Markov Processes - Foundations of Stochastic Models of Finance

Ivan E. Perez

January 17, 2019

Welcome to the Foundations of Stochastic Models of Finance series. This is a small set of blogposts that will let me build up to the traditional short derivation of BSM-OPM. These posts are meant to communicate an intuitive feel of the underlying principles and assumptions we need to get to BM-OPM.

The Markov Chain:

The Markov chain is a sequence of Random Variables with the Markov Property.

The Markov Property:

The Markov Property states that future and past states are independent.

$$P\left\{X_{n+1}=j|X_n=i,X_{n-1}=i_{n-1},...,X_0=i_0\right\}=p_{ij}$$

In English: The conditional probability of the next state given all past states including the present state is a preset probability.

$$p_{ij}=\{X_{n+1}=j|X_n=i_n\}$$
$$\forall n,i_0,...,i_{n-1},i_n,j$$

Where

$$i,j\in\{1,2,...,M\}$$

represents discrete state space.

Transition Graphs and Matrices

Now, that we know Markov Processes probabilites are not "informed" by any past or future state, we can describe transitions from State A to State B using a transition and set probabilies.

Let P be the $M\times M$ Transition Matrix containing the transition probabilities from state $i\to j$. (i.e. each element in the matrix, is giving me a probability that you can move from state i to state j).

$$P=\begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1M} \\ p_{21} & p_{22} & \cdots & p_{2M} \\ \vdots & \vdots & \ddots & \vdots \\ p_{M1} & p_{M2} & \cdots & p_{MM} \end{bmatrix}$$

Lets give the transition matrix some values.

$$P=\begin{bmatrix} 0 & 0.5 & 0 & 0 \\ 0.5 & 0 & 0.25 & 0.25 \\ 0 & 0.4 & 0.4 & 0.2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Transition graphs are just small plots that we can use to intuitively display the transition matrix. In this case, moving from state 1 to state 2 has a 50% chance of occuring.

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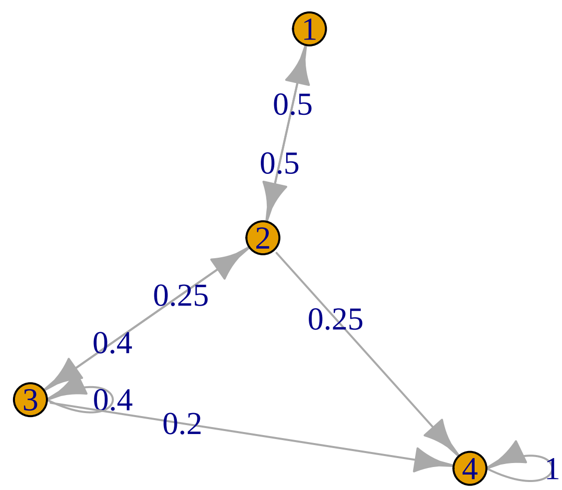


Classification of States

Accessible States:

There is a directed path in the transition grapsh/matrix such that the transition probability from state i to j is non-zero.

Communicable States:

There is a non-zero probability from state i to j and there is a non-zero probabiltiy moving from j to i.

Recurrent States:

For every state j that is accessible from i the state i is also accessible from j.

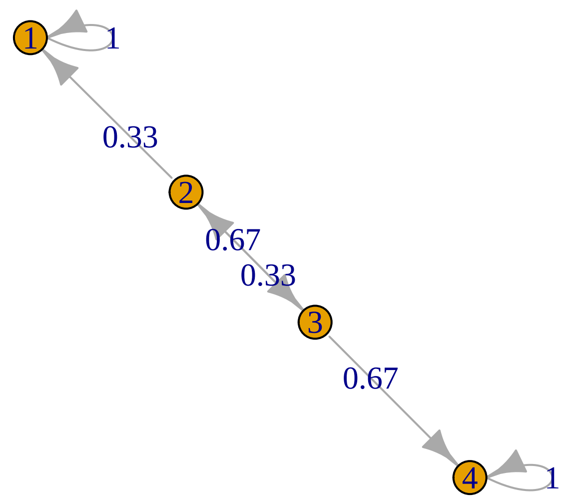
Absorbing States:

The probability of transitioning from i to j and j to i is 1.

Example: Gamblers Ruin

A classic problem that explains why the house alway. Lets say, I start with two dollars (State = 2), and my opponent starts with 3 dollars. I'm pretty good so 2/3 times I can beat him, so 1/3 times he beats me. when I beat him he gives me a dollar. I will quit when I'm down to one dollar and he will quit when he's broke. What is the probability of me winning?

Lets construct a transition graph and matrix to describe my sit.



Now lets show the transition matrix GR.

$$GR = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/3 & 0 & 2/3 & 0 \\ 0 & 1/3 & 0 & 2/3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

I can find the probability of winning by adding up probabilities of winning by adding the markov chain.

So the probability of winning is $P\{p_{23} * p_{34}\} = (2/3) * (2/3) = 4/9$.

What can we take away from this? We can extended this to build someething like a blackjack odds calculator. But, the most important thing is to remember that when we hear the word Markov Property. **The Markov Property simply tells us, the probability of the next movement is independent of any of the past(and future) states.**

Sounds eerily like the disclaimer of every investment product.