Gamma Function and Its Usage in Probability

Definition

The Gamma function extends the factorial function to real numbers. It is defined as:

$\Gamma(n) = \int^{\infty}_{0} x^{n-1} e^{-x} dx$

For positive integers:

$\Gamma(n) = (n - 1)!$

Example:
$\Gamma(5) = 24$

Usage in Probability

The Gamma function mainly appears as a **normalization constant** in continuous probability distributions.

1. Gamma Distribution

$f(x) = (1 / (\Gamma(k)\ \theta^{k}))\ x^{k-1} e^{-x/\theta}$

Used to model:

- Waiting times
- Lifetime and reliability analysis

2. Beta Distribution

$f(x) = (\Gamma(\alpha + \beta) / (\Gamma(\alpha)\ \Gamma(\beta)))\ x^{(\alpha - 1)}(1 - x)^{(\beta - 1)}$

Used in:

- Bayesian inference
- Modeling probabilities between 0 and 1

3. Chi-Squared and t-Distributions

Both are defined using the Gamma function and are widely used in hypothesis testing.

Hessian Matrix and Its Usage in AI

Definition

The Hessian matrix is the matrix of second-order partial derivatives of a function.

For a function $f(x_1, x_2, \ldots, x_n)$:

$H[i, j] = \partial^2 f / (\partial x_i \, \partial x_j)$

It describes the **curvature** of the function.

Usage in Artificial Intelligence

1. Optimization

AI models minimize a loss function $L(\theta)$.

First-order method (Gradient Descent):
$\theta = \theta - \eta \, \nabla L$

Second-order method (using Hessian):
$\theta = \theta - H^{-1} \, \nabla L$

Using the Hessian can improve convergence speed.

2. Critical Point Detection

Hessian eigenvalues determine:

- All positive → Minimum
- All negative → Maximum
- Mixed signs → Saddle point

Conclusion

- The Gamma function is essential in probability for defining and normalizing distributions.
- The Hessian matrix is important in AI for understanding loss function curvature and improving optimization methods.