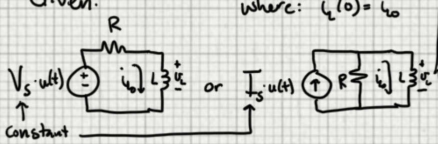
The following are equations for simple RL, RC, & RLC ckts. They take into account initial conditions. These are for DC driven ckts that have a turn on time of $t=0$. Therefore, $t \geq 0$ for $v_c(t)$ & $i_L(t)$.

Given: where: $i_L(0) = i_{L0}$



we know the O.D.E. is

$$i_L' + \frac{R}{L} i_L = \frac{R}{L} I_s \qquad \text{where } I_s = \frac{V_s}{R}$$

The Homogeneous Solution:

$$i_{LH} = K_1 e^{s_1 t}$$

where the characteristic eqn is:

$$s + \frac{R}{L} = 0 \rightarrow s = -\frac{R}{L}$$

$$\therefore i_{LH} = K_1 e^{-\frac{R}{L}t}$$

The Particular Solution:

i_{Lp} will be of the form of the driving function. Since the driving function is constant then so will i_{Lp}.

$$i_{Lp} = A \quad \& \quad i_{Lp}' = 0$$

Plugging back into equation

$$0 + \frac{R}{L} \cdot A = \frac{R}{L} I_s \quad \therefore A = I_s$$

General Solution:

$$i_L(t) = K_1 e^{-\frac{R}{L}t} + I_s$$

using initial condition:

$$i_L(0) = i_{L0} = K_1 + I_s$$

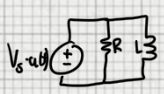
$$\hookrightarrow K_1 = (i_{L0} - I_s)$$

$$\therefore \boxed{i_L(t) = (i_{L0} - I_s) e^{-\frac{R}{L}t} + I_s} \quad \text{for } t \geq 0$$

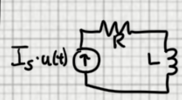
given $v_L(t) = L i_L'(t)$

$$\boxed{v_L(t) = -R(i_{L0} - I_s) e^{-\frac{R}{L}t}}$$

Remember you don't want:

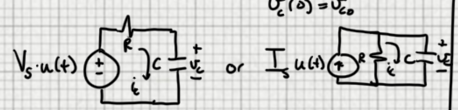


because @ steady state the inductor becomes a wire and the current goes to infinity



because @ $t=0$ the current of the inductor is being forced to instantaneously go from 0 to some value. The voltage across it would have to be infinite.

Given: where $v_c(0) = v_{c0}$



we know the O.D.E.

$$v_c' + \frac{1}{RC} v_c = \frac{V_s}{RC} \qquad \text{where } V_s = R \cdot I_s$$

Homogeneous Solution:

$$v_{cH} = K_1 e^{-\frac{t}{RC}}$$

Particular Solution:

$$v_{cp} = V_s$$

General Solution

$$v_c(t) = K_1 e^{-\frac{t}{RC}} + V_s$$

using i.c.

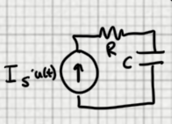
$$v_c(0) = v_{c0} = K_1 + V_s \rightarrow K_1 = (v_{c0} - V_s)$$

$$\therefore \boxed{v_c(t) = (v_{c0} - V_s) e^{-\frac{t}{RC}} + V_s} \quad \text{for } t \geq 0$$

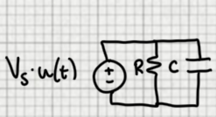
since $i_c(t) = C v_c'(t)$

$$\boxed{i_c(t) = -\frac{(v_{c0} - V_s)}{R} e^{-\frac{t}{RC}}}$$

Remember you Don't want:

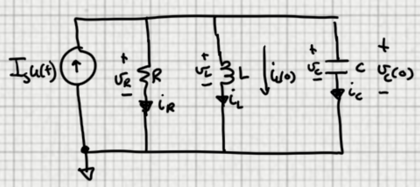


because the current source will always be charging up the capacitor and the voltage across it will go to infinity



because @ $t=0$ the voltage across the capacitor jumps instantaneously from 0 to some value. The current going to the capacitor would have to infinite.

RLC parallel
ckt analysis of DC
current input w/ initial
Conditions.



Given for $t \geq 0$,
Driving function is DC current
w/ value of "I_s"
and $i_L(0) = i_{L_0}$ & $v_C(0) = v_{C_0}$

We know the O.D.E. is

$$i_L'' + \underbrace{\frac{1}{RC}}_{2\sigma} i_L' + \underbrace{\frac{1}{LC}}_{\omega_n^2} i_L = \frac{I_s}{LC}$$

we also know the particular solution is set by the driving function, so its form does not change w/ R, L, or C

Solving for Particular solution we find i_{Lp} is of the same form as the driving function; constant.

$\therefore \; i_{Lp} = A \quad \& \quad i_{Lp}' = i_{Lp}'' = 0$

$\therefore \; 0 + \frac{1}{RC} \cdot 0 + \frac{1}{LC} \cdot A = \frac{I_s}{LC} \rightarrow A = I_s$

$\therefore \; i_{Lp} = I_s$

Now solving for the 3 cases of the homogeneous solution. Before starting let's define some terms.

σ, damping coefficient, $\left(\frac{1}{s}\right)$
ω_n, natural frequency, $\left(\frac{rad}{s}\right)$
ω_d, damped frequency, $\left(\frac{rad}{s}\right)$
ζ, neper frequency, (unitless)

---

general forms of Homogeneous solutions

1) overdamped when $\sigma^2 > \omega_n^2$

$$i_{LH1} = K_1 e^{-(\sigma - \sqrt{\sigma^2 - \omega_n^2})t} + K_2 e^{-(\sigma + \sqrt{\sigma^2 - \omega_n^2})t}$$

ω_n

2) critically damped when $\sigma^2 = \omega_n^2$

$$i_{LH2} = (K_1 + K_2 t) e^{-\sigma t}$$

3) underdamped when $\sigma^2 < \omega_n^2$

$$i_{LH3} = e^{-\sigma t}\left[K_1 \cos(\omega_d t) + K_2 \sin(\omega_d t)\right]$$

ω_d

what is ω_d, damped frequency?
it is an oscillation of the system that oscillates slower than the natural frequency.
The form is given by:

$$\omega_d = \omega_n \sqrt{1 - \frac{\sigma^2}{\omega_n^2}}$$

since the neper ratio is

$$\zeta = \frac{\sigma}{\omega_n}$$

$$\omega_d = \omega_n \sqrt{1 - \zeta^2}$$

note: ζ must be less than 1

Therefore, the 3 general Solutions are:

1) $K_1 e^{-\overbrace{(\sigma - \sqrt{\sigma^2 - \omega_n^2})}^{\Delta} t} + K_2 e^{-\overbrace{(\sigma + \sqrt{\sigma^2 - \omega_n^2})}^{\bigstar} t} + I_s$

2) $e^{-\sigma t}(K_1 + K_2 t) + I_s$

3) $e^{-\sigma t}\left[K_1 \cos(\omega_d t) + K_2 \sin(\omega_d t)\right]$

Let's solve using i.c.

1) $i_{L1}(0) = i_{L_0} = K_1 + K_2 + I_s \rightarrow K_1 = i_{L_0} - (K_2 + I_s)$

since $v_C = v_L = L i_L'$

$v_{C1}(0) = v_{C_0} = L\left[K_1(-\Delta) + K_2(-\bigstar)\right]$

$\therefore \; \frac{v_{C_0}}{L} = (-i_{L_0} + K_2 + I_s)\Delta - K_2 \bigstar$

$\therefore \; K_2(\Delta - \bigstar) = \frac{v_{C_0}}{L} + \Delta(i_{L_0} - I_s)$

$\hookrightarrow K_2 = \frac{v_{C_0} + \Delta L(i_{L_0} - I_s)}{L(\Delta - \bigstar)}$

$K_1 = (i_{L_0} - I_s) - \frac{v_{C_0} + \Delta L(i_{L_0} - I_s)}{L(\Delta - \bigstar)}$

$K_1 = \frac{\cancel{\Delta L(i_{L_0} - I_s)} - \bigstar L(i_{L_0} - I_s) - v_{C_0} - \cancel{\Delta L(i_{L_0} - I_s)}}{L(\Delta - \bigstar)}$

$K_1 = \frac{-(v_{C_0} + \bigstar L(i_{L_0} - I_s))}{L(\Delta - \bigstar)}$

$\therefore$ for case 1:

$$i_{L1}(t) = \frac{-(v_{C_0} + \bigstar L(i_{L_0} - I_s))}{L(\Delta - \bigstar)} \cdot e^{-\Delta t} + \frac{(v_{C_0} + \Delta L(i_{L_0} - I_s))}{L(\Delta - \bigstar)} e^{-\bigstar t} + I_s$$

where

$\Delta = \sigma - \sqrt{\sigma^2 - \omega_n^2}$ & $\bigstar = \sigma + \sqrt{\sigma^2 - \omega_n^2}$

$\sigma = \frac{1}{2RC} \quad \omega_n^2 = \frac{1}{LC}$

For Case 2, critically damped

$\sigma^2 = \omega_n^2$

then: $i_{L2} = e^{-\sigma t}(K_1 + K_2 t) + I_s$

$\therefore \; i_{L2}(0) = i_{L_0} = K_1 + I_s$

$K_1 = i_{L_0} - I_s$

again $v_C = v_L = L i_L'$

$\frac{v_{C2}(0)}{L} = \frac{v_{C_0}}{L} = -\sigma K_1 + K_2$

$K_2 = \frac{v_{C_0}}{L} + \sigma(i_{L_0} - I_s)$

$\therefore$ for case 2 $\quad \sigma^2 = \omega_n^2$
$\left(\frac{1}{2RC}\right)^2 = \frac{1}{LC}$

$$i_{L2}(t) = e^{-\sigma t}\left[i_{L_0} - I_s + t\left(\frac{v_{C_0} + \sigma L(i_{L_0} - I_s)}{L}\right)\right] + I_s$$

where $\sigma = \frac{1}{2RC}$

For case 3, underdamped

$i_{L3}(t) = e^{-\sigma t}\left[K_1 \cos(\omega_d t) + K_2 \sin(\omega_d t)\right] + I_s$

$i_{L3}(0) = i_{L_0} = K_1 + I_s \rightarrow K_1 = i_{L_0} - I_s$

$\frac{v_{C3}(0)}{L} = \frac{v_{C_0}}{L} = -\sigma K_1 + K_2 \omega_d$

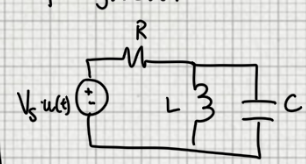
$\therefore \; K_2 = \frac{v_{C_0} + \sigma L(i_{L_0} - I_s)}{\omega_d L}$

$\therefore$ for case 3 $\quad \sigma^2 < \omega_n^2$
$\left(\frac{1}{2RC}\right)^2 < \frac{1}{LC}$

$$i_{L3}(t) = e^{-\sigma t}\Big[(i_{L_0} - I_s)\cos(\omega_d t) + \ldots$$
$$\ldots \left(\frac{v_{C_0} + \sigma L(i_{L_0} - I_s)}{\omega_d L}\right)\sin(\omega_d t)\Big] + I_s$$

where $\sigma = \frac{1}{2RC} \quad \omega_d = \frac{1}{\sqrt{LC}}\sqrt{1 - \frac{\left(\frac{1}{2RC}\right)^2}{\frac{1}{LC}}}$

If given:

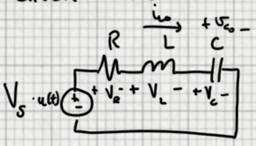


The same equations apply. The only consideration is the source transform where

$$I_s = \frac{V_s}{R}$$

just replace and go

Given the ckt:



where:

$i_L(0) = i_{L_o}$

$v_C(0) = v_{C_o}$

V_s is DC

we know the ODE is:

$$v_C'' + \underbrace{\frac{R}{L}}_{2\sigma} v_C' + \underbrace{\frac{1}{LC}}_{\omega_n^2} v_C = \frac{V_s}{LC}$$

we know the particular solution must be of the same form as the driving function:

$\therefore v_{Cp} = A$ & $v_{Cp}' = v_{Cp}'' = 0$

plugging back in we get:

$$0 + \frac{R}{L}\cdot 0 + \frac{A}{LC} = \frac{V_s}{LC} \quad \therefore A = V_s$$

$\therefore v_{Cp} = V_s$

Homogeneous solution has 3 cases:

1) $\sigma^2 > \omega_n^2$

$$\therefore v_{CH1} = K_1 e^{-\underbrace{(\sigma - \sqrt{\sigma^2-\omega_n^2})}_{\Delta}t} + K_2 e^{-\underbrace{(\sigma + \sqrt{\sigma^2-\omega_n^2})}_{\bigstar}t}$$

2) $\sigma^2 = \omega_n^2$

$$\therefore v_{CH2} = e^{-\sigma t}(K_1 + K_2 t)$$

3) $\sigma^2 < \omega_n^2$

$$\therefore v_{CH3} = e^{-\sigma t}(K_1 \cos(\omega_d t) + K_2 \sin(\omega_d t))$$

where:

σ = damping coeffecient $= \frac{1}{2RC}$

ω_n = natural frequency $= \frac{1}{LC}$

ω_d = damped frequency $= \omega_n\sqrt{1-\zeta^2}$

ζ = neper frequency $= \frac{\sigma}{\omega_n}$

↳ where $0 < \frac{\sigma}{\omega_n} < 1$

$\therefore \omega_d = \frac{1}{\sqrt{LC}}\sqrt{1 - \frac{L}{4R^2C}}$

Recall we just set:

$\Delta = \sigma - \sqrt{\sigma^2 - \omega_n^2}$ & $\bigstar = \sigma + \sqrt{\sigma^2-\omega_n^2}$

Case 1 solution

$$v_C(t) = K_1 e^{-\Delta t} + K_2 e^{-\bigstar t} + V_s$$

$v_C(0) = v_{C_o} = K_1 + K_2 + V_s \rightarrow K_1 = v_{C_o} - (K_2 + V_s)$

since $i_L = i_C = C v_C'$ | $\therefore K_1 = v_C -$

$$\frac{i_L(t)}{C} = -K_1 \Delta e^{-\Delta t} - K_2 \bigstar e^{-\bigstar t}$$

$$\frac{i_L(0)}{C} = \frac{i_{L_o}}{C} = -K_1\Delta - K_2\bigstar$$

$$\therefore \frac{i_{L_o}}{C} = -(v_{C_o} - (K_2 + V_s))\Delta - \bigstar K_2$$

$$K_2 = \left(\frac{i_{L_o}}{C} + (v_C - V_s)\Delta\right) / (\Delta - \bigstar)$$

$$\therefore K_2 = \frac{\frac{i_{L_o}}{C} + (v_{C_o} - V_s)\Delta}{(\Delta - \bigstar)}$$

$$K_2 = \frac{i_{L_o} + \Delta C(v_{C_o} - V_s)}{C(\Delta - \bigstar)}$$

$$K_1 = (v_{C_o} - V_s) - \left(\frac{i_{L_o} + \Delta C(v_{C_o} - V_s)}{C(\Delta - \bigstar)}\right)$$

$$K_1 = \frac{\cancel{\Delta C(v_{C_o} - V_s)} - \bigstar C(v_C - V_s) - i_{L_o} - \cancel{\Delta C(v_{C_o} - V_s)}}{C(\Delta - \bigstar)}$$

$$K_1 = -\frac{(i_{L_o} + \bigstar C(v_C - V_s))}{C(\Delta - \bigstar)}$$

$\therefore$ For Case 1 $\sigma^2 > \omega_n^2$

$$v_C(t) = -\frac{(i_{L_o} + \bigstar C(v_{C_o} - V_s))}{C(\Delta - \bigstar)} e^{-\Delta t} + \ldots$$

$$\ldots \frac{(i_{L_o} + \Delta C(v_{C_o} - V_s))}{C(\Delta - \bigstar)} e^{-\bigstar t} + V_s$$

For Case 2 $\sigma^2 = \omega_n^2$

$$v_{C2}(t) = e^{-\sigma t}(K_1 + K_2 t) + V_s$$

$v_{C2}(0) = v_{C_o} = K_1 + V_s \rightarrow K_1 = v_{C_o} - V_s$

using $i_L = i_C = C v_C'$

$$\frac{i_L(t)}{C} = -\sigma e^{-\sigma t}(K_1 + K_2 t) + e^{-\sigma t}(K_2)$$

$$\frac{i_L(0)}{C} = \frac{i_{L_o}}{C} = -\sigma K_1 + K_2$$

$$\therefore K_2 = \frac{i_{L_o}}{C} + \sigma(v_{C_o} - V_s)$$

$\therefore$ For Case 2 $\sigma^2 = \omega_n^2$

$$v_C(t) = e^{-\sigma t}\left[(v_{C_o} - V_s) + \left(\frac{i_{L_o}}{C} + \sigma(v_{C_o} - V_s)\right)t\right]$$

For Case 3 $\sigma^2 < \omega_n^2$

$$v_C(t) = e^{-\sigma t}\left[K_1\cos(\omega_d t) + K_2 \sin(\omega_d t)\right] + V_s$$

$v_C(0) = v_{C_o} = K_1 + V_s \rightarrow K_1 = v_{C_o} - V_s$

&

$$\frac{i_L(t)}{C} = e^{-\sigma t}\left[-K_1\omega_d \sin(\omega_d t) + K_2\omega_d\cos(\omega_d t)\right]\ldots$$

$$\ldots - \sigma e^{-\sigma t}\left[K_1\cos(\omega_d t) + K_2 \sin(\omega_d t)\right]$$

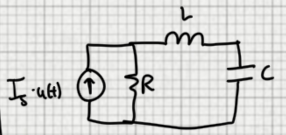
$$\frac{i_L(0)}{C} = \frac{i_{L_o}}{C} = K_2\omega_d - \sigma K_1$$

$$\therefore K_2 = \frac{i_{L_o}}{\omega_d C} + \frac{\sigma(v_{C_o} - V_s)}{\omega_d}$$

$\therefore$ For Case 3 $\sigma^2 < \omega_n^2$

$$v_C(t) = e^{-\sigma t}\left[(v_{C_o} - V_s)\cos(\omega_d t) + \left(\frac{i_{L_o} + \sigma C(v_{C_o} - V_s)}{\omega_d C}\right)\sin(\omega_d t)\right] + V_s$$

Note:
the same equations apply for the ckt:

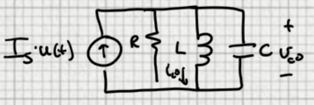


The source transformation yields

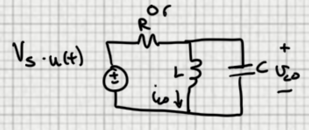
$V_s = I_s \cdot R$

Quick Reference

Given



or



The O.D.E. is:

$$i_L'' + \underbrace{\frac{1}{RC}}_{2\sigma} i_L' + \underbrace{\frac{1}{LC}}_{\omega_n^2} i_L = \frac{I_S}{LC} = \frac{V_S}{RLC}$$

$$\sigma = \frac{1}{2RC} \qquad \omega_n = \frac{1}{\sqrt{LC}} \qquad \omega_d = \frac{1}{\sqrt{LC}}\sqrt{1 - \frac{L}{4R^2C}}$$

where: $0 < \frac{L}{4R^2C} < 1$

$$\bigstar - \Delta = 2\sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

$$\Delta = \frac{1}{2RC} - \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}} \qquad \bigstar = \frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}} \qquad \bigstar \cdot \Delta = \frac{1}{LC}$$

Case 1 $\sigma^2 > \omega_n^2$

$$i_L(t) = \frac{v_{C0} + \bigstar L(i_{L0} - I_S)}{L(\bigstar - \Delta)} \cdot e^{-\Delta t} - \frac{v_{C0} + \Delta L(i_{L0} - I_S)}{L(\bigstar - \Delta)} \cdot e^{-\bigstar t} + I_S$$

$$v_C(t) = -\left(\frac{\Delta v_{C0} + \frac{1}{C}(i_{L0} - I_S)}{(\bigstar - \Delta)}\right) e^{-\Delta t} + \left(\frac{\bigstar v_{C0} + \frac{1}{C}(i_{L0} - I_S)}{(\bigstar - \Delta)}\right) e^{-\bigstar t}$$

Case 2 $\sigma^2 = \omega_n^2$

$$i_L(t) = e^{-\frac{t}{2RC}}\left[(i_{L0} - I_S) + \left(\frac{v_{C0}}{L} + \frac{1}{2RC}(i_{L0} - I_S)\right) \cdot t\right] + I_S$$

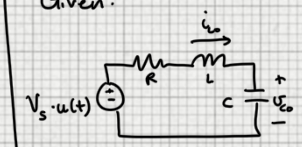
$$v_C(t) = \frac{L}{2RC} e^{-\frac{t}{2RC}}\left[\frac{2RC}{L} \cdot v_{C0} - \left(\frac{v_{C0}}{L} + \frac{1}{2RC}(i_{L0} - I_S)\right) \cdot t\right]$$

Case 3 $\sigma^2 < \omega_n^2$

$$i_L(t) = e^{-\frac{t}{2RC}}\left[(i_{L0} - I_S)\cos(\omega_d \cdot t) + \left(\frac{v_{C0}}{\omega_d \cdot L} + \frac{(i_{L0} - I_S)}{2\omega_d RC}\right)\sin(\omega_d \cdot t)\right] + I_S$$

$$v_C(t) = e^{-\frac{t}{2RC}}\left[v_{C0}\cos(\omega_d t) - \frac{L}{2RC}\left(\frac{v_{C0}}{\omega_d L} + \frac{(i_{L0} - I_S)}{2\omega_d RC} + 2\omega_d RC(i_{L0} - I_S)\right)\sin(\omega_d t)\right]$$

Given:



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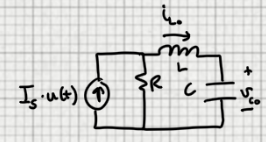


The O.D.E. is:

$$v_C'' + \underbrace{\frac{R}{L}}_{2\sigma} v_C' + \underbrace{\frac{1}{LC}}_{\omega_n^2} v_C = \frac{V_S}{LC} = \frac{R}{LC} \cdot I_S$$

$\therefore$

$$\sigma = \frac{R}{2L} \qquad \omega_n = \frac{1}{\sqrt{LC}} \ \therefore\ \omega_n^2 = \frac{1}{LC}$$

$$\zeta = \frac{\sqrt{LC} \cdot R}{2L} \ \therefore\ \zeta^2 = \frac{R^2C}{4L} \rightarrow \omega_d = \frac{1}{\sqrt{LC}}\sqrt{1 - \frac{R^2C}{4L}}$$

where: $0 < \frac{R^2C}{4L} < 1$

$$\Delta = \frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} \qquad \bigstar = \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$$\Delta \cdot \bigstar = \frac{1}{LC} \quad \& \quad \bigstar - \Delta = 2\sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

Case 1 $\sigma^2 > \omega_n^2$

$$v_C(t) = \frac{i_{L0} + \bigstar C(v_{C0} - V_S)}{C(\bigstar - \Delta)} \cdot e^{-\Delta t} - \frac{i_{L0} + \Delta C(v_{C0} - V_S)}{C(\bigstar - \Delta)} \cdot e^{-\bigstar t} + V_S$$

$$i_L(t) = -\left(\frac{i_{L0} + \frac{1}{L}(v_{C0} - V_S)}{(\bigstar - \Delta)}\right) \cdot e^{-\Delta t} + \left(\frac{\bigstar i_{L0} + \frac{1}{L}(v_{C0} - V_S)}{(\bigstar - \Delta)}\right) \cdot e^{-\bigstar t}$$

Case 2 $\sigma^2 = \omega_n^2$

$$v_C(t) = e^{-\frac{R}{2L} t}\left[(v_{C0} - V_S) + \left(\frac{i_{L0}}{C} + \frac{R}{2L}(v_{C0} - V_S)\right) \cdot t\right] + V_S$$

$$i_L(t) = \left(\frac{RC}{2L}\right) e^{-\frac{R}{2L} \cdot t}\left[\frac{2L}{RC} \cdot i_{L0} - \left(\frac{i_{L0}}{C} + \frac{R}{2L}(v_{C0} - V_S)\right) \cdot t\right]$$

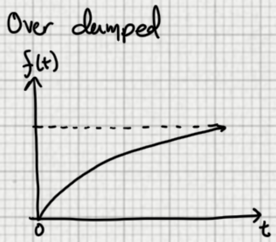
Case 3 $\sigma^2 < \omega_n^2$

$$v_C(t) = e^{-\frac{R}{2L} t}\left[(v_{C0} - V_S)\cos(\omega_d t) + \left(\frac{i_{L0}}{\omega_d C} + \frac{R(v_{C0} - V_S)}{2\omega_d L}\right) \cdot \sin(\omega_d t)\right] + V_S$$

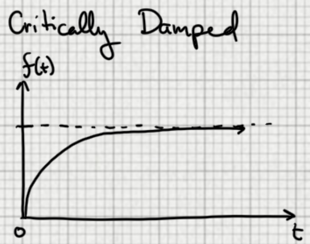
$$i_L(t) = e^{-\frac{R}{2L} \cdot t}\left[i_{L0}\cos(\omega_d t) - \frac{RC}{2L}\left(\frac{i_{L0}}{\omega_d C} + \frac{R(v_{C0} - V_S)}{2\omega_d L} + \frac{2\omega_d L}{R}(v_{C0} - V_S)\right) \cdot \sin(\omega_d t)\right]$$

These solutions are for $t \geq 0$ for simple RLC ckt's driven by DC sources that turn on @ $t = 0$. Initial conditions need to be found or given.

Over damped



Critically Damped



Under damped