic characteristics of individual members of the superstructure. Equations of motion:

For (a): $m_i\ddot{y}_i + k_{ij}y_i = F_i$ *These equations are in matrix form and can only be solved with the help of a computer program.*

For (b): $m_i\ddot{x}_1 + C_x\dot{x}_1 + k_x x_1 = F_x(t)$
$m_i\ddot{z}_1 + C_z\dot{z}_1 + k_z z_1 = F_z(t)$
$I_\psi\ddot{\psi} + C_\psi\dot{\psi} + k_\psi\psi = F_x(t) \cdot H$

Figure 2-8. Model B. Multi-lumped mass model of table top (uncoupled superstructure).



Model C

This model is an improvement over Model A due to the incorporation of interactive capability between structure and soil which was neglected in Model A. The model is shown in two parts, Figure a for the coupled horizontal and rocking mode and Figure b for vertical mode. The model can be analyzed as shown without much loss of analytical accuracy; however, a coupled mode of both of these modes can also be studied without additional difficulty. The method of parameter calculation also does not change. The drawback in this model becomes apparent when the natural frequency of individual elements is required or it is necessary to calculate the vibration response at some other points in the structure. Equations of motion:

For (a):

$$m_u\ddot{x}_2 + k_h(x_2 - x_1 - \psi H) = F_{x2}(t)$$

$$m_l\ddot{x}_1 + C_{x1}\dot{x}_1 + k_x x_1 - k_h(x_2 - x_1 - \psi H) = 0$$

$$I_\psi\ddot{\psi} + C_\psi\dot{\psi} + m_u\ddot{x}_2 H + k_\psi\psi = F_{x2}(t)\cdot H$$

For (b):

$$m_u\ddot{z}_2 + k_v(z_2 - z_1) = F_{z2}(t)$$

$$m_l\ddot{z}_1 + C_{z1}\dot{z}_1 + k_v(z_1 - z_2) + k_z z_1 = 0$$

Figure 2-9. Model C. Two-lumped mass model of table top with coupled soil-structure interaction.

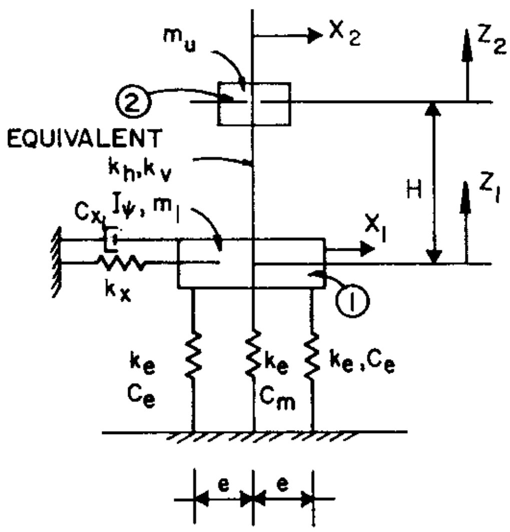
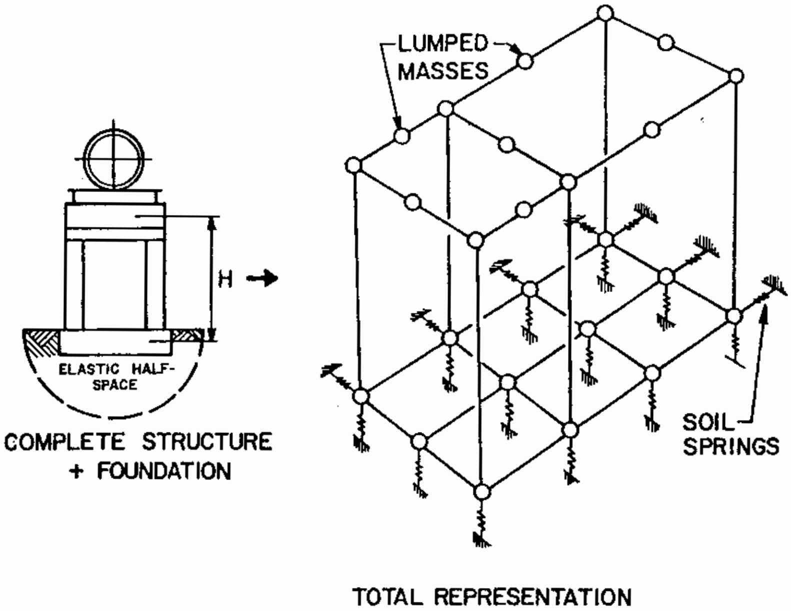


Figure 2-10. Model C-Alternate form. Coupled lateral, rocking, and vertical oscillation representations of table top for computer program application.



Model D

This model is an improvement in relation to the previous model, not only with respect to degree of reliability of results, but also with regard to availability of sufficient information at all points of interest. The approach is based on lumping the masses at the points of maximum displacement occurring for any direction of motion. The foundation slab is modeled using finite-element procedures and is supported by soil-springs at the node points. The interaction of the soil's stiffness with the foundation structure is obtained through the use of the elastic half-space theory. The calculation of stiffness for the structure is quite complex and is generally done through the use of computer programs. An average damping constant may be used when performing the response calculation for all modes.

Figure 2-11. Model D. Multi-lumped mass model of table top with coupled soil-structure interaction.

3 Development of Information, Trial Sizing, and Design Checklist

The design of a dynamically loaded structure requires that certain site and loading parameters be known even before preliminary sizing of the structure can be completed. These design conditions and requirements may be generally classified into three groups: machine properties and requirements, soil parameters, and environmental requirements. Therefore, the required design information includes not only geometrical constraints of the actual machine to be supported but also includes detailed knowledge of the structural supports. These supports are in turn related to the particular site conditions and can be of three types: soil supports, piles, or piers. Structures supporting dynamic machines are generally soil supported, or may be supported by piles if the soil is of low-bearing capacity. Characteristics of these foundation support types are further described in Chapters 4 and 5. Some foundations lying close to bedrock or resting on thick deposits of overconsolidated clay have been supported on piers, but this is a relatively unusual situation.

Machine Properties and Requirements

Machines causing dynamic loads on structures are of many types but may be classified in either of two large groups: centrifugal or reciprocating machines. In either case, a periodic time-dependent loading function is transmitted through the structure into the foundation. In order to design the structure, a number of machine geometrical and performance factors are required. These factors may be supplied by the machine manufacturer or may be available in sales catalogues or engineering handbooks. Often, the information is not available, and the designer must either perform some preliminary calculations or make some assumptions. The required machine properties and parameters include the following:

—Outline drawing of machine assembly
—Functions of machine
—Weight of machine and its rotor components
—Location of center of gravity both vertically and horizontally
—Speed ranges of machine and components or frequency of unbalanced primary and secondary forces
—Magnitude and direction of unbalanced forces both vertically and horizontally and their points of application
—Limits imposed on the foundation with respect to differential deflection between points on the plan area of foundation
—Foundation requirements

The physical size of the structure depends on the required base dimensions for the machine. Often, appurtenances such as platforms and piping supports require increases in base dimensions. The outline for the machine base generally specifies minimum dimensions and locates specific areas that must be left clear for machine attachments. For example, in turbines, certain regions under and over the machine must be left clear for condensors and piping.

Machine function includes information on the overall purpose and critical nature of the machine. Should the machine be of an extremely critical importance to overall operations, then a more conservative design approach is recommended. For example, shut down of a small pump may not affect plant production. However, if a large centrifugal compressor is to be shut down, a multimillion dollar operation may be affected. The designer must set the level of conservativeness balanced against possible unnecessary expense.

The weights of the machine and its components are provided by the manufacturer and serve to give a preliminary indication of soil support feasibility. The weight of rotors and speed in centrifugal machines determine the magnitude of possible machine unbalanced forces. The center of gravity location in the horizontal and vertical planes is often provided. When not available, calculations or assumptions may be needed. Basically, the machine is set on the foundation in such a way as to avoid eccentricities between the resultant of all loads and the support center of resistance, that is, the centroid of the pile group if pile supported or the center of resistance of the supporting soil if soil supported.

The speed range and frequencies of primary and secondary forces are required in the dynamic analysis in order to check for possible resonance. The designer is generally only interested in the operating frequencies, although in many machines, there will be particular speeds briefly attained during start-up or shut down where the assembly will be in resonance with the machine frequency. A temporary resonance condition may be tolerated in such cases especially when significant damping is available.

The magnitude and direction of unbalanced forces are often not available from the machine manufacturer. Some claim that their centrifugal machines are perfectly balanced, a condition that may be approached initially at the manufacturing plant. However, after a few years of use and due to normal wear, some eccentricity will exist regardless of initial machine and installation workmanship. Eccentricity criteria useful in designing structures supporting centrifugal machines are given in Table 3-1 and Figure 3-1. For reciprocating machines, the unbalanced forces, which are generally of considerable magnitude, are provided by the machine manufacturer.

Limits on differential deflection allowed between points of the foundation are set to avoid possible damage to piping and other appurtenances that connect to the machine. In some high pressure (50,000 psi) piping, differential deflection limits are approximately less than 0.0001 in. This is generally the case for machines with very rigid (thick) attached piping.

Foundation requirements refer to minimum depth of foundation, as dictated by expansive soils, frost action, fluctuating water table, piping clearance, or paving elevation. The top layer of weathered soil is often not recommended for supporting foundations since firm, undisturbed soil is required. Also, the bearing strength required for the soil may dictate placing the bottom of the foundation at a deeper level for soil-supported structures. The recommendations of the geotechnical consultant must be integrated into the design process (ref. 3).

Soil Parameters

Knowledge of the soil formation and its representative properties is required for static and dynamic analysis. In the case of a sand or clay formation, the information is to be obtained from field borings and laboratory tests. These are usually performed by the geotechnical consultants. Chapter 4 describes procedures for proper evaluation of these parameters and discusses other soil-related problems. The following parameters are generally required:

—density of soil, γ
—Poisson's ratio, ν
—shear modulus of soil, G, at several levels of strain (or magnitudes of bearing pressure)

Table 3–1
Design Eccentricities for Centrifugal Machines

For operating speeds up to 3,000 rpm, we have the following:

Operating Speed*	Eccentricity, in. (Double Amplitude)
750	.014–.032
1,500	.008
3,000	.002

*Major, A., *Vibration Analysis and Design of Foundations for Machines and Turbines*, Akademiai Kiado, Budapest, Collet's Holdings Limited, London, 1962.

The following are the modified API standards† for centrifugal compressors:

$$e\,(\text{mil}) = \alpha\sqrt{12{,}000/\text{rpm}} < 1.0\,(\text{mil}),$$

where α = 0.5 at installation time,
= 1.0 after several years of operation,
rpm = operating machine speed, rev/min,
1 mil = 0.001 in.

†American Petroleum Institute Standard for Centrifugal Compressors #617, Section 2.18.4, as modified by the parameter α.

For gear units, we have the following:**

Maximum continuous speed, rpm	Double amplitude including runout, mils	
	Shop test unloaded	Shop test loaded
Up to 8,000	2.0	1.5
8,000 to 12,000	1.5	1.0
Over 12,000	Less than 1.5	Less than 1.0

**American Institute Petroleum Standard, "High Speed, Special Purpose Gear Units for Refinery Service," API Standard 613, August 1968.

(Table 3-1 continued on page 48)

For motors, we have the following:

Integral Horsepower Electric Motors‡

Speed, rpm*	Peak-to-peak displacement amplitude, in.
3,000–4,000	0.0010
1,500–2,999	0.0015
1,000–1,499	0.0020
999 and below	0.0025

*For alternating-current motors, use the highest synchronous speed. For direct-current motors, use the highest rated speed. For series and universal motors, use the operating speed.
‡National Electrical Manufacturers Association Standard, "MG1-12.05, Dynamic Balance of Motor," December 1971.

Large Induction Motors***

Speed, rpm	Peak-to-peak displacement amplitude, in.
3,000 and above	0.0010
1,500–2,999	0.0020
1,000–1,499	0.0025
999 and below	0.0030

***National Electrical Manufacturers Association Standard, "MG1-20.52, Balance of Machines," July 1969.

Form-wound Squirrel-cage Induction Motors†††

Synchronous speed, rpm	Peak-to-peak displacement amplitude, in. Motor on elastic mount	Motor on rigid mount
720 to 1,499	0.002	0.0025
1,500 to 3,000	0.0015	0.002
3,000 and above	0.001	0.001

†††American Petroleum Institute Standard, "Recommended Practice for Form-wound Squirrel-cage Induction Motors," API Standard 541.

—coefficient of subgrade reaction of soil, if the above parameters are not accurately known
—the foundation depth and the bearing pressure at which the above parameters are applicable
—other information required for the static design of the footing

The soil density γ and Poisson's ratio ν are normally reported by the geotechnical consultant. Chapter 4 lists the typical range of values for these parameters. Often, mass density ρ is reported in lieu of soil density γ, but if one is given, the other is known, since

$$\rho = \gamma/g$$

where g is the gravitational constant.

The soil shear modulus G is not usually reported unless specifically requested. Since this parameter is a controlling

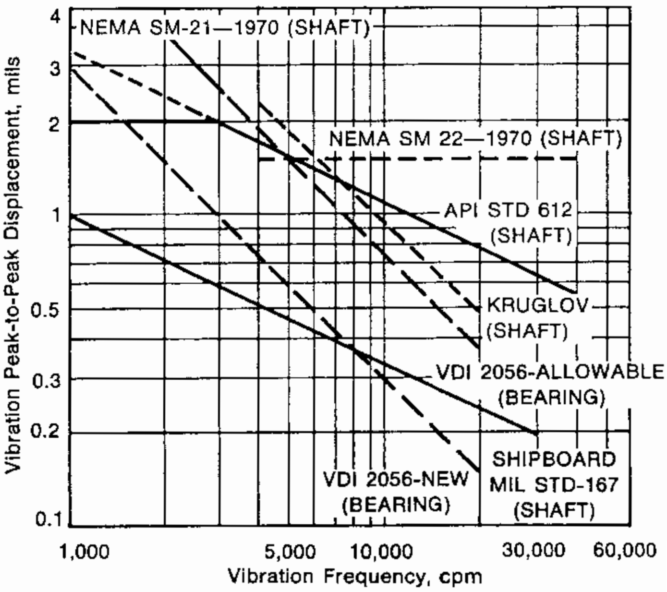


Figure 3-1. Classification of severity of machinery vibration. From C.M. Harris, and C.E. Crede, *Shock and Vibration Handbook.* Used with permission of McGraw-Hill, 1976.

factor in the calculation of the half-space spring constants it is desirable that the most reliable value be available. This often means that field testing will be necessary to obtain G since inaccurate values of this parameter will result in a worthless analysis. Therefore, Chapter 4 describes methods and procedures which should be followed in the calculation of G. The soil strain level has an important effect on the value of the shear modulus and should be accounted for in its calculation as discussed in Chapter 4.

The coefficient of subgrade reaction may be used in lieu of G or directly in a Barkan analysis (ref. 2). However, since the elastic half-space theory (ref. 1) is recommended here, its availability may be used as a check on the order of magnitude of the soil moduli as calculated using the half-space theory.

The soil parameters discussed above generally vary with depth and effective bearing pressure. Therefore, the specific values to be used in design should correspond to the actual bearing pressure and foundation depth used. In particular, the shear modulus is sensitive to strain level and since dynamic loads often produce low strain levels, the shear modulus used in analysis should correspond to the actual dynamic strain level expected.

In addition to the information listed above, the design engineer, either by himself or with the help of the geotechnical consultant, should establish the layout for the foundation structure. There are two common types of foundations used: concrete block footing placed directly on the soil or rock, and concrete footing supported by piles or piers. The preference of one system over the other should be decided by taking into consideration: relative economy, settlement, bearing capacity of the soil, vibration isolation, and the level of the underground water table. Pile- or pier-supported footings are the exception and are used only where poor soil conditions are found.

Environmental Conditions

There can be several situations during which a machine installation is in the vicinity of vibration sources such as quarry blasting, vehicular traffic, construction pile driving, or the location is in a continental zone where seismic occurrence is possible. The design engineer must then establish the severity of the situation and, if required, should seek the help of a vibration measurement consultant. The information requested should include the character of the vibration and the attenuation at the installation site. The effects of seismic forces can be determined through information and procedures described in References 4, 5, and 6.

Trial Sizing of a Block Foundation

The design of a block foundation for a centrifugal or reciprocating machine starts with the preliminary sizing of the block. This initial sizing phase is based on a number of guidelines that are partially derived from empirical and practical experience sources. Initial sizing is only preliminary; it does not constitute a final design. A block foundation design can only be considered complete when a dynamic analysis and check is performed and the foundation is predicted to behave in an acceptable manner as illustrated in Chapter 6. However, the following guidelines for initial trial sizing have been found to result in acceptable configurations:

1. The bottom of the block foundation should be above the water table when possible. It should not be resting on previously backfilled soil nor on a specially sensitive (to vibration) soil. The recommendations of the geotechnical consultant are usually followed with respect to depth of structures supporting dynamic or vibratory machines. Sometimes, the soil quality is poor, and the geotechnical consultant may recommend using piles or piers.

2. The following items apply to block-type foundations resting on soil:
   a. A rigid block-type foundation resting on soil should have a mass of two to three times the mass of the supported machine for centrifugal machines. However, when the machine is reciprocating, the mass of the foundation should be three to five times the mass of the machine.
   b. The top of the block is usually kept 1 ft above the finished floor or pavement elevation to prevent damage from surface water runoff.
   c. The vertical thickness of the block should not be less than 2 ft, or as dictated by the length of anchor bolts used. The vertical thickness may also be governed by the other dimensions of the block in order that the foundation be considered rigid. The thickness is seldom less than one fifth the least dimension or one tenth the largest dimension.
   d. The foundation should be wide to increase damping in the rocking mode. The width should be at least 1 to 1.5 times the vertical distance from the base to the machine centerline.
   e. Once the thickness and width have been selected, the length is determined according to (a) above, provided that sufficient plan area is available to support the machine plus 1-ft clearance from the edge of the machine base to the edge of the block for maintenance purposes.
   f. The length and width of the foundation are adjusted so that the center of gravity of the machine plus equipment coincides with the center of gravity of the foundation. The combined center of gravity should coincide with the center of resistance of the soil.
   g. For large reciprocating machines, it may be desirable to increase the embedded depth in soil such that 50 to 80% of the depth is soil-embedded. This will increase the lateral restraint and the damping ratios for all modes of vibration.
   h. Should the dynamic analysis predict resonance with the acting frequency, the mass of the foundation is increased or decreased so that, generally, the modified structure is overtuned or undertuned for reciprocating and centrifugal machines, respectively.

3. The following guidelines only apply to block foundations supported on piles:
   a. The pile cap mass should be 1.5 to 2.5 times and 2.5 to 4 times the mass of the machine for centrifugal and reciprocating machines, respectively.
   b. The thickness, width, and length of the block is selected as in 2(b) through 2(f).
   c. The number and size of piles are selected such that no single element carries over one half of its allowable design load.
   d. The piles are arranged so that the centroid of

the pile group coincides with the center of gravity of the combined structure and machine loads.

e. Piles are battered away from the pile cap to carry any transverse and longitudinal unbalanced forces. Vertical piles provide small resistance to horizontal loads, and the batter piles are usually designed to carry all such horizontal forces as axial loads.
f. When piers are used, bells may be desirable to increase their overall capacity.
g. If resonance conditions are predicted to occur, modifications are necessary as described in 2(h) above.
h. Piles and piers must be properly anchored to the slab for adequate rigidity and for meeting the design conditions assumed during the analysis phase.

Trial Sizing of Elevated Foundations (Table Tops)

Preliminary sizing and geometrical member arrangement constitute the initial design phase for elevated foundations. Although this preliminary phase is often based on the experience of the designer, suggested guidelines can be useful in arriving at a satisfactory final design. It should be emphasized that the general guidelines for trial sizing are only useful in the initial phase and are no substitute for a thorough dynamic analysis and check as described in Chapter 7. These general guidelines include the following:

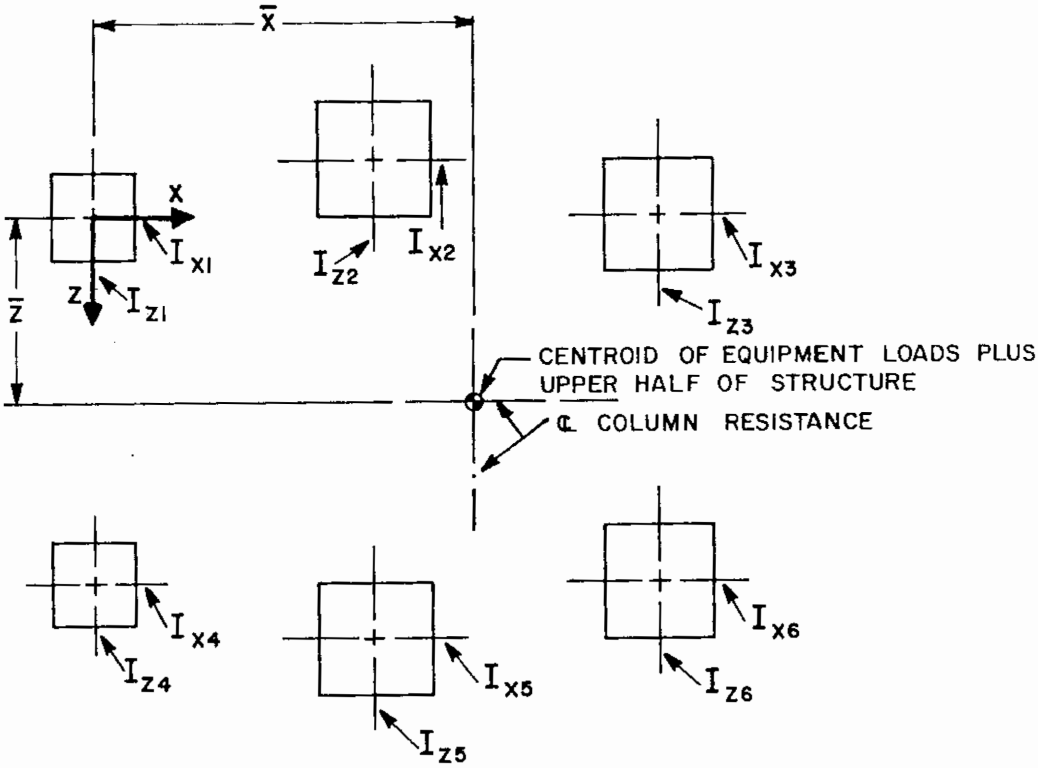
1. The designer should carefully analyze equipment size and clearance requirements to assure that sufficient space is allocated to equipment, anchor bolts, piping, and clearance for installation, maintenance and operation, that is, physical space limits and requirements should be clearly identified and considered.

2. The bottom of the foundation mat should be placed no higher than the minimum founding depth recommended by the soil consultant. This generally includes considering the location of adequate bearing strata, water table, depth of frost penetration, paving elevation, and special local soil conditions. However, in very poor soils, the geotechnical consultant may recommend the use of piles. The mat thickness t should not be less than

$$t = 0.07\,L^{4/3}$$

where L is the average of two adjacent spans between columns.

3. All columns should be stressed almost equally when subjected to vertical load. Thus, the column areas should be proportional to the load caried by the column, and P_i/A_i should be fairly constant for all columns where P_i and A_i are the axial load and cross-sectional area of any column. The columns should be capable of carrying six times the vertical load. Column spacing should preferably be less than 12 ft. The intermediate columns should be located preferably under the couplings or the gear box.
4. The beam depth should be a minimum of one fifth of the clear span, and the beam width is normally equal to the width of the column consistent with anchor bolt requirements for spacing, embedded depth, and edge distance. The beams should not deflect over 0.02 in. when subjected to static loads.
5. The flexural stiffness of the beams should be at least twice the flexural stiffness of the columns.
6. The total mass of the structure including the mat should be no less than three times the mass of the supported machine for centrifugal machines and five times the mass of the machine for reciprocating-type machines.
7. The mass of the top half of the structure should not be less than the mass of the supported machine.
8. The maximum static-bearing pressure for soil-supported foundations should not exceed one half of the allowable soil pressure. For pile-supported foundations, the heaviest loaded pile should not carry over one half of its allowable load.
9. The center of resistance of the soil should be within 1 ft of all superimposed loads for soil-supported foundations. For pile-supported foundations, the centroid of the piles should be within 1 ft of the superimposed loads.
10. The center of column resistance should coincide with the center of gravity of the equipment plus the top half of the structure loads in the longitudinal as well as the transverse directions, that is, the column moments of inertia should be "balanced" about the centroid of the equipment as shown in Figure 3-2.
11. All the columns should deflect equally in the vertical, lateral, and longitudinal directions when subjected to equivalent static machine loads acting in those directions. These equivalent loads are often assumed to be 0.5, 0.3, and 0.1 of the total load for the vertical, transverse, and longitudinal directions, respectively, with the vertical dead load acting in all conditions. Chapter 7 gives a further



$$\bar{X} = \frac{\sum_{i=1}^{n} X_i I_{Xi}}{\sum_{i=1}^{n} I_{Xi}}$$

$$\bar{Z} = \frac{\sum_{i=1}^{n} Z_i I_{Zi}}{\sum_{i=1}^{n} I_{Zi}}$$

Figure 3-2. Center of column resistance.

description of these approximate equivalent static loadings. The maximum horizontal deflections for these equivalent static loadings should be less than 0.02 in. in all cases.

12. The columns and beams should be checked for individual member resonance with the machine-acting frequency. The lowest natural frequency of the columns is approximately given by

$$f_n = \frac{44800\,(f'_c)^{1/4}}{\sqrt{pL}}$$

where f'_c is the concrete strength in psi, p is the actual column axial stress in psi and is usually in the 40-300 psi range, L is the column height in inches, and f_n is in rpm.

13. When piles or piers are recommended by the geotechnical consultant, additional guidelines 3(a) through 3(h) given above for block foundations may be used in arriving at a trial configuration.

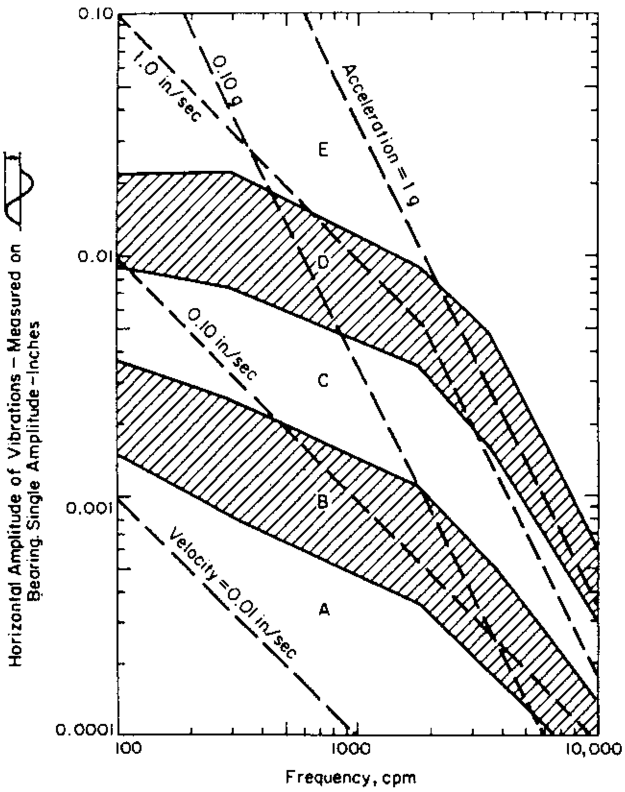
Checklist for Design

Once the proposed structure is modeled (see Chapter 2), trial sizes are selected, and an analysis is performed (see Chapters 6 and 7). The predicted behavior of the proposed structure is checked or compared against certain design requirements. These design requirements include: (1) the usual static strength checks against soil, structural failures, and excessive deformations; (2) comparison to limiting dynamic behavior including maximum amplitude of vibration, maximum velocity and acceleration, maximum magnification factor, maximum dynamic load factor, possible resonance conditions, and maximum transmissibility factor; (3) inspection of all modes of oscillation including coupled modes; (4) consideration of possible fatigue failures in the machine, structure, or connections; (5) consideration of environmental demands such as physiological and psychological effects on people, effect on adjoining sensitive equipment, possible damage to the structure, and possible resonance of individual structural components.

Figures 3-3 through 3-7 are used in checking on the dynamic behavior of the proposed structure. Suitable limits of vibration amplitude, velocity, and acceleration, which are all functions of operating frequency, are given in these figures. Table 3-2 lists appropriate limits of peak velocity for certain categories of machine operation.

In order to account for the relative importance of the machine to overall plant operation, increased factors of

(text continued on page 54)



EXPLANATION OF CASES

E DANGEROUS. SHUT IT DOWN NOW TO AVOID DANGER.

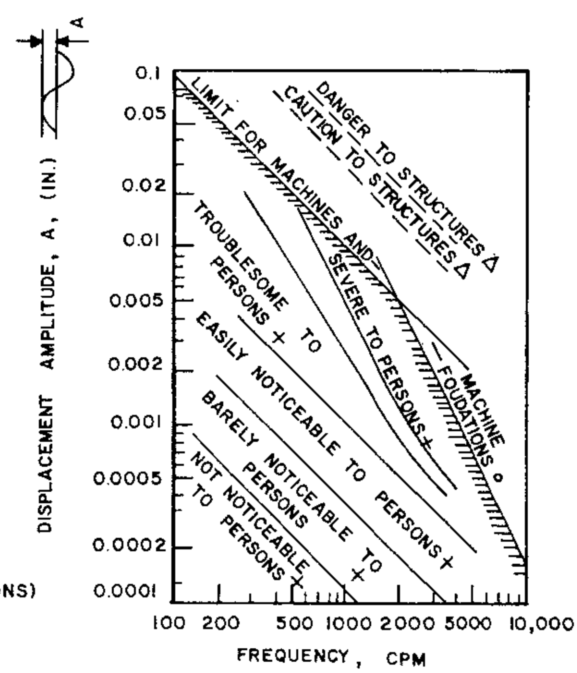
D FAILURE IS NEAR. CORRECT WITHIN TWO DAYS TO AVOID BREAKDOWN.

C FAULTY. CORRECT WITHIN 10 DAYS TO SAVE MAINTENANCE DOLLARS.

B MINOR FAULTS. CORRECTION WASTES DOLLARS.

A NO FAULTS. TYPICAL NEW EQUIPMENT.

Figure 3-3. Vibration performance of rotating machines (see ref. 9).



+ FROM REIHER AND MEISTER (1931) — (STEADY STATE VIBRATIONS)
o FROM RAUSH (1943)—(STEADY STATE VIBRATIONS)
Δ FROM CRANDELL (1949) (DUE TO BLASTING)

Figure 3-4. General limits of vibration amplitude for a particular frequency (see ref. 10).

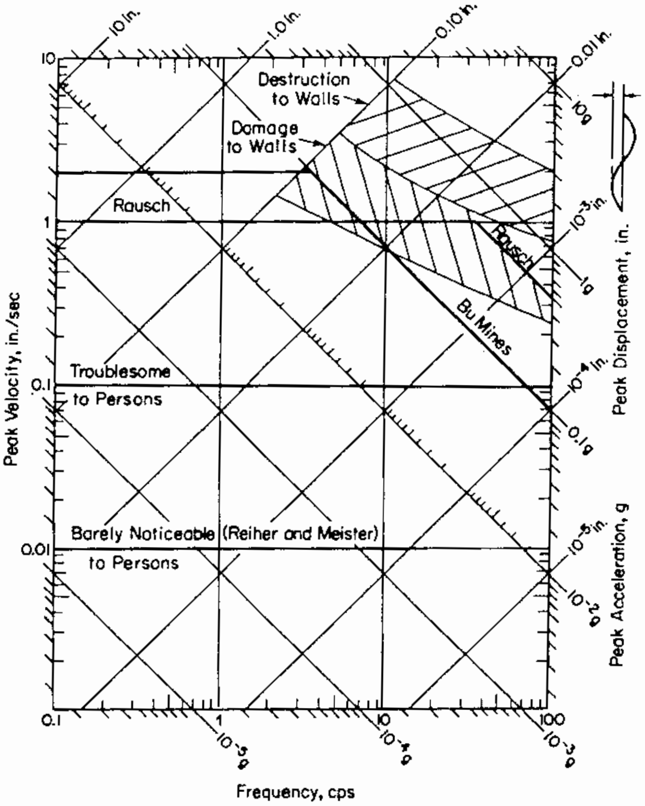


Figure 3-5. Response spectra for allowable vibration at facility (see ref. 1).

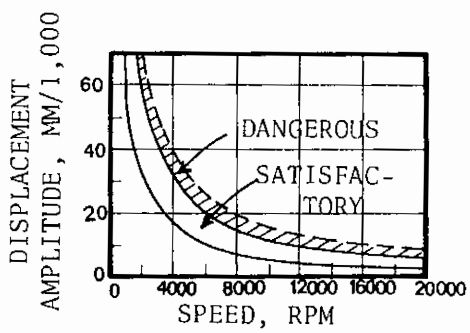


Figure 3-6. Vibration standards of high-speed machines (see ref. 11).

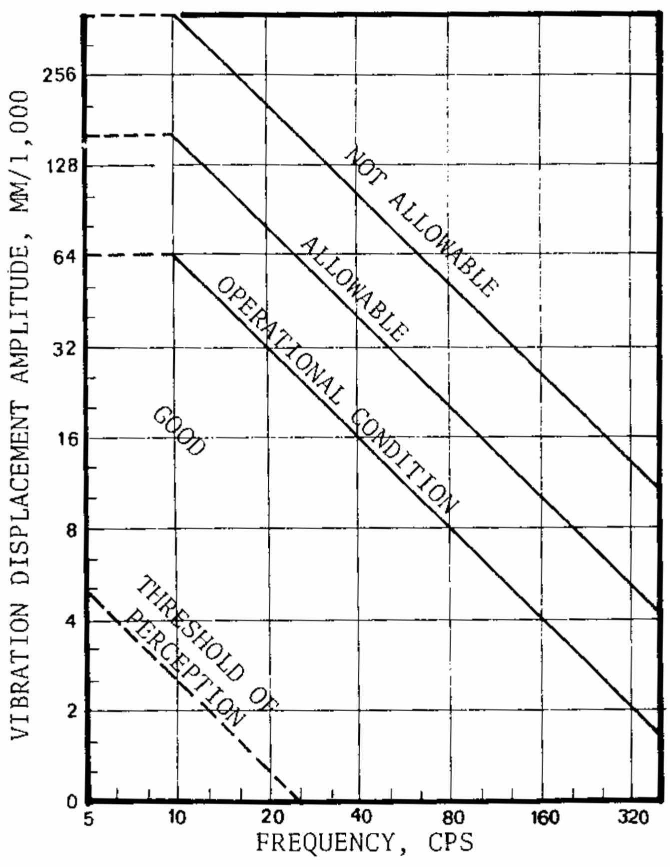


Figure 3-7. Turbomachinery bearing vibration limits (see ref. 12).

Table 3–2

General Machinery-Vibration-Severity Data*

Horizontal Peak Velocity (in./sec.)	Machine Operation
<0.005	Extremely smooth
0.005–0.010	Very smooth
0.010–0.020	Smooth
0.020–0.040	Very good
0.040–0.080	Good
0.080–0.160	Fair
0.160–0.315	Slightly rough
0.315–0.630	Rough
>0.630	Very rough

*After Baxter and Bernhard (ref. 8).

safety (service factor) have been proposed. These factors are applied to the computed maximum amplitude of vibration to obtain an effective vibration amplitude.

Use of a service factor* (or additional factor of safety) has been proposed to account for the relative importance of the machine to overall plant operation. The actual vibration amplitude is multiplied by the service factor to obtain an effective vibration amplitude. The effective vibration amplitude (rather than the actual vibration amplitude) is then used to determine the adequacy of the installation.

Single-stage centrifugal pump, electric motor, fan . . 1.0
Typical chemical processing equipment, noncritical 1.0
Turbine, turbogenerator, centrifugal compressor . . . 1.6
Centrifuge, stiff-shaft†; multistage centrifugal pump 2.0
Miscellaneous equipment, characteristics unknown 2.0
Centrifuge, shaft-suspended, on shaft near basket . . 0.5
Centrifuge, link-suspended, slung 0.3
Effective vibration = measured single amplitude vibration, inches multiplied by the service factor.
Machine tools are excluded. Values are for bolted-down equipment; when not bolted, multiply the service factor by 0.4 and use the product as a service factor.
Caution: Vibration is measured on the bearing housing, except as stated.

The design checklist and design procedure may be summarized as follows. Note that most of the steps listed in the design checklist are implicitly considered in a computer analysis (see chapter 7). The checklist is appropriate for analysis of block-type foundations.

Design Checklist

Design Conditions	Procedures
Static Conditions	
Static Bearing Capacity	Proportion footing area for 50% of allowable soil pressure.
Static Settlement	Settlement must be uniform; center of gravity of footing and machine loads should be within 5% of any linear dimension.
Bearing capacity: Static plus Dynamic Loads	The sum of static plus modified dynamic load should not create a bearing pressure greater than 75% of the allowable soil pressure given in the soil report.
Settlement: Static plus Repeated Dynamic Loads	The combined center of gravity of the dynamic loads and the static loads should be within 5% of the linear dimension from the center of gravity of footing. In the case of rocking motion, the axis of rocking should coincide with principal axis of the footing. The magnitude of the resulting settlement should be less than the permissible deflecting capability of the connected piping system.
Limiting Dynamic Conditions	
Vibration Amplitude at Operating Frequency	The maximum single amplitude of motion of the foundation system as calculated from Table 1-4 should lie in zone A or B of Figure 3-3 for the given acting frequency. Where unbalanced forces are caused by

* From Blake (ref. 9).
† Horizontal displacement on basket housing.

machines operating at different frequencies, the total displacement amplitudes to be compared at the lower acting frequency, are taken as the sum of all displacement amplitudes.

Velocity	$2\pi f$ (cps) $\times$ (Displacement Amplitude as calculated in the condition above) should be compared against the limiting values in Table 3-2 and Figure 3-3 at least for the case of "good" operation. The resultant velocity where two machines operate at different frequencies is calculated by the RMS (root mean square) method, $V = [(\omega_1 A_1)^2 + (\omega_2 A_2)^2]^{0.5}$, where V = resultant velocity, in/sec, ω_1, ω_2 = operating frequencies for machines 1 and 2, respectively, rad/sec, and A_1, A_2 = vibration displacement, in., for machines 1 and 2, respectively.
Acceleration (Note: not necessary if the two conditions above are satisfied)	$4\pi^2 f^2 \times$ (Displacement Amplitude as calculated above) should be tested for zone B of Figure 3-3.
Magnification Factor (Note: applicable to machines generating unbalanced forces)	The calculated values of M or M_r (Table 1-4) should be less than 1.5 at resonance frequency.
Dynamic Load Factor, *DLF* (Note: applicable to foundation subjected to quarry blasting or seismic shock waves)	The value of *DLF* is to be obtained from refs. 1 and 7. The duration of shockwave may be taken as 0.1 to 0.5 sec.
Resonance	The acting frequencies of the machine should not be within ±20% of the resonance frequency (damped or undamped).
Transmissibility Factor (Note: usually applied to high-frequency spring mounted machines)	The value of transmissibility is to be obtained from Table 1-4 and should be less than 3%.
Resonance of Individual Structural Components (Superstructure Without the Footing)	The resonance condition with the lowest natural frequency shall be avoided by maintaining the frequency ratio either less than 0.5 or greater than 1.5.

Possible Modes Of Vibration

Vertical Oscillation	This mode is possible if the force acts in this direction.
Horizontal Translation	This mode is possible if the force acts in this direction.
Rocking Oscillation	This mode is possible when the point of application of horizontal force is above mass center of foundation.
Torsional Oscillation	This mode is possible when the horizontal forces form a couple in horizontal plane.
Coupled Mode	The horizontal translation and rocking oscillation are usually coupled. If $\sqrt{f^2_{nx} + f^2_{n\psi}}/f_{nx}f_{n\psi} \leq 2/3f$, then the coupling effect may be ignored; the horizontal translation and the rocking oscillation modes can be treated alone, and the results can be combined. See nomenclature on page 98 for definition of terms.

Fatigue Failures

Machine Components	Limits stated in Figure 3-4 and/or Table 3-2 are to be followed. In case machine components are very delicate, then the machine should be mounted on springs with an added inertia block.
Connections	Same as the machine components condition above and check stresses using AISC code when connectors are bolts or welds (ref. 13).
Supporting Structure	For steel structures, use the connections condition above. For a concrete footing, if reversal of stresses takes place and the amplitude is very high such that the peak stress reversal is over 50% of the allowable stress, the

	main and the shear reinforcement (if any) should be designed for the stress reversal condition (ref. 14).
Environmental Demands	
Physiological Effects on Persons	If the machine is located inside a building, use the procedure given in the transmissibility factor condition above and use the limits indicated in Figures 3-4 through 3-7. The concept of physical isolation of the supporting structure is another alternative. The amplitude of vibration in any direction should fall below the zone "troublesome to persons" for the specific acting frequency as determined from Figure 3-4.
Psychological Effects on Persons	Use the procedures indicated in the condition immediately above. In case the facility is located very close to people not connected with machine operations, use acoustic barriers.
Sensitive Equipment Nearby	Physically isolate the support system from the sensitive equipment.
Damage to Structure	Use the limits indicated in Figures 3-4 and 3-5 to avoid structural damage.

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4 Geotechnical Considerations

Notation for Chapter 4

A_L = area within hysteresis loop
A_Y = age of soil deposit
A_T = crosshatched area under hysteresis loop
A_z = amplitude of displacement
B_x, B_z = mass ratios
B_θ, B_ψ = inertia ratios
$\bar{c}$ = effective stress cohesion
D = geometric damping ratio
D_m = material damping ratio
D_r = relative density
D_t = total damping ratio
D_{10} = equivalent grain diameter for which 10% of sample is smaller
d_c = characteristic depth
E = Young's modulus
e = void ratio
f = frequency in cycles per second
f_n = undamped natural frequency
f_{mr} = resonant frequency with damping included
f_o = fundamental resonant frequency
G = shear modulus of soil
G_{max} = shear modulus at very low-strain amplitude
G_{1000} = shear modulus after 1000 min of consolidation
g = gravitational constant
H = thickness of soft stratum
h = depth of embedment or borehole spacing
I_θ, I_ψ = mass moment of inertia
K_0 = at rest earth pressure coefficient
K_2 = shear modulus factor
k = plasticity factor
k_x = spring constant for horizontal excitation
k_z = spring constant for vertical excitation
k_θ = spring constant for torsional excitation
k_ψ = spring constant for pure rocking excitation
L_R = length of Rayleigh wave
M = mass of footing plus load vibrating in phase
m = relative density term
N = number of stress cycles
N_a = number of log cycles of time required for re-establishment of soil fabric
n = relative density term
n_ψ = correction factor for B_ψ
OCR = overconsolidation ratio
Q_0 = unbalanced vertical force
q_d = dynamic bearing stress
q_0 = static bearing stress
r_0 = effective radius of footing
s_u = undrained shear strength
t_s = time for shear wave to pass from penetrometer to geophone
u = pore fluid pressure
W = weight of foundation plus load vibrating in phase
z = depth coordinate
z_1, z_2 = displacement amplitudes for two successive cycles
α = embedment factor for damping
γ = shear strain amplitude
γ_r = reference shear strain
γ_s = unit weight of soil
ΔG = increase in shear modulus
$\Delta\sigma_v$ = vertical stress due to static load
ε_p = permanent axial (vertical) strain
η = embedment factor for stiffness
$\lambda = \sigma_d/\sigma_c$
ν = Poisson's ratio
ρ = mass density
σ_c = confining pressure
σ_d = vertical dynamic stress
$\bar{\sigma}_h$ = horizontal effective stress
$\bar{\sigma}_o$ = octahedral normal effective stress
σ_v = vertical total stress
$\bar{\sigma}_v$ = vertical effective stress
τ = shear stress
τ_{max} = shear stress related to shear strain through G_{max}
$\bar{\phi}$ = effective stress angle of internal friction

Dynamically loaded foundations induce strains in the supporting soil, which, in turn, require the elements of the foundation (footing, piles, etc.) to deform in a manner compatible with the deformation of the soil. It is therefore necessary to adopt a model that will predict the response of the soil to imposed dynamic loadings in order to allow the structural designer to include the effects of foundation deformation in a global structural analysis. Several models are available to accomplish this prediction, but the model that is most widely accepted is the "elastic halfspace model." The halfspace model, which is used in this book for shallow foundations, presumes that a circular footing rests upon the surface of an elastic halfspace (the soil) extending to an infinite depth, which is homogeneous and isotropic and whose stress-strain properties can be defined by two elastic constants, usually shear modulus (G) and Poisson's ratio (ν). With the elastic halfspace model it is possible to predict the response (e.g., deformation at a point vs. time) of the soil and, therefore, of the footing, to harmonic vertical forces, rocking moments, twisting moments, horizontal shears, and combinations of such loads applied to the footing, which is considered to be rigid. The model provides for dissipation of energy through radiation or "geometric" damping. Exact mathematical expressions have been derived from halfspace models which define the response of footings to harmonic loading of any of the above types. These expressions indicate that the stiffness of the soil and the amount of damping that occurs is a function not only of elastic properties of the soil (halfspace), but also of the frequency of loading. Since it is inconvenient to include this so-called frequency-dependent response in most algorithms for structural analysis, lumped parameter approximations of the halfspace model have been developed that allow the soil to be represented by linear spring constants, which resist applied loads in vertical, horizontal, twisting, and rocking modes, and dashpot constants, which simulate viscous damping in the halfspace in the respective modes. Both spring and damping constants are frequency independent. It is possible, therefore, to assign spring constants and damping ratios to the soil and determine deformation-time relationships using the fundamental lumped parameter relationships introduced in Chapter 1. In those relationships, the mass is taken to be that of the footing and any load vibrating in phase with the footing. None of the soil mass is included.

The user of the elastic halfspace model must make a judgment whether such a model can sufficiently describe the soil-foundation system at a given site, particularly with regard to the assumption of isotropy and homogeneity. If it can not, alternative models should be used. Common alternatives, which are actually modifications of the elastic halfspace model, include techniques for predicting footing response in strongly stratified soils, response of footings (caps) supported by piles, and response of footings embedded beneath the surface of the ground. Each of these techniques is discussed in this chapter and Chapter 5.

The basic elastic halfspace model is valid only for isolated foundations. No valid theory has been developed to permit the precise calculation of the response of two or more footings situated near each other, although finite-element modeling of the soil itself can be useful in this respect. Empirically, however, the soil spring constants for the individual footings are usually reduced by placing footings near each other, and the geometric damping is always reduced. These points should be kept in mind when applying the techniques described here.

For design purposes, the assumption of frequency independence of stiffness and damping is valid for low frequencies. Specifically, "low frequency" exists when the driving frequency f is less than about $(1/\pi r_0)(Gg/\gamma)^{0.5}$ where r_0 is the radius of a circular footing or is $(BL/\pi)^{0.5}$ for translatory motion or $(BL^3/3\pi)^{0.25}$ for rocking motion of a rectangular footing, G is the low-amplitude shear modulus of the soil (discussed later), γ is the total unit weight of the soil, and g is the gravitational constant. B and L are the plan dimensions of a rectangular footing, where L is the dimension perpendicular to the axis of rocking.

Spring constants for the halfspace model are evaluated under such conditions as the elastic static spring constants. Expressions for the theoretical spring constants for vertical, horizontal, rocking and torsional motion are given in Table 4-1. Note that it is possible to evaluate the spring constants for both circular and rectangular

Table 4–1
Equivalent Spring Constants for Rigid Circular and Rectangular Footings
(after ref. 19)

Mode of Vibration	Circular Footing	Rectangular Footing
Vertical	$k_z = \dfrac{4Gr_0}{1-\nu}\eta_z$	$k_z = \dfrac{G}{1-\nu}\beta_z\sqrt{BL}\,\eta_z$
Horizontal	$k_x = \dfrac{32(1-\nu)Gr_0}{7-8\nu}\eta_x$	$k_x = 2(1+\nu)G\beta_x\sqrt{BL}\,\eta_x$
Rocking	$k_\psi = \dfrac{8Gr_0^3}{3(1-\nu)}\eta_\psi$	$k_\psi = \dfrac{G}{1-\nu}\beta_\psi BL^2\eta_\psi$
Torsional	$k_\theta = \dfrac{16Gr_0^3}{3}$	No solution available; Use r_0 from Table 4-2

footings and for footings resting on the surface or embedded beneath the surface. Embedment effects are discussed further in the latter part of this chapter.

In Table 4-1, ν is the Poisson's ratio of the soil; the η-factors are embedment coefficients defined in Table 4-2; and the β-factors are geometry factors defined in Figure 4-1.

The damping ratios can be computed from the mass or inertia ratios as indicated in Table 4-3. The weights W in the mass ratio expressions are the total weights of the footing plus the load supported by the footing vibrating in phase with it, including machinery. The moments of inertia I_ψ and I_θ are the rocking and twisting mass moments of inertia about the axis of rotation, which is an axis in the plane of the base of the foundation perpendicular to the plane of rocking for rocking motion and an axis perpendicular to the foundation and to the plane of twisting for torsional motion. The factors α are damping ratio coefficients to account for the increased geometric damping that occurs due to effective embedment. Equations for evaluation of α are given in Table 4-4. The factor n_ψ is an inertia ratio correction factor for rocking, given in Table 4-5.

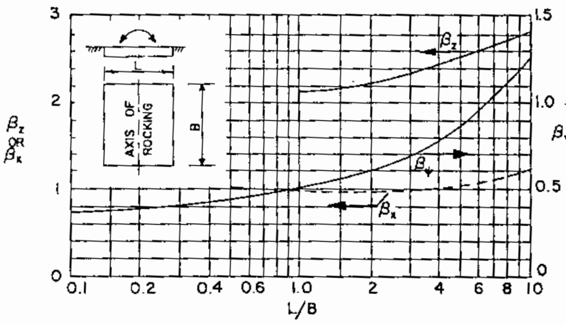


Figure 4-1. Coefficients β_z, β_x, and β_ψ for rectangular footings (after ref. 19).

Table 4–2
Embedment Coefficients for Spring Constants (after ref. 18)

Mode of Vibration	r_0 for Rectangular Foundation	Coefficient
Vertical	$\sqrt{BL/\pi}$	$\eta_z = 1 + 0.6(1-\nu)(h/r_0)$
Horizontal	$\sqrt{BL/\pi}$	$\eta_x = 1 + 0.55(2-\nu)(h/r_0)$
Rocking	$\sqrt[4]{BL^3/3\pi}$	$\eta_\psi = 1 + 1.2(1-\nu)(h/r_0) + 0.2(2-\nu)(h/r_0)^3$
Torsional	$\sqrt[4]{BL(B^2+L^2)/6\pi}$	None available

Notes: h is the depth of foundation embedment below grade; L is horizontal dimension perpendicular to axis of rocking; B is remaining horizontal dimension.

Table 4-6 gives typical expressions for mass moments of inertia of simple volumes. Generally, mass moments of inertia for most machines and foundations can be adequately represented by these expressions.

It is obvious that a primary objective of foundation design is to maximize geometric damping, consistent with economy. It is observed that the damping ratios for a halfspace (Table 4-3) increase with increasing foundation size (r_0) and decrease with increasing weight (W) or mass moment of inertia (I). It can be concluded, therefore, that, ideally, foundations should be as wide and shallow as is practicable.

Evaluation of Soil Parameters

In order to evaluate the spring constants k_z, k_x, k_θ, and k_ψ and the corresponding damping ratios required for computer analysis of dynamic structure-soil interaction problems, it is necessary to determine relevant values for the soil parameters G (shear modulus), ν (Poisson's ratio), ρ (mass density) and D_m (internal or material damping ratio). Furthermore, factors not needed directly in the computer analysis, such as the permanent settlement that will occur beneath the foundation of a structure supporting vibrating machinery, also need to be considered before the structure is constructed. The objective of this chapter is to describe procedures by which the soil parameters can be evaluated and permanent settlements estimated. In addition, certain other special problems will be discussed. It will be assumed throughout that the reader possesses a basic knowledge of elementary soil mechanics and understands foundation engineering terminology.

This chapter is intended primarily to be a source of information for those interested in analyzing or designing structures to support vibrating machinery. As a result, the emphasis is on procedures or criteria developed from research rather than on fundamental aspects of soil behavior. For the reader who wishes to pursue a more detailed treatment of behavior of dynamically loaded soil, papers by Richart (ref. 11) and Seed and Idriss (ref. 14) and texts by Richart, Hall and Woods (ref. 12) and Wu (ref. 20) provide execellent points of departure.

The discipline of soil dynamics is a relatively new one, and many problems faced by the practicing engineer have either not yet been treated in a rigorous manner, have been considered only for ideal conditions which may not occur in geological materials, or have been studied in some detail but without dissemination of criteria to the profession in general. Examples of such problems are the assessment of permanent settlements beneath near-surface vibratory loads, the response of soil

(text continued on page 62)

Table 4–3
Equivalent Damping Ratio for Rigid Circular and Rectangular Footings (after ref. 12)

Mode of Vibration	Mass (or Inertia) Ratio	Damping Ratio D
Vertical	$B_z = \frac{(1-\nu)}{4}\frac{W}{\gamma r_0^3}$	$D_z = \frac{0.425}{\sqrt{B_z}}\alpha_z$
Horizontal	$B_x = \frac{7-8\nu}{32(1-\nu)}\frac{W}{\gamma r_0^3}$	$D_x = \frac{0.288}{\sqrt{B_x}}\alpha_x$
Rocking	$B_\psi = \frac{3(1-\nu)}{8}\frac{I_\psi}{\rho r_0^5}$	$D_\psi = \frac{0.15\,\alpha_\psi}{(1+n_\psi B_\psi)\sqrt{n_\psi B_\psi}}$
Torsional	$B_\theta = \frac{I_\theta}{\rho r_0^5}$	$D_\theta = \frac{0.50}{1+2B_\theta}$

Table 4–4
Effect of Depth of Embedment on Damping Ratio (after ref. 18)

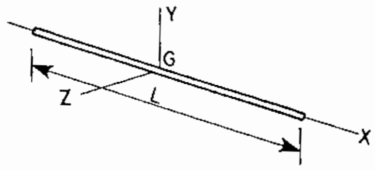
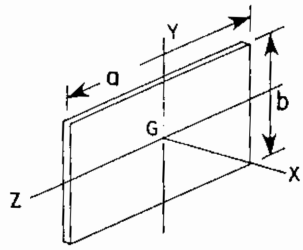
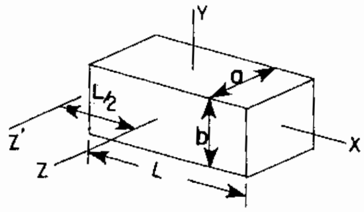
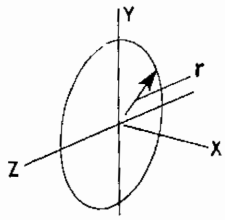
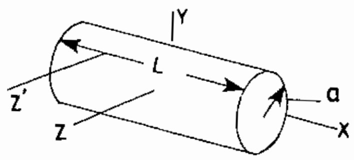
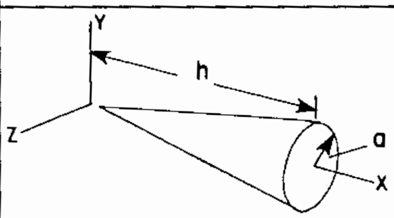
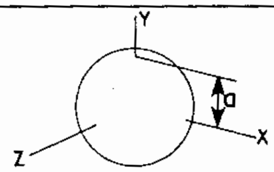
Mode of Vibration	Damping Ratio Embedment Factor
Vertical	$\alpha_z = \frac{1 + 1.9(1-\nu)\frac{h}{r_0}}{\sqrt{\eta_z}}$
Horizontal	$\alpha_x = \frac{1 + 1.9(2-\nu)\frac{h}{r_0}}{\sqrt{\eta_x}}$
Rocking	$\alpha_\psi = \frac{1 + 0.7(1-\nu)(h/r_0) + 0.6(2-\nu)(h/r_o)^3}{\sqrt{\eta_\psi}}$

Table 4–5
Values of n_ψ for Various Values of β_ψ*

B_ψ	5	3	2	1	0.8	0.5	0.2
n_ψ	1.079	1.110	1.143	1.219	1.251	1.378	1.600

* After Richart, Hall, and Woods (ref. 12). Reprinted by permission of Prentice-Hall, Inc., Englewood Cliffs, NJ.

Table 4–6
Mass Moments of Inertia of Common Geometric Shapes

SLENDER ROD		$I_y = I_z = \frac{1}{12} mL^2$
THIN RECTANGULAR PLATE		$I_x = \frac{1}{12} m(a^2 + b^2)$ $I_y = \frac{1}{12} ma^2$ $I_z = \frac{1}{12} mb^2$
RECTANGULAR PRISM		$I_x = \frac{1}{12} m(a^2 + b^2)$ $I_y = \frac{1}{12} m(a^2 + L^2)$ $I_z = \frac{1}{12} m(b^2 + L^2)$ $I_{z'} = I_z + mL^2/4$
THIN DISK		$I_x = \frac{1}{2} mr^2$ $I_y = I_z = \frac{1}{4} mr^2$
CIRCULAR CYLINDER		$I_x = \frac{1}{2} ma^2$ $I_y = I_z = \frac{1}{12} m(3a^2 + L^2)$ $I_{z'} = I_z + mL^2/4$
CIRCULAR CONE		$I_x = \frac{3}{10} ma^2$ $I_y = I_z = \frac{3}{5} m(\frac{1}{4} a^2 + h^2)$
SPHERE		$I_x = I_y = I_z = \frac{2}{5} ma^2$

beneath flexible mats loaded by vibrating machinery, and the response of pile groups. Nonetheless, rational analysis of structures for which such problems are inherent still must be conducted. Therefore, the authors have chosen to present procedures for parameter evaluation and for prediction of foundation behavior that are based on rigorous dynamic solutions verified by observations, where such procedures exist, and to present procedures based on related static or dynamic models, modified empirically by experience, where they do not exist. It is fully anticipated that, as further research is done, many of the procedures described in this chapter will become outdated and will be replaced by more accurate ones.

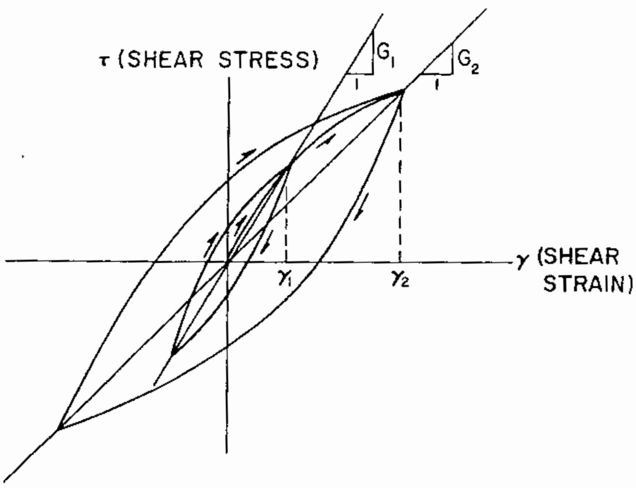


Figure 4-2. Hypothetical Shear Stress-Strain curve for soil.

Shear Modulus

Unbalanced loads in vibrating machinery produce shear strains in the supporting soil that are usually of a much smaller magnitude than the strains produced by static loading. The mechanism governing the stress-strain behavior of soils at small strains involves mainly the stress-deformation characteristics of the soil particle contacts and is not controlled by the relative slippage of particles associated with larger strains. As a consequence, the stress-deformation behavior of soil is much stiffer at very small strain levels than at usual static strain levels. It is therefore inappropriate to obtain a shear modulus directly from a static stress-deformation test, such as a laboratory triaxial compression test or a field plate bearing test, unless the stresses and strains in the soil can be measured accurately for very small values of strain. Special techniques, which will be described subsequently, must therefore be employed.

Even at very small values of strain, the stress-strain relationship is nonlinear. Therefore, it becomes expedient to define the shear modulus as an equivalent linear modulus having the slope of the line joining the extremities of a closed loop stress-strain curve (Figure 4-2). It is obvious that the shear modulus so defined is strain dependent and that in order to conduct an equivalent linear analysis of the type described in previous chapters it is necessary to know the approximate strain amplitude in the soil. For conditions of controlled applied strain, the ordinates of shear stress in Figure 4-2, and therefore G, will vary (usually decrease) with number of cycles applied until a stable condition is reached. Henceforth, only the stable equivalent linear value of shear modulus will be considered and will be termed "*the* shear modulus," since the response of the structure after at least several thousand cycles of load is the condition of principal interest to the structural designer.

Richart (ref. 11) has described at least eight variables that influence the shear modulus of soil:

1. Amplitude of dynamic strain
2. Mean effective stress (octahedral normal effective stress) and length of time since the stress was applied
3. Void ratio
4. Grain characteristics and structure of the soil
5. Stress history
6. Frequency of vibrations
7. Degree of saturation
8. Temperature

Regarding Variable 1, the amplitude of strain due only to the dynamic component of loading should be considered when evaluating the shear modulus in normal practice, as superimposed static strain levels have a relatively minor effect on G unless the static strains are of a very large magnitude not usually present in a foundation for vibrating machinery.

The first five variables are considered directly or indirectly in the procedures for obtaining G which follow. Variables 6-8 are generally of secondary importance, although the temperature of the soil may alter G considerably near the freezing point of the soil.

Soil shear modulus may be determined by field measurements, laboratory measurements, and use of published correlations that relate shear modulus to other more easily measured properties.

Field Procedures. Two widely used *in situ* procedures are the steady-state oscillator test (ref. 12) and the cross-hole test (ref. 16). The steady-state oscillator test is illustrated in Figure 4-3. An oscillator is placed on the surface of a site to be investigated, and Rayleigh (sur-

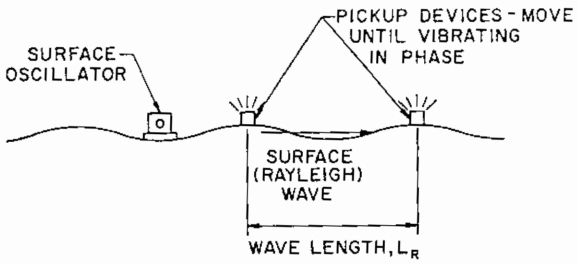


Figure 4-3. Surface Oscillator Test (after ref. 12).

face) waves are created. Pickup devices (e.g., accelerometers) are moved relative to each other and to the oscillator until they are found to be vibrating in phase, and the distance between the pickup devices is then measured. If no closer spacing can be found such that the pickup devices remain in phase, the spacing between the devices is one Rayleigh wave length. Since Rayleigh and shear wave velocities are nearly equal for soils with Poisson's ratios greater than about 0.35 (i.e., for most soils),

$$G_{max} = \rho v_s^2 \simeq \rho f^2 L_R^2 \qquad (4\text{-}1)$$

where G_{max} = shear modulus (at the very low strain level occurring in the test),
ρ = total mass density of the soil; soil unit weight/acceleration of gravity,
f = frequency of oscillation in cycles per unit of time,
L_R = measured wave length,
v_s = shear wave velocity.

The process is normally repeated with the pickup devices in several positions relative to the oscillator and with different oscillator frequencies to obtain an average value for G_{max} at a given location.

Since most sites in or near developed areas have considerable low-frequency background noise, steady-state tests with small oscillators are generally limited to high-frequency vibrations (generally greater than 100 cps). In such a case the Rayleigh wave length will seldom be more than a few feet. Oscillator tests measure an average shear modulus in a zone of soil from the surface to a depth of about one Rayleigh wave length. Hence, it is apparent that such high-frequency tests are limited to very shallow depths, usually no more than 2 to 5 feet. Furthermore, soil borings should also be made in conjunction with the oscillator test in order to assess whether the measured shear modulus represents a single layer or more than one layer of soil. Low-frequency tests can be conducted on quiet sites, but to be accurate, they often must be conducted with very heavy oscillators, which generally renders the method uneconomical relative to crosshole testing as described below.

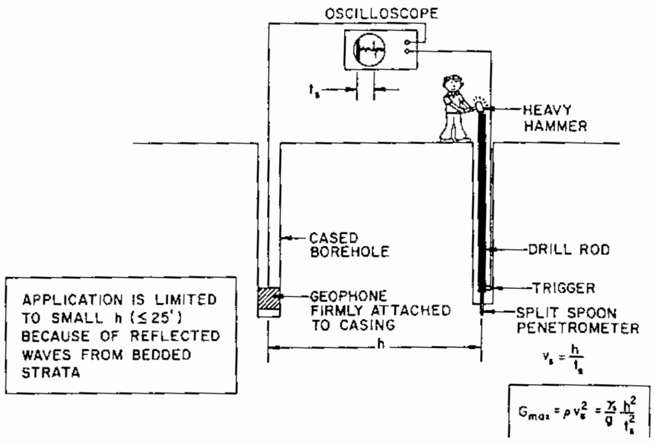


Figure 4-4. Crosshole Test (after ref. 16).

The crosshole test is depicted in Figure 4-4. Test equipment simply consists of a device to create a shear wave below the surface, another device a known distance away and at the same level to sense the passage of the shear wave, and an instrument to measure the time required for the wave to traverse the distance between the two devices. The shear modulus is then computed from the equation given in Figure 4-4. The shear wave velocity is measured because the shear wave is not influenced to any great degree by the presence of a water table, whereas the compression wave travels at its velocity in water, regardless of soil type, below the water table.

Crosshole tests permit an accurate assessment of the variation of shear modulus with depth to relatively large distances below the ground surface. Since they can be made an integral part of standard boring operations, the soils encountered can be inspected as a means of verifying qualitatively the validity of the results, and water table locations can be conveniently determined.

It should be pointed out here that the values of G_{max} obtained from oscillator or crosshole tests occur at values of strain amplitude that may be somewhat less than those which will occur under a prototype foundation. The value of shear modulus G to be used in the structure-soil interaction analysis will, therefore, probably be slightly less than G_{max}. Adjustments to G_{max} can be made by applying the correction suggested by Hardin and Drnevich described in detail in the section on saturated clays, or a simple procedure suggested by Whitman (ref. 18) can be adopted. Whitman suggests that for vertical motion the structure-soil system be analyzed assuming that G can vary from 0.7 G_{max} to G_{max} and that for rocking motion the system be analyzed assuming that

G can vary from 0.5 G_{max} to G_{max}. Simply stated, the system should meet the design criteria for any shear modulus value within the staged ranges.

Laboratory Procedures. The shear modulus can be evaluated from very low-amplitude cyclic triaxial compression or simple shear tests, although such tests must be run with such precision that their use is limited mainly to research. However, another type of test, the resonant column test, is also an accurate means of obtaining G_{max}. It has found generally wide acceptance among practicing engineers because of its relative simplicity. In the most common type of resonant column test, a solid, cylindrical column of soil is excited either longitudinally or torsionally at low amplitude within a cell in which an appropriate confining pressure has been applied. The exciting frequency is varied and the amplitude of deformation in the soil is monitored at each exciting frequency in order to determine the resonant frequency of the soil column. Simple elasticity equations for vibrating rods are then used to compute G_{max}. The reader is referred to Richart, Hall, and Woods (ref. 12) for a complete description of the resonant column test and the appropriate elasticity equations.

Field and/or laboratory shear modulus determinations should be performed for each specific project wherever possible. As a means of checking field and laboratory measurements, published correlations between G_{max} and G and the various factors listed earlier, developed through resonant column and very low-amplitude cyclic triaxial and simple shear tests, should be consulted. These correlations can also be used as guidelines for calculating the shear modulus value in the absence of direct measurements; however, shear modulus obtained only from published correlations must be considered to be relatively uncertain, and the structure-soil system should be analyzed assuming that the shear modulus (or spring constant) can vary within a large range both above and below the calculated value.

Several significant published correlations for shear modulus are summarized in the following sections:

Published Correlations: Sands and Gravels. Hardin and Richart (ref. 8) published criteria for the shear modulus of dry or saturated sand derived from resonant column tests conducted at or below a shear strain level that would occur under most foundations for vibrating machinery. Two expressions for G were obtained. For round grained sands where the void ratio is equal to or less than 0.80,

$$G \text{ (psi)} = \frac{2630(2.17-e)^2}{1+e} (\bar{\sigma}_o)^{0.5} \tag{4-2}$$

and for angular sands,

$$G \text{ (psi)} = \frac{1230(2.97-e)^2}{1+e} (\bar{\sigma}_o)^{0.5} \tag{4-3}$$

where e = void ratio,
$\bar{\sigma}_o$ = octahedral (or mean) normal effective stress in psi, given by Equation 4-4.

$$\bar{\sigma}_o = 0.333\ (\bar{\sigma}_v + 2\ \bar{\sigma}_h), \tag{4-4}$$

where $\bar{\sigma}_v$ = vertical effective stress in psi,
$\bar{\sigma}_h$ = horizontal effective stress in psi,
$= K_o\bar{\sigma}_v$

In order to obtain $\bar{\sigma}_0$, the vertical effective stress in the soil in question is computed by summing component stresses due to geostatic and applied loads (specifically considered later), and the horizontal effective stress is then computed by multiplying the vertical effective stress by the earth pressure coefficient at rest, K_0, which according to Brooker and Ireland (ref. 2), is a function of the plasticity index of the soil and the overconsolidation ratio (OCR). The overconsolidation ratio for an element of soil may be defined as the ratio of the maximum past effective vertical stress (approximately equal to the preconsolidation pressure indicated by a one-dimensional consolidation test) to the present total vertical stress minus the free pore fluid pressure $\bar{\sigma}_v$. Curves relating K_0 to plasticity index and overconsolidation ratio, developed by Brooker and Ireland, are given in Figure 4-5.

Seed and Idriss (ref. 14) have presented a slightly different correlation:

$$G \text{ (psi)} = 83.3\ K_2\ (\bar{\sigma}_0)^{0.5} \tag{4-5}$$

where K_2 is a factor which depends on relative density (D_r) and shear strain amplitude, as shown in Figure 4-6.

In situ relative density is often determined for sands by conducting standard peneration tests and correlating penetration resistance (or "blow count") to relative density. Typically, an approximate correlation made by Gibbs and Holtz (ref. 6) is utilized for saturated soils (Figure 4-7). Similar correlations are also given by Gibbs and Holtz for air-dry and moist sands. Care should be exercised in applying the correlation of Figure 4-7, particularly for "dirty" sands (sands containing more than 5% fines), since considerable data scatter is evident and since the correlation is based only on tests for two soils. Factors not explicitly included in the Gibbs and Holtz correlations, such as angularity and gradation, are known to have an effect on penetration resistance.

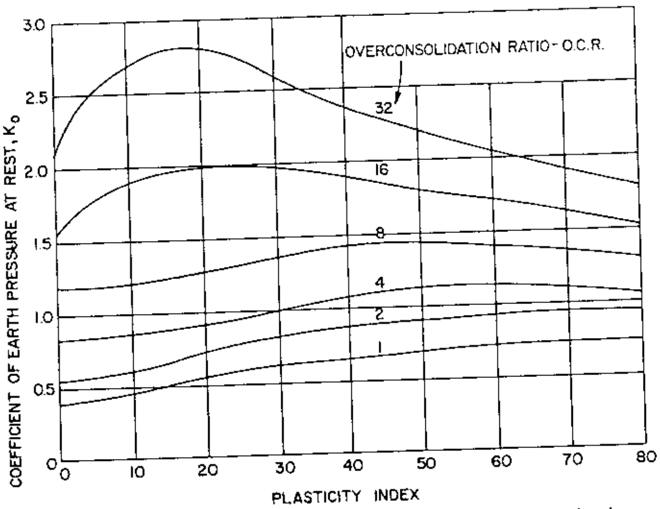
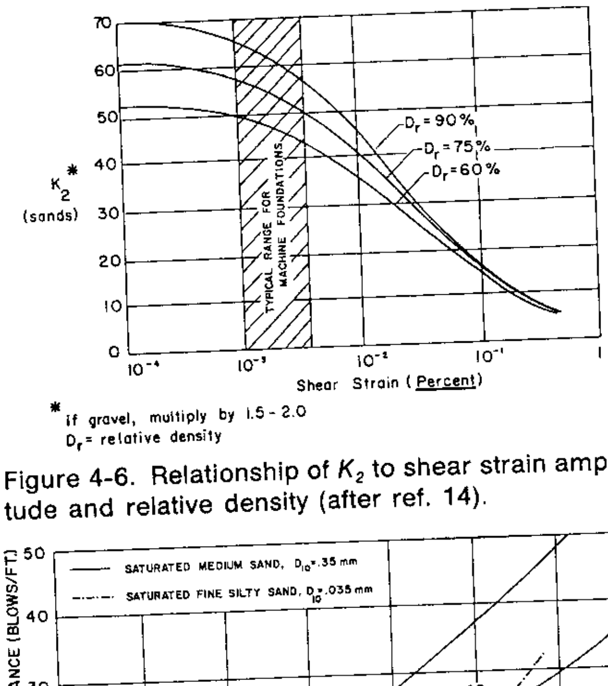


Figure 4-5. Relationship of K_o to Plasticity Index and OCR (after ref. 2). Reproduced by permission of the National Research Council of Canada from the Canadian Geotechnical Journal, Vol. 2 (1965), pp. 14.



* if gravel, multiply by 1.5 - 2.0
D_r = relative density

Figure 4-6. Relationship of K_2 to shear strain amplitude and relative density (after ref. 14).

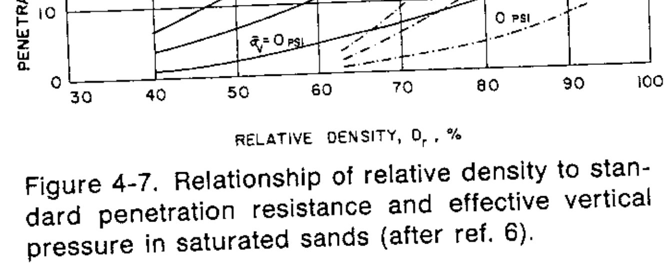


Figure 4-7. Relationship of relative density to standard penetration resistance and effective vertical pressure in saturated sands (after ref. 6).

Thus, if it is economically justifiable, it is recommended that a "calibration" be made by conducting a series of penetration tests and recovering undisturbed samples of sand near the test points. The relative density can then be assessed from the measured dry density of the samples. In such a case extreme care must be taken to determine the maximum and minimum dry densities according to appropriate published American Society for Testing and Materials procedures. In effect, a correlation similar to the one shown in Figure 4-7 is thus developed specifically for the soils at the site being investigated.

Example. Standard penetration tests are conducted on a stratum of clean, medium, round-grained sand 30 ft below the soil surface, which is at the base of a very large 20-ft-deep excavation. The penetration resistance is 16 blows per foot. The unit weight of the sand being tested and overlying soil is 117 pcf, and the void ratio of the soil being tested is 0.65. The water table is situated 10 ft below the bottom of the excavation, and the sand is known to have been normally consolidated prior to the time the excavation was made. What is the *in situ* shear modulus at the point where the penetration test is conducted?

Using Equation (4-2):

$\bar{\sigma}_v = 10(117) + 20(117\text{-}62.4) = 2262 \text{ psf} = 15.7 \text{ psi}$

Maximum past $\bar{\sigma}_v$ (assuming constant water table position) $= 30(117) + 20(117\text{-}62.4) = 4602 \text{ psf} = 32.0 \text{ psi}$

$$OCR = \frac{32.0}{15.7} = 2.0$$

K_o (Figure 4-5, plasticity index = 0) = 0.6

$\bar{\sigma}_0 = 0.333\ (15.7)\ [1 + 2(0.6)] = 11.5 \text{ psi}$

$$G = \frac{2630(2.17 - 0.65)^2}{1.65}\ (11.5)^{0.5} = 12{,}490 \text{ psi}$$

Using Equation (4-5):

D_r (Figure 4-7) = 75%

K_2 (shear strain at right-hand edge of crosshatched area, Figure 4-6) = 50

$G = 83.3\ (50)\ (11.5)^{0.5} = 14{,}120 \text{ psi}$

As is demonstrated in this example, the Hardin-Richart and Seed-Idriss approaches usually yield somewhat different values for G.

Published Correlations: Saturated Clays. Hardin and Drnevich (ref. 7) obtained detailed correlations between the shear modulus of clays (and sands) and (1) void ratio, (2) effective octahedral normal stress, (3) overconsolidation ratio, and (4) strain amplitude by conducting laboratory resonant column tests on specimens of natural soil that had been sampled with thin-walled sampling devices. The results are expressed as

$$G_{max}\ (\text{psi}) = 1230\ \frac{(2.973 - e)^2}{(1 + e)}\ (OCR)^k\ (\bar{\sigma}_o)^{0.5} \quad (4\text{-}6)$$

where k = plasticity constant determined from Table 4-7.

The value of G_{max} in Equation (4-6) corresponds to the shear modulus at a shear strain amplitude of about $0.25 \times 10^{-4}\%$, which is generally below the amplitude appropriate for use in analysis of foundations for vibrating machinery. In order to obtain the value of G corresponding to the correct strain magnitude, Equation (4-7) must be employed.

$$G = G_{max}/(1 + \gamma/\gamma_r) \qquad (4\text{-}7)$$

where γ is the desired shear strain magnitude expressed as a percentage and γ_r is a "reference strain" defined by Equation (4-8):

$$\gamma_r = (\tau_{max}/G_{max}) \times 100 \qquad (4\text{-}8)$$

If the effective vertical stress $\bar{\sigma}_v$, the effective stress coefficient of earth pressure at rest K_0, the effective cohesion $\bar{c}$, and the effective stress angle of internal friction $\bar{\phi}$ are known, τ_{max} can be evaluated by using the Mohr-Coulomb failure theory. The quantity τ_{max} is defined as the shear stress on a horizontal plane at failure for the case where failure is induced by application of shear stresses on the horizontal and vertical faces of a soil element, from which, for an initial anisotropic state of stress:

$$\tau_{max} = \left[\left(\frac{1+K_o}{2}\bar{\sigma}_v \sin\bar{\phi} + \bar{c}\cos\bar{\phi}\right)^2 - \left(\frac{1-K_o}{2}\bar{\sigma}_v\right)^2\right]^{0.5} \qquad (4\text{-}9)$$

The quantity $\bar{\sigma}_v$ is usually evaluated as the total vertical geostatic stress (σ_v) minus the pressure in the free pore fluid (u), which is a valid definition for sands, normally consolidated clays, and lightly overconsolidated clays. However, in heavily overconsolidated clays $\bar{\sigma}_v$ may exceed $\sigma_v - u$, which creates more difficulty in applying the results of the calculations. The OCR term in Equation (4-6) is an empirical parameter which minimizes this effect if $\bar{\sigma}_0$ (Equation (4-6)) and $\bar{\sigma}_v$ (Equation (4-9)) are always defined as total stress minus free pore fluid pressure. Therefore, $\bar{\sigma}_0$ and $\bar{\sigma}_v$ can be evaluated from results of routine borings and laboratory tests.

The value of K_0 again can be determined from Figure 4-5. The shear strength parameters $\bar{c}$ and $\bar{\phi}$ can be evaluated from static, consolidated-undrained strength tests in which pore pressures are measured or from static, consolidated-drained strength tests.

Example. A sample of saturated clay is recovered from a depth of 15 ft. Its total unit weight, as well as that for the overlying soil, is measured to be 125 pcf. Piezometer tests in the borehole indicated a piezometric surface (water table) depth of 5 ft. A consolidation test is conducted on the sample, which yields an indicated preconsolidation pressure of 4000 psf and an initial void ratio of 0.60. The plasticity index of the clay is measured to be 30.

Table 4–7
Values of k

Plasticity Index	k
0	0
20	0.18
40	0.30
60	0.41
80	0.48
≥ 100	0.50

A set of consolidated-undrained triaxial compression tests is then conducted on the soil in which confining pressures are in the order of $\bar{\sigma}_v$, and $\bar{c}$ and $\bar{\phi}$ are found to be 3 psi and 20 degrees, respectively. What is G at a shear strain amplitude of $5 \times 10^{-3}\%$?

$$\bar{\sigma}_v = 15(125) - 10(62.4) = 1251 \text{ psf} = 8.69 \text{ psi}$$

$$OCR = \frac{4000 \text{ psf}}{1251 \text{ psf}} = 3.2$$

$$K_0 = 0.9 \text{ (Figure 4-5)}$$

$$\bar{\sigma}_0 = 0.333\,\bar{\sigma}_v\,(1 + 2K_0) = 8.1 \text{ psi}$$

$$k = 0.24 \text{ (interpolated from Table 4-7)}$$

$$G_{max} = 1230\,[(2.973-0.60)^2/1.60]\,(3.2)^{0.24}\,(8.1)^{0.5} = 16{,}288 \text{ psi}$$

$$\tau_{max} = [(1.9/2)\,8.7 \sin 20° + 3\cos 20°)^2 - ((0.1/2)\,8.7)^2]^{0.5} = 5.63 \text{ psi}$$

G is desired at a shear strain level of $5 \times 10^{-3}\%$:

$$\gamma_r = (\tau_{max}/G_{max}) \times 100 = (5.63/16{,}288) \times 100 = 0.0346\%$$

$$G = 16{,}288/(1 + 0.005/0.0346) = 16{,}288/1.14 = 14{,}288 \text{ psi}$$

Note that G is about 88% of G_{max}, which is a typical reduction for shear strains in the usual range for machine foundations.

Seed and Idriss (ref. 14) have shown that the shear moduli calculated from the Hardin-Drnevich equations underestimated in the *in situ* shear moduli by factors of 4 and 14 for two soft normally consolidated marine clays with void ratios exceeding 1.6. This effect has been observed by others as well, leading to the conclusion that the Hardin-Drnevich equations should not be expected to correlate well with *in situ* moduli for marine soils with high void ratios and for soils of otherwise relatively high sensitivity. For such soils it is advisable that the shear modulus be determined directly from *in situ* tests.

Alternatively, the results of laboratory resonant column tests can be employed for sensitive clays, provided the effect of sampling disturbance can be estimated in a rational way. Anderson and Woods (ref. 1) have shown that all soils experience a slow, time-dependent increase in shear modulus as measured in a resonant column device after an initial disturbance. This increase apparently occurs as a result of reestablishment of a stable soil fabric following sampling or other disturbing processes. The indicated shear modulus generally plots in a straight line as a function of the logarithm of time after initial confinement in a cell. For cohesive soils, Anderson and Woods have found that

$$(\Delta G/G_{1000})\ (\%) = 2 \exp\,(1.7 - 0.25 s_u + 0.37 e), \quad (4\text{-}10)$$

where $\Delta G(\%)$ = the increase in shear modulus per log cycle of time in min,

G_{1000} = shear modulus obtained after sample has consolidated in a cell for 1000 min (approximately the value that would be predicted using the Hardin-Drnevich equations),

s_u = undrained shear strength of the soil in kg/cm^2 or tons/ft,^{2}

e = initial void ratio of the soil.

Factors such as magnitude of confining pressure do not play a major role in the rate of time-dependent increase in the value of shear modulus, or "secondary gain." Examination of Equation (4-10) indicates that the rate of secondary gain increases with increasing void ratio and decreasing shear strength, which is consistent with the observation of Seed and Idriss that the Hardin-Drnevich equations, which do not account for sampling disturbance, underpredict the shear modulus in soft clays with high void ratios.

Equation (4-10) can be used to convert in an approximate fashion the shear modulus measured by short time-of-consolidation laboratory resonant column tests, and possibly that predicted by the Hardin-Drnevich equation, into an equivalent value representative of the undisturbed soil. The value of G_{max} can be obtained from Equation (4-10) as

$$G_{max} = \left[\frac{N_a \dfrac{\Delta G}{G_{1000}}(\%)}{100} + 1 \right] G_{1000}, \quad (4\text{-}11)$$

where N_a is the number of log cycles of time in min (beginning at 1000 min) required for the soil fabric to reestablish itself as it existed *in situ* prior to sampling, using 1000 min as the point of reference. The correction can also be applied directly to G with little error. For soils of the type being considered, the appropriate time required for complete fabric reestablishment depends mainly upon the mineralogical properties of the soil. Very little data exist relative to this point. However, if all the fabric reestablishment is assumed to be due to thixotropy, as suggested by Anderson and Woods, experimental relationships developed by Skempton and Northey (ref. 13) can be used to obtain order of magnitude values for the time in question. For soils whose sensitivity is less than about 8 (moderate sensitivity), Skempton and Northey indicated that disturbed soils appear to regain their *in situ* strength (and by implication their *in situ* structure) in a period approximately equal to the geological age of the deposit. Using the above observation as a criterion, N_a can be defined by the following equation:

$$N_a = 2.72 + \log_{10} A_y, \quad (4\text{-}12)$$

where A_y = age of deposit in years.

Equation (4-12) does not apply to heavily overconsolidated clays, whose structure is not significantly changed by the sampling process, nor does it apply to soils whose sensitivity exceeds about 8. For the latter class of soils, no reliable methods exist to predict N_a; therefore, only *in situ* tests should be used to obtain the shear modulus. Furthermore, such soils should be avoided altogether as founding strata for structures supporting vibrating equipment whenever possible.

Example. A deposit of slightly overconsolidated clay on an industrial site adjacent to an estuary is known to be 10,000 years old. Soil recovered from the deposit from a depth of 30 ft has a shear strength of 500 psf (0.25 kg/cm^2) as determined by unconfined compression testing. The soil has an *in situ* void ratio of 1.6 and a sensitivity of 6. Resonant column tests yield a value of G of 2000 psi. What is the corrected *in situ* shear modulus?

To correct the computed shear modulus, use Equations (4-10), (4-12), and (4-11):

$$\begin{aligned}(\Delta G/G_{1000})\ (\%) &= 2 \exp\,[1.7 - 0.25(0.25) \\ &\quad + 0.37(1.6)] \\ &= 18.6\%\end{aligned}$$

$$\begin{aligned}N_a &= 2.72 + \log_{10}(10{,}000) \\ &= 6.72\end{aligned}$$

$$G = [6.72(18.6/100 + 1]\ 2000$$
$$= 2.25(2000) = 4500 \text{ psi}$$

Seed and Idriss suggest that shear modulus be correlated directly to undrained strength in an attempt to circumvent the sampling disturbance problem, primarily because strength is affected much less than modulus by sampling. The shear modulus G was correlated to undrained shear strength s_u for a number of clays by Seed and Idriss. For low strain levels s_u, as obtained by standard static tests, was correlated with G obtained from *in situ* seismic tests. For high-strain levels s_u was correlated with G obtained from a variety of laboratory tests, whose results were corrected by multiplying the value of G obtained in the test by a factor of 2.5 to account for the effects of sampling disturbance. The results of the correlations, which are shown in Figure 4-8, yield a simple procedure for estimating G from standard static laboratory tests.

Calculation of Shear Modulus for Structure-Soil Interaction Analysis

Since the shear modulus is a function of effective confining pressure ($\bar{\sigma}_0$) for both sands and clays, the value of the shear modulus is influenced by both the geostatic stress and the net bearing stress produced by the structure. Therefore, the static vertical stress produced in the soil by the structure should be added to the geostatic vertical effective stress when calculating $\bar{\sigma}_v$ and $\bar{\sigma}_0$ and subsequently G from Equations (4-2), (4-3), (4-5), (4-6), and (4-9) or when computing confining pressures for resonant column tests. The principal exception to this statement is that only geostatic effective stresses need be considered for overconsolidated clays which remain overconsolidated after being loaded by the structure. Furthermore, static machine and foundation weights can often be neglected in practice when evaluating Equation (4-9), since their effects on shear modulus reduction are small.

For reasonably uniform soils, it is sufficient to evaluate the shear modulus for purposes of calculating soil spring constants at only one characteristic depth d_c below the ground surface, as shown in Figure 4-9. For freedraining soils, it is usually adequate to set $\bar{\sigma}_v$ in the various equations for shear modulus equal to $\bar{\sigma}_{v\min}$, also defined in Figure 4-9. Since K_0 is needed in the calculation of $\bar{\sigma}_0$, *OCR*, which is needed to obtain K_0 should be computed by including the effect of the vertical stresses from the structure, that is, by setting $\bar{\sigma}_v$ equal to $\bar{\sigma}_{v\min}$ (Figure 4-9) in the definition of *OCR*.

If the soil is an overconsolidated clay which remains overconsolidated after loading, the imposed stresses have little effect on G. Hence, values of G measured *in situ* or computed only from geostatic stresses need be used. Uniform deposits of such soils generally possess a reasonably uniform shear modulus with depth so that it is usually not necessary to evaluate the modulus at a critical depth.

When the founding soil consists of a normally consolidated clay (which is a rare circumstance) or a clay which will become normally consolidated after the load has been applied and excess pore pressures dissipate, the shear modulus will increase with time, and two shear

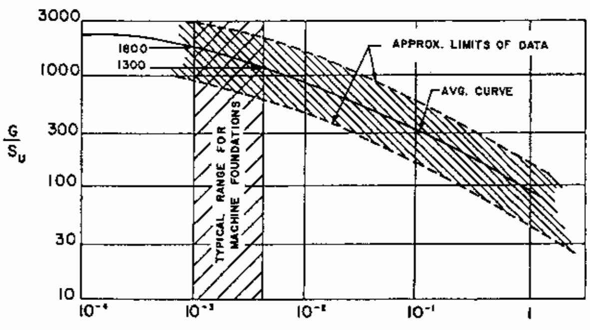


Figure 4-8. *In situ* shear modulus for saturated clay (after ref. 14).

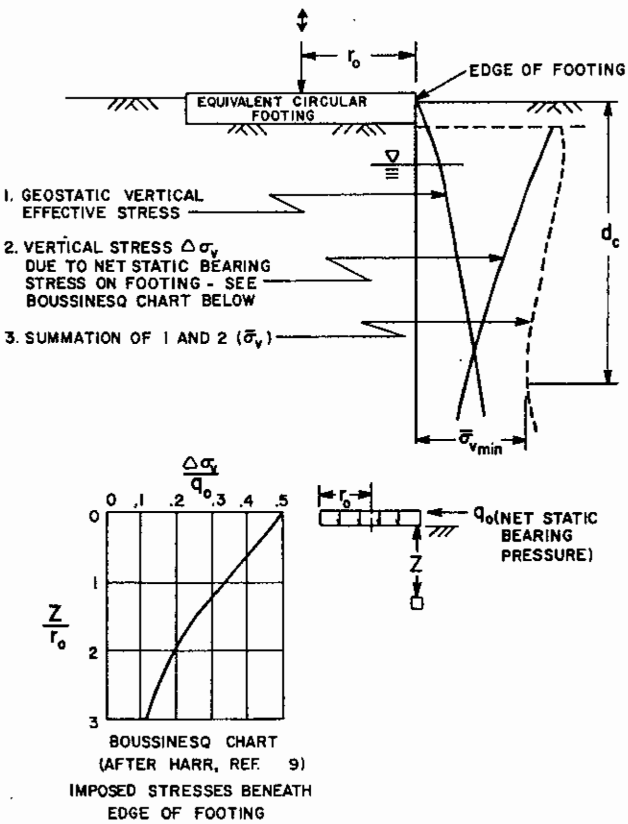


Figure 4-9. Determination of $\bar{\sigma}_{v\min}$.

modulus values and consequently two spring constant values should be obtained and used in the analysis: first, for the unconsolidated-undrained condition which exists immediately after the static load is applied, and, second, for the fully consolidated condition which occurs at some time in the future. The shear modulus for the former condition should be evaluated using only geostatic stress conditions (in which case d_c corresponds to the depth of the base of the footing) and soil properties that exist just prior to placement of the structure. The shear modulus for the latter condition should be evaluated by the method described above or by assessing the new undrained shear strength at depth d_c beneath the edge of the footing at the end of consolidation and using Figure 4-8.

Example. Determine k_z and k_ψ for the footing shown in Figure 4-10.

Assuming that the unit weight of the concrete and unit weight of displaced soil are equal, the net static bearing stress at the base of the footing is

$$25{,}000/\pi 4^2 = 497 \text{ psf}$$

Using the Boussinesq chart in Figure 4-9, the plot of induced vertical stress at the edge of the footing is made (Line A). The geostatic vertical effective stress is plotted as Line B. Finally, the stresses from Lines A and B are added to produce Line C. It is observed that d_c is at the base of the footing and that $\bar{\sigma}_{v\min} = 490 \text{ psf} = 3.40 \text{ psi}$. Since the sand is normally consolidated, $K_0 = 0.4$ (Figure 4-5), and

$$\bar{\sigma}_0 = 0.333(\bar{\sigma}_v + 2\bar{\sigma}_h)$$
$$= 0.333\,[3.40 + (2)(0.4)(3.40)]$$
$$= 2.04 \text{ psi}$$

Using the Seed-Idriss correlation, and assuming a shear strain level of $3 \times 10^{-3}\%$,

$$G = 83.3K_2(\bar{\sigma}_0)^{0.5}$$
$$= 83.3(52)\ (2.04)^{0.5}$$
$$= 6187 \text{ psi}$$

Assuming that embedment is effective (footing is cast against undisturbed soil or dense backfill is provided),

$$k_z = [4Gr_0/(1-\nu)][1 + 0.6(1-\nu)(h/r_0)]$$
$$= [4(6187)4(12)/0.6][1 + 0.6(0.6)(2/4)]$$
$$= 2.34 \times 10^6 \text{ lb/in.}$$

and
$$k_\psi = [8Gr_0^3/3(1-\nu)][1 + 1.2(1-\nu)(h/r_0) + 0.2(2-\nu)(h/r_0)^3]$$
$$= [8(6187)48^3/3(0.6)][1 + 1.2(0.6)(2/4) + 0.2(1.6)(2/4)^3]$$
$$= 4.26 \times 10^9 \text{ in.-lb/rad.}$$

Typical ranges of values of low-strain-amplitude shear modulus for several soils are given in Table 4-8 for the purpose of general information. Table 4-8 should never be used as a substitute for a rational determination of modulus values.

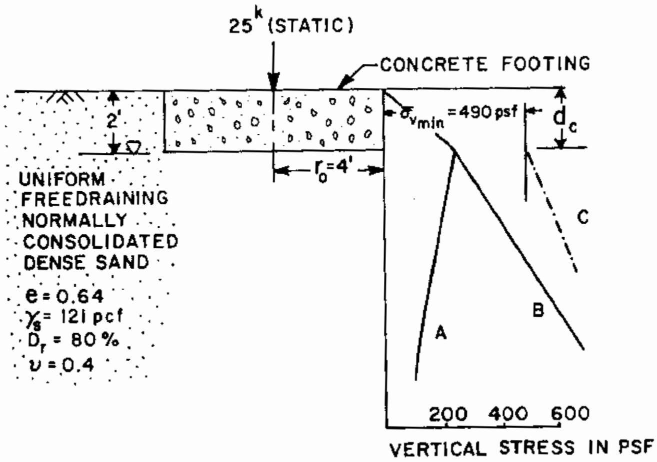


Figure 4-10. Footing for example calculations of soil spring constants.

Table 4–8

Typical Values for Low-Strain-Amplitude Shear Modulus

Soil Type	Shear Modulus (psi)
Soft Clay	3,000– 5,000
Stiff Clay	10,000–20,000
Very Stiff to Hard Clay	≥20,000
Medium Dense Sand*	5,000–15,000
Dense Sand*	10,000–20,000
Medium Dense Gravel*	15,000–25,000
Dense Gravel*	20,000–40,000

*For shallow depths.

Selection of Shear Strain Magnitude for Computing Approximate Shear Modulus Beneath Footings

Since the shear modulus is a function of shear strain magnitude, it is necessary to obtain an estimate of the appropriate value of shear strain magnitude to use in calculations of soil spring constants. The authors recommend the following approximate procedure for vertical loading, which is based on an analogy with static conditions:

1. Select a shear strain amplitude in the range of the crosshatched area of Figures 4-6 and 4-8, and compute G.
2. Conduct the structure-soil interaction analysis and determine the transmissibility factor T_r from Table 1-4 for the forcing frequency desired.
3. Multiply the unbalanced vertical force by the transmissibility factor and divide the result by the contact area of the footing to obtain the dynamic bearing stress q_d.

4. The approximate average shear strain γ in a central block of soil below the footing of dimensions $2r_0 \times 2r_0$ (horizontal) $\times$ $4r_0$ (vertical) is given by Eq. (4-13).

$$\gamma(\%) = 12\, q_d/G \tag{4-13}$$

Equation (4-13) presumes q_d to induce the same strains in the soil as it would if it were acting as a static stress (ref. 10). Of course, the strains due to a dynamic bearing stress emanate as waves, making Equation (4-13) nonrigorous. Nonetheless, Equation (4-13) will yield order-of-magnitude strain levels that are sufficiently accurate for most analyses. Therefore Equation (4-13) may be used to verify the assumed value of G (Step 1). If the assumed and computed shear strains differ significantly, these four steps should be repeated iteratively, using the value of γ computed on the preceding trial to obtain G for the present trial, until the strains close to within an acceptable difference.

If a particularly precise analysis is warranted, the approach described above should be abandoned in favor of a more comprehensive technique, such as the finite element method, in which complete modeling of a relevant volume of the soil and its constitutive relationships is considered.

It should be pointed out here that the shear strain magnitude beneath a footing should be taken as that produced only by the dynamic component of the footing load. The static shear strain should be neglected, since it in effect only provides a nonzero strain level about which the dynamic strain is cycled. The small strain shear modulus relative to that nonzero reference is generally about the same as the small strain modulus relative to a reference level of zero strain.

Damping Ratio

Damping in a soil-foundation system consists of a geometric component, which is a measure of energy radiated away from the immediate region of the foundation, and material damping within the soil, which is a measure of energy lost as a result of hysteresis effects. Geometric damping ratios have been shown at the beginning of this chapter to be related to the mass or inertia ratio of the system through the use of elastic halfspace theory. Relationships between mass and inertia ratios and the geometric damping ratio are shown for the four uncoupled modes of motion in Figure 4-11.

Material damping is defined in Figure 4-12. It is seen to be proportional to the ratio of A_L, the area of the soil hysteresis loop in simple shear (energy lost), to A_T, the crosshatched area (energy input). Material damping ratios can be obtained as a part of resonant column testing. After the soil has been vibrating in a steady-state condition, the exciter is stopped and the soil vibrations are monitored as they decay. The displacement-time relationship is essentially sinusoidal, but with the amplitudes decreasing with time. If two successive amplitudes are z_1 and z_2, then

$$D_m = (\ln [z_1/z_2])\,[4\pi^2 + (\ln [z_1/z_2])^2\,]^{-0.5} \tag{4-14}$$

Additional procedures are described by Richart, Hall and Woods (ref. 12).

Seed and Idriss (ref. 14) have shown that material damping in soils is primarily a function of strain amplitude and soil type. Figure 4-13 gives typical values for material damping proposed by Seed and Idriss.

In order to obtain the total soil-foundation system damping ratio D_t, the geometric and material damping ratios may be added directly. Since the material damping ratio is significant relative to the geometric damping ratio in rotational modes, the total damping ratio, rather than the geometric damping ratio, should be used when analyzing the response of foundations in those

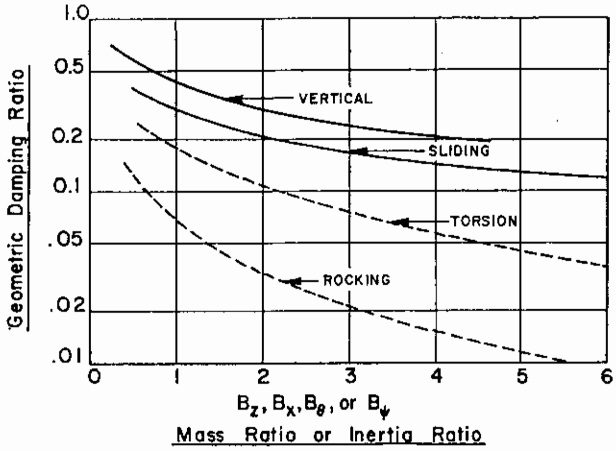


Figure 4-11. Geometric damping ratios for four modes of loading. After Richart, Hall, and Woods, ref. 12. Reprinted by permission of Prentice-Hall, Inc., Englewood Cliffs, NJ.

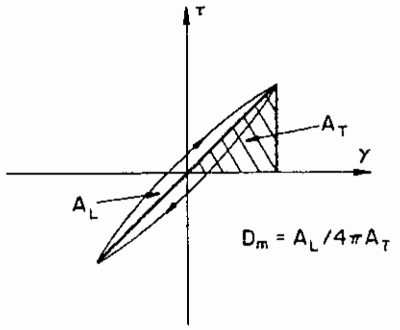


Figure 4-12. Definition of material damping (after ref. 7).

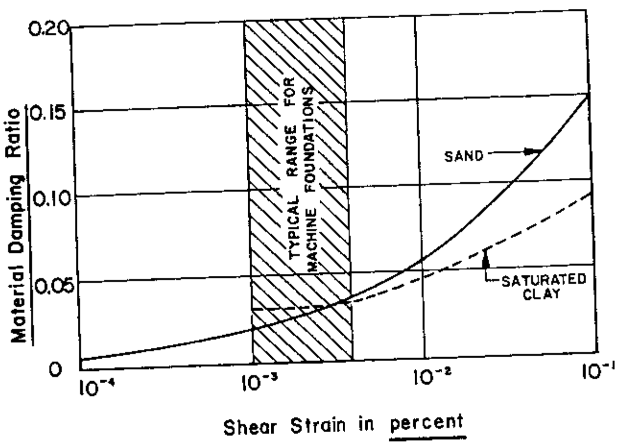


Figure 4-13. Material damping ratios for sands and clays (after ref. 14).

modes. On the other hand, material damping is small compared to geometric damping in the translatory modes and may often be disregarded. The exclusion of material damping from an analysis will result in amplitudes that are too high, especially at or near resonance, and resonant frequencies that, for rotating mass excitation, are slightly too low.

Example. The inertia ratio B_ψ for a machine-foundation system undergoing rocking oscillations and resting on a saturated clay subgrade is determined to be 1.3. How does material damping influence the deformation at resonance and the resonant frequency?

According to Figure 4-11, the geometric damping ratio of the system is 0.05. If the material damping within the clay is neglected, the amplitude magnification factor for rotating mass excitation (Equation 1-43) is approximately 10 at resonance. Further,

$$f_{mr} = (1/\sqrt{1 - 2D^2}) f_n$$
$$= 1.0025 f_n$$

If material damping is considered, according to Equation 1-43, its value should be approximately 0.03 in the typical range for machine foundations. Hence,

$$D_t = D + D_m$$
$$= 0.05 + 0.03$$
$$= 0.08$$

Using the value thus obtained for total system damping in Equation 1-43, the amplitude magnification factor is reduced to approximately 6. Also, using the total system damping factor of 0.08,

$$f_{mr} = 1.0065 f_n$$

Consideration of material damping thus reduces the rotational amplitude at resonance by about 40%, but has only a minor effect upon the resonant frequency. Other effects, discussed in Chapter 5, may also have an influence on f_{mr} and amplitude of motion in the rocking mode.

Whitman (ref. 18) has proposed that the effects of geometric and material damping be combined and that the total damping ratios be computed as follows:

Vertical Translation Mode:

$$D_t = 0.49\ (M/\rho r_o^3)^{-0.5} \tag{4-15}$$

Horizontal Translation Mode:

$$D_t = 0.31\ (M/\rho r_o^3)^{-0.5} \tag{4-16}$$

Rocking Mode:

$$D_t = 0.05 + 0.1\ ([I_\psi/\rho r_o^5]^{0.5}\ [1 + (I_\psi/4\rho r_o^5)])^{-1} \tag{4-17}$$

where M = mass of foundation plus mass of structure or machine vibrating in phase with the foundation,
r_0 = effective foundation radius,
I_ψ = mass moment of inertia of foundation plus that part of structure or machine vibrating in phase with the foundation about a horizontal axis through the base of the foundation perpendicular to the plane of rocking,
ρ = total mass density of the soil (unit weight/acceleration of gravity).

When the foundation being analyzed is a rigid footing within a multi-degree-of-freedom structure and to which vibrating machinery is not directly attached, it is conservative to calculate D_t from the above equations assuming that M is the mass of the foundation alone. Under the same circumstances, when the geometric damping ratio is determined from Figure 4-11, mass and inertia ratio arguments may be computed from the mass properties of the foundation alone.

Whitman's proposed values for total damping ratios can be seen to be somewhat lower than the geometric damping ratios given by Figure 4-11. This apparent anomaly is due to the fact that Whitman's expressions represent lower envelopes to available test data and implies that geometric damping in a prototype may be less than that predicted by halfspace theory, possibly because of layering and boundary effects.

Selection of Poisson's Ratio and Soil Density

Soil-foundation interaction problems are relatively insensitive to the values chosen for ν and ρ. Generally, Poisson's ratio can be selected based on the predominant soil type using Table 4-9. It is also possible to obtain a value for Poisson's ratio by measuring independently the shear modulus (G) in a laboratory torsional resonant

Table 4–9
Typical Values for Poisson's Ratio

Soil Type	ν
Saturated Clay	0.45–0.50
Partially Saturated Clay	0.35–0.45
Dense Sand or Gravel	0.4 –0.5
Medium Dense Sand or Gravel	0.3 –0.4
Silt	0.3 –0.4

column test and Young's modulus (E) in a laboratory longitudinal resonant column test. Assuming isotropy,

$$\nu = (E/2G) - 1 \tag{4-18}$$

Soil mass density values should always be calculated from the total unit weight rather than the buoyant unit weight because the density term in the mass and inertia ratio equations always represents soil undergoing vibration. Total weights are used because both the solid and liquid phases vibrate.

Effect of Footing Embedment

Analytical expressions have been given in Tables 4-2 and 4-4 that are to be employed as multipliers to the equivalent spring constant and geometric damping ratio values whenever the footing is embedded. A foundation should be considered as "embedded," however, only if it is cast against undisturbed soil or if it is formed and backfilled carefully with a high compactive effort using soils with low shrink-swell potential. Casual backfilling is ineffective.

Stokoe and Richart (ref. 15) studied the response of model circular footings embedded in a dense, dry, poorly graded sand to vertical, horizontal, and rocking excitation. Two sets of tests were conducted: one in which the embedded footings were cast against the soil and one in which a small air gap existed between the soil and the sides of the footing. The results, which were relatively consistent for all modes of loading, are summarized in Figures 4-14 and 4-15.

It is evident that proper embedment had a significant effect on both total damping and resonant frequency, while embedment without adequate lateral support was essentially ineffective.

The benefits of proper footing embedment are especially pronounced for the rocking mode, since D_t can be increased by several times by embedding the footing to a depth equal to or greater than its equivalent radius. It is also desirable to embed pile caps in competent soil,

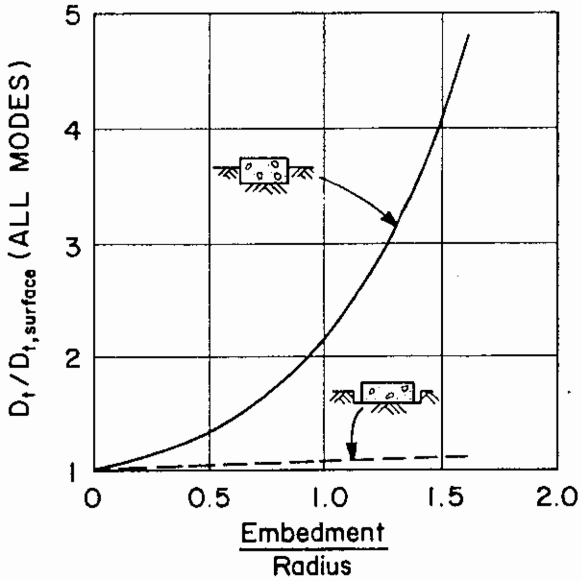


Figure 4-14. Effect of embedment on damping (after ref. 15).

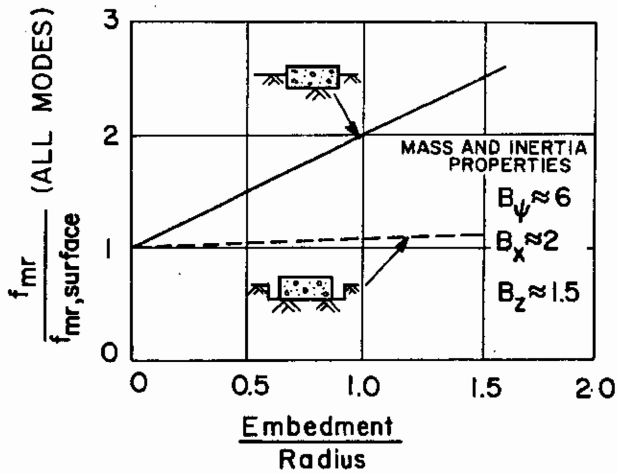


Figure 4-15. Effect of embedment on resonant frequency (after ref. 15).

since the response of pile foundations is sharply peaked at resonance in all modes. Cap embedment in cases where significant lateral loading occurs in the piles is especially important, since resonant displacements can often be excessive without the damping and load transfer provided by the cap. This problem will be addressed in greater detail in Chapter 5.

Effect of Stiff Underlying Stratum

If the subgrade consists of a softer soil overlying an appreciably stiffer soil or bedrock within two to three footing diameters of the base of the footing, the response will be altered significantly. A stiff stratum interface will reflect a considerable proportion of the energy that would ordinarily be radiated away from the foundation, in effect producing a lower geometric damping ratio than would occur in a homogeneous subgrade. This effect is offset to a degree by the fact that the effective lumped spring constant is increased due to the presence of the stiff stratum.

Quantitative research in the vertical response of foundations on layered media was described by Warburton (ref. 17). Little usable information is available for the rocking and sliding modes at present, making the use of special modeling techniques such as the finite-element method attractive for important problems in rocking and sliding. Torsional response in a layered system is discussed by Richart, Hall and Woods (ref. 12).

Table 4-10 presents approximate relationships for calculating k_z for various values of H/r_0, where H is the distance from the base of the footing to the stiff soil contact. The stiff soil is assumed to be fully rigid.

Establishment of a lumped damping ratio in this case is difficult; however, when $5 \leq W/\gamma_s r_0^3 \leq 10$ and $1 \leq H/r_0 \leq 2$, damping in the vertical mode will be on the order of 15% of that which would occur were the subgrade a halfspace. Thus, the geometric damping ratio, for vertical motion in the ranges defined above, may be expressed approximately as

$$D \approx 0.06/\sqrt{B_z} \qquad (4\text{-}19)$$

It is evident that material damping in the overlying soil will be important. Thus, in the analysis of a single- or multi-degree-of-freedom structure using lumped spring constants and damping ratios to represent the soil, the total damping ratio should be used.

The response of a single-degree-of-freedom system undergoing vertical oscillations in layered soils can be studied in an approximate way using Tables 4-11 and 4-12, which give theoretical values for displacement at resonance and resonant frequency, respectively, and consider only geometric damping. At frequencies below about 60% of resonance and above about 140% of resonance, dynamic displacements will generally not exceed the static displacement produced by the dynamic unbalanced load by more than about 50%. Hence, it is good practice to operate machines in those two zones, although harmonic resonant frequencies may be detectable in light structures at about 3, 5, 7, . . ., times the fundamental resonant frequency.

Example. A rotating mass machine has an unbalanced vertical force of 250 lb at 1000 rpm. It weighs 60,000 lb and is supported on a rigid block foundation weighing 80,000 lb. The equivalent radius of the foundation is 5 ft and the foundation rests on the surface of a stratum of dense sand having a shear modulus of 10,000 psi and a unit weight of 112 pcf. The sand overlies bedrock at a depth of 10 ft. What is the resonant frequency of the system and the amplitude of vertical displacement if material damping is neglected within the soil? How do these values compare with those that would be calculated if the bedrock were absent? Assume $\nu = 0.25$.

Table 4–10
Approximate k_z Variation with Stratum Thickness (after Richart, Hall, and Woods, ref. 12)

H/r_0	$k_z/[4Gr_0/(1-\nu)]$
1	2.1
2	1.6
∞	1.0

Table 4–11
Resonant Vertical Displacement for Stratum Overlying Rigid Base: $\nu = 0.25$ (after Richart, Hall, and Woods, ref. 12)

H/r_0		$A_{z,\ resonance}$ / $[(3/16)/(Q_0/Gr_0)]$			
	$W/\gamma_s r_0^3 =$	5	10	20	30
1		5.8	11	21	29
2		8.0	16	31	41
∞		1.2	1.6	2.2	2.7

W = weight of footing plus machinery vibrating in phase.
Q_0 = unbalanced vertical force.
G = shear modulus of soft soil.
γ_s = total unit weight of soft soil.
ν = Poisson's ratio of soft soil.

Table 4–12
Resonant Frequency in Vertical Mode for Stratum Overlying Rigid Base: $\nu = 0.25$ (after Richart, Hall, and Woods, ref. 12)

$W/\gamma_s r_0^3$		$f_0/\sqrt{Gg/\gamma_s r_0^2}$		
	$H/r_0 =$	1	2	∞
5		0.21	0.15	—
10		0.16	0.12	0.10
20		0.12	0.094	0.080
30		0.097	0.080	0.064

f_0 = fundamental resonant frequency (in cps if g is in linear units/sec^2).

$$W = 60{,}000 + 80{,}000 = 140{,}000 \text{ lb.}$$
$$W/\gamma_s r_0^3 = 140{,}000/112(5)^3 = 10.0$$
$$H/r_0 = 10/5 = 2$$

When bedrock is present, from Table 4-12,

$$f_0 = 0.12[(10{,}000)(144)(32.2)/(112)(25)]^{0.5}$$
$$= 15.4 \text{ cps}$$
$$= 924 \text{ rpm}$$

Then, $Q_0 = (924/1000)^2 250 = 213$ lb. From Table 4-11,

$$A_{z,\ \text{resonance}} = 16(3/16)(Q_0/Gr_0)$$
$$= 3(213)/(10{,}000)(5)(12)$$
$$= 0.00107 \text{ in.}$$

When bedrock is absent, from Table 4-12,

$$f_0 = 0.1[(10{,}000)(144)(32.2)/(112)(25)]^{0.5}$$
$$= 12.9 \text{ cps}$$
$$= 774 \text{ rpm}$$

Then $Q_0 = (774/1000)^2 250 = 150$ lb.

From Table 4-11,

$$A_{z,\ \text{resonance}} = 1.6(3/16)\ (Q_0/Gr_0)$$
$$= 3(150)/10(10{,}000)(5)(12)$$
$$= 0.000075 \text{ in.}$$

The resonant frequencies will increase only slightly when material damping is considered. Hence, the resonant frequencies computed above would usually be adequate for design purposes. Because of material damping, and because even a bedrock boundary is not truly rigid, the displacement amplitude for the layered system is overestimated by using Table 4-11.

Effect of Stratum of Loose Granular Soil

It is considered to be generally poor practice to situate footings supporting loads from vibrating machinery on cohesive soils which will consolidate under the static load or on granular soils having a relative density of less than 70–75%. In granular soils having low relative density, problems with permanent settlement due to compaction can occur. In cases where soils with adequate relative densities are not found, the soils should be stabilized mechanically by vibroflotation,(TM) terra probing,(TM) dynamic compaction, vibro-replacement, removal and replacement, or similar techniques (ref. 21) to increase the relative density to the appropriate value. Chemical stabilization is not generally economical but may be used in certain cases. When stabilization is not economical, a carefully designed deep foundation system should be used.

Permanent settlement due to vibration is generally not a problem in clays because cyclic stresses transmitted to the subgrade in a well-designed facility are seldom sufficient to generate pore pressure of a magnitude which would either affect consolidation or reduce the shear strength of the clay to the point where cyclic shear failure (fatigue) could occur. Similarly, liquefaction or cyclic mobility will seldom occur in waterbearing sands.

The designer is occasionally faced with a stratigraphic situation in which the majority of the soil profile consists of either clays or very dense sands, but which contains one or more layers of silt, sand, or gravel having a relative density of less than 70–75%. If such layers occur at a significant depth below the ground surface, the only practical alternative to a shallow foundation system may be a pile or pier foundation, which would be expensive, and which, if not properly designed and installed, could lead to worse performance than the shallow foundation alternate. Therefore, it becomes incumbent on the designer to assess the permanent settlement that would occur beneath the shallow foundation due to vibration-induced compaction in the looser layers. Unfortunately, no rational methods are known to the authors to have been published regarding estimation of permanent settlements of this type. Therefore, an approximate method which has been found to provide a reasonable estimate of the permanent settlement due to vibration is described below.

1. Undisturbed samples are recovered from near the middle of the subject stratum. The *in situ* effective octahedral normal stress is estimated (Equation 4-4), and the specimen is subjected to a drained, controlled stress cyclic triaxial test in which the applied cyclic or "dynamic" stress σ_d, is a simple percentage of the confining pressure σ_c, which is set equal to the *in situ* octahedral normal effective stress. The test is conducted at about 5 cps continuously for several days, and a plot of number of stress cycles (log scale) versus permanent axial strain is made for the value of σ_d/σ_c (λ) used in the test. Typical results are shown in Figure 4-16. According to D'Appolonia (ref. 4), the relationship will be linear, so the test results can be extrapolated to the number of cycles anticipated for any period of service for the machine.
2. Repeat the procedure described above for other samples with varying values of σ_d (and therefore λ).
3. Determine the mean dynamic stress amplitude transmitted to the soil by the vibrating equipment by using Table 1-4 for a range of shear moduli, and divide the transmitted forces so obtained by the contact area of the base of the foundation.

4. For the largest value of transmitted force compute σ_d for the prototype as the vertical stress at the center of the subject layer directly beneath the cen- of the foundation assuming that the dynamic contact stress amplitude computed in Step 3 is distributed within the soil according to Boussinesq theory. Limited evidence (e.g., ref. 5) indicates that the use of Boussinesq theory, though nonrigorous in a dynamic sense, is sufficiently accurate considering the magnitude of uncertainty of the other variables.
5. Compute λ for the value of σ_d obtained in Step 4, and determine the permanent vertical (axial) strain in the layer for the number of stress cycles to be applied in the field using a graph similar to that shown in Fig. 4-16.
6. Compute the permanent settlement as the product of the vertical strain and the layer thickness.

The procedure outlined above will usually give conservative results. Caution should be used in its application, however, particularly where reflective interfaces (e.g., soil-rock interfaces) appear near the layer being studied. If the layer being studied is thick, it should be subdivided into two or more horizontal sublayers; settlement should be ascertained in each sublayer separately; and the settlements should be summed.

In the absence of permanent strain soil test data, a mathematical relationship developed for dry sand undergoing a relatively few stress cycles by Cuellar *et al.* (ref. 3) can be used. Cuellar's relationship should be applied with caution, although results apparently will be very conservative at the large number of stress cycles for which settlement is usually computed and for the state of stress encountered beneath a vibrating footing. The study reported by Cuellar *et al.* revealed that permanent strain is primarily a function of initial relative density, magnitude of shearing strain, and number of cycles of applied load, which may be related by

$$\varepsilon_p = -(1/3m)\ln\,[1 - m\gamma^n N] \qquad (4\text{-}20)$$

where ε_p = permanent vertical strain (negative sign indicates settlement),
N = number of stress cycles,

and

$$m = -33.33\,D_r^2 + 61.66\,D_r - 20 \qquad (4\text{-}21)$$

$$n = -0.95\,D_r^2 + 2.33\,D_r + 0.54 \qquad (4\text{-}22)$$

in which D_r is the relative density expressed as a ratio (not a percentage).

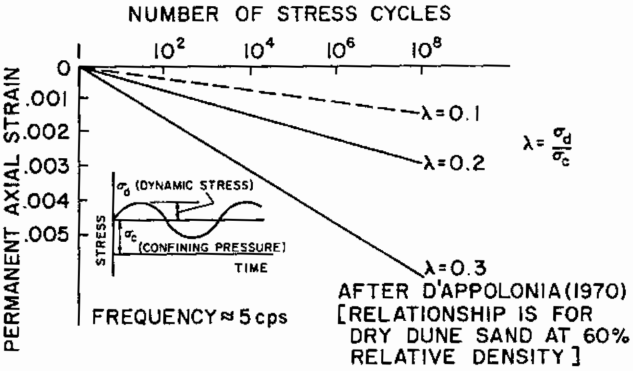


Figure 4-16. Effect of number and amplitude of stress cycles on permanent strain (after ref. 4).

The quantity γ is the maximum shearing strain amplitude (in percent) in simple shear. Since a simple shear condition is not achieved beneath a vibrating footing, it is recommended that γ be taken as the amplitude of the octahedral shearing strain at the location of and for the stress state recommended for the procedure involving the direct use of laboratory data. Therefore,

$$\gamma\,(\%) = (\sqrt{2}/3G)\,(\sigma_d)\,(100) = 47.1\,(\sigma_d/G) \qquad (4\text{-}23)$$

The permanent settlement of the footing is again the product of the permanent vertical strain and the layer thickness.

Settlement in granular soil is, of course, accompanied by a decrease in void ratio, with a resulting increase in shear modulus. For example, according to the Hardin and Richart correlation, a decrease in void ratio from 0.70 to 0.65 yields a 10% increase in G. This, in turn, leads to a 5% increase in the natural frequency of a single-degree-of-freedom system undergoing vertical oscillations. Corresponding resonance shifts can occur in the rotary modes, which can have an effect on the long-term performance of the foundation.

Remarks. The procedures described in this chapter for evaluating soil properties for lumped spring constants and damping ratios should be considered approximate. This fact, coupled with the fact that the frequency independent expressions for spring constants and damping ratios given earlier are themselves approximate, makes it prudent to conduct structure-soil interaction analyses for several combinations of spring and damping values within a reasonable range of uncertainty to assure that resonant conditions are not encountered or, if resonant conditions are encountered, to assure that structural velocities and/or displacements remain within allowable limits.

It is impossible to make a general statement about the range of spring constants and damping ratios that should be employed. However, in most cases it will be sufficient to vary the spring constant and damping ratio in any mode about ±25% with respect to the computed value if high-quality, direct field and/or laboratory measurements of the soil shear modulus and material damping ratio are made and ±50% if correlative methods are used. The above ranges are suggested for essentially uniform sites. If stratigraphic situations are encountered for which no adequate theoretical model exists (such as a footing resting on a hard, compacted, shallow fill overlying a softer subgrade or on a multilayered soil system) spring constants may be assumed to be based on average soil properties, but the uncertainty arising from such an assumption requires that the ranges suggested above be increased by a factor of about 1.5 to 2.0, depending on the degree of soil variability. It is evident that, when an uncertainty pertaining to the model occurs, soil properties should be obtained by direct measurements rather than by correlative methods in order to obtain the best possible values for the inputs and, therefore, reduce the range of spring constants and damping ratios that should be considered in the analysis.

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5 Foundations

Notation for Chapter 5

A = cross-sectional area of pile
A' = corrected contact area
A_e = contact area for discrete element
A_z = amplitude of vertical displacement
c_u = undrained cohesion
c_x = damping factor for horizontal excitation
$c_{x\psi}$ = cross-damping factor
c_z = damping factor for vertical excitation
c_ψ = damping factor for pure rocking excitation
D_t = total damping ratio
D_x = damping ratio for horizontal excitation
D_z = damping ratio for vertical excitation
E = Young's modulus (specifically for pile material when a p subscript is included)
e = eccentricity of load
FS = factor of safety
f = superscript pertaining to pile cap
f_{ij}, f'_{ij} = stiffness and damping terms for piles
G = shear modulus for soil (sometimes subscripted with s to distinguish from other material)
g = gravitational constant; pertaining to group of piles when used as superscript
h = depth of embedment of pile cap
I = moment of inertia
k_b = spring constant for embedded portion of pile
k_f = spring constant for freestanding portion of pile
K_E = relative stiffness
k_r = coefficient of subgrade reaction
k_x = spring constant for horizontal excitation
k_z = spring constant for vertical excitation
k_{ze} = corrected value of k_z
$k_{x\psi}$ = Cross-stiffness term
k_ψ = spring constant for pure rocking excitation
L_b = embedded length of a pile
L_f = freestanding length of a pile
l = length of a fully embedded pile
l_s = relative stiffness dimension for mat
M = mass of cap plus in-phase load on cap
M_r = amplitude magnification factor
m = equivalent vibrating mass
m_c = mass of pile cap
m_e = mass of eccentric load
N = number of piles in group
P = load on pile
Q_0 = unbalanced force
r_0 = effective radius; pile radius
S = pile spacing (center-to-center)
$\bar{S}_{u1}, \bar{S}_{u2}$ = stiffness and damping constants for embedded cap, horizontal excitation
$\bar{S}_1, \bar{S}_2$ = stiffness and damping constants for embedded cap, vertical excitation
$\bar{S}_{\psi1}, \bar{S}_{\psi2}$ = stiffness and damping constants for embedded cap, pure rocking excitation
t = thickness of mat
v_c = compression wave velocity in pile
v_s = shear wave velocity in soil
W_{ep} = equivalent weight of pile
x_r = dimension defined in Figure 5-6
z_c = dimension defined in Figure 5-6
1 = superscript indicating single pile
α_A = axial displacement influence factor
α_L = lateral displacement influence factor
β = direction angle
γ_s = unit weight of soil
γ_p = unit weight of pile material
$\delta = h/r_0$
ν = Poisson's ratio of soil (sometimes subscripted with s to distinguish from other material)
ν_m = Poisson's ratio of mat material

Foundations for dynamically loaded structures usually fall into one of three categories: footings, mats, and deep foundations (i.e., piles and piers). The choice depends on structural loading and geometry and upon the quality of the near-surface soil. In general, the procedures outlined in Chapter 4 can be used directly to analyze footings, which are considered to be essentially rigid and must be designed structurally to be so. When a footing becomes large enough to be classified as flexible (e.g., a "mat" supporting several columns), it is necessary to model it as a series of discrete elements, connected by springs and damping elements and supported by soil springs and damping elements. Since the soil is being loaded not only by a given discrete element but also by surrounding elements, assignment of damping and stiffness values should be based on a somewhat different rationale than for assignment of stiffness and damping values for rigid footings. Deep foundations do not behave as surface foundations and must be treated by special methods. Prediction of the response of pile groups is often complicated by the fact that a rigid cap is placed over the heads of the piles, providing surface stiffness and damping to the pile group.

The understanding of the dynamic behavior of mat, pile, and pier foundations is at best in its infancy. Finite-element modeling encompassing both the soil and the structure represents the most complete and rational mathematical approach to the analysis of such foundations presently available. However, some aspects of soil-structure interaction, such as clastic behavior at the interface between the soil and the structure and absorptive behavior of fictitious boundaries, are difficult to model in a finite-element scheme. Furthermore, the output from such an analysis is not in the form that will allow the designer to easily assess the relative effects of the variables in the problem. Therefore, soil-structure finite-element analyses should be conducted only by experts. The cost of such an analysis usually precludes its use on all but the most important structures or in cases where uncertainties arising from simpler modes of analysis cannot be resolved by other means. A description of the details of dynamic finite-element modeling for soils is beyond the scope of this book.

An approximate procedure is described in this chapter for obtaining equivalent vertical soil spring constants and damping ratios for use in a discrete-element analysis of a large mat or slab in which the soil is not explicitly included in the model, but is instead represented by springs and dashpots, as when a rigid block foundation is being analyzed. The procedure is based on an empirical analogy with static load conditions, since little usable theoretical guidance is available with respect to dynamic response of the soil beneath a flexible mat. In essence, static spring constants are used, a technique which can be partially justified on the basis that at low frequency the dynamic spring constant and the static spring constant do not differ by a large amount and that at high frequency, where the spring constants do differ appreciably, their contribution to the response of the system is subordinate to mass effects.

Several approximate procedures based both on mathematical solutions and on empirical data from vibratory load tests for analyzing the response of pile and pier foundations are also described in this chapter. The mathematical solutions are predicated on the assumption that both the pile and the soil can be represented as elastic materials. As with rigid footings in an elastic halfspace, the problem of modeling flexible piles is reduced to a one-degree-of-freedom problem, allowing expressions to be obtained for equivalent spring and damping constants, which can in turn be input conveniently into a structural analysis program.

The reader is advised to use the procedures presented in this chapter with caution and considerable judgment, as none has been verified in the field in more than a few simple cases. The procedures are therefore presented not in the context of criteria but rather in the context of best available information for the designer.

Modification of Foundation Response

Because of the uncertainties in the mathematical models used to compute foundation response and the uncertainties involved in determining relevant soil inputs to the model, foundation behavior cannot be predicted with the reliability that can be expected for elements of the superstructure. The recommended ranges for foundation response analysis described at the end of Chapter 4 are reflections of these uncertainties. Note that it is often impossible to design a structure, particularly a multi-degree-of-freedom structure, which will not contain one or more foundation elements that will not potentially resonate (or nearly resonate) in some mode, considering the ranges of stiffness and damping that must be designed for. The capable designer, anticipating this problem, will, therefore, plan foundation elements that can be corrected or "tuned" if a vibration problem arises during operation (ref. 9). Foundation response can best be altered during operation by changing the resonant frequency of the foundation. Thus, provisions for subtracting or adding mass to the foundation may be included in final designs. Provisions for altering stiffness are also advisable. Generally, it is easiest to increase, rather than decrease, stiffness once the structure has gone into operation. For example, grout holes can be cast into footings or mats resting on a sandy subgrade so the subgrade can be grouted if necessary and, therefore, stiffened. Consideration may also be given to installing piles and attaching the piles to the footing if a considerable change in foundation stiffness is required during operation. Hence, footing reinforcement in the initial design may be established so the footing can also be employed later as a pile cap, if necessary. Increasing the bearing area of a footing is an effective means of increasing stiffness in all modes. Where space will permit later increases in footing area, original footing reinforcement may be designed to accommodate additional moments that will occur with increased area. Rebalancing or remounting of vibrating machinery to reduce unbalanced forces should always be considered before modifications are made to foundation elements, since the cost of machine modifications is often less than the cost of tuning a foundation.

When rocking is the primary mode of motion, permanent settlements of soil near the periphery of footings can occur after a machine has been in operation for some time, leaving the footing supported on a "fulcrum" of soil near the axis of rocking for the majority of the load cycle (ref. 9). This phenomenon is especially pronounced where the subgrade is cohesionless and the shear strength of the soil, therefore, is low near the edges of the footing due to lack of confinement. The result is a significant decrease in stiffness in the rocking mode. In such a case, grout holes should be cast into the foundation to permit future closing of the space between the footing and soil with low-pressure grout and reestablishment of the original stiffness properties.

Vertical Spring and Damping Constants for Flexible Mats

Agarwal and Hudson (ref. 1) have indicated that the vertical displacements in the vicinity of a static vertical point load in the interior of an elastic, rectangular, prismatic mat supported on an elastic subgrade are not influenced by mat dimensions, as long as the overall horizontal mat dimensions exceed three times a relative stiffness dimension l_s in both directions, where l_s is defined as follows:

$$l_s = [Et^3/12\,(1-\nu_m^2)\,k_r],^{0.25} \qquad (5\text{-}1)$$

where E = Young's modulus of mat material,
t = thickness of mat,
ν_m = Poisson's ratio of mat material,
k_r = coefficient of subgrade reaction of the soil. (kN/m³)
$K_r = P/S$; kN/m²/m

Equivalent criteria for dynamic loading have not been established; however, if it is assumed that the static criterion holds for dynamic loading, then it follows that the soil reaction outside of an area 3 l_s by 3 l_s in the interior of a mat has little or no effect on the response of a relatively smaller discrete element situated at the center of that area. Since the soil spring constant depends on the area of soil being loaded and since a flexible mat is not effective in distributing a load applied to a small area to the entire bearing area of the mat, it is obvious that spring constants for the various discrete elements employed in solving the mat-supported structure problem should not always be calculated using the overall dimensions (or equivalent radius) of the mat. It is also obvious that it is, in general, improper to assume that r_0 obtained from the contact area of a discrete element is appropriate for determining k_z, since doing so would be equivalent to an assumption of Winkler springs, which will result in a system model where stiffness is directly dependent on the choice of element size.

It appears reasonable, therefore, to define a flexible mat as one whose outside dimensions exceed 3 l_s in each direction and which is loaded by columns or machines over a relatively small portion of its area. It is recommended that when a structure is supported on such a mat that a discrete-element model, rather than a rigid block model, be used for the foundation to accommodate its several modes of vibration. To obtain an approximate value for k_z for the individual elements, the effective loaded area of the real (not discretized) mat in the vicinity of the element should be computed assuming that the area of soil loaded due to the vertical motion of the element is 3 l_s by 3 l_s in plan. This area will normally be greater than the area of the element. Utilization of this area for elements near the edges of the mat will lead to errors there; however, considering the lack of information on the subject of lumped soil spring constants for flexible mats, further sophistication is unwarranted.

Using Equation (5-1) and assuming that k_r is that for a loaded area 3 l_s by 3 l_s,

$$r_0 = 0.8t\ [(E/G_s)\ (1-\nu_s)/(1-\nu_m^2)]^{0.333}, \qquad (5\text{-}2)$$

where G_s and ν_s are properties of the soil and r_0 is the effective radius to be used in computing k_z.

The value of k_z can thus be computed from the half-space equation for every element using the value of r_0 given by Equation (5-2) and an appropriately estimated shear modulus and Poisson's ratio for the soil. The value of the spring constant must then be corrected to include only the reaction against the contact area of the element.

This is not a straightforward process, however. For example, if vibrating loads are spaced at such a distance so that effective loaded areas do not overlap (e.g., spacing greater than r_0), then

$$k_{ze} = A'k_z \qquad (5\text{-}3)$$

where $$A' = A_e/\pi r_0^2 \leq 1 \qquad (5\text{-}3a)$$

in which A_e is the contact area of the discrete element and k_{ze} is the appropriate vertical spring constant for the individual element. Equations (5-3) and (5-3a) yield higher soil spring constants than would be obtained by taking the mat as a whole and dividing that value among the discrete elements in proportion to their areas. This is a valid representation for very flexible mats with widely spaced loads because uneven contact stress distributions cause the soil to behave more stiffly in the vicinity of applied loads.

When loads are more closely spaced, k_{ze} is usually less than the value given by Equation (5-3) because a more uniform contact pressure distribution in the primary modes is likely. Its precise value is very difficult to determine and will, in fact, vary from mode to mode, depending on the phase differences between adjoining elements and the spacing of the loads. As a limiting value, it can be assumed that the soil responds as if the mat were rigid; hence,

$$k_{ze} = (A_e/A_m)\,k_z \qquad (5\text{-}4)$$

in which A_m is the area of the entire mat and k_z is the vertical spring constant for the entire mat. In most cases, k_{ze} will lie between the values given by Equations (5-3) and (5-4) and judgment should be exercised in its selection. Most usage in the past has favored Equation (5-4).

The soil geometric damping ratio is more difficult to evaluate in a flexible mat than the spring constant because of phase differences in various parts of the mat. An upper limit for the geometric damping ratio for the soil will be that computed from halfspace theory assuming that a section of mat and in-phase loads supported thereupon having a radius equal to r_0 computed from Equation (5-2) vibrates as a rigid body. However, interference from surrounding sections of the mat makes the use of the upper limit value entirely unreasonable and the correct value virtually unpredictable using the halfspace approach. A value of $D_t = 0.15$ for discrete elements is suggested in the absence of any other criteria if the subgrade is homogeneous.

Example. A 100-ft-square mat is 3 ft thick. Its Young's modulus is 3×10^6 psi, and its Poisson's ratio is 0.2. The mat rests upon a clay subgrade having a shear modulus of 8000 psi and a Poisson's ratio of 0.4. What value of k_{ze} should be used for a 10-ft-square discrete element?

Using Equation (5-3):

$$r_0 = 0.8t\,[\,(E/G_s)\,(1 - \nu_s)\,/\,(1 - \nu_m^2)\,]^{0.333}$$

$$= 0.8(3)\,[\,((3 \times 10^6)/(8 \times 10^3))\,(0.6/0.96)\,]^{0.333}$$

$$= 14.8 \text{ ft}$$

$$k_z = 4Gr_0/(1 - \nu_s) = 4(8000)\,(14.8)\,(12)/0.6$$

$$= 9.5 \times 10^6 \text{ lb/in.}$$

$$k_{ze} = [\,(10 \times 10)/(\pi \times 14.8^2)\,]\,k_z$$

$$= 1.38 \times 10^6 \text{ lb/in.}$$

Using Equation (5-4):

$$r_0 = (100 \times 100/\pi)^{0.5} = 56.4 \text{ ft}$$

$$k_z = (4 \times 8000 \times 56.4 \times 12)/(1 - 0.4)$$

$$= 36 \times 10^6 \text{ lb/in.}$$

$$k_{ze} = (36 \times 10^6) \times (10 \times 10)/(100 \times 100)$$

$$= 0.36 \times 10^6 \text{ lb/in.}$$

Deep Foundations

Pile or pier foundations (which hereafter will be termed simply "pile" foundations) are often used to support vibratory loads when soil conditions at a site indicate that shallow foundations will result in unacceptable permanent settlements. Present understanding of the behavior of pile foundations under vibratory loading is relatively poor, but it is known that the use of piles decreases geometric damping, increases the resonant frequency of the foundation, and influences deformation near resonance. Since in some cases, particularly in the lateral load mode, the effect of piles can be adverse, piles should not be used without some understanding of their behavior.

In this chapter approximate procedures developed by Novak and his associates for analyzing the response characteristics of single piles and pile groups in the uncoupled vertical, horizontal, and pure rocking modes are described. The solutions are based on the assumption of elastic, fully embedded vertical piles interacting with uniform elastic soil. Furthermore, the pile tip is assumed to be fixed against motion, except in the case of vertical response, where the tip can be fixed or "relaxed."

The limitations of applying Novak's procedures to real geological materials are obvious; however, they have provided reasonable predictions for the response of small pile groups in relatively simple soil profiles, and furthermore, they represent the current (1979) state-of-the-art in the practical analytical treatment of pile-soil interaction. The solutions for spring and damping constants developed by Novak are frequency dependent; however, approximate frequency independent expressions have also been developed for both the spring constant and damping ratio. These will be described herein. Solutions for the stiffness and damping of torsionally loaded individual piles and for coupled rocking and sliding, obtained by Novak and his associates, are also available (refs. 10 and 14), but are not included here.

Novak's various procedures do not permit calculation of the stresses induced in the pile material, although such stresses can be important, particularly during lateral loading. Ghazzaly, Hwong and O'Neill (ref. 5) describe a numerical algorithm for making stress computations in a pile undergoing harmonic lateral loading. A detailed description of that algorithm is beyond the scope of this text.

Both the spring constants and damping ratios should be obtained experimentally from full-sized groups of test piles whenever possible, especially if batter piles are contained in the foundation or if short friction piles are to be employed. (Novak's procedures for pile group analysis do not consider batter piles directly in expressions for overall group stiffness and damping; however, Saul's procedure, referenced at the end of this chapter, among others, does allow consideration of batter piles in an approximate fashion.) Furthermore, according to Novak and Grigg (ref. 13), the "apparent" shear modulus to be used for analyzing laterally loaded piles should be that value backcalculated from elastic beam on foundation theory (e.g., ref. 17) using the results of a static load test after several cycles of load have been applied. The initial slope of the free-head load versus displacement curve can be used conveniently in the calculations. The reasons for using this approach are that the soils very near the surface control the load deformation properties of the pile. Field surveys and laboratory tests are often conducted to obtain only a mean site value for G, which is generally appropriate only for a depth greater than that of the soil effective in resisting lateral motion. Also, since a gap often forms behind a laterally loaded pile, the use of a shear modulus obtained from load test data provides a convenient empirical correction to Novak's elastic solution, which is not rigorous for such a case. In a series of large-scale model tests in sand, Novak and Grigg (ref. 13) determined that the apparent shear modulus from static lateral pile tests was about one fourth to one fifth the mean modulus to a depth of about 30 pile diameters, obtained for the site from oscillator tests.

Simple methods of interpreting field load test data are discussed briefly at the end of this chapter. Approximate procedures for estimating damping ratios and spring constants based on results of several vibration tests reported in the literature for vertically excited friction piles are also given.

Two important rules should be followed relative to the sizing and construction of deep foundations. First, the factor of safety relative to the ultimate static axial load on a pile should exceed 3 in order to restrict soil stresses to a magnitude that will preclude the "resonant driving" effect, wherein the soil around the pile resonates with the pile and loses its ability to carry an appreciable load. If the static load is too large (e.g., more than one half the static capacity of the soil), plunging or load shedding can result as the soil resonates with the pile.

Second, the pile cap should be buried in competent soil (preferably dense granular soil) whenever possible in order to take advantage of the damping afforded by the pile cap.

Vertical Motion

The expression for the effective spring constant for a single end-bearing ("fixed-tip") pile undergoing vertical motion, k_z^1, given by Novak (ref. 10) is

$$k_z^1 = (E_pA/r_0)f_{18,1} \tag{5-5}$$

where E_p is the Young's modulus of the pile material, A is the cross-sectional area of the pile, r_0 is the equivalent radius of the pile, and $f_{18,1}$ is a factor given by Figure 5-1a (concrete piles) or Figure 5-1b (timber piles) as a function of ratios of pile penetration l to radius r_0 and v_s/v_c (shear wave velocity in soil above tip/compression wave velocity in pile). Note that the factors given in Figures 5-1a and 5-1b are for fixed tip piles (end bearing or combined friction and end bearing piles). For friction piles these factors will be in error by a relatively small amount for values l/r_0 greater than 60 and for values for v_s/v_c greater than about 0.03 for timber and concrete piles. For friction piles having lower values of l/r_0 and/or v_s/v_c, the procedure for "relaxed tip" piles (ref. 11) or an empirical approach, both described later, should be employed. For steel piles Novak and Grigg (ref. 13) have given a value of $f_{18,1} = 0.030$ where $v_s/v_c = 0.033$ (medium stiff soil) and $l/r_0 > 80$.

The effective geometric damping constant for vertical motion in a single pile c_z^1 is given by Equation (5-6):

$$c_z^1 = (E_pA/v_s)\, f_{18,2} \tag{5-6}$$

where v_s is the shear wave velocity of the soil through which the pile is driven ($\sqrt{G_s g/\gamma_s}$) and $f_{18,2}$ is a factor given by Figures 5-1a and 5-1b. The damping factors $f_{18,2}$ are also approximately valid for friction piles in the ranges described previously relative to the stiffness factors $f_{18,1}$, although the error in the damping factor is somewhat greater. For a steel pile in medium stiff soil ($v_s/v_c = 0.033$) and for $l/r_0 > 80$, $f_{18,2} \approx 0.045$.

For relatively short friction piles (that is, piles with a "relaxed tip"), Novak (ref. 10) suggests computing k_z^1 and c_z^1 as follows:

$$k_z^1 = (E_pA/r_0)\, f'_{18,1} \tag{5-5a}$$

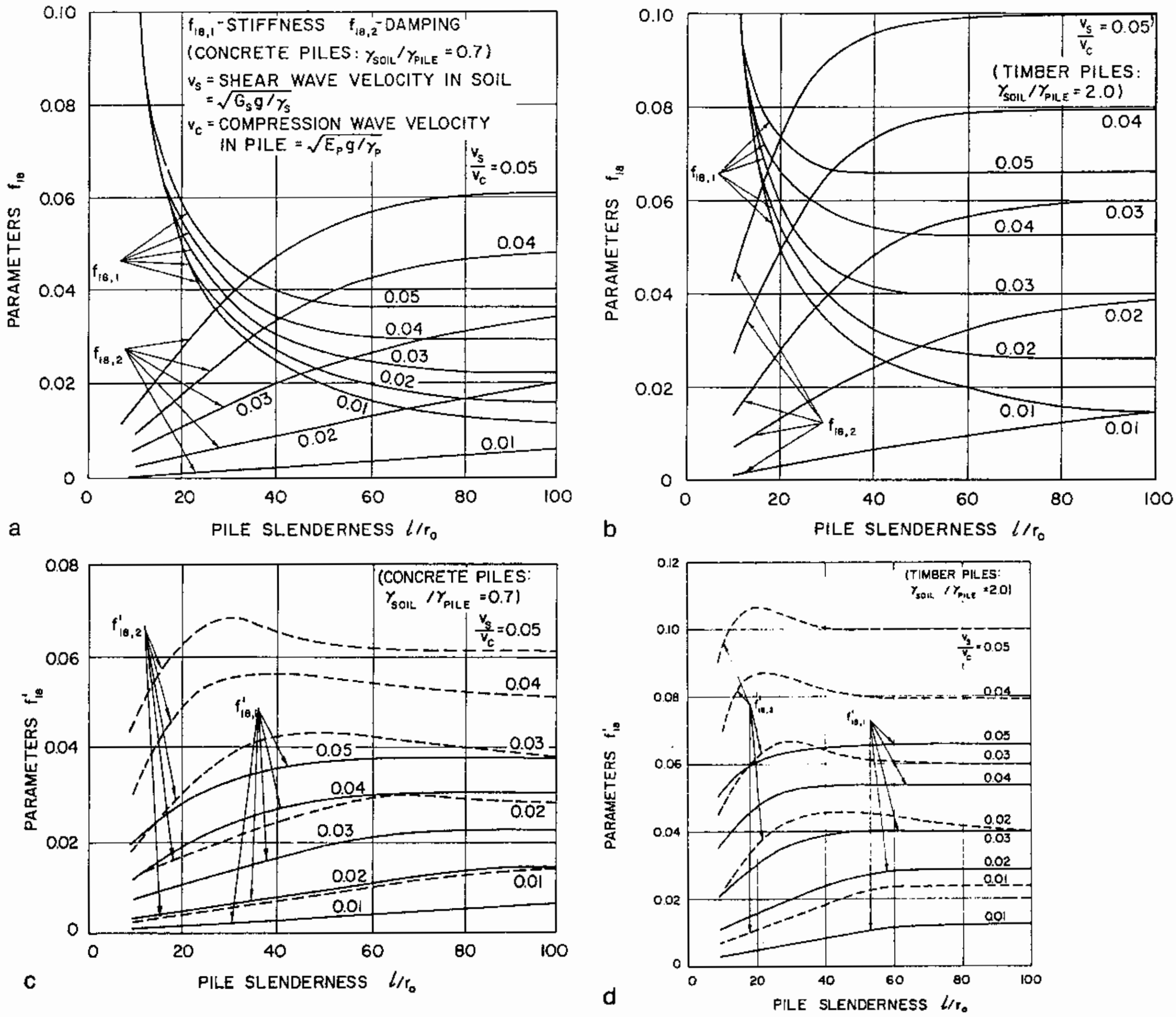


Figure 5-1. Stiffness and damping factors for vertically excited piles (after refs. 10 and 11). a. Fixed-tip concrete piles. b. Fixed-tip timber piles. c. Relaxed-tip concrete piles. d. Relaxed-tip timber piles. Figures 5-1a and 5-1b reproduced by permission of the National Research Council of Canada from the Canadian Geotechnical Journal, Vol. 11 (1974), p. 586.

and $c_z^1 = (E_p A/v_s) f'_{18,2}$ (5-6a)

where $f'_{18,1}$ and $f'_{18,2}$ are stiffness and damping factors, respectively, given in Figures 5-1c and 5-1d for concrete and timber piles. The values in Figures 5-1a, b, c, and d are most appropriate for use for a_0 ($= 2\pi f r_0/\sqrt{G_s g/\gamma_s}$) between 0.1 and 0.8 where f is the machine frequency. Stevens (ref. 20) suggests that they are valid even for a_0 as low as 0.05, which means that reasonable results can be anticipated for slender piles and low forcing frequencies.

The geometric damping ratio for a single pile supporting a structure can be computed from the damping constant by using Equation (5-7):

$$D_z^1 = c_z^1/2\sqrt{k_z^1 m_c} \quad (5\text{-}7)$$

where m_c is the mass of the cap plus machinery or portion of the structure vibrating in phase with the cap. It may also be appropriate to include part of the pile mass in m_c, but for fully embedded piles, the ratio of that mass to the mass of the supported weight is usually small enough to be neglected.

Pile Groups

Most piles are installed in groups or clusters. The group stiffness will not in general be the simple sum of the stiffnesses of the individual piles. A similar statement can be made relative to damping. If several distinct groups are used to support a vibrating superstructure, it is reasonable to obtain an equivalent spring constant and damping ratio for each cluster. These equivalent constants may then be applied to the structure at the appropriate supports for purposes of analysis of structural response. An example of this approach is a frame whose columns are supported by separate pile groups.

Often, however, a block or rigid mat foundation will be supported by a single large group of piles. In such a case it is often convenient to represent the structure-foundation system by Model C, described in Figure 2-9, Chapter 2, where the entire group is modeled by a single spring constant and a single damping ratio in each mode of vibration.

Whenever piles are used to support a flexible mat, when large torsional moments are applied to the superstructure, or when batter piles are present, it is usually necessary to employ a multidegree-of-freedom computer model (Fig. 2-11) and to represent each individual pile in the system by assigning linear support spring constants and dashpots to the point in the structure where the pile head is located. These spring constants represent uncoupled vertical stiffness and horizontal stiffnesses in two perpendicular directions. When batter piles are present, axial stiffness and damping parameters may be computed from the "vertical" motion equations, and lateral parameters may be computed from the "horizontal" motion equations. Input values for stiffness and damping can then be obtained by taking components in the coordinate directions. The horizontal spring constants (described later) are strongly dependent on the manner in which the piles are fixed to the structure. For block foundations and table top structures containing a thick mat (which also serves as the pile cap), the pile heads are normally assumed to be fixed rigidly to the structure. While frame analysis programs allow the inclusion of uncoupled rotational stiffness terms at the pile head, it usually is necessary to include rotational springs for rocking about transverse pile axes only when large rocking moments are present and when batter piles are not present in the foundation. It is rarely necessary to include a torsional stiffness constant for an individual pile, since torsional moments applied to the structure are resisted almost completely by couples produced by lateral reactions at the pile heads. While the piles are represented individually in this type of analysis, it is still desirable to include the softening effect of group action in the model. This can be done by calculating the stiffness for the pile group as a whole in the appropriate mode, distributing the stiffness equally among the piles (presuming that all piles have the same size and penetration), and using the distributed stiffness values in the analysis. In other words, it is not appropriate to use the values given by Eqs. 5-5 or 5-5a or the corresponding equations presented in the following section on single pile horizontal response.

The damping ratio to be assigned each pile for each mode of translatory motion in a multi-degree-of-freedom-foundation representation should include a consideration of group action, which will result in a higher damping ratio for each pile than would be obtained using the single pile damping ratio. Typically, each pile in the system would be assigned a damping ratio equal to the group damping ratio in a given mode divided by the square root of the number of piles in the group.

In the remainder of this section simple equations are given for the determination of group stiffness k_z^g and damping D_z^g for a group of vertical piles oscillating in the vertical mode. Similar expressions will be presented in the next section for horizontal translatory stiffness and damping for vertical pile groups. Note that in evaluating group effects for the latter mode it is necessary to estimate the direction of primary horizontal motion. Since no general solutions are available for the assessment of group effects in groups containing batter piles, some judgement is necessary to apply these expressions to batter pile foundations.

Novak and Grigg (ref. 13) have argued that the deflection factors proposed by Poulos (ref. 15) for groups of statically loaded piles may also be applied to a pile group undergoing steady-state vibration. Hence, Novak and Grigg propose that

$$k_z^g = \frac{\sum_1^N k_z^1}{\sum_1^N \alpha_A} \tag{5-8}$$

where N = number of piles in group

α_A = axial displacement interaction factor for a typical reference pile in the group relative to itself and to all other piles in the group, assuming the reference pile and all other piles carry the same load.

The factor α_A can be evaluated using Figure 5-2.

The equivalent geometric damping ratio for the group is given by

$$D_z^g = \frac{\sum_1^N c_z^1}{2\sqrt{\sum_1^N k_z^1\, m_c}\ \sqrt{\sum_1^N \alpha_A}} \tag{5-9}$$

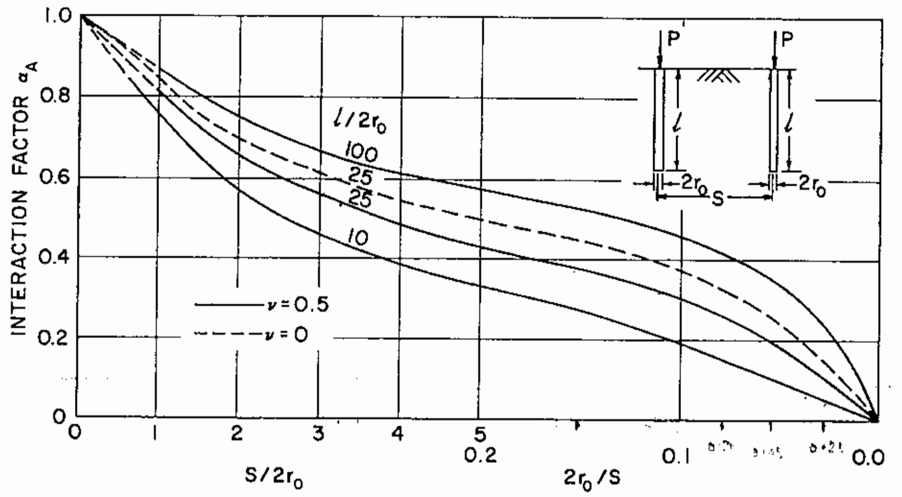


Figure 5-2. α_A as a function of pile length and spacing (after ref. 15).

where m_c is defined as in Equation (5-7), that is, the total mass of the cap plus machinery.

If the pile cap is not in contact with the ground, Equations (5-8) and (5-9) can be used directly in the structural analysis. Embedment of the pile cap, however, has a favorable effect on the response of the group and should be employed whenever possible. It is good practice to assume that embedment is effective only in the development of side friction between the cap and soil and only when dense granular backfill is used, since soil beneath the base of the cap is likely to be of poor quality and can settle away from the cap. Likewise, cohesive backfill can shrink away from the sides and become ineffective.

Novak and Beredugo (ref. 12) have given expressions for calculating stiffness and geometric damping constants for the embedded cap, which are added to the stiffness and damping values obtained in Equations (5-8) and (5-9) to arrive at total system stiffness and damping for a group of piles. Those expressions are given here as Equations (5-10) (stiffness, k_z^f) and (5-11) (damping, c_z^f) with only the side resistance component considered. Expressions for caps or footings in complete contact at the base can be found in Novak and Beredugo (ref. 12).

$$k_z^f = G_s\, h\, \bar{S}_1 \tag{5-10}$$

$$c_z^f = h r_o \sqrt{G_s\, \gamma_s / g}\;\; \bar{S}_2 \tag{5-11}$$

In Equations (5-10) and (5-11), h is the depth of embedment of the cap, r_0 is the equivalent radius of the cap, G_s and γ_s are the shear modulus and total unit weight of the backfill, and $\bar{S}_1$ and $\bar{S}_2$ are constants given in Table 5-1, in which ν_s is the Poisson's ratio of the backfill soil.

Table 5–1
Frequency Independent Constants for Embedded Pile Cap with Side Resistance (after refs. 3 and 12)*

ν_s	$\bar{S}_1$	$\bar{S}_2$	$\bar{S}_{u1}$	$\bar{S}_{u2}$	$\bar{S}_{\psi 1}$	$\bar{S}_{\psi 2}$
0.0	2.7	6.7	3.6	8.2	2.5	1.8
0.25	2.7	6.7	4.0	9.1	2.5	1.8
0.4	2.7	6.7	4.1	10.6	2.5	1.8

*Values are appropriate for $a_0 = 2\pi f r_0 / \sqrt{G_s g/\gamma_s}$ in the range $0 < a_0 \leq 2$ where f is the machine frequency. $\bar{S}_{u1}$ is not constant for $\nu_s > 0.43$. Reproduced by permission of the National Research Council of Canada from the *Canadian Geotechnical Journal*, Vol. 9 (1972), p. 495.

An interesting comparison from a theoretical study of the vertical response of a machine and its foundation when the foundation is embedded and when it is placed on the surface without embedment was developed by Novak. The foundation consisted of a solid rectangular block of concrete (16 ft long, 10 ft wide, 8 ft high) embedded 2 ft and not embedded. It was supported on eight 35-ft long fixed-tip timber piles in a medium stiff clay. The machine supported on the block weighed 10 tons and the operating speed was varied. The response of the pile foundation system is shown in Figure 5-3. In that figure, A_z is the static deflection that is produced by the unbalanced force at a given frequency, M is the mass of the footing (cap) plus machine, and m_e and e are the unbalanced mass and its eccentricity within the machine. The effects of the relatively small amount of embedment are evident.

Also shown in Figure 5-3 are the response curves for the machine and rectangular block footing without piles. The shapes of the various curves show vividly important aspects of pile-supported structures: (1) damping is very low compared to soil-supported footings and (2) the use of piling increases the resonant frequency and, in this case, increases displacement amplitude at resonance. Damping can be increased by embedding the pile cap. Material damping was not considered in this particular analysis.

Example. Determine the equivalent spring constant and damping ratio for vertical motion for the pile group depicted in Figure 5-4. Assume that Novak's solution for fixed-tip piles is valid.

$$v_s = (G_s g/\gamma_s)^{0.5} = [5000\ (144)\ (32.2)/110]^{0.5}$$
$$= 459 \text{ ft/sec} = 5510 \text{ in./sec}$$

$$v_c = (E_p g/\gamma_p)^{0.5} = \{[(3.5 \times 10^6)/150]\ (144)\ (32.2)\}^{0.5}$$
$$= 10{,}400 \text{ ft/sec}$$

$$v_s/v_c = 0.044;\ m_c = (70{,}000 + 12 \times 12 \times 8 \times 150)/386$$
$$= 629 \text{ (lb-sec}^2\text{)/in.}$$

$$r_o = [(20 \times 20)/\pi]^{0.5} = 11.3 \text{ in.}$$
$$l/r_o = 90\ (12)/11.3 = 95.6$$

$$f_{18,1} = 0.033 \qquad f_{18,2} = 0.053$$

$$k_z^1 = (E_p A/r_o)\ f_{18,1} = [3.5 \times 10^6\ (20)^2/11.3]\ (0.033)$$
$$= 4.0 \times 10^6 \text{ lb/in.}$$

$$c_z^1 = (E_p A/v_s)\ f_{18,2} = [3.5 \times 10^6\ (20)^2/5510]\ (0.053)$$
$$= 1.3 \times 10^4 \text{ lb-sec/in.}$$

$$k_z^f = G_s\ h\ \bar{S}_1 = 6000\ (6 \times 12)\ (2.7)$$
$$= 1.2 \times 10^6 \text{ lb/in.}$$

$$r_o(\text{cap}) = (12 \times 12/\pi)^{0.5} = 6.77 \text{ ft} = 81.2 \text{ in.}$$

$$c_z^f = h r_o\ (G_s \gamma_s/g)^{0.5}\ \bar{S}_2$$
$$= (6 \times 12)(81.2)[6000(120)/(1728)(386)]^{0.5}(6.7)$$
$$= 4.1 \times 10^4 \text{ lb-sec/in.}$$

Adjust for group effects. Assuming any pile in the group is a reference pile with $r_0 = 11.3$ in., $S/2r_0$ to an adjacent corner pile is $72/22.6 = 3.19$ and to the diagonally opposite pile is $1.414\ (3.19) = 4.50$. Using Figure 5-2:

$l/2r_o = 90(12)/22.6 = 48$
Use $\nu = 0.5$ (nearest to actual value of 0.4):
$\alpha_A = 1$ (reference pile)

$\alpha_A = 0.58$ (adjacent corner piles) (2 piles)
$\alpha_A = 0.5$ (opposite corner pile)

$$\sum_1^4 \alpha_A = 1 + 2\ (0.58) + 0.5 = 2.66$$

$$\sum_1^4 k_z^1 \Big/ \sum_1^4 \alpha_A = \frac{4(4.0 \times 10^6)}{2.66} = 6.0 \times 10^6 \text{ lb/in.}$$
(combined stiffness of the piles)

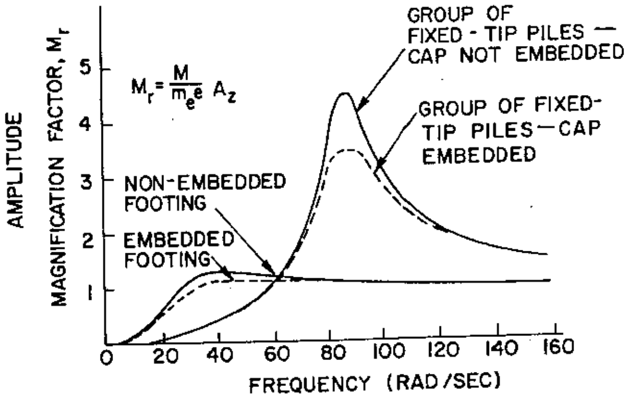
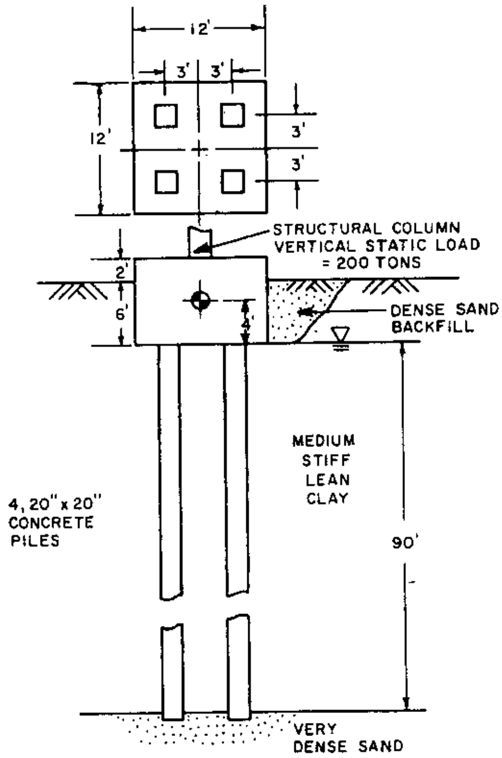


Figure 5-3. Effect of type of support on vertical response of a foundation (after ref. 10). Reproduced by permission of the National Research Council of Canada from the Canadian Geotechnical Journal, Vol. 11 (1974), p. 589.



MATERIAL PROPERTIES

LEAN CLAY:	G_s = 5000 psi, ν_s = 0.4 γ_s = 110 pcf
BACKFILL:	G_s = 6000 psi, γ_s = 120 pcf
RIGID CAP:	γ_{CAP} = 150 pcf
PILES:	$E_p = 3.5 \times 10^6$ psi γ_p = 150 pcf

(ASSUME 35T OF THE STATIC COLUMN LOAD VIBRATES WITH THE CAP IN VERTICAL MODE)

Figure 5-4. Pile group for example problem.

Adding in the stiffness of cap due to side resistance:

$$k_z^g = 6.0 \times 10^6 + 1.2 \times 10^6 = 7.2 \times 10^6 \text{ lb/in.}$$

$$c_z^g = \sum_1^4 c_z^1 \Big/ \left(\sum_1^4 \alpha_A\right)^{0.5} + c_z^f$$

$$= \frac{4(1.3 \times 10^4)}{(2.66)^{0.5}} + 4.1 \times 10^4$$

$$= (3.2 + 4.1) \times 10^4 = 7.3 \times 10^4 \text{ lb-sec/in.}$$

$$D_z^g = c_z^g/[2(k_z^g m_c)^{0.5}]$$
$$= (7.3 \times 10^4)/2(7.2 \times 10^6 \times 629)^{0.5} = 0.54$$

Note that over half of the damping is produced by the embedded cap. Material damping is not included.

Horizontal Motion

The solutions for pure horizontal motion of vertical piles (ref. 10) follow the same logic as those for vertical motion. A notable exception is that a reduced value for G_s should be used to represent the action of soil against pile as described earlier. For a single pile,

$$k_x^1 = (E_p I/r_o^3)\, f_{11,1} \qquad (l/r_o \geqslant 25) \tag{5-12}$$

$$c_x^1 = (E_p I/r_o^2\, v_s)\, f_{11,2} \qquad (l/r_o \geqslant 25) \tag{5-13}$$

where I is the moment of inertia of the pile cross-section about a centroidal axis perpendicular to the direction of translation. The subscript x denotes horizontal motion, and $f_{11,1}$ and $f_{11,2}$ are factors for fixed-head piles given in Table 5-2.

For a group of piles,

$$k_x^g = \sum_1^N k_x^1 \Big/ \sum_1^N \alpha_L \tag{5-14}$$

$$D_x^g = \sum_1^N c_x^1 \Big/ \left[2\sqrt{\sum_1^N k_x^1\, m_c}\sqrt{\sum_1^N \alpha_L}\,\right] \tag{5-15}$$

where α_L is a displacement factor for lateral motion defined in an analagous fashion to α_A. Approximate values for α_L may be obtained from Figure 5-5.

Finally, the stiffness and geometric damping characteristics of the cap are represented as follows:

$$k_x^f = G_s h \bar{S}_{u1} \tag{5-16}$$

$$c_x^f = h r_o \sqrt{G_s\, \gamma_s/g}\;\; \bar{S}_{u2} \tag{5-17}$$

where $\bar{S}_{u1}$ and $\bar{S}_{u2}$ are factors given in Table 5-1.

Uncoupled Rocking Motion

Novak (ref. 10) has presented expressions for the uncoupled stiffness and geometric damping constants for single piles and for pile groups undergoing pure rocking. In summary,

$$k_\psi^1 = (E_p I/r_o)\, f_{7,1} \tag{5-18}$$

$$c_\psi^1 = (E_p I/v_s)\, f_{7,2} \tag{5-19}$$

where I is the moment of inertia of the pile cross-section about the axis of rotation, and $f_{7,1}$ and $f_{7,2}$ are factors given in Table 5-2.

For a pile group,

$$k_\psi^g = \sum_1^N [k_\psi^1 + k_z^1 x_r^2 + k_x z_c^2 - 2 z_c k_{x\psi}^1] + k_\psi^f \tag{5-20}$$

where x_r and z_c are defined in Figure 5-6, and k_z^1 and k_x^1 are stiffness constants for single piles defined in Equations (5-5) and (5-12), respectively. In addition,

$$k_{x\psi}^1 = (E_p I/r_o^2)\, f_{9,1} \tag{5-21}$$

where $f_{9,1}$ is obtained from Table 5-2; and

$$k_\psi^f = G_s r_o h^2 \bar{S}_{\psi 1} + G_s r_o^2 h\, [(\delta^2/3) + (z_c/r_o)^2 - \delta(z_c/r_o)]\, \bar{S}_{u1} \tag{5-22}$$

where $\delta = h/r_o$, and $\bar{S}_{\psi 1}$ and $\bar{S}_{u1}$ are defined in Table 5-1. Note that it is necessary to include $k_{x\psi}^1$, a cross-stiffness term, in the solution. Note also that interaction factors (α) are not included in the solution for group stiffness because group action in pure rocking is not as prevalent as in the translational modes and because adequate studies have not been conducted to established appropriate interaction factors for the rocking mode.

Finally,

$$c_\psi^g = \sum_1^N [c_\psi^1 + c_z^1 x_r^2 + c_x^1 z_c^2 - 2 z_c c_{x\psi}^1] + c_\psi^f \tag{5-23}$$

where c_z^1 and c_x^1 are damping constants for individual piles given by Equations (5-6) and (5-13), respectively, and

$$c_{x\psi}^1 = (E_p I/r_o v_s)\, f_{9,2} \tag{5-24}$$

The factor $f_{9,2}$ is evaluated in Table 5-2. Further,

$$c_\psi^f = \delta r_o^4 \sqrt{G_s\, \gamma_s/g}\, \{\bar{S}_{\psi 2} + [(\delta^2/3) + (z_c/r_o)^2 - \delta(z_c/r_o)]\, \bar{S}_{u2}\} \tag{5-25}$$

where $\bar{S}_{\psi 2}$ and $\bar{S}_{u2}$ are defined in Table 5-1.

Table 5–2
Values of $f_{11,1}$; $f_{11,2}$; $f_{7,1}$; $f_{7,2}$; $f_{9,1}$; $f_{9,2}$ for $l/r_0 > 25$ (after ref. 10)*

			Concrete Piles (γ_s/γ_p = 0.7)				
ν_s	v_s/v_c	$f_{11,1}$	$f_{11,2}$	$f_{7,1}$	$f_{7,2}$	$f_{9,1}$	$f_{9,2}$
0.4	0.01	0.0036	0.0084	0.202	0.139	−0.0194	−0.0280
	0.03	0.0185	0.0438	0.349	0.243	−0.0582	−0.0848
	0.05	0.0397	0.0942	0.450	0.314	−0.0970	−0.1410
0.25	0.01	0.0032	0.0076	0.195	0.135	−0.0181	−0.0262
	0.03	0.0166	0.0395	0.337	0.235	−0.0543	−0.0793
	0.05	0.0358	0.0850	0.435	0.304	−0.0905	−0.1321

			Timber Piles (γ_s/γ_p = 2)				
ν_s	v_s/v_c	$f_{11,1}$	$f_{11,2}$	$f_{7,1}$	$f_{7,2}$	$f_{9,1}$	$f_{9,2}$
0.4	0.01	0.0082	0.0183	0.265	0.176	−0.0336	−0.0466
	0.03	0.0425	0.0949	0.459	0.305	−0.1010	−0.1400
	0.05	0.0914	0.2040	0.592	0.394	−0.1680	−0.2330
0.25	0.01	0.0074	0.0165	0.256	0.169	−0.0315	−0.0434
	0.03	0.0385	0.0854	0.444	0.293	−0.0945	−0.1301
	0.05	0.0828	0.1838	0.573	0.379	−0.1575	−0.2168

*Values are appropriate for a_0 = 0.3 (See Table 5-1), but are approximately valid (±10%) for $0.1 \le a_0 \le 0.8$. Reproduced by permission of the National Research Council of Canada from the *Canadian Geotechnical Journal*, Vol. 11 (1974), p. 584.

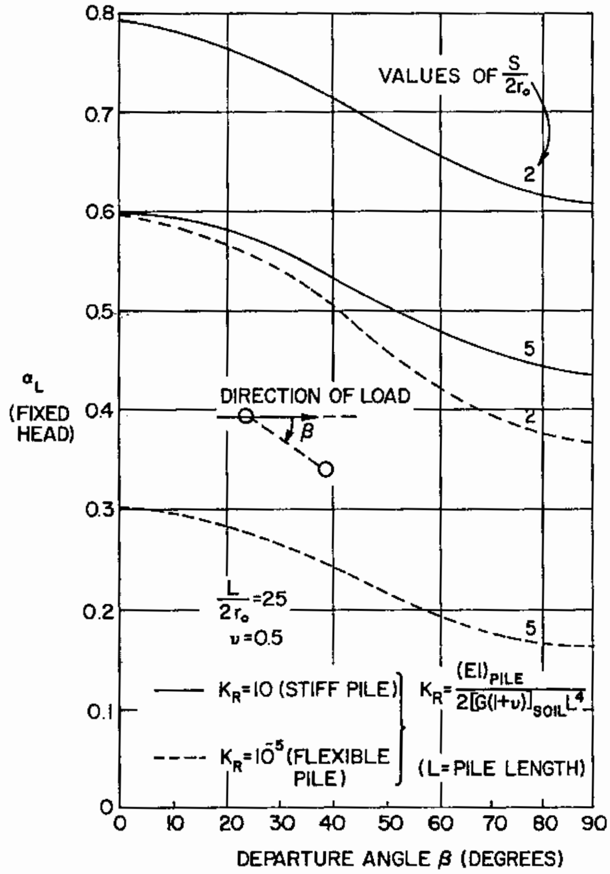


Figure 5-5. Graphical solution for α_L (after ref. 16).

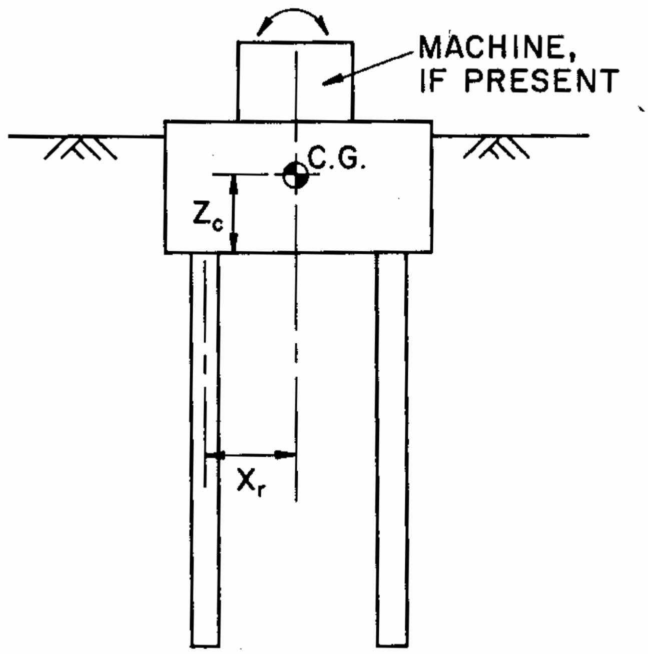


Figure 5-6. Definition of x_r and z_c.

The geometric damping ratio can then be computed as the ratio of the damping constant to the critical damping constant for rocking motion.

Testing Methods and Empirical Correlations Based on Tests

Vibration tests can be conducted on single piles or pile groups by placing a variable frequency steady-state oscillator on a rigid cap atop the pile or piles and measuring the amplitude of deformation in the direction of loading over a range of driving frequencies. Such tests can be conducted most easily for the case in which the applied load is vertical. They are beneficial in determining both the resonant frequency and the damping ratio for friction piles either singly or in groups, particularly where a group contains batter piles.

Results of vertical steady-state vibration tests on large, partially embedded single piles reported by Hart (ref. 6) are shown in Figure 5-7, in which

L_f = freestanding length of pile
L_e = embedded length of pile
f_r = resonant frequency
FS = factor of safety based on ratio of ultimate static capacity to static weight on the test pile
Q_o = unbalanced force at any frequency
k_z = stiffness of pile, defined by Equation (5-28).

The shape of the curves is characteristic for piles without massive, buried pile caps: they have a sharp resonance peak indicative of low total damping.

For a partially buried single pile, the spring constant k_z can be evaluated by combining the spring constants for the embedded and freestanding portions (ref 2 and 20):

$$k_f = AE_p/L_f \tag{5-26}$$

$$k_b = AE_p/L_e \tag{5-27}$$

where k_f is the vertical stiffness constant for the freestanding portion, k_b is the vertical stiffness constant for the embedded portion, and the remaining factors are defined in Figure 5-8. It then follows that

$$k_z = k_f k_b/(k_f + k_b) \tag{5-28}$$

Alternatively, k_z can be obtained by numerical pile load-settlement synthesis techniques (ref. 20). In fact, use of Equation 5-28 or numerical synthesis techniques is preferable to the use of Equations 5-5 or 5-5a when a_0 is less than 0.05.

Based on analysis of test results (ref. 4, 6, 7), the lumped mass m of the pile can be taken approximately as

$$m = M + (W_{ep}/g) \tag{5-29}$$

in which W_{ep} and M are defined in Figure 5-8.

The approximate resonant frequency, then, is given by Equation (5-30):

$$f_r = \frac{1}{2\pi}\sqrt{k_z/m} \tag{5-30}$$

It is possible to compute the approximate total damping ratio from steady-state vibration tests. In the vertical mode,

$$D_t = [(Q_o/k_z)/A_z]_{\text{resonance}} \times 0.5 \tag{5-31}$$

From Hart's test (Figure 5-7), D_t can be seen to be approximately 0.02 for the steel pile and 0.035 for the concrete pile. Lacking direct test data, D_t for a single pile can be calculated as Novak's D_z (geometric damping only) or may conservatively be taken to be approximately 0.025. With D_t known, the amplitude of vibration at resonance is given by Equation (5-32):

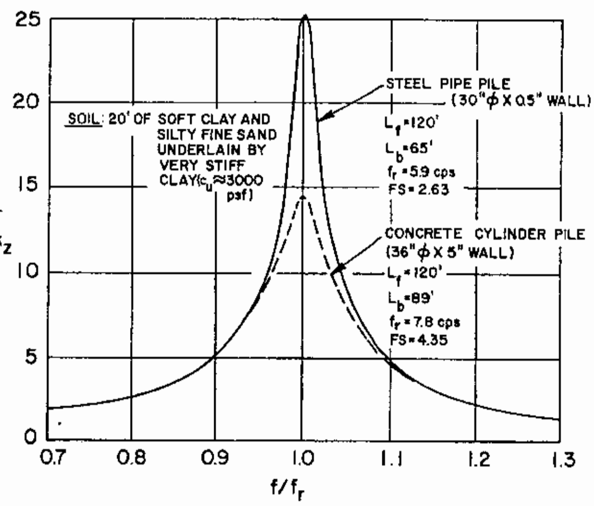


Figure 5-7. Vertical response curves for two partially embedded friction piles (after ref. 6).

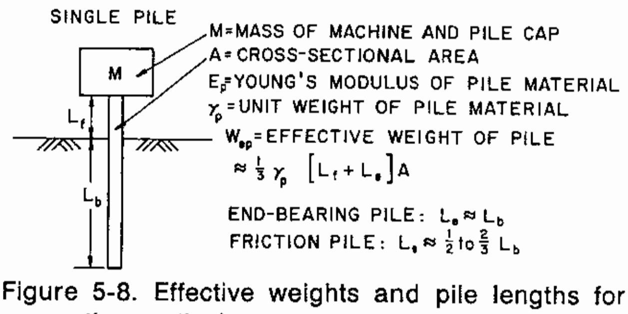


Figure 5-8. Effective weights and pile lengths for computing vertical response of partially embedded piles (after ref. 2).

$$(A_z)_{\text{resonance}} = (Q_o/k_z)\,(1/2D_t) \qquad (5\text{-}32)$$

where Q_0 is the unbalanced force at the resonant frequency computed from Equation (5-30).

When vertical piles are present in small groups (2-6 piles) with suspended caps and when the spacing exceeds three times the effective diameter of the piles, analysis of test data presented by Maxwell, Fry and Poplin (ref. 7) indicates that the vertical response of the group can be predicted using the equations just presented for single piles with the modifications indicated in Figure 5-9. The relationship of group damping to single pile damping for groups containing 2-6 piles is similar to that described in the section on group effects. If the cap is adequately embedded, its damping ratio should be included in the group damping.

Equivalent vibrating lengths and weights of piling and mass properties of the pile cap can be used to compute resonant frequencies for pile groups containing six degrees of freedom in numerical techniques described by Saul (ref. 18) and Singh, Donovan, and Jobsis (ref. 19). Saul's method does not permit computation of amplitudes of motion at resonance because damping is not considered; however, the method developed by Singh, Donovan, and Jobsis does permit inclusion of a single overall damping factor. These methods, which require the off-line use of a separate digital computer program, are useful in studying the effects of batter piles on resonant frequency in the horizontal translatory and rocking modes.

Although the various correlative techniques outlined in this section are useful, uncertainties relative to their application should be emphasized. For example, soil stratification and the state of residual stress within a pile after driving can have a profound effect on k_z (or L_e) and, therefore, on f_r and A_z at resonance. Residual stresses generally make the stiffness of a vertically loaded pile much greater for very small loads, in the normal range of unbalanced dynamic loads, than would be implied from using L_e of 0.5 to 1.0 times L_b. Whenever possible, therefore, both vibratory load tests, to measure f_r and A_z at resonance, and very low amplitude static load tests (to no more than 10% of the frictional capacity of the pile), to measure k_z, should be conducted on test piles or pile groups for purposes of obtaining k and D for designing prototype foundations.

Comparison of Theory and Measured Behavior

Few published case studies are available which compare the results of the methods described in this chapter with performance of foundations. One such study, conducted by plucking 90-ft-long concrete-filled step taper friction piles both horizontally and vertically and singly and in a four-pile square group is reported in ref. 8. The piles in the group were spaced three butt diameters on center, and the cap was suspended above the soil. The piles were driven through layers of very stiff, overconsolidated saturated clay with a few sand seams. The shear modulus of the soil layers as measured by crosshole tests varied from about 5000 psi to about 30,000 psi, with an average of 20,000 psi. The stiffness of the piles, which vibrated at a relatively low frequency, was computed using static numerical synthesis procedures based on methods described in refs. 15 and 16, as opposed to using Equations (5-5) and (5-12), because a_0 was very low. Damping constants were determined from the procedures developed by Novak and his associates. The measured and predicted spring constants and damping ratios for several repeated tests are shown in Figures 5-10 and 5-11. Note that damping in these fully embedded step taper piles was considerably higher than that measured by Hart for partially embedded prismatic piles.

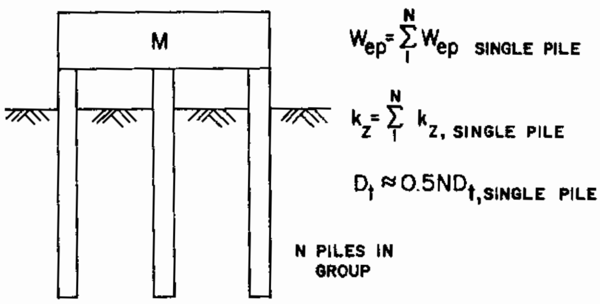


Figure 5-9. Effective weights and lengths for vertically loaded piles in a group.

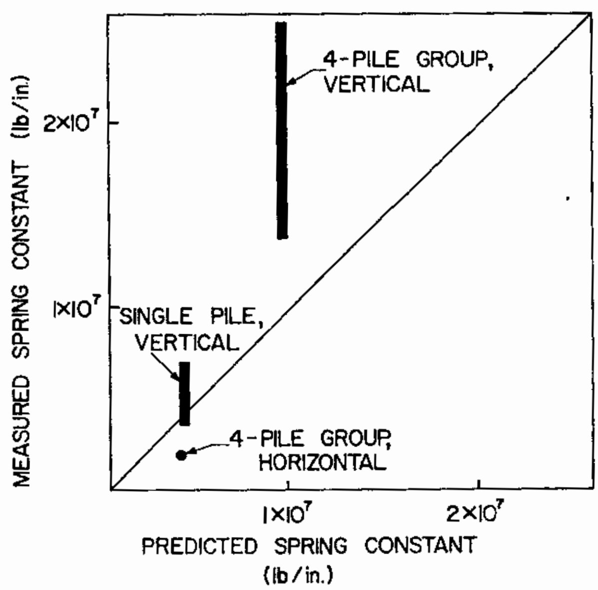


Figure 5-10. Comparison of measured and predicted stiffness for piles in multiple tests (after ref. 8).

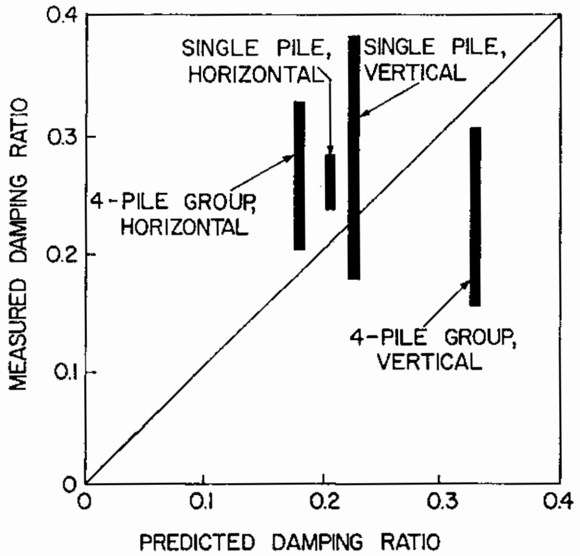


Figure 5-11. Comparison of measured and predicted damping for piles in multiple tests (after ref. 8).

The trend in the tests was that measured stiffness exceeded computed stiffness in the vertical mode in the initial test but the difference in measured and computed stiffnesses decreased as further tests were conducted, possibly due to relief of residual driving stresses in the piles. No such trend could be observed for stiffness in the horizontal mode or with the damping ratio in either mode. Note also that little difference in measured damping existed between single-pile vibration and group vibration. Figures 5-10 and 5-11 serve to underscore previous statements concerning the uncertainties in dynamic response analysis of foundations and the desirability of providing for foundation tuning capability.

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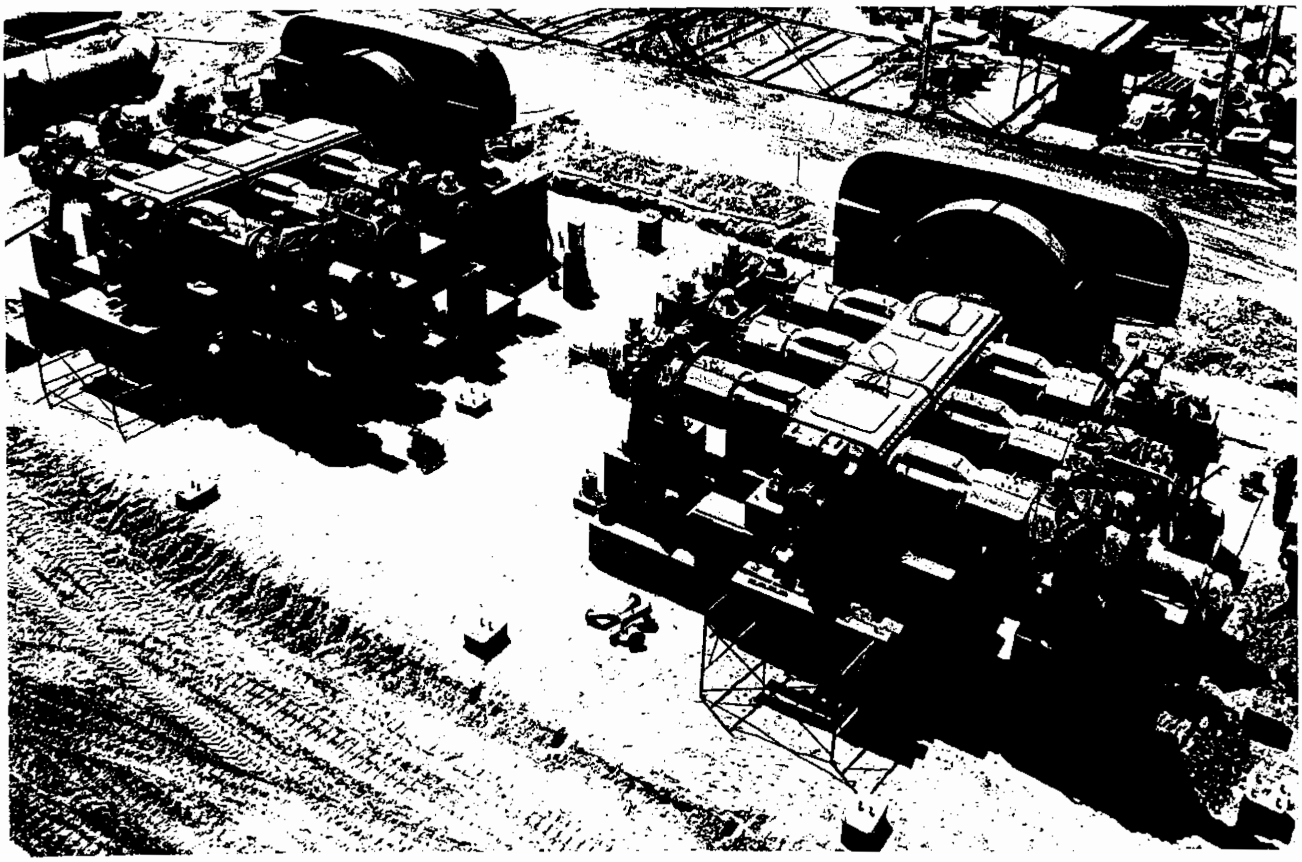
6 Design Examples: Block Foundations

Three block foundation design examples are presented in this chapter. These examples use the theory and information developed in previous chapters. The selected foundations are typical and commonly used in many industrial plants. The examples follow a standard format which includes a series of steps so that a thorough design is accomplished without the danger of missing any necessary check. Cross references have been made to other parts of this book at each design step in order to illustrate the utilization of previously derived formulae.



Horizontally mounted reciprocating compressors on block foundation.

Example 1: Foundation Design for Reciprocating Compressor (Footing Embedment Effect Included) (ref. 1)

A. Introduction

Reciprocating compressors are relatively heavy machines and generate vibrating forces of substantial magnitude at low operating frequencies. The operating frequencies usually lie very close to the natural frequencies of the foundation in the various vibrating modes, thus creating resonance conditions in the foundation system. The magnitude of vibration amplitude at resonance condition becomes a controlling criteria because of the closeness of operating and natural frequencies. Therefore, inclusion of the effects of internal and geometrical damping during oscillation becomes an important consideration, and this can only be accomplished by using the elastic half-space theory.

In this theory, the footing is assumed to rest on the surface of the elastic half-space and to have simple geometrical areas of contact, usually circular, but other shapes such as rectangular or long strip can also be handled with some simplification, as decribed in Reference 4. This theory includes the dissipation of energy throughout the half-space by "geometric damping" and allows calculation of a finite amplitude of vibration at the "resonant frequency," (ref. 1). The method is an analytical procedure which provides a rational means of evaluating the spring and damping constants for incorporation into lumped-parameter, mass-spring-dashpot vibrating systems, as described in Chapter 4. Recently, this theory has been extended to account for the effect of depth of embedment of the foundation on the values of spring constant and the damping ratio. The information presented in Chapter 4 is used in the examples that follow.

Reciprocating machines. Machinery involving crank mechanisms, such as piston-type compressors and pumps, internal combustion engines, and pumps, produce reciprocating forces. A single cylinder engine is inherently unbalanced; however, in multicylinder engines and compressors, it is possible to select the size of cylinders and to arrange them in such a manner that the resulting unbalanced forces are minimized (ref. 5). Unbalanced forces and couples for different crank arrangements but of equal cylinder bore and stroke are given in Table 6-1. However, note that in addition to the primary frequency either the vertical or horizontal forces and couples may generate a secondary frequency which depends upon the orientation of the machine.

Vibration modes. A rigid block foundation supporting a vibrating machine can experience up to six modes of vibration as shown in Figure 6-1. Three modes are trans-

Table 6–1
Unbalanced Forces and Couples for Different Crank Arrangements (ref. 5)

Crank Arrangements	Forces		Couples	
	Primary	Secondary	Primary	Secondary
Single crank	F' without counterwts. $(0.5)F'$ with counterwts.	F''	None	None
Two cranks at 180° In-line cylinders	0	$2F''$	$F'D$ without counterwts. $\frac{F'}{2}D$ with counterwts.	None
Opposed cylinders	0	0	Nil	Nil
Two cranks at 90°	$(1.41)F'$ without counterwts. $(0.707)F'$ with counterwts.	0	$(1.41)F'D$ without counterwts. $(0.707)F'D$ without counterwts.	$F''D$
Two cylinders on one crank Cylinders at 90°	F' without counterwts. 0 with counterwts.	$(1.41)F''$	Nil	Nil
Two cylinders on one crank Opposed cylinders	$2F'$ without counterwts. F' with counterwts.	0	None	Nil
Three cranks at 120°	0	0	$(3.46)F'D$ without counterwts. $(1.73)F'D$ with counterwts.	$(3.46)F''D$
Four cylinders Cranks at 180°	0	0	0	0
Cranks at 90°	0	0	$(1.41)F'D$ without counterwts. $(0.707)F'D$ with counterwts.	$4.0F''D$
Six cylinders	0	0	0	0

r = crank radius (in.)
L = connecting-rod length (in.)
D = cylinder-center distance (in.)
W = recip. wt. of one cylinder (lb.)
$F' = (0.0000284)\, rW\, (\text{rpm})^2$ = Primary

$F'' = \frac{r}{L} F'$ = Secondary

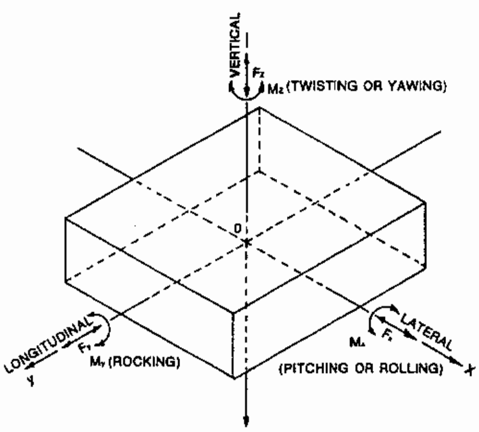


Figure 6-1. Six vibration modes of a block-type foundation (Translational modes: vertical, longitudinal, lateral. Rotational modes: twisting, rocking, pitching.)

latory (the vertical, lateral and longitudinal modes) and three modes are rotational (twisting or yawing, rocking, and pitching or rolling modes). The vertical and twisting vibration modes are usually independent. However, if rocking oscillation caused by the eccentric location of vertical and horizontal forces on the foundation is possible, then the vertical and twisting motions are always coupled with the rocking motion. In many practical problems, vertical, lateral, and rocking modes exhibit the greatest influence on the overall motion and are generally considered independent of each other in the analytical solution. The results can then be superimposed. In some special problems, where the center of gravity (C.G.) of the foundation system along the vertical axis is substantially higher than the center of resistance offered by the soil to horizontal forces, coupled modes (rocking and lateral) must be considered. A rheological representation of the modes of oscillation is shown in Figure 6-2.

B. Machine Parameters

The following information is supplied by the machine vendor: (refer also to Chapter 3 for development of information).

Vertical reciprocating compressor (four cylinders) and auxiliary equipment:

Compressor	28,115 lbs.
Gas Coolers	4,350 lbs.
Snubbers	7,010 lbs.
Motor	18,000 lbs. (rotor weight = 6,000 lbs.)
Total Machine Weight	57,475 lbs.

Dynamic forces (Figures 6-3 and 6-4)

a. Compressor speed, primary (operating) = 585 rpm, secondary = 1,170 rpm

Max. Vertical	Primary Force F_z	= 1,329 lbs.
Max. Vertical	Secondary Force F_z	= 553 lbs.
Max. Horizontal	Primary Force F_x	= 725 lbs.
Max. Rocking	Primary Moment T_ψ	= 11,304 lbs.-ft
Max. Pitching	Primary Moment T_ϕ	= 34,000 lbs.-ft
Max. Pitching	Secondary Moment T_ϕ	= 12,350 lbs.-ft

b. Motor speed = 585 rpm. Motor dynamic forces are negligible.

Note: Superposition of primary and secondary forces results in a non-harmonic forcing function a period equal to that of the primary motion. For simplicity, one may assume a single harmonic forcing function with the maximum amplitude equal to the sum of the amplitudes of the primary and secondary forces acting with the primary frequency. In this example however, a complete analysis is performed using primary and secondary forcing functions separately.

C. Soil and Foundation Parameters

The required soil and foundation parameters are obtained from the soil report and the facility's plot plan.

Plant Grade El.	= 100′-0″
Top of Foundation El.	= 100′-6″
Recommended Foundation Base El.	= 95′-6″
Soil Stratum is Medium Dense Silty Sand with Gravel	
Soil Density (γ)	= 117 pcf
Shear Modulus (G)	= 14,000 psi
Poisson's Ratio (ν)	= 0.35
Soil Internal Damping Ratio ($D_{\psi i}$)	= 0.05
Static Allowable Bearing Capacity, S_{all}	= 2.5 ksf
Permanent Settlement of Soil	= 0.2 in. at 2.5 ksf

D. Selection of a Foundation Configuration

Trial sizing of the supporting block follows the suggested guidelines 1 and 2(a) through 2(g) given in Chapter 3.

Try a shallow and wide footing such that the combined center of gravity of mass of machines and of footing coincides in plan with centroid of the contact area of footing (Figure 6-4). It is also recommended that at least 80% of the footing thickness should be embedded in the soil to restrain the translation movement of the footing. Note that the effective foundation embedded depth h is taken as 3 ft, i.e., the full 4 ft embedded depth minus the top 1 ft layer.

Concrete Footing Trial Outline (see Figure 6-4)

Weight of the footing (W_F) = 324,843 lbs.

Total static load (W) = machine weight + weight of footing = 382,318 lbs.

Weight of footing/weight of machine = 324,843/57,475 = 5.65 > 5 O.K.

Actual soil pressure = 382,318/15.75(27.5) = 883 psf $< 0.5\, S_{all} = 1{,}250$ psf

Thus trial area of footing is O.K.

All other guidelines for trial sizing are checked and found satisfactory.

A dynamic analysis check is then performed on the trial foundation. The various steps in this procedure are listed in Table 6-2.

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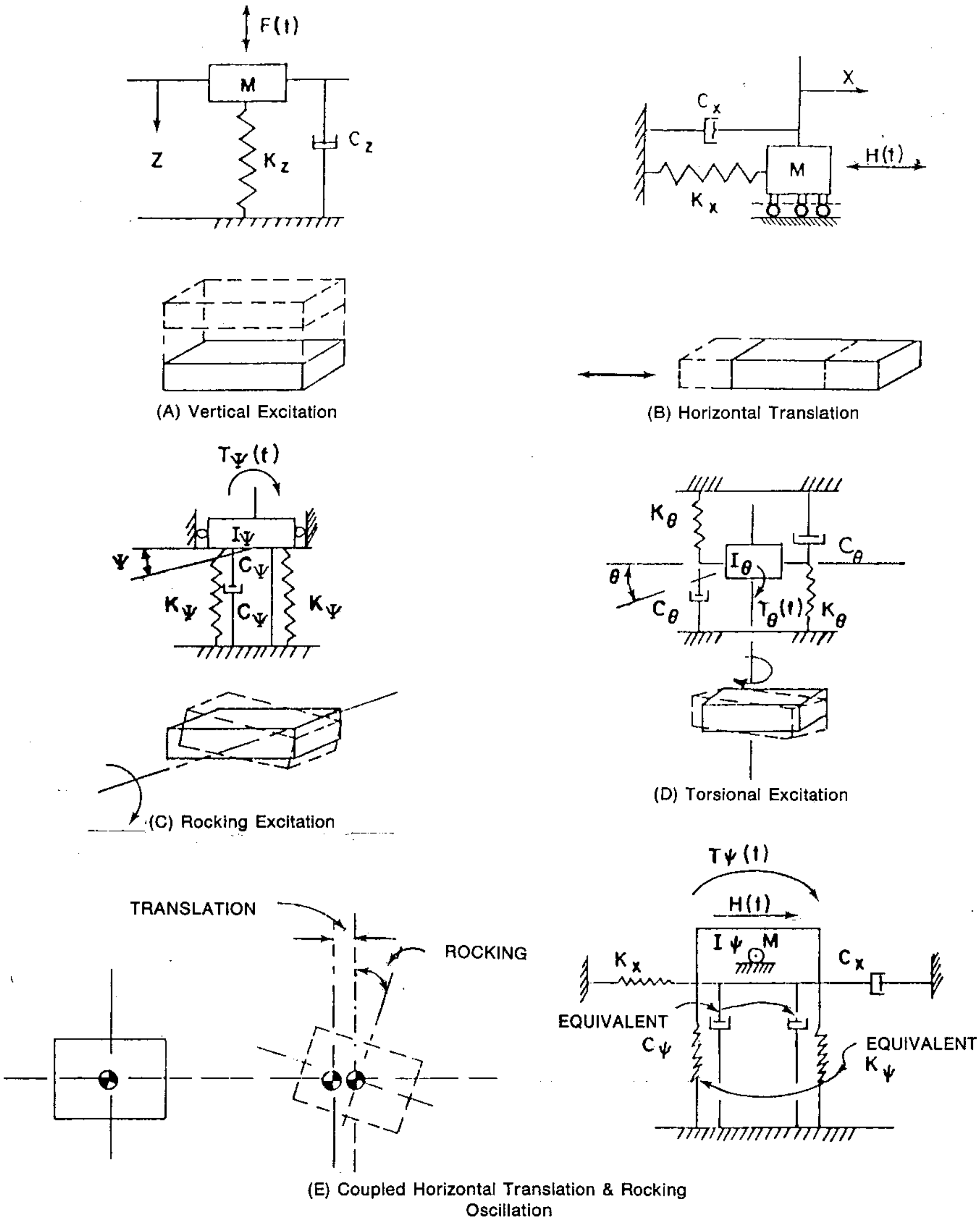


Figure 6-2. Modes of oscillation.

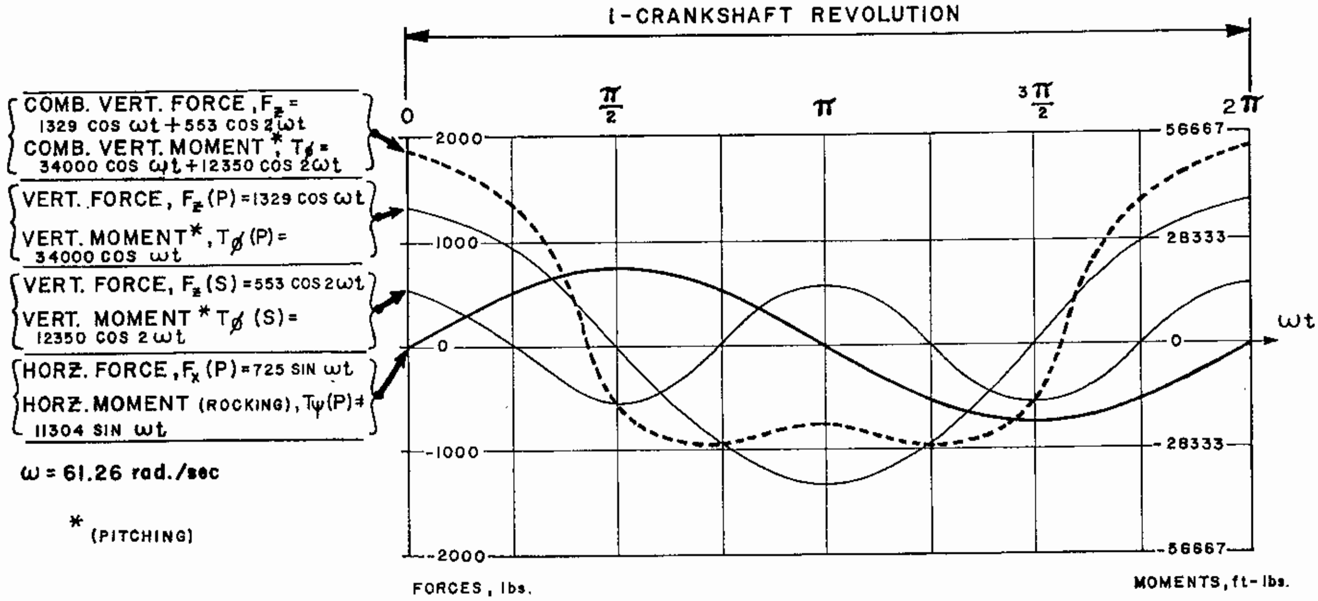


Figure 6-3. Unbalanced vertical and horizontal forces and moments.

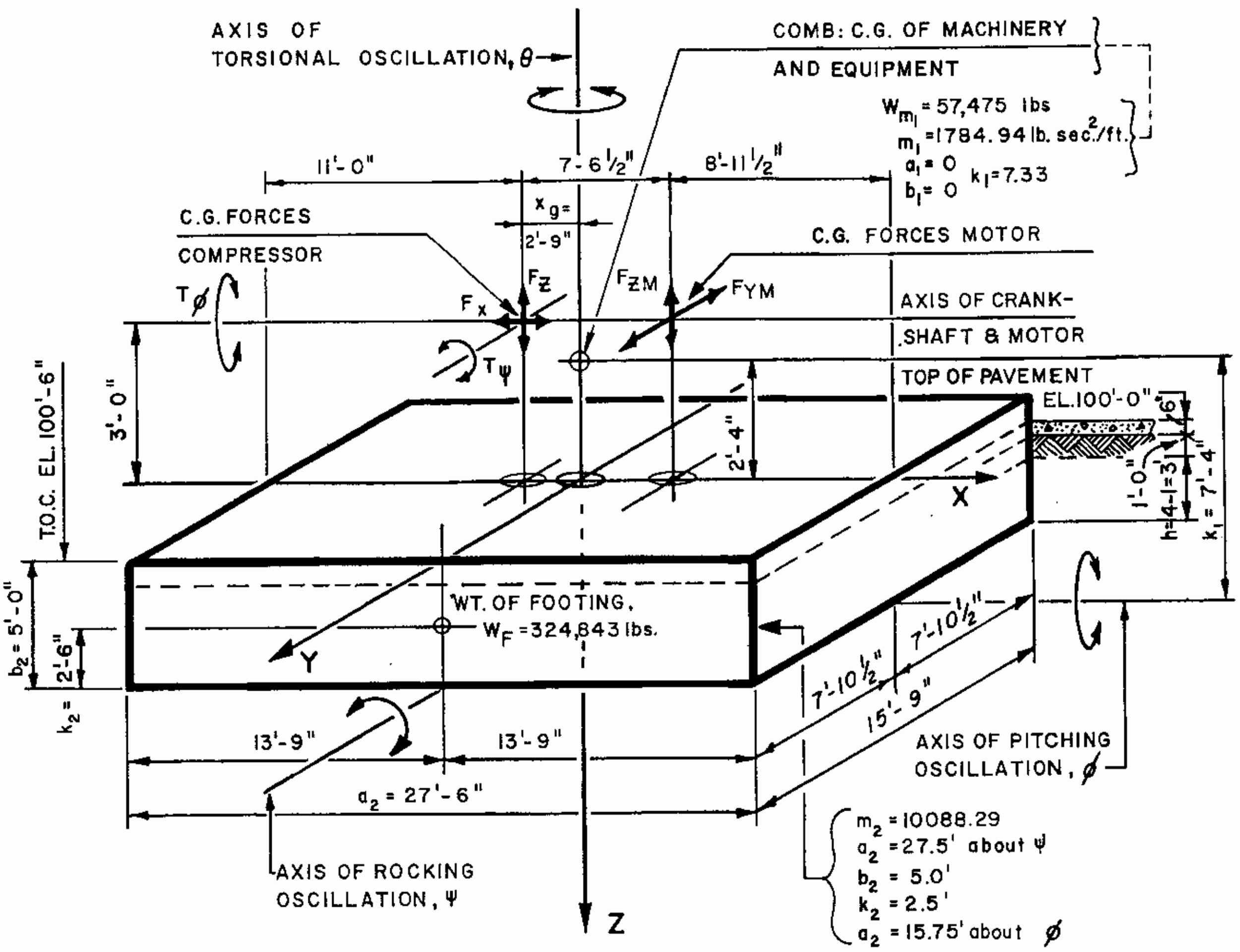


Figure 6-4. Foundation layout for reciprocating machine example problem.

Table 6–2
Dynamic Analysis of the Trial Foundation

Step	Parameter	Reference Source	Vertical Excitation (z-Direction)	Horizontal Excitation (x-Direction)	Rocking Excitation (ψ-Direction)	Pitching Excitation (ϕ-Direction)
1	Mass, and mass moment of inertia	$m = W/g$ $I_{\psi,\phi} = \sum_i^n \left[\frac{m_i}{12}(a_i^2 + b_i^2) + m_i k_i^2\right]$ Table 4-6	$m = 382{,}318/32.2$ $= 11{,}873.2$ lbs.-sec²/ft	$m = 382{,}318/32.2$ $= 11{,}873.2$ lbs.-sec²/ft	I_ψ (machine) = 95,902.9 I_ψ (footing) = 719,841.5 815,744.4 lbs.-sec²-ft	I_ϕ (machine) = 95,902.9 I_ϕ (footing) = 292,612.9 388,515.8 lbs.-sec²-ft
2	**Spring Constant** (a) Eqvl. radius (rect. ftg), r_o	Table 4-2	$r_o = 11.74$ ft	$r_o = 11.74$ ft	$L = 27.50$ ft $B = 15.75$ ft $r_o = 13.65$ ft	$L = 15.75$ ft $B = 27.50$ ft $r_o = 10.33$ ft
	(b) Embedment factor for spring constant	Table 4-2	$Eff.h = 4.5 - 1.5 = 3$ $\eta_z = 1.100$	$\eta_x = 1.232$	$\eta_\psi = 1.175$	$\eta_\phi = 1.235$
	(c) Spring constant coefficient	Figure 4-1	$\beta_z = 2.15$	$\beta_x = 0.95$	$\beta_\psi = 0.58$	$\beta_\phi = 0.45$
	(d) Eqvl. spring constant (rect. footing)	Table 4-1	$k_z = 152.66 \times 10^6$ lbs./ft	$k_x = 132.59 \times 10^6$ lbs./ft	$k_\psi = 25.176 \times 10^9$ lbs.-ft/rad	$k_\phi = 11.758 \times 10^9$ lbs.-ft/rad
3	**Damping Ratio** (a) Embedment factor	Table 4-4	$\alpha_z = 1.254$	$\alpha_x = 1.623$	$\alpha_\psi = 1.024$	$\alpha_\phi = 1.041$
	(b) Mass ratio	Table 4-3	$B_z = 0.328$	$B_x = 0.408$	$B_\psi = 0.115$	$B_\phi = 0.222$
	(c) Effective damping coefficient	Table 4-5			$n_\psi = 1.6$	$n_\phi = 1.581$
	(d) Geometrical damping ratio	Table 4-3	$D_{zg} = 0.931$	$D_{xg} = 0.732$	$D_{\psi g} = 0.302$	$D_{\phi g} = 0.195$
	(e) Internal damping	soil data	$D_{zi} = 0.05$	$D_{xi} = 0.05$	$D_{\psi i} = 0.05$	$D_{\phi i} = 0.05$
	(f) Total damping	$(d) + (e)$	$D_z = 0.981$	$D_x = 0.782$	$D_\psi = 0.352$	$D_\phi = 0.245$
4	(a) Natural freq. (rpm)	$\frac{60}{2\pi}\sqrt{\frac{k}{m}}$	$f_{nz} = \frac{60}{2\pi}\sqrt{\frac{152.66 \times 10^6}{11{,}873.2}}$ $= 1{,}082.8$	$f_{nx} = \frac{60}{2\pi}\sqrt{\frac{132.59 \times 10^6}{11{,}873.2}}$ $= 1{,}009.1$	$f_{n\psi} = \frac{60}{2\pi}\sqrt{\frac{25.176 \times 10^9}{815{,}744.4}}$ $= 1{,}677.6$	$f_{n\phi} = \frac{60}{2\pi}\sqrt{\frac{11.758 \times 10^9}{388{,}515.8}}$ $= 1{,}611.2$
	(b) Resonance freq. (rpm)	Table 1-4 (rotating mass)	$2D^2 > 1$, resonance not possible	$2D^2 > 1$, resonance not possible	$f_{m\psi} = 1{,}934.3$	$f_{m\phi} = 1{,}717.6$
	(c) Vibrating force (max. absolute ampl.)	machine parameter	$F_{zo}(P) = 1{,}329$ lbs. @ $\omega t = 0, 2n\pi$ $F_{zo}(P) = 0$ @ $\omega t = (2n+1)\pi/2$ $F_{zo}(S) = 553$ lbs. @ $\omega t = 0, n\pi/2$	$F_{xo}(P) = 725$ lbs. @ $\omega t = (2n+1)\pi/2$ $F_{xo}(P) = 0$ @ $\omega t = n\pi$	Primary @ $\omega t = (2n+1)\pi/2$ $T_{\psi o}(P) = 11{,}304$ $+ F_{zo}(P)\,X_g = 0$ $+ F_{xo}(P)\,X_g =$ $725 \times 8 = 5{,}800$ 17,104 lbs.-ft. Secondary @ $\omega t = (2n+1)\pi/2$ $F_{zo}(S)\,Z_g = 553 \times 2.75$ $= 1{,}521$ lbs-ft	$T_{\phi o}(P) = 34{,}000$ lbs.-ft @ $\omega t = 0, 2n\pi$ $T_{\phi o}(S) = 12{,}350$ lbs.-ft @ $\omega t = 0, n\pi/2$
	(d) Magnification factor	Table 1-4	$M_z(P) = 0.784$ @ 585 RPM $M_z(S) = 0.470$ @ 1,170 RPM $M_{z\text{Max}}$ not possible	$M_x(P) = 0.890$ @ 585 RPM $M_{x\text{Max}}$ not possible	$M_\psi(P) = 1.111$ @ 585 RPM $M_\psi(S) = 1.454$ @ 1,170 RPM	$M_\phi(P) = 1.128$ @ 585 RPM $M_\phi(S) = 1.690$ @ 1,170 RPM
	(e) Displacement response	Table 1-4	$Z = \Sigma M_z F_{zo}/k_z$ $= \frac{.784 \times 1329 + .470 \times 553}{152.66 \times 10^6}$ $= 8.528 \times 10^{-6}$ ft $= 0.10233 \times 10^{-3}$ in.	$X = M_x F_{xo}/k_x$ $= \frac{.890 \times 725}{132.59 \times 10^6}$ $= 4.867 \times 10^{-6}$ ft $= 0.058398 \times 10^{-3}$ in.	$\psi = \Sigma M_\psi T_\psi/k_\psi$ $= (1.111 \times 17{,}104$ $+ 1.454 \times 1{,}521)$ $\div 25{,}176 \times 10^9$ $= 0.8428 \times 10^{-6}$ rad	$\phi = \Sigma M_\phi T_\phi/k_\phi$ $= (1.128 \times 34{,}000$ $+ 1.690 \times 12{,}350)$ $\div 11.758 \times 10^9$ $= 5.0369 \times 10^{-6}$ rad
	(f) Total displacement response		$Z_t = Z + \psi \cdot R_{hx} + \phi \cdot R_{hy}$ $= 0.10233 \times 10^{-3}$ $+ 0.8428 \times 10^{-6} \times 165$ $+ 5.0369 \times 10^{-6} \times 94.5$ $= 0.717 \times 10^{-3}$ in.*	$X_t = X + \psi \cdot R_v$ $= .058398 \times 10^{-3}$ $+ 0.8428 \times 10^{-6} \times 88$ $= 0.133 \times 10^{-3}$ in. $Y_t = \phi \cdot R_v$ $= 5.0369 \times 10^{-6} \times 88$ $= 0.4432 \times 10^{-3}$ in.		
	(g) Transmissibility factor	Table 1-4	$T_r(P) = 1.142$ $T_r(S) = 1.102$	$T_r(P) = 1.201$	$T_r(P) = 1.144$ $T_r(S) = 1.620$	$T_r(P) = 1.146$ $T_r(S) = 1.793$
	(h) Transmitted force		$P_V = \Sigma T_r F_{zo}$ $= 1.142 \times 1{,}329$ $+ 1.102 \times 553$ $= 2{,}127.1$ lbs.	$P_H = T_r F_{xo}$ $= 1.201 \times 725$ $= 870.7$ lbs.	$P_{T\psi} = \Sigma T_r \cdot T_{\psi o}$ $= 1.144 \times 17{,}104$ $+ 1.620 \times 1{,}521$ $= 22{,}035.9$ lbs.-ft	$P_{T\phi} = \Sigma T_r \cdot T_\phi$ $= 1.146 \times 34{,}000$ $+ 1.793 \times 12{,}350$ $= 61{,}107.6$ lbs.-ft

*Vertical forces and moments, and horz. forces and moments are out of phase by 90 degrees and, hence, are not maximum at the same time. However, this combination gives conservative results.

E. Dynamic Analysis

Forces generated by the reciprocating compressor (Figure 6-4) will result in vertical (z), lateral (x), rocking (ψ), and pitching (ϕ) oscillation of the footing, while the forces generated by the motor only will tend to oscillate the footing in the vertical (z), longitudinal (y), torsional (θ), and pitching (ϕ) modes. However, the magnitude of the forces generated by the motor are small and, consequently, the resulting vibration response is negligible. Therefore, a dynamic analysis is only performed for the compressor forces. If an analysis is required for the motor forces, the tables and figures provided are applicable.

F. Check of Design Criteria—Static Conditions (See Chapter 3 for Design Checklist)

1. Static bearing capacity: proportion footing area for 50% of allowable soil pressure. From D above, 883 < 1,250 psf, O.K.

2. Static settlement must be uniform; C.G. of footing and machine loads should be within 5% of each linear dimension. The center of gravity of machine loads and footing coincide.

3. Bearing capacity: static plus dynamic loads. The magnification factor (Table 1-4) should preferably be less than 1.5. In the example problem, this factor is less than 1.5 for the primary operating frequency but is slightly higher for the secondary operating speed.

The sum of static and modified dynamic loads should not create bearing pressures greater than 75% of the allowable soil pressure given for the static load conditions.

Transmitted Dynamic Vert. Force, $P_v = 2{,}127.1$ lbs.
Transmitted Moments $P_{T\psi} = {}^*22{,}035.9 + 2{,}127.1$ $(2.75) + {}^*870.7\ (8.0) = 34{,}851.0$ lbs. ft
$P_{T\phi} = 61{,}107.6$ lbs.-ft

* Out of Phase by 90°, but this combination is conservative.

$$p_{bdyn} = \frac{2{,}127.1}{(15.75)\,(27.5)} \mp \frac{34{,}851.0\ (6)}{15.75\ (27.5)^2}$$

$$\mp \frac{61{,}107.6\ (6)}{27.5\ (15.75)^2} = -\,66.4,\ 76.2\ \text{psf}$$

Total static plus dynamic bearing pressure = 883 + 76.2 = 959.2 psf < 0.75 (2,500) is O.K. and 883 − 66.4 = 816.6 psf (no uplift) is also O.K.

4. Settlement: static plus repeated dynamic loads. The combined C.G. of the dynamic loads and the static loads should be within six in. of the footing center of gravity. For rocking and pitching motion, the axes of rocking and pitching should coincide with the principal axes of the footing. The magnitude of the resulting settlement should be less than the permissible deflecting capability of the connected piping system. In this example, dynamic forces are small compared to static loads; therefore, settlement caused by dynamic loads will be negligible.

Limiting Dynamic Conditions

1. Vibration amplitude at operating frequency. The maximum amplitude of motion for the foundation should lie in Zones A or B of Figure 3-3 for the given acting frequency. The maximum vibration amplitudes are 0.000717 in. and 0.000443 in. in the vertical and horizontal directions, respectively. These amplitudes fall in Zone A of Figure 3-3 at the operating frequency of 585 rpm and are, therefore, acceptable.

2. Velocity is equal to $2\pi f$ (cps) $\times$ displacement amplitude as determined in (1) above. This velocity should be compared to the limiting values of Table 3-2 and Figure 3-3 for at least the "good" condition. Velocity $= 2\pi\ (585/60)\ (0.000717) = 0.0439$ in./sec. This velocity falls in the "good" range of Figure 3-3 and is, therefore, acceptable.

3. Acceleration is equal to $4\pi^2 f^2 \times$ displacement amplitude as determined in (1) above. This check is only necessary if conditions (1) or (2) are not satisfied. The acceleration should fall in Zones A or B of Figure 3-3.

4. Magnification factors should be less than 1.5. In this example-problem the magnification factors are less than 1.5 for the primary operating frequency and slightly larger for the secondary frequency.

5. Resonance. The acting frequencies of the machine should have at least a difference of ±20 with the resonance frequencies; that is, $f < 0.8\ f_m$ or $f > 1.2\ f_m$. In this example, resonance cannot occur in the vertical and horizontal modes (due to the large amount of damping in those directions). In the rocking mode, $0.8\ f_m = 0.8$ $(1{,}934.3) = 1{,}547.4$ rpm and $1.2\ f_n = 2{,}321.2$ rpm. Since the primary and secondary machine frequencies are 585 rpm and 1,170 rpm, no resonance will occur in the rocking mode; therefore the design is judged acceptable. In the pitching mode, $0.8\ f_n = 0.8\ (1{,}717.6) =$ $1{,}374.1$ rpm and $1.2\ f_n = 1.2\ (1{,}717.6) = 2{,}061.1$ rpm. The primary and secondary machine frequencies also fall outside of these ranges and, therefore, no resonance conditions are possible.

6. Transmissibility factor (usually considered for high-frequency machines mounted on springs). The transmissiblity factors should normally be less than 1. In the example, the T_r values are greater than 1 indicating that the dynamic forces are amplified.

7. Possible vibration modes: Vertical and horizontal oscillation are possible modes since the force may act in

those directions. Rocking (ψ) oscillation is also possible since the horizontal forces act above the C.G. of the foundation. Pitching (ϕ) oscillation must also be considered in this example since unbalanced moments are provided by the machine manufacturer. However, twisting or yawing oscillation is not considered since the horizontal forces do not form a couple in the horizontal plane. The horizontal translation and the rocking modes need not be coupled if:

$$\sqrt{f_n{}^2{}_x + f_n{}^2{}_\psi}/(f_{nx} \times f_{n\psi}) < 2/(3f)$$

Using values from Table 6-2 for $f_{nx} = 1009.1$, $f_{n\psi} = 1{,}677.6$ and f (primary frequency) $= 585$ rpm, and substituting in the above expression,

$$\sqrt{(1{,}009.1)^2 + (1{,}677.6)^2}/(1{,}009.1 \times 1{,}677.6)$$
$$= 1.156 \times 10^{-3}$$

and $2/(3 \times 585) = 1.140 \times 10^{-3}$, which appears to be a border line case (within 1%). Hence, uncoupled mode analysis is O.K.

Environmental Demands

1. Physiological effects on persons. If the machine is located in a building, Figures 3-4, 3-5 and 3-7 are used to test the adequacy of the installation. In the example, Figure 3-4 indicates vibrations to be "barely noticeable to persons" at the operating frequency of 585 cpm for a maximum vibration amplitude of 0.000717 in.

2. Psychological effects on persons. Use the same procedures as in (1). When the machine is located close to people not connected with machine operations, an acoustical barrier may be necessary.

3. Damage to structure. Use the limits given in Figure 3-4. The example check shows no danger.

4. Resonance of structural components (superstructure above the footing). Avoid resonance with the lowest natural frequency by keeping the ratio of operating frequency to natural frequency less than 0.5 or greater than 1.5. No other structural components are involved in the example.

Conclusion. The foundation is predicted to perform in an acceptable manner. The static and the dynamic analysis confirm the adequacy of the proposed foundation configuration and, therefore, the design as proposed is acceptable.

Nomenclature—Example 1

a_i = Width of section i, ft
B = Length of rectangular foundation block, ft
$B_z, B_x, B_\psi, B_\theta, B_\phi$ = Mass (or inertia) ratio; vertical horizontal, rocking torsional and pitching vibration mode
b_i = Depth of section i, ft
D = Damping ratio = C/C_c
$D_{zg}, D_{xg}, D_{\psi g}, D_{\phi g}$ = Geometric damping ratios; vertical, horizontal, rocking and pitching modes
D_i = Internal damping ratio
$F(t)$ = Excitation force, lbs.
F_o = Amplitude of excitation force, lbs.
F_x = Maximum horizontal dynamic force, lbs.
F_z = Maximum vertical dynamic force, lbs.
f = Operating speed of machine, rpm
f_c = Critical speed of the machine, rpm
f_n = Natural frequency, rpm
f_m = Resonant frequency, rpm
$f_{m\psi}$ = Resonant frequency in the rocking mode, rpm
$f_{m\phi}$ = Resonant frequency in the pitching mode, rpm
f_{nx} = Natural frequency in the x-direction, rpm
f_{nz} = Natural frequency in the z-direction, rpm
$f_{n\psi}$ = Natural frequency in the rocking direction, rpm
$f_{n\phi}$ = Natural frequency in the pitching direction, rpm
G = Shear modulus of soil, psi
g = Acceleration of gravity = 32.2 ft/sec^2
H = Dynamic horizontal force, lbs.
h = Effective foundation embedment depth, ft
I_ψ, I_θ, I_ϕ = Mass moment of inertia in the ψ (rocking), θ (twisting) and ϕ (pitching) directions, lbs. sec^2-ft
i = Segment ($i = 1, 2, 3 \ldots$)
k = Spring constant, kips/ft
k_i = Distance from center of mass to base of footing for segment i, ft
$k_z, k_x, k_\psi, k_\theta, k_\phi$ = Equivalent spring constants: vertical, horizontal, rocking, torsional and pitching modes
L = Width of base of machine foundation block, ft
M = Dynamic magnification factor
m = Mass, lb. sec^2/ft
m_i = Mass of segment i, lbs. sec^2/ft
n_ψ, n_ϕ = Rocking and pitching mass ratio factors for geometric damping
P_H = Force transmitted in the horizontal direction, lbs.
P_v = Force transmitted in the vertical direction, lbs.
$P_{T\psi}$ = Transmitted rocking moment, ft-lbs.
$P_{T\phi}$ = Transmitted pitching moment, ft-lbs.
p_{bdyn} = Bearing pressure due to transmitted dynamic force, psf
r = Ratio of operating frequency to natural frequency = f/f_n

r_o = Equivalent radius for rectangular footing, ft
R_{hx} = Horizontal distance from center to edge of footing in the x-direction, ft
R_{hy} = Horizontal distance from center to edge of footing in the y-direction, ft
R_v = Vertical distance from base to horizontal machine load, ft
S_{all} = Allowable soil pressure, ksf
T_ψ = Unbalanced rocking moment, ft-lbs.
T_ϕ = Unbalanced pitching moment, ft-lbs.
T = Transmissibility factor
$T_r\ (P)$ = Transmissibility factor for primary operating frequency
$T_r\ (S)$ = Transmissibility factor for secondary operating frequency
W = Total weight of machine plus foundation, lbs.
W_F = Weight of foundation, lbs.
X = Displacement response in the horizontal x-direction, in.
X_t = Total displacement response in the horizontal x-direction, in.
Z = Displacement response in the vertical z-direction, in.
Z_t = Total displacement response in the vertical z-direction, in.
$\alpha_z, \alpha_x, \alpha_\psi, \alpha_\phi$ = Damping ratio embedment factor; vertical, horizontal, rocking and pitching modes
β = Phase angle, rad
$\beta_z, \beta_x, \beta_\psi, \beta_\phi$ = Spring coefficients; vertical, horizontal rocking and pitching modes
γ = Soil density, pcf
$\eta_z, \eta_x, \eta_\psi, \eta_\phi$ = Spring constant embedment factors; vertical, horizontal, rocking and pitching modes
ν = Poisson's ratio of soil
ρ = Mass density of soil $=\gamma/g$ lbs. sec²/ft.⁴
ω = Frequency of excitation force, rad/sec
ω_n = Natural circular frequency, rad/sec

Example 2: Design of a Foundation Block for a Centrifugal Machine (ref. 2)

A. Machine Parameters

The machine parameters necessary for the design of the foundation are defined in Chapter 3. The following data are required (all terms are defined where they first occur):

Compressor:

Weight (W_C)	= 35,270 lbs.
Rotor Weight (W_R)	= 2,100 lbs.
Operating Speed (f)	= 6,949 rpm
(ω)	= 727.7 rad/sec
Critical Speed (f_c)	= 1st ~ 3,400 rpm
	2nd ~ 9,000 rpm
Eccentricity of Unbalanced Mass $\bar{e}$	= 0.0015 in. (provided by manufacturer for the static condition)

The dynamic eccentricity at operating speed may then be calculated from

$$e = \bar{e}/[1 - (f/f_c)^2] = 0.000472 \text{ in.}$$

Often, the manufacturer may claim a zero eccentricity for the rotor components. A design value can nevertheless be selected from Table 3-1 as in Example 3 which follows.

Centrifugal Force F_0 $= (W_R/g)\, e\, \omega^2$	= 1,359 lbs.

Turbine:

Weight (W_T)	= 16,000 lbs.
Rotor Weight (W_R)	= 545 lbs.
Operating Speed (f)	= 6,949 rpm
ω	= 727.7 rad/sec
Critical Speed (f_c)	= 1st ~ 2,000 rpm
	= 2nd ~ 9,020 rpm
Eccentricity of Unbalanced Mass, $\bar{e}$	= 0.0015 in. (which is again given by the manufacturer)

Dynamic eccentricity at operating speed,
$e = 0.0015/[1 - (6{,}949/2{,}000)^2] = 0.0001354$ in.

Centrifugal force F_0 = 101 lbs.
Total centrifugal force F_0 = 1,359 + 101 = 1,460 lbs.

Base plate: weight (W_B) = 5,000 lbs.
Total machine weight (W_M) = $W_c + W_T + W_B$
56,270 lbs.

B. Soil and Foundation Parameters

The soil parameters are obtained from the soil report prepared for the plant facility. The factors which are considered in the preparations of the soil report are listed in Chapter 3. In this example the following information is obtained from the soil report:

Plant Grade El. 100′-0″
Top of Foundation El. 101′-0″
Recommended Foundation base, El. 98′-0″
Soil is Medium Firm Clay

Soil Density (γ)	= 125 pcf
Shear Modulus (G)	= 6,500 psi
Poisson's Ratio (ν)	= 0.45

Soil Internal Damping Ratio ($D_{\psi i}$) = 0.05
Static Allowable Bearing Capacity (S_{all}) = 1.5 ksf
Also, the settlement at the allowable bearing pressure is negligible.

C. Selection of a Foundation Configuration

A trial configuration is selected following the guidelines listed in Chapter 3 under Trial Sizing of a Block Foundation.

A shallow and wide footing is desired such that the combined center of gravity of the machines and of the foundation coincides closely with the centroid of the area of the foundation. A foundation block configuration which satisfies this requirement is shown in Figure 6-5.

Concrete Footing Trial Outline:
Weight of the footing (W_F) = 100,500 lbs.
Total static load (W) = machine weight + weight of footing = 156,770 lbs.
Actual soil pressure = 156,770/12.5 (20) = 627 psf $<$ 0.5 (S_{all}) = 750 psf
Thus, area of footing is O.K.

Weight of footing/weight of machine = 100,500/56,270 = 1.78 which is close to the suggested minimum of 2.0 for well-balanced centrifugal machines. All other guidelines for trial sizing are satisfied and the dynamic analysis is then performed.

D. Dynamic Analysis

The axis of rotation of the shaft is located 6 ft above the bottom of the foundation. The dynamic force acting at the axis of shaft is of the form $F = m_e e \omega^2 \sin \omega t$ (see Figure 6-5) which will excite the structure in three different modes, viz., vertical, horizontal, and rocking. Since the machine will operate at a constant speed in the steady-state condition, the amplitude $m_e e \omega^2$ is constant. Thus, formulas associated with a sinusoidal force of constant amplitude F_0 are used in the dynamic analysis ($F = F_0 \sin \omega t$). A complete dynamic check is performed in Table 6-3.

E. Check of Design Criteria

The foundation block is checked for the design criteria as described in Chapter 3.

Static Conditions

1. Static bearing capacity. Proportion of footing area for 50% of allowable soil pressure. From C above, 627 $<$ 750 psf is O.K.

2. Static settlement must be uniform; C.G. of footing and machine loads should be within 5% of each linear dimension. The center of gravity of machine loads and foundation coincide and is O.K.

3. Bearing capacity: static plus dynamic loads. The magnification factor (Table 1-4) should preferably be less than 1.5. The sum of static and modified dynamic loads should be within 6 in. of the footing C.G. For 75% of the allowable soil pressure given for the static load condition = 627 + 374/(12.5 x 20) $\pm$ 1420(6)/20 $(12.5)^2$ = 632 or 626 psf $<$ 0.75 (1,500) psf is O.K.

4. Settlement: static plus repeated dynamic loads. The combined C.G. of the dynamic loads and the static loads should be within 6 in. of the footing C.G. For rocking motion the axis of rocking should coincide with the principal axis of the footing. The magnitude of the resulting settlement should be less than the permissible deflecting capacity of the connected piping system. In this example, dynamic forces are small compared to static loads; therefore, settlement caused by dynamic loads will be negligible.

Limiting Dynamic Conditions

1. Vibration amplitude at operating frequency. The maximum amplitude of motion for the foundation system should lie in Zone A or B of Figure 3-3 for the given acting frequency. Vibration amplitude (vertical) Z_t = 0.000019 in. at 6,949 rpm. From Figure 3-3 this amplitude is within the safe allowable limits. Vibration amplitude (horizontal) at center of bearing area X_t = 0.000018 in. at 6,949 rpm. The amplitude falls in Zone A in Figure 3-3 and is, therefore, acceptable.

2. Velocity equals $2\pi f$ (cps) $\times$ displacement amplitude as calculated in (1) above. Compare with the limiting values in Table 3-2 and Figure 3-3 at least for the "good" condition. Velocity equals 727.7 (0.000019) = 0.0138 in./sec. From Table 3-2 this velocity falls in the "smooth operation" range and is, therefore, acceptable.

3. Acceleration equals $4\pi^2 f^2 \times$ (displacement amplitude, as calculated in (1) above). Should be tested for Zone B in Figure 3-3. Note: This check is not necessary if conditions (1) and (2) are satisfied, which they were in this example.

4. Magnification factor (applicable to machines generating unbalanced forces). The calculated values of M and M_r (Table 1-4) should be less than 1.5 at resonance frequency. In the example, M in all modes is less than 1.5.

5. Resonance. The acting frequencies of the machine should have at least a difference of $\pm 20\%$ with the resonance frequency of Equations of Table 1-4. ($0.8 f_{mr} \gtrless f \gtrless 1.2 f_{mr}$). In this example, there is no resonance frequency in the vertical mode. In the horizontal mode, 1.2 $\times$ 1,221.4 $<$ 6,949. In the rocking mode, 1.2 $\times$ 1,686.1 $<$ 6,949. Therefore, a resonance condition does not occur.

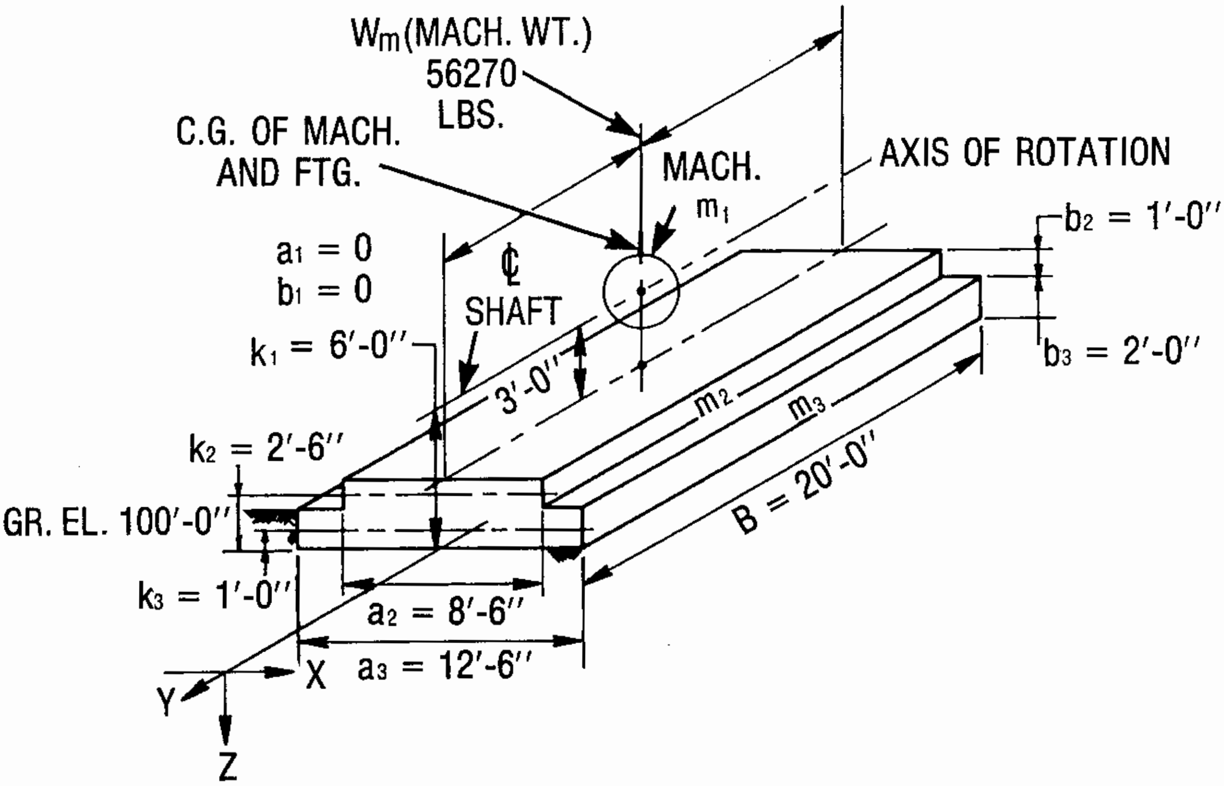


Figure 6-5. Foundation layout for centrifugal machine example problem.

Table 6–3

Dynamic Analysis (Three Modes of Oscillation are Possible)

Step No.	Parameter	Source	Vertical Excitation	Horizontal Excitation	Rocking Oscillation
1	Equivalent radius, r_o	Table 4-2	$r_o = 8.92$ ft	$r_o = 8.92$ ft	$r_o = 8.02$ ft
2	Mass, and mass moment of inertia	$m = W/g$ $I_\psi = \sum_i^n \left[\frac{m_i}{12}(a_i^2+b_i^2) + m_i k_i^2\right]$ Table 4-6	$m = 4{,}869$ lbs. sec²/ft	$m = 4{,}869$ lbs. sec²/ft	I_ψ (Machine) = 62,910 I_ψ (Footing) = 43,218 $I_\psi = 106{,}128$ lbs. sec²-ft
3	Mass ratio	Table 4-3	$B_z = 0.24$	$B_x = 0.35$	$B_\psi = 0.166$
4	Geometric damping ratio Internal damping	Table 4-3 Soil data	$D_z = 0.868$ Negligible	$D_x = 0.494$ Negligible	$D_\psi = 0.311$ $D_{\psi i} = 0.050$ } = 0.361
5	Spring coefficient	Figure 4-1	$\beta_z = 2.15$	$\beta_x = 0.95$	$\beta_\psi = 0.46$
6	Equivalent spring constant	Table 4-1	$k_z = 57.85 \times 10^6$ lbs./ft	$k_{xs} = 40.77 \times 10^6$ lbs./ft	$k_{\psi s} = 2{,}446.36 \times 10^6$ lbs.-ft/rad
7	Natural frequency, f_n	$\frac{60}{2\pi}\sqrt{\frac{k}{m}}$	$f_{nz} = 1{,}040.9$ rpm	$f_{nx} = 873.9$ rpm	$f_{n\psi} = 1{,}449.8$ rpm
8	Resonance frequency, f_{mr}	Table 1-4	Resonance not possible	$f_{mx} = 1{,}221.4$ rpm	$f_{m\psi} = 1{,}686.1$
9	Magnification factor, M	Table 1-4	$M = 0.022$	$M = 0.016$	$M = 0.045$
10	Dynamic force	Centrifugal force	$V_o = 1{,}460$ lbs.	$H_o = 1{,}460$ lbs.	$T_o = H_o k_o = 1{,}460\ (6)$ $= 8{,}760$ lbs.-ft
11 (a)	Vibration amplitude	Table 1-4	$Z = 0.56 \times 10^{-6}$ ft	$X = 0.57 \times 10^{-6}$ ft	$\psi = 0.1611 \times 10^{-6}$ rad
11 (b)	Component of rocking oscillation		At edge of ftg. $= \psi R_h$ $= 0.1611 \times 10^{-6}\ (6.25)$ $= 1.007 \times 10^{-6}$ ft	At center of bearing $= \psi R_v$ $= 0.1611 \times 10^{-6}\ (6)$ $= 0.967 \times 10^{-6}$ ft	
11 (c)	Resultant vibration amplitude	11 (a) + 11 (b)	$Z_t = 1.567 \times 10^{-6}$ $= 0.000019$ in.	$X_t = 1.537 \times 10^{-6}$ ft $= 0.000018$ in.	
12	Transmissibility factor T_r and force transmitted, P_o	Table 1-4	$T_r = 0.256$ $P_v = 374$ lbs.	$T_r = 0.127$ $P_H = 185$ lbs.	$T_r = 0.162$ $P_M = 1{,}420$ lbs.-ft

6. Transmissibility factor (usually applied only to high-frequency spring-mounted machines). The value of transmissibility is calculated by equations of Table 1-4 and should normally be less than 1 for spring-mounted machines having an inertia block. In the example, T_r is less than 1 indicating that dynamic forces are not amplified.

Possible Vibration Modes

1 and 2. Vertical oscillation or horizontal translation are possible modes as the force acts in either direction.

3. Rocking oscillation is possible since the point of horizontal force application is above the foundation mass C.G.

4. Torsional oscillation. Since horizontal forces do not form a couple in the horizontal plane, this mode is not possible.

5. Coupled modes. The horizontal translation and rocking oscillation are usually coupled. The coupled modes may be considered as in example 3 which follows.

Fatigue Failures

1. Machine components. Follow limits in Figure 3-4 and/or Table 3-2.

2. Connections. Same as (1) but check stresses using AISC code (ref. 13 of Chapter 3) when connectors are bolts or welds.

3. Supporting structures. Use (2) for structural steel. For concrete footing, if reversal of stresses takes place and the amplitude is very high (such that the peak stress reversal is over 50% of the allowable stress), the main and the shear reinforcement (if any) should be designed for the stress reversal condition.

In this example the amplitude of the dynamic forces is not large enough to produce any significant stress increase over the stresses caused only by the static loads.

Environmental Demands

1. Physiological effects on persons. If the machine is located in a building, use the procedure given in condition Enviromental Demands under "Limiting Dynamic Conditions", and use the limits from Figure 3-4. In the example, Figure 3-4 indicates no discomfort to people.

2. Psychological effects on persons. Use same procedures as (1). If the facility is located close to people not connected with machine operations, use acoustic barriers. In the example, the machine is located away from habitable areas.

3. Damage to structures. Use limits in Figure 3-4 or 3-5. Example check shows no danger.

4. Resonance of structural components (superstructures above the footing). Avoid resonance with lowest natural structural frequency by keeping the frequency ratio either less than 0.5 or greater than 1.5. In this example, no structural components are involved.

Thus, the trial design is acceptable and may be used to support the machine.

Nomenclature—Example 2

A = Dynamic amplitude
a_i = Width of section i, ft
B = Length of rectangular foundation block, ft
$B_z, B_x, B_\psi, B_\theta$ = Mass (or inertia) ratio; vertical, horizontal, rocking and torsional vibration modes
b_i = Depth of section i, ft
D = Damping ratio
$D_z, D_x, D_\psi, D_\theta$ = Damping ratios; vertical, horizontal, rocking and torsional modes
D_i = Internal damping ratio
e = Eccentricity of unbalanced mass to axis of rotation at operating speed, in.
$\bar{e}$ = Eccentricity of the machine's unbalanced mass, in.
F = Excitation force
F_o = Amplitude of excitation force, lbs.
f = Operating speed of the machine, rpm
f_c = Critical speed of the machine, rpm
f_n = Natural frequency, rpm
f_m = Resonant frequency for constant force-amplitude excitation, rpm
f_{mr} = Resonant frequency for rotating mass-type excitation, rpm
f_{mx} = Resonant frequency in the horizontal direction, rpm
$f_{m\psi}$ = Resonant frequency in the rocking direction, rpm
G = Shear modulus, psi
g = Acceleration of gravity, ft./sec^2
H_o = Dynamic horizontal force, lbs.
I_ψ = Mass moment of inertia, lbs.-sec^2-ft
i = Segment (1, 2, . . .)
k = Spring constant
k_g = Distance from center of rotor axis to footing, ft
k_i = Distance from center of mass to base of footing for segment i, ft
$k_z, k_{xs}, k_{\psi s}, k_{\theta s}$ = Equivalent spring constants; vertical, horizontal, rocking and torsional modes
L = Width at base of machine foundation block, ft
M_r = Magnification factor
$M_{r\,max}$ = Maximum magnification factor
M = Dynamic magnification factor
m = Total mass, lb-sec^2/ft

m_e = Unbalanced mass
m_i = Mass of segment i
n = Number of segments
P_o = Force transmitted through spring mounts
r = Ratio of operating frequency to natural frequency, f/f_n
r_o = Equivalent radius for rectangular footing, ft
R_h = Horizontal distance from center to edge of footing, ft
R_v = Vertical distance from base to center of rotor axis, ft
S_{all} = Allowable soil bearing capacity, ksf
T_o = Unbalanced torque, ft-lbs.
T_r = Transmissibility factor
t = Time, sec
V_o = Dynamic vertical force, lbs.
W = Total weight of machine plus footing, lbs.
W_B = Base plate weight, lbs.
W_C = Compressor weight, lbs.
W_F = Weight of footing, lbs.
W_M = Total machine weight $= W_C + W_T + W_B$, lbs.
W_R = Rotor weight, lbs.
W_T = Turbine weight, lbs.
X_t = Total displacement response in the horizontal x-direction, in.
Z_t = Total displacement response in the vertical z-direction, in.
$\beta_z, \beta_x, \beta_\psi$ = Spring coefficients; vertical, horizontal rocking modes
γ = Soil density, pcf
ν = Poisson's ratio
ρ = Mass density $= \gamma/g$, lbs.-sec^2/ft^4
ω = Frequency of excitation force, rad/sec
ω_n = Natural circular frequency, rad/sec

Example 3: Foundation Design for Centrifugal Machines with Different Operating Frequencies and Supported on an Inertia Block

In some plant facilities, due to environmental considerations or poor soil conditions, it becomes necessary to limit the propagation and amplitude of the machine vibrations transmitted to the foundation (ref. 3). In those circumstances, the use of an inertia block supported on springs is recommended as a vibration isolator; see Figure 6-6. This type of supporting system requires that the piping which is connected to the machines be jointed with flexible couplings in order to absorb without distress the resulting large movements of the inertia block. This movement may be caused either due to a sudden surge condition during the operation of the centrifugal machine or when a resonance condition occurs temporarily at start-up or shutdown of the machine. The latter condition generally is more severe since an inertia block spring system generally has negligible damping resistance.

The inertia block spring suspended foundation is not recommended for heavy machines with large unbalanced forces. However, this type of system may be used when the machines are located on an elevated steel-framed structure.

In this example problem, a foundation system for a gas turbine/generator set is investigated, i.e., the machine consists of an electric generator powered by a gas turbine. Both machines run at different operating frequencies and the step-down from the higher to the lower frequency is accomplished through a gear box. Foundations for this type of machine have been discussed in Chapter 2, and the various steps required to complete the dynamic analysis are given below:

A. Machine Parameter

1. Generator:

Weight (W_M)	= 28,150 lbs.
Rotor Weight (W_R)	= 9,460 lbs.
Operating Speed (f)	= 1,800 rpm
ω	= 188.5 rad/sec
Critical Speed f_c	= 2,200 rpm
Eccentricity of Unbalanced Mass, e	$\simeq$ .001 in. (Table 3-1)
Centrifugal Force $F_0 = (W_R/g)\, e\, \omega^2 =$	871 lbs.

2. Turbine:

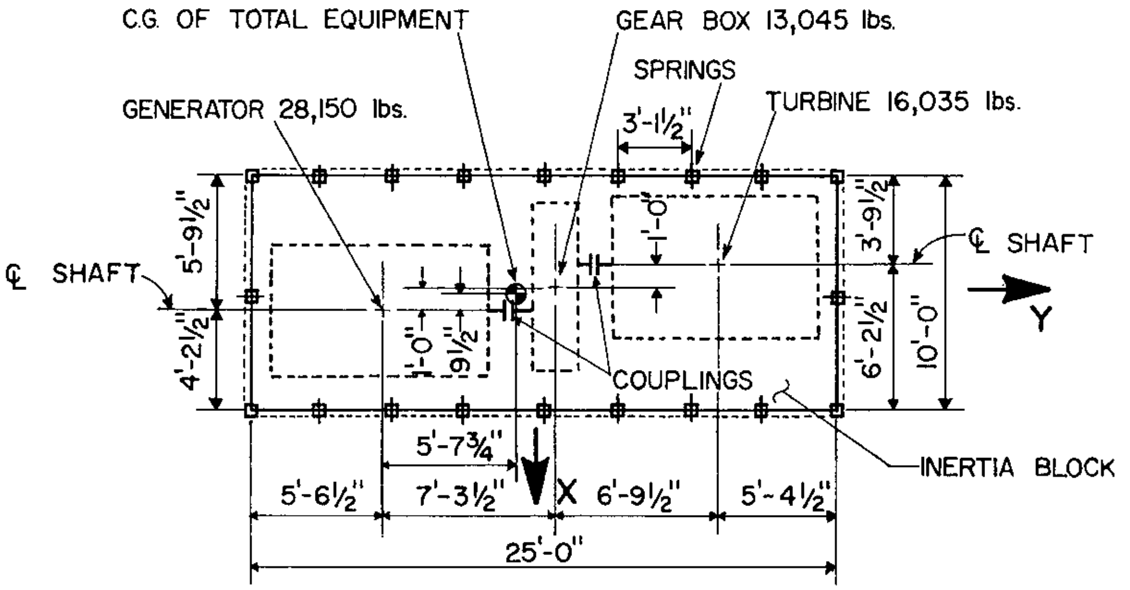
Weight (W_T)	= 16,305 lbs.
Rotor Weight (W_R)	= 567 lbs.
Operating Speed (f)	= 8,990 rpm
ω	= 941.43 rad/sec
Critical Speed (f_c)	= 1st ~ 2,885 rpm 2nd ~ 11,670 rpm
Eccentricity of Unbalanced Mass at Operating Speed, e	$= 0.5\sqrt{12{,}000/8{,}990}$ mil (see Table 3-1) = .00057 in.
Centrifugal force F_0	$= (567/386) \times .00057 \times (941.43)^2$ = 742 lbs.

3. Gear Box:

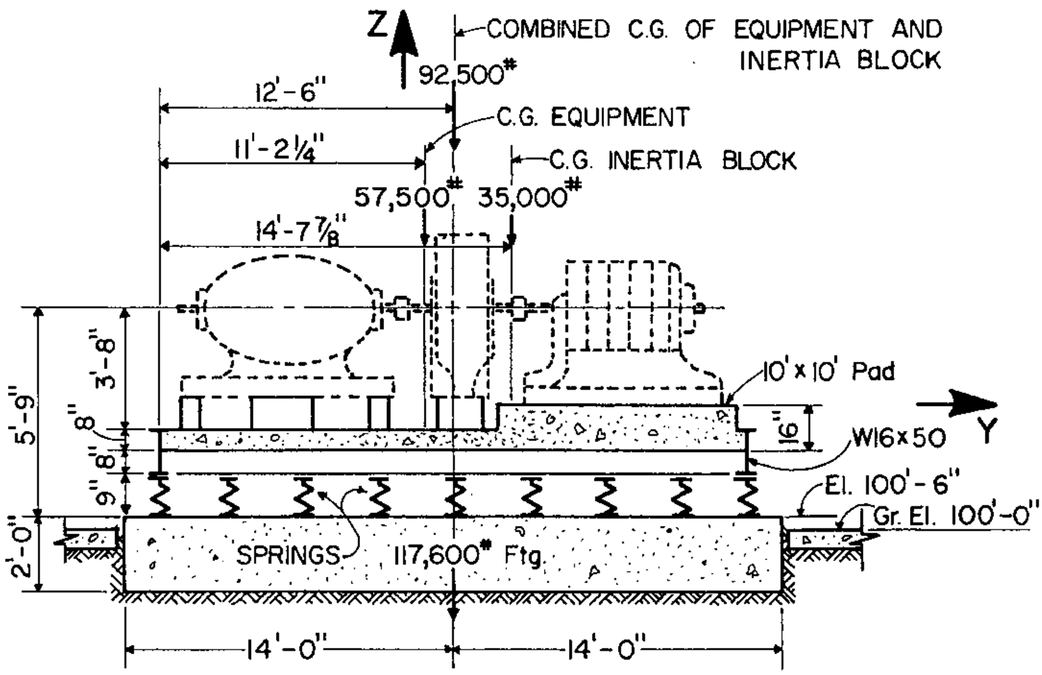
Weight (W_G) = 13,045 lbs.

Unbalanced forces generated by gear box and couplings are assumed negligible. Total Machine weight $(W) = W_M + W_T + W_G = 57{,}500$ lbs.

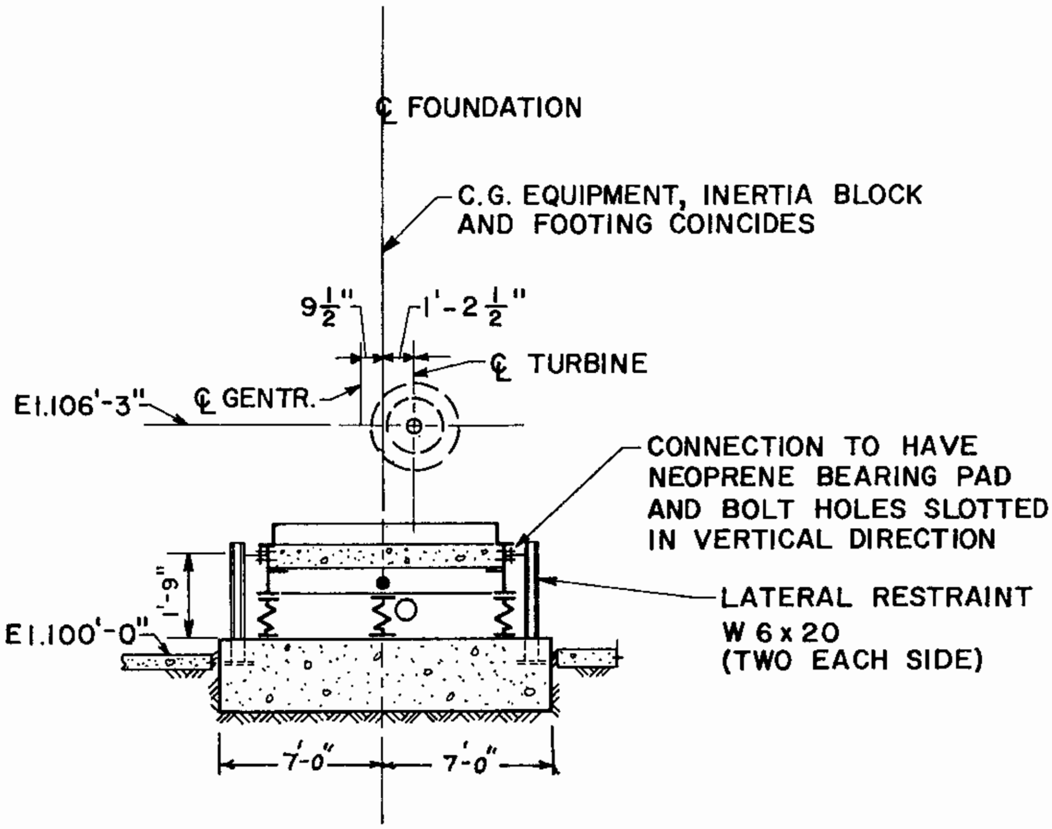
A layout of the equipment is shown in Figure 6-6.



(A) PLAN – EQUIPMENT LAYOUT



(B) FOUNDATION AND EQUIPMENT LOADS



(C) DETAIL LATERAL RESTRAINT

Figure 6-6. Foundation configuration for centrifugal machine with an inertia block.

4. Center of Gravity of Unbalanced Forces:

The unbalanced forces generated by the equipment are assumed to be acting at the center of gravity of the machine loads and perpendicular to their shaft axis. The shaft axes are shown in Figure 6-6B. The forces of the two machines, when combined, are given by:

$$F(t) = 871 \sin 188.5t + 742 \sin 941.4t$$

The plot of individual force functions, as well as the combination of the individual force functions, is given in Figure 6-7.

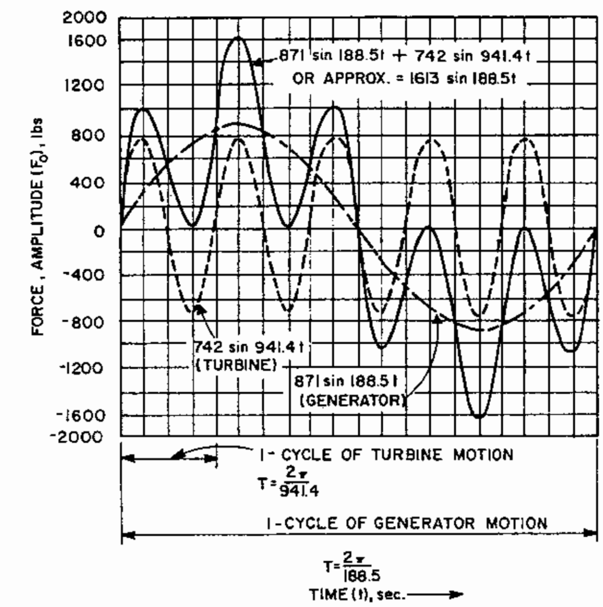


Figure 6-7. Plot of unbalanced centrifugal force.

B. Soil and Foundation Parameter

Soil is Soft Silty Clay
Soil Density (γ) = 110 pcf
Shear Modulus (G) = 3,500 psi
Poisson's ratio (ν) = 0.35
Soil Internal Damping Ratio, ($D_{\psi i}$) = 0.05
Static allowable bearing capacity (S_{all}) = 1.0 ksf

Settlement of soil for a 14 ft by 28 ft footing at 1 ksf bearing pressure = 0.125 in.

Water table fluctuates and is 3 ft below grade at certain times of the year.

C. Selection of a Foundation Configuration

The guidelines listed under Trial Sizing of a Block Foundation in Chapter 3 are followed in selecting an initial configuration. Because the water table is 3 ft below grade, it is recommended that the footing be located at a shallow depth in order to avoid construction complications and a relatively large contact surface area be used. In order to achieve uniform settlement, it is necessary that the center of gravity of the equipment plus the inertia block coincide with the center of gravity of the footing. A trial proportioning of the inertia block and the footing is shown in Figure 6-6.

1. Inertia Block Trial Outline (Figure 6-6A):

Center of gravity of total equipment (y-direction)

$$= \frac{16{,}305\ (14.0833) + 13{,}045\ (7.2917)}{16{,}305 + 13{,}045 + 28{,}150}$$

= 5.608 ft

= 5 ft − 7¾ in. from centerline generator,

or 11 ft − 2¼ in. from the left edge of the inertia block. Center of gravity of total equipment (x-direction)

$$= \frac{16{,}305\ (2.0) + 13{,}045\ (1.0)}{16{,}305 + 13{,}045 + 28{,}150}$$

= 0.7940 ft

= 9½ in. from centerline generator.

The inertia block has a uniform thickness of 8 in. (100 psf) and an additional thickness of 8 in. (100 psf) of dimension 10 ft × 10 ft in the region under the turbine.

The center of gravity of inertia block (y-direction)

$$= \frac{10 \times 10 \times 100\ (20.0) + 25 \times 10 \times 100\ (12.5)}{25 \times 10 \times 100 + 10 \times 10 \times 100}$$

= 14.6429 ft

= 14 ft — 7¾ in. (from left edge of inertia block).

The center of gravity of inertia block (x-direction) = 5 ft — 0 in. (from bottom edge of inertia block).

From Figure 6-6B, C,

Combined C.G. of equipment and inertia block,

$$(y\text{-direction}) = \frac{57{,}500 \times (11.2292) + 35{,}000\ (14.6429)}{57{,}500 + 35{,}000}$$

= 12 ft — 6 in. from left edge of the inertia block.

Combined C.G. of equipment and inertia block, (x-direction) = 5 ft − 0 in. from bottom edge of inertia block.

2. Footing Trial Outline:

The footing plan dimensions should be larger than the inertia block in order to accommodate the supporting springs of the inertia block and its lateral supports. Also, the resultant bearing pressure on the soil should be less than 50% of the allowable soil bearing pressure in order to minimize possible foundation settlements. A trial concrete slab size 14 ft wide by 28 ft long and 2 ft thick is then analyzed. The footing center of gravity is made to coincide with the combined center of gravity of the equipment and the inertia block.

Weight of the footing (W_F) = 117,600 lbs.

Total static load (W) = equipment weight + inertia block + footing weight = 210,100 lbs.

Actual soil pressure = 210,100/14(28) $\simeq$ 0.5 S_{all} = 534 psf

Thus, area of footing is O.K.

D. Dynamic Analysis

A mathematical model of this foundation was previously discussed as Model 3 of Chapter 2. We have the following parameter calculations:

Selection of Springs for Inertia Block

1. Vertical Direction. Try for transmissibility factor (T_r) = .02.

From the transmissibility equation of Table 1-4, and assuming damping to be negligible in the springs, $D = 0$, then the resulting equation is

$0.02 = 1/|1 - r^2|$

or $r = f/f_n = 7.0$

For f = 1800 rpm, f_n = 257.14 rpm

For f = 8990 rpm, f_n = 1284.29 rpm

A natural frequency (f_n) = 257.14 will be used, since a higher natural frequency will require a large number of springs.

Mass (m_1) of inertia block + equipment

$$= \frac{(57{,}500 + 35{,}000)}{386}$$

= 239.64 lbs.-sec²/in.

Total spring constant (k_{z1})

$$= \left(\frac{[257.14 \times 2\pi]}{60}\right)^2 \times 239.64$$

= 173,762 lbs./in.

Try 20 spring with a spring constant for each

$$\text{spring} = \frac{173{,}762}{20} = 8{,}688.1 \text{ lbs./in.}$$

There is a commercially available spring of 8,800 lbs. force for a 1-in. deflection. The dimensions of the springs are: height = 9.0 in., width = 5.25 in., length = 13.0 in., and maximum solid load is 10,912 lbs. for a 1.24-in. deflection.

Use 20 springs for total k_{z1} = 176,000 lbs./in.

Then, $f_n = (60/2\pi)\ \sqrt{176{,}000/239.64} = 258.8$ rpm.

2. Horizontal Direction (Figure 6-6C). Lateral restraint is provided by W6 × 20 vertical posts, two on

each side. The posts are fully fixed at their bottom and are connected to the inertia block by the provision of slotted holes in the vertical direction so that the oscillation of the vertical springs is not effected. A neoprene bearing pad layer is inserted in the connection in order to absorb high-frequency vibrations. The spring constant in the lateral direction is given by:

$k_{x1} = 3EI_x/l^3$

where $I_x = 4 \times 41.5 = 166$ in.[4]

$l = 21.0$ in.

$E = 30 \times 10^6$ psi

$\therefore k_{x1} = 1.6132 \times 10^6$ lbs./in.

Mass (m_1) of the inertia block and equipment 239.64 lbs.-sec²/in.

$\therefore f_n = (60/2\pi)\sqrt{1{,}613{,}200/239.64} = 783.5$ rpm

3. Rocking Oscillation about Point 0 (Figure 6-6C). Two rows of springs, each containing ten springs, are located at a distance of 60 in. on either side of an axis passing through point 0. Due to this arrangement, the inertia block is capable of rocking about that axis. The spring constant for the rocking oscillation k_ψ is, thus, a function of the vertical spring constant and is given by:

$k_\psi = 2k_v e^2$

where k_v is spring constant of each row ($10 \times 8{,}800 = 88{,}000$ lbs/in.), e is the distance from the axis to the row (= 60 in.).

$\therefore k_\psi = 2 \times 88{,}000 \times 60 \times 60$

$= 633.6 \times 10^6$ lbs.-in./rad.

In order to calculate the natural frequency, the mass moment of inertia for the inertia block and equipment must be calculated and is given by:

$$I_\psi = \sum_i^n m_i k_i^2$$

Machines = $(57{,}500/386)\ (60)^2 = 536{,}269.0$

Inertia block = $(25{,}000/386)\ (12)^2$

$+ (10{,}000/386)\ (20)^2 = 19{,}689.0$

$I_\psi = 536{,}269 + 19{,}689 = 555{,}958$ lbs.-in.-sec²

$\therefore f_n = (60/2\pi)\sqrt{(633.6 \times 10^6)/555{,}958}$

$= 322.4$ rpm

Therefore, the natural frequencies of the inertia block-equipment-spring system are 258.8, 783.5, and 322.4 rpm in the vertical, lateral, and rocking modes. Table 6-4 lists all computations for the single-degree-of-freedom system.

E. Dynamic Analysis as a Multi-Mass System

The calculation of the natural frequencies for a two-mass model (see Figure 1-20) is given by:

$$\omega_{1,2}^2 = \tfrac{1}{2}\left[\frac{k_1}{m_1} + \frac{k_1 + k_2}{m_2} \mp \sqrt{\left(\frac{k_1}{m_1} - \frac{k_1 + k_2}{m_2}\right)^2 + \frac{4\,k_1^2}{m_1 m_2}}\right]$$

The terms with subscript 1 stand for inertia block plus equipment, and 2 for the footing. The calculations for the footing only (14′ × 28′ block) is performed in Table 6-5.

1. Vertical Oscillation:

$k_{z1} = 176{,}000$ lbs./in., $m_1 = 239.64$ lbs.-sec²/in.

$k_{z2} = 3.07 \times 10^6$ lbs./in., $m_2 = 304.35$ lbs.-sec²/in.

where k_{z2} (Table 6-5) and m_2 are the spring constant and mass of the foundation block, respectively.

$\omega_1^2 = 691.85$ or $f_{n1} = 251.2$ rpm

$\omega_2^2 = 10{,}707.93$ or $f_{n2} = 988.2$ rpm

2. Horizontal Oscillation:

$k_{x1} = 1.6132 \times 10^6$ lbs./in., $m_1 = 239.64$ lbs.-sec²/in.

$k_{x2} = 2.3583 \times 10^6$ lbs./in., $m_2 = 304.35$ lbs.-sec²/in.

$\omega_1^2 = 3{,}133.31$, $f_{n1} = 534.5$ rpm

$\omega_2^2 = 16{,}647.57$, $f_{n2} = 1232.1$ rpm

F. Discussion of Dynamic Analysis

1. Natural frequencies. The values calculated for a single-mass model and a two-mass model reveal that for the vertical mode there is no difference in the calculated frequency when either model is used. This is because the natural frequency of the model, including the inertia block, has a natural frequency of less than half the natural frequency of the footing in the vertical mode (258.8 vs. 717.5 rpm, Table 6-5, respectively). Therefore, the fundamental frequency of the coupled model has a small difference with the lowest frequency calculated as individual uncoupled models (251.2 rpm vs. 258.8 rpm, respectively). This fact can be demonstrated by using Southwell-Dunkerley's formula,

$$\frac{1}{f_e^2} = \frac{1}{f_1^2} + \frac{1}{f_2^2} + \frac{1}{f_2^3} + \ldots$$

substituting, $f_1 = 258.8$, $f_2 = 717.5$, then, $f_e = 243.5$ rpm, which differs less than 3.2% from the calculated coupled model frequency and 6.3% from the calculated single mass frequency. Similarly, for the rocking mode, the fundamental frequency calculated by using $f_1 = 322.4$ and $f_2 = 883.2$ into the Southwell-Dunkerley's formula will be 302.85 rpm, which differs by 6.5% from the calculated frequency of the single mass model. If the coupled equations of horizontal and rocking modes are solved, then the lowest frequency is found to be close to 250 rpm. Therefore, due to the significant difference in

Table 6–4
Dynamic Analysis of Single-Degree-of-Freedom System
(Machine plus Inertia Block Only)

Step No.	Parameter	Source	Vertical Excitation	Horizontal Excitation	Rocking Oscillation
1.	Natural Frequency f_n	$\frac{60}{2\pi}\sqrt{\frac{k}{m}}$	258.8 rpm	783.5 rpm	322.4 rpm
2.	Magnification factor M	Table 1-4	$(f = 1{,}800) = 0.021$	$= 0.234$	$= 0.033$
			$(f = 8{,}990) = 0.001$	$= 0.008$	$= 0.001$
3.	Dynamic force $F(t)$	Figure 6-7	$F_0\ (f = 1{,}800) = 871$	871.0	$F_0 \times h = 871 \times 60 = 52{,}260$
			$F_0\ (f = 8{,}990) = 742$ (lbs.)	742.0 (lbs.)	$F_0 \times h = 742 \times 60 = 44{,}520$ (lbs.-in.)
4.	Vibration amplitude	Table 1-4	(a) $Z\ (f = 1{,}800) = 1.039 \times 10^{-4}$	$X = 1.263 \times 10^{-4}$	$\psi = 2.722 \times 10^{-6}$
			(b) $Z\ (f = 8{,}990) = 4.21 \times 10^{-8}$ (in.)	$X = 3.680 \times 10^{-6}$ (in.)	$\psi = 7.027 \times 10^{-8}$ (radians)
5.	Components of rocking oscillation	$f = 1{,}800$	$\psi \times R_H$ (c) $2.722 \times 10^{-6} \times 60 = 1.634 \times 10^{-4}$	$\psi \times R_v = 2.722 \times 10^{-6} \times 60 = 1.634 \times 10^{-4}$	—
		$f = 8{,}990$	(d) $7.027 \times 10^{-8} \times 60 = 4.216 \times 10^{-6}$ (in.)	$= 7.027 \times 10^{-8} \times 60 = 4.216 \times 10^{-6}$ (in.)	—
6.	Resulting vibration amplitude	(a)+(b) +(c)+(d)	$Z_t = 2.757 \times 10^{-4}$ (in.)	$X_t = 2.976 \times 10^{-4}$ (in.)	—
7.	Transmissibility factor T_r and force transmitted P_0	Table 1-4	$T_r\ (f = 1{,}800) = 0.021$ $Pv = 18.30$	$T_r = 0.234,\ P_H = 203.80$	$T_r = 0.033,\ P_M = 1{,}725.0$
			$T_r\ (f = 8{,}990) = 0.001$ $P_v = 0.74$	$T_r = 0.008,\ P_H = 5.94$	$T_r = 0.001,\ P_M = 44.5$
			Total 19.04 (lbs.)	Total 209.74 (lbs.)	Total 1,769.5 (lbs.-in.)

the natural periods of the two mass elements, it is permissible to assume that the individual elements act independently of each other, i.e., in the vertical and rocking modes the inertia block and the bottom footing masses can be analyzed on the basis of an equivalent one-degree-of-freedom or uncoupled system.

However, in the horizontal mode, both mass elements have nearly equal natural frequency; the inertia block has a natural frequency of 783.5 rpm, and the footing has a natural frequency of 628.9 rpm. The frequencies of the two-mass coupled mode are 534.5 and 1,232.0 rpm for the inertia block and the footing, respectively. From Dunkerley's formula, the lowest frequency is 490.5 rpm, which is quite low compared to 783.5 rpm obtained by considering the inertia block plus the machine as an individual element. Therefore, a coupled model investigation is justified in the lateral direction.

2. Response calculations: Because the equations of motion of Model 3 for these foundations are linear, the dynamic response generated by each of the two components 871 sin 188.5t and 742 sin 944.4t of the excitation force can be combined using the principle of superposition. This procedure has been used in Steps 4 to 6 of Table 6-4. However, in that table, the inertia block element was considered to be acting independent of the footing. This uncoupling was found to be justified for the vertical and rocking modes, but for the horizontal mode, an analysis based on coupling of m_1 and m_2 is required. The following equations give the response values and consider the effects described above:

$$x_1(t) = \Sigma \frac{\left(\frac{k_1 + k_2}{m_1} - \omega^2\right) F \sin(\omega t - \phi_1)}{m_1(\omega^2 - \omega_1^2)(\omega^2 - \omega_2^2)}$$

Table 6–5
Dynamic Analysis of Footing Only (14′ × 28′ × 2′-0″)

Step No.	Parameter	Source	Vertical Excitation	Horizontal Excitation	Rocking Excitation
1.	Equivalent radius r_o	Table 4-2	$L = 14.0'$, $B = 28.0'$, $r_o = 11.17$ ft	$r_o = 11.17$	$r_o = 9.50$
2.	Mass and mass moment of inertia	$m = W/g$; $I_\psi = \sum_i^n [m_i(a_i^2 + b_i^2)/12 + m_i k_i^2]$	$W = 210{,}100$ lbs. $m = 210{,}100/32.2 = 6{,}524.8$ lbs.-sec²/ft	$m = 6{,}524.8$ lbs.-sec²/ft	I_ψ (Machine) = 107,441.0; I_ψ (Inertia block) = 27,023.6; I_ψ (Footing) = 64,521.8; (Summation) ΣI_ψ 198,986.4 lbs.-sec²-ft
3.	Mass ratio	Table 4-3	$B_z = 0.223$	$B_x = 0.277$	$B_\psi = 0.183$
4.	Geometric damping ratio / Internal damping	Table 4-3 / soil data	$D_z = 0.900$; Negligible	$D_x = 0.547$; Negligible	$D_\psi = 0.214$, $D_{\psi i} = 0.050$ } = 0.264
5.	Spring coefficient	Figure 4-1	$\beta_z = 2.40$	$\beta_x = 1.05$	$\beta_\psi = 0.40$
6.	Equivalent spring constant	Table 4-1	$k_z = 36.84 \times 10^6$ lbs./ft	$k_{xs} = 28.30 \times 10^6$ lbs./ft	$k_{\psi s} = 1{,}702.12 \times 10^6$ lbs.-ft/rad
7.	Natural frequency f_n	$(60/2\pi)\sqrt{k/m}$	$f_{nz} = 717.50$ rpm	$f_{nx} = 628.90$ rpm	$f_{n\psi} = 883.20$ rpm
8.	Resonance frequency f_{mr}	Table 1-4	Resonance not possible	$f_{mx} = 992.40$ rpm	$f_{m\psi} = 954.00$ rpm

$$x_2(t) = \Sigma \frac{k_1 F \sin(\omega t - \phi_2)}{m_1 m_2 (\omega^2 - \omega_1^2)(\omega^2 - \omega_2^2)}$$

In these equations, ω (operating speeds) = 188.5 and 941.4 rad/sec.

ω_1^2 (square of the mass m_1 circular frequency) = 3,133.31

ω_2^2 (square of the mass m_2 circular frequency) = 16,647.57

m_1 (total mass of inertia block and equipment) = 239.64 lbs.-sec²/in.

m_2 (mass of the footing) = 304.45 lbs.-sec²/in.

k_1 (horizontal spring constant between m_1 and m_2) $= k_{x1} = 1.6132 \times 10^6$ lbs./in.

k_2 (horizontal spring constant of soil) $= k_{x2} = 2.3575 \times 10^6$ lbs./in.

F (amplitude of the dynamic forces) = 871 and 742 lbs.

ϕ_1 and ϕ_2 are phase difference and $= 0, \pi$

Substituting the above parameters,

$x_1(t) = 1.336 \times 10^{-4} \sin 188.5t + 3.521 \times 10^{-6} \sin 941.4t$ in.

$x_2(t) = 2.682 \times 10^{-5} \sin 188.5t + 0.0022 \times 10^{-5} \sin 941.4t$ in.

Comparing the amplitudes of $x_1(t)$ with the values of x in Step 4 of Table 6-4, it may be observed that the response values do not change significantly, using either of the two assumptions. Therefore, the assumption of independent behavior of the inertia block in all modes of oscillation is a valid step. Furthermore, in Figure 6-7, which shows the plot of the total centrifugal force, it is found that the total forcing function follows approximately the path of the curve: sin 188.5t, and has an amplitude of 1613 lbs. Using this function in the response equations,

$x_1(t) = 2.474 \times 10^{-4} \sin 188.5t$ in.

$x_2(t) = 5.831 \times 10^{-5} \sin 188.5t$ in.

The above values are much higher than the response values calculated by the summation of the individual forcing functions. Therefore, the analysis of Table 6-4 gives more accurate results.

Another point worth discussing is the exclusion of the damping term when solving the coupled equations of motion of Model 3. Since the natural frequencies of the modes, $f_n = 534.5$ and 1,232.1 rpm are quite different from the operating frequencies, $f = 1{,}800$ and 8,990 rpm, the influence of damping on the response values is negligible for all practical purposes.

3. Transmissibility factor: From the foregoing discussion, it can be stated that the factors calculated in Table 6-4, considering the footing as a stiff support for the inertia block, is a valid assumption. Transmissibility factors were not calculated for the footing because the forces transmitted from the inertia block were of very small magnitude. However, for design purposes the values obtained for the inertia block are assumed to be transmitted to the soil without any amplification or reduction.

G. Check of Design Criteria (as listed in Chapter 3)

1. Static Conditions:
   (a). Static bearing capacity. Proportion footing area for 50% of allowable soil pressure. From C above, 534 psf $\simeq$ 500 psf (allowable).
   (b). Static settlement must be uniform; C.G. of footing, inertia block and machine loads coincides, and thus settlement will be uniform.
   (c). Bearing capacity: static plus dynamic loads.
   $p_b = 534 + [19/14\ (28)]$
   $\pm\ [(1770/12)\ 6/28\ (14)^2]$
   $= 535$ psf < 0.75 (1000) psf is O.K.
   (d). Settlement: static plus repeated dynamic loads. The increase in pressure due to dynamic loads is less than 1 psf and thus would not create uneven settlement.
2. Limiting Dynamic Conditions (refer to Table 6-4):
   (a). Vibration amplitude at operating frequency. Inertia block: Z_t (vertical vibration amplitube) = .00028 in. at f = 1800 rmp. From Figure 3-3, this falls within the safe allowable limits. x_t (horizontal vibration at centerline of bearings) = .00030 in. at f = 1800 rpm. From Figure 3-3, the amplitude falls in zone A and, therefore, is acceptable.
   Footing: The dynamic forces transmitted through the inertia block are very small and thus vibration amplitude is also negligible.
   (b). Velocity equals $2\pi f$ (cps) $\times$ displacement amplitude as calculated in (a) above. Velocity = 2π (1800) (1/60) 0.0003 = 0.0565 in./sec. From Table 3-2 this velocity falls in the "good operation" range and is, therefore, acceptable.
   Velocity check by RMS (root mean square) method, when response involves more than one frequency: Using response values of m_1 of Model 3,

$$\text{Velocity} = \sqrt{(188.5 \times 1.336 \times 10^{-4})^2 + (941.4 \times 3.52 \times 10^{-6})^2}$$
$$= 0.0254$$
< 0.0565 in./sec. calculated above, thus is O.K.

   (c). Acceleration: $4\pi^2\ (30)^2\ (0.0003) = 10.66$ in./sec^2.
   (d). Magnification factor: From Table 6-4 this value is less than 1.5 for all modes of oscillation and, thus, is acceptable.
   (e). Resonance condition: The natural frequencies (Table 6-4) in all modes of oscillations for the inertia block and footing are less than 0.8(1800). Therefore, no resonance condition occurs at the lower operating speed. This ratio is also true for the critical speed of the machine rotor. Thus, the foundation is classified as low-tuned or under-tuned.
   (f). Transmissibility factor: This factor is less than 5% in the vertical and rocking modes of the inertia block and, thus, meets the normal limitation. However, in the horizontal mode when acted on by the lower frequency (f = 1800), T_r was found to be 1.234 > 0.05 normally used. This happened due to the use of a structural member (W6 $\times$ 20) as a vibration isolator. Use of structural member as a lateral restraint is a required feature in this type of system in order to maintain the stability of the inertia block in case of failure of the springs. In any case, the lateral force transmitted to the footing is small and can easily be absorbed by the lateral soil in contact with the footing.
3. Possible Vibration Modes:
   (a) and (b). Vertical oscillation or horizontal translation is a possible mode as the force acts in either direction.
   (c). Rocking oscillation is possible since the point of horizontal force application is above the foundation mass C.G.
   (d). Torsional oscillation is possible as the forces generated by the two machines are of different frequencies. However, it is estimated that the natural frequency of this mode would be too low compared to the acting frequency such that the response values would not be of much significance. In case an analysis is required, then the following steps are given:
   (1). Mass moment of inertia about the vertical axis through center of gravity.

$$\text{Machines: } (16{,}305/386)\ [(85.5)^2 + (14.5)^2]$$
$$+ (28{,}150/386)\ [(83.5)^2 + (9.5)^2]$$
$$+ (13{,}045/386)\ [(2.5)^2 + (4)^2]$$
$$= 833{,}475.0 \text{ lbs.-in. sec}^2$$

Inertia block: $\frac{25,000}{386}\left(\frac{120 \times 120}{12} + \frac{300 \times 300}{12}\right)$

$$+ \frac{10,000}{386}(90)^2 = 773,316.0 \text{ lbs.-in. sec}^2$$

$$I_\theta = 833,475.0 + 773,316.0$$
$$= 1,606,791.0 \text{ lbs.-in. sec}^2$$

(2). Spring constant of vertical posts (W6 × 20) using the weak axis.

$$k_l \text{ (2 posts)} = \frac{3EI_y}{l^3}$$
$$= \frac{3\times30\times10^6\times2\times13.3}{(21)^3}$$
$$= 258,503 \text{ lbs./in.}$$

$$k_\theta = 2\,k_l e^2 = 2 \times 258,503\ (60)^2$$
$$= 1.8612 \times 10^9 \text{ lbs.-in./rad}$$

$(e = 60 \text{ in.})$

$$f_n = \frac{60}{2\pi}\sqrt{\frac{1.8612 \times 10^9}{1.6068 \times 10^6}}$$
$$= 325.0 \text{ rpm}$$

(3). Forcing function: The centrifugal force of the generator which is in phase with the peaks of the turbine's centrifugal force at $f = 1,800$ rpm will not form any significant torque couple. The other peaks of the turbine force will form a torque couple; i.e., four out of six peaks (Figure 6-7). Conservatively, it may be assumed that the turbine centrifugal force will form a torsional moment and may be given as:

$$T_\theta = 742 \times 60 \sin 941.4t$$
$$= 44,520 \sin 941.4t \text{ lbs.-in.}$$

(4). Magnification factor $M = 1/(r^2 - 1) = 0.0013$

(5). Transmissibility factor $T_r = 0.0013$

(6). Response value of inertia block (longitudinal direction):

$$M\ (T_\theta/k_\theta)\ e = 0.0013 \times [44,520/(1.8612 \times 10^9)] \times 60$$
$$= 1.87 \times 10^{-6} \text{ in. (negligible)}$$

(7). Force transmitted to the foregoing:

$$(T_\theta \times T_r)/2e = (44,520 \times 0.0013)/2\ (60)$$
$$= 0.482 \text{ lbs. (negligible)}$$

Therefore, torsional mode oscillations are not significant.

(e). Coupled modes: The degrees of freedom for each of the masses were found to be acting independent of each other. Because of the linearity in the equations of motion, the principle of superposition is used to find the total response. Possible fatigue failure checks and environmental demands are also found to be satisfactory and the foundation is judged to be adequate.

Nomenclature—Example 3

A = Dynamic amplitude
a_i = Width of section i, ft
B = Length of rectangular foundation block, ft
$B_z, B_x, B_\psi, B_\Theta$ = Mass (or inertia) ratio: vertical mode, horizontal, rocking, and torsional vibration modes
b_i = Depth of section i, ft
D = Damping ratio
$D_z, D_x, D_\psi, D_\Theta$ = Damping ratios: vertical, horizontal, rocking, and torsional modes
D_i = Internal damping ratio
e = Eccentricity of unbalanced mass to axis of rotation at operating speed, in. or half the distance between the vertical springs for calculating the equivalent value of k_ψ and k_Θ
E = Modulus of elasticity, psi
F = Excitation force, lbs.
F_0 = Amplitude of excitation force, lbs.
f = Operating speed of the machine, rpm
f_c = Critical speed of the machine, rpm
f_e = Equivalent fundamental frequency, rpm
f_m = Resonant frequency for constant force-amplitude excitation, rpm
$f_{mx}, f_{mz}, f_{m\psi}$ = Resonant frequency in horizontal (x) vertical (z), and rocking (ψ) modes.
f_n = Natural frequency, rpm
f_{n1}, f_{n2} = Natural frequencies of masses m_1, m_2 in coupled model, rpm
f_1, f_2 = Natural frequencies of masses m_1, m_2 in uncoupled model, rpm

G = Shear modulus, psi
g = Acceleration of gravity, ft/sec^2
H_0 = Dynamic horizontal force, lbs.
I_x, I_y = Moment of Inertia of vertical post of strong axis (x) and weak axis (y), in.^{4}
I_ψ, I_Θ = Mass moment of inertia of inertia block in rocking (ψ) and torsional (Θ) modes, lbs.-in.-sec^2
i = Segment (1, 2, . . .)
k = Spring constant
k_g = Distance from center of rotor axis to base of footing, ft
k_i = Distance from center of mass to base of footing for segment i, ft
$k_z, k_{xs}, k_{\psi s}, k_{\Theta s}$ = Equivalent spring constants: vertical, horizontal, rocking, and torsional modes
L = Width at base of machine foundation block, ft
l = Height of vertical post, in.
M_r = Magnification factor
$M_{r\,max}$ = Maximum magnification factor
M = Dynamic magnification factor
m = Total mass
m_e = Unbalanced mass
m_i = Mass of segment i
n = Number of segments
P_H, P_V = Force transmitted through springs in horizontal (x), vertical (z) directions, lbs.
P_M = Moment transmitted to springs in rocking oscillation (ψ), lbs.-in.
P_o = Force transmitted through spring mounts, lbs.
R_h = Horizontal distance from center to edge of footing, ft
R_v = Vertical distance from base to center of rotor axis, ft
r = Ratio of operating frequency to natural frequency, f/f_n
r_0 = Equivalent radius for rectangular footing, ft
S_{all} = Allowable soil bearing capacity, ksf
T_0 = Unbalanced torque, ft-lbs.
TR, T_r = Transmissibility factor
t = Time, sec
V_0 = Dynamic vertical force, lbs.
W = Total weight of machine plus footing, lbs.
W_G = Weight of gear box, lbs.
W_M = Weight of generator, lbs.
W_T = Weight of turbine, lbs.
W_0 = Total weight of machines, lbs.
W_R = Weight of rotor, lbs.
X, Z, ψ = Displacement amplitude in horizontal (x), vertical (z), and rocking (ψ) modes
X_t, Z_t = Total vibration amplitude in horizontal (x) direction at machine axis level and in vertical (z) direction at footing level, in.
Y_1, Y_2 = Displacement response of masses m_1, m_2 in coupled model, in.
$\beta_z, \beta_x, \beta_\psi$ = Spring coefficients: vertical horizontal and rocking modes
γ = Soil Density, pcf
μ = Ratio of unbalanced mass to total mass $= m_e/m$
ν = Poisson's ratio
ρ = Mass density $= \gamma/g$, lbs.-sec^2/ft^4
ω = Frequency of excitation force, rad/sec
ω_n = Natural frequency, rad/sec

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7 Computer Analysis and Applications: Elevated Foundation

The availability of electronic digital computers having great calculating speed and analytical power has resulted in substantial advancement in the engineering art of analysis and design of structures supporting dynamic machines. Increasing machine weight and speed coupled with large costs have made the rule-of-thumb approach and hand computation either unsafe or too conservative for many structures. Modern computer programs yield, among other factors, the natural frequencies, the deformations, and the forces in the

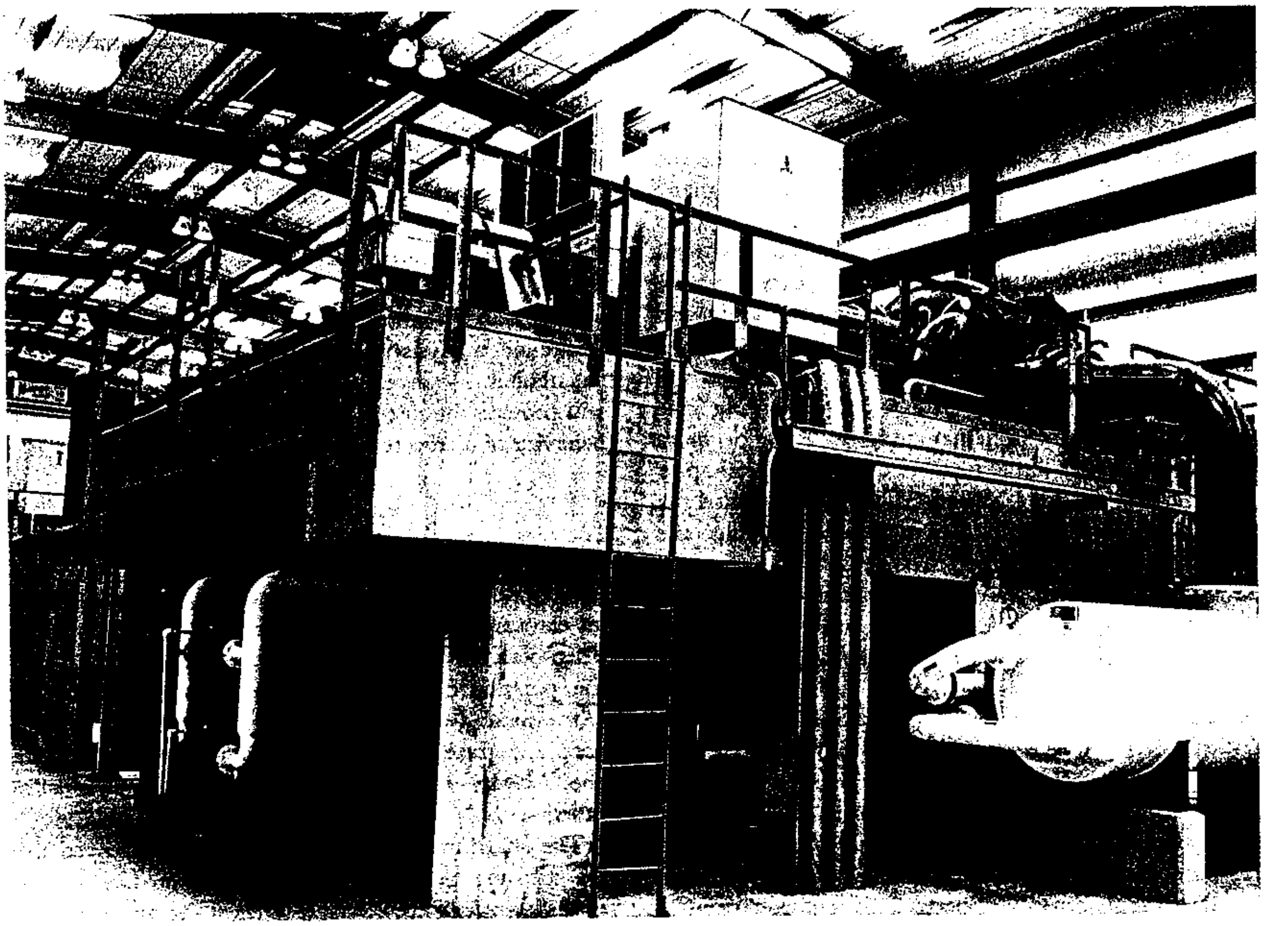


Table Top Compressor Unit. Courtesy of Big Three Industries, Inc., Channelview, Texas. Photo by Engineering Media Center, University of Houston.

structure (ref. 1). These quantities were either ignored, conservatively assumed, or calculated in a simplified approximate way in precomputer times. For example, an equivalent static strength analysis of a structure supporting a centrifugal machine is usually made for the following loading conditions:

1. Total vertical load plus 0.5 of the full load acting in the vertical direction.
2. Total vertical load plus 0.3 of the full load acting in the transverse direction.
3. Total vertical load plus 0.1 of the full load acting in the longitudinal direction.

These approximate machine load factors (0.5, 0.3, 0.1) are fairly accurate for an equivalent static analysis when the ratio of machine acting frequency to natural frequency r in the specified direction is less than 1, greater than 1, and much greater than 1, respectively, as shown in Figure 1-36. However, the factors 0.5, 0.3, and 0.1 are derived for a machine with an acting frequency of 1,800 rpm and considered very stiff in the longitudinal direction. The selected ratios of 0.5, 0.3, and 0.1 are generally consistent with a highest rigidity in the vertical direction, a not-so-high rigidity in the transverse direction, and no dynamic load component in the longitudial direction. Even though the ratios 0.5, 0.3 and 0.1 are approximate, they are useful in that being conservative, a safe structure will result. Since the *strength* is usually not a controlling design factor, many designers have traditionally used equivalent static loads in their strength check. A dynamic analysis, coupled with the help of the computer, will give the true dynamic forces that act on the structure in addition to the natural frequencies and displacements.

In many cases, the structure or soil parameters are known only within certain limits. For example, the shear modulus of the soil may vary by 25% or more at points below the foundation as described in Chapter 4. The effect of these variations may be studied by making additional computer runs and varying the parameter in question. Thus, the behavior of the structure may be predicted for probable *ranges* of parameter values. This feature of computer use is important since the possible variation of some parameters may be rather wide and strongly affect the results.

Computer coding and software applications for the solution of structures supporting dynamic loads are considered in this chapter. An analysis of a dynamically loaded structure is performed to obtain the following information:

1. Forces and deflections in members and joints for all static loading conditions. This will determine if the structure is statically safe or if deformations exceed tolerable limits. For structures supporting dynamic machines, the members are usually very large and massive and the stresses and deflections will be well within tolerable limits. This situation is a direct result of initial trial sizing of the structure where the mass of the supporting structure is made several times the mass of the machine as described in the section on Trial Sizing of Elevated Foundations (Table Tops) of Chapter 3.
2. A dynamic analysis is also performed to determine the natural frequencies or eigenvalues of the structure, the mode shapes or eigenvectors, and the displacements and member forces at a number of time intervals.

The dynamic analysis technique used in most computer programs is called a normal mode (or modal) technique which results in the calculation of the frequencies and mode shapes which in turn are used for the response calculations. The method is termed normal because the equations of motion (one per dynamic degree of freedom) are transformed to a new coordinate system called normal coordinates, resulting in uncoupled linear equations leading to a relatively efficient solution process.

The primary purpose of the dynamic analysis is to ascertain possible resonance conditions, that is, to determine if any of the structure natural frequencies coincide with the machine acting frequency or any of its critical speeds. A true dynamic analysis is sometimes replaced by a static analysis by using the Rayleigh method to calculate the lowest natural frequencies. The calculation of the Raleigh frequencies is a very simple and inexpensive feature when used with a computer static analysis, and some designers will only perform a static computer analysis with Rayleigh natural frequencies calculation. However, only a complete dynamic analysis will provide the necessary information for predicting the behavior of a structure supporting time-dependent loads.

Example Problem

An example of computer coding for the solution of an elevated foundation is given in the following pages. This example has been selected to illustrate the use of the popular computer software package, STRUDL (*Str*uctural *D*esign *L*anguage, part of the MIT-developed *I*ntegrated *C*ivil *E*ngineering *S*ystem, ICES) applied to the analysis of an elevated foundation (also called a table top).

The structure shown in Figure 7-1 is analyzed using the software package mentioned above. The structure trial dimensions are selected to meet certain preliminary criteria as described in Chapter 3 under Trial Sizing of Elevated Foundations (Table Tops).

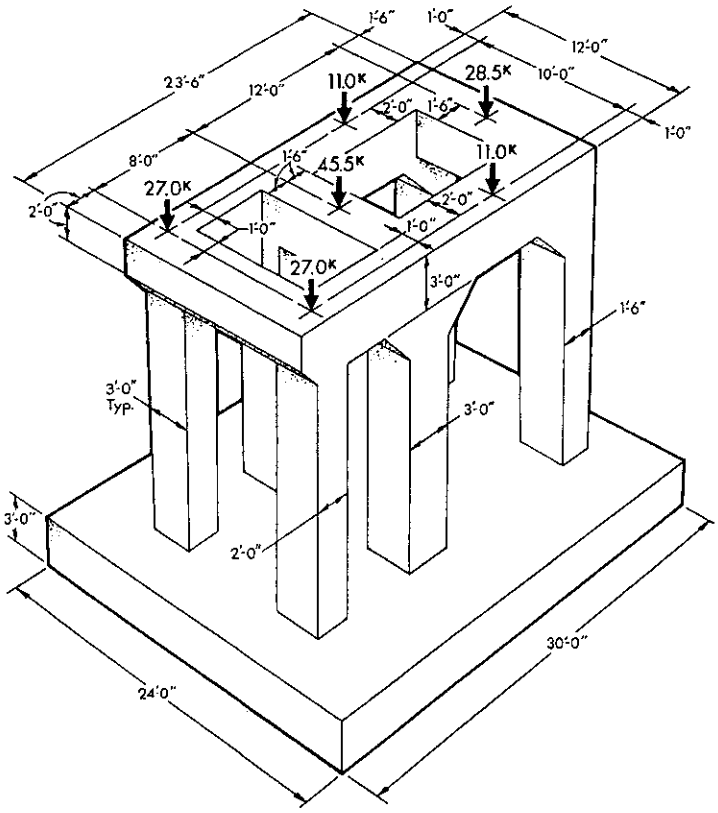


Figure 7-1. Typical elevated pedestal foundation (table top).

A. Machine Parameters

Total machine weight = 150,000 lbs.

Turbine speed = 6,949 rpm or ω = 727.7 rad/sec

Compressor speed = 6,949 rpm or ω = 727.7 rad/sec

Turbine rotor weight = 159 lbs.

Amplitude of turbine force = $(W/g)\,e\omega^2$

$= [159/(32.2 \times 12)] \times (1.0\sqrt{12{,}000/6{,}949}/1{,}000) \times (727.7)^2$

$= 286$ lbs

where e is obtained from Table 3-1.

Compressor rotor weight = 4,328 lbs.

Amplitude of compressor force = $(W/g)\,e\omega^2$

$= [4{,}328/(32.2 \times 12)] \times (1.0\sqrt{12{,}000/6{,}949}/1{,}000) \times (727.7)^2$

$= 7{,}794$ lbs.

B. Soil Parameters

Shear modulus at the expected bearing pressure, see Chapter 4,

Shear Modulus, $G = 6{,}500$ psi

Poisson's ratio, $\nu = 0.45$

Coefficient of subgrade reaction, $k_r = 120$ lbs./in.^{3}

Soil density $\gamma = 115$ pcf

Allowable bearing capacity = 2,000 psf

Predicted static settlement = 0.2 in. at 2,000 psf

C. Selection of Foundation Configuration

Selection of a trial configuration is accomplished by following the guidelines described under Trial Sizing of Elevated Foundation (Table Tops) of Chapter 3:

1. Machine and piping requirements dictate the plan arrangement of the top of the foundation, as shown in Figure 7-1.
2. A mat foundation is recommended by the soil consultant. The column spans are 12 ft and 8 ft; thus, the mat thickness is at least

$$t = 0.07\ (10)^{4/3} = 1.51 \text{ ft}$$

Try a 3-ft mat. The thickness of the mat will also be at least one tenth of its largest dimension to assure rigid behavior. The relative stiffness dimension, Equation (5-1), is

$$l_s = \left[\frac{Et^3}{12\,(1 - \nu_m^2)\,k_r} \right]^{0.25}$$

The modulus of elasticity of concrete is 3,122,000 psi, and its Poisson's ratio is 0.17; therefore,

$$l_s = \left[\frac{3{,}122{,}000 \times (36)^3}{12\,(1 - 0.17^2)\,120} \right]^{0.25}$$

$$= 101.0 \text{ in.} = 8.42 \text{ ft}$$

A flexible mat is one whose outside dimensions exceed $3l_s$ or 25.3 ft in each direction and is loaded over a small area, as described in Chapter 5. Therefore, a 24 ft by 30 ft by 3 ft mat may be considered as rigid since the load is spread out over most of the mat.

3–5. The cross-sectional dimensions of columns and beams are selected according to these guidelines.

6. The ratio of mass of structure to mass of machine is

   $423{,}000/150{,}000 = 2.82 \sim 3$ O.K.

7. The mass of the top half of the structure is 0.8 times the mass of the machine (the ratio should preferably exceed 1.0).

8. The maximum static pressure is

   $(423{,}000 + 150{,}000)/(24 \times 30) = 796$ psf
   $\leq 0.5 \times 2{,}000$ psf O.K.

9. The center of resistance of the soil is found to coincide with the centroid of all superimposed loads (structure plus machine).

10. The center of column resistance found as shown in Figure 3-2 coincides with the center of gravity of the equipment plus the top half of the structure.

11. Column and beam deflections are checked in the computer analysis that follows.

12. Column resonance check shows no column resonance with the acting machine frequency (6,949 rpm). For example, for all the columns,

   $p = 44.34$ psi
   $L = 168$ in. (clear height of columns)
   $f_n = 44{,}800\,(3{,}000)^{0.25}/\sqrt{44.34 \times 168}$
   $= 3{,}842$ rpm

Thus, the trial design is judged satisfactory and the dynamic analysis for the proposed configuration is then performed.

The idealized computer model is shown in Figure 7-2 where numbered joints have been located at member intersections and at other points of interest such as loading points. The structure is idealized as a Model 6 type D described in Chapter 2. The global coordinate axes are selected according to the right-hand rule with axis Y being vertical; each member is also numbered (numbers within circle in Figure 7-2), and springs are placed at joints in contact with the soil in the vertical and horizontal directions. These springs represent the resistance that the supporting soil offers to displacement, and the equivalent spring stiffness is calculated using soil properties as described in 4 below.

A flow chart of the steps that occur during the computer analysis is given in Figure 7-3 and represents a typical analysis regardless of the software package being

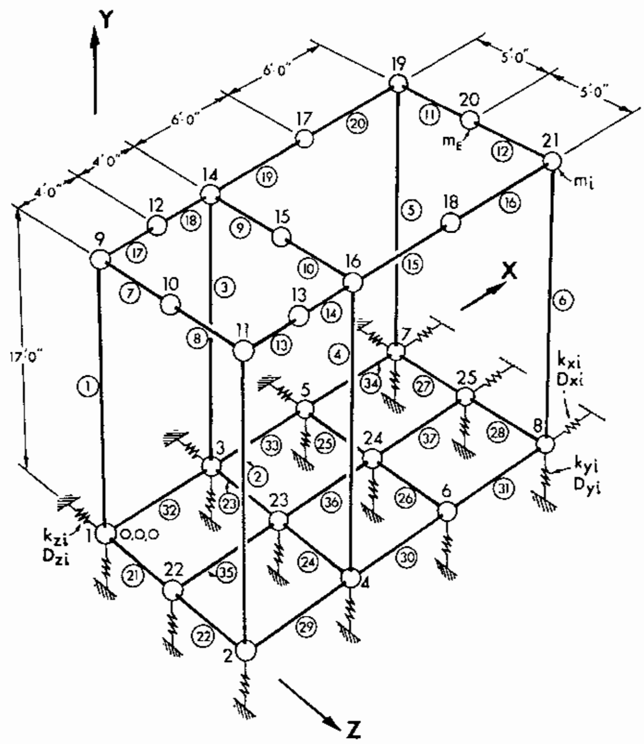


Figure 7-2. Computer model of elevated pedestal foundation.

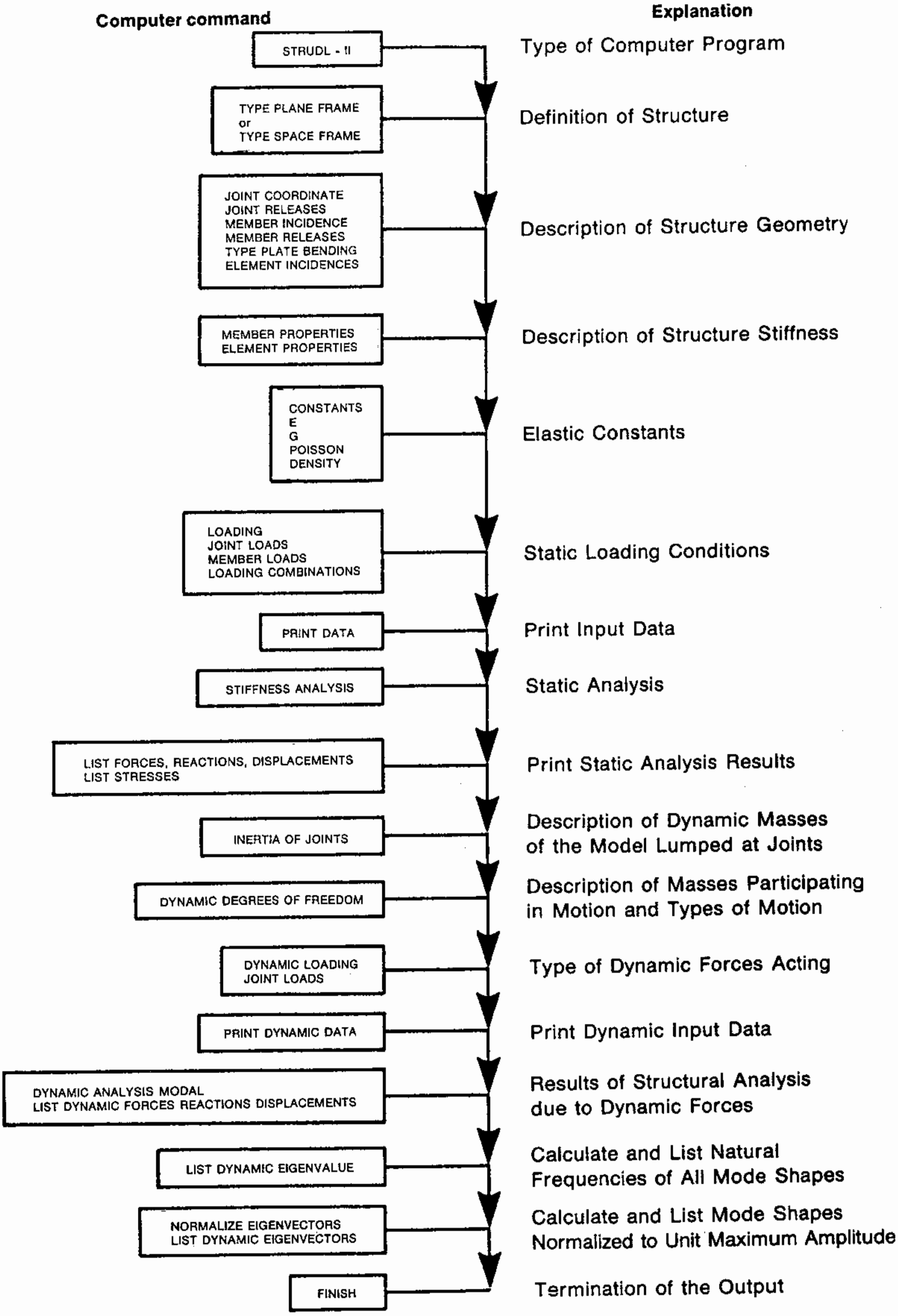


Figure 7-3. Computer program flow chart.

used. The chart lists the specific stages in an ICES-STRUDL analysis, and these are described in detail. Other software packages would include similar stages, but the exact commands and their order would be different.

Example—STRUDL Coding

STRUDL is a command structured language where the user can describe the structure in simple, almost conversational statements. The commands are given in logical order, that is, the geometry of the structure, the topology or connectivity, the member properties, the material constants, and the loads are described prior to the analysis. A fixed format is not required in either the exact order of the commands or in a precise alignment on a computer card.

A summary of STRUDL commands is given in Appendix B, and the reader is referred to the ICES-STRUDL user's manuals for additional information (ref. 2 and 3). The steps in coding a problem for STRUDL solution are given below (lines in capital letters are actual commands—one per computer card). These steps are noted in the printout on pages 121-157.*

1. Computer program and structure definition:
   STRUDL 'EXAMPLE' 'STATIC AND
   DYNAMIC ANALYSIS OF A†
   TABLE TOP'†
   TYPE SPACE FRAME

2. Geometry of the structure (Figures 7-1 and 7-2). Units are defined since the default internal units are inches and pounds. Each joint is described by its number and its x y z coordinates. Numbering of joints should be selected so that the difference in joint number at each end of all members is a minimum. The term SUPPORT identifies the joint as a support. Note that support joints have been identified under each column and midway across the slab edge and transverse equivalent "beams."

UNITS	FEET		KIPS	
JOINT COORDINATES				
1	0.0	0.0	0.0	SUPPORT
2	0.0	0.0	10.0	SUPPORT
3	8.0	0.0	0.0	SUPPORT
4	8.0	0.0	10.0	SUPPORT
	—	—	—	
15	8.0	17.0	5.0	
16	8.0	17.0	10.0	
	—	—	—	
25	20.0	0.0	5.0	SUPPORT

3. Structure topology which indicates the connectivity of the members in the structure. Each command is of the form IJK and means that member I goes from the start joint J to the end joint K. The positive sense of the forces acting on the member follows the right-hand rule when the first (axial) axis is oriented from start to end of member.

MEMBER INCIDENCES		
1	1	9
2	2	11
3	3	14
—	—	—
—	—	—
37	24	25

4. Restraint conditions at the joints. Since a supported joint is assumed rigidly supported (fixed), it is necessary to release these restraints and describe the stiffness (force per unit displacement) of springs attached to the joints. The spring stiffness is a function of the supporting soil properties and the bearing area around the joint that acts against the soil for this rigid mat as described above during trial sizing of the structure. Chapter 5 gives procedures for calculating the spring constants of rigid or flexible mats. For example, at joints 1 and 2, the contact area is (2.5 + 7) ft along Z times (5 + 4) ft along X = 85.5 sq ft. The total foundation contact area is 24 ft times 30 ft = 720 sq ft. The total foundation spring stiffness in the vertical direction (see Chapter 4) is

$$k_z = G\beta_z \sqrt{BL}\,\eta_z/(1-\nu)$$

or:

$$k_z = \frac{6{,}500 \times 144 \times 2.2 \times \sqrt{24 \times 30}}{(1-.45) \times 1{,}000} = 100{,}462 \text{ kips/ft}$$

where the terms G, ν, and η_z (equal to unity in this example) are defined in Chapter 4. Therefore, for nodes 1 and 2, $k_{y1} = k_{y2} = 100{,}462 \times 85.5/720 = 11{,}930$ kips/ft. Note that the Y direction in the computer example of Figure 7-2 is the *vertical* soil direction previously denoted as the Z-direction in the soil-spring constant equations of Chapter 4. At the risk of some confusion, the Y-direction is selected vertical in the computer analysis due to certain global-local axes advantages. Also, rotational restraints are assumed to be non-existent since they are generally negligible. Further discussion of the calculation of the spring constants in the case of larger flexible mats is presented in Chapter 5.

* The symbol † denotes continuation of the previous line on a single card.

```
UNITS KIPS FEET
JOINT RELEASES
1, 2 MOMENT  X Y Z KFX 8649. KFY 11930.
    KFZ 8649.†
  — — — —
  — — — —
25 MOMENT  X Y Z KFX 4046. KFY 5581.
    KFZ 4046.†
```

The third command listed above means that joints 1 and 2 have X, Y, and Z rotational freedom and linear springs with 8,649 kips/ft in the global X- and Z-directions (horizontal) and 11,930 kips/ft in the Y-global direction (vertical). The horizontal springs are placed at exterior joints although equivalent horizontal springs could have also been distributed among all joints in contact with the soil. No significant difference would be detected in the results.

5. Cross-sectional properties for the member. The cross-sectional area and the moments of inertia for each member about its own *local* coordinate axis are given. The local x-axis is always directed along the member while the local z-axis is parallel to the global Z-axis for columns and is horizontal for beams, *provided* that the global Y-axis is set vertical. For further details on local-global axes relationships, the reader is referred to the MIT STRUDL User's Manual, Volume I (ref. 2). Modifications are necessary if the local Z- and Y-axes are not parallel to the global axes. The areas AY and AZ are disregarded and the analysis will, therefore, assume the members to have no shear stiffness. For frames consisting of relatively long and shallow members, no significant difference can be detected by neglecting the shear stiffness of the members.

```
MEMBER PROPERTIES
1, 2 PRIS  AX  6.0  IX  4.7  IY  4.5  IZ  2.0
— — —
35, 36, 37  PRIS  AX  12.5  IX  17.9  IY  26.  IZ  6.5
```

The second command means that members 1 and 2 are prismatic, have cross-sectional areas of 6 sq ft, torsional moments of inertia of 4.7 ft^4, moments of inertia around the local y-axis of 4.5 ft^4, and moments of inertia of 2 ft^4 around the local z-axis. Note that for these 3 ft $\times$ 2 ft columns the local z-axis is parallel to the global Z-axis and, therefore, $I_z = 3 \times 2^3/12 = 2.0$ ft^4.

6. Material Properties. The material properties include the modulus of elasticity E, the shear modulus of elasticity G, Poisson's ratio, and the material density. These constants are 3,122 ksi, 1,334 ksi, 0.17, and 0.0868 lbs./cu in., respectively. Density is specified in lbs./cu in. for convenience; however, some STRUDL packages treat density as *mass* density, and units and magnitude of *mass* density must be entered.

```
UNITS        KIPS       INCHES
CONSTANTS
E            3122.      ALL
G            1334.      ALL
POISSON      0.17       ALL
UNITS        POUNDS     INCHES
CONSTANTS
DENSITY      0.0868     ALL
```

7. Static Loadings. A number of basic loading conditions and combinations of the basic loadings are considered.

```
UNITS      KIPS      FEET
LOADING    1         'FULL LOAD ACTING
   IN THE VERTICAL DIRECTION'†
JOINT LOADS
1, 2       FORCE     Y       – 38.4
———        ———       ———     ———
25         FORCE     Y       – 15.0
```

(other loadings and combinations follow)

```
RAYLEIGH LOADING 3 'FULL LOAD ACTING
   IN THE TRANSVERSE DIRECTION'†
JOINT LOADS
1, 2       FORCE     Z       38.4
———        ———       ———     ———
25         FORCE     Z       15.0
```

A loading for which the RAYLEIGH frequency is to be calculated is denoted as shown above for loading 3. This loading includes the weights of the structure adjacent to each joint and the weight of the machine applied in the direction of the desired vibration mode, in this case, the transverse of Z-global direction. Thus, the analysis will include a Rayleigh-Ritz calculation of the natural frequency for the structure in the transverse direction. Loading combinations as follows would be added by some designers in a static check as discussed previously. These loadings are quasi-static, that is, conservative equivalent static loadings which may be used for design check of deflections and member forces.

Total vertical load + 0.3 $\times$ total load acting in the transverse direction (Z)

Total vertical load + 0.1 $\times$ total load acting in the longitudinal direction (X)

Total vertical load + 0.5 $\times$ total load acting in the vertical direction (Y)

These are denoted Loading Combinations 4 through 6, respectively, in the computer printout.

```
LOADING COMBINATION 4 'FULL
    VERTICAL LOAD PLUS 0.3†
    FULL TRANSVERSE LOAD' —†
    COMBINE 1 1.0 3 0.3
—— —— —— —— —— ——
LOADING COMBINATION 6 'FULL
    VERTICAL LOAD PLUS 0.5†
    FULL VERTICAL LOAD' —†
    COMBINE 1 1.0 1 0.5
```

The first command specifies that loading combination 4 consists of loading 1 times 1 plus loading 3 times 0.3,

8. Listing of all data. A printout of all internal data is requested with the command PRINT DATA ALL.

9. Geometry plotting. Structure geometry plots are requested as a further check of the input data.

10. Static Analysis. The following command is used to build and invert the structural stiffness matrix and to solve the problem for all loadings.

```
STIFFNESS ANALYSIS (REDUCE BAND ROOT)
```

The command within parentheses is optional and generally results in a more efficient algorithm for large problems.

11. Output of results. Results are printed by using the LIST commands, such as

```
UNITS KIPS INCHES CYCLES SECONDS
LIST RAYLEIGH
OUTPUT BY MEMBER
LOADING LIST 4, 5, 6
LIST FORCES DISPLACEMENTS
    REACTIONS ALL†
```

Results for loadings 4 through 6 are requested. This step completes the static analysis. The dynamic analysis includes the following additional steps:

12. Mass acting at each joint. The structure's mass and the machine mass are taken to act at the structural joints in the three linear directions only. Inclusion of rotational inertia has a negligible effect on the results. The structure mass may be computed internally and automatically lumped at each joint and the machine mass is then added at the machine support joints.

```
UNITS POUNDS INCHES
INERTIA OF JOINTS LUMPED
INERTIA OF JOINT ADD 9, 11 LINEAR ALL
    69.88†
INERTIA OF JOINT ADD 15 LINEAR ALL
    117.75†
INERTIA OF JOINT ADD 17, 18 LINEAR ALL
    28.47†
INERTIA OF JOINT ADD 20 LINEAR ALL 73.76
```

Note that the added machine mass at joints 9 and 11 is $27{,}000/(32.2 \times 12) = 69.88$ lbs.-sec^2/in., and similarly for joints 15, 17, 18, and 20, see Figures 7-1 and 7-2. The structural mass is included through the INERTIA OF JOINTS LUMPED command.

13. Dynamic degrees of freedom—only translation modes are considered. The commands that accomplish this are

```
DYNAMIC DEGREES OF FREEDOM
JOINT 1 TO 25 DISPLACEMENT X Y Z
```

14. Damping ratio. The damping ratio for each degree of freedom is given by

```
DAMPING RATIO 0.10   75
```

Where total average damping of 0.10 has been specified for the X, Y, Z translatory degrees of freedom at each of the 25 joints, or $3 \times 25 = 75$ times. Note that, realistically, those degrees of freedom associated with foundation movement have a damping ratio in the 0.15–0.20 range, whereas damping for the remaining degrees of freedom may only be in the 0.05–0.10 range. Therefore, 0.10 is used throughout as an average value. Chapter 5 gives a further discussion on the choice of soil damping ratios.

15. Dynamic forcing function. Forcing functions in the vertical (Y-direction) and in the transverse (Z-direction) are applied at the joints where they occur. These forcing functions include a force amplitude equal to the unbalanced machine force, a frequency given by the acting machine frequency (in radians) and a phase angle of 1.5707 radians (90°) for the transverse functions, i.e., the transverse function is 90° out of phase to the vertical function in this centrifugal machine. The dynamic forcing functions are applied at the centerline of the centrifugal machine shaft, joints 10, 15, and 20. One half of the turbine force acts at joint 10 or $0.5 \times 286 \sin 727.7t$, one half of the turbine and compressor forces act at joint 15 or $0.5 \times 286 \sin 727.7t + 0.5 \times 7{,}794 \sin 727.7t$, and one half of the compressor force acts at joint 20 or $0.5 \times 7{,}794 \sin 727.7t$.

```
UNITS RADIANS SECONDS POUNDS
    INCHES†
DYNAMIC LOADING 7 'CENTRIFUGAL
    FORCES'†
JOINT 10 LOAD FORCE Y FUN SIN AMPL 143.
    FREQ 727.7†
```

```
JOINT 10 LOAD FORCE Z FUN SIN AMPL 143.
    FREQ 727.7 PHASE 1.5707†
JOINT 15 LOAD FORCE Y FUN SIN AMPL
    4040. FREQ 727.7†
JOINT 15 LOAD FORCE Z FUN SIN AMPL
    4040. FREQ 727.7 PHASE 1.5707†
JOINT 20 LOAD FORCE Y FUN SIN AMPL
    3897. FREQ 727.7†
JOINT 20 LOAD FORCE Z FUN SIN AMPL
    3897. FREQ 727.7 PHASE 1.5707†
```

16. Time Periods. The time span and time increments for the dynamic analysis must be specified. The integration time periods should include, as a minimum, 12 steps per single complete operating frequency cycle, in order to achieve a $\pm 5\%$ accuracy in the results (ref. 4), that is, for a frequency of 727.7 radians/sec, the integration time periods should not be greater than T (one cycle) $= 2\pi/727.7 = 0.00863$ sec; then,

$$\Delta t = T/12 = 0.00863/12 = 0.0007194 \text{ sec}$$

The following command includes 10 complete cycles of machine operation with 12 steps in each cycle. However, 3 complete cycles of operation may be sufficient to study the response of the structure.

```
INTEGRATE FROM 0.0 to 0.0863 AT 0.0007194
```

17. Listing of dynamic data. A printout of all dynamic data is obtained with the command

```
PRINT DYNAMIC DATA ALL
```

18. Dynamic analysis. The actual dynamic analysis is obtained with the following command with the part within parentheses being optional. Only the first 20 modes are included in the analysis to reduce computing time with negligible loss of accuracy.

```
DYNAMIC ANALYSIS MODAL
    (REDUCE BAND ROOT) 20†
```

19. Output of dynamic analysis. The natural frequencies, the modes (normalized), the displacements, and forces for each time increment are requested with this command. The first 20 modes are requested to conserve paper but more may be printed.

```
UNITS KIPS INCHES CYCLES SECONDS
LIST DYNAMIC EIGENVALUES 20
NORMALIZE EIGENVECTORS
LIST DYNAMIC EIGENVECTORS 20
LIST DYNAMIC DISPLACEMENTS ALL
LIST DYNAMIC FORCES ALL
```

20. End of analysis. The last command in the job is

```
FINISH
```

(text continued on page 157)

Computer Printout in ICES-STRUDL

(1)

```
STRUDL 'EXAMPLE'  'STATIC AND DYNAMIC ANALYSIS OF A TABLE-TOP'

        ********************************************
        *                                          *
        *                                          *
        *              ICES STRUDL-II              *
        *                                          *
        *      THE STRUCTURAL DESIGN LANGUAGE      *
        *                                          *
        *   CIVIL ENGINEERING SYSTEMS LABORATORY   *
        *   MASSACHUSETTS INSTITUTE OF TECHNOLOGY  *
        *        CAMBRIDGE, MASSACHUSETTS          *
        *        V2 M2        JUNE, 1972           *
        *                                          *
        *                                          *
        *                                          *
        ********************************************
```

(2)

```
TYPE SPACE FRAME
UNITS FEET KIPS
JOINTS COORDINATES
1   0.0   0.0   0.0   SUPPORT
2   0.0   0.0  10.0   SUPPORT
3   8.0   0.0   0.0   SUPPORT
4   8.0   0.0  10.0   SUPPORT
```

```
5  14.0  0.0  0.0  SUPPORT
6  14.0  0.0 10.0  SUPPORT
7  20.0  0.0  0.0  SUPPORT
8  20.0  0.0 10.0  SUPPORT
9   0.0 17.0  0.0
10  0.0 17.0  5.0
11  0.0 17.0 10.0
12  4.0 17.0  0.0
13  4.0 17.0 10.0
14  8.0 17.0  0.0
15  8.0 17.0  5.0
16  8.0 17.0 10.0
17 14.0 17.0  0.0
18 14.0 17.0 10.0
19 20.0 17.0  0.0
20 20.0 17.0  5.0
21 20.0 17.0 10.0
22  0.0  0.0  5.0  SUPPORT
23  8.0  0.0  5.0  SUPPORT
24 14.0  0.0  5.0  SUPPORT
25 20.0  0.0  5.0  SUPPORT
$
MEMBER INCIDENCES
1    1    9
2    2    11
3    3    14
4    4    16
5    7    19
6    8    21
7    9    10
8    10   11
9    14   15
10   15   16
11   19   20
12   20   21
13   11   13
14   13   16
15   16   18
16   18   21
17   9    12
18   12   14
19   14   17
20   17   19
21   1    22
22   22   2
23   3    23
24   23   4
```

(3)

```
25   5   24
26   24  6
27   7   25
28   25  8,
29   2   4
30   4   6
31   6   8
32   1   3
33   3   5
34   5   7
35   22  23
36   23  24
37   24  25
$
JOINT RELEASES
$ JOINT RELEASES INCLUDE EQUIVALENT SPRING STIFFNESS FOR SOIL
1,2 MOMENT X Y Z KFX 8649.0  KFY 11930.  KFZ 8649.0
3,4 MOMENT X Y Z KFX 6727.0  KFY  9279.  KFZ 6727.0
5,6 MOMENT X Y Z KFX 5766.0  KFY  7953.  KFZ 5766.0
7,8 MOMENT X Y Z KFX 7688.0  KFY 10604.  KFZ 7688.0
22  MOMENT X Y Z  KFX 4552.0  KFY  6279.0  KFZ  4552.0
23  MOMENT X Y Z  KFX 3541.0  KFY  4884.0  KFZ  3541.0
24  MOMENT X Y Z  KFX  3035.0  KFY  4186.0  KFZ  3035.0
25  MOMENT X Y Z  KFX  4046.0  KFY  5581.0  KFZ  4046.0
$
MEMBER PROPERTIES
$ UNITS OF AREA ARE FT.SQ. AND UNITS OF MOM.OF INERTIA ARE FT.X 4
1,2  PRISMATIC AX 6.0 IX 4.7 IY 4.5 IZ 2.0
3,4  PRISMATIC AX 9.0 IX 11.4 IY 6.8 IZ 6.8
5,6  PRISMATIC AX 4.5 IX 2.3 IY 3.8 IZ 0.8
7,8  PRISMATIC AX 9.0 IX 11.1 IY 6.8 IZ 6.8
9,10  PRISMATIC AX 6.0 IX 4.7 IY 2.0 IZ 4.5
11,12  PRISMATIC AX 4.5 IX 2.3 IY 0.8 IZ 3.8
13,14,17,18  PRISMATIC AX 6.0 IX 4.7 IY 2.0 IZ 4.5
15,16,19,20  PRISMATIC AX 7.5 IX 7.8 IY 3.91 IZ 5.6
21,22 PRISMATIC AX 22.5 IX 44.41 IY 1687.5 IZ 11.7
23,24 PRISMATIC AX 17.5 IX 34.54 IY 1312.5 IZ 9.1
25,26 PRISMATIC AX 15.0 IX 29.61 IY 1125. IZ 7.8
27,28 PRISMATIC AX 20.0 IX 39.48 IY 1500. IZ 10.4
29 TO 34 PRISMATIC AX 23.8 IX 46.23 IY 1140. IZ 12.4
35 TO 37 PRISMATIC AX 12.5 IX 24.33 IY 600. IZ 6.5
$
UNITS INCHES KIPS
CONSTANTS
   E  3122.0  ALL
   G  1334.   ALL
   POISSON  0.17  ALL
```

(4)

(5)

(6)

```
UNITS POUNDS INCHES
CONSTANTS DENSITY 0.0868  ALL
$
UNITS KIPS FEET
LOADING 1  'FULL LOAD ACTING IN THE VERTICAL DIRECTION'
JOINT LOADS
1,2 FORCE Y -38.4
3,4 FORCE Y -34.4
5,6 FORCE Y -21.4
7,8 FORCE Y -33.2
9,11 FORCE Y -38.5
10 FORCE Y -6.8
12,13 FORCE Y -3.6
14,16 FORCE Y -16.9
15 FORCE Y -50.0
17,18 FORCE Y -17.8
19,21 FORCE Y -9.8
20 FORCE Y -31.9
22  FORCE Y -16.9
23  FORCE Y -13.1
24  FORCE Y -11.3
25  FORCE Y -15.0
LOADING 2  'FULL LOAD ACTING IN THE LONGITUDINAL DIRECTION'
JOINT LOADS
1,2 FORCE X 38.4
3,4 FORCE X 34.4
5,6 FORCE X 21.4
7,8 FORCE X 33.2
9,11 FORCE X 38.5
10 FORCE X 6.8
12,13 FORCE X 3.6
14,16 FORCE X 16.9
15 FORCE X 50.0
17,18 FORCE X 17.8
19,21 FORCE X 9.8
20 FORCE X 31.9
22 FORCE X 16.9
23 FORCE X 13.1
24 FORCE X 11.3
25 FORCE X 15.0
RAYLEIGH LOADING 3  'FULL LOAD ACTING IN THE TRANSVERSE DIRECTION'
JOINT LOADS
1,2 FORCE Z 38.4
3,4 FORCE Z 34.4
5,6 FORCE Z 21.4
7,8 FORCE Z 33.2
9,11 FORCE Z 38.5
```

(7)

```
10 FORCE Z 6.8

12,13 FORCE Z 3.6

14,16 FORCE Z 16.9

15 FORCE Z 50.0

17,18 FORCE Z 17.8

19,21 FORCE Z 9.8

20 FORCE Z 31.9

22 FORCE Z 16.9

23 FORCE Z 13.1

24 FORCE Z 11.3

25 FORCE Z 15.0

LOADING COMBINATION 4 'FULL VERTICAL LOAD PLUS 0.3 FULL TRANSVERSE LOAD' -
   COMBINE 1 1.0 3 0.3

LOADING COMBINATION 5 'FULL VERTICAL LOAD PLUS 0.1 FULL LONGITUDINAL LOAD' -
   COMBINE 1 1.0 2 0.1

LOADING COMBINATION 6 'FULL VERTICAL LOAD PLUS 0.5 FULL VERTICAL LOAD' -
   COMBINE 1 1.0 1 0.5

$

PRINT DATA ALL
```

(8)

```
******************************************
*  PROBLEM DATA FROM INTERNAL STORAGE  *
******************************************

JOB ID - EXAMPLE      JOB TITLE - STATIC AND DYNAMIC ANALYSIS OF A TABLE-TOP

ACTIVE UNITS - LENGTH      WEIGHT      ANGLE      TEMPERATURE      TIME
               FEET        KIP         RAD        DEGF             SEC

          ********** STRUCTURAL DATA **********

ACTIVE STRUCTURE TYPE - SPACE   FRAME

ACTIVE COORDINATE AXES   X Y Z
```

JOINT COORDINATES--/ STATUS---/

JOINT	X	Y	Z	CONDITION		
1	0.0	0.0	0.0	SUPPORT	ACTIVE	GLOBAL
2	0.0	0.0	10.000	SUPPORT	ACTIVE	GLOBAL
3	8.000	0.0	0.0	SUPPORT	ACTIVE	GLOBAL
4	8.000	0.0	10.000	SUPPORT	ACTIVE	GLOBAL
5	14.000	0.0	0.0	SUPPORT	ACTIVE	GLOBAL
6	14.000	0.0	10.000	SUPPORT	ACTIVE	GLOBAL
7	20.000	0.0	0.0	SUPPORT	ACTIVE	GLOBAL
8	20.000	0.0	10.000	SUPPORT	ACTIVE	GLOBAL
9	0.0	17.000	0.0		ACTIVE	GLOBAL
10	0.0	17.000	5.000		ACTIVE	GLOBAL
11	0.0	17.000	10.000		ACTIVE	GLOBAL
12	4.000	17.000	0.0		ACTIVE	GLOBAL
13	4.000	17.000	10.000		ACTIVE	GLOBAL
14	8.000	17.000	0.0		ACTIVE	GLOBAL
15	8.000	17.000	5.000		ACTIVE	GLOBAL
16	8.000	17.000	10.000		ACTIVE	GLOBAL
17	14.000	17.000	0.0		ACTIVE	GLOBAL
18	14.000	17.000	10.000		ACTIVE	GLOBAL
19	20.000	17.000	0.0		ACTIVE	GLOBAL
20	20.000	17.000	5.000		ACTIVE	GLOBAL
21	20.000	17.000	10.000		ACTIVE	GLOBAL
22	0.0	0.0	5.000	SUPPORT	ACTIVE	GLOBAL
23	8.000	0.0	5.000	SUPPORT	ACTIVE	GLOBAL
24	14.000	0.0	5.000	SUPPORT	ACTIVE	GLOBAL
25	20.000	0.0	5.000	SUPPORT	ACTIVE	GLOBAL

JOINT RELEASES-------------------------------------/ELASTIC SUPPORT RELEASES------------------------------

JOINT	FORCE	MOMENT	THETA 1	THETA 2	THETA 3	KFX	KFY	KFZ	KMX	KMY	KMZ
1		X Y Z	0.0	0.0	0.0	8648.996	11929.996	8648.996	0.0	0.0	0.0
2		X Y Z	0.0	0.0	0.0	8648.996	11929.996	8648.996	0.0	0.0	0.0
3		X Y Z	0.0	0.0	0.0	6726.996	9278.996	6726.996	0.0	0.0	0.0

4	X Y Z	0.0	0.0	0.0	6726.996	9278.996	6726.996	0.0	0.0	0.0
5	X Y Z	0.0	0.0	0.0	5765.996	7952.996	5765.996	0.0	0.0	0.0
6	X Y Z	0.0	0.0	0.0	5765.996	7952.996	5765.996	0.0	0.0	0.0
7	X Y Z	0.0	0.0	0.0	7687.996	10603.996	7687.996	0.0	0.0	0.0
8	X Y Z	0.0	0.0	0.0	7687.996	10603.996	7687.996	0.0	0.0	0.0
22	X Y Z	0.0	0.0	0.0	4551.996	6278.996	4551.996	0.0	0.0	0.0
23	X Y Z	0.0	0.0	0.0	3540.999	4883.996	3540.999	0.0	0.0	0.0
24	X Y Z	0.0	0.0	0.0	3034.999	4185.996	3034.999	0.0	0.0	0.0
25	X Y 7	0.0	0.0	0.0	4045.999	5580.996	4045.999	0.0	0.0	0.0

MEMBER INCIDENCES----------/ LENGTH------/ RELEASES---------------------/ STATUS--/

MEMBER	START	END	LOCAL COORD.	START FORCE	START MOMENT	END FORCE	END MOMENT	STATUS		
1	1	9	17.000					ACTIVE	SPACE	FRAME
2	2	11	17.000					ACTIVE	SPACE	FRAME
3	3	14	17.000					ACTIVE	SPACE	FRAME
4	4	16	17.000					ACTIVE	SPACE	FRAME
5	7	19	17.000					ACTIVE	SPACE	FRAME
6	8	21	17.000					ACTIVE	SPACE	FRAME
7	9	10	5.000					ACTIVE	SPACE	FRAME
8	10	11	5.000					ACTIVE	SPACE	FRAME
9	14	15	5.000					ACTIVE	SPACE	FRAME
10	15	16	5.000					ACTIVE	SPACE	FRAME
11	19	20	5.000					ACTIVE	SPACE	FRAME
12	20	21	5.000					ACTIVE	SPACE	FRAME
13	11	13	4.000					ACTIVE	SPACE	FRAME
14	13	16	4.000					ACTIVE	SPACE	FRAME
15	16	18	6.000					ACTIVE	SPACE	FRAME
16	18	21	6.000					ACTIVE	SPACE	FRAME
17	9	12	4.000					ACTIVE	SPACE	FRAME
18	12	14	4.000					ACTIVE	SPACE	FRAME
19	14	17	6.000					ACTIVE	SPACE	FRAME
20	17	19	6.000					ACTIVE	SPACE	FRAME
21	1	22	5.000					ACTIVE	SPACE	FRAME
22	22	2	5.000					ACTIVE	SPACE	FRAME
23	3	23	5.000					ACTIVE	SPACE	FRAME
24	23	4	5.000					ACTIVE	SPACE	FRAME
25	5	24	5.000					ACTIVE	SPACE	FRAME
26	24	6	5.000					ACTIVE	SPACE	FRAME
27	7	25	5.000					ACTIVE	SPACE	FRAME
28	25	8	5.000					ACTIVE	SPACE	FRAME
29	2	4	8.000					ACTIVE	SPACE	FRAME
30	4	6	6.000					ACTIVE	SPACE	FRAME
31	6	8	6.000					ACTIVE	SPACE	FRAME
32	1	3	8.000					ACTIVE	SPACE	FRAME
33	3	5	6.000					ACTIVE	SPACE	FRAME
34	5	7	6.000					ACTIVE	SPACE	FRAME
35	22	23	8.000					ACTIVE	SPACE	FRAME
36	23	24	6.000					ACTIVE	SPACE	FRAME
37	24	25	6.000					ACTIVE	SPACE	FRAME

MEMBER PROPERTIES--/

MEMBER/SEG	TYPE	SEG.L	COMP	AX/YD	AY/ZD	AZ/YC	IX/ZC	IY/EY	IZ/EZ	SY	SZ
1	PRISMATIC			6.000	0.0	0.0	4.700	4.500	2.000	0.0	0.0
				0.0	0.0	0.0	0.0	0.0	0.0		
2	PRISMATIC			6.000	0.0	0.0	4.700	4.500	2.000	0.0	0.0
				0.0	0.0	0.0	0.0	0.0	0.0		
3	PRISMATIC			9.000	0.0	0.0	11.400	6.800	6.800	0.0	0.0
				0.0	0.0	0.0	0.0	0.0	0.0		
4	PRISMATIC			9.000	0.0	0.0	11.400	6.800	6.800	0.0	0.0
				0.0	0.0	0.0	0.0	0.0	0.0		
5	PRISMATIC			4.500	0.0	0.0	2.300	3.800	0.800	0.0	0.0
				0.0	0.0	0.0	0.0	0.0	0.0		
6	PRISMATIC			4.500	0.0	0.0	2.300	3.800	0.800	0.0	0.0
				0.0	0.0	0.0	0.0	0.0	0.0		
7	PRISMATIC			9.000	0.0	0.0	11.100	6.800	6.800	0.0	0.0
				0.0	0.0	0.0	0.0	0.0	0.0		
8	PRISMATIC			9.000	0.0	0.0	11.100	6.800	6.800	0.0	0.0
				0.0	0.0	0.0	0.0	0.0	0.0		
9	PRISMATIC			6.000	0.0	0.0	4.700	2.000	4.500	0.0	0.0
				0.0	0.0	0.0	0.0	0.0	0.0		
10	PRISMATIC			6.000	0.0	0.0	4.700	2.000	4.500	0.0	0.0
				0.0	0.0	0.0	0.0	0.0	0.0		
11	PRISMATIC			4.500	0.0	0.0	2.300	0.800	3.800	0.0	0.0
				0.0	0.0	0.0	0.0	0.0	0.0		
12	PRISMATIC			4.500	0.0	0.0	2.300	0.800	3.800	0.0	0.0
				0.0	0.0	0.0	0.0	0.0	0.0		
13	PRISMATIC			6.000	0.0	0.0	4.700	2.000	4.500	0.0	0.0
				0.0	0.0	0.0	0.0	0.0	0.0		
14	PRISMATIC			6.000	0.0	0.0	4.700	2.000	4.500	0.0	0.0
				0.0	0.0	0.0	0.0	0.0	0.0		
15	PRISMATIC			7.500	0.0	0.0	7.800	3.910	5.600	0.0	0.0
				0.0	0.0	0.0	0.0	0.0	0.0		

16	PRISMATIC	7.500	0.0	0.0	7.800	3.910	5.600	0.0	0.0
		0.0	0.0	0.0	0.0	0.0	0.0		
17	PRISMATIC	6.000	0.0	0.0	4.700	2.000	4.500	0.0	0.0
		0.0	0.0	0.0	0.0	0.0	0.0		
18	PRISMATIC	6.000	0.0	0.0	4.700	2.000	4.500	0.0	0.0
		0.0	0.0	0.0	0.0	0.0	0.0		
19	PRISMATIC	7.500	0.0	0.0	7.800	3.910	5.600	0.0	0.0
		0.0	0.0	0.0	0.0	0.0	0.0		
20	PRISMATIC	7.500	0.0	0.0	7.800	3.910	5.600	0.0	0.0
		0.0	0.0	0.0	0.0	0.0	0.0		
21	PRISMATIC	22.500	0.0	0.0	44.410	1687.498	11.700	0.0	0.0
		0.0	0.0	0.0	0.0	0.0	0.0		
22	PRISMATIC	22.500	0.0	0.0	44.410	1687.498	11.700	0.0	0.0
		0.0	0.0	0.0	0.0	0.0	0.0		
23	PRISMATIC	17.500	0.0	0.0	34.540	1312.498	9.100	0.0	0.0
		0.0	0.0	0.0	0.0	0.0	0.0		
24	PRISMATIC	17.500	0.0	0.0	34.540	1312.498	9.100	0.0	0.0
		0.0	0.0	0.0	0.0	0.0	0.0		
25	PRISMATIC	15.000	0.0	0.0	29.610	1124.999	7.800	0.0	0.0
		0.0	0.0	0.0	0.0	0.0	0.0		
26	PRISMATIC	15.000	0.0	0.0	29.610	1124.999	7.800	0.0	0.0
		0.0	0.0	0.0	0.0	0.0	0.0		
27	PRISMATIC	20.000	0.0	0.0	39.480	1499.998	10.400	0.0	0.0
		0.0	0.0	0.0	0.0	0.0	0.0		
28	PRISMATIC	20.000	0.0	0.0	39.480	1499.998	10.400	0.0	0.0
		0.0	0.0	0.0	0.0	0.0	0.0		
29	PRISMATIC	23.800	0.0	0.0	46.230	1139.999	12.400	0.0	0.0
		0.0	0.0	0.0	0.0	0.0	0.0		
30	PRISMATIC	23.800	0.0	0.0	46.230	1139.999	12.400	0.0	0.0
		0.0	0.0	0.0	0.0	0.0	0.0		
31	PRISMATIC	23.800	0.0	0.0	46.230	1139.999	12.400	0.0	0.0
		0.0	0.0	0.0	0.0	0.0	0.0		
32	PRISMATIC	23.800	0.0	0.0	46.230	1139.999	12.400	0.0	0.0
		0.0	0.0	0.0	0.0	0.0	0.0		
33	PRISMATIC	23.800	0.0	0.0	46.230	1139.999	12.400	0.0	0.0
		0.0	0.0	0.0	0.0	0.0	0.0		
34	PRISMATIC	23.800	0.0	0.0	46.230	1139.999	12.400	0.0	0.0
		0.0	0.0	0.0	0.0	0.0	0.0		
35	PRISMATIC	12.500	0.0	0.0	24.330	599.999	6.500	0.0	0.0
		0.0	0.0	0.0	0.0	0.0	0.0		
36	PRISMATIC	12.500	0.0	0.0	24.330	599.999	6.500	0.0	0.0
		0.0	0.0	0.0	0.0	0.0	0.0		
37	PRISMATIC	12.500	0.0	0.0	24.330	599.999	6.500	0.0	0.0

STRUDL DATA SET

CONSTRAINT DICTIONARY------------/

NAME	RETRIEVAL
AX	TABULAR
AY	TABULAR
AZ	TABULAR
IX	TABULAR
IY	TABULAR
IZ	TABULAR
SY	TABULAR
SZ	TABULAR
YD	TABULAR
ZD	TABULAR
FLTK	TABULAR
WBTK	TABULAR
YD/AFL	TABULAR
RY	TABULAR
RZ	TABULAR
COMP	TABULAR
YC	TABULAR
ZC	TABULAR
WEIGHT	TABULAR

********** LOADING DATA **********

LOADING - 1 FULL LOAD ACTING IN THE VERTICAL DIRECTION STATUS - ACTIVE

MEMBER AND ELEMENT LOADS--/
MEMBER/ELEMENT

JOINT LOADS---------------------------------------/ -----------------------------------/

JOINT	STEP	FORCE X	Y	Z	MOMENT X	Y	Z
1		0.0	-38.400	0.0	0.0	0.0	0.0
2		0.0	-38.400	0.0	0.0	0.0	0.0
3		0.0	-34.400	0.0	0.0	0.0	0.0
4		0.0	-34.400	0.0	0.0	0.0	0.0
5		0.0	-21.400	0.0	0.0	0.0	0.0
6		0.0	-21.400	0.0	0.0	0.0	0.0
7		0.0	-33.200	0.0	0.0	0.0	0.0
8		0.0	-33.200	0.0	0.0	0.0	0.0
9		0.0	-38.500	0.0	0.0	0.0	0.0
10		0.0	-6.800	0.0	0.0	0.0	0.0
11		0.0	-38.500	0.0	0.0	0.0	0.0

12		0.0	-3.600	0.0	0.0	0.0	0.0
13		0.0	-3.600	0.0	0.0	0.0	0.0
14		0.0	-16.900	0.0	0.0	0.0	0.0
15		0.0	-50.000	0.0	0.0	0.0	0.0
16		0.0	-16.900	0.0	0.0	0.0	0.0
17		0.0	-17.800	0.0	0.0	0.0	0.0
18		0.0	-17.800	0.0	0.0	0.0	0.0
19		0.0	-9.800	0.0	0.0	0.0	0.0
20		0.0	-31.900	0.0	0.0	0.0	0.0
21		0.0	-9.800	0.0	0.0	0.0	0.0
22		0.0	-16.900	0.0	0.0	0.0	0.0
23		0.0	-13.100	0.0	0.0	0.0	0.0
24		0.0	-11.300	0.0	0.0	0.0	0.0
25		0.0	-15.000	0.0	0.0	0.0	0.0

JOINT DISPLACEMENTS-----------------------------/ -----------------------------/

JOINT	STEP	DISP. X	Y	Z	ROT. X	Y	Z

JOINT FORCE ASSUMPTIONS ---/

JOINT	THETA 1	2	3	FORCE X	Y	Z	MOMENT X	Y	Z

NO ASSUMPTIONS GIVEN FOR THIS LOADING

MEMBER FORCE ASSUMPTIONS ---------------------/ ---------------------------------/

MEMBER	COMPONENT	DISTANCE	VALUE	COMPONENT	DISTANCE	VALUE

NO ASSUMPTIONS GIVEN FOR THIS LOADING

LOADING - 2 FULL LOAD ACTING IN THE LONGITUDINAL DIRECTION STATUS - ACTIVE

MEMBER AND ELEMENT LOADS---/
MEMBER/ELEMENT

JOINT LOADS---/ ----------------------------------/

JOINT	STEP	FORCE X	Y	Z	MOMENT X	Y	Z
1		38.400	0.0	0.0	0.0	0.0	0.0
2		38.400	0.0	0.0	0.0	0.0	0.0
3		34.400	0.0	0.0	0.0	0.0	0.0
4		34.400	0.0	0.0	0.0	0.0	0.0
5		21.400	0.0	0.0	0.0	0.0	0.0
6		21.400	0.0	0.0	0.0	0.0	0.0
7		33.200	0.0	0.0	0.0	0.0	0.0
8		33.200	0.0	0.0	0.0	0.0	0.0
9		38.500	0.0	0.0	0.0	0.0	0.0
10		6.800	0.0	0.0	0.0	0.0	0.0
11		38.500	0.0	0.0	0.0	0.0	0.0
12		3.600	0.0	0.0	0.0	0.0	0.0
13		3.600	0.0	0.0	0.0	0.0	0.0
14		16.900	0.0	0.0	0.0	0.0	0.0
15		50.000	0.0	0.0	0.0	0.0	0.0
16		16.900	0.0	0.0	0.0	0.0	0.0
17		17.800	0.0	0.0	0.0	0.0	0.0
18		17.800	0.0	0.0	0.0	0.0	0.0
19		9.800	0.0	0.0	0.0	0.0	0.0
20		31.900	0.0	0.0	0.0	0.0	0.0
21		9.800	0.0	0.0	0.0	0.0	0.0
22		16.900	0.0	0.0	0.0	0.0	0.0
23		13.100	0.0	0.0	0.0	0.0	0.0
24		11.300	0.0	0.0	0.0	0.0	0.0
25		15.000	0.0	0.0	0.0	0.0	0.0

JOINT DISPLACEMENTS-----------------------------/ -----------------------------/

JOINT	STEP	DISP. X	Y	Z	ROT. X	Y	Z

JOINT FORCE ASSUMPTIONS ---/

JOINT	THETA 1	2	3	FORCE X	Y	Z	MOMENT X	Y	Z

NO ASSUMPTIONS GIVEN FOR THIS LOADING

MEMBER FORCE ASSUMPTIONS ---------------------/ ---------------------------------/

MEMBER	COMPONENT	DISTANCE	VALUE	COMPONENT	DISTANCE	VALUE

NO ASSUMPTIONS GIVEN FOR THIS LOADING

LOADING - 3 FULL LOAD ACTING IN THE TRANSVERSE DIRECTION STATUS - ACTIVE

MEMBER AND ELEMENT LOADS---/
MEMBER/ELEMENT

JOINT LOADS---/ ----------------------------------/

JOINT	STEP	FORCE X	Y	Z	MOMENT X	Y	Z
1		0.0	0.0	38.400	0.0	0.0	0.0
2		0.0	0.0	38.400	0.0	0.0	0.0
3		0.0	0.0	34.400	0.0	0.0	0.0
4		0.0	0.0	34.400	0.0	0.0	0.0

5	0.0	0.0	21.400	0.0	0.0	0.0
6	0.0	0.0	21.400	0.0	0.0	0.0
7	0.0	0.0	33.200	0.0	0.0	0.0
8	0.0	0.0	33.200	0.0	0.0	0.0
9	0.0	0.0	38.500	0.0	0.0	0.0
10	0.0	0.0	6.800	0.0	0.0	0.0
11	0.0	0.0	38.500	0.0	0.0	0.0
12	0.0	0.0	3.600	0.0	0.0	0.0
13	0.0	0.0	3.600	0.0	0.0	0.0
14	0.0	0.0	16.900	0.0	0.0	0.0
15	0.0	0.0	50.000	0.0	0.0	0.0
16	0.0	0.0	16.900	0.0	0.0	0.0
17	0.0	0.0	17.800	0.0	0.0	0.0
18	0.0	0.0	17.800	0.0	0.0	0.0
19	0.0	0.0	9.800	0.0	0.0	0.0
20	0.0	0.0	31.900	0.0	0.0	0.0
21	0.0	0.0	9.800	0.0	0.0	0.0
22	0.0	0.0	16.900	0.0	0.0	0.0
23	0.0	0.0	13.100	0.0	0.0	0.0
24	0.0	0.0	11.300	0.0	0.0	0.0
25	0.0	0.0	15.000	0.0	0.0	0.0

JOINT DISPLACEMENTS--------------------------------/ -----------------------------------/

JOINT STEP, DISP. X Y Z ROT. X Y Z

JOINT FORCE ASSUMPTIONS --/

JOINT THETA 1 2 3 FORCE X Y Z MOMENT X Y Z

NO ASSUMPTIONS GIVEN FOR THIS LOADING

MEMBER FORCE ASSUMPTIONS ---------------------/ ----------------------------------/

MEMBER COMPONENT DISTANCE VALUE COMPONENT DISTANCE VALUE

NO ASSUMPTIONS GIVEN FOR THIS LOADING

LOADING - 4 FULL VERTICAL LOAD PLUS 0.3 FULL TRANSVERSE LOAD STATUS - ACTIVE

COMBINATION GIVEN - 1 1.000 3 0.300

LOADING - 5 FULL VERTICAL LOAD PLUS 0.1 FULL LONGITUDINAL LOAD STATUS - ACTIVE

COMBINATION GIVEN - 1 1.000 2 0.100

LOADING - 6 FULL VERTICAL LOAD PLUS 0.5 FULL VERTICAL LOAD STATUS - ACTIVE

COMBINATION GIVEN - 1 1.000 1 0.500

****
* END OF DATA FROM INTERNAL STORAGE *
****

$ PLOTS ARE REQUESTED TO CHECK ON THE STRUCTURE GEOMETRY

UNITS FEET

PLOT DEVICE PRINTER

PLOT PLANE Y EQUALS 0.0

(9)

PLANE IDENTIFIED BY - PLANE Y EQUALS 0.0

IN PLANE JOINTS

JOINT	COORDINATES X	Y	Z
1	0.0	0.0	0.0
2	0.0	0.0	10.0000
3	8.0000	0.0	0.0
4	8.0000	0.0	10.0000
5	14.0000	0.0	0.0
6	14.0000	0.0	10.0000
7	20.0000	0.0	0.0
8	20.0000	0.0	10.0000
22	0.0	0.0	5.0000
23	8.0000	0.0	5.0000
24	14.0000	0.0	5.0000
25	20.0000	0.0	5.0000

IN PLANE MEMBERS
MEMBER INCIDENCES

MEMBER	START	END
21	1	22
32	1	3
22	22	2
29	2	4
23	3	23
33	3	5
24	23	4
30	4	6
25	5	24
34	5	7
26	24	6
31	6	8
27	7	25
28	25	8
35	22	23
36	23	24
37	24	25

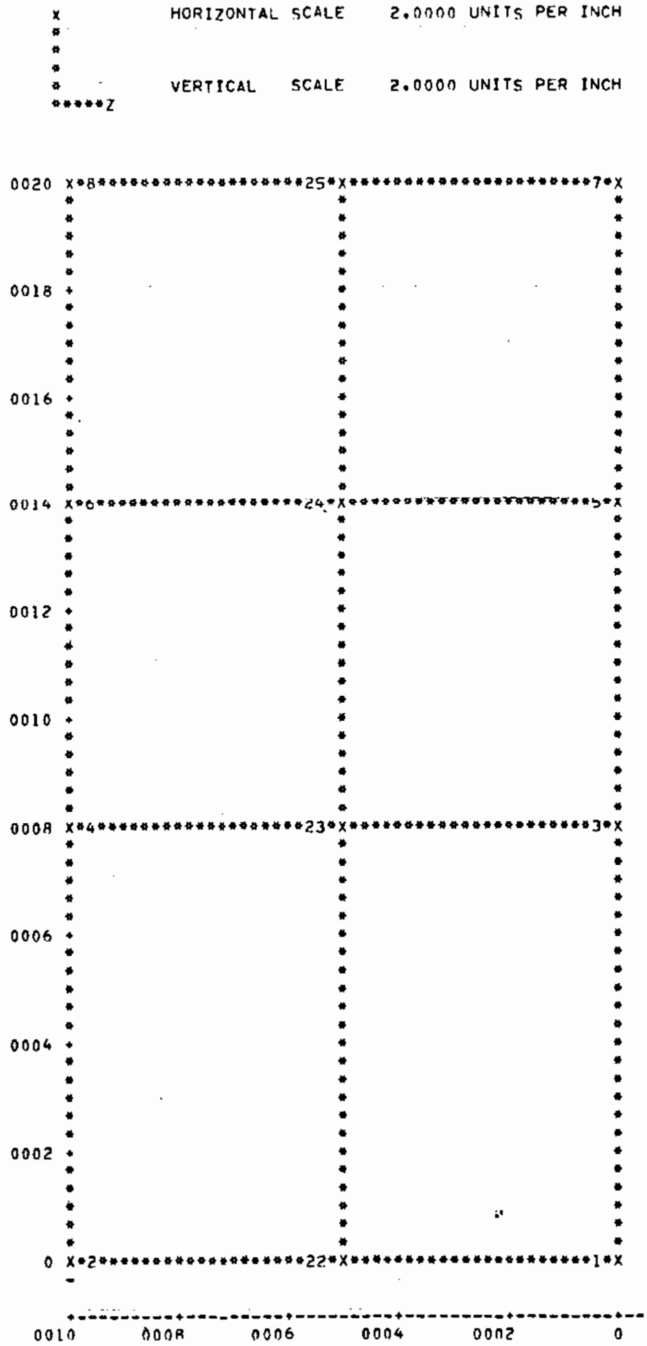
OUT OF PLANE MEMBERS
MEMBER INCIDENCES

MEMBER	START	END
1	1	9
2	2	11
3	3	14
4	4	16
5	7	19
6	8	21

ORIENTATION

HORIZONTAL SCALE 2.0000 UNITS PER INCH

VERTICAL SCALE 2.0000 UNITS PER INCH



PLOT PLANE Y EQUALS 17.0

PLANE IDENTIFIED BY - PLANE Y EQUALS 17.000

IN PLANE JOINTS

JOINT	COORDINATES X	Y	Z
9	0.0	17.0000	0.0
10	0.0	17.0000	5.0000
11	0.0	17.0000	10.0000
12	4.0000	17.0000	0.0
13	4.0000	17.0000	10.0000
14	8.0000	17.0000	0.0
15	8.0000	17.0000	5.0000
16	8.0000	17.0000	10.0000
17	14.0000	17.0000	0.0
18	14.0000	17.0000	10.0000
19	20.0000	17.0000	0.0
20	20.0000	17.0000	5.0000
21	20.0000	17.0000	10.0000

HORIZONTAL SCALE 2.0000 UNITS PER INCH

VERTICAL SCALE 2.0000 UNITS PER INCH

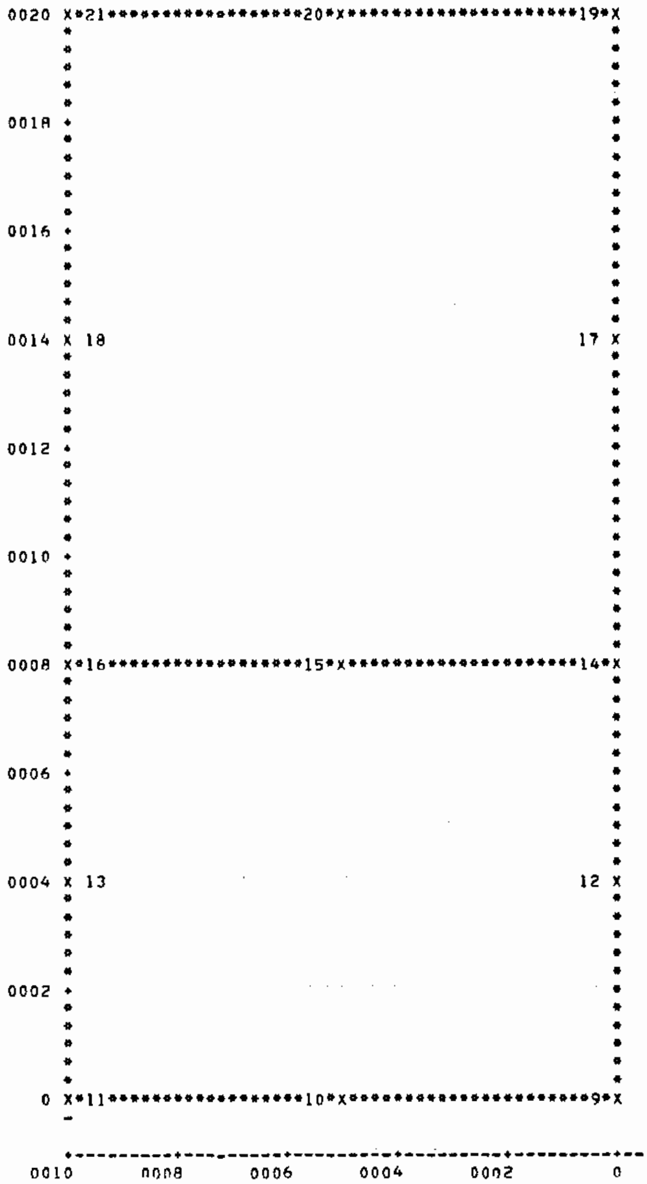
ORIENTATION X ***** Z

IN PLANE MEMBERS
MEMBER INCIDENCES

MEMBER	START	END
7	9	10
17	9	12
8	10	11
13	11	13
18	12	14
14	13	16
9	14	15
19	14	17
10	15	16
15	16	18
20	17	19
16	18	21
11	19	20
12	20	21

OUT OF PLANE MEMBERS
MEMBER INCIDENCES

MEMBER	START	END
1	1	9
2	2	11
3	3	14
4	4	16
5	7	19
6	8	21



PLOT PLANE Z EQUALS 0.0

PLANE IDENTIFIED BY - PLANE Z EQUALS 0.0

IN PLANE JOINTS

JOINT	COORDINATES X	Y	Z
1	0.0	0.0	0.0
3	8.0000	0.0	0.0
5	14.0000	0.0	0.0
7	20.0000	0.0	0.0
9	0.0	17.0000	0.0
12	4.0000	17.0000	0.0
14	8.0000	17.0000	0.0
17	14.0000	17.0000	0.0
19	20.0000	17.0000	0.0

IN PLANE MEMBERS
MEMBER INCIDENCES

MEMBER	START	END
1	1	9
32	1	3
3	3	14
33	3	5
34	5	7
5	7	19
17	9	12
18	12	14
19	14	17
20	17	19

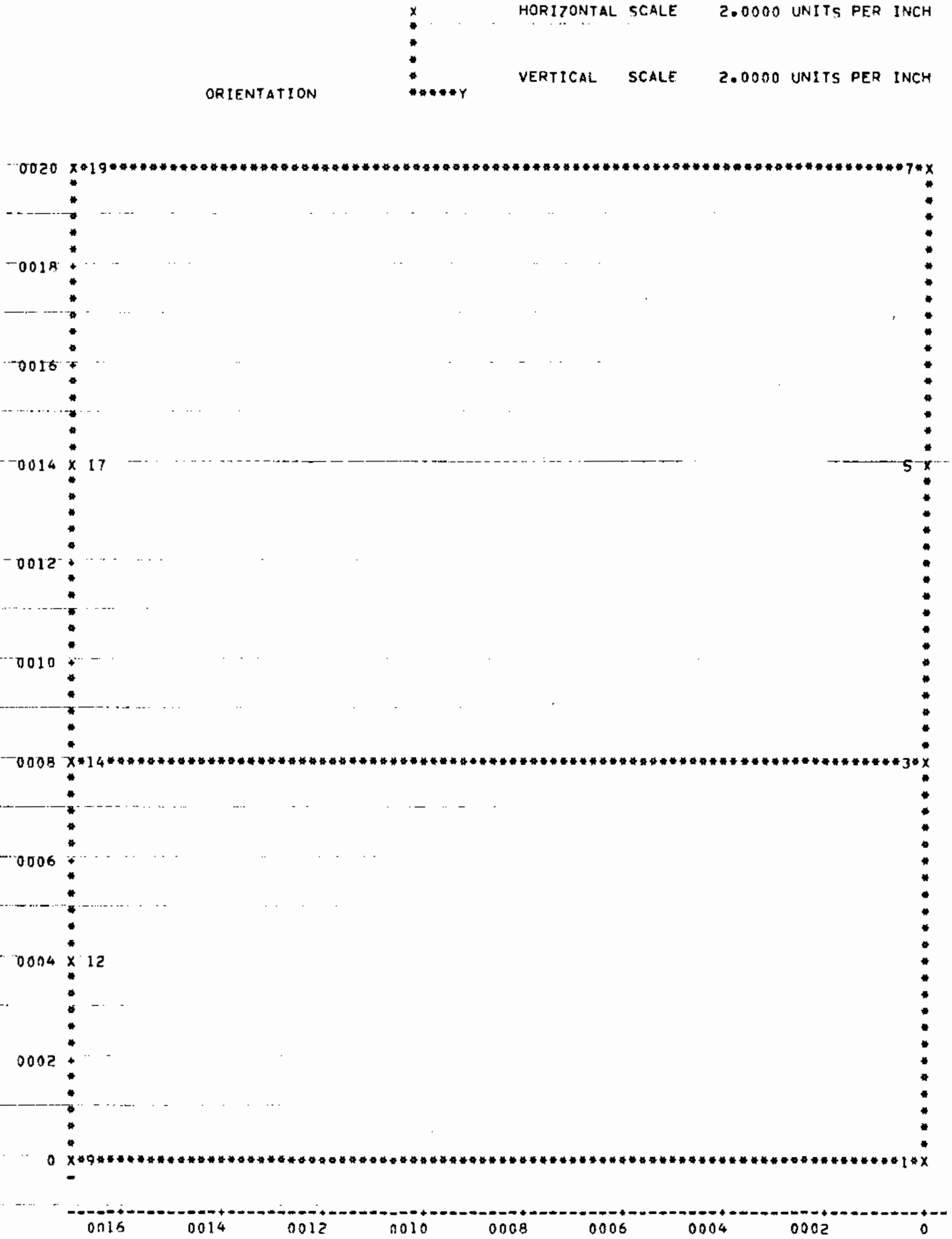
OUT OF PLANE MEMBERS
MEMBER INCIDENCES

MEMBER	START	END
21	1	22
23	3	23
25	5	24
27	7	25
7	9	10
9	14	15
11	19	20

HORIZONTAL SCALE 2.0000 UNITS PER INCH

VERTICAL SCALE 2.0000 UNITS PER INCH

ORIENTATION



PLOT PLANE Z EQUALS 10.0

PLANE IDENTIFIED BY - PLANE Z EQUALS 10.000

IN PLANE JOINTS

JOINT	COORDINATES X	Y	Z
2	0.0	0.0	10.0000
4	8.0000	0.0	10.0000
6	14.0000	0.0	10.0000
8	20.0000	0.0	10.0000
11	0.0	17.0000	10.0000
13	4.0000	17.0000	10.0000
16	8.0000	17.0000	10.0000
18	14.0000	17.0000	10.0000
21	20.0000	17.0000	10.0000

IN PLANE MEMBERS
MEMBER INCIDENCES

MEMBER	START	END
2	2	11
29	2	4
4	4	16
30	4	6
31	6	8
6	8	21
13	11	13
14	13	16
15	16	18
16	18	21

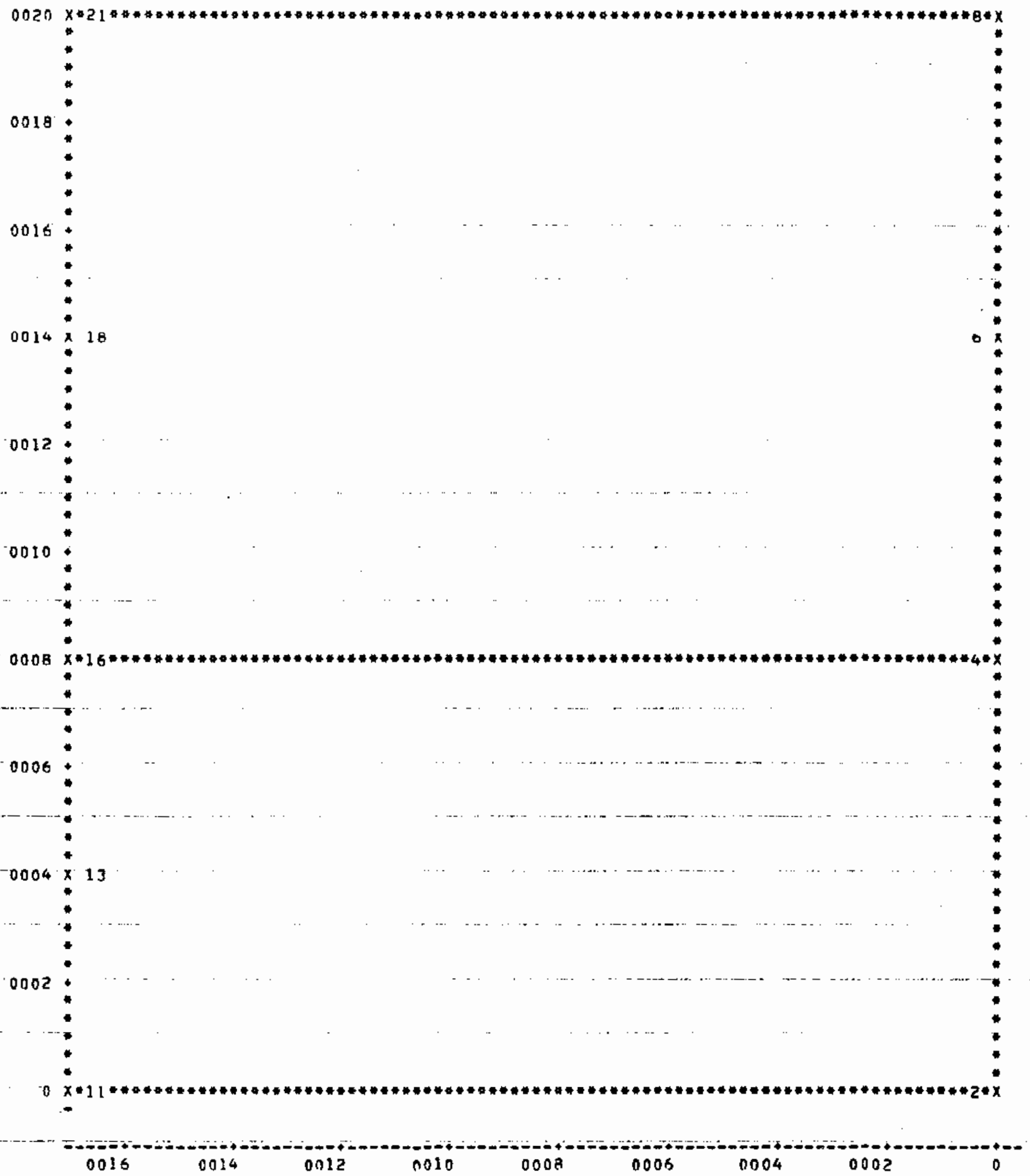
OUT OF PLANE MEMBERS
MEMBER INCIDENCES

MEMBER	START	END
22	22	2
24	23	4
26	24	6
28	25	8
8	10	11
10	15	16
12	20	21

ORIENTATION X / Y

HORIZONTAL SCALE 2.0000 UNITS PER INCH

VERTICAL SCALE 2.0000 UNITS PER INCH



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STIFFNESS ANALYSIS                                                        (10)
UNITS KIPS INCHES CYCLES SECONDS                                          (11)
LIST RAYLEIGH

    ******************************
    *RESULTS OF RAYLEIGH ANALYSIS*
    ******************************

JOB ID - EXAMPLE      JOB TITLE - STATIC AND DYNAMIC ANALYSIS OF A TABLE-TOP

ACTIVE UNITS -  LENGTH        WEIGHT        ANGLE       TEMPERATURE      TIME
                 INCH           KIP           CYC           DEGF           SEC

RAYLEIGH NATURAL FREQUENCY =      3.9850

OUTPUT BY MEMBER
LOADING LIST 4,5,6
LIST FORCES DISPLACEMENTS REACTIONS ALL

    ****************************
    *RESULTS OF LATEST ANALYSES*
    ****************************

    PROBLEM - EXAMPLE    TITLE - STATIC AND DYNAMIC ANALYSIS OF A TABLE-TOP

    ACTIVE UNITS   INCH KIP   CYC   DEGF SEC
    ACTIVE STRUCTURE TYPE   SPACE    FRAME
    ACTIVE COORDINATE AXES   X Y Z

    MEMBER FORCES
```

MEMBER	LOADING	JOINT	FORCES AXIAL	FORCES SHEAR Y	FORCES SHEAR Z	MOMENTS TORSION	MOMENTS MOMENT Y	MOMENTS MOMENT Z
1								
	4	1	22.7012482	-0.4683557	-12.5214949	-3.5564518	1399.6728516	-56.4777374
		9	-22.7012482	0.4683557	12.5214949	3.5564518	1154.7106934	-39.0668945
	5	1	35.6751099	2.4360933	-0.1276147	0.1945127	66.7783051	244.0932312
		9	-35.6751099	-2.4360933	0.1276147	-0.1945127	-40.7449493	252.8695984
	6	1	63.2339020	-0.4561379	-0.2769509	0.3132216	113.3043671	-63.5768738
		9	-63.2339020	0.4561379	0.2769509	-0.3132216	-56.8063507	-29.4753265
2								
	4	2	61.6106262	-0.1398282	-12.1522217	-3.9740801	1248.5998535	-28.2914429
		11	-61.6106262	0.1398282	12.1522217	3.9740801	1230.4519043	-0.2335469
	5	2	35.6751099	2.4360933	0.1276148	-0.1945106	-66.7783051	244.0932312
		11	-35.6751099	-2.4360933	-0.1276148	0.1945106	40.7449493	252.8695984
	6	2	63.2339020	-0.4561379	0.2769509	-0.3132207	-113.3043671	-63.5768738
		11	-63.2339020	0.4561379	-0.2769509	0.3132207	56.8063507	-29.4753265
3								
	4	3	31.1257324	-1.0835218	-14.2669010	-6.5465736	1801.5737305	-58.9640808
		14	-31.1257324	1.0835218	14.2669010	6.5465736	1108.8728027	-162.0743561
	5	3	57.4260101	8.5714283	2.1368227	-6.3765850	-37.1990051	997.6184082
		14	-57.4260101	-8.5714283	-2.1368227	6.3765850	-398.7128906	750.9543457
	6	3	83.1856079	-0.1123674	3.2094212	0.5992888	-56.3248596	87.7592926
		14	-83.1856079	0.1123674	-3.2094212	-0.5992888	-598.3969727	-110.6822357
4								
	4	4	79.7884216	0.9336989	-18.5461273	-7.3456202	1876.6728516	175.9764862
		16	-79.7884216	-0.9336989	18.5461273	7.3456202	1906.7358398	14.4979982
	5	4	57.4260101	8.5714283	-2.1368227	6.3765850	37.1990204	997.6184082
		16	-57.4260101	-8.5714283	2.1368227	-6.3765850	398.7128906	750.9543457
	6	4	83.1856079	-0.1123674	-3.2094212	-0.5992853	56.3248749	87.7592926
		16	-83.1856079	0.1123674	3.2094212	0.5992853	598.3969727	-110.6822357
5								
	4	7	15.7968073	0.3037781	-9.1984711	-9.7853155	1131.3947754	18.6266327
		19	-15.7968073	-0.3037781	9.1984711	9.7853155	745.0932617	43.3441467
	5	7	37.8488159	2.0874596	1.3083725	-1.6638927	-34.3237305	198.6451721
		19	-37.8488159	-2.0874596	-1.3083725	1.6638927	-232.5844116	227.1965942
	6	7	50.0053711	0.5685053	2.0146761	-0.0761758	-59.8608704	37.0961456
		19	-50.0053711	-0.5685053	-2.0146761	0.0761758	-351.1330566	78.8789825
6								
	4	8	50.8770142	0.4542289	-11.8847094	-9.6837492	1211.2097168	30.8348846
		21	-50.8770142	-0.4542289	11.8847094	9.6837492	1213.2697754	61.8278809
	5	8	37.8488159	2.0874596	-1.3083725	1.6638899	34.3237152	198.6451721
		21	-37.8488159	-2.0874596	1.3083725	-1.6638899	232.5844116	227.1965942

	6	8	50.0053711	0.5685053	-2.0146761	0.0761767	59.8608704	37.0961456
		21	-50.0053711	-0.5685053	2.0146761	-0.0761767	351.1330566	78.8789825
7								
	4	9	-1.1590748	-17.2346497	-0.3151071	-31.5249939	16.9988403	-1265.6818848
		10	1.1590748	17.2346497	0.3151071	31.5249939	1.9075861	231.6033478
	5	9	0.2333486	3.3999968	0.3399994	-0.0000021	-12.0425339	-25.2678986
		10	-0.2333486	-3.3999968	-0.3399994	0.0000021	-8.3574286	229.2676697
	6	9	-0.2086138	5.0999956	-0.0000000	-0.0000027	-2.8613806	-41.4052124
		10	0.2086138	-5.0999956	0.0000000	0.0000027	2.8613806	347.4047852
8								
	4	10	0.8809227	-24.0346375	-0.3151071	-31.5249939	-1.9075861	-231.6033478
		11	-0.8809227	24.0346375	0.3151071	31.5249939	20.8140106	-1210.4748535
	5	10	0.2333486	-3.3999977	-0.3399994	-0.0000021	8.3574286	-229.2676697
		11	-0.2333486	3.3999977	0.3399994	0.0000021	12.0425339	25.2678986
	6	10	-0.2086138	-5.0999956	-0.0000000	-0.0000027	-2.8613806	-347.4047852
		11	0.2086138	5.0999956	0.0000000	0.0000027	2.8613815	41.4052124
9								
	4	14	-5.4169369	1.4864140	0.0159638	-8.5884562	-1.4062576	-947.1843262
		15	5.4169369	-1.4864140	-0.0159638	8.5884562	0.4484282	1036.3691406
	5	14	2.3373747	24.9999847	2.4999981	-0.0000006	-67.3990479	463.5261230
		15	-2.3373747	-24.9999847	-2.4999981	0.0000006	-82.6006775	1036.4733887
	6	14	3.1245861	37.4999847	0.0000000	-0.0000003	-0.6726444	695.4455566
		15	-3.1245861	-37.4999847	-0.0000000	0.0000003	0.6726437	1554.5537109
10								
	4	15	9.5830498	-48.5135651	0.0159638	-8.5884562	-0.4484282	-1036.3691406
		16	-9.5830498	48.5135651	-0.0159638	8.5884562	-0.5094008	-1874.4448242
	5	15	2.3373747	-24.9999847	-2.4999981	-0.0000006	82.6006775	-1036.4733887
		16	-2.3373747	24.9999847	2.4999981	0.0000006	67.3991089	-463.5261230
	6	15	3.1245861	-37.4999847	0.0000000	-0.0000003	-0.6726437	-1554.5537109
		16	-3.1245861	37.4999847	-0.0000000	0.0000003	0.6726432	-695.4455566
11								
	4	19	-3.4308796	-1.2279253	-0.9489564	-8.3734407	57.0611572	-795.8110352
		20	3.4308796	1.2279253	0.9489564	8.3734407	-0.1237821	722.1354980
	5	19	0.7468583	15.9499941	1.5949984	0.0000005	-43.9149780	233.7839203
		20	-0.7468583	-15.9499941	-1.5949984	-0.0000005	-51.7849426	723.2155762
	6	19	2.0311737	23.9249878	0.0000000	0.0000009	0.1856556	352.2958984
		20	-2.0311737	-23.9249878	-0.0000000	-0.0000009	-0.1856564	1083.2028809
12								
	4	20	6.1391087	-33.1278992	-0.9489564	-8.3734407	0.1237821	-722.1354980
		21	-6.1391087	33.1278992	0.9489564	8.3734407	56.8135986	-1265.5388184
	5	20	0.7468583	-15.9499903	-1.5949984	0.0000005	51.7849426	-723.2155762
		21	-0.7468583	15.9499903	1.5949984	-0.0000005	43.9149475	-233.7839203
	6	20	2.0311737	-23.9249725	0.0000000	0.0000009	0.1856564	-1083.2028809
		21	-2.0311737	23.9249725	-0.0000000	-0.0000009	-0.1856573	-352.2958984
13								
	4	11	0.4549353	-0.9240214	0.2786918	19.9775391	-24.7880707	-31.2914429
		13	-0.4549353	0.9240214	-0.2786918	-19.9775391	11.4108896	-13.0615692
	5	11	1.7539034	-6.2248697	0.3609633	66.0128021	-12.2370491	-252.8695984
		13	-1.7539034	6.2248697	-0.3609633	-66.0128021	-5.0891943	-45.9242859
	6	11	0.4561379	0.3839194	0.0683372	98.2115479	-3.1746016	29.4753113
		13	-0.4561379	-0.3839194	-0.0683372	-98.2115479	-0.1055872	-11.0471859
14								
	4	13	0.4549353	-4.5240183	1.3586903	19.9775391	-11.4108896	13.0615692
		16	-0.4549353	4.5240183	-1.3586903	-19.9775391	-53.8062744	-230.2144775
	5	13	2.1139030	-9.8248701	0.3609633	66.0128021	5.0891943	45.9242859
		16	-2.1139030	9.8248701	-0.3609633	-66.0128021	-22.4154358	-517.5180664
	6	13	0.4561379	-5.0160770	0.0683372	98.2115479	0.1055872	11.0471859
		16	-0.4561379	5.0160770	-0.0683372	-98.2115479	-3.3857765	-251.8189087
15								
	4	16	-0.4947273	9.8508663	-2.5343895	52.2685242	46.9700470	207.1280365
		18	0.4947273	-9.8508663	2.5343895	-52.2685242	135.5059814	502.1345215
	5	16	~~-2.2675343~~	~~5.7011518~~	~~0.5615151~~	~~1.1994839~~	-38.6071320	-233.4361572
		18	2.2675343	-5.7011518	-0.5615151	-1.1994839	-1.8219614	643.9189453
	6	16	0.5685053	15.3195801	-0.0164978	1.1628866	2.1138477	362.5012207
		18	-0.5685053	-15.3195801	0.0164978	-1.1628866	-0.9260067	740.5087891
16								
	4	18	-0.4947273	-7.9491081	2.8056021	52.2685242	-135.5059814	-502.1345215
		21	0.4947273	7.9491081	-2.8056021	-52.2685242	-66.4972992	-70.2012939
	5	18	-0.4875373	-12.0988264	0.5615151	1.1994839	1.8219614	-643.9189453
		21	0.4875373	12.0988264	-0.5615151	-1.1994839	-42.2510529	-227.1965942
	6	18	0.5685053	-11.3803854	-0.0164978	1.1628866	0.9260067	-740.5087891
		21	-0.5685053	11.3803854	0.0164978	-1.1628866	0.2618340	-78.8789825
17								
	4	9	0.1532485	1.4359131	0.1875754	-110.9712372	-20.5552826	70.5918579
		12	-0.1532485	-1.4359131	-0.1875754	110.9712372	11.5516748	-1.6680193
	5	9	1.7539034	-6.2248697	-0.3609633	-66.0128632	12.2370415	-252.8695984
		12	-1.7539034	6.2248697	0.3609633	66.0128632	5.0891943	-45.9242859
	6	9	0.4561379	0.3839196	-0.0683372	-98.2115479	3.1746025	29.4753265
		12	-0.4561379	-0.3839196	0.0683372	98.2115479	0.1055868	-11.0471859
18								
	4	12	0.1532485	-2.1640844	1.2675734	-110.9712372	-11.5516748	1.6680193
		14	-0.1532485	2.1640844	-1.2675734	110.9712372	-49.2919006	-105.5440521
	5	12	2.1139030	-9.8248701	-0.3609633	-66.0128632	-5.0891943	45.9242859
		14	-2.1139030	9.8248701	0.3609633	66.0128632	22.4154358	-517.5180664
	6	12	0.4561379	-5.0160770	-0.0683372	-98.2115479	-0.1055868	11.0471859
		14	-0.4561379	5.0160770	0.0683372	98.2115479	3.3857756	-251.8189087
19								
	4	14	1.2527342	10.5752373	-2.5123930	50.7180176	44.1515808	276.2067871
		17	-1.2527342	-10.5752373	2.5123930	-50.7180176	136.7406769	485.2104492
	5	14	-2.2675352	5.7011518	-0.5615151	-1.1994877	38.6071167	-233.4361572
		17	2.2675352	-5.7011518	0.5615151	1.1994877	1.8219805	643.9189453
	6	14	0.5685053	15.3195801	0.0164977	-1.1628933	-2.1138420	362.5012207
		17	-0.5685053	-15.3195801	-0.0164977	1.1628933	0.9260055	740.5087891

20								
	4	17	1.2527342	-7.2247372	2.8275986	50.7180176	-136.7406769	-485.2104492
		19	-1.2527342	7.2247372	-2.8275986	-50.7180176	-66.8463593	-34.9706879
	5	17	-0.4875376	-12.0988264	-0.5615151	-1.1994877	-1.8219805	-643.9189453
		19	0.4875376	12.0988264	0.5615151	1.1994877	42.2510834	-227.1965942
	6	17	0.5685053	-11.3803854	0.0164977	-1.1628933	-0.9260055	-740.5087891
		19	-0.5685053	11.3803854	-0.0164977	1.1628933	-0.2618312	-78.8789825
21								
	4	1	2.5722189	-31.2175751	-0.1492021	-36.8824615	37.4164581	-1326.2836914
		22	-2.5722189	31.2175751	0.1492021	36.8824615	-28.4643402	-546.7700195
	5	1	-0.8523622	-8.4504757	-0.6549445	-49.7070160	37.2952881	2.7630424
		22	0.8523622	8.4504757	0.6549445	49.7070160	2.0014019	-509.7917480
	6	1	-0.9759381	-15.5776978	0.0872470	-16.8474274	52.0229950	-49.9329834
		22	0.9759381	15.5776978	-0.0872470	16.8474274	-57.2578125	-884.7290039
22								
	4	22	-3.8734703	-10.4473114	-0.2655314	-14.4192839	47.8794250	632.8684082
		2	3.8734703	10.4473114	0.2655314	14.4192839	-31.9475403	-1259.7067871
	5	22	-0.8523620	8.4504757	0.6549445	49.7069855	-2.0014057	509.7917480
		2	0.8523620	-8.4504757	-0.6549445	-49.7069855	-37.2952881	-2.7630262
	6	22	-0.9759381	15.5776978	-0.0872470	16.8474121	57.2578125	884.7287598
		2	0.9759381	-15.5776978	0.0872470	-16.8474121	-52.0229950	49.9329987
23								
	4	3	1.6871014	-30.5602570	-0.4913376	-168.1846619	72.0997314	-1276.7749023
		23	-1.6871014	30.5602570	0.4913376	168.1846619	-42.6194916	-556.8403320
	5	3	-0.8341884	-11.4810495	-1.9511776	-93.2221069	54.3481293	-101.3010406
		23	0.8341884	11.4810495	1.9511776	93.2221069	62.7225647	-587.5620117
	6	3	-1.2359715	-16.8940887	0.0172989	31.4012604	27.2372284	-142.2630463
		23	1.2359715	16.8940887	-0.0172989	-31.4012604	-28.2751617	-871.3828125
24								
	4	23	-3.3350630	-8.0347996	-0.5144027	-210.0530396	-4.9192572	605.0034180
		4	3.3350630	8.0347996	0.5144027	210.0530396	35.7834320	-1087.0898438
	5	23	-0.8341884	11.4810495	1.9511776	93.2221069	-62.7225647	587.5620117
		4	0.8341884	-11.4810495	-1.9511776	-93.2221069	-54.3480988	101.3010406
	6	23	-1.2359715	16.8940887	-0.0172989	-31.4012756	28.2751617	871.3830566
		4	1.2359715	-16.8940887	0.0172989	31.4012756	-27.2372284	142.2630463
25								
	4	5	0.8726366	-14.4131622	-1.0863199	-154.2997284	77.8192444	-602.0170898
		24	-0.8726366	14.4131622	1.0863199	154.2997284	-12.6400490	-262.7724609
	5	5	-0.6855827	-2.9729900	-0.0173113	34.4197845	-0.2396955	91.2842407
		24	0.6855827	2.9729900	0.0173113	-34.4197845	1.2783785	-269.6635742
	6	5	-1.1254892	-3.3824186	0.0138888	35.9213104	5.7231550	156.2539215
		24	1.1254892	3.3824186	-0.0138888	-35.9213104	-6.5564795	-359.1989746
26								
	4	24	-2.3732882	-9.9032688	-1.1048384	-202.1947327	-3.8980465	216.1595459
		6	2.3732882	9.9032688	1.1048384	202.1947327	70.1883698	-810.3557129
	5	24	-0.6855827	2.9729910	0.0173114	-34.4197693	-1.2783823	269.6635742
		6	0.6855827	-2.9729910	-0.0173114	34.4197693	0.2396970	-91.2842407
	6	24	-1.1254892	3.3824186	-0.0138888	-35.9212799	6.5564795	359.1989746
		6	1.1254892	-3.3824186	0.0138888	35.9212799	-5.7231550	-156.2539215
27								
	4	7	1.0371208	-26.9805298	-0.4244760	-37.7019348	20.6600189	-1127.5646973
		25	-1.0371208	26.9805298	0.4244760	37.7019348	4.8085461	-491.2670898
	5	7	-0.9313248	-8.0229244	-0.5303620	-54.6304016	15.0185099	11.9981470
		25	0.9313248	8.0229244	0.5303620	54.6304016	16.8032074	-493.3735352
	6	7	-1.5886860	-10.5390100	-0.1186792	-38.3853149	13.8409948	38.8234863
		25	1.5886860	10.5390100	0.1186792	38.3853149	-6.7202454	-671.1640625
28								
	4	25	-3.1553698	-12.9285221	-0.2662370	13.4786081	13.7688932	403.6186523
		8	3.1553698	12.9285221	0.2662370	-13.4786081	2.2053432	-1179.3298340
	5	25	-0.9313248	8.0229244	0.5303620	54.6304169	-16.8032227	493.3735352
		8	0.9313248	-8.0229244	-0.5303620	-54.6304169	-15.0185022	-11.9981394
	6	25	-1.5886860	10.5390100	0.1186792	38.3853607	6.7202454	671.1640625
		8	1.5886860	-10.5390100	-0.1186792	-38.3853607	-13.8409948	-38.8234863
29								
	4	2	-0.8897111	7.4278393	-2.6610632	-11.1071587	35.9216156	13.8721590
		4	0.8897111	-7.4278393	2.6610632	11.1071587	219.5404816	699.2004395
	5	2	-1.1874189	1.9948902	-0.9836210	-69.5413666	37.4897919	-194.3862305
		4	1.1874189	-1.9948902	0.9836210	69.5413666	56.9378052	385.8955078
	6	2	-0.3655444	10.0114088	-1.2570610	-63.3713684	52.3362122	80.4242401
		4	0.3655444	-10.0114088	1.2570610	63.3713684	68.3416748	880.6708984
30								
	4	4	-0.2358506	-27.9603424	6.6682806	778.4753418	-247.9782867	-1085.2297363
		6	0.2358506	27.9603424	-6.6682806	-778.4753418	-232.1380463	-927.9150391
	5	4	3.5713120	-23.7786865	0.3154470	68.9586792	-8.9663000	-1290.2917480
		6	-3.5713120	23.7786865	-0.3154470	-68.9586792	-13.7458897	-421.7751465
	6	4	-0.4598483	-24.9645386	0.7111034	135.2165527	-40.5051880	-999.8315430
		6	0.4598483	24.9645386	-0.7111034	-135.2165527	-10.6942759	-797.6157227
31								
	4	6	0.1874470	9.8383856	-2.3531675	-31.8803101	161.9496613	725.7202148
		8	-0.1874470	-9.8383856	2.3531675	31.8803101	7.4784050	-17.3562775
	5	6	1.1613836	3.3797274	-0.3730666	-22.3255768	13.5061979	387.3552246
		8	-1.1613836	-3.3797274	0.3730666	22.3255768	13.3546124	-144.0146637
	6	6	-0.4467912	10.5969954	-0.4191979	-21.0373840	16.4174347	761.6943359
		8	0.4467912	-10.5969954	0.4191979	21.0373840	13.7648191	1.2892208
32								
	4	1	0.4023183	5.9207020	-0.9849809	73.3881073	-33.8600159	93.3601685
		3	-0.4023183	-5.9207020	0.9849809	-73.3881073	128.4181671	475.0270996
	5	1	-1.1874189	1.9948902	0.9836210	69.5413666	-37.4898071	-194.3862305
		3	1.1874189	-1.9948902	-0.9836210	-69.5413666	-56.9378052	385.8955078
	6	1	-0.3655444	10.0114088	1.2570610	63.3713837	-52.3362122	80.4243011
		3	0.3655444	-10.0114088	-1.2570610	-63.3713837	-68.3416748	880.6708984

33	4	3	-0.3772805	-5.3257103	5.7201433	598.1862793	-193.9714203	-247.8784790
		5	0.3772805	5.3257103	-5.7201433	-598.1862793	-217.8789825	-135.5727386
	5	3	3.5713120	-23.7786865	-0.3154467	-68.9586182	8.9662838	-1290.2907715
		5	-3.5713120	23.7786865	0.3154467	68.9586182	13.7458935	-421.7751465
	6	3	-0.4598483	-24.9645386	-0.7111034	-135.2165527	40.5051727	-999.8315430
		5	0.4598483	24.9645386	0.7111034	135.2165527	10.6942759	-797.6157227
34	4	5	-0.7831686	4.2909365	-1.7942352	-3.8307295	140.0596771	289.8723145
		7	0.7831686	-4.2909365	1.7942352	3.8307295	-10.8747015	19.0752869
	5	5	1.1613836	3.3797274	0.3730668	22.3255768	-13.5062056	387.3552246
		7	-1.1613836	-3.3797274	-0.3730668	-22.3255768	-13.3546162	-144.0146637
	6	5	-0.4467912	10.5969954	0.4191979	21.0373840	-16.4174347	761.6943359
		7	0.4467912	-10.5969954	-0.4191979	-21.0373840	-13.7648191	1.2891808
35	4	22	-0.1153864	2.2996998	-0.2962773	86.0985565	-19.4151001	-22.4631653
		23	0.1153864	-2.2996998	0.2962773	-86.0985565	47.8577271	243.2343445
	5	22	-0.5835522	1.3648109	-0.0000001	-0.0000425	0.0000049	-99.4139252
		23	0.5835522	-1.3648109	0.0000001	0.0000425	0.0000078	230.4358521
	6	22	-0.1730796	3.4495478	0.0000000	-0.0000653	-0.0000017	-33.6948547
		23	0.1730796	-3.4495478	-0.0000000	0.0000653	0.0000011	364.8513184
36	4	23	-0.1382998	-4.6729994	0.1350319	134.2614136	-0.3189836	-201.3659821
		24	0.1382998	4.6729994	-0.1350319	-134.2614136	-9.4033117	-135.0902405
	5	23	1.8383026	-6.4187880	-0.0000001	0.0000564	0.0000062	-416.8801270
		24	-1.8383026	6.4187880	0.0000001	-0.0000564	0.0000022	-45.2728577
	6	23	-0.2074497	-7.0094986	0.0000000	0.0000816	-0.0000009	-302.0488281
		24	0.2074497	7.0094986	-0.0000000	-0.0000816	0.0000007	-202.6353607
37	4	24	-0.1571361	1.8306208	-0.1022773	87.6484222	25.9413910	182.9852295
		25	0.1571361	-1.8306208	0.1022773	-87.6484222	-18.5774384	-51.1805267
	5	24	0.6171364	0.0673831	-0.0000001	0.0000715	0.0000001	114.1124268
		25	-0.6171364	-0.0673831	0.0000001	-0.0000715	0.0000049	-109.2608032
	6	24	-0.2357042	2.7459345	-0.0000000	0.0000967	0.0000000	274.4777832
		25	0.2357042	-2.7459345	0.0000000	-0.0000967	0.0000001	-76.7706757

JOINT LOADS - SUPPORTS

JOINT		LOADING	FORCES X FORCE	Y FORCE	Z FORCE	MOMENTS X MOMENT	Y MOMENT	Z MOMENT
1	GLOBAL	4	1.0198765	35.8043518	-22.4542236	-0.0000000	0.0000000	0.0000000
		5	-6.8085537	67.6194916	0.0036440	-0.0000000	0.0000000	0.0000000
		6	0.0033467	115.2675476	0.0041723	-0.0000000	-0.0000000	0.0000000
2	GLOBAL	4	-1.0154142	117.8857269	-22.4597931	-0.0000000	0.0000000	0.0000000
		5	-6.8085537	67.6194916	-0.0036440	-0.0000000	0.0000000	0.0000000
		6	0.0033467	115.2675476	-0.0041723	-0.0000000	-0.0000000	0.0000000
3	GLOBAL	4	0.7952607	23.7190399	-16.1946564	-0.0000000	0.0000000	0.0000000
		5	-5.3015184	54.5713501	0.0035663	-0.0000000	-0.0000000	0.0000000
		6	0.0007644	82.9155426	0.0052840	-0.0000000	0.0000000	0.0000000
4	GLOBAL	4	-0.7942414	86.8349915	-16.2017059	0.0000000	0.0000000	0.0000000
		5	-5.3015184	54.5713501	-0.0035663	0.0000000	-0.0000000	0.0000000
		6	0.0007644	82.9155426	-0.0052840	0.0000000	-0.0000000	0.0000000
5	GLOBAL	4	0.6804323	16.6034698	-13.0617371	-0.0000000	0.0000000	0.0000000
		5	-4.5326090	45.5854187	0.0029310	-0.0000000	-0.0000000	0.0000000
		6	-0.0008316	64.2790985	0.0048117	-0.0000000	0.0000000	0.0000000
6	GLOBAL	4	-0.6815410	69.1019897	-13.0681553	0.0000000	0.0000000	0.0000000
		5	-4.5326090	45.5854187	-0.0029310	0.0000000	0.0000000	0.0000000
		6	-0.0008316	64.2790985	-0.0048117	0.0000000	-0.0000000	0.0000000
7	GLOBAL	4	0.9038666	17.7253113	-16.3271027	-0.0000000	-0.0000000	0.0000000
		5	-6.0384760	59.6461487	0.0039816	-0.0000000	-0.0000000	0.0000000
		6	-0.0030350	78.6692963	0.0067920	-0.0000000	-0.0000000	0.0000000
8	GLOBAL	4	-0.9079132	87.1671143	-16.3361664	0.0000000	0.0000000	0.0000000
		5	-6.0384760	59.6461487	-0.0039816	0.0000000	-0.0000000	0.0000000
		6	-0.0030350	78.6692963	-0.0067920	0.0000000	0.0000000	0.0000000
22	GLOBAL	4	0.0009429	39.9699554	-11.8119631	0.0000000	-0.0000000	0.0000000
		5	-3.5834408	35.1657562	-0.0000000	-0.0000000	-0.0000000	0.0000000
		6	0.0014144	59.9549255	0.0000000	-0.0000000	-0.0000000	0.0000000
23	GLOBAL	4	0.0001519	28.6527557	-8.5208502	0.0000000	-0.0000000	0.0000000
		5	-2.7904997	28.2784882	-0.0000000	-0.0000000	-0.0000000	0.0000000
		6	0.0002278	42.9791412	-0.0000000	-0.0000000	-0.0000000	0.0000000
24	GLOBAL	4	-0.0003180	22.3135071	-6.8732300	0.0000000	-0.0000000	0.0000000
		5	-2.3857870	23.7321472	0.0000000	-0.0000000	-0.0000000	-0.0000000
		6	-0.0004770	33.4702606	-0.0000000	-0.0000000	-0.0000000	0.0000000
25	GLOBAL	4	-0.0011027	27.2213745	-8.5902138	0.0000000	0.0000000	0.0000000
		5	-3.1778603	30.9784546	0.0000000	-0.0000000	-0.0000000	0.0000000
		6	-0.0016541	40.8320770	-0.0000000	-0.0000000	-0.0000000	0.0000000

JOINT DISPLACEMENTS - SUPPORTS

JOINT	LOADING	DISPLACEMENTS X DISP	DISPLACEMENTS Y DISP	DISPLACEMENTS Z DISP	ROTATIONS X ROT	ROTATIONS Y ROT	ROTATIONS Z ROT
1	GLOBAL						
	4	-0.0014150	-0.0360144	0.0311540	0.0001115	0.0000037	0.0000079
	5	0.0094465	-0.0680162	-0.0000051	-0.0000032	0.0000000	-0.0000067
	6	-0.0000046	-0.1159439	-0.0000058	-0.0000053	0.0000000	0.0000122
2	GLOBAL						
	4	0.0014088	-0.1185775	0.0311617	0.0001185	0.0000037	0.0000083
	5	0.0094465	-0.0680162	0.0000051	0.0000032	-0.0000000	-0.0000067
	6	-0.0000046	-0.1159439	0.0000058	0.0000053	-0.0000000	0.0000122
3	GLOBAL						
	4	-0.0014186	-0.0306745	0.0288889	0.0001108	0.0000038	0.0000116
	5	0.0094572	-0.0705740	-0.0000064	-0.0000039	-0.0000000	-0.0000012
	6	-0.0000014	-0.1072299	-0.0000094	-0.0000059	0.0000000	0.0000198
4	GLOBAL						
	4	0.0014168	-0.1122988	0.0289015	0.0001186	0.0000038	0.0000148
	5	0.0094572	-0.0705740	0.0000064	0.0000039	0.0000000	-0.0000012
	6	-0.0000014	-0.1072299	0.0000094	0.0000059	-0.0000000	0.0000198
5	GLOBAL						
	4	-0.0014161	-0.0250524	0.0271836	0.0001062	0.0000038	0.0000124
	5	0.0094331	-0.0687822	-0.0000061	-0.0000034	-0.0000000	0.0000050
	6	0.0000017	-0.0969884	-0.0000100	-0.0000049	0.0000000	0.0000212
6	GLOBAL						
	4	0.0014184	-0.1042655	0.0271970	0.0001127	0.0000038	0.0000159
	5	0.0094331	-0.0687822	0.0000061	0.0000034	0.0000000	0.0000050
	6	0.0000017	-0.0969884	0.0000100	0.0000049	-0.0000000	0.0000212
7	GLOBAL						
	4	-0.0014108	-0.0200588	0.0254846	0.0001062	0.0000038	0.0000104
	5	0.0094253	-0.0674984	-0.0000062	-0.0000036	-0.0000000	0.0000012
	6	0.0000047	-0.0890260	-0.0000106	-0.0000050	0.0000000	0.0000158
8	GLOBAL						
	4	0.0014171	-0.0986426	0.0254987	0.0001129	0.0000038	0.0000106
	5	0.0094253	-0.0674984	0.0000062	0.0000036	0.0000000	0.0000012
	6	0.0000047	-0.0890260	0.0000106	0.0000050	-0.0000000	0.0000158
22	GLOBAL						
	4	-0.0000025	-0.0763879	0.0311387	0.0001066	0.0000037	0.0000082
	5	0.0094467	-0.0672064	0.0000000	0.0000000	0.0000000	-0.0000064
	6	-0.0000037	-0.1145818	-0.0000000	0.0000000	-0.0000000	0.0000123
23	GLOBAL						
	4	-0.0000005	-0.0703999	0.0288761	0.0001049	0.0000038	0.0000130
	5	0.0094566	-0.0694803	0.0000000	0.0000000	0.0000000	-0.0000004
	6	-0.0000008	-0.1055998	0.0000000	0.0000000	-0.0000000	0.0000195
24	GLOBAL						
	4	0.0000013	-0.0639661	0.0271759	0.0001030	0.0000038	0.0000139
	5	0.0094331	-0.0680329	-0.0000000	0.0000000	0.0000000	0.0000047
	6	0.0000019	-0.0959491	0.0000000	0.0000000	0.0000000	0.0000209
25	GLOBAL						
	4	0.0000033	-0.0585302	0.0254777	0.0001017	0.0000038	0.0000107
	5	0.0094252	-0.0666084	-0.0000000	-0.0000000	0.0000000	0.0000016
	6	0.0000049	-0.0877952	0.0000000	-0.0000000	0.0000000	0.0000161

JOINT DISPLACEMENTS - FREE JOINTS

JOINT	LOADING	DISPLACEMENTS X DISP	DISPLACEMENTS Y DISP	DISPLACEMENTS Z DISP	ROTATIONS X ROT	ROTATIONS Y ROT	ROTATIONS Z ROT
9	GLOBAL						
	4	-0.0155484	-0.0377312	0.2132522	0.0001252	0.0000045	0.0000101
	5	0.0306374	-0.0707142	0.0000035	0.0000028	-0.0000000	-0.0000056
	6	-0.0208277	-0.1207261	-0.0000031	0.0000042	-0.0000001	0.0000164
10	GLOBAL						
	4	-0.0138929	-0.0811531	0.2132694	0.0001089	0.0000043	0.0000110
	5	0.0306438	-0.0713737	-0.0000000	-0.0000000	-0.0000000	-0.0000056
	6	-0.0208394	-0.1217296	-0.0000000	-0.0000000	-0.0000000	0.0000164
11	GLOBAL						
	4	-0.0122218	-0.1232369	0.2132563	0.0001195	0.0000046	0.0000118
	5	0.0306374	-0.0707142	-0.0000035	-0.0000028	0.0000000	-0.0000056
	6	-0.0208277	-0.1207261	0.0000031	-0.0000042	0.0000001	0.0000164
12	GLOBAL						
	4	-0.0155511	-0.0348670	0.2117366	0.0001307	0.0000055	0.0000092
	5	0.0306062	-0.0717963	0.0000730	0.0000061	-0.0000003	-0.0000029
	6	-0.0208358	-0.1158606	0.0000341	0.0000091	-0.0000002	0.0000159
13	GLOBAL						
	4	-0.0122299	-0.1196138	0.2116911	0.0001185	0.0000057	0.0000120
	5	0.0306062	-0.0717963	-0.0000730	-0.0000061	0.0000003	-0.0000029
	6	-0.0208358	-0.1158606	-0.0000341	-0.0000091	0.0000002	0.0000159
14	GLOBAL						
	4	-0.0155539	-0.0322438	0.2101692	0.0001363	0.0000043	0.0000078
	5	0.0305686	-0.0734693	0.0000520	0.0000094	0.0000006	-0.0000103
	6	-0.0208439	-0.1114240	0.0000695	0.0000141	-0.0000000	0.0000125
15	GLOBAL						
	4	-0.0139022	-0.0775968	0.2102897	0.0001038	0.0000044	0.0000083
	5	0.0310218	-0.0767840	0.0000000	-0.0000000	-0.0000000	-0.0000103
	6	-0.0208532	-0.1163952	-0.0000000	-0.0000000	-0.0000000	0.0000125
16	GLOBAL						
	4	-0.0122380	-0.1163216	0.2100765	0.0001175	0.0000044	0.0000088
	5	0.0305686	-0.0734693	-0.0000520	-0.0000094	-0.0000006	-0.0000103
	6	-0.0208439	-0.1114240	-0.0000695	-0.0000141	0.0000000	0.0000125

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17       GLOBAL
                 4   -0.0155806   -0.0288897    0.2080399    0.0001340    0.0000064    0.0000111
                 5    0.0306170   -0.0754688    0.0000557    0.0000094   -0.0000003    0.0000036
                 6   -0.0208560   -0.1057499    0.0000744    0.0000141    0.0000000    0.0000184
18       GLOBAL
                 4   -0.0122274   -0.1121102    0.2079408    0.0001152    0.0000064    0.0000135
                 5    0.0306170   -0.0754688   -0.0000557   -0.0000094    0.0000003    0.0000036
                 6   -0.0208560   -0.1057499   -0.0000744   -0.0000141   -0.0000000    0.0000184
19       GLOBAL
                 4   -0.0156074   -0.0216517    0.2044226    0.0001317    0.0000080    0.0000182
                 5    0.0306274   -0.0713151    0.0000222    0.0000095    0.0000007    0.0000102
                 6   -0.0208682   -0.0940685    0.0000602    0.0000142    0.0000000    0.0000289
20       GLOBAL
                 4   -0.0139078   -0.0656620    0.2045244    0.0001022    0.0000027    0.0000193
                 5    0.0313185   -0.0742726   -0.0000000   -0.0000000   -0.0000000    0.0000102
                 6   -0.0208617   -0.0984930    0.0000000   -0.0000000   -0.0000000    0.0000289
21       GLOBAL
                 4   -0.0122169   -0.1037729    0.2043423    0.0001128    0.0000080    0.0000203
                 5    0.0306274   -0.0713151   -0.0000222   -0.0000095   -0.0000007    0.0000102
                 6   -0.0208682   -0.0940685   -0.0000602   -0.0000142   -0.0000000    0.0000289
```

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$
$ DYNAMIC ANALYSIS FOLLOWS
$
UNITS POUNDS INCHES
INERTIA OF JOINTS LUMPED                                                 (12)
INERTIA OF JOINT ADD 9,11 LINEAR ALL 69.88
INERTIA OF JOINT ADD 15 LINEAR ALL 117.75
INERTIA OF JOINT ADD 17,18 LINEAR ALL 28.47
INERTIA OF JOINT ADD 20 LINEAR ALL 73.76
DYNAMIC DEGREES OF FREEDOM                                               (13)
JOINT 1 TO 25 DISPLACEMENT X Y Z
DAMPING RATIO 0.10 75                                                    (14)
UNITS RADIANS SECONDS POUNDS INCHES
DYNAMIC LOADING 7 'CENTRIFUGAL FORCES'                                   (15)
JOINT 10 LOAD FORCE Y FUNCTION SINE AMPLITUDE 143. FREQ 727.7
JOINT 10 LOAD FORCE Z FUNCTION SINE AMPLITUDE 143. FREQ 727.7 PHASE 1.5707
JOINT 15 LOAD FORCE Y FUNCTION SINE AMPLITUDE 4040. FREQ 727.7
JOINT 15 LOAD FORCE Z FUNCTION SINE AMPLITUDE 4040. FREQ 727.7 PHASE 1.5707
JOINT 20 LOAD FORCE Y FUNCTION SINE AMPLITUDE 3897. FREQ 727.7
JOINT 20 LOAD FORCE Z FUNCTION SINE AMPLITUDE 3897. FREQ 727.7 PHASE 1.5707
INTEGRATE FROM 0.0 TO 0.0863 AT 0.0007194                                (16)
PRINT DYNAMIC DATA ALL                                                   (17)

****************************************
*  PROBLEM DATA FROM INTERNAL STORAGE  *
****************************************

JOB ID - EXAMPLE      JOB TITLE - STATIC AND DYNAMIC ANALYSIS OF A TABLE-TOP

ACTIVE UNITS - LENGTH      WEIGHT      ANGLE      TEMPERATURE      TIME
               INCH        LB          RAD        DEGF             SEC

        ********** DYNAMIC STRUCTURAL DATA **********


JOINT DYNAMIC DEGREES OF FREEDOM---/ JOINT INERTIAS-------------------------------------------------/
                                                  FORCE                                 MOMENT
JOINT     FORCE     MOMENT             X            Y            Z            X            Y            Z

          ****** CONDENSATION IS STATIC ******          ****** INERTIA OF JOINTS IS LUMPED ******

1         X Y Z
2         X Y Z
```

3	X Y Z						
4	X Y Z						
5	X Y Z						
6	X Y Z						
7	X Y Z						
8	X Y Z						
9	X Y Z	69.87999	69.87999	69.87999	0.0	0.0	0.0
10	X Y Z						
11	X Y Z	69.87999	69.87999	69.87999	0.0	0.0	0.0
12	X Y Z						
13	X Y Z						
14	X Y Z						
15	X Y Z	117.75000	117.75000	117.75000	0.0	0.0	0.0
16	X Y Z						
17	X Y Z	28.46999	28.46999	28.46999	0.0	0.0	0.0
18	X Y Z	28.46999	28.46999	28.46999	0.0	0.0	0.0
19	X Y Z						
20	X Y Z	73.75999	73.75999	73.75999	0.0	0.0	0.0
21	X Y Z						
22	X Y Z						
23	X Y Z						
24	X Y Z						
25	X Y Z						

DYNAMIC DAMPING MATRIX---/

MODE	DAMPING RATIO	MODE	DAMPING RATIO	MODE	DAMPING RATIO	MODE	DAMPING RATIO	MODE	DAMPING RATIO
1	0.10000	2	0.10000	3	0.10000	4	0.10000	5	0.10000
6	0.10000	7	0.10000	8	0.10000	9	0.10000	10	0.10000
11	0.10000	12	0.10000	13	0.10000	14	0.10000	15	0.10000
16	0.10000	17	0.10000	18	0.10000	19	0.10000	20	0.10000
21	0.10000	22	0.10000	23	0.10000	24	0.10000	25	0.10000
26	0.10000	27	0.10000	28	0.10000	29	0.10000	30	0.10000
31	0.10000	32	0.10000	33	0.10000	34	0.10000	35	0.10000
36	0.10000	37	0.10000	38	0.10000	39	0.10000	40	0.10000
41	0.10000	42	0.10000	43	0.10000	44	0.10000	45	0.10000
46	0.10000	47	0.10000	48	0.10000	49	0.10000	50	0.10000
51	0.10000	52	0.10000	53	0.10000	54	0.10000	55	0.10000
56	0.10000	57	0.10000	58	0.10000	59	0.10000	60	0.10000
61	0.10000	62	0.10000	63	0.10000	64	0.10000	65	0.10000
66	0.10000	67	0.10000	68	0.10000	69	0.10000	70	0.10000
71	0.10000	72	0.10000	73	0.10000	74	0.10000	75	0.10000

DYNAMIC STIFFNESS MATRIX---/

****** MATRIX IS TO BE COMPUTED FROM MEMBER PROPERTIES ******

********** DYNAMIC LOADING DATA **********

LOADING - 7 CENTRIFUGAL FORCES STATUS - ACTIVE

JOINT LOADS---/

JOINT	DIRECTION	TIME	LOAD	TIME	LOAD	TIME	LOAD	TIME	LOAD

10	FORCE Y	FUNCTION: SIN	AMPLITUDE:	1.4300E 02	FREQUENCY:	7.2770E 02	PHASE ANGLE:	0.0
10	FORCE Z	FUNCTION: SIN	AMPLITUDE:	1.4300E 02	FREQUENCY:	7.2770E 02	PHASE ANGLE:	1.5707E 00
15	FORCE Y	FUNCTION: SIN	AMPLITUDE:	4.0400E 03	FREQUENCY:	7.2770E 02	PHASE ANGLE:	0.0
15	FORCE Z	FUNCTION: SIN	AMPLITUDE:	4.0400E 03	FREQUENCY:	7.2770E 02	PHASE ANGLE:	1.5707E 00
20	FORCE Y	FUNCTION: SIN	AMPLITUDE:	3.8970E 03	FREQUENCY:	7.2770E 02	PHASE ANGLE:	0.0
20	FORCE Z	FUNCTION: SIN	AMPLITUDE:	3.8970E 03	FREQUENCY:	7.2770E 02	PHASE ANGLE:	1.5707E 00

INTEGRATION PERIODS-------------------------------/

INITIAL	FINAL	INCREMENT
0.0	0.08630	0.00072

****
* END OF DATA FROM INTERNAL STORAGE *
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DYNAMIC ANALYSIS MODAL 20
  INTERPOLATION FOR DYNAMIC RESPONSE QUANTITIES
  IS PERFORMED USING A DISPLACEMENT SPECTRUM

(18)

UNITS KIPS INCHES CYCLES SECONDS

LIST DYNAMIC EIGENVALUES 20

(19)

****
RESULTS OF LATEST ANALYSES
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PROBLEM - EXAMPLE   TITLE - STATIC AND DYNAMIC ANALYSIS OF A TABLE-TOP

ACTIVE UNITS   INCH KIP   CYC   DEGF SEC

ACTIVE STRUCTURE TYPE   SPACE   FRAME

ACTIVE COORDINATE AXES   X Y Z

EIGENVALUES

MODE	EIGENVALUE	FREQUENCY	PERIOD
1	1.355696D 01	3.681978D 00	2.715931D-01
2	2.211166D 01	4.702304D 00	2.126617D-01
3	5.984095D 01	7.735694D 00	1.292709D-01
4	1.270442D 02	1.127139D 01	8.872022D-02
5	1.693498D 02	1.301345D 01	7.684360D-02
6	1.704098D 02	1.305411D 01	7.660421D-02
7	3.066904D 02	1.751258D 01	5.710182D-02
8	3.360195D 02	1.833084D 01	5.455289D-02
9	4.337265D 02	2.082610D 01	4.801667D-02
10	7.565536D 02	2.750552D 01	3.635634D-02
11	8.137907D 02	2.852702D 01	3.505449D-02
12	1.068460D 03	3.268731D 01	3.059292D-02
13	1.163706D 03	3.411314D 01	2.931421D-02
14	1.740107D 03	4.171459D 01	2.397243D-02
15	2.426024D 03	4.925469D 01	2.030264D-02
16	2.571863D 03	5.071354D 01	1.971860D-02
17	3.785540D 03	6.152674D 01	1.625310D-02
18	4.256958D 03	6.524537D 01	1.532676D-02
19	4.475587D 03	6.689983D 01	1.494772D-02
20	4.622681D 03	6.799030D 01	1.470798D-02

NORMALIZE EIGENVECTORS

LIST DYNAMIC EIGENVECTORS 20

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RESULTS OF LATEST ANALYSES
****

PROBLEM - EXAMPLE   TITLE - STATIC AND DYNAMIC ANALYSIS OF A TABLE-TOP

ACTIVE UNITS   INCH KIP   CYC   DEGF SEC

ACTIVE STRUCTURE TYPE   SPACE   FRAME

ACTIVE COORDINATE AXES   X Y Z

EIGENVECTORS

MODE   1

JOINT		DISPLACEMENT X DISP.	Y DISP.	Z DISP.	ROTATION X ROT.	Y ROT.	Z ROT.
1	GLOBAL	-0.0067923	0.2124984	0.0792493	0.0037039	0.0001132	-0.0000051
2	GLOBAL	0.0067923	-0.2124984	0.0792493	0.0037039	0.0001132	0.0000051
3	GLOBAL	-0.0068172	0.2100704	0.0683671	0.0036886	0.0001137	-0.0000519
4	GLOBAL	0.0068172	-0.2100704	0.0683671	0.0036886	0.0001137	0.0000519
5	GLOBAL	-0.0068133	0.2037185	0.0601735	0.0035246	0.0001136	-0.0000615
6	GLOBAL	0.0068133	-0.2037185	0.0601735	0.0035246	0.0001136	0.0000615

7	GLOBAL	-0.0068001	0.2016964	0.0520057	0.0035207	0.0001133	-0.0000068
8	GLOBAL	0.0068001	-0.2016964	0.0520057	0.0035207	0.0001133	0.0000068
9	GLOBAL	-0.0098958	0.2202079	0.9999302	0.0039341	0.0001690	-0.0000282
10	GLOBAL	0.0000000	-0.0000000	0.9999995	0.0035381	0.0001629	0.0000000
11	GLOBAL	0.0098958	-0.2202079	0.9999302	0.0039341	0.0001690	0.0000282
12	GLOBAL	-0.0098804	0.2180931	0.9908612	0.0039948	0.0002043	-0.0000506
13	GLOBAL	0.0098805	-0.2180931	0.9908612	0.0039948	0.0002043	0.0000506
14	GLOBAL	-0.0098642	0.2163134	0.9814661	0.0040554	0.0001680	-0.0000128
15	GLOBAL	0.0000000	0.0000000	0.9822227	0.0033801	0.0001626	-0.0000000
16	GLOBAL	0.0098642	-0.2163134	0.9814661	0.0040554	0.0001680	0.0000128
17	GLOBAL	-0.0099592	0.2142941	0.9681619	0.0039758	0.0002373	-0.0000450
18	GLOBAL	0.0099592	-0.2142941	0.9681619	0.0039758	0.0002373	0.0000450
19	GLOBAL	-0.0100490	0.2104947	0.9480168	0.0038963	0.0002767	-0.0000497
20	GLOBAL	0.0000000	-0.0000000	0.9486380	0.0033142	0.0001129	-0.0000000
21	GLOBAL	0.0100490	-0.2104947	0.9480168	0.0038963	0.0002767	0.0000497
22	GLOBAL	0.0000000	0.0000000	0.0791737	0.0034562	0.0001132	-0.0000000
23	GLOBAL	0.0000000	-0.0000000	0.0682918	0.0034034	0.0001136	-0.0000000
24	GLOBAL	0.0000000	-0.0000000	0.0601170	0.0033343	0.0001135	0.0000000
25	GLOBAL	0.0000000	0.0000000	0.0519509	0.0032877	0.0001133	0.0000000

MODE 2

JOINT		/----DISPLACEMENT---- X DISP.	Y DISP.	Z DISP.	//----ROTATION---- X ROT.	Y ROT.	Z ROT.
1	GLOBAL	0.1141280	0.2520703	-0.0000238	0.0000440	-0.0000002	-0.0024443
2	GLOBAL	0.1141280	0.2520703	0.0000238	-0.0000440	0.0000002	-0.0024443
3	GLOBAL	0.1143430	0.0306087	-0.0000023	0.0000011	-0.0000003	-0.0023728
4	GLOBAL	0.1143430	0.0306087	0.0000023	-0.0000011	0.0000003	-0.0023728
5	GLOBAL	0.1137926	-0.1061810	0.0000100	-0.0000242	-0.0000001	-0.0016053
6	GLOBAL	0.1137926	-0.1061810	-0.0000100	0.0000242	0.0000001	-0.0016053
7	GLOBAL	0.1135614	-0.2192960	0.0000138	-0.0000360	-0.0000001	-0.0016290
8	GLOBAL	0.1135614	-0.2192960	-0.0000138	0.0000360	0.0000001	-0.0016290
9	GLOBAL	0.9840969	0.2652970	0.0001235	-0.0000124	-0.0000000	-0.0027126
10	GLOBAL	0.9844028	0.2657518	-0.0000000	0.0000000	-0.0000000	-0.0027126
11	GLOBAL	0.9840969	0.2652970	-0.0001235	0.0000124	0.0000000	-0.0027126
12	GLOBAL	0.9835075	0.1503842	0.0011290	-0.0000065	-0.0000210	-0.0022718
13	GLOBAL	0.9835075	0.1503842	-0.0011290	0.0000065	0.0000210	-0.0022718
14	GLOBAL	0.9827755	0.0284256	0.0001294	-0.0000007	0.0000835	-0.0030046
15	GLOBAL	0.9930775	0.0285451	0.0000000	-0.0000000	-0.0000000	-0.0030046
16	GLOBAL	0.9827755	0.0284256	-0.0001294	0.0000007	-0.0000835	-0.0030046
17	GLOBAL	0.9840539	-0.1270545	0.0001671	0.0000148	-0.0000405	-0.0015566
18	GLOBAL	0.9840539	-0.1270545	-0.0001671	-0.0000148	0.0000405	-0.0015566
19	GLOBAL	0.9844893	-0.2314746	-0.0003848	0.0000303	0.0000998	-0.0015984
20	GLOBAL	0.9999995	-0.2329970	-0.0000000	-0.0000000	-0.0000000	-0.0015984
21	GLOBAL	0.9844893	-0.2314746	0.0003848	-0.0000303	-0.0000998	-0.0015984
22	GLOBAL	0.1141202	0.2496134	-0.0000000	0.0000000	-0.0000000	-0.0024069
23	GLOBAL	0.1143250	0.0307146	-0.0000000	-0.0000000	0.0000000	-0.0022308
24	GLOBAL	0.1137908	-0.1048773	-0.0000000	-0.0000000	-0.0000000	-0.0016196
25	GLOBAL	0.1135581	-0.2175590	-0.0000000	-0.0000000	0.0000000	-0.0015967

MODE 3

JOINT		/----DISPLACEMENT---- X DISP.	Y DISP.	Z DISP.	//----ROTATION---- X ROT.	Y ROT.	Z ROT.
1	GLOBAL	0.1417329	-0.0112100	-0.2808349	-0.0003631	-0.0023626	0.0000622
2	GLOBAL	-0.1417329	0.0112100	-0.2808349	-0.0003631	-0.0023626	-0.0000622
3	GLOBAL	0.1417809	0.0064624	-0.0540366	0.0001191	-0.0023631	0.0000324
4	GLOBAL	-0.1417809	-0.0064624	-0.0540366	0.0001191	-0.0023631	-0.0000324
5	GLOBAL	0.1414983	0.0262196	0.1159251	0.0005198	-0.0023587	0.0003433
6	GLOBAL	-0.1414983	-0.0262196	0.1159251	0.0005198	-0.0023587	-0.0003433
7	GLOBAL	0.1414254	0.0507389	0.2857042	0.0010257	-0.0023576	0.0001933
8	GLOBAL	-0.1414254	-0.0507389	0.2857042	0.0010257	-0.0023576	-0.0001933
9	GLOBAL	0.4060833	-0.0158627	-0.7106385	-0.0007704	-0.0067492	0.0000255
10	GLOBAL	0.0000000	0.0000000	-0.7108562	-0.0000114	-0.0067775	-0.0000000
11	GLOBAL	-0.4060833	0.0158627	-0.7106385	-0.0007704	-0.0067492	-0.0000255
12	GLOBAL	0.4053364	-0.0020500	-0.3695758	-0.0003041	-0.0072886	0.0003814
13	GLOBAL	-0.4053364	0.0020500	-0.3695758	-0.0003041	-0.0072886	-0.0003814
14	GLOBAL	0.4044307	0.0045544	-0.0287171	0.0001621	-0.0067165	-0.0002749
15	GLOBAL	0.0000000	-0.0000000	-0.0288151	0.0000328	-0.0067525	-0.0000000
16	GLOBAL	-0.4044307	-0.0045544	-0.0287171	0.0001621	-0.0067165	0.0002749
17	GLOBAL	0.4055614	0.0207648	0.4840401	0.0010292	-0.0072804	0.0005615
18	GLOBAL	-0.4055614	-0.0207648	0.4840401	0.0010292	-0.0072804	-0.0005615
19	GLOBAL	0.4057521	0.0622181	0.9971092	0.0018963	-0.0069046	0.0004316
20	GLOBAL	-0.0000000	-0.0000000	0.9999995	0.0006073	-0.0066915	-0.0000000
21	GLOBAL	-0.4057521	-0.0622181	0.9971092	0.0018963	-0.0069046	-0.0004316
22	GLOBAL	-0.0000000	0.0000000	-0.2807355	-0.0000823	-0.0023619	-0.0000000
23	GLOBAL	-0.0000000	0.0000000	-0.0540228	0.0001200	-0.0023625	-0.0000000
24	GLOBAL	-0.0000000	-0.0000000	0.1158892	0.0004012	-0.0023581	0.0000000
25	GLOBAL	-0.0000000	0.0000000	0.2856015	0.0007171	-0.0023568	0.0000000

MODE 4

JOINT		/----DISPLACEMENT---- X DISP.	Y DISP.	Z DISP.	//----ROTATION---- X ROT.	Y ROT.	Z ROT.
1	GLOBAL	-0.1357258	0.9512756	0.0000647	0.0001767	-0.0000002	-0.0005171
2	GLOBAL	-0.1357258	0.9512756	-0.0000647	-0.0001767	0.0000002	-0.0005171

3	GLOBAL	-0.1360005	0.8794618	0.0000861	0.0002113	-0.0000000	-0.0011704
4	GLOBAL	-0.1360005	0.8794618	-0.0000861	-0.0002113	0.0000000	-0.0011704
5	GLOBAL	-0.1357959	0.7569069	0.0000824	0.0001545	0.0000000	-0.0019545
6	GLOBAL	-0.1357959	0.7569069	-0.0000824	-0.0001545	-0.0000000	-0.0019545
7	GLOBAL	-0.1356713	0.6186250	0.0000829	0.0001600	-0.0000000	-0.0018612
8	GLOBAL	-0.1356713	0.6186250	-0.0000829	-0.0001600	0.0000000	-0.0018612
9	GLOBAL	-0.0617591	0.9922456	-0.0000611	-0.0001987	0.0000036	-0.0007531
10	GLOBAL	-0.0617608	0.9999998	-0.0000000	0.0000000	0.0000000	-0.0007531
11	GLOBAL	-0.0617591	0.9922456	0.0000611	0.0001987	-0.0000036	-0.0007531
12	GLOBAL	-0.0613237	0.9523607	-0.0007635	-0.0004410	0.0000161	-0.0008887
13	GLOBAL	-0.0613237	0.9523607	0.0007635	0.0004410	-0.0000161	-0.0008887
14	GLOBAL	-0.0608373	0.9116666	-0.0006962	-0.0006833	-0.0000283	-0.0007281
15	GLOBAL	-0.0646018	0.9512376	-0.0000000	0.0000000	0.0000000	-0.0007281
16	GLOBAL	-0.0608373	0.9116666	0.0006962	0.0006833	0.0000283	-0.0007281
17	GLOBAL	-0.0610613	0.8215370	-0.0007474	-0.0006150	0.0000132	-0.0018863
18	GLOBAL	-0.0610613	0.8215370	0.0007474	0.0006150	-0.0000132	-0.0018863
19	GLOBAL	-0.0609848	0.6565737	-0.0002659	-0.0005467	-0.0000425	-0.0023556
20	GLOBAL	-0.0670850	0.6833196	0.0000000	0.0000000	0.0000000	-0.0023556
21	GLOBAL	-0.0609848	0.6565737	0.0002659	0.0005467	0.0000425	-0.0023556
22	GLOBAL	-0.1357332	0.9442801	-0.0000000	-0.0000000	-0.0000000	-0.0005266
23	GLOBAL	-0.1359981	0.8703532	-0.0000000	-0.0000000	-0.0000000	-0.0012052
24	GLOBAL	-0.1357968	0.7522643	0.0000000	-0.0000000	-0.0000000	-0.0019286
25	GLOBAL	-0.1356719	0.6126020	0.0000000	0.0000000	-0.0000000	-0.0018910

MODE 5

JOINT		/----DISPLACEMENT----			//----ROTATION----		/
		X DISP.	Y DISP.	Z DISP.	X ROT.	Y ROT.	Z ROT.
1	GLOBAL	0.9988843	-0.4129394	0.0000667	-0.0000119	0.0000004	0.0051520
2	GLOBAL	0.9988843	-0.4129394	-0.0000668	0.0000119	-0.0000004	0.0051520
3	GLOBAL	0.9999954	0.0787779	0.0000578	0.0000362	0.0000000	0.0049613
4	GLOBAL	0.9999954	0.0787779	-0.0000578	-0.0000362	-0.0000000	0.0049613
5	GLOBAL	0.9991596	0.4351376	0.0000490	0.0000525	0.0000003	0.0049780
6	GLOBAL	0.9991596	0.4351376	-0.0000491	-0.0000525	-0.0000003	0.0049780
7	GLOBAL	0.9980285	0.7966839	0.0000259	0.0001081	0.0000005	0.0050278
8	GLOBAL	0.9980285	0.7966839	-0.0000259	-0.0001081	-0.0000005	0.0050278
9	GLOBAL	0.0282955	-0.4348981	0.0000401	0.0000379	0.0000004	0.0052981
10	GLOBAL	0.0283761	-0.4370804	-0.0000000	0.0000000	-0.0000000	0.0052981
11	GLOBAL	0.0282955	-0.4348981	-0.0000401	-0.0000379	-0.0000004	0.0052981
12	GLOBAL	0.0282294	-0.1753971	0.0002104	-0.0000581	-0.0000029	0.0054684
13	GLOBAL	0.0282294	-0.1753971	-0.0002104	0.0000581	0.0000029	0.0054684
14	GLOBAL	0.0281319	0.0849489	-0.0001319	-0.0001541	0.0000217	0.0053189
15	GLOBAL	0.0306260	0.0920303	0.0000000	0.0000000	-0.0000000	0.0053189
16	GLOBAL	0.0281319	0.0849489	0.0001319	0.0001541	-0.0000217	0.0053189
17	GLOBAL	0.0285549	0.4766068	-0.0003179	-0.0004447	-0.0000062	0.0053206
18	GLOBAL	0.0285549	0.4766068	0.0003179	0.0004447	0.0000062	0.0053206
19	GLOBAL	0.0287905	0.8416854	-0.0007488	-0.0007354	0.0000289	0.0049296
20	GLOBAL	0.0328009	0.8815361	-0.0000000	0.0000000	-0.0000000	0.0049296
21	GLOBAL	0.0287905	0.8416854	0.0007488	0.0007354	-0.0000289	0.0049296
22	GLOBAL	0.9989050	-0.4119033	-0.0000000	0.0000000	-0.0000000	0.0051520
23	GLOBAL	0.9999999	0.0772559	-0.0000000	0.0000000	-0.0000000	0.0049821
24	GLOBAL	0.9991839	0.4340099	-0.0000000	0.0000000	-0.0000000	0.0049619
25	GLOBAL	0.9980538	0.7924210	-0.0000000	0.0000000	-0.0000000	0.0050130

MODE 6

JOINT		/----DISPLACEMENT----			//----ROTATION----		/
		X DISP.	Y DISP.	Z DISP.	X ROT.	Y ROT.	Z ROT.
1	GLOBAL	-0.0221089	-0.2949927	0.9996825	-0.0048784	0.0003682	0.0000038
2	GLOBAL	0.0221089	0.2949927	0.9996825	-0.0048784	0.0003682	-0.0000038
3	GLOBAL	-0.0225003	-0.2909883	0.9641493	-0.0047706	0.0003749	0.0000708
4	GLOBAL	0.0225003	0.2909883	0.9641493	-0.0047706	0.0003749	-0.0000708
5	GLOBAL	-0.0229821	-0.2794129	0.9368019	-0.0046408	0.0003835	0.0001563
6	GLOBAL	0.0229821	0.2794129	0.9368019	-0.0046408	0.0003835	-0.0001563
7	GLOBAL	-0.0231594	-0.2697898	0.9090357	-0.0044969	0.0003862	0.0000754
8	GLOBAL	0.0231595	0.2697899	0.9090357	-0.0044969	0.0003862	-0.0000754
9	GLOBAL	0.0101910	-0.3080194	-0.0126071	-0.0050918	-0.0001803	0.0000617
10	GLOBAL	0.0000000	-0.0000000	-0.0126181	-0.0051546	-0.0001646	0.0000000
11	GLOBAL	-0.0101910	0.3080194	-0.0126071	-0.0050918	-0.0001803	-0.0000617
12	GLOBAL	0.0101242	-0.3022175	0.0001028	-0.0049588	-0.0003155	0.0001457
13	GLOBAL	-0.0101242	0.3022175	0.0001028	-0.0049588	-0.0003155	-0.0001457
14	GLOBAL	0.0100460	-0.2985236	0.0144308	-0.0048259	-0.0002477	-0.0000511
15	GLOBAL	0.0000000	0.0000000	0.0145719	-0.0050501	-0.0001273	0.0000000
16	GLOBAL	-0.0100460	0.2985236	0.0144308	-0.0048259	-0.0002477	0.0000511
17	GLOBAL	0.0101557	-0.2983022	0.0399194	-0.0046460	-0.0003974	0.0001584
18	GLOBAL	-0.0101557	0.2983022	0.0399194	-0.0046460	-0.0003974	-0.0001584
19	GLOBAL	0.0101984	-0.2769642	0.0656312	-0.0044661	-0.0002960	0.0003157
20	GLOBAL	-0.0000000	0.0000000	0.0661758	-0.0046910	-0.0001069	0.0000000
21	GLOBAL	-0.0101984	0.2769642	0.0656312	-0.0044661	-0.0002960	-0.0003157
22	GLOBAL	0.0000000	-0.0000000	0.9999998	-0.0049306	0.0003686	0.0000000
23	GLOBAL	0.0000000	0.0000000	0.9644235	-0.0048685	0.0003750	0.0000000
24	GLOBAL	0.0000000	0.0000000	0.9371206	-0.0046715	0.0003829	0.0000000
25	GLOBAL	0.0000000	0.0000000	0.9093761	-0.0045153	0.0003860	0.0000000

MODE 7

JOINT		/----DISPLACEMENT----			//----ROTATION----		/
		X DISP.	Y DISP.	Z DISP.	X ROT.	Y ROT.	Z ROT.
1	GLOBAL	0.4619697	0.1026532	-0.8489224	0.0020031	-0.0076982	-0.0002470
2	GLOBAL	-0.4619697	-0.1026532	-0.8489224	0.0020031	-0.0076982	0.0002470

3	GLOBAL	0.4621285	0.0461054	-0.1098231	0.0007171	-0.0076992	-0.0002716
4	GLOBAL	-0.4621285	-0.0461054	-0.1098231	0.0007171	-0.0076992	0.0002716
5	GLOBAL	0.4623657	-0.0152219	0.4447166	-0.0004191	-0.0077036	-0.0009738
6	GLOBAL	-0.4623657	0.0152219	0.4447166	-0.0004191	-0.0077036	0.0009738
7	GLOBAL	0.4621059	-0.0841920	0.9992144	-0.0017586	-0.0077005	-0.0005645
8	GLOBAL	-0.4621059	0.0841920	0.9992144	-0.0017586	-0.0077005	0.0005645
9	GLOBAL	-0.0688224	0.1209738	0.3347217	0.0030519	0.0013073	-0.0002315
10	GLOBAL	0.0000000	-0.0000000	0.3352479	0.0014984	0.0010669	0.0000000
11	GLOBAL	0.0688224	-0.1209738	0.3347217	0.0030519	0.0013073	0.0002315
12	GLOBAL	-0.0681350	0.0799024	0.2061988	0.0018638	0.0035192	-0.0010993
13	GLOBAL	0.0681350	-0.0799024	0.2061988	0.0018638	0.0035192	0.0010993
14	GLOBAL	-0.0673107	0.0525383	0.0509307	0.0006757	0.0023527	0.0003516
15	GLOBAL	-0.0000000	-0.0000000	0.0518339	0.0009756	0.0005064	0.0000000
16	GLOBAL	0.0673107	-0.0525383	0.0509307	0.0006757	0.0023527	-0.0003516
17	GLOBAL	-0.0695602	-0.0024613	-0.2404357	-0.0015924	0.0048511	-0.0015154
18	GLOBAL	0.0695602	0.0024613	-0.2404357	-0.0015924	0.0048511	0.0015154
19	GLOBAL	-0.0709832	-0.1135046	-0.5523883	-0.0038605	0.0033811	-0.0012085
20	GLOBAL	0.0000001	0.0000000	-0.5606936	-0.0009074	0.0000840	0.0000000
21	GLOBAL	0.0709832	0.1135048	-0.5523883	-0.0038605	0.0033811	0.0012085
22	GLOBAL	-0.0000000	-0.0000000	-0.8495282	0.0015032	-0.0077006	0.0000000
23	GLOBAL	-0.0000000	-0.0000000	-0.1098380	0.0007453	-0.0077047	0.0000000
24	GLOBAL	-0.0000000	0.0000000	0.4451560	-0.0001740	-0.0077078	0.0000000
25	GLOBAL	-0.0000000	0.0000000	0.9999997	-0.0011113	-0.0077028	0.0000000

MODE 8

JOINT		/------DISPLACEMENT------			//------ROTATION------/		
		X DISP.	Y DISP.	Z DISP.	X ROT.	Y ROT.	Z ROT.
1	GLOBAL	-0.4209461	-0.5031327	-0.0001302	0.0000374	-0.0000015	0.0032842
2	GLOBAL	-0.4209460	-0.5031327	0.0001302	-0.0000374	0.0000015	0.0032842
3	GLOBAL	-0.4209600	-0.1290688	0.0000263	-0.0000213	-0.0000021	0.0038368
4	GLOBAL	-0.4209600	-0.1290688	-0.0000263	0.0000213	0.0000021	0.0038368
5	GLOBAL	-0.4225176	0.3012768	0.0001572	-0.0000274	-0.0000016	0.0074400
6	GLOBAL	-0.4225175	0.3012768	-0.0001572	0.0000274	0.0000016	0.0074400

LIST DYNAMIC DISPLACEMENTS ALL

****
RESULTS OF LATEST ANALYSES
****

PROBLEM - EXAMPLE TITLE - STATIC AND DYNAMIC ANALYSIS OF A TABLE-TOP

ACTIVE UNITS INCH KIP CYC DEGF SEC

ACTIVE STRUCTURE TYPE SPACE FRAME

ACTIVE COORDINATE AXES X Y Z

LOADING - 7 CENTRIFUGAL FORCES

RESULTANT JOINT DISPLACEMENTS- SUPPORTS

JOINT		TIME	/------DISPLACEMENT------			//------ROTATION------/		
			X DISP.	Y DISP.	Z DISP.	X ROT.	Y ROT.	Z ROT.
1	GLOBAL	0.0	0.0	0.0	0.0	0.0	0.0	0.0
		0.00072	0.0000000	0.0000000	0.0000000	0.0000000	-0.0000000	0.0000000
		0.00144	0.0000000	0.0000000	0.0000000	0.0000000	-0.0000000	0.0000000
		0.00216	0.0000000	0.0000002	0.0000000	0.0000000	-0.0000000	0.0000000
		0.00288	0.0000001	0.0000005	0.0000001	0.0000000	-0.0000000	0.0000000
		0.00360	0.0000001	0.0000012	0.0000002	0.0000000	-0.0000000	0.0000000
		0.00432	0.0000002	0.0000022	0.0000004	0.0000000	-0.0000000	0.0000000
		0.00504	0.0000004	0.0000037	0.0000005	0.0000000	-0.0000000	0.0000000
		0.00576	0.0000006	0.0000056	0.0000006	0.0000001	-0.0000000	0.0000000
		0.00647	0.0000008	0.0000079	0.0000005	0.0000001	-0.0000000	0.0000000
		0.00719	0.0000010	0.0000104	0.0000003	0.0000001	-0.0000000	0.0000000
		0.00791	0.0000013	0.0000130	0.0000001	0.0000001	-0.0000000	0.0000001
		0.00863	0.0000016	0.0000158	-0.0000001	0.0000001	-0.0000000	0.0000001
		0.00935	0.0000020	0.0000186	-0.0000002	0.0000000	-0.0000000	0.0000001
		0.01007	0.0000024	0.0000213	-0.0000003	0.0000000	-0.0000000	0.0000001
		0.01079	0.0000030	0.0000240	-0.0000003	-0.0000000	-0.0000000	0.0000001
		0.01151	0.0000036	0.0000266	-0.0000003	-0.0000000	-0.0000000	0.0000001
		0.01223	0.0000044	0.0000292	-0.0000003	-0.0000001	-0.0000000	0.0000001
		0.01295	0.0000052	0.0000320	-0.0000003	-0.0000001	-0.0000000	0.0000000
		0.01367	0.0000061	0.0000349	-0.0000003	-0.0000001	-0.0000000	0.0000000

0.01439	0.0000070	0.0000381	-0.0000004	-0.0000000	-0.0000000	0.0000000
0.01511	0.0000080	0.0000413	-0.0000006	-0.0000000	-0.0000000	0.0000001
0.01583	0.0000090	0.0000444	-0.0000008	0.0000000	-0.0000000	0.0000001
0.01655	0.0000099	0.0000470	-0.0000010	0.0000000	-0.0000000	0.0000001
0.01727	0.0000108	0.0000488	-0.0000013	0.0000001	-0.0000000	0.0000001
0.01798	0.0000117	0.0000497	-0.0000015	0.0000001	-0.0000000	0.0000001
0.01870	0.0000125	0.0000497	-0.0000017	0.0000001	-0.0000000	0.0000001
0.01942	0.0000134	0.0000493	-0.0000018	0.0000000	-0.0000000	0.0000001
0.02014	0.0000142	0.0000491	-0.0000018	0.0000000	-0.0000000	0.0000001
0.02086	0.0000149	0.0000496	-0.0000019	-0.0000000	-0.0000000	0.0000000
0.02158	0.0000157	0.0000515	-0.0000020	-0.0000000	-0.0000000	-0.0000000
0.02230	0.0000164	0.0000548	-0.0000021	-0.0000000	-0.0000000	-0.0000000
0.02302	0.0000170	0.0000593	-0.0000024	-0.0000000	-0.0000000	-0.0000000
0.02374	0.0000176	0.0000645	-0.0000027	0.0000000	-0.0000000	-0.0000000
0.02446	0.0000182	0.0000692	-0.0000030	0.0000001	-0.0000000	-0.0000000
0.02518	0.0000186	0.0000727	-0.0000033	0.0000001	-0.0000000	0.0000000
0.02590	0.0000189	0.0000742	-0.0000035	0.0000001	-0.0000000	0.0000000
0.02662	0.0000191	0.0000734	-0.0000037	0.0000001	-0.0000000	0.0000001
0.02734	0.0000192	0.0000705	-0.0000038	0.0000001	-0.0000000	0.0000001
0.02806	0.0000192	0.0000662	-0.0000037	0.0000000	-0.0000000	0.0000000
0.02878	0.0000191	0.0000616	-0.0000036	-0.0000000	-0.0000000	0.0000000
0.02950	0.0000189	0.0000575	-0.0000034	-0.0000001	-0.0000000	0.0000000
0.03021	0.0000186	0.0000549	-0.0000032	-0.0000001	-0.0000000	-0.0000000
0.03093	0.0000183	0.0000539	-0.0000031	-0.0000001	-0.0000000	-0.0000000
0.03165	0.0000178	0.0000543	-0.0000030	-0.0000001	-0.0000000	-0.0000000
0.03237	0.0000172	0.0000555	-0.0000030	-0.0000000	-0.0000000	-0.0000000
0.03309	0.0000165	0.0000565	-0.0000030	0.0000000	-0.0000000	0.0000000
0.03381	0.0000156	0.0000565	-0.0000029	0.0000001	-0.0000000	0.0000000
0.03453	0.0000146	0.0000550	-0.0000028	0.0000001	-0.0000000	0.0000001
0.03525	0.0000135	0.0000518	-0.0000026	0.0000001	-0.0000000	0.0000001
0.03597	0.0000123	0.0000473	-0.0000023	0.0000000	-0.0000000	0.0000001
0.03669	0.0000109	0.0000424	-0.0000018	-0.0000000	-0.0000000	0.0000001
0.03741	0.0000095	0.0000379	-0.0000013	-0.0000000	-0.0000000	0.0000000
0.03813	0.0000080	0.0000347	-0.0000007	-0.0000001	-0.0000000	0.0000000
0.03885	0.0000064	0.0000331	-0.0000002	-0.0000001	0.0000000	-0.0000000
0.03957	0.0000048	0.0000332	0.0000003	-0.0000001	0.0000000	-0.0000001
0.04029	0.0000031	0.0000344	0.0000007	-0.0000000	0.0000000	-0.0000001
0.04101	0.0000014	0.0000357	0.0000011	-0.0000000	0.0000000	-0.0000001
0.04173	-0.0000004	0.0000361	0.0000014	0.0000001	0.0000000	-0.0000000
0.04244	-0.0000022	0.0000347	0.0000017	0.0000001	0.0000000	-0.0000000
0.04316	-0.0000041	0.0000311	0.0000021	0.0000001	0.0000000	0.0000000
0.04388	-0.0000060	0.0000252	0.0000025	0.0000001	0.0000000	0.0000000
0.04460	-0.0000080	0.0000176	0.0000030	0.0000001	0.0000000	0.0000000
0.04532	-0.0000100	0.0000093	0.0000035	0.0000000	0.0000000	0.0000000
0.04604	-0.0000121	0.0000014	0.0000041	-0.0000000	0.0000000	0.0000000
0.04676	-0.0000140	-0.0000049	0.0000047	-0.0000001	0.0000000	-0.0000000
0.04748	-0.0000160	-0.0000091	0.0000052	-0.0000001	0.0000000	-0.0000000
0.04820	-0.0000179	-0.0000111	0.0000056	-0.0000001	0.0000000	-0.0000001
0.04892	-0.0000197	-0.0000114	0.0000059	-0.0000000	0.0000000	-0.0000001
0.04964	-0.0000215	-0.0000109	0.0000061	0.0000000	0.0000000	-0.0000001
0.05036	-0.0000232	-0.0000107	0.0000062	0.0000000	0.0000000	-0.0000000
0.05108	-0.0000249	-0.0000119	0.0000062	0.0000001	0.0000000	-0.0000000
0.05180	-0.0000265	-0.0000152	0.0000063	0.0000001	0.0000000	0.0000000
0.05252	-0.0000281	-0.0000206	0.0000064	0.0000001	0.0000000	0.0000000
0.05324	-0.0000295	-0.0000276	0.0000066	0.0000000	0.0000000	0.0000000
0.05395	-0.0000309	-0.0000355	0.0000068	-0.0000000	0.0000000	0.0000000
0.05467	-0.0000321	-0.0000429	0.0000070	-0.0000001	0.0000000	0.0000000
0.05539	-0.0000332	-0.0000490	0.0000072	-0.0000001	0.0000000	-0.0000000
0.05611	-0.0000341	-0.0000531	0.0000074	-0.0000001	0.0000000	-0.0000000
0.05683	-0.0000349	-0.0000551	0.0000074	-0.0000001	0.0000000	-0.0000000
0.05755	-0.0000355	-0.0000554	0.0000073	-0.0000001	0.0000000	-0.0000001
0.05827	-0.0000360	-0.0000549	0.0000071	-0.0000000	0.0000000	-0.0000000
0.05899	-0.0000363	-0.0000548	0.0000069	0.0000000	0.0000000	-0.0000000
0.05971	-0.0000365	-0.0000559	0.0000066	0.0000001	0.0000000	0.0000000
0.06043	-0.0000366	-0.0000589	0.0000063	0.0000001	0.0000000	0.0000000
0.06115	-0.0000366	-0.0000637	0.0000061	0.0000001	0.0000000	0.0000000
0.06187	-0.0000364	-0.0000698	0.0000059	0.0000001	0.0000000	0.0000001
0.06259	-0.0000360	-0.0000760	0.0000058	0.0000000	0.0000000	0.0000001
0.06331	-0.0000355	-0.0000814	0.0000058	-0.0000000	0.0000000	0.0000000
0.06403	-0.0000349	-0.0000850	0.0000057	-0.0000001	0.0000000	0.0000000
0.06475	-0.0000341	-0.0000862	0.0000056	-0.0000001	0.0000000	-0.0000000
0.06547	-0.0000331	-0.0000849	0.0000053	-0.0000001	0.0000000	-0.0000000
0.06618	-0.0000320	-0.0000817	0.0000050	-0.0000001	0.0000000	-0.0000000
0.06690	-0.0000307	-0.0000777	0.0000046	-0.0000000	0.0000000	-0.0000000
0.06762	-0.0000293	-0.0000739	0.0000041	0.0000000	0.0000000	-0.0000000
0.06834	-0.0000279	-0.0000715	0.0000037	0.0000001	0.0000000	0.0000000

0.04748	-0.0000471	-0.0000680	0.0000706	-0.0000001	-0.0000002	-0.0000001
0.04820	-0.0000474	-0.0000356	0.0000601	-0.0000002	-0.0000002	0.0000000
0.04892	-0.0000482	-0.0000050	0.0000334	-0.0000002	-0.0000001	0.0000002
0.04964	-0.0000493	0.0000152	-0.0000026	-0.0000001	0.0000001	0.0000003
0.05036	-0.0000500	0.0000192	-0.0000382	-0.0000000	0.0000002	0.0000003
0.05108	-0.0000501	0.0000055	-0.0000640	0.0000001	0.0000002	0.0000002
0.05180	-0.0000491	-0.0000224	-0.0000730	0.0000002	0.0000003	0.0000001
0.05252	-0.0000472	-0.0000572	-0.0000630	0.0000002	0.0000002	-0.0000000
0.05324	-0.0000445	-0.0000897	-0.0000367	0.0000002	0.0000001	-0.0000002

		0.05395	-0.0000417	-0.0001111	-0.0000013	0.0000002	-0.0000001	-0.0000003
		0.05467	-0.0000390	-0.0001158	0.0000336	0.0000001	-0.0000002	-0.0000003
		0.05539	-0.0000372	-0.0001023	0.0000584	0.0000000	-0.0000003	-0.0000002
		0.05611	-0.0000362	-0.0000741	0.0000663	-0.0000000	-0.0000003	-0.0000001
		0.05683	-0.0000363	-0.0000385	0.0000550	-0.0000001	-0.0000002	0.0000000
		0.05755	-0.0000371	-0.0000048	0.0000272	-0.0000001	-0.0000001	0.0000002
		0.05827	-0.0000381	0.0000183	-0.0000098	-0.0000001	0.0000001	0.0000003
		0.05899	-0.0000389	0.0000249	-0.0000464	-0.0000000	0.0000002	0.0000003
		0.05971	-0.0000389	0.0000137	-0.0000729	0.0000000	0.0000003	0.0000002
		0.06043	-0.0000380	-0.0000121	-0.0000826	0.0000001	0.0000003	0.0000001
		0.06115	-0.0000362	-0.0000451	-0.0000730	0.0000001	0.0000002	0.0000000
		0.06187	-0.0000337	-0.0000762	-0.0000468	0.0000001	0.0000001	-0.0000001
		0.06259	-0.0000310	-0.0000966	-0.0000106	0.0000001	-0.0000000	-0.0000002
		0.06331	-0.0000286	-0.0001005	0.0000252	-0.0000000	-0.0000002	-0.0000002
		0.06403	-0.0000270	-0.0000867	0.0000510	-0.0000001	-0.0000002	-0.0000002
		0.06475	-0.0000263	-0.0000584	0.0000598	-0.0000001	-0.0000003	-0.0000001
		0.06547	-0.0000267	-0.0000230	0.0000492	-0.0000002	-0.0000002	0.0000001
		0.06618	-0.0000278	0.0000103	0.0000219	-0.0000002	-0.0000001	0.0000002
		0.06690	-0.0000292	0.0000328	-0.0000148	-0.0000001	0.0000001	0.0000003
		0.06762	-0.0000304	0.0000387	-0.0000512	-0.0000001	0.0000002	0.0000003
		0.06834	-0.0000309	0.0000266	-0.0000776	0.0000000	0.0000003	0.0000002
		0.06906	-0.0000304	-0.0000001	-0.0000871	0.0000001	0.0000003	0.0000001
		0.06978	-0.0000291	-0.0000340	-0.0000771	0.0000001	0.0000002	-0.0000000
		0.07050	-0.0000270	-0.0000660	-0.0000504	0.0000002	0.0000001	-0.0000001
		0.07122	-0.0000248	-0.0000872	-0.0000143	0.0000001	-0.0000000	-0.0000002
		0.07194	-0.0000230	-0.0000918	0.0000216	0.0000001	-0.0000001	-0.0000002
		0.07266	-0.0000219	-0.0000783	0.0000475	-0.0000000	-0.0000002	-0.0000002
		0.07338	-0.0000217	-0.0000502	0.0000566	-0.0000001	-0.0000002	-0.0000000
		0.07410	-0.0000226	-0.0000146	0.0000462	-0.0000001	-0.0000002	0.0000001
		0.07482	-0.0000242	0.0000191	0.0000193	-0.0000001	-0.0000001	0.0000002
		0.07554	-0.0000260	0.0000423	-0.0000170	-0.0000001	0.0000001	0.0000003
		0.07626	-0.0000275	0.0000491	-0.0000529	-0.0000000	0.0000002	0.0000003
		0.07697	-0.0000283	0.0000381	-0.0000789	0.0000000	0.0000003	0.0000003
		0.07769	-0.0000281	0.0000126	-0.0000878	0.0000001	0.0000003	0.0000002
		0.07841	-0.0000270	-0.0000201	-0.0000773	0.0000001	0.0000002	0.0000000
		0.07913	-0.0000252	-0.0000509	-0.0000500	0.0000001	0.0000001	-0.0000001
		0.07985	-0.0000231	-0.0000710	-0.0000134	0.0000001	-0.0000000	-0.0000002
		0.08057	-0.0000214	-0.0000746	0.0000230	0.0000000	-0.0000002	-0.0000002
		0.08129	-0.0000204	-0.0000603	0.0000493	-0.0000001	-0.0000003	-0.0000002
		0.08201	-0.0000204	-0.0000315	0.0000587	-0.0000001	-0.0000003	-0.0000001
		0.08273	-0.0000213	0.0000045	0.0000487	-0.0000002	-0.0000002	0.0000001
		0.08345	-0.0000230	0.0000384	0.0000219	-0.0000002	-0.0000001	0.0000002
		0.08417	-0.0000248	0.0000614	-0.0000143	-0.0000001	0.0000000	0.0000003
		0.08489	-0.0000264	0.0000678	-0.0000502	-0.0000001	0.0000002	0.0000003
		0.08561	-0.0000272	0.0000562	-0.0000761	0.0000000	0.0000003	0.0000002
		0.08630	-0.0000270	0.0000310	-0.0000850	0.0000001	0.0000003	0.0000001
20	GLOBAL	0.0	0.0	0.0	0.0	0.0	0.0	0.0
		0.00072	-0.0000000	0.0000020	0.0000110	-0.0000000	0.0000000	0.0000000
		0.00144	-0.0000000	0.0000152	0.0000363	-0.0000000	0.0000000	0.0000000
		0.00216	-0.0000000	0.0000463	0.0000616	-0.0000000	0.0000000	0.0000000
		0.00288	-0.0000000	0.0000958	0.0000769	-0.0000000	0.0000001	0.0000000
		0.00360	-0.0000000	0.0001568	0.0000805	-0.0000000	0.0000001	0.0000001
		0.00432	-0.0000000	0.0002174	0.0000752	-0.0000000	0.0000001	0.0000001
		0.00504	0.0000000	0.0002640	0.0000634	0.0000000	0.0000000	0.0000002
		0.00576	0.0000001	0.0002851	0.0000450	0.0000000	-0.0000000	0.0000002
		0.00647	0.0000001	0.0002746	0.0000206	0.0000000	-0.0000001	0.0000003
		0.00719	0.0000001	0.0002339	-0.0000048	0.0000000	-0.0000002	0.0000002
		0.00791	-0.0000000	0.0001714	-0.0000229	0.0000000	-0.0000002	0.0000002
		0.00863	-0.0000003	0.0001007	-0.0000260	0.0000000	-0.0000002	0.0000000
		0.00935	-0.0000006	0.0000372	-0.0000117	0.0000000	-0.0000001	-0.0000001
		0.01007	-0.0000011	-0.0000058	0.0000155	0.0000000	-0.0000001	-0.0000003
		0.01079	-0.0000018	-0.0000204	0.0000466	0.0000000	0.0000000	-0.0000004
		0.01151	-0.0000025	-0.0000056	0.0000729	0.0000000	0.0000001	-0.0000004
		0.01223	-0.0000032	0.0000322	0.0000882	0.0000000	0.0000002	-0.0000003
		0.01295	-0.0000040	0.0000817	0.0000900	0.0000000	0.0000002	-0.0000001
		0.01367	-0.0000048	0.0001294	0.0000784	0.0000000	0.0000001	0.0000001
		0.01439	-0.0000056	0.0001635	0.0000559	0.0000000	0.0000001	0.0000003
		0.01511	-0.0000065	0.0001763	0.0000268	0.0000000	-0.0000000	0.0000004
		0.01583	-0.0000075	0.0001669	-0.0000024	0.0000000	-0.0000001	0.0000005
		0.01655	-0.0000086	0.0001405	-0.0000243	0.0000000	-0.0000001	0.0000005
		0.01727	-0.0000098	0.0001072	-0.0000330	0.0000000	-0.0000002	0.0000003
		0.01798	-0.0000113	0.0000789	-0.0000264	0.0000000	-0.0000002	0.0000002
		0.01870	-0.0000128	0.0000661	-0.0000073	0.0000000	-0.0000001	0.0000000
		0.01942	-0.0000144	0.0000748	0.0000182	0.0000000	-0.0000001	-0.0000001
		0.02014	-0.0000161	0.0001048	0.0000424	0.0000000	0.0000000	-0.0000002
		0.02086	-0.0000179	0.0001499	0.0000585	0.0000000	0.0000001	-0.0000001
		0.02158	-0.0000196	0.0001990	0.0000626	0.0000000	0.0000001	-0.0000001
		0.02230	-0.0000214	0.0002396	0.0000537	0.0000000	0.0000001	0.0000000
		0.02302	-0.0000232	0.0002606	0.0000343	0.0000000	0.0000000	0.0000001
		0.02374	-0.0000250	0.0002556	0.0000094	0.0000000	-0.0000000	0.0000002
		0.02446	-0.0000269	0.0002247	-0.0000144	0.0000000	-0.0000001	0.0000002
		0.02518	-0.0000289	0.0001741	-0.0000307	0.0000000	-0.0000001	0.0000001
		0.02590	-0.0000310	0.0001152	-0.0000348	0.0000000	-0.0000001	0.0000000
		0.02662	-0.0000332	0.0000610	-0.0000255	0.0000000	-0.0000001	-0.0000001
		0.02734	-0.0000354	0.0000231	-0.0000055	-0.0000000	-0.0000000	-0.0000002
		0.02806	-0.0000375	0.0000086	0.0000195	-0.0000000	0.0000000	-0.0000003
		0.02878	-0.0000395	0.0000181	0.0000425	-0.0000000	0.0000001	-0.0000003
		0.02950	-0.0000414	0.0000458	0.0000568	-0.0000000	0.0000001	-0.0000002
		0.03021	-0.0000430	0.0000809	0.0000583	-0.0000000	0.0000001	-0.0000001
		0.03093	-0.0000445	0.0001107	0.0000463	0.0000000	0.0000001	0.0000001
		0.03165	-0.0000457	0.0001242	0.0000236	0.0000000	0.0000000	0.0000002
		0.03237	-0.0000467	0.0001145	-0.0000040	0.0000000	-0.0000000	0.0000003
		0.03309	-0.0000477	0.0000816	-0.0000296	0.0000000	-0.0000001	0.0000003

0.03381	-0.0000486	0.0000315	-0.0000465	0.0000000	-0.0000001	0.0000003
0.03453	-0.0000494	-0.0000246	-0.0000501	0.0000000	-0.0000001	0.0000001
0.03525	-0.0000502	-0.0000737	-0.0000396	0.0000000	-0.0000001	-0.0000000
0.03597	-0.0000510	-0.0001044	-0.0000175	-0.0000000	-0.0000000	-0.0000002
0.03669	-0.0000517	-0.0001097	0.0000101	-0.0000000	0.0000000	-0.0000003
0.03741	-0.0000523	-0.0000894	0.0000361	-0.0000000	0.0000001	-0.0000003
0.03813	-0.0000527	-0.0000495	0.0000535	-0.0000000	0.0000001	-0.0000003
0.03885	-0.0000529	-0.0000013	0.0000579	-0.0000000	0.0000002	-0.0000002
0.03957	-0.0000529	0.0000422	0.0000482	0.0000000	0.0000001	-0.0000001
0.04029	-0.0000528	0.0000693	0.0000270	0.0000000	0.0000001	0.0000000
0.04101	-0.0000524	0.0000728	-0.0000000	0.0000000	0.0000000	0.0000001
0.04173	-0.0000520	0.0000521	-0.0000257	0.0000000	-0.0000001	0.0000002
0.04244	-0.0000516	0.0000128	-0.0000434	0.0000000	-0.0000001	0.0000001
0.04316	-0.0000511	-0.0000344	-0.0000484	0.0000000	-0.0000001	0.0000000
0.04388	-0.0000507	-0.0000768	-0.0000397	0.0000000	-0.0000001	-0.0000001
0.04460	-0.0000502	-0.0001031	-0.0000197	-0.0000000	-0.0000000	-0.0000002
0.04532	-0.0000497	-0.0001067	0.0000060	-0.0000000	0.0000000	-0.0000003
0.04604	-0.0000491	-0.0000871	0.0000303	-0.0000000	0.0000001	-0.0000003
0.04676	-0.0000484	-0.0000505	0.0000465	-0.0000000	0.0000001	-0.0000002
0.04748	-0.0000476	-0.0000075	0.0000501	-0.0000000	0.0000001	-0.0000001
0.04820	-0.0000466	0.0000290	0.0000402	0.0000000	0.0000001	0.0000000
0.04892	-0.0000454	0.0000481	0.0000192	0.0000000	0.0000001	0.0000002
0.04964	-0.0000442	0.0000431	-0.0000072	0.0000000	-0.0000000	0.0000002
0.05036	-0.0000430	0.0000140	-0.0000321	0.0000000	-0.0000001	0.0000003
0.05108	-0.0000419	-0.0000329	-0.0000487	0.0000000	-0.0000001	0.0000002
0.05180	-0.0000408	-0.0000861	-0.0000527	0.0000000	-0.0000001	0.0000001
0.05252	-0.0000398	-0.0001325	-0.0000431	0.0000000	-0.0000001	-0.0000000
0.05324	-0.0000390	-0.0001606	-0.0000225	0.0000000	-0.0000000	-0.0000002
0.05395	-0.0000381	-0.0001633	0.0000034	-0.0000000	0.0000000	-0.0000002
0.05467	-0.0000373	-0.0001402	0.0000275	-0.0000000	0.0000001	-0.0000003

LIST DYNAMIC FORCES ALL

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*RESULTS OF LATEST ANALYSES*
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PROBLEM - EXAMPLE   TITLE - STATIC AND DYNAMIC ANALYSIS OF A TABLE-TOP

ACTIVE UNITS INCH KIP CYC DEGF SEC

ACTIVE STRUCTURE TYPE SPACE FRAME

ACTIVE COORDINATE AXES X Y Z

LOADING - 7   CENTRIFUGAL FORCES

MEMBER FORCES

MEMBER	JOINT	TIME	FORCE AXIAL	FORCE SHEAR-Y	FORCE SHEAR-Z	MOMENT TORSIONAL	MOMENT BENDING-Y	MOMENT BENDING-Z
1	1	0.0	0.0	0.0	0.0	0.0	0.0	0.0
		0.0007	-0.0000048	0.0000009	-0.0003077	-0.0006793	0.0193290	0.0001064
		0.0014	-0.0001502	0.0000078	-0.0021204	-0.0045415	0.1298373	0.0010491
		0.0022	-0.0008723	0.0000967	-0.0055061	0.0021757	0.3169094	0.0089364
		0.0029	-0.0026475	0.0006098	-0.0090810	0.0497976	0.4692348	0.0489272
		0.0036	-0.0055652	0.0021775	-0.0109826	0.1451673	0.4597294	0.1676926
		0.0043	-0.0091749	0.0053453	-0.0097662	0.2422276	0.2024531	0.4088115
		0.0050	-0.0126441	0.0100831	-0.0051750	0.2623003	-0.2983776	0.7784010
		0.0058	-0.0150404	0.0156145	0.0016432	0.1586174	-0.9366371	1.2301559
		0.0065	-0.0156170	0.0206603	0.0087142	-0.0330958	-1.5520172	1.6776514
		0.0072	-0.0141106	0.0238722	0.0139952	-0.2147561	-1.9815874	2.0209589
		0.0079	-0.0109225	0.0242475	0.0160011	-0.2977197	-2.0986624	2.1748085
		0.0086	-0.0069765	0.0214669	0.0142484	-0.2579504	-1.8469486	2.0945234
		0.0094	-0.0032276	0.0160976	0.0092973	-0.1346154	-1.2519779	1.7953176
		0.0101	-0.0000862	0.0095301	0.0024694	0.0131671	-0.4124475	1.3549767
		0.0108	0.0029029	0.0035914	-0.0045336	0.1459336	0.5161754	0.8944716
		0.0115	0.0069477	-0.0000216	-0.0100539	0.2492253	1.3506975	0.5433398
		0.0122	0.0134842	-0.0002491	-0.0128379	0.3157126	1.9169483	0.4036994
		0.0129	0.0233904	0.0030436	-0.0122392	0.3334895	2.0837765	0.5232136
		0.0137	0.0361552	0.0090360	-0.0083136	0.2921811	1.7941494	0.8815742
		0.0144	0.0493199	0.0161635	-0.0018346	0.1959464	1.0883017	1.3932562
		0.0151	0.0585164	0.0225332	0.0058038	0.0677934	0.1083144	1.9278841
		0.0158	0.0582820	0.0264420	0.0128107	-0.0587701	-0.9217333	2.3441181
		0.0165	0.0435641	0.0268340	0.0173683	-0.1520869	-1.7428083	2.5266008
		0.0173	0.0115493	0.0235494	0.0181152	-0.1901968	-2.1293535	2.4144821
		0.0180	-0.0367126	0.0173131	0.0145969	-0.1643412	-1.9591026	2.0147142
		0.0187	-0.0954685	0.0095037	0.0075082	-0.0816327	-1.2601366	1.3988113
		0.0194	-0.1546631	0.0017872	-0.0013882	0.0338514	-0.2167616	0.6850468
		0.0201	-0.2017426	-0.0042960	-0.0096884	0.1459095	0.8728356	0.0100942

0.0209	-0.2245857	-0.0076921	-0.0150105	0.2185102	1.6778612	-0.5034372
0.0216	-0.2148601	-0.0080577	-0.0156949	0.2300679	1.9318886	-0.7754113
0.0223	-0.1708440	-0.0057957	-0.0113286	0.1809601	1.5166245	-0.7854892
0.0230	-0.0987706	-0.0019110	-0.0029219	0.0903067	0.5058544	-0.5757641
0.0237	-0.0121053	0.0022596	0.0073264	-0.0140928	-0.8462275	-0.2389016
0.0245	0.0712815	0.0053769	0.0166154	-0.1054264	-2.1709404	0.1052532
0.0252	0.1334266	0.0064184	0.0223093	-0.1630204	-3.0870762	0.3394802
0.0259	0.1609224	0.0049191	0.0226663	-0.1740922	-3.3095379	0.3786311
0.0266	0.1485652	0.0010825	0.0173256	-0.1352504	-2.7329550	0.1914678
0.0273	0.1008728	-0.0042706	0.0074222	-0.0546026	-1.4660358	-0.1908666
0.0281	0.0309492	-0.0099020	-0.0046933	0.0477584	0.1931738	-0.6821794
0.0288	-0.0430544	-0.0144615	-0.0160553	0.1434393	1.8347139	-1.1628685
0.0295	-0.1028926	-0.0168270	-0.0238498	0.2036114	3.0459433	-1.5107460
0.0302	-0.1351778	-0.0163981	-0.0261531	0.2082131	3.5231562	-1.6334105
0.0309	-0.1348916	-0.0132673	-0.0224533	0.1527282	3.1552162	-1.4937925
0.0317	-0.1064528	-0.0082195	-0.0138126	0.0500126	2.0561314	-1.1218300
0.0324	-0.0620576	-0.0025516	-0.0026147	-0.0732809	0.5365614	-0.6088666
0.0331	-0.0178125	0.0022466	0.0080514	-0.1843800	-0.9787885	-0.0857356
0.0338	0.0111764	0.0048718	0.0152166	-0.2537850	-2.0661697	0.3100063
0.0345	0.0153296	0.0045472	0.0168457	-0.2635705	-2.4187574	0.4693832
0.0353	-0.0065727	0.0012355	0.0124112	-0.2121474	-1.9315252	0.3406337
0.0360	-0.0470366	-0.0043389	0.0030603	-0.1142247	-0.7311332	-0.0565261
0.0367	-0.0920634	-0.0108668	-0.0086824	0.0036574	0.8576906	-0.6350861
0.0374	-0.1252760	-0.0167902	-0.0195982	0.1103845	2.4002466	-1.2625608
0.0381	-0.1327874	-0.0207085	-0.0266569	0.1782112	3.4701328	-1.7948704
0.0388	-0.1074482	-0.0217469	-0.0278502	0.1895702	3.7651587	-2.1133680
0.0396	-0.0512703	-0.0197861	-0.0227453	0.1414018	3.1893568	-2.1552343
0.0403	0.0246629	-0.0154916	-0.0126063	0.0460653	1.8790712	-1.9291058
0.0410	0.1028151	-0.0101311	-0.0000533	-0.0716006	0.1665686	-1.5118484
0.0417	0.1636801	-0.0052299	0.0116442	-0.1806604	-1.5086260	-1.0272942
0.0424	0.1909446	-0.0021571	0.0194452	-0.2523929	-2.7167110	-0.6127096
0.0432	0.1759590	-0.0017555	0.0213514	-0.2681967	-3.1517191	-0.3820063
0.0439	0.1203042	-0.0041160	0.0169395	-0.2247097	-2.7125740	-0.3955632
0.0446	0.0357315	-0.0085572	0.0074735	-0.1346219	-1.5297461	-0.6446533
0.0453	-0.0585579	-0.0138147	-0.0044329	-0.0230206	0.0695632	-1.0545225
0.0460	-0.1406271	-0.0183884	-0.0155131	0.0795900	1.6485701	-1.5051384
0.0468	-0.1913031	-0.0209509	-0.0227196	0.1455911	2.7774534	-1.8639250
0.0475	-0.1990318	-0.0207075	-0.0240403	0.1576292	3.1484165	-2.0216646
0.0482	-0.1628079	-0.0176098	-0.0190410	0.1131400	2.6585636	-1.9218721
0.0489	-0.0923983	-0.0123664	-0.0089883	0.0248761	1.4381838	-1.5759506
0.0496	-0.0057799	-0.0062495	0.0034826	-0.0826466	-0.1833984	-1.0600643
0.0504	0.0755457	-0.0007519	0.0150696	-0.1797815	-1.7658215	-0.4947641
0.0511	0.1321319	0.0028147	0.0226878	-0.2396772	-2.8761072	-0.0128437
0.0518	0.1515903	0.0036681	0.0243002	-0.2454099	-3.2058105	0.2759289
0.0525	0.1316923	0.0017621	0.0194690	-0.1944841	-2.6552515	0.3138095
0.0532	0.0807498	-0.0022065	0.0094797	-0.0995474	-1.3618317	0.1101354
0.0540	0.0151788	-0.0069890	-0.0029992	0.0148790	0.3354899	-0.2614970
0.0547	-0.0450876	-0.0111214	-0.0146303	0.1188472	1.9863243	-0.6830968
0.0554	-0.0820982	-0.0133175	-0.0223014	0.1849468	3.1494265	-1.0238590
0.0561	-0.0848028	-0.0128116	-0.0239618	0.1956756	3.5113602	-1.1750259
0.0568	-0.0520563	-0.0095615	-0.0191732	0.1482078	2.9711323	-1.0787773
0.0576	0.0071037	-0.0042578	-0.0092281	0.0552283	1.6680489	-0.7435029
0.0583	0.0759878	0.0018625	0.0031962	-0.0584202	-0.0549131	-0.2414348
0.0590	0.1348978	0.0073457	0.0147559	-0.1623694	-1.7424574	0.3106650
0.0597	0.1666211	0.0109089	0.0223366	-0.2288369	-2.9485493	0.7826039
0.0604	0.1612729	0.0117796	0.0238890	-0.2400339	-3.3556337	1.0653410
0.0611	0.1191541	0.0098989	0.0189778	-0.1928909	-2.8594131	1.0998516
0.0619	0.0508364	0.0059332	0.0088986	-0.0998598	-1.5965433	0.8925995
0.0626	-0.0256107	0.0010977	-0.0036865	0.0147252	0.0941772	0.5139714
0.0633	-0.0893140	-0.0031831	-0.0154116	0.1202179	1.7572880	0.0779917
0.0640	-0.1223509	-0.0056452	-0.0231620	0.1887751	2.9487534	-0.2870134
0.0647	-0.1145977	-0.0055287	-0.0248863	0.2023600	3.3524981	-0.4733293
0.0655	-0.0665862	-0.0027801	-0.0201502	0.1576115	2.8654375	-0.4225446
0.0662	0.0104773	0.0019341	-0.0102534	0.0667346	1.6249018	-0.1410898
0.0669	0.0975522	0.0074065	0.0021183	-0.0457738	-0.0285124	0.3015915
0.0676	0.1727132	0.0122128	0.0136127	-0.1496944	-1.6426172	0.7917194
0.0683	0.2169008	0.0150934	0.0211101	-0.2171941	-2.7752924	1.2016945
0.0691	0.2189416	0.0152911	0.0225605	-0.2302963	-3.1134558	1.4242640
0.0698	0.1784958	0.0127541	0.0175346	-0.1857020	-2.5574064	1.4011860
0.0705	0.1061509	0.0081497	0.0073404	-0.0956635	-1.2479639	1.1387234
0.0712	0.0206274	0.0026853	-0.0053255	0.0153261	0.4709445	0.7058378
0.0719	-0.0561589	-0.0022152	-0.0170992	0.1171321	2.1441956	0.2155629
0.0727	-0.1048666	-0.0052937	-0.0248499	0.1820861	3.3272734	-0.2053994
0.0734	-0.1137981	-0.0057951	-0.0265184	0.1924221	3.7053585	-0.4507114
0.0741	-0.0819209	-0.0036696	-0.0216688	0.1450461	3.1777182	-0.4629220
0.0748	-0.0190440	0.0004142	-0.0116055	0.0523646	1.8848696	-0.2488200
0.0755	0.0568886	0.0052492	0.0009757	-0.0610446	0.1711287	0.1224958
0.0763	0.1246619	0.0094148	0.0127094	-0.1650222	-1.5074005	0.5383068
0.0770	0.1655026	0.0116592	0.0204621	-0.2319134	-2.7048168	0.8727052
0.0777	0.1681096	0.0112372	0.0221712	-0.2439827	-3.1047049	1.0205936
0.0784	0.1316625	0.0081120	0.0173978	-0.1981723	-2.6047907	0.9261352
0.0791	0.0659992	0.0029678	0.0074442	-0.1069291	-1.3442373	0.5979997
0.0799	-0.0110763	-0.0029716	-0.0049948	0.0051231	0.3336520	0.1073403
0.0806	-0.0786306	-0.0082681	-0.0165538	0.1077752	1.9734955	-0.4310033
0.0813	-0.1182788	-0.0116529	-0.0240994	0.1733237	3.1298971	-0.8890014
0.0820	-0.1191799	-0.0123644	-0.0255697	0.1839937	3.4869080	-1.1594982
0.0827	-0.0809986	-0.0103503	-0.0205283	0.1367048	2.9428062	-1.1846991
0.0834	-0.0140399	-0.0062804	-0.0102823	0.0439034	1.6375246	-0.9714625
0.0842	0.0634851	-0.0013653	0.0024667	-0.0697622	-0.0846828	-0.5893115
0.0849	0.1303340	0.0029688	0.0143458	-0.1740321	-1.7673559	-0.1516985
0.0856	0.1679377	0.0054632	0.0222129	-0.2411245	-2.9640503	0.2144531
0.0863	0.1662481	0.0054188	0.0240490	-0.2537157	-3.3592262	0.3999426

9	0.0	0.0	0.0	0.0	0.0	0.0	0.0
	0.0007	0.0000048	-0.0000009	0.0003077	0.0006793	0.0434500	0.0000767
	0.0014	0.0001502	-0.0000078	0.0021204	0.0045415	0.3027334	0.0005363
	0.0022	0.0008723	-0.0000967	0.0055061	-0.0021757	0.8063324	0.0107893
	0.0029	0.0026475	-0.0006098	0.0090810	-0.0497976	1.3832798	0.0754817
	0.0036	0.0055652	-0.0021775	0.0109826	-0.1451673	1.7807217	0.2765229
	0.0043	0.0091749	-0.0053453	0.0097662	-0.2422276	1.7898540	0.6816210
	0.0050	0.0126441	-0.0100831	0.0051750	-0.2623003	1.3540697	1.2785530
	0.0058	0.0150404	-0.0156145	-0.0016432	-0.1586174	0.6014304	1.9552021
	0.0065	0.0156170	-0.0206603	-0.0087142	0.0330958	-0.2256873	2.5370388
	0.0072	0.0141106	-0.0238722	-0.0139952	0.2147561	-0.8734281	2.8489609
	0.0079	0.0109225	-0.0242475	-0.0160011	0.2977197	-1.1655569	2.7716789
	0.0086	0.0069765	-0.0214669	-0.0142484	0.2579504	-1.0597239	2.2847137
	0.0094	0.0032276	-0.0160976	-0.0092973	0.1346154	-0.6446701	1.4885998
	0.0101	0.0000862	-0.0095301	-0.0024694	-0.0131671	-0.0913200	0.5891740
	0.0108	-0.0029029	-0.0035914	0.0045336	-0.1459336	0.4086818	-0.1618238
	0.0115	-0.0069477	0.0000216	0.0100539	-0.2492253	0.7002973	-0.5477443
	0.0122	-0.0134842	0.0002491	0.0128379	-0.3157126	0.7019855	-0.4545187
	0.0129	-0.0233904	-0.0030436	0.0122392	-0.3334895	0.4130168	0.0976863
	0.0137	-0.0361552	-0.0090360	0.0083136	-0.2921811	-0.0981795	0.9617660
	0.0144	-0.0493199	-0.0161635	0.0018346	-0.1959464	-0.7140502	1.9040956
	0.0151	-0.0585164	-0.0225332	-0.0058038	-0.0677934	-1.2922935	2.6688881
	0.0158	-0.0582820	-0.0264420	-0.0128107	0.0587701	-1.6916513	3.0500517
	0.0165	-0.0435641	-0.0268340	-0.0173683	0.1520869	-1.8003206	2.9475393
	0.0173	-0.0115493	-0.0235494	-0.0181152	0.1901968	-1.5661469	2.3895988
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	0.0201	0.2017426	0.0042960	0.0096884	-0.1459095	1.1036072	-0.8864706
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	0.0216	0.2148601	0.0080577	0.0156949	-0.2300679	1.2698803	-0.8683652
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2

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0.0604	0.1518341	0.0125500	-0.0275289	-0.2683791	3.7325115	1.1211872
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0.0619	0.0447116	0.0076685	-0.0111410	-0.1897097	1.8261776	1.0450983
0.0626	-0.0279120	0.0028038	0.0026133	-0.0727320	0.0136111	0.6732175
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0.0647	-0.1081573	-0.0054924	0.0251487	0.2312927	-3.3799725	-0.4449067
0.0655	-0.0605184	-0.0032883	0.0195168	0.2270266	-2.7974043	-0.4449238
0.0662	0.0144133	0.0011728	0.0081973	0.1583487	-1.4065180	-0.1921639
0.0669	0.0982497	0.0067643	-0.0058003	0.0437465	0.4180190	0.2529323
0.0676	0.1700080	0.0120426	-0.0187447	-0.0861143	2.1842823	0.7772118
0.0683	0.2116039	0.0156334	-0.0271830	-0.1965947	3.4153566	1.2451277
0.0691	0.2126081	0.0166003	-0.0288670	-0.2583483	3.7775927	1.5350094
0.0698	0.1729946	0.0146952	-0.0233563	-0.2551656	3.1703730	1.5715761
0.0705	0.1031497	0.0104245	-0.0121373	-0.1882904	1.7534122	1.3459759
0.0712	0.0211330	0.0049135	0.0017742	-0.0760604	-0.0958354	0.9180224
0.0719	-0.0520813	-0.0003958	0.0146418	0.0510333	-1.8831863	0.4000270
0.0727	-0.0981194	-0.0041311	0.0230090	0.1585360	-3.1301193	-0.0734109
0.0734	-0.1060147	-0.0053578	0.0246263	0.2172717	-3.5020685	-0.3815407
0.0741	-0.0750334	-0.0038293	0.0190547	0.2111663	-2.8981972	-0.4498227
0.0748	-0.0147682	-0.0000524	0.0077843	0.1415580	-1.4786301	-0.2698466
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0.0763	0.1215478	0.0094374	-0.0190496	-0.1024381	2.1766958	0.5434784
0.0770	0.1595408	0.0123479	-0.0274087	-0.2117993	3.4366875	0.9312655
0.0777	0.1609322	0.0126567	-0.0289879	-0.2720361	3.8219061	1.1424923
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0.0791	0.0619947	0.0052938	-0.0119716	-0.1982974	1.8200512	0.8102350
0.0799	-0.0115704	-0.0007144	0.0021114	-0.0841904	-0.0302424	0.3222001
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0.0827	-0.0746884	-0.0105395	0.0200473	0.2071873	-2.8886185	-1.1750517
0.0834	-0.0101614	-0.0067857	0.0089012	0.1373529	-1.4885435	-0.9969364
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2   11

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		0.0259	-0.1704661	-0.0055310	0.0241740	0.1835783	1.4299021	0.7156269
		0.0266	-0.1558908	-0.0024260	0.0192685	0.1852671	0.9671945	0.1993732
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		0.0338	0.0038623	-0.0051895	0.0282440	0.2333034	2.3489199	0.7237929
		0.0345	0.0010298	-0.0057344	0.0300294	0.2909418	2.3458281	0.5987377
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		0.0417	-0.1609153	0.0046704	0.0194148	0.1075382	1.6366215	0.0133262
		0.0424	-0.1861150	0.0009203	0.0273230	0.2196814	2.0292225	0.3099052
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		0.0439	-0.1161356	0.0015323	0.0228681	0.2834819	1.3265877	-0.1526800
		0.0446	-0.0346225	0.0056642	0.0115266	0.2186113	0.3907565	-0.7806276
		0.0453	0.0553983	0.0109938	-0.0024577	0.1073492	-0.6470851	-1.4600973
		0.0460	0.1329052	0.0160037	-0.0154262	-0.0199649	-1.5184269	-2.0006618
		0.0468	0.1797312	0.0192534	-0.0239871	-0.1288640	-1.9970255	-2.2487812
		0.0475	0.1851503	0.0197694	-0.0259216	-0.1900645	-1.9593678	-2.1294622
		0.0482	0.1485896	0.0173076	-0.0207802	-0.1873291	-1.4178228	-1.6663866
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		0.0496	-0.0039574	0.0062857	0.0034487	-0.0115575	0.5017731	-0.2316473
		0.0504	-0.0818794	0.0004271	0.0159085	0.1131442	1.3701553	0.3756408
		0.0511	-0.1355174	-0.0037338	0.0239641	0.2181658	1.8588963	0.6964701
		0.0518	-0.1532420	-0.0052369	0.0253892	0.2745757	1.8418159	0.6590314
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		0.0590	-0.1299041	-0.0067258	0.0170018	0.1015682	1.4925480	1.1220341
		0.0597	-0.1585165	-0.0109428	0.0256485	0.2099305	1.9400806	1.4580059
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		0.0633	0.0876176	0.0018543	-0.0153498	-0.0586628	-1.3771248	-0.5909840
		0.0640	0.1174811	0.0049319	-0.0236117	-0.1697893	-1.8191338	-0.8054841
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		0.0842	-0.0639103	0.0018108	0.0049375	-0.0220177	0.6592904	0.2490609
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0.0129	0.2940490	0.0801644	-0.0322799	-2.0794678	6.2966166	8.1741552
0.0137	0.1844137	0.0549632	-0.0450314	-1.7464209	6.5617065	6.8247643
0.0144	0.1125942	0.0280785	-0.0367714	-1.0484066	4.7167606	5.1281042
0.0151	0.1237318	0.0051747	-0.0093970	-0.1515182	1.1839027	3.4480152
0.0158	0.2289145	-0.0093005	0.0294059	0.7260553	-3.1065102	2.1198969
0.0165	0.4008889	-0.0132748	0.0680910	1.3783331	-6.9441710	1.3619585
0.0173	0.5827353	-0.0074577	0.0945525	1.6618328	-9.1843033	1.2269411
0.0180	0.7063757	0.0050747	0.0997583	1.5251255	-9.0993156	1.6045837
0.0187	0.7147284	0.0198860	0.0804855	1.0219870	-6.6112337	2.2647877
0.0194	0.5802948	0.0324171	0.0402940	0.3044502	-2.3251553	2.9228106
0.0201	0.3143063	0.0391188	-0.0114582	-0.4142265	2.6404457	3.3090458
0.0209	-0.0360802	0.0382321	-0.0621501	-0.9248680	6.9654989	3.2299871
0.0216	-0.4011045	0.0300708	-0.0992309	-1.0921583	9.4885807	2.6099453
0.0223	-0.7076420	0.0167845	-0.1134979	-0.8960678	9.5199480	1.5063648
0.0230	-0.8984147	0.0016990	-0.1015251	-0.4274780	7.0299797	0.0958116
0.0237	-0.9457358	-0.0115834	-0.0666420	0.1529038	2.6658201	-1.3669329
0.0245	-0.8564020	-0.0201081	-0.0181777	0.6678832	-2.4100122	-2.6088829
0.0252	-0.6671323	-0.0223053	0.0308881	0.9691125	-6.8212175	-3.4051428
0.0259	-0.4326499	-0.0183701	0.0673068	0.9679318	-9.3435383	-3.6351957
0.0266	-0.2102970	-0.0101871	0.0811093	0.6535566	-9.2387877	-3.3139782
0.0273	-0.0455328	-0.0008153	0.0683764	0.0984693	-6.4670458	-2.5883503
0.0281	0.0380844	0.0063317	0.0324459	-0.5539211	-1.7150888	-1.6995277
0.0288	0.0424068	0.0083972	-0.0168373	-1.1227770	3.7729521	-0.9213085
0.0295	-0.0085163	0.0038432	-0.0656947	-1.4391947	8.5234404	-0.4906589
0.0302	-0.0769686	-0.0071291	-0.1001524	-1.3968735	11.2333660	-0.5486313
0.0309	-0.1228997	-0.0226079	-0.1099156	-0.9866772	11.1322527	-1.1065264
0.0317	-0.1151260	-0.0394389	-0.0912844	-0.3029035	8.2021475	-2.0453472
0.0324	-0.0397344	-0.0540118	-0.0482705	0.4810566	3.1913977	-3.1479053
0.0331	0.0961722	-0.0631660	0.0084066	1.1602097	-2.5830355	-4.1553392
0.0338	0.2661157	-0.0649903	0.0641679	1.5544195	-7.5899172	-4.8338079
0.0345	0.4307856	-0.0593169	0.1044820	1.5575218	-10.4966946	-5.0351591
0.0353	0.5478945	-0.0477867	0.1187695	1.1654434	-10.5287800	-4.7362919
0.0360	0.5830092	-0.0334587	0.1032535	0.4762831	-7.6787996	-4.0466833
0.0367	0.5187370	-0.0200542	0.0619930	-0.3364450	-2.7101793	-3.1811733
0.0374	0.3599780	-0.0110174	0.0058172	-1.0691795	3.0459757	-2.4041166
0.0381	0.1338136	-0.0086360	-0.0505592	-1.5401802	8.0468388	-1.9592829
0.0388	-0.1161515	-0.0134551	-0.0924733	-1.6352282	10.9520063	-2.0042696
0.0396	-0.3397897	-0.0241468	-0.1092430	-1.3363562	10.9846706	-2.5674047
0.0403	-0.4924479	-0.0378754	-0.0970253	-0.7264740	8.1421556	-3.5390577
0.0410	-0.5459325	-0.0510613	-0.0598265	0.0319727	3.1981440	-4.6996317
0.0417	-0.4956186	-0.0603361	-0.0083854	0.7396186	-2.5050192	-5.7764912
0.0424	-0.3616733	-0.0634293	0.0427679	1.2131977	-7.4186201	-6.5142450
0.0432	-0.1838911	-0.0597470	0.0792806	1.3345032	-10.2067137	-6.7397060
0.0439	-0.0113255	-0.0504909	0.0908923	1.0815744	-10.1086044	-6.4046440
0.0446	0.1106387	-0.0382946	0.0742337	0.5334885	-7.1462526	-5.5958080
0.0453	0.1523511	-0.0264798	0.0337506	-0.1523109	-2.1199875	-4.5100069
0.0460	0.1077963	-0.0181378	-0.0194940	-0.7835790	3.6062222	-3.4012556
0.0468	-0.0038909	-0.0152853	-0.0708076	-1.1857061	8.4718580	-2.5138292
0.0475	-0.1438166	-0.0183180	-0.1058376	-1.2488632	11.1400690	-2.0188131
0.0482	-0.2642226	-0.0259107	-0.1144512	-0.9572873	10.8579950	-1.9707508
0.0489	-0.3219426	-0.0353877	-0.0934860	-0.3929802	7.6605225	-2.2957850
0.0496	-0.2907124	-0.0434632	-0.0476374	0.2869975	2.3630190	-2.8143044
0.0504	-0.1689039	-0.0471495	0.0117549	0.8933961	-3.6550379	-3.2918434
0.0511	0.0194972	-0.0445860	0.0696971	1.2561665	-8.8177528	-3.5043240
0.0518	0.2310083	-0.0355639	0.1115117	1.2706022	-11.7735806	-3.3000669
0.0525	0.4142063	-0.0216025	0.1267267	0.9259447	-11.7566042	-2.6417665
0.0532	0.5235810	-0.0055600	0.1118471	0.3088793	-8.7913780	-1.6173630
0.0540	0.5318830	0.0091184	0.0712652	-0.4190827	-3.6862221	-0.4171994
0.0547	0.4375798	0.0193087	0.0160679	-1.0644732	2.1835470	0.7163395
0.0554	0.2653843	0.0230315	-0.0389389	-1.4536686	7.2443857	1.5529985
0.0561	0.0597058	0.0199695	-0.0791914	-1.4797649	10.1464052	1.9356012
0.0568	-0.1271863	0.0115440	-0.0942544	-1.1318474	10.1251669	1.8215504
0.0576	-0.2490695	0.0005312	-0.0805900	-0.4985749	7.2061634	1.2940025
0.0583	-0.2779875	-0.0096767	-0.0424727	0.2549910	2.1971636	0.5398325
0.0590	-0.2114947	-0.0159695	0.0091948	0.9311283	-3.5295172	-0.1997580
0.0597	-0.0730006	-0.0163365	0.0598422	1.3520088	-8.4063425	-0.6927690
0.0604	0.0948347	-0.0103871	0.0951753	1.4069719	-11.0923347	-0.7772599
0.0611	0.2418435	0.0005621	0.1050423	1.0818844	-10.8341198	-0.4033225
0.0619	0.3240478	0.0138548	0.0861861	0.4628273	-7.6694717	0.3548610
0.0626	0.3155766	0.0262291	0.0431046	-0.2860366	-2.4138823	1.3212805
0.0633	0.2158186	0.0346616	-0.0130162	-0.9685075	3.5430336	2.2623234
0.0640	0.0495389	0.0372138	-0.0674058	-1.4059448	8.6233702	2.9536905
0.0647	-0.1396977	0.0335303	-0.1056104	-1.4857130	11.4772606	3.2387066
0.0655	-0.3012106	0.0249262	-0.1173794	-1.1907291	11.3457136	3.0700731
0.0662	-0.3907927	0.0140222	-0.0994251	-0.6036745	8.2646198	2.5217123
0.0669	-0.3825866	0.0040282	-0.0563249	0.1154204	3.0560265	1.7679033
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0.0683	-0.0963393	-0.0026133	0.0566735	1.1901741	-7.9919910	0.5419340
0.0691	0.1125151	0.0028381	0.0970280	1.2578478	-10.8899164	0.4361137
0.0698	0.2989416	0.0128158	0.1113062	0.9585546	-10.8188076	0.7574748
0.0705	0.4175197	0.0246210	0.0960323	0.3746174	-7.8078775	1.4248133

		0.0712	0.4408600	0.0349861	0.0556060	-0.3350154	-2.6732769	2.2575302
		0.0719	0.3666867	0.0409523	0.0011093	-0.9776955	3.2011395	3.0237179
		0.0727	0.2182248	0.0406691	-0.0526730	-1.3789978	8.2349014	3.5022259
		0.0734	0.0378010	0.0338985	-0.0912187	-1.4296970	11.0746117	3.5420408
		0.0741	-0.1246181	0.0220906	-0.1041607	-1.1151857	10.9566240	3.1030645
		0.0748	-0.2248862	0.0080059	-0.0880597	-0.5194606	7.9115925	2.2673035
		0.0755	-0.2365214	-0.0050123	-0.0473177	0.1973841	2.7559719	1.2174835
		0.0763	-0.1577734	-0.0139235	0.0070126	0.8421244	-3.1270933	0.1888191
		0.0770	-0.0120234	-0.0167950	0.0602027	1.2403107	-8.1588936	-0.5932040
		0.0777	0.1585984	-0.0133128	0.0978097	1.2832022	-10.9887295	-0.9723478
		0.0784	0.3046486	-0.0048593	0.1095608	0.9571283	-10.8564367	-0.9025089
		0.0791	0.3828066	0.0058610	0.0921195	0.3473976	-7.7966471	-0.4590942
		0.0799	0.3676924	0.0155485	0.0499917	-0.3843259	-2.6299877	0.1821550
		0.0806	0.2588910	0.0211942	-0.0056724	-1.0431509	3.2565203	0.7931361
		0.0813	0.0812921	0.0208851	-0.0600598	-1.4530449	8.2810106	1.1552916
		0.0820	-0.1213772	0.0143157	-0.0986601	-1.5039492	11.0907412	1.1188011
		0.0827	-0.2980840	0.0028674	-0.1111527	-1.1812840	10.9247999	0.6431982
		0.0834	-0.4040304	-0.0107631	-0.0941738	-0.5699438	7.8184328	-0.1914369
		0.0842	-0.4125713	-0.0232894	-0.0522212	0.1683037	2.5941162	-1.2057619
		0.0849	-0.3223141	-0.0317168	0.0035458	0.8379821	-3.3583574	-2.1691399
		0.0856	-0.1575125	-0.0341451	0.0582902	1.2620792	-8.4537220	-2.8614368
		0.0863	0.0310494	-0.0305204	0.0964117	1.3338833	-11.2779131	-3.1301794
3	14	0.0	0.0	0.0	0.0	0.0	0.0	0.0
		0.0007	0.0002123	-0.0000019	0.0015553	0.0019988	0.2101146	0.0001899
		0.0014	0.0044363	-0.0000075	0.0120985	0.0270053	1.6142435	0.0005586
		0.0022	0.0255420	0.0001087	0.0369658	0.1264631	4.9088507	-0.0158727
		0.0029	0.0831275	0.0008333	0.0740476	0.3319557	9.8635731	-0.1151969
		0.0036	0.1934381	0.0028684	0.1138172	0.5647638	15.3324909	-0.3930402
		0.0043	0.3577871	0.0063761	0.1432559	0.6203490	19.7000732	-0.8717831
		0.0050	0.5553125	0.0102259	0.1512737	0.2890622	21.5087128	-1.3981037
		0.0058	0.7438354	0.0120323	0.1329493	-0.4589667	19.9807739	-1.6481562
		0.0065	0.8701954	0.0090547	0.0910938	-1.3894205	15.2564440	-1.2507629
		0.0072	0.8879137	-0.0006521	0.0351137	-2.1198044	8.3196983	0.0538078
		0.0079	0.7756323	-0.0174258	-0.0219958	-2.3466387	0.6848325	2.2981577
		0.0086	0.5480148	-0.0396872	-0.0674105	-1.9989376	-6.0351791	5.2405777
		0.0094	0.2537333	-0.0640907	-0.0918984	-1.2225723	-10.5494871	8.3871708
		0.0101	-0.0390394	-0.0861348	-0.0920048	-0.2519156	-12.1881351	11.0882568
		0.0108	-0.2642741	-0.1011940	-0.0705286	0.7041437	-11.0143309	12.7004166
		0.0115	-0.3793103	-0.1057310	-0.0353900	1.4904547	-7.7476320	12.7698622
		0.0122	-0.3784484	-0.0983194	0.0025754	1.9819469	-3.5460749	11.1746082
		0.0129	-0.2940490	-0.0801644	0.0322799	2.0794678	0.2884794	8.1793737
		0.0137	-0.1844137	-0.0549632	0.0450314	1.7464209	2.6246967	4.3877220
		0.0144	-0.1125942	-0.0280785	0.0367714	1.0484066	2.7846041	0.5999123
		0.0151	-0.1237318	-0.0051747	0.0093970	0.1515182	0.7330778	-2.3923721
		0.0158	-0.2289145	0.0093005	-0.0294059	-0.7260553	-2.8922853	-4.0172043
		0.0165	-0.4008889	0.0132748	-0.0680910	-1.3783331	-6.9463978	-4.0700169
		0.0173	-0.5827353	0.0074577	-0.0945525	-1.6618328	-10.1044130	-2.7483044
		0.0180	-0.7063757	-0.0050747	-0.0997583	-1.5251255	-11.2513781	-0.5693408
		0.0187	-0.7147284	-0.0198860	-0.0804855	-1.0219870	-9.8078070	1.7919617
		0.0194	-0.5802948	-0.0324171	-0.0402940	-0.3044502	-5.8948193	3.6902781
		0.0201	-0.3143063	-0.0391188	0.0114582	0.4142265	-0.3029751	4.6711941
		0.0209	0.0360802	-0.0382321	0.0621501	0.9248680	5.7131166	4.5693626
		0.0216	0.4011045	-0.0300708	0.0992309	1.0921583	10.7545376	3.5244942
		0.0223	0.7076420	-0.0167845	0.1134979	0.8960678	13.6336117	1.9176693
		0.0230	0.8984147	-0.0016990	0.1015251	0.4274780	13.6811428	0.2507923
		0.0237	0.9457358	0.0115834	0.0666420	-0.1529038	10.9291515	-0.9960868
		0.0245	0.8564020	0.0201081	0.0181777	-0.6678832	6.1182728	-1.4931669
		0.0252	0.6671323	0.0223053	-0.0308881	-0.9691125	0.5200544	-1.1451406
		0.0259	0.4326499	0.0183701	-0.0673068	-0.9679318	-4.3870583	-0.1123038
		0.0266	0.2102970	0.0101871	-0.0811093	-0.6535566	-7.3075066	1.2358189
		0.0273	0.0455328	0.0008153	-0.0683764	-0.0984693	-7.4817371	2.4220285
		0.0281	-0.0380844	-0.0063317	-0.0324459	0.5539211	-4.9038620	2.9912033
		0.0288	-0.0424068	-0.0083972	0.0168373	1.1227770	-0.3381418	2.6343384
		0.0295	0.0085163	-0.0038432	0.0656947	1.4391947	4.8782921	1.2746696
		0.0302	0.0769686	0.0071291	0.1001524	1.3968735	9.1977253	-0.9056998
		0.0309	0.1228997	0.0226079	0.1099156	0.9866772	11.2905378	-3.5054760
		0.0317	0.1151260	0.0394389	0.0912844	0.3029035	10.4198742	-6.0001907
		0.0324	0.0397344	0.0540118	0.0482705	-0.4810566	6.6557798	-7.8704910
		0.0331	-0.0961722	0.0631660	-0.0084066	-1.1602097	0.8680839	-8.7305298
		0.0338	-0.2661157	0.0649903	-0.0641679	-1.5544195	-5.5003271	-8.4242134
		0.0345	-0.4307856	0.0593169	-0.1044820	-1.5575218	-10.8176317	-7.0654955
		0.0353	-0.5478945	0.0477867	-0.1187695	-1.1654434	-13.7001982	-5.0121832
		0.0360	-0.5830092	0.0334587	-0.1032535	-0.4762831	-13.3849163	-2.7788897
		0.0367	-0.5187370	0.0200542	-0.0619930	0.3364450	-9.9363852	-0.9098852
		0.0374	-0.3599780	0.0110174	-0.0058172	1.0691795	-4.2326946	0.1565589
		0.0381	-0.1338136	0.0086360	0.0505592	1.5401802	2.2672386	0.1975298
		0.0388	0.1161515	0.0134551	0.0924733	1.6352282	7.9125538	-0.7405754
		0.0396	0.3397897	0.0241468	0.1092430	1.3363562	11.3009052	-2.3585472
		0.0403	0.4924479	0.0378754	0.0970253	0.7264740	11.6510143	-4.1875305
		0.0410	0.5459325	0.0510613	0.0598265	-0.0319727	9.0064516	-5.7168703
		0.0417	0.4956186	0.0603361	0.0083854	-0.7396186	4.2156363	-6.5320730
		0.0424	0.3616733	0.0634293	-0.0427679	-1.2131977	-1.3060360	-6.4253387
		0.0432	0.1838911	0.0597470	-0.0792806	-1.3345032	-5.9665346	-5.4486713
		0.0439	0.0113255	0.0504909	-0.0908923	-1.0815744	-8.4334211	-3.8954926
		0.0446	-0.1106387	0.0382946	-0.0742337	-0.5334885	-7.9974289	-2.2162924
		0.0453	-0.1523511	0.0264798	-0.0337506	0.1523109	-4.7651281	-0.8918617
		0.0460	-0.1077963	0.0181378	0.0194940	0.7835790	0.3705510	-0.2988615
		0.0468	0.0038909	0.0152853	0.0708076	1.1857061	5.9728975	-0.6043690
		0.0475	0.1438166	0.0183180	0.1058376	1.2488632	10.4508152	-1.7180500
		0.0482	0.2642226	0.0259107	0.1144512	0.9572873	12.4900532	-3.3150215
		0.0489	0.3219426	0.0353877	0.0934860	0.3929802	11.4106159	-4.9233036
		0.0496	0.2907124	0.0434632	0.0476374	-0.2869975	7.3549986	-6.0521870

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0.0525	-0.4142063	0.0216025	-0.1267267	-0.9259447	-14.0956545	-1.7651501
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0.0561	-0.0597058	-0.0199695	0.0791914	1.4797649	6.0086555	2.1381807
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4 4

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		0.0058	0.7188652	0.0136748	-0.1099312	-0.4092340	-17.8612823	-1.7571020
		0.0065	0.8534225	0.0134517	-0.0946802	-1.3596315	-15.6248188	-1.6341553
		0.0072	0.8879548	0.0073866	-0.0633646	-2.1384077	-10.9812412	-0.7083515
		0.0079	0.7971013	-0.0059009	-0.0216743	-2.4368639	-4.7620850	1.1593809

0.0086	0.5885954	-0.0259529	0.0214746	-2.1689749	1.7781553	3.8466463
0.0094	0.3038222	-0.0500446	0.0564417	-1.4603386	7.2982454	6.9315729
0.0101	0.0062460	-0.0735660	0.0756844	-0.5264807	10.7333698	9.7634640
0.0108	-0.2382730	-0.0912580	0.0760438	0.4348801	11.5827293	11.6415186
0.0115	-0.3824636	-0.0988221	0.0596939	1.2682371	10.0313854	12.0350847
0.0122	-0.4126640	-0.0942210	0.0333240	1.8364897	6.8616667	10.7537289
0.0129	-0.3524008	-0.0782285	0.0058305	2.0203943	3.1868620	8.0079517
0.0137	-0.2529566	-0.0541776	-0.0145441	1.7613077	0.1103060	4.3544407
0.0144	-0.1742508	-0.0270997	-0.0225278	1.1072626	-1.5693512	0.5538437
0.0151	-0.1630471	-0.0025122	-0.0169218	0.2159511	-1.5283537	-2.6179094
0.0158	-0.2363133	0.0148656	-0.0004694	-0.6918622	0.0505901	-4.5575933
0.0165	-0.3752971	0.0221746	0.0211363	-1.3978224	2.5586023	-4.9742651
0.0173	-0.5319106	0.0190089	0.0407326	-1.7398052	5.1205111	-3.9447260
0.0180	-0.6448715	0.0074103	0.0514295	-1.6465282	6.8099756	-1.8745098
0.0187	-0.6600024	-0.0087157	0.0484137	-1.1561747	6.8906002	0.6201413
0.0194	-0.5479696	-0.0246211	0.0304469	-0.4140317	5.0328159	2.8764439
0.0201	-0.3138701	-0.0359264	0.0005563	0.3623415	1.4408598	4.3534365
0.0209	0.0039371	-0.0397082	-0.0344754	0.9494188	-3.1641827	4.7612572
0.0216	0.3447314	-0.0351645	-0.0658262	1.1915674	-7.7164049	4.1172104
0.0223	0.6421506	-0.0236880	-0.0851025	1.0484180	-11.0896978	2.7161369
0.0230	0.8414730	-0.0083677	-0.0868011	0.5964770	-12.4125338	1.0289974
0.0237	0.9126108	0.0069072	-0.0700004	-0.0077021	-11.3197842	-0.4369968
0.0245	0.8556442	0.0184824	-0.0387631	-0.5795717	-8.0682135	-1.2773991

FINISH

(20)

Interpretation Of Results

Computer output results include the following:

Member Forces—Static Loads. Axial forces, shear forces, torques, and moments at each member end are tabulated for the design loading conditions, loading 4 (full vertical load plus 0.3 full transverse load), loading 5 (full vertical load plus 0.1 full longitudinal load), and loading 6 (full vertical load plus 0.5 full vertical load). These approximate equivalent static loading conditions yield conservative estimates of the maximum dynamic plus gravity loads, forces, and deflections as described previously. The members and deflections are then checked for the maximum applied loads. In the usual case, this step includes selection of the longitudinal reinforcement, which is often the minimum code value and which is otherwise determined by the largest axial force and moment. Transverse reinforcement is also selected during this stage using the tabulated maximum shear and torque values.

Displacements—Static Loads. Displacements of the joints, both support joints as well as free joints, are tabulated. Certain tolerable limits on deflections may be established based on attached piping or other equipment, and the deflection values are checked against the tolerable limits. Note that the tabulated values include the effect of gravity loads. The incremental deflections due to the 0.3, 0.5, and 0.1 equivalent static loads over and above the deflections due to gravity only are very small and, therefore, negligible in this example problem.

Dynamic Eigenvalues—Natural Frequencies. The first 20 natural frequencies (out of a total of 75 arising from 25 joints with 3 translatory dynamic degrees of freedom each) are printed. Each frequency corresponds to a mode shape discussed below. The first frequency (transverse mode) is given as 3.682 cycles/sec, section 19 of the computer printout, which compares to a Rayleigh-calculated frequency of 3.985 cycles/sec, section 11 of the computer printout. In general, Rayleigh-obtained natural frequencies are close to but higher than the corresponding true natural frequency as obtained from a dynamic analysis. Since the acting machine frequency is 727.7 radians/sec (see 15 above), which equals 6,949 rpm or 115.8 cycles/sec, the first 20 natural frequencies for the structure are well below the acting frequency, and the structure is said to be low tuned (undertuned). The undesirable range of natural frequencies is between 0.8–1.2 of the acting machine frequency as given in item 2(f) of the checklist table in Chapter 3. This check assures that no resonance condition will be encountered during machine operation. Additional checks are performed, not necessarily during the computer analysis phase, but during the preliminary design to assure that no resonance condition exists between the natural frequencies of vibration of individual columns and beams with the acting machine frequency, as in item 12 of the trial sizing procedure above.

Dynamic Eigenvectors—Mode Shapes. The first 20 eigenvectors or mode shapes which are normalized to a maximum unit value are listed. These serve to identify the physical direction for each mode. For example, for Mode 1 joints 9 through 21 show a near-unity Z displacement. Therefore, the first mode occurs in the transverse Z-direction. This is as expected since the structural stiffness is the lowest in the transverse direction. In general, the modes are ordered according to the stiffness of the structure in each direction. Other listed values give the deformation of the structure and a plot of the structure vibrating at its first frequency

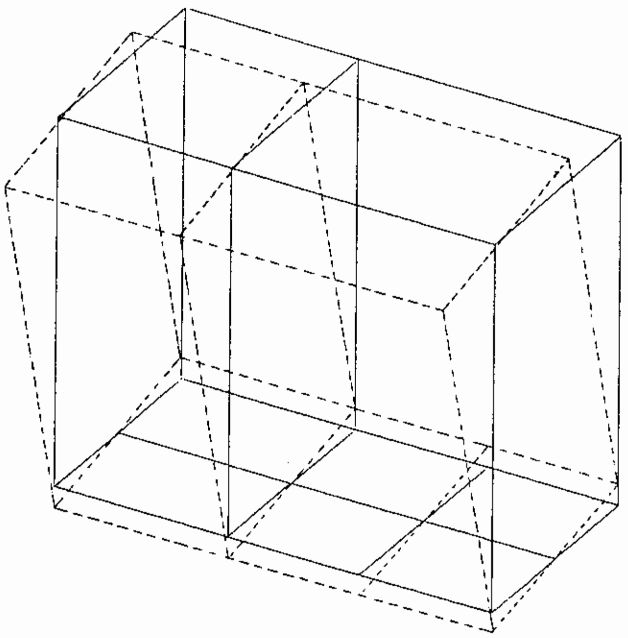

Figure 7-4. Structure vibrating at the first frequency mode (transverse).

mode may be obtained either manually or using a built-in STRUDL plot package; see Figure 7-4. The second mode shows joints 9 through 21 with near unity longitudinal displacements; thus, the second mode occurs in the longitudinal X-direction. In particular, all modes occurring in the direction of the applied dynamic forces (transverse and vertical in this example) are important for response studies. Some of the higher modes are coupled since they result from combination of basic lower modes.

Dynamic Displacements. Displacement of all joints in the structure at each time period are listed. The upper joints show the largest displacements. For example, joint 20 has a maximum displacement of 0.0002851 in. (0.007 mm) in the Y-direction occurring at time equal to 0.00576 sec. This maximum amplitude of displacement is then located on Figure 3-3 at the given machine-acting frequency. The point falls in zone B (minor faults, correction wastes dollars) which indicates satisfactory performance. Figures 3-6 and 3-7 also indicate that for a maximum displacement of 0.007 mm and a machine speed of 6,949 rpm (115.8 cps), the predicted structure behavior is satisfactory. The amplitude of vibration at all other joints, being smaller, is also satisfactory. Note that many structures supporting centrifugal machines show a largest amplitude of vibration in the transverse direction. However, in this example, the representation of the soil support as springs results in the amplitude of vibration being the largest in the vertical Y-direction.

Maximum Velocity. The maximum velocity is

$$(6{,}949 \times 2\pi \times 0.0002851)/60 = 0.207 \text{ in./sec}$$

This maximum velocity falls in the "slightly rough" range of Table 3-2. However, the combination of velocity and machine speed fall within the acceptable zone B of Figure 3-3. The designer may consider increasing the base dimensions and re-analyzing the revised structure so that all design criteria are met.

Dynamic Forces. The dynamic forces acting at the ends of each member at each time increment are listed. The forces are very small, for example, the dynamic axial force is 1,132 lbs. in column member 5 at 0.0237 sec. The moments are likewise very small, and the structure is considered adequate for supporting these small dynamic loads.

Since the structure was dimensioned initially to meet the design requirements listed in Chapter 3, including providing a sufficiently large soil bearing area, the proposed design is satisfactory and the structure meets its intended purpose. Note that all items listed in the design checklist of Chapter 3 are explicitly considered during the initial trial sizing phase or implicitly considered during the computer analysis. The design checklist serves as a reminder of important factors to be considered during a step-by-step hand calculation.

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Appendix A

Solution of Multi-Degree-of-Freedom System

Introduction

A multidegree-of-freedom system may be defined as a system in which more than one independent motion is possible. These independent motions may either be associated with a single mass, or a single independent motion may be associated with each of the several masses in a system. In the former type of system, the motion of the mass may either be coupled or uncoupled. The coupled motions of a single mass are described by the equations of motion where determination of the response (vibration displacements) of the system involves the solution of a set of simultaneous equations. An example of this type of system is shown in Figures 1-45 and 1-46. The response values x and θ (associated with a single mass) appear in both equations, and the pair is said to be coupled. The characteristic of uncoupled motions is described by the model of Figure 1-47. The response values x and θ appear individually in the equations of motion, and each of those equations can be independently solved.

An example of a multidegree-of-freedom system having several masses associated with one type of independent motion is shown in Figures 1-50 and 1-51. The independent motions x_1 and x_2 in the vertical direction are associated with masses m_1 and m_2, respectively. These types of systems always undergo coupled motions, and that effect is due to the coupling of masses m_1 and m_2 through the spring k_c. Thus, the two equations of motion for this system have to be solved simultaneously since x_1 and x_2 appear in both of them. Note that it is not necessary for each mass to have only one type of motion. If the masses are capable of oscillating in the horizontal direction, then each of the masses m_1 and m_2 will have two types of motions, i.e., vertical and horizontal motion. In fact, a two-dimensional planar system has up to three degrees of freedom (two linear and one rotational direction) per mass. For a spatial system, each mass can oscillate in six directions (three rectilinear and three rotational directions) and has six degrees of freedom. Therefore, the number of degrees of freedom is not necessarily equal to the number of lumped masses.

Dynamic Analysis

A complete dynamic analysis of a system is normally performed in two stages. The initial stage of investigation involves the determination of the natural frequencies and the mode shapes of the system. The natural frequencies and mode shapes provide information about the dynamic characteristic of the system. For instance, the lowest value of the natural frequencies (generally called fundamental frequency) indicates the relative degree of stiffness built into the system. In addition, it is also possible to compare the lowest natural frequency with the frequency of the acting dynamic force so that a possible resonance condition may be prevented. This requirement can be achieved when the ratio of the operating frequency to the lowest natural frequency does not fall within a given range (normally the undesirable range is at least 0.8-1.2). The determination of the mode shapes in a multidegree system has further significance. The mode shapes (see definition of *first (fundamental) mode* of the Terminology section in chapter 1) gives the deflection pattern that the system assumes when it is left to vibrate after termination of the disturbing force. Generally, it is the first mode which dominates the vibrating shape, and the higher mode overrides (when superimposed) that shape. The first mode will also indicate the particular mass or masses which will have the maximum amplitude of oscillation in a given direction from their state of rest (static equilibrium). This serves to indicate the relative degree of structural stiffness among various points of the system. This examination of the mode shapes in the vibrating system is considered a valuable step in adjusting the vibration amplitudes at critical points by varying the stiffness, mass, and damping resistance of the system. A practical example can be observed in the operation of a washing machine during the rinse cycle. If the load is accidentally lumped to one side of the drum during centrifugal motion, severe vibration beyond a predetermined magnitude can occur. Special sensor cut-off switches are then activated in order to prevent damage to the machine. Activation of the contact switches is actually set according to the fundamental mode of the spindle-drum assembly.

The next stage of analysis is a response calculation of the system caused by the dynamic force. This solution is quite tedious and time-consuming for multidegree-of-freedom systems; however, for a system with three degrees of freedom, the response calculation can be accomplished by using hand calculators. Use of computer programs is recommended for systems with more than three degrees of freedom. This part of the analysis gives the displacement, velocity, and acceleration of the masses and also the internal forces in all members of the system.

Determination of Natural Frequencies and Mode Shapes

Natural frequencies and mode shapes are obtained by employing one of following methods. A three-degrees-of-freedom system shown as Model 6-C (part a) of Chapter 2 (Figure 2-9) is used as an example. The parameters for that model are

$m_u = 9.662$ kips-sec²/ft
$k_h = 6{,}160.2$ kips/ft
$m_l = 17.80$ kips-sec²/ft
$I\psi = 2{,}843.3$ kips-sec²-ft
$k_z = 72{,}835.0$ kips/ft
$k_\psi = 14{,}703{,}707$ kips-ft/rad
$D_z = 0.50$
$D_\psi = 0.20$
$H = 17.0$ ft

Determinant Equation Method

All natural frequencies and mode shapes are obtained at the same time in this method. The general form of the equations of motion for a three-degree-of-freedom system is given by (ref. 3, Chapter 1):

$$\left.\begin{aligned} &m_{11}\ddot{x}_1 + m_{12}\ddot{x}_2 + m_{13}\ddot{x}_3 + c_{11}\dot{x}_1 + c_{12}\dot{x}_2 \\ &\quad + c_{13}\dot{x}_3 + k_{11}x_1 + k_{12}x_2 + k_{13}x_3 = F_1(t) \\ &m_{21}\ddot{x}_1 + m_{22}\ddot{x}_2 + m_{23}\ddot{x}_3 + c_{21}\dot{x}_1 + c_{22}\dot{x}_2 \\ &\quad + c_{23}\dot{x}_3 + k_{21}x_1 + k_{22}x_2 + k_{23}x_3 = F_2(t) \\ &m_{31}\ddot{x}_1 + m_{32}\ddot{x}_2 + m_{33}\ddot{x}_3 + c_{31}\dot{x}_1 + c_{32}\dot{x}_2 \\ &\quad + c_{33}\dot{x}_3 + k_{31}x_1 + k_{32}x_2 + k_{33}x_3 = F_3(t) \end{aligned}\right\} \tag{A1-1}$$

The natural frequencies depend on the mass (m_{ij}) and stiffness (k_{ij}) terms; therefore, the damping (c_{ij}) and applied forces (F_j) terms are omitted.

Equations (A1-1) are then reduced to

$$\left.\begin{aligned} &m_{11}\ddot{x}_1 + m_{12}\ddot{x}_2 + m_{13}\ddot{x}_3 \\ &\quad + k_{11}x_1 + k_{12}x_2 + k_{13}x_3 = 0 \\ &m_{21}\ddot{x}_1 + m_{22}\ddot{x}_2 + m_{23}\ddot{x}_3 \\ &\quad + k_{21}x_1 + k_{22}x_2 + k_{23}x_3 = 0 \\ &m_{31}\ddot{x}_1 + m_{32}\ddot{x}_2 + m_{33}\ddot{x}_3 \\ &\quad + k_{31}x_1 + k_{32}x_2 + k_{33}x_3 = 0 \end{aligned}\right\} \tag{A1-2}$$

It is assumed that the free-vibration motion of masses is simple harmonic (see definition of *modes* in the Terminology section of Chapter 1), which is expressed for a multidegrees-of-freedom system as

$$x_i = A_i \sin(\omega t + \phi) \tag{A1-3}$$
where $i = 1, 2, 3$

Substituting Equation (A1-3) into Equations (A1-2) and omitting the sine term, the following set of equations is obtained:

$$\left.\begin{aligned} &(k_{11} - m_{11}\omega^2)A_1 + (k_{12} - m_{12}\omega^2)A_2 \\ &\quad + (k_{13} - m_{13}\omega^2)A_3 = 0 \\ &(k_{21} - m_{21}\omega^2)A_1 + (k_{22} - m_{22}\omega^2)A_2 \\ &\quad + (k_{23} - m_{23}\omega^2)A_3 = 0 \\ &(k_{31} - m_{31}\omega^2)A_1 + (k_{32} - m_{32}\omega^2)A_2 \\ &\quad + (k_{33} - m_{33}\omega^2)A_3 = 0 \end{aligned}\right\} \tag{A1-4}$$

Equations (A1-4) are a set of algebraic equations and have a nontrivial solution for A_i only if the determinant

$$\left.\begin{aligned} &(a)\ \omega_1 = 22.63 \\ &(b)\ \omega_2 = 66.94 \\ &(c)\ \omega_3 = 76.68 \end{aligned}\right\} \quad \text{(A1-13)}$$

The mode shapes are obtained by substituting the above frequency values in any (A1-8) equation. For Equation A1-8(a) and (b) using the given values of masses and spring stiffnesses,

$$\left.\begin{aligned} &(a)\ (78{,}995.2 - 17.80\omega_n^2)A_{1n} - 6{,}160.2A_{2n} \\ &\quad + 104{,}723.4A_{\psi n} = 0 \\ &(b)\ -6{,}160.2A_{1n} + (6{,}160.2 - 9.662\omega_n^2)A_{2n} \\ &\quad - 104{,}723.4A_{\psi n} = 0 \end{aligned}\right\} \quad \text{(A1-14)}$$

In the equations above, the subscript n takes the values of 1, 2, and 3 and gives the amplitudes A's for each of the three modes. Since the right sides of Equations (A1-14) are zero, unique values of the A's are not obtained. However, it is possible to obtain the relative values of all amplitudes, in other words, the ratio of any two amplitudes. When one amplitude is assigned an arbitrary value, then all others are fixed in magnitude. A set of such amplitudes defines the mode shape, and therefore, the modes shapes are not dependent upon the absolute true values of amplitude. Using $n = 1$ and substituting $\omega_1^2 = 512.0$ in Equations (A1-14),

$$\left.\begin{aligned} &(a)\ 69{,}881.6A_{11} - 6{,}160.2A_{21} \\ &\quad + 104{,}723.4A_{\psi 1} = 0 \\ &(b)\ - 6{,}160.2A_{11} + 1{,}213.3A_{21} \\ &\quad - 104{,}723.4A_{\psi 1} = 0 \end{aligned}\right\} \quad \text{(A1-15)}$$

Assuming an arbitrary value of $A_{11} = +1$ and solving Equations (A1-15) simultaneously for A_{21} and $A_{\psi 1}$, the amplitudes of the first mode are obtained:

$$A_{11} = +1, \quad A_{21} = +12.8809, \quad A_{\psi 1} = +0.0904$$

The notation adopted is that the first subscript of the A identifies the mass, or point on the structure at which the amplitude occurs, and the second subscript designates the mode. Using the value of $n = 2$, and substituting $\omega_2^2 = 4{,}481.0$ in Equations (A1-14) yields,

$$\left.\begin{aligned} &(a)\ 766.60A_{12} - 6{,}160.2A_{22} \\ &\quad + 104{,}723.4A_{\psi 2} = 0 \\ &(b)\ - 6{,}160.2A_{12} - 37{,}135.2A_{22} \\ &\quad - 104{,}723.4A_{\psi 2} = 0 \end{aligned}\right\} \quad \text{(A1-16)}$$

Again setting $A_{12} = +1.0$ and solving Equations (a) and (b) of (A1-16) simultaneously for A_{22} and $A_{\psi 2}$, the amplitudes of the second mode are obtained:

$$A_{12} = +1.0, \quad A_{22} = -0.16, \quad A_{\psi 2} = -0.002091$$

Similarly, the amplitude for the third mode can be obtained by following the procedures given above for the first two modes. Those amplitudes using $\omega_3^2 = 5{,}880.0$ are:

$$A_{13} = +1.0, \quad A_{23} = -0.560248, \quad A_{\psi 3} = +0.212150$$

The computation of natural frequencies and mode shapes were performed using small electronic hand calculators with eight significant digits. However, in order to solve the cubic equation (Equation (A1-11)), 15 significant digits are required to obtain more "exact" results. This type of accuracy is normally not available in small hand calculators and if sufficient number of significant digits are not retained in the calculation of frequency, then large errors are present in the mode shape amplitude results. Therefore, it is important that the values of mode shape amplitudes be checked using the orthogonality conditions. The orthogonality condition of normal modes (see definition in the Terminology section of Chapter 1) is expressed by the following equation (ref. 1, Chapter 2):

$$\sum_{r=1}^{m} M_r A_{rl} A_{rn} = 0 \quad \text{(A1-17)}$$

where l and n identify any two normal modes of the system, and the subscript r refers to the rth mass out of a total of m masses.

Expanding the series of Equation (A1-17) for the first and second mode, i.e., $l = 1$ and $n = 2$; then

$$M_1A_{11}A_{12} + M_2A_{21}A_{22} + M_3A_{31}A_{32} = 0 \quad \text{(A1-18)}$$

Substituting an appropriate value for each term,

$| k_{ij} - m_{ij}\omega^2 | = 0$; thus,

$$\Delta(\omega) = \begin{vmatrix} (k_{11}-m_{11}\omega^2) & (k_{12}-m_{12}\omega^2) & (k_{13}-m_{13}\omega^2) \\ (k_{21}-m_{21}\omega^2) & (k_{22}-m_{22}\omega^2) & (k_{23}-m_{23}\omega^2) \\ (k_{31}-m_{31}\omega^2) & (k_{32}-m_{32}\omega^2) & (k_{33}-m_{33}\omega^2) \end{vmatrix} = 0 \quad \text{(A1-5)}$$

The expansion of determinant (A1-5) gives the characteristic equation of the system. This equation will be of third degree in the frequency parameter (ω^2) and has three roots representing the three basic frequencies of the system. Having determined the three natural frequencies, the mode shapes are obtained by making use of Equations (A1-4). For each of the three values of ω^2, the ratios of (A_1/A_1), (A_2/A_1), and (A_3/A_1) are evaluated and yield the three mode shapes for the system.

Example. For part (a) of Model 6-C, the equations of motion without damping and applied forces terms are:

$$\left.\begin{aligned} &(a)\ m_l\ddot{x}_1 + (k_x + k_h)x_1 - k_hx_2 + k_hH\psi = 0 \\ &(b)\ m_u\ddot{x}_2 - k_hx_1 + k_hx_2 - k_hH\psi = 0 \\ &(c)\ m_uH\ddot{x}_2 + I_\psi\ddot{\psi} + k_\psi\psi = 0 \end{aligned}\right\} \quad \text{(A1-6)}$$

Substituting the assumed solution of the form

$$\left.\begin{aligned} &(a)\ x_1 = A_1 \sin(\omega t + \phi) \\ &(b)\ x_2 = A_2 \sin(\omega t + \phi) \\ &(c)\ \psi = A_\psi \sin(\omega t + \phi) \end{aligned}\right\} \quad \text{(A1-7)}$$

into Equations (A1-6),

$$\left.\begin{aligned} &(a)\ (k_x + k_h - m_l\omega^2)A_1 - k_hA_2 + k_hH\,A_\psi = 0 \\ &(b)\ - k_hA_1 + (k_h - m_u\omega^2)A_2 - k_hH\,A_\psi = 0 \\ &(c)\ - m_uH\omega^2A_2 + (k_\psi - I_\psi\omega^2)A_\psi = 0 \end{aligned}\right\} \quad \text{(A1-8)}$$

The frequencies of the system are given by the condition $\Delta = 0$, where Δ is the determinant of the square matrix in Equation (A1-8). The expansion for a determinant

$$| a_{ij} | = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \quad \text{(A1-9)}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) + a_{12}(a_{23}a_{31} - a_{21}a_{33}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

In order to reduce the amount of computation, the values of a_{ij} are calculated by substituting the actual values of masses and stiffnesses.

$$\begin{aligned} a_{11} &= (k_x + k_h - m_l\omega^2) = (78{,}995.2 - 17.80\,\omega^2) \\ a_{12} &= -k_h = -6{,}160.2 \\ a_{13} &= k_hH = 6{,}160.2 \times 17.0 = 104{,}723.4 \\ a_{21} &= -k_h = -6{,}160.2 \\ a_{22} &= (k_h - m_u\omega^2) = (6{,}160.2 - 9.662\,\omega^2) \\ a_{23} &= -k_hH = -6{,}160.2 \times 17.0 = -104{,}723.4 \\ a_{31} &= 0 \\ a_{32} &= -m_uH\omega^2 = -164.254\,\omega^2 \\ a_{33} &= (k_\psi - I_\psi\omega^2) = (14{,}703{,}707 - 2{,}843.3\,\omega^2) \end{aligned}$$

Substituting these values in the expansion of Equation (A1-9) and setting $\omega^2 = z$,

$$\begin{aligned} &(78{,}995.2 - 17.80z)[9.0577775 \times 10^{10} \\ &- 17.515296 \times 10^6 z - 142.06721 \times 10^6 z \\ &+ 27{,}471.964z^2 - 17.201237 \times 10^6 z] \\ &- 6{,}160.2[9.0577775 \times 10^{10} - 17.515296 \times 10^6 z] \\ &+ 104{,}723.4[1.011837 \times 10^6 z] = 0 \qquad \text{(A1-10)} \end{aligned}$$

Rearranging Equation (A1-10) results in a cubic equation in z,

$$\begin{aligned} &z^3 - 10{,}872.992z^2 + 31.653102 \times 10^6 z \\ &- 1.3491464 \times 10^{10} = 0 \qquad \text{(A1-11)} \end{aligned}$$

Solution of the cubic equation yields three roots:

$$\left.\begin{aligned} &(a)\ z_1 = 512.0 \\ &(b)\ z_2 = 4{,}481.0 \\ &(c)\ z_3 = 5{,}880.0 \end{aligned}\right\} \quad \text{(A1-12)}$$

The square root of the z's is then the natural frequencies (in rad/sec) of the system,

$M_1 = m_l = 17.80, \quad M_2 = m_u = 9.662,$
$M_3 = I_\psi = 2{,}843.3$
$A_{11} = +1.0, \quad A_{12} = +1.0, \quad A_{21} = +12.8809,$
$A_{22} = -0.1600, \quad A_{31} = A_{\psi 1} = +0.0904,$
$A_{32} = A_{\psi 2} = -0.002091$ on the left side of equation (A1-18),

$$17.80 \times (1)(1) + 9.662(12.8809)(-0.16) + 2{,}843.3(+0.0904)(-0.002091) = -2.650298,$$

which is not zero and thus indicates errors in the results calculated above. Similarly, the orthogonality condition for the first and third modes ($l = 1$, $n = 3$) is checked by expanding Equation (A1-17) into a series:

$$M_1A_{11}A_{13} + M_2A_{21}A_{23} + M_3A_{31}A_{33} = 0 \quad \text{(A1-19)}$$

Substituting $A_{13} = +1.0$, $A_{23} = -0.560248$, $A_{\psi 3} = +0.212150$, and the values of other terms which are given above into the left side of Equation (A1-19):

$$17.80 \times (1)(1) + 9.662(12.8809)(-0.560248) + 2{,}843.3(+0.0904)(+0.212150) = +2.6040,$$

which is again not zero, thus confirming that errors are present in the frequency calculations. Therefore, it is desirable that natural frequencies be calculated using digital computers for a system having more than two degrees of freedom.

Stodola-Vianello Method

Calculation of the natural frequencies and characteristic shapes become cumbersome even with hand calculators for cases where the degrees of freedom exceed more than two, as is evident from the above example. The reason is that round-off errors are relatively important when the equations include terms of very large and very small numbers. Accuracy is lost in rounding off significant figures during the process of solving for the roots of the characteristic equation. It, therefore, has become common practice to resort to numerical, iterative (i.e., trial and error) procedures, such as the Stodola-Vianello method (ref. 1, Chapter 2). The various steps in the solution are as follows:

(1) Assume a characteristic shape, i.e., a set of A_i values (see Equation (A1-4)); (2) using one of Equations (A1-4), solve for ω^2; (3) using the remaining (N-1) equations, obtain a new shape by solving for the (N-1) A_i's in terms of the Nth A; and (4) use the new computed shape as the revised assumed shape in the next cycle starting at (2) again. The procedure is to be repeated until the computed shape is the same or very close to the last assumed shape. In Step (1) it is usually convenient to assign a unit value to the first value of A.

The procedure outlined above can best be described by applying the method to the solution of Equations (A1-8).

The equations of motion (A1-8) can be rewritten as:

$$\left.\begin{aligned}
&(a)\ -\omega^2 m_l A_1 + (k_x + k_h)A_1 - k_hA_2 + k_hHA_\psi = 0\\
&(b)\ -\omega^2 m_u A_2 - k_hA_1 + k_hA_2 - k_hHA_\psi = 0\\
&(c)\ -\omega^2 I_\psi A_\psi - \omega^2 m_u HA_2 + k_\psi A_\psi = 0
\end{aligned}\right\} \quad \text{(A1-20)}$$

Multiplying equation (A1-20b) by H and adding to Equation (A1-20c) in order to eliminate the term $\omega^2 m_u HA_2$ from Equation (A1-20c), the resulting equations are rearranged in a convenient form:

$$\left.\begin{aligned}
&(a)\ \omega_n^2 A_{1n} = \frac{(k_x + k_h)}{m_l}A_{1n} - \frac{k_h}{m_l}A_{2n} + \frac{k_hH}{m_l}A_{\psi n}\\
&(b)\ \omega_n^2 A_{2n} = -\frac{k_h}{m_u}A_{1n} + \frac{k_h}{m_u}A_{2n} - \frac{k_hH}{m_u}A_{\psi n}\\
&(c)\ \omega_n^2 A_{\psi n} = \frac{k_hH}{I_\psi}A_{1n} - \frac{k_hH}{I_\psi}A_{2n} + \frac{(k_\psi + k_hH^2)}{I_\psi}A_{\psi n}
\end{aligned}\right\} \quad \text{(A1-21)}$$

where the added subscript n indicates that the equations apply to any mode. Substituting the values of m_l, m_u, I_ψ, k_x, k_h, k_ψ, and H in the Equations (A1-21),

$$\left.\begin{aligned}
&(a)\ \omega_n^2 A_{1n} = 4{,}437.93A_{1n} - 346.08A_{2n} + 5{,}883.34\,A_{\psi n}\\
&(b)\ \omega_n^2 A_{2n} = -637.57A_{1n} + 637.57A_{2n} - 10{,}838.69\,A_{\psi n}\\
&(c)\ \omega_n^2 A_{\psi n} = 36.83A_{1n} - 36.83A_{2n} + 5{,}797.49\,A_{\psi n}
\end{aligned}\right\} \quad \text{(A1-22)}$$

The iteration procedure converges on that of the highest, or third mode. The following steps are used:

1. Assume values of A's (amplitudes of the mode) such that the amplitude of mass 1 = +1.0, i.e., $A_{1n} = +1.0$;
2. Substitute the values of A's in the right side of Equation (A1-22a) to compute the value of ω_n^2;
3. Substitute the value of ω_n^2 and trial values of A's in the right side of Equation (A1-22b) to compute the value of A_{2n};
4. Substitute the value of ω_n^2 and the trial values of A's in the right side of Equation (A1-22c) to obtain the value of $A_{\psi n}$;

Use the new values of A_{2n}, $A_{\psi n}$ along with the value of $A_{1n} = +1.0$ as new trial values of the A's and follow Steps 2 to 4. This process is to be continued until convergence is achieved, i.e., the difference between previous and new trial values is negligible. For example, the amplitude values for mode shape three, i.e., $n = 3$

$$A_{13} = +1.0, \qquad A_{23} = -1.5, \qquad A_{\psi 3} = +0.5,$$

Substituting these in Equation (A1-22a),

$$\omega_3^2\,(+1) = 4{,}437.93(+1.0) - 346.08(-1.5) + 5{,}883.34(+0.5)$$
$$\therefore\ \omega_3^2 = 7{,}898.72$$

Substituting this value for ω_3^2 and the trial values of A's in the right side of Equation (A1-22b),

$$7{,}898.72(A_{2n}) = -637.57(+1.0) + 637.57(-1.5) - 10{,}838.69(+0.5)$$

or $A_{2n} = -0.8879$

Finally, from Equation (A1-22c),

$$7{,}898.72(A_{\psi 3}) = 36.83(+1.0) - 36.83(-1.5) - 5{,}797.49(+0.5)$$

or $A_{\psi 3} = +0.3786$

Therefore, the first estimate of ω_3^2 is 7,898.72, and the next set of trial values of A's are + 1.0, −0.8879, and +0.3786. This procedure is to be repeated from Step 2 to 4 until convergence is achieved, as shown in Table A1-1. The value of A's computed are in fact the amplitude ratios with respect to A_{13} which is arbitrarily taken = +1.0.

Proceeding further to the second mode, the orthogonality conditions given by Equation (A1-17) are used in order to reduce the number of equations by one. Expanding the equation for the second and third mode, i.e., $l = 2$ and $n = 3$:

$$\sum_{r=1}^{3} M_r A_{rl} A_{rn} = M_1 A_{12} A_{13} + M_2 A_{22} A_{23} + M_3 A_{32} A_{33} = 0$$

Substituting $M_1 = m_l = 17.80$, $M_2 = m_u = 9.662$, $M_3 = I_\psi = 2{,}843.3$, $A_{13} = +1.0$, $A_{23} = -0.6011$, $A_{33} = A_{\psi 3} = +0.2412$, in the series.

$$17.80(A_{12})(+1.0) + 9.662(A_{22})(-0.6011) + 2{,}843.3(A_{32})(+0.2412) = 0$$

or $A_{32} = -0.025955\,A_{12} + 0.008469 A_{22}$, (A1-23)

which is equal to $A_{\psi 2}$.

Substituting the expression for $A_{\psi 2}$ into Equations (a) and (b) of (A1-22) and using $n = 2$, yields

$$\left.\begin{aligned}(a)\ \omega_2^2 A_{12} &= 4{,}285.23A_{12} - 296.25A_{22}\\ (b)\ \omega_2^2 A_{22} &= -356.25A_{12} + 545.74A_{22}\end{aligned}\right\} \qquad \text{(A1-24)}$$

Equation (A1-24) is iterated in a similar fashion as was done for Equation (A1-22) in order to find ω_2^2, A_{12} and A_{22}. Then, Equation (A1-23) is used to obtain $A_{\psi 2}$. As a first trial, assume $A_{12} = +1.0$ and $A_{22} = +0.5$. Using Equation (A1-24a),

$$\omega_2^2\,(+1) = 4{,}285.23(+1.0) - 296.25(+0.5)$$

or $\omega_2^2 = 4{,}434.36$

Using Equation (A1-24b),

$$4434.36(A_{22}) = -356.25(+1.0) + 545.74(+0.5)$$

or $A_{22} = -0.1419$

Subsequent cycles of iterations are given in Table A1-1, where it may be seen that the values converge rapidly in four cycles. In the fourth cycle, Equation (A1-23) is used to obtain $A_{\psi 2}$.

The determination of the first mode is made directly from the orthogonality conditions by applying the conditions to the first and the second modes, and then, to the first and the third modes. Expanding Equation (A1-17),

Table A1–1

Stodola-Vianello Procedure for Model Shown in Figure 2-9

	Third Mode					
	Trial Values			Computed Values		
Trial No.	A_{13}	A_{23}	$A_{\psi 3}$	ω_3^2 Equation (A1-22a)	A_{23} Equation (A1-22b)	$A_{\psi 3}$ Equation (A1-22c)
1.	+1.0	−1.5	+0.5	7,898.72	−0.8879	+0.3786
2	+1.0	−0.8879	+0.3786	6,972.65	−0.7611	+0.3248
3	+1.0	−0.7611	+0.3248	6,612.24	−0.7022	+0.2946
4	+1.0	−0.7022	+0.2946	6,414.18	−0.6670	+0.2760
5	+1.0	−0.6670	+0.2760	6,291.64	−0.6444	+0.2641
6	+1.0	−0.6444	+0.2641	6,214.73	−0.6293	+0.2561
7	+1.0	−0.6293	+0.2561	6,162.44	−0.6190	+0.2507
8	+1.0	−0.6190	+0.2507	6,127.11	−0.6120	+0.2469
9	+1.0	−0.6120	+0.2469	6,102.33	−0.6069	+0.2443
10	+1.0	−0.6069	+0.2443	6,085.27	−0.6035	+0.2425
11	+1.0	−0.6035	+0.2425	6,073.50	−0.6011	+0.2412

	Second Mode				
	Trial Values		Computed Values		
Trial No.	A_{12}	A_{22}	ω_2^2 Equation (A1-24a)	A_{22} Equation (A1-24b)	$A_{\psi 2}$ Equation (A1-23)
1	+1.0	−0.5	4,433.36	−0.1419	
2	+1.0	−0.1419	4,327.27	−0.1002	
3	+1.0	−0.1002	4,314.91	−0.0952	
4	+1.0	−0.0952	4,313.44	−0.0946	−0.02676

Summary	ω^2	A_1	A_2	A_ψ	ω (radians/sec.)	f (cps)
Third Mode	6,073.50	+1.0	− 0.6011	+0.2412	77.93	12.40
Second Mode	4,313.44	+1.0	− 0.0946	−0.0268	65.68	10.45
First Mode	515.68	+1.0	12.6894	+0.0815	22.71	3.61

$$\left.\begin{aligned}
&\text{(a)}\ \sum_{r=1}^{3} M_r A_{r1} A_{r2} = M_1 A_{11} A_{12} + M_2 A_{21} A_{22} + M_3 A_{31} A_{32} \\
&\text{(b)}\ \sum_{r=1}^{3} M_r A_{r1} A_{r3} = M_1 A_{11} A_{13} + M_2 A_{21} A_{23} + M_3 A_{31} A_{33}
\end{aligned}\right\} \quad \text{(A1-25)}$$

Substituting the values of $M_1 = m_l = 17.80$, $M_2 = m_u = 9.662$, $M_3 = I_\psi = 2{,}843.3$, $A_{12} = +1.0$, $A_{22} = -0.0946$, $A_{32} = A_{\psi 2} = -0.02676$, $A_{13} = +1.0$, $A_{23} = -0.6011$, $A_{33} = A_{\psi 3} = +0.2412$, Equations (A1-25a) and (b) yield

$$\left.\begin{aligned}
&\text{(a)}\ 17.80(A_{11})(1.0) + 9.662(A_{21})(-0.0946) + 2843.3(A_{31})(-0.02676) = 0 \\
&\text{(b)}\ 17.80(A_{11})(+1.0) + 9.662(A_{21})(-0.6011) + 2843.3(A_{31})(+0.2412) = 0
\end{aligned}\right\} \quad \text{(A1-26)}$$

Assuming $A_{11} = +1.0$, these two equations are solved simultaneously to provide

$$A_{21} = 12.6894,\ A_{31} = A_{\psi 1} = +0.08151$$

The value of ω_1^2 may be computed by making use of any one of Equations (A1-22). Using Equation (A1-22a),

$$\omega_1^2(+1.0) = 4{,}437.93(+1.0) - 346.08(+12.6894) + 5{,}883.34(+0.08151)$$

$$\omega_1^2 = 525.93 \quad \text{(A1-27)}$$

If Equations (A1-22b) and (c) are used and the last computed mode is substituted, ω_1^2 is found to be 517.70 and 515.68, which for normal purposes is considered to be an accurate solution. A summary of the complete solution is given in Table A1-1.

Steady-State Response Analysis

Calculation of the maximum amplitude of vibration for a steady-state condition is often the main item of interest in an engineering dynamics problem. The amount of computation work is quite extensive when a response analysis is required for a system with more than three degrees of freedom. In those cases, computer programs such as ICES-STRUDL, NASTRAN, ANSYS, or NISA may be used. Hence, the investigation of the three-degrees-of-freedom system considered above is extended to the calculation of the mode shapes and frequencies. There is a variety of methods for finding the response in this type of problem; however, a modal analysis technique is used here. This technique has become the current state-of-the-art (ref. 1, Chapter 2). This method of analysis consists of calculating the response for each normal mode individually and then superimposing the individual responses to yield the total solution. There are some limitations on the applicability of this method: The system has to be linearly elastic and the dynamic forces acting on the masses must follow the same time variation, i.e., if the applied forces are harmonic, then all the forces must have the same acting frequency. However, these restrictions can be relaxed if numerical methods are used in the solution of the modal equations. For a lumped multimass system having j masses M damping constant C_r associated with mass r, s springs, and N normal modes the modal equations of motion for the nth mode is derived by the use of the Lagrange equation.

At any instant in the system, the total kinetic energy is

$$K = \sum_{r=1}^{J} \tfrac{1}{2} M_r \left(\sum_{n=1}^{N} \dot{a}_{rn}\right)^2, \qquad \text{(A1-28)}$$

the total strain energy in the springs is

$$U = \sum_{g=1}^{s} \tfrac{1}{2} k_g \left(\sum_{n=1}^{N} \Delta_{gn}\right)^2, \qquad \text{(A1-29)}$$

the total energy dissipated by the dampers is

$$D = -\sum_{r=1}^{j} C_r \sum_{n=1}^{N} \dot{a}_{rn}\, a_{rn} \qquad \text{(A1-30)}$$

and the total work in terms of displacement is

$$W_e = \sum_{r=1}^{J} F_r \sum_{n=1}^{N} a_{rn} \qquad \text{(A1-31)}$$

In equations A-1.28 to A-1.31, a_{rn} and $\dot{a}_{rn}$ are respectively the displacement and velocity component of mass r associated with the nth mode, Δ_{gn} is the distortion of spring g (i.e., the relative displacement of its ends) in the nth mode, and k_g is the stiffness of that spring. These equations are based on the fact that any displacement or velocity is equal to the sum of the modal components.

The squared series in Equation A-1.28 is equivalent to the sum of the squares of all modal components of a_{rn} plus twice the sum of all cross products of these components. When summed over all masses, the total of these cross products must be zero, according to the orthogonality condition given by Equation A-1.32.

$$\sum_{r=1}^{J} M_r\, \dot{a}_{rn}\, \dot{a}_{rm} = 0 \qquad \text{(A1-32)}$$

This orthogonality condition is true for the displacement and for the velocity vectors. Thus Equation A-1.28 may be written as

$$K = \sum_{r=1}^{j} \tfrac{1}{2} M_r \sum_{n=1}^{N} \dot{a}_{rn}^2 \qquad \text{(A1-33)}$$

Similarly, by the same reasoning the cross product terms of the series in Equation A-1.29 will also be zero and thus, is reduced to

$$U = \sum_{g=1}^{s} \tfrac{1}{2} k_g \sum_{n=1}^{N} \Delta_{gn}^2 \qquad \text{(A1-34)}$$

For each mode it is convenient to select a modal displacement X_n so that all individual mass displacements may be expressed in terms of this one variable. X_n is usually taken as the displacement of one arbitrary selected mass. Thus

$$\text{(a) } a_{rn} = X_n \left(\frac{a_{rn}}{X_n}\right) = X_n A_{rn}$$

$$\text{(b) } \dot{a}_{rn} = \dot{X}_n \left(\frac{\dot{a}_{rn}}{\dot{X}_n}\right) = \dot{X}_n A_{rn}$$

$$\text{(c) } \Delta_{gn} = X_n \left({}^{\Delta_{gn}}/X_n\right) = X_n A_{\Delta_{gn}} \qquad \text{(A1-35)}$$

where A_{rn} and $A_{\Delta gn}$ are constants for a given mode. The resulting equations of K, U, D, and W_e may therefore be written as

$$\text{(a) } K = \sum_{r=1}^{j} \tfrac{1}{2} M_r \sum_{n=1}^{N} \dot{X}_n^2 A_{rn}^2$$

$$\text{(b) } U = \sum_{g=1}^{s} \tfrac{1}{2} k_g \sum_{n=1}^{N} X_n^2 A^2_{\Delta gn}$$

$$\text{(c) } D = -\sum_{r=1}^{j} C_r \sum_{n=1}^{N} \dot{X}_n X_n A_{rn}^2$$

$$\text{(d) } W_e = \sum_{r=1}^{j} F_r \sum_{n=1}^{N} X_n A_{rn} \qquad \text{(A1-36)}$$

The Lagrange equation for a conservative system is given as

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}_i}\right) - \frac{\partial K}{\partial q_i} + \frac{\partial U}{\partial q_i} - \frac{\partial D}{\partial q_i} = \frac{\partial W_e}{\partial q_i} \qquad \text{(A1-37)}$$

where q_i and $\dot{q}_i$ are the generalized coordinates of the system, which in Equation A-1.36 are X_n and $\dot{X}_n$. Substitution of expressions of Equation A-1.36 into A-1.37 leads to the equation of motion

$$\ddot{X}_n \sum_{r=1}^{j} M_r A_{rn}^2 + \dot{X}_n \sum_{r=1}^{j} C_r A_{rn}^2 + X_n \sum_{g=1}^{j} k_g A^2_{\Delta gn}$$

$$= \sum_{r=1}^{j} F_r A_{rn} \qquad \text{(A1-38)}$$

These equations of motion are analogous to the single-degree-of-freedom system equation and are associated as follows:

$$\sum_{r=1}^{j} M_r A_{rn}^2 = \text{equivalent mass}$$

$$\sum_{r=1}^{j} C_r A_{rn}^2 = \text{equivalent damping constant}$$

$$\sum_{g=1}^{s} k_g A^2_{\Delta gn} = \text{equivalent spring constant}$$

$$\sum_{r=1}^{j} F_r A_{rn} = \text{equivalent force}$$

where X_n = displacement of nth mode,
$F_r = F_{r1}[f(t)]$ = forcing function acting on mass r,
A_{rn} = constant of mode n at mass r,
$A_{\Delta gn}$ = distortion of spring g of mode n.
$n = 1, N$

Solution of Equation (A1-38) for a normal mode n is

$$X_n = \frac{f(t) \sum_{r=1}^{j} F_{r1} A_{rn}}{\omega_n^2 \sum_{r=1}^{j} M_r A_{rn}^2} (M)_n \qquad \text{(A1-39)}$$

where M is the magnification factor given in Table 1-4.

The total displacement for the mass r is given by superimposing the N modes:

$$X_r(t) = \sum_{n=1}^{N} A_{rn} (M)_n \frac{\sum_{r=1}^{j} F_{r1} A_{rn}}{\omega_n^2 \sum_{r=1}^{j} M_r A_{rn}^2} f(t) \qquad \text{(A1-40)}$$

The parameters F_{r1}, M_r, and $f(t)$ are specified for each particular problem, and if the modal frequencies and characteristic shapes are available, then the solution of Equation (A1-40) follows.

Example. The response of a three-degree-of-freedom system shown in part (a) for Model 6-C in Chapter 2 (Figure 2-9) subjected to dynamic loads will be determined. The modal frequencies and mode shapes were previously obtained and the results listed in Table A1-1 are used. The equations of motion containing damping terms and the forcing functions for the system are as follows:

$$\left.\begin{aligned}
&\text{(a) } m_l\ddot{x}_1 + C_{x1}\dot{x}_1 + k_x x_1 - k_h(x_2 - x_1 - \psi H) = 0 \\
&\text{(b) } m_u\ddot{x}_2 + k_h(x_2 - x_1 - \psi H) = F_{x2}(t) \\
&\text{(c) } I_\psi\ddot{\psi} + C_\psi\dot{\psi} + m_u\ddot{x}_2 H + k_\psi\psi = F_{x2}(t)\,H
\end{aligned}\right\} \text{(A1-41)}$$

The following values of the forcing functions are given for this example as:

$$F_{x1}(t) = F_{11} f(t) = 0$$

$$F_{x2}(t) = F_{21} f(t) = 8.080 \sin 727.7t \text{ kips}$$

$$T_\psi(t) = F_{31} f(t) = 137.360 \sin 727.7t \text{ kips-ft,}$$

where the acting frequency is $\omega = 727.7$ rad/sec

The values for the masses are

$$\left\{\begin{aligned}
m_l &= 17.80 \text{ kips-sec}^2/\text{ft} \\
m_u &= 9.662 \text{ kips-sec}^2/\text{ft} \\
I_\psi &= 2843.3 \text{ kips-sec}^2\text{-ft}
\end{aligned}\right.$$

Table A1–2

Modal Response Analysis

Mode n	Mass Point r (1)	Mode Amplitude A_{rn} (2)	Forcing Function F_{r1} (3)	$F_{r1}A_{rn}$ (4)	Mass M_r (5)	$M_rA^2_{rn}$ (6)	ω^2_n (7)	$\dfrac{\sum_{r=1}^{3} F_{r1}A_{rn}}{\omega^2_n \sum_{r=1}^{3} M_rA^2_{rn}}$ (8)	$(M)_n$ Table 1-4	Equation: A-1-40 (Max. Response)
	1	+ 1.0000	0	0.0	17.80	17.80		113.725		$x_1 = 0.000975(1)$ (138.50×10^{-6})
	2	+12.6894	8.080	102.530	9.662	1,555.78	515.68	511.68(1,592.47) $= 138.50 \times 10^{-6}$	0.000975	$+ 0.008180(1)$ (-51.70×10^{-6})
1	3	+ 0.0815	137.360	11.195	2,843.3	18.89				$+ 0.011590(1)$ (24.93×10^{-6})
	Σ			113.725		1,592.47				$= 0.0011 \times 10^{-6}$ ft
	1	+ 1.0000	0	0.0	17.80	17.80		−4.445		$x_2 = 0.000975\ (12.6894)$ (138.50×10^{-6})
	2	− 0.0946	8.080	−0.764	9.662	0.0865	4,313.44	4,313.44(19.93) $= -51.70 \times 10^{-6}$	0.008180	$+0.008180\ (-0.0946)$ (-51.70×10^{-6})
2	3	− 0.0268	137.360	−3.681	2,843.3	2.04				$+0.011590\ (-0.6011)$ (24.93×10^{-6})
	Σ			−4.445		19.93				$= 1.580 \times 10^{-6}$ ft
	1	1.0000	0	0.0	17.80	17.80		28.274		$x_3 = 0.000975(0.0815)$ (138.50×10^{-6})
	2	− 0.6011	8.080	−4.857	9.662	3.49	6,073.50	6,073.71(186.71) $= 24.93 \times 10^{-6}$	0.011590	$+ .008180(-0.0268)$ (-51.70×10^{-6})
3	3	+ 0.2412	137.360	33.131	2,843.3	165.42				$+ .011590(+0.2412)$ (24.93×10^{-6})
	Σ			28.274		186.71				$= 0.092 \times 10^{-6}$ rad

The inclusion of damping in a multidegree-of-freedom system is accomplished by assuming a certain percentage of critical damping in each mode. This factor is normally accounted for while computing the value of the magnification factor (M) for that mode, which are then superimposed. In multi-degrees-of-freedom problems requiring a steady-state response analysis, damping is generally assumed in terms of the damping ratios rather than as damping coefficients such as C_{x1} and C_{ψ} used in Equation (A1-41). These damping coefficients are included in the equations of motion, when it is required that a free-vibration (transient) analysis be obtained. Therefore, in this example problem, damping ratio (D) for various modes are assumed as follows:

Mode 1, $D = 0.0$ (no damping in the structure)
Mode 2, $D = 0.50$ (damping in the lateral mode)
Mode 3, $D = 0.20$ (damping in the rocking mode)
(A1-42)

For the sake of clarity, Equation (A1-40) is expanded as

$$x_1(t) = A_{11}(M)_1\left[\frac{F_{11}A_{11}+F_{21}A_{21}+F_{31}A_{31}}{\omega_1^2(M_1A^2_{11}+M_2A^2_{21}+M_3A^2_{31})}\right]f(t)$$
$$+ A_{12}(M)_2\left[\frac{F_{11}A_{12}+F_{21}A_{22}+F_{31}A_{32}}{\omega_2^2(M_1A^2_{12}+M_2A^2_{22}+M_3A^2_{32})}\right]f(t)$$
$$+ A_{13}(M)_3\left[\frac{F_{11}A_{13}+F_{21}A_{23}+F_{31}A_{33}}{\omega_3^2(M_1A^2_{13}+M_2A^2_{23}+M_3A^2_{33})}\right]f(t) \quad \text{(A1-43)}$$

Similar expansions can also be written for x_2 and x_3. Numerical solutions of these expansions are tabulated in Table A1-2 where x_1 corresponds to the deflection at mass m_l (at the base of the footing), x_2 corresponds to the mass m_u (at the top of the structure), and x_3 is the rotation of the mass I_ψ. The function $f(t)$ in this example problem is equal to sin ωt, which when superimposed with the maximum displacements, gives the final solution:

(a) $x_1 = 0.0011 \times 10^{-6} \sin 727.7t$ ft
(b) $x_2 = 1.5800 \times 10^{-6} \sin 727.7t$ ft
(c) $\psi = 0.09195 \times 10^{-6} \sin 727.7t$ rad
(A1-44)

Appendix B

Summary of ICES-STRUDL Commands

The following ICES-STRUDL commands are described in refs. 2 and 3 of Chapter 7. Some special symbols are used to denote options. Among these are:

—Underlines: The portion of the word which is required is underlined and the rest of the word is optional.

—Braces { }: A set of braces indicates that a choice exists. Any one, sometimes more than one choice, can be made.

—Parentheses () : Any item in a parentheses may be omitted. The element inside a parentheses is optional, therefore, the meaning of the command does not change if the item is omitted or included.

—Asterisk *: An asterisk located outside and in front of a set of braces indicates that more than one choice may be made.

—Arrow → : An arrow located in front of an element inside a set of braces indicates that if the user does not make any choice, the element indicated by the arrow will be assumed (default value).

For convenience, the format of the "list" element is given first. The symbols v, v_1 . . . v_n denote decimal values.

where: list = * { alphalist / integerlist / n_1 TO n_2 }

alphalist = 'a_1' ('a_2') . . .

integerlist = i_1 (i_2) . . .

1. EJECT Command

    EJECT

2. FINISH Command

    FINISH

3. DEBUG Command

DEBUG { OFF / MAP / REGISTERS / → COMMON / POOL / ALL }

4. STRUDL Command

STRUDL { 'a₁' ('title') / RESTORE 'a₁' }

5. SAVE Command

SAVE ('a₁')

6. CHANGE ID Command

CHANGE ID 'a₁' ('title')

7. UNITS Command

UNITS * { length unit / force unit / angular unit / temperature unit / times unit / mass unit }

Elements:

length unit = { INCHES / FEET / FT / CENTIMETERS / CMS / METERS / M / MILLIMETERS / MM }

force units = { POUNDS / LBS / KIPS / TONS / MTONS / NEWTON / N / KN / MN }

angular unit = { CYCLES / RADIANS / DEGREES }

temperature unit = { FAHRENHEIT / CENTIGRADE }

time unit = { SECONDS / MINUTES / HOURS }

mass unit = { LBM / SLUG / KILOGRAM / KG / KGM }

8. Input Mode Command

{ → ADDITIONS / CHANGES / DELETIONS }

9. TYPE Command

TYPE { PLANE { TRUSS / FRAME / GRID } { → XY / XZ / YZ } / SPACE { TRUSS / FRAME } }

10. Identification Mode Command

SET ELEMENTS { INTEGER / UNRESTRICTED }

11. TIME Command

TIME BEGIN

TIME PRINT

12. SCAN Command

SCAN { → ON / OFF }

13. DUMP Command

DUMP { → ON (i) / OFF / COMMON / POOL / TIME }

14. JOINT COORDINATES Command

{ JOINT / NODE } COORDINATES

{ i_1 / 'a_1' } ([XCOORD] v_x [YCOORD] v_y [ZCOORD] v_z) { → FREE / SUPPORT }

.
.
.

{ i_n / 'a_n' } ([XCOORD] v_x [YCOORD] v_y [ZCOORD] v_z) { → FREE / SUPPORT }

15. JOINT RELEASES Command

{ JOINT / NODE } RELEASES (* { force releases / moment releases }), (angle specs), (elastic support specs)

list (* { force releases / moment releases }), (angle specs), (elastic support specs)

.
.
.

list (* { force releases / moment releases }), (angle specs), (elastic support specs)

force releases = FORCE * { X / Y / Z }

moment releases = MOMENT * { X / Y / Z }

angle specs = ([TH1] v_1 [TH2] v_2 [TH3] v_3)

elastic support specs = ([KFX] v_4 [KFY] v_5 [KFZ] v_6 [KMX] v_7 [KMY] v_8 [KMZ] v_9)

16. MEMBER INCIDENCES Command

MEMBER INCIDENCES

{ i_1 / 'a_1' } starting joint, end joint

.
.
.

{ i_n / 'a_n' } starting joint, end joint

17. MEMBER RELEASES Command

MEMBER RELEASES *({ START *{ force releases / moment releases } / END *{ force releases / moment releases } })

list *({ START *{ force releases / moment releases } / END *{ force releases / moment releases } })

.
.
.

list *({ START *{ force releases / moment releases } / END *{ force releases / moment releases } })

force releases = FORCE *{ X / Y / Z }

moment releases = MOMENT *{ X / Y / Z }

18. MEMBER ECCENTRICITIES Command

MEMBER ECCENTRICITIES (eccentric specs)

list (eccentric specs)

.
.
.

list (eccentric specs)

eccentric specs = ({ GLOBAL / → LOCAL }) *{ START [X] v_1 [Y] v_2 [Z] v_3 / END [X] v_4 [Y] v_5 [Z] v_6 }

19. MEMBER END SIZE Command

MEMBER END (JOINT) SIZE (size specs)

list (size specs)

. . .

list (size specs)

size specs = [START] v_1 [END] v_2

```
list           ( * {FORCE     [X] v1, [Y] v2, [Z] v3} )
                   {MOMENT    [X] v1, [Y] v2, [Z] v3}
 .
 .
 .
list           ( * {FORCE     [X] v1, [Y] v2, [Z] v3} )
                   {MOMENT    [X] v1, [Y] v2, [Z] v3}
```

25. JOINT DISPLACEMENTS Command

```
{JOINT} DISPLACEMENTS   ( * {DISPLACEMENTS   [X] v1, [Y] v2, [Z] v3} )
{NODE }                     {ROTATIONS       [X] v1, [Y] v2, [Z] v3}

list                    ( * {DISPLACEMENTS   [X] v1, [Y] v2, [Z] v3} )
                            {ROTATIONS       [X] v1, [Y] v2, [Z] v3}
 .
 .
 .
list                    ( * {DISPLACEMENTS   [X] v1, [Y] v2, [Z] v3} )
                            {ROTATIONS       [X] v1, [Y] v2, [Z] v3}
```

26. MEMBER LOADS Command

```
MEMBER LOADS, (direction specs), (type specs)
list, (direction specs), (type specs)
 .
 .
 .
list, (direction specs), (type specs)

                        {  FORCE   {X}  }
                        {          {Y}  }
                        {          {Z}  }
directions specs   =    {               }  (GLOBAL)
                        {  MOMENT  {X}  }
                        {          {Y}  }
                        {          {Z}  }

type specs   =   {CONCENTRATED (FRACTIONAL) [P] v1, [L] v2                   }
                 {UNIFORM (FRACTIONAL) [W] v3, ([LA] v4, [LB] v5)            }
                 {LINEAR (FRACTIONAL) [WA] v6, [WB] v7 ([LA] v8, [LB] v9)    }
```

27. MEMBER TEMPERATURE LOADS Command

```
MEMBER TEMPERATURE(LOADS) (temp specs)
list, (temp specs)
 .
 .
 .
list, (temp specs)

temp specs  =  (FRACTIONAL) ([LA] v1, [LB] v2,)   * {[AXIAL] v3                   }
                                                    {(BENDING) [Y] v4 [Z] v5      }
```

28. MEMBER DISTORTIONS Command

MEMBER DISTORTIONS (place types)
list (place types)
.
.
.
list (place types)

place types = { CONCENTRATED (FRACTIONAL) [L] v_1 / UNIFORM (FRACTIONAL) [LA] v_2 [LB] v_3 } dist data

dist data = { DISPLACEMENT [X] v_4, [Y] v_5, [Z] v_6 / ROTATION [X] v_4, [Y] v_5 [Z] v_6 }

29. MEMBER END LOAD Command

MEMBER END (LOADS) (*{ START / END } { FORCE / MOMENT } [X] v_1 [Y] v_2 [Z] v_3)
list (*{ START / END } { FORCE / MOMENT } [X] v_1 [Y] v_2 [Z] v_3)
.
.
.
list (*{ START / END } { FORCE / MOMENT } [X] v_1 [Y] v_2 [Z] v_3)

30. JOINT FORCES Command

JOINT FORCES (force specs, (angle specs), loading specs)
list, (force specs, (angle specs), loading specs)
.
.
.
list (force specs, (angle specs), loading specs)

force specs = {
FORCE X or FX
FORCE Y or FY
FORCE Z or FZ
MOMENT X or MX
MOMENT Y or MY
MOMENT Z or MZ
}

angle specs = *{
TH1 Θ_1
TH2 Θ_2
TH3 Θ_3
}

loading specs = { $v_1 \ldots v_n$ / 'loading$_1$' v_1 . . . 'loading$_n$' v_n }

20. MEMBER PROPERTIES Command

MEMBER PROPERTIES ({ PRISMATIC, (Section values) | VARIABLE | TABLE 'table name' ('section name') | FLEXIBILITY | STIFFNESS })

list ({ (PRISMATIC), section values | variable specs | table specs | (FLEXIBILITY), matrix specs | (STIFFNESS), matrix specs })

.
.
.
.
.

list ({ (PRISMATIC), section values | variable specs | table specs | (FLEXIBILITY), matrix specs | (STIFFNESS), matrix specs })

section values = { [AX] v_1 [AY] v_2 [AZ] v_3
[IX] v_4 [IY] v_5 [IZ] v_6
[SY] v_7 [SZ] v_8 [YD] v_9
[ZD] v_{10} [YC] v_{11}
[ZC] v_{12} [EY] v_{13} [EZ] v_{14} }

(VARIABLE)

SEGMENTS i_1 (AND i_2) { TABLE 'table name' 'section name' | section values } { LENGTH | XC | YC | ZC } v_1

.
.
.
.
.
.

SEGMENTS i_1 (AND i_2) { TABLE 'table name' 'section name' | section values } { LENGTH | XC | YC | ZC } v_1

matrix specs = a series of commands, the first of which contains the "list" element and, if not specified in the heading, the word FLEXIBILITY or STIFFNESS.

(MATRIX) COLUMNS (i_1) (i_2) (i_3) (i_4) (i_5) (i_6)

ROW 1 v_{11} v_{12} v_{13} v_{14} v_{15} v_{16}

.
.
.

ROW 6 v_{61} v_{62} v_{63} v_{64} v_{65} v_{66}

21. CONSTANTS Command

CONSTANTS (constant description)

constant description

.
.
.

constant description

$$\text{constant description} = \left\{\begin{array}{l} \text{E} \\ \text{G} \\ \text{CTE} \\ \text{DENSITY} \\ \text{BETA} \\ \text{POISSON} \end{array}\right\} \left\{\begin{array}{l} v_1 \text{ list, } v_2 \text{ list, } v_3 \text{ list} \ldots \\ v_1 \text{ ALL} \\ v_1 \text{ ALL BUT } v_2 \text{ list, } v_3 \text{ list} \ldots \end{array}\right\}$$

22. LOADING Command

$$\text{LOADING} \left\{\begin{array}{c} i_1 \\ \text{`}a_1\text{'} \end{array}\right\} (\text{`title'})$$

23. LOADING COMBINATION Command

$$\text{LOADING COMBINATION} \left\{\begin{array}{c} i_1 \\ \text{`}a_1\text{'} \end{array}\right\} (\text{`title'}) \ (\text{COMBINE loading specs})$$

$$\text{loading specs} = \left\{\begin{array}{c} i_2 \\ \text{`}a_2\text{'} \end{array}\right\} v_2 \quad \left( \left\{\begin{array}{c} i_3 \\ \text{`}a_3\text{'} \end{array}\right\} v_3 \right) \ldots\ldots$$

24. JOINT LOADS Command

$$\left\{\begin{array}{l} \text{JOINT} \\ \text{NODE} \end{array}\right\} \text{LOADS} \qquad \left({}^{*} \left\{\begin{array}{ll} \text{FORCE} & [\text{X}]\ v_1,\ [\text{Y}]\ v_2,\ [\text{Z}]\ v_3 \\ \text{MOMENT} & [\text{X}]\ v_1,\ [\text{Y}]\ v_2,\ [\text{Z}]\ v_3 \end{array}\right\} \right)$$

31. MEMBER FORCES Command

MEMBER FORCES (force specs, distance specs, loading specs)

list, (force specs, distance specs, loading specs)

.
.
.

list, (force specs, distance specs, loading specs)

force specs = { FORCE X or FX | FORCE Y or FY | FORCE Z or FZ | MOMENT X or MX | MOMENT Y or MY | MOMENT Z or MZ }

distance specs = { DISTANCE v | FRACTION v }

loading specs = { $v_1 \ldots v_n$ | 'loading$_1$' $v_1 \ldots$ 'loading$_n$' v_n }

32. ACTIVE-INACTIVE Command

{ ACTIVE | INACTIVE } { MEMBERS | JOINTS | LOADINGS } { ALL | ALL BUT list | list }

33. LOADING LIST Command

LOADING LIST { ALL | ALL BUT list | list }

34. ANALYSIS Commands

{ STIFFNESS | DETERMINATE | PRELIMINARY } (ANALYSIS) ([NJP] i) (REDUCE BAND ROOT)

35. COMBINE Command

COMBINE { i | 'a' } (loading specs)

36. PRINT Command

PRINT, type, (component specs)

type = {
- DATA
- STRUCTURAL DATA
- LOADING DATA
- DESIGN (DATA)
- MEMBER { LENGTH / RELEASES / CONSTANTS / INCIDENCES / PROPERTIES / STATUS / END (CONDITIONS) }
- JOINT { STATUS / COORDINATES / RELEASES }
- APPLIED { MEMBER LOADS / JOINT LOADS / JOINT DISPLACEMENTS }
- FORCE ASSUMPTIONS

}

component specs = {
- ALL (active and inactive) (joints and members)
- JOINTS list
- MEMBERS list

}

joints and members = { JOINTS / MEMBERS } (AND { MEMBERS / JOINTS })

active and inactive = { ACTIVE / INACTIVE } (AND { INACTIVE / ACTIVE })

37. OUTPUT Command

OUTPUT {
- DECIMAL i
- BY { LOADING / JOINT / MEMBER }

}

38. LIST Command

LIST * { FORCES / DISTORTIONS / LOADS / REACTIONS / DISPLACEMENTS } (component specs) component specs = { ALL (active and inactive) (joints and members) | MEMBERS list | JOINTS list }

active and inactive = { ACTIVE | INACTIVE } (AND { INACTIVE | ACTIVE })

members and joints = { MEMBERS | JOINTS } (AND { JOINTS | MEMBERS })

39. SECTION Command

SECTION section specs (MEMBER list)

section specs = (FRACTIONAL) { NS i v_1 . . . v_i | DS v_1 v_2 (NS i) }

40. Internal Member Results Command

LIST, output type, { ALL (MEMBERS) | MEMBER list } (SECTION section specs)

output type = { SECTION FORCE | SECTION STRESS | FORCE ENVELOPE | STRESS ENVELOPE | MAXIMUM STRESS | MAXIMUM STRESS EACH LOADING }

41. PLOT DEVICE Command

PLOT DEVICE { → PRINTER | PLOTER | SCOPE } (*{ LENGTH v_1 | WIDTH v_1 | COLUMNS i | ROWS i })

42. PLOT Command

PLOT { PLANE | DIAGRAM | ENVELOPE } (identification specs)

For PLOT PLANE

identification specs = {
(TRUE (VIEW) | { XY | XZ | YZ } PROJECTION { JOINTS id_1, id_2, id_3 | MEMBERS id_1, id_2 })
{ XY | XZ | YZ } THROUGH (JOINT) 'id'
{ X | Y | Z } EQUALS v_1
}

id = { 'a' / i }

For PLOT DIAGRAM and PLOT ENVELOPE

identification specs = { FORCE (* { X / Y / Z }) / MOMENT (* { X / Y / Z }) } { ALL (MEMBERS) / MEMBERS 'list' }

PLOT FORMAT 'format type'

→ NORMAL

ORIENTATION { STANDARD / NON STANDARD }

SIZE (FRACTIONAL) * { HORIZONTAL v_1 / VERTICAL v_1 }

SCALE { EQUAL / MAXIMUM / LENGTH (* { X / Y / Z }) v_1 / FORCE (* { X / Y / Z }) v_1 / MOMENT (* { X / Y / Z }) v_1 } (UNITS PER INCH)

LENGTH (* { X / Y / Z }) v_1

format type = {

SCALE / INTERVAL { FORCE (* { X / Y / Z }) v_1 (UNITS) ; MOMENT (* { X / Y / Z }) v_1 }

POSITION [HORIZONTAL] v_1 [VERTICAL] v_1

POSITION INCREMENT (FRACTIONAL) * { (i) HORIZONTAL v_1 / (i) VERTICAL v_1 }

MARGINS * { ALL v_1 / TOP v_1 / BOTTOM v_1 / LEFT v_1 / RIGHT v_1 }

DATA { →STANDARD / MINIMUM }

SEGMENT (CHARACTER) 'a'

AXIS (CHARACTER) 'a'

}

(43 THROUGH 58 REFER TO DYNAMIC ANALYSIS)

43. INERTIA Command

General form:

INERTIA (OF) ({ JOINTS / NODES }) { { list / ALL / ALL BUT list / ADD list } * { LINEAR / ANGULAR } (* { [X] v_1 / [Y] v_2 / [Z] v_3 } / ALL v_4) ; { LUMPED / CONSISTENT } }

Elements:

$v_1, \ldots, v_4$ = values of inertias expressed as lumped masses. Note that mass, not weight, is input.

44. STORE TIME Command

General form:

STORE TIME (HISTORY) ({ FORCE / DISPLACEMENT / VELOCITY / ACCELERATION } { → TRANSLATION / ROTATION }) 'Name' (IN) –

{ → USER / SUBSYSTEM 'password' } (DATA SET) (FACTOR s) (DUMP)

v_1 t_1 v_2 t_2 . . . v_n t_n

Elements:

'name' = the identifier (up to 8 characters) which is given to the time history record.

'password' = the password for the subsystem data set.

s = the scale factor to be applied to the record prior to storage; i.e. the value of the time history at t_i will be stored as $v_1 \times s$. s is set equal to 1.0 if omitted.

45. STORE RESPONSE Command

General form:

STORE RESPONSE (SPECTRA) { DISPLACEMENT / VELOCITY / ACCELERATION } (VS) { → FREQUENCY / PERIOD } 'Name' (IN) –

{ → USER / SUBSYSTEM 'password' } (DATA SET) (DUMP)

DAMPING { RATIO / PERCENT } v_1 (FACTOR s_1)

r_{11} f_{11} r_{12} f_{12} . . . r_{1n} f_{1n}

.
.
.

DAMPING { RATIO / PERCENT } v_l (FACTOR s_l)

r_{l1} f_{l1} r_{l2} f_{l2} . . . r_{lm} f_{lm}

(END (OF RESPONSE SPECTRA))

46. DYNAMIC LOADING Command

General form:

DYNAMIC LOADING { i_1 / 'a' } ('title')

47. JOINTS LOAD Command

General form:

{ JOINTS / NODES } list (LOADS) { FORCE / MOMENT } { X / Y / Z } (load specs.)

where

(load specs.) = { FILE 'Name' ([FACTOR] v_1) / (FACTOR v_1) (time history) / (function specs.) }

(time history) = f_1 f_2 . . . f_n

(function specs) =

FUNCTION { SINE / COSINE } [AMPLITUDE] v_2 [FREQUENCY] v_3 ([PHASE] v_4)

48. SUPPORT ACCELERATION Command

General form:

SUPPORT (i_1) (ACCELERATIONS)

{ { TRANSLATION / DISPLACEMENT } / ROTATION } { X / Y / Z } (load specs.)

.
.
.

{ { TRANSLATION / DISPLACEMENT } / ROTATION } { X / Y / Z } (load specs.)

where

(load specs) = same as for JOINT LOAD command.

49. TIME POINTS Command

General form:

TIME (POINTS) t_1 t_2 . . . t_n

50. END OF DYNAMIC LOADING Command

END (OF DYNAMIC LOADING)

51. RAYLEIGH Commands

a) RAYLEIGH LOADING { i_1 / 'a_1' } ('title')

b) LIST RAYLEIGH (NATURAL FREQUENCY)

52. DYNAMIC DEGREES OF FREEDOM Command

DYNAMIC DEGREES (OF FREEDOM)

{ JOINTS / NODES } list *{ DISPLACEMENTS / ROTATIONS } *{ X / Y / Z }

.
.
.

{ JOINTS / NODES } list *{ DISPLACEMENTS / ROTATIONS } *{ X / Y / Z }

53. DYNAMIC ANALYSIS Command

DYNAMIC ANALYSIS { →TRIDIAGONALIZATION / ITERATION } ([NJP] i_1) (solution specs.) (REDUCE BAND ROOT)

where

(solution specs.) = { { EIGENVALUE (ONLY) / MODAL } (frequency specs.) / PHYSICAL ([BETA] v_1) }

(frequency specs.) = { (MINIMUM (FREQUENCY) v_2) MAXIMUM (FREQUENCY) v_3 (i_2) / CRITICAL (FREQUENCY) v_4 (i_3) }

54. PRINT DYNAMIC Command

General form:

PRINT DYNAMIC (type specs.) (component specs.)

where

(type specs.) = *{
DATA
STRUCTURAL (DATA)
LOADING DATA
NORMAL (MODES)
DEGREES (OF FREEDOM)
JOINT (INERTIAS)
MATRICES
LOADS
SUPPORT (ACCELERATIONS)
INITIAL (CONDITIONS)
INTEGRATION (PERIODS)
}

(component specs.) = {
ALL (active and inactive) (joints and members)
JOINTS list
MEMBERS list
}

(active and inactive) = ({ ACTIVE | INACTIVE } (AND { INACTIVE | ACTIVE }))

(joints and members) = ({ JOINTS | MEMBERS } (AND { MEMBERS | JOINTS }))

55. NORMALIZE EIGENVECTORS Command

General form:

NORMALIZE EIGENVECTORS

56. LIST DYNAMIC Command

General form:

LIST DYNAMIC (type specs.) (BY (TIME)) (component specs.)

(type specs.) = * { (MAXIMUM) { FORCES | DISTORTIONS | LOADS | REACTIONS | DISPLACEMENTS | STRESSES / PRINCIPAL { STRAINS | STRESSES } | VELOCITIES | ACCELERATION | EIGENVALUES (i_1) | EIGENVECTORS (i_2) } }

component specs.) = ({ ALL (active and inactive) (joints and members) | JOINTS list | MEMBERS list })

(joints and members) = ({ JOINTS | MEMBERS } (AND { MEMBERS | JOINTS }))

(active and inactive) = ({ ACTIVE | INACTIVE } (AND { INACTIVE | ACTIVE }))

57. INTEGRATE Command

General form:

INTEGRATE (FROM) t_{11} (TO) t_{12} (AT) t_{13}
(FROM) t_{n1} (TO) t_{n2} (AT) t_{n3}

58. DAMPING Command

General Form:

$$\underline{\text{DAM}}\text{PING} \left\{ \begin{array}{l} \rightarrow \underline{\text{RAT}}\text{IOS} \\ \underline{\text{PERC}}\text{ENTS} \end{array} \right\} v_1\ (i_1)\ \ldots\ldots\ v_n\ (i_n)$$

59. PARAMETER Command

$\underline{\text{PAR}}\text{AMETER}$ ('parameter') {parameter specs}

'parameter' {parameter specs}

. .

. .

'parameter' {parameter specs}

Elements:

'parameter' = alphameric parameter name (up to 8 characters)

$$\text{parameter specs} = \left\{ \begin{array}{l} v_1\ (\underline{\text{FOR}})\ (\underline{\text{MEM}}\text{BERS})\ \text{list} \\ v_1\ \underline{\text{ALL}} \\ v_1\ \underline{\text{ALL}}\ \underline{\text{BUT}}\ v_2\ (\underline{\text{FOR}})\ (\underline{\text{MEM}}\text{BERS})\ \text{list} \end{array} \right\}$$

60. CHECK CODE Command

CHECK (CODE $(\underline{\text{FOR}})$) $\underline{\text{MEM}}\text{BERS}$ list

Index

NOTES

NOTES

Design of Structures and Foundations for Vibrating Machines

This text brings together traditional and new concepts and procedures for analyzing and designing dynamically loaded structures. With *Design of Structures and Foundations for Vibrating Machines,* practicing engineers and students now have a text which integrates theories of vibration, geotechnical engineering (including soil dynamics and half-space theory), computer coding and applications, and structural analysis and design. The many concepts and procedures used in the design of structures supporting dynamic machines and ultimately supported by the soil until now have been unavailable in a single source.

The design process in this field has gradually evolved from an approximate rule-of-thumb procedure to a scientifically sound procedure. In *Design of Structures and Foundations for Vibrating Machines,* state-of-the-art techniques are employed in actual design problems by using simplified step-by-step routines. In addition, at every step of investigation a brief description explains the physical meaning of the parameters used and the role they play in the design process.

The introductory chapter reviews fundamentals. Chapter 2 describes alternatives of modeling dynamically loaded systems. Chapter 3 considers and lists the information necessary for design. Chapters 4 and 5 describe the geotechnical aspects of the problem, and Chapter 5 specifically considers flexible mats and foundations. Finally, Chapters 6 and 7 include examples of different types of structures supporting dynamic machines.

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