Code

Intuitive Probabilistic Derivation of Black Scholes - Option Pricing Formula

Today I will be debuting my first mathematics blog post. Hopefuly it will be part of my series on all the creative derivations of the Black Scholes Merton Option Pricing Model (BSM-OPM). This is the most simple one I've found, that uses only logic of compounding prices, and Log Normal Distributions.

Lets start by defining features of Stock Prices and Random Variables:

1. Future Value of a Stock Price is the price today compounded continuosly into the future $F_0^T = S_0 e^{rT}$. If you want more info check out any version of Hull in the chapter titled "Properties of Stock Options".
2. Random Variables of a Log Normal Distribution x_L:
    a. are continuously distributed over $0 < x_L < +\infty$.
    b. follow probability density function $f_L(x_L)$.
    c. can map to the Standard Normal Random Variable $x_N = ln[x_L]$.
3. Conventions for means and variances of these distributions:
    a. Log Normal Distribution has Mean and Variance μ_L and σ_L^2
    b. Standard Normal Distribution has Mean and Variance μ_N and σ_N^2.
4. We can caluclate the log normal mean μ_L:

    $$\mu_L = \int_0^\infty x_L f_L(x_L) dx_L$$

    $$\mu_L = e^{\mu_N + \frac{1}{2}\sigma_N^2}$$

5. Now lets introduce a 'truncated' also called Partial Distribution of x_L which we will call μ_k, where $k < x_L < +\infty$ and $k > 0$.

    $$\mu_k = \int_k^\infty (x_L - k) f_L(x_L) dx_L$$

    Which evaluates to:

    $$\mu_k = e^{\mu_N + \frac{1}{2}\sigma_N^2} \phi(\frac{-ln(k) + \mu_N + \sigma_N^2}{\sigma_N}) - k\phi(\frac{-ln(k) + \mu_N}{\sigma_N})$$

    Where $\phi(y)$ is the Cummulative Distribution Function (CDF) of a standard normal random variable.

    $$\phi(y) = \int_{-\infty}^y f_N(y_N) dy_N$$

Hopefully I've introduced the tools we'll be using to make a rational argument for the accepted BS-OPM.

Justification of BS-OPM:

1. Making the analogy that a Partial Normal Distribution model the expectation of a call option we can say:
    i. A *European Call* at strike k can be surmised as $C_0^T(k)$.
    ii. The expected price of a Stock in the future is the Mean of a Log normal distribution. $\mu_L = S_0 e^{rT}$.
    iii. Standard Normal Variance and be expressed as a stationtionary volatility weighted by time T.

$$\sigma_N^2 = \sigma^2 T.$$

iv. putting i. ii. and iii. together we can express the expected future stock price as $e^{\mu_N \frac{1}{2}\sigma^2 T} = S_0 e^{rT}$.

v. Solving for we get: $\mu_N = ln(S_0) + (r - \frac{1}{2}\sigma^2)T$

2. Now the price of the Call is expressed as a partial distribution of my stock price with Strike K.

$$\mu_L(K) = e^{ln(S_0)+rT}\phi(\frac{-ln(K)+ln(S_0)+rT+\frac{1}{2}\sigma^2 T}{T\sqrt{\sigma}}) - K\phi(\frac{-ln(K)+ln(S_0)+rT-\frac{1}{2}\sigma^2 T}{T\sqrt{\sigma}})$$

This expectation of the partial distribution function is an expectation of the future.

$$\mu_L = C_0^T(K)e^{rT} = S_0 e^{rT}\phi(\frac{ln(\frac{S_0}{K})+(r+\frac{1}{2}\sigma^2)T}{T\sqrt{\sigma}}) - K\phi(\frac{ln(\frac{S_0}{K})+(r-\frac{1}{2}\sigma^2)T}{T\sqrt{\sigma}})$$

Dividing both sides by discounting back today by dividing by e^{rT}.

$$C_0^T(K) = S_0\phi(\frac{ln(\frac{S_0}{K})+(r+\frac{1}{2}\sigma^2)T}{T\sqrt{\sigma}}) - Ke^{-rT}\phi(\frac{ln(\frac{S_0}{K})+(r-\frac{1}{2}\sigma^2)T}{T\sqrt{\sigma}})$$

3. Comparing it to BS-OPM:

$$C(S_0, 0) = \phi(d_1)S_0 - \phi(d_2)Ke^{-rT}$$

$$d_1 = \frac{1}{\sigma\sqrt{T}}[ln(\frac{S_0}{K}) + (r + \frac{\sigma^2}{2})T]$$

$$d_2 = d_1 - \sigma\sqrt{T} \rightarrow \frac{1}{\sigma\sqrt{T}}[ln(\frac{S_0}{K}) + (r - \frac{\sigma^2}{2})T]$$

Reflections:

This intuitive approach to solving the BS-OPM shows that given the “No Arbitrage Condition”" and a log normal distribution of stock returns we can express the no arbitrage stock price as an expectation μ_L of the log normal distribution. When working with a call, you know it has no value below the strike price. We model that into the expectation using a partial distribution of the random variable. Next I’ll do the more classical SDE and the Euler Style Numerical approasch to solving BS-OPM.

Reference: Intuitive Proof of Black Scholes Option Pricing Formula By Alexei Kruoglov (https://arxiv.org/ftp/physics/papers/0612/0612022.pdf)