# SLCO4A - Aula Prática 12

**Sumário**



## Transformada de Laplace

Tranformada de Laplace expressa um sinal no domínio de frequência complexa $s$ (domínio $s$), ou seja, um sinal é descrito por uma função $F(s)$.

A transformada de Laplace é denotada por

$$F(s) = L\{f(t)\} = \int_{-\infty}^{\infty} f(t)e^{-\mathrm{st}}\,\mathrm{dt}$$

### Comandos laplace e ilaplace

**Exercício 1**: Compute e a transformada de Laplace unilateral das seguintes funções

a) $f(t) = e^{-t}$

```
syms t s
f=exp(-t);
F=laplace(f, s) % Caso precise forçar, coloque um S
```

F =

$$\frac{1}{s+1}$$

b) $f(t) = \delta(t)$

```
f=dirac(t);
F=laplace(f,s)
```

F = $1$

c) $f(t) = u(t) = 1$, $\forall\, t > 0$

```
f=1;
F=laplace(f, s)
```

F =

$$\frac{1}{s}$$

**Exercício 2**: Compute e a transformada inversa de Laplace das seguintes funções:

a) $F(s) = \dfrac{1}{s+1}$

```
F=1/(s+1);
f=ilaplace(F)
```

f = $e^{-t}$

b) $F(s) = 1$

```
F=1;
f=ilaplace(F, t)
```

f = $\delta(t)$

c) $F(s) = \dfrac{1}{s}$

```
F=1/s;
f=ilaplace(F, t)
```

f = $1$

## Pares da transformada de Laplace

**Exercício 3**: Compute e a transformada de Laplace ou transformada inversa de Laplace das seguintes funções

a) $x(t) = \dfrac{d^n\delta(t)}{\mathrm{dt}^n}$, $n = 4$

```
syms t s
n=4;
x=diff(dirac(t), n);
X=laplace(x)
```

X = $s^4$

b) $x(t) = e^{-\mathrm{at}}u(t)$

```
syms a
```

```
x=exp(-a*t);
X=laplace(x)
```

X =

$$\frac{1}{a+s}$$

c) $X(s) = \dfrac{1}{s-a}$

```
X=1/(s-a);
x=ilaplace(X, t)
```

x = $e^{a\,t}$

d) $x(t) = e^{-j\omega_0 t}$

```
syms w0
x=exp(-j*w0*t);
laplace(x)
```

ans =

$$\frac{1}{s+w_0\,\mathrm{i}}$$

e) $X(s) = \dfrac{s}{s^2+\omega^2}$

```
syms w
X=s/(s^2+w^2)
```

X =

$$\frac{s}{s^2+w^2}$$

```
ilaplace(X, t)
```

ans = $\cos(t\,w)$

g) $x(t) = \mathrm{sen}(\omega t)$

```
x=sin(w*t);
laplace(x)
```

ans =

$$\frac{w}{s^2+w^2}$$

```
x=cos(2*t);
laplace(x)
```

ans =

$$\frac{s}{s^2+4}$$

h) $x(t) = e^{-\mathrm{at}}\cos(\omega t)u(t)$

```
x=exp(-a*t)*cos(w*t)*heaviside(t);
laplace(x)
```

ans =

$$\frac{a+s}{(a+s)^2+w^2}$$

i) $X(s) = \dfrac{\omega}{(s+a)^2+\omega^2}$

```
X=w/((s+a)^2+w^2);
ilaplace(X, t)
```

ans = $\mathrm{e}^{-a\,t}\sin(t\,w)$

j) $x(t) = t^5u(t)$, $t^nu(t) \Longleftrightarrow \dfrac{n!}{(s+a)^{n+1}}$

```
x=(t^5)*heaviside(t);
X=laplace(x, s)
```

X =

$$\frac{120}{s^6}$$

k) $X(s) = \dfrac{10!}{(s+a)^{n+1}}$

```
X=factorial(10)/(s+a)^11
```

X =

$$\frac{3628800}{(a+s)^{11}}$$

```
ilaplace(X, t)
```

ans = $t^{10}\,\mathrm{e}^{-a\,t}$

l) $x(t) = t\cos(\omega t)$

```
x=t*cos(w*t);
laplace(x)
```

ans =

$$\frac{2\,s^2}{{(s^2+w^2)}^2}-\frac{1}{s^2+w^2}$$

```
simplify(laplace(x))
```

ans =

$$\frac{s^2-w^2}{{(s^2+w^2)}^2}$$

```
laplace(t)
```

ans =

$$\frac{1}{s^2}$$

m) $X(s)=\frac{1}{(s+a)^6},\ \frac{t^{n-1}}{(n-1)!}e^{-\mathrm{at}}u(t)\Longleftrightarrow\frac{1}{(s+a)^n}$

```
ilaplace(1/(s+a)^1,t)
```

ans = $\mathrm{e}^{-a\,t}$

```
ilaplace(1/(s+a)^2,t)
```

ans = $t\,\mathrm{e}^{-a\,t}$

```
ilaplace(1/(s+a)^6,t)
```

ans =

$$\frac{t^5\,\mathrm{e}^{-a\,t}}{120}$$

# Propriedades da transformada de Laplace

## Linearidade

$L\,\{a_1x_1(t)+a_2x_2(t)\}=a_1X_1(s\,)+a_2X_2(s\,)$

**Exercício 4:** Considere os seguintes sinais $x_1(t)=e^{-t}u(t),\ x_2(t)=\cos(t)u(t),\ a_1=3$ e $a_2=4$.

```
x1=exp(-t);
x2=cos(t)*heaviside(t);
a1=3;
a2=4;
laplace(a1*x1+a2*x2)
```

ans =

$$\frac{3}{s+1}+\frac{4\,s}{s^2+1}$$

```
a1*laplace(x1, s)+a2*laplace(x2, s)
```

ans =

$$\frac{3}{s+1}+\frac{4\,s}{s^2+1}$$

## Deslocamento temporal

$L\left\{x(t-t_0)u(t-t_0)\right\}=e^{-st_0}X(s)$

**Exercício 5:** Considere o sinal $x(t)=\cos(t)u(t)$, $t_0=2$.

```
x=cos(t)*heaviside(t);
t0=2;
X=laplace(cos(t-t0)*heaviside(t-t0))
```

X =

$$\frac{\mathrm{e}^{-2\,s}\sin(2)\ (\cos(2)+s\sin(2))}{s^2+1}-\frac{\cos(2)\,\mathrm{e}^{-2\,s}\ (\sin(2)-s\cos(2))}{s^2+1}$$

```
simplify(X)
```

ans =

$$\frac{s\,\mathrm{e}^{-2\,s}}{s^2+1}$$

```
X=laplace(x);
exp(-s*t0)*X
```

ans =

$$\frac{s\,\mathrm{e}^{-2\,s}}{s^2+1}$$

## Deslocamento em frequência complexa

$L\left\{e^{s_0t}x(t)\right\}=X(s-s_0)$

**Exercício 6:** Considere o sinal $x(t)=t^3u(t)$, $s_0=2$.

```
x=t^3;
s0=2;
f=x*exp(s0*t);
L=laplace(f)
```

L =

$$\frac{6}{(s-2)^4}$$

```
X=laplace(x);
```

```
R=subs(X, s, s-2)
```

R =

$$\frac{6}{(s-2)^4}$$

## Escalonamento temporal

$L\{x(\mathrm{bt})\} = \frac{1}{b}X\left(\frac{s}{b}\right)$

**Exercício 7:** Considere o sinal $x(t) = e^{-2t}u(t),\ x(\mathrm{bt})$

```
syms b
laplace(exp(-2*b*t))
```

ans =

$$\frac{1}{2\,b+s}$$

```
x=exp(-2*t);
X=laplace(x);
R=1/b*subs(X, s, s/b)
```

R =

$$\frac{1}{b\,\left(\frac{s}{b}+2\right)}$$

```
simplify(R)
```

ans =

$$\frac{1}{2\,b+s}$$

## Diferenciação no domínio s

$L\{(-t)^n x(t)\} = \frac{d^n X(s)}{\mathrm{ds}^n}$

**Exercício 8:** Considere o sinal $x(t) = e^t u(t), n = 5.$

```
syms t s
x=exp(t);
f=(-t)^5*x;
laplace(f)
```

ans =

$$-\frac{120}{(s-1)^6}$$

```
X=laplace(x);
diff(X,5)
```

ans =

$$-\frac{120}{(s-1)^6}$$

## Integração na frequência complexa

$$L\left\{\frac{x(t)}{t}\right\} = \int_s^{\infty} X(u)\mathrm{du}$$

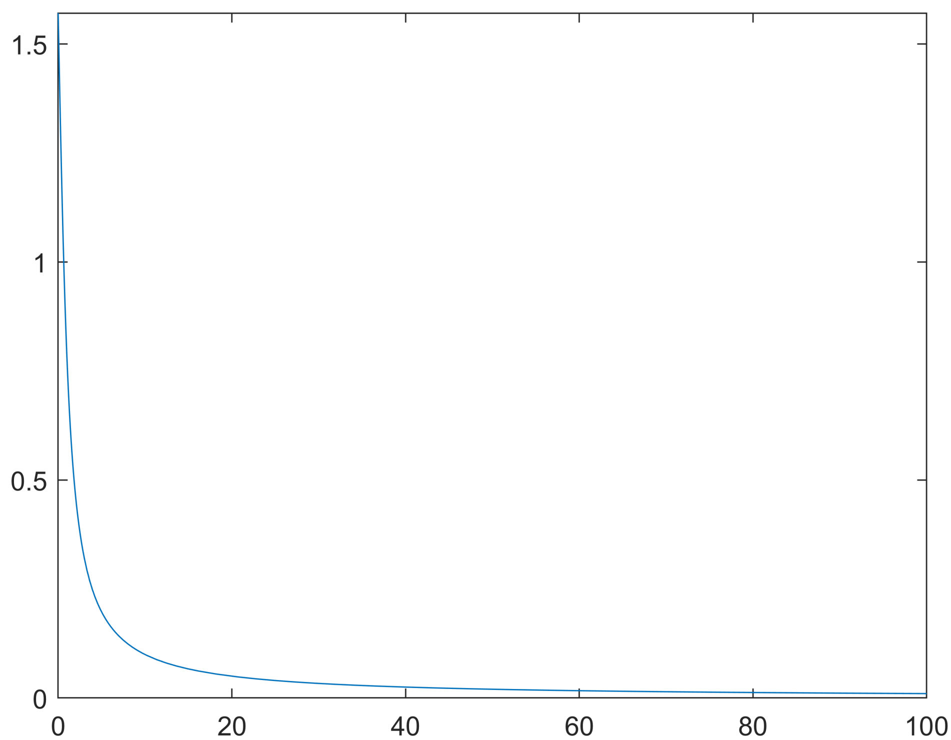
**Exercício 9:** Considere o sinal $x(t) = \mathrm{sen}(t)u(t)$

```
x=sin(t);
L=laplace(x/t, s)
```

L =

$$\mathrm{atan}\left(\frac{1}{s}\right)$$

```
fplot(L, [0 100])
```

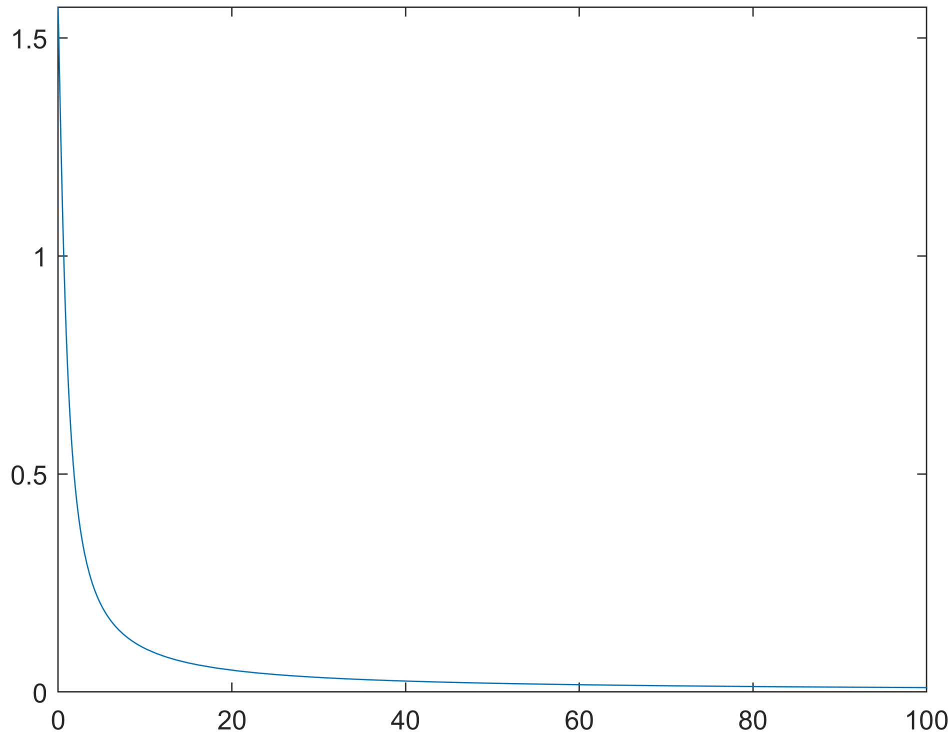



```
syms u
X=laplace(x, u);
R=int(X, u, s, inf)
```

```
R =
```

$$\frac{\pi}{2} - \operatorname{atan}(s)$$

```
fplot(R, [0 100])
```




## Diferenciação no domínio do tempo

$$L\left\{\frac{\mathrm{dx}(t)}{\mathrm{dt}}\right\} = s\,X(s) - x(0)$$

$$L\left\{\frac{d^n x(t)}{\mathrm{dt}^n}\right\} = s\,L\left\{\frac{d^{n-1} x(t)}{\mathrm{dt}^{n-1}}\right\} - x^{(n-1)}(0)$$

**Exercício 10:** Considere o sinal $x(t) = \cos(t)$, $x(0) = 1$

```
x=cos(t);
L=diff(x, t);
laplace(L, s)
```

```
ans =
```

$$-\frac{1}{s^2+1}$$

```
X=laplace(x);
x0=1;
R=s*X-x0;
```

```
simplify(R)
```

ans =

$$-\frac{1}{s^2+1}$$

## Integração no domínio do tempo

$$L\left\{\int_{-\infty}^{t} x(\tau)d\tau\right\} = \frac{X(s)}{s} + \frac{\int_{-\infty}^{0} x(\tau)d\tau}{s}$$

**Exercício 11:** Considere o sinal $x(t) = e^{3t}$

```
syms tau
x=exp(3*tau);
le=int(x, tau, -inf, t);
Left=laplace(le, s)
```

Left =

$$\frac{1}{3\ (s-3)}$$

```
X=laplace(x, s);
xr=int(x, tau, -inf, 0);
Right=X/s+xr/s
```

Right =

$$\frac{1}{s\ (s-3)} + \frac{1}{3\,s}$$

```
simplify(Right)
```

ans =

$$\frac{1}{3\ (s-3)}$$

## Teorema do valor final

$$\lim_{t \longrightarrow \infty} x(t) = \lim_{s \longrightarrow 0} s\,X(s)$$

**Exercício 12:** Considere o sinal $x(t) = (1 - e^{-t})u(t)$

```
x=(1-exp(-t));
limit(x, t, inf)
```

ans = $1$

```
X=laplace(x, s);
limit(s*X, s, 0)
```

ans = $1$

## Expansão em frações parciais

$$X(s) = \frac{B(s)}{A(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0},\ m < n$$

supondo que $\lambda_i$ são as raízes de $A(s)$

- Se as raízes forem distintas $A(s) = a_n \Pi_{i=1}^{n} s - \lambda_i$

$$X(s) = \frac{c_1}{s - \lambda_1} + \frac{c_2}{s - \lambda_2} + \cdots + \frac{c_n}{s - \lambda_n}$$

e os coeficientes são calculados como

$$c_i = \lim_{s \longrightarrow \lambda_i} (s - \lambda_i) X(s)$$

ou

$$c_i = [(s - \lambda_i) X(s)]_{s=\lambda_i}$$

**Exercício 13:** Expresse em frações parciais o sistema dado por

$$X(s) = \frac{s^2 + 3s + 1}{s^3 + 5s^2 + 2s - 8}$$

```
A=[1 5 2 -8];
r=roots(A)
```

```
r = 3×1
   -4.0000
   -2.0000
    1.0000
```

```
syms s
X=(s^2+3*s+1)/(s^3+5*s^2+2*s-8)
```

X =

$$\frac{s^2 + 3\,s + 1}{s^3 + 5\,s^2 + 2\,s - 8}$$

```
c1=limit((s-r(1))*X, s, r(1))
```

c1 =

$$\frac{1}{2}$$

```
c2=limit((s-r(2))*X, s, r(2))
```

c2 =

$\frac{1}{6}$

```
c3=limit((s-r(3))*X, s, r(3))
```

c3 =

$\frac{1}{3}$

```
X1=simplify((s-r(1))*X);
X2=simplify((s-r(2))*X);
X3=simplify((s-r(3))*X);
c1=subs(X1, s, r(1))
```

c1 =

$\frac{1}{2}$

```
c2=subs(X2, s, r(2))
```

c2 =

$\frac{1}{6}$

```
c3=subs(X3, s, r(3))
```

c3 =

$\frac{1}{3}$

## Comando residue

```
[R,P,K] = residue(B,A)
```

$$X(s) = \frac{r_1}{s-p_1} + \frac{r_2}{s-p_2} + \cdots + \frac{r_n}{s-p_n} + K$$

$R = [r_1\, r_2\, r_3 \cdots r_n]$, $P = [p_1\, p_2\, p_3 \cdots p_n]$ e $K$é o resíduo da divisão dos dois polinômios.

**Exercício 14:** Expresse em frações parciais por meio do comando `residue`, o sistema dado por

$$X(s) = \frac{s^2+3s+1}{s^3+5s^2+2s-8}$$

```
num=[1 3 1];
den=[1 5 2 -8];
[R, P, K] = residue(num, den)
```

R = 3×1

```
    0.5000
    0.1667
    0.3333
P = 3×1
   -4.0000
   -2.0000
    1.0000
K =

     []
```