Signal & Systems

Equivalent pure tone sinusoidal signals.

3 ways to represent:

Trig. form w/ magnitude & phase information stored in the coeffecient

(I) $a_n \cos(\omega_0 \cdot n \cdot t) + b_n \sin(\omega_0 \cdot n \cdot t)$

where "n" is the harmonic of the fundamental frequency.

Note: in order to plot a real signal the amplitude must be real. If the signal has a complex number for the amplitude you must find magnitude info of complex number and plot WRT freq. and then plot phase info WRT to freq.

Some do not like this form due to looking @ 2 trig functions to find info for the same harmonic frequency. The other form is:

(II) $A_n \cos(\omega_0 \cdot n \cdot t + \theta_n)$

or you can look @ as

$A_n \sin(\omega_0 n t + \theta_n)$

but not both @ the same time.

They are equivalent.

Where A_n is only magnitude info for a harmonic & θ_n is only phase info for a harmonic.

Some even prefer the complex exponential form:

(III) $C_n e^{j\omega_0 \cdot n \cdot t}$

here the first 2 forms are combined.

phase & magnitude info are combined in the coefficient and the info is found @ one term for each harmonic.

You need to understand all of these are equivalent representations of each other.

$f_n(t) = a_n \cos(\omega_0 \cdot n \cdot t) + b_n \cdot \sin(\omega_0 n t)$

or

$f_n(t) = A_n \cos(\omega_0 n t + \theta_n)$

or

$f_n(t) = C_n e^{j\omega_0 n t}$

if these are equivalent we must be able to go from one to the other.

Proof of eqn's to go from: (I) ⇄ (II)

Using trig identities we know

$\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)$

$\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta)$

Notice in (I) there is no phase shift, like in (II)

∴ $A_n \cos(\omega_0 n t + \theta_n)$

is equivalent to

$A_n(\cos(\omega_0 n t)\cos(\theta_n) - \sin(\omega_0 n t)\sin(\theta_n))$

↳ $\underbrace{[A_n \cdot \cos(\theta_n)]}_{b_n} \cos(\omega_0 n t) + \underbrace{[-A_n \sin(\theta_n)]}_{a_n} \sin(\omega_0 n t)$

leave the negative sign so that to go from (II) → (I) we get

$b_n = A_n \cos(\theta_n)$
$a_n = -A_n \sin(\theta_n)$
} be mindful of you θ_n

now to derive (I) → (II)

$$(b_n)^2 + (a_n)^2 = A_n^2 \cos^2(\theta_n) + A_n^2 \sin^2(\theta_n) = A_n^2[\cos^2(\theta_n) + \sin^2(\theta_n)] = A_n^2$$

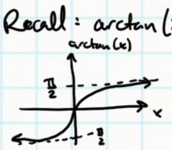
∴ $A_n = \sqrt{(b_n)^2 + (a_n)^2}$

we also see

$$\frac{b_n}{a_n} = \frac{-A_n \sin(\theta_n)}{A_n \cos(\theta_n)} = -\tan(\theta_n)$$

$$\theta_n = \arctan\left(\frac{-b_n}{a_n}\right) = -\arctan\left(\frac{b_n}{a_n}\right)$$

Recall: arctan(x) is an odd function



Note: we must be careful to find the right phase. When a_n changes sign this will shift the phase by π

∴
$$\theta_n = \begin{cases} -\arctan\left(\frac{b_n}{a_n}\right), & \text{when } a_n > 0 \\ \pi - \arctan\left(\frac{b_n}{a_n}\right), & \text{when } a_n < 0 \end{cases}$$

To go to complex form and back Euler's representation is used in different forms.

$e^{j\theta} = \cos(\theta) + j\sin(\theta)$ & $e^{-j\theta} = \cos(\theta) - j\sin(\theta)$

$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2}$ $\sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{j2}$

To go from (I) → (III)

$a_n \cos(\omega_0 n t) + b_n \sin(\omega_0 n t)$

$$a_n\left(\frac{e^{j\omega_0 n t}}{2} + \frac{e^{-j\omega_0 n t}}{2}\right) + b_n\left(\frac{e^{j\omega_0 n t}}{j2} - \frac{e^{-j\omega_0 n t}}{j2}\right)$$

$$\left(\frac{a_n}{2} - j\frac{b_n}{2}\right)e^{j\omega_0 n t} + \left(\frac{a_n}{2} + j\frac{b_n}{2}\right)e^{-j\omega_0 n t}$$

∴ $C_n = \left(\frac{a_n}{2} - j\frac{b_n}{2}\right)$ $C_{-n} = \left(\frac{a_n}{2} + j\frac{b_n}{2}\right)$

we see: $C_n = C_{-n}^*$ where "*" is the complex conjugate.

To go from (III) → (I)

∴ $a_n = (C_n + C_{-n})$

$b_n = j(C_n - C_{-n})$

(II) → (III) we use Euler representation

$$A_n \cos(\omega_0 n t + \theta_n) \to \frac{A_n}{2}\left(e^{j(\omega_0 n t + \theta_n)} + e^{-j(\omega_0 n t + \theta_n)}\right)$$

↳ $\left[\frac{A_n}{2} e^{j\theta_n}\right] e^{j\omega_0 n t} + \left[\frac{A_n}{2} e^{-j\theta_n}\right] e^{-j\omega_0 n t}$

∴ $C_n = \left[\frac{A_n}{2} e^{j\theta_n}\right]$ & $C_{-n} = \left[\frac{A_n}{2} e^{-j\theta_n}\right]$

---

from (III) → (II)

$$C_n e^{j\omega_0 n t} + C_{-n} e^{-j\omega_0 n t} \to A_n \cos(\omega_0 n t + \theta_n)$$

$A_n = |C_n| + |C_{-n}|$

where "| |" is the magnitude of a complex number.

ex: $z = \alpha + j\beta$ ∴ $|z| = \sqrt{\alpha^2 + \beta^2}$

$\theta_n = \angle C_n = -\angle C_{-n}$

where "∠" is the phase of some complex number.

ex: $z = \alpha + j\beta$

then

$$\angle z = \begin{cases} \arctan\left(\frac{\beta}{\alpha}\right), & \text{if } \beta \text{ is "+" or "-"}, \alpha \text{ is "+"} \\ \pi - \arctan\left(\frac{\beta}{\alpha}\right), & \text{if } \beta \text{ is "+"}, \alpha \text{ is "-"} \\ -\pi + \arctan\left(\frac{\beta}{\alpha}\right), & \text{if } \beta \text{ is "-"}, \alpha \text{ is "-"} \end{cases}$$

Equation Sheet

3 forms

I $a_n \cos(\omega_0 n t) + b_n \sin(\omega_0 n t)$

II $A_n \cos(\omega_0 n t + \theta_n)$

III $C_n e^{j\omega_0 n t}$

I → II

$$A_n = \sqrt{a_n^2 + b_n^2}$$

$$\theta_n = \begin{cases} -\arctan\left(\frac{b_n}{a_n}\right) & \text{if } a_n > 0 \text{ regardless of } b_n \text{ sign} \\ \pi - \arctan\left(\frac{b_n}{a_n}\right) & \text{if } a_n < 0 \text{ regardless of } b_n \text{ sign} \end{cases}$$

I → III

$$C_n = \left(\frac{a_n}{2} - j\frac{b_n}{2}\right) \quad \& \quad C_{-n} = \left(\frac{a_n}{2} + j\frac{b_n}{2}\right)$$

in complex form "n" harmonics are both "+" and "-"
but in trig form it's just "n"
no negative a_{-n} or b_{-n}
just a_n & b_n is how we defined it.

II → I

$$a_n = A_n \cos(\theta_n)$$

$$b_n = -A_n \sin(\theta_n) \text{ or } A_n \sin(-\theta_n)$$

Recall if you are given
a $A_n \sin(\omega_0 n t + \phi_n)$
this is the same as

$$A_n \cos(\omega_0 n t + (\phi_n - \tfrac{\pi}{2}))$$

and θ_n in this case is

$\theta_n = \phi_n - \frac{\pi}{2}$ & A_n is still the same

II → III

$$C_n = A_n e^{j\theta_n} = A_n[\cos(\theta_n) + j\sin(\theta_n)]$$

$$C_{-n} = A_n e^{-j\theta_n} = A_n[\cos(\theta_n) - j\sin(\theta_n)]$$

III → I

$$a_n = [C_n + C_{-n}]$$

$$b_n = j[C_n - C_{-n}]$$

III → II

$$A_n = |C_n| + |C_{-n}|$$

$$\theta_n = \angle C_n = -\angle C_{-n}$$