PRICING MODEL CALIBRATION THROUGH STOCHASTIC OPTIMIZATION

STOCHASTIC OPTIMIZATION TERM PROJECT

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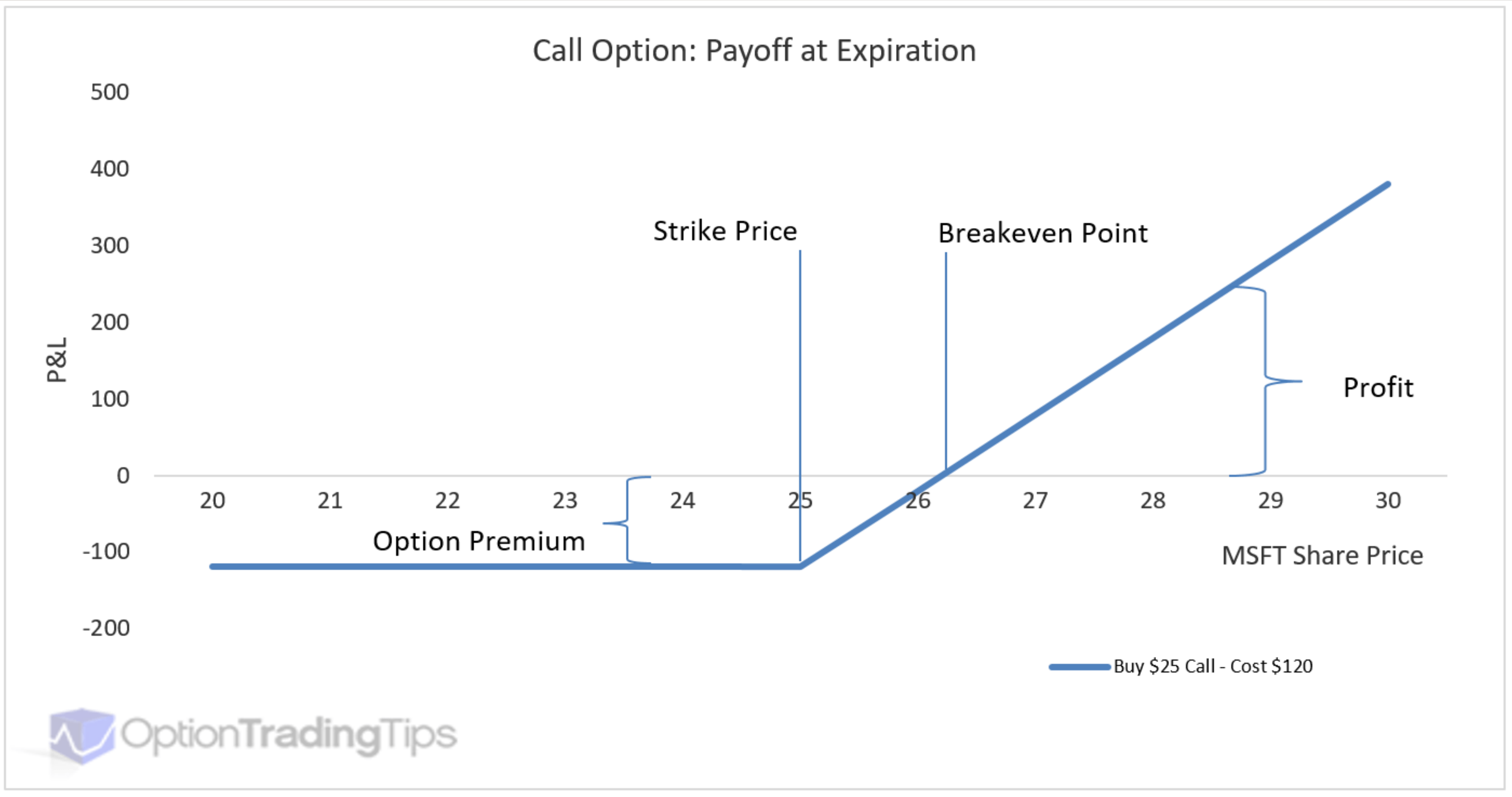
CALL OPTION EXAMPLE[1]



Suppose You Speculate that $MSFT will be at or above 28 USD/Share in 1 month:
You could:

1) BUY 100 Shares of $MFST for 25 USD/Share and sell at the end of the month.
Cost = 100*25USD= 2500 USD
Profit if $MFST > 25 USD

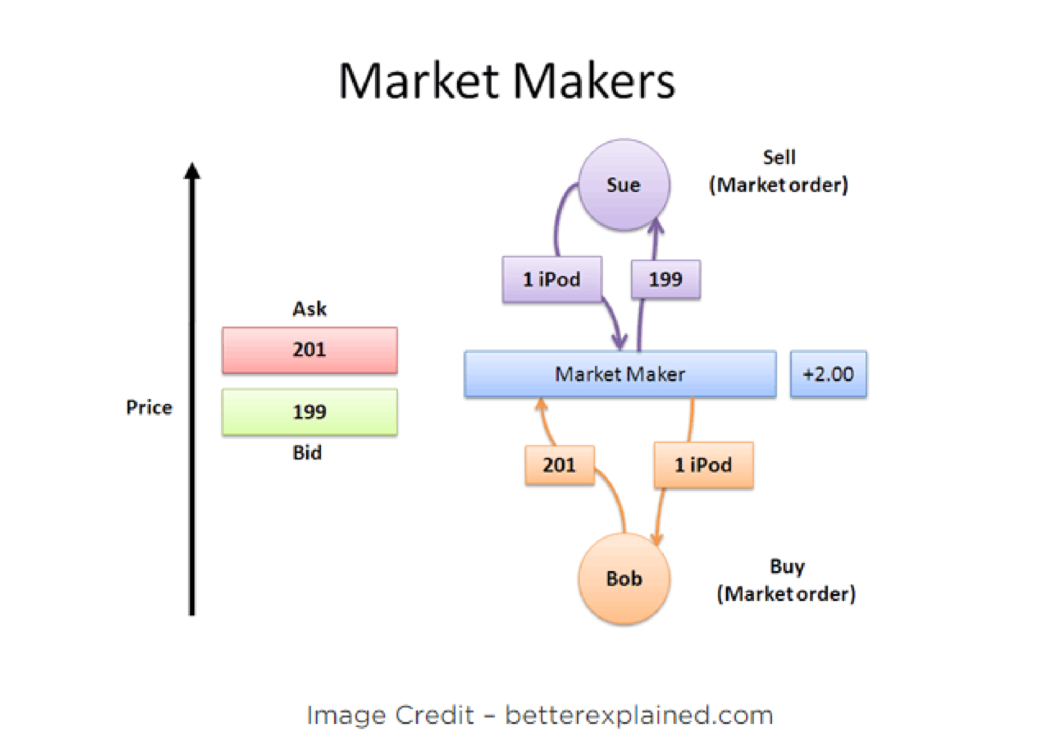
2) BUY a 25 USD Strike Price Call Option for $120/contract
Notational 100Shares/Contract = $1.2 per share of MSFT.
Cost = 120 USD
Profit if $MFST > 26.2 USD

1. https://www.optiontradingtips.com/options101/payoff-diagrams.html

INTRODUCTION AND BACKGROUND

Market Participants in derivatives markets are the ultimate deciders on the value of an option and its underlying.

Market Making firms (e.g., Jane Street, 2Sigma, Renaissance Trading) create value for their clients by buying and selling large quantities of assets to market participants and using **Quantitative Models** to **Set and Estimate Prices** and capture return in excess of risk.



Popular models for Pricing of Call Options and Predicting prices:

1) Black-Scholes-Merton Model,

2) Heston Model.

Both describe the dynamics of Call Options as dependent on risk free rate r, μ, and volatility σ, v_t.

We will Attempt to apply the Stochastic Optimization ideas in this course to estimate optimal parameters in BSM $\{r, \sigma\}$ and Heston$\{\mu, a, b, \xi\}$ Models.

2. https://betterexplained.com/articles/what-you-should-know-about-the-stock-market/

BLACK-SCHOLES-MERTON MODEL

The BSM Formula states that the price of a European Call Option is dependent on the time to expiration ,$\tau = T - t$, and the price of the underlying at time, t, Yielding the SDE:

$$dS(t) = rS(t)dt + \sigma S(t)dW(t)$$

Taking the results from solution under the risk neutral measure[3] we get:

$$S(t) = S(0)\exp\left\{\sigma W(t) + \left(r - \frac{1}{2}\sigma^2\right)t\right\}$$

The well known solution to BSM Equation for the Call[4]:

$$c(t,x) = xN\big(d_+(T-t,x)\big) - Ke^{-r(T-t)}N(d_-(T-t,x))$$

Where, $\tau = T - t$, $N(d_+)$ is the risk adjusted probability that the option will be exercised, and $\mathrm{N}(d_-)$ is the factor by which the present value of the option is greater than the current stock price

$$d_+(\tau,x) = \frac{1}{\sigma\sqrt{\tau}}\left[\log\left(\frac{x}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)\right]$$

$$d_-(\tau,x) = d_+ - \sigma\sqrt{\tau}$$

$$N(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-\frac{z^2}{2}}dz = \frac{1}{\sqrt{2\pi}}\int_{y}^{\infty} e^{-\frac{z^2}{2}}dz$$

$$N'(y) = \frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}$$

3. S.E. Shreve, *Stochastic Calculus for Finance II: Continuous-Time Models*, Springer, New York, 2004, p. 219
4. Ref: S.E. Shreve, *Stochastic Calculus for Finance II: Continuous-Time Models*, Springer, New York, 2004, p. 158-159

METHODOLOGY

1. Observations:

a. End of Trading day Price data
b. End of Trading day Call Option data

$$X_n(t) = \begin{bmatrix} S_n \\ C_{n,K_1} \\ \vdots \\ C_{n,K_2} \end{bmatrix}(t)$$

2. Cost Function: The Expectation of the Squared Difference between Estimates and Observations.

$$\min_{\theta \epsilon (\mathbb{R}^n)^n} J(\theta) = \mathbb{E}[(X_n - \hat{X}_n)^2]$$

3. Iterative Scheme:

$$\theta_{n+1} = \theta_n + \epsilon Y_n$$

$$Y_n = \mathbb{E}[\nabla J(\theta_n)] = \mathbb{E}\left[\frac{1}{N}\left(\frac{\partial}{\partial\theta} X(\theta_n)\left(\hat{X}(\theta) - X_{obs}(\theta)\right)\right)\right]$$

4. Adaptation of Models for gradient estimation:

$$\frac{\partial \hat{S}(t)}{\partial \sigma} = (W(t) - \sigma t) S_0 \exp\left\{\sigma W(t) + \left(r - \frac{1}{2}\sigma^2\right) t\right\}$$

$$\frac{\partial \hat{S}(t)}{\partial r} = t S_0 \exp\left\{\sigma W(t) + \left(r - \frac{1}{2}\sigma^2\right) t\right\}$$

$$\frac{\partial \hat{C}}{\partial r} = S_0 N'(d_+) \frac{\partial d_+}{\partial r} - \tau\left(K e^{-2r\tau} N(d_-)\right) + K e^{-r\tau} N'(d_-) \frac{\partial d_-}{\partial r}$$

$$\frac{\partial \hat{C}}{\partial \sigma} = S_0 N'(d_+) \frac{\partial d_+}{\partial \sigma} - K e^{-r\tau} N'(d_-) \frac{\partial d_-}{\partial \sigma}$$

DATA DESCRIPTION

Empirical Data
Source of Data: Bloomberg Terminal[5]

Dates Surveyed: 10/22/2019 – 11/23/2019

Products: AMD 12/20/2019 Call, AMD 11/15/2019 Call, CHTR 1/17/2019 Call, CHTR 6/19/2020

25 Strike Prices {K} numbered $K_1, \ldots, K_{25}$.

For each Call in {K} there exists an associated final bid and asking price and a end of Day Stock Price.

Volume, last trade price, IVM is included but not treated as feature

Date	Strike	Ticker	Bid	Ask	Last	IVM	Volm	AMD_Close
10/22/2019	25.5	AMD 11/15/19 C25.5	6.20	6.30	0	62.47	0	31.51
10/22/2019	26	AMD 11/15/19 C26	5.75	5.85	6	61.84	50	31.51
10/22/2019	26.5	AMD 11/15/19 C26.5	5.35	5.40	0	62.17	0	31.51
10/22/2019	27	AMD 11/15/19 C27	4.90	5	5.20	61.63	13	31.51

Simulated Data
Purpose: verify convergence of BSM and Heston Model IPA:
Given a set of initial parameters , $r, \sigma, \mu, a, b, \xi$, we generate $S(t), C(S(t), K)$ and v_t from Heston SDE.
BSM:

$$S_{t+\Delta t} = S_t \exp\left\{\sigma W(t) + \left(r - \frac{1}{2}\sigma^2\right)\Delta t\right\}$$

$$c(\tau, S_t) = S_t N\big(d_+(\tau, S_t)\big) - Ke^{-r\tau} N(d_-(\tau, S_t))$$

Heston[6]:

$$S_{t+\Delta t} = S_t \exp\left\{\sqrt{v_t} W_2 + \left(\mu - \frac{1}{2} v_t\right)\Delta t\right\}$$

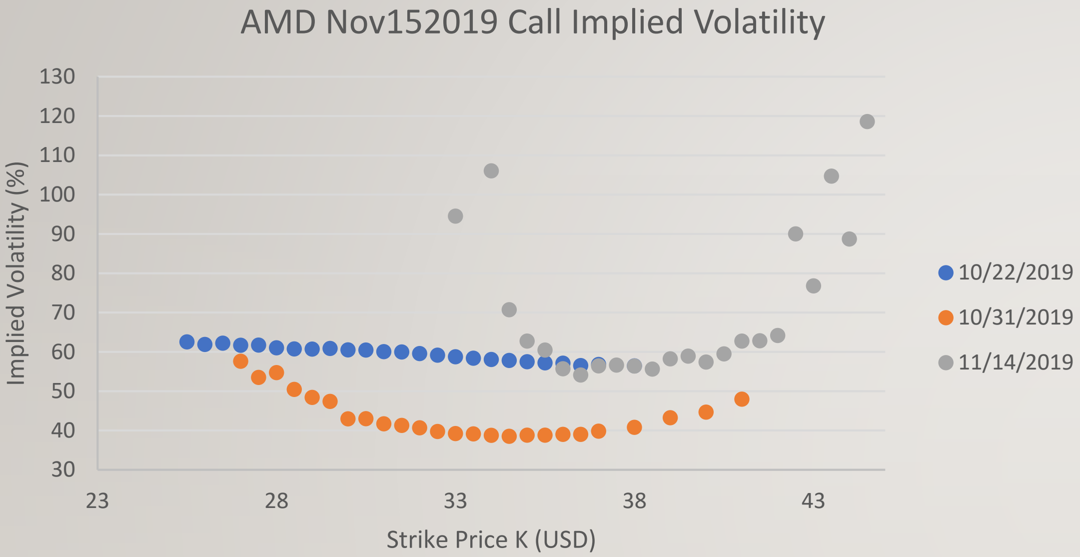
$$v_{t+1} = \left(\sqrt{v_t} + \frac{\xi}{2}\sqrt{\Delta t} W_1\right)^2 - a(v_t - b)\Delta t - \frac{\xi^2}{4}\Delta t$$

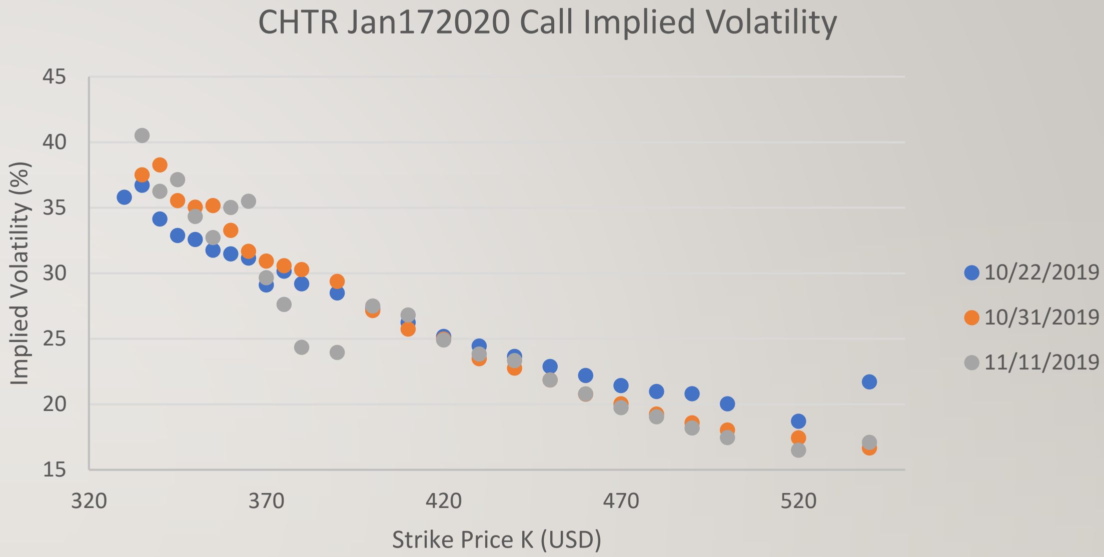
$$W_1 = Z_1;\ W_2 = \ \rho Z_1 + \sqrt{1 - \rho^2} Z_2$$

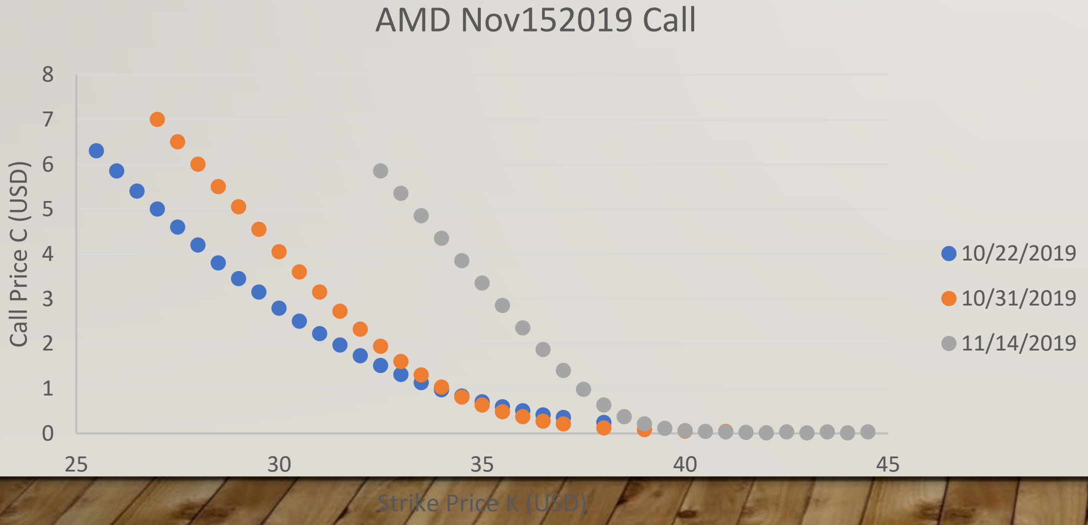
S0	35	t	W1	W2	vt	St	C(K=35)
a	0.5	0	0	0	0.5	35	0
b	0.06	1	1.037879	1.037879	0.324103	62.62969	27.62969
xi	0.06	2	-0.07639	-0.07639	0.188547	56.24388	21.24388
v0	0.06	3	-0.65967	-0.65967	0.106579	42.39291	7.392914
mu	0.098	4	0.099043	0.099043	0.084338	45.78818	10.78818
dt	1	5	0.118983	0.118983	0.073355	50.11953	15.11953
rho	1	6	-0.73378	-0.73378	0.054338	43.68456	8.684557

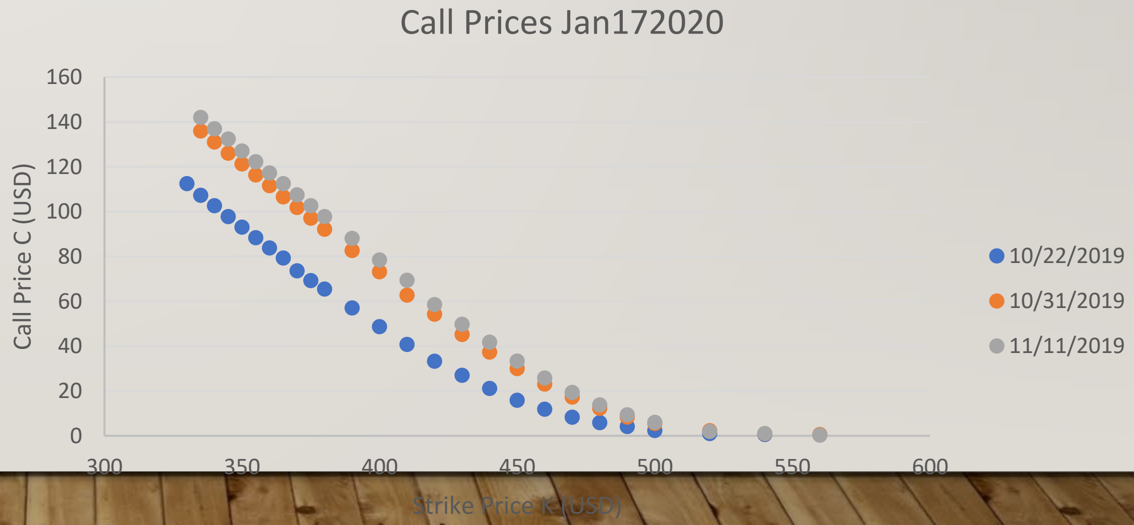
5. Bloomberg L.P., “Options Chain of AMD, CHTR. 10/22/2019 to 11/23/2019," *Bloomberg* 2019.
6. https://www.quantconnect.com/tutorials/introduction-to-options/local-volatility-and-stochastic-volatility

CALL STRIKE VS CALL PRICE AND IMPLIED VOLATILITY

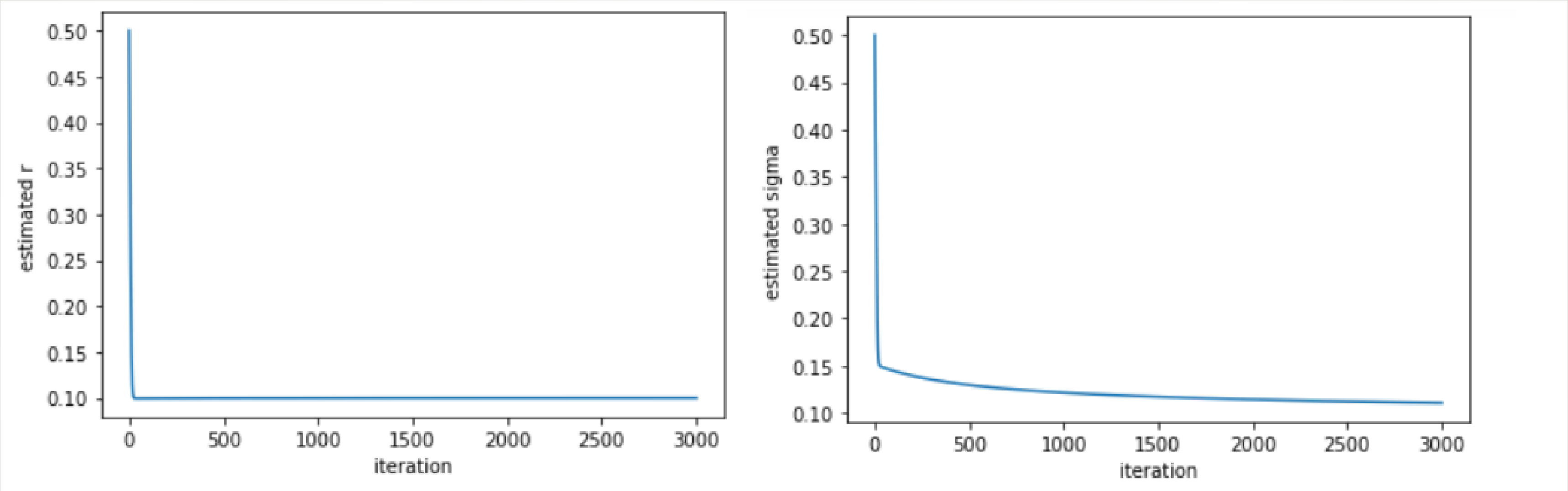








BLACK-SCHOLES-MERTON – PRELIMINARY EXPERIMENTS



HESTON MODEL[7]

The Heston Model describes Price dynamics with a similar equation as BSM:

$$dS_t = \mu S_t dt + \sqrt{v_t} S_t dW_t^1$$

But is driven by a variance process , v_t, That is assumed to move like a Cox Ingersoll Ross Process (CIR)[8]:

$$dv_t = a(b - v_t)dt + \xi\sqrt{v_t} dW_t^2$$

The Brownian motions are assumed to have constant correlation, ρ, over the time horizon of interest:

$$\langle dW_t^1, dW_t^2 \rangle = \rho$$

While the $S(t)$, can be expressed similar to as before, one can Implement Milstein discretization[5] to describe the variance process iteratively:

$$v_{t+1} = \left(\sqrt{v_t} + \frac{\xi}{2}\sqrt{\Delta t} Z\right)^2 - a(v_t - b)\Delta t - \frac{\xi^2}{4}\Delta t$$

- **How can we apply IPA to find parameters $\boldsymbol{\mu, a, b, \xi}$ that would match observed price dynamics?**

7. J. Gatheral , *The Volatility Surface: A Practicioner's Guide*, Wiley, New Jerset, 2006, p. 15-24
8. S.E. Shreve, *Stochastic Calculus for Finance II: Continuous-Time Models*, Springer, New York, 2004, p. 266

METHODOLOGY

We can try to estimate the gradient of $\mathbb{E}[Y_n|\mathcal{F}_{n-1}]$ using the IPA derivative:

$$\mathbb{E}[Y_n|\mathcal{F}_{n-1}] = \frac{d}{d\theta}\mathbb{E}\left[h\big(X(\theta)\big)\right]$$

And we can argue the exchange of derivative and expectation, by the 3 assumptions in Theorem 8.1:

Theorem 8.1: Let $\Theta \subset \mathbb{R}$ be an open and connected set, such that $X(\theta)$ is a measurable mapping on a common underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ $Let\ \theta_0 \in \Theta\ If$:
i) The sample path (or stochastic derivative) of $dX(\theta)/d\theta$ exists w.p.1 at θ_0.
ii) The mapping $h : S \to \mathbb{R}$ is differentiable
iii) The mapping $h\big(X(\theta)\big)$ is Lipschitz continuous on Θ w.p.1. then,

$$\frac{d}{d\theta}\mathbb{E}\left[h\big(X(\theta)\big)\right] = \mathbb{E}\left[\frac{\partial X(\theta)}{\partial \theta}\frac{dh\big(X(\theta)\big)}{dX(\theta)}\right]$$

We can simplify the math, by taking the IPA derivative of the log of incremental returns $L = \log\frac{S_{t+\Delta t}}{S_t}$.

$$\mathbb{E}\left[\frac{d}{d\theta}\log\frac{S_{t+\Delta t}}{S_t}\middle|S_t\right] = \frac{d}{d\theta}\left(\sqrt{v_t}W_t^2 + \Delta t\mu - \Delta t\frac{1}{2}v_t\right) = \mathbb{E}\begin{bmatrix}\frac{\partial L}{\partial \mu}\\ \frac{\partial L}{\partial a}\\ \frac{\partial L}{\partial b}\\ \frac{\partial L}{\partial \xi}\end{bmatrix} =$$

Expectation of Partial Derivative	Result	Dependence
$\mathbb{E}[\partial L(\theta)/\partial\mu]$	Δt	Linear in time
$\mathbb{E}[\partial L(\theta)/\partial a]$	$-\frac{1}{2}(v_t - b)\Delta t^2$	Quadratic in time, and the long term variance
$\mathbb{E}[\partial L(\theta)/\partial b]$	$\frac{1}{2}a\Delta t^2$	Quadratic in time, and the mean reversion rate
$\mathbb{E}[\partial L(\theta)/\partial\xi]$	$\frac{1}{4}\xi\Delta t^2$	Quadratic in time, and linear on vol of vol

s

SENSITIVITY ANALYSIS OF HESTON MODEL

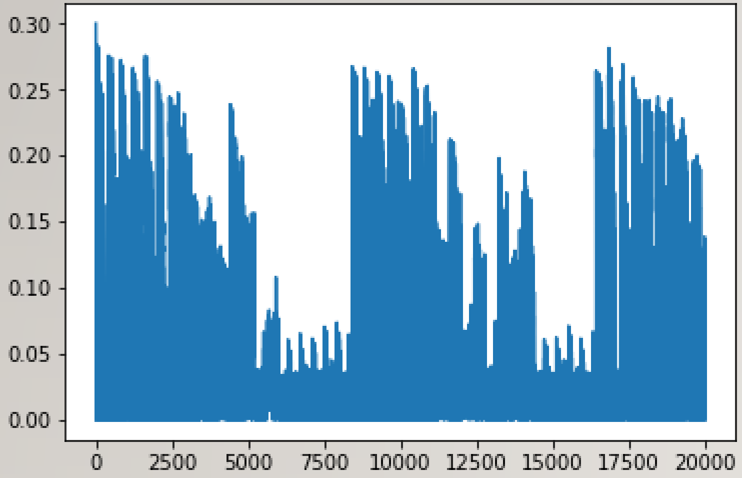
We can say that the derivative of the price process is:

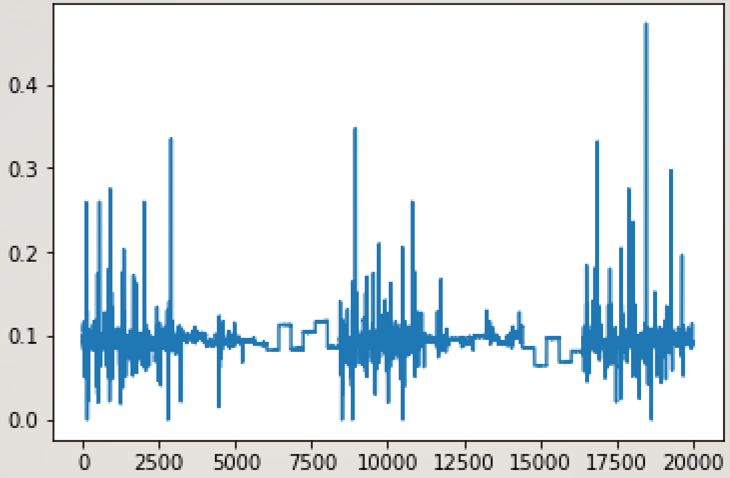
$$\mathbb{E}[\frac{d}{d\theta}S_{t+\Delta t}|S_t] = \begin{bmatrix} \Delta t S_t \exp\left\{\left(\mu - \frac{1}{2}v_t\right)\Delta t\right\} \\ -\frac{1}{2}(E[v_t|v_{t-1}] - b)\Delta t^2 S_t \exp\left\{\left(\mu - \frac{1}{2}v_t\right)\Delta t\right\} \\ \frac{1}{2}a\Delta t^2 S_t \exp\left\{\left(\mu - \frac{1}{2}v_t\right)\Delta t\right\} \\ \frac{1}{4}\xi\Delta t^2 S_t \exp\left\{\left(\mu - \frac{1}{2}v_t\right)\Delta t\right\} \end{bmatrix}$$

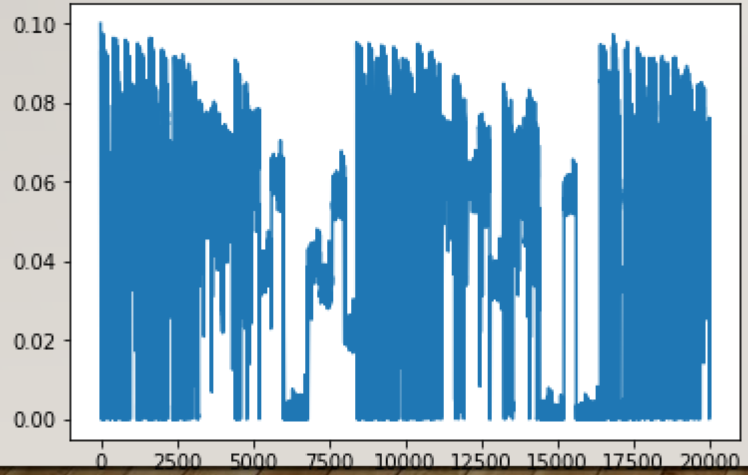
- Which affords the final IPA of

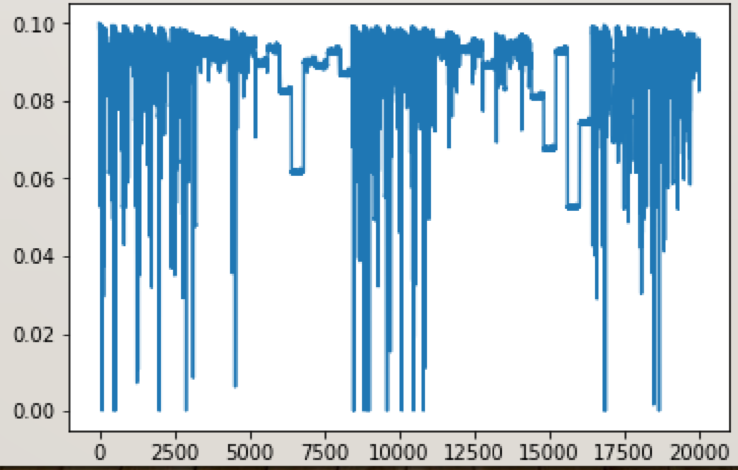
$$\frac{d}{d\theta}\mathbb{E}\big[h\big(X(\theta)\big)\big] = \mathbb{E}\left[\frac{\partial X(\theta)}{\partial\theta}\frac{dh\big(X(\theta)\big)}{dX(\theta)}\right] = \begin{bmatrix} \Delta t \mathrm{S_t} \exp\left\{\left(\mu - \frac{1}{2}v_t\right)\Delta t\right\} 2\big(\mathbb{E}[\hat{S}_{t+\Delta t}|S_t] - S_{t,obs}\big) \\ -\frac{1}{2}(\mathbb{E}[v_t|v_{t-1}] - b)\Delta t^2 S_t \exp\left\{\left(\mu - \frac{1}{2}v_t\right)\Delta t\right\} 2(\mathbb{E}[S_{t+\Delta t}|S_t] - S_{t,obs}) \\ \frac{1}{2}a\Delta t^2 S_t \exp\left\{\left(\mu - \frac{1}{2}v_t\right)\Delta t\right\} 2\big(\mathbb{E}[S_{t+\Delta t}|S_t] - S_{t,obs}\big) \\ \frac{1}{4}\xi\Delta t^2 S_t \exp\left\{\left(\mu - \frac{1}{2}v_t\right)\Delta t\right\} 2\big(\mathbb{E}[S_{t+\Delta t}|S_t] - S_{t,obs}\big) \end{bmatrix}$$

EXPERIMENTS









Initial parameters:

$$\{\mu_0, a_0, b_0, \xi_0\} = \{0.3, 0.1, 0.1, 0.1\}$$

Optimal parameters:

$$\{\mu^*, a^*, b^*, \xi^*\} = \{0.07, 0.2, 0.05, 0.01\}$$

4. CONCLUSION

A. The IPA Estimator for the Black Scholes Merton formula was found to converge

B. The IPA Estimator for the Heston Model, can be derived recursively.

C. Experimental data is inconclusive for Heston Model Calibration.