Chapter 1 Solutions

1.1-1. (a) $E = \int_0^2 (1)^2 dt + \int_2^3 (-1)^2 dt = 3$

(b) $E = \int_0^2 (-1)^2 dt + \int_2^3 (1)^2 dt = 3$

(c) $E = \int_0^2 (2)^2 dt + \int_2^3 (-2)^2 dt = 12$

(d) $E = \int_3^5 (1)^2 dt + \int_5^6 (-1)^2 dt = 3$

Comments: Changing the sign of a signal does not change its energy. Doubling a signal quadruples its energy. Shifting a signal does not change its energy. Multiplying a signal by a constant K increases its energy by a factor K^2.

1.1-2.

$$E_x = \int_0^1 t^2 dt = \frac{1}{3}\, t^3\Big|_0^1 = \frac{1}{3}, \qquad E_{x_1} = \int_{-1}^0 (-t)^2 dt = \frac{1}{3}\, t^3\Big|_{-1}^0 = \frac{1}{3},$$

$$E_{x_2} = \int_0^1 (-t)^2 dt = \frac{1}{3}\, t^3\Big|_0^1 = \frac{1}{3}, \qquad E_{x_3} = \int_1^2 (t-1)^2 dt = \int_0^1 x^2 dx = \frac{1}{3},$$

$$E_{x_4} = \int_0^1 (2t)^2 dt = \frac{4}{3}\, t^3\Big|_{-1}^0 = \frac{4}{3}$$

1.1-3. (a)

$$E_x = \int_0^2 (1)^2 dt = 2, \qquad E_y = \int_0^1 (1)^2 dt + \int_1^2 (-1)^2 dt = 2,$$

$$E_{x+y} = \int_0^1 (2)^2 dt = 4, \qquad E_{x-y} = \int_1^2 (2)^2 dt = 4$$

Therefore $E_{x \pm y} = E_x + E_y$.

(b)

$$E_x = \int_0^{2\pi} \sin^2 t\, dt = \frac{1}{2}\int_0^{2\pi} (1) dt - \frac{1}{2}\int_0^{2\pi} \cos(2t) dt = \pi + 0 = \pi$$

$$E_y = \int_0^{2\pi} (1)^2\, dt = 2\pi$$

$$E_{x+y} = \int_0^{2\pi} (\sin t + 1)^2\, dt = \int_0^{2\pi} \sin^2(t) dt + 2\int_0^{2\pi} \sin(t) dt + \int_0^{2\pi} (1)^2 dt = \pi + 0 + 2\pi = 3\pi$$

In both cases (a) and (b), $E_{x+y} = E_x + E_y$. Similarly we can show that for both cases $E_{x-y} = E_x + E_y$.

(c) As seen in part (a),

$$E_x = \int_0^{\pi} \sin^2 t\, dt = \pi/2$$

Furthermore,

$$E_y = \int_0^{\pi} (1)^2\, dt = \pi$$

Thus,

$$E_{x+y} = \int_0^{\pi} (\sin t{+}1)^2\, dt = \int_0^{\pi} \sin^2(t)dt{+}2\int_0^{\pi} \sin(t)dt{+}\int_0^{\pi} (1)^2 dt = \pi/2{+}2(2){+}\pi\frac{3\pi}{2}{+}4$$

Additionally,

$$E_{x-y} = \int_0^{\pi} (\sin t - 1)^2\, dt = \pi/2 - 4 + \pi = \frac{3\pi}{2} - 4$$

In this case, $E_{x+y} \neq E_{x-y} \neq E_x + E_y$. Hence, we cannot generalize the conclusions observed in parts (a) and (b).

1.1-4. $P_x = \frac{1}{4}\int_{-2}^{2}(t^3)^2 dt = 64/7$

(a) $P_{-x} = \frac{1}{4}\int_{-2}^{2}(-t^3)^2 dt = 64/7$

(b) $P_{2x} = \frac{1}{4}\int_{-2}^{2}(2t^3)^2 dt = 4(64/7) = 256/7$

(c) $P_{cx} = \frac{1}{4}\int_{-2}^{2}(ct^3)^2 dt = 64c^2/7$

Comments: Changing the sign of a signal does not affect its power. Multiplying a signal by a constant c increases the power by a factor c^2.

1.1-5. (a) Power of a sinusoid of amplitude C is $C^2/2$ [Eq. (1.4a)] regardless of its frequency ($\omega \neq 0$) and phase. Therefore, in this case $P = 5^2 + (10)^2/2 = 75$.

(b) Power of a sum of sinusoids is equal to the sum of the powers of the sinusoids [Eq. (1.4b)]. Therefore, in this case $P = \frac{(10)^2}{2} + \frac{(16)^2}{2} = 178$.

(c) $(10 + 2\sin 3t)\cos 10t = 10\cos 10t + \sin 13t - \sin 3t$. Hence from Eq. (1.4b) $P = \frac{(10)^2}{2} + \frac{1}{2} + \frac{1}{2} = 51$.

(d) $10\cos 5t \cos 10t = 5(\cos 5t{+}\cos 15t$. Hence from Eq. (1.4b) $P = \frac{(5)^2}{2} + \frac{(5)^2}{2} = 25$.

(e) $10\sin 5t \cos 10t = 5(\sin 15t{-}\sin 5t$. Hence from Eq. (1.4b) $P = \frac{(5)^2}{2} + \frac{(-5)^2}{2} = 25$.

(f) $e^{j\alpha t}\cos\omega_0 t = \frac{1}{2}\left[e^{j(\alpha+\omega_0)t} + e^{j(\alpha-\omega_0)t}\right]$. Using the result in Prob. 1.1-5, we obtain $P = (1/4) + (1/4) = 1/2$.

1.1-6. First, $x(t) = \begin{cases} \frac{2A}{T}t & 0 \leq t < \frac{T}{2} \\ 0 & \frac{T}{2} \leq t < T \\ x(t+T) & \forall t \end{cases}$. Next, $P_x = \frac{1}{T}\int_0^{T/2}\left(\frac{2A}{T}t\right)^2 dt = \frac{4A^2}{T^2}\int_0^{T/2} t^2 dt = \frac{4A^2}{T^2}\frac{t^3}{3}\Big|_0^{T/2} = \frac{4A^2}{T^2}\frac{T^3}{3(8)} = \frac{A^2}{6}$. Since power is finite, energy must be infinite. Thus,

$$P_x = \frac{A^2}{6} \text{ and } E_x = \infty.$$

1.1-7. (a) i. By definition, $E[Tx_1(t)] = \int_{t=-\infty}^{\infty} (Tx_1(t))^2\,dt = \int_{t=-\infty}^{\infty} T^2 x_1^2(t)dt = T^2 \int_{t=-\infty}^{\infty} x_1^2(t)dt = T^2 E[x_1(t)]$.

$$E[Tx_1(t)] = T^2 E[x_1(t)].$$

ii. By definition, $E[x_1(t-T)] = \int_{t=-\infty}^{\infty} (x_1(t-T))^2\,dt$. Substituting $t' = t - T$ and $dt' = dt$ yields $\int_{t'=\infty}^{\infty} x_1^2(t')dt' = E[x_1(t)]$.

$$E[x_1(t-T)] = E[x_1(t)].$$

iii. By definition, $E[x_1(t) + x_2(t)] = \int_{t=-\infty}^{\infty} (x_1(t) + x_2(t))^2\,dt = \int_{t=-\infty}^{\infty} (x_1^2(t) + 2x_1(t)x_2(t) + x_2^2(t))\,dt$. However, $x_1(t)$ and $x_2(t)$ are non-overlapping so their product $x_1(t)x_2(t)$ must be zero. Thus, $E[x_1(t) + x_2(t)] = \int_{t=-\infty}^{\infty} (x_1^2(t) + x_2^2(t))\,dt = \int_{t=-\infty}^{\infty} x_1^2(t)dt + \int_{t=-\infty}^{\infty} x_2^2(t)dt = E[x_1(t)] + E[x_2(t)]$.

$$\text{If } (x_1(t) \neq 0) \Rightarrow (x_2(t) = 0) \text{ and } (x_2(t) \neq 0) \Rightarrow (x_1(t) = 0),$$

$$\text{Then, } E[x_1(t) + x_2(t)] = E[x_1(t)] + E[x_2(t)].$$

iv. By definition, $E[x_1(Tt)] = \int_{t=-\infty}^{\infty} x_1^2(Tt)dt$.
First, consider the case $T > 0$. Substituting $t' = Tt$ and $dt' = Tdt$ yields $E[x_1(Tt)] = \int_{t'=-\infty}^{\infty} x_1^2(t')\frac{dt'}{T} = \frac{1}{T}\int_{t'=-\infty}^{\infty} x_1^2(t')dt' = \frac{E[x_1(t)]}{T} = \frac{E[x_1(t)]}{|T|}$.
Next, consider the case $T < 0$. Substituting $t' = Tt$ and $dt' = Tdt$ yields $E[x_1(Tt)] = \int_{t'=\infty}^{-\infty} x_1^2(t')\frac{dt'}{T} = \frac{-1}{T}\int_{t'=-\infty}^{\infty} x_1^2(t')dt' = \frac{E[x_1(t)]}{-T}$. For $T < 0$, we know $T = -|T|$. Making this substitution yields $E[x_1(Tt)] = \frac{E[x_1(t)]}{|T|}$.
Since energy is the same whether $T < 0$ or $T > 0$, we know

$$E[x_1(Tt)] = \frac{E[x_1(t)]}{|T|}.$$

(b) To begin, notice that signal $y(t) = t(u(t) - u(t-1))$ has energy equal to $E[y(t)] = \int_0^1 t^2 dt = 1/3$.
To determine $E[x(t)]$, consider dividing $x(t)$ into three non-overlapping pieces: a first piece $x_a(t)$ from $(-2 \le t < -1)$, a second piece $x_b(t)$ from $(-1 \le t < 0)$, and a third piece $x_c(t)$ from $(0 \le t < 3)$. Since the pieces are non-overlapping, the total energy $E[x(t)] = E[x_a(t)] + E[x_a(t)] + E[x_a(t)]$.
Using the properties of energy, we know that shifting or reflecting a signal does not affect its energy. Notice that $y(t/3)$ is the same as a flipped and shifted version of $x_c(t)$. Thus, $E[x_c(t)] = E[y(t/3)] = 3(1/3) = 1$. Also, it is possible to combine $x_a(t)$ with a flipped and shifted version of $x_b(t)$ to equal a flipped and shifted version of $2y(t/2)$. Thus, $E[x_a(t) + x_b(t)] = E[2y(t/2)] = 4(2)(1/3) = 8/3$.
Thus,

$$E[x(t)] = 11/3.$$

1.1-8. (a)

$$P_x = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} x(t)x^*(t)\,dt = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}\sum_{k=m}^{n}\sum_{r=m}^{n} D_k D^*{}_r e^{j(\omega_k - \omega_r)t}\,dt$$

The integrals of the cross-product terms (when $k \neq r$) are finite because the integrands are periodic signals (made up of sinusoids). These terms, when divided by $T \to \infty$, yield zero. The remaining terms ($k = r$) yield

$$P_x = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \sum_{k=m}^{n} |D_k|^2 \, dt = \sum_{k=m}^{n} |D_k|^2$$

(b) i.

$$\begin{aligned} x(t) &= 5 + 10\cos(100t + \pi/3) \\ &= 5 + 5e^{j(100t+\frac{\pi}{3})} + 5e^{-j(100t+\frac{\pi}{3})} \\ &= 5 + 5e^{j\pi/3}e^{j100t} + 5e^{-j\pi/3}e^{-j100t} \end{aligned}$$

Hence,

$$P_x = 5^2 + |5e^{j\pi/3}|^2 + |5e^{-j\pi/3}|^2 = 25 + 25 + 25 = 75.$$

Thought of another way, note that $D_0 = 5, D_{\pm 1} = 5$ and thus $P_x = 5^2 + 5^2 + 5^2 = 75$.

ii.

$$\begin{aligned} x(t) &= 10\cos(100t + \pi/3) + 16\sin(150t + \pi/5) \\ &= 5e^{j\pi/3}e^{j100t} + 5e^{-j\pi/3}e^{-j100t} - j8e^{j\pi/5}e^{j150t} + j8e^{-j\pi/5}e^{-j150t} \end{aligned}$$

Hence,

$$P_x = |5e^{j\pi/3}|^2 + |5e^{-j\pi/3}|^2 + |-j8e^{j\pi/5}|^2 + |j8e^{-j\pi/5}|^2 = 25+25+64+64 = 178.$$

Thought of another way, note that $D_{\pm 1} = 5$ and $D_{\pm 2} = 8$. Hence, $P_x = 5^2 + 5^2 + 8^2 + 8^2 = 178$.

iii. $(10 + 2\sin 3t)\cos 10t = 10\cos 10t + \sin 13t - \sin 3t$. In this case, $D_{\pm 1} = 5$, $D_{\pm 2} = 0.5$ and $D_{\pm 3} = 0.5$. Hence, $P = 5^2 + 5^2 + (0.5)^2 + (0.5)^2 + (0.5)^2 + (0.5)^2 = 51$

iv. $10\cos 5t \cos 10t = 5(\cos 5t + \cos 15t)$. In this case, $D_{\pm 1} = 2.5$ and $D_{\pm 2} = 2.5$. Hence, $P = (2.5)^2 + (2.5)^2 + (2.5)^2 + (2.5)^2 = 25$

v. $10\sin 5t \cos 10t = 5(\sin 15t - \sin 5t)$. In this case, $D_{\pm 1} = 2.5$ and $D_{\pm 2} = 2.5$. Hence, $P = (2.5)^2 + (2.5)^2 + (2.5)^2 + (2.5)^2 = 25$

vi. $e^{j\alpha t}\cos\omega_0 t = \frac{1}{2}\left[e^{j(\alpha+\omega_0)t} + e^{j(\alpha-\omega_0)t}\right]$. In this case, $D_{\pm 1} = 0.5$. Hence, $P = (1/2)^2 + (1/2)^2 = 1/2$.

1.1-9. First, notice that $x(t) = x^2(t)$ and that the area of each pulse is one. Since $x(t)$ has an infinite number of pulses, the corresponding energy must also be infinite. To compute the power, notice that N pulses requires an interval of width $\sum_{i=0}^{N} 2(i+1) = N^2 + 3N$. As $N \to \infty$, power is computed by the ratio of area to width, or $P = \lim_{N\to\infty} \frac{N}{N^2+3N} = 0$. Thus,

$$P = 0 \text{ and } E = \infty.$$

1.2-1. Refer to Figure S1.2-1.

1.2-2. Refer to Figure S1.2-2.

1.2-3. (a) $x_1(t)$ can be formed by shifting $x(t)$ to the left by 1 plus a time-inverted version of $x(t)$ shifted to left by 1. Thus,

$$x_1(t) = x(t+1) + x(-t+1) = x(t+1) + x(1-t).$$

integrands are periodic signals (made up of sinusoids). These terms, when divided by $T \to \infty$, yield zero. The remaining terms ($k = r$) yield

$$P_x = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \sum_{k=m}^{n} |D_k|^2\, dt = \sum_{k=m}^{n} |D_k|^2$$

(b) i.

$$\begin{aligned} x(t) &= 5 + 10\cos(100t + \pi/3) \\ &= 5 + 5e^{j(100t+\frac{\pi}{3})} + 5e^{-j(100t+\frac{\pi}{3})} \\ &= 5 + 5e^{j\pi/3}e^{j100t} + 5e^{-j\pi/3}e^{-j100t} \end{aligned}$$

Hence,

$$P_x = 5^2 + |5e^{j\pi/3}|^2 + |5e^{-j\pi/3}|^2 = 25 + 25 + 25 = 75.$$

Thought of another way, note that $D_0 = 5, D_{\pm 1} = 5$ and thus $P_x = 5^2 + 5^2 + 5^2 = 75$.

ii.

$$\begin{aligned} x(t) &= 10\cos(100t + \pi/3) + 16\sin(150t + \pi/5) \\ &= 5e^{j\pi/3}e^{j100t} + 5e^{-j\pi/3}e^{-j100t} - j8e^{j\pi/5}e^{j150t} + j8e^{-j\pi/5}e^{-j150t} \end{aligned}$$

Hence,

$$P_x = |5e^{j\pi/3}|^2 + |5e^{-j\pi/3}|^2 + |-j8e^{j\pi/5}|^2 + |j8e^{-j\pi/5}|^2 = 25+25+64+64 = 178.$$

Thought of another way, note that $D_{\pm 1} = 5$ and $D_{\pm 2} = 8$. Hence, $P_x = 5^2 + 5^2 + 8^2 + 8^2 = 178$.

iii. $(10 + 2\sin 3t)\cos 10t = 10\cos 10t + \sin 13t - \sin 3t$. In this case, $D_{\pm 1} = 5$, $D_{\pm 2} = 0.5$ and $D_{\pm 3} = 0.5$. Hence, $P = 5^2 + 5^2 + (0.5)^2 + (0.5)^2 + (0.5)^2 + (0.5)^2 = 51$

iv. $10\cos 5t\cos 10t = 5(\cos 5t + \cos 15t)$. In this case, $D_{\pm 1} = 2.5$ and $D_{\pm 2} = 2.5$. Hence, $P = (2.5)^2 + (2.5)^2 + (2.5)^2 + (2.5)^2 = 25$

v. $10\sin 5t\cos 10t = 5(\sin 15t - \sin 5t)$. In this case, $D_{\pm 1} = 2.5$ and $D_{\pm 2} = 2.5$. Hence, $P = (2.5)^2 + (2.5)^2 + (2.5)^2 + (2.5)^2 = 25$

vi. $e^{j\alpha t}\cos\omega_0 t = \frac{1}{2}\left[e^{j(\alpha+\omega_0)t} + e^{j(\alpha-\omega_0)t}\right]$. In this case, $D_{\pm 1} = 0.5$. Hence, $P = (1/2)^2 + (1/2)^2 = 1/2$.

1.1-9. First, notice that $x(t) = x^2(t)$ and that the area of each pulse is one. Since $x(t)$ has an infinite number of pulses, the corresponding energy must also be infinite. To compute the power, notice that N pulses requires an interval of width $\sum_{i=0}^{N} 2(i+1) = N^2 + 3N$. As $N \to \infty$, power is computed by the ratio of area to width, or $P = \lim_{N\to\infty} \frac{N}{N^2+3N} = 0$. Thus,

$$P = 0 \text{ and } E = \infty.$$

1.2-1. Refer to Figure S1.2-1.

1.2-2. Refer to Figure S1.2-2.

1.2-3. (a) $x_1(t)$ can be formed by shifting $x(t)$ to the left by 1 plus a time-inverted version of $x(t)$ shifted to left by 1. Thus,

$$x_1(t) = x(t+1) + x(-t+1) = x(t+1) + x(1-t).$$

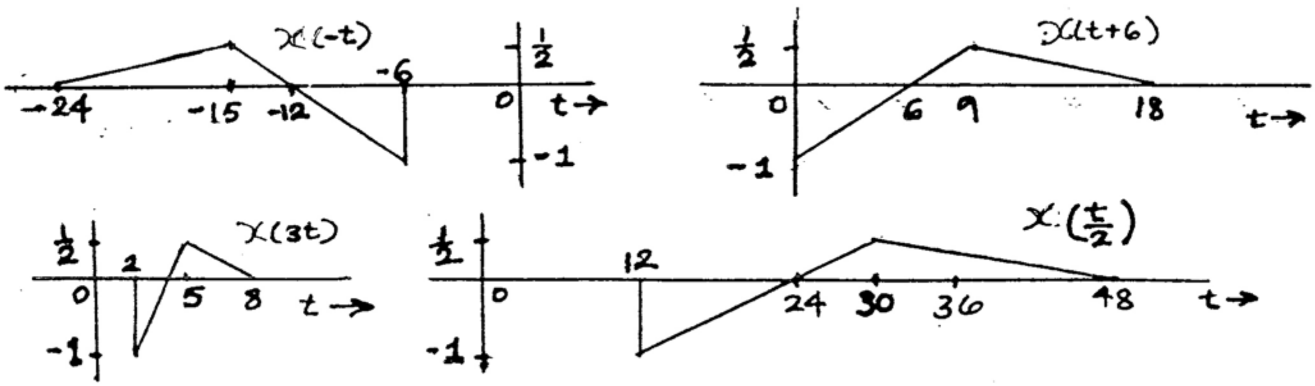


Figure S1.2-1

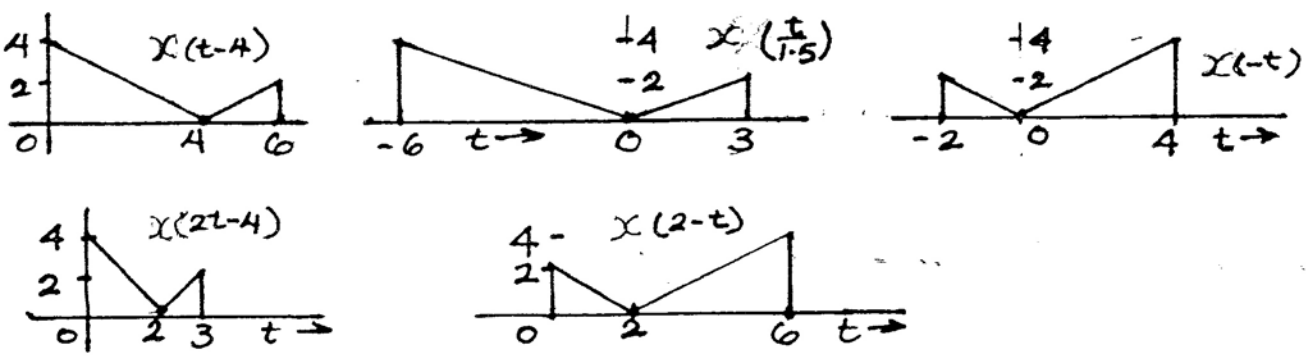


Figure S1.2-2

(b) $x_2(t)$ can be formed by time-expanding $x(t)$ by factor 2 to obtain $x(t/2)$ Now, left-shift $x(t/2)$ by unity to obtain $x(\frac{t+1}{2})$. We now add to this a time-inverted version of $x(\frac{1-t}{2})$ to obtain $x_2(t)$. Thus,

$$x_2(t) = x(\frac{t+1}{2}) + x(\frac{1-t}{2}).$$

(c) Observe that $x_3(t)$ is composed of two parts:
First, a rectangular pulse to form the base is constructed by time-expanding $x_2(t)$ by a factor of 2. This is obtained by replacing t with $t/2$ in $x_2(t)$. Thus, we obtain $x_2(t/2) = x(\frac{t+2}{4}) + x(\frac{2-t}{4})$.
Second, the two triangles on top of the rectangular base are constructed by time-expanded (factor of 2) and shifted versions of $x(t)$ according to $x(t/2) + x(-t/2)$. Thus,

$$x_3(t) = x(\frac{t+2}{4}) + x(\frac{2-t}{4}) + x(t/2) + x(-t/2).$$

(d) $x_4(t)$ can be obtained by time-expanding $x_1(t)$ by a factor 2 and then multiplying it by 4/3 to obtain $\frac{4}{3}x_1(t/2) = \frac{4}{3}\left[x(\frac{t+2}{2}) + x(\frac{2-t}{2})\right]$. From this, we subtract a rectangular pedestal of height 1/3 and width 4. This is obtained by time-expanding $x_2(t)$ by 2 and multiplying it by 1/3 to yield $\frac{1}{3}x_2(t/2) = \frac{1}{3}\left[x(\frac{t+2}{4}) + x(\frac{2-t}{4})\right]$. Hence,

$$x_4(t) = \frac{4}{3}\left[x(\frac{t+2}{2}) + x(\frac{2-t}{2})\right] - \frac{1}{3}\left[x(\frac{t+2}{4}) + x(\frac{2-t}{4})\right].$$

(e) $x_5(t)$ is a sum of three components: (i) $x_2(t)$ time-compressed by a factor 2, (ii) $x(t)$ left-shifted by 1.5, and (iii) $x(t)$ time-inverted and then right shifted by 1.5. Hence,

$$x_5(t) = x(t+0.5) + x(0.5-t) + x(t+1.5) + x(1.5-t).$$

1.2-4.

$$E_{-x} = \int_{-\infty}^{\infty} [-x(t)]^2\,dt = \int_{-\infty}^{\infty} x^2(t)\,dt = E_x$$

$$E_{x(-t)} = \int_{-\infty}^{\infty} [x(-t)]^2\,dt = \int_{-\infty}^{\infty} x^2(x)\,dx = E_x$$

$$E_{x(t-T)} = \int_{-\infty}^{\infty} [x(t-T)]^2\,dt = \int_{-\infty}^{\infty} x^2(x)\,dx = E_x,$$

$$E_{x(at)} = \int_{-\infty}^{\infty} [x(at)]^2\,dt = \frac{1}{a}\int_{-\infty}^{\infty} x^2(x)\,dx = E_x/a$$

$$E_{x(at-b)} = \int_{-\infty}^{\infty} [x(at-b)]^2\,dt = \frac{1}{a}\int_{-\infty}^{\infty} x^2(x)\,dx = E_x/a,$$

$$E_{x(t/a)} = \int_{-\infty}^{\infty} [x(t/a)]^2\,dt = a\int_{-\infty}^{\infty} x^2(x)\,dt = aE_x$$

$$E_{ax(t)} = \int_{-\infty}^{\infty} [ax(t)]^2\,dt = a^2\int_{-\infty}^{\infty} x^2(t)\,dt = a^2E_x$$

Comment: Multiplying a signal by constant a increases the signal energy by a factor a^2.

1.2-5. (a) Calling $y(t) = 2x(-3t+1) = t(u(-t-1) - u(-t+1))$, MATLAB is used to sketch $y(t)$.

```
>> t = [-1.5:.001:1.5]; y = inline('t.*((t<=-1)-(t<=1))');
>> plot(t,y(t),'k-'); axis([-1.5 1.5 -1.1 1.1]);
>> xlabel('t'); ylabel('2x(-3t+1)');
```

(b) Since $y(t) = 2x(-3t+1)$, $0.5*y(-t/3+1/3) = 0.5(2)x(-3(-t/3+1/3)+1) = x(t)$. MATLAB is used to sketch $x(t)$.

```
>> y = inline('t.*((t<=-1)-(t<=1))');
>> t = [-3:.001:5]; x = 0.5*y(-t/3+1/3);
>> plot(t,x,'k-'); axis([-3 5 -0.6 0.6]);
>> xlabel('t'); ylabel('x(t)');
```

1.2-6. MATLAB is used to compute each sketch. Notice that the unit step is in the exponent of the function $x(t)$.

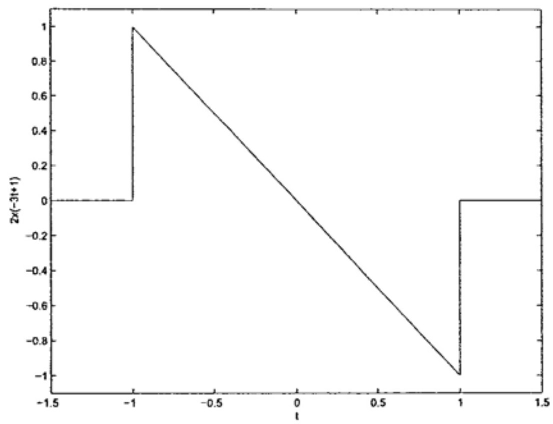


Figure S1.2-5a: Plot of $2x(-3t+1)$.

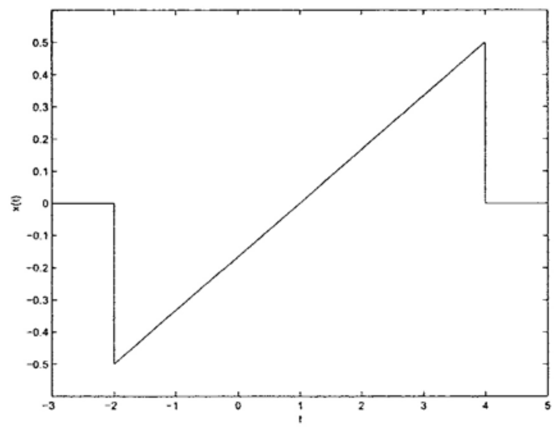


Figure S1.2-5b: Plot of $x(t) = 0.5y(-t/3+1/3)$.

(a)
```
>> t = [-1:.001:1];
>> x = inline('2.^(-t.*(t>=0))');
>> plot(t,x(t),'k'); axis([-1 1 0 1.1]);
>> xlabel('t'); ylabel('x(t)');
```

(b)
```
>> plot(t,0.5*x(1-2*t),'k'); axis([-1 1 0 1.1]);
>> xlabel('t'); ylabel('y(t)');
```

1.3-1. (a) False. Figure 1.11b is an example of a signal that is continuous-time but digital.

(b) False. Figure 1.11c is discrete-time but analog.

(c) False. e^{-t} is neither an energy nor a power signal.

(d) False. $e^{-t}u(t)$ has infinite duration but is an energy signal.

(e) False. $u(t)$ is a power signal that is causal.

(f) True. A periodic signal, by definition, exists for all t.

1.3-2. (a) True. Every bounded periodic signal is a power signal.

(b) False. Signals with bounded power are not necessarily periodic. For example, $x(t) = \cos(t)u(t)$ is non-periodic but has a bounded power of $P_x = 0.25$.

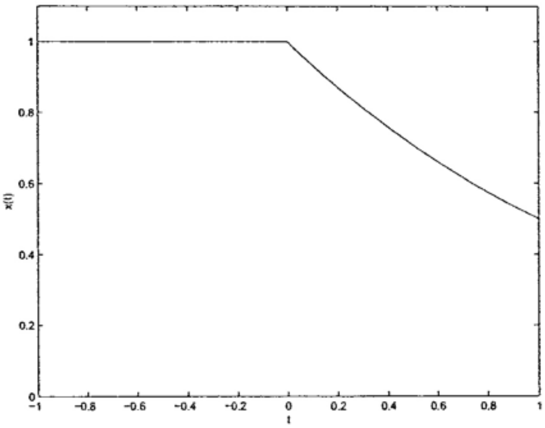


Figure S1.2-6a: Plot of $x(t) = 2^{-tu(t)}$.

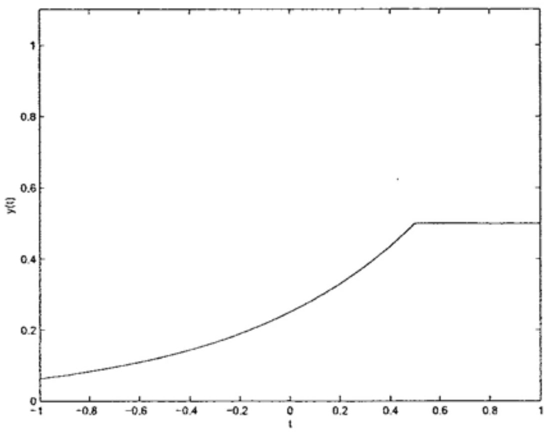


Figure S1.2-6b: Plot of $y(t) = 0.5x(1-2t)$.

(c) True. If an energy signal $x(t)$ has energy E, then the energy of $x(at)$ is $\frac{E}{a}$ (a real and positive).

(d) False. If a power signal $x(t)$ has power P, then the power of $x(at)$ is generally not $\frac{P}{a}$. A counter-example provides a simple proof. Consider the case of $x(t) = u(t)$, which has $P = 0.5$. Letting $a = 2$, $x(at) = x(2t) = u(2t) = u(t)$, which still has power $P = 0.5$ and not $P/a = P/2$.

1.3-3. (a) For periodicity, $x_1(t) = \cos(t) = \cos(t + T_1) = x_1(t + T_1)$. Since cosine is a 2π-periodic function, $T_1 = 2\pi$. Similarly, $x_2(t) = \sin(\pi t) = \sin(\pi t + \pi T_2) = \sin(\pi(t + T_2)) = x_2(t + T_2)$. Thus, $\pi T_2 = 2\pi k$. The smallest possible value is $T_2 = 2$. Thus,

$$T_1 = 2\pi \text{ and } T_2 = 2.$$

(b) Periodicity requires $x_3(t) = x_3(t+T_3)$ or $\cos(t)+\sin(\pi t) = \cos(t+T_3)+\sin(\pi t + \pi T_3)$. This requires $T_3 = 2\pi k_1$ and $\pi T_3 = 2\pi k_2$ for some integers k_1 and k_2. Combining, periodicity thus requires $T_3 = 2\pi k_1 = 2k_2$ or $\pi = k_1/k_2$. However, π is irrational. Thus, no suitable k_1 and k_2 exist, and $x_3(t)$ cannot be periodic.

(c)

$$
\begin{aligned}
P_{x_1} &= \tfrac{1}{2\pi}\int_0^{2\pi}\cos^2(t)dt \\
&= \tfrac{1}{2\pi}\left(0.5(t+\sin(t)\cos(t))|_{t=0}^{2\pi}\right) \\
&= \tfrac{1}{2\pi}\tfrac{1}{2}2\pi = \tfrac{1}{2} \\
P_{x_2} &= \tfrac{1}{2}\int_0^2\sin^2(\pi t)dt \\
&= \tfrac{1}{2}\left(\tfrac{1}{2\pi}(\pi t-\sin(\pi t)\cos(\pi t))|_{t=0}^{2}\right) \\
&= \tfrac{1}{2}\tfrac{1}{2\pi}2\pi = \tfrac{1}{2} \\
P_{x_3} &= \lim_{t\to\infty}\tfrac{1}{2T}\int_{-T}^{T}\left(\cos(t)+\sin(\pi t)\right)^2 dt \\
&= \lim_{t\to\infty}\tfrac{1}{2T}\int_{-T}^{T}\left(\cos^2(t)+\sin^2(t)+\cos(t)\sin(\pi t)\right)dt \\
&= P_{x_1}+Px_2+\lim_{t\to\infty}\tfrac{1}{2T}\int_{-T}^{T}0.5\left(\sin(\pi t-t)+\sin(\pi t+t)\right)dt \\
&= P_{x_1}+Px_2+0 = 1
\end{aligned}
$$

Thus,

$$P_{x_1} = P_{x_2} = \frac{1}{2} \text{ and } P_{x_3} = 1.$$

1.3-4. No, $f(t) = \sin(\omega t)$ is not guaranteed to be a periodic function for an arbitrary constant ω. Specifically, if ω is purely imaginary then $f(t)$ is in the form of hyperbolic sine, which is not a periodic function. For example, if $\omega = \jmath$ then $f(t) = \jmath\sinh(t)$. Only when ω is constrained to be real will $f(t)$ be periodic.

1.3-5. (a) $E_{y_1} = \int_{-\infty}^{\infty} y_1^2(t)dt = \int_{-\infty}^{\infty}\frac{1}{9}x^2(2t)dt$. Performing the change of variable $t' = 2t$ yields $\int_{-\infty}^{\infty}\frac{1}{9}x^2(t')\frac{dt'}{2} = \frac{E_x}{18}$. Thus,

$$E_{y_1} = \frac{E_x}{18} \approx \frac{1.0417}{18} = 0.0579.$$

(b) Since $y_2(t)$ is just a $(T_{y_2} = 4)$-periodic replication of $x(t)$, the power is easily obtained as

$$P_{y_2} = \frac{E_x}{T_{y_2}} = \frac{E_x}{4} \approx= 0.2604.$$

(c) Notice, $T_{y_3} = T_{y_2}/2 = 2$. Thus, $P_{y_3} = \frac{1}{2}\int_{T_{y_3}} y_3^2(t)dt = \frac{1}{2}\int_{T_{y_3}}\frac{1}{9}y_2(2t)dt$. Performing the change of variable $t' = 2t$ yields $P_{y_3} = \frac{1}{2}\int_{T_{y_2}}\frac{1}{9}y_2(t')\frac{dt'}{2} = \frac{1}{36}\int_0^4 x(t')dt' = \frac{E_x}{36}$. Thus,

$$P_{y_3} = \frac{E_x}{36} \approx 0.0289.$$

1.3-6. For all parts, $y_1(t) = y_2(t) = t^2$ over $0 \le t \le 1$.

(a) To ensure $y_1(t)$ is even, $y_1(t) = t^2$ over $-1 \le t \le 0$. Since $y_1(t)$ is $(T_1 = 2)$-periodic, $y_1(t) = y_1(t+2)$ for all t. Thus, $y_1(t) = \begin{cases} t^2 & -1 \le t \le 1 \\ y_1(t+2) & \forall t \end{cases}$. $P_{y_1} = \frac{1}{T_1}\int_{-1}^{1}(t^2)^2 dt = 0.5\frac{t^5}{5}\Big|_{t=-1}^{1} = 1/5$. Thus,

$$P_{y_1} = 1/5.$$

A sketch of $y_1(t)$ over $-3 \le t \le 3$ is created using MATLAB.

```
>> t = [-3:.001:3]; mt = mod(t,2);
>> y_1 = (mt<=1).*(mt.^2) + (mt>1).*((mt-2).^2);
>> plot(t,y_1,'k'); xlabel('t'); ylabel('y_1(t)'); axis tight;
```

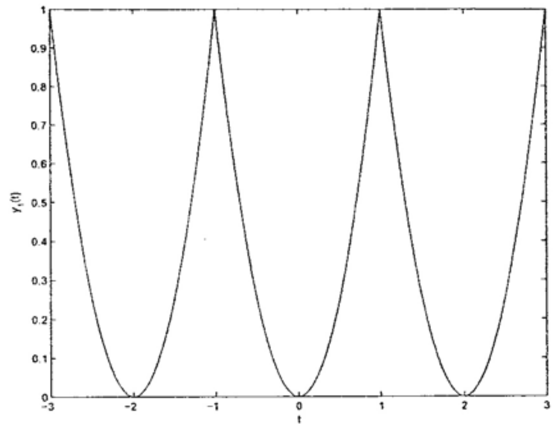


Figure S1.3-6a: Plot of $y_1(t)$

(b) Let

$$y_2(t) = \begin{cases} k & 1 \le t < 1.5 \\ t^2 & 0 \le t < 1 \\ -y_2(-t) & \forall t \\ y_2(t+3) & \forall t \end{cases}.$$

With this form, $y_2(t)$ is odd and ($T_2 = 3$)-periodic. The constant k is determined by constraining the power to be unity, $P_{y_2} = 1 = \frac{1}{3}\left(k^2 + \frac{2}{5}t^5\big|_{t=0}^{1}\right)$. Solving for k yields $k^2 = 3 - 2/5 = 13/5$ or $k = \sqrt{13/5}$. Thus,

$$y_2(t) = \begin{cases} \sqrt{13/5} & 1 \le t < 1.5 \\ t^2 & 0 \le t < 1 \\ -y_2(-t) & \forall t \\ y_2(t+3) & \forall t \end{cases}.$$

A sketch of $y_2(t)$ over $-3 \le t \le 3$ is created using MATLAB.

```
>> t = [-3:.001:3]; mt = mod(t,3);
>> y_2 = (mt<1).*(mt.^2)-(mt>=2).*((mt-3).^2)+...
        ((mt>=1)&(mt<1.5))*sqrt(13/5)-...
        ((mt>=1.5)&(mt<2))*sqrt(13/5);
>> plot(t,y_2,'k'); xlabel('t'); ylabel('y_2(t)'); axis([-3 3 -2 2]);
```

(c) Define $y_3(t) = y_1(t) + jy_2(t)$. To be periodic, $y_3(t)$ must equal $y_3(t+T_3)$ for some value T_3. This implies that $y_1(t) = y_1(t+T_3)$ and $y_2 = y_2(t+T_3)$. Since $y_1(t)$ is ($T_1 = 2$)-periodic, T_3 must be an integer multiple of T_1. Similarly, since $y_2(t)$ is ($T_2 = 3$)-periodic, T_3 must be an integer multiple of T_2. Thus, periodicity of $y_3(t)$ requires $T_3 = T_1k_1 = 2k_1 = T_2k_2 = 3k_2$, which is satisfied letting $k_1 = 3$ and $k_2 = 2$. Thus,

$$y_3(t) \text{ is periodic with } T_3 = 6.$$

(d) Noting $y_3(t)y_3^*(t) = y_1^2(t) + y_2^2(t)$, $P_{y_3} = \frac{1}{T_3}\int_{T_3}\left(y_1^2(t) + y_2^2(t)\right)dt = P_{y_1} + P_{y_2}$. Thus,

$$P_{y_3} = 1 + \frac{1}{5} = \frac{6}{5}.$$

1.4-1. Refer to Figure S1.4-1.

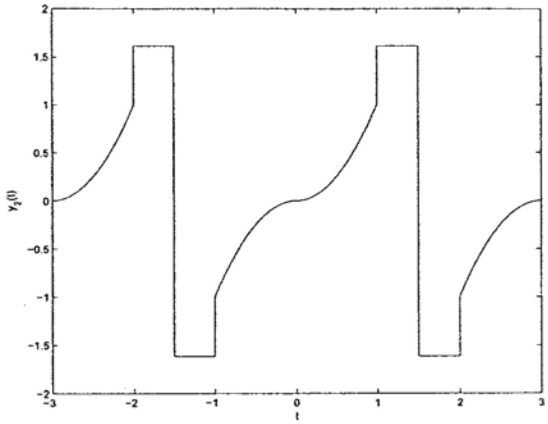


Figure S1.3-6b: Plot of $y_2(t)$

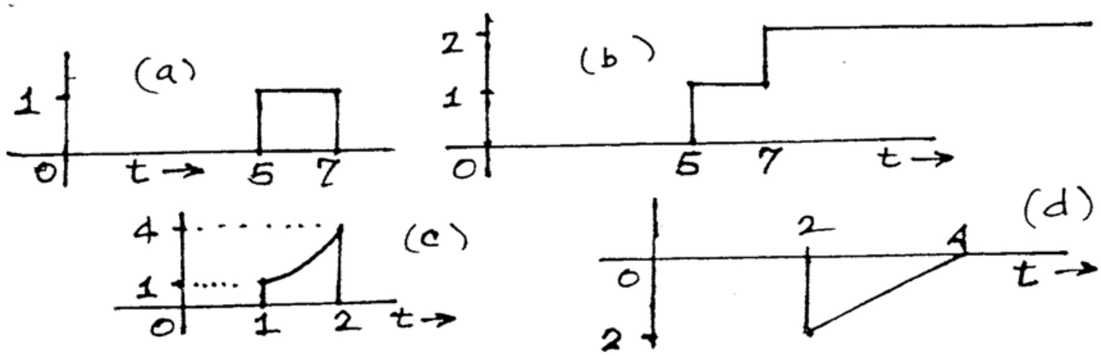


Figure S1.4-1

1.4-2.

$$
\begin{aligned}
x_1(t) &= (4t+1)[u(t+1)-u(t)] + (-2t+4)[u(t)-u(t-2)] \\
&= (4t+1)u(t+1) - 6tu(t) + 3u(t) + (2t-4)u(t-2)
\end{aligned}
$$

$$
\begin{aligned}
x_2(t) &= t^2[u(t)-u(t-2)] + (2t-8)[u(t-2)-u(t-4)] \\
&= t^2u(t) - (t^2-2t+8)u(t-2) - (2t-8)u(t-4)
\end{aligned}
$$

1.4-3. Using the fact that $f(x)\delta(x) = f(0)\delta(x)$, we have

(a) 0
(b) $\frac{2}{9}\delta(\omega)$
(c) $\frac{1}{2}\delta(t)$
(d) $-\frac{1}{5}\delta(t-1)$
(e) $\frac{1}{2-j3}\delta(\omega+3)$
(f) $k\delta(\omega)$ (use L' Hôpital's rule)

1.4-4. In these problems remember that impulse $\delta(x)$ is located at $x = 0$. Thus, an impulse $\delta(t-\tau)$ is located at $\tau = t$, and so on.

(a) The impulse is located at $\tau = t$ and $x(\tau)$ at $\tau = t$ is $x(t)$. Therefore

$$\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau = x(t).$$

(b) The impulse $\delta(\tau)$ is at $\tau = 0$ and $x(t-\tau)$ at $\tau = 0$ is $x(t)$. Therefore

$$\int_{-\infty}^{\infty} \delta(\tau)x(t-\tau)\,d\tau = x(t).$$

Using similar arguments, we obtain

(c) 1

(d) 0

(e) e^3

(f) 5

(g) $x(-1)$

(h) $-e^2$

1.4-5. For sketches, refer to Figure S1.4-5.

(a) Recall that the derivative of a function at the jump discontinuity is equal to an impulse of strength equal to the amount of discontinuity. Hence, dx/dt contains impulses $4\delta(t+4)$ and $2\delta(t-2)$. In addition, the derivative is -1 over the interval $(-4,\ 0)$, and is 1 over the interval $(0,\ 2)$. The derivative is zero for $t < -4$ and $t > 2$. The result is sketched in Figure S1.4-5(a).

(b) Using the procedure in part (a), Figure S1.4-5(b) depicts d^2x/dt^2 for the signal in Figure P1.4-2a.

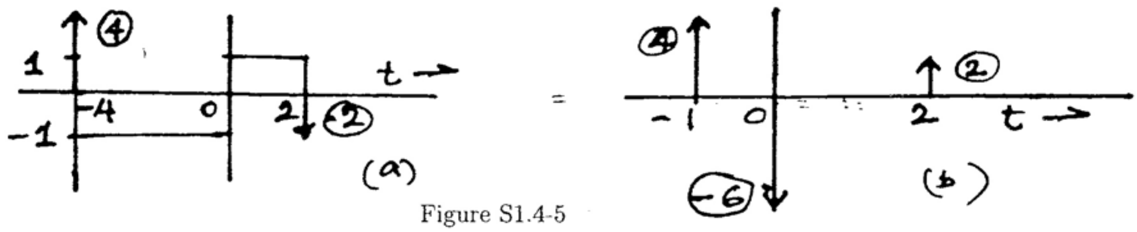


Figure S1.4-5

1.4-6. For sketches, refer to Figure S1.4-6.

(a) Recall that the area under an impulse of strength k is k. Over the interval $0 \le t \le 1$, we have

$$y(t) = \int_0^t 1\,dx = t \qquad 0 \le t \le 1.$$

Over the interval $0 \le t < 3$, we have

$$y(t) = \int_0^1 1\,dx + \int_1^t (-1)\,dx = 2 - t \qquad 1 \le t < 3.$$

At $t = 3$, the impulse (of strength unity) yields an additional term of unity. Thus (assuming $\epsilon \to 0$),

$$y(t) = \int_0^1 1\,dx + \int_1^{3-\epsilon} (-1)\,dx + \int_{3-\epsilon}^t \delta(x-3)\,dx = 1 + (-2) + 1 = 0 \qquad t > 3$$

(b)

$$y(t) = \int_0^t [1-\delta(x-1)-\delta(x-2)-\delta(x-3)+\cdots]\, dx = tu(t)-u(t-1)-u(t-2)-u(t-3)-\ldots$$

Figure S1.4-6

1.4-7. Changing the variable t to $-x$, we obtain

$$\int_{-\infty}^{\infty} \phi(t)\delta(-t)\, dt = -\int_{\infty}^{-\infty} \phi(-x)\delta(x)\, dx = \int_{-\infty}^{\infty} \phi(-x)\delta(x)\, dx = \phi(0).$$

This shows that

$$\int_{-\infty}^{\infty} \phi(t)\delta(t)\, dt = \int_{-\infty}^{\infty} \phi(t)\delta(-t)\, dt = \phi(0).$$

Therefore

$$\delta(t) = \delta(-t).$$

1.4-8. Letting $at = x$, we obtain (for $a > 0$)

$$\int_{-\infty}^{\infty} \phi(t)\delta(at)\, dt = \frac{1}{a}\int_{-\infty}^{\infty} \phi(\frac{x}{a})\delta(x)\, dx = \frac{1}{a}\phi(0)$$

Similarly for $a < 0$, we show that this integral is $-\frac{1}{a}\phi(0)$. Therefore

$$\int_{-\infty}^{\infty} \phi(t)\delta(at)\, dt = \frac{1}{|a|}\phi(0) = \frac{1}{|a|}\int_{-\infty}^{\infty} \phi(t)\delta(t)\, dt$$

Therefore

$$\delta(at) = \frac{1}{|a|}\delta(t)$$

1.4-9.

$$\begin{aligned}\int_{-\infty}^{\infty} \dot{\delta}(t)\phi(t)\, dt &= \phi(t)\delta(t)|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \dot{\phi}(t)\delta(t)\, dt \\ &= 0 - \int \dot{\phi}(t)\delta(t)\, dt = -\dot{\phi}(0)\end{aligned}$$

1.4-10. For sketches, refer to Figure S1.4-10.

(a) $s_{1,2} = \pm j3$

(b) $e^{-3t}\cos 3t = 0.5[e^{-(3+j3)t} + e^{-(3-j3)t}]$. Therefore the frequencies are $s_{1,2} = -3 \pm j3$.

(c) Using the argument in (b), we find the frequencies $s_{1,2} = 2 \pm j3$

(d) $s = -2$

(e) $s = 2$

(f) $5 = 5e^{0t}$ so that $s = 0$.

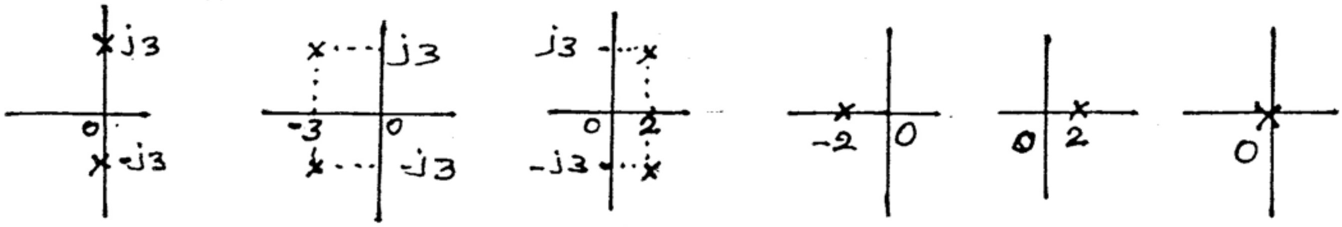


Figure S1.4-10

1.5-1. For sketches, refer to Figure S1.5-1.

(a) $x_e(t) = 0.5[u(t) + u(-t)] = 0.5$ and $x_o(t) = 0.5[u(t) - u(-t)]$.

(b) $x_e(t) = 0.5[tu(t) - tu(-t)] = 0.5|t|$ and $x_o(t) = 0.5[tu(t) + tu(-t)] = 0.5t$.

(c) $x_e(t) = 0.5[\sin\omega_0 t + \sin(-\omega_0 t)] = 0$ and $x_o(t) = 0.5[\sin\omega_0 t - \sin(-\omega_0 t)] = \sin\omega_0 t$.

(d) $x_e(t) = 0.5[\cos\omega_0 t + \cos(-\omega_0 t)] = \cos\omega_0 t$ and $x_o(t) = 0.5[\cos\omega_0 t - \cos(-\omega_0 t)] = 0$.

(e) $\cos(\omega_0 t + \theta) = \cos\omega_0 t\cos\theta - \sin\omega_0 t\sin\theta$. Hence $x_e(t) = \cos\omega_0 t\cos\theta$ and $x_o(t) = -\sin\omega_0 t\sin\theta$

(f) $x_e(t) = 0.5[\sin\omega_0 t\, u(t) + \sin(-\omega_0 t)u(-t)] = 0.5[\sin\omega_0 t\, u(t) - \sin\omega_0 t\, u(-t)]$ and $x_o(t) = 0.5[\sin\omega_0 t\, u(t) - \sin(-\omega_0 t)u(-t)] = 0.5[\sin\omega_0 t\, u(t) + \sin\omega_0 t\, u(-t)] = 0.5\sin\omega_0 t$.

(g) $x_e(t) = 0.5[\cos\omega_0 t\, u(t) + \cos(-\omega_0 t)u(-t)] = 0.5[\cos\omega_0 t\, u(t) + \cos\omega_0 t\, u(-t)] = 0.5\cos\omega_0 t$ and $x_o(t) = 0.5[\cos\omega_0 t\, u(t) - \cos(-\omega_0 t)u(-t)] = 0.5[\cos\omega_0 t\, u(t) - \cos\omega_0 t\, u(-t)]$.

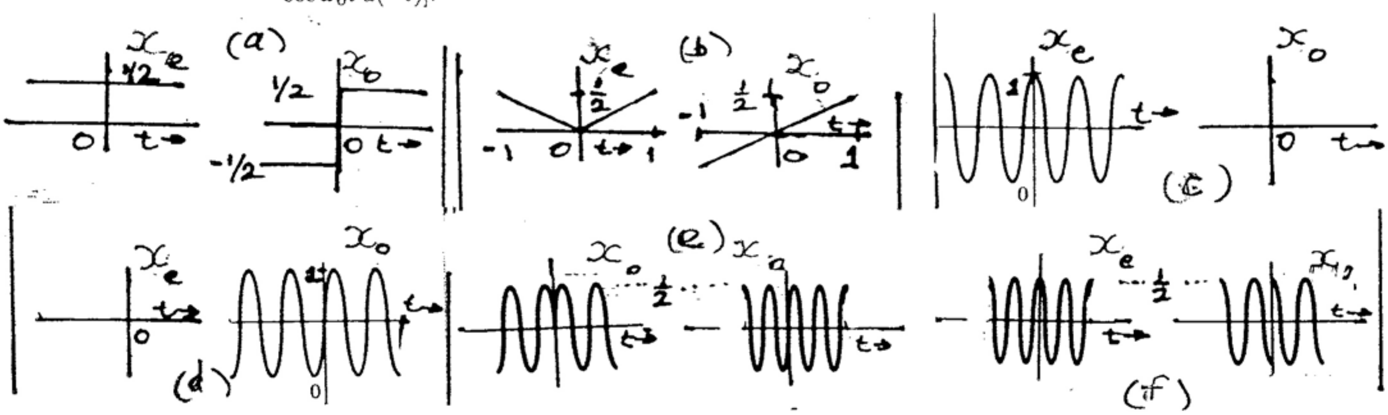


Figure S1.5-1

1.5-2. (a)

$$x_e(t) = \frac{1}{2}[e^{-2t}u(t) + e^{2t}u(-t)]$$

$$x_0(t) = \frac{1}{2}[e^{-2t}u(t) - e^{2t}u(-t)]$$

(b) $E_{x_e} = \int_{-\infty}^{\infty} x_e^2(t)dt$. Because $e^{-2t}u(t)$ and $e^{2t}u(-t)$ are disjoint in time, the cross-product term in $x_e^2(t)$ is zero. Hence,

$$E_{x_e} = \int_{-\infty}^{\infty} x_e^2(t)dt = \frac{1}{4}\left[\int_0^{\infty} e^{-4t}dt + \int_{-\infty}^{0} e^{4t}dt\right] = \frac{1}{8}.$$

Using a similar argument, we have

$$E_{x_o} = \frac{1}{8}.$$

Also,

$$E_x = \int_0^{\infty} e^{-4t}dt = \frac{1}{4}.$$

Hence,

$$E_x = E_{x_e} + E_{x_o}.$$

(c) To generalize this result, we first consider causal $x(t)$. In this case, $x(t)$ and $x(-t)$ are disjoint. Moreover, energy or $x(t)$ is identical to that of $x(-t)$. Hence,

$$E_{x_e} = \frac{1}{4}\left[\int_0^{\infty} |x(t)|^2 dt + \int_{-\infty}^{0} |x(-t)|^2 dt\right] = \frac{1}{2}E_x.$$

Using a similar argument, it follows that $E_{x_o} = \frac{1}{2}E_x$. Hence, for causal signals,

$$E_x = E_{x_e} + E_{x_o}$$

Identical arguments hold for anti-causal signals. Thus, for anti-causal signal $x(t)$

$$E_x = E_{x_e} + E_{x_o}$$

Now, every signal can be expressed as a sum of a causal and an anti-causal signal. Also, the signal energy is equal to the sum of energies of the causal and the anti-causal components. Hence, it follows that for a general case

$$E_x = E_{x_e} + E_{x_o}$$

1.5-3. (a)

$$\begin{aligned} x_e(t)x_o(t) &= \frac{1}{4}[x(t) + x(-t)][x(t) - x(-t)] \\ &= \frac{1}{4}[|x(t)|^2 - |x(-t)|^2] \end{aligned}$$

Since the areas under $|x(t)|^2$ and $|x(-t)|^2$ are identical, it follows that

$$\int_{-\infty}^{\infty} x_e(t)x_o(t)dt = 0$$

(b)

$$\int_{-\infty}^{\infty} x_e(t)dt = \frac{1}{2}\int_{-\infty}^{\infty} x(t)dt + \frac{1}{2}\int_{-\infty}^{\infty} x(-t)dt$$

Because the areas under $x(t)$ and $x(-t)$ are identical, it follows that

$$\int_{-\infty}^{\infty} x_e(t)dt = \int_{-\infty}^{\infty} x(t)dt.$$

1.5-4. $x_o(t) = 0.5(x(t) - x(-t)) = 0.5(\sin(\pi t)u(t) - \sin(-\pi t)u(-t)) = 0.5\sin(\pi t)(u(t) + u(-t))$. Since $\sin(0) = 0$, this reduces to $x_o(t) = 0.5\sin(\pi t)$, which is a $(T = 2)$-periodic signal. Therefore,

$$x_o(t) = 0.5\sin(\pi t) \text{ is a periodic signal.}$$

1.5-5. $x_e(t) = 0.5(x(t) + x(-t)) = 0.5(\cos(\pi t)u(t) + \cos(-\pi t)u(-t)) = 0.5\cos(\pi t)(u(t) + u(-t))$. Written another way, $x_e(t) = \begin{cases} 0.5\cos(\pi t) & t \neq 0 \\ 1 & t = 0 \end{cases}$. Since there exists no $T \neq 0$ such that $x_e(t+T) = x_e(t)$,

$$x_e(t) \text{ is not a periodic function.}$$

It is worth pointing out that sometimes the unit step is defined as $u(t) = \begin{cases} 1 & t > 0 \\ 0.5 & t = 0 \\ 0 & t < 0 \end{cases}$. Using this alternate definition, $x_e(t)$ is periodic.

1.5-6. (a) Using the figure, $x(t) = (t+1)(u(t+1) - u(t)) + (-t+1)(u(t) - u(t-1))$. MATLAB is used to plot $v(t) = 3x\left(-\frac{1}{2}(t+1)\right)$.

```
>> x = inline('(t+1).*((t>=-1)&(t<0))+(-t+1).*((t>=0)&(t<1))');
>> t = [-5:.001:5]; v = 3*x(-0.5*(t+1));
>> plot(t,v,'k-'); xlabel('t'); ylabel('v(t)');
```

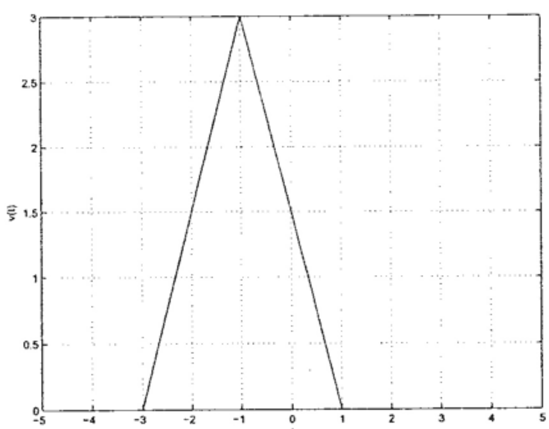


Figure S1.5-6a: Plot of $v(t) = 3x\left(-\frac{1}{2}(t+1)\right)$.

(b) Since $v(t)$ is finite duration, $P_v = 0$. Signal is unaffected by shifting, so $v(t)$ is shifted to start at $t = 0$. By symmetry, the energy of the first half is equal to the energy of the second half. Thus, $E_v = 2\int_0^2 \left(\frac{3}{2}t\right)^2 dt = 2\frac{9}{4}\frac{t^3}{3}\Big|_{t=0}^{t=2} = \frac{24}{2} = 12$. Thus,

$$E_v = 12 \text{ and } P_v = 0.$$

(c) Using 1.5-6a, $v(t) = (1.5t+4.5)(u(t+3)-u(t+1))+(-1.5t+1.5)(u(t+1)-u(t-1))$. MATLAB is used to determine and plot $v_e(t)$.

```
>> v_str = ['(1.5*t+4.5).*((t>=-3)&(t<-1))+',...
            '(-1.5*t+1.5).*((t>=-1)&(t<1))'];
>> v = inline(v_str); t = [-4:.001:4]; v_e = 0.5*(v(t)+v(-t));
>> plot(t,v_e,'k'); xlabel('t'); ylabel('v_e(t)');
>> axis([-4 4 -.25 1.75]);
```

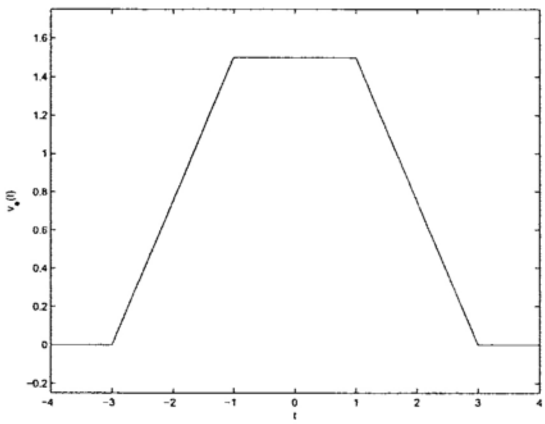


Figure S1.5-6c: Plot of $v_e(t)$.

Thus,

$$v_e(t) = \begin{cases} 3t/4 + 9/4 & -3 \le t < -1 \\ 1.5 & -1 \le t < 1 \\ -3t/4 + 9/4 & 1 \le t < 3 \\ 0 & \text{otherwise} \end{cases}.$$

(d) Using 1.5-6a, $v(t) = (1.5t+4.5)(u(t+3)-u(t+1))+(-1.5t+1.5)(u(t+1)-u(t-1))$. MATLAB is used to create the four desired plots.

```
>> v_str = ['(1.5*t+4.5).*((t>=-3)&(t<-1))+',...
            '(-1.5*t+1.5).*((t>=-1)&(t<1))'];
>> v = inline(v_str);
>> t = [-6:.001:1]; a = 2; b = 3; ax = [-6 1 -.5 9.5];
>> subplot(221); plot(t,v(a*t+b),'k-'); grid;
>> xlabel('t'); ylabel('v(at+b)'); axis(ax);
>> subplot(222); plot(t,v(a*t)+b,'k-'); grid;
>> xlabel('t'); ylabel('v(at)+b'); axis(ax);
>> subplot(223); plot(t,a*v(t+b),'k-'); grid;
>> xlabel('t'); ylabel('av(t+b)'); axis(ax);
>> subplot(224); plot(t,a*v(t)+b,'k-'); grid;
>> xlabel('t'); ylabel('av(t)+b'); axis(ax);
```

(e) Using 1.5-6a, $v(t) = (1.5t+4.5)(u(t+3)-u(t+1))+(-1.5t+1.5)(u(t+1)-u(t-1))$. MATLAB is used to create the four desired plots.

```
>> v_str = ['(1.5*t+4.5).*((t>=-3)&(t<-1))+',...
            '(-1.5*t+1.5).*((t>=-1)&(t<1))'];
>> v = inline(v_str);
>> t = [-3:.001:3]; a = -3; b = -2; ax = [-3 3 -11.5 3.5];
```

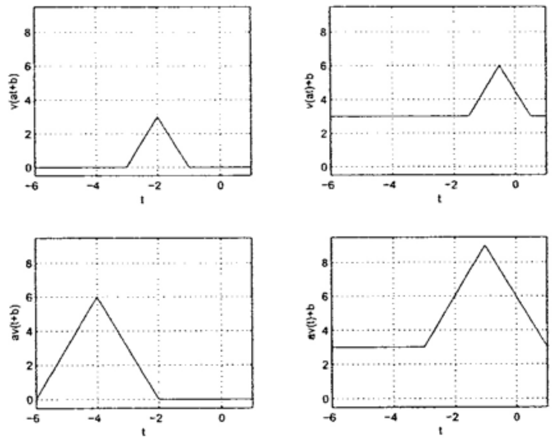


Figure S1.5-6d: Plots of $v(2t+3)$, $v(2t)+3$, $2v(t+3)$, and $2v(t)+3$.

```
>> subplot(221); plot(t,v(a*t+b),'k-'); grid;
>> xlabel('t'); ylabel('v(at+b)'); axis(ax);
>> subplot(222); plot(t,v(a*t)+b,'k-'); grid;
>> xlabel('t'); ylabel('v(at)+b'); axis(ax);
>> subplot(223); plot(t,a*v(t+b),'k-'); grid;
>> xlabel('t'); ylabel('av(t+b)'); axis(ax);
>> subplot(224); plot(t,a*v(t)+b,'k-'); grid;
>> xlabel('t'); ylabel('av(t)+b'); axis(ax);
```

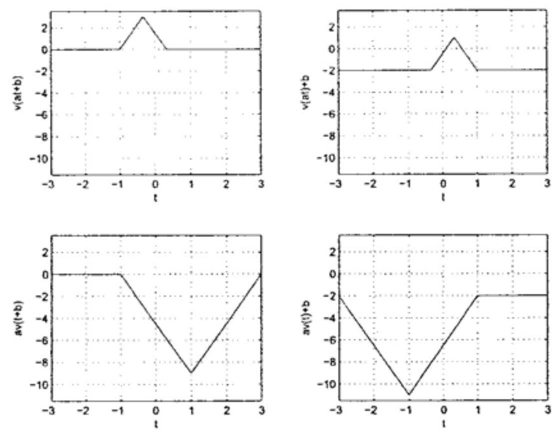


Figure S1.5-6e: Plots of $v(-3t-2)$, $v(-3t)-2$, $-3v(t-2)$, and $-3v(t)-2$.

1.5-7. (a) Using the figure, $y(t) = t(u(t) - u(t-1)) + (u(t-1) - u(t-2))$. MATLAB is used to plot $y_o(t) = \frac{y(t)-y(-t)}{2}$.

```
>> y = inline('t.*((t>=0)&(t<1))+((t>=1)&(t<2))');
>> t = [-5:.001:5]; y_o = (y(t)-y(-t))/2;
>> plot(t,y_o,'k-'); xlabel('t'); ylabel('y_o(t)'); axis([-5 5 -.6 .6]);
```

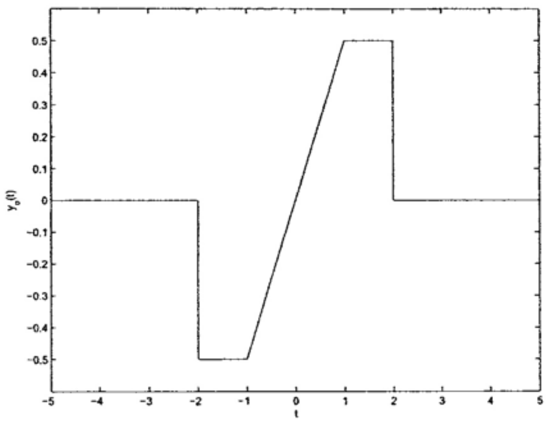


Figure S1.5-7a: Plot of $y_o(t) = \frac{y(t)-y(-t)}{2}$.

Thus,

$$y_o(t) = \begin{cases} -1/2 & -2 \le t < -1 \\ t/2 & -1 \le t < 1 \\ 1/2 & 1 \le t < 2 \\ 0 & \text{otherwise} \end{cases}.$$

(b) Since $y(t) = 0.2x(-2t-3)$, $5y(-0.5t-1.5) = 5(0.2)x(-2(-0.5t-1.5)-3) = x(t)$. MATLAB is used to sketch $x(t)$.

```
>> y = inline('t.*((t>=0)&(t<1))+((t>=1)&(t<2))');
>> t = [-8:.001:0]; x = 5*y(-0.5*t-1.5);
>> plot(t,x,'k-'); xlabel('t'); ylabel('x(t)'); axis([-8 0 -.5 5.5]);
```

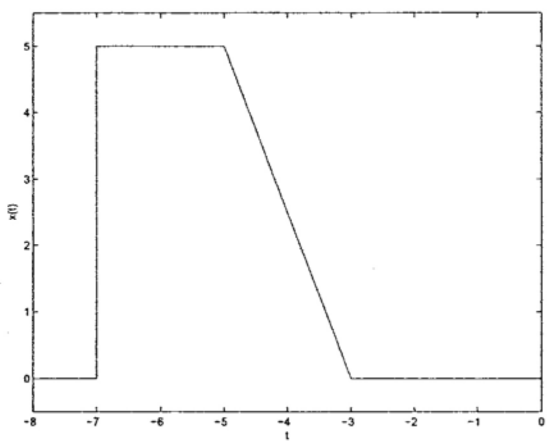


Figure S1.5-7b: Plot of $x(t) = 5y(-0.5t - 1.5)$.

Thus,

$$x(t) = \begin{cases} 5 & -7 \le t < -5 \\ -5(t+3)/2 & -5 \le t < -3 \\ 0 & \text{otherwise} \end{cases}.$$

1.5-8. Let the graphed signal be named $y(t)$.

(a) Since $y(t) = -0.5x(-3t+2)$, $-2y(-t/3+2/3) = -2(-0.5)x(-3(-t/3+2/3)+2) = x(t)$. MATLAB is used to sketch $x(t)$.

```
>> y = inline('((t>=-1)&(t<0))+(-t+1).*((t>=0)&(t<1))');
>> t = [-2:.001:6]; x = -2*y(-t/3+2/3);
>> plot(t,x,'k-'); xlabel('t'); ylabel('x(t)'); axis([-2 6 -2.5 0.5]);
```

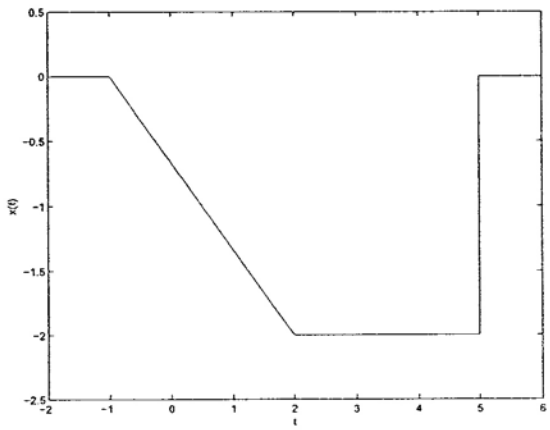


Figure S1.5-8a: Plot of $x(t) = -2y(-t/3+2/3)$.

Thus,

$$x(t) = \begin{cases} -2(t+1)/3 & -1 \leq t < 2 \\ -2 & 2 \leq t < 5 \\ 0 & \text{otherwise} \end{cases}.$$

(b) The even portion of $x(t)$ is $x_e(t) = 0.5(x(t) + x(-t))$.

```
>> x = inline('-2*(t+1)/3.*((t>=-1)&(t<2))-2*((t>=2)&(t<5))');
>> t = [-6:.001:6]; x_e = 0.5*(x(t)+x(-t));
>> plot(t,x_e,'k-'); xlabel('t'); ylabel('x_e(t)'); axis([-6 6 -1.5 0.5]);
```

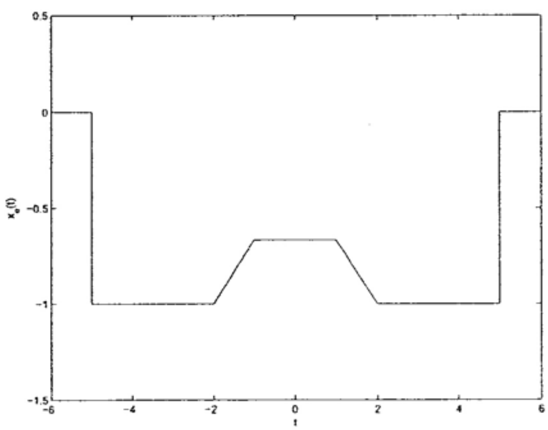


Figure S1.5-8b: Plot of $x_e(t)$.

Thus,

$$x_e(t) = \begin{cases} -1 & 2 \leq |t| < 5 \\ (-|t|-1)/3 & 1 \leq |t| < 2 \\ -2/3 & |t| < 1 \\ 0 & \text{otherwise} \end{cases} .$$

(c) The odd portion of $x(t)$ is $x_o(t) = 0.5(x(t) - x(-t))$.

```
>> x = inline('-2*(t+1)/3.*((t>=-1)&(t<2))-2*((t>=2)&(t<5))');
>> t = [-6:.001:6]; x_o = 0.5*(x(t)-x(-t));
>> plot(t,x_o,'k-'); xlabel('t'); ylabel('x_e(t)'); axis([-6 6 -1.5 1.5]);
```

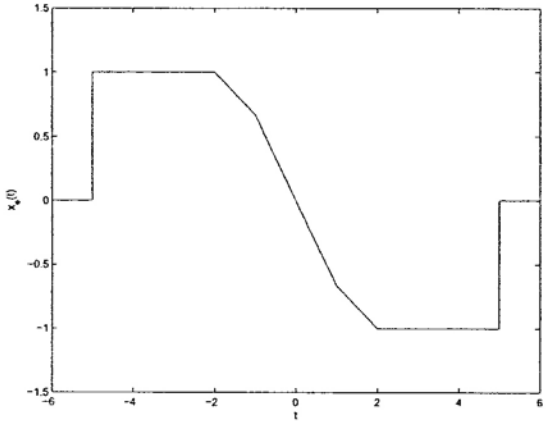


Figure S1.5-8c: Plot of $x_o(t)$.

Thus,

$$x_o(t) = \begin{cases} 1 & -5 \leq t < -2 \\ (-t+1)/3 & -2 \leq t < -1 \\ -2t/3 & -1 \leq t < 1 \\ -(t+1)/3 & 1 \leq t < 2 \\ -1 & 2 \leq t < 5 \\ 0 & \text{otherwise} \end{cases} .$$

1.5-9. Notice, $w_{cs}^*(-t) = 0.5(w(-t) + w^*(t))^* = 0.5(w^*(-t) + w(t)) = w_{cs}(t)$. In Cartesian form, this becomes $w_{cs}^*(-t) = x(-t) - jy(-t) = x(t) + jy(t) = w_{cs}(t)$. Equating the real portions yields $x(-t) = x(t)$, and equating the imaginary portions yields $-y(-t) = y(t)$. Thus, by definition, the real portion of $w_{cs}(t)$ is even and the imaginary portion of $w_{cs}(t)$ is odd.

1.5-10. Notice, $-w_{ca}^*(-t) = -0.5(w(-t) - w^*(t))^* = 0.5(-w^*(-t) + w(t)) = w_{ca}(t)$. In Cartesian form, this becomes $-w_{ca}^*(-t) = -x(-t) + jy(-t) = x(t) + jy(t) = w_{ca}(t)$. Equating the real portions yields $-x(-t) = x(t)$, and equating the imaginary portions yields $y(-t) = y(t)$. Thus, by definition, the real portion of $w_{ca}(t)$ is odd and the imaginary portion of $w_{ca}(t)$ is even.

1.5-11. Complex signal $w(t)$ is defined over $(0 \leq t \leq 1)$.

(a) Assigning certain properties to $w(t)$ allows us to plot $w(t)$ over $(-1 \leq t \leq 1)$.

i. If $w(t)$ is even, $w(t) = w(-t)$, it is even in both the real and imaginary components. Thus, the graph folds back on itself and appears unchanged.

Consider, for example, point (2,1), which now corresponds to both $t = 1$ and $t = -1$.

ii. If $w(t)$ is odd, $w(t) = -w(-t)$, it is odd in both the real and imaginary components. Thus, the graph reflects about both the real and imaginary axes.

iii. If $w(t)$ is conjugate symmetric, $w(t) = w^*(-t)$, it is even in the real component and odd in the imaginary component. Thus, the graph reflects about the real axis.

iv. If $w(t)$ is conjugate antisymmetric, $w(t) = -w^*(-t)$, it is odd in the real component and even in the imaginary component. Thus, the graph reflects about the imaginary axis.

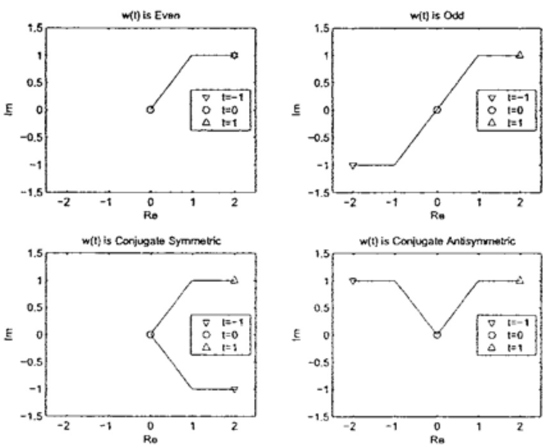


Figure S1.5-11a: Plots of $w(t)$.

(b) Since $w(t)$ is only given over $(0 \leq t \leq 1)$, $w(3t)$ can be determined only for $(0 \leq t \leq 1/3)$. Since the function does not change, only the time at which it occurs, the complex-plane graph of $w(3t)$ looks identical to the original complex-plane graph of $w(t)$ with the exception that the points are assigned different times. For example, point (2,1) occurs now at $t = 1/3$.

1.5-12. (a) We know $x(t) = t^2(1+j)$ over $(1 \leq t \leq 2)$. Since $x(t)$ is skew-Hermitian, $x(t) = -x^*(-t)$ and thus $x(t) = -t^2(1-j)$ over $(-2 \leq t \leq -1)$. To minimize energy, $x(t)$ is set to zero everywhere else. Thus,

$$x(t) = \begin{cases} t^2(1+j) & 1 \leq t \leq 2 \\ -t^2(1-j) & -2 \leq t \leq -1 \\ 0 & \text{otherwise} \end{cases}.$$

(b) MATLAB is used to sketch $y(t) = \text{Real}\{x(t)\}$.

```
>> x_str = ['(t.^2*(1+j)).*((t>=1)&(t<=2))+',...
            '(-t.^2*(1-j)).*((t>=-2)&(t<=-1))'];
>> x = inline(x_str);
>> t = [-2.5:.001:2.5]; plot(t,real(x(t)),'k');
>> xlabel('t'); ylabel('y(t)'); axis([-2.5 2.5 -4.5 4.5]);
```

(c) MATLAB is used to sketch $z(t) = \text{Real}\{jx(-2t+1)\}$.

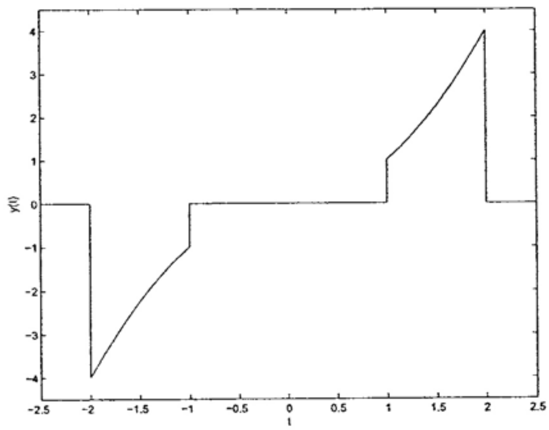


Figure S1.5-12b: Plot of $y(t) = \text{Real}\{x(t)\}$.

```
>> x_str = ['(t.^2*(1+j)).*((t>=1)&(t<=2))+',...
           '(-t.^2*(1-j)).*((t>=-2)&(t<=-1))'];
>> x = inline(x_str);
>> t = [-2.5:.001:2.5]; plot(t,real(j*x(-2*t+1)),'k');
>> xlabel('t'); ylabel('z(t)'); axis([-2.5 2.5 -4.5 4.5]);
```

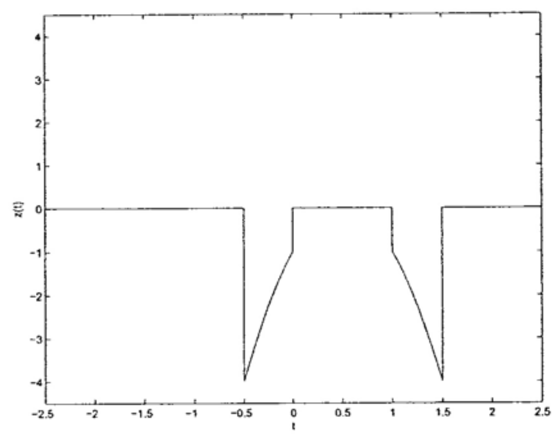


Figure S1.5-12c: Plot of $z(t) = \text{Real}\{jx(-2t+1)\}$.

(d) Since $x(t)$ is finite duration, $P_x = 0$. Using symmetry, $E_x = 2\int_1^2 (t^2(1+j))(t^2(1-j))dt = 2\int_1^2 2t^4 dt = \frac{4t^5}{5}\Big|_{t=1}^{2} = \frac{4}{5}(32-1) = \frac{124}{5}$. Thus,

$$E_x = \frac{124}{5} = 24.8 \text{ and } P_x = 0.$$

1.6-1. If $x(t)$ and $y(t)$ are the input and output, respectively, of an ideal integrator, then

$$\dot{y}(t) = x(t)$$

and

$$y(t)=\int_{-\infty}^{t}x(\tau)\,d\tau=\int_{-\infty}^{0}x(\tau)\,d\tau+\int_{0}^{t}x(\tau)\,d\tau=\underbrace{y(0)}_{\text{zero-input}}+\underbrace{\int_{0}^{t}x(\tau)\,d\tau}_{\text{zero-state}}$$

1.6-2. From Newton's law

$$x(t)=M\frac{dv}{dt}$$

and

$$v(t)=\frac{1}{M}\int_{-\infty}^{t}x(\tau)\,d\tau=\frac{1}{M}\int_{-\infty}^{0}x(\tau)\,d\tau+\frac{1}{M}\int_{0}^{t}x(\tau)\,d\tau=v(0)+\frac{1}{M}\int_{0}^{t}x(\tau)\,d\tau$$

1.7-1. Only (b), (f), and (h) are linear. All the remaining are nonlinear. This can be verified by using the procedure discussed in Example 1.9.

1.7-2. (a) The system is time-invariant because the input $x(t)$ yields the output $y(t)=x(t-2)$. Hence, if the input is $x(t-T)$, the output is $x(t-T-2)=y(t-T)$, which makes the system time-invariant.

(b) The system is time-varying. The input $x(t)$ yields the output $y(t)=x(-t)$. Thus, the output is obtained by changing the sign of t in $x(t)$. Therefore, when the input is $x(t-T)$, the output is $x(-t-T)=x(-[t+T])=y(t+T)$, which represents the original output advanced by T (not delayed by T).

(c) The system is time-varying. The input $x(t)$ yields the output $y(t)=x(at)$, which is a scaled version of the input. Thus, the output is obtained by replacing t in the input with at. Thus, if the input is $x(t-T)$ ($x(t)$ delayed by T), the output is $x(at-T)=x(a[t-\frac{T}{a}])$, which is $x(at)$ delayed by T/a (not T). Hence the system is time-varying.

(d) The system is time-varying. The input $x(t)$ yields the output $y(t)=tx(t)$. For the input $x(t-T)$, the output is $tx(t-T)$, which is not $tx(t)$ delayed by T. Hence the system is time-varying.

(e) The system is time-varying. The output is a constant, given by the area under $x(t)$ over the interval $|t|\le 5$. Now, if $x(t)$ is delayed by T, the output, which is the area under the delayed $x(t)$, is another constant. But this output is not the same as the original output delayed by T. Hence the system is time-varying.

(f) The system is time-invariant. The input $x(t)$ yields the output $y(t)$, which is the square of the second derivative of $x(t)$. If the input is delayed by T, the output is also delayed by T. Hence the system is time-invariant.

1.7-3. We construct the table below from the first three rows of data. Let r_j denote the jth row.

Row	$x(t)$	$q_1(0)$	$q_2(0)$	$y(t)$
r_1	0	1	-1	$e^{-t}u(t)$
r_2	0	2	1	$e^{-t}(3t+2)u(t)$
r_3	$u(t)$	-1	-1	$2u(t)$
$r_4=\frac{1}{3}(r_1+r_2)$	0	1	0	$(t+1)e^{-t}u(t)$
$r_5=\frac{1}{2}(r_1+r_3)$	$\frac{1}{2}u(t)$	0	-1	$(\frac{1}{2}e^{-t}+1)u(t)$
$r_6=(r_4+r_5)$	$\frac{1}{2}u(t)$	1	-1	$(1.5e^{-t}+te^{-t}+1)u(t)$
$r_7=2(r_6+r_1)$	$u(t)$	0	0	$(e^{-t}+2te^{-t}+2)u(t)$

In our case, the input $x(t) = u(t+5) - u(t-5)$. From r_7 and the superposition and time-invariance properties, we have

$$\begin{aligned} y(t) &= r_7(t+5) - r_7(t-5) \\ &= \left[e^{-(t+5)} + 2(t+5)e^{-(t+5)} + 2\right]u(t+5) - \left[e^{-(t-5)} + 2(t-5)e^{-(t-5)} + 2\right]u(t-5) \end{aligned}$$

1.7-4. If the input is $kx(t)$, the new output $y(t)$ is

$$y(t) = k^2x^2(t)/\left(k\frac{dx}{dt}\right) = k[x^2(t)/\left(\frac{dx}{dt}\right)]$$

Hence the homogeneity is satisfied. If the input-output pair is denoted by $x_i \rightarrow y_i$, then

$$x_1 \rightarrow y_1 = (x_1)^2/(\dot{x}_1) \quad \text{and} \quad x_2 \rightarrow y_2 = (x_2)^2/(\dot{x}_2)$$

$$\text{But} \quad x_1 + x_2 \rightarrow (x_1 + x_2)^2/(\dot{x}_1 + \dot{x}_2) \neq y_1 + y_2$$

1.7-5. From the hint it is clear that when $v_c(0) = 0$, the capacitor may be removed, and the circuit behaves as shown in Figure S1.7-5. It is clearly zero-state linear. To show that it is zero-input nonlinear, consider the circuit with $x(t) = 0$ (zero-input). The current $y(t)$ has the same direction (shown by arrow) regardless of the polarity of v_c (because the input branch is a short). Thus the system is zero-input nonlinear.

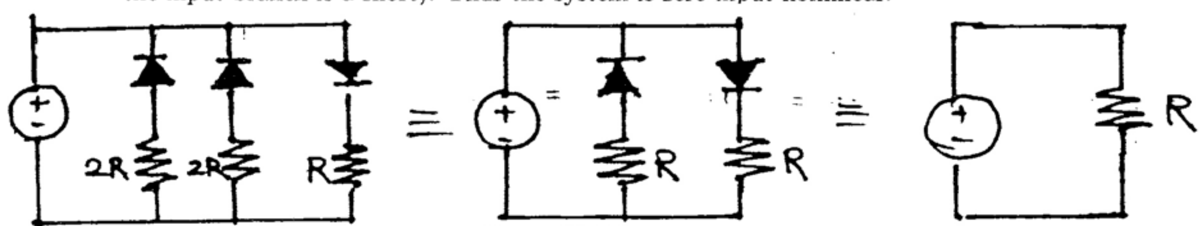


Figure S1.7-5

1.7-6. The solution is trivial. The input is a current source, which has infinite impedance. Hence, as far as the output $y(t)$ is concerned, the circuit behaves as shown in Figure S1.7-6. The nonlinear elements are irrelevant in computing the output $y(t)$, and the output $y(t)$ satisfies the linearity conditions. Yet, the circuit is nonlinear because it contains nonlinear elements.

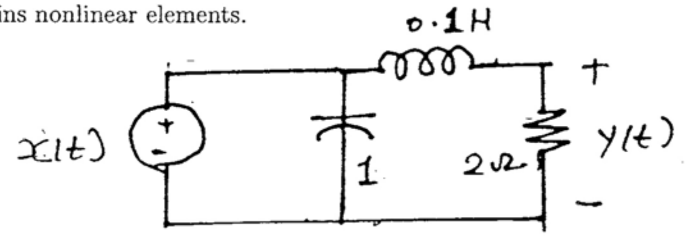


Figure S1.7-6

1.7-7. (a) $y(t) = x(t-2)$. Thus, the output $y(t)$ always starts after the input by 2 seconds (see Figure S1.7-7a). Clearly, the system is causal.

(b) $y(t) = x(-t)$. The output $y(t)$ is obtained by time inversion in the input. Thus, if the input starts at $t = 0$, the output starts before $t = 0$ (see Figure S1.7-7b). Hence, the system is not causal.

(c) $y(t) = x(at)$, $a > 1$. The output $y(t)$ is obtained by time compression of the input by factor a. Hence, the output can start before the input (see Figure S1.7-7c), and the system is not causal.

(d) $y(t) = x(at)$, $a < 1$. The output $y(t)$ is obtained by time expansion of the input by factor $1/a$. Hence, the output can start before the input (see Figure S1.7-7d), and the system is not causal.

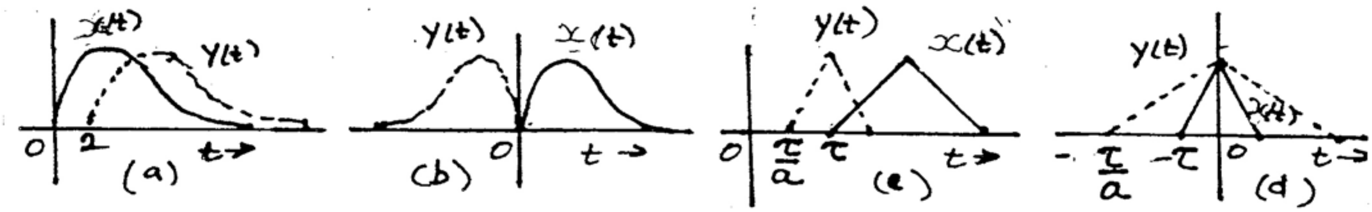


Figure S1.7-7

1.7-8. (a) Invertible because the input can be obtained by taking the derivative of the output. Hence, the inverse system equation is $y(t) = dx/dt$.

(b) Not invertible for even values of n, because the sign information is lost. However, the system is invertible for odd values of n. The inverse system equation is $y(t) = [x(t)]^{1/n}$.

(c) Not invertible because differentiation operation irretrievable loses the constant part of $x(t)$.

(d) The system $y(t) = x(3t - 6) = x(3[t - 2])$ represents an operation of signal compression by factor 3, and then time delay by 2 seconds. Hence, the input can be obtained from the output by first advancing the output by 2 seconds, and then time-expanding by factor 3. Hence, the inverse system equation is $y(t) = x(\frac{t}{3} + 2)$. Although the system is invertible, it is not realizable because it involves the operation of signal compression and signal advancing (which makes it noncausal). However, if we can accept time delay, we can realize a noncausal system.

(e) Not invertible because cosine is a multiple valued function, and $\cos^{-1}[x(t)]$ is not unique.

(f) Invertible. $x(t) = \ln y(t)$.

1.7-9. (a) Yes, the system is linear. Begin assuming $y_1(t) = r(t)x_1(t)$ and $y_2(t) = r(t)x_2(t)$. Applying $ax_1(t) + bx_2(t)$ to the system yields $y(t) = r(t)\,(ax_1(t) + bx_2(t)) = ar(t)x_1(t) + br(t)x_2(t) = ay_1(t) + by_2(t)$.

(b) Yes, the system is memoryless. By inspection, it is clear that the system only depends on the current input.

(c) Yes, the system is causal. Since the system is memoryless, the system cannot depend on future values and must be causal.

(d) No, the system is not time-invariant. Since the system function depends on the independent variable t, it is unlikely that the system is time-invariant. To explicitly verify, let $y(t) = r(t)x(t)$. Next, delay $x(t)$ by τ to obtain a new input $x_2 = x(t-\tau)$. Applying $x_2(t)$ to the system yields $y_2(t) = r(t)x_2(t) = r(t)x(t-\tau) \neq r(t-\tau)x(t-\tau) = y(t-\tau)$. Since, the system operator and the time-shift operator do not commute, the system is not time-invariant.

1.7-10. Using the sifting property, this system operation is rewritten as $y(t) = 0.5\,(x(t) - x(-t))$.

(a) This system extracts the odd portion of the input.

(b) Yes, the system is BIBO stable. If the input is bounded, then the output is necessarily bounded. That is, if $|x(t)| \leq M_x < \infty$, then $|y(t)| = |0.5\,(x(t) - x(-t))\,| \leq 0.5\,(|x(t)| + |-x(-t)|) \leq M_x < \infty$.

(c) Yes, the system is linear. Let $y_1(t) = 0.5(x_1(t) - x_1(-t))$ and $y_2(t) = 0.5(x_2(t) - x_2(-t))$. Applying $ax_1(t) + bx_2(t)$ to the system yields $y(t) = 0.5\,(ax_1(t) + bx_2(t) - (ax_1(-t) + bx_2(-t))) = 0.5a(x_1(t) - x_1(-t)) + 0.5b(x_2(t) - x_2(-t)) = ay_1(t) + by_2(t)$.

(d) No, the system is not memoryless. For example, at time $t = 1$ the output $y(1) = 0.5(x(1) - x(-1))$ depends on a past value of the input, $x(-1)$.

(e) No, the system is not causal. For example, at time $t = -1$ the output $y(-1) = 0.5(x(-1) - x(1))$ depends on a future value of the input, $x(1)$.

(f) No, the system is not time-invariant. For example, let the input be $x(t) = t(u(t+1) - u(t-1))$. Since this input is already odd, the output is just the input, $y(t) = x(t)$. Shifting by a non-zero τ, $x(t-\tau)$ is not odd, and the output is not $y(t-\tau) = x(t-\tau)$. Thus, the system cannot be time-invariant.

1.7-11. (a) No, the system is not BIBO stable. The system returns the time-delayed derivative, or slope, of the input signal. A square-wave is a bounded signal which, due to point discontinuities, has infinite slope at certain instants in time. Thus, a bounded input may not result in a bounded output, and the system cannot be BIBO stable.

(b) Yes, the system is linear. Begin assuming $y_1(t) = \frac{d}{dt}x_1(t-1)$ and $y_2(t) = \frac{d}{dt}x_2(t-1)$. Applying $ax_1(t) + bx_2(t)$ to the system yields $y(t) = \frac{d}{dt}\,(ax_1(t-1) + bx_2(t-1)) = a\frac{d}{dt}x_1(t-1) + b\frac{d}{dt}x_2(t-1) = ay_1(t) + by_2(t)$.

(c) No, the system is not memoryless. By inspection, it is clear that the system depends on a past value of the input. For example, at $t = 0$, the output $y(0)$ depends on the time-derivative of $x(-1)$, a past value.

(d) Yes, the system is causal. By inspection, it is clear that the system does not depend on future values.

(e) Yes, the system is time-invariant. To explicitly verify, let $y(t) = \frac{d}{dt}x(t-1)$. Next, delay $x(t)$ by τ to obtain a new input $x_2 = x(t-\tau)$. Applying $x_2(t)$ to the system yields $y_2(t) = \frac{d}{dt}x_2(t) = \frac{d}{dt}x(t-1-\tau) = y(t-\tau)$. Since, the system operator and the time-shift operator commute, the system is time-invariant. In more loose terms, the derivative operator returns the delayed slope of a signal independent of when that signal is applied.

1.7-12. (a) Yes, the system is BIBO stable. From the definition of the system, we know that $y(t)$ is either $x(t)$ or 0. Correspondingly, if $|x(t)| < \infty$ then $|y(t)| < \infty$, and the system must be BIBO stable.

(b) No, the system is not linear. Consider two signals: $x_1(t) = 1$ and $x_2(t) = \cos(t)$. The corresponding outputs of these individual signals is $y_1(t) = 1$ and $y_2(t) = \begin{cases} \cos(t) & \text{if } \cos(t) > 0 \\ 0 & \text{if } \cos(t) \leq 0 \end{cases}$. However, if we create a third input $x_3(t) = x_1(t) + x_2(t)$, the system output is $y_3(t) = 1 + \cos(t) \neq y_1(t) + y_2(t)$. Since superposition does not apply, the system cannot be linear.

(c) Yes, the system is memoryless. By inspection, it is clear that the system only depends on the current input.

(d) Yes, the system is causal. Since the system is memoryless, the system cannot depend on future values and must be causal.

(e) Yes, the system is time-invariant. Consider delaying $x(t)$ by τ to obtain a new input $x_2 = x(t-\tau)$. Applying $x_2(t)$ to the system yields $y_2(t) = \begin{cases} x(t-\tau) & \text{if } x(t-\tau) > 0 \\ 0 & \text{if } x(t-\tau) \leq 0 \end{cases} = y(t-\tau)$. Since the system operator and the time-shift operator commute, the system is time-invariant.

1.7-13. (a) No, Bill is not correct. The $x_1(3t)$ term represents a compression rather than the necessary dilation. One way to construct $x_2(t)$ is $x_2(t) = 2x_1(t/3) - x_1(t-1)$. However, this form is not unique; $x_2(t) = 2x_1(t) + x_1(t-1) + 2x_1(t-2)$ also works and may be more useful.

(b) The output $y_1(t)$ is given for the input signal $x_1(t)$. The expression $x_2(t) = 2x_1(t)+x_1(t-1)+2x_1(t-2)$ forms $x_2(t)$ from a superposition of scaled and shifted copies of $x_1(t)$. Since the system is linear and time invariant, the operations of scaling, summing, and shifting commute with the system operator. Thus, $y_2(t) = 2y_1(t) + y_1(t-1) + 2y_1(t-2)$. Notice, it is not true that $y_2(t) = 2y_1(t/3) - y_1(t-1)$; linearity and time-invariance cannot help with the time-scaling operation. MATLAB is used to plot $y_2(t)$.

```
>> t = [-1:.001: 4]; y_1 = inline('t.*((t>=0)&(t<1))+(t>=1)');
>> y_2 = 2*y_1(t)+y_1(t-1)+2*y_1(t-2);
>> plot(t,y_2,'k-'); xlabel('t'); ylabel('y_2(t)'); axis([-1 4 -.5 5.5]);
```

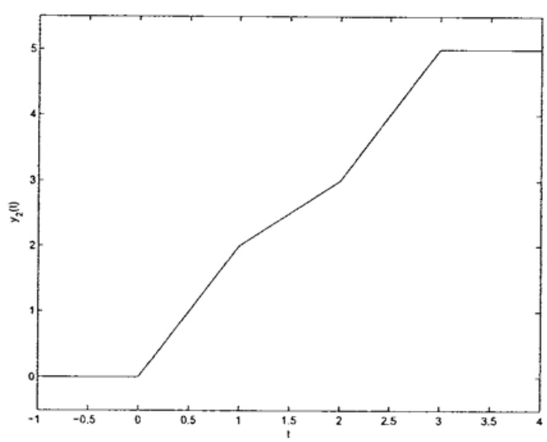


Figure S1.7-13b: Plot of $y_2(t) = 2y_1(t) + y_1(t-1) + 2y_1(t-2)$.

1.8-1. The loop equation for the circuit is

$$3y_1(t) + Dy_1(t) = x(t) \quad \text{or} \quad (D+3)y_1(t) = x(t) \tag{1}$$

Also

$$Dy_1(t) = y_2(t) \Longrightarrow y_1(t) = \frac{1}{D}y_2(t) \tag{2}$$

Substitution of (2) in (1) yields

$$\frac{(D+3)}{D}y_2(t) = x(t) \quad \text{or} \quad (D+3)y_2(t) = Dx(t)$$

1.8-2. The currents in the resistor, capacitor and inductor are $2y_2(t)$, $Dy_2(t)$ and $(2/D)y_2(t)$, respectively. Therefore

$$(D + 2 + \frac{2}{D})y_2(t) = x(t)$$

or

$$(D^2 + 2D + 2)y_2(t) = Dx(t) \tag{1}$$

Also

$$y_1(t) = Dy_2(t) \quad \text{or} \quad y_2(t) = \frac{1}{D}y_1(t) \tag{2}$$

Substituting of (2) in (1) yields

$$\frac{D^2 + 2D + 2}{D}y_1(t) = Dx(t)$$

or

$$(D^2 + 2D + 2)y_1(t) = D^2x(t)$$

1.8-3. The freebody diagram for the mass M is shown in Figure 1.8-3. From this diagram it follows that

$$M\ddot{y} = B(\dot{x} - \dot{y}) + K(x - y)$$

or

$$(MD^2 + BD + K)y(t) = (BD + K)x(t)$$

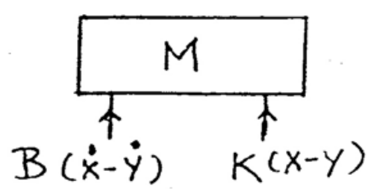


Figure S1.8-3

1.8-4. The loop equation for the field coil is

$$(DL_f + R_f)i_f(t) = x(t) \tag{1}$$

If $T(t)$ is the torque generated, then

$$T(t) = K_f i_f(t) = (JD^2 + BD)\theta(t) \tag{2}$$

Substituting of (1) in (2) yields

$$\frac{K_f}{DL_f + R_f} x(t) = (JD^2 + BD)\theta(t)$$

or

$$(JD^2 + BD)(DL_f + R_f)\theta(t) = K_f x(t)$$

1.8-5.

$$[q_i(t) - q_0(t)]\triangle t = A\triangle h$$

or

$$\dot{h}(t) = \frac{1}{A}[q_i(t) - q_0(t)] \tag{1}$$

But

$$q_0(t) = Rh(t) \tag{2}$$

Differentiation of (2) yields

$$\dot{q}_0(t) = R\dot{h}(t) = \frac{R}{A}[q_i(t) - q_0(t)]$$

and

$$\left(D + \frac{R}{A}\right) q_0(t) = \frac{R}{A} q_i(t)$$

or

$$(D + a)q_0(t) = aq_i(t) \qquad a = \frac{R}{A} \tag{3}$$

and

$$q_0(t) = \frac{a}{D + a} q_i(t)$$

substituting this in (1) yields

$$\dot{h}(t) = \frac{1}{A}\left(1 - \frac{a}{D + a}\right) q_i(t) = \frac{D}{A(D + a)} q_i(t)$$

or

$$(D + a)h(t) = \frac{1}{A} q_i(t)$$

1.8-6. (a) The order of the system is zero; there are no energy storage components such as capacitors or inductors.

(b) Using KVL on the left loop yields $x(t) = R_1 y_1(t) + R_2(y_1(t) - y_2(t)) = 3y_1(t) - 2y_2(t)$. KVL on the middle loop yields $0 = R_2(y_2(t) - y_1(t)) + R_3 y_2(t) + R_4(y_2(t) - y_3(t)) = -2y_1(t) + 9y_2(t) - 4y_3(t)$. Finally, KVL on the right loop yields $R_4(y_3(t) - y_2(t)) + (R_5 + R_6)y_3(t) = -4y_2(t) + 15y_3(t)$. Combining together yields

$$\begin{bmatrix} 3 & -2 & 0 \\ -2 & 9 & -4 \\ 0 & -4 & 15 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \begin{bmatrix} x(t) \\ 0 \\ 0 \end{bmatrix}.$$

(c) Cramer's rule suggests

$$y_3(t) = \frac{\begin{vmatrix} 3 & -2 & x(t) \\ -2 & 9 & 0 \\ 0 & -4 & 0 \end{vmatrix}}{\begin{vmatrix} 3 & -2 & 0 \\ -2 & 9 & -4 \\ 0 & -4 & 15 \end{vmatrix}}.$$

MATLAB computes the denominator determinant.

```
>> det([3 -2 0;-2 9 -4;0 -4 15])
ans = 297
```

The numerator determinant is easy computed by hand as $0 + 0 + 8x(t) = 8x(t)$. Thus,

$$y_3(t) = \frac{8}{297}x(t) = \frac{8}{297}(2 - |\cos(t)|)\,u(t-1).$$

1.10-1. From Figure P1.8-2, we obtain

$$x(t) = q_1/2 + \dot{q}_1 + q_2$$

Moreover, the capacitor voltage $q_1(t)$ equals the voltage across the inductor, which is $\frac{1}{2}\dot{q}_2$. Hence, the state equations are

$$\dot{q}_1 = -q_1/2 - q_2 - x \quad \text{and} \quad \dot{q}_2 = 2q_1$$

1.10-2. The capacitor current $C\dot{q}_3 = \frac{1}{2}\dot{q}_3$ is $q_1 - q_2$. Therefore

$$\dot{q}_3 = 2q_1 - 2q_2 \tag{1}$$

The two loop equations are

$$2q_1 + \dot{q}_1 + q_3 = x \Longrightarrow \dot{q}_1 = -2q_1 - q_3 + x \tag{2}$$

$$-q_3 + \frac{1}{3}\dot{q}_2 + q_2 = 0 \Longrightarrow \dot{q}_2 = -3q_2 + 3q_3 \tag{3}$$

Equations (1), (2) and (3) are the state equations.

For the 2Ω resistor: current is q_1, voltage is $2q_1$.

For the 1H inductor: current is q_1, voltage is $\dot{q}_1 = x(t) - 2q_1 - q_3$.

For the capacitor: current is $q_1 - q_2$, voltage is q_3.

For the $\frac{1}{3}$H inductor: current is q_2, voltage is $\frac{1}{3}\dot{q}_2 = -q_2 + q_3$.

For the 1Ω resistor: current is q_2 and voltage is q_2.

At the instant t, $q_1 = 5$, $q_2 = 1$, $q_3 = 2$ and $x = 10$. Substituting these values in the above results yields

2Ω resistor: current 5A, voltage 10A.

1H capacitor: current 5A, voltage $10 - 10 - 2 = -2$V.

The capacitor: current $5 - 1 = 4$A, voltage 2V.

The $\frac{1}{3}$H inductor: current 1A, voltage $-1 + 2 = 1$V.

The 1Ω resistor: current 1A, voltage 1V.

Chapter 2 Solutions

2.2-1. The characteristic polynomial is $\lambda^2+5\lambda+6$. The characteristic equation is $\lambda^2+5\lambda+6=0$. Also $\lambda^2+5\lambda+6=(\lambda+2)(\lambda+3)$. Therefore the characteristic roots are $\lambda_1=-2$ and $\lambda_2=-3$. The characteristic modes are e^{-2t} and e^{-3t}. Therefore

$$y_0(t) = c_1 e^{-2t} + c_2 e^{-3t}$$

and

$$\dot{y}_0(t) = -2c_1 e^{-2t} - 3c_2 e^{-3t}$$

Setting $t=0$, and substituting initial conditions $y_0(0)=2$, $\dot{y}_0(0)=-1$ in this equation yields

$$\left.\begin{aligned} c_1 + c_2 &= 2 \\ -2c_1 - 3c_2 &= -1 \end{aligned}\right\} \Longrightarrow \quad \begin{aligned} c_1 &= 5 \\ c_2 &= -3 \end{aligned}$$

Therefore

$$y_0(t) = 5e^{-2t} - 3e^{-3t}$$

2.2-2. The characteristic polynomial is $\lambda^2+4\lambda+4$. The characteristic equation is $\lambda^2+4\lambda+4=0$. Also $\lambda^2+4\lambda+4=(\lambda+2)^2$, so that the characteristic roots are -2 and -2 (repeated twice). The characteristic modes are e^{-2t} and te^{-2t}. Therefore

$$y_0(t) = c_1 e^{-2t} + c_2 t e^{-2t}$$

and

$$\dot{y}_0(t) = -2c_1 e^{-2t} - 2c_2 t e^{-2t} + c_2 e^{-2t}$$

Setting $t=0$ and substituting initial conditions yields

$$\left.\begin{aligned} 3 &= c_1 \\ -4 &= -2c_1 + c_2 \end{aligned}\right\} \Longrightarrow \quad \begin{aligned} c_1 &= 3 \\ c_2 &= 2 \end{aligned}$$

Therefore

$$y_0(t) = (3+2t)e^{-2t}$$

2.2-3. The characteristic polynomial is $\lambda(\lambda+1)=\lambda^2+\lambda$. The characteristic equation is $\lambda(\lambda+1)=0$. The characteristic roots are 0 and -1. The characteristic modes are 1 and e^{-t}. Therefore

$$y_0(t) = c_1 + c_2 e^{-t}$$

and

$$\dot{y}_0(t) = -c_2 e^{-t}$$

Setting $t=0$, and substituting initial conditions yields

$$\left.\begin{aligned} 1 &= c_1 + c_2 \\ 1 &= -c_2 \end{aligned}\right\} \Longrightarrow \quad \begin{aligned} c_1 &= 2 \\ c_2 &= -1 \end{aligned}$$

Therefore

$$y_0(t) = 2 - e^{-t}$$

2.2-4. The characteristic polynomial is λ^2+9. The characteristic equation is $\lambda^2+9=0$ or $(\lambda+j3)(\lambda-j3)=0$. The characteristic roots are $\pm j3$. The characteristic modes are e^{j3t} and e^{-j3t}. Therefore

$$y_0(t) = c\cos(3t+\theta)$$

and

$$\dot{y}_0(t) = -3c\sin(3t+\theta)$$

Setting $t=0$, and substituting initial conditions yields

$$\left.\begin{array}{r} 0 = c\cos\theta \\ 6 = -3c\sin\theta \end{array}\right\} \Longrightarrow \left.\begin{array}{r} c\cos\theta = 0 \\ c\sin\theta = -2 \end{array}\right\} \Longrightarrow \begin{array}{l} c = 2 \\ \theta = -\pi/2 \end{array}$$

Therefore

$$y_0(t) = 2\cos(3t-\frac{\pi}{2}) = 2\sin 3t$$

2.2-5. The characteristic polynomial is $\lambda^2+4\lambda+13$. The characteristic equation is $\lambda^2+4\lambda+13=0$ or $(\lambda+2-j3)(\lambda+2+j3)=0$. The characteristic roots are $-2\pm j3$. The characteristic modes are $c_1e^{(-2+j3)t}$ and $c_2e^{(-2-j3)t}$. Therefore

$$y_0(t) = ce^{-2t}\cos(3t+\theta)$$

and

$$\dot{y}_0(t) = -2ce^{-2t}\cos(3t+\theta) - 3ce^{-2t}\sin(3t+\theta)$$

Setting $t=0$, and substituting initial conditions yields

$$\left.\begin{array}{r} 5 = c\cos\theta \\ 15.98 = -2c\cos\theta - 3c\sin\theta \end{array}\right\} \Longrightarrow \left.\begin{array}{r} c\cos\theta = 5 \\ c\sin\theta = -8.66 \end{array}\right\} \Longrightarrow \begin{array}{l} c = 10 \\ \theta = -\pi/3 \end{array}$$

Therefore

$$y_0(t) = 10e^{-2t}\cos(3t-\frac{\pi}{3})$$

2.2-6. The characteristic polynomial is $\lambda^2(\lambda+1)$ or $\lambda^3+\lambda^2$. The characteristic equation is $\lambda^2(\lambda+1)=0$. The characteristic roots are 0, 0 and -1 (0 is repeated twice). Therefore

$$y_0(t) = c_1 + c_2t + c_3e^{-t}$$

and

$$\begin{aligned} \dot{y}_0(t) &= c_2 - c_3e^{-t} \\ \ddot{y}_0(t) &= c_3e^{-t} \end{aligned}$$

Setting $t=0$, and substituting initial conditions yields

$$\left.\begin{array}{r} 4 = c_1 + c_3 \\ 3 = c_2 - c_3 \\ -1 = c_3 \end{array}\right\} \Longrightarrow \begin{array}{l} c_1 = 5 \\ c_2 = 2 \\ c_3 = -1 \end{array}$$

Therefore

$$y_0(t) = 5 + 2t - e^{-t}$$

2.2-7. The characteristic polynomial is $(\lambda+1)(\lambda^2+5\lambda+6)$. The characteristic equation is $(\lambda+1)(\lambda^2+5\lambda+6)=0$ or $(\lambda+1)(\lambda+2)(\lambda+3)=0$. The characteristic roots are -1, -2 and -3. The characteristic modes are e^{-t}, e^{-2t} and e^{-3t}. Therefore

$$y_0(t) = c_1e^{-t} + c_2e^{-2t} + c_3e^{-3t}$$

and

$$\dot{y}_0(t) = -c_1e^{-t} - 2c_2e^{-2t} - 3c_3e^{-3t}$$

$$\ddot{y}_0(t) \;=\; c_1e^{-t} + 4c_2e^{-2t} + 9c_3e^{-3t}$$

Setting $t = 0$, and substituting initial conditions yields

$$\left.\begin{array}{r} 2 = c_1 + c_2 + c_3 \\ -1 = -c_1 - 2c_2 - 3c_3 \\ 5 = c_1 + 4c_2 + 9c_3 \end{array}\right\} \Longrightarrow \quad \begin{array}{l} c_1 = 6 \\ c_2 = -7 \\ c_3 = 3 \end{array}$$

Therefore

$$y_0(t) = 6e^{-t} - 7e^{-2t} + 3e^{-3t}$$

2.2-8. The zero-input response for a LTIC system is given as $y_0(t) = 2e^{-t} + 3$. Since two modes are visible, the system must have, at least, the characteristic roots $\lambda_1 = 0$ and $\lambda_2 = -1$.

(a) No, it is not possible for the system's characteristic equation to be $\lambda + 1 = 0$ since the required mode at $\lambda = 0$ is missing.

(b) Yes, it is possible for the system's characteristic equation to be $\sqrt{3}(\lambda^2 + \lambda) = 0$ since this equation has the two required roots $\lambda_1 = 0$ and $\lambda_2 = -1$.

(c) Yes, it is possible for the system's characteristic equation to be $\lambda(\lambda+1)^2 = 0$. This equation supports a general zero-input response of $y_0(t) = c_1 + c_2e^{-t} + c_3te^{-t}$. By letting $c_1 = 3$, $c_2 = 2$, and $c_3 = 0$, the observed zero-input response is possible.

2.3-1. The characteristic equation is $\lambda^2 + 4\lambda + 3 = (\lambda + 1)(\lambda + 3) = 0$. The characteristic modes are e^{-t} and e^{-3t}. Therefore

$$y_n(t) = c_1e^{-t} + c_2e^{-3t}$$
$$\dot{y}_n(t) = -c_1e^{-t} - 3c_2e^{-3t}$$

Setting $t = 0$, and substituting $y(0) = 0$, $\dot{y}(0) = 1$, we obtain

$$\left.\begin{array}{r} 0 = c_1 + c_2 \\ 1 = -c_1 - 3c_2 \end{array}\right\} \Longrightarrow \quad \begin{array}{l} {}_1 = \frac{1}{2} \\ c_2 = -\frac{1}{2} \end{array}$$

Therefore

$$y_n(t) = \frac{1}{2}(e^{-t} - e^{-3t})$$

$$h(t) = [P(D)y_n(t)]u(t) = [(D+5)y_n(t)]u(t) = [\dot{y}_n(t) + 5y_n(t)]u(t) = (2e^{-t} - e^{-3t})u(t)$$

2.3-2. The characteristic equation is $\lambda^2 + 5\lambda + 6 = (\lambda + 2)(\lambda + 3) = 0$. and

$$y_n(t) = c_1e^{-2t} + c_2e^{-3t}$$
$$\dot{y}_n(t) = -2c_1e^{-2t} - 3c_2e^{-3t}$$

Setting $t = 0$, and substituting $y(0) = 0$, $\dot{y}(0) = 1$, we obtain

$$\left.\begin{array}{r} 0 = c_1 + c_2 \\ 1 = -2c_1 - 3c_2 \end{array}\right\} \Longrightarrow \quad \begin{array}{l} c_1 = 1 \\ c_2 = -1 \end{array}$$

Therefore

$$y_n(t) = e^{-2t} - e^{-3t}$$

and

$$[P(D)y_n(t)]u(t) = [\ddot{y}_n(t) + 7\dot{y}_n(t) + 11y_n(t)]u(t) = (e^{-2t} + e^{-3t})u(t)$$

Hence

$$h(t) = b_n\delta(t) + [P(D)y_n(t)]u(t) = \delta(t) + (e^{-2t} + e^{-3t})u(t)$$

2.3-3. The characteristic equation is $\lambda + 1 = 0$ and

$$y_n(t) = ce^{-t}$$

In this case the initial condition is $y_n^{n-1}(0) = y_n(0) = 1$. Setting $t = 0$, and using $y_n(0) = 1$, we obtain $c = 1$, and

$$y_n(t) = e^{-t}$$

$$P(D)y_n(t) = [-\dot{y}_n(t) + y_n(t)]u(t) = 2e^{-t}u(t)$$

Hence

$$h(t) = b_n\delta(t) + [P(D)y_n(t)]u(t) = -\delta(t) + 2e^{-t}u(t)$$

2.3-4. The characteristic equation is $\lambda^2 + 6\lambda + 9 = (\lambda + 3)^2 = 0$. Therefore

$$y_n(t) = (c_1 + c_2 t)e^{-3t}$$

$$\dot{y}_n(t) = [-3(c_1 + c_2 t) + c_2]e^{-3t}$$

Setting $t = 0$, and substituting $y_n(0) = 1$, $\dot{y}_n(0) = 1$, we obtain

$$\left.\begin{aligned} 0 &= c_1 \\ 1 &= -3c_1 + c_2 \end{aligned}\right\} \Longrightarrow \quad \begin{aligned} c_1 &= 0 \\ c_2 &= 1 \end{aligned}$$

and

$$y_n(t) = te^{-3t}$$

Hence

$$h(t) = [P(D)y_n(t)]u(t) = [2\dot{y}_n(t) + 9y_n(t)]u(t) = (2 + 3t)e^{-3t}u(t)$$

2.4-1.

$$\begin{aligned} A_c &= \int_{-\infty}^{\infty} c(t)\,dt \\ &= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x(\tau)g(t-\tau)\,d\tau\right]dt \\ &= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x(\tau)\,d\tau\right]g(t-\tau)\,dt \\ &= A_x\int_{-\infty}^{\infty} g(t-\tau)\,dt \\ &= A_x A_g \end{aligned}$$

This property can be readily verified from Examples 2.7 and 2.8. For Example 2.6, we note that

$$\int_{-\infty}^{\infty} e^{-at}\,dt = \frac{1}{a}$$

Use of this result yields $A_x = 1$, $A_h = 0.5$, and $A_y = 1 - 0.5 = 0.5 = A_x A_h$ For example 2.8, $A_x = 2$, $A_g = 1.5$, and

$$\begin{aligned} A_c &= \int_{-1}^{1} -\frac{1}{6}(t+1)^2\,dt + \int_{1}^{2}\frac{2}{3}t\,dt + \int_{2}^{4} -\frac{1}{6}(t^2 - 2t - 8)\,dt \\ &= \frac{4}{9} + 1 + \frac{14}{9} = 3 = A_x A_g \end{aligned}$$

2.4-2.

$$\begin{aligned} x(at) * g(at) &= \int_{-\infty}^{\infty} x(a\tau)g[a(t-\tau)]\,d\tau \\ &= \frac{1}{a}\int_{-\infty}^{\infty} x(w)g(at-w)\,dw \\ &= \frac{1}{a}c(at) \qquad a \geq 0 \end{aligned}$$

When $a < 0$, the limits of integration become from ∞ to $-\infty$, which is equivalent to the limits from $-\infty$ to ∞ with a negative sign. Hence, $x(at) * g(at) = |\frac{1}{a}|c(at)$.

2.4-3. Let $x(t) * g(t) = c(t)$. Using the time scaling property in Prob. 2.4-2 with $a = -1$, we have $x(-t) * g(-t) = c(-t)$. Now, if $x(t)$ and $g(t)$ are both even functions of t, then $x(t) = x(-t)$ and $g(-t) = g(t)$. Clearly $c(t) = c(-t)$. Using a parallel argument, we can show that if both functions are odd, $c(t) = c(-t)$, indicating that $c(t)$ is even. But if one is odd and the other is even, $c(t) = -c(-t)$, indicating that $c(t)$ is odd.

2.4-4.

$$\begin{aligned} e^{-at}u(t) * e^{-bt}u(t) &= \int_0^t e^{-a\tau}e^{-b(t-\tau)}\,d\tau = e^{-bt}\int_0^t e^{(b-a)\tau}\,d\tau \\ &= \frac{e^{-bt}}{b-a}e^{(b-a)\tau}\Big|_0^t = \frac{e^{-bt}}{b-a}[e^{(b-a)t}-1] = \frac{e^{-at}-e^{-bt}}{a-b} \end{aligned}$$

Because both functions are causal, their convolution is zero for $t < 0$. Therefore

$$e^{-at}u(t) * e^{-bt}u(t) = \left(\frac{e^{-at}-e^{-bt}}{a-b}\right)u(t)$$

2.4-5. (i)

$$\begin{aligned} u(t) * u(t) &= \int_0^t u(\tau)u(t-\tau)\,d\tau = \int_0^t d\tau = \tau\Big|_0^t = t \quad \text{for } t \geq 0 \\ &= 0 \qquad \text{for } t < 0 \end{aligned}$$

Therefore

$$u(t) * u(t) = tu(t)$$

(ii) Because both functions are causal

$$\begin{aligned} e^{-at}u(t) * e^{-at}u(t) = \int_0^t e^{-a\tau}e^{-a(t-\tau)}\,d\tau &= e^{-at}\int_0^t d\tau \\ &= te^{-at} \qquad t \geq 0 \end{aligned}$$

and

$$e^{-at}u(t) * e^{-at}u(t) = te^{-at}u(t)$$

(iii) Because both functions are causal

$$tu(t) * u(t) = \int_0^t \tau u(\tau)u(\tau - t)\,d\tau$$

The range of integration is $0 \leq \tau \leq t$. Therefore $\tau > 0$ and $\tau - t > 0$ so that

$u(\tau) = u(\tau - t) = 1$ and

$$tu(t) * u(t) = \int_0^t \tau\, d\tau = \frac{t^2}{2} \qquad t \geq 0$$

and

$$tu(t) * u(t) = \frac{1}{2}t^2 u(t)$$

2.4-6. (i)

$$\sin tu(t) * u(t) = \left(\int_0^t \sin\tau\, u(\tau)u(t-\tau)\, d\tau\right) u(t)$$

Because τ and $t - \tau$ are both nonnegative (when $0 \leq \tau \leq t$), $u(\tau) = u(t-\tau) = 1$, and

$$\sin t\, u(t) * u(t) = \left(\int_0^t \sin\tau\, d\tau\right) u(t) = (1 - \cos t)u(t)$$

(ii) Similarly

$$\cos t\, u(t) * u(t) = \left(\int_0^t \cos\tau\, d\tau\right) u(t) = \sin t\, u(t)$$

2.4-7. In this problem, we use Table 2.1 to find the desired convolution.

(a) $y(t) = h(t) * x(t) = e^{-t}u(t) * u(t) = (1 - e^{-t})u(t)$

(b) $y(t) = h(t) * x(t) = e^{-t}u(t) * e^{-t}u(t) = te^{-t}u(t)$

(c) $y(t) = e^{-t}u(t) * e^{-2t}u(t) = (e^{-t} - e^{-2t})u(t)$

(d) $y(t) = \sin 3tu(t) * e^{-t}u(t)$
Here we use pair 12 (Table 2.1) with $\alpha = 0$, $\beta = 3$, $\theta = -90°$ and $\lambda = -1$. This yields

$$\phi = \tan^{-1}\left[\frac{-3}{-1}\right] = -108.4°$$

and

$$\begin{aligned}\sin 3t\, u(t) * e^{-t}u(t) &= \frac{(\cos 18.4°)e^{-t} - \cos(3t + 18.4°)}{\sqrt{10}} u(t) \\ &= \frac{0.9486e^{-t} - \cos(3t + 18.4°)}{\sqrt{10}} u(t)\end{aligned}$$

2.4-8. (a)

$$\begin{aligned}y(t) = (2e^{-3t} - e^{-2t})u(t) * u(t) &= 2e^{-3t}u(t) * u(t) - e^{-2t}u(t) * u(t) \\ &= \left[\frac{2(1 - e^{-3t})}{3} - \frac{1 - e^{-2t}}{2}\right] u(t) \\ &= \left(\frac{1}{6} - \frac{2}{3}e^{-3t} + \frac{1}{2}e^{-2t}\right) u(t)\end{aligned}$$

(b)

$$\begin{aligned}(2e^{-3t} - e^{-2t})u(t) * e^{-t}u(t) &= 2e^{-3t}u(t) * e^{-t}u(t) - e^{-2t}u(t) * e^{-t}u(t) \\ &= \left[\frac{2(e^{-t} - e^{-3t})}{2} - \frac{e^{-t} - e^{-2t}}{1}\right] u(t) \\ &= (e^{-2t} - e^{-3t})u(t)\end{aligned}$$

(c)

$$
\begin{aligned}
y(t) = (2e^{-3t} - e^{-2t})u(t) * e^{-2t}u(t) &= 2e^{-3t}u(t) * e^{-2t}u(t) - e^{-2t}u(t) * e^{-2t}u(t) \\
&= \left[\frac{2(e^{-2t} - e^{-3t})}{1} - te^{-2t}\right] u(t) \\
&= [(2-t)e^{-2t} - 2e^{-3t}]u(t)
\end{aligned}
$$

2.4-9.

$$
\begin{aligned}
y(t) = (1-2t)e^{-2t}u(t) * u(t) &= e^{-2t}u(t) * u(t) - 2te^{-2t}u(t) * u(t) \\
&= \left[\left(\frac{1-e^{-2t}}{2}\right) - \left(\frac{1}{2} - \frac{1}{2}e^{-2t} - te^{-2t}\right)\right] u(t) \\
&= te^{-2t}u(t)
\end{aligned}
$$

2.4-10. (a) For $y(t) = 4e^{-2t}\cos 3t\, u(t) * u(t)$, We use pair 12 with $\alpha = 2$, $\beta = 3$, $\theta = 0$, $\lambda = 0$. Therefore

$$
\phi = \tan^{-1}\left[\frac{-3}{2}\right] = -56.31^\circ
$$

and

$$
\begin{aligned}
y(t) &= 4\left[\frac{\cos(56.31^\circ) - e^{-2t}\cos(3t + 56.31^\circ)}{\sqrt{4+9}}\right] u(t) \\
&= \frac{4}{\sqrt{13}}\left[0.555 - e^{-2t}\cos(3t + 56.31^\circ)\right] u(t)
\end{aligned}
$$

(b) For $y(t) = 4e^{-2t}\cos 3tu(t) * e^{-t}u(t)$, we use pair 12 with $\alpha = 2$, $\beta = 3$, $\theta = 0$, and $\lambda = -1$. Therefore

$$
\phi = \tan^{-1}\left[\frac{-3}{1}\right] = -71.56^\circ
$$

and

$$
\begin{aligned}
y(t) &= 4\left[\frac{\cos(71.56^\circ)e^{-t} - e^{-2t}\cos(3t + 71.56^\circ)}{\sqrt{10}}\right] u(t) \\
&= \frac{4}{\sqrt{10}}\left[0.316e^{-t} - e^{-2t}\cos(3t + 71.56^\circ)\right] u(t) \\
&= 4\left[e^{-t} - \frac{1}{\sqrt{10}}e^{-2t}\cos(3t + 71.56^\circ)\right] u(t)
\end{aligned}
$$

2.4-11. (a) $y(t) = e^{-t}u(t) * e^{-2t}u(t) = (e^{-t} - e^{-2t})u(t)$

(b) $e^{-2(t-3)}u(t) = e^{6}e^{-2t}u(t)$, and $y(t) = e^{6}\left[e^{-t}u(t) * e^{-2t}u(t)\right] = e^{6}(e^{-t} - e^{-2t})u(t)$

(c) $e^{-2t}u(t-3) = e^{-6}e^{-2(t-3)}u(t-3)$. Now from the result in part (a) and the shift property of the convolution [Eq. (2.34)]: $y(t) = e^{-6}\left[e^{-(t-3)}u(t) - e^{-2(t-3)}\right]u(t-3)$

(d) $x(t) = u(t) - u(t-1)$. Now $y_1(t)$, the system response to $u(t)$ is given by

$$
y_1(t) = e^{-t}u(t) * u(t) = (1 - e^{-t})u(t)
$$

The system response to $u(t-1)$ is $y_1(t-1)$ because of time-invariance property. Therefore the response $y(t)$ to $x(t) = u(t) - u(t-1)$ is given by

$$
y(t) = y_1(t) - y_1(t-1) = (1 - e^{-t})u(t) - [1 - e^{-(t-1)}]u(t-1)
$$

The response is shown in Figure S2.4-11d.

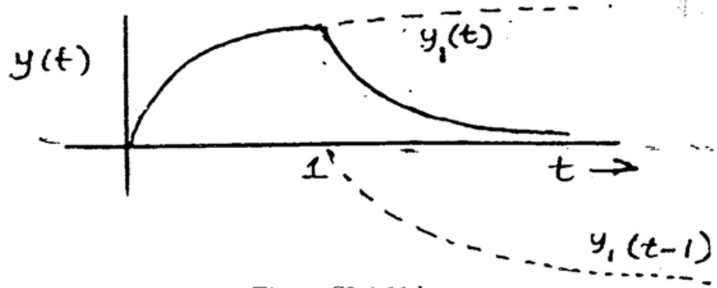


Figure S2.4-11d

2.4-12. (a)

$$
\begin{aligned}
y(t) &= [-\delta(t) + 2e^{-t}u(t)] * e^{t}u(-t) \\
&= -\delta(t) * e^{t}u(-t) + 2e^{-t}u(t) * e^{t}u(-t) \\
&= -e^{t}u(-t) + [e^{-t}u(t) + e^{t}u(-t)] \\
&= e^{-t}u(t)
\end{aligned}
$$

(b) Refer to Figure S2.4-12b.

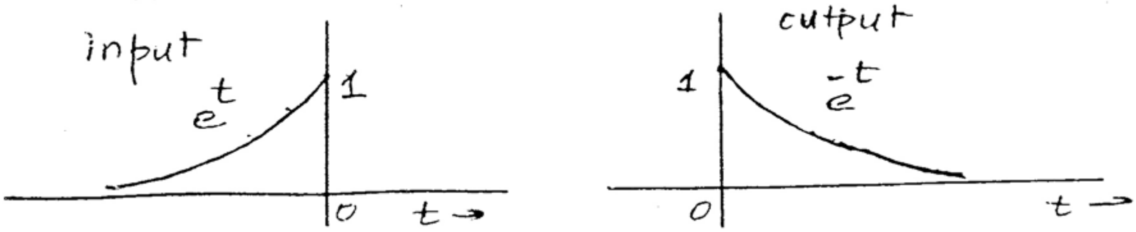
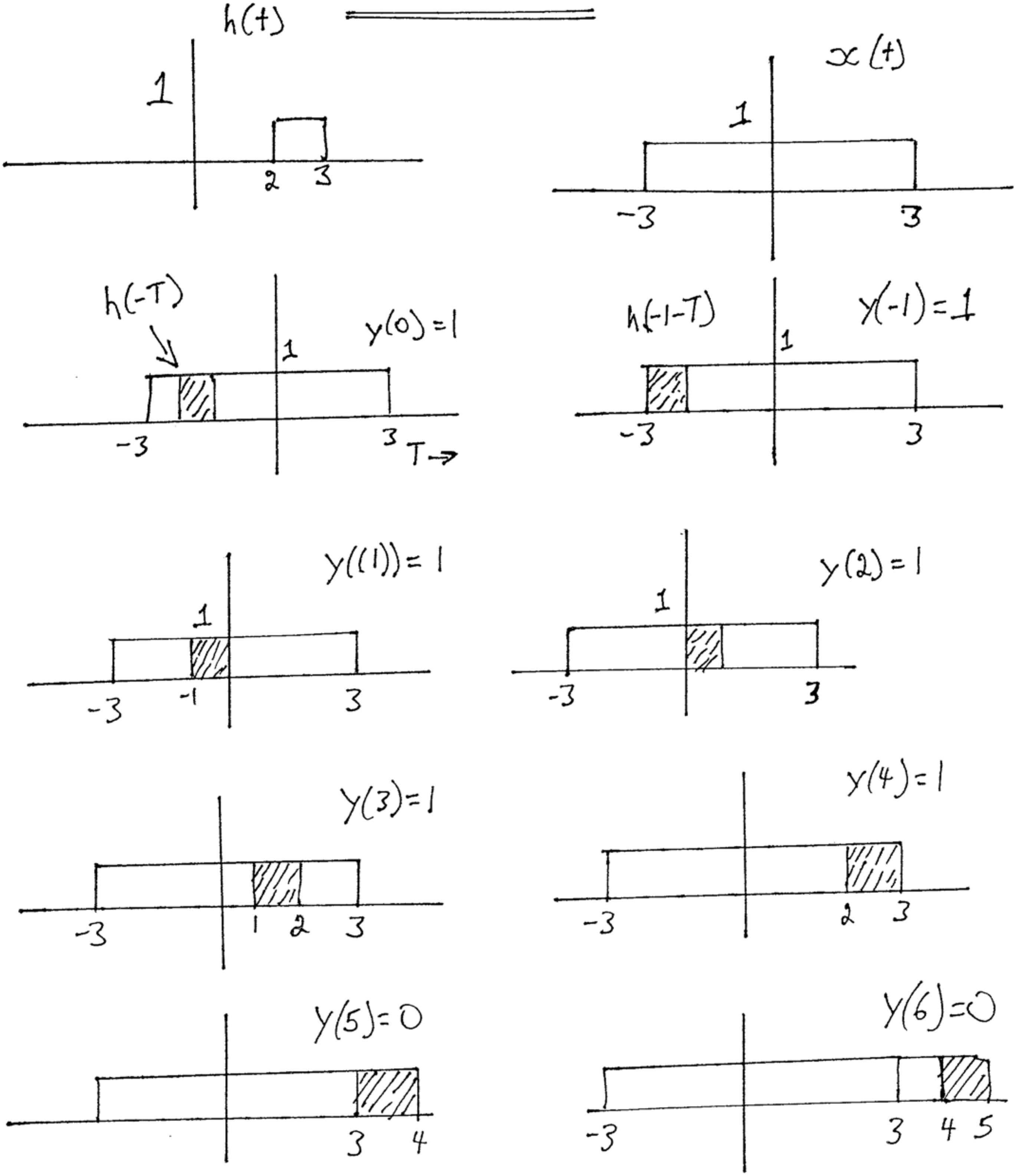


Figure S2.4-12b

2.4-13. Refer to Figure S2.4-13.

2.4-14. The output has the term $e^{-3t}u(t)$ that is not in the input. Hence, the input should have the term $e^{-3t}u(t)$. There is also a possibility of an impulse term in the input

FIGURE S2.4-13



that will result in a term of the form $e^{-2t}u(t)$ in the output. Let us try

$$x(t) = a\delta(t) + be^{-3t}u(t)$$

This yields the output

$$\begin{aligned} y(t) &= 2e^{-2t}u(t)\left[a\delta(t) + be^{-3t}u(t)\right] \\ &= 2e^{-2t}u(t) + 2b\left[e^{-2t} - e^{-3t}\right]u(t) \\ &= (2a+2b)e^{-2t} - 2be^{-3t}u(t) \end{aligned}$$

Matching the coefficients of similar terms yields

$$\left.\begin{aligned} 2a + 2b &= 4 \\ -2b &= 6 \end{aligned}\right\} \implies \begin{aligned} a &= 5 \\ b &= -3 \end{aligned}$$

Hence $\quad y(t) = 5\delta(t) - 3e^{-3t}u(t).$

2.4-15.

$$\frac{1}{t^2+1} * u(t) = \int_{-\infty}^{\infty} \frac{1}{\tau^2+1} u(t-\tau)\, d\tau$$

Because $u(t-\tau) = 1$ for $\tau < t$ and is 0 for $\tau > t$, we need integrate only up to $\tau = t$.

$$\frac{1}{t^2+1} * u(t) = \int_{-\infty}^{t} \frac{1}{\tau^2+1}\, d\tau = \tan^{-1}\tau\Big|_{-\infty}^{t} = \tan^{-1} t + \frac{\pi}{2}$$

Figure S2.4-15 shows $\frac{1}{t^2+1}$ and $c(t)$ (the result of the convolution)

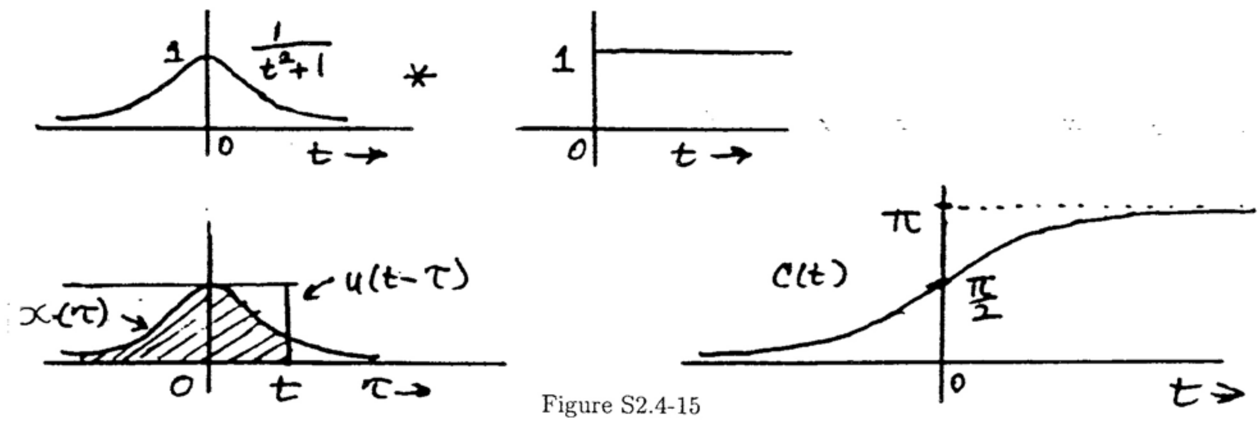


Figure S2.4-15

2.4-16. For $t < 2\pi$ (see Figure S2.4-16)

$$c(t) = x(t) * g(t) = \int_0^t \sin\tau\, d\tau = 1 - \cos t \qquad 0 \le t \le 2\pi$$

For $t \ge 2\pi$, the area of one cycle is zero and

$$x(t) * g(t) = 0 \qquad t \ge 2\pi \quad \text{and} \quad t < 0$$

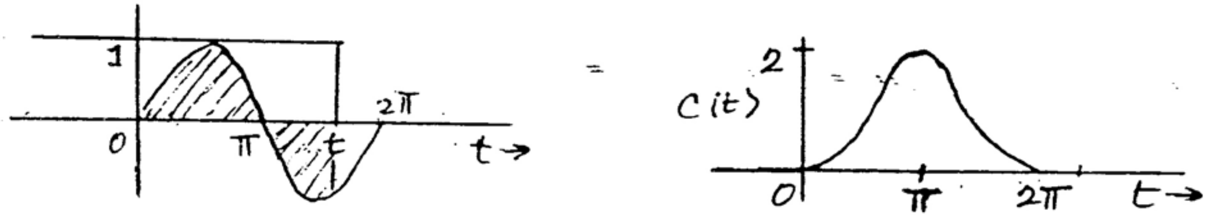


Figure S2.4-16

2.4-17. For $0 \le t \le 2\pi$ (see Figure S2.4-17a)

$$x(t) * g(t) = \int_0^t \sin\tau \, d\tau = 1 - \cos t \qquad 0 \le t \le 2\pi$$

For $2\pi \le t \le 4\pi$ (Figure S2.4-17b)

$$x(t) * g(t) = \int_{t-2\pi}^{2\pi} \sin\tau \, d\tau = \cos t - 1 \qquad 2\pi \le t \le 4\pi$$

For $t > 4\pi$ (also for $t < 0$), $\quad x(t) * g(t) = 0$. Figure S2.4-17c shows $c(t)$.

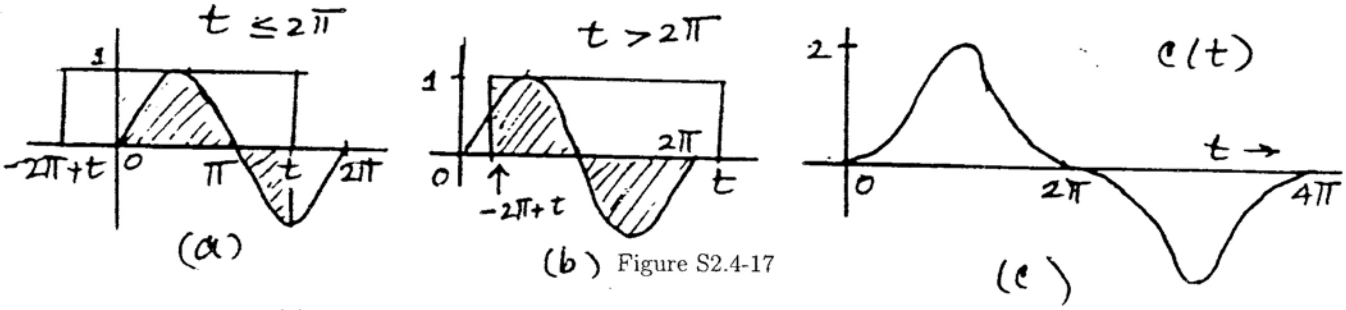


Figure S2.4-17

2.4-18. (a)

$$c(t) = \int_{2+t}^{2.5+t} AB \, d\tau = \tfrac{AB}{2} \qquad 0 \le t \le 0.5$$
$$c(t) = \int_{2+t}^{3} AB \, d\tau = AB(1-t) \qquad 0.5 \le t \le 1$$
$$c(t) = \int_{2}^{2.5+t} AB \, d\tau = AB(t+0.5) \qquad -0.5 \le t \le 0$$
$$c(t) = 0 \quad t \ge 1 \quad \text{or} \quad t \le -0.5$$

(b)

$$c(t) = \int_{1.5+t}^{2.5} AB \, d\tau = AB(1-t) \qquad 0 \le t \le 1$$
$$c(t) = \int_{1.5}^{2.5+t} AB \, d\tau = AB(t+1) \qquad -1 \le t \le 0$$
$$c(t) = 0 \qquad \text{for} \quad |t| \ge 1$$

(c)

$$c(t) = \int_{-1+t}^{2+t} d\tau = 3 \qquad t > -1$$
$$c(t) = \int_{-2}^{2+t} d\tau = t + 4 \qquad -1 \ge t \ge -4$$
$$c(t) = 0 \qquad t \le -4$$

(d)

$$\begin{aligned}
c(t) &= \int_t^{3+t} e^{-\tau}\,d\tau = e^{-t}(1-e^{-3}) = 0.95e^{-t} \qquad t \geq 0 \\
&= \int_0^{3+t} e^{-\tau}\,d\tau = 1-e^{-(3+t)} = 1-0.0498e^{-t} \qquad 0 \geq t \geq -3 \\
&= 0 \qquad t \leq -3
\end{aligned}$$

(e)

$$c(t) = \int_{-\infty}^{-1+t} \tfrac{1}{\tau^2+1}\,d\tau = \tan^{-1}(t-1)+\tfrac{\pi}{2} \quad t \leq 1$$

$$c(t) = \int_{-\infty}^{0} \tfrac{1}{\tau^2+1}\,d\tau = \tan^{-1}\tau\Big|_{-\infty}^{0} = \tfrac{\pi}{2} \quad t \geq 1$$

(f)

$$c(t) = \int_0^t e^{-\tau}\,d\tau = 1-e^{-t} \qquad 0 \leq t \leq 3$$

$$c(t) = \int_{t-3}^t e^{-\tau}\,d\tau = e^{-(t-3)}-e^{-t} \qquad t \geq 3$$

$$c(t) = 0 \qquad t \leq 0$$

(g) This problem is more conveniently solved by inverting $x_1(t)$ rather than $x_2(t)$

$$c(t) = \int_t^{t+1}(\tau-t)\,d\tau = \tfrac{1}{2} \qquad t \geq 0$$

$$c(t) = \int_0^{t+1}(\tau-t)\,d\tau = \tfrac{1}{2}(1-t^2) \qquad 0 \geq t \geq -1$$

$$c(t) = 0 \qquad \text{for} \quad t \geq 0$$

(h) $x_1(t) = e^t, \quad x_2(t) = e^{-2t}, \quad x_1(\tau) = e^{\tau}, \quad x_2(t-\tau) = e^{-2(t-\tau)}.$

$$\begin{aligned}
c(t) &= \int_{-1+t}^{0} e^{\tau}e^{-2(t-\tau)}\,d\tau = e^{-2t}\int_{-1+t}^{0} e^{3\tau}\,d\tau = \frac{1}{3}[e^{-2t}-e^{t-3}] \qquad 0 \leq t \leq 1 \\
c(t) &= \int_{-1+t}^{t} e^{\tau}e^{-2(t-\tau)}\,d\tau = e^{-2t}\int_{-1+t}^{t} e^{3\tau}\,d\tau = \frac{1}{3}[e^{t}-e^{t-3}] \qquad 0 \geq t \geq -1 \\
c(t) &= \int_{-2}^{t} e^{\tau}e^{-2(t-\tau)}\,d\tau = e^{-2t}\int_{-2}^{t} e^{3\tau}\,d\tau = \frac{1}{3}[e^{t}-e^{-2(t+3)}] \qquad -1 \geq t \geq -2 \\
c(t) &= 0 \qquad t \leq -2
\end{aligned}$$

2.4-19.

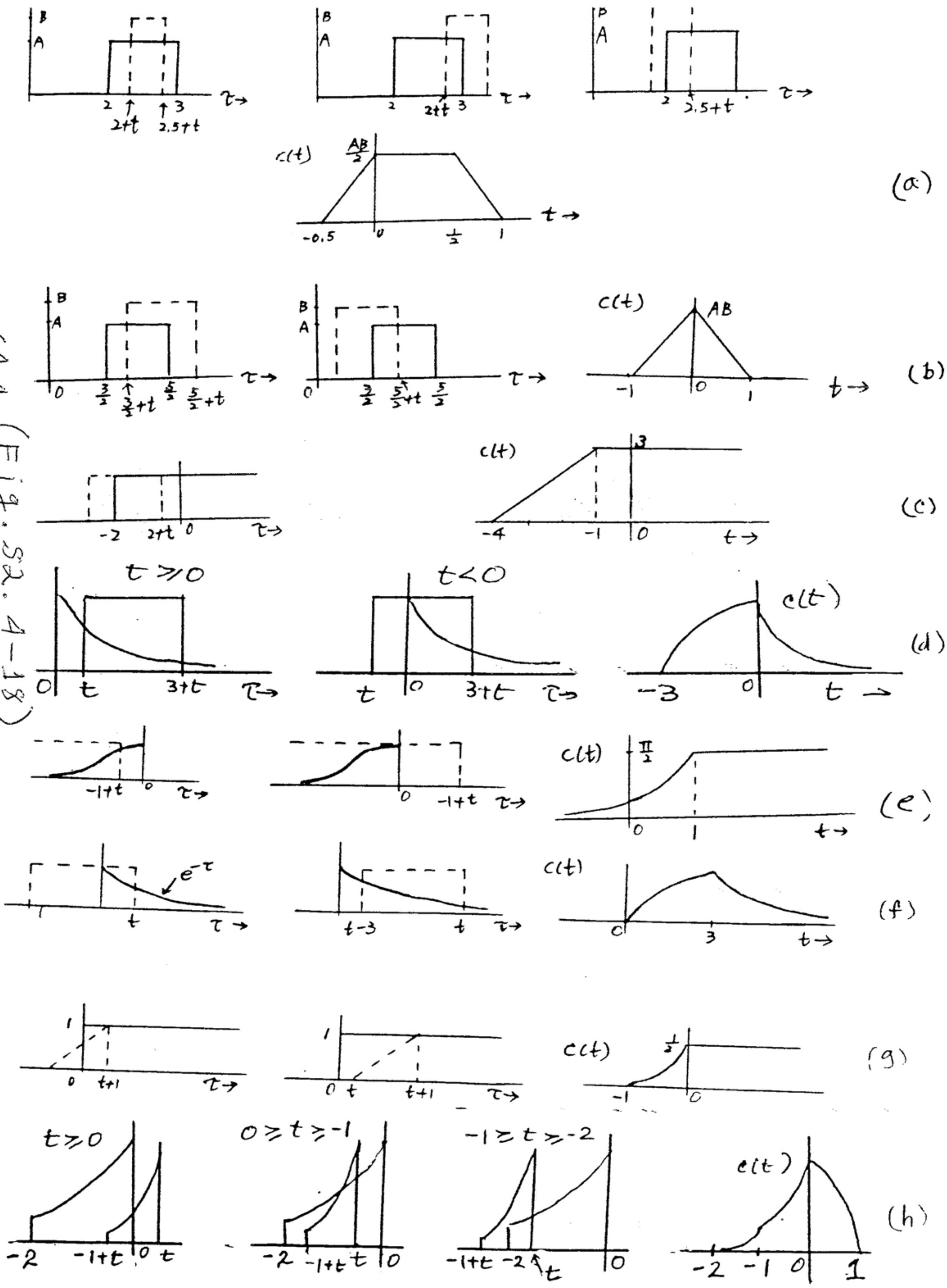
$$\dot{x}(t) = \delta(t) - \delta(t-2)$$

By inspection, we find

$$\int_0^t W(\tau)d\tau = \Delta\left(\frac{t-1}{2}\right)$$

Therefore

$$\begin{aligned}
x(t) * W(t) &= [\delta(t)-\delta(t-2)] * \Delta\left(\frac{t-1}{2}\right) \\
&= \Delta\left(\frac{t-1}{2}\right) - \Delta\left(\frac{t-3}{2}\right)
\end{aligned}$$



S4.d (Fig. S2.4-18)

Figure S2.4-19 shows $x(t) * W(t)$

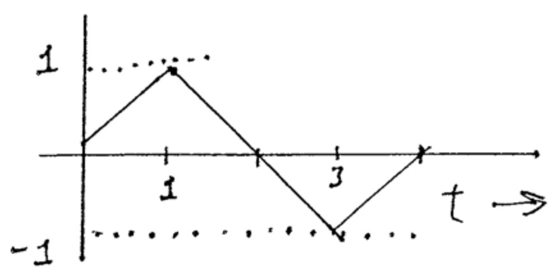


Figure S2.4-19

2.4-20. The unit impulse response of an ideal delay of T seconds is $h(t) = \delta(t - T)$. Using Eq. (2.48), we obtain

$$H(s) = \int_{-\infty}^{\infty} \delta(t - T)e^{-st}dt = e^{-sT}$$

For an input $x(t) = e^{st}$, the output of the delay is $y(t) = s^{s(t-T)}$. Hence, according to Eq. (2.49)

$$H(s) = \frac{e^{s(t-T)}}{e^{st}} = e^{-sT}$$

2.4-21. $y(t) = x(t) * h(t)$. For $t < -1$, $y(t) = 0$;

For $-1 \leq t < 0$, $y(t) = \int_{-1}^{t}(\tau+1)d\tau = \tau^2/2 + \tau\Big|_{\tau=-1}^{t} = t^2/2 + t + 1/2$.

For $t \geq 0$, $y(t) = \int_{-1}^{0}(\tau+1)d\tau = 1/2$.

Thus,

$$y(t) = \begin{cases} 0 & t < -1 \\ t^2/2 + t + 1/2 & -1 \leq t < 0 \\ 1/2 & t \geq 0 \end{cases}.$$

2.4-22. Using the graph of the system response, $h(t) = (-t/2+1)(u(t) - u(t-2))$. $y(1) = \int_{-\infty}^{\infty} h_{\text{total}}(\tau)x(1-\tau)d\tau$. Since $x(t)$ is causal, the upper limit of the integral is one. Furthermore, since $h(t)$ is causal, the total response $h_{\text{total}}(t) = h(t) * h(t)$ is also causal, which makes the lower limit of the integral zero. Over $[0,1]$, $x(t) = u(t) = 1$. Thus, $y(1) = \int_0^1 h_{\text{total}}(\tau)d\tau$. To compute $y(1)$, it is only necessary to know $h_{\text{total}}(t)$ up to $t = 1$.

Over $(0 \leq t < 2)$, $h_{\text{total}}(t) = \int_0^t(-\tau/2+1)(-(t-\tau)/2+1)d\tau = \int_0^t(-\tau/2+1)(\tau/2+1-t/2)d\tau = \int_0^t\left(-\tau^2/4 + \tau(1-1+t/2)/2 + (1-t/2)\right)d\tau = -\frac{t^3}{12} + \frac{t^3}{8} + (1-t/2)t = \frac{t^3}{24} - \frac{t^2}{2} + t$.

Thus,

$$y(1) = \int_0^1 (\tau^3/24 - \tau^2/2 + \tau)d\tau = \frac{\tau^4}{96} - \frac{\tau^3}{6} + \frac{\tau^2}{2}\Big|_{\tau=0}^{1} = \frac{1}{96} - \frac{1}{6} + \frac{1}{2} = \frac{11}{32} = 0.34375.$$

2.4-23. (a) Using KVL, $x(t) = v_L(t) + y(t)$. Also, $i_C(t) = C\frac{dy}{dt}$ and $v_L(t) = L\frac{di_L}{dt} = L\frac{di_C}{dt} = LC\frac{d^2y}{dt^2}$. Combining yields

$$\frac{d^2y}{dt^2} + \frac{1}{LC}y(t) = \frac{1}{LC}x(t).$$

(b) The characteristic equation is

$$\lambda^2 + \frac{1}{LC} = 0.$$

The characteristic roots are

$$\lambda_{1,2} = \frac{\pm j}{\sqrt{LC}}.$$

(c) The form of the zero-input response is $y_0(t) = c_1e^{\lambda_1 t} + c_2e^{\lambda_2 t}$. Using $\lambda_1 = -\lambda_2$, $i_c(0) = 0 = C\frac{dy}{dt}\Big|_{t=0} = C(c_1\lambda_1 + c_2\lambda_2) = C(c_1\lambda_1 - c_2\lambda_1) = C\lambda_1(c_1 - c_2)$. Thus, $c_1 = c_2$. Also, $v_c(0) = 1 = y(0) = c_1 + c_2$. Combining yields $2c_1 = 2c_2 = 1$ or $c_1 = c_2 = 0.5$. The zero-input response is thus $y_0(t) = 0.5(e^{jt/\sqrt{LC}} + e^{-jt/\sqrt{LC}})$. Using Euler's identity yields

$$y_0(t) = \cos(t/\sqrt{LC}).$$

(d) MATLAB is used to plot $y_0(t)$ for a short time after $t \geq 0$.

```
>> t = linspace(0,6*pi,201); y_0 = cos(t); plot(t,y_0,'k');
>> axis tight; xlabel('t(LC)^{-1/2}'); ylabel('y_0(t)');
```

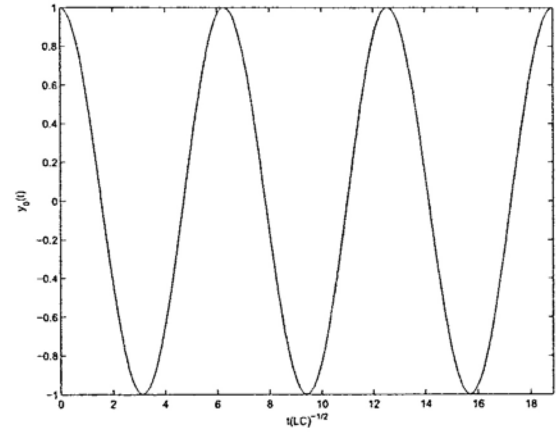


Figure S2.4-23d: Plot of $y_0(t)$

Since $y_0(t)$ is a non-decaying sinusoid, it continues forever; the initial conditions never die out.

(e) Since $L = C = 1$, $\lambda_{1,2} = \pm j$. Let $\tilde{y}_0(t) = \tilde{c}_1 e^{jt} + \tilde{c}_2 e^{-jt}$. Using $\tilde{y}_0(0) = 0 = \tilde{c}_1 + \tilde{c}_2$, we know $\tilde{c}_1 = -\tilde{c}_2$. Combining with $\tilde{y}_0^{(1)}(0) = 1 = j\tilde{c}_1 - j\tilde{c}_2$, we know $2j\tilde{c}_1 = 1$ or $\tilde{c}_1 = -j0.5$. Thus, $\tilde{c}_2 = j0.5$ and $\tilde{y}_0(t) = \frac{e^{jt} - e^{-jt}}{2j} = \sin(t)$. From this, the system response is determined to be $h(t) = \frac{1}{LC}\sin(t)u(t) = \sin(t)u(t)$.

Next, the zero-state response is computed as $x(t) * h(t) = \int_0^t \sin\tau e^{-(t-\tau)} d\tau = \left(e^{-t}\int_0^t \text{Imag}\left(e^{j\tau}e^{\tau}\right) d\tau\right) u(t) = \left(\text{Imag}\left(e^{-t}\int_0^t e^{\tau(1+j)} d\tau\right)\right) u(t) = \left(\text{Imag}\left(e^{-t}\left.\frac{e^{\tau(1+j)}}{1+j}\right|_{\tau=0}^{t}\right)\right) u(t) = \left(\text{Imag}\left(e^{-t}\frac{e^{t(1+j)}-1}{1+j}\right)\right) u(t) = \left(\text{Imag}\left(\frac{e^{jt}-e^{-t}}{1+j}\right)\right) u(t) = \left(\text{Imag}\left(0.5e^{jt} - j0.5e^{jt} - 0.5(1-j)e^{-t}\right)\right) u(t) = \left(0.5\sin(t) - 0.5\cos(t) + 0.5e^{-t}\right) u(t)$.

Summing the zero-state response and the zero-input response calculated in 2.4-23c yields the total response, $y(t) = x(t) * h(t) + y_0(t) = \left(0.5\sin(t) - 0.5\cos(t) + 0.5e^{-t} + \cos(t)\right) u(t)$. Thus,

$$y(t) = \left(0.5\sin(t) + 0.5\cos(t) + 0.5e^{-t}\right) u(t).$$

2.4-24. (a) MATLAB is used to sketch $h_1(t)$ and $h_2(t)$.

```
>> h1 = inline('(1-t).*((t>=0)-(t>=1))');
>> h2 = inline('t.*((t>=-2)-(t>=2))');
>> t = linspace(-2.5,2.5,501);
>> subplot(211),plot(t,h1(t),'k');
>> axis([-2.5 2.5 -2.5 2.5]); xlabel('t'); ylabel('h_1(t)');
>> subplot(212),plot(t,h2(t),'k');
>> axis([-2.5 2.5 -2.5 2.5]); xlabel('t'); ylabel('h_2(t)');
```

(b) For a parallel connection, $h_p(t) = h_1(t) + h_2(t)$. MATLAB is used to plot $h_p(t)$.

```
>> h1 = inline('(1-t).*((t>=0)-(t>=1))');
>> h2 = inline('t.*((t>=-2)-(t>=2))');
>> t = linspace(-2.5,2.5,501); hp = h1(t)+h2(t);
```

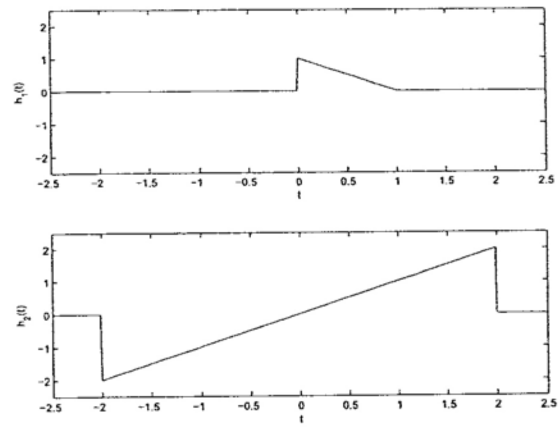


Figure S2.4-24a: Plots of $h_1(t)$ and $h_2(t)$.

```
>> plot(t,hp,'k');
>> axis([-2.5 2.5 -2.5 2.5]); xlabel('t'); ylabel('h_p(t)');
```

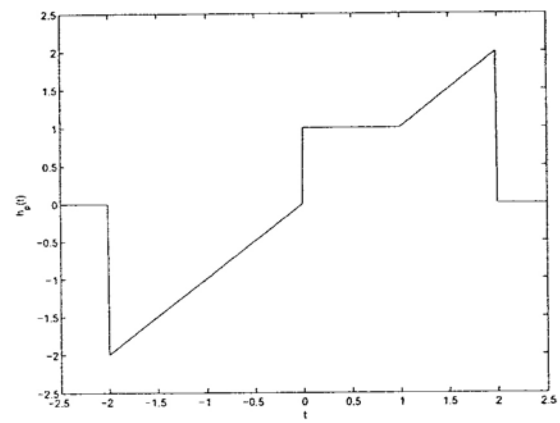


Figure S2.4-24b: Plot of $h_p(t)$.

(c) For a series connection, $h_s(t) = h_1(t) * h_2(t)$.
For $(t < -2)$, $h_s(t) = 0$.
For $(-2 \leq t < -1)$, $h_s(t) = \int_0^{t+2}(1-\tau)(t-\tau)d\tau = \int_0^{t+2}\left(t - \tau(t+1) + \tau^2\right)d\tau = t\tau - (t+1)\tau^2/2 + \tau^3/3\big|_{\tau=0}^{t+2} = t(t+2) - (t+1)(t+2)^2/2 + (t+2)^3/3 = -t^3/6 + t^2/2 + 2t + 2/3$.
For $(-1 \leq t < 2)$, $h_s(t) = \int_0^1(1-\tau)(t-\tau)d\tau = \int_0^1\left(t - \tau(t+1) + \tau^2\right)d\tau = t\tau - (t+1)\tau^2/2 + \tau^3/3\big|_{\tau=0}^{1} = t - (t+1)/2 + 1/3 = t/2 - 1/6$.
For $(2 \leq t < 3)$, $h_s(t) = \int_{t-2}^1(1-\tau)(t-\tau)d\tau = \int_{t-2}^1\left(t - \tau(t+1) + \tau^2\right)d\tau = t\tau - (t+1)\tau^2/2 + \tau^3/3\big|_{\tau=t-2}^{1} = t/2 - 1/6 - \left(t(t-2) - (t+1)(t-2)^2/2 + (t-2)^3/3\right) = t/2 - 1/6 - \left(-t^3/6 + t^2/2 + 2t - 14/3\right) = t^3/6 - t^2/2 - 3t/2 + 9/2$.
For $(t > 3)$, $h_s(t) = 0$.

Combining all pieces yields

$$h_s(t) = \begin{cases} -t^3/6 + t^2/2 + 2t + 2/3 & -2 \le t < -1 \\ t/2 - 1/6 & -1 \le t < 2 \\ t^3/6 - t^2/2 - 3t/2 + 9/2 & 2 \le t < 3 \\ 0 & \text{otherwise} \end{cases}.$$

MATLAB is used to plot $h_s(t)$.

```
>> t = linspace(-2.5,3.5,501);
>> hs = (-t.^3/6+t.^2/2+2*t+2/3).*((t>=-2)&(t<-1));
>> hs = hs+(t/2-1/6).*((t>=-1)&(t<2));
>> hs = hs+(t.^3/6-t.^2/2-3*t/2+9/2).*((t>=2)&(t<3));
>> plot(t,hs,'k'); xlabel('t'); ylabel('h_s(t)');
```

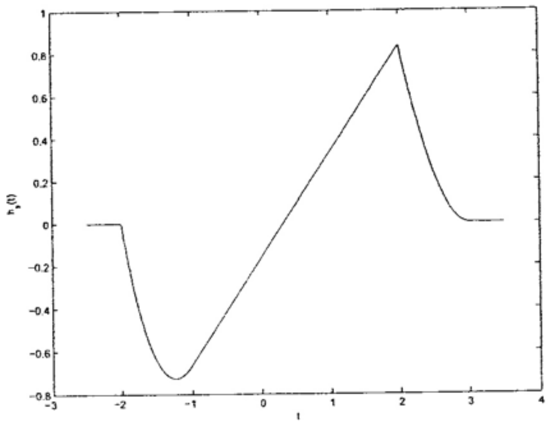


Figure S2.4-24c: Plot of $h_s(t)$.

2.4-25. (a) Using KVL, $x(t) = RC\dot{y}(t) + y(t)$ or $\dot{y}(t) + \frac{1}{RC}y(t) = \frac{1}{RC}x(t)$. The characteristic root is $\lambda = \frac{-1}{RC}$.

The zero-input response has form $y_0(t) = c_1 e^{-t/(RC)}$. Using the IC, $y_0(0) = 2 = c_1$. Thus, $y_0(t) = 2e^{-t/(RC)}$.

The zero-state response is $x(t) * h(t)$, where $h(t) = b_0\delta(t) + [P(D)\tilde{y}_0(t)]u(t)$. For this first-order system, $\tilde{y}_0(t) = \tilde{c}_1 e^{-t/(RC)}$ and $\tilde{y}_0(0) = 1 = \tilde{c}_1$. Using $\tilde{y}_0(t) = e^{-t/(RC)}$, $b_0 = 0$, and $P(D) = \frac{1}{RC}$, the impulse response is $h(t) = \frac{1}{RC}e^{-t/(RC)}u(t)$. Thus, the zero-state response is $\left(\int_0^t \frac{1}{RC}e^{-\tau/(RC)}d\tau\right)u(t) = \left(-e^{-\tau/(RC)}\Big|_{\tau=0}^{t}\right)u(t) = \left(1 - e^{-t/(RC)}\right)u(t)$.

For $t \ge 0$, the total response is the sum of the zero-input response and the zero state response,

$$y(t) = \left(1 + e^{-t/(RC)}\right)u(t).$$

(b) From 2.4-25a, we know the zero-input response is $y_0(t) = y_0(0)e^{-t/(RC)}$. Since the system is time-invariant, the unit step response from 2.4-25a is shifted by one to provide the response to $x(t) = u(t-1)$. Thus, the zero-state response is $\left(1 - e^{-(t-1)/(RC)}\right)u(t-1)$. Summing the two parts together and evaluating at

$t = 2$ yields $y(2) = 1/2 = y_0(0)e^{-2/(RC)} + (1 - e^{-1/(RC)})$. Solving for $y_0(0)$ yields

$$y_0(0) = e^{1/(RC)} - 0.5e^{2/(RC)}.$$

2.4-26. Notice, $x(2t)$ is a compressed version of $x(t)$. The convolution $y(t) = x(t) * x(2t)$ has several distinct regions.

For $t < 0$ and $t \geq 3/2$, $y(t) = 0$.

For $0 \leq t < 1/2$, $y(t) = \int_0^t 2\tau(t-\tau)d\tau = t\tau^2 - 2\tau^3/3\Big|_{\tau=0}^{t} = t^3/3$.

For $1/2 \leq t < 1$, $y(t) = \int_0^{1/2} 2\tau(t-\tau)d\tau = t\tau^2 - 2\tau^3/3\Big|_{\tau=0}^{1/2} = t/4 - 1/12$.

For $1 \leq t < 3/2$, $y(t) = \int_{t-1}^{1/2} 2\tau(t-\tau)d\tau = t\tau^2 - 2\tau^3/3\Big|_{\tau=t-1}^{1/2} = t/4 - 1/12 - (t^3 - 2t^2 + t - 2t^3/3 + 2t^2 - 2t + 2/3) = -t^3/3 + 5t/4 - 3/4$.

Thus,

$$y(t) = \begin{cases} t^3/3 & 0 \leq t < 1/2 \\ t/4 - 1/12 & 1/2 \leq t < 1 \\ -t^3/3 + 5t/4 - 3/4 & 1 \leq t < 3/2 \\ 0 & \text{otherwise} \end{cases}.$$

MATLAB is used to plot $y(t)$.

```
>> t = linspace(-1/2,2,251);
>> y = (t.^3/3).*((t>=0)&(t<1/2));
>> y = y+(t/4-1/12).*((t>=1/2)&(t<1));
>> y = y+(-t.^3/3+5*t/4-3/4).*((t>=1)&(t<3/2));
>> plot(t,y,'k'); xlabel('t'); ylabel('y(t)');
```

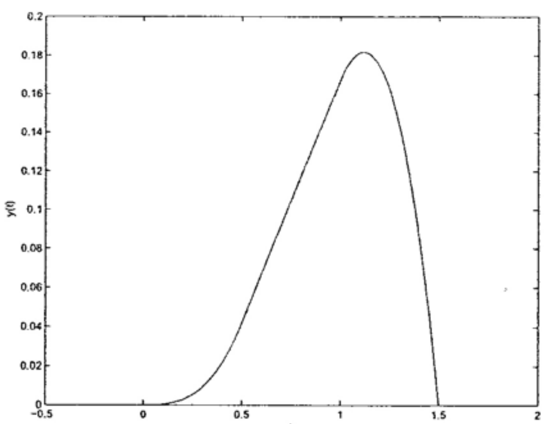


Figure S2.4-26: Plot of $y(t) = x(t) * x(2t)$.

2.4-27. Notice, $v_R(t) = v_{L_1}(t) = v_{L_2}(t) = v(t)$.

(a) KCL at the top node gives $x(t) = y(t) + i_{L_1}(t) + i_R$. Since $v(t) = L_2\dot{y}(t)$, we know $i_R(t) = v(t)/R = \frac{L_2}{R}\dot{y}(t)$. Thus, $x(t) = y(t) + \frac{L_2}{R}\dot{y}(t) + i_{L_1}(t)$. Differentiating this expression yields $\dot{x}(t) = \dot{y}(t) + \frac{L_2}{R}\ddot{y}(t) + i_{L_1}^{(1)}(t)$. However, $i_{L_1}^{(1)}(t) = v(t)/L_1 =$

$\frac{L_2}{L_1}\dot{y}(t)$. Thus, $\dot{x}(t) = \dot{y}(t) + \frac{L_2}{R}\ddot{y}(t) + \frac{L_2}{L_1}\dot{y}(t)$ or

$$\ddot{y}(t) + \left(\frac{R}{L_1} + \frac{R}{L_2}\right)\dot{y}(t) = \frac{R}{L_2}\dot{x}(t).$$

(b) The characteristic equation is $\lambda^2 + \left(\frac{R}{L_1} + \frac{R}{L_2}\right)\lambda = 0$ which yields characteristic roots of $\lambda_1 = 0$ and $\lambda_2 = -\left(\frac{R}{L_1} + \frac{R}{L_2}\right)$.

(c) The zero-input response has form $y_0(t) = c_1 + c_2e^{\lambda_2 t}$. Each inductor has an initial current of one amp each. Thus, $y_0(0) = 1 = c_1 + c_2$. The initial resistor current is $i_R(0) = -i_{L_1}(0) - i_{L_2}(0) = -2$ and the initial resistor voltage is $v(0) = i_R(0)R = -2R$. Thus, $\dot{y}_0(0) = -\frac{2R}{L_2} = \lambda_2 c_2$. Solving yields $c_2 = \frac{2L_1}{L_1+L_2}$ and $c_1 = 1 - c_2 = \frac{L_2-L_1}{L_1+L_2}$. Thus,

$$y_0(t) = \frac{L_2 - L_1}{L_1 + L_2} + \frac{2L_1}{L_1 + L_2}e^{-tR/L_1 - tR/L_2}.$$

2.4-28. Since the system step response is $s(t) = e^{-t}u(t) - e^{-2t}u(t)$, the system impulse response is $h(t) = \frac{d}{dt}s(t) = -e^{-t}u(t) + \delta(t) + 2e^{-2t}u(t) - \delta(t) = (2e^{-2t} - e^{-t})u(t)$. The input $x(t) = \delta(t-\pi) - \cos(\sqrt{3})u(t)$ is just a sum of a shifted delta function and a scaled step function. Since the system is LTI, the output is quickly computed using just $h(t)$ and $s(t)$. That is,

$$y(t) = h(t-\pi) - \cos(\sqrt{3})s(t) = (2e^{-2(t-\pi)} - e^{-(t-\pi)})u(t-\pi) - \cos(\sqrt{3})(e^{-t} - e^{-2t})u(t).$$

2.4-29. Since $x(t)$ is $(T = 2)$-periodic, the convolution $y(t) = x(t) * h(t)$ is also $(T = 2)$-periodic. Thus, it is sufficient to evaluate $y(t)$ over any interval of length two.

For $0 \le t < 1/2$, $y(t) = \int_0^t \tau d\tau + \int_{t+1}^{3/2} \tau d\tau = \left.\frac{\tau^2}{2}\right|_{\tau=0}^{t} + \left.\frac{\tau^2}{2}\right|_{\tau=t+1}^{3/2} = t^2/2 + 9/8 - (t^2/2 + t + 1/2) = -t + 5/8$.

For $1/2 \le t < 1$, $y(t) = \int_0^t \tau d\tau = t^2/2$.

For $1 \le t < 3/2$, $y(t) = \int_{t-1}^t \tau d\tau = \left.\frac{\tau^2}{2}\right|_{\tau=t-1}^{t} = t^2/2 - (t^2/2 - t + 1/2) = t - 1/2$.

For $3/2 \le t < 2$, $y(t) = \int_{t-1}^{3/2} \tau d\tau = \left.\frac{\tau^2}{2}\right|_{\tau=t-1}^{3/2} = 9/8 - (t^2/2 - t + 1/2) = -t^2/2 + t + 5/8$.

Combining,

$$y(t) = \begin{cases} -t + 5/8 & 0 \le t < 1/2 \\ t^2/2 & 1/2 \le t < 1 \\ t - 1/2 & 1 \le t < 3/2 \\ -t^2/2 + t + 5/8 & 3/2 \le t < 2 \\ y(t+2) & \forall t \end{cases}.$$

MATLAB is used to plot $y(t)$ over $(-3 \le t \le 3)$. This interval includes three periods of the $(T = 2)$-periodic function $y(t)$.

```
>> t = linspace(-3,3,601); tm = mod(t,2);
>> y = (-tm+5/8).*((tm>=0)&(tm<1/2));
>> y = y+(tm.^2/2).*((tm>=1/2)&(tm<1));
>> y = y+(tm-1/2).*((tm>=1)&(tm<3/2));
>> y = y+(-tm.^2/2+tm+5/8).*((tm>=3/2)&(tm<2));
>> plot(t,y,'k'); xlabel('t'); ylabel('y(t)');
```

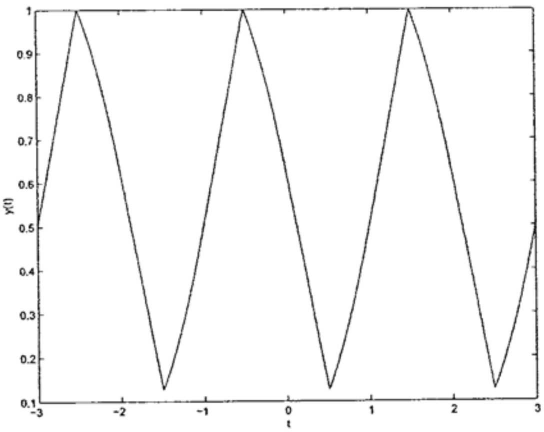


Figure S2.4-29: Plot of $y(t) = x(t) * h(t)$ over $(-3 \leq t \leq 3)$.

2.4-30. (a) Using KVL, $x(t) = i(t)R + v_{C_1}(t) + y(t) = RC_2\dot{y}(t) + v_{C_1}(t) + y(t)$. Differentiating yields $\dot{x}(t) = RC_2\ddot{y}(t) + \dot{v}_{C_1}(t) + \dot{y}(t) = RC_2\ddot{y}(t) + \frac{1}{C_1}i(t) + \dot{y}(t) = RC_2\ddot{y}(t) + \frac{C_2}{C_1}\dot{y}(t) + \dot{y}(t)$. Thus,

$$\ddot{y}(t) + \left(\frac{1}{RC_1} + \frac{1}{RC_2}\right)\dot{y}(t) = \frac{1}{RC_2}\dot{x}(t).$$

(b) Since $R = 1$, $C_1 = 1$, and $C_2 = 2$, the differential equation becomes $\ddot{y}(t) + 3/2\dot{y}(t) = 1/2\dot{x}(t)$.
The characteristic equation is $\lambda^2 + 3/2\lambda = 0$, and the characteristic roots are $\lambda_1 = 0$ and $\lambda_2 = -3/2$. Thus, the form of the zero-input response is $y_0(t) = c_1 + c_2e^{-3t/2}$. Using the first IC, $y(0) = 1 = c_1 + c_2$. The initial voltage across the resistor is $v_R(0) = -3$ which yields $i_R(0) = -3/R = -3$. Also, $i_R(0) = -3 = i_{C_2}(0) = C_2\dot{y}(0) = 2\dot{y}(0)$. Thus, $\dot{y}(0) = -3/2 = -3c_2/2$. Solving yields $c_2 = 1$ and $c_1 = 0$. Thus,

$$y_0(t) = e^{-3t/2}.$$

The zero-state response is $x(t) * h(t)$, where $h(t) = b_0\delta(t) + [P(D)\tilde{y}_0(t)]u(t)$. For this second-order system, $\tilde{y}_0(t) = \tilde{c}_1 + \tilde{c}_2e^{-3t/2}$, $\tilde{y}_0(0) = 0 = \tilde{c}_1 + \tilde{c}_2$ and $\tilde{y}_0^{(1)}(0) = 1 = -3\tilde{c}_2/2$. Thus, $\tilde{c}_2 = -2/3$ and $\tilde{c}_1 = 2/3$. Using $b_0 = 0$, and $P(D) = 0.5D$, the impulse response is $h(t) = 0.5D(\tilde{y}_0(t))u(t) = 0.5D(2/3 - 2/3e^{-3t/2})u(t) = 0.5(-2/3(-3/2)e^{-3t/2})u(t) = 0.5e^{-3t/2}u(t)$. Using $x(t) = 4te^{-3t/2}u(t)$, the zero-state response is $\left(\int_0^t(4\tau e^{-3\tau/2})(0.5e^{-3(t-\tau)/2})d\tau\right)u(t) = \left(2e^{-3t/2}\int_0^t \tau d\tau\right)u(t) = \left(2e^{-3t/2}t^2/2\right)u(t)$. Thus,

$$x(t) * h(t) = t^2e^{-3t/2}u(t).$$

Since the input is driving a natural mode, resonance is expected; thus, the t^2 term seems sensible.
For $(t \geq 0)$, the total response is the sum of the zero-input response and the zero-state response.

$$y(t) = y_0(t) + x(t) * h(t) = \left(e^{-3t/2} + t^2e^{-3t/2}\right)u(t).$$

2.4-31. Since $h(t)$ is only provided for over $(0 \leq t < 0.5)$, it is not possible to determine with certainty whether or not the system is causal or stable. However, when looking at $h(t)$ the waveform appears to have a DC offset. This apparent DC offset can be very troubling if $h(t)$ is truly an impulse response function. If a DC offset is present, the system is neither causal nor stable. Imagine, a non-causal, unstable heart! Something is probably wrong.

One simple explanation is that a blood-filled heart always has some ventricular pressure. Unless removed, this relaxed-state pressure would likely appear as a DC offset to any measurements. It would likely be most appropriate to subtract this offset when trying to measure the impulse response function.

Another problem is that the impulse response function is most appropriate in the study of linear, time-invariant systems. It is quite unlikely that the heart is either linear or time-invariant. Even if the impulse response could be reliably measured at a particular time, it might not provide much useful information.

2.4-32. (a) $x(t) * x(-t) = \int_{-\infty}^{\infty} x(\tau)x(-(t-\tau))d\tau = \int_{-\infty}^{\infty} x(\tau)x(\tau - t)d\tau = r_{xx}(t)$.

(b) Since $r_{xx}(t)$ is an even function, we only need to compute $r_{xx}(t)$ for either $t \geq 0$ or $t \leq 0$. In either case, the autocorrelation function is computed by convolving the original signal with its reflection.

For $t < -2$, $r_{xx}(t) = 0$.

For $-2 \leq t < -1$, $r_{xx}(t) = \int_0^{t+2} \tau d\tau = 0.5\tau^2\big|_{\tau=0}^{t+2} = t^2/2 + 2t + 2$.

For $-1 \leq t < 0$, $r_{xx}(t) = \int_0^{t+1} \tau(\tau - t)d\tau + \int_{t+1}^{1} \tau d\tau + \int_1^{t+2} d\tau = \left(\frac{\tau^3}{3} - t\frac{\tau^2}{2}\Big|_{\tau=0}^{t+1}\right) + \frac{\tau^2}{2}\Big|_{\tau=t+1}^{1} + \tau\big|_{\tau=1}^{t+2} = \frac{t^3+3t^2+3t+1}{3} - \frac{t^3+2t^2+t}{2} + \frac{1}{2} - \frac{t^2+2t+1}{2} + (t+2-1) = -t^3/6 - t^2/2 + t/2 + 4/3$.

Combining and using $r_{xx}t = r_{xx}-t$ yields

$$r_{xx}(t) = \begin{cases} t^2/2 + 2t + 2 & -2 \leq t < -1 \\ -t^3/6 - t^2/2 + t/2 + 4/3 & -1 \leq t < 0 \\ t^3/6 - t^2/2 - t/2 + 4/3 & 0 \leq t < 1 \\ t^2/2 - 2t + 2 & 1 \leq t < 2 \\ 0 & \text{otherwise} \end{cases}$$

MATLAB is used to plot the result.

```
>> t = linspace(-2.5,2.5,501);
>> rxx = (t.^2/2+2*t+2).*((t>=-2)&(t<-1));
>> rxx = rxx+(-t.^3/6-t.^2/2+t/2+4/3).*((t>=-1)&(t<0));
>> rxx = rxx+(t.^3/6-t.^2/2-t/2+4/3).*((t>=0)&(t<1));
>> rxx = rxx+(t.^2/2-2*t+2).*((t>=1)&(t<2));
>> plot(t,rxx,'k'); xlabel('t'); ylabel('r_{xx}(t)');
```

2.4-33. (a) KCL at the negative terminal of the op-amp yields $\frac{x(t)-0}{R} + C\dot{y}(t) = 0$. Thus,

$$\dot{y}(t) = -\frac{1}{RC}x(t).$$

(b) The zero-state response is $y(t) = x(t) * h(t)$, where $h(t) = b_0\delta(t) + [P(D)\tilde{y}_0(t)]u(t)$. This is a first order system with $\lambda = 0$, thus $\tilde{y}_0(t) = \tilde{c}_1 e^{\lambda t} = \tilde{c}_1$. Since $\tilde{y}_0(0) = 1 = \tilde{c}_1$, $b_0 = 0$, and $P(D) = -\frac{1}{RC}$, the impulse response is $h(t) = -\frac{1}{RC}u(t)$.

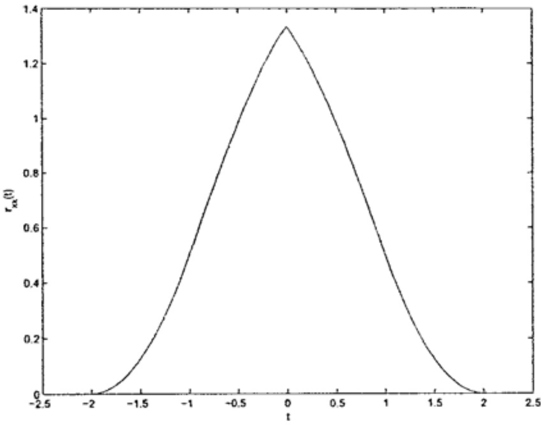


Figure S2.4-32b: Plot of $r_{xx}(t)$.

Thus,

$$y(t) = \left(\int_0^t -\frac{1}{RC}d\tau\right)u(t) = -\frac{t}{RC}u(t).$$

Notice, $|y(t)|$ ramps toward infinity as time increases. Intuitively, this makes sense; a DC input to an integrator should output an unbounded ramp function.

2.4-34. The system response to $u(t)$ is $g(t)$ and the response to step $u(t-\tau)$ is $g(t-\tau)$. The input $x(t)$ is made up of step components. The step component at τ has a height $\triangle f$ which can be expressed as

$$\triangle f = \frac{\triangle f}{\triangle\tau}\triangle\tau = \dot{x}(\tau)\triangle\tau$$

The step component at $n\triangle\tau$ has a height $\dot{x}(n\triangle\tau)\triangle\tau$ and it can be expressed as $[\dot{x}(n\triangle\tau)\triangle\tau]u(t-n\triangle\tau)$. Its response $\triangle y(t)$ is

$$\triangle y(t) = [\dot{x}(n\triangle\tau)\triangle\tau]g(t-n\triangle\tau)$$

The total response due to all components is

$$\begin{aligned} y(t) &= \lim_{\triangle\tau\to 0}\sum_{n=-\infty}^{\infty}\dot{x}(n\triangle\tau)g(t-n\triangle\tau)\triangle\tau \\ &= \int_{-\infty}^{\infty}\dot{x}(\tau)g(t-\tau)\,d\tau = \dot{x}(\tau)*g(\tau) \end{aligned}$$

2.4-35. Consider the input $x(t) = e^{j\omega_o t}$. Letting $s = j\omega_o$ in Eq. (2.47), the system response is found as

$$y(t) = H(j\omega_o)e^{j\omega_o t}$$

Using Eq. (2.40), the system response to input $\hat{x}(t) = \cos\omega_o t = Re[e^{j\omega_o t}]$ is $\hat{y}(t)$, where

$$\begin{aligned} \hat{y}(t) &= Re[H(j\omega_o)e^{j\omega_o t}] \\ &= Re\left\{|H(j\omega_o)|e^{j[\omega_o t+\angle H(j\omega_o)]}\right\} \\ &= |H(j\omega_o)|\cos[\omega_o t+\angle H(j\omega_o)] \end{aligned}$$

Where $H(j\omega)$ is $H(s)|_{s=j\omega}$ in Eq. (2.48). Hence

$$H(j\omega) = \int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}\,d\tau$$

2.4-36. An element of length $\triangle\tau$ at point $n\triangle\tau$ has a charge (Figure S2.4-36). A point x is at a distance $x - n\triangle\tau$ from this charge. The electric field at point x due to the charge $Q(n\triangle\tau)\triangle\tau$ is

$$\triangle E = \frac{Q(n\triangle\tau)\triangle\tau}{4\pi\epsilon(x - n\triangle\tau)^2}$$

The total field due to the charge along the entire length is

$$\begin{aligned} E(x) &= \lim_{\triangle\tau\to 0} \sum_{n=-\infty}^{\infty} \frac{Q(n\triangle\tau)\triangle\tau}{4\pi\epsilon(x - n\triangle\tau)^2} \\ &= \int_{-\infty}^{\infty} \frac{Q(\tau)}{4\pi\epsilon(x-\tau)^2}\,d\tau = Q(x) * \frac{1}{4\pi\epsilon x} \end{aligned}$$

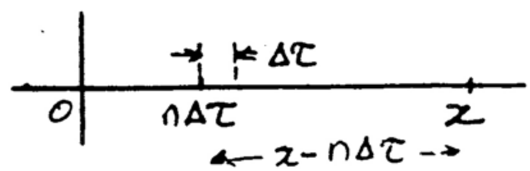


Figure S2.4-36

2.4-37. (a) KCL at the negative terminal of the op-amp yields $\frac{x(t)-0}{R_{in}} + \frac{y(t)-0}{R_f} + i_c(t) = 0$. Also, $i_c(t) = C\dot{y}(t)$. Thus, $\frac{x(t)}{R_{in}} + \frac{y(t)}{R_f} + C\dot{y}(t) = 0$ or

$$\dot{y}(t) + \frac{1}{CR_f}y(t) = \frac{-1}{CR_{in}}x(t).$$

The characteristic equation is $\lambda + \frac{1}{CR_f} = 0$, and the characteristic root is

$$\lambda = \frac{-1}{CR_f}.$$

(b) The zero-state response is $y(t) = x(t)*h(t)$, where $h(t) = b_0\delta(t) + [P(D)\tilde{y}_0(t)]u(t)$. This is a first order system with $\lambda = \frac{-1}{CR_f}$, thus $\tilde{y}_0(t) = \tilde{c}_1 e^{\lambda t}$. Since $\tilde{y}_0(0) = 1 = \tilde{c}_1$, $b_0 = 0$, and $P(D) = \frac{-1}{CR_{in}}$, the impulse response is $h(t) = \frac{-1}{CR_{in}}e^{-t/(CR_f)}u(t)$. Thus, $y(t) = \left(\int_0^t \frac{-1}{CR_{in}}e^{-\tau/(CR_f)}d\tau\right)u(t) = \left(\frac{R_f}{R_{in}}e^{-\tau/(CR_f)}\Big|_{\tau=0}^{t}\right)u(t) = \frac{R_f}{R_{in}}(e^{-t/(CR_f)} - 1)u(t)$.

$$y(t) = \frac{R_f}{R_{in}}(e^{-t/(CR_f)} - 1)u(t).$$

Notice, $y(t)$ approaches $\frac{-R_f}{R_{in}}$ as time increases; unlike a true integrator, the "lossy" integrator provides a bounded output in response to a DC input.

(c) For this system, the characteristic root is only affected by C and R_f. Using 10% resistors, the resistor R_f is generally expected to lie in the range $(0.9R_f, 1.1R_f)$. Using 25% capacitors, the capacitor C is generally expected to lie in the range $(.25C, 1.25C)$. Since $\lambda = \frac{-1}{CR_f}$, the characteristic root is expected to lie in the range $(\lambda/[(0.9)(0.75)], \lambda/[(1.1)(1.25)])$. Thus,

The characteristic root is expected within the interval $(1.48\lambda, 0.73\lambda)$.

2.4-38. Identify the output of the first op-amp as $v(t)$.

(a) KCL at the negative terminal of the first op-amp yields $\frac{x(t)}{R_1} + C_1\dot{v}(t) = 0$ or $\frac{1}{R_1C_1}x(t) = -\dot{v}(t)$. KCL at the negative terminal of the second op-amp yields $\frac{v(t)}{R_2} + \frac{y(t)}{R_3} + C_2\dot{y}(t) = 0$ or $v(t) = -\frac{R_2}{R_3}y(t) - R_2C_2\dot{y}(t)$. Substituting this expression for $v(t)$ into the first expression yields $\frac{1}{R_1C_1}x(t) = \frac{R_2}{R_3}\dot{y}(t) + R_2C_2\ddot{y}(t)$. Thus,

$$\ddot{y}(t) + \frac{1}{R_3C_2}\dot{y}(t) = \frac{1}{R_1R_2C_1C_2}x(t).$$

The characteristic equation is $\lambda^2 + \frac{1}{R_3C_2}\lambda = 0$ and the characteristic roots are

$$\lambda_1 = 0 \text{ and } \lambda_2 = -\frac{1}{R_3C_2}.$$

Substituting $C_1 = C_2 = 10\mu\text{F}$, $R_1 = R_2 = 100k\Omega$, and $R_3 = 50k\Omega$ yields

$$\ddot{y}(t) + 2\dot{y}(t) = x(t), \lambda_1 = 0, \text{ and } \lambda_2 = -2.$$

Since one root lies on the ω-axis, the circuit is not BIBO stable. In particular, a DC input results in an unbounded output.

(b) The zero-input response has form $y_0(t) = c_1 + c_2e^{-2t}$. Each op-amp has an initial output of one volt. Thus, $y_0(0) = 1 = c_1 + c_2$. KCL at the negative terminal of the second op-amp yields $\frac{1}{R_2} + \frac{1}{R_3} + C_2\dot{y}_0(0) = 0$ or $\dot{y}_0(0) = -\frac{1}{R_2C_2} - \frac{1}{R_3C_2} = -1-2 = -3$. Thus, $\dot{y}_0(0) = -3 = -2c_2$. Thus, $c_2 = 3/2$ and $c_1 = 1-3/2 = -1/2$ and

$$y_0(t) = -1/2 + 3/2e^{-2t}.$$

(c) The zero-state response is $y(t) = x(t)*h(t)$, where $h(t) = b_0\delta(t) + [P(D)\tilde{y}_0(t)]u(t)$. This is a second order system with $\lambda_1 = 0$ and $\lambda_2 = -2$, so $\tilde{y}_0(t) = \tilde{c}_1 + \tilde{c}_2e^{-2t}$. Solving $\tilde{y}_0(0) = 0 = \tilde{c}_1 + \tilde{c}_2$ and $\tilde{y}_0^{(1)}(t) = 1 = -2\tilde{c}_2$ yields $\tilde{c}_2 = -1/2$ and $\tilde{c}_1 = 1/2$. Since $b_0 = 0$ and $P(D) = 1$, $h(t) = \left(1/2 - e^{-2t}/2\right)u(t)$.

Next, $y(t) = x(t) * h(t) = \left(\int_0^t (1/2 - e^{-2\tau}/2)d\tau\right)u(t) = \left(\tau/2 + e^{-2\tau}/4\Big|_{\tau=0}^{t}\right)u(t) = \left(t/2 + e^{-2t}/4 - 1/4\right)u(t)$.

$$y(t) = \left(t/2 + e^{-2t}/4 - 1/4\right)u(t).$$

As expected, the DC nature of the unit step input results in an unbounded output.

(d) For this system, $\lambda_1 = 0$ is not affected by the components and λ_2 is only affected by C_2 and R_3. Using 10% resistors, the resistor R_3 is generally expected to lie in the range $(0.9R_3, 1.1R_3)$. Using 25% capacitors, the capacitor C_2 is generally expected to lie in the range $(.25C_2, 1.25C_2)$. Since $\lambda_2 = -\frac{1}{R_3C_2}$, the characteristic root is expected to lie in the range $(\lambda_2/[(0.9)(0.75)], \lambda_2/[(1.1)(1.25)])$. Thus,

$$\lambda_1 \text{ is unaffected and } \lambda_2 \text{ is expected to lie within } (-2.9630, -1.4545).$$

2.4-39.

2.4-40. (a) Yes, the system is causal since $h(t) = 0$ for $(t < 0)$.

(b) To compute the zero-state response $y_1(t)$, the convolution of two rectangular pulses is required: a pulse of amplitude $\jmath$ and width two and a pulse of amplitude one and a width of one. The convolution involves several regions.
For $t < 0$, $y_1(t) = 0$.
For $0 \le t < 1$, $y_1(t) = \int_0^t \jmath dt = \jmath t$.
For $1 \le t < 2$, $y_1(t) = \int_{t-1}^t \jmath dt = \jmath(t-(t-1)) = \jmath$.
For $2 \le t < 3$, $y_1(t) = \int_{t-1}^2 \jmath dt = \jmath(2-(t-1)) = \jmath(3-t)$.
For $t \ge 0$, $y_1(t) = 0$.
Thus,

$$y_1(t) = \begin{cases} \jmath t & 0 \le t < 1 \\ \jmath & 1 \le t < 2 \\ \jmath(3-t) & 2 \le t < 3 \\ 0 & \text{otherwise} \end{cases}.$$

(c) To compute $y_2(t)$, first note that $x_2(t) = 2x_1(t-1) - x_1(t-2)$. Using the system properties of linearity and time-invariance, the output $y_2(t)$ is given by

$$y_2(t) = 2y_1(t-1) - y_1(t-2).$$

2.5-1.

$$\lambda^2 + 7\lambda + 12 = (\lambda+3)(\lambda+4)$$

The natural response is

$$y_n(t) = K_1e^{-3t} + K_2e^{-4t}$$

(a) For $x(t) = u(t) = e^{0t}u(t)$, $\quad y_\phi(t) = H(0) = \frac{P(0)}{Q(0)} = \frac{1}{6}$

$$\begin{aligned} y(t) &= K_1e^{-3t} + K_2e^{-4t} + \frac{1}{6} \\ \dot{y}(t) &= -3K_1e^{-3t} - 4K_2e^{-4t} \end{aligned}$$

Setting $t = 0$ and substituting initial conditions, we obtain

$$\left.\begin{aligned} 0 &= K_1 + K_2 + \tfrac{1}{6} \\ 1 &= -3K_1 - 4K_2 \end{aligned}\right\} \Longrightarrow \quad \begin{aligned} K_1 &= -\tfrac{2}{3} \\ K_2 &= \tfrac{1}{2} \end{aligned}$$

and

$$y(t) = -\frac{2}{3}e^{-3t} + \frac{1}{2}e^{-4t} + \frac{1}{6} \qquad t \ge 0$$

(b) $x(t) = e^{-t}u(t)$, $\quad y_\phi(t) = H(-1) = \frac{P(-1)}{Q(-1)} = \frac{1}{6}$

$$\begin{aligned} y(t) &= K_1e^{-3t} + K_2e^{-4t} + \frac{1}{6}e^{-t} \\ \dot{y}(t) &= -3K_1e^{-3t} - 4K_2e^{-4t} - \frac{1}{6}e^{-t} \end{aligned}$$

Setting $t = 0$, and substituting initial conditions yields

$$\left.\begin{array}{l} 0 = K_1 + K_2 + \frac{1}{6} \\ 1 = -3K_1 - 4K_2 - \frac{1}{6} \end{array}\right\} \Longrightarrow \qquad \begin{array}{l} K_1 = -\frac{1}{2} \\ K_2 = -\frac{2}{3} \end{array}$$

and

$$y(t) = \frac{1}{2}e^{-3t} - \frac{2}{3}e^{-4t} + \frac{1}{6}e^{-t} \qquad t \geq 0$$

(c) $\quad x(t) = e^{-2t}u(t), \quad y_\phi(t) = H(-2) = 0$

$$\begin{aligned} y(t) &= K_1e^{-3t} + K_2e^{-4t} \\ \dot{y}(t) &= -3K_1e^{-3t} - 4K_2e^{-4t} \end{aligned}$$

Setting $t = 0$, and substituting initial conditions yields

$$\left.\begin{array}{l} 0 = K_1 + K_2 \\ 1 = -3K_1 - 4K_2 \end{array}\right\} \Longrightarrow \qquad \begin{array}{l} K_1 = 1 \\ K_2 = -1 \end{array}$$

and

$$y(t) = e^{-3t} - e^{-4t} \qquad t \geq 0$$

2.5-2. $\quad \lambda^2 + 6\lambda + 25 = (\lambda + 3 - j4)(\lambda + 3 + j4)$ characteristic roots are $-3 \pm j4$

$$y_n(t) = Ke^{-3t}\cos(4t + \theta)$$

For $x(t) = u(t), \quad y_\phi(t) = H(0) = \frac{3}{25}$ so that

$$\begin{aligned} y(t) &= Ke^{-3t}\cos(4t + \theta) + \frac{3}{25} \\ \dot{y}(t) &= -3Ke^{-3t}\cos(4t + \theta) - 4Ke^{-3t}\cos(4t + \theta) \end{aligned}$$

Setting $t = 0$, and substituting initial conditions yields

$$\left.\begin{array}{l} 0 = K\cos\theta + \frac{3}{25} \\ 2 = -3K\cos\theta - 4K\sin\theta \end{array}\right\} \Longrightarrow \quad \left.\begin{array}{l} K\cos\theta = \frac{-3}{25} \\ K\sin\theta = \frac{-41}{100} \end{array}\right\} \Longrightarrow \quad \begin{array}{l} K = 0.427 \\ \theta = -106.3 \end{array}$$

and

$$y(t) = 0.427e^{-3t}\cos(4t - 106.3°) + \frac{3}{25} \qquad t \geq 0$$

2.5-3. Characteristic polynomial is $\lambda^2 + 4\lambda + 4 = (\lambda + 2)^2$. The roots are -2 repeated twice.

$$y_n(t) = (K_1 + K_2t)e^{-2t}$$

(a) $\quad$ For $x(t) = e^{-3t}u(t), \quad y_\phi(t) = H(-3) = -2e^{-3t}$

$$\begin{aligned} y(t) &= (K_1 + K_2t)e^{-2t} - 2e^{-3t} \\ \dot{y}(t) &= -2(K_1 + K_2t)e^{-2t} + K_2e^{-2t} + 6e^{-3t} \end{aligned}$$

Setting $t = 0$, and substituting initial conditions yields

$$\left.\begin{array}{l} \frac{9}{4} = K_1 - 2 \\ 5 = -2K_1 + K_2 + 6 \end{array}\right\} \Longrightarrow \qquad \begin{array}{l} K_1 = \frac{17}{4} \\ K_2 = \frac{15}{2} \end{array}$$

and

$$y(t) = (\frac{17}{4} + \frac{15}{2}t)e^{-2t} - 2e^{-3t} \qquad t \geq 0$$

(b) $\quad x(t) = e^{-t}u(t), \quad y_\phi(t) = H(-1)e^{-t} = 0$

$$\begin{aligned} y(t) &= (K_1 + K_2 t)e^{-2t} \\ \dot{y}(t) &= -2(K_1 + K_2 t)e^{-2t} + K_2 e^{-2t} \end{aligned}$$

Setting $t = 0$, and substituting initial conditions yields

$$\left.\begin{aligned} \tfrac{9}{4} &= K_1 \\ 5 &= -2K_1 + K_2 \end{aligned}\right\} \Longrightarrow \qquad \begin{aligned} K_1 &= \tfrac{9}{4} \\ K_2 &= \tfrac{19}{2} \end{aligned}$$

and

$$y(t) = (\frac{9}{4} + \frac{19}{2}t)e^{-2t} \qquad t \geq 0$$

2.5-4. Because $\quad (\lambda^2 + 2\lambda) = \lambda(\lambda + 2)$, the characteristic roots are 0 and -2.

$$y_n(t) = K_1 + K_2 e^{-2t}$$

In this case $x(t) = u(t)$. The input itself is a characteristic mode. Therefore

$$y_\phi(t) = \beta t$$

But $y_\phi(t)$ satisfied the system equation

$$(D^2 + 2D)y_\phi(t) = (D + 1)y(t) = \ddot{y}_\phi(t) + 2\dot{y}_\phi(t) = \dot{x}(t) + x(t)$$

Substituting $x(t) = u(t)$ and $y_\phi(t) = \beta t$, we obtain

$$0 + 2\beta = 0 + 1 \quad \Longrightarrow \quad \beta = \frac{1}{2}$$

Therefore $y_\phi(t) = \frac{1}{2}t$.

$$\begin{aligned} y(t) &= K_1 + K_2 e^{-2t} + \frac{1}{2}t \\ \dot{y}(t) &= -2K_2 e^{-2t} + \frac{1}{2} \end{aligned}$$

Setting $t = 0$, and substituting initial conditions yields

$$\left.\begin{aligned} 2 &= K_1 + K_2 \\ 1 &= -2K_2 + \tfrac{1}{2} \end{aligned}\right\} \Longrightarrow \qquad \begin{aligned} K_1 &= \tfrac{9}{4} \\ K_2 &= -\tfrac{1}{4} \end{aligned}$$

and

$$y(t) = \frac{9}{4} - \frac{1}{4}e^{-2t} + \frac{1}{2}t \qquad t \geq 0$$

2.5-5. The natural response $\quad y_n(t)$ is found in Prob. 2.5-1:

$$y_n(t) = K_1 e^{-3t} + K_2 e^{-4t}$$

The input $x(t) = e^{-3t}$ is a characteristic mode. Therefore

$$y_\phi(t) = \beta t e^{-3t}$$

Also $y_\phi(t)$ satisfies the system equation:

$$(D^2 + 7D + 12)y_\phi(t) = (D + 2)x(t)$$

or $$\ddot{y}_\phi(t) + 7\dot{y}_\phi(t) + 12y_\phi(t) = \dot{x}(t) + 2x(t)$$

Substituting $x(t) = e^{-3t}$ and $y_\phi(t) = \beta te^{-3t}$ in this equation yields

$$(9\beta t - 6\beta)e^{-3t} + 7(-3\beta t + \beta)e^{-3t} + 12\beta te^{-3t} = -3e^{-3t} + 2e^{-3t}$$

or $$\beta e^{-3t} = -e^{-3t} \quad \Longrightarrow \quad \beta = -1$$

Therefore
$$\begin{aligned} y(t) &= K_1e^{-3t} + K_2e^{-4t} - te^{-3t} \\ \dot{y}(t) &= -3K_1e^{-3t} - 4K_2e^{-4t} + 3te^{-3t} - e^{-3t} \end{aligned}$$

Setting $t = 0$, and substituting initial conditions yields

$$\left.\begin{aligned} 0 &= K_1 + K_2 \\ 1 &= -3K_1 - 4K_2 - 1 \end{aligned}\right\} \Longrightarrow \qquad \begin{aligned} K_1 &= 2 \\ K_2 &= -2 \end{aligned}$$

and
$$\begin{aligned} y(t) &= 2e^{-3t} - 2e^{-4t} - te^{-3t} \qquad t \geq 0 \\ &= (2-t)e^{-3t} - 2e^{-4t} \qquad t \geq 0 \end{aligned}$$

2.6-1. (a) $\lambda^2 + 8\lambda + 12 = (\lambda + 2)(\lambda + 6)$
Both roots are in LHP. The system is BIBO stable and also asymptotically stable.

(b) $\lambda(\lambda^2 + 3\lambda + 2) = \lambda(\lambda + 1)(\lambda + 2)$
Roots are $0, -1, -2$. One root on imaginary axis and none in RHP. The system is BIBO ustable and marginally stable.

(c) $\lambda^2(\lambda^2 + 2) = \lambda^2(\lambda + j\sqrt{2})(\lambda - j\sqrt{2})$
Roots are 0 (repeated twice) and $\pm j\sqrt{2}$. Multiple roots on imaginary axis. The system is BIBO unstable and asymptotically unstable.

(d) $(\lambda + 1)(\lambda^2 - 6\lambda + 5) = (\lambda + 1)(\lambda - 1)(\lambda - 5)$
Roots are -1, 1 and 5. Two roots in RHP. The system is BIBO unstable and asymptotically unstable.

2.6-2. (a) $(\lambda + 1)(\lambda^2 + 2\lambda + 5)^2 = (\lambda + 1)(\lambda + 1 - j2)^2(\lambda + 1 + j2)^2$
Roots -1, $-1 \pm j2$ (repeated twice) are all in LHP. The system is BIBO stable and asymptotically stable.

(b) $(\lambda + 1)(\lambda^2 + 9) = (\lambda + 1)(\lambda + j3)(\lambda - j3)$
Roots are -1, $\pm j3$. Two (simple) roots on imaginary axis, none in RHP. The system is BIBO unstable and marginally stable.

(c) $(\lambda + 1)(\lambda^2 + 9)^2 = (\lambda + 1)(\lambda + j3)^2(\lambda - j3)^2$
Roots are -1 and $\pm j3$ repeated twice. Multiple roots on imaginary axis. The system is BIBO unstable and asymptotically unstable.

(d) $(\lambda^2 + 1)(\lambda^2 + 4)(\lambda^2 + 9) = (\lambda + j1)(\lambda - j1)(\lambda + j2)(\lambda - j2)(\lambda + j3)(\lambda - j3)$
The roots are $\pm j1$, $\pm j2$ and $\pm j3$. All roots are simple and on imaginary axis. None in RHP. The system is BIBO unstable and marginally stable.

2.6-3. (a) Because $u(t) = e^{0t}u(t)$, the characteristic root is 0.

(b) The root lies on the imaginary axis, and the system is marginally stable.

(c) $\int_0^\infty h(t)\,dt = \infty$
The system is BIBO unstable.

(d) The integral of $\delta(t)$ is $u(t)$. The system response to $\delta(t)$ is $u(t)$. Clearly, the system is an ideal integrator.

2.6-4. Assume that a system exists that violates Eq. (2.64) and yet produces a bounded output for every bounded input. The response at $t = t_1$ is

$$y(t_1) = \int_0^\infty h(\tau)x(t_1 - \tau)\,d\tau$$

Consider a bounded input $x(t)$ such that at some instant t_1

$$x(t_1 - \tau) = \begin{cases} 1 & \text{if } h(\tau) > 0 \\ -1 & \text{if } h(\tau) < 0 \end{cases}$$

In this case

$$h(\tau)x(t_1 - \tau) = |h(\tau)|$$

and

$$y(t_1) = \int_0^\infty |h(\tau)|\,d\tau = \infty$$

This violates the assumption.

2.6-5. (a) For this convolution, there are several regions. For $(t < -2)$ and $(t \geq 4)$, $y(t) = 0$.
For $(-2 \leq t < 0)$, $y(t) = \int_0^{t+2} \tau d\tau = (t+2)^2/2 = t^2/2 + 2t + 2$.
For $(-2 \leq t < 0)$, $y(t) = \int_0^2 \tau d\tau = 2$.
For $(-2 \leq t < 0)$, $y(t) = \int_{t-2}^2 \tau d\tau = 2^2/2 - (t-2)^2/2 = 2 - t^2/2 - 2t - 2 = -t^2/2 + 2t$.
Combining yields

$$y(t) = \begin{cases} t^2/2 + 2t + 2 & -2 \leq t < 0 \\ 2 & 0 \leq t < 2 \\ -t^2/2 + 2t & 2 \leq t < 4 \\ 0 & \text{otherwise} \end{cases}.$$

MATLAB is used to plot the result.

```
>> t = linspace(-3,5,401);
>> y = (t.^2/2+2*t+2).*((t>=-2)&(t<0));
>> y = y+2.*((t>=0)&(t<2));
>> y = y+(-t.^2/2+2*t).*((t>=2)&(t<4));
>> plot(t,y,'k'); axis([-3 5 -.5 2.5]);
>> xlabel('t'); ylabel('y(t)');
```

(b) Yes, the system is stable since $\int h(t) = 4 < \infty$.
No, the system is not causal since $h(t) \neq 0$ for all $t < 0$.

2.6-6. Yes, the system is stable since $h(t)$ is absolutely integrable. That is, $\int_{-\infty}^{\infty} h(t)dt = \int_0^1 1dt = 1 < \infty$.
Yes, the system is causal since $h(t) = 0$ for $t < 0$.s

2.6-7. Expanding

$$h(t) = \sum_{i=0}^{\infty} (0.5)^i \delta(t - i)$$

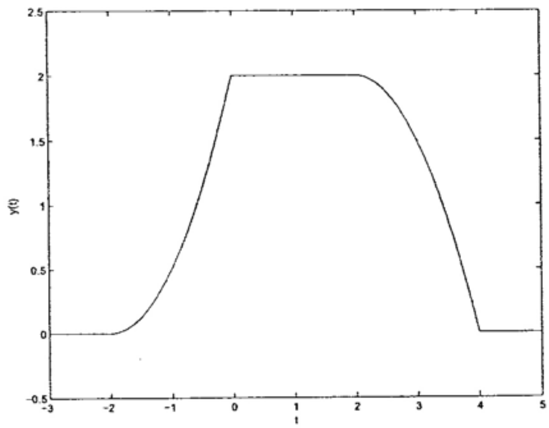


Figure S2.6-5a: Plot of $y(t) = x(t) * h(t)$.

yields

$$h(t) = (\delta(t) + 0.5\delta(t-1) + 0.25\delta(t-2) + 0.125\delta(t-3) + \cdots)$$

(a) Yes, the system is causal since $h(t) = 0$ for $t < 0$.

(b) Yes, the system is stable since the impulse response is absolutely integrable. That is, $\int_{-\infty}^{\infty} \sum_{i=0}^{\infty} (0.5)^i \delta(t-i)dt = \sum_{i=0}^{\infty}(0.5)^i \int_{-\infty}^{\infty} \delta(t-i) = \sum_{i=0}^{\infty}(0.5)^i = \frac{1-0}{1-0.5} = 2 < \infty$.

2.7-1. (a) The time-constant (rise-time) of the system is $T_h = 10^{-5}$. The rate of pulse communication $< \frac{1}{T_h} = 10^5$ pulses/sec. The channel cannot transmit million pulses/second.

(b) The bandwidth of the channel is

$$B = \frac{1}{T_h} = 10^5 \,\text{Hz}$$

The channel can transmit audio signal of bandwidth 15 kHz readily.

2.7-2.

$$T_h = \frac{1}{B} = \frac{1}{10^4} = 10^{-4} = 0.1 \,\text{ms}$$

The received pulse width $= (0.5 + 0.1) = 0.6$ ms. Each pulse takes up 0.6 ms interval. The maximum pulse rate (to avoid interference between successive pulses) is

$$\frac{1}{0.6 \times 10^{-3}} \simeq 1667 \,\text{pulses/sec}$$

2.7-3. Using Eqs. (2.67) and (2.68)

(a)

$$T_r = T_h = -\frac{1}{\lambda} = 10^{-4}$$

(b) The bandwidth $f_c = 1/T_h = 1/T_r = 10^4$.

(c) The pulse transmission rate is $f_c = 10^4$ pulses/sec.

2.7-4. (a) For a causal system with finite duration $h(t)$, the rise time is exactly equal to the time when the signal is last non-zero. That is,

$$T_r = 4 \text{ seconds}.$$

(b) The impulse response function $h(t)$ is consistent with a channel that has the following three characteristics: 1) a channel with delay from input to output (for example, signal propagation delay), 2) a channel with low-pass character (pulse dispersion that results in a $\delta(t)$ input spreading into a square pulse), and 3) a channel with two signal paths (for example, a primary signal path and an echo path).

For systems with predominantly low-pass character, digital information can be transmitted without significant interference at a rate of $\mathcal{F}_c = \frac{1}{T_r} = 1/4$. However, this estimate is too conservative for the present system. Notice that $h(t) = 0$ for $0 \leq t < 1$, corresponding to a transmission delay in the primary signal path. The remaining portion of $h(t)$ has a width of three, so it is therefore practical to transmit at rates of $\mathcal{F}_c = 1/3$. By clever interleaving of data, it is possible to transmit at rates of $\mathcal{F}_c = 1/2$. Consider transmitting the binary sequence $\{b_0, b_1, b_2, b_3, \ldots\}$ using a ($t = 1$)-spaced delta train weighted by the pulse sequence $\{b_0, b_1, 0, 0, b_2, b_3, 0, 0, \ldots\}$. The output is the series of non-overlapping unit-duration pulses given by $\{b_0, b_1, b_0, b_1, b_2, b_3, b_2, b_3, \ldots\}$. The effective transmission rate is 0.5 bits per unit time.

(c) The resulting convolution $y(t) = x(t) * h(t)$ has many regions.
For $t < 1$, $y(t) = 0$.
For $1 \leq t < 2$, $y(t) = \int_1^t (-1)dt = 1 - t$.
For $2 \leq t < 3$, $y(t) = \int_1^2 (-1)dt = -1$.
For $3 \leq t < 4$, $y(t) = \int_{t-2}^2 (-1)dt + \int_3^t (-1)dt = (t-2) - 2 + 3 - t = -1$.
For $4 \leq t < 5$, $y(t) = \int_3^4 (-1)dt = -1$.
For $5 \leq t < 6$, $y(t) = \int_{t-2}^4 (-1)dt = (t-2) - 4 = t - 6$.
For $6 \leq t$, $y(t) = 0$.
Thus,

$$y(t) = \begin{cases} 1-t & 1 \leq t < 2 \\ -1 & 2 \leq t < 5 \\ t-6 & 5 \leq t < 6 \\ 0 & \text{otherwise} \end{cases}.$$

MATLAB is used to plot the result.

```
>> t = [0:.01:10]; y = zeros(size(t));
>> y = y + (1-t).*((t>=1)&(t<2));
>> y = y + (-1).*((t>=2)&(t<5));
>> y = y + (t-6).*((t>=5)&(t<6));
>> plot(t,y,'k'); axis([0 10 -1.2 .2]);
>> xlabel('t'); ylabel('y(t)');
```

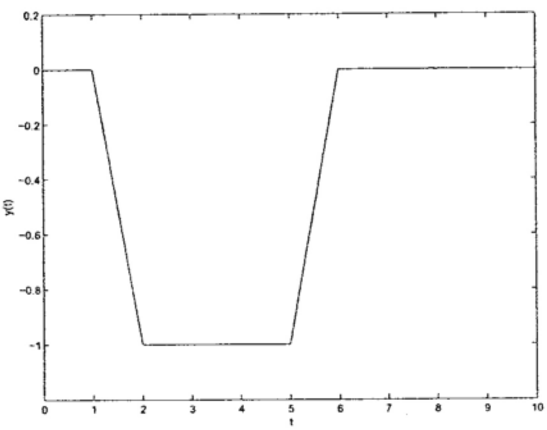


Figure S2.7-4c: Plot of $y(t) = x(t) * h(t)$.

Chapter 3 Solutions

3.1-1. (a) $E_x = (3)^2 + 2(2)^2 + 2(1)^2 = 19$

(b) $E_x = (3)^2 + 2(2)^2 + 2(1)^2 = 19$

(c) $E_x = 2(3)^2 + 2(6)^2 + 2(9)^2 = 252$

(d) $E_x = 2(2)^2 + 2(4)^2 = 40$

3.1-2. (a)

$$P_x = \frac{1}{N_0 + 1} \sum_{n=-3}^{3} |x[n]|^2 = \frac{1}{6}\left[2(1)^2 + 2(2)^2 + 3^2\right] = \frac{19}{6}$$

(b)

$$P_x = \frac{1}{12}\left[2(1)^2 + 2(2)^2 + 2(3)^2\right] = \frac{7}{3}$$

(c)

$$P_x = \frac{1}{N_0} \sum_{n=0}^{N_0-1} a^n = \frac{1}{N_0}\left[\frac{a^{N_0} - 1}{a - 1}\right] = \frac{a^{N_0} - 1}{N_0(a-1)}$$

3.1-3.

$$\left|\mathcal{D}e^{jr\frac{2\pi}{N_0}n}\right|^2 = |\mathcal{D}|^2$$

and

$$\begin{aligned} P_x = \frac{1}{N_0}|x[n]|^2 &= \frac{1}{N_0} \sum_{n=0}^{N_0-1} \left| \sum_{r=0}^{N_0-1} \mathcal{D}_r e^{jr\Omega_0 n} \right|^2 \\ &= \frac{1}{N_0} \sum_{n=0}^{N_0-1} \sum_{r=0}^{N_0-1} \mathcal{D}_r e^{jr\Omega_0 n} \sum_{m=0}^{N_0-1} \mathcal{D}_m^* e^{-jm\Omega_0 n} \end{aligned}$$

Interchanging the order of summation yields

$$P_x = \frac{1}{N_0} \sum_{r=0}^{N_0-1} \sum_{m=0}^{N_0-1} \mathcal{D}_r \mathcal{D}_m^* \left[\sum_{n=0}^{N_0-1} e^{j(r-m)\Omega_0 n}\right]$$

The summation within square brackets is N_0 when $r = m$ and 0 otherwise. Hence

$$P_x = \sum_{r=0}^{N_0-1} \mathcal{D}_r \mathcal{D}_r^* = \sum_{r=0}^{N_0-1} |\mathcal{D}_r|^2$$

3.1-4. (a)

$$x[n] = 0.8^n u[n] = \underbrace{\frac{1}{2}\left\{0.8^n u[n] + 0.8^{-n} u[-n]\right\}}_{x_e[n]} + \underbrace{\frac{1}{2}\left\{0.8^n u[n] - 0.8^{-n} u[-n]\right\}}_{x_o[n]}$$

$$E_x = \sum_{n=0}^{\infty} (0.8)^{2n} = \sum_{n=0}^{\infty} 0.64^n = \frac{1}{1-0.64} = 2.78$$

(b) To find the energy of the even component $x_e[n]$, we observe that both terms $0.8^n u[n]$ and $0.8^{-n} u[-n]$ are nonzero at $n = 0$. Hence, the two terms are not disjoint, and the energy E_{x_e} is not the sum of the energies of the two terms. For this reason, we rearrange $x_e[n]$ as

$$x_e[n] = \delta[n] + \frac{1}{2}\left\{0.8^n u[n-1] + 0.8^{-n} u[n-1]\right\}$$

All the above three terms are disjoint. Hence E_{x_e} is the sum of energies of the three terms. Thus

$$\begin{aligned} E_{x_e} &= 1 + \frac{1}{4}\sum_{n=1}^{\infty} 0.64^n + \frac{1}{4}\sum_{n=-1}^{-\infty} 0.64^{-n} \\ &= 1 + \frac{1}{2}\sum_{n=1}^{\infty} 0.64^n = 1 + \frac{1}{2}\left[\frac{0.64}{1-0.64}\right] = 1.89 \end{aligned}$$

The energy of $x_o[n]$ is

$$E_{x_o} = \frac{1}{4}\left[\sum_{n=1}^{\infty} 0.64^n + \sum_{n=-1}^{-\infty} 0.64^{-n}\right] = \frac{1}{2}\sum_{n=1}^{\infty} 0.64^n = 0.8$$

Hence

$$E_{x_e} + E_{x_o} = 1.89 + 0.89 = 2.78 = E_x$$

(c) Let us first consider a causal signal $x[n]$, which can be expressed as

$$x[n] = \underbrace{x[0]\delta[n] + \frac{1}{2}\left\{x[n]u[n-1] + x[-n]u[-n-1]\right\}}_{x_e[n]} + \underbrace{\frac{1}{2}\left\{x[n]u[n-1] - x[-n]u[-n-1]\right\}}_{x_o[n]}$$

The energy of the even component $x_e[n]$ is

$$E_{x_e} = x^2[0] + \frac{1}{4}\sum_{n=1}^{\infty} |x[n]|^2 + \frac{1}{4}\sum_{n=-1}^{-\infty} |x[-n]|^2 = x^2[0] + \frac{1}{2}\sum_{n=1}^{\infty} |x[n]|^2$$

Similarly

$$E_{x_o} = \frac{1}{2}\sum_{n=1}^{\infty} |x[n]|^2$$

Hence

$$E_{x_e} + E_{x_o} = x^2[0] + \sum_{n=1}^{\infty} |x[n]|^2 = E_x$$

Similarly, we can show that this result applies to anticausal signals also.
A general signal is made up of causal and anticausal components, which are disjoint. Hence the energy of a signal is the sum of energies of the causal and anticausal components. Moreover the energy of each causal and anticausal component is equal to the sum of the respective even and odd components. Also the sum of the even components of the causal and anticausal signal equals the even component of $x[n]$. Same is true of odd components. Hence, it follows that the energy of $x[n]$ is the sum of energies of even and odd components of $x[n]$.

3.1-5. (a) As seen in Prob 3.1-4, the even and odd component of a causal $x[n]$ are

$$x[n] = \underbrace{x[0]\delta[n] + \frac{1}{2}\{x[n]u[n-1] + x[-n]u[-n-1]\}}_{x_e[n]} + \underbrace{\frac{1}{2}\{x[n]u[n-1] - x[-n]u[-n-1]\}}_{x_o[n]}$$

If $x[0] = 0$, then

$$E_{x_e} = \frac{1}{4}\sum_{n=1}^{\infty} |x[n]|^2 + \frac{1}{4}\sum_{n=-1}^{-\infty} |x[-n]|^2 = \frac{1}{2}\sum_{n=1}^{\infty} |x[n]|^2 = 0.5E_x$$

Similarly, we can show that $E_{x_o} = 0.5E_x$.
If $x[0] \neq 0$, then it is clear that

$$E_{x_e} = E_{x_o} + x^2[0] \neq E_{x_o}$$

(b) Observe that $x[0]\delta[n]$ is disjoint with $x_o[n]$. Hence, the product $x[0]\delta[n]x_o[n]$ is zero. Moreover, $u[n-1]$ and $u[-n-1]$ is disjoint. Hence the cross-product terms involving product $u[n-1]u[-n-1]$ are zero. Hence

$$\sum_{n=-\infty}^{\infty} x_e[n]x_o^*[n] = \frac{1}{4}\sum_{n=-\infty}^{\infty} \left\{|x[n]|^2u[n-1] - |x[-n]|^2u[-n-1]\right\}$$

The two terms within the curly brackets are equal. Hence

$$\sum_{n=-\infty}^{\infty} x_e[n]x_o^*[n] = 0$$

Using similar argument, we can show that this result holds for anticausal signal.
A general signal is made up of causal and anticausal components, which are disjoint. Hence it follows that

$$\sum_{n=-\infty}^{\infty} x_e[n]x_o^*[n] = 0$$

3.2-1. (a) The energy of $x[-n]$ is E_1 given by

$$E_1 = \sum_{n=-\infty}^{\infty} |x[-n]|^2$$

Setting $n = -m$, we obtain

$$E_1 = \sum_{m=\infty}^{-\infty} |x[m]|^2 = \sum_{m=-\infty}^{\infty} |x[m]|^2 = E_x$$

(b) The energy E_2 of $x[n-m]$ is given by

$$E_2 = \sum_{n=-\infty}^{\infty} |x[n-m]|^2 = \sum_{r=-\infty}^{\infty} |x[r]|^2 = E_x$$

(c) The energy E_3 of $x[m-n]$ is given by

$$E_3 = \sum_{n=-\infty}^{\infty} |x[m-n]|^2 = \sum_{r=\infty}^{-\infty} |x[r]|^2 = \sum_{r=-\infty}^{\infty} |x[r]|^2 = E_x$$

(d) The energy E_4 of $Kx[n]$ is given by

$$E_4 = \sum_{n=-\infty}^{\infty} |Kx[n]|^2 = K^2 \sum_{n=-\infty}^{\infty} |x[n]|^2 = K^2 E_x$$

3.2-2. Because $|-x[n]|^2 = |x[n]|^2$, it follows that the power of $-x[n]$ is P_x.
For parts b, c, d and e, using the arguments in Prob. 3.2-1, we show
Power of $x[-n] = P_x$
Power of $x[n-m] = P_x$
Power of $cx[n] = c^2 P_x$
Power of $x[m-n] = P_x$
This shows the time-shift or time-inversion operation does not affect the power of a signal. Same is the case with sign change. Multiplication of a signal by a constant c causes c^2-fold increase in power.

3.2-3. Figure S3.2-3 shows all the signals.

3.2-4. Figure S3.2-4 shows all the signals.

3.3-1. (a) $P_x = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{-N}^{N} (1)^{2n} = 1$

(b) $P_x = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{-N}^{N} (-1)^{2n} = 1$

(c) $P_x = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{0}^{N} (1)^2 = 0.5$

(d) $P_x = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{0}^{N} (-1)^2 = 0.5$

(e) $N_0 = \frac{2\pi}{\frac{\pi}{3}} = 6$,

$P_x = \frac{1}{6} \sum_0^5 (\cos[\frac{\pi}{3}n + \frac{\pi}{6}])^2 = \frac{1}{6}\left[\left(\frac{\sqrt{3}}{2}\right)^2 + 0^2 + \left(-\frac{\sqrt{3}}{2}\right)^2 + \left(-\frac{\sqrt{3}}{2}\right)^2 + 0^2 + \left(\frac{\sqrt{3}}{2}\right)^2\right] =$
0.5

3.3-2. (a) Trivial

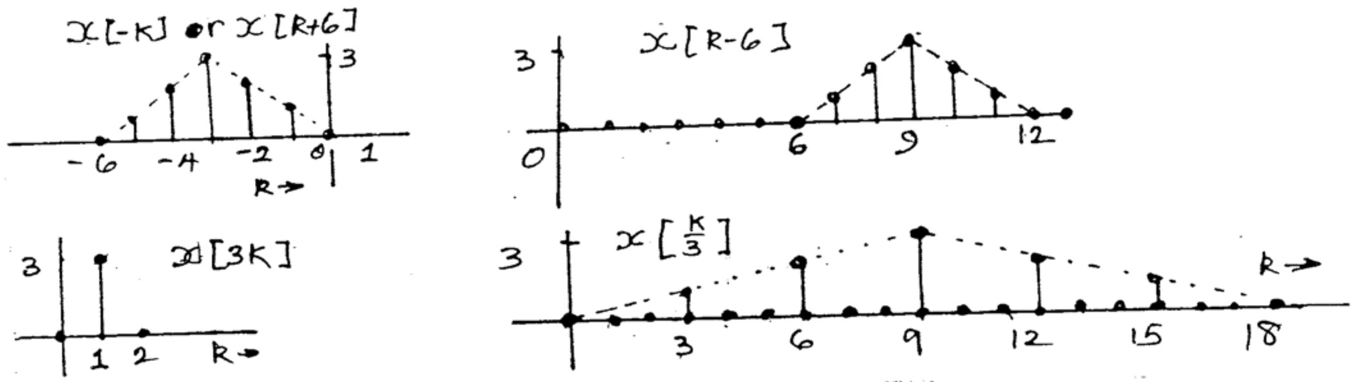


Figure S3.2-3

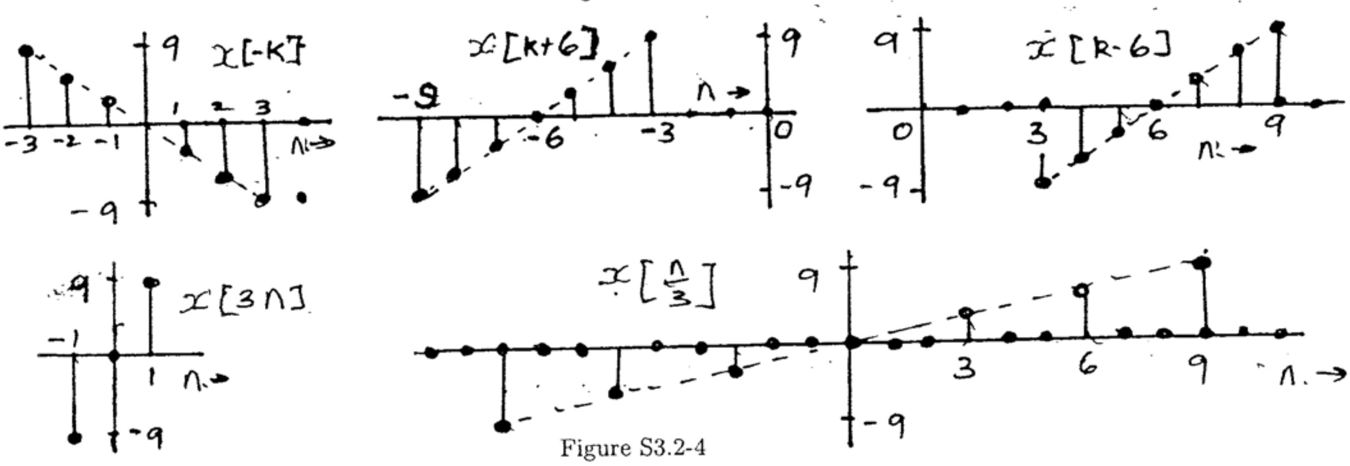


Figure S3.2-4

(b) Because $\sin\frac{\pi n}{3} = 0$ for $n = 0$, the result follows.

(c) Because $n(n-1) = 0$ for $n = 0$ and $n = 1$, the result follows.

(d) Because $\sin\frac{\pi n}{2} = 0$ for even n and $u[n] + (-1)^n u[n] = 0$ for odd n, the result follows.

(e) Because $\cos\frac{\pi n}{2} = 0$ for odd n and $u[n] + (-1)^{n+1} = 0$ for even n, the result follows.

3.3-3. Figure S3.3-3 shows all the signals.

3.3-4. (a) $x[n] = (n+3)\,(u[n+3] - u[n]) + (-n+3)\,(u[n] - u[n-4])$

(b) $x[n] = n\,(u[n] - u[n-4]) + (-n+6)\,(u[n-4] - u[n-7])$

(c) $x[n] = n\,(u[n+3] - u[n-4])$

(d) $x[n] = -2n\,(u[n+2] - u[n]) + 2n\,(u[n] - u[n-3])$

In all four cases, $x[n]$ may be represented by several other (slightly different) expressions. For instance, in case (a), we may also use $x[n] = (n+3)\,(u[n+3] - u[n-1]) + (-n+3)\,(u[n-1] - u[n-4])$. Moreover because $x[n] = 0$ at $n = \pm 3$, $u[n+3]$ and $u[n-4]$ may be replaced with $u[n+2]$ and $u[n-3]$, respectively. Similar observations apply to other cases also.

3.3-5. (a) $e^{-0.5n} = (0.6065)^n$,

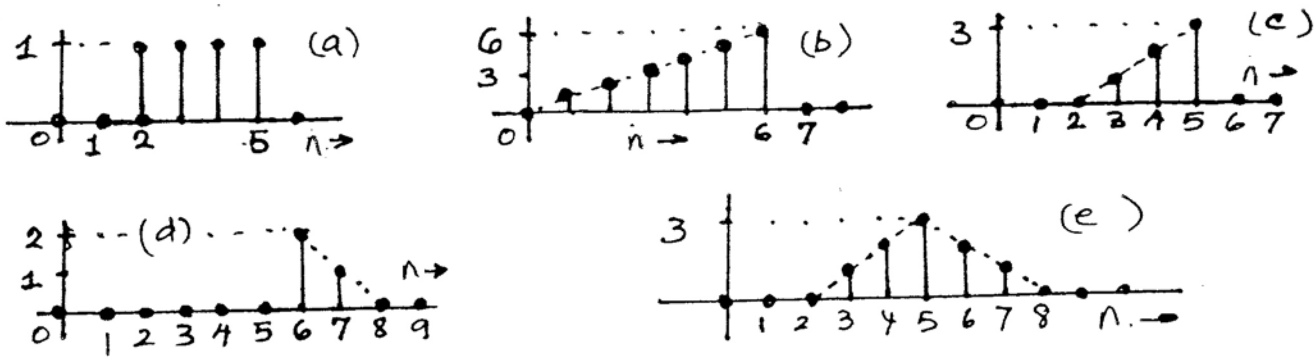


Figure S3.3-3

(b) $e^{0.5n} = (1.6487)^n$,
(c) $e^{-j\pi n} = (e^{-j\pi})^n = (-1)^n$,
(d) $e^{j\pi n} = (e^{j\pi})^n = (-1)^n$

Figure S3.3-5 shows locations of λ and γ in each case. By plotting these signals, we observe that when $\gamma > 1$ (λ in RHP), the signal grows exponentially. When $\gamma < 1$ (λ in LHP), the signal decays exponentially. When $|\gamma| = 1$ (λ on imaginary axis), the signal amplitude is constant.

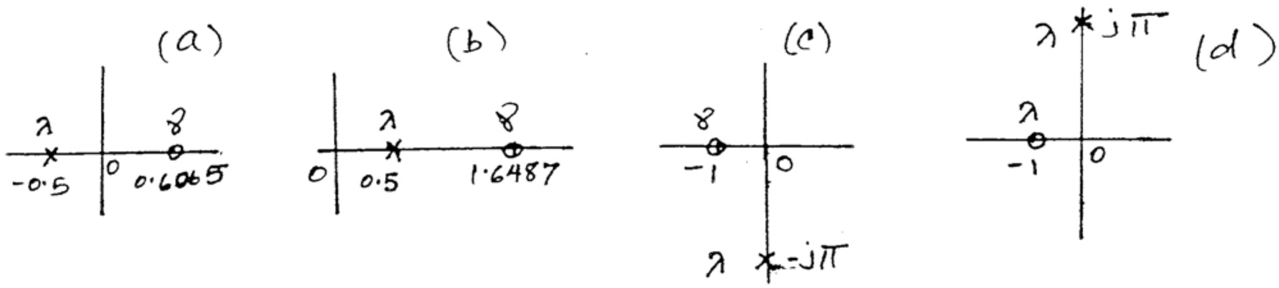


Figure S3.3-5

3.3-6. (a) $e^{-(1+j\pi)n} = (e^{-1}e^{-j\pi})^n = \left(-\frac{1}{e}\right)^n$
(b) $e^{-(1-j\pi)n} = (e^{-1}e^{j\pi})^n = \left(-\frac{1}{e}\right)^n$
(c) $e^{(1+j\pi)n} = (e\,e^{j\pi})^n = (-e)^n$
(d) $e^{(1-j\pi)n} = (e\,e^{-j\pi})^n = (-e)^n$
(e) $e^{-(1+j\frac{\pi}{3})n} = (e^{-1})^n\, e^{-j\frac{\pi}{3}n} = (\frac{1}{e})^n[\cos\frac{\pi}{3}n - j\sin\frac{\pi}{3}n]$
(f) $e^{(1-j\frac{\pi}{3})n} = (e^1)^n\, e^{-j\frac{\pi}{3}n} = e^n[\cos\frac{\pi}{3}n - j\sin\frac{\pi}{3}n]$

3.3-7.

$$x[n] = \underbrace{\frac{1}{2}\{x[n] + x[-n]\}}_{x_e[n] \quad (even)} + \underbrace{\frac{1}{2}\{x[n] - x[-n]\}}_{x_o[n] \quad (odd)}$$

(a)

$$u[n] = \underbrace{\frac{1}{2}\{u[n] + u[-n]\}}_{x_e} + \underbrace{\frac{1}{2}\{u[n] - u[-n]\}}_{x_o}$$

(b)

$$nu[n] = \underbrace{\frac{1}{2}\{nu[n] - nu[-n]\}}_{x_e} + \underbrace{\frac{1}{2}\{nu[n] + nu[-n]\}}_{x_o}$$

(c) $\sin\left(\frac{\pi n}{4}\right)$ is an odd function. It has no even component.

(d) $\cos\left(\frac{\pi n}{4}\right)$ is an even function. It has no odd component.

Figure S3.3-7 shows the even and odd components for the four cases.

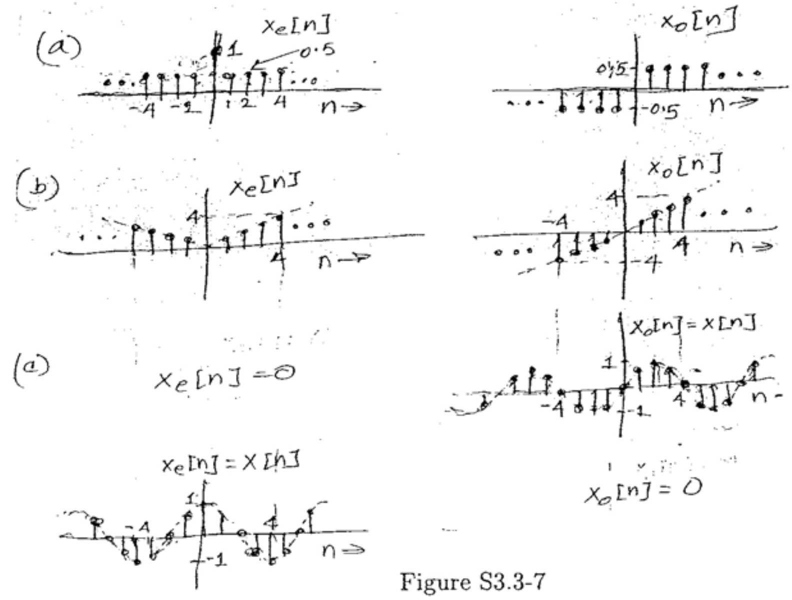


Figure S3.3-7

3.4-1. Because $y[n] = y[n-1] + x[n]$,

$$y[n] - y[n-1] = x[n]$$

Realization of this equation is shown in Figure S3.4-1.

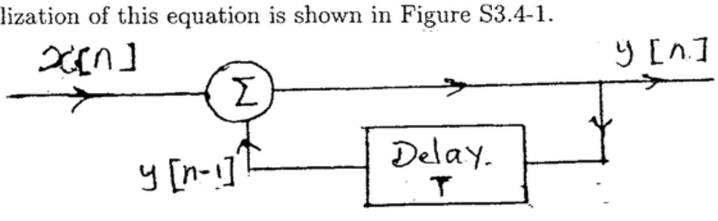


Figure S3.4-1

3.4-2. The net growth rate of the native population is $3.3 - 1.3 = 2\%$ per year. Assuming the immigrants enter at a uniform rate throughout the year, their birth and death rate will be $(3.3/2)\%$ and $(1.3/2)\%$, respectively of the immigrants at the end of the year. The population $p[n]$ at the beginning of the kth year is $p[n-1]$ plus the net increase in the native population plus $i[n-1]$, the immigrants entering during $(n-1)$st year plus the net increase in the immigrant population for the year $(n-1)$.

$$\begin{aligned} p[n] &= p[n-1] + \frac{3.3-1.3}{100}p[n-1] + i[n-1] + \frac{3.3-1.3}{2\times 100}i[n-1] \\ &= 1.02p[n-1] + 1.01i[n-1] \end{aligned}$$

or

$$p[n] - 1.02p[n-1] = 1.01i[n-1]$$

or

$$p[n+1] - 1.02p[n] = 1.01i[n]$$

3.4-3. (a)

$$y[n] = x[n] + x[n-1] + x[n-2] + x[n-3] + x[n-4]$$

(b) Refer to Figure S3.4-3b.

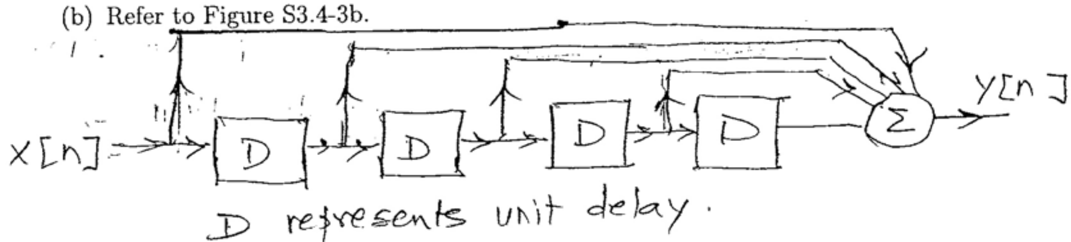


Figure S3.4-3b

3.4-4. The input $x[n] = u[n]$, which has a constant value of unity for all $n \geq 0$. Also $y[n] - y[n-1] = Tu[n]$. Hence the difference between two successive output values is always constant of value T. Clearly $y[n]$ must be a ramp with a possible constant component. Thus

$$y[n] = (nT + c)u[n]$$

To find the value of unknown constant c, we let $n = 0$ and obtain

$$y[0] = c$$

But from the input equation $y[n] - y[n-1] = Tu[n]$, we find $y[0] = T$ [remember that $y[-1] = 0$]. Hence

$$y[n] = (n+1)Tu[n] \simeq nTu[n] \qquad \text{for} \quad T \to 0$$

3.4-5. The differential equation is

$$\frac{d^2y}{dt^2} + a_1\frac{dy}{dt} + a_0y(t) = x(t)$$

We use the notation $y[n]$ to represent $y(nT)$, $x[n]$ to represent $x(nT), \cdots$ etc. and assume that T is small enough so that the assumption $T \to 0$ may be made. We have

$$\begin{aligned} y(t) &= y[n] \\ \frac{dy}{dt} &\simeq \frac{y[n]-y[n-1]}{T} \\ \frac{d^2y}{dt^2} &\simeq \frac{\frac{y[n]-y[n-1]}{T} - \frac{y[n-1]-y[n-2]}{T}}{T} \end{aligned}$$

$$= \frac{y[n] - 2y[n-1] + y[n-2]}{T^2}$$

Hence, the given differential equation can be approximated by the following differential equation

$$A_1 y[n] + A_2 y[n-1] + A_3 y[n-2] = Bx[n]$$

Substituting approximate difference expressions in the differential equation yields

$$y[n] - 2y[n-1] + y[n-2] + a_1 T\{y[n] - y[n-1]\} + a_0 T^2 y[n] = T^2 x[n]$$

or

$$(1 + a_1 T + a_0 T^2) y[n] - (2 + a_1 T) y[n-1] + y[n-2] = T^2 x[n]$$

3.4-6. The node equation at the kth node is $i_1 + i_2 + i_3 = 0$, or

$$\frac{v[n-1] - v[n]}{R} + \frac{v[n+1] - v[n]}{R} - \frac{v[n]}{aR} = 0$$

Therefore

$$a(v[n-1] + v[n+1] - 2v[n]) - v[n] = 0$$

or

$$v[n+1] - \left(2 + \frac{1}{a}\right) v[n] + v[n-1] = 0$$

that is

$$v[n+2] - \left(2 + \frac{1}{a}\right) v[n+1] + v[n] = 0$$

3.4-7. (a) True; all finite power signals have infinite energy, and therefore cannot be energy signals. Energy signals and power signals are mutually exclusive.

(b) False; a signal with infinite energy need not be a power signal. For example, the signal $x[n] = 2^n u[n]$ has infinite energy and infinite power. Thus, it is neither an energy signal nor a power signal.

(c) True; the system is causal. Even though the input is scaled by $(n+1)$, the current output only depends on the current input. Another way to see this is to rewrite the expression as $y[n] = nx[n] + x[n]$.

(d) False; the system is not causal. The current output depends on a future input value. To help see this, substitute $n' = n - 1$ to yield $y[n'] = x[n'+1]$; the output at time n' requires the future input value at time $n' + 1$.

(e) False; a signal $x[n]$ with energy E does not guarantee that signal $x[an]$ has energy $\frac{E}{|a|}$. Although this statement is true for continuous-time signals, it is not true for discrete-time signals. Remember, the discrete operation $x[an]$ results in a loss of information and thus a likely loss in energy. For example, consider $x[n] = \delta[n-1]$, which has energy $E = 1$. The signal $y[n] = x[2n] = 0$ has zero energy, not $E/2 = 1/2$.

3.4-8. Notice, $y_1[n] = -\delta[n] + \delta[n-1] + 2\delta[n-2]$. Furthermore, $x_2[n] = x_1[n-1] - 2x_1[n-2]$. Since the system is LTI,

$$y_2[n] = y_1[n-1] - 2y_1[n-2].$$

MATLAB is used to plot the result.

```
>> y1 = inline('-(n==0)+(n==1)+2*(n==2)'); n = [-2:8];
>> stem(n,y1(n-1)-2*y1(n-2),'k'); axis([-2 8 -4.5 4.5]);
>> xlabel('n'); ylabel('y_2[n]');
```

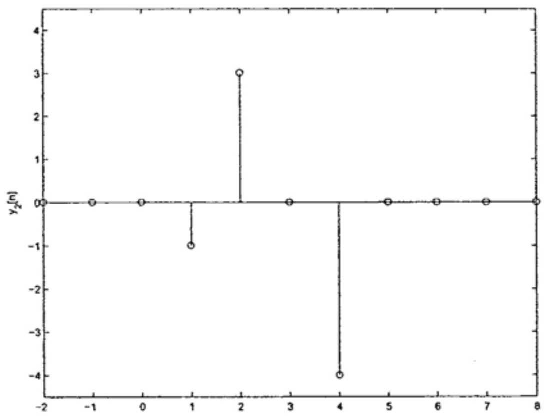


Figure S3.4-8: Plot of $y_2[n] = y_1[n-1] - 2y_1[n-2]$.

3.4-9. Using the sifting property, this system operation is rewritten as $y[n] = 0.5\,(x[n] + x[-n])$.

(a) This system extracts the even portion of the input.

(b) Yes, the system is BIBO stable. If the input is bounded, then the output is necessarily bounded. That is, if $|x[n]| \leq M_x < \infty$, then $|y[n]| = |0.5\,(x[n] + x[-n])| \leq 0.5\,(|x[n]| + |x[-n]|) \leq M_x < \infty$.

(c) Yes, the system is linear. Let $y_1[n] = 0.5(x_1[n] + x_1[-n])$ and $y_2[n] = 0.5(x_2[n] + x_2[-n])$. Applying $ax_1[n] + bx_2[n]$ to the system yields $y[n] = 0.5\,(ax_1[n] + bx_2[n] + (ax_1[-n] + bx_2[-n])) = 0.5a(x_1[n]+x_1[-n]) + 0.5b(x_2[n] + x_2[-n]) = ay_1[n] + by_2[n]$.

(d) No, the system is not memoryless. For example, at time $n = 1$ the output $y[1] = 0.5(x[1] + x[-1])$ depends on a past value of the input, $x[-1]$.

(e) No, the system is not causal. For example, at time $n = -1$ the output $y[-1] = 0.5(x[-1] + x[1])$ depends on a future value of the input, $x[1]$.

(f) No, the system is not time-invariant. For example, let the input be $x_1[n] = u[n+10] - u[n-11]$. Since this input is already even, the output is just the input, $y_1[n] = x_1[n]$. Shifting by a non-zero integer N, the signal $x_2[n] = x_1[n-N]$ is not even, and the output is $y_2[n] \neq y_1[n-N] = x_1[n-N]$. Thus, the system cannot be time-invariant.

3.4-10. It is convenient to substitute $n' = n + 1$ and rewrite the system expression as $y[n'] = x[n'-1]/x[n']$.

(a) No, the system is not BIBO stable. Input values of zero can result in unbounded outputs. For example, at $n' = 0$ the bounded input $x[n'] = \delta[n']$ yields an unbounded output $y[1] = 1/0 = \infty$.

(b) No, the system is not memoryless. The current output relies on a past input. For example, at $n' = 0$, the output $y[n']$ requires both the current input $x[n']$ and a stored past input $x[n'-1]$.

(c) Yes, the system is causal. The current output $y[n']$ does not depend on any future value of the input.

3.4-11. The operation $y(t) = x(2t)$ is a one-to-one mapping, where no information is lost. Any one-to-one mapping is invertible. In this case, $x(t)$ is recovered by taking $y(t/2)$.

Since every other sample of $x[n]$ is removed in the operation $y[n] = x[2n]$, one half of $x[n]$ is lost and the process is not invertible. Thought of another way, the operation $y[n] = x[2n]$ is not a one-to-one mapping; many different signals $x[n]$ map to the same signal $y[n]$, which makes inversion impossible.

3.4-12. Notice that $y_1[n]$ is obtained by multiplying $x[n]$ by the repeating sequence $\{\sin(\frac{\pi}{2}n+1)\} = \{\ldots, 0.5403, -0.8415, -0.5403, 0.8415, \ldots\}$. In this particular case, $x[n]$ is recovered by simply multiplying $y[n]$ by the inverse repeating sequence $\left\{\frac{1}{\sin(\frac{\pi}{2}n+1)}\right\} = \{\ldots, 1.8508, -1.1884, -1.8508, 1.1884, \ldots\}$.

In the case second, however, $y_2[n]$ is obtained by multiplying $x[n]$ by the repeating sequence $\{\sin(\frac{\pi}{2}(n+1))\} = \{\ldots, 0, -1, 0, 1, \ldots\}$. Since the sequence includes zeros, information is lost and the original sequence cannot be recovered.

3.4-13. Using the definition of the ramp function, the system expression is rewritten as $y[n] = nx[n]u[n]$.

(a) No, the system is not BIBO stable. For example, if the input is a unit step $x[n] = u[n]$, then the output is a ramp function $y[n] = r[n]$, which grows unbounded with time.

(b) Yes, the system is linear. Let $y_1[n] = nx_1[n]u[n]$ and $y_2[n] = nx_2[n]u[n]$. Applying $ax_1[n] + bx_2[n]$ to the system yields $y[n] = n\left(ax_1[n] + bx_2[n]\right)u[n] = anx_1[n]u[n] + bnx_2[n]u[n] = ay_1[n] + by_2[n]$.

(c) Yes, the system is memoryless. The current output only depends on the current input multiplied by a known (time-varying) scale factor.

(d) Yes, the system is causal. All memoryless systems are causal. The output does not depend on future values of the input or output.

(e) No, the system is not time-invariant. For example, applying $x_1[n] = \delta[n]$ yields the output $y_1[n] = n\delta[n]u[n] = 0$. Applying $x_2[n] = \delta[n-1]$ yields the output $y_2[n] = n\delta[n-1]u[n] = \delta[n-1]$. Note, $x_2[n] = x_1[n-1]$ but $y_2[n] \neq y_1[n-1]$. Shifting the input does not produce a corresponding shift in the output.

3.4-14. (a) Position measurements $x[n]$ are in meters. The difference $x[n] - x[n-1]$ is the change in position, in meters, per frame of film. Since the camera operates at 60 frames per second, dimensional analysis requires $k = \frac{60}{\text{seconds}}$.

(b) Since $v[n] = k(x[n] - x[n-1])$ is an estimate of the velocity $v(t) = \frac{d}{dt}x(t)$, it is sensible to use $a[n] = k(v[n] - v[n-1])$ as an estimate of the acceleration $a(t) = \frac{d}{dt}v(t)$. Combining estimates yields $a[n] = k(k(x[n] - x[n-1]) - k(x[n-1] - x[n-2]))$ or

$$a[n] = k^2\left(x[n] - 2x[n-1] + x[n-2]\right) = 3600\left(x[n] - 2x[n-1] + x[n-2]\right).$$

This estimate of acceleration has two primary advantages. First, it is simple to calculate. Second, it is a causal, stable, LTI system and therefore enjoys the properties of such systems.

There are several shortcomings of the estimate as well. Of particular significance, the estimate $a[n]$ lags the actual acceleration $a(t)$. One way to see this is that

the estimate $a[n]$ depends only on current and past values. A more balanced estimate of $a(t)$ is a shifted version $a_2[n] = a[n+1] = k^2(x[n+1] - 2x[n] + x[n-1])$. While this may fix the problem of lag, the new system is no longer causal. Both cases estimate derivatives using first-order differences; there are more sophisticated (and more complex) methods to more accurately estimate derivatives.
By substituting $\delta[n]$ for $x[n]$, the impulse response is

$$h[n] = k^2(\delta[n] - 2\delta[n-1] + \delta[n-2]).$$

3.4-15. There are many ways to solve this problem. For this solution, let $d[n]$ designate the distance from the student's destination, which alternates between home and the exam location, just before the student changes his mind. For even-valued n the destination is home, and for odd-valued n the destination is the exam location.
Just before changing direction, the student is a distance of $d[n]$ miles from his destination. Turning around, his next destination is therefore a distance of $2-d[n]$ miles. The student travels one-half of this distance, which leaves $\frac{2-d[n]}{2}$ miles remaining. Thus, a difference equation description of this problem is $d[n+1] = \frac{2-d[n]}{2} = 1 - 0.5d[n] = u[n] - 0.5d[n]$. Rearranging and shifting by one yields

$$d[n] + 0.5d[n-1] = u[n-1]$$

For this description, $d[0] = 0$. This auxiliary condition simply states that before the student first decides to go to the exam, he is at home.

3.5-1. (a)

$$y[n+1] = 0.5y[n] \qquad (1)$$

Setting $n = -1$ and substituting $y[-1] = 10$, yields

$$y[0] = 0.5(10) = 5$$

Setting $n = 0$, and substituting $y[0] = 5$, yields

$$y[1] = 0.5(5) = 2.5$$

Setting $n = 1$ in (1), and substituting $y[1] = 2.5$, yields

$$y[2] = 0.5(2.5) = 1.25$$

(b)

$$y[n+1] = -2y[n] + x[n+1] \qquad (2)$$

Setting $n = -1$, and substituting $y[-1] = 0$, $x[0] = 1$, yields

$$y[0] = 0 + 1 = 1$$

Setting $n = 0$, and substituting $y[0] = 1$, $x[1] = \frac{1}{e}$, yields

$$y[1] = -2(1) + \frac{1}{e} = -2 + \frac{1}{e} = -1.632$$

Setting $n = 1$ in (1), and substituting $y[1] = -2 + \frac{1}{e}$, $x[2] = \frac{1}{e^2}$, yields

$$y[2] = -2(-2 + \frac{1}{e}) + \frac{1}{e^2} = 4 - \frac{2}{e} + \frac{1}{e^2} = 3.399$$

3.5-2.

$$y[n] = 0.6y[n-1] + 0.16[n-2]$$

Setting $n = 0$, and substituting $y[-1] = -25$, $y[-2] = 0$, yields

$$y[0] = 0.6(-25) + 0.16(0) = -15$$

Setting $n = 1$, and substituting $y[-1] = 0$, $y[0] = -15$, yields

$$y[1] = 0.6(-15) + 0.16(-25) = -13$$

Setting $n = 2$, and substituting $y[1] = -13$, $y[0] = -15$, yields

$$y[2] = 0.6(-13) + 0.16(-15) = -10.2$$

3.5-3. This equation can be expressed as

$$y[n+2] = -\frac{1}{4}y[n+1] - \frac{1}{16}y[n] + x[n+2]$$

Setting $n = -2$, and substituting $y[-1] = y[-2] = 0$, $x[0] = 100$, yields

$$y[0] = -\frac{1}{4}(0) - \frac{1}{16}(0) + 100 = 100$$

Setting $n = -1$, and substituting $y[-1] = 0$, $y[0] = 100$, $x[1] = 100$, yields

$$y[1] = -\frac{1}{4}(100) - \frac{1}{16}(0) + 100 = 75$$

Setting $n = 0$, and substituting $y[0] = 100$, $y[1] = 75$, $x[2] = 100$, yields

$$y[2] = -\frac{1}{4}(75) - \frac{1}{16}(100) + 100 = 75$$

3.5-4.

$$y[n+2] = -3y[n+1] - 2y[n] + x[n+2] + 3x[n+1] + 3x[n]$$

Setting $n = -2$, and substituting $y[-1] = 3$, $y[-2] = 2$, $x[-1] = x[-2] = 0$, $x[0] = 1$, yields

$$y[0] = -3(3) - 2(2) + 1 + 3(0) + 3(0) = -12$$

Setting $n = -1$, and substituting $y[0] = -12$, $y[-1] = 3$, $x[-1] = 0$, $x[0] = 1$, $x[1] = 3$, yields

$$y[1] = -3(-12) - 2(3) + 3 + 3(1) + 3(0) = 36$$

Proceeding along same lines, we obtain

$$y[2] = -3(36) - 2(-12) + 9 + 3(3) + 3(1) = -63$$

3.5-5.

$$y[n] = -2y[n-1] - y[n-2] + 2x[n] - x[n-1]$$

Setting $n = 0$, and substituting $y[-1] = 2$, $y[-2] = 3$, $x[0] = 1$, $x[-1] = 0$, yields

$$y[0] = -2(2) - 3 + 2(1) - 0 = -5$$

Setting $n = 1$, and substituting $y[0] = -5$, $y[-1] = 2$, $x[0] = 1$, $x[1] = \frac{1}{3}$, yields

$$y[1] = -2(-5) - (2) + 2(\frac{1}{3}) - 1 = 7.667$$

Setting $n = 2$, and substituting $y[1] = 7.667$, $y[0] = -5$, $x[1] = \frac{1}{3}$, $x[2] = \frac{1}{9}$, yields

$$y[2] = -2(7.667) - (-5) + 2(\frac{1}{9}) - \frac{1}{3} = -5.445$$

3.6-1.

$$(E^2 + 3E + 2)y[n] = 0$$

The characteristic equation is $\gamma^2 + 3\gamma + 2 = (\gamma + 1)(\gamma + 2) = 0$. Therefore

$$y[n] = c_1(-1)^n + c_2(-2)^n$$

Setting $n = -1$ and -2 and substituting initial conditions yields

$$\left.\begin{array}{l} 0 = -c_1 - \frac{1}{2}c_2 \\ 1 = c_1 + \frac{1}{4}c_2 \end{array}\right\} \Longrightarrow \quad \begin{array}{l} c_1 = 2 \\ c_2 = -4 \end{array}$$

$$y[n] = 2(-1)^n - 4(-2)^n \qquad n \geq 0$$

3.6-2.

$$(E^2 + 2E + 1)y[n] = 0$$

The characteristic equation is $\gamma^2 + 2\gamma + 1 = (\gamma + 1)^2 = 0$.

$$y[n] = (c_1 + c_2 k)(-1)^n$$

Setting $n = -1$ and -2 and substituting initial conditions yields

$$\left.\begin{array}{l} 1 = -c_1 + c_2 \\ 1 = c_1 - 2c_2 \end{array}\right\} \Longrightarrow \quad \begin{array}{l} c_1 = -3 \\ c_2 = -2 \end{array}$$

$$y[n] = -(3 + 2n)(-1)^n$$

3.6-3.

$$(E^2 - 2E + 2)y[n] = 0$$

The characteristic equation is $\gamma^2 - 2\gamma + 2 = (\gamma - 1 - j1)(\gamma - 1 + j1) = 0$. The roots are $1 \pm j1 = \sqrt{2}e^{\pm j\pi/4}$.

$$y[n] = c(\sqrt{2})^n \cos(\frac{\pi}{4}n + \theta)$$

Setting $n = -1$ and -2 and substituting initial conditions yields

$$1 \quad = \quad \frac{c}{\sqrt{2}} \cos(-\frac{\pi}{4} + \theta) = \frac{c}{\sqrt{2}}(\frac{1}{\sqrt{2}} \cos\theta + \frac{1}{\sqrt{2}} \sin\theta)$$

$$0 = \frac{c}{2}\cos(-\frac{\pi}{2}+\theta) = \frac{c}{2}\sin\theta$$

Solution of these two simultaneous equations yields

$$\left.\begin{array}{l} c\cos\theta = 2 \\ c\sin\theta = 0 \end{array}\right\} \Longrightarrow \quad \begin{array}{l} c = 2 \\ \theta = 0 \end{array}$$

$$y[n] = 2(\sqrt{2})^n \cos(\frac{\pi}{4}n)$$

3.6-4. The equation can be expressed in terms of advance operation notation as

$$E^N y[n] = \left(b_0 E^N + b_1 E^{N-1} + \cdots + b_N\right) x[n]$$

The characteristic equation is

$$\gamma^N = 0$$

Hence, all the N characteristic roots are zero. Therefore, the zero-input component is zero and the total response in given by the zero-state component.

3.6-5. (a) By definition, any element in the Fibonacci sequence is the sum of the previous two. Thus, $f[n] = f[n-1] + f[n-2]$. Written in standard form, this yields

$$f[n] - f[n-1] - f[n-2] = 0.$$

This is a somewhat unusual system in the fact that it has no input. In the lingo of signals and systems, $f[n]$ is a zero-input response that is completely driven by the auxiliary conditions.

(b) The characteristic equation is $\gamma^2 - \gamma - 1 = 0$. This yields two characteristic roots

$$\gamma_1 = \frac{1+\sqrt{5}}{2} \approx 1.618 \text{ and } \gamma_2 = \frac{1-\sqrt{5}}{2} \approx -0.618.$$

Since one characteristic root is in the right-half plane, the system is not stable.

(c) To determine a particular Fibonacci number, it is convenient to determine a closed form expression for $f[n]$. Since $f[n]$ is a zero-input response, it has form $f[n] = c_1\gamma_1^n + c_2\gamma_2^n$. The auxiliary equations yield $f[1] = 0 = c_1\gamma_1 + c_2\gamma_2$ and $f[2] = 1 = c_1\gamma_1^2 c_2\gamma_2^2$. Solving yields $c_1 = \frac{-\gamma_2}{\gamma_1\gamma_2(\gamma_2-\gamma_1)} = \frac{\sqrt{5}-1}{2\sqrt{5}} \approx 0.2764$ and $c_2 = \frac{\gamma_1}{\gamma_1\gamma_2(\gamma_2-\gamma_1)} = \frac{\sqrt{5}+1}{2\sqrt{5}} \approx 0.7236$.

MATLAB is used to solve for the requested values of $f[n]$.

```
>> gamma1 = (1+sqrt(5))/2; gamma2 = (1-sqrt(5))/2;
>> c1 = -gamma2/(gamma1*gamma2*(gamma2-gamma1));
>> c2 = gamma1/(gamma1*gamma2*(gamma2-gamma1));
>> f50 = c1*gamma1^50+c2*gamma2^50
f50 = 7.7787e+009
>> f1000 = c1*gamma1^1000+c2*gamma2^1000
f1000 = 2.6864e+208
```

Thus,

$$f[50] = \frac{\sqrt{5}-1}{2\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{50} + \frac{\sqrt{5}+1}{2\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{50} \approx 7.7787(10^9)$$

and

$$f[1000] = \frac{\sqrt{5}-1}{2\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{1000} + \frac{\sqrt{5}+1}{2\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{1000} \approx 2.6864(10^{208})$$

3.6-6.

$$v(n+2) - 2.5v(n+1) + v(n) = 0$$

The auxiliary conditions are $v(0) = 100$, $v(N) = 0$.

$$(E^2 - 2.5E + 1)v[n] = 0$$

The characteristic equation is $\gamma^2 - 2.5\gamma + 1 = (\gamma - 0.5)(\gamma - 2) = 0$.

$$v(n) = c_1(0.5)^n + c_2(2)^n$$

Setting $n = 0$ and N, and substituting $v(0) = 100$, $v(N) = 0$, yields

$$\left.\begin{array}{rcl} 100 &=& c_1 + c_2 \\ 0 &=& c_1(0.5)^N + c_2(2)^N \end{array}\right\} \Longrightarrow \begin{array}{rcl} c_1 &=& \frac{100(2)^N}{2^N-(0.5)^N} \\ c_2 &=& \frac{100(0.5)^N}{(0.5)^N-2^N} \end{array}$$

$$v[n] = \frac{100}{2^N - (0.5)^N}[2^N(0.5)^n - (0.5)^N(2)^n] \qquad n = 0, 1, \cdots, N$$

3.6-7. Since we are looking for the zero-input response, the term $\sqrt{3}x[n-8]$ is irrelevant and the equation becomes $y_0[n] + y_0[n-1] + 0.25y_0[n-2] = 0$. The characteristic equation for this second-order system is $\gamma^2 + \gamma + 0.25 = 0$. This yields a repeated root at $\gamma = -0.5$, and the zero-input response has form $y_0[n] = c_1(-0.5)^n + c_2 n(-0.5)^n$. The pair of equations $y_0[-1] = 1 = -2c_1 + 2c_2$ and $y_0[1] = 1 = -c_1/2 - c_2/2$ are solved using MATLAB.

```
>> c = [-2 2;-1/2 -1/2]\[1;1]
c = -1.2500 -0.7500
```

Thus,

$$y_0[n] = -1.25(0.5)^n - 0.75n(0.5)^n.$$

3.7-1. (a)

$$(E+2)y[n] = x[n]$$

The characteristic equation is $\gamma + 2 = 0$. The characteristic root is -2. Also $a_1 = 2$, $b_1 = 1$. Therefore

$$h[n] = \frac{1}{2}\delta[n] + c(-2)^n \tag{1}$$

We need one value of $h[n]$ to determine c. This is determined by iterative solution of

$$(E+2)h[n] = \delta[n]$$

or

$$h[n+1] + 2h[n] = \delta[n]$$

Setting $n = -1$, and substituting $h[-1] = \delta[-1] = 0$, yields

$$h[0] = 0$$

Setting $n = 0$ in Eq. (1) and using $h[0] = 0$ yields

$$0 = \frac{1}{2} + c \quad \Longrightarrow \quad c = -\frac{1}{2}$$

Therefore

$$h[n] = \frac{1}{2}\delta[n] - \frac{1}{2}(-2)^n u[n]$$

(b) The characteristic root is -2, $b_1 = 0$, $a_1 = 2$. Therefore

$$h[n] = c(-2)^n \tag{2}$$

We need one value of $h[n]$ to determine c. This is done by solving iteratively

$$h[n+1] + 2h[n] = \delta[n+1]$$

Setting $n = -1$, and substituting $h[-1] = 0$, $\delta[0] = 1$, yields

$$h[0] = 1$$

Setting $n = 0$ in Eq. (2) and using $h[0] = 0$ yields

$$1 = c$$

and

$$h[n] = (-2)^n u[n]$$

3.7-2. Characteristic equation is $\gamma^2 - 6\gamma + 9 = (\gamma - 3)^2 = 0$. Also $a_2 = 9$, $b_2 = 0$. Therefore

$$h[n] = (c_1 + c_2 n)3^n u[n] \tag{1}$$

We need two values of $h[n]$ to determine c_1 and c_2. This is found from iterative solution of

$$(E^2 - 6E + 9)h[n] = E\delta[n]$$

or

$$h[n+2] - 6h[n+1] + 9h[n] = \delta[n+1] \tag{2}$$

Also $h[-1] = h[-2] = \delta[-1] = 0$ and $\delta[0] = 1$. Setting $n = -2$ in (2) yields

$$h[0] - 6(0) + 9(0) = 0 \quad \Longrightarrow \quad h[0] = 0$$

Setting $n = -1$ in (2) yields

$$h[1] - 6(0) + 9(0) = 1 \quad \Longrightarrow \quad h[1] = 1$$

Setting $n = 0$ and 1 in Eq. (1) and substituting $h[0] = 0$, $h[1] = 1$ yields

$$\left.\begin{array}{l} 0 = c_1 \\ 1 = 3(c_1 + c_2) \end{array}\right\} \Longrightarrow \quad \begin{array}{l} c_1 = 0 \\ c_2 = \frac{1}{3} \end{array}$$

and

$$h[n] = \frac{1}{3}n(3)^n u[n]$$

3.7-3.

$$(E^2 - 6E + 25)y[n] = (2E^2 - 4E)x[n]$$

The characteristic roots are $5e^{\pm j0.923}$, $b_2 = 0$. Therefore

$$h[n] = c(5)^n \cos(0.923n + \theta)u[n] \tag{1}$$

We need two values of $h[n]$ to determine c and θ. This is done by solving iteratively

$$h[n] - 6h[n-1] + 25h[n-2] = 2\delta[n] - 4\delta[n-1] \tag{2}$$

Setting $n = 0$ yields

$$h[0] - 6(0) + 25(0) = 2(1) - 4(0) \quad \Longrightarrow \quad h[0] = 2$$

Setting $n = 1$ in (2) yields

$$h[1] - 6(2) + 25(0) = 2(0) - 4 \quad \Longrightarrow \quad h[1] = 8$$

Setting $n = 0$, 1 in (1) and substituting $h[0] = 2$, $h[1] = 8$ yields

$$\begin{aligned} &2 = c\cos\theta \\ &8 = 5c\cos(0.923 + \theta) = 3.017c\cos\theta - 3.987c\sin\theta \end{aligned}$$

Solution of these two equations yields

$$\left.\begin{aligned} c\cos\theta &= 2 \\ c\sin\theta &= -0.4931 \end{aligned}\right\} \Longrightarrow \quad \begin{aligned} c &= 2.061 \\ \theta &= -0.244\,\text{rad} \end{aligned}$$

and

$$h[n] = 2.061(5)^n \cos(0.923n - 0.244)u[n]$$

3.7-4. (a)

$$y[n] = b_0 x[n] + b_1 x[n-1] + \cdots + b_N x[n-N]$$

Letting $x[n] = \delta[n]$ and $y[n] = h[n]$ yields

$$h[n] = b_0\delta[n] + b_1\delta[n-1] + \cdots + b_N\delta[n-N]$$

(b) From the result in part (a), we can immediately write $h[n]$ for this case as

$$h[n] = 3\delta[n] - 5\delta[n-1] - 2\delta[n-3]$$

3.8-1.

$$\begin{aligned} y[n] &= (-2)^n u[n-1] * e^{-n}u[n+1] \\ &= \sum_{m=-\infty}^{\infty} (-2)^m u[m-1]e^{-(n-m)}u[n-m-1] \end{aligned}$$

However, $u[m-1]=0$ for $m<1$ and $u[n-m+1]=0$ for $m>n+1$. Hence the summation limits may be restricted for $1 \leq m \leq n+1$.

$$
\begin{aligned}
y[n]=e^{-n}\sum_{m=1}^{n+1}(-2e)^m &= e^{-n}\left[\frac{(-2e)^{n+2}+2e}{-2e-1}\right] \\
&= \frac{2e^2}{2e+1}\left[(-2)^{n+1}-e^{-(n+1)}\right]u[n]
\end{aligned}
$$

We can also obtain this answer by using the convolution Table and the shift property of convolution. If we advance impulse response $h[n]$ by one unit and delay the input by one unit, the convolution remains unchanged according to the shift property. Hence we should obtain the convolution by using

$$
h[n]=(-2)^{n+1}u[n] \qquad \text{and} \qquad x[n]=e^{-(n-1)}u[n]
$$

Thus the desired convolution is given by

$$
\begin{aligned}
y[n] &= (-2)^{n+1}u[n]*e^{-(n-1)}u[n] \\
&= -2e\left\{(-2)^n u[n]*e^{-n}u[n]\right\}
\end{aligned}
$$

From the convolution Table, we obtain

$$
\begin{aligned}
y[n] &= -2e\left[\frac{(-2)^{n+1}-e^{-(n+1)}}{-1-e^{-1}}\right]u[n] \\
&= \frac{2e^2}{2e+1}\left[(-2)^{n+1}-e^{-(n+1)}\right]u[n]
\end{aligned}
$$

which confirms earlier result.

3.8-2.

$$
y[n]=\frac{1}{2}\left[\delta[n-2]-(-2)^{n+1}\right]u[n-3]*3^{n-1}u[n+2]
$$

Because $\delta[n-2]u[n-3]=0$, we have

$$
y[n]=-\frac{1}{2}(-2)^{n+1}u[n-3]*3^{n-1}u[n+2]
$$

If we advance the first term by 3 units and delay the second term by 2 units, the resulting convolution yields $y[n+1]$. Hence

$$
\begin{aligned}
y[n+1] &= -\frac{1}{2}(-2)^{n+4}u[n]*3^{n-3}u[n] \\
&= -8(-2)^n u[n]*\frac{1}{9}3^n u[n] \\
&= -\frac{8}{9}(-2)^n * 3^n u[n]
\end{aligned}
$$

From convolution Table, we obtain

$$
\begin{aligned}
y[n+1] &= -\frac{8}{9}\left[\frac{(-2)^{n+1}-(3)^{n+1}}{-2-3}\right]u[n] \\
&= \frac{8}{45}\left[(-2)^{n+1}-(3)^{n+1}\right]u[n]
\end{aligned}
$$

$$\text{and} \qquad y[n] = \frac{8}{45}\left[(-2)^n - (3)^n\right] u[n-1]$$

3.8-3. Here let us delay $x[n]$ by one unit and advance $h[n]$ by one unit to obtain $y[n]$.

$$\begin{aligned}
y[n] &= 3^{n+1}u[n] * \left[2^{n-1} + 3(-5)^{n+1}\right] u[n] \\
&= 3^{n+1}u[n] * 2^{n-1}u[n] + 3(3)^{n+1}u[n] * (-5)^{n+1}u[n] \\
&= \frac{3}{2}\left\{3^n u[n] * 2^n u[n]\right\} - 45\left\{3^n u[n] * (-5)^n u[n]\right\} \\
&= \frac{3}{2}\left[\frac{3^{n+1} - 2^{n+1}}{1}\right] u[n] - 45\left[\frac{3^{n+1} - (-5)^{n+1}}{8}\right] u[n] \\
&= \left[\frac{99}{8}(3)^n - 3(2)^n - 225(-5)^n\right] u[n]
\end{aligned}$$

3.8-4. Here let us delay the input $x[n]$ by 3 units and advance $h[n]$ by 4 units. The resulting convolution yields $y[n-3+4] = y[n+1]$.

$$\begin{aligned}
y[n+1] &= 3^{-(n+1)}u[n] * 3(n+2)(2)^{n+1}u[n] \\
&= 2\left\{3^{-n}u[n] * (n2^n u[n] + 2(2)^n u[n])\right\} \\
&= 4\left\{3^{-n}u[n] * 2^n u[n]\right\} + 2\left\{3^{-n}u[n] * n2^n u[n]\right\}
\end{aligned}$$

From convolution Table, we obtain

$$y[n+1] = -\frac{20}{3}\left[3^{-(n+1)} - 2^{(n+1)}\right] u[n] + \frac{6}{25}\left[3^{-n} + (5n-1)2^n\right] u[n]$$

Hence

$$\begin{aligned}
y[n] &= -\frac{20}{3}\left[3^{-n} - 2^n\right] u[n-1] + \frac{6}{25}\left[3^{-(n-1)} + (5n-6)2^{(n-1)}\right] u[n-1] \\
&= -\frac{446}{75}\left\{3^{-n}u[n-1] - 2^n u[n-1]\right\} + \frac{3}{5}n2^n u[n-1]
\end{aligned}$$

3.8-5. Here, we advance $x[n]$ by one unit and leave $h[n]$ unchanged. The resulting convolution is $y[n-1]$. Hence

$$\begin{aligned}
y[n-1] &= 2^{n+1}u[n] * 3^n \cos\left(\frac{\pi n}{3} - 0.5\right) u[n] \\
&= 2\left\{2^n u[n] * 3^n \cos\left(\frac{\pi n}{3} - 0.5\right) u[n]\right\}
\end{aligned}$$

Use Table of convolution pair 10 with

$$\begin{aligned}
R &= \left[(3)^2 + (2)^2 - 2(3)(2)(0.5)\right]^{1/2} = \sqrt{7} \\
\phi &= \tan^{-1}\left[\frac{3\sqrt{3}/2}{1.5-2}\right] = 1.761 \quad \text{radians}
\end{aligned}$$

Hence

$$y[n-1] = 2\left\{\frac{1}{\sqrt{7}}\left[(3)^{n+1}\cos\left[\frac{\pi}{3}(n+1) - 2.26\right] - 2^{n+1}\cos(2.261)\right]\right\} u[n]$$

Hence

$$y[n] = \left\{ \frac{2}{\sqrt{7}}(3)^n \cos\left[\frac{\pi n}{3} - 2.26\right] + 1.273(2)^n \right\} u[n-1]$$

3.8-6. #1

$$\delta[n-k] * x[n] = x[n-k]$$

$$\delta[n-k] * x[n] = \sum_{m=0}^{\infty} x[m]\delta[n-m-k]$$

$\delta[n-m-k] = 1$ for $m = n-k$ and is zero for all other values of m. Hence the right-side sum is given by $x[n-k]$.

#2

$$\gamma^n u[n] * u[n] = \sum_{m=0}^{n} \gamma^m u[n-m]$$

Because $u[n-m] = 1$ for all $0 \le m \le n$, we have

$$\gamma^n u[n] * u[n] = \sum_{m=0}^{n} \gamma^m = \frac{\gamma^{n+1} - 1}{\gamma - 1} u[n] \qquad \gamma \neq 1$$

We multiply the result with $u[n]$ because the convolution is zero for $n < 0$.

#3

$$u[n] * u[n] = \sum_{m=0}^{n} u[m]u[n-m]$$

Over the range $0 \le m \le n$, $u[m] = u[n-m] = 1$. Hence

$$u[n] * u[n] = \sum_{m=0}^{n} 1 = (n+1)u[n]$$

3.8-7. #4

$$\begin{aligned}
\gamma_1^n u[n] * \gamma_2^n u[n] &= \sum_{m=0}^{n} \gamma_1^m \gamma_2^{n-m} \\
&= \gamma_2^n \sum_{m=0}^{n} (\gamma_1/\gamma_2)^m \\
&= \gamma_2^n \left[\frac{(\gamma_1/\gamma_2)^{n+1} - 1}{1 - (\gamma_1/\gamma_2)}\right] \qquad \gamma_1 \neq \gamma_2 \\
&= \left[\frac{\gamma_1^{n+1} - \gamma_2^{n+1}}{\gamma_1 - \gamma_2}\right] u[n] \qquad \gamma_1 \neq \gamma_2
\end{aligned}$$

We multiply the result by $u[n]$ because convolution of two causal sequences is zero for $n < 0$.

#5

$$nu[n] * u[n] = \sum_{m=0}^{n} m = \frac{n(n+1)}{2} u[n]$$

#6

$$\begin{aligned}
\gamma^n u[n] * nu[n] &= \sum_{m=0}^{n} \gamma^m (n-m) \\
&= n\sum_{m=0}^{n} \gamma^m - \sum_{m=0}^{n} m\gamma^m \\
&= n\frac{\gamma^{n+1}-1}{\gamma-1} - \frac{\gamma + [n(\gamma-1)-1]\gamma^{n+1}}{(\gamma-1)^2} \\
&= \left[\frac{\gamma(\gamma^n-1)+n(1-\gamma)}{(1-\gamma)^2}\right] u[n]
\end{aligned}$$

3.8-8. #7

$$\begin{aligned}
nu[n] * nu[n] &= \sum_{m=0}^{n} m(n-m) \\
&= n\sum_{m=0}^{n} m - \sum_{m=0}^{n} m^2 \\
&= \frac{n(n+1)}{2} - \frac{n(n+1)(2n+1)}{6} \\
&= \frac{n(n^2-1)}{6} u[n]
\end{aligned}$$

#8

$$\begin{aligned}
\gamma^n u[n] * \gamma^n u[n] &= \sum_{m=0}^{n} \gamma^m \gamma^{n-m} \\
&= \gamma^n \sum_{m=0}^{n} 1 \\
&= (n+1)\gamma^n u[n]
\end{aligned}$$

3.8-9. #9

$$\begin{aligned}
n\gamma_1^n u[n] * \gamma_2^n u[n] &= \sum_{m=0}^{n} m\gamma_1^m \gamma_2^{n-m} \\
&= \gamma_2^2 \sum_{m=0}^{n} n(\gamma_1/\gamma_2)^m \\
&= \gamma_2^n \frac{(\gamma_1/\gamma_2) + [(n\gamma_1/\gamma_2) - n - 1](\gamma_1/\gamma_2)^{n+1}}{[(\gamma_1/\gamma_2)-1]^2} \\
&= \frac{\gamma_1\gamma_2}{(\gamma_1-\gamma_2)^2}\left[\gamma_2^n - \gamma_1^n + \frac{\gamma_1-\gamma_2}{\gamma_2} n\gamma_1^n\right] u[n] \qquad \gamma_1 \neq \gamma_2
\end{aligned}$$

#11 Let

$$c[n] = \gamma_1^n u[n] * \gamma_2^n u[-(n+1)]$$

$$= \sum_{m=-\infty}^{\infty} \gamma_1^m \gamma_2^{n-m} u[m]u[-(n-m+1)]$$

Consider $c[n]$ for $n \geq 0$. In this case $-(n-m+1) \geq 0$ for $m \geq n+1$. Therefore $u[-(n-m+1)] = 0$ for $m < n+1$ and equal to 1 for $m \geq n+1$. Also $u[m] = 1$ for $m \geq n+1$ (for positive n). Hence

$$c[n] = \gamma_2^n \sum_{m=n+1}^{\infty} \left(\frac{\gamma_1}{\gamma_2}\right)^m = \frac{\gamma_1^{n+1}}{\gamma_2 - \gamma_1} \qquad \text{for} \quad n \geq 0 \quad \text{and} \quad |\gamma_2| > |\gamma_1|$$

When $n \leq -1$, $-(n-m+1) \geq 0$ for $m \geq 0$. Hence $u[m]u[-(n-m+1)] = 1$ for $m \geq 0$ and zero otherwise. Hence

$$c[n] = \gamma_2^n \sum_{m=0}^{\infty} \left(\frac{\gamma_1}{\gamma_2}\right)^m = \frac{\gamma_2^{n+1}}{\gamma_2 - \gamma_1} \qquad \text{for} \quad n \leq -1 \quad \text{and} \quad |\gamma_2| > |\gamma_1|$$

Therefore

$$c[n] = \frac{\gamma_1^{n+1}}{\gamma_2 - \gamma_1} u[n] + \frac{\gamma_2^{n+1}}{\gamma_2 - \gamma_1} u[-(n+1)] \qquad |\gamma_2| > |\gamma_1|$$

3.8-10. The characteristic root is -2. Therefore

$$y_0[n] = c(-2)^n$$

Setting $n = -1$ and substituting $y[-1] = 10$, yields

$$10 = -\frac{c}{2} \quad \Longrightarrow \quad c = -20$$

Therefore

$$y_0[n] = -20(-2)^n \qquad n \geq 0$$

For this system $h[n]$, the unit impulse response is found in Prob. 3.7-1b to be

$$h[n] = (-2)^n u[n]$$

The zero-state response is

$$y[n] = e^{-n}u[n] * (-2)^n u[n]$$

This is found by using the convolution Table to be

$$\begin{aligned} y[n] &= \frac{e}{2e+1}[e^{-(n+1)} - (-2)^{n+1}]u[n] \\ &= \frac{e}{2e+1}[\frac{1}{e}(e)^{-n} + 2(-2)^n]u[n] \\ &= \left[\frac{1}{2e+1}(e)^{-n} + \frac{2e}{2e+1}(-2)^n\right] u[n] \end{aligned}$$

$$\text{Total Response} = y_0[n] + y[n]$$

$$= \quad [-20(-2)^n + \frac{1}{2e+1}(e)^{-n} + \frac{2e}{2e+1}(-2)^n]u[n]$$
$$= \quad \frac{1}{2e+1}[-(38e+20)(-2)^n + (e)^{-n}]u[n]$$

3.8-11. (a)

$$\begin{aligned} y[n] &= 2^n u[n] * (0.5)^n u[n] \\ &= \frac{2^{n+1} - (0.5)^{n+1}}{2 - 0.5} u[n] = \frac{2}{3}[2^{n+1} - (0.5)^{n+1}]u[n] \end{aligned}$$

(b)

$$x[n] = 2^{(n-3)}u[n] = 2^{-3}2^n u[n] = \frac{1}{8}2^n u[n]$$

From the result in part (a), it follows that

$$y[n] = \frac{1}{8}\frac{2}{3}[2^{n+1} - (0.5)^{n+1}]u[n] = \frac{1}{12}[2^{n+1} - (0.5)^{n+1}]u[n]$$

(c)

$$x[n] = 2^n u[n-2] = 4\{2^{(n-2)}u[n-2]\}$$

Note that $2^{(n-2)}u[n-2]$ is the same as the input $2^n u[n]$ in part (a) delayed by 2 units. Therefore from the shift property of the convolution, its response will be the same as in part (a) delayed by 2 units. The input here is $4\{2^{(n-2)}u[n-2]\}$. Therefore

$$y[n] = 4\frac{2}{3}[2^{n+1-2} - (0.5)^{n+1-2}]u[n-2] = \frac{8}{3}[2^{n-1} - (0.5)^{n-1}]u[n-2]$$

3.8-12. For $x[n] = u[n]$

$$y[n] = u[n] - 2u[n-1]$$

The highest order difference is one. Hence, this is a first-order system.
This is a nonrecursive system, whose output at any instant depends only on the input, Initial conditions are irrelevant for the finding the response.

3.8-13. (a) Figure S3.8-13a represents the system in Figure P3.8-13 in a more convenient fashion. A parallel connection requires the individual impulse responses to be added, and a series connection requires the individual impulse responses to be convolved. Thus, the overall impulse response is given by

$$h[n] = h_1[n] * h_2[n] + (h_1[n] * h_5[n] + h_4[n]) * h_3[n]$$

(b) First, we shall simplify expressions for $h_1[n]$ and $h_2[n]$ by using the facts that $u[n-1] = u[n] - \delta[n]$ and $(0.9)^{n-1} = \frac{1}{0.9}(0.9)^n$ and $(0.5)^{n-1} = \frac{1}{0.5}(0.5)^n$. Now

$$h_1[n] = 0.9^n u[n] - \frac{0.5}{0.9}(0.9)^n(u[n] - \delta[n]) = \frac{4}{9}0.9^n u[n] + \frac{5}{9}\delta[n]$$

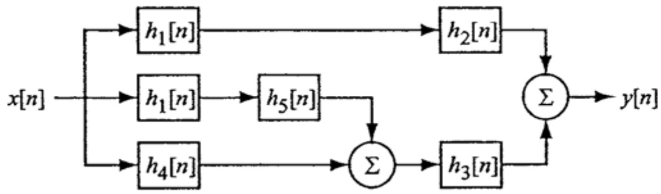


Figure S3.8-13a: Interconnection of discrete-time systems.

Similarly

$$h_2[n] = 0.5^n u[n] - \frac{0.9}{0.5}(0.5)^n(u[n] - \delta[n]) = -0.8(0.5)^n u[n] + 1.8\delta[n]$$

Recall that $x[n] * \delta[n] = x[n]$ and $\delta[n] * \delta[n] = \delta[n]$. Hence

$$\begin{aligned} h_1[n] * h_2[n] &= \frac{4}{9}(0.9)^n u[n] * (-0.8)(0.5)^n u[n] \\ &\quad -0.8(\frac{5}{9})(0.5)^n u[n] + 1.8(\frac{4}{9})(0.9)^n u[n] + \delta[n] \\ &= -\frac{16}{45}\left[\frac{0.9^{n+1} - 0.5^{n+1}}{0.4}\right] u[n] - \frac{4}{9}(0.5)^n u[n] + \frac{4}{5}(0.9)^n u[n] + \delta[n] \\ &= \delta[n] \end{aligned}$$

Thus the cascade of the two systems is an identity system.

3.8-14. (a) From Eq. (3.70b), for a causal system

$$g[n] = \sum_{k=0}^{n} h[k]$$

Let $k = n - m$. The limits of summation change from $m = n$ to $m = 0$.

$$g[n] = \sum_{m=n}^{0} h[n-m]$$

In summation, we can sum from either direction. Hence

$$g[n] = \sum_{m=0}^{n} h[n-m]$$

(b) When the system is noncausal,

$$g[n] = \sum_{k=-\infty}^{\infty} h[k]$$

Let $k = n - m$. In this case, when $k = -\infty$, $m = \infty$ and when $k = n$, $m = 0$. Hence

$$g[n] = \sum_{m=\infty}^{0} h[n-m] = \sum_{m=0}^{\infty} h[n-m]$$

3.8-15. The equation describing this situation is [see Eq. (3.9b)]

$$(E-a)y[n] = Ex[n] \qquad a = 1 + r = 1.01$$

The initial condition $y[-1] = 0$. Hence there is only zero-state component. The input is $500u[n] - 1500\delta[n-4]$ because at $n = 4$, instead of depositing the usual \$500, she withdraws \$1000. To find $h[n]$, we solve iteratively

$$(E-a)h[n] = E\delta[n]$$

or

$$h[n+1] - ah[n] = \delta[n+1]$$

Setting $n = -1$ and substituting $h[-1] = 0$, $\delta[0] = 1$, yields

$$h[0] = 1$$

Also, the characteristic root is a and $b_0 = 0$. Therefore

$$h[n] = ca^n u[n]$$

Setting $n = 0$ and substituting $h[0] = 1$ yields

$$1 = c$$

Therefore

$$h[n] = (a)^n u[n] = (1.01)^n u[n]$$

The (zero-state) response is

$$\begin{aligned}
y[n] &= (1.01)^n u[n] * x[n] \\
&= (1.01)^n u[n] * \{500u[n] - 1500\delta[n-4]\} \\
&= 500(1.01)^n u[n] * u[n] - 1500(1.01)^{n-4}u[n-4] \\
&= \frac{500}{0.01}[(1.01)^{n+1} - 1]u[n] - 1500(1.01)^{n-4}u[n-4] \\
&= 50000[(1.01)^{n+1} - 1]u[n] - 1500(1.01)^{n-4}u[n-4]
\end{aligned}$$

3.8-16. This problem is identical to the savings account problem with negative initial deposit (loan). If M is the initial loan, then $y[0] = -M$. If $y[n]$ is the loan balance, then [see Eq. (3.9b)]

$$y[n+1] - ay[n] = x[n+1] \qquad a = 1 + r$$

or

$$(E-a)y[n] = Ex[n]$$

The characteristic root is a, and the impulse response for this system is found in Prob. 3.8-15 to be

$$h[n] = a^n u[n]$$

This problem can be solved in two ways.

First method: We may consider the loan of M dollars as an a negative input $-M\delta[n]$. The monthly payment of P starting at $n = 1$ also is an input. Thus the total input is $x[n] = -M\delta[n] + Pu[n-1]$ with zero initial conditions. Because $u[n] = \delta[n] + u[n-1]$, we can express the input in a more convenient form as $x[n] = -(M+P)\delta[n] + Pu[n]$.

The loan balance (response) $y[n]$ is

$$\begin{aligned}
y[n] &= h[n]*x[n] \\
&= a^n u[n]*\{-(M+P)\delta[n]+Pu[n]\} \\
&= -(M+P)a^n u[n] + Pa^n u[n]*u[n] \\
&= -(M+P)a^n u[n] + P\left[\frac{a^{n+1}-1}{a-1}\right]u[n] \\
&= -Ma^n u[n] - P\left[a^n - \frac{a^{n+1}-1}{a-1}\right]u[n] \\
&= \left\{-Ma^n + P\left[\frac{a^n-1}{a-1}\right]\right\}u[n]
\end{aligned}$$

Also $a = 1+r$ and $a-1 = r$ where r is the interest rate per dollar per month. At $n = N$, the loan balance is zero. Therefore

$$y[N] = -Ma^N + P\left[\frac{a^N-1}{r}\right] = 0$$

or

$$P = \frac{ra^N}{a^N-1}M = \frac{rM}{1+a^{-N}} = \frac{rM}{1+(1+r)^{-N}}$$

Second method: In this approach, the initial condition is $y[0] = -M$, and the input is $x[n] = Pu[n-1]$ because the monthly payment of P starts at $n = 1$. The characteristic root is a, and The zero-input response is

$$y_0[n] = ca^n u[n]$$

Setting $n = 0$, and substituting $y_0[0] = -M$, yields $c = -M$ and

$$y_0[n] = -Ma^n u[n]$$

The zero-state response $y[n]$ is

$$y[n] = h[n]*x[n] = h[n]*Pu[n-1] = Pa^n u[n]*u[n-1]$$

Here we use shift property of convolution. If we let

$$x[n] = a^n u[n]*u[n] = [\frac{a^{n+1}-1}{a-1}]u[n]$$

The shift property yields

$$Pa^n u[n]*u[n-1] = x[n-1] = P\left[\frac{a^n-1}{a-1}\right]u[n-1]$$

The total balance is $y_0[n]+y[n]$

$$y_0[n]+y[n] = -Ma^n u[n] + P\left[\frac{a^n-1}{a-1}\right]u[n-1]$$

For $n > 1$, $u[n] = u[n-1] = 1$. Therefore

$$\text{Loan balance} = -Ma^n + P\left[\frac{a^n - 1}{a-1}\right] \qquad n > 1$$

which confirms the result obtained by the first method. From here on the procedure is identical to that of the first method.

3.8-17. We use the result in Prob. 3.8-16. In this problem $r = 0.015$, $a = 1.015$, $P = 500$, $M = 10000$. Therefore

$$500 = 10000\frac{(1.015)^N(0.015)}{(1.015)^N - 1}$$

or

$$\begin{aligned}(1.015)^N &= 1.42857\\ N\ln(1.015) &= \ln(1.42857)\\ N &= \frac{\ln(1.42857)}{\ln(1.015)} = 23.956\end{aligned}$$

Hence $N = 23$ payments are needed. The residual balance (remainder) at the 23rd payment is

$$y[23] = -10000(1.015)^{23} + 500[\frac{(1.015)^{23} - 1}{0.015}] = -471.2$$

3.8-18. (a) The two strips corresponding to $u[n]$ and $u[n]$ (after inversion) are shown in the Figure S3.8-18a for no shift ($n = 0$) and for one shift ($n = 1$). We see that if $c[n] = u[n] * u[n]$ then

$$c[0] = 1, \quad c[1] = 2, \quad c[2] = 3, \quad c[3] = 4, \quad c[4] = 5, \quad c[5] = 6, \cdots, \; c[n] = n+1$$

Hence

$$u[n] * u[n] = (n+1)u[n]$$

(b) The appropriate strips for the two functions $u[n] - u[n-m]$ and $u[n]$ are shown in Figure S3.8-18b. The upper strip corresponding to $u[n] - u[n-m]$ has first m slots with value 1 and all the remaining slots have value 0. The lower (inverted) strip corresponding to $u[n]$ has all slot values of 1. From this figure it follows that

$$c[0] = 1, \quad c[1] = 2, \quad c[2] = 3, \cdots, c[m-1] = m$$

$$c[m] = c[m+1] = \cdots = m$$

$$\text{Hence} \qquad c[n] = (n+1)u[n] - (n-m+1)u[n-m]$$

3.8-19. From Figure S3.8-19 we observe that

n	$y[n]$	
0	$0+1+2+3+4+5 = 15$	$y[n] = 0 \quad k \geq 6$
1	$1+2+3+4+5 = 15$	
2	$2+3+4+5 = 14$	$y[n] = 15 \quad n < 0$
3	$3+4+5 = 12$	
4	$4+5 = 9$	
5	5	
6	0	

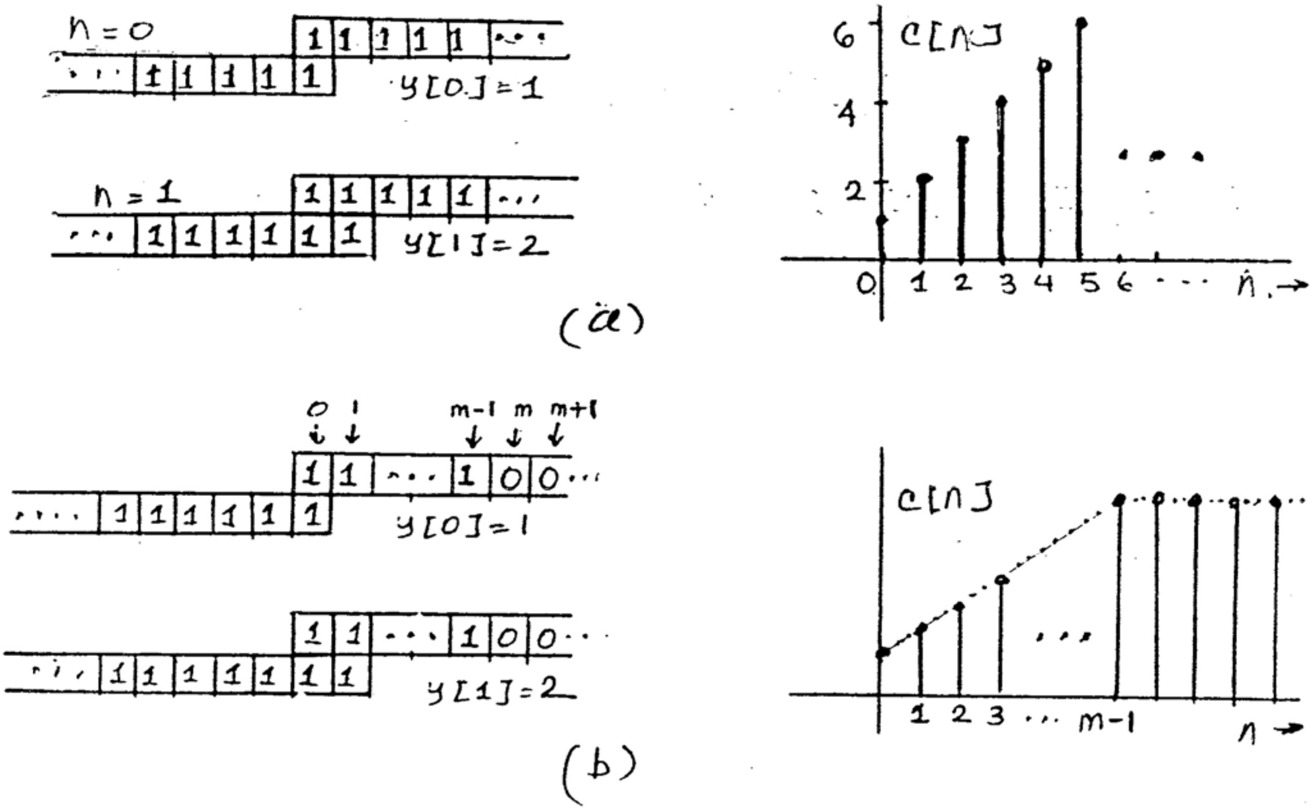


Figure S3.8-18

3.8-20. From Figure S3.8-20, we observe the following values for $y[n]$:

n	$y[n]$	n	$y[n]$
0	$5\times 5+5\times 5=50$	± 11	$0\times 0+5\times 4=20$
± 1	$5\times 4+0=20$	± 12	$0\times 0+5\times 3=15$
± 2	$5\times 3+0=15$	± 13	$0\times 0+5\times 2=10$
± 3	$5\times 2+0=10$	± 14	$0\times 0+5\times 1=5$
± 4	$5\times 1+0=5$	± 15	$0\times 0+0\times 0=0$
± 5	$5\times 0+0=0$	± 16	0
$\cdots$	$\cdots$	± 17	0
± 9	$0\times 0+0\times 0=0$	± 18	0
± 10	$0\times 0+5\times 1=5$		

Observe that

$$y[n]=0 \qquad 5\le |n|\le 9 \quad \text{and} \quad |n|\ge 15$$

3.8-21. (a) From Figure S3.8-21, we observe the following values of $y[n]$:

n	0	± 1	± 2	± 3	± 4	± 5	± 6	± 7	$\lvert n\rvert>7$
$y[n]$	7	6	5	4	3	2	1	0	0

(b) The answer is identical to that of (a). This is because when we lay the tapes $x[m]$ and $g[-m]$ together, the situation is identical to that in (a).

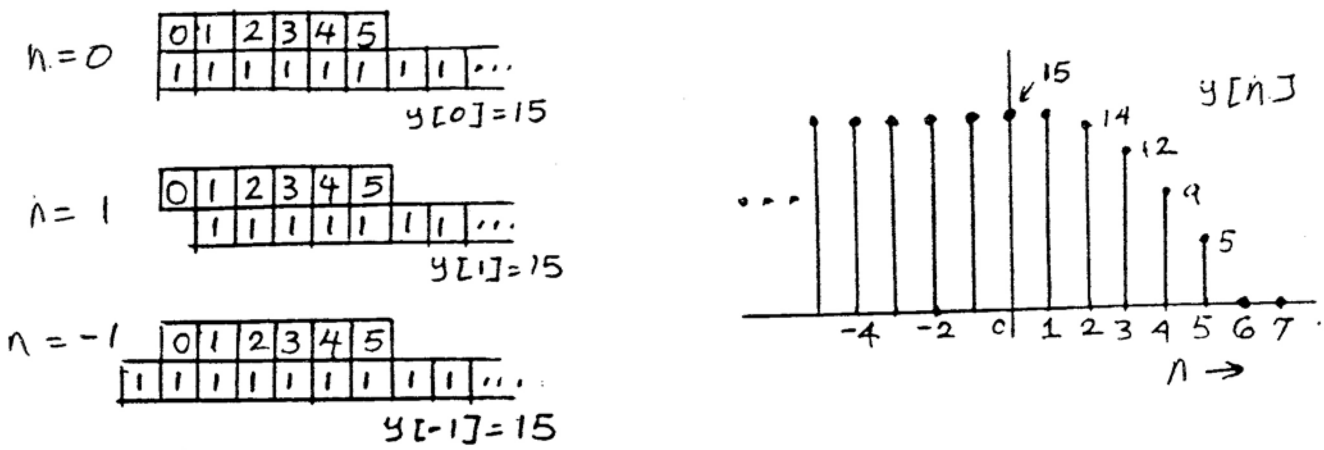


Figure S3.8-19

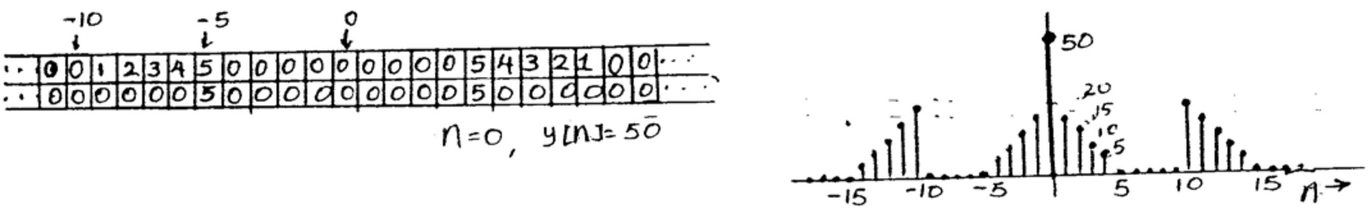


Figure S3.8-20

3.8-22. (a)

$$
\begin{aligned}
8 &= h[0] \\
12 &= h[1] + h[0] \Longrightarrow h[1] = 12 - 8 = 4 \\
14 &= h[2] + h[1] + h[0] \Longrightarrow h[2] = 2 \\
15 &= h[3] + h[2] + h[1] + h[0] \Longrightarrow h[3] = 1 \\
15.5 &= h[4] + h[3] + h[2] + h[1] + h[0] \Longrightarrow h[4] = 0.5 \\
15.75 &= h[5] + h[4] + h[3] + h[2] + h[1] + h[0] \Longrightarrow h[5] = 0.25
\end{aligned}
$$

(b)

$$
\mathbf{H} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 2 & 1 \end{bmatrix} \Longrightarrow \mathbf{H}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}
$$

$$
\mathbf{f} = \mathbf{H}^{-1}\mathbf{y} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 7/3 \\ 43/9 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/3 \\ 1/9 \end{bmatrix}
$$

Hence the input sequence is: $(1,\ 1/3,\ 1/9,\cdots)$

3.8-23. Using the sifting property, the zero-input response is rewritten as $y_0[n] = 2u[n] + (1/3)^n u[n]$.

(a) Since the system is second-order, it has two characteristic roots. Both roots appear in the zero-input response, and are easily identified as $\gamma_1 = 1$ and $\gamma_2 = 1/3$

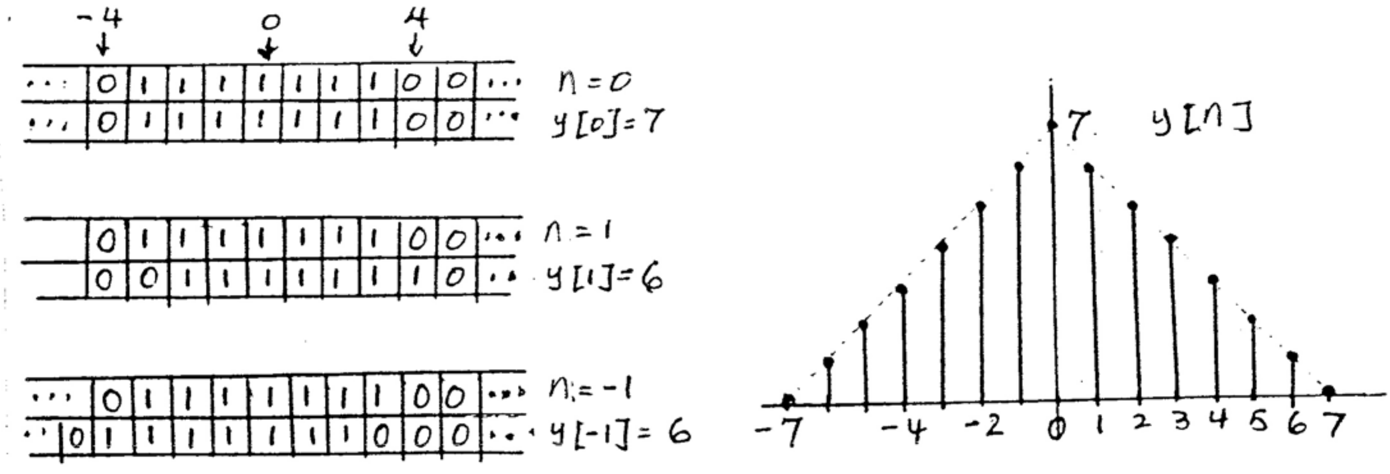


Figure S3.8-21

Let $a_0 = 1$ (standard form), so $\gamma^2 + a_1\gamma + a_2 = (\gamma - 1)(\gamma - 1/3) = \gamma_2 - 4/3\gamma + 1/3$. Thus,

$$a_0 = 1, a_1 = -4/3, \text{ and } a_2 = 1/3.$$

(b) To produce a strong response, the input should be close to a natural mode. Since the system has two bounded modes, any linear combination of these modes will produce a strong response. That is, a strong response is generated in response to

$$x[n] = (c_1\gamma_1^n + c_2\gamma_2^n)\,u[n] = (c_1 + c_2(1/3)^n)\,u[n],$$

where c_1 and c_2 are arbitrary constants (except $c_1 = c_2 = 0$).

(c) A causal, bounded, and infinite duration input is conveniently chosen in the form $x[n] = \gamma^n u[n]$, where $|\gamma| \leq 1$. To produce a weak response, the input should be far from the system's natural modes. In the complex plane, the farthest point γ from the modes $\gamma_1 = 1$ and $\gamma_2 = 1/3$ with $|\gamma| \leq 1$ is $\gamma = -1$. Thus, a relatively weak response is generated in response to

$$x[n] = c_1(-1)^n u[n],$$

where c_1 is an arbitrary non-zero constant.

3.8-24. (a) MATLAB is used to plot $h_1[n]$ and $h_2[n]$.

```
>> n = [-10:10]; h1 = (n==-2)-(n==2);
>> h2 = n.*((n>=-4)-(n>=4));
>> subplot(211); stem(n,h1,'k'); axis([-10 10 -5 5]);
>> xlabel('n'); ylabel('h_1[n]');
>> subplot(212); stem(n,h2,'k'); axis([-10 10 -5 5]);
>> xlabel('n'); ylabel('h_2[n]');
```

(b) For systems connected in parallel, the total impulse response is just the sum of the individual impulse responses

$$h_p[n] = h_1[n] + h_2[n].$$

MATLAB easily computes $h_p[n]$.

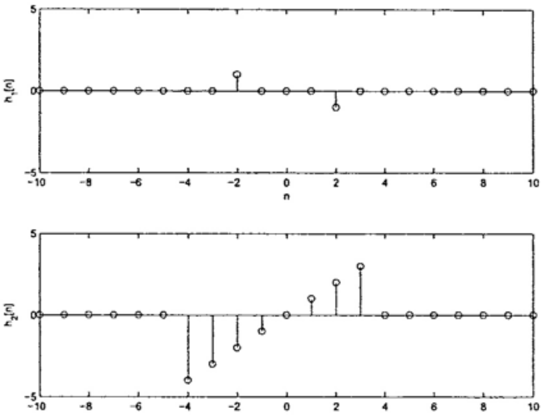


Figure S3.8-24a: Plot of $h_1[n]$ and $h_2[n]$.

```
>> n = [-10:10]; h1 = (n==-2)-(n==2);
>> h2 = n.*((n>=-4)-(n>=4)); hp = h1+h2;
>> stem(n,hp,'k'); axis([-10 10 -5 5]);
>> xlabel('n'); ylabel('h_p[n]');
```

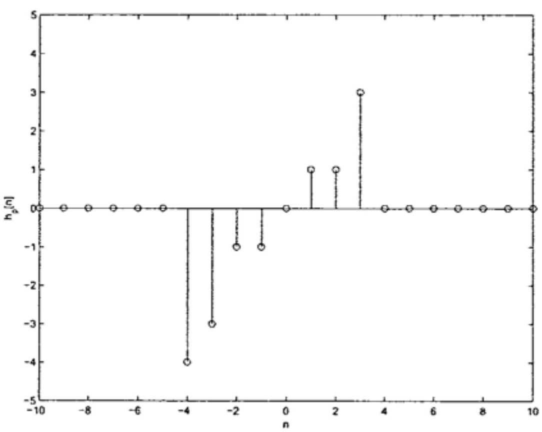


Figure S3.8-24b: Plot of $h_p[n]$.

(c) For systems connected in series, the total impulse response is just the convolution of the individual impulse responses

$$h_s[n] = h_1[n] * h_2[n].$$

MATLAB easily computes $h_s[n]$.

```
>> n = [-10:10]; h1 = (n==-2)-(n==2);
>> h2 = n.*((n>=-4)-(n>=4));
>> n = [-20:20]; hs = conv(h1,h2);
>> stem(n,hs,'k'); axis([-10 10 -5 5]);
>> xlabel('n'); ylabel('h_s[n]');
```

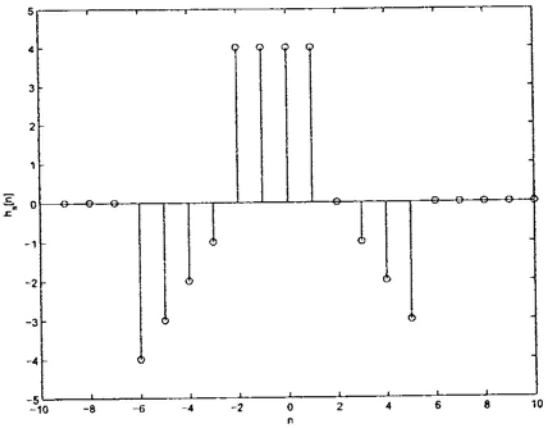


Figure S3.8-24c: Plot of $h_s[n]$.

3.8-25. (a) $(z^3 + z^2 + z + 1)^2 = z^6 + 2z^5 + 3z^4 + 4z^3 + 3z^2 + 2z + 1$. MATLAB is used to compute the convolution $[1111] * [1111]$.

```
>> conv([1 1 1 1],[1 1 1 1])
ans = 1 2 3 4 3 2 1
```

Notice, the coefficients of the polynomial expansion are the same as the terms from the convolution.

(b) It is possible to expand the product of two polynomial expressions by convolving the coefficients from each ordered polynomial expression. Repeated convolution can be used to expand the product of more than two polynomial expressions. Care must be used to include all coefficient terms, including zeros, from the highest power to the lowest, and coefficients need to be ordered in a consistent manner.

(c) $(z^{-4} - 2z^{-3} + 3z^{-2})^4 = ((z^{-4} - 2z^{-3} + 3z^{-2})^2)^2$. First, use MATLAB and convolution to compute $(z^{-4} - 2z^{-3} + 3z^{-2})^2$.

```
>> temp = conv([1 -2 3],[1 -2 3]);
```

Convolution is used to square this intermediate result and obtain $(z^{-4} - 2z^{-3} + 3z^{-2})^4$.

```
>> conv(temp,temp)
ans = 1 -8 36 -104 214 -312 324 -216 81
```

Recognizing that the highest power of z^{-1} in the result must be 16, the expansion is

$$z^{-16} - 8z^{-15} + 36z^{-14} - 104z^{-13} + 214z^{-12} - 312z^{-11} + 324z^{-10} - 216z^{-9} + 81z^{-8}.$$

(d) First, use convolution to expand $(z^5 + 2z^4 + 3z^2 + 5)^2$.

```
>> temp = conv([1 2 0 3 0 5],[1 2 0 3 0 5]);
```

Convolution is again used to multiply the resulting polynomial by $(13 - 5z^{-2} + z^{-4})$. It is important to order the coefficients in descending powers of z to be compatible with the first result.

```
>> conv(temp,[13 0 -5 0 1])
ans = 13 52 47 58 137 104 321 -44 257 10 204 0 -95 0 25
```

Recognizing that the highest power of z in the result must be 10, the expansion is

$$13z^{10}+52z^9+47z^8+58z^7+137z^6+104z^5+321z^4-44z^3+257z^2+10z+204-95z^{-2}+25z^{-4}.$$

3.8-26. (a) Actually, only a first-order difference equation is necessary to describe this system. On refill n, the amount of sugar $y[n]$ is equal to the amount added $x[n]$ plus whatever was still in the mug. Since Joe drinks 2/3 of his cup of coffee before each refill, one-third of the sugar from the previous cup remains. Thus, $y[n] = y[n-1]/3 + x[n]$. Thus,

$$a_1 = -1/3, a_2 = 0, b_0 = 1, b_1 = 0, \text{ and } b_2 = 0.$$

In standard form, the difference equation is

$$y[n] - y[n-1]/3 = x[n].$$

(b) Since Joe adds two teaspoons of sugar each time he fills his cup,

$$x[n] = 2u[n].$$

(c) The total solution to a difference equation is the sum of the zero-input response and the zero-state response. Since Joe starts with a clean mug, $y[-1] = 0$ and the zero-input response is necessarily zero, $y_0[n] = 0$. Thus, the total solution is just the zero-state solution.

To obtain the zero-state solution, the impulse response $h[n]$ is needed. In this case, $h[n] = \frac{0}{1/3} + \tilde{y}_0[n]u[n]$, where $\tilde{y}_0[n] = c\gamma^n$. To determine c, input $x[n] = \delta[n]$ into the original difference equation to yield $h[n] - h[n-1]/3 = \delta[n]$. Since $h[n]$ is causal, $h[0] - h[-1]/3 = h[0] = \delta[0] = 1 = \tilde{y}_0[0]u[0] = c$. Thus, $h[n] = 3^{-n}u[n]$. The zero-state solution is $x[n] * h[n] = \left(\sum_{k=0}^{n} 2(3)^{-n}\right) u[n] = 2\frac{1-3^{-(n+1)}}{1-1/3}u[n] = (3 - 3^{-n})\, u[n]$.

Thus,

$$y[n] = \left(3 - 3^{-n}\right) u[n].$$

(d)

$$\lim_{n\to\infty} y[n] = \lim_{n\to\infty} \left(3 - 3^{-n}\right) = 3.$$

That is, after many cups of coffee, Joe's mug reaches a steady-state of three teaspoons of sugar.

One way to make Joe's coffee remain a constant for all non-negative n is begin at the steady-state value of three and then add two teaspoons of sugar at each refill. That is,

$$x[n] = 2u[n] + \delta[n].$$

The added $\delta[n]$ "jump starts" the sugar content of the first cup to the steady-state value.

If Joe desires a steady value of two teaspoons, which is two-thirds the original steady-state value, the input is simply scaled by 2/3. That is, the steady level

$y[n] = 2u[n]$ is achieved using the input

$$x[n] = \frac{4}{3}u[n] + \frac{2}{3}\delta[n].$$

3.8-27. (a) The characteristic equation is $j\gamma + 0.5 = 0$. Thus, the characteristic root of the system is $\gamma = j0.5$. The impulse response has form $h[n] = \frac{b_N}{a_N}\delta[n] + c\gamma^n u[n] = c(j0.5)^n u[n]$. To determine c, evaluate the original difference equation according to $jh[0] + 0.5h[-1] = jh[0] = -5\delta[0] = -5$. Thus, $h[0] = j5 = c$. Taken together,

$$h[n] = j5(j0.5)^n u[n].$$

(b) First, compute the zero-input response $y_0(t) = c(j0.5)^n$. To find c, use the initial condition $y_0[-1] = j = c(j0.5)^{-1}$ Thus, $c = j^2 0.5 = -0.5$ and $y_0[n] = -0.5(j0.5)^n$.

The zero-state response is $h[n] * x[n] = \left(\sum_{k=0}^{n-5} j5(j0.5)^k\right) u[n-5] = \left(j5\frac{1-(j0.5)^{n-4}}{1-j0.5}\right) u[n-5]$.

Adding the zero-input and zero-state responses yields the total response for $n \geq 0$,

$$y[n] = -0.5(j0.5)^n + \left(j5\frac{1-(j0.5)^{n-4}}{1-j0.5}\right) u[n-5].$$

3.8-28. (a) MATLAB is used to sketch the function $h[n] = n(u[n-2] - u[n+2])$.

```
>> n = [-5:5]; h = n.*((n>=2)-(n>=-2));
>> stem(n,h,'k'); axis([-5 5 -2.5 2.5]);
>> xlabel('n'); ylabel('h[n]');
```

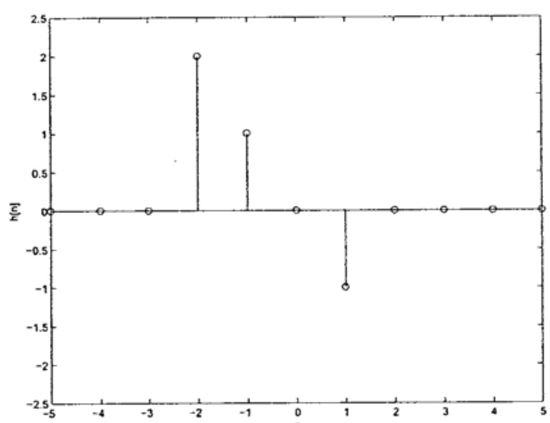


Figure S3.8-28a: Plot of $h[n] = n(u[n-2] - u[n+2])$.

(b) Using Figure S3.8-28a, write $h[n] = 2\delta[n+2] + \delta[n+1] - \delta[n-1]$. A difference equation representation immediately follows from this form,

$$y[n] = 2x[n+2] + x[n+1] - x[n-1].$$

3.8-29. See 3.M-3

3.8-30. For this solution, consider the signals $x[n] = \delta[n] - \delta[n-1] = [1, -1]$, $y[n] = z[n] = \delta[n] + \delta[n-1] = [1, 1]$. In this case, $x[n]\,(y[n] * z[n]) = [1, -2]$.

(a) Not equivalent. By counter-example, $(x[n] * y[n])z[n] = [1] \neq [1, -2] = x[n]\,(y[n] * z[n])$.

(b) Not equivalent. By counter-example, $(x[n]y[n]) * (x[n]z[n]) = [1, -2, 1] \neq [1, -2] = x[n]\,(y[n] * z[n])$.

(c) Not equivalent. By counter-example, $(x[n]y[n]) * z[n] = [1, 0, -1] \neq [1, -2] = x[n]\,(y[n] * z[n])$.

(d) True. None of the above expressions is equivalent to $x[n]\,(y[n] * z[n])$.

3.9-1.

$$(E+2)y[n] = Ex[n]$$

The characteristic equation is $\gamma + 2 = 0$, and the characteristic root is -2. Therefore

$$y_n(t) = B(-2)^n$$

For $x[n] = e^{-n}u[n] = r^n$ with $r = e^{-1}$

$$\begin{aligned} y_\phi[n] &= H[e^{-1}]e^{-n} = \frac{e^{-1}}{e^{-1}+2}e^{-n} = \frac{1}{2e+1}e^{-n} \\ y[n] &= B(-2)^n + \frac{1}{2e+1}e^{-n} \qquad n \geq 0 \end{aligned}$$

Setting $n = 0$, and substituting $y[0] = 1$ yields

$$1 = B + \frac{1}{2e+1} \quad \Longrightarrow \quad B = \frac{2e}{2e+1}$$

and

$$y[n] = \frac{1}{2e+1}[2e(-2)^n + e^{-n}] \qquad n \geq 0$$

3.9-2.

$$y[n] + 2y[n-1] = x[n-1] \tag{1}$$

We solve this equation iteratively to obtain $y[0]$. Setting $n = 0$, and substituting $y[-1] = 0$, $x[-1] = 0$, we get

$$y[0] + 2(0) = 0 \quad \Longrightarrow \quad y[0] = 0$$

The system equation can be expressed as

$$(E+2)y[n] = x[n]$$

The characteristic root is -2. Therefore

$$y_n[n] = B(-2)^n$$

For $x[n] = e^{-n}u[n] = r^n u[n]$ with $r = e^{-1}$,

$$y_\phi[n] = H[r]r^n = H[e^{-1}]e^{-n} = \frac{1}{e^{-1}+2}e^{-n} = \frac{e}{2e+1}e^{-n}$$

Therefore

$$y[n] = B(-2)^n + \frac{e}{2e+1}e^{-n} \qquad n \geq 0$$

Setting $n = 0$ and substituting $y[0] = 0$ yields

$$0 = B + \frac{e}{2e+1} \quad \Longrightarrow \quad B = \frac{-e}{2e+1}$$

and

$$y[n] = \frac{e}{2e+1}[-(-2)^n + e^{-n}] \qquad n \geq 0$$

3.9-3.

$$(E^2 + 3E + 2)y[n] = (E^2 + 3E + 3)x[n]$$

The characteristic equation is $\gamma^2 + 3\gamma + 2 = (\gamma + 1)(\gamma + 2) = 0$. Therefore

$$y_n[n] = B_1(-1)^n + B_2(-2)^n$$

For $x[n] = 3^n$

$$y_\phi[n] = H[3]3^n = \frac{(3)^2 + 3(3) + 3}{(3)^2 + 3(3) + 2}3^n = (\frac{21}{20})3^n$$

The total response

$$y[n] = B_1(-1)^n + B_2(-2)^n + (\frac{21}{20})3^n \qquad n \geq 0$$

(a) Setting $n = 0$, 1, and substituting $y[0] = 1$, $y[1] = 3$, yields

$$\left.\begin{array}{l} 1 = B_1 + B_2 + \frac{21}{20} \\ 3 = -B_1 - 2B_2 + \frac{63}{20} \end{array}\right\} \Longrightarrow \quad \begin{array}{l} B_1 = -\frac{1}{4} \\ B_2 = \frac{1}{5} \end{array}$$

$$y[n] = -\frac{1}{4}(-1)^n + \frac{1}{5}(-2)^n + \frac{21}{20}(3)^n \qquad n \geq 0$$

(b) We solve system equation iteratively to find $y[0]$ and $y[1]$. We are given $y[-1] = y[-2] = 1$. System equation is

$$y[n+2] + 3y[n+1] + 2y[n] = x[n+2] + 3x[n+1] + 3x[n]$$

Setting $n = -2$, we obtain

$$y[0] + 3(1) + 2(1) = (3)^0 + 3(0) + 3(0) \quad \Longrightarrow \quad y[0] = -4$$

Setting $n = -1$, we obtain

$$y[1] + 3[-4] + 2(1) = (3)^1 + 3(3)^0 + 3(0) \quad \Longrightarrow \quad y[1] = 16$$

Also

$$y[n] = B_1(-1)^n + B_2(-2)^n + \frac{21}{20}(3)^n \qquad n \geq 0$$

Setting $n = 1$, 2, and substituting $y[0] = -4$, $y[1] = 16$, yields

$$\left.\begin{array}{rcl} -4 &=& B_1 + B_2 + \frac{21}{20} \\ 16 &=& -B_1 - 2B_2 + \frac{63}{20} \end{array}\right\} \Longrightarrow \quad \begin{array}{rcl} B_1 &=& \frac{11}{4} \\ B_2 &=& -\frac{39}{5} \end{array}$$

and

$$y[n] = \frac{11}{4}(-1)^n - \frac{39}{5}(-2)^n + \frac{21}{20}(3)^n \qquad n \geq 0$$

3.9-4.

$$\gamma^2 + 2\gamma + 1 = (\gamma + 1)^2 = 0$$

The roots are -1 repeated twice.

$$y_n[n] = (B_1 + B_2 n)(-1)^n$$

Also the system equation is $(E^2 + 2E + 1)y[n] = (2E^2 - E)x[n]$, and $x[n] = (\frac{1}{3})^n$. Therefore

$$y_\phi[n] = H[\frac{1}{3}]3^{-n} = -\frac{1}{16}(3)^{-n} \qquad n \geq 0$$

The total response

$$y[n] = (B_1 + B_2 n)(-1)^n - \frac{1}{16}(3)^{-n} \qquad n \geq 0$$

Setting $n = 0$, 1, and substituting $y[0] = 2$, $y[1] = -\frac{13}{3}$, yields

$$\left.\begin{array}{rcl} 2 & = & B_1 - \frac{1}{16} \\ -\frac{13}{3} & = & -(B_1 + B_2) - \frac{1}{48} \end{array}\right\} \Longrightarrow \begin{array}{rcl} B_1 & = & \frac{33}{16} \\ B_2 & = & \frac{9}{4} \end{array}$$

$$y[n] = (\frac{33}{16} + \frac{9}{4}n)(-1)^n - \frac{1}{16}(3)^{-n} \qquad n \geq 0$$

3.9-5. (a)

$$y[n] = \sum_{k=0}^{n} k$$

Hence

$$y[n+1] - y[n] = n + 1$$

Hence, the required solution for $y[n]$ is the solution of the above difference equation. To find the initial condition, we know that $y[0] = 0$.
We can rewrite the difference equation as

$$(E - 1)y[n] = n + 1$$

Hence the natural response is

$$y_c[n] = B(1)^n = B$$

From Table 3.2, we find the forced response form as

$$y_\phi[n] = c_0 + c_1 n$$

However, c_0 is inadmissible because it is also the characteristic mode. Hence, we assume

$$y_\phi[n] = c_1 n + c_2 n^2$$

Substitution of this expression in the difference equation yields

$$y_\phi[n+1] - y_\phi[n] = n+1$$

or

$$c_1(n+1) + c_2(n+1)^2 - c_1 n - c_2 n^2 = n+1$$

Equating coefficients of similar power yields

$$\left.\begin{aligned} c_1 + c_2 &= 1 \\ 2c_2 &= 1 \end{aligned}\right\} \Longrightarrow c_1 = c_2 = \frac{1}{2}$$

Hence

$$y[n] = B + \frac{n(n+1)}{2}$$

Setting $n=0$ and using the initial condition $y[0]=0$ yields

$$0 = B$$

Hence

$$y[n] = \sum_{k=0}^{n} k = \frac{n(n+1)}{2}$$

(b)

$$y[n] = \sum_{k=0}^{n} k^3$$

$$y[n+1] - y[n] = (n+1)^3$$

The natural response is

$$y_c[n] = B$$

The initial condition is $y[0] = \sum_{m=0}^{0} k^3 = 0$. The forced response , as seen from Table 3.2, is

$$y_\phi[n] = c_1 n + c_2 n^2 + c_3 n^3 + c_4 n^4$$

Substitution $y_\phi[n]$ in the difference equation yields

$$c_1(n+1)+c_2(n+1)^2+c_3(n+1)^3+c_4(n+1)^4-c_1n-c_2n^2-c_3n^3-c_4n^4 = (n+1)^3$$

Equating coefficients of similar powers on two sides yields

$$\left.\begin{aligned} c_1 + c_2 + c_3 + c_4 &= 1 \\ 2c_2 + 3c_3 + 4c_4 &= 3 \\ 3c_3 + 6c_4 &= 3 \\ 4c_4 &= 1 \end{aligned}\right\} \Longrightarrow \begin{aligned} c_1 &= 0 \\ c_2 &= 1/4 \\ c_3 &= 1/2 \\ c_4 &= 1/4 \end{aligned}$$

Hence

$$y[n] = B + \frac{1}{4}n^2 + \frac{1}{2}n^3 + \frac{1}{4}n^4$$

Also $y[n] = 0$. This yields $B = 0$, and

$$y[n] = \sum_{k=0}^{n} k^3 = \frac{1}{4}n^2 + \frac{1}{2}n^3 + \frac{1}{4}n^4 = \frac{n^2(n+1)^2}{4}$$

3.9-6.

$$y[n] = \sum_{k=0}^{n} kr^k$$

Therefore

$$y[n+1] - y[n] = (n+1)r^{n+1}$$

The natural response (for characteristic root $\gamma = 1$) is

$$y_c[n] = B$$

The forced response, as seen from Table 3.2, is

$$y_\phi[n] = (c_0 + c_1 n)r^n$$

Substitution of $y_\phi[n]$ is the difference equation yields

$$(c_0 + c_1 n + c_1)r^{n+1} - (c_0 + c_1 n)r^n = (n+1)r^{n+1}$$

Equating the coefficients of similar powers on both sides yields

$$\left.\begin{array}{r} c_0 + c_1 - \frac{c_0}{r} = 1 \\ c_1 - \frac{c_1}{r} = 1 \end{array}\right\} \Longrightarrow \begin{array}{l} c_1 = \frac{r}{r-1} \\ c_0 = \frac{-r}{(r-1)^2} \end{array}$$

Hence

$$y[n] = B + \left[\frac{-r}{(r-1)^2} + \frac{rn}{r-1}\right] r^n$$

Moreover $y[0] = 0$. Hence

$$0 = B - \frac{r}{(r-1)^2} \Longrightarrow B = \frac{r}{(r-1)^2} \qquad r \neq 1$$

Hence

$$y[n] = \frac{r}{(r-1)^2} + \left[\frac{-r}{(r-1)^2} + \frac{rn}{r-1}\right] r^n = \frac{r + [n(r-1) - 1]r^{n+1}}{(r-1)^2} \qquad r \neq 1$$

3.9-7.

$$\gamma^2 - \gamma + 0.16 = (\gamma - 0.2)(\gamma - 0.8)$$

The roots are 0.2 and 0.8.

$$y_n[n] = B_1(0.2)^n + B_2(0.8)^n$$

Because the input is a mode

$$y_\phi[n] = cn(0.2)^n$$

But $y_\phi[n]$ satisfies the system equation, that is,

$$y_\phi[n+2] - y_\phi[n+1] + 0.16 y_\phi[n] = x[n+1]$$

and

$$c(n+2)(0.2)^{n+2} - c(n+1)(0.2)^{n+1} + 0.16cn(0.2)^n = (0.2)^{n+1}$$

This yields

$$-0.12c(0.2)^n = 0.2(0.2)^n$$

Therefore

$$c = -\frac{5}{3}$$

and

$$\begin{aligned} y_\phi[n] &= -\frac{5}{3}n(0.2)^n \\ y[n] &= B_1(0.2)^n + B_2(0.8)^n - \frac{5}{3}n(0.2)^n \qquad n \geq 0 \end{aligned}$$

Setting $n = 0$, 1, and substituting initial conditions $y[0] = 1$, $y[1] = 2$, yields

$$\left.\begin{array}{l} 1 = B_1 + B_2 \\ 2 = 0.2B_1 + 0.8B_2 - \frac{1}{3} \end{array}\right\} \Longrightarrow \quad \begin{array}{l} B_1 = -\frac{-23}{9} \\ B_2 = \frac{32}{9} \end{array}$$

$$y[n] = -\frac{23}{9}(0.2)^n + \frac{32}{9}(0.8)^n - \frac{5}{3}(0.2)^n \qquad n \geq 0$$

3.9-8.

$$y[n+2] - y[n+1] + 0.16y[n] = x[n+1]$$

We solve this equation iteratively for $x[n] = \cos(\frac{\pi n}{2} + \frac{\pi}{3})$, $y[-1] = y[-2] = 0$, to find $y[0]$ and $y[1]$. Remember also that $x[n] = 0$ for $n < 0$. Setting $n = -2$ in the equation yields

$$y[0] - 0 + 0.16(0) = 0 \quad \Longrightarrow \quad y[0] = 0$$

Setting $n = -1$ in the equation yields

$$y[1] - 0 + 0.16(0) = \cos\frac{\pi}{3} = 0.5 \quad \Longrightarrow \quad y[1] = 0.5$$

Therefore $y[0] = 0$ and $y[1] = 0.5$. For the input $x[n] = \cos(\frac{\pi n}{2} + \frac{\pi}{3})$.

$$y_\phi[n] = c\cos(\frac{\pi n}{2} + \frac{\pi}{3} + \phi)$$

But $y_\phi[n]$ satisfies the system equation, that is,

$$y_\phi[n+2] - y_\phi[n+1] + 0.16y_\phi[n] = x[n+1]$$

or

$$c\cos[\frac{\pi}{2}(n+2)+\frac{\pi}{3}+\phi] - c\cos[\frac{\pi}{2}(n+1)+\frac{\pi}{3}+\phi] + 0.16c\cos(\frac{\pi}{2}n+\frac{\pi}{3}+\phi) = \cos[\frac{\pi}{2}(n+1)+\frac{\pi}{3}]$$

or

$$-c\cos(\frac{\pi n}{2} + \frac{\pi}{3} + \phi) + c\sin(\frac{\pi n}{2} + \frac{\pi}{3} + \phi) + 0.16c\cos(\frac{\pi n}{2} + \frac{\pi}{3} + \phi) = \cos(\frac{\pi}{2}n + \frac{\pi}{3} + \frac{\pi}{2})$$

or

$$1.306c\cos(\frac{\pi n}{2} + \frac{\pi}{3} + \phi - 2.27) = \cos(\frac{\pi n}{2} + \frac{\pi}{3} + \frac{\pi}{2})$$

Therefore

$$1.306c = 1 \qquad \Longrightarrow \qquad c = 0.765$$

$$\phi - 2.27 = \frac{\pi}{2} \qquad \Longrightarrow \qquad \phi = 3.84 = -2.44\text{rad}$$

Therefore

$$\begin{aligned} y_\phi[n] &= 0.765\cos(\frac{\pi n}{2} + \frac{\pi}{3} - 2.44) \\ &= 0.765\cos(\frac{\pi n}{2} - 1.393) \end{aligned}$$

$$y[n] = B_1(0.2)^n + B_2(0.8)^n + 0.765\cos(\frac{\pi n}{2} - 1.393)$$

Setting $n = 0$, 1, and substituting $y[0] = 0$, $y[1] = 0.5$, yields

$$\left.\begin{aligned} 0 &= B_1 + B_2 + 0.1354 \\ 0.5 &= 0.2B_1 + 0.8B_2 + 0.753 \end{aligned}\right\} \Longrightarrow \quad \begin{aligned} B_1 &= 0.241 \\ B_2 &= -0.377 \end{aligned}$$

$$y[n] = 0.241(0.2)^n - 0.377(0.8)^n + 0.765\cos(\frac{\pi n}{2} - 1.393)$$

3.10-1. Assume that a system exists that violates (9.61), and yet produces bounded output for every bounded input. The system response at $n = k_1$ is

$$y[n_1] = \sum_{m=0}^{\infty} h[m]x[n_1 - m]$$

Consider a bounded input $x[n]$ such that

$$x[n_1 - m] = \begin{cases} 1 & \text{if} \quad h[m] > 0 \\ -1 & \text{if} \quad h[m] < 0 \end{cases}$$

In this case

$$h[m]x[n_1 - m] = |h[m]|$$

and

$$y[n_1] = \sum_{m=0}^{\infty} |h[m]| = \infty$$

This violates the assumption.

3.10-2. (a)

$$\gamma^2 + 0.6\gamma - 1.6 = (\gamma - 0.2)(\gamma + 0.8)$$

Roots are 0.2 and -0.8. Both are inside the unit circle. The system is BIBO stable and asymptotically stable.

(b)

$$\gamma^2 + 3\gamma + 2 = (\gamma + 2)(\gamma + 1)$$

Roots are -1 and -2. One root outside the unit circle and the other is on the unit circle. The system is BIBO unstable and asymptotically unstable.

(c)

$$(\gamma - 1)^2(\gamma + \frac{1}{2})$$

Roots are 1 (repeated twice) and -0.5. Repeated root on unit circle. The system is BIBO unstable and asymptotically unstable.

(d)

$$\gamma^2 + 2\gamma + 0.96 = (\gamma + 0.8)(\gamma + 1.2)$$

Roots are -0.8 and -1.2. One root (-1.2) is outside the the unit circle. The system is BIBO unstable and asymptotically unstable.

(e)

$$\gamma^2 + \gamma - 2 = (\gamma + 0.5 + j1.5)(\gamma + 05 - j1.5)$$

Roots are $-0.5 \pm j1.5$. Both roots outside the unit circle. The system is BIBO unstable and asymptotically unstable.

(f)

$$(\gamma^2 - 1)(\gamma^2 + 1) = (\gamma + 1)(\gamma - 1)(\gamma + j1)(\gamma - j1)$$

Roots are ± 1, $\pm j1$. All the roots are simple and on unit circle. The system is BIBO unstable and marginally stable.

3.10-3. The system $\mathcal{S}_1$ is asymptotically (and BIBO) unstable. The system $\mathcal{S}_2$ is BIBO and asymptotically stable. If we cascade the two systems, the impulse response of the composite system is

$$h[n] = 2^n u[n] * (\delta[n] - 2\delta[n-1]) = 2^n u[n] - 2(2)^{n-1} u[n-1] = \delta[n]$$

The composite system is BIBO stable. However, the system $\mathcal{S}_1$ will burn (or saturate) out because its output contains the signal of the form 2^n.

3.10-4. (a) To be unstable, a causal mode must have magnitude greater than one. That is, at least one characteristic root must be outside the unit circle. By this criteria,

Systems C, D, and H are unstable.

(b) To be real, the characteristic modes need to be either real or in complex-conjugate pairs. By this criteria,

Systems A, B, C, F, H, and I are real.

(c) Oscillatory modes include sinusoids, decaying sinusoids, or exponentially growing sinusoids. Unless the characteristic roots are all real and positive, the corresponding natural mode(s) will exhibit oscillatory behavior. By this criteria,

Systems A, B, C, D, E, F, G, and H have oscillatory natural modes.

(d) To have a mode that decays at a rate of 2^{-n}, at least one characteristic root needs to lie on the circle of radius one-half centered at the origin. By this criteria,

Systems A, C, E, and I have at least one mode that decays by 2^{-n}

(e) For a second-order system with two finite roots to only have one mode, one characteristic root needs to be located at the origin. By this criteria,

Systems E, G, and I have only one mode.

3.10-5. Notice, the system response can be written more simply as $h[n] = \delta[n] + \left(\frac{1}{3}\right)^n u[n-1] = \left(\frac{1}{3}\right)^n u[n]$.

(a) Yes, the system is stable since the impulse response function is absolutely summable. That is, $\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n = \frac{1-0}{1-1/3} = 3/2 < \infty$.
Yes, the system is causal since $h[n] = 0$ for $n < 0$.

(b) MATLAB is used to plot $x[n]$.

```
>> n = [-5:5]; x = (n>=3)-(n>=-3);
>> stem(n,x,'k'); axis([-5 5 -1.2 1.2]);
>> xlabel('n'); ylabel('x[n]');
```

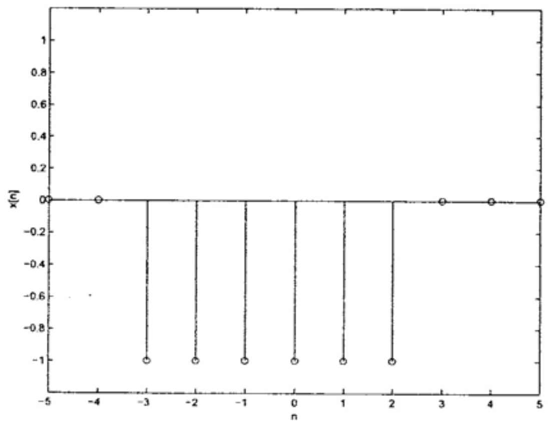


Figure S3.10-5b: Plot of $x[n] = u[n-3] - u[n+3]$.

(c) The zero-state response is computed as $y[n] = x[n] * h[n]$. This convolution involves several regions.
For $(n < -3)$, $y[n] = 0$.
For $(-3 \leq n < 2)$, $y[n] = \sum_{k=-3}^{n} -(1/3)^{n-k} = -(1/3)^n \sum_{k=-3}^{n} 3^k = -(1/3)^n \frac{3^{-3}-3^{n+1}}{1-3} = \frac{3^{-(n+3)}-3}{2}$.
For $(n \geq 2)$, $y[n] = \sum_{k=-3}^{2} -(1/3)^{n-k} = -(1/3)^n \sum_{k=-3}^{2} 3^k = -(1/3)^n \frac{3^{-3}-3^3}{1-3} = -\frac{728}{54}(3)^{-n}$.
Combining yields

$$y[n] = \begin{cases} 0 & n < -3 \\ \frac{3^{-(n+3)}-3}{2} & -3 \leq n < 2 \\ -\frac{728}{54}(3)^{-n} & n \geq 2 \end{cases}.$$

MATLAB is used to plot the result.

```
>> n = [-10:10];
>> y = (3.^(-(n+3))-3)/2.*((n>=-3)&(n<2));
>> y = y+(-3.^(-n)*728/54).*(n>=2);
>> stem(n,y,'k'); axis([-10 10 -2 .5]);
>> xlabel('n'); ylabel('y[n]');
```

3.10-6. (a) No, the system is not causal since $h[n] \neq 0$ for $(n < 0)$.

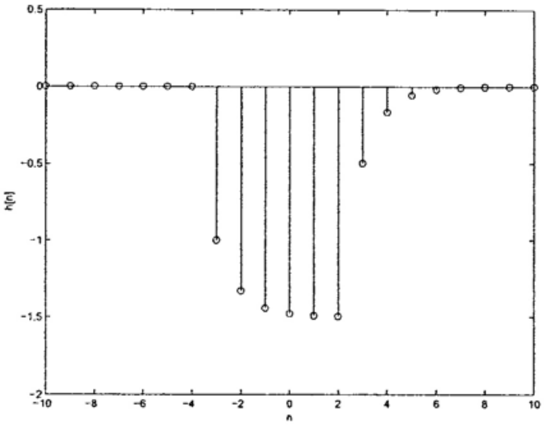


Figure S3.10-5c: Plot of $y[n] = x[n] * h[n]$.

(b) $\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=-\infty}^{\infty} |\left(\frac{1}{2}\right)^{|n|}| = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^{|n|} = \sum_{n=-\infty}^{-1} 0.5^{-n} + \sum_{n=0}^{\infty} 0.5^{n} = \sum_{n=-\infty}^{-1} 2^{n} + \sum_{n=0}^{\infty} 0.5^{n} = \frac{0-2^0}{1-2} + \frac{0.5^0-0}{1-0.5} = 1 + 2 = 3.$

$$\sum_{n=-\infty}^{\infty} |h[n]| = 3.$$

Since $h[n]$ is absolutely summable, the system is BIBO stable.

(c) $P_x = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{n=N} |x[n]|^2 = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{n=N} 9u[n-5] = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=5}^{n=N} 9 \lim_{N\to\infty} \frac{1}{2N+1} 9(N-5+1) = \lim_{N\to\infty} \frac{9N-36}{2N+1} = \frac{9}{2}$. Since power is finite, energy must be infinite.

$$E_x = \infty \text{ and } P_x = \frac{9}{2}.$$

(d) $y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$. Thus, $y[10] = \sum_{k=-\infty}^{\infty} h[k]x[10-k] = \sum_{k=-\infty}^{\infty} 0.5^{|k|} 3u[10-5-k] = \sum_{k=-\infty}^{-1} 3(2^k) + \sum_{k=0}^{5} 3(0.5^k) = 3\frac{0-2^0}{1-2} + 3\frac{0.5^0-0.5^6}{1-0.5} = 3 + 3\frac{63}{32} = \frac{285}{32} \approx 8.91$. Thus,

$$y[10] = \frac{285}{32}.$$

3.M-1. To accommodate upsampling, the inline function needs to be modified so that it assigns a zero for non-integer inputs.

```
>> f = inline('(n==fix(n)).*exp(-n/5).*cos(pi*n/5).*(n>=0)','n');
```

The added term `(n==fix(n))` is one if n is an integer and zero if n is not an integer. The modified function is easy to test.

```
>> n = (-10:10); stem(n,f(n/2),'k');
>> ylabel('f[n/2]'); xlabel('n'); axis([-10.5 10.5 -0.5 1.1]);
```

As hoped, `f[n/2]` inserts a zero between every sample of `f[n]`, which corresponds to the desired upsample by two operation.

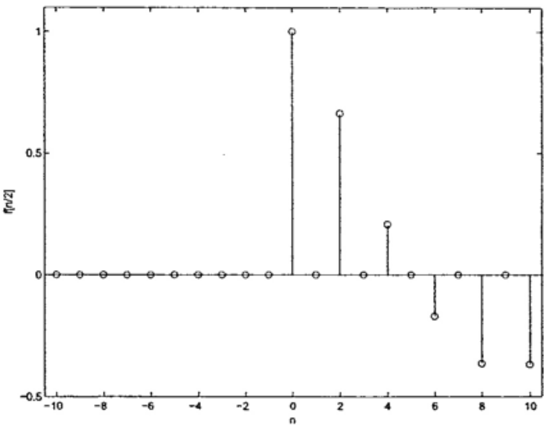


Figure S3.M-1: Plot of $f[n/2]$.

3.M-2. There are many ways to solve this problem. For this solution, let $d[n]$ designate the distance from the student's destination, which alternates between home and the exam location, just before the student changes his mind. For even-valued n the destination is home, and for odd-valued n the destination is the exam location.

(a) Just before changing direction, the student is a distance of $d[n]$ miles from his destination. Turning around, his next destination is therefore a distance of $2-d[n]$ miles. The student travels one-half of this distance, which leaves $\frac{2-d[n]}{2}$ miles remaining. Thus, a difference equation description of this problem is $d[n+1] = \frac{2-d[n]}{2} = 1 - 0.5d[n] = u[n] - 0.5d[n]$. Rearranging and shifting by one yields

$$d[n] + 0.5d[n-1] = u[n-1].$$

For this description, $d[0] = 0$. This auxiliary condition simply states that before the student first decides to go to the exam, he is at home.

(b) MATLAB is used to iteratively simulate the difference equation.

```
>> n = [0:20]; d = zeros(size(n));
>> for index = find(n>0), d(index) = 1-0.5*d(index-1); end; d
d =        0 1.0000 0.5000 0.7500 0.6250 0.6875 0.6563 0.6719
     0.6641 0.6680 0.6660 0.6670 0.6665 0.6667 0.6666 0.6667
     0.6667 0.6667 0.6667 0.6667 0.6667
```

As time increases, the remaining distance to the destination (alternating home and exam) reaches a steady-state value of two-thirds of a mile. That is,

$$\lim_{n\to\infty} d[n] = 2/3.$$

Changing the problem so that the student travels two-thirds the remaining distance each time changes the result. In this case, the difference equation is $d[n] + \frac{1}{3}d[n-1] = \frac{2}{3}u[n]$.

```
>> n = [0:20]; d = zeros(size(n));
>> for index = find(n>0), d(index) = (2-d(index-1))/3; end; d
d =        0 0.6667 0.4444 0.5185 0.4938 0.5021 0.4993 0.5002
    0.4999 0.5000 0.5000 0.5000 0.5000 0.5000 0.5000 0.5000
    0.5000 0.5000 0.5000 0.5000 0.5000
```

In this case, the steady-state remaining distance is $\lim_{n\to\infty} d[n] = 1/2$.

(c) The closed form, or total, solution is the sum of the zero-input and zero-state responses. Since the auxiliary condition is $d[0] = 0$, the zero-input response is just zero. To compute the zero-state response, write the difference equation as $d[n] + 0.5d[n] = u[n-1] = x[n]$, where the "input" $x[n]$ is just a shifted unit step. In this way, $h[n] = (-1/2)^n u[n]$. The final solution is $d[n] = h[n] * x[n] = \left(\sum_{k=0}^{n-1}(-0.5)^k\right) u[n-1] = \frac{1-(-0.5)^n}{1-(-0.5)} u[n-1]$. Thus,

$$d[n] = \frac{2}{3}\left(1 - (-0.5)^n\right) u[n-1].$$

MATLAB is used to evaluate $d[n]$.

```
>> n = [0:20]; d = 2/3*(1-(-0.5).^n).*(n>0)
d =        0 1.0000 0.5000 0.7500 0.6250 0.6875 0.6563 0.6719
      0.6641 0.6680 0.6660 0.6670 0.6665 0.6667 0.6666 0.6667
      0.6667 0.6667 0.6667 0.6667 0.6667
```

These results are identical to the iterative solution, which provides good evidence that the solution is correct.

3.M-3. (a) Given $r_{xy}[k] = \sum_{n=-\infty}^{\infty} x[n]y[n-k]$, substituting $m = -n+k$ yields $r_{xy}[k] = \sum_{m=-\infty}^{\infty} y[-m]x[k-m] = y[-n] * x[n]$. Thus,

$$r_{xy}[k] = x[n] * y[-n].$$

Similarly, $r_{yx}[k] = y[n] * x[-n]$. In general, $x[n] * y[-n] \neq y[n] * x[-n]$. Thus,

$$r_{xy}[k] \neq r_{yx}[k].$$

It is true, however, that $r_{xy}[k] = r_{yx}[-k]$.

(b) Yes, cross-correlation indicates similarity between signals as a function of the *shift* between the two functions. That is, when the shift k aligns two similar signals, the two signals constructively interact and $r_{xy}[k]$ becomes large. A large negative correlation means that the first signal is very similar to the negative of the first signal.

i.
```
function [rxy,k] = crosscorr(x,y,nx,ny)
% function [rxy,k] = crosscorr(x,y,nx,ny)

% Ensure inputs are column vectors:
x=x(:); y=y(:);
% Reverse y and compute rxy using the conv command:
rxy = conv(x,flipud(y));
% Compute shifts:
k = [nx(1)-ny(end):nx(end)-ny(1)];
```

ii.
```
>> nx = [0:20]; x = (nx>=5)-(nx>=10);
>> ny = [-20:10]; y = (ny<=-15)-(ny<=-10)+(ny==2);
>> [rxy,k] = crosscorr(x,y,nx,ny);
>> subplot(221); stem(nx,x,'k');
>> xlabel('n'); ylabel('x[n]'); axis([0 20 -1.1 1.1]);
>> subplot(222); stem(ny,y,'k');
>> xlabel('n'); ylabel('y[n]'); axis([-20 10 -1.1 1.1]);
>> subplot(212); stem(k,rxy,'k');
```

```
>> xlabel('k'); ylabel('r_{xy}[k]');
```

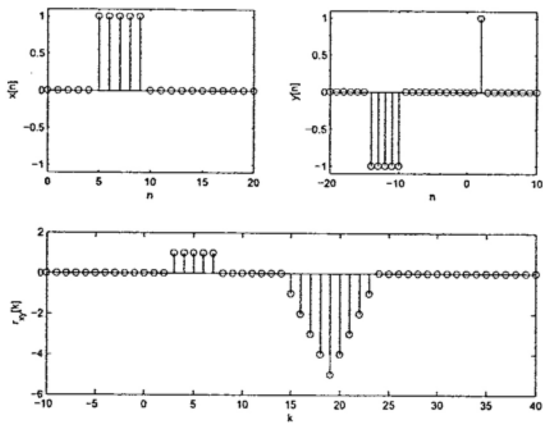


Figure S3.M-3(b)ii: Plots of $x[n]$, $y[n]$, and $r_{xy}[k]$.

The largest magnitude of $r_{xy}[k]$ is five and occurs at $k = 19$. Signal $x[n]$ has a unit pulse of width five starting at $n = 5$. Signal $y[n]$ has a similar feature: a negative unit pulse of width five starting at $n = -14$. These two similar features are separated by a shift $k = 5 - (-14) = 19$. Since these are the most similar features between the two signals, it seems very sensible that the autocorrelation function has a large, negative value at shift $k = 19$.

3.M-4. (a)

```
function [y] = filtermax(x,N)
% function [y] = filtermax(x,N)

M = length(x); x = x(:); x = [zeros(N-1,1);x]; y = zeros(M,1); for
m = 1:M,
    y(m) = max(x([m:m+(N-1)]));
end
```

(b)

```
>> n = [0:44]; x = cos(pi*n/5)+(n==30)-(n==35);
>> y1 = filtermax(x,4);
>> y2 = filtermax(x,8);
>> y3 = filtermax(x,12);
>> subplot(221); stem(n,x,'k'); axis([0 44 -2.1 2.1]);
>> xlabel('n'); ylabel('x[n]');
>> subplot(222); stem(n,y1,'k'); axis([0 44 -2.1 2.1]);
>> xlabel('n'); ylabel('y[n] for N=4')
>> subplot(223); stem(n,y2,'k'); axis([0 44 -2.1 2.1]);
>> xlabel('n'); ylabel('y[n] for N=8')
>> subplot(224); stem(n,y3,'k'); axis([0 44 -2.1 2.1]);
>> xlabel('n'); ylabel('y[n] for N=12')
```

The plots are consistent with the expected behavior of a max filter. The output, which is always greater than or equal to the input, emphasizes large input values. Larger values of N cause particular maximum values to persist longer. The max filter is very sensitive to large, positive outliers, such as that caused by the added $\delta[n-30]$.

Also notice that the max filter is an FIR filter. Thus, a sinusoidal input reaches steady-state in after $N-1$ samples. Furthermore, since a sinusoidal input does

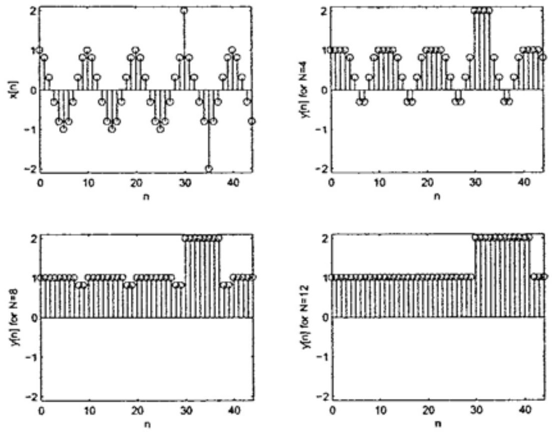


Figure S3.M-4b: Plots for N-point max filters.

not result in a sinusoidal output, the max filter cannot be a LTI system (the max filter is TI but not linear).

3.M-5. (a)
```
function [y] = filtermin(x,N)
% function [y] = filtermin(x,N)

M = length(x); x = x(:); x = [zeros(N-1,1);x]; y = zeros(M,1); for
m = 1:M,
    y(m) = min(x([m:m+(N-1)]));
end
```

(b)
```
>> n = [0:44]; x = cos(pi*n/5)+(n==30)-(n==35);
>> y1 = filtermin(x,4);
>> y2 = filtermin(x,8);
>> y3 = filtermin(x,12);
>> subplot(221); stem(n,x,'k'); axis([0 44 -2.1 2.1]);
>> xlabel('n'); ylabel('x[n]');
>> subplot(222); stem(n,y1,'k'); axis([0 44 -2.1 2.1]);
>> xlabel('n'); ylabel('y[n] for N=4')
>> subplot(223); stem(n,y2,'k'); axis([0 44 -2.1 2.1]);
>> xlabel('n'); ylabel('y[n] for N=8')
>> subplot(224); stem(n,y3,'k'); axis([0 44 -2.1 2.1]);
>> xlabel('n'); ylabel('y[n] for N=12')
```

The plots are consistent with the expected behavior of a min filter. The output, which is always less than or equal to the input, emphasizes highly negative input values. Larger values of N cause particular minimum values to persist longer. The min filter is very sensitive to large, negative outliers, such as that caused by the added $-\delta[n-35]$.

Also notice that the min filter is an FIR filter. Thus, a sinusoidal input reaches steady-state in after $N-1$ samples. Furthermore, since a sinusoidal input does not result in a sinusoidal output, the min filter cannot be a LTI system (the min filter is TI but not linear).

3.M-6. (a)
```
function [y] = filtermedian(x,N)
% function [y] = filtermedian(x,N)
```

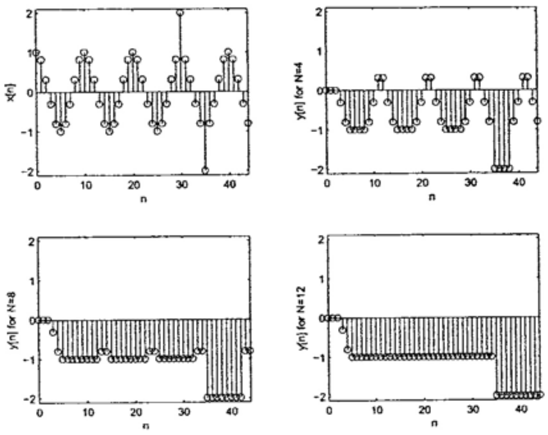


Figure S3.M-5b: Plots for N-point min filters.

```
M = length(x); x = x(:); x = [zeros(N-1,1);x]; y = zeros(M,1); for
m = 1:M,
    y(m) = median(x([m:m+(N-1)]));
end
```

(b)
```
>> n = [0:44]; x = cos(pi*n/5)+(n==30)-(n==35);
>> y1 = filtermedian(x,4);
>> y2 = filtermedian(x,8);
>> y3 = filtermedian(x,12);
>> subplot(221); stem(n,x,'k'); axis([0 44 -2.1 2.1]);
>> xlabel('n'); ylabel('x[n]');
>> subplot(222); stem(n,y1,'k'); axis([0 44 -2.1 2.1]);
>> xlabel('n'); ylabel('y[n] for N=4')
>> subplot(223); stem(n,y2,'k'); axis([0 44 -2.1 2.1]);
>> xlabel('n'); ylabel('y[n] for N=8')
>> subplot(224); stem(n,y3,'k'); axis([0 44 -2.1 2.1]);
>> xlabel('n'); ylabel('y[n] for N=12')
```

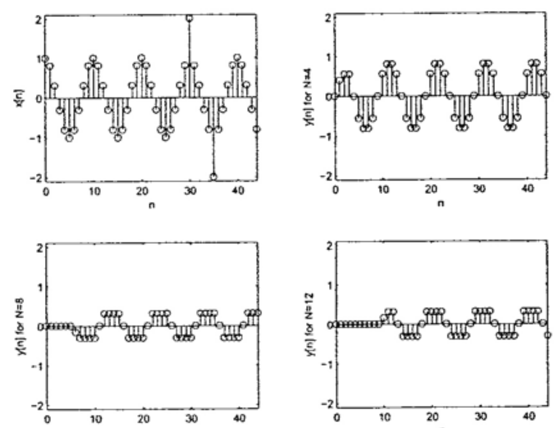


Figure S3.M-6b: Plots for N-point median filters.

The plots are consistent with the expected behavior of a median filter. The output magnitude tends to be smaller than the input magnitude. Unlike the max or min filters, the median filter is not sensitive to outliers in the input data. Also notice that the median filter is an FIR filter. Thus, a sinusoidal input reaches steady-state in after $N-1$ samples. Furthermore, since a sinusoidal input does not result in a sinusoidal output, the median filter cannot be a LTI system (the median filter is TI but not linear).

3.M-7. (a) Replacing $h[n]$ with $y[n]$ and $\delta[n]$ with $x[n]$ yields the desired difference equation,

$$y[n] = \sum_{k=-(N-1)}^{N-1} \left(1 - \left|\frac{k}{N}\right|\right) x[n-k].$$

(b) To make the system causal, $h[n]$ must be right-shifted by at least $(N-1)$. Since the system is time-invariant, shifting $h[n]$ causes the output to be delayed (shifted) by the same $(N-1)$ amount.

(c)
```
function [a,b] = interpfilter(N)
%function [a,b] = interpfilter(N)

a = 1; b = conv(ones(N,1),ones(N,1))/N;
```

(d)
```
>> n = [0:9]; x = cos(n);
>> N = 10; nup = [0:N*length(n)-1];
>> xup = [x;zeros(N-1,length(x))]; xup = xup(:);
>> [a,b] = interpfilter(N);
>> y = filter(b,a,xup);
>> subplot(311),stem(n,x,'k');
>> xlabel('n'); ylabel('x[n]');
>> subplot(312),stem(nup,xup,'k');
>> xlabel('n'); ylabel('x_{up}[n]');
>> subplot(313),stem(nup,y,'k');
>> xlabel('n'); ylabel('y[n]');
```

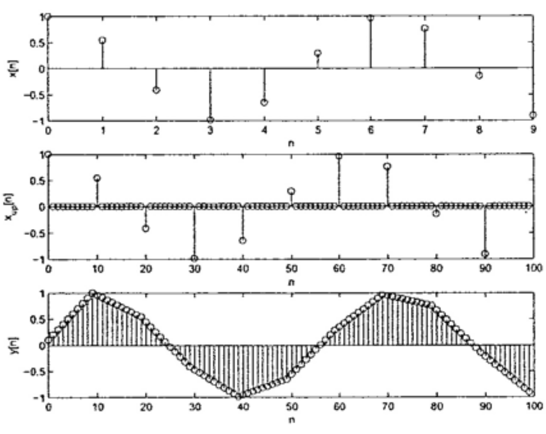


Figure S3.M-7d: Plots for $N = 10$ interpolation filter.

Indeed, the filter produces the desired linear interpolation of the up-sampled input. As expected from a causal implementation of the interpolation filter, an $(N-1)$ delay is visible in the output.

3.M-8. (a) First, rewrite the impulse response as $h[n] = (u[n] - u[n-N])/N = \frac{1}{N}\sum_{k=0}^{N-1}\delta[n-k]$. Replacing $h[n]$ with $y[n]$ and $\delta[n]$ with $x[n]$ yields the desired difference equation,

$$y[n] = \frac{1}{N}\sum_{k=0}^{N-1} x[n-k].$$

(b)
```
function [a,b] = filterma(N)
% function [a,b] = filterma(N)

a = 1; b = ones(N,1)/N
```

(c)
```
>> n = [0:44]; x = cos(pi*n/5)+(n==30)-(n==35);
>> N = 4; [a,b] = filterma(N);
>> y1 = filter(b,a,x);
>> N = 8; [a,b] = filterma(N);
>> y2 = filter(b,a,x);
>> N = 12; [a,b] = filterma(N);
>> y3 = filter(b,a,x);
>> subplot(221); stem(n,x,'k'); axis([0 44 -2.1 2.1]);
>> xlabel('n'); ylabel('x[n]');
>> subplot(222); stem(n,y1,'k'); axis([0 44 -2.1 2.1]);
>> xlabel('n'); ylabel('y[n] for N=4');
>> subplot(223); stem(n,y2,'k'); axis([0 44 -2.1 2.1]);
>> xlabel('n'); ylabel('y[n] for N=8');
>> subplot(224); stem(n,y3,'k'); axis([0 44 -2.1 2.1]);
>> xlabel('n'); ylabel('y[n] for N=12');
```

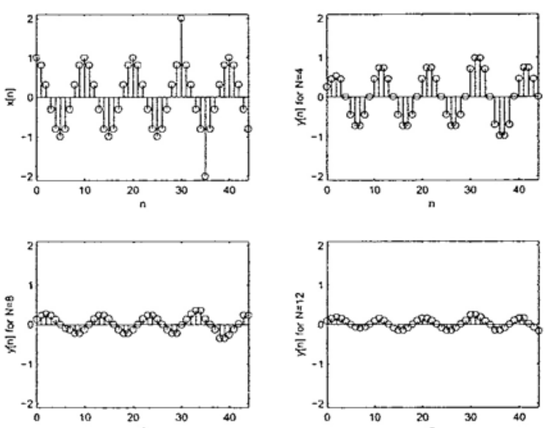


Figure S3.M-8c: Plots for N-point moving average filters.

The plots are consistent with the expected behavior of a moving average filter. The output is a low-pass version of the input. Larger values of N average over a wider window; thus, larger N results in greater attenuation of the input sinusoid. Large outliers have high frequency content and are significantly attenuated.

Also notice that the moving average filter is an FIR filter. Thus, a sinusoidal input reaches steady-state in after $N-1$ samples. This particular filter is a LTI filter, so a sinusoidal input will result in a steady-state sinusoidal output.

(d) The total impulse response of a cascade of two N-point moving average filters is $h_{\text{cascade}}[n] = \left(\frac{u[n]-u[n-N]}{N}\right) * \left(\frac{u[n]-u[n-N]}{N}\right) = \left(\sum_{k=0}^{n} N^{-2}\right)(u[n]-u[n-N]) + \left(\sum_{k=n-(N-1)}^{N-1} N^{-2}\right)(u[n-N]-u[n-(2N-1)]) \;=\; \frac{n+1}{N^2}(u[n]-u[n-N]) + \frac{2N-n-1}{N^2}(u[n-N]-u[n-(2N-1)])$. That is, $h_{\text{cascade}}[n]$ is a triangle-shaped function with width $2N-1$ and maximum height $1/N$.

The impulse response of a causal linear interpolation filter is $h_{\text{lininterp}}[n] = \sum_{k=-(N-1)}^{N-1}\left(1-\left|\frac{k}{N}\right|\right)\delta(n-(N-1)-k)$. This function is identical to $h_{\text{cascade}}[n]$ except that it has a maximum height of 1. Thus, the cascade of moving average filters is a factor $1/N$ different than the linear interpolation filter.

$$N h_{\text{cascade}}[n] = h_{\text{lininterp}}[n].$$

MATLAB is used to implement a linear interpolation filter using a cascade of two moving average filters.

```
>> n = [0:9]; x = cos(n);
>> N = 10; nup = [0:N*length(n)-1];
>> xup = [x;zeros(N-1,length(x))]; xup = xup(:);
>> [a,b] = filterma(N);
>> y = N*filter(b,a,filter(b,a,xup));
>> subplot(311),stem(n,x,'k');
>> xlabel('n'); ylabel('x[n]');
>> subplot(312),stem(nup,xup,'k');
>> xlabel('n'); ylabel('x_{up}[n]');
>> subplot(313),stem(nup,y,'k');
>> xlabel('n'); ylabel('y[n]'); axis tight;
```

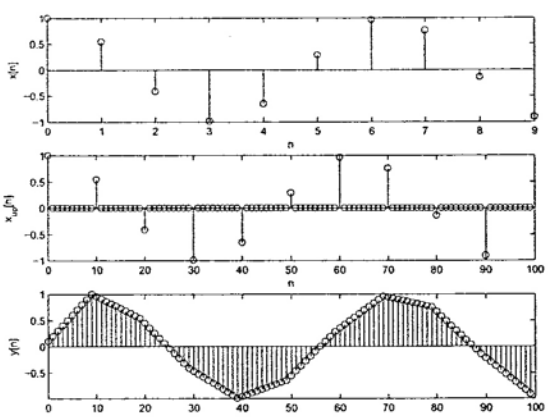


Figure S3.M-8d: Linear interpolation using cascaded moving average filters.

Figure S3.M-8d demonstrates that linear interpolation is possible using a cascade of two moving average filters. As expected, a delay of $N-1$ is visible in the output.

Chapter 4 Solutions

4.1-1. (a)

$$x(t) = u(t) - u(t-1)$$

$$\begin{aligned} X(s) = \int_0^1 e^{-st}\,dt &= -\frac{e^{-st}}{s}\bigg|_0^1 \\ &= -\frac{1}{s}[e^{-s} - 1] \\ &= \frac{1}{s}[1 - e^{-s}] \end{aligned}$$

Note that the result is valid for all values of s; hence the region of convergence is the entire s-plane. The abscissa of convergence is $\sigma_0 = -\infty$.

(b)

$$x(t) = te^{-t}u(t)$$

$$\begin{aligned} X(s) = \int_0^\infty te^{-t}e^{-st}\,dt &= \int_0^\infty te^{-(s+1)}\,dt \\ &= -\frac{e^{-(s+1)t}}{(s+1)^2}\left[-(s+1)t - 1\right]_0^\infty \\ &= \frac{1}{(s+1)^2} \end{aligned}$$

provided that $e^{-(s+1)\infty} = 0$ or Re $(s+1) > 0$. Hence the abscissa of convergence is Re $(s) > -1$ or $\sigma_0 > -1$.

(c)

$$x(t) = t\cos\omega_0 t\, u(t)$$

$$\begin{aligned} X(s) &= \int_0^\infty t\cos\omega_0 te^{-st}\,dt \\ &= \frac{1}{2}\left\{\int_0^\infty [te^{(j\omega_0 - s)t} + te^{-(j\omega_0+s)t}]\,dt\right\} \\ &= \frac{1}{2}\left[\frac{1}{(s-j\omega_0)^2} + \frac{1}{(s+j\omega_0)^2}\right] \qquad \text{Re } (s) > 0 \end{aligned}$$

$$= \frac{s^2 - \omega_0^2}{(s^2 + \omega_0^2)^2}$$

(d)

$$x(t) = (e^{2t} - 2e^{-t})u(t)$$

$$\begin{aligned}
X(s) &= \int_0^\infty (e^{2t} - 2e^{-t})e^{-st}\,dt \\
&= \int_0^\infty e^{2t}e^{-st}\,dt - 2\int_0^\infty e^{-t}e^{-st}\,dt \\
&= \int_0^\infty e^{-(s-2)t}\,dt - 2\int_0^\infty e^{-(s+1)t}\,dt \\
&= \frac{1}{s-2} - \frac{2}{s+1}
\end{aligned}$$

We get the first term only if Re $s > 2$, and we get the second term only if Re $(s) > -1$. Both conditions will be satisfied if Re $(s) > 2$ or $\sigma_0 > 2$. Hence:

$$X(s) = \frac{1}{s-2} - \frac{2}{s+1} \qquad \text{for} \quad \sigma_0 > 2$$

(e)

$$x(t) = \cos\omega_1 t \cos\omega_2 t\, u(t) = \left[\frac{1}{2}\cos(\omega_1 + \omega_2)t + \frac{1}{2}\cos(\omega_1 - \omega_2)t\right] u(t)$$

$$\begin{aligned}
X(s) &= \frac{1}{2}\int_0^\infty \cos(\omega_1 + \omega_2)te^{-st}\,dt + \frac{1}{2}\int_0^\infty \cos(\omega_1 - \omega_2)te^{-st}\,dt \\
&= \frac{1}{2}\left[\frac{s}{s^2 + (\omega_1 + \omega_2)^2} + \frac{s}{s^2 + (\omega_1 - \omega_2)^2}\right]
\end{aligned}$$

provided that Re $(s) > 0$.

(f)

$$x(t) = \cosh(at)u(t)$$

$$\begin{aligned}
X(s) &= \frac{1}{2}\left[\int_0^\infty e^{at}e^{-st}\,dt + \int_0^\infty e^{-at}e^{-st}\,dt\right] \\
&= \frac{1}{2}\left[\int_0^\infty e^{-(s-a)t}\,dt + \int_0^\infty e^{-(s+a)t}\,dt\right] \\
&= \frac{s}{s^2 - a^2} \qquad\qquad \text{Re } s > |a|
\end{aligned}$$

(g)

$$x(t) = \sinh(at)u(t)$$

$$X(s) = \frac{1}{2}\left[\int_0^\infty e^{-(s-a)t}\,dt - \int_0^\infty e^{-(s+a)t}\,dt\right]$$

$$= \frac{a}{s^2 - a^2} \qquad \text{Re } s > |a|$$

(h)

$$\begin{aligned}
x(t) &= e^{-2t}\cos(5t+\theta)u(t) \\
&= \frac{1}{2}\left[e^{-2t+j(5t+\theta)} + e^{-2t-j(5t+\theta)}\right] \\
&= \frac{1}{2}e^{j\theta}e^{-(2-j5)t} + \frac{1}{2}e^{-j\theta}e^{-(2+j5)t}
\end{aligned}$$

Hence $\quad X(s) = \frac{1}{2}e^{j\theta}\left(\frac{1}{s+2-j5}\right) + \frac{1}{2}e^{-j\theta}\left(\frac{1}{s+2+j5}\right)$

This is valid if Re $(s) > -2$ for both terms; hence

$$X(s) = \frac{(s+2)\cos\theta - 5\sin\theta}{s^2 + 4s + 29}$$

4.1-2. (a)

$$X(s) = \int_0^1 te^{-st}\,dt = \frac{e^{-st}}{s}(-st-1)\Big|_0^1 = \frac{1}{s^2}(1 - e^{-s} - se^{-s})$$

(b)

$$X(s) = \int_0^{\pi} \sin t\, e^{-st}\,dt = \frac{e^{-st}}{s^2+1}(-s\sin t - \cos t)\Big|_0^{\pi} = \frac{1+e^{-\pi s}}{s^2+1}$$

(c)

$$\begin{aligned}
X(s) &= \int_0^1 \frac{t}{e}e^{-st}\,dt + \int_1^{\infty} e^{-t}e^{-st}\,dt = \frac{1}{e}\int_0^1 te^{-st}\,dt + \int_1^{\infty} e^{-(s+1)t}\,dt \\
&= \frac{e^{-st}}{es}(-st-1)\Big|_0^1 - \frac{1}{s+1}e^{-(s+1)}\Big|_1^{\infty} \\
&= \frac{1}{es^2}(1 - e^{-s} - se^{-s}) + \frac{1}{s+1}e^{-(s+1)}
\end{aligned}$$

4.1-3. (a)

$$\begin{aligned}
X(s) &= \frac{2s+5}{s^2+5s+6} = \frac{2s+5}{(s+2)(s+3)} = \frac{1}{s+2} + \frac{1}{s+3} \\
x(t) &= (e^{-2t} + e^{-3t})u(t)
\end{aligned}$$

(b)

$$X(s) = \frac{3s+5}{s^2+4s+13}$$

Here $A = 3$, $B = 5$, $a = 2$, $c = 13$, $b = \sqrt{13-4} = 3$.

$$r = \sqrt{\frac{117+25-60}{13-4}} = 3.018 \qquad \theta = \tan^{-1}(\tfrac{1}{9}) = 6.34°$$

$$x(t) = 3.018e^{-2t}\cos(3t + 6.34°)u(t)$$

(c)

$$X(s) = \frac{(s+1)^2}{s^2 - s - 6} = \frac{(s+1)^2}{(s+2)(s-3)}$$

This is an improper fraction with $b_n = b_2 = 1$. Therefore

$$\begin{aligned} X(s) &= 1 + \frac{a}{s+2} + \frac{b}{s-3} = 1 - \frac{0.2}{s+2} + \frac{3.2}{s-3} \\ x(t) &= \delta(t) + (3.2e^{3t} - 0.2e^{-2t})u(t) \end{aligned}$$

(d)

$$X(s) = \frac{5}{s^2(s+2)} = \frac{k}{s} + \frac{2.5}{s^2} + \frac{1.25}{s+2}$$

To find k set $s = 1$ on both sides to obtain

$$\frac{5}{3} = k + 2.5 + \frac{5}{12} \quad \Longrightarrow \quad k = -1.25$$

and

$$\begin{aligned} X(s) &= -\frac{1.25}{s} + \frac{2.5}{s^2} + \frac{1.25}{s+2} \\ x(t) &= 1.25(-1 + 2t + e^{-2t})u(t) \end{aligned}$$

(e)

$$X(s) = \frac{2s+1}{(s+1)(s^2+2s+2)} = \frac{-1}{s+1} + \frac{As+B}{s^2+2s+2}$$

Multiply both sides by s and let $s \to \infty$. This yields

$$0 = -1 + A \quad \Longrightarrow \quad A = 1$$

Setting $s = 0$ on both sides yields

$$\frac{1}{2} = -1 + \frac{B}{2} \quad \Longrightarrow \quad B = 3$$

$$X(s) = -\frac{1}{s+1} + \frac{s+3}{s^2+2s+2}$$

In the second fraction, $A = 1$, $B = 3$, $a = 1$, $c = 2$, $b = \sqrt{2-1} = 1$.

$$r = \sqrt{\frac{2+9-6}{2-1}} = \sqrt{5} \qquad \theta = \tan^{-1}(\frac{-2}{1}) = -63.4^\circ$$

$$x(t) = [-e^{-t} + \sqrt{5}e^{-t}\cos(t - 63.4^\circ)]u(t)$$

(f)

$$X(s) = \frac{s+2}{s(s+1)^2} = \frac{2}{s} + \frac{k}{s+1} - \frac{1}{(s+1)^2}$$

To compute k, multiply both sides by s and let $s \to \infty$. This yields

$$0 = 2 + k + 0 \quad \Longrightarrow \quad k = -2$$

and

$$\begin{aligned} X(s) &= \frac{2}{s} - \frac{2}{s+1} - \frac{1}{(s+1)^2} \\ x(t) &= [2 - (2+t)e^{-t}]u(t) \end{aligned}$$

(g)

$$X(s) = \frac{1}{(s+1)(s+2)^4} = \frac{1}{s+1} + \frac{k_1}{s+2} + \frac{k_2}{(s+2)^2} + \frac{k_3}{(s+2)^3} - \frac{1}{(s+2)^4}$$

Multiplying both sides by s and let $s \to \infty$. This yields

$$0 = 1 + k_1 \quad \Longrightarrow \quad k_1 = -1$$

$$\frac{1}{(s+1)(s+2)^4} = \frac{1}{s+1} - \frac{1}{s+2} + \frac{k_2}{(s+2)^2} + \frac{k_3}{(s+2)^3} - \frac{1}{(s+2)^4}$$

Setting $s = 0$ and -3 on both sides yields

$$\begin{aligned} \frac{1}{16} &= 1 - \frac{1}{2} + \frac{k_2}{4} + \frac{k_3}{8} - \frac{1}{16} &&\Longrightarrow \quad 4k_2 + 2k_3 = -6 \\ -\frac{1}{2} &= -\frac{1}{2} + 1 + k_2 - k_3 - 1 &&\Longrightarrow \quad k_2 - k_3 = 0 \end{aligned}$$

Solving these two equations simultaneously yields $k_2 = k_3 = -1$. Therefore

$$\begin{aligned} X(s) &= \frac{1}{s+1} - \frac{1}{s+2} - \frac{1}{(s+2)^2} - \frac{1}{(s+2)^3} - \frac{1}{(s+2)^4} \\ x(t) &= [e^{-t} - (1 + t + \frac{t^2}{2} + \frac{t^3}{6})e^{-2t}]u(t) \end{aligned}$$

Comment: This problem could be tackled in many ways. We could have used Eq. (B.47b), or after determining first two coefficients by Heaviside method, we could have cleared fractions. Also instead of letting $s = 0$ and -3, we could have selected any other set of values. However, in this case these values appear most suitable for numerical work.

(h)

$$X(s) = \frac{s+1}{s(s+2)^2(s^2+4s+5)} = \frac{(1/20)}{s} + \frac{k}{s+2} + \frac{(1/2)}{(s+2)^2} + \frac{As+B}{s^2+4s+5}$$

Multiplying both sides by s and let $s \to \infty$ yields

$$0 = \frac{1}{20} + k + A \quad \Longrightarrow \quad k + A = -\frac{1}{20}$$

Setting $s = 1$ and -1 yields

$$\begin{aligned} \tfrac{2}{90} &= \tfrac{1}{20} + \tfrac{k}{3} + \tfrac{1}{18} + \tfrac{A+B}{10} &&\Longrightarrow \quad 20k + 6A + 6B = -5 \\ 0 &= -\tfrac{1}{20} + k + \tfrac{1}{2} + \tfrac{-A+B}{2} &&\Longrightarrow \quad 20k - 10A + 10B = -9 \end{aligned}$$

Solving these three equations in k, A and B yields $k = -\frac{1}{4}$, $A = \frac{1}{5}$ and $B = -\frac{1}{5}$. Therefore

$$X(s) = \frac{1/20}{s} - \frac{1/4}{s+2} + \frac{(1/2)}{(s+2)^2} + \frac{1}{5}(\frac{s-1}{s^2+4s+5})$$

For the last fraction in parenthesis on the right-hand side $A = 1$, $B = -1$, $a = 2$,

$c = 5$, $b = \sqrt{5-4} = 1$.

$$r = \sqrt{\frac{5+1+4}{5-4}} = \sqrt{10} \qquad \theta = \tan^{-1}(\frac{3}{1}) = 71.56°$$

$$x(t) = [\frac{1}{20} - \frac{1}{4}(1-2t)e^{-2t} + \frac{\sqrt{10}}{5}e^{-2t}\cos(t + 71.56°)]u(t)$$

(i)

$$X(s) = \frac{s^3}{(s+1)^2(s^2+2s+5)} = \frac{k}{s+1} - \frac{1/4}{(s+1)^2} + \frac{As+B}{s^2+2s+5}$$

Multiply both sides by s and let $s \to \infty$ to obtain

$$1 = k + A$$

Setting $s = 0$ and 1 yields

$$0 = k - \tfrac{1}{4} + \tfrac{B}{5} \qquad \Longrightarrow \qquad 20k + 4B = 5$$

$$\tfrac{1}{32} = \tfrac{k}{2} - \tfrac{1}{16} + \tfrac{A+B}{8} \qquad \Longrightarrow \qquad 16k + 4A + 4B = 3$$

Solving these three equations in k, A and B yields $k = \frac{3}{4}$, $A = \frac{1}{4}$ and $B = -\frac{5}{2}$.

$$X(s) = \frac{3/4}{s+1} - \frac{1/4}{(s+1)^2} + \frac{1}{4}(\frac{s-10}{s^2+2s+5})$$

For the last fraction in parenthesis, $A = 1$, $B = -10$, $a = 1$, $c = 5$, $b = \sqrt{5-1} = 2$.

$$r = \sqrt{\frac{5+100+20}{5-1}} = 5.59 \qquad \theta = \tan^{-1}(\frac{11}{4}) = 70°$$

Therefore

$$\begin{aligned} x(t) &= [(\frac{3}{4} - \frac{1}{4}t)e^{-t} + \frac{5.59}{4}e^{-t}\cos(2t + 70°)]u(t) \\ &= [\frac{1}{4}(3-t) + 1.3975\cos(2t + 70°)]e^{-t}u(t) \end{aligned}$$

4.2-1. (a)

$$x(t) = u(t) - u(t-1)$$

and

$$\begin{aligned} X(s) &= \mathcal{L}[u(t)] - \mathcal{L}[u(t-1)] \\ &= \frac{1}{s} - e^{-s}\frac{1}{s} \\ &= \frac{1}{s}(1 - e^{-s}) \end{aligned}$$

(b)

$$x(t) = e^{-(t-\tau)}u(t-\tau)$$

$$X(s) = \frac{1}{s+1}e^{-s\tau}$$

(c)

$$x(t) = e^{-(t-\tau)}u(t) = e^{\tau}e^{-t}u(t)$$

$$\text{Therefore} \qquad X(s) = e^{\tau}\frac{1}{s+1}$$

(d)

$$x(t) = e^{-t}u(t-\tau) = e^{-\tau}e^{-(t-\tau)}u(t-\tau)$$

Observe that $e^{-(t-\tau)}u(t-\tau)$ is $e^{-t}u(t)$ delayed by τ. Therefore

$$X(s) = e^{-\tau}\left(\frac{1}{s+1}\right)e^{-s\tau} = \left(\frac{1}{s+1}\right)e^{-(s+1)\tau}$$

(e)

$$\begin{aligned} x(t) &= te^{-t}u(t-\tau) = (t-\tau+\tau)e^{-(t-\tau+\tau)}u(t-\tau) \\ &= e^{-\tau}\left[(t-\tau)e^{-(t-\tau)}u(t-\tau) + \tau e^{-(t-\tau)}u(t-\tau)\right] \end{aligned}$$

Therefore

$$\begin{aligned} X(s) &= e^{-\tau}\left[\frac{1}{(s+1)^2}e^{-s\tau} + \frac{\tau}{(s+1)}e^{-s\tau}\right] \\ &= \frac{e^{-(s+1)\tau}\left[1+\tau(s+1)\right]}{(s+1)^2} \end{aligned}$$

(f)

$$x(t) = \sin\omega_0(t-\tau)u(t-\tau)$$

Note that this is $\sin\omega_0 t$ shifted by τ; hence

$$X(s) = \left(\frac{\omega_0}{s^2+\omega_0^2}\right)e^{-s\tau}$$

(g)

$$x(t) = \sin\omega_0(t-\tau)u(t) = [\sin\omega_0 t\cos\omega_0\tau - \cos\omega_0 t\sin\omega_0\tau]u(t)$$

$$X(s) = \frac{\omega_0\cos\omega_0\tau - s\sin\omega_0\tau}{s^2+\omega_0^2}$$

(h)

$$\begin{aligned} x(t) &= \sin\omega_0 t\, u(t-\tau) = \sin[\omega_0(t-\tau+\tau)]u(t-\tau) \\ &= \cos\omega_0\tau\sin[\omega_0(t-\tau)]u(t-\tau) + \sin\omega_0\tau\cos[\omega_0(t-\tau)]u(t-\tau) \end{aligned}$$

Therefore

$$X(s) = \left[\cos\omega_0\tau\left(\frac{\omega_0}{s^2+\omega_0^2}\right) + \sin\omega_0\tau\left(\frac{s}{s^2+\omega_0^2}\right)\right]e^{-s\tau}$$

4.2-2. (a)

$$x(t) = t[u(t) - u(t-1)] = tu(t) - (t-1)u(t-1) - u(t-1)$$

$$X(s) = \frac{1}{s^2} - \frac{1}{s^2}e^{-s} - \frac{1}{s}e^{-s}$$

(b)

$$x(t) = \sin t\, u(t) + \sin(t-\pi)\, u(t-\pi)$$

$$X(s) = \frac{1}{s^2+1}(1 + e^{-\pi s})$$

(c)

$$\begin{aligned} x(t) &= t[u(t) - u(t-1)] + e^{-t}u(t-1) \\ &= tu(t) - (t-1)u(t-1) - u(t-1) + e^{-1}e^{-(t-1)}u(t-1) \end{aligned}$$

Therefore

$$X(s) = \frac{1}{s^2}(1 - e^{-s} - se^{-s}) + \frac{e^{-s}}{e(s+1)}$$

4.2-3. (a)

$$X(s) = \frac{(2s+5)e^{-2s}}{s^2+5s+6} = \hat{X}(s)e^{-2s}$$

It is clear that $x(t) = \hat{x}(t-2)$.

$$\begin{aligned} \hat{X}(s) &= \frac{2s+5}{s^2+5s+6} = \frac{2s+5}{(s+2)(s+3)} = \frac{1}{s+2} + \frac{1}{s+3} \\ \hat{x}(t) &= (e^{-2t} + e^{-3t})u(t) \\ x(t) &= \hat{x}(t-2) = [e^{-2(t-2)} + e^{-3(t-2)}]u(t-2) \end{aligned}$$

(b)

$$X(s) = \frac{s}{s^2+2s+2}e^{-3s} + \frac{2}{s^2+2s+2} = X_1(s)e^{-3s} + X_2(s)$$

where

$$X_1(s) = \frac{s}{s^2+2s+2} \quad \begin{cases} A = 1,\ B = 0,\ a = 1,\ c = 2,\ b = 1 \\ r = \sqrt{2},\ \theta = \tan^{-1}(1) = \pi/4 \end{cases}$$

$$\begin{aligned} x_1(t) &= \sqrt{2}e^{-t}\cos(t + \frac{\pi}{4}) \\ X_2(s) &= \frac{2}{s^2+2s+2} \quad \text{and} \quad x_2(t) = 2e^{-t}\sin t \end{aligned}$$

Also

$$\begin{aligned} x(t) &= x_1(t-3) + x_2(t) \\ &= \sqrt{2}e^{-(t-3)}\cos(t - 3 + \frac{\pi}{4})u(t-3) + 2e^{-t}\sin t\, u(t) \end{aligned}$$

(c)

$$\begin{aligned} X(s) &= \frac{(e)e^{-s}}{s^2-2s+5} + \frac{3}{s^2-2s+5} \\ &= e\frac{1}{s^2-2s+5}e^{-s} + \frac{3}{s^2-2s+5} \\ &= eX_1(s)e^{-s} + X_2(s) \end{aligned}$$

where

$$X_1(s) = \frac{1}{s^2-2s+2} \quad \text{and} \quad x_1(t) = \frac{1}{2}e^t \sin 2t\, u(t)$$

$$X_2(s) = \frac{3}{s^2-2s+2} \quad \text{and} \quad x_2(t) = \frac{3}{2}e^t \sin 2t\, u(t)$$

Therefore

$$\begin{aligned} x(t) &= ex_1(t-1) + x_2(t) \\ &= \frac{e}{2}e^{(t-1)} \sin 2(t-1)u(t-1) + \frac{3}{2}e^t \sin 2t\, u(t) \end{aligned}$$

(d)

$$\begin{aligned} X(s) = \frac{e^{-s}+e^{-2s}+1}{s^2+3s+2} &= (e^{-s}+e^{-2s}+1)\left[\frac{1}{s^2+3s+2}\right] \\ &= (e^{-s}+e^{-2s}+1)\left[\frac{1}{s+1} - \frac{1}{s+2}\right] \end{aligned}$$

$$X(s) = (e^{-s}+e^{-2s}+1)\hat{X}(s)$$

where $\hat{X}(s) = \frac{1}{s+1} - \frac{1}{s+2}$ and $\hat{x}(t) = (e^{-t} - e^{-2t})u(t)$

Moreover

$$\begin{aligned} x(t) &= \hat{x}(t-1) + \hat{x}(t-2) + \hat{x}(t) \\ &= [e^{-(t-1)} - e^{-2(t-1)}]u(t-1) + [e^{-(t-2)} - e^{-2(t-2)}]u(t-2) + (e^{-t} - e^{-2t})u(t) \end{aligned}$$

4.2-4. (a)

$$g(t) = x(t) + x(t-T_0) + x(t-2T_0) + \cdots$$

and

$$\begin{aligned} G(s) &= X(s) + X(s)e^{-sT_0} + X(s)e^{-2sT_0} + \cdots \\ &= X(s)[1 + e^{-sT_0} + e^{-2sT_0} + e^{-3sT_0} + \cdots \\ &= \frac{X(s)}{1-e^{-sT_0}} \qquad |e^{-sT_0}| < 1 \text{ or } \operatorname{Re} s > 0 \end{aligned}$$

(b)

$$\begin{aligned} x(t) &= u(t) - u(t-2) \quad \text{and} \quad X(s) = \frac{1}{s}(1-e^{-2s}) \\ G(s) &= \frac{X(s)}{1-e^{-8s}} = \frac{1}{s}\left(\frac{1-e^{-2s}}{1-e^{-8s}}\right) \end{aligned}$$

4.2-5. Pair 2

$$u(t) = \int_{0^-}^{t} \delta(\tau)\, d\tau \Longleftrightarrow \frac{1}{s}(1) = \frac{1}{s}$$

Pair 3

$$tu(t) = \int_{0^-}^{t} u(\tau)\, d\tau \Longleftrightarrow \frac{1}{s}(\frac{1}{s}) = \frac{1}{s^2}$$

Pair 4: Use successive integration of $tu(t)$

Pair 5: From frequency-shifting (4.23), we have

$$u(t) \Longleftrightarrow \frac{1}{s} \quad \text{and} \quad e^{\lambda t}u(t) \Longleftrightarrow \frac{1}{s-\lambda}$$

Pair 6: Because

$$tu(t) \Longleftrightarrow \frac{1}{s^2} \quad \text{and} \quad te^{\lambda t}u(t) \Longleftrightarrow \frac{1}{(s-\lambda)^2}$$

Pair 7: Apply the same argument to $t^2u(t)$, $t^3u(t)$, ..., and so on.
Pair 8a:

$$\cos bt\, u(t) = \frac{1}{2}(e^{jbt} + e^{-jbt})u(t) \Longleftrightarrow \frac{1}{2}\left(\frac{1}{s-jb} + \frac{1}{s+jb}\right) = \frac{s}{s^2+b^2}$$

Pair 8b: Same way as the pair 8a.
Pair 9a: Application of the frequency-shift property (4.23) to pair 8a $\cos bt\, u(t) \Longleftrightarrow \frac{s}{s^2+b^2}$ yields

$$e^{-at}\cos bt\, u(t) \Longleftrightarrow \frac{s+a}{(s+a)^2+b^2}$$

Pair 9b: Similar to the pair 9a.
Pairs 10a and 10b: Recognize that

$$re^{-at}\cos(bt+\theta) = re^{-at}[\cos\theta\cos bt - \sin\theta\sin bt]$$

Now use results in pairs 9a and 9b to obtain pair 10a. Pair 10b is equivalent to pair 10a.

4.2-6. (a) (i)

$$\begin{aligned} \frac{dx}{dt} &= \delta(t) - \delta(t-2) \\ sX(s) &= 1 - e^{-2s} \\ X(s) &= \frac{1}{s}(1-e^{-2s}) \end{aligned}$$

(ii)

$$\begin{aligned} \frac{dx}{dt} &= \delta(t-2) - \delta(t-4) \\ sX(s) &= e^{-2s} - e^{-4s} \\ X(s) &= \frac{1}{s}(e^{-2s} - e^{-4s}) \end{aligned}$$

(b)

$$\begin{aligned} \frac{dx}{dt} &= u(t) - 3u(t-2) + 2u(t-3) \\ sX(s) &= \frac{1}{s} - \frac{3}{s}e^{-2s} + \frac{2}{s}e^{-3s} \quad [x(0^-)=0] \\ X(s) &= \frac{1}{s^2}(1 - 3e^{-2s} + 2e^{-3s}) \end{aligned}$$

4.2-7. $X(s) = e^{-3}e^{-s}\left[\frac{s^2}{(s+1)(s+2)}\right] = e^{-3}e^{-s}\left[1 + \frac{-3s-2}{(s+1)(s+2)}\right] = e^{-3}e^{-s}\left[1 + \frac{1}{(s+1)} + \frac{-4}{(s+2)}\right]$.
Thus,

$$x(t) = e^{-3}\left[\delta(t-1) + e^{-(t-1)}u(t-1) - 4e^{-2(t-1)}u(t-1)\right].$$

4.2-8. First, note that the n^{th} derivative of $\frac{1}{s+a}$ is $\frac{(-1)^n n!}{(s+1)^{n+1}}$. Thus, rewrite the transform as $X(s) = \frac{1}{(s+1)^{13}} = \frac{1}{12!}\frac{12!}{(s+1)^{13}} = \frac{1}{12!}\frac{d^{12}}{ds^{12}}\left(\frac{1}{s+1}\right)$. Since $\sigma > -1$, the time-domain signal $x(t)$ must be right sided. Repeated use of the differentiation in s property provides the resulting inverse transform.

$$x(t) = \frac{1}{12!}(-t)^{12}e^{-t}u(t) = \frac{t^{12}}{12!}e^{-t}u(t).$$

4.2-9. (a) Using the differentiation in s property,

$$\mathcal{L}\left[tx(t)\right] = -\frac{d}{ds}X(s).$$

(b) $y(t) = tx(t) = t\frac{1}{t}u(t) = u(t)$. Thus, $Y(s) = \int_{-\infty}^{\infty} u(t)e^{-st}dt = \int_0^{\infty} e^{-st}dt = \frac{e^{-st}}{-s}\Big|_{t=0}^{\infty}$. For $\sigma > 0$, this simplifies to $Y(s) = \frac{1}{s}$.

(c) Combining the previous two parts yields $-\frac{d}{ds}X(s) = \frac{1}{s}$. Thus,

$$X(s) = -\int \frac{1}{s}ds = -\ln(s).$$

4.3-1. (a)

$$\begin{aligned}
(s^2+3s+2)Y(s) &= s(\frac{1}{s}) \\
Y(s) &= \frac{1}{s^2+3s+2} = \frac{1}{s+1} - \frac{1}{s+2} \\
y(t) &= (e^{-t} - e^{-2t})u(t)
\end{aligned}$$

(b)

$$(s^2Y(s) - 2s - 1) + 4(sY(s) - 2) + 4Y(s) = (s+1)\frac{1}{s+1}$$

or

$$(s^2+4s+4)Y(s) = 2s + 10$$

and

$$\begin{aligned}
Y(s) &= \frac{2s+10}{s^2+4s+4} = \frac{2s+10}{(s+2)^2} = \frac{2}{s+2} + \frac{6}{(s+2)^2} \\
y(t) &= (2+6t)e^{-2t}u(t)
\end{aligned}$$

(c)

$$(s^2Y(s) - s - 1) + 6(sY(s) - 1) + 25Y(s) = (s+2)\frac{25}{s} = 25 + \frac{50}{s}$$

or

$$(s^2+6s+25)Y(s) = s + 32 + \frac{50}{s} = \frac{s^2+32s+50}{s}$$

and

$$\begin{aligned}
Y(s) &= \frac{s^2+32s+50}{s(s^2+6s+25)} = \frac{2}{s} + \frac{-s+20}{s^2+6s+25} \\
y(t) &= [2 + 5.836e^{-3t}\cos(4t - 99.86^\circ)]u(t)
\end{aligned}$$

4.3-2. (a) All initial conditions are zero. The zero-input response is zero. The entire response found in Prob4.3-2a is zero-state response, that is

$$y_{zs}(t) = (e^{-t} - e^{-2t})u(t)$$

$$y_{zi}(t) = 0$$

(b) The Laplace transform of the differential equation is

$$(s^2Y(s) - 2s - 1) + 4(sY(s) - 2) + 4Y(s) = (s+1)\frac{1}{s+1}$$

or

$$(s^2 + 4s + 4)Y(s) - (2s + 9) = 1$$

or

$$(s^2 + 4s + 4)Y(s) = \underbrace{2s+9}_{\text{i.c. terms}} + \underbrace{1}_{\text{input}}$$

$$\begin{aligned}
Y(s) &= \underbrace{\frac{2s+9}{s^2+4s+4}}_{\text{zero-input}} + \underbrace{\frac{1}{s^2+4s+4}}_{\text{zero-state}} \\
&= \underbrace{\frac{2}{s+2} + \frac{5}{(s+2)^2}}_{\text{zero-input}} + \underbrace{\frac{1}{(s+2)^2}}_{\text{zero-state}} \\
y(t) &= \underbrace{(2+5t)e^{-2t}}_{\text{zero-input}} + \underbrace{te^{-2t}}_{\text{zero-state}}
\end{aligned}$$

(c) The Laplace transform of the equation is

$$(s^2Y(s) - s - 1) + 6(sY(s) - 1) + 25Y(s) = 25 + \frac{50}{s}$$

or

$$(s^2 + 6s + 25)Y(s) = \underbrace{s+7}_{\text{i.c. terms}} + \underbrace{25 + \frac{50}{s}}_{\text{input}}$$

$$\begin{aligned}
Y(s) &= \underbrace{\frac{s+7}{s^2+6s+25}}_{\text{zero-input}} + \underbrace{\frac{25s+50}{s(s^2+6s+25)}}_{\text{zero-state}} \\
&= \left(\frac{s+7}{s^2+6s+25}\right) + \left(\frac{2}{s} + \frac{-2s+13}{s^2+6s+25}\right) \\
y(t) &= \underbrace{[\sqrt{2}e^{-3t}\cos(4t - \frac{\pi}{4})]}_{\text{zero-input}} + \underbrace{[2 + 5.154e^{-3t}\cos(4t - 112.83^\circ)]}_{\text{zero-state}}
\end{aligned}$$

4.3-3. (a) Laplace transform of the two equations yields

$$\begin{aligned}
(s+3)Y_1(s) - 2Y_2(s) &= \frac{1}{s} \\
-2Y_1(s) + (2s+4)Y_2(s) &= 0
\end{aligned}$$

Using Cramer's rule, we obtain

$$Y_1(s) = \frac{s+2}{s(s^2+5s+4)} = \frac{s+2}{s(s+1)(s+4)} = \frac{1/2}{s} - \frac{1/3}{s+1} - \frac{1/6}{s+4}$$

$$Y_2(s) = \frac{1}{s(s^2+5s+4)} = \frac{1}{s(s+1)(s+4)} = \frac{1/4}{s} - \frac{1/3}{s+1} + \frac{1/12}{s+4}$$

and

$$\begin{aligned} y_1(t) &= \left(\frac{1}{2} - \frac{1}{3}e^{-t} - \frac{1}{6}e^{-4t}\right)u(t) \\ y_2(t) &= \left(\frac{1}{4} - \frac{1}{3}e^{-t} + \frac{1}{12}e^{-4t}\right)u(t) \end{aligned}$$

If $H_1(s)$ and $H_2(s)$ are the transfer functions relating $y_1(t)$ and $y_2(t)$, respectively to the input $x(t)$, thus

$$H_1(s) = \frac{s+2}{s^2+5s+4} \quad \text{and} \quad H_2(s) = \frac{1}{s^2+5s+4}$$

(b) The Laplace transform of the equations are

$$\begin{aligned} (s+2)Y_1(s) - (s+1)Y_2(s) &= 0 \\ -(s+1)Y_1(s) + (2s+1)Y_2(s) &= 0 \end{aligned}$$

Application of Cramer's rule yields

$$\begin{aligned} Y_1(s) &= \frac{s+1}{s(s^2+3s+1)} = \frac{s+1}{s(s+0.382)(s+2.618)} = \frac{1}{s} - \frac{0.724}{s+0.382} - \frac{0.276}{s+2.618} \\ Y_2(s) &= \frac{s+2}{s(s^2+3s+1)} = \frac{s+2}{s(s+0.382)(s+2.618)} = \frac{2}{s} - \frac{1.894}{s+0.382} - \frac{0.1056}{s+2.618} \end{aligned}$$

$$H_1(s) = \frac{s+1}{s^2+3s+1} \quad \text{and} \quad H_2(s) = \frac{s+2}{s^2+3s+1}$$

$$\begin{aligned} y_1(t) &= (1 - 0.724e^{-0.382t} - 0.276e^{-2.618t})u(t) \\ y_2(t) &= (2 - 1.894e^{-0.382t} - 0.1056e^{-2.618t})u(t) \end{aligned}$$

4.3-4. At $t = 0$, the inductor current $y_1(0) = 4$ and the capacitor voltage is 16 volts. After $t = 0$, the loop equations are

$$\begin{aligned} 2\frac{dy_1}{dt} - 2\frac{dy_2}{dt} + 5y_1(t) - 4y_2(t) &= 40 \\ -2\frac{dy_1}{dt} - 4y_1(t) + 2\frac{dy_2}{dt} + 4y_2(t) + \int_{-\infty}^{t} y_2(\tau)\,d\tau &= 0 \end{aligned}$$

If

$$\begin{aligned} y_1(t) &\Longleftrightarrow Y_1(s), \quad \tfrac{dy_1}{dt} = sY_1(s) - 4 \\ y_2(t) &\Longleftrightarrow Y_2(s), \quad \tfrac{dy_2}{dt} = sY_2(s) \end{aligned}$$

$$\int_{-\infty}^{t} y_2(\tau)\,d\tau \Longleftrightarrow \frac{1}{s}Y_2(s) + \frac{16}{s}$$

Laplace transform of the loop equations are

$$\begin{aligned} 2(sY_1(s) - 4) - 2sY_2(s) + 5Y_1(s) - 4Y_2(s) &= \frac{40}{s} \\ -2(sY_1(s) - 4) - 4Y_1(s) + 2sY_2(s) + 4Y_2(s) + \frac{1}{s}Y_2(s) + \frac{16}{s} &= 0 \end{aligned}$$

Or

$$\begin{aligned}(2s+5)Y_1(s) - (2s+4)Y_2(s) &= 8 + \frac{40}{s} \\ -(2s+4)Y_1(s) + (2s+4+\frac{1}{s})Y_2(s) &= -8 - \frac{16}{s}\end{aligned}$$

Cramer's rule yields

$$\begin{aligned}Y_1(s) &= \frac{4(6s^2+13s+5)}{s(s^2+3s+2.5)} = \frac{8}{s} + \frac{16s+28}{s^2+3s+2.5} \\ y_1(t) &= [8 + 17.89e^{-1.5t}\cos(\frac{t}{2} - 26.56°)]u(t) \\ Y_2(s) &= \frac{20(s+2)}{(s^2+3s+2.5)} \\ y_2(t) &= 20\sqrt{2}e^{-1.5t}\cos(\frac{t}{2} - \frac{\pi}{4})u(t)\end{aligned}$$

4.3-5. (a) $\frac{5s+3}{s^2+11s+24}$

(b) $\frac{3s^2+7s+5}{s^3+6s^2-11s+6}$

(c) $\frac{3s+2}{s(s^3+4)}$

(d) $\frac{1}{s+1}$

4.3-6. (a) $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 8y(t) = \frac{dx}{dt} + 5x(t)$

(b) $\frac{d^3y}{dt^3} + 8\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 7y(t) = \frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 5x(t)$

(c) $\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 5y(t) = 5\frac{d^2x}{dt^2} + 7\frac{dx}{dt} + 2x(t)$

4.3-7. (a) (i) $X(s) = \frac{10}{s}$

$$\begin{aligned}Y(s) &= \frac{10(2s+3)}{s(s^2+2s+5)} = \frac{6}{s} + \frac{-6s+8}{s^2+2s+5} \\ y(t) &= [6 + 9.22e^{-t}\cos(2t - 130.6°)]u(t)\end{aligned}$$

(ii) $x(t) = u(t-5)$ and $X(s) = \frac{1}{s}e^{-5s}$

$$\begin{aligned}Y(s) &= \frac{2s+3}{s(s^2+2s+5)}e^{-5s} = \left[\frac{0.6}{s} + \frac{1}{10}\left(\frac{-6s+8}{s^2+2s+5}\right)\right]e^{-5s} \\ y(t) &= \frac{1}{10}\{6 + 9.22e^{-(t-5)}\cos[2(t-5) - 130.6°]\}u(t-5)\end{aligned}$$

(b) $\ddot{y}(t) + 2\dot{y}(t) + 5y(t) = 2\dot{x}(t) + 3x(t)$

4.3-8. (a) $X(s) = \frac{1}{s(s+1)}$

$$Y(s) = \frac{1}{(s+1)(s^2+9)} = \frac{0.1}{s+1} + \frac{re^{j\theta}}{s+j3} + \frac{re^{-j\theta}}{s-j3} \qquad r = \frac{1}{3\sqrt{10}},\ \theta = -161.56°$$

$$y(t) = 0.1e^{-t} + \frac{1}{3\sqrt{10}}\cos(3t - 161.56°)$$

(b)

$$\ddot{y}(t) + 9y(t) = \dot{x}(t)$$

4.3-9. (a) (i) $X(s) = \frac{1}{s+3}$ and

$$\begin{aligned} Y(s) &= \frac{s+5}{(s+3)(s^2+5s+6)} = \frac{s+5}{(s+2)(s+3)^2} = \frac{3}{s+2} - \frac{3}{s+3} - \frac{2}{(s-3)^2} \\ y(t) &= (3e^{-2t} - 3e^{-3t} - 2te^{-3t})u(t) \end{aligned}$$

(ii) $X(s) = \frac{1}{s+4}$

$$\begin{aligned} Y(s) &= \frac{s+5}{(s+2)(s+3)(s+4)} = \frac{3/2}{s+2} - \frac{2}{s+3} + \frac{1/2}{(s+4)} \\ y(t) &= \frac{3}{2}e^{-2t} - 2e^{-3t} + \frac{1}{2}e^{-4t}u(t) \end{aligned}$$

(iii) The input here is the input in (ii) delayed by 5 secs. Therefore $X(s) = \frac{1}{s+4}e^{-5s}$

$$\begin{aligned} Y(s) &= \frac{s+5}{(s+2)(s+3)(s+4)}e^{-5s} = [\frac{3/2}{s+2} - \frac{2}{s+3} + \frac{1/2}{(s+4)}]e^{-5s} \\ y(t) &= \frac{3}{2}e^{-2(t-5)} - 2e^{-3(t-5)} + \frac{1}{2}e^{-4(t-5)}]u(t-5) \end{aligned}$$

(iv) The input here is equal to the input in (ii) multiplied by e^{20} because $e^{-4(t-5)} = e^{20}e^{-4t}$. Therefore the output is equal to the output in (ii) multiplied by e^{20}.

$$y(t) = e^{20}[\frac{3}{2}e^{-2t} - 2e^{-3t} + \frac{1}{2}e^{-4t}]u(t)$$

(v) The input here is equal to the input in (iii) multiplied by e^{-20} because $e^{-4t}u(t-5) = e^{-20}e^{-4(t-5)}u(t-5)$. Therefore

$$y(t) = e^{-20}[\frac{3}{2}e^{-2(t-5)} - 2e^{-3(t-5)} + \frac{1}{2}e^{-4(t-5)}]u(t-5)$$

(b) $(D^2 + 2D + 5)y(t) = (2D+3)x(t)$

4.3-10. Although this problem can be solved with Laplace transforms, it is easier to solve in the time domain. Since the system step response is $s(t) = e^{-t}u(t) - e^{-2t}u(t)$, the system impulse response is $h(t) = \frac{d}{dt}s(t) = -e^{-t}u(t) + \delta(t) + 2e^{-2t}u(t) - \delta(t) = (2e^{-2t} - e^{-t})u(t)$. The input $x(t) = \delta(t-\pi) - \cos(\sqrt{3})u(t)$ is just a sum of a shifted delta function and a scaled step function. Since the system is LTI, the output is quickly computed using just $h(t)$ and $s(t)$. That is,

$$y(t) = h(t-\pi) - \cos(\sqrt{3})s(t) = (2e^{-2(t-\pi)} - e^{-(t-\pi)})u(t-\pi) - \cos(\sqrt{3})(e^{-t} - e^{-2t})u(t).$$

4.3-11. (a) Let $H(s)$ be the system transfer function.

$$Y(s) = X(s)H(s)$$

Consider an input $x_1(t) = \dot{x}(t)$. Then $X_1(s) = sX(s)$. If the output is $y_1(t)$ and its transform is $Y_1(s)$, then

$$Y_1(s) = X_1(s)H(s) = sX(s)H(s) = sY(s)$$

This shows that $y_1(t) = dy/dt$.

(b) Using similar argument we show that for the input $\int_0^t x(\tau)\,d\tau$, the output is $\int_0^t y(\tau)\,d\tau$. Because $u(t)$ is an integral of $\delta(t)$, the unit step response is the integral of the unit impulse response $h(t)$.

4.3-12. (a) (i)

$$H(s) = \frac{s+5}{s^2+3s+2} = \frac{s+5}{(s+1)(s+2)}$$

Both characteristic roots, -1 and -2 are in LHP. Hence the system is BIBO (and asymptotically) stable.

(ii)

$$H(s) = \frac{s+5}{s^2(s+2)}$$

The characteristic roots are 0, 0, -2. There are repeated roots on imaginary axis. Hence the system is BIBO (and asymptotically) unstable.

(iii)

$$H(s) = \frac{s(s+2)}{s+5}$$

Although the characteristic root -5 is in LHP, because $M > N$, the system is BIBO unstable.

(iv)

$$H(s) = \frac{s+5}{s(s+2)}$$

The roots are 0, and -2. One of the roots is on the imaginary axis which makes the system BIBO unstable (but marginally stable).

(v)

$$H(s) = \frac{s+5}{s^2-2s-3} = \frac{s+5}{(s-3)(s+1)}$$

The roots are 3 and -1. One root in RHP makes system BIBO unstable (and also asymptotically unstable).

(b) (i)

$$(D^2+3D+2)y(t) = (D+3)x(t)$$

or

$$(D+1)(D+2)y(t) = (D+3)x(t)$$

The system transfer function is

$$H(s) = \frac{s+3}{(s+1)(s+2)}$$

The characteristic roots are -1 and -2 (both in LHP). Hence the system is asymptotically and BIBO stable.

(ii)

$$(D^2+3D+2)y(t)=(D+1)x(t)$$

or
$$(D+1)(D+2)y(t)=(D+1)x(t)$$

The system transfer function is

$$H(s)=\frac{s+1}{(s+1)(s+2)}=\frac{1}{s+2}$$

The characteristic roots are -1 and -2 (both in LHP). The only pole of H(s) is at -2. Hence the system is asymptotically and BIBO stable.

(iii)

$$(D^2+D-2)y(t)=(D-1)x(t)$$

or
$$(D-1)(D+2)y(t)=(D-1)x(t)$$

The system transfer function is

$$H(s)=\frac{s-1}{(s-1)(s+2)}=\frac{1}{s+2}$$

The system's characteristic roots at 1 and -2 makes system asymptotically unstable. But the only pole of H(s) is at -2, which makes system BIBO stable.

(iv)

$$(D^2-3D+2)y(t)=(D-1)x(t)$$

or
$$(D-1)(D-2)y(t)=(D-1)x(t)$$

The system transfer function is

$$H(s)=\frac{s-1}{(s-1)(s-2)}=\frac{1}{s-2}$$

The characteristic roots are 1 and 2. The only pole of H(s) is at 2. Hence the system is asymptotically and BIBO unstable.

4.4-1. Figure S4.4-1 shows the transformed network. The loop equations are

$$\begin{aligned}(1+\frac{1}{s})Y_1(s)-\frac{1}{s}Y_2(s) &= \frac{1}{(s+1)^2}\\ -\frac{1}{s}Y_1(s)+(s+1+\frac{1}{s})Y_2(s) &= 0\end{aligned}$$

$$\begin{bmatrix}\frac{s+1}{s} & -\frac{1}{s}\\ -\frac{1}{s} & \frac{s^2+s+1}{s}\end{bmatrix}\begin{bmatrix}Y_1(s)\\ Y_2(s)\end{bmatrix}=\begin{bmatrix}\frac{1}{(s+1)^2}\\ 0\end{bmatrix}$$

Cramer's rule yields

$$Y_2(s) = \frac{1}{(s+1)^2(s^2+2s+2)}=\frac{1}{(s+1)^2}-\frac{1}{s^2+2s+2}$$

$$v_0(t) = y_2(t) = (te^{-t} - \frac{1}{2}e^{-t}\sin t)u(t)$$

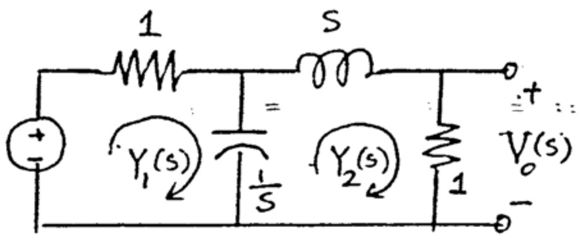


Figure S4.4-1

4.4-2. Before the switch is opened, the inductor current is 5A, that is $y(0) = 5$. Figure S4.4-2b shows the transformed circuit for $t \geq 0$ with initial condition generator. The current $Y(s)$ is given by

$$Y(s) = \frac{(10/s)+5}{3s+2} = \frac{5s+10}{s(3s+2)} = \frac{5}{3}[\frac{3}{s} - \frac{2}{s+(2/3)}]$$
$$y(t) = (5 - \frac{10}{3}e^{-2t/3})u(t)$$

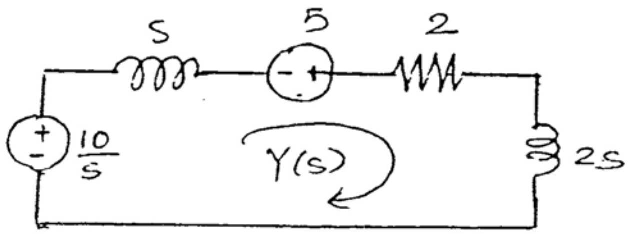


Figure S4.4-2

4.4-3. The impedance seen by the source $x(t)$ is

$$Z(s) = \frac{Ls(1/Cs)}{Ls+(1/Cs)} = \frac{Ls}{LCs^2+1} = \frac{Ls{\omega_0}^2}{s^2+{\omega_0}^2}$$

The current $Y(s)$ is given by

$$Y(s) = \frac{X(s)}{Z(s)} = \frac{s^2+{\omega_0}^2}{Ls{\omega_0}^2}X(s)$$

(a)

$$X(s) = \frac{As}{s^2+{\omega_0}^2}, \qquad Y(s) = \frac{A}{L{\omega_0}^2} \quad \text{and} \quad y(t) = \frac{A}{L{\omega_0}^2}\delta(t)$$

(b)

$$X(s) = \frac{A\omega_0}{s^2+{\omega_0}^2}, \qquad Y(s) = \frac{A}{L\omega_0 s} \quad \text{and} \quad y(t) = \frac{A}{L\omega_0}u(t)$$

4.4-4. At $t = 0$, the steady-state values of currents y_1 and y_2 is $y_1(0) = 2$, $y_2(0) = 1$. Figure S4.4-4 shows the transformed circuit for $t \geq 0$ with initial condition generators. The loop equations are

$$\begin{aligned}(s+2)Y_1(s) - Y_2(s) &= 2 + \frac{6}{s} \\ -Y_1(s) + (s+2)Y_2(s) &= 1\end{aligned}$$

Cramer's rule yields

$$\begin{aligned}Y_1(s) &= \frac{2s^2 + 11s + 12}{s(s+1)(s+3)} = \frac{4}{s} - \frac{3/2}{s+1} - \frac{1/2}{s+3} \\ Y_2(s) &= \frac{s^2 + 4s + 6}{s(s+1)(s+3)} = \frac{2}{s} - \frac{3/2}{s+1} + \frac{1/2}{s+3}\end{aligned}$$

$$\begin{aligned}y_1(t) &= (4 - \frac{3}{2}e^{-t} - \frac{1}{2}e^{-3t})u(t) \\ y_2(t) &= (2 - \frac{3}{2}e^{-t} + \frac{1}{2}e^{-3t})u(t)\end{aligned}$$

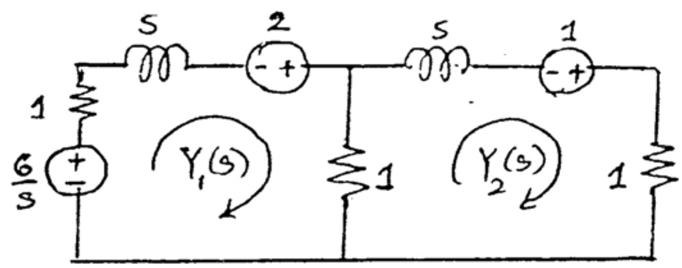


Figure S4.4-4

4.4-5. The current in the 2H inductor at $t = 0$ is 10A. The transformed circuit with initial condition generators is shown in Figure S4.4-5 for $t \geq 0$.

$$Y_1(s) = \frac{\frac{10}{s} + 20}{3s + \frac{1}{s} + 1} = \frac{20s + 10}{3s^2 + s + 1} = \frac{20}{s}[\frac{s + 0.5}{s^2 + \frac{1}{3s} + \frac{1}{3}}]$$

Here $A = 1$, $B = 0.5$, $a = \frac{1}{6}$, $c = \frac{1}{3}$, $b = \frac{\sqrt{11}}{6}$

$$r = \sqrt{\frac{15}{11}} = 1.168 \qquad \theta = \tan^{-1}(\frac{-2}{\sqrt{11}}) = -31.1°$$

$$\begin{aligned}y_1(t) &= \frac{20}{3}(1.168)e^{-t/6}\cos(\frac{\sqrt{11}}{6}t - 31.1°)u(t) \\ &= 7.787e^{-t/6}\cos(\frac{\sqrt{11}}{6}t - 31.1°)u(t)\end{aligned}$$

The voltage $v_s(t)$ across the switch is

$$
\begin{aligned}
V_s(s) &= (s+\frac{1}{s})Y(s) = (\frac{s^2+1}{s})(\frac{20s+10}{3s^2+s+1}) = \frac{20}{s}\frac{(s^2+1)(s+0.5)}{s(s^2+\frac{1}{3s}+\frac{1}{3})} \\
&= \frac{20}{3}\left[1+\frac{3/2}{s}+\frac{1}{6}\frac{-8s+1}{s^2+1/3s+1/3}\right] \\
v_s(t) &= \frac{20}{3}\delta(t)+[10+9.045e^{-t/6}\cos(\frac{\sqrt{11}}{6}t-152.2°)]u(t)
\end{aligned}
$$

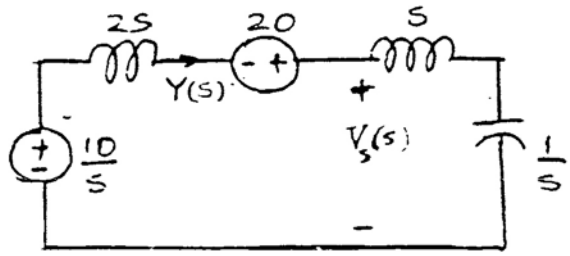


Figure S4.4-5

4.4-6. Figure S4.4-6 shows the transformed circuit with mutually coupled inductor replaced by their equivalents (see Figure 4.14b). The loop equations are

$$
\begin{aligned}
(s+1)Y_1(s) - 2sY_2(s) &= \frac{100}{s} \\
-2sY_1(s) + (4s+1)Y_2(s) &= 0
\end{aligned}
$$

Cramer's rule yields

$$Y_2(s) = \frac{40}{(s+0.2)}$$

and

$$v_0(t) = y_2(t) = 40e^{-t/5}u(t)$$

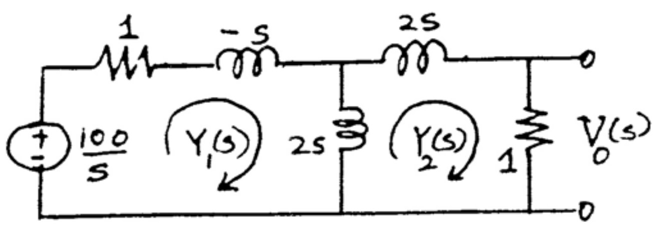


Figure S4.4-6

4.4-7. Figure S4.4-7 shows the transformed circuit with parallel form of initial condition generators. The admittance $W(s)$ seen by the source is

$$W(s) = \frac{13}{s} + s + 4 = \frac{s^2+4s+13}{s}$$

The voltage across terminals a b is

$$V_{ab}(s) = \frac{I(s)}{W(s)} = \frac{\frac{1}{s}+3}{\frac{s^2+4s+13}{s}} = \frac{3s+1}{s^2+4s+13}$$

Also

$$V_0(s) = \frac{1}{2}V_{ab}(s) = \frac{3s+1}{2(s^2+4s+13)}$$

and $$v_0(t) = 1.716e^{-2t}\cos(3t+29°)u(t)$$

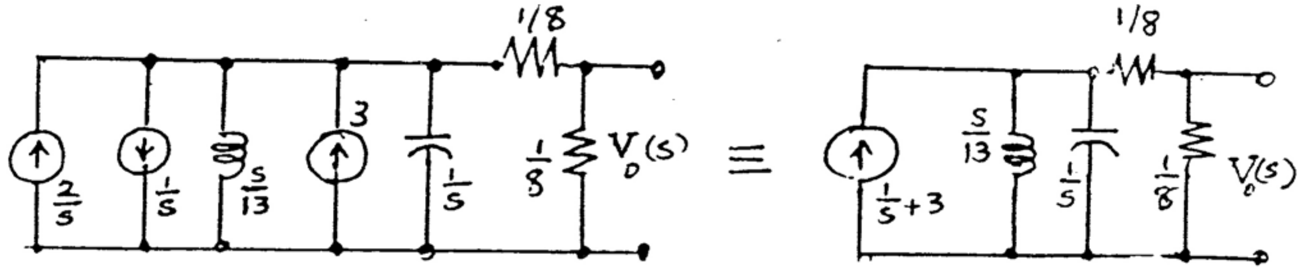


Figure S4.4-7

4.4-8. The capacitor voltage at $t = 0$ is 10 volts. The inductor current is zero. The transformed circuit with initial condition generators is shown for $t > 0$ in Figure S4.4-8. To determine the current $Y(s)$, we determine $Z_{ab}(s)$, the impedance seen across terminals ab:

$$Z_{ab}(s) = \frac{1}{1 + \left(\frac{1}{2+\frac{s+2}{s+3}}\right)} = \frac{3s+8}{4s+11}$$

Also
$$\begin{aligned} Y(s) &= \frac{\frac{90}{s}}{\frac{5}{s} + \left(\frac{3s+8}{4s+11}\right)} \\ &= \frac{90(4s+11)}{3s^2+28s+55} \\ &= \frac{30(4s+11)}{s^2+\frac{28}{3}s+\frac{55}{3}} \\ &= \frac{30(4s+11)}{(s+2.8)(s+6.53)} \\ &= -\frac{1.61}{s+2.8} + \frac{121.61}{s+6.53} \end{aligned}$$

and $$y(t) = [121.61e^{-6.53t} - 1.61e^{-2.8t}]u(t)$$

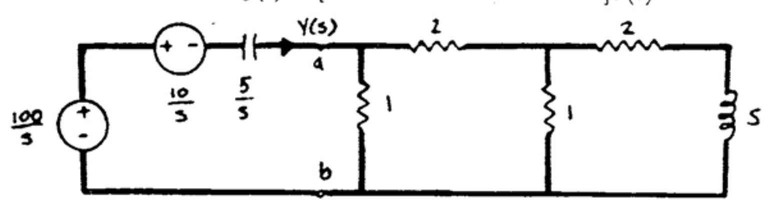


Figure S4.4-8

4.4-9. Figure S4.4-9 shows the transformed circuit (with noninverting op amp replaced by its equivalent as shown in Figure 4.16) from Figure S4.4-9a

$$V_0(s) = KV_1(s) = K\frac{1}{Cs}R + \frac{1}{Cs}X(s) = \frac{Ka}{s+a} \qquad a = \frac{1}{RC}$$

Therefore $$H(s) = \frac{Ka}{s+a} \qquad a = \frac{1}{RC}, \quad K = 1 + \frac{R_b}{R_a}$$

Similarly for the circuit in Figure P4.4-9b, we can show (see Figure S4.4-9)

$$H(s) = \frac{Ks}{s+a}$$

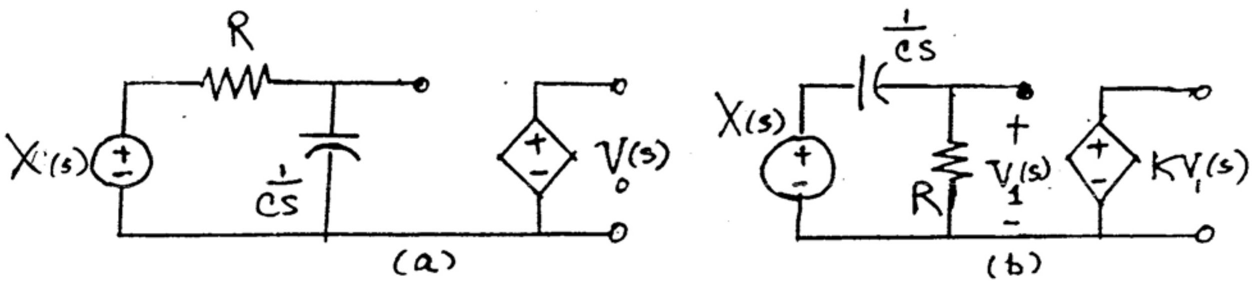


Figure S4.4-9

4.4-10. Figure S4.4-10 shows the transformed circuit. The op amp input voltage is $V_x(s) \simeq 0$. The loop equations are

$$\begin{aligned} I_1(s) + (\frac{6}{s} + 1)[I_1(s) - I_2(s)] &= X(s) \\ -\frac{3}{s} I_2(s) + (\frac{6}{s} + \frac{3}{2})[I_1(s) - I_2(s)] &= 0 \end{aligned}$$

Cramer's rule yields

$$I_1(s) = \frac{s(s+6)}{s^2+8s+12} X(s), \qquad I_2(s) = \frac{s(s+4)}{s^2+8s+12}$$

$$Y(s) = -\frac{1}{2}[I_1(s) - I_2(s)] = \frac{-s}{s^2+8s+12} X(s)$$

The transfer function $$H(s) = \frac{-s}{s^2+8s+12}$$

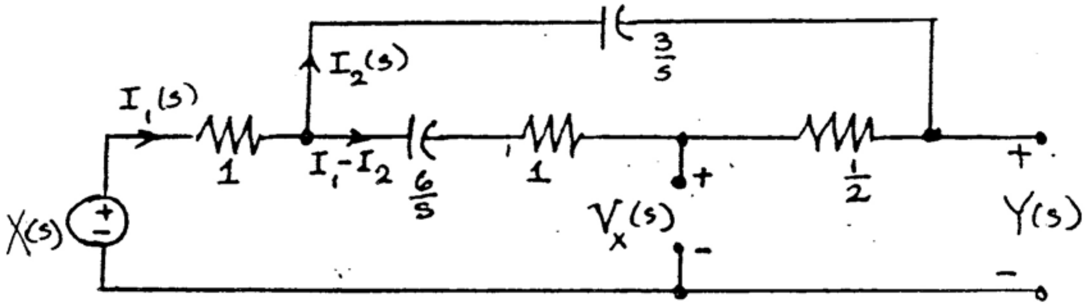


Figure S4.4-10

4.4-11. (a) (i)

$$\begin{aligned} Y(s) &= \frac{6s^2+3s+10}{s(2s^2+6s+5)} \\ y(0^+) &= \lim_{s\to\infty} sY(s) = 3 \end{aligned}$$

$$y(\infty) = \lim_{s\to 0} sY(s) = 2$$

(ii)

$$\begin{aligned}
Y(s) &= \frac{6s^2+3s+10}{(s+1)(2s^2+6s+5)} \\
y(0^+) &= \lim_{s\to\infty} sY(s) = 3 \\
y(\infty) &= \lim_{s\to 0} sY(s) = 0
\end{aligned}$$

(b) (i)

$$Y(s) = \frac{s^2+5s+6}{s^2+3s+2}$$

This Y(s) is not strictly proper. We can express it as

$$Y(s) = 1 + \frac{2s+4}{s^2+3s+2}$$

Hence
$$y(0^+) = \lim_{s\to\infty} \frac{s(2s+4)}{s^2+3s+2} = 2$$

and
$$y(\infty) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} \frac{s^3+5s^2+6s}{s^2+3s+2} = 0$$

(ii)

$$Y(s) = \frac{s^3+4s^2+10s+7}{s^2+2s+3}$$

Because Y(s) is improper, we shall find its strictly proper component.

$$Y(s) = (s+2) + \frac{s+1}{s^2+2s+3}$$

Hence
$$y(0^+) = \lim_{s\to\infty} s\left(\frac{s+1}{s^2+2s+3}\right) = 1$$

$$y(\infty) = \lim_{s\to 0} s\left(\frac{s^3+4s^2+10s+7}{s^2+2s+3}\right) = 0$$

4.5-1. (a) At first glance, we are tempted to answer the question in affirmative. Let us verify the reality.

(b) The loop equations are

$$4I_1 - 2I_2 = X(s)$$

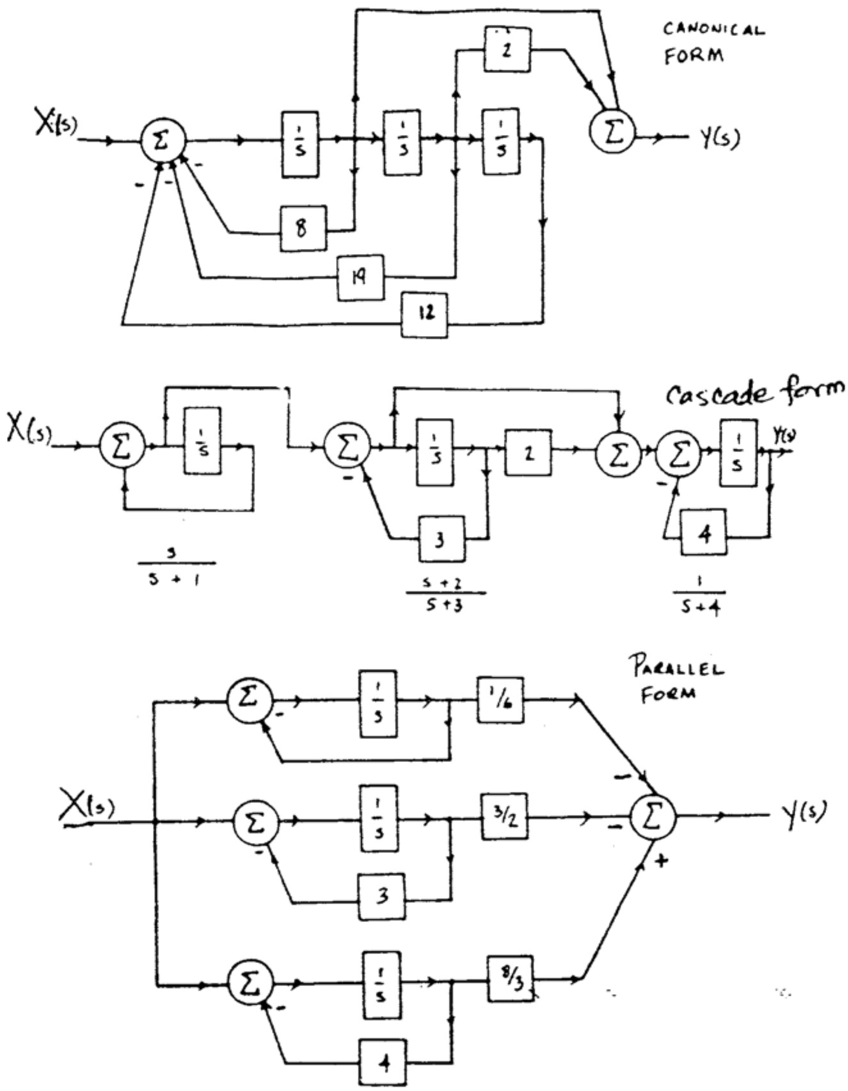

CANONICAL FORM
X(s)
Y(s)
8
19
12
2
Cascade form
3
4
PARALLEL FORM
1/6
3/2
8/3

FIGURE S4.5-1

$$-2I_1 + 4I_2 = 0$$

Cramer's rule yields $$I_2(s) = \frac{1}{6}X(s)$$

and $$Y(s) = I_2(s) = \frac{1}{6}X(s)$$

Therefore $H(s) = \frac{1}{6}$ not $\frac{1}{4}$.

(c) In this case ($R_3 = R_4 = 20000$)

$$\begin{aligned} 4I_1 - 2I_2 &= X(s) \\ -2I_1 + 40002I_2 &= 0 \end{aligned}$$

Cramer's rule yields

$$\begin{aligned} I_2(s) &= \frac{1}{80002}X(s) \\ Y(s) &= 20000I_2(s) = \frac{20000}{80002}X(s) = 0.249994X(s) \end{aligned}$$

In this case $H(s)$ is very close to 1/4. This is because the second ladder section causes a negligible load on the first. Let $R_3 = R_4 = R$. In this case, as $R \to \infty$, we observe that $H(s) \to 1/4$. The second ladder causes no loading in this case. The Cascade rule applies only when the successive subsystems do not load the preceding subsystems.

4.5-2. The transfer function of the two paths are e^{-st} and $ae^{-s(T+\tau)}$. The two paths are in parallel. Hence the transfer function of this communication channel is

$$\begin{aligned} H(s) &= e^{-sT} + ae^{-s(T+\tau)} \\ &= e^{-sT}(1 + ae^{-s\tau}) \end{aligned}$$

For distortionless transmission, it is adequate to undo only the term $(1 + ae^{-s\tau})$ in H(s) because e^{-sT} represents pure delay. Clearly, we need an equalizer with transfer function

$$H_{eq}(s) = \frac{1}{1 + ae^{-sT}}$$

Comparing this form with the transfer function of the feedback system in Eq. (4.59) or Figure 4.18d, it is immediately obvious that such an equalizer can be realized by the following system

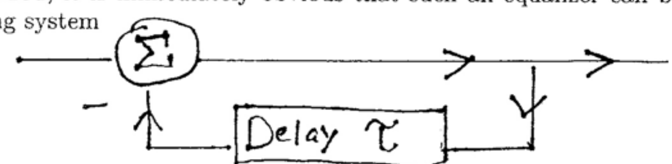
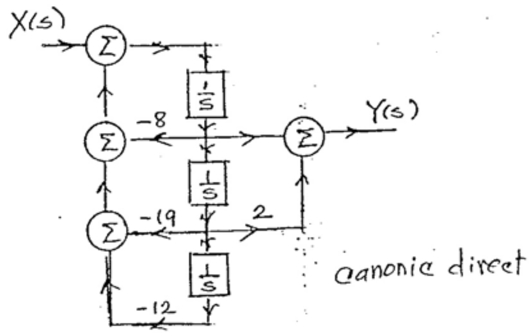


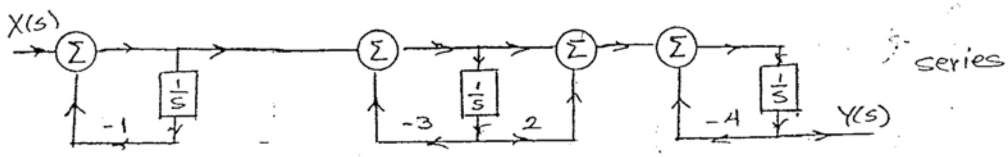
Figure S4.5-2

When this equalizer is placed in cascade with the communication channel, the effective transfer function is given by

$$H_c(s) = \frac{e^{-sT}(1 + ae^{-s\tau})}{1 + ae^{-s\tau}} = e^{-sT}$$

The effective system represents a pure delay of T seconds, which makes it distortionless. Moreover, the equalizer is realizable.


X(s)
-8
Y(s)
-19
2
-12
canonic direct


X(s)
-1
-3
2
-4
Y(s)
series

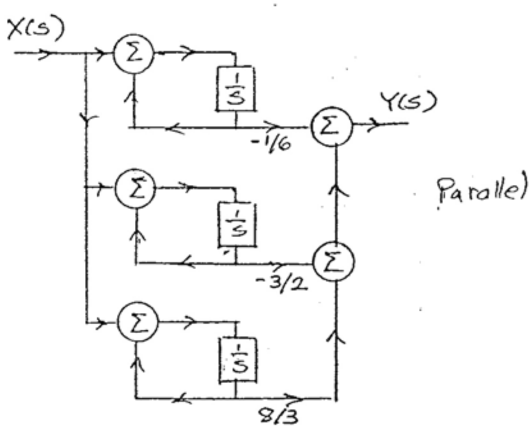

X(s)
Y(s)
-1/6
Parallel
-3/2
8/3

Figure S4.6-1

4.5-3. (a) The system transfer function is

$$H(s) = \frac{\frac{1}{s-1}}{1+\frac{2}{s-1}} = \frac{1}{s+1}$$

The system is BIBO stable.

(b) The system transfer function is

$$H(s) = \frac{\frac{K}{s(s+2)(s+4)}}{1+\frac{K}{s(s+2)(s+4)}} = \frac{1}{s^3+6s^2+8s+K}$$

(i) We can verify that for $K = 10$, all the roots are in LHP and hence the system is BIBO stable.

(ii) For $K = 50$, we can verify that two roots are in RHP and on LHP. Hence the system is BIBO stable.

(iii) For $K = 48$, we verify that two roots are on imaginary axis at $\pm j\sqrt{8}$ and one is in the LHP. Hence the system is BIBO unstable (but marginally stable).

4.6-1.

$$H(s) = \frac{s^2+2s}{s^3+8s^2+19s+12} = \left(\frac{s}{s+1}\right)\left(\frac{s+2}{s+3}\right)\left(\frac{1}{s+4}\right) = \frac{-1/6}{s+1} - \frac{3/2}{s+3} + \frac{8/3}{s+4}$$

$$\text{Also} \qquad H(s) = \frac{b_0s^3+b_1s^2+b_2s+b_3}{s^3+a_1s^2+a_2s+a_3}$$

with $a_3 = 12$, $a_2 = 19$, $a_1 = 8$, and $b_3 = 0$, $b_2 = 2$, $b_1 = 1$, $b_0 = 0$. Figure S4.6-1 shows the canonical, series and parallel realizations.

4.6-2. The transposed version of the realizations for the transfer function in Prob. 4.6-1 are shown in Figure S4.6-2.

4.6-3. (a)

$$\begin{aligned} H(s) &= \frac{3s(s+2)}{(s+1)(s^2+2s+2)} = \frac{3s^2+6s}{s^3+3s^2+4s+2} \\ &= \left(\frac{3s}{s+1}\right)\left(\frac{s+2}{s^2+2s+2}\right) = -\frac{3}{s+1} + \frac{6s+6}{s^2+2s+2} \end{aligned}$$

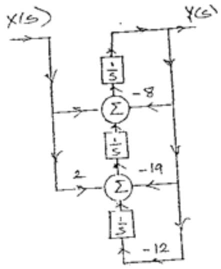
For the canonical form, we have $a_3 = 2$, $a_2 = 4$, $a_1 = 3$, and $b_3 = 0$, $b_2 = 6$, $b_1 = 3$, $b_0 = 0$. Figure S4.6-3a shows a canonical, cascade and parallel realizations.

(b)

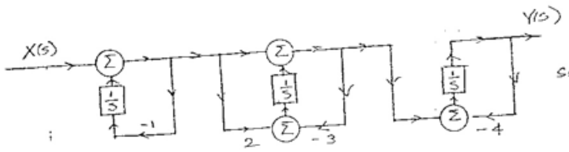
$$\begin{aligned} H(s) &= \frac{2s-4}{(s+2)(s^2+4)} = \frac{2s-4}{s^3+2s^2+4s+8} \\ &= \frac{2(s-2)}{s^3+2s^2+4s+8} = \left(\frac{s-2}{s+2}\right)\left(\frac{2}{s^2+4}\right) = -\frac{1}{s+2} + \frac{s}{s^2+4} \end{aligned}$$

For a canonical forms, we have $a_3 = 8$, $a_2 = 4$, $a_1 = 2$, and $b_3 = -4$, $b_2 = 2$, $b_1 = 0$, $b_0 = 0$. Figure S4.6-3b shows a canonical, cascade and parallel realizations.

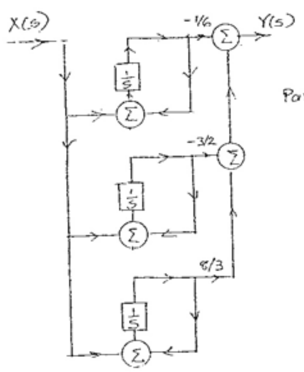
FIG. S4.6-2



Canonic direct

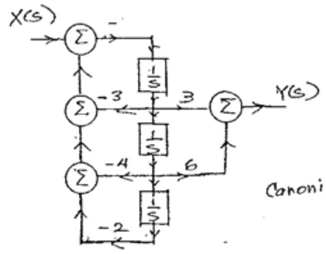


series

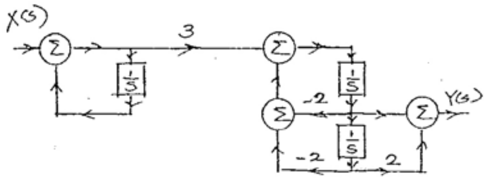


Parallel

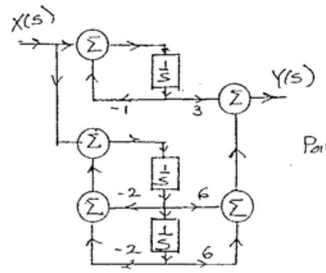
Fig S4.6-3a



Canonic direct

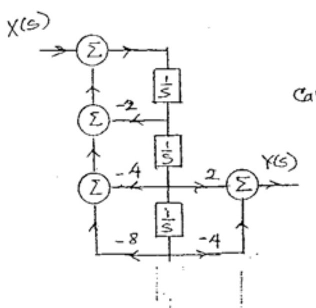


series

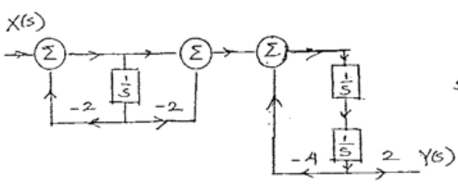


Parallel

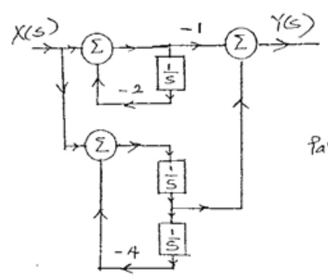
Fig. S4.6-3b



Canonic direct

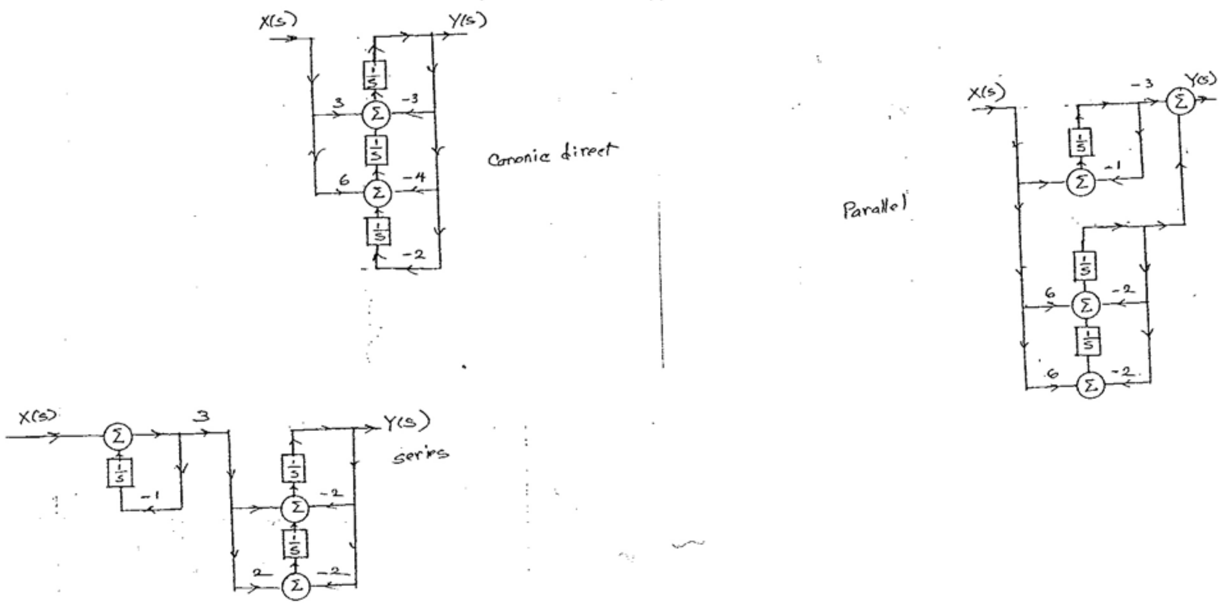


series



Parallel

Fig. S 4.6-4a



4.6-4. The transposed version of the realizations for the transfer function in Prob. 4.6-3 are shown in Figure S4.6-4.

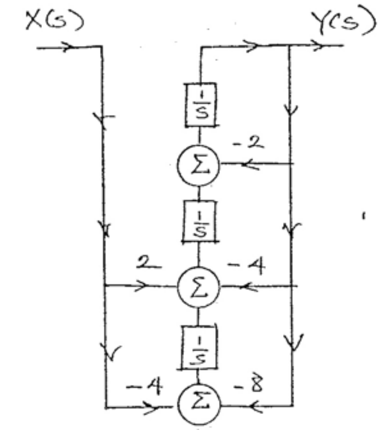
4.6-5.

$$\begin{aligned} H(s) &= 7\frac{2s+3}{5(s^4+7s^3+16s^2+12s)} = \frac{0.4s+0.6}{s^4+7s^3+16s^2+12s} \\ &= \left(\frac{1}{s}\right)\left(\frac{1}{s+2}\right)\left(\frac{1}{s+2}\right)\left(\frac{0.4s+0.6}{s+3}\right) = \frac{\frac{1}{20}}{s} - \frac{\frac{1}{4}}{s+2} + \frac{\frac{1}{10}}{(s+2)^2} + \frac{\frac{1}{5}}{s+3} \end{aligned}$$

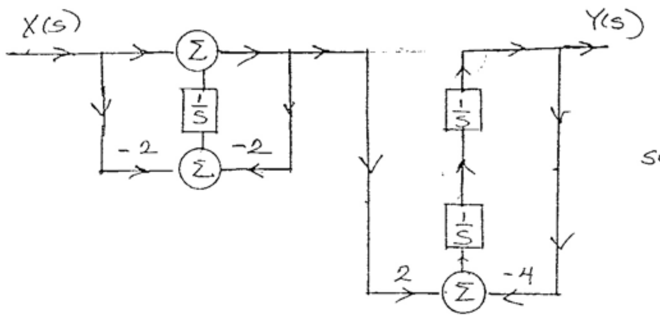
Figure S4.6-5 shows a canonical, cascade and parallel realizations.

4.6-6. The transposed version of the realizations for the transfer function in Prob. 4.6-5 are shown in Figure S4.6-6.

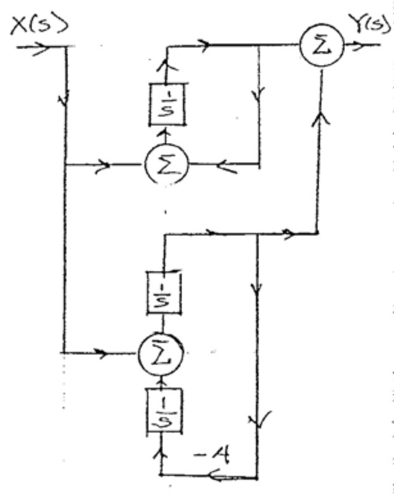
Fig. S 4.6-4b



canonic direct



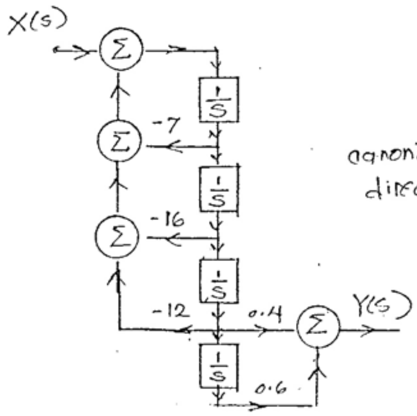
series



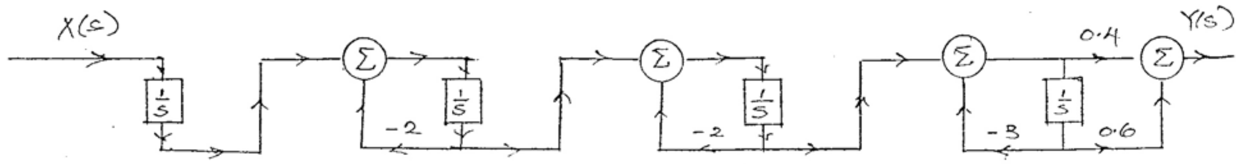
Parallel

FIG. S 4.6-4b

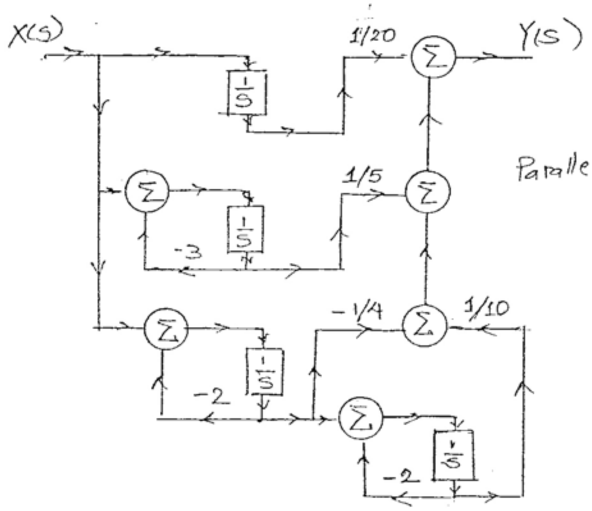
Fig S4.6-5



canonic direct

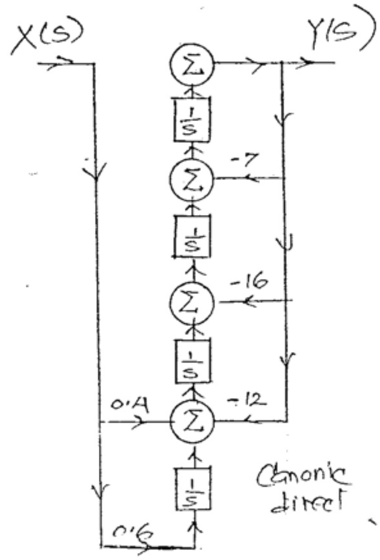


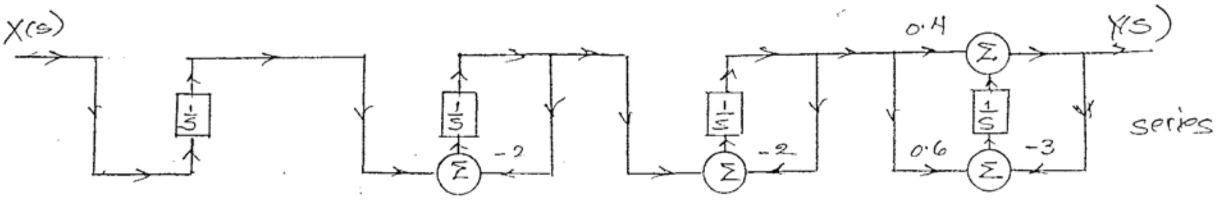
series

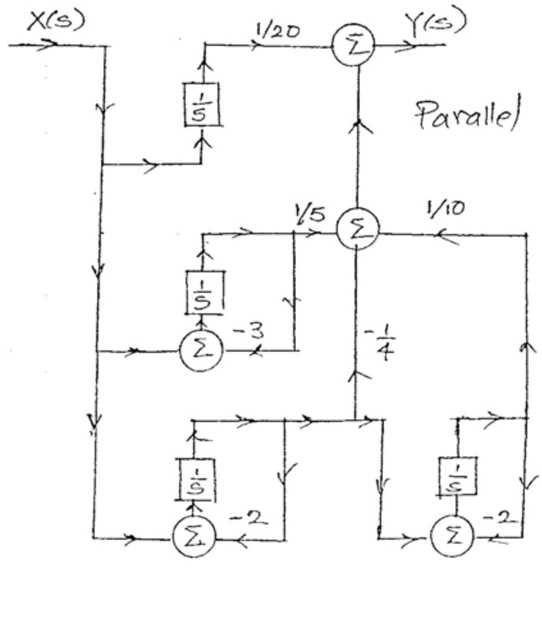


Parallel

Fig. S 4.6-6


X(s)
Y(s)
1/s
-7
-16
-12
0.4
0.6
Canonic direct


X(s)
Y(s)
1/s
-2
-2
-3
0.4
0.6
series


X(s)
Y(s)
1/20
1/5
1/10
1/s
-3
-1/4
-2
-2
Parallel

4.6-7.

$$H(s) = \frac{s(s+1)(s+2)}{(s+5)(s+6)(s+8)} = \frac{s^3+3s^2+2s}{s^3+19s^2+118s+240} = 1 - \frac{20}{s+5} + \frac{60}{s+6} - \frac{56}{s+8}$$

For a canonical form $a_3 = 24$, $a_2 = 118$, $a_1 = 19$, and $b_3 = 0$, $b_2 = 2$, $b_1 = 3$, $b_0 = 1$. Figure S4.6-7 shows a canonical, cascade and parallel realizations.

4.6-8. The transposed version of the realizations for the transfer function in Prob. 4.6-7 are shown in Figure S4.6-8.

4.6-9.

$$\begin{aligned} H(s) &= \frac{s^3}{(s+1)^2(s+2)(s+3)} = \frac{s^3}{s^4+7s^3+17s^2+17s+6} \\ &= \left(\frac{s}{s+1}\right)\left(\frac{s}{s+1}\right)\left(\frac{s}{s+2}\right)\left(\frac{1}{s+3}\right) = -\frac{8}{s+2} + \frac{\frac{27}{4}}{s+3} + \frac{\frac{9}{4}}{s+1} - \frac{\frac{1}{2}}{(s+1)^2} \end{aligned}$$

Figure S4.6-9 shows a canonical, cascade and parallel realizations.

4.6-10. The transposed version of the realizations for the transfer function in Prob. 4.6-9 are shown in Figure S4.6-10.

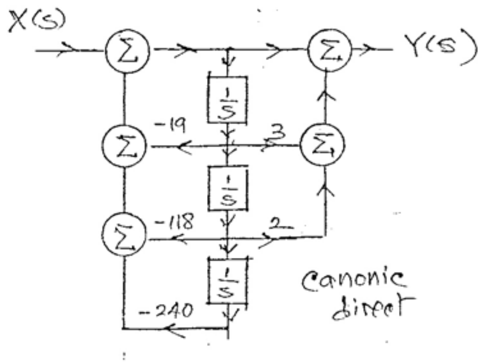
4.6-11.

$$\begin{aligned} H(s) &= \frac{s^3}{(s+1)(s^2+4s+13)} = \frac{s^3}{s^3+5s^2+17s+13} \\ &= \left(\frac{s}{s+1}\right)\left(\frac{s^2}{s^2+4s+13}\right) = -\frac{0.1}{s+1} + \frac{s^2-0.9s+1.3}{s^2+4s+13} = 1 - \frac{0.1}{s+1} - \frac{4.9s+11.7}{s^2+4s+13} \end{aligned}$$

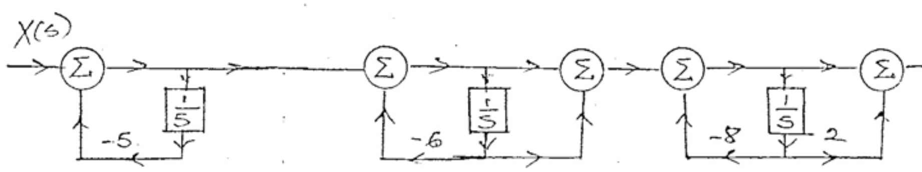
Figure S4.6-11 shows a canonical, cascade and parallel realizations.

4.6-12. The transposed version of the realizations for the transfer function in Prob. 4.6-11 are shown in Figure S4.6-12.

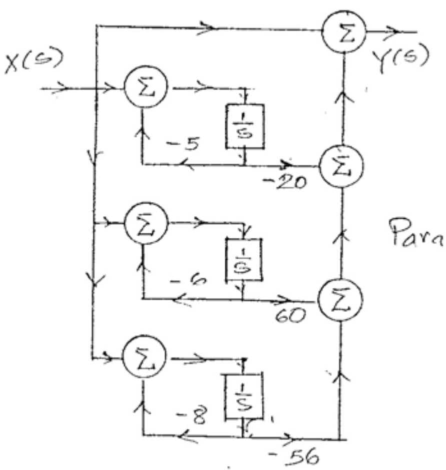
Fig S 4.6-7



canonic direct

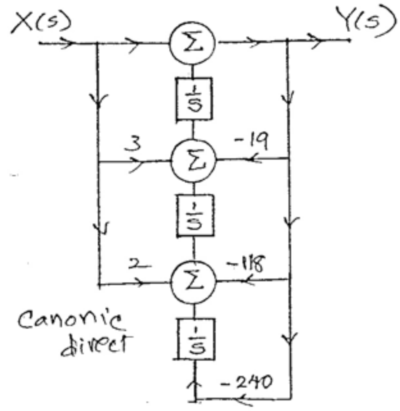


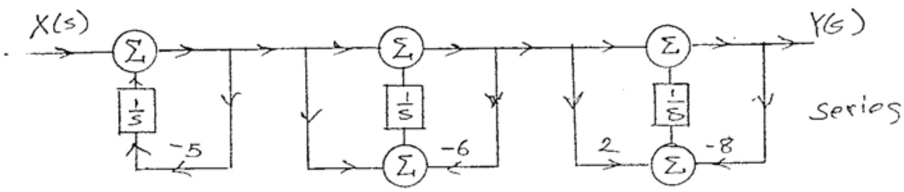
series



Parallel

Fig. S4.6-8.


X(s)
Y(s)
Σ
1/s
3
-19
2
-118
-240
Canonic direct


X(s)
Y(s)
Σ
1/s
-5
-6
2
-8
series

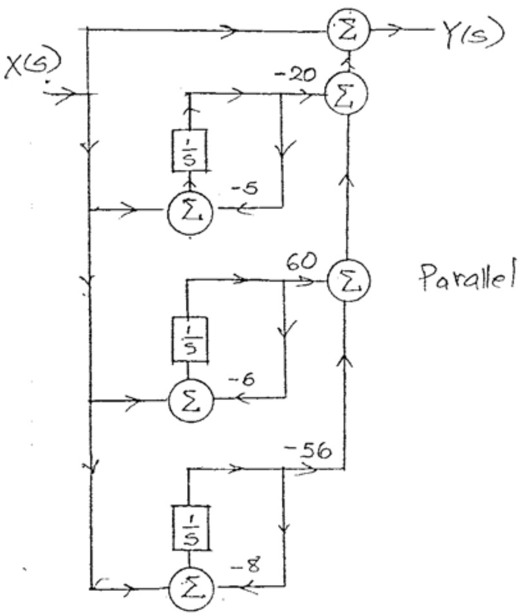

X(s)
Y(s)
Σ
1/s
-20
-5
60
-6
-56
-8
Parallel

Fig. S 4.6-9

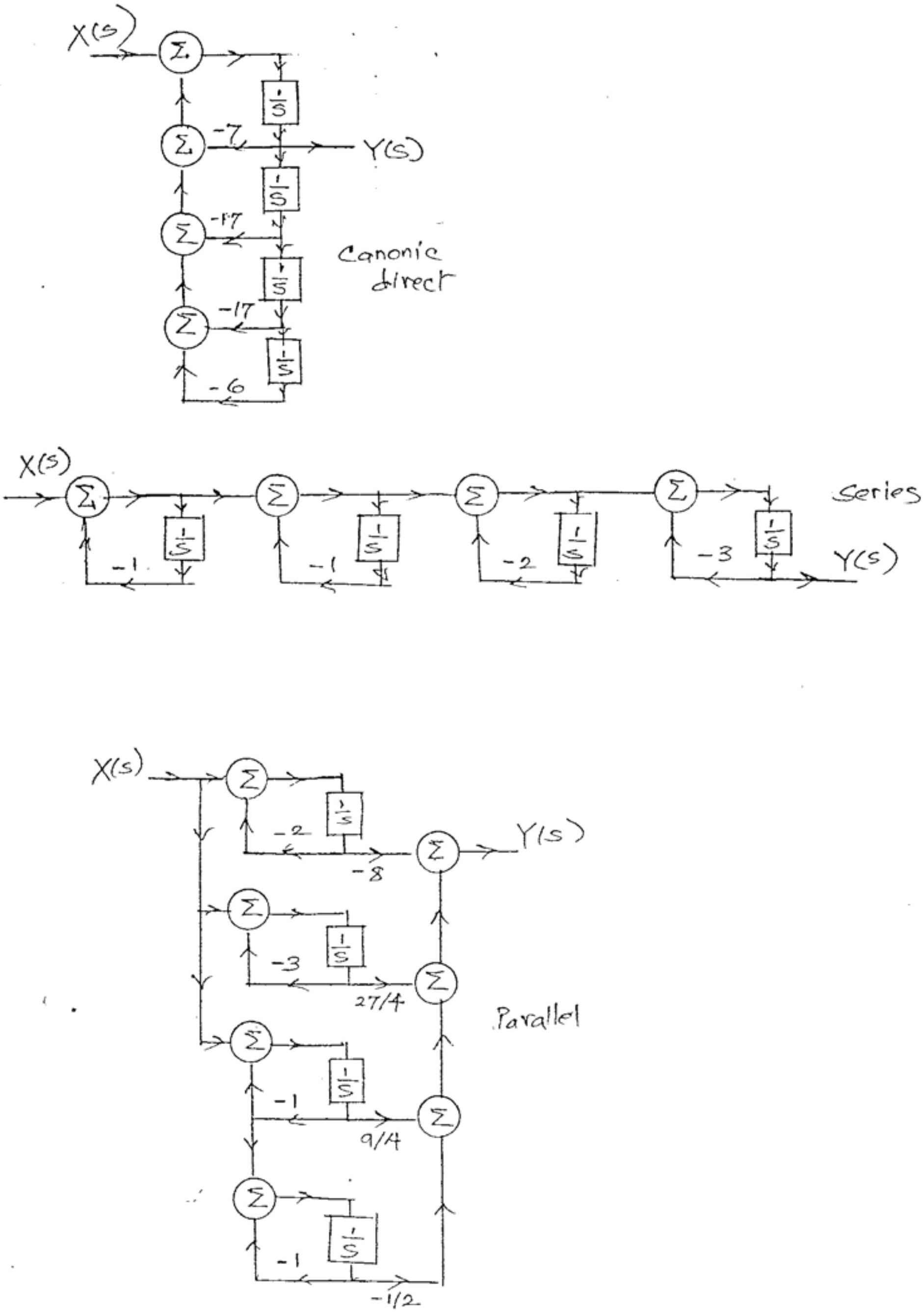
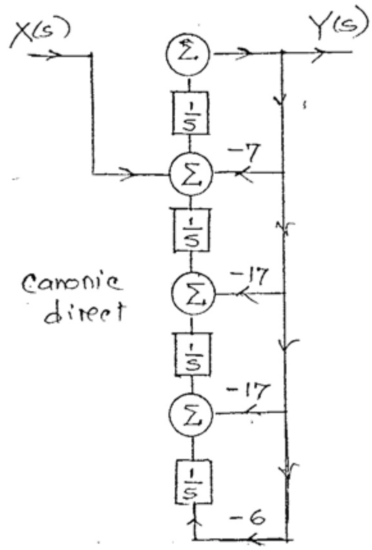
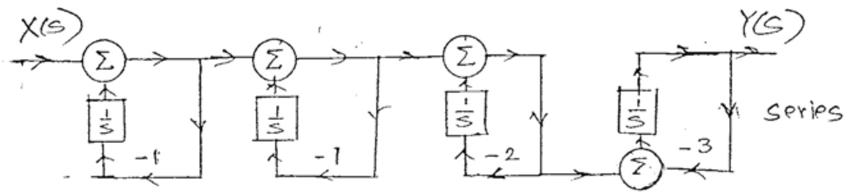

X(s)
Y(S)
-7
-17
-17
-6
Canonic direct
Series
-1
-1
-2
-3
Parallel
-2
-8
-3
27/4
-1
9/4
-1
-1/2

Fig. S 4.6-10


X(s)
Y(s)
1/s
-7
-17
-17
-6
Canonic direct


X(s)
Y(s)
1/s
-1
-1
-2
-3
series

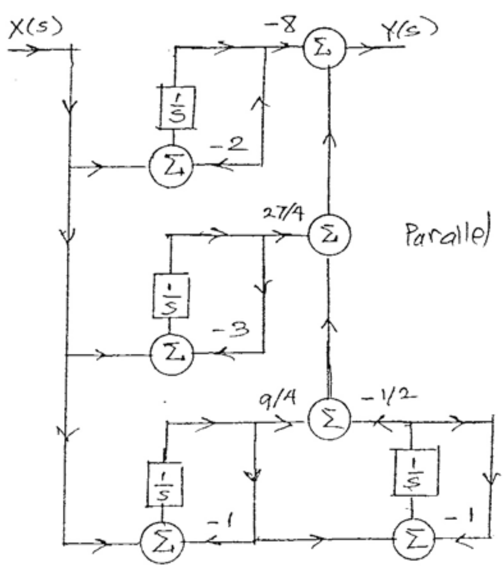
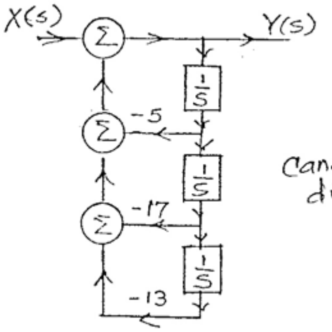
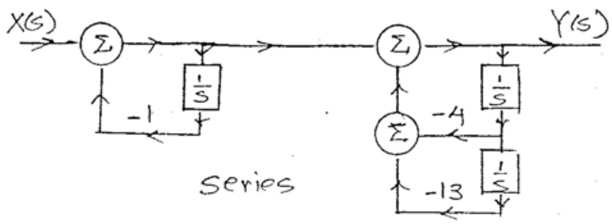

X(s)
Y(s)
-8
1/s
-2
27/4
Parallel
-3
9/4
-1/2
-1
-1

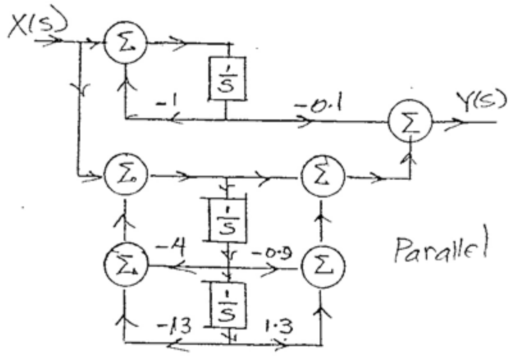
Fig. S4.6-11



canonic direct

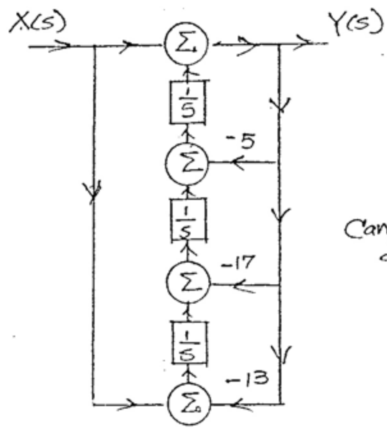


series

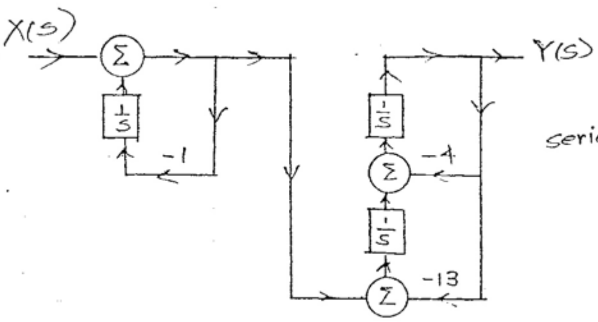


Parallel

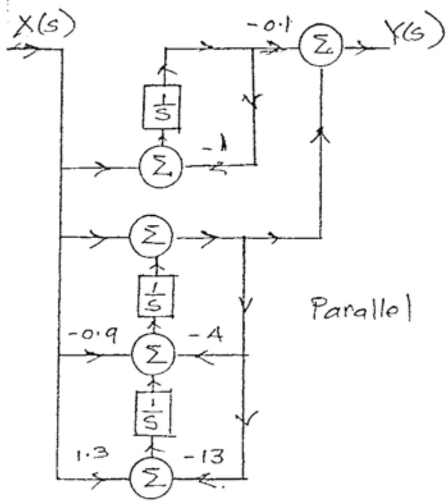
Fig. S 4.6-12



Canonic direct



series



Parallel

4.6-13. Application of Eq. (4.59) to Figure P4.6-13a yields

$$H_1(s) = \frac{\frac{1}{(s+a)^2}}{1 + \frac{b^2}{(s+a)^2}} = \frac{1}{(s+a)^2 + b^2}$$

Figure P4.6-13b is also a feedback loop with forward gain $G(s) = \frac{1}{s+a}$ and the loop gain $\frac{b^2}{(s+a)^2}$. Therefore

$$H_2(s) = \frac{\frac{1}{s+a}}{1 + \frac{b^2}{(s+a)^2}} = \frac{s+a}{(s+a)^2 + b^2}$$

The output in Figure P4.6-13c is the same of $B - aA$ times the output of Figure P4.6-13a and A times the output of Figure P4.6-13b. Therefore its transfer function is

$$\begin{aligned} H(s) &= (B - aA)H_1(s) + AH_2(s) \\ &= \frac{B - aA}{(s+a)^2 + b^2} + \frac{A(s+a)}{(s+a)^2 + b^2} \\ &= \frac{As + B}{(s+a)^2 + b^2} \end{aligned}$$

4.6-14. These transfer functions are readily realized by using the arrangement in Figure 4.28 by a proper choice of $Z_f(s)$ and $Z(s)$.

(i) In Figure S4.6-14a

$$\begin{aligned} Z_f(s) &= \frac{\frac{R_f}{C_f s}}{R_f + \frac{1}{C_f s}} = \frac{1}{C_f(s+a)} \qquad a = \frac{1}{R_f C_f} \\ Z(s) &= R \end{aligned}$$

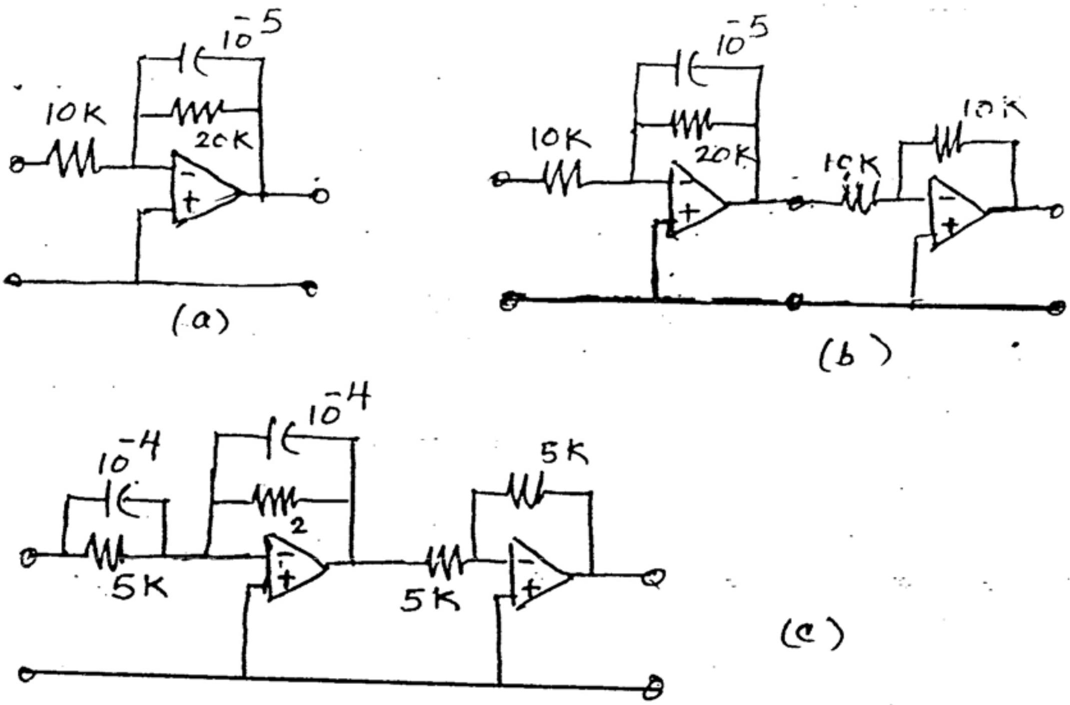


Figure S4.6-14

and
$$H(s) = -\frac{Z_f(s)}{Z(s)} = -\frac{k}{s+a} \qquad k = \frac{1}{RC_f}, \quad a = \frac{1}{R_f C_f}$$

Choose $R = 10,000$, $R_f = 20,000$ and $C_f = 10^{-5}$. This yields $k = 10$ and $a = 5$. Therefore

$$H(s) = \frac{-10}{s+5}$$

(ii) This is same as (i) followed by an amplifier of gain -1 as shown in Figure S4.6-14b.

(iii) For the first stage in Figure S4.6-14c (see Exercise E4.13, Figure 4.32b),

$$Z_f(s) = \frac{1}{C_f(s+a)} \qquad a = \frac{1}{R_f C_f}$$
$$Z(s) = \frac{1}{C(s+b)} \qquad b = \frac{1}{RC}$$

and
$$H(s) = -\frac{Z_f(s)}{Z(s)} = -\frac{C}{C_f}\left(\frac{s+b}{s+a}\right)$$

Choose $C = C_f = 10^{-4}$, $R = 5000$, $R_f = 2000$. This yields

$$H(s) = -\left(\frac{s+2}{s+5}\right)$$

This is followed by an op amp of gain -1 as shown in Figure S4.6-14c. This yields

$$H(s) = \frac{s+2}{s+5}$$

4.6-15. One realization is given in Figure S4.6-14c. For the other realization, we express $H(s)$ as

$$H(s) = \frac{s+2}{s+5} = 1 - \frac{3}{s+5}$$

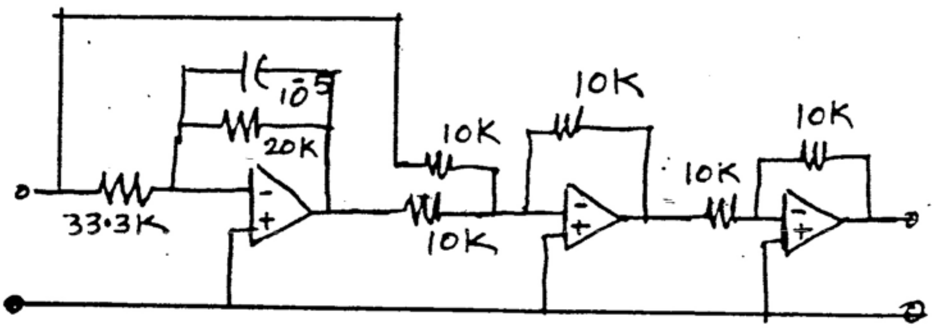


Figure S4.6-15

We realize $H(s)$ as a parallel combination of $H_1(s) = 1$ and $H_2(s) = -3/(s+5)$ as shown in Figure S4.6-15. The second stage serves as a summer for which the inputs are the input and output of the first stage. Because the summer has a gain -1, we need a third stage of gain -1 to obtain the desired transfer functions.

4.6-16. Canonical realization of $H(s)$ is shown in Figure S4.6-16. Observe that this is identical to $H(s)$ in Example 4.22 with a minor difference. Hence the op amp circuit in Figure 4.31c can be used for our purpose with appropriate changes in the element values. The last summer input resistors now are $\frac{100}{3}\,k\Omega$ and $\frac{100}{7}\,k\Omega$ instead of $50\,k\Omega$ and $20\,k\Omega$.

4.6-17. We follow the procedure in Example 4.22 with appropriate modifications. In this case $a_2 = 13$, $a_1 = 4$, and $b_2 = 2$, $b_1 = 5$, and $b_0 = 1$ (in Example 4.22, we have $a_2 = 10$, $a_1 = 4$, and $b_2 = 5$, $b_1 = 2$, and $b_0 = 0$). Because b_0 is nonzero here, we have one more feedforward connection. Figure S4.6-17 shows the development of the suitable realization.

4.7-1. (a) In this case,

$$H(j\omega) = \frac{\omega_c}{j\omega + \omega_c} \quad \Longrightarrow |H(j\omega)| = \frac{\omega_c}{\sqrt{\omega^2 + \omega_c^2}}$$

The dc gain is $H(0) = 1$ and the gain at $\omega = \omega_c$ is $1/\sqrt{2}$, which is -3 dB below the dc gain. Hence, the 3-dB bandwidth is ω_c. Also the dc gain is unity. Hence, the gain-bandwidth product is ω_c.

We could derive this result indirectly as follows. The system is a lowpass filter with a single pole at $\omega = \omega_c$. The dc gain is $H(0) = 1$ (0 dB). Because, there is a single pole at ω_c (and no zeros), there is only one asymptote starting at $\omega = \omega_c$ (at a rate -20 dB/dec.). The break point is ω_c, where there is a correction of -3 dB. Hence, the amplitude response at ω_c is 3 dB below 0 dB (the dc gain). Thus, the 3-dB bandwidth of this filter is ω_c.

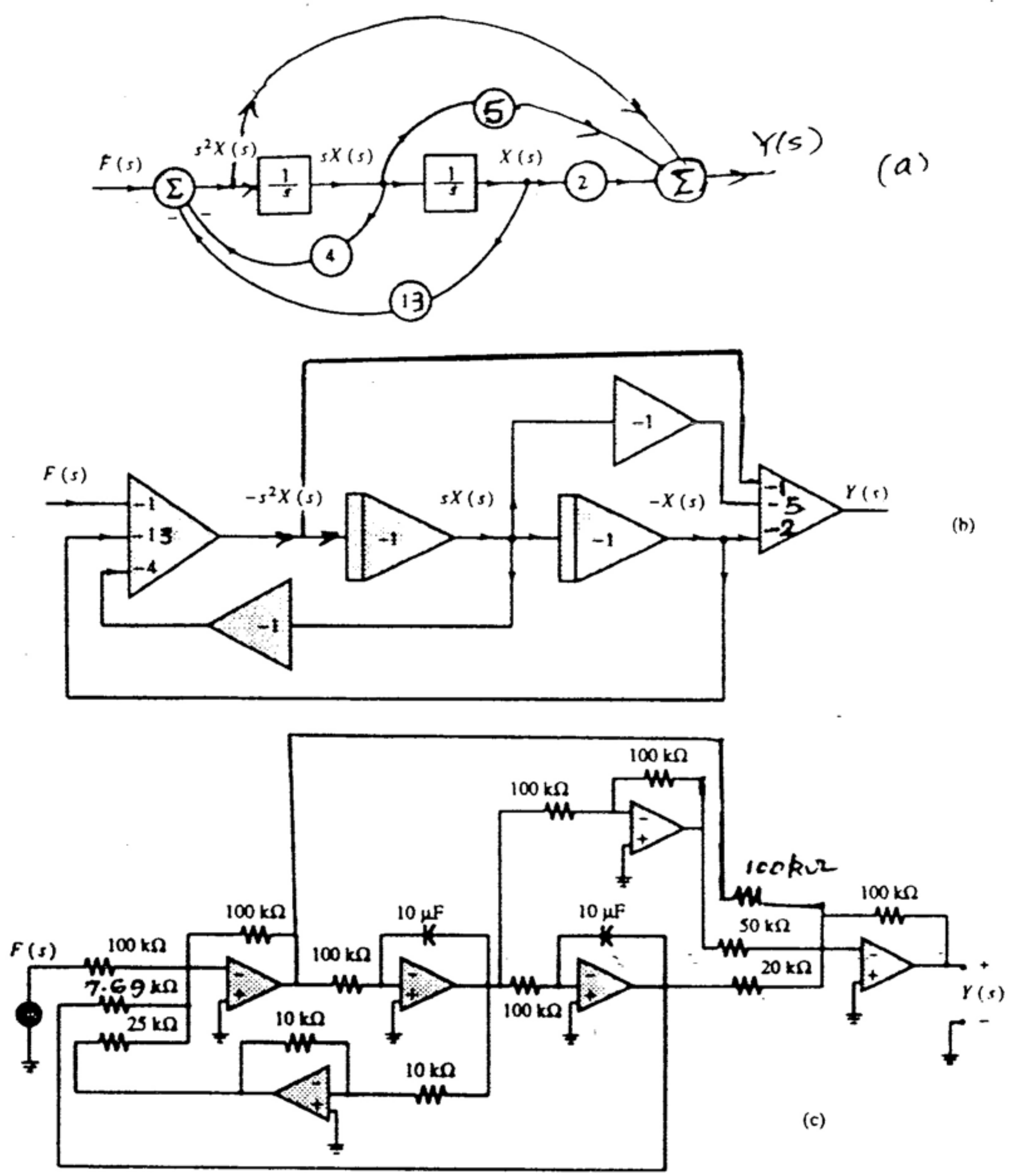


FIGURE S4.6-16

(b) The transfer function of this system is

$$H(s) = \frac{G(s)}{1 + G(s)H(s)} = \frac{\frac{\omega_c}{s+\omega_c}}{1 + \frac{9\omega_c}{s+\omega_c}} = \frac{\omega_c}{s + 10\omega_c}$$

We use the same argument as in part (a) to deduce that the dc gain is 0.1 and the 3-dB bandwidth is $10\omega_c$. Hence, the gain-bandwidth product is ω_c.

(c) The transfer function of this system is

$$H(s) = \frac{G(s)}{1 - G(s)H(s)} = \frac{\frac{\omega_c}{s+\omega_c}}{1 - \frac{0.9\omega_c}{s+\omega_c}} = \frac{\omega_c}{s + 0.1\omega_c}$$

We use the same argument as in part (a) to deduce that the dc gain is 10 and the 3-dB bandwidth is $0.1\omega_c$. Hence, the gain-bandwidth product is ω_c.

(d) Included in previous parts.

4.8-1.

$$\begin{aligned}
H(j\omega) &= \frac{j\omega + 2}{(j\omega)^2 + 5j\omega + 4} = \frac{j\omega + 2}{(4 - \omega^2) + j5\omega} \\
|H(j\omega)| &= \sqrt{\frac{\omega^2 + 4}{(4 - \omega^2)^2 + (5\omega)^2}} = \sqrt{\frac{\omega^2 + 4}{\omega^4 + 17\omega^2 + 16}} \\
\angle H(j\omega) &= \tan^{-1}(\frac{\omega}{2}) - \tan^{-1}(\frac{5\omega}{4 - \omega^2})
\end{aligned}$$

(a) $x(t) = 5\cos(2t + 30°)$. Here $\omega = 2$ and

$$\begin{aligned}
|H(j2)| &= \sqrt{\frac{2}{25}} = \frac{\sqrt{2}}{5} \\
\angle H(j\omega) &= \tan^{-1} - \tan^{-1}(\infty) = 45° - 90° = -45°
\end{aligned}$$

$$y(t) = 5\frac{\sqrt{2}}{5}\cos(2t + 30^\circ - 45^\circ) = \sqrt{2}\cos(2t - 15^\circ)$$

(b) $x(t) = 10\sin(2t + 45^\circ)$

$$y(t) = 10(\frac{\sqrt{2}}{5})\sin(2t + 45^\circ - 45^\circ) = 2\sqrt{2}\sin 2t$$

(c) $x(t) = 10\cos(3t + 40^\circ)$. Here $\omega = 3$

$$|H(j\omega)| = \sqrt{\frac{13}{250}} = 0.228 \quad \text{and} \quad \angle H(j3) = 56.31^\circ - 108.43^\circ = -52.12^\circ$$

Therefore

$$y(t) = 10(0.228)\cos(3t + 40^\circ - 52.12^\circ) = 2.28\cos(3t - 12.12^\circ)$$

4.8-2.

$$\begin{aligned} H(j\omega) &= \frac{j\omega + 3}{(j\omega + 2)^2} \\ |H(j\omega)| &= \frac{\sqrt{\omega^2 + 9}}{\omega^2 + 4} \quad \text{and} \quad \angle H(j\omega) = \tan^{-1}(\frac{\omega}{3}) - \tan^{-1}(\frac{\omega}{2}) \end{aligned}$$

(a) $x(t) = 10u(t) = 10e^{j0t}u(t)$. Here $\omega = 0$ and $H(j0) = 1$. Therefore

$$y(t) = 1 \times 10e^{j0t}u(t) = 10u(t)$$

(b) $x(t) = \cos(2t + 60^\circ)\,u(t)$. Here $\omega = 2$

$$|H(j2)| = \frac{\sqrt{13}}{8} \quad \text{and} \quad \angle H(j2) = 33.69^\circ - 90^\circ = -56.31^\circ$$

Therefore

$$y(t) = \frac{\sqrt{13}}{8}\cos(2t + 60° - 56.31°)u(t) = \frac{\sqrt{13}}{8}\cos(2t + 3.69°)u(t)$$

(c) $x(t) = \sin(3t - 45°)u(t)$ Here $\omega = 3$ and

$$|H(j3)| = \frac{\sqrt{18}}{13} \quad \text{and} \quad \angle H(j3) = 45° - 112.62° = -67.62°$$

Therefore

$$y(t) = \frac{\sqrt{18}}{13}\sin(3t - 45° - 67.62°)u(t) = \frac{\sqrt{18}}{13}\sin(3t - 112.62°)u(t)$$

(d) $x(t) = e^{j3t}u(t)$

$$y(t) = H(j3)e^{j3t} = |H(j3)|e^{j[3t + \angle H(j3)]}u(t) = \frac{\sqrt{18}}{13}e^{j[3t - 67.62°]}u(t)$$

4.8-3.

$$\begin{aligned}
H(j\omega) &= \frac{-(j\omega - 10)}{j\omega + 10} = \frac{10 - j\omega}{10 + j\omega} \\
|H(j\omega)| &= \sqrt{\frac{\omega^2 + 100}{\omega^2 + 100}} = 1 \\
\angle H(j\omega) &= \tan^{-1}(-\frac{\omega}{10}) - \tan^{-1}(\frac{\omega}{10}) = -2\tan^{-1}(\frac{\omega}{10})
\end{aligned}$$

(a) $x(t) = e^{j\omega t}$

$$y(t) = H(j\omega)e^{j\omega t} = |H(j\omega)|e^{j[\omega t + \angle H(j\omega)]} = e^{j[\omega t - 2\tan^{-1}(\omega/10)]}$$

(b) $x(t) = \cos(\omega t + \theta)$

$$y(t) = \cos[\omega t + \theta - 2\tan^{-1}(\frac{\omega}{10})]$$

(c) $x(t) = \cos t$. Here $\omega = 1$

$$\begin{aligned} |H(j1)| &= 1 \\ \angle H(j\omega) &= -2\tan^{-1}(\frac{1}{10}) = -11.42^\circ \\ y(t) &= \cos(t - 11.42^\circ) \end{aligned}$$

(d) $x(t) = \sin 2t$. Here $\omega = 2$

$$\begin{aligned} |H(j2)| &= 1 \\ \angle H(j2) &= -2\tan^{-1}(\frac{2}{10}) = -22.62^\circ \\ y(t) &= \sin(2t - 22.62^\circ) \end{aligned}$$

(e) $x(t) = \cos 10t$. Here $\omega = 10$

$$\begin{aligned} |H(j10)| &= 1 \\ \angle H(j10) &= -2\tan^{-1}(\frac{10}{10}) = -90^\circ \\ y(t) &= \cos(10t - 90^\circ) = \sin 10t \end{aligned}$$

(f) $x(t) = \cos 100t$. Here $\omega = 100$

$$|H(j100)| = 1$$

$$\begin{aligned}\angle H(j100) &= -2\tan^{-1}(\frac{100}{10}) = -168.58^\circ \\ y(t) &= \cos(100t - 168.58^\circ)\end{aligned}$$

4.8-4. (a) From the graph, the two system zeros are at $s = \pm j1.5$. Thus, $s^2 + b_1 s + b_2 = (s+j1.5)(s-j1.5) = s^2+2.25$. The two system poles are at $s = -1 \pm j0.5$. Thus, $s^2 + a_1 s + a_2 = (s+1+j0.5)(s+1-j0.5) = s^2 + 2s + 1.25$. At DC, the system function is $H(j0) = -1 = k\frac{b_2}{a_2} = k\frac{2.25}{1.25} = k\frac{9}{5}$. Therefore,

$$k = -\frac{5}{9}, b_1 = 0, b_2 = \frac{9}{4}, a_1 = 2, \text{ and } a_2 = \frac{5}{4}.$$

(b) The DC gain is given as $S(j0) = -1$. Thus, the input of 4 just becomes -4. To compute the output to $\cos(t/2 + \pi/3)$, $H(j0.5)$ is required. Graphically, $|H(j0.5)| = |k|\frac{(1)(2)}{(1)(\sqrt{2})} = \frac{10}{9\sqrt{2}}$ and $\angle H(j0.5) = \pi - \pi/2 + \pi/2 - (0 + \pi/4) = 3\pi/4$. Thus, the output to $\cos(t/2 + \pi/3$ is just $\frac{10}{9\sqrt{2}}\cos(t/2 + \pi/3 + 3\pi/4)$. Thus, the output to $x(t) = 4 + \cos(t/2 + \pi/3)$ is

$$y(t) = -4 + \frac{10}{9\sqrt{2}} \cos(t/2 + 13\pi/12) \approx -4 + 0.7857 \cos(t/2 + 3.4034).$$

4.9-1. (a) The transfer function can be expressed as

$$H(s) = \frac{100}{2 \times 20} \frac{s(\frac{s}{100} + 1)}{(\frac{s}{2} + 1)(\frac{s}{20} + 1)} = 2.5 \frac{s(\frac{s}{100} + 1)}{(\frac{s}{2} + 1)(\frac{s}{20} + 1)}$$

The amplitude response: The horizontal axis where the asymptotes begin is 2.5, which is 7.96 db. We draw the asymptotes at $\omega = 1$ (20 dB/dec.), 2 (-20 dB/dec.), 20 (-20 dB/dec.), and 100 (20 dB/dec.) as shown in Figure S4.9-1a. The corrections are applied at various points as discussed in Examples 4.25 and 4.26. to obtain the Bode plot for amplitude response. We follow the similar procedure for phase response.

(b) The transfer function can be expressed as

$$H(s) = \frac{10 \times 20}{100} \frac{(\frac{s}{10} + 1)(\frac{s}{20} + 1)}{s^2(\frac{s}{100} + 1)} = 2 \frac{(\frac{s}{10} + 1)(\frac{s}{20} + 1)}{s^2(\frac{s}{100} + 1)}$$

The amplitude response: The horizontal axis where the asymptotes begin is 2, which is 6 db. Asymptotes start at $\omega = 1$ (-40 dB/dec.), 10 (20 dB/dec.), 20 (20 dB/dec.), and 100 (-20 dB/dec.). The corrections are applied at various points as discussed in Examples 4.25 and 4.26. to obtain the Bode plot.

(c) The transfer function can be expressed as

$$H(s) = \frac{10 \times 200}{400 \times 1000} \frac{(\frac{s}{10} + 1)(\frac{s}{200} + 1)}{(\frac{s}{20} + 1)^2(\frac{s}{1000} + 1)} = \frac{1}{200} \frac{(\frac{s}{10} + 1)(\frac{s}{200} + 1)}{(\frac{s}{20} + 1)^2(\frac{s}{1000} + 1)}$$

The amplitude response: The horizontal axis where the asymptotes begin is

FIG. S4.9-1

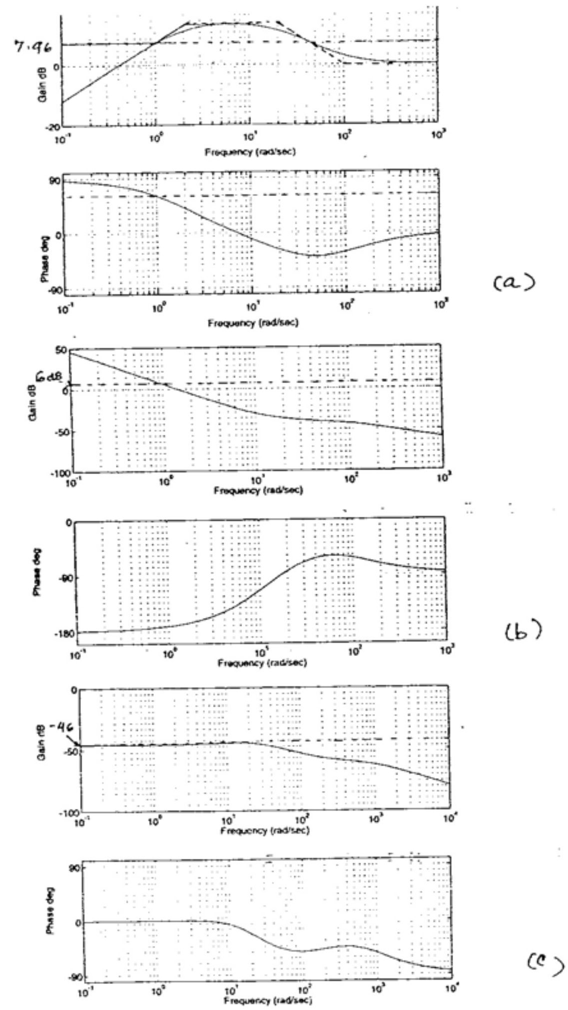


Fig. S4.9-1

1/200, which is -46 db. Asymptotes start at $\omega = 10$ (20 dB/dec.), 20 (-40 dB/dec.), 200 (20 dB/dec.), and 1000 (-20 dB/dec.). The corrections are applied at various points as discussed in Examples 4.25 and 4.26. to obtain the Bode plot.

4.9-2. (a) The transfer function can be expressed as

$$H(s) = \frac{1}{16}\frac{s^2}{(\frac{s}{1}+1)(\frac{s^2}{16}+\frac{s}{4}+1)}$$

The amplitude response: The horizontal axis where the asymptotes begin is 1/16, which is -24 dB. Asymptotes start at $\omega = 1$ (40 dB/dec.), 1 (-20 dB/dec.), 4 (-40 dB/dec.). The corrections are applied at various points as discussed in Examples 4.25 and 4.26. to obtain the Bode plot.

(b) The transfer function can be expressed as

$$H(s) = \frac{1}{100} \frac{s}{(\frac{s}{1}+1)(\frac{s^2}{100}+0.1414s+1)}$$

The amplitude response: The horizontal axis where the asymptotes begin is 1/100, which is -40 dB. Asymptotes start at $\omega = 1$ (20 dB/dec.), 1 (-20 dB/dec.), 10 (-40 dB/dec.). The corrections are applied at various points as discussed in Examples 4.25 and 4.26. to obtain the Bode plot.

(c) The transfer function can be expressed as

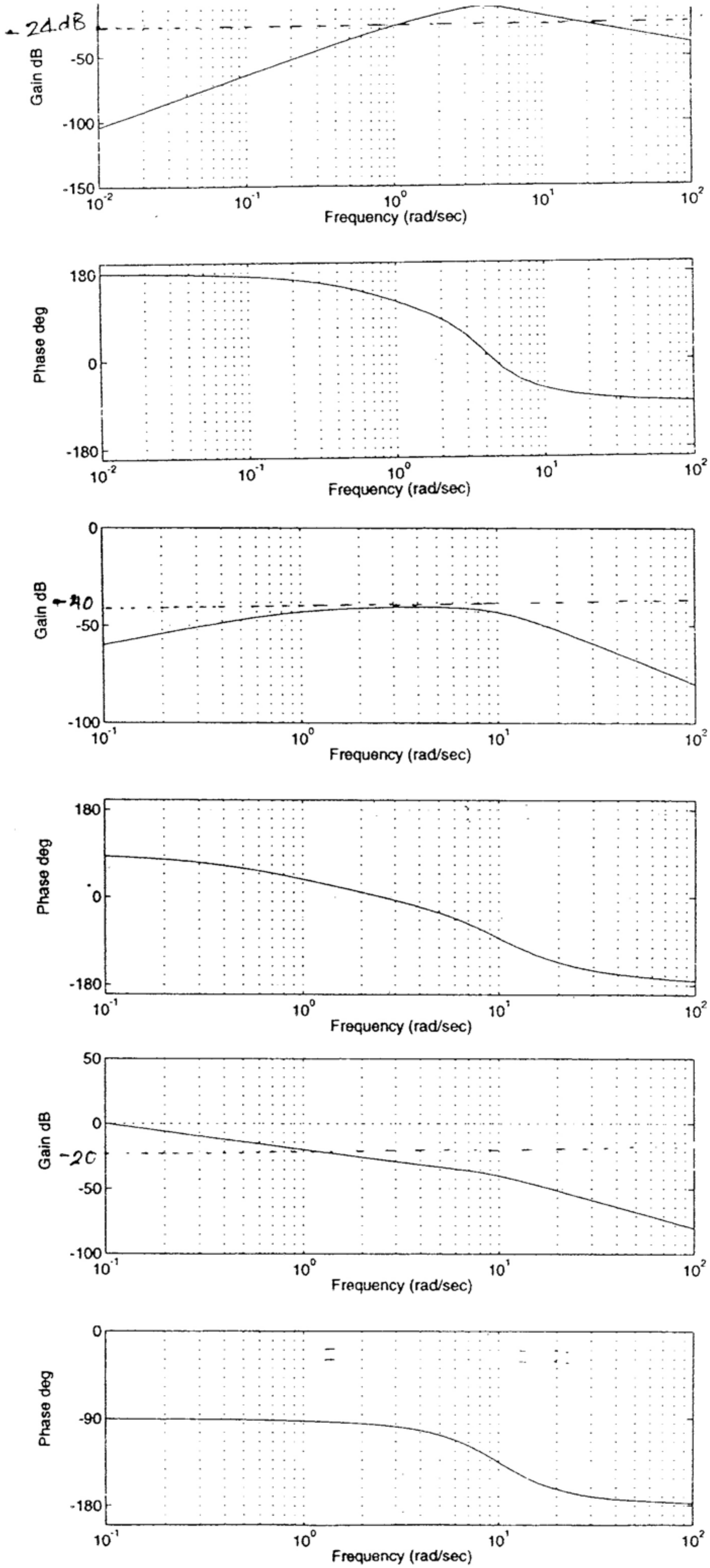
$$H(s) = \frac{10}{100} \frac{\frac{s}{10}+1}{s(\frac{s^2}{100}+0.1414s+1)}$$

The amplitude response: The horizontal axis where the asymptotes begin is 1/10, which is -20 dB. Asymptotes start at $\omega = 1$ (-20 dB/dec.), 10 (20 dB/dec.), 10 (-40 dB/dec.). The corrections are applied at various points as discussed in Examples 4.25 and 4.26. to obtain the Bode plot.

4.9-3. To rise at 20dB per decade, a zero must be present before $\omega = 0.1$. At $\omega = 30$, the magnitude response begins to fall at -20dB per decade. This requires two poles at that frequency: one pole to counteract the previous zero and another pole to cause the -20dB per decade slope. The magnitude response level out at $\omega = 500$, which requires the action of a zero. Thus, a second order system should be sufficient.

$$H(s) = k\frac{s(s+500)}{(s+30)^2}.$$

To determine the constant k, notice that $20\log(|H(j1)|) = 10$ or $|H(j1)| = \sqrt{10} \approx$


-24dB
Gain dB
-50
-100
-150
10^{-2} 10^{-1} 10^{0} 10^{1} 10^{2}
Frequency (rad/sec)
Phase deg
180
0
-180
Frequency (rad/sec)
0
-40
Gain dB
-50
-100
10^{-1} 10^{0} 10^{1} 10^{2}
Frequency (rad/sec)
180
Phase deg
0
-180
Frequency (rad/sec)
50
Gain dB
0
-20
-50
-100
Frequency (rad/sec)
0
Phase deg
-90
-180
Frequency (rad/sec)

(a)

(b)

(c)

FIG. S4.9-2

yields $y(t) = RC_1\dot{v}_{C_1}(t) + v_{C_1}(t)$. In transform domain, this becomes $Y(S) = V_{C_1}(s)\,(1 + RC_1 s)$ or $V_{C_1}(s) = \frac{Y(s)}{1+RC_1 s}$. Combining the KCL and KVL equations yields $\frac{X(s)-C_2 sY(s)}{C_1 s} = \frac{Y(s)}{1+RC_1 s}$. Simplifying yields $Y(s)\left(RC_1C_2s^2 + C_1 s + C_2 s\right) = X(s)\,(RC_1 s + 1)$. Thus,

$$H(s) = \frac{Y(s)}{X(s)} = \frac{RC_1 s + 1}{RC_1C_2s^2 + (C_1 + C_2)s}.$$

(b) Notice, $H(s)$ has two poles, one at zero and another at a negative, real number. It also has two zeros, one at infinity and another at a negative, real number. Only plots B and D show evidence of a finite zero as well as a finite pole. Of these, only plot B can have the necessary pole at zero. Thus,

Plot B is the only plot consistent with the system.

(c) At low frequencies, $H(j\omega) \approx \frac{1}{(C_1+C_2)j\omega}$. Thus, R doesn't affect $|H(j\omega)$ at very low frequencies.

(d) At high frequencies, $H(j\omega) \approx \frac{RC_-1j\omega}{-RC-1C_2\omega^2} = \frac{-j}{C_2\omega}$. Thus, R doesn't affect $|H(j\omega)$

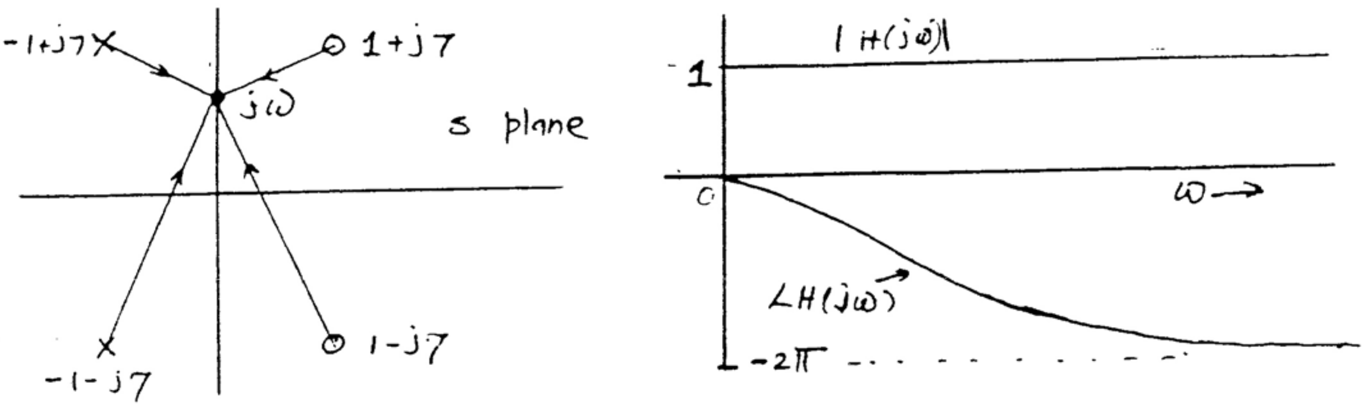


FIG. 4.10-1

at very high frequencies.

4.10-1. We plot the poles $-1 \pm j7$ and $1 \pm j7$ in the s-plane. To find response at some frequency ω, we connect all the poles and zeros to the point $j\omega$ as shown in Figure S4.10-1. Note that the product of distances from the zeros is equal to the product of the distances from the poles for all values of ω. Therefore $|H(j\omega)| = 1$. Graphical argument shows that $\angle H(j\omega)$ (sum of the angles from the zeros$-$ sum of the angles from poles) starts at zero for $\omega = 0$ and then reduces continuously (becomes negative) as ω increases. As $\omega \to \infty$, $\angle H(\omega) \to -2\pi$.

4.10-2. (a) If r and d are the distances of the zero and pole, respectively from $j\omega$, then the amplitude response $|H(j\omega)|$ is the ratio r/d corresponding to $j\omega$. This ratio is 0.5 for $\omega = 0$. Therefore, the dc gain is 0.5. Also the ratio $r/d = 1$ for $\omega = \infty$. Thus, the gain is unity at $\omega = \infty$. Also the angles of the line segments connecting the zero and pole to the point $j\omega$ are both zero for $\omega = 0$, and are both $\pi/2$ for $\omega = \infty$. Therefore $\angle H(j\omega) = 0$ at $\omega = 0$ and $\omega = \infty$. In between the angle is positive as shown in Figure S4.10-2a.

(b) In this case the ratio r/d is 2 for $\omega = 0$. Therefore, the dc gain is 2. Also the ratio $r/d = 1$ for $\omega = \infty$. Thus, the gain is unity at $\omega = \infty$. Also the angles

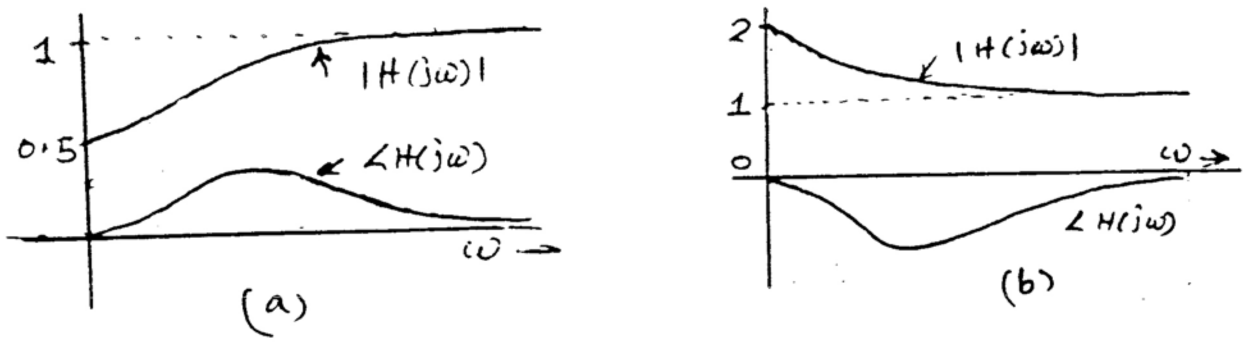


Figure S4.10-2

of the line segments connecting the zero and pole to the point $j\omega$ are both zero for $\omega = 0$, and are both $\pi/2$ for $\omega = \infty$. Therefore $\angle H(j\omega) = 0$ at $\omega = 0$ and $\omega = \infty$. In between the angle is negative as shown in Figure S4.10-2b.

4.10-3. The poles are at $-a \pm j10$. Moreover zero gain at $\omega = 0$ and $\omega = \infty$ requires that there be a single zero at $s = 0$. This clearly causes the gain to be zero at $\omega = 0$. Also because there is one excess pole over zero, the gain for large values of ω is $1/\omega$, which approaches 0 as $\omega \to \infty$. therefore, the suitable transfer function is

$$H(s) = \frac{s}{(s + a + j10)(s + a - j10)} = \frac{s}{s^2 + 2as + (100 + a^2)}$$

The amplitude response is high in the vicinity of $\omega = 10$ provided a is small. Smaller the a, more pronounced the gain in the vicinity of $\omega = 10$. For $a = 0$, the gain at $\omega = 10$ is ∞.

4.10-4. Cynthia is correct. Although the system is all-pass and has $|H(j\omega)| = 1$, the phase response is not zero. Thus, the output generally has different phase than the input. Furthermore, the output can also include transient components that would not be present in the original input.

4.10-5. Both Amy and Jeff are correct. By definition, a zero is any value s that forces $H(s) = 0$

and a pole is any value s that forces $H(s) = \infty$. Thus, the system $H(s) = s = \frac{1}{s^{-1}}$ has both a zero at $s = 0$ and a pole at $s = \infty$. Remember, a rational system function always has the same number of poles and zeros; if $H(s) = s$ has an obvious zero at $s = 0$ there must be a matching pole somewhere, even if it is not finite. By similar argument, the system $H(s) = \frac{1}{s}$ has a pole at $s = 0$ and a zero at $s = \infty$.

4.10-6. At high frequencies, the highest powers of s dominate both the numerator and denominator of $H(s)$. That is, $\lim_{s\to\infty} H(s) = \lim_{s\to\infty} \frac{b_0 s^M}{s^N}$.

Lowpass and bandpass filters both require $\lim_{s\to\infty} H(s) = 0$, which ensures a response of zero at high frequencies. Only $H(s)$ that are strictly proper ($M < N$) yield the required $\lim_{s\to\infty} H(s) = 0$.

Highpass and bandstop filters both require $\lim_{s\to\infty} H(s) = k$, where k is some finite, non-zero constant. Only $H(s)$ that are proper ($M = N$) yield the required $\lim_{s\to\infty} H(s) = b_0 = k$.

The case $M > N$ is not considered, since such systems are not physically practical.

4.10-7. At high frequencies, the highest powers of s dominate both the numerator and denominator of $H(s)$. That is, $\lim_{s\to\infty} H(s) = \lim_{s\to\infty} \frac{b_0 s^M}{s^N}$. Thus, the log magnitude response at high frequencies is given by $\lim_{\omega\to\infty} \log|H(\jmath\omega)| = \log(b_0) + M\log(\omega) - N\log(\omega)$. The fastest attenuation as a function of frequency requires M to be as small as possible. Thus, for a given N, the attenuation rate of an all-pole lowpass filter ($M = 0$) is faster than the attenuation rate of any filter with a finite number of zeros ($M \neq 0$).

4.10-8. No, it is not possible for such a system to function as a lowpass filter. For any choice of ($[k, b_1, b_2, a_1, a_2] \in \mathcal{R}$), the system function $H(s) = k\frac{s^2+b_1 s+b_2}{s^2+a_1 s+a_2}$ is proper. Thus, the system function always has high-frequency gain of k. For $k \neq 0$, the system cannot be lowpass. Furthermore, for $k = 0$ the system becomes a useless "nopass filter" (again, not lowpass).

4.10-9. Nick is more correct than his professor. A cascade of two identical filters, each with system response $H(\jmath\omega)$, gives a total response of $H^2(\jmath\omega)$. Since realizable filters, such as Butterworth filters, are not ideal, the cascade system will tend to have a faster transition band and greater stopband attenuation. In a sense, the resulting fourth-order system really does provide "twice the filtering" of the original second-order system.

Unfortunately, there are also problems with Nick's approach. Simply cascading a designed lowpass filter twice has negative consequences. For example, the cutoff frequency shifts to a lower frequency than desired. As the cascaded RC example in

MATLAB Session 4 suggests, a cascade of low-order filters is inferior to a carefully designed, equivalent-order filter. In general, a fourth-order Butterworth filter performs better than a cascade of two second-order Butterworth filters.

4.10-10. (a) Using tables,

$$H(s) = \frac{1}{s} - \frac{1}{s}e^{-s} = e^{-s/2}\,\frac{e^{s/2} - e^{-s/2}}{s}$$

Substituting $s = j\omega$ yields the frequency response

$$H(j\omega) = e^{-j\omega/2}\,\frac{e^{j\omega/2} - e^{-j\omega/2}}{j\omega} = e^{-j\omega/2}\,\frac{\sin(\omega/2)}{\omega/2} = e^{-j\omega/2}\,\mathrm{sinc}\left(\frac{\omega}{2}\right)$$

The sinc type of frequency response (with a linear phase shift of $-\omega/2$) represents a lowpass system.

(b) Since $h(t)$ is finite duration, the system has no finite poles. There are, however, an infinite number of finite zeros for $\mathrm{sinc}(\omega/2)$ at $\omega = 2\pi k$ or $s = j2\pi k$, where k is any non-zero integer.

(c) In transform-domain, the inverse system is given by the reciprocal of $H_c(s)$. Thus,

$$H_c^{-1}(s) = \frac{1}{H_c(s)} = e^{s/2}\,\frac{\frac{s}{j2}}{\sin\left(\frac{s}{j2}\right)}$$

The inverse system has no finite zeros and an infinite number of finite poles. Since poles lie on the ω-axis, the inverse system cannot be asymptotically stable. The same approach does not work in the time-domain. That is, $h_c^{-1}(t) \neq \frac{1}{h(t)}$. The impulse response needs to be obtained from an inverse Laplace transform of $H_c^{-1}(s)$. Unfortunately, it is difficult to take the inverse Laplace transform of $H_c^{-1}(s)$; no closed form solution for $h_c^{-1}(t)$ is known to exist.
It is possible to approximate $h_c^{-1}(t)$. Consider the following idea. Replace the denominator $\sin\left(\frac{s}{j2}\right)$ with a truncated Taylor series expansion. The result is a rational approximation to $H_c^{-1}(s)$ that can be inverted using partial fraction expansion techniques. Although not perfect, the result can perform reasonably for many low-frequency inputs.

4.10-11. No, the suggested lowpass to highpass transformation $H_{\mathrm{HP}}(s) = 1 - H_{\mathrm{LP}}(s)$ does not work in general. Although it is possible to relate the ideal magnitude responses according to $|H_{\mathrm{HP}}(j\omega)| = 1 - |H_{\mathrm{LP}}(j\omega)|$, the phase information contained in $H(s)$ generally makes $H_{\mathrm{HP}}(s) \neq 1 - H_{\mathrm{LP}}(s)$.

As an example, consider an ideal lowpass filter described by

$$H_{\mathrm{LP}}(j\omega) = \begin{cases} -1 & |\omega| \leq \omega_c \\ 0 & |\omega| > \omega_c \end{cases}.$$

The transformation $1 - H_{\mathrm{LP}}(s)$ is clearly not highpass.

$$1 - H_{\mathrm{LP}}(s) = \begin{cases} 2 & |\omega| \leq \omega_c \\ 1 & |\omega| > \omega_c \end{cases}.$$

4.10-12. (a) Yes, it is possible for the system to output $y(t) = \sin(100\pi t)u(t)$ in response to $x(t) = \cos(100\pi t)u(t)$. Noting $Y(s) = \frac{100\pi}{s^2+(100\pi)^2}$ and $X(s) = \frac{s}{s^2+(100\pi)^2}$, one way to obtain $y(t)$ from $x(t)$ is using the system $H(s) = Y(s)/X(s) = \frac{100\pi}{s^2+(100\pi)^2}\frac{s^2+(100\pi)^2}{s} = \frac{100\pi}{s}$.

(b) Yes, it is possible for the system to output $y(t) = \sin(100\pi t)u(t)$ in response to $x(t) = \sin(50\pi t)u(t)$. Noting $Y(s) = \frac{100\pi}{s^2+(100\pi)^2}$ and $X(s) = \frac{s}{s^2+(50\pi)^2}$, one way to obtain $y(t)$ from $x(t)$ is using the system $H(s) = Y(s)/X(s) = \frac{100\pi}{s^2+(100\pi)^2}\frac{s^2+(50\pi)^2}{s} = \frac{100\pi(s^2+(50\pi)^2)}{s(s^2+(100\pi)^2)}$.

(c) Yes, it is possible for the system to output $y(t) = \sin(100\pi t)$ in response to $x(t) = \cos(100\pi t)$. To do this, the system must have $H(\jmath 100\pi) = e^{-\jmath\pi/2}$. That is, the magnitude response at $\omega = 100\pi$ must be unity, and the phase response at $\omega = 100\pi$ must be $-\pi/2$.

(d) No, it is not possible for the system to output $y(t) = \sin(100\pi t)$ in response to $x(t) = \sin(50\pi t)$. In an LTI system, an everlasting sinusoidal input of frequency 50π cannot produce a different frequency output.

4.11-1. (a) Let $x_1(t) = x(t)u(t) = e^t u(t)$ and $x_2(t) = x(t)u(-t) = u(-t)$. Then $X_1(s)$ has a region of convergence $\sigma > 1$. And $X_2(s)$ has a region $\sigma < 0$. Hence there is no common region of convergence for $X(s) = X_1(s) + X_2(s)$.

(b) $x_1(t) = e^{-t}u(t)$, and $X_1(s) = \frac{1}{s+1}$ converges for $\sigma > -1$. Also $x_2(t) = u(-t)$, and $X_2(s) = -\frac{1}{s}$ converges for $\sigma < 0$. Therefore, the strip of convergence is

$$-1 < \sigma < 0$$

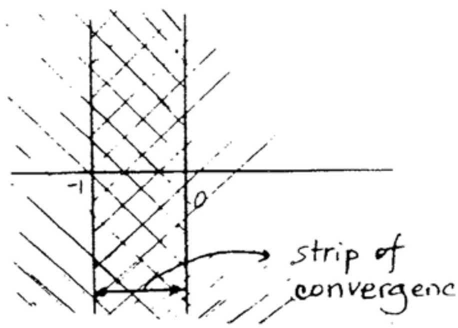


Figure S4.11-1b

(c)

$$\frac{1}{t^2+1}e^{-st} \to 0 \left.\right\} \quad \begin{array}{l}\text{as } t \to \infty \text{ if Re } s \geq 0 \\ \text{as } t \to -\infty \text{ if Re } s \leq 0\end{array}$$

Hence the convergence occurs at $\sigma = 0$ ($j\omega$-axis)

(d)

$$x(t) = \frac{1}{1+e^t}$$

$$\frac{1}{1+e^t}e^{-st} \to 0 \left.\right\} \quad \begin{array}{l}\text{as } t \to \infty \text{ if Re } s > -1 \\ \text{as } t \to -\infty \text{ if Re } s < 0\end{array}$$

Hence the region of convergence is $-1 < \sigma < 0$

(e)

$$x(t) = e^{-kt^2}$$

$$e^{-kt^2}e^{-st} \to 0 \left.\begin{matrix} \\ \\ \end{matrix}\right\} \begin{matrix} \text{as } t \to \infty \text{ for any value of } s \\ \text{as } t \to -\infty \text{ for any value of } s \end{matrix}$$

Hence the region of convergence is the entire s-plane.

4.11-2. (a)

$$x(t) = e^{-|t|} = e^{-t}u(t) + e^{t}u(-t) = x_1(t) + x_2(t)$$

$$X_1(s) = \frac{1}{s+1} \qquad \sigma > -1$$

$$x_2(-t) = e^{-t}u(t) \quad \text{and} \quad X_2(-s) = \frac{1}{s+1}$$

$$\text{and} \qquad X_2(s) = \frac{1}{-s+1} \qquad \sigma < 1$$

$$\text{Hence:} \quad X(s) = X_1(s) + X_2(s) = \frac{1}{s+1} + \frac{1}{-s+1} = \frac{-2}{s^2-1} \qquad -1 < \sigma < 1$$

(b)

$$x(t) = e^{-|t|}\cos t = e^{-t}\cos t\, u(t) + e^{t}\cos t\, u(-t) = x_1(t) + x_2(t)$$

$$\text{Hence} \quad X_1(s) = \frac{s+1}{(s+1)^2+1} \quad \text{and} \quad X_2(-s) = \frac{s+1}{(s+1)^2+1} \quad \sigma < 1$$

$$X(s) = X_1(s) + X_2(s) = \frac{s+1}{(s+1)^2+1} - \frac{s-1}{(s-1)^2+1} = \frac{4-2s^2}{s^4-4} \qquad -1 < \sigma < 1$$

(c)

$$x(t) = e^{t}u(t) + e^{2t}u(-t); \quad X_1(s) = \frac{1}{s-1} \quad \sigma > 1 \quad \text{and} \quad X_2(-s) = \frac{1}{s+2}$$

$$X_2(s) = \frac{1}{-s+2} \qquad \sigma < 2.$$

$$\text{Hence} \qquad X(s) = X_1(s) + X_2(s) = \frac{-1}{(s-1)(s-2)} \quad 1 < \sigma < 2$$

(d)

$$x(t) = e^{-tu(t)} = \begin{cases} e^{-t} & \text{for} \quad t > 0 \\ 1 & \text{for} \quad t < 0 \end{cases}$$

$$x_1(t) = e^{-t}u(t), \quad x_2(t) = u(-t). \quad \text{Hence} \quad X_1(s) = \frac{1}{s+1} \quad \sigma > -1$$

$$\text{and} \quad X_2(-s) = \frac{1}{s}, \quad X_2(s) = \frac{-1}{s} \quad \sigma < 0$$

$$\text{and hence:} \qquad X(s) = \frac{1}{s+1} - \frac{1}{s} = \frac{-1}{s(s+1)} \qquad -1 < \sigma < 0$$

(e)

$$x(t) = e^{tu(-t)} = \begin{cases} x_1(t) = 1 & \text{for} \quad t > 0 \\ x_2(t) = e^t & \text{for} \quad t < 0 \end{cases}$$

$$X_1(s) = \frac{1}{s} \qquad \sigma > 0$$

$$X_2(-s) = \frac{1}{s+1} \quad X_2(s) = \frac{1}{-s+1} \quad \sigma < 1$$

and hence: $$X(s) = \frac{1}{s} - \frac{1}{s-1} = \frac{-1}{s(s-1)} \qquad 0 < \sigma < 1$$

(f)

$$x(t) = \cos\omega_0 t\, u(t) + e^t u(-t) = x_1(t) + x_2(t)$$

$$X_1(s) = \frac{s}{s^2 + \omega_0^2} \qquad \sigma > 0$$

and $$X_2(-s) = \frac{1}{s+1}, \qquad X_2(s) = \frac{1}{1-s} \qquad \sigma < 1$$

$$X(s) = X_1(s) + X_2(s) = \frac{-(s + \omega_0^2)}{(s-1)(s^2 + \omega_0^2)} \quad 0 < \sigma < 1$$

4.11-3. (a)

$$\begin{aligned} X(s) &= \frac{2s+5}{(s+2)(s+3)} & -3 < \sigma < -2 \\ &= \frac{1}{s+2} + \frac{1}{s+3} & -3 < \sigma < -2 \end{aligned}$$

The pole -2 lies to the right, and the pole -3 lies to the left of the region of convergence; hence the first term represents causal and the second term represents anticausal signal:

$$x(t) = e^{-3t}u(t) - e^{-2t}u(-t)$$

(b)

$$\begin{aligned} X(s) &= \frac{2s-5}{(s-2)(s-3)} & 2 < \sigma < 3 \\ &= \frac{1}{s-2} + \frac{1}{s-3} & 2 < \sigma < 3 \end{aligned}$$

The pole at -2 lies to the left and that at 3 lies to the right of the region of convergence; hence

$$x(t) = e^{2t}u(t) - e^{3t}u(-t)$$

(c)

$$\begin{aligned} X(s) &= \frac{2s+3}{(s+1)(s+2)} & \sigma > -1 \\ &= \frac{1}{s+1} + \frac{1}{s+2} & \sigma > -1 \end{aligned}$$

Both poles lie to the left of the region of convergence, and

$$x(t) = (e^{-t} + e^{-2t})u(t)$$

(d)

$$\begin{aligned} X(s) &= \frac{2s+3}{(s+1)(s+2)} \qquad \sigma < -2 \\ &= \frac{1}{s+1} + \frac{1}{s+2} \qquad \sigma < -2 \end{aligned}$$

Both poles lie to the right of the region of convergence, and hence:

$$x(t) = -(e^{-t} + e^{-2t})u(-t)$$

(e)

$$\begin{aligned} X(s) &= \frac{3s^2 - 2s - 17}{(s+1)(s+3)(s-5)} \qquad -1 < \sigma < 5 \\ &= \frac{1}{s+1} + \frac{1}{s+3} + \frac{1}{s-5} \end{aligned}$$

The poles -1 and -3 lie to the left of the region of convergence, whereas the pole 5 lies to the right:

$$x(t) = (e^{-t} + e^{-3t})u(t) - e^{5t}u(-t)$$

4.11-4.

$$\frac{2s^2 - 2s - 6}{(s+1)(s-1)(s+2)} = \frac{1}{s+1} - \frac{1}{s-1} + \frac{2}{s+2}$$

(a) Re $s > 1$: All poles to the left of the region of convergence. Therefore

$$x(t) = (e^{-t} - e^{t} + 2e^{-2t})u(t)$$

(b) Re $s < -2$: All poles to the right of the region of convergence. Therefore

$$x(t) = (-e^{-t} + e^{t} - 2e^{-2t})u(-t)$$

(c) $-1 <$ Re $s < 1$: Poles -1 and -2 to the left and pole 1 to the right of the region of convergence. Therefore

$$x(t) = (e^{-t} + 2e^{-2t})u(t) + e^{t}u(-t)$$

(d) $-2 <$ Re $s < -1$: Poles -1 and 1 are to the right and pole -2 is to the left of the region of convergence. Therefore

$$x(t) = 2e^{-2t}u(t) + [-e^{-t} + e^{t}]u(-t)$$

4.11-5. (a)

$$x(t) = e^{-\frac{|t|}{2}}, \qquad H(s) = \frac{1}{s+1} \quad \sigma > -1$$

And $$X(s) = \frac{1}{s+0.5} - \frac{1}{s-0.5} \qquad -\frac{1}{2} < \sigma < \frac{1}{2}$$

hence: $$Y(s) = H(s)X(s) = \frac{1}{s+1}\left[\frac{1}{s+0.5} - \frac{1}{s-0.5}\right] \qquad -\frac{1}{2} < \sigma < \frac{1}{2}$$

$$\begin{aligned} Y(s) &= \frac{-2}{s+1} + \frac{2}{s+0.5} + \frac{\frac{2}{3}}{s+1} - \frac{\frac{2}{3}}{s-0.5} \\ &= \frac{-\frac{4}{3}}{s+1} + \frac{2}{s+0.5} - \frac{\frac{2}{3}}{s-0.5} \qquad -\frac{1}{2} < \sigma < \frac{1}{2} \end{aligned}$$

The poles -1 and -0.5, which are to the left of the strip of convergence, yield the causal signal, and the pole 0.5, which is to the right of the strip of convergence, yields the anticausal signal. Hence

$$y(t) = \left(-\frac{4}{3}e^{-t} + 2e^{-t/2}\right)u(t) + \frac{2}{3}e^{t/2}u(-t)$$

(b)

$$x(t) = e^{t}u(t) + e^{2t}u(-t)$$

$$\begin{aligned} X(s) &= \frac{1}{s-1} - \frac{1}{s-2} \qquad 1 < \sigma < 2 \\ &= \frac{-1}{(s-1)(s-2)} \end{aligned}$$

And $$H(s) = \frac{1}{s+1} \qquad \sigma > -1$$

Hence: $$Y(s) = H(s)X(s) = \frac{-1}{(s+1)(s-1)(s-2)} \qquad 1 < \sigma < 2$$

$$Y(s) = \frac{-1/6}{s+1} + \frac{1/2}{s-1} - \frac{1/3}{s-2} \qquad 1 < \sigma < 2$$

Hence $$y(t) = \left(-\frac{1}{6}e^{-t} + \frac{1}{2}e^{t}\right)u(t) + \frac{1}{3}e^{2t}u(-t)$$

(c)

$$x(t) = e^{-t/2}u(t) + e^{-t/4}u(-t)$$

$$X(s) = \frac{1}{s+0.5} - \frac{1}{s+0.25} = \frac{-\frac{1}{4}}{(s+0.5)(s+0.25)} \qquad -\frac{1}{2} < \sigma < \frac{1}{4}$$

Also $$H(s) = \frac{1}{s+1} \qquad \sigma > -1$$

Hence: $$\begin{aligned} Y(s) = H(s)X(s) &= \frac{-\frac{1}{4}}{(s+1)(s+0.5)(s+0.25)} \qquad -\tfrac{1}{2} < \sigma < \tfrac{1}{4} \\ &= \frac{-\frac{2}{3}}{s+1} + \frac{2}{s+0.5} - \frac{\frac{4}{3}}{s+0.25} \qquad -\tfrac{1}{2} < \sigma < \tfrac{1}{4} \end{aligned}$$

and $$y(t) = \left(-\frac{2}{3}e^{-t} + 2e^{-\frac{t}{2}}\right)u(t) + \frac{4}{3}e^{-\frac{t}{4}}u(-t)$$

(d)

$$x(t) = e^{2t}u(t) + e^t u(-t) = x_1(t) + x_2(t)$$

$$\begin{aligned} X_1(s) &= \frac{1}{s-2} & \sigma > 2 \\ X_2(s) &= \frac{-1}{s-1} & \sigma < 1 \end{aligned}$$

$$\text{and} \qquad H(s) = \frac{1}{s+1} \qquad \sigma > -1$$

In this case, there is no region of convergence that is common to $X_1(s)$ and $X_2(s)$. However, each of $X_1(s)$ and $X_2(s)$ have a region of convergence that is common to $H(s)$. Hence the output can be computed by finding the system response to $x_1(t)$ and $x_2(t)$ separately, and then adding these two components. This means we need not worry about the common region of convergence for $X_1(s)$ and $X_2(s)$. Thus:

$$Y(s) = Y_1(s) + Y_2(s) \qquad \text{where}$$

$$\begin{aligned} Y_1(s) = X_1(s)H(s) &= \frac{1}{(s+1)(s-2)} & \sigma > 2 \\ &= \frac{-\frac{1}{3}}{s+1} + \frac{\frac{1}{3}}{s-2} & \sigma > 2 \end{aligned}$$

Observe that both the poles (-1 and 2) are to the left of the region of convergence, hence both terms are causal, and:

$$y_1(t) = \left(-\frac{1}{3}e^{-t} + \frac{1}{3}e^{2t}\right)u(t)$$

$$\begin{aligned} Y_2(s) = X_2(s)H(s) &= \frac{-1}{(s+1)(s-1)} & -1 < \sigma < 1 \\ &= \frac{\frac{1}{2}}{s+1} - \frac{\frac{1}{2}}{s-1} & -1 < \sigma < 1 \end{aligned}$$

The poles -1 and 1 are to the left and the right, respectively, of the strip of convergence. Hence the first term yields causal signal and the second yields anticausal signal. Hence

$$y_2(t) = -\frac{1}{2}e^{-t}u(t) + \frac{1}{2}e^t u(-t)$$

$$\text{Therefore} \quad y(t) = y_1(t) + y_2(t) = \left(\frac{1}{6}e^{-t} + \frac{1}{3}e^{2t}\right)u(t) + \frac{1}{2}e^t u(-t)$$

(e)

$$x(t) = e^{-\frac{t}{4}}u(t) + e^{-\frac{t}{2}}u(-t) = x_1(t) + x_2(t)$$

$$X(s) = X_1(s) + X_2(s)$$

$$\text{where}\quad \begin{aligned} X_1(s) &= \frac{1}{s+0.25} \qquad \sigma > -\frac{1}{4} \\ X_2(s) &= \frac{-1}{s+0.5} \qquad \sigma < -\frac{1}{2} \\ H(s) &= \frac{1}{s+1} \qquad \sigma > -1 \end{aligned}$$

Here also, we have no common region of convergence, for $X_1(s)$ and $X_2(s)$ as in part d. Let $Y(s) = Y_1(s) + Y_2(s)$ where:

$$\begin{aligned} Y_1(s) &= \frac{1}{(s+1)(s+0.25)} \qquad \sigma > -\frac{1}{4} \\ &= \frac{-\frac{4}{3}}{s+1} + \frac{\frac{4}{3}}{s+0.25} \qquad \sigma > -\frac{1}{4} \end{aligned}$$

$$y_1(t) = \left(-\frac{4}{3}e^{-t} + \frac{4}{3}e^{-\frac{t}{4}}\right) u(t)$$

$$\begin{aligned} Y_2(s) &= \frac{-1}{(s+1)(s+0.5)} \qquad -1 < \sigma < -\frac{1}{2} \\ &= \frac{2}{s+1} - \frac{2}{s+0.5} \qquad -1 < \sigma < -\frac{1}{2} \end{aligned}$$

$$\text{and}\qquad y_2(t) = 2e^{-t}u(t) + 2e^{-\frac{t}{2}}u(-t)$$

$$\text{Hence}\quad y(t) = y_1(t) + y_2(t) = \left(\frac{2}{3}e^{-t} + \frac{4}{3}e^{-\frac{t}{4}}\right) u(t) + 2e^{-\frac{t}{2}}u(-t)$$

(f)

$$x(t) = e^{-3t}u(t) + e^{-2t}u(-t) = x_1(t) + x_2(t)$$

$$X(s) = X_1(s) + X_2(s)$$

$$\text{where}\quad \begin{aligned} X_1(s) &= \frac{1}{s+3} \qquad \sigma > -3 \\ X_2(s) &= \frac{-1}{s+2} \qquad \sigma < -2 \\ H(s) &= \frac{1}{s+1} \qquad \sigma > -1 \end{aligned}$$

In this case, there is a common region of convergence for $X_1(s)$ and $H(s)$, but there is no region of convergence common to $X_2(s)$ and $H(s)$. Hence the output $y_1(t)$ will be finite but $y_2(t)$ will be ∞.

4.11-6.

$$\begin{aligned} \mathcal{L}\left(r_{xx}(t)\right) &= \int_{-\infty}^{\infty} r_{xx}(t)e^{-st}dt \\ &= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} x(\tau)x(\tau+t)d\tau\right)e^{-st}dt \\ &= \int_{-\infty}^{\infty} x(\tau)\left(\int_{-\infty}^{\infty} x(\tau+t)e^{-st}dt\right)d\tau \\ &= \int_{-\infty}^{\infty} x(\tau)e^{s\tau}X(s)d\tau \\ &= X(s)\int_{-\infty}^{\infty} x(\tau)e^{-\tau(-s)}d\tau \\ R_{xx}(s) &= X(s)X(-s) \end{aligned}$$

4.11-7. For $\sigma < 0$, we know that $\mathcal{L}^{-1}\left[\frac{2}{s}\right] = -2u(-t)$. Additionally, $\mathcal{L}^{-1}[1/2] = \delta/2$. Using properties, $\mathcal{L}^{-1}[s(1/2)] = \frac{d}{dt}(\delta(t)/2)$. Thus,

$$x(t) = -2u(-t) + \frac{d}{dt}(\delta(t)/2).$$

The function $\frac{d}{dt}(\delta(t)/2)$ is called the "unit doublet". Like $\delta(t)$, the unit doublet is not a physically realizable signal. It is a mathematical constructions that is useful, among other things, in finding function deriyatives. Refer to the topic of generalized derivatives.

4.11-8. (a) Yes, $x(t)$ can be left-sided. To be left-sided and absolutely integrable, the signal's region of convergence must: 1) be left-sided, 2) include the ω-axis, and 3) not include any poles. With a pole at $s = \pi$, it is possible to achieve all three necessary conditions. For example, $x(t) = e^{\pi t}u(-t)$ has a pole at $s = \pi$, is absolutely integrable, and is left-sided.

(b) No, $x(t)$ cannot be right-sided. To be right-sided and absolutely integrable, the signal's region of convergence must: 1) be right-sided, 2) include the ω-axis, and 3) not include any poles. The ω-axis cannot be included in a region of convergence that is to the right of the known pole at $s = \pi$.

(c) Yes, $x(t)$ can be two-sided. To be two-sided and absolutely integrable, the signal must: 1) have at least one pole in the right-half plane, 2) have at least one pole in the left-half plane, and 3) have a region of convergence that includes the ω-axis. With a pole at $s = \pi$, these conditions are possible. For example, $x(t) = e^{\pi t}u(-t) + e^{-\pi t}u(t)$ has a pole at $s = \pi$ (and another at $s = -\pi$), is absolutely integrable, and is two-sided.

(d) No, $x(t)$ cannot be finite duration. To be finite duration, the signal's region of convergence must include all finite values of s. However, since a pole is present at $s = \pi$, this point cannot be included in the region of convergence. Thought of another way, a pole at $s = \pi$ implies a signal component of either $e^{\pi t}u(t)$ or $e^{\pi t}u(-t)$, both of which are infinite in duration.

4.11-9. (a) $X_1(s) = \int_{-\infty}^{\infty} x_1(t)e^{-st}dt = \int_{-\infty}^{\infty}(\jmath + e^{\jmath t})u(t)e^{-st}dt = \int_0^{\infty} \jmath e^{-st} + e^{t(\jmath - s)}dt = \left(\frac{\jmath}{-s}e^{-st} + \frac{e^{t(\jmath-s)}}{\jmath-s}\right)\Big|_{t=0}^{\infty}$. For $\sigma > 0$, this simplifies to $X_1(s) = \frac{\jmath}{-s}(0 - e^0) + \frac{0-e^0}{\jmath-s}$. Thus,

$$X_1(s) = \frac{\jmath}{s} + \frac{1}{s-\jmath} \text{ for } \sigma > 0.$$

(b) $X_2(s) = \int_{-\infty}^{\infty} x_2(t)e^{-st}dt = \int_{-\infty}^{\infty} \jmath\cosh(t)u(-t)e^{-st}dt = \int_{-\infty}^{0} \jmath\frac{e^t+e^{-t}}{2}e^{-st}dt = \int_{-\infty}^{0} \jmath\frac{e^{t(1-s)}+e^{t(-1-s)}}{2}dt = \left(\frac{\jmath e^{t(1-s)}}{2(1-s)} + \frac{\jmath e^{t(-1-s)}}{2(-1-s)}\right)\Big|_{t=-\infty}^{0}$. For $\sigma < -1$, this simplifies to $X_2(s) = \frac{\jmath(e^0-0)}{2(1-s)} + \frac{\jmath(e^0-0)}{2(-1-s)} = \frac{-0.5\jmath}{s-1} + \frac{-0.5\jmath}{s+1}$. Thus,

$$X_2(s) = \frac{-\jmath s}{s^2-1} \text{ for } \sigma < -1.$$

(c) $X_3(s) = \int_{-\infty}^{\infty} x_3(t)e^{-st}dt = \int_{-\infty}^{\infty}\left(e^{\jmath(\frac{\pi}{4})}u(-t+1) + \jmath\delta(t-5)\right)e^{-st}dt = \int_{-\infty}^{1} e^{\jmath(\frac{\pi}{4})}e^{-st}dt + \int_{-\infty}^{\infty} \jmath\delta(t-5)e^{-st}dt = e^{\jmath(\frac{\pi}{4})}\frac{e^{-st}}{-s}\Big|_{t=-\infty}^{1} + \jmath e^{-5s}$. For $\sigma < 0$, this simplifies to $X_3(s) = e^{\jmath(\frac{\pi}{4})}\frac{e^{-s}-0}{-s} + \jmath e^{-5s}$. Thus,

$$X_3(s) = -e^{\jmath(\frac{\pi}{4})}\frac{e^{-s}}{s} + \jmath e^{-5s} \text{ for } \sigma < 0.$$

(d) $X_4(s) = \int_{-\infty}^{\infty} x_4(t)e^{-st}dt = \int_{-\infty}^{\infty}(\jmath^t u(-t) + \delta(t-\pi))e^{-st}dt = \int_{-\infty}^{0} e^{t\jmath\pi/2}e^{-st}dt + \int_{-\infty}^{\infty}\delta(t-\pi)e^{-st}dt = \left.\frac{e^{t(\jmath\pi/2-s)}}{\jmath\pi/2-s}\right|_{t=-\infty}^{0} + \jmath e^{-s\pi}$. For $\sigma < 0$, this simplifies to $X_4(s) = \frac{1-0}{\jmath\pi/2-s} + e^{-s\pi}$. Thus,

$$X_4(s) = e^{-s\pi} - \frac{1}{s-\jmath\pi/2} \text{ for } \sigma < 0.$$

4.11-10. (a) To be bounded amplitude, the region of convergence must include the ω-axis. The transfer function has two poles, at $s = \pm 1$, that must be excluded from the region of convergence. Thus, the region of convergence must be

$$-1 < \sigma < 1.$$

(b) Rewrite $H(s)$ as $2^s\frac{s}{(s-1)(s+1)} = e^{\ln(s)}\left(\frac{0.5}{s-1} + \frac{0.5}{s+1}\right)$. Using $-1 < \sigma < 1$, the time-shifting property, and a table of Laplace transform pairs, the inverse transform is found to be

$$h(t) = 0.5e^{-(t+\ln(2))}u(t+\ln(2)) - 0.5e^{t+\ln(2)}u(-(t+\ln(2))).$$

4.M-1. Using program MS4P3:

```
>> MS4P3(20)
ans = 524288          0    -2621440           0     5570560
            0   -6553600           0     4659200           0
     -2050048          0      549120           0      -84480
            0       6600           0        -200           0
            1
```

Thus,

$$\begin{aligned} C_{20}(x) &= 524288x^{20} - 2621440x^{18} + 5570560x^{16} - 6553600x^{14} + 4659200x^{12} + \\ &\quad -2050048x^{10} + 549120x^{8} - 84480x^{6} + 6600x^{4} - 200x^{2} + 1 \end{aligned}$$

4.M-2. (a) MATLAB is used for the design. To evaluate filter performance, the magnitude response is plotted over the frequency range ($0 \le f \le 10$kHz).

```
>> N = 12; omega_c = 2*pi*5000;
>> poles = roots([(j*omega_c)^(-2*N),zeros(1,2*N-1),1]);
>> B_poles = poles(find(real(poles)<0));
>> subplot(121), plot(real(B_poles),imag(B_poles),'xk');
>> xlabel('Real'); ylabel('Imag');
>> axis([-4e4 0 -4e4 4e4]); axis equal;
>> A = poly(B_poles); A = A/A(end); B = 1;
>> f = linspace(0,10000,1001);
>> Hmag_B = abs(polyval(B,j*2*pi*f)./polyval(A,j*2*pi*f));
>> subplot(122), plot(f,Hmag_B,'k');
>> xlabel('f [Hz]'); ylabel('|H_{LP}(j2\pi f)|');
```

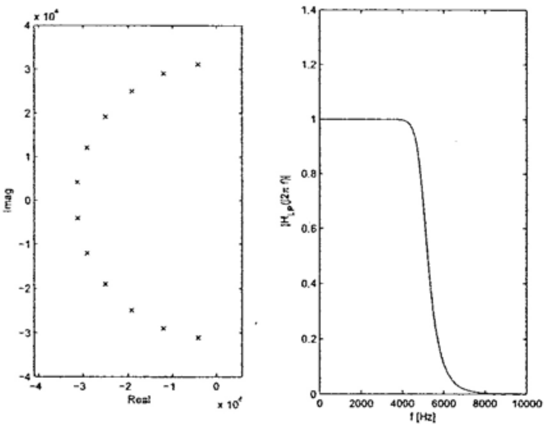


Figure S4.M-2a: Order-12 Butterworth LPF with $\omega_c = 2\pi 5000$.

The resulting figures are consistent with a Butterworth design; the poles lie on a semicircle in the left-half s-plane, and the magnitude response exhibits smooth monotonic roll-off.

(b) Modifying program MS4P2, the Sallen-Key component values and magnitude response plots are easily found.

```
>> omega_0 = 5000*2*pi; f = linspace(0,10000,200);
>> psi = [7.5:15:90]*pi/180; Hmag_SK = zeros(6,200);
>> for stage = 1:6,
>>     Q = 1/(2*cos(psi(stage)));
>>     disp(['Stage ',num2str(stage),...
>>           ' (Q = ',num2str(Q),...
>>           '):  R1 = R2 = ',num2str(100000)]);
>>     disp(['         C1 = ',num2str(2*Q/(omega_0*100000)),...
>>           ', C2 = ',num2str(1/(2*Q*omega_0*100000))]);
>>     B = omega_0^2; A = [1 omega_0/Q omega_0^2];
>>     Hmag_SK(stage,:) = abs(polyval(B,j*2*pi*f)./polyval(A,j*2*pi*f));
>> end
>> plot(f,Hmag_SK,'k',f,prod(Hmag_SK),'k:')
>> xlabel('f [Hz]'); ylabel('Magnitude Responses')
Stage 1 (Q = 0.50431):  R1 = R2 = 100000
        C1 = 3.2106e-010, C2 = 3.1559e-010
Stage 2 (Q = 0.5412):  R1 = R2 = 100000
        C1 = 3.4454e-010, C2 = 2.9408e-010
Stage 3 (Q = 0.63024):  R1 = R2 = 100000
        C1 = 4.0122e-010, C2 = 2.5253e-010
Stage 4 (Q = 0.82134):  R1 = R2 = 100000
        C1 = 5.2288e-010, C2 = 1.9377e-010
Stage 5 (Q = 1.3066):  R1 = R2 = 100000
        C1 = 8.3178e-010, C2 = 1.2181e-010
Stage 6 (Q = 3.8306):  R1 = R2 = 100000
        C1 = 2.4387e-009, C2 = 4.1548e-011
```

The resulting resistor and capacitor values are realistic. Each Sallen-Key stage implements a complex-conjugate pair of poles. The flattest magnitude response corresponds to the pair of poles that are furthest from the ω-axis, or Stage 1.

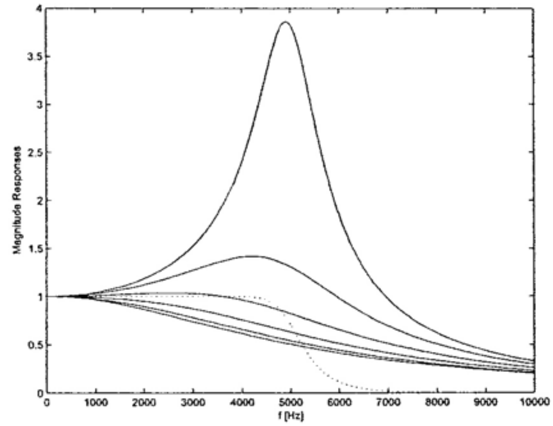


Figure S4.M-2b: Order-12 Butterworth LPF Sallen-Key Stage Responses.

The most peaked magnitude response corresponds to the pair of poles that are closest to the ω-axis, or Stage 6. The remaining stages are ordered in between. The dashed curve is the total magnitude response, and it is exactly the same as the one shown in Figure S4.M-2a.

4.M-3. (a) MATLAB is used for the design. To evaluate filter performance, the magnitude response is plotted over the frequency range ($0 \le f \le 10$kHz).

```
>> omega_c = 2*pi*5000; R = 3; N = 12;
>> epsilon = sqrt(10^(R/10)-1);
>> k = [1:N]; xi = 1/N*asinh(1/epsilon); phi = (k*2-1)/(2*N)*pi;
>> C_poles = omega_c*(-sinh(xi)*sin(phi)+j*cosh(xi)*cos(phi));
>> subplot(121), plot(real(C_poles),imag(C_poles),'xk');
>> xlabel('Real'); ylabel('Imag');
>> axis([-4e4 0 -4e4 4e4]); axis equal;
>> A = poly(C_poles);
>> B = A(end)/sqrt(1+epsilon^2);
>> omega = linspace(0,2*pi*10000,2001);
>> Hmag_C = abs(MS4P1(B,A,omega));
>> subplot(122); plot(omega/2/pi,abs(Hmag_C),'k'); grid
>> xlabel('f [Hz]'); ylabel('|H_{LP}(j2\pi f)|');
```

The resulting figures are consistent with a Chebyshev design; the poles lie on an ellipse in the left-half s-plane, passband ripples are equal in height and never exceed $R = 3$dB, there are a total of $N = 12$ maxima and minima in the passband, and the gain rapidly and monotonically decreases after the cutoff frequency of $f_c = 5$kHz.

(b) Modifying program MS4P2, the Sallen-Key component values and magnitude response plots are easily found.

```
>> omega_c = 2*pi*5000; R = 3; N = 12;
>> epsilon = sqrt(10^(R/10)-1);
>> k = [1:N]; xi = 1/N*asinh(1/epsilon); phi = (k*2-1)/(2*N)*pi;
>> C_poles = omega_c*(-sinh(xi)*sin(phi)+j*cosh(xi)*cos(phi));
>> C_poles = C_poles(find(imag(C_poles)>0)); % Quadrant 2 poles
>> f = linspace(0,10000,501); Hmag_SK = zeros(6,501);
```

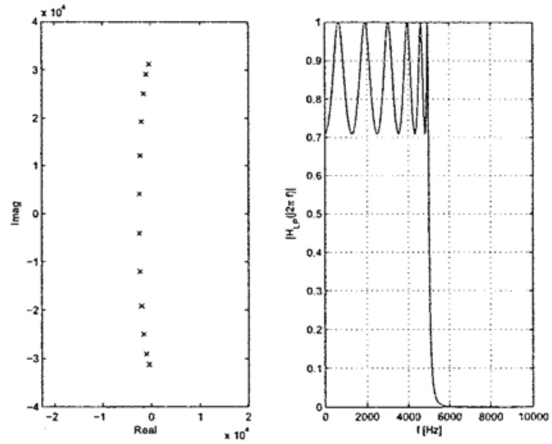


Figure S4.M-3a: Order-12 Chebyshev LPF with $\omega_c = 2\pi 5000$ and $R = 3$dB.

```
>> for stage = 1:6,
>>     omega_0 = abs(C_poles(stage));
>>     psi = pi-angle(C_poles(stage));
>>     Q = 1/(2*cos(psi));
>>     disp(['Stage ',num2str(stage),...
>>            ' (Q = ',num2str(Q),...
>>            '):  R1 = R2 = ',num2str(100000)]);
>>     disp(['         C1 = ',num2str(2*Q/(omega_0*100000)),...
>>            ', C2 = ',num2str(1/(2*Q*omega_0*100000))]);
>>     B = omega_0^2; A = [1 omega_0/Q omega_0^2];
>>     Hmag_SK(stage,:) = abs(polyval(B,j*2*pi*f)./polyval(A,j*2*pi*f));
>> end
>> plot(f,20*log10(Hmag_SK),'k',f,20*log10(prod(Hmag_SK)),'k:')
>> xlabel('f [Hz]'); ylabel('Magnitude Responses [dB]')
>> axis([0 10000 -40 40]);
Stage 1 (Q = 51.7057):  R1 = R2 = 100000
        C1 = 3.311e-008, C2 = 3.0961e-012
Stage 2 (Q = 16.4408):  R1 = R2 = 100000
        C1 = 1.1293e-008, C2 = 1.0445e-011
Stage 3 (Q = 8.885):  R1 = R2 = 100000
        C1 = 7.0991e-009, C2 = 2.2482e-011
Stage 4 (Q = 5.247):  R1 = R2 = 100000
        C1 = 5.4474e-009, C2 = 4.9466e-011
Stage 5 (Q = 2.8635):  R1 = R2 = 100000
        C1 = 4.6778e-009, C2 = 1.4262e-010
Stage 6 (Q = 1.0262):  R1 = R2 = 100000
        C1 = 4.359e-009, C2 = 1.0348e-009
```

The resulting resistor and capacitor values are realistic. Each Sallen-Key stage implements a complex-conjugate pair of poles. The most peaked magnitude response corresponds to the pair of poles that are closest to the ω-axis, or Stage 1. The least peaked magnitude response corresponds to the pair of poles that are furthest from the ω-axis, or Stage 6. The remaining stages are ordered in between. The dashed curve is the total magnitude response, and within a gain error of 3dB is exactly the same as the one shown in Figure S4.M-3a. The gain

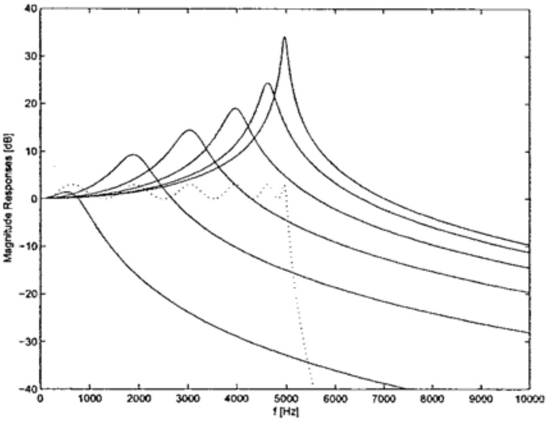


Figure S4.M-3b: Order-12 Chebyshev LPF Sallen-Key Stage Responses.

error occurs since the Salley-Key stages are constrained to unity gain at dc, yet the Chebyshev filter requires gain $\frac{1}{\sqrt{1+\epsilon^2}}$ at dc. This error is easily corrected by adding a gain stage to the circuit.

4.M-4. (a) Using MATLAB, the Sallen-Key component values are easily found.

```
>> omega_0 = 4000*2*pi; NS = 4;
>> psi = [90/(2*NS):90/NS:90]*pi/180;
>> Q = 1./(2*cos(psi));
>> R1 = 1e9/omega_0*ones(1,NS); R2 = R1;
>> C1 = 2*Q./(omega_0*R1); C2 = 1./(2*omega_0*Q.*R2);
>> for stage = 1:NS,
>>     disp(['Stage ',num2str(stage),...
>>             ' (Q = ',num2str(Q(stage)),...
>>             '):  R1 = R2 = ',num2str(R1(stage))]);
>>     disp(['          C1 = ',num2str(C1(stage)),...
>>             ', C2 = ',num2str(C2(stage))]);
>> end
Stage 1 (Q = 0.5098):  R1 = R2 = 39788.7358
        C1 = 1.0196e-009, C2 = 9.8079e-010
Stage 2 (Q = 0.60134):  R1 = R2 = 39788.7358
        C1 = 1.2027e-009, C2 = 8.3147e-010
Stage 3 (Q = 0.89998):  R1 = R2 = 39788.7358
        C1 = 1.8e-009, C2 = 5.5557e-010
Stage 4 (Q = 2.5629):  R1 = R2 = 39788.7358
        C1 = 5.1258e-009, C2 = 1.9509e-010
```

The resulting resistor and capacitor values are realistic.

(b) The transformed Sallen-Key circuit is shown in Figure S4.M-4b. Name the node between capacitors $v(t)$. In transform domain, KCL at the positive terminal of the op-amp yields

$$\frac{Y(s)-V(s)}{\frac{1}{sC_2'}} = -\frac{Y(s)-0}{R_2'}.$$

Solving for $V(s)$ yields

$$V(s) = Y(s)\frac{1+R_2'C_2's}{R_2'C_2's}.$$

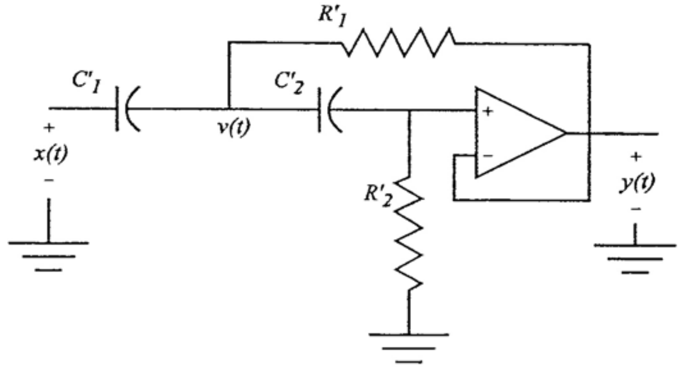


Figure S4.M-4b: $RC-CR$ transformed Sallen-Key circuit.

KCL at node $V(s)$ yields

$$\frac{X(s)-V(s)}{\frac{1}{C_1's}}+\frac{Y(s)-V(s)}{R_1'}+\frac{Y(s)-V(s)}{\frac{1}{C_2's}}=0.$$

Rearranging yields

$$V(s)\left[C_1's+1/R_1'+C_2's\right]=C_1'sX(s)+Y(s)\left[C_2's+1/R_1'\right].$$

Substituting the previous expression for $V(s)$ yields

$$Y(s)\frac{1+R_2'C_2's}{R_2'C_2's}\left[C_1's+1/R_1'+C_2's\right]=C_1'sX(s)+Y(s)\left[C_2's+1/R_1'\right].$$

Rearranging yields

$$Y(s)\left[\frac{1+R_2'C_2's}{R_2'C_2's}\frac{1+R_1'C_1's+R_1'C_2's}{R_1'}-\frac{1+R_1'C_2's}{R_1'}\right]=X(s)\left[C_1's\right].$$

Following simplification, we get

$$H(s)=\frac{Y(s)}{X(s)}=\frac{s^2}{s^2+s\left(\frac{1}{R_1'C_2'}+\frac{1}{R_2'C_1'}\right)+\frac{1}{R_1'R_2'C_1'C_2'}}.$$

(c) MATLAB is used to transform the Butterworth LPF from 4.M-4a.

```
>> R1p = 1./(C1*omega_0); R2p = 1./(C2*omega_0);
>> C1p = 1./(R1*omega_0); C2p = 1./(R2*omega_0);
>> for stage = 1:NS,
>>     disp(['Stage ',num2str(stage),...
>>             ' (Q = ',num2str(Q(stage)),...
>>             '):  C1'' = C2'' = ',num2str(C1p(stage))]);
>>     disp(['        R1'' = ',num2str(R1p(stage)),...
>>             ', R2'' = ',num2str(R2p(stage))]);
>> end
Stage 1 (Q = 0.5098):  C1' = C2' = 1e-009
        R1' = 39024.2064, R2' = 40568.2432
Stage 2 (Q = 0.60134):  C1' = C2' = 1e-009
```

```
       R1' = 33083.1247, R2' = 47853.5056
Stage 3 (Q = 0.89998):  C1' = C2' = 1e-009
       R1' = 22105.4372, R2' = 71617.8323
Stage 4 (Q = 2.5629):  C1' = C2' = 1e-009
       R1' = 7762.3973, R2' = 203950.3311
```

The resulting resistor and capacitor values are realistic.

MATLAB also conveniently computes magnitude responses and pole locations. From $H(s)$, it is clear that all zeros are at zero.

```
>> Hmag_SK = zeros(NS,200); Poles = zeros(NS,2);
>> f = linspace(0,omega_0/pi,200);
>> for stage = 1:NS,
>>     B = [1 0 0];
>>     A = [1,(1/(R2p(stage)*C2p(stage))+1/(R2p(stage)*C1p(stage))),...
>>          1/(R1p(stage)*R2p(stage)*C1p(stage)*C2p(stage))];
>>     Poles(stage,:) = (roots(A)).';
>>     Hmag_SK(stage,:) = abs(polyval(B,j*2*pi*f)./polyval(A,j*2*pi*f));
>> end
>> subplot(121), plot(real(Poles(:)),imag(Poles(:)),'kx',0,0,'ko');
>> axis(omega_0*[-1.1, .1 -1.1 1.1]); axis equal;
>> xlabel('Re(s) = \sigma'); ylabel('Im(s) = \omega');
>> subplot(122), plot(f,Hmag_SK,'k',f,prod(Hmag_SK),'k:');
>> xlabel('f [Hz]'); ylabel('Magnitude Responses');
```

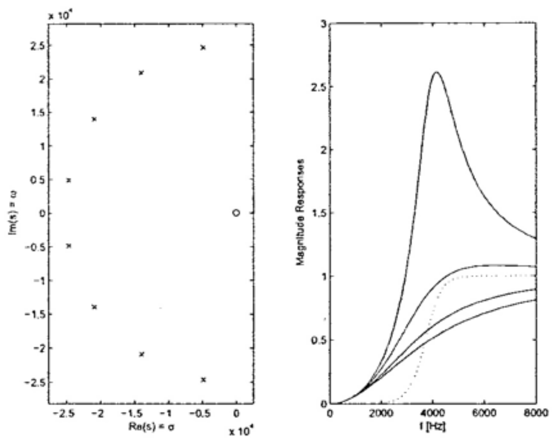


Figure S4.M-4c: Order-8 Butterworth HPF with $\omega_c = 2\pi 4000$.

The overall magnitude response plot looks like a highpass Butterworth filter; the cutoff is correctly located at ω_c and the response is smooth and monotonic. Interestingly, the Butterworth HPF poles look identical to the Butterworth LPF poles. The only difference is seen in the zeros; all the zeros of the LPF are infinite, and all the zeros of the HPF are located at $s = 0$.

4.M-5. (a) Using MATLAB, the Sallen-Key component values are easily found.

```
>> omega_0 = 1500*2*pi; NS = 8;
>> psi = [90/(2*NS):90/NS:90]*pi/180;
>> Q = 1./(2*cos(psi));
>> R1 = 1e9/omega_0*ones(1,NS); R2 = R1;
```

```
>> C1 = 2*Q./(omega_0*R1); C2 = 1./(2*omega_0*Q.*R2);
>> for stage = 1:NS,
>>     disp(['Stage ',num2str(stage),...
>>             ' (Q = ',num2str(Q(stage)),...
>>             '):  R1 = R2 = ',num2str(R1(stage))]);
>>     disp(['        C1 = ',num2str(C1(stage)),...
>>             ', C2 = ',num2str(C2(stage))]);
>> end
Stage 1 (Q = 0.50242):  R1 = R2 = 106103.2954
        C1 = 1.0048e-009, C2 = 9.9518e-010
Stage 2 (Q = 0.5225):  R1 = R2 = 106103.2954
        C1 = 1.045e-009, C2 = 9.5694e-010
Stage 3 (Q = 0.56694):  R1 = R2 = 106103.2954
        C1 = 1.1339e-009, C2 = 8.8192e-010
Stage 4 (Q = 0.64682):  R1 = R2 = 106103.2954
        C1 = 1.2936e-009, C2 = 7.7301e-010
Stage 5 (Q = 0.78815):  R1 = R2 = 106103.2954
        C1 = 1.5763e-009, C2 = 6.3439e-010
Stage 6 (Q = 1.0607):  R1 = R2 = 106103.2954
        C1 = 2.1214e-009, C2 = 4.714e-010
Stage 7 (Q = 1.7224):  R1 = R2 = 106103.2954
        C1 = 3.4449e-009, C2 = 2.9028e-010
Stage 8 (Q = 5.1011):  R1 = R2 = 106103.2954
        C1 = 1.0202e-008, C2 = 9.8017e-011
```

The resulting resistor and capacitor values are realistic.

(b) See the solution to problem 4.M-4b.

(c) MATLAB is used to transform the Butterworth LPF from 4.M-5a.

```
>> R1p = 1./(C1*omega_0); R2p = 1./(C2*omega_0);
>> C1p = 1./(R1*omega_0); C2p = 1./(R2*omega_0);
>> for stage = 1:NS,
>>     disp(['Stage ',num2str(stage),...
>>             ' (Q = ',num2str(Q(stage)),...
>>             '):  C1'' = C2'' = ',num2str(C1p(stage))]);
>>     disp(['        R1'' = ',num2str(R1p(stage)),...
>>             ', R2'' = ',num2str(R2p(stage))]);
>> end
Stage 1 (Q = 0.50242):  C1' = C2' = 1e-009
        R1' = 105592.379, R2' = 106616.6839
Stage 2 (Q = 0.5225):  C1' = C2' = 1e-009
        R1' = 101534.5231, R2' = 110877.6498
Stage 3 (Q = 0.56694):  C1' = C2' = 1e-009
        R1' = 93574.7524, R2' = 120309.2608
Stage 4 (Q = 0.64682):  C1' = C2' = 1e-009
        R1' = 82018.9565, R2' = 137259.8455
Stage 5 (Q = 0.78815):  C1' = C2' = 1e-009
        R1' = 67311.218, R2' = 167251.6057
Stage 6 (Q = 1.0607):  C1' = C2' = 1e-009
        R1' = 50016.7472, R2' = 225082.7957
Stage 7 (Q = 1.7224):  C1' = C2' = 1e-009
        R1' = 30800.1609, R2' = 365514.6265
```

```
Stage 8 (Q = 5.1011):   C1' = C2' = 1e-009
        R1' = 10399.9416, R2' = 1082497.3575
```

The resulting resistor and capacitor values are reasonably realistic.

MATLAB also conveniently computes magnitude responses and pole locations. From $H(s)$, it is clear that all zeros are at zero.

```
>> Hmag_SK = zeros(NS,200); Poles = zeros(NS,2);
>> f = linspace(0,omega_0/pi,200);
>> for stage = 1:NS,
>>     B = [1 0 0];
>>     A = [1,(1/(R2p(stage)*C2p(stage))+1/(R2p(stage)*C1p(stage))),...
>>          1/(R1p(stage)*R2p(stage)*C1p(stage)*C2p(stage))];
>>     Poles(stage,:) = (roots(A)).';
>>     Hmag_SK(stage,:) = abs(polyval(B,j*2*pi*f)./polyval(A,j*2*pi*f));
>> end
>> subplot(121), plot(real(Poles(:)),imag(Poles(:)),'kx',0,0,'ko');
>> axis(omega_0*[-1.1, .1 -1.1 1.1]); axis equal;
>> xlabel('Re(s) = \sigma'); ylabel('Im(s) = \omega');
>> subplot(122), plot(f,Hmag_SK,'k',f,prod(Hmag_SK),'k:');
>> xlabel('f [Hz]'); ylabel('Magnitude Responses');
```

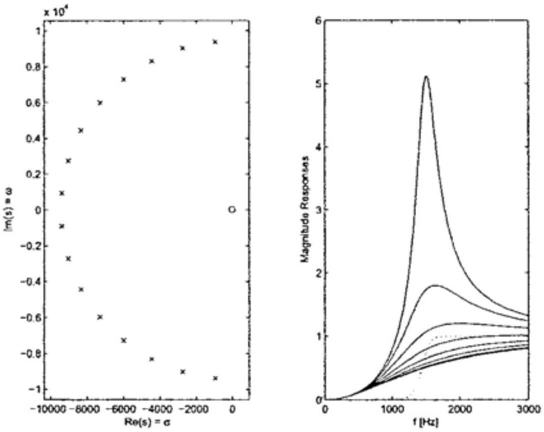


Figure S4.M-5c: Order-16 Butterworth HPF with $\omega_c = 2\pi 1500$.

The overall magnitude response plot looks like a highpass Butterworth filter; the cutoff is correctly located at ω_c and the response is smooth and monotonic. Interestingly, the Butterworth HPF poles look identical to the Butterworth LPF poles. The only difference is seen in the zeros; all the zeros of the LPF are infinite, and all the zeros of the HPF are located at $s = 0$.

4.M-6. (a) Using MATLAB, the Sallen-Key component values are easily found.

```
>> omega_c = 2*pi*4000; R = 3; N = 8;
>> epsilon = sqrt(10^(R/10)-1);
>> k = [1:N]; xi = 1/N*asinh(1/epsilon); phi = (k*2-1)/(2*N)*pi;
>> C_poles = omega_c*(-sinh(xi)*sin(phi)+j*cosh(xi)*cos(phi));
>> C_poles = C_poles(find(imag(C_poles)>0)); % Quadrant 2 poles
>> f = linspace(0,10000,501); Hmag_SK = zeros(6,501);
>> R1 = zeros(N/2,1); R2 = R1; C1 = R1; C2 = R1; Q = R1; omega_0 = R1;
```

```
>> for stage = 1:N/2,
>>     omega_0(stage) = abs(C_poles(stage));
>>     psi = pi-angle(C_poles(stage));
>>     Q(stage) = 1/(2*cos(psi));
>>     R1(stage) = 1e9/omega_c; R2(stage) = R1(stage);
>>     C1(stage) = 2*Q(stage)./(omega_0(stage)*R1(stage));
>>     C2(stage) = 1./(2*omega_0(stage)*Q(stage).*R2(stage));
>>     disp(['Stage ',num2str(stage),...
>>             ' (Q = ',num2str(Q(stage)),...
>>             '): R1 = R2 = ',num2str(R1(stage))]);
>>     disp(['        C1 = ',num2str(C1(stage)),...
>>             ', C2 = ',num2str(C2(stage))]);
>>     B = omega_0(stage)^2; A = [1 omega_0(stage)/Q omega_0(stage)^2];
>>     Hmag_SK(stage,:) = abs(polyval(B,j*2*pi*f)./polyval(A,j*2*pi*f));
>> end
Stage 1 (Q = 22.8704):  R1 = R2 = 39788.7358
        C1 = 4.6343e-008, C2 = 2.215e-011
Stage 2 (Q = 6.8251):  R1 = R2 = 39788.7358
        C1 = 1.6274e-008, C2 = 8.7339e-011
Stage 3 (Q = 3.0798):  R1 = R2 = 39788.7358
        C1 = 1.0874e-008, C2 = 2.8659e-010
Stage 4 (Q = 1.0337):  R1 = R2 = 39788.7358
        C1 = 9.2182e-009, C2 = 2.1569e-009
```

The resulting resistor and capacitor values are realistic.

(b) See the solution to problem 4.M-4b.

(c) MATLAB is used to transform the Chebyshev LPF from 4.M-6a.

```
>> R1p = 1./(C1*omega_c); R2p = 1./(C2*omega_c);
>> C1p = 1./(R1*omega_c); C2p = 1./(R2*omega_c);
>> for stage = 1:N/2,
>>     disp(['Stage ',num2str(stage),...
>>             ' (Q = ',num2str(Q(stage)),...
>>             '): C1'' = C2'' = ',num2str(C1p(stage))]);
>>     disp(['        R1'' = ',num2str(R1p(stage)),...
>>             ', R2'' = ',num2str(R2p(stage))]);
>> end
Stage 1 (Q = 22.8704):  C1' = C2' = 1e-009
        R1' = 858.5676, R2' = 1796313.2499
Stage 2 (Q = 6.8251):  C1' = C2' = 1e-009
        R1' = 2444.9937, R2' = 455567.9755
Stage 3 (Q = 3.0798):  C1' = C2' = 1e-009
        R1' = 3659.1916, R2' = 138833.4048
Stage 4 (Q = 1.0337):  C1' = C2' = 1e-009
        R1' = 4316.3108, R2' = 18446.8849
```

The resulting resistor and capacitor values possibly realistic; however, there is a fairly large dynamic range between the largest and smallest resistors.

MATLAB also conveniently computes magnitude responses and pole locations. By necessity, the transformation really stretches out the passband; it is therefore important to plot the magnitude response over a broad range of frequencies. To facilitate a reasonable plot, the magnitude response is plotted using both log-magnitude and log-frequency scales. From $H(s)$, it is clear that all zeros are at zero.

```
>> Hmag_SK = zeros(N/2,5001); Poles = zeros(N/2,2);
>> f = logspace(2,5,5001);
>> for stage = 1:N/2,
>>     B = [1 0 0];
>>     A = [1,(1/(R2p(stage)*C2p(stage))+1/(R2p(stage)*C1p(stage))),...
>>          1/(R1p(stage)*R2p(stage)*C1p(stage)*C2p(stage))];
>>     Poles(stage,:) = (roots(A)).';
>>     Hmag_SK(stage,:) = abs(polyval(B,j*2*pi*f)./polyval(A,j*2*pi*f));
>> end
>> subplot(121), plot(real(Poles(:)),imag(Poles(:)),'kx',0,0,'ko');
>> axis equal; ax = axis; axis([1.1*ax]);
>> xlabel('Re(s) = \sigma'); ylabel('Im(s) = \omega');
>> subplot(122),
>> semilogx(f,20*log10(Hmag_SK),'k',f,20*log10(prod(Hmag_SK)),'k:')
>> xlabel('f [Hz]'); ylabel('Magnitude Responses [dB]'); axis tight
>> axis([100 1e5 -40 40]);
```

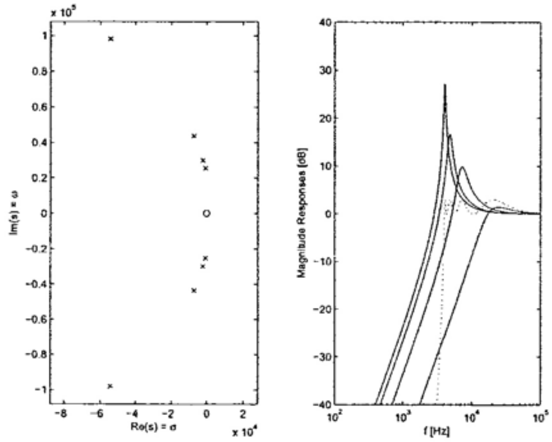


Figure S4.M-6c: Order-8 Chebyshev HPF with $\omega_c = 2\pi 4000$ and $R = 3$dB.

The pole locations of the transformed Chebyshev filter are dramatically different than the pole locations of the original LPF. The zeros, as expected, are all concentrated at $s = 0$. The overall magnitude response plot looks like a highpass Chebyshev filter; passband ripples are equal in height and never exceed $R = 3$dB, there are a total of $N = 8$ maxima and minima in the passband, and the cutoff is correctly located at $\omega_c = 2\pi 4000$.

4.M-7. Factored form is used to plot roots, and standard transfer function form is used to compute magnitude response plots.

(a) Order-6 Butterworth LPF with $\omega_c = 2\pi 3500$.

```
>> omega_c = 2*pi*3500;
>> [z,p,k] = butter(6,omega_c,'s');
>> subplot(121),plot(real(p),imag(p),'kx',...
   real(z),imag(z),'ko');
```

```
>> axis(omega_c*[-1.1 0.1 -1.1 1.1]); axis equal;
>> xlabel('Re(s) = \sigma'); ylabel('Im(s) = \omega');
>> f = linspace(0,7000,501);
>> [B,A] = butter(6,omega_c,'s');
>> HLP = polyval(B,j*2*pi*f)./polyval(A,j*2*pi*f);
>> subplot(122),plot(f,20*log10(abs(HLP)),'k');
>> axis([0 7000 -40 2])
>> xlabel('f [Hz]'); ylabel('|H_{LP}(j\omega)|');
```

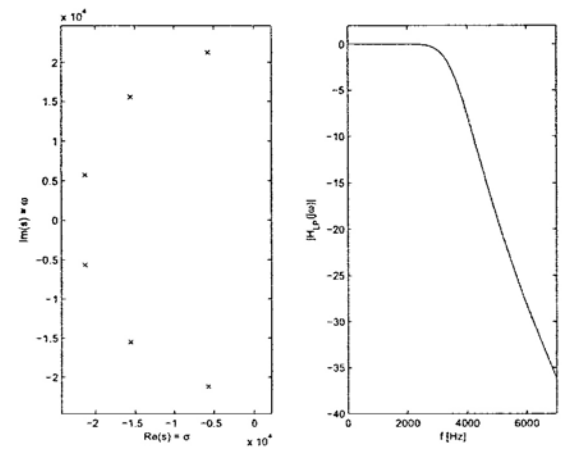


Figure S4.M-7a: Order-6 Butterworth LPF with $\omega_c = 2\pi 3500$.

(b) Order-6 Butterworth HPF with $\omega_c = 2\pi 3500$.

```
>> omega_c = 2*pi*3500;
>> [z,p,k] = butter(6,omega_c,'high','s');
>> subplot(121),plot(real(p),imag(p),'kx',...
    real(z),imag(z),'ko');
>> axis(omega_c*[-1.1 0.1 -1.1 1.1]); axis equal;
>> xlabel('Re(s) = \sigma'); ylabel('Im(s) = \omega');
>> f = linspace(0,7000,501);
>> [B,A] = butter(6,omega_c,'high','s');
>> HHP = polyval(B,j*2*pi*f)./polyval(A,j*2*pi*f);
>> subplot(122),plot(f,20*log10(abs(HHP)),'k');
>> axis([0 7000 -40 2])
>> xlabel('f [Hz]'); ylabel('|H_{HP}(j\omega)|');
```

(c) Order-6 Butterworth BPF with passband between 2kHz and 4kHz. Notice that the command `butter` requires the parameter $N = 3$ to be used to obtain a $(2N = 6)$-order bandpass filter.

```
>> omega_c = [2*pi*2000,2*pi*4000];
>> [z,p,k] = butter(3,omega_c,'s');
>> subplot(121),plot(real(p),imag(p),'kx',...
    real(z),imag(z),'ko');
>> axis(omega_c(2)*[-1.1 0.1 -1.1 1.1]); axis equal;
>> xlabel('Re(s) = \sigma'); ylabel('Im(s) = \omega');
>> f = linspace(0,7000,501);
>> [B,A] = butter(3,omega_c,'s');
```

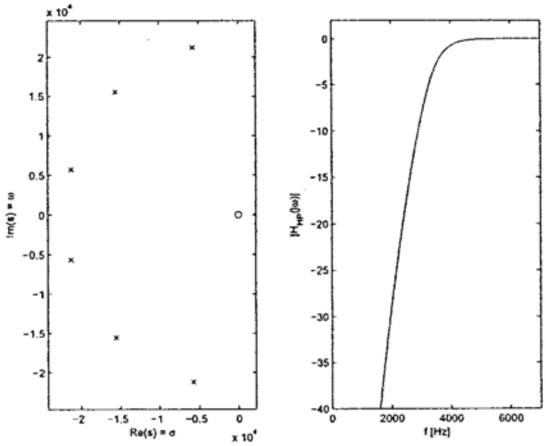


Figure S4.M-7b: Order-6 Butterworth HPF with $\omega_c = 2\pi 3500$.

```
>> HBP = polyval(B,j*2*pi*f)./polyval(A,j*2*pi*f);
>> subplot(122),plot(f,20*log10(abs(HBP)),'k');
>> axis([0 7000 -40 2])
>> xlabel('f [Hz]'); ylabel('|H_{BP}(j\omega)|');
```

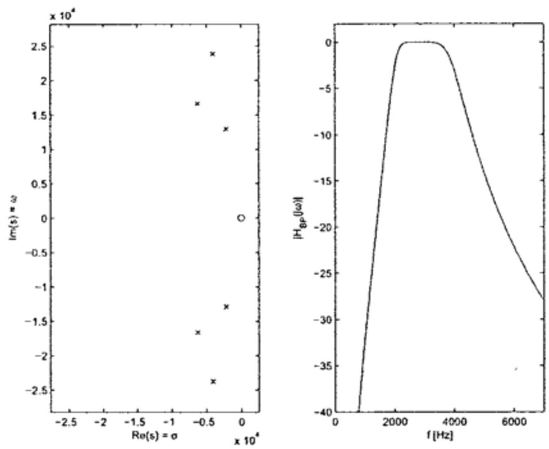


Figure S4.M-7c: Order-6 Butterworth BPF with passband between 2kHz and 4kHz.

(d) Order-6 Butterworth BSF with stopband between 2kHz and 4kHz. Notice that the command butter requires the parameter $N = 3$ to be used to obtain a $(2N = 6)$-order bandstop filter.

```
>> omega_c = [2*pi*2000,2*pi*4000];
>> [z,p,k] = butter(3,omega_c,'stop','s');
>> subplot(121),plot(real(p),imag(p),'kx',...
    real(z),imag(z),'ko');
>> axis(omega_c(2)*[-1.1 0.1 -1.1 1.1]); axis equal;
>> xlabel('Re(s) = \sigma'); ylabel('Im(s) = \omega');
>> f = linspace(0,7000,501);
>> [B,A] = butter(3,omega_c,'stop','s');
```

```
>> HBS = polyval(B,j*2*pi*f)./polyval(A,j*2*pi*f);
>> subplot(122),plot(f,20*log10(abs(HBS)),'k');
>> axis([0 7000 -40 2])
>> xlabel('f [Hz]'); ylabel('|H_{BS}(j\omega)|');
```

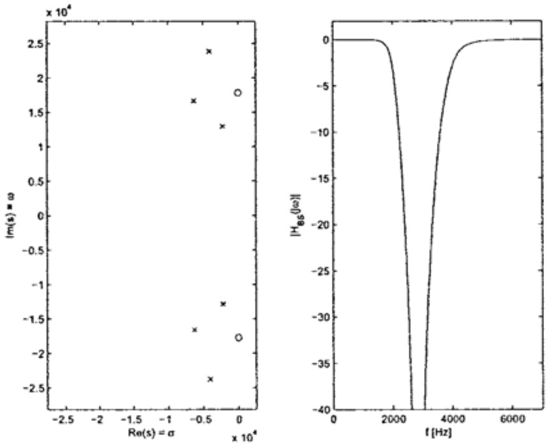


Figure S4.M-7d: Order-6 Butterworth BSF with stopband between 2kHz and 4kHz.

4.M-8. Factored form is used to plot roots, and standard transfer function form is used to compute magnitude response plots.

(a) Order-6 Chebyshev Type I LPF with $\omega_c = 2\pi 3500$.

```
>> omega_c = 2*pi*3500;
>> [z,p,k] = cheby1(6,3,omega_c,'s');
>> subplot(121),plot(real(p),imag(p),'kx',...
    real(z),imag(z),'ko');
>> axis equal; axis(1.1*axis);
>> xlabel('Re(s) = \sigma'); ylabel('Im(s) = \omega');
>> f = linspace(0,7000,501);
>> [B,A] = cheby1(6,3,omega_c,'s');
>> HLP = polyval(B,j*2*pi*f)./polyval(A,j*2*pi*f);
>> subplot(122),plot(f,20*log10(abs(HLP)),'k');
>> axis([0 7000 -40 2])
>> xlabel('f [Hz]'); ylabel('|H_{LP}(j\omega)|');
```

(b) Order-6 Chebyshev Type I HPF with $\omega_c = 2\pi 3500$.

```
>> omega_c = 2*pi*3500;
>> [z,p,k] = cheby1(6,3,omega_c,'high','s');
>> subplot(121),plot(real(p),imag(p),'kx',...
    real(z),imag(z),'ko');
>> axis equal; axis(1.1*axis);
>> xlabel('Re(s) = \sigma'); ylabel('Im(s) = \omega');
>> f = linspace(0,7000,501);
>> [B,A] = cheby1(6,3,omega_c,'high','s');
>> HHP = polyval(B,j*2*pi*f)./polyval(A,j*2*pi*f);
>> subplot(122),plot(f,20*log10(abs(HHP)),'k');
>> axis([0 7000 -40 2])
```

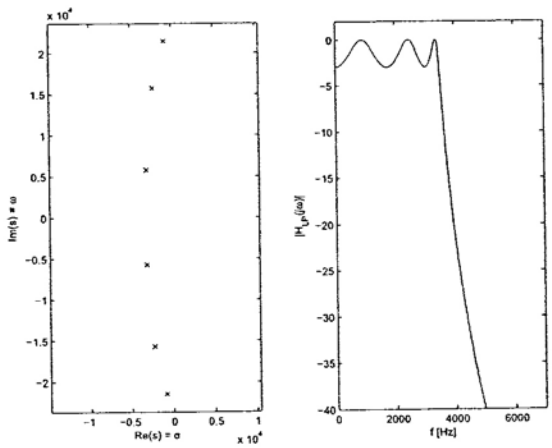


Figure S4.M-8a: Order-6 Chebyshev Type I LPF with $\omega_c = 2\pi 3500$.

```
>> xlabel('f [Hz]'); ylabel('|H_{HP}(j\omega)|');
```

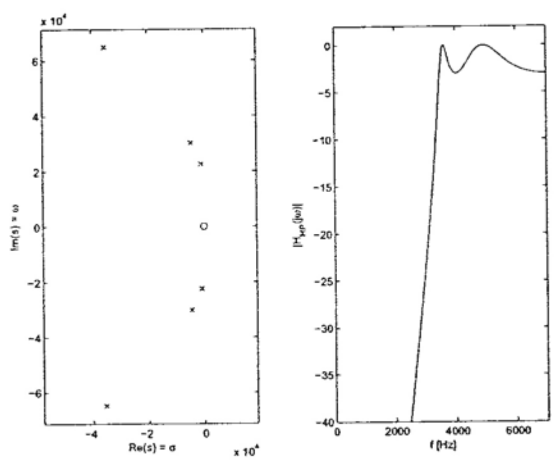


Figure S4.M-8b: Order-6 Chebyshev Type I HPF with $\omega_c = 2\pi 3500$.

(c) Order-6 Chebyshev Type I BPF with passband between 2kHz and 4kHz. Notice that the command `cheby1` requires the parameter $N = 3$ to be used to obtain a $(2N = 6)$-order bandpass filter.

```
>> omega_c = [2*pi*2000,2*pi*4000];
>> [z,p,k] = cheby1(3,3,omega_c,'s');
>> subplot(121),plot(real(p),imag(p),'kx',...
     real(z),imag(z),'ko');
>> axis equal; axis(1.1*axis);
>> xlabel('Re(s) = \sigma'); ylabel('Im(s) = \omega');
>> f = linspace(0,7000,501);
>> [B,A] = cheby1(3,3,omega_c,'s');
>> HBP = polyval(B,j*2*pi*f)./polyval(A,j*2*pi*f);
>> subplot(122),plot(f,20*log10(abs(HBP)),'k');
>> axis([0 7000 -40 2])
```

```
>> xlabel('f [Hz]'); ylabel('|H_{BP}(j\omega)|');
```

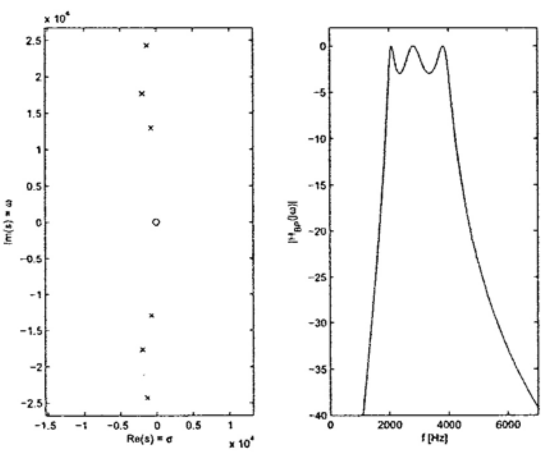


Figure S4.M-8c: Order-6 Chebyshev Type I BPF with passband between 2kHz and 4kHz.

(d) Order-6 Chebyshev Type I BSF with stopband between 2kHz and 4kHz. Notice that the command cheby1 requires the parameter $N = 3$ to be used to obtain a $(2N = 6)$-order bandstop filter.

```
>> omega_c = [2*pi*2000,2*pi*4000];
>> [z,p,k] = cheby1(3,3,omega_c,'stop','s');
>> subplot(121),plot(real(p),imag(p),'kx',...
    real(z),imag(z),'ko');
>> axis equal; axis(1.1*axis);
>> xlabel('Re(s) = \sigma'); ylabel('Im(s) = \omega');
>> f = linspace(0,7000,501);
>> [B,A] = cheby1(3,3,omega_c,'stop','s');
>> HBS = polyval(B,j*2*pi*f)./polyval(A,j*2*pi*f);
>> subplot(122),plot(f,20*log10(abs(HBS)),'k');
>> axis([0 7000 -40 2])
>> xlabel('f [Hz]'); ylabel('|H_{BS}(j\omega)|');
```

To demonstrate the effect of decreasing the passband ripple, consider magnitude response plots for Chebyshev Type I LPFs with $R_p = \{0.1, 1.0, 3.0\}$.

```
>> omega_c = 2*pi*3500; f = linspace(0,7000,501);
>> [B,A] = cheby1(6,.1,omega_c,'s');
>> HLP1 = polyval(B,j*2*pi*f)./polyval(A,j*2*pi*f);
>> [B,A] = cheby1(6,1,omega_c,'s');
>> HLP2 = polyval(B,j*2*pi*f)./polyval(A,j*2*pi*f);
>> [B,A] = cheby1(6,3,omega_c,'s');
>> HLP3 = polyval(B,j*2*pi*f)./polyval(A,j*2*pi*f);
>> plot(f,20*log10(abs(HLP1)),'k-',...
      f,20*log10(abs(HLP2)),'k--',...
      f,20*log10(abs(HLP3)),'k:');
>> axis([0 7000 -40 2])
>> xlabel('f [Hz]'); ylabel('|H_{LP}(j\omega)|');
>> legend('R_p = 0.1','R_p = 1.0','R_p = 3.0',0);
```

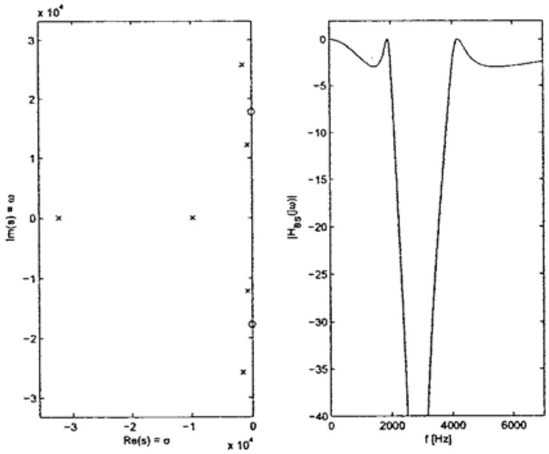


Figure S4.M-8d: Order-6 Chebyshev Type I BSF with stopband between 2kHz and 4kHz.

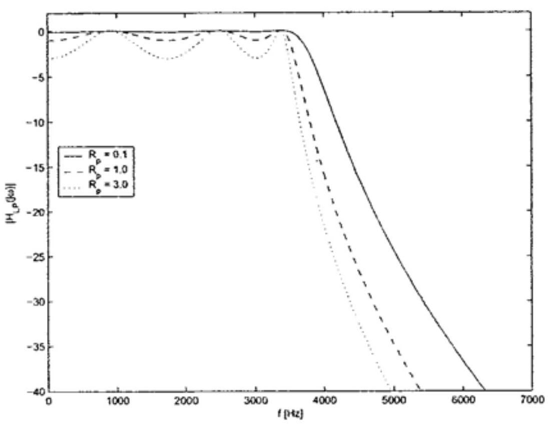


Figure S4.M-8d: Changing R_p for a Chebyshev Type I filter.

Thus, reducing the allowable passband ripple R_p tends to broaden the transition bands of the filter.

4.M-9. Factored form is used to plot roots, and standard transfer function form is used to compute magnitude response plots.

(a) Order-6 Chebyshev Type II LPF with $\omega_c = 2\pi 3500$.

```
>> omega_c = 2*pi*3500;
>> [z,p,k] = cheby2(6,20,omega_c,'s');
>> subplot(121),plot(real(p),imag(p),'kx',...
    real(z),imag(z),'ko');
>> axis equal; axis(1.1*axis);
>> xlabel('Re(s) = \sigma'); ylabel('Im(s) = \omega');
>> f = linspace(0,7000,501);
>> [B,A] = cheby2(6,20,omega_c,'s');
>> HLP = polyval(B,j*2*pi*f)./polyval(A,j*2*pi*f);
>> subplot(122),plot(f,20*log10(abs(HLP)),'k');
>> axis([0 7000 -40 2])
```

```
>> xlabel('f [Hz]'); ylabel('|H_{LP}(j\omega)|');
```

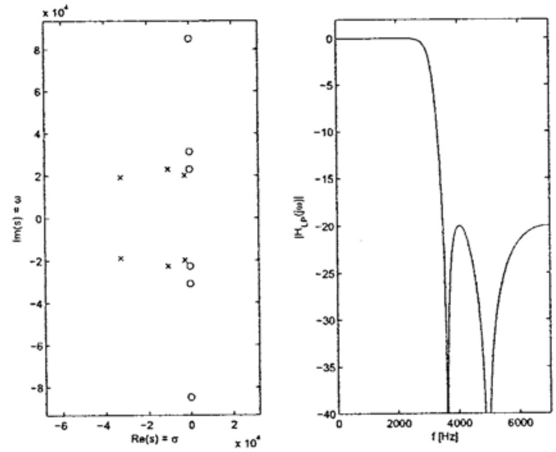


Figure S4.M-9a: Order-6 Chebyshev Type II LPF with $\omega_c = 2\pi 3500$.

(b) Order-6 Chebyshev Type II HPF with $\omega_c = 2\pi 3500$.

```
>> omega_c = 2*pi*3500;
>> [z,p,k] = cheby2(6,20,omega_c,'high','s');
>> subplot(121),plot(real(p),imag(p),'kx',...
    real(z),imag(z),'ko');
>> axis equal; axis(1.1*axis);
>> xlabel('Re(s) = \sigma'); ylabel('Im(s) = \omega');
>> f = linspace(0,7000,501);
>> [B,A] = cheby2(6,20,omega_c,'high','s');
>> HHP = polyval(B,j*2*pi*f)./polyval(A,j*2*pi*f);
>> subplot(122),plot(f,20*log10(abs(HHP)),'k');
>> axis([0 7000 -40 2])
>> xlabel('f [Hz]'); ylabel('|H_{HP}(j\omega)|');
```

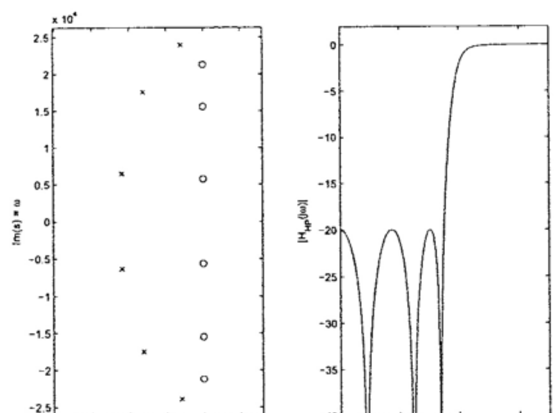


Figure S4.M-9b: Order-6 Chebyshev Type II HPF with $\omega_c = 2\pi 3500$.

(c) Order-6 Chebyshev Type II BPF with passband between 2kHz and 4kHz. Notice that the command cheby2 requires the parameter $N = 3$ to be used to obtain a $(2N = 6)$-order bandpass filter.

```
>> omega_c = [2*pi*2000,2*pi*4000];
>> [z,p,k] = cheby2(3,20,omega_c,'s');
>> subplot(121),plot(real(p),imag(p),'kx',...
    real(z),imag(z),'ko');
>> axis equal; axis(1.1*axis);
>> xlabel('Re(s) = \sigma'); ylabel('Im(s) = \omega');
>> f = linspace(0,7000,501);
>> [B,A] = cheby2(3,20,omega_c,'s');
>> HBP = polyval(B,j*2*pi*f)./polyval(A,j*2*pi*f);
>> subplot(122),plot(f,20*log10(abs(HBP)),'k');
>> axis([0 7000 -40 2])
>> xlabel('f [Hz]'); ylabel('|H_{BP}(j\omega)|');
```

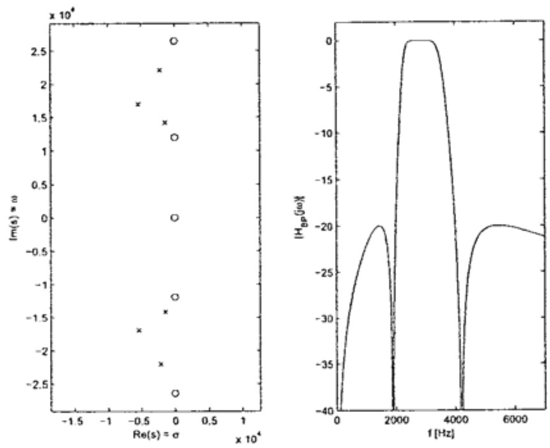


Figure S4.M-9c: Order-6 Chebyshev Type II BPF with passband between 2kHz and 4kHz.

(d) Order-6 Chebyshev Type II BSF with stopband between 2kHz and 4kHz. Notice that the command cheby2 requires the parameter $N = 3$ to be used to obtain a $(2N = 6)$-order bandstop filter.

```
>> omega_c = [2*pi*2000,2*pi*4000];
>> [z,p,k] = cheby2(3,20,omega_c,'stop','s');
>> subplot(121),plot(real(p),imag(p),'kx',...
    real(z),imag(z),'ko');
>> axis equal; axis(1.1*axis);
>> xlabel('Re(s) = \sigma'); ylabel('Im(s) = \omega');
>> f = linspace(0,7000,501);
>> [B,A] = cheby2(3,20,omega_c,'stop','s');
>> HBS = polyval(B,j*2*pi*f)./polyval(A,j*2*pi*f);
>> subplot(122),plot(f,20*log10(abs(HBS)),'k');
>> axis([0 7000 -40 2])
>> xlabel('f [Hz]'); ylabel('|H_{BS}(j\omega)|');
```

To demonstrate the effect of increasing R_s, consider magnitude response plots for Chebyshev Type II LPFs with $R_s = \{10, 20, 30\}$.

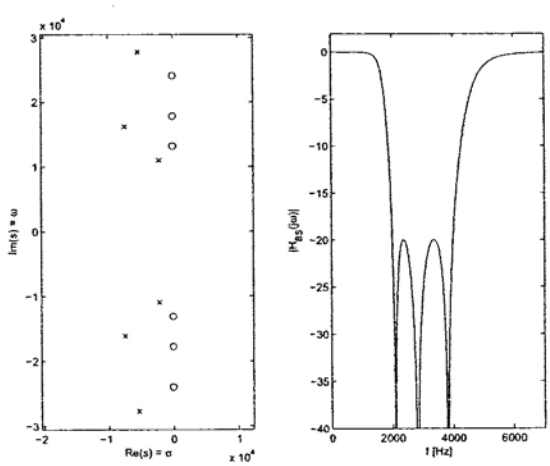


Figure S4.M-9d: Order-6 Chebyshev Type II BSF with stopband between 2kHz and 4kHz.

```
>> omega_c = 2*pi*3500; f = linspace(0,7000,501);
>> [B,A] = cheby2(6,10,omega_c,'s');
>> HLP1 = polyval(B,j*2*pi*f)./polyval(A,j*2*pi*f);
>> [B,A] = cheby2(6,20,omega_c,'s');
>> HLP2 = polyval(B,j*2*pi*f)./polyval(A,j*2*pi*f);
>> [B,A] = cheby2(6,30,omega_c,'s');
>> HLP3 = polyval(B,j*2*pi*f)./polyval(A,j*2*pi*f);
>> plot(f,20*log10(abs(HLP1)),'k-',...
       f,20*log10(abs(HLP2)),'k--',...
       f,20*log10(abs(HLP3)),'k:');
>> axis([0 7000 -40 2])
>> xlabel('f [Hz]'); ylabel('|H_{LP}(j\omega)|');
>> legend('R_s = 10','R_s = 20','R_s = 30',0);
```

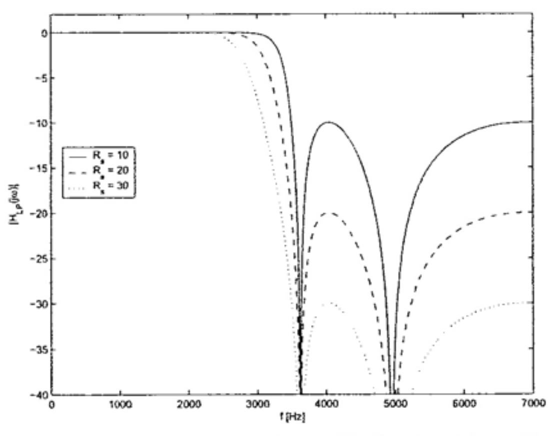


Figure S4.M-9d: Changing R_s for a Chebyshev Type II filter.

Thus, increasing R_s tends to broaden the transition bands of the filter.

4.M-10. Factored form is used to plot roots, and standard transfer function form is used to compute magnitude response plots.

(a) Order-6 Elliptic LPF with $\omega_c = 2\pi 3500$.

```
>> omega_c = 2*pi*3500;
>> [z,p,k] = ellip(6,3,20,omega_c,'s');
>> subplot(121),plot(real(p),imag(p),'kx',...
   real(z),imag(z),'ko');
>> axis equal; axis(1.1*axis);
>> xlabel('Re(s) = \sigma'); ylabel('Im(s) = \omega');
>> f = linspace(0,7000,2001);
>> [B,A] = ellip(6,3,20,omega_c,'s');
>> HLP = polyval(B,j*2*pi*f)./polyval(A,j*2*pi*f);
>> subplot(122),plot(f,20*log10(abs(HLP)),'k');
>> axis([0 7000 -40 2])
>> xlabel('f [Hz]'); ylabel('|H_{LP}(j\omega)|');
```

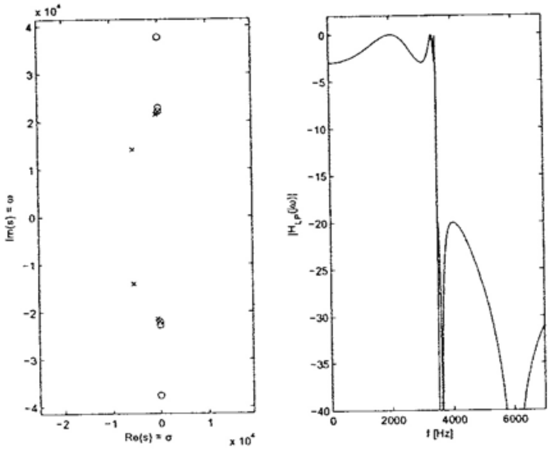


Figure S4.M-10a: Order-6 Elliptic LPF with $\omega_c = 2\pi 3500$.

(b) Order-6 Elliptic HPF with $\omega_c = 2\pi 3500$.

```
>> omega_c = 2*pi*3500;
>> [z,p,k] = ellip(6,3,20,omega_c,'high','s');
>> subplot(121),plot(real(p),imag(p),'kx',...
   real(z),imag(z),'ko');
>> axis equal; axis(1.1*axis);
>> xlabel('Re(s) = \sigma'); ylabel('Im(s) = \omega');
>> f = linspace(0,7000,2001);
>> [B,A] = ellip(6,3,20,omega_c,'high','s');
>> HHP = polyval(B,j*2*pi*f)./polyval(A,j*2*pi*f);
>> subplot(122),plot(f,20*log10(abs(HHP)),'k');
>> axis([0 7000 -40 2])
>> xlabel('f [Hz]'); ylabel('|H_{HP}(j\omega)|');
```

(c) Order-6 Elliptic BPF with passband between 2kHz and 4kHz. Notice that the command `ellip` requires the parameter $N = 3$ to be used to obtain a $(2N = 6)$-order bandpass filter.

```
>> omega_c = [2*pi*2000,2*pi*4000];
>> [z,p,k] = ellip(3,3,20,omega_c,'s');
```

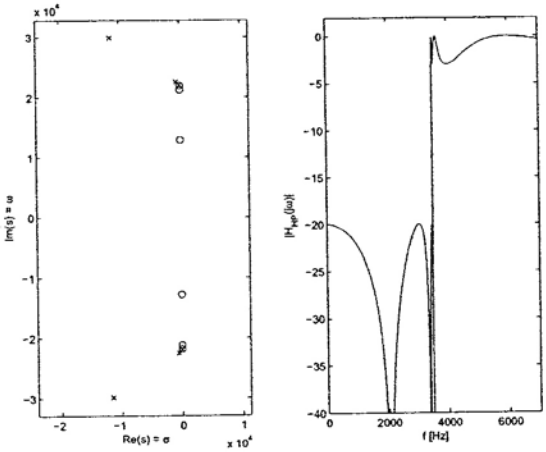


Figure S4.M-10b: Order-6 Elliptic HPF with $\omega_c = 2\pi 3500$.

```
>> subplot(121),plot(real(p),imag(p),'kx',...
   real(z),imag(z),'ko');
>> axis equal; axis(1.1*axis);
>> xlabel('Re(s) = \sigma'); ylabel('Im(s) = \omega');
>> f = linspace(0,7000,2001);
>> [B,A] = ellip(3,3,20,omega_c,'s');
>> HBP = polyval(B,j*2*pi*f)./polyval(A,j*2*pi*f);
>> subplot(122),plot(f,20*log10(abs(HBP)),'k');
>> axis([0 7000 -40 2])
>> xlabel('f [Hz]'); ylabel('|H_{BP}(j\omega)|');
```

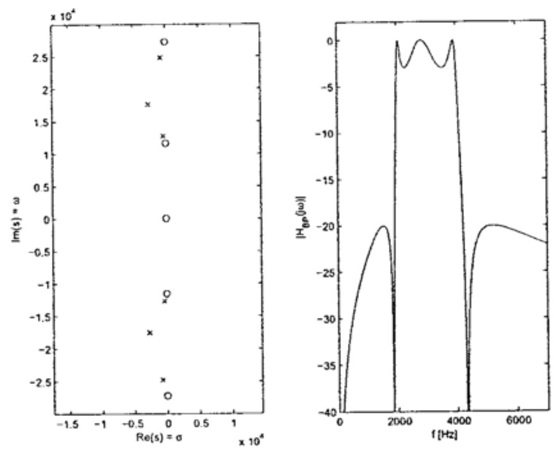


Figure S4.M-10c: Order-6 Elliptic BPF with passband between 2kHz and 4kHz.

(d) Order-6 Elliptic BSF with stopband between 2kHz and 4kHz. Notice that the command `ellip` requires the parameter $N = 3$ to be used to obtain a $(2N = 6)$-order bandstop filter.

```
>> omega_c = [2*pi*2000,2*pi*4000];
>> [z,p,k] = ellip(3,3,20,omega_c,'stop','s');
```

```
>> subplot(121),plot(real(p),imag(p),'kx',...
    real(z),imag(z),'ko');
>> axis equal; axis(1.1*axis);
>> xlabel('Re(s) = \sigma'); ylabel('Im(s) = \omega');
>> f = linspace(0,7000,2001);
>> [B,A] = ellip(3,3,20,omega_c,'stop','s');
>> HBS = polyval(B,j*2*pi*f)./polyval(A,j*2*pi*f);
>> subplot(122),plot(f,20*log10(abs(HBS)),'k');
>> axis([0 7000 -40 2])
>> xlabel('f [Hz]'); ylabel('|H_{BS}(j\omega)|');
```

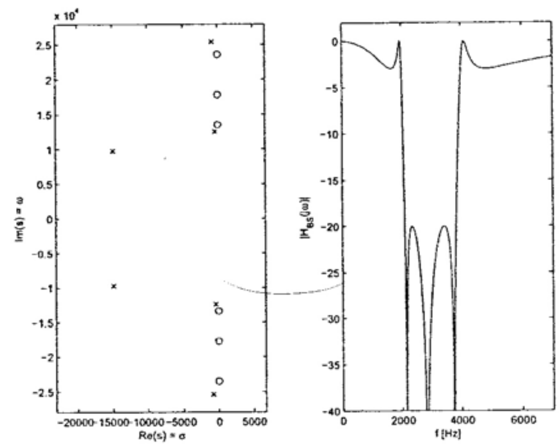


Figure S4.M-10d: Order-6 Elliptic BSF with stopband between 2kHz and 4kHz.

4.M-11. First, the recursion relation $C_N(x) = 2xC_{N-1}(x) - C_{N-2}(x)$ is rewritten as $C_{N+1}(x) = 2xC_N(x) - C_{N-1}(x)$ or $C_{N+1} + C_{N-1} = 2xC_N(x)$.

Letting $\gamma = \cosh^{-1}(x)$ and using Euler's formula, we know $C_N(x) = \cosh\left(N\cosh^{-1}(x)\right) = \cosh(N\gamma) = \frac{e^{N\gamma}+e^{-N\gamma}}{2}$. Thus, $C_{N+1} + C_{N-1} = \frac{e^{(N+1)\gamma}+e^{-(N+1)\gamma}}{2} + \frac{e^{(N-1)\gamma}+e^{-(N-1)\gamma}}{2} = \frac{e^{N\gamma}(e^{\gamma}+e^{-\gamma})+e^{-N\gamma}(e^{\gamma}+e^{-\gamma})}{2} = 2\frac{(e^{\gamma}+e^{-\gamma})}{2}\frac{(e^{N\gamma}+e^{-N\gamma})}{2} = 2\cosh(\gamma)\cosh(N\gamma)$. Replacing γ yields $C_{N+1} + C_{N-1} = 2\cosh(\cosh^{-1}(x))\cosh(N\cosh^{-1}(x)) = 2xC_N(x)$. Thus,

$$C_{N+1} + C_{N-1} = 2xC_N(x) \text{ or } C_N(x) = 2xC_{N-1}(x) - C_{N-2}(x).$$

4.M-12. Note that $p_k = \sigma_k + j\omega_k = \omega_c \sinh(\xi)\sin(\phi_k) + j\omega_c\cosh(\xi)\cos(\phi_k)$. From the real portion, we know $\sigma_k = \omega_c\sinh(\xi)\sin(\phi_k)$ or $\sin(\phi_k) = \frac{\sigma_k}{\omega_c\sinh(\xi)}$. From the imaginary portion, we know $\omega_k = \omega_c\cosh(\xi)\cos(\phi_k)$ or $\cos(\phi_k) = \frac{\omega_k}{\omega_c\cosh(\xi)}$. From trigonometry, we know $1 = \cos^2(\phi_k) + \sin^2(\phi_k)$. Thus,

$$\left(\frac{\omega_k}{\omega_c\cosh(\xi)}\right)^2 + \left(\frac{\sigma_k}{\omega_c\sinh(\xi)}\right)^2 = 1.$$

This is the equation of an ellipse. Since the Chebyshev poles $p_k = \sigma_k + j\omega_k$ satisfy the equation, they must lie on the ellipse.

Chapter 5 Solutions

5.1-1. Given the fact that the time-domain signal is finite in duration, the region of convergence should include the entire z-plane, except possibly $z = 0$ or $z = \infty$. $X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=0}^{7}(-1)^n z^{-n} = \sum_{n=0}^{7}(-1/z)^n = \frac{1-(-1/z)^8}{1-(-1/z)}$. Thus,

$$X(z) = \frac{1-z^{-8}}{1+z^{-1}};\ \text{ROC } |z| > 0.$$

In this form, $X(z)$ appears to have eight finite zeros and one finite pole. The eight zeros are the eight roots of unity, or $z = e^{2\pi k/8}$ for $k = (0, 1, \ldots, 7)$. The apparent pole is at $z = -1$. However, there is also a zero $z = -1$ ($k = 4$) that cancels this pole. Thus, there are actually no finite poles and only seven finite zeros, $z = e^{2\pi k/8}$ for $k = (0, 1, 2, 3, 5, 6, 7)$. MATLAB is used to plot the zeros in the complex plane; the unit circle is also plotted for reference.

```
>> k = [0:3,5:7]; zz = exp(j*2*pi*k/8);
>> ang = linspace(0,2*pi,201);
>> plot(real(zz),imag(zz),'ko',cos(ang),sin(ang),'k');
>> xlabel('Re(z)'); ylabel('Im(z)');
>> axis([-1.1 1.1 -1.1 1.1]); axis equal; grid;
```

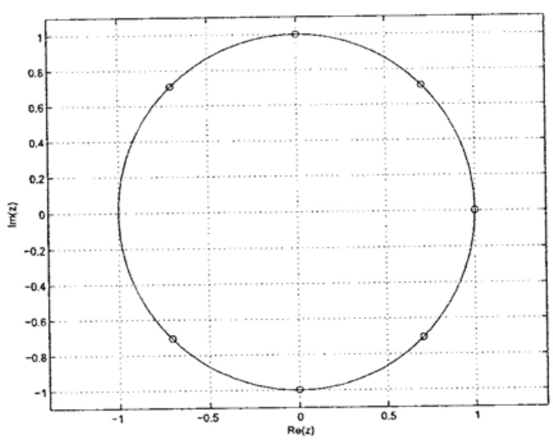


Figure S5.1-1: Pole-zero plot for $x[n] = (-1)^n(u[n] - u[n-8])$.

5.1-2. (a)

$$X[z] = \sum_{n=m}^{\infty} z^{-n} = z^{-m} + z^{-(m+1)} + z^{-(m+2)} + \cdots$$

$$\begin{aligned} &= z^{-m}\left[1+\frac{1}{z}+\frac{1}{z^2}+\cdots\right] \\ &= z^{-m}\left(\frac{1}{1-\frac{1}{z}}\right) = \frac{z}{z^m(z-1)} \end{aligned}$$

(b) $\gamma^n \sin \pi n u[n] = 0$ for all n.
Hence

$$X[z] = 0$$

(c) $\gamma^n \cos \pi n u[n] = (-\gamma)^n u[n]$
Hence

$$X(z) = \frac{z}{z+\gamma}$$

(d) $\gamma^n \sin \frac{\pi n}{2} u[n]$ is a sequence

$$0,\ \gamma^1,\ 0,\ -\gamma^3,\ 0,\ \gamma^5,\ 0,\ -\gamma^7,\ \cdots$$

Hence

$$\begin{aligned} X(z) &= \left(\frac{\gamma}{z}+\frac{\gamma^5}{z^5}+\frac{\gamma^9}{z^9}+\cdots\right) - \left(\frac{\gamma^3}{z^3}+\frac{\gamma^7}{z^7}+\frac{\gamma^{11}}{z^{11}}+\cdots\right) \\ &= \frac{\gamma}{z}\left[1+\left(\frac{\gamma}{z}\right)^4+\left(\frac{\gamma}{z}\right)^8+\cdots\right] - \left(\frac{\gamma}{z}\right)^3\left[1+\left(\frac{\gamma}{z}\right)^4+\left(\frac{\gamma}{z}\right)^8+\cdots\right] \\ &= \frac{\gamma}{z}\left[\frac{1}{1-\left(\frac{\gamma}{z}\right)^4}\right] - \left(\frac{\gamma}{z}\right)^3\left[\frac{1}{1-\left(\frac{\gamma}{z}\right)^4}\right] \qquad \left|\frac{\gamma}{z}\right| < 1 \\ &= \frac{\gamma}{z}\left[1-\left(\frac{\gamma}{z}\right)^2\right]\left[\frac{1}{1-\left(\frac{\gamma}{z}\right)^4}\right] \qquad |z| > |\gamma| \\ &= \frac{\gamma z}{z^2+\gamma^2} \end{aligned}$$

(e) $\gamma^n \cos \frac{\pi n}{2} u[n]$ is a sequence

$$1,\ 0,\ -\gamma^2,\ 0,\ \gamma^4,\ 0,\ -\gamma^6,\ 0,\ \gamma^8,\ \cdots$$

Hence

$$\begin{aligned} X[z] &= 1-\frac{\gamma^2}{z^2}+\frac{\gamma^4}{z^4}-\frac{\gamma^6}{z^6}+\frac{\gamma^8}{z^8}-\cdots \\ &= \left[1+\left(\frac{\gamma}{z}\right)^4+\left(\frac{\gamma}{z}\right)^8+\cdots\right] - \left(\frac{\gamma}{z}\right)^2\left[1+\left(\frac{\gamma}{z}\right)^4+\left(\frac{\gamma}{z}\right)^8+\cdots\right] \\ &= \left(1-\frac{\gamma^2}{z^2}\right)\left(\frac{1}{1-\frac{\gamma^4}{z^4}}\right) \\ &= \frac{z^2}{z^2+\gamma^2} \qquad |z| > |\gamma| \end{aligned}$$

(f)

$$\sum_{k=0}^{\infty} 2^{2k}\delta[n-2k] = \delta[n] + 4\delta[n-2] + 16\delta[n-4] + \cdots$$

$$X[z] = 1 + \frac{4}{z^2} + \frac{16}{z^4} + \frac{64}{z^6} + \cdots$$

Geometric progression with common ratio $\frac{4}{z^2}$. Hence

$$X[z] = \frac{1}{1 - \frac{4}{z^2}} = \frac{z^2}{z^2 - 4} \qquad |z| > 2$$

(g)

$$\begin{aligned} X[z] &= \sum_{n=1}^{\infty} \gamma^{n-1} z^{-n} = \frac{1}{\gamma} \sum_{n=1}^{\infty} \left(\frac{\gamma}{z}\right)^n \\ &= \frac{1}{\gamma}\left[\frac{\gamma}{z} + \left(\frac{\gamma}{z}\right)^2 + \left(\frac{\gamma}{z}\right)^3 + \cdots\right] \\ &= \frac{1}{\gamma}\left[-1 + \left(1 + \frac{\gamma}{z} + \left(\frac{\gamma}{z}\right)^2 + \left(\frac{\gamma}{z}\right)^3 + \cdots\right)\right] \\ &= \frac{1}{\gamma}\left[-1 + \frac{1}{1 - \frac{\gamma}{z}}\right] = \frac{1}{z - \gamma} \end{aligned}$$

(h) $x[n] = n\gamma^n u[n]$ is a sequence

$$0,\ \gamma,\ 2\gamma^2,\ 3\gamma^3,\ \cdots$$

Hence

$$X[z] = \frac{\gamma}{z} + 2\left(\frac{\gamma}{z}\right)^2 + 3\left(\frac{\gamma}{z}\right)^3 + \cdots$$

Using the results in Sec. B.7-4 with $n \to \infty$ and $|z| > |\gamma|$, we obtain

$$X[z] = \frac{\gamma/z}{[(\gamma/z) - 1]^2} = \frac{\gamma z}{(z - \gamma)^2} \qquad |z| > |\gamma|$$

(i)

$$x[n] = nu[n]$$

and

$$X[z] = \sum_{n=0}^{\infty} n z^{-n}$$

Using the results in Sec. B.7.4 with $|z| > 1$, we obtain

$$X[z] = \frac{(1/z)}{[(1/z) - 1]^2} = \frac{z}{(z - 1)^2} \qquad |z| > 1$$

(j)

$$X[z] = \sum_{n=0}^{\infty} \frac{\gamma^n}{n!} z^{-n} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{\gamma}{z}\right)^n$$

Recall that

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

Therefore

$$X[z] = e^{\gamma/z}$$

(k)

$$\begin{aligned}
x[n] &= \left[z^{n-1} - (-2)^{n-1}\right] u[n] \\
X[z] &= \frac{1}{2}\left[\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n + \sum_{n=0}^{\infty}\left(\frac{-2}{z}\right)^n\right] \\
&= \frac{1}{2}\left[\frac{1}{1-\frac{2}{z}} + \frac{1}{1+\frac{2}{z}}\right] \\
&= \frac{1}{2}\left[\frac{z}{z-2} + \frac{z}{z+2}\right] \\
&= \frac{z^2}{z^2-4}
\end{aligned}$$

(l)

$$X[z] = \sum_{n=0}^{\infty}\frac{1}{n!}(\ln\alpha)^n z^{-n} = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\ln\alpha}{z}\right)^n$$

From the result in part (j) it follows that

$$X[z] = e^{\ln\alpha/z} = (e^{\ln\alpha})^{1/z} = \alpha^{1/z}$$

5.1-3. Note that the signal $x[n] = nu[n]$ has z-transform $X(z) = \sum_{n=0}^{\infty} nz^{-n} = \frac{z}{(z-1)^2}$. Thus, $\sum_{n=0}^{\infty} n(-3/2)^{-n}$ is easily found by evaluating the $X(z)|_{z=-3/2}$. That is, $\sum_{n=0}^{\infty} n(-3/2)^{-n} = \left.\frac{z}{(z-1)^2}\right|_{z=-3/2} = \frac{-3/2}{(-3/2-1)^2} = -\frac{3/2}{25/4} = -\frac{12}{50} = -\frac{6}{25}$. Thus,

$$\sum_{n=0}^{\infty} n(-3/2)^{-n} = -\frac{6}{25} = -0.24.$$

5.1-4. (a)

$$u[n] - u[n-2] = \delta[n] + \delta[n-1]$$

Hence

$$u[n] - u[n-2] \Longleftrightarrow 1 + \frac{1}{z} = \frac{z+1}{z}$$

(b)

$$\gamma^{n-2}u[n-2] = \frac{1}{\gamma^2}\left\{\gamma^n u[n] - \delta[n] - \gamma\delta[n-1]\right\}$$

Hence

$$\gamma^{n-2}u[n-2] \Longleftrightarrow \frac{1}{\gamma^2}\left[\frac{z}{z-\gamma} - 1 - \frac{\gamma}{z}\right] = \frac{1}{z(z-\gamma)}$$

(c)

$$x[n] = (2)^{n+1}u[n-1] + (e)^{n-1}u[n] = 4(2)^{n-1}u[n-1] + \frac{1}{e}(e)^n u[n]$$

Therefore

$$X[z] = \frac{4}{z-2} + \frac{1}{e}\frac{z}{z-e}$$

(d)

$$x[n] = \left[(2)^{-n}\cos\frac{\pi n}{3}\right]u[n-1] = (2)^{-n}\cos\frac{\pi n}{3}u[n] - \delta[n]$$

Therefore

$$X[z] = \frac{z(z-0.25)}{z^2-0.5z+0.25} - 1 = \frac{0.25(z-1)}{z^2-0.5z+0.25}$$

(e)

$$x[n] = n\gamma^n u[n-1] = n\gamma^n u[n] - 0 = n\gamma^n u[n]$$

Therefore

$$X[z] = \frac{\gamma z}{(z-\gamma)^2}$$

(f) Because $n(n-1)(n-2) = 0$ for $n = 0$, 1, and 2

$$x[n] = n(n-1)(n-2)2^{n-3}u[n-m] = n(n-1)(n-2)(2)^{n-3}u[n]$$

$n = 0$, 1, or 2. Therefore

$$x[n] = (2)^{-3}\{n(n-1)(n-2)2^n u[n]\}$$

and

$$X[z] = (2)^{-3}\left[\frac{3!(2)^3 z}{(z-2)^4}\right] = \frac{6z}{(z-2)^4}$$

(g)

$$x[n] = (-1)^n n u[n]$$

$$\begin{aligned}
X[z] &= 0 - \frac{1}{z} + \frac{2}{z^2} - \frac{3}{z^3} + \frac{4}{z^4} - \frac{5}{z^5} + \cdots \\
&= \sum_{n=0}^{\infty} 2n\left(\frac{1}{z^2}\right)^n - \sum_{n=0}^{\infty}(2n+1)\left(\frac{1}{z^2}\right)^{2n-1} \\
&= 2\sum_{n=0}^{\infty} n\left(\frac{1}{z^2}\right)^n - \frac{2}{z}\sum_{n=0}^{\infty} n\left(\frac{1}{z^2}\right)^n - \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z^2}\right)^n
\end{aligned}$$

Using the entries in Sec. B.7-4, we obtain

$$\begin{aligned}
X[z] &= \frac{2z^2}{(z^2-1)^2} - \frac{2}{z}\frac{z^2}{(z^2-1)^2} - \frac{z}{z^2-1} \\
&= \frac{-z^3+2z^2-z}{(z^2-1)^2} \\
&= \frac{-z(z-1)^2}{(z+1)^2(z-1)^2} \\
&= \frac{-z}{(z+1)^2}
\end{aligned}$$

(h)

$$x[n] = \sum_{k=0}^{\infty} k\delta[n-2k+1]$$

$$X[z] = \sum_{n=0}^{\infty} x[n]z^{-n} = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty} k\delta[n-2k+1]\right) z^{-n}$$

Interchanging the order of summation and noting that

$$\delta[n-2k+1] = \begin{cases} 1 & n = 2k-1 \\ 0 & n \neq 2k-1 \end{cases}$$

We obtain

$$\begin{aligned} X[z] &= \sum_{k=0}^{\infty} k\left(\sum_{n=0}^{\infty} \delta[n-2k+1]z^{-n}\right) \\ &= \sum_{k=0}^{\infty} kz^{-(2k-1)} \\ &= z\sum_{k=0}^{\infty} k\left(\frac{1}{z^2}\right)^k \\ &= z\frac{1/z^2}{\left[(1/z)^2-1\right]^2} \\ &= \frac{z^3}{\left(z^2-1\right)^2} \end{aligned}$$

5.1-5. (a)

$$\begin{aligned} \frac{X[z]}{z} &= \frac{z-4}{(z-2)(z-3)} = \frac{2}{z-2} - \frac{1}{z-3} \\ X[z] &= 2\frac{z}{z-2} - \frac{z}{z-3} \\ x[n] &= \left[2(2)^n - (3)^n\right]u[n] \end{aligned}$$

(b)

$$\begin{aligned} \frac{X[z]}{z} &= \frac{z-4}{z(z-2)(z-3)} = \frac{-2/3}{z} + \frac{1}{z-2} - \frac{1/3}{z-3} \\ X[z] &= -\frac{2}{3} + \frac{z}{z-2} - \frac{1}{3}\frac{z}{z-3} \\ x[n] &= -\frac{2}{3}\delta[n] + \left[(2)^n - \frac{1}{3}(3)^n\right]u[n] \end{aligned}$$

(c)

$$\begin{aligned} \frac{X[z]}{z} &= \frac{e^{-2}-2}{(z-e^{-2})(z-2)} = \frac{1}{z-e^{-2}} - \frac{1}{z-2} \\ X[z] &= \frac{z}{z-e^{-2}} - \frac{z}{z-2} \end{aligned}$$

$$x[n] = \left[e^{-2n} - 2^n\right] u[n]$$

(d)

$$X[z] = \frac{(z-1)^2}{z^3} = \frac{z^2 - 2z + 1}{z^3} = \frac{1}{z} - \frac{2}{z^2} + \frac{1}{z^3}$$

Hence

$$x[n] = \delta[n-1] - 2\delta[n-2] + \delta[n-3]$$

(e)

$$\begin{aligned}
\frac{X[z]}{z} &= \frac{2z+3}{(z-1)(z-2)(z-3)} = \frac{5/2}{z-1} - \frac{7}{z-2} + \frac{9/2}{z-3} \\
X[z] &= \frac{5}{2}\frac{z}{z-1} - 7\frac{z}{z-2} + \frac{9}{2}\frac{z}{z-3} \\
x[n] &= \left[\frac{5}{2} - 7(2)^n + \frac{9}{2}(3)^n\right] u[n]
\end{aligned}$$

(f)

$$\frac{X[z]}{z} = \frac{-5z+22}{(z+1)(z-2)^2} = \frac{3}{z+1} + \frac{k}{z-2} + \frac{4}{(z-2)^2}$$

Multiply both sides by z and let $z \to \infty$. This yields

$$0 = 3 + k + 0 \implies k = -3$$

$$\begin{aligned}
X[z] &= 3\frac{z}{z+1} - 3\frac{z}{z-2} + 4\frac{z}{(z-2)^2} \\
x[n] &= \left[3(-1)^n - 3(2)^n + 2n(2)^n\right] u[n]
\end{aligned}$$

(g)

$$\frac{X[z]}{z} = \frac{1.4z + 0.08}{(z-0.2)(z-0.8)^2} = \frac{1}{z-0.2} + \frac{k}{z-0.8} + \frac{2}{(z-0.8)^2}$$

Multiply both sides by z and let $z \to \infty$. This yields

$$0 = 1 + k \implies k = -1$$

$$\begin{aligned}
X[z] &= \frac{z}{z-0.2} - \frac{z}{z-0.8} + 2\frac{z}{(z-0.8)^2} \\
x[n] &= \left[(0.2)^n - (0.8)^n + \frac{5}{2}n(0.8)^n\right] u[n]
\end{aligned}$$

(h) We use pair 12c with $A = 1$, $B = -2$, $a = -0.5$, $|\gamma| = 1$. Therefore

$$r = \sqrt{4} = 2 \qquad \beta = \cos^{-1}\left(\frac{0.5}{1}\right) = \frac{\pi}{3} \qquad \theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{3}$$

$$x[n] = 2(1)^n \cos\left(\frac{\pi n}{3} + \frac{\pi}{3}\right) u[n] = 2\cos\left(\frac{\pi n}{3} + \frac{\pi}{3}\right) u[n]$$

(i)

$$\frac{X[z]}{z} = \frac{2z^2 - 0.3z + 0.25}{z(z^2 + 0.6z + 0.25)} = \frac{1}{z} + \frac{Az + B}{z^2 + 0.6z + 25}$$

Multiply both sides by z and let $z \to \infty$. This yields

$$2 = 1 + A \implies A = 1$$

Setting $z = 1$ on both sides yields

$$\frac{1.95}{1.85} = 1 + \frac{1 + B}{1.85} \implies B = -0.9$$

$$X[z] = 1 + \frac{z(z - 0.9)}{z^2 + 0.6z + 0.25}$$

For the second fraction on right side, we use pair 12c with $A = 1$, $B = -0.9$, $a = 0.3$, and $|\gamma| = 0.5$. This yields

$$r = \sqrt{10} \qquad \beta = \cos^{-1}(\frac{-0.3}{0.5}) = 2.214 \qquad \theta = \tan^{-1}(\frac{1.2}{0.4}) = 1.249$$

$$x[n] = \delta[n] + \sqrt{10}(0.5)^n \cos(2.214n + 1.249)u[n]$$

(j)

$$\frac{X[z]}{z} = \frac{2(3z - 23)}{(z - 1)(z^2 - 6z + 25)} = \frac{-2}{z - 1} + \frac{Az + B}{z^2 - 6z + 25}$$

Multiply both sides by z and let $z \to \infty$. This yields

$$0 = -2 + A \implies A = 2$$

Set $z = 0$ on both sides to obtain

$$\frac{46}{25} = 2 + \frac{B}{25} \implies B = -4$$

$$X[z] = -2 + \frac{z}{z - 1} + \frac{z(2z - 4)}{z^2 - 6z + 25}$$

For the second fraction on the right-hand side, we use pair 12c with $A = 2$, $B = -4$, $a = -3$, and $|\gamma| = 5$.

$$r = \frac{\sqrt{17}}{2} \qquad \beta = \cos^{-1}(\frac{3}{5}) = 0.927 \qquad \theta = \tan^{-1}(\frac{-1}{4}) = -0.25$$

$$x[n] = \left[-2 + \frac{\sqrt{17}}{2}(5)^n \cos(0.927n - 0.25)\right] u[n]$$

(k)

$$\frac{X[z]}{z} = \frac{3.83z + 11.34}{(z - 2)(z^2 - 5z + 25)} = \frac{1}{z - 2} + \frac{Az + B}{z^2 - 5z + 25}$$

Multiply both sides by z and let $z \to \infty$. This yields

$$0 = 1 + A \implies A = -1$$

Setting $z = 0$ on both sides yields

$$\frac{11.34}{-50} = -\frac{1}{2} + \frac{B}{25} \implies B = 6.83$$

$$X[z] = \frac{z}{z-2} + \frac{z(-z+6.83)}{z^2-5z+25}$$

For the second fraction on right-hand side, use pair 12c with $A = -1$, $B = 6.83$, $a = -2.5$, and $|\gamma| = 5$.

$$r = \sqrt{2} \qquad \beta = \cos^{-1}(0.5) = \frac{\pi}{3} \qquad \theta = \tan^{-1}(\frac{-4.33}{-4.33}) = -\frac{3\pi}{4}$$

$$x[n] = \left[(2)^n + \sqrt{2}(5)^n \cos(\frac{\pi}{3}n - \frac{3\pi}{4})\right] u[n]$$

(l)

$$\frac{X[z]}{z} = \frac{z(-2z^2+8z-7)}{(z-1)(z-2)^3} = \frac{1}{z-1} + \frac{k_1}{z-2} + \frac{k_2}{(z-2)^2} + \frac{2}{(z-2)^3}$$

Multiply both sides by z and let $z \to \infty$. This yields

$$-2 = 1 + k_1 \implies k_1 = -3$$

Set $z = 0$ on both sides to obtain

$$0 = -1 + \frac{3}{2} + \frac{k_2}{4} - \frac{1}{4} \implies k_2 = -1$$

$$\begin{aligned} X[z] &= \tfrac{z}{z-1} - 3\tfrac{z}{z-2} - \tfrac{z}{(z-2)^2} + 2\tfrac{z}{(z-2)^3} \\ x[n] &= \left[1 - 3(2)^n - \tfrac{n}{2}(2)^n + \tfrac{1}{4}n(n-1)(2)^n\right] u[n] \end{aligned}$$

5.1-6. (a) Long division of $2z^3 + 13z^2 + z$ by $z^3 + 7z^2 + 2z + 1$ yields

$$X[z] = 2 - \frac{1}{z} + \frac{4}{z^2} + \cdots$$

Therefore $x[0] = 2$, $x[1] = -1$, $x[2] = 4$.

(b) Long division of $2z^4 + 16z^3 + 17z^2 + 3z$ by $z^3 + 7z^2 + 2z + 1$ yields

$$X[z] = 2z + 2 - \frac{1}{z} + \frac{4}{z^2} + \cdots$$

Therefore $x[-1] = 2$, $x[0] = 2$, $x[1] = -1$, $x[2] = 4$.

5.1-7.

$$X[z] = \frac{\gamma z}{z^2 - 2\gamma z + \gamma^2}$$

Long division yields

$$\frac{\gamma z}{z^2 - 2\gamma z + \gamma^2} = \frac{\gamma}{z} + 2\left(\frac{\gamma}{z}\right)^2 + 3\left(\frac{\gamma}{z}\right)^3 + \cdots$$

Therefore $x[0] = 0$, $x[1] = \gamma$, $x[2] = 2\gamma^2$, $x[3] = 3\gamma^3, \cdots$, and

$$x[n] = n\gamma^n u[n]$$

5.1-8. (a) We can express

$$X[z] = X[0] + \frac{X[1]}{z} + \frac{X[2]}{z^2} + \cdots$$

Let $X_n[z]$ and $X_d[z]$ be the numerator and the denominator polynomials of $X[z]$ with powers M and N, respectively. If $M = N$, then the long division of X_n with X_d in power series of z^{-1} yields $x[0]$ = a nonzero constant. If $N - M = 1$, the term $x[0] = 0$, but $x[1]$, $x[2]$, $\cdots$ are generally nonzero. In general, if $N - M = m$, then the long division show that all $x[0]$, $x[1]$, $\cdots$, $x[m-1]$ are zero. Only the terms from $x[m]$ on are generally nonzero. The difference $N - M$ indicates that the first $N - M$ samples of $x[n]$ are zero.

(b) In this case the first four samples of $x[n]$ are zero. Hence $N - M = 4$.

5.2-1. (a) $X[z] = 1 + \frac{1}{z} + \frac{1}{z^2} + \cdots + \frac{1}{z^{(m-1)}}$. This sum can be found using the result in Sec. B.7-4, as

$$X[z] = \frac{(1/z)^{m-1} - 1}{(1/z) - 1} = \frac{1 - z^{-m}}{1 - z^{-1}}$$

(b)

$$\begin{aligned} x[n] &= u[n] - u[n-m] \\ X[z] &= \frac{z}{z-1} - z^{-m}\frac{z}{z-1} = \frac{1 - z^{-m}}{1 - z^{-1}} \end{aligned}$$

5.2-2.

$$x[n] = \delta[n-1] + 2\delta[n-2] + 3\delta[n-3] + 4\delta[n-4] + 3\delta[n-5] + 2\delta[n-6] + \delta[n-7]$$

Therefore

$$\begin{aligned} X[z] &= \frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3} + \frac{4}{z^4} + \frac{3}{z^5} + \frac{2}{z^6} + \frac{1}{z^7} \\ &= \frac{z^6 + 2z^5 + 3z^4 + 4z^3 + 3z^2 + 2z + 1}{z^7} \end{aligned}$$

Alternate Method:

$$\begin{aligned} x[n] &= n\{u[n] - u[n-5]\} + (-n+8)\{u[n-5] - u[n-9]\} \\ &= nu[n] - 2nu[n-5] + nu[n-9] + 8u[n-5] - 8u[n-9] \\ &= nu[n] - 2\{(n-5)u[n-5] + 5u[n-5]\} \\ &\quad +(n-9)u[n-9] + 9u[n-9] + 8u[n-5] - 8u[n-9] \\ &= nu[n] - 2(n-5)u[n-5] + (n-9)u[n-9] - 2u[n-5] + u[n-9] \end{aligned}$$

Therefore

$$\begin{aligned} X[z] &= \frac{z}{(z-1)^2} - \frac{2z}{z^5(z-1)^2} + \frac{z}{z^9(z-1)^2} - \frac{2z}{z^5(z-1)} + \frac{z}{z^9(z-1)} \\ &= \frac{z}{z^9(z-1)^2}\left[z^9 - 2z^4 + 1 - 2z^4(z-1) + (z-1)\right] \end{aligned}$$

$$= \frac{1}{z^7(z-1)^2}\left[z^8 - 2z^4 + 1\right]$$

Reader may verify that the two answers are identical.

5.2-3. (a)

$$x[n] = n^2\gamma^n u[n]$$

Repeated application of Eq. (5.21) to $\gamma^n u[n] \Longleftrightarrow \frac{z}{z-\gamma}$ yields

$$\begin{aligned} n\gamma^n u[n] &\Longleftrightarrow \frac{\gamma z}{(z-\gamma)^2} \\ n^2\gamma^n u[n] &\Longleftrightarrow \frac{\gamma z(z+\gamma)}{(z-\gamma)^3} \end{aligned}$$

Let $\gamma = 1$ $\quad n^2 u[n] \Longleftrightarrow \frac{z(z+1)}{(z-1)^3}$

(b) Consider

$$\gamma^n u[n] \Longleftrightarrow \frac{z}{z-\gamma}$$

Application of multiplication property to this pair yields

$$n\gamma^n u[n] \Longleftrightarrow -z\frac{d}{dz}\left(\frac{z}{z-\gamma}\right) = \frac{\gamma z}{(z-\gamma)^2}$$

One more application of multiplication property to this pair yields

$$n^2\gamma^n u[n] \Longleftrightarrow -z\frac{d}{dz}\left[\frac{\gamma z}{(z-\gamma)^2}\right] = \frac{\gamma z(z+\gamma)}{(z-\gamma)^3}$$

(c) Application of Eq. (5.21) to $n^2\gamma^n u[n] \Longleftrightarrow \frac{\gamma z(z+\gamma)}{(z-\gamma)^3}$ (found in part a) yields

$$n^3\gamma^n u[n] = -z\frac{d}{dz}\left[\frac{\gamma z(z+\gamma)}{(z-\gamma)^3}\right] = \frac{\gamma z(z^2 + 4\gamma z + \gamma^2)}{(z-\gamma)^3}$$

Now setting $\gamma = 1$ in this result yields

$$n^3 u[n] = \frac{z(z^2+4z+1)}{(z-1)^3}$$

(d)

$$\begin{aligned} x[n] &= a^n\{u[n] - u[n-m]\} \\ &= a^n u[n] - a^m a^{(n-m)} u[n-m] \\ X[z] &= \frac{z}{z-a} - \frac{a^m z}{z-a} z^{-m} = \frac{z}{z-a}\left[1 - \left(\frac{a}{z}\right)^m\right] \end{aligned}$$

(e)

$$\begin{aligned} x[n] = ne^{-2n}u[n-m] &= (n-m+m)e^{-2(n-m+m)}u[n-m] \\ &= e^{-2m}(n-m)e^{-2(n-m)}u[n-m] + me^{-2m}e^{-2(n-m)}u[n-m] \\ X[z] &= e^{-2m}\frac{e^{-2}z}{(z-e^{-2})^2}z^{-2} + me^{-2m}\left(\frac{z}{z-e^{-2}}\right)z^{-2} \end{aligned}$$

$$= \frac{e^{-2m}}{z(z-e^{-2})^2}\left[\frac{1}{e^2}(1-m)+mz\right]$$

(f)

$$\begin{aligned} x[n] &= (n-2)(0.5)^{n-3}u[n-4] \\ &= \frac{1}{2}(n-4+2)(0.5)^{n-4}u[n-4] \\ &= \frac{1}{2}(n-4)(0.5)^{n-4}u[n-4] + (0.5)^{n-4}u[n-4] \end{aligned}$$

Application of shift property yields

$$x[n] \Longleftrightarrow \frac{1}{2}\left[\frac{0.5z}{z^4(z-0.5)^2}\right] + \frac{z}{z^4(z-0.5)}$$

or

$$X[z] = \frac{0.25}{z^3(z-0.5)^2} + \frac{1}{z^3(z-0.5)} = \frac{z-0.25}{z^3(z-0.5)^2}$$

5.2-4. Pair2:

$$u[n] = \delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3] + \cdots$$

$$u[n] \Longleftrightarrow 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots = \frac{1}{1-\frac{1}{z}} = \frac{z}{z-1}$$

Repeated application of Eq. (5.21) to pair 2 yields pair 3, 4, and 5. Application of Eq. (5.20) to pair 2 yields pair 6, and application of time-delay (5.15b) to pair 6 yields pair 7. Repeated application of Eq. (5.21) to pair 7 yields pair 8 and 9.

5.2-5. There is a typographic error in the book problem: $\cos(\pi n/A)$ should be $\cos(\pi n/4)$. Given (Pair 11b with $|\gamma| = 1$)

$$\sin\beta n u[n] \Longleftrightarrow \frac{z\sin\beta}{z^2 - (2\cos\beta)z + 1}$$

$$\cos\beta n u[n] = -\sin\left(\beta n - \frac{\pi}{2}\right)u[n]$$

and

$$\begin{aligned} \cos\frac{\pi}{4}nu[n] &= -\sin\left(\beta n - \frac{\pi}{2}\right)u[n] \\ &= -\sin\left[\frac{\pi}{4}(n-2)\right]u[n] \\ &= -\sin\left[\frac{\pi}{4}(n-2)\right]u[n-2] + \sin\left(-\frac{\pi}{2}\right)\delta[n] + \sin\left(-\frac{\pi}{4}\right)\delta[n-1] \\ &= -\sin\left[\frac{\pi}{4}(n-2)\right]u[n-2] + \delta[n] + \frac{1}{\sqrt{2}}\delta[n-1] \end{aligned}$$

From Pair 11b, and shift property

$$\sin\left[\frac{\pi}{4}(n-2)\right]u[n-2] \Longleftrightarrow \frac{z/\sqrt{2}}{z^2(z^2-\sqrt{2}z+1)} = \frac{1}{\sqrt{2}z(z^2-\sqrt{2}z+1)}$$

Therefore

$$\cos\frac{\pi}{4}nu[n] \Longleftrightarrow \frac{-1}{\sqrt{2}z(z^2-\sqrt{2}z+1)} + 1 + \frac{1}{\sqrt{2}z} = \frac{z(z-\frac{1}{\sqrt{2}})}{(z^2-\sqrt{2}z+1)}$$

5.2-6. Application of time reversal to pair 6 yiels

$$\beta^{-n}u[-n] \Longleftrightarrow \frac{1/z}{1/z-\beta} = \frac{1}{1-\beta z} \qquad |z| < 1/\beta$$

Moreover

$$\beta^{-n}u[-n-1] = \beta^{-n}u[-n] - \delta[n]$$

Hence

$$\beta^{-n}u[-n-1] \Longleftrightarrow \frac{1}{1-\beta z} - 1 = \frac{\beta z}{1-\beta z} \qquad |z| < 1/\beta$$

Setting $\beta = 1/\gamma$, we obtain

$$\gamma^n u[-n-1] = \frac{-z}{z-\gamma} \qquad |z| < |\gamma|$$

5.2-7. (a)

$$(-1)^n x[n] \Longleftrightarrow \sum_{n=0}^{\infty} \frac{(-1)^n x[n]}{z^n} = \sum_{n=0}^{\infty} \frac{x[n]}{(-z)^n} = X[-z]$$

(b) Application of (a) to pair 6 yields

$$(-1)^n \gamma^n u[n] = (-\gamma)^n u[n] \Longleftrightarrow \frac{-z}{-z-\gamma} = \frac{z}{z+\gamma}$$

(c) (i)

$$2^{n-1}u[n] = \frac{1}{2}2^n u[n] \Longleftrightarrow \frac{1}{2}\frac{z}{z-2}$$

Application of (b) to this result yields

$$(-2)^{n-1}u[n] = -\frac{1}{2}(-2)^n u[n] \Longleftrightarrow -\frac{1}{2}\frac{z}{z+2}$$

Hence

$$\left[2^{n-1} - (-2)^{n-1}\right]u[n] \Longleftrightarrow \frac{1}{2}\left[\frac{z}{z-2} + \frac{z}{z+2}\right] = \frac{z^2}{z^2-4}$$

(ii) To find

$$\gamma^n \cos\pi n u[n] = (-\gamma)^n u[n]$$

Hence

$$\gamma^n \cos\pi n u[n] \Longleftrightarrow \frac{z}{z+\gamma}$$

5.2-8. (a)

$$\sum_{k=0}^{n} x[k] = \sum_{k=0}^{n} x[k]u[n-m] = x[n] * u[n]$$

Application of time-convolution property to this result yields

$$\sum_{k=0}^{n} x[k] \Longleftrightarrow \frac{zX[z]}{z-1}$$

(b) Let $x[n] = \delta[n]$, which yields $X[z] = 1$. Application of the result in part (a) yields

$$\sum_{k=0}^{n} \delta[k] = u[n] \Longleftrightarrow \frac{z}{z-1}$$

5.2-9. (a) In the time-domain, $\frac{z^2}{(z-0.75)^2}$ is a convolution of two causal, decaying exponentials. Thus, either plot 1 or plot 13 is possible. Using the IVT, the initial value is $\lim_{z\to\infty} \frac{z^2}{(z-0.75)^2} = 1$. Thus,

$$\text{Plot 13 corresponds to } \frac{z^2}{(z-0.75)^2}.$$

(b) In the time-domain, $\frac{z^2-0.9z/\sqrt{2}}{z^2-0.9\sqrt{2}z+0.81}$ is a decaying sinusoid. Thus, either plot 8 or plot 9 is possible. Using the IVT, the initial value is $\lim_{z\to\infty} \frac{z^2-0.9z/\sqrt{2}}{z^2-0.9\sqrt{2}z+0.81} = 1$. Thus,

$$\text{Plot 9 corresponds to } \frac{z^2-0.9z/\sqrt{2}}{z^2-0.9\sqrt{2}z+0.81}.$$

(c) Note that $\sum_{k=0}^{4} z^{-2k} = 1 + z^{-2} + z^{-4} + z^{-6} + z^{-8}$. Thus, the time-domain signal is $\delta[n] + \delta[n-2] + \delta[n-4] + \delta[n-6] + \delta[n-8]$ and

$$\text{Plot 18 corresponds to } \sum_{k=0}^{4} z^{-2k}.$$

(d) By inspection, $\frac{z^{-5}}{1-z^{-1}}$ corresponds to a unit step that is shifted to the right by five. Thus,

$$\text{Plot 10 corresponds to } \frac{z^{-5}}{1-z^{-1}}.$$

(e) Using synthetic division on $\frac{z^2}{z^4-1}$ yields $(z^{-2} + z^{-6} + z^{-10} + \ldots)$. In the time-domain, the first non-zero term therefore occurs at $n = 2$. Thus,

$$\text{Plot 15 corresponds to } \frac{z^2}{z^4-1}.$$

(f) In the time-domain, $\frac{0.75z}{(z-0.75)^2}$ is a convolution of two causal, decaying exponentials. Thus, either plot 1 or plot 13 is possible. Using the IVT, the initial value is $\lim_{z\to\infty} \frac{0.75z}{(z-0.75)^2} = 0$. Thus,

$$\text{Plot 1 corresponds to } \frac{0.75z}{(z-0.75)^2}.$$

(g) In the time-domain, $\frac{z^2-z/\sqrt{2}}{z^2-\sqrt{2}z+1}$ has the form of a sinusoid. Thus, either plot 3 or plot 4 is possible. Using the IVT, the initial value is $\lim_{z\to\infty} \frac{z^2-z/\sqrt{2}}{z^2-\sqrt{2}z+1} = 1$. Thus,

$$\text{Plot 4 corresponds to } \frac{z^2-z/\sqrt{2}}{z^2-\sqrt{2}z+1}.$$

(h) The apparent repeated root at $z=1$ of $\frac{z^{-1}-5z^{-5}+4z^{-6}}{5(1-z^{-1})^2}$ suggests a signal that grows linearly in the time-domain. Thus, plot 6 or plot 12 are possible. To distinguish between the two, first determine whether or not a root at $z=1$ really exists. First, $\lim_{z\to 1} \frac{z^{-1}-5z^{-5}+4z^{-6}}{5(1-z^{-1})^2} = \lim_{z\to 1} \frac{z^5-5z+4}{5(z^3-z^2)^2} = \lim_{z\to 1} \frac{z^5-5z+4}{5(z^6-2z^5+z^4)} = 0/0$. This is indeterminant so use L'Hospital's rule: $\lim_{z\to 1} \frac{z^{-1}-5z^{-5}+4z^{-6}}{5(1-z^{-1})^2} = \lim_{z\to 1} \frac{\frac{d}{dz}(z^5-5z+4)}{\frac{d}{dz}(5z^6-10z^5+5z^4)} = \lim_{z\to 1} \frac{5z^4-5}{30z^5-50z^4+20z^3} = 0/0$. This too is indeterminant so use L'Hospital's rule again: $\lim_{z\to 1} \frac{z^{-1}-5z^{-5}+4z^{-6}}{5(1-z^{-1})^2} = \lim_{z\to 1} \frac{\frac{d^2}{dz^2}(z^5-5z+4)}{\frac{d^2}{dz^2}(5z^6-10z^5+5z^4)} = \lim_{z\to 1} \frac{20z^3}{150z^4-200z^3+60z^2} = 20/10 = 2$. Since $\lim_{z\to 1} \frac{z^{-1}-5z^{-5}+4z^{-6}}{5(1-z^{-1})^2} = 2 \neq \infty$, no root exists at $z=1$ and

$$\text{Plot 6 corresponds to } \frac{z^{-1}-5z^{-5}+4z^{-6}}{5(1-z^{-1})^2}.$$

(i) In the time-domain, $\frac{z}{z-1.1}$ is a growing exponential due to the root outside the unit circle. Thus, plot 16 or plot 17 is possible. Using the IVT, the initial value is $\lim_{z\to\infty} \frac{z}{z-1.1} = 1$. Thus,

$$\text{Plot 16 corresponds to } \frac{z}{z-1.1}.$$

(j) In the time-domain, $\frac{0.25z^{-1}}{(1-z^{-1})(1-0.75z^{-1})}$ is the convolution of a decaying sinusoid and a unit step. Thus, plot 7 or plot 11 are possible. Using the IVT, the initial value is $\lim_{z\to\infty} \frac{0.25z^{-1}}{(1-z^{-1})(1-0.75z^{-1})} = 0$. Thus,

$$\text{Plot 7 corresponds to } \frac{0.25z^{-1}}{(1-z^{-1})(1-0.75z^{-1})}.$$

5.3-1.

$$y[n+1] - \gamma y[n] = x[n+1]$$

with $y[0] = -M$, $x[n] = Pu[n-1]$

$$X[z] = \frac{P}{z-1}$$

$$y[n] \Longleftrightarrow Y[z] \qquad y[n+1] \Longleftrightarrow zY[z] + Mz$$

The z-transform of the system equation is

$$\begin{aligned} zY[z] + Mz - \gamma Y[z] &= \frac{Pz}{z-1} \\ (z-\gamma)Y[z] &= -M + \frac{Pz}{z-1} \end{aligned}$$

and

$$\begin{aligned} Y[z] &= \frac{-Mz}{z-\gamma} + \frac{Pz}{(z-\gamma)(z-1)} \\ \frac{Y[z]}{z} &= \frac{-M}{z-\gamma} + \frac{P}{(z-\gamma)(z-1)} = \frac{-M}{z-\gamma} + \frac{P}{\gamma-1}\left[\frac{1}{z-\gamma} - \frac{1}{z-1}\right] \\ Y[z] &= -M\frac{z}{z-\gamma} + \frac{P}{\gamma-1}\left[\frac{z}{z-\gamma} - \frac{z}{z-1}\right] \\ y[n] &= \left[-M\gamma^n + \frac{P(\gamma^n - 1)}{r}\right]u[n] \qquad r = \gamma - 1 \end{aligned}$$

The loan balance is zero for $n = N$, that is, $y[N] = 0$. Setting $n = N$ in the above equation we obtain

$$y[N] = \left[-M\gamma^N + \frac{P(\gamma^N - 1)}{r}\right] = 0$$

This yields

$$P = \frac{r\gamma^N}{\gamma^N - 1}M = \frac{rM}{1-(1+r)^{-N}}$$

5.3-2. Because part (b) requires us to separate the response into zero-input and zero-state components, we shall start with the delay operator form of the equations, as

$$y[n] + 2y[n-1] = x[n]$$

To determine the initial condition $y[-1]$, we set $n = 0$ in this equation and substitute $y[0] = 1$ to obtain

$$1 + 2y[-1] = x[0] = e \quad \Longrightarrow y[-1] = (e-1)/2$$

The z-transform of the delay form of equation yields

$$Y[z] + 2\left[\frac{1}{z}Y[z] + \frac{e-1}{2}\right] = \frac{ez}{z-e^{-1}}$$

Rearranging the terms yields

$$\frac{Y[z]}{z} = \frac{1}{z+2}\left[(1-e) + \frac{ez}{z-e^{-1}}\right]$$

The term $(1-e)$ on the right-hand side is due to the initial condition, and hence represents the zero-input component. the second term on the right-hand side represents the zero-state component of the response. Thus

$$\frac{Y[z]}{z} = \frac{1-e}{z+2} + \left[\frac{ez}{(z-e^{-1})(z+2)}\right]$$

$$= \frac{1-e}{z+2} + \frac{2e^2}{(2e+1)(z+2)} + \frac{e}{(2e+1)(z-e^{-1})}$$

Hence

$$Y[z] = (1-e)\frac{z}{z+2} + \frac{2e^2}{2e+1}\,\frac{z}{z+2} + \frac{e}{2e+1}\,\frac{z}{z-e^{-1}}$$

The first term on the right-hand side is the zero-input component and the remaining two terms represent the zero-state component. Thus

$$y[n] = \underbrace{(1-e)(-2)^n u[n]}_{\text{zero-input comp.}} + \underbrace{\frac{2e^2}{2e+1}(-2)^n u[n] + \frac{e}{2e+1}e^{-n}u[n]}_{\text{zero-state comp.}}$$

The total response is

$$y[n] = \frac{1}{2e+1}\left[(e+1)(-2)^n + e^{-(n-1)}\right]u[n]$$

5.3-3. (a) The system equation in delay form is

$$2y[n] - 3y[n-1] + y[n-2] = 4x[n] - 3x[n-1]$$

Also

$$y[n] \iff Y[z] \quad y[n-1] \iff \frac{1}{z}Y[z] \quad y[n-2] \iff \frac{1}{z^2}Y[z] + 1$$

$$x[n] \iff X[z] = \frac{z}{z-0.25} \quad x[n-1] \iff \frac{1}{z-0.25}$$

The z-transform of the equation is

$$2Y[z] - \frac{3}{z}Y[z] + \frac{1}{z^2}Y[z] + 1 = \frac{4z}{z-0.25} - \frac{3}{z-0.25} = \frac{4z-3}{z-0.25}$$

or

$$\left(2 - \frac{3}{z} + \frac{1}{z^2}\right)Y[z] = -1 + \frac{4z-3}{z-0.25} = \frac{3z-2.75}{z-0.25}$$

and

$$\begin{aligned}
\frac{Y[z]}{z} &= \frac{z(3z-2.75)}{(2z^2-3z+1)(z-0.25)} \\
&= \frac{z(3z-2.75)}{2(z-0.5)(z-1)(z-0.25)} \\
&= \frac{5/2}{z-1/2} + \frac{1/3}{z-1} - \frac{4/3}{z-0.25} \\
y[n] &= \left[\frac{1}{3} + \frac{5}{2}(0.5)^n - \frac{4}{3}(0.25)^n\right]u[n] \\
&= \left[\frac{1}{3} + \frac{5}{2}(2)^{-n} - \frac{4}{3}(4)^{-n}\right]u[n]
\end{aligned}$$

5.3-4 For initial conditions $y[0], y[1]$, we require equation in advance form:

$$2y[n+2] - 3y[n+1] + y[n] = 4x[n+2] - 3x[n+1]$$

Also: $y[n] \Longleftrightarrow Y[z]$

$$y[n+1] \Longleftrightarrow zY[z] - \frac{3}{2}z$$

$$y[n+2] \Longleftrightarrow z^2Y[z] - \frac{3}{2}z^2 - \frac{35}{4}z$$

$$x[n] \Longleftrightarrow X[z] = \frac{z}{z-0.25} \qquad x[n+1] \Longleftrightarrow zX[z] - z = \frac{0.25z}{z-0.25}$$

AND $x[n+2] \Longleftrightarrow z^2X[z] - z^2 - \frac{1}{4}z = \dfrac{z}{16(z-0.25)}$

The z-transform of the equation is:

$$2\left[z^2Y[z] - \frac{3}{2}z^2 - \frac{35}{4}z\right] - 3\left[zY[z] - \frac{3}{2}z\right] + Y[z] = \frac{-z/2}{z-0.25}$$

or $(2z^2 - 3z + 1)Y[z] = \dfrac{z(3z^2 + 12.25z - 3.75)}{(z-0.25)}$

(cont.)

and

$$
\begin{aligned}
\frac{Y[z]}{z} &= \frac{3z^2+12.25z-3.75}{2(z-0.25)(z-1)(z-0.5)} = \frac{46/3}{z-1} - \frac{4/3}{z-0.25} - \frac{25/2}{z-0.5} \\
Y[z] &= \frac{46}{3}\frac{z}{z-1} - \frac{4}{3}\frac{z}{z-0.25} - \frac{25}{2}\frac{z}{z-0.5} \\
y[n] &= \left[\frac{46}{3} - \frac{4}{3}(0.25)^n - \frac{25}{2}(0.5)^n\right] u[n]
\end{aligned}
$$

5.3-5. (a) System equation in delay form is

$$4y[n] + 4y[n-1] + y[n-2] = x[n-1]$$

Also

$$
\begin{aligned}
y[n] &\Longleftrightarrow Y[z] \quad y[n-1] \Longleftrightarrow \frac{1}{z}Y[z] \quad y[n-2] \Longleftrightarrow \frac{1}{z^2}Y[z] + 1 \\
x[n] &\Longleftrightarrow \frac{z}{z-1} \quad x[n-1] \Longleftrightarrow \frac{1}{z-1} \quad (x[-1]=0)
\end{aligned}
$$

The z-transform of the system equation is

$$4Y[z] + \frac{4}{z}Y[z] + \frac{1}{z^2}Y[z] + 1 = \frac{1}{z-1} \tag{1}$$

$$\frac{4z^2+4z+1}{z^2}Y[z] = \frac{2-z}{z-1} \tag{2}$$

and

$$
\begin{aligned}
\frac{Y[z]}{z} &= \frac{z(2-z)}{4(z-1)(z^2+z+0.25)} = \frac{z(2-z)}{4(z-1)(z+0.5)^2} \\
&= \frac{1}{4}\left[\frac{4/9}{z-1} - \frac{13/9}{z+0.5} + \frac{5/6}{(z+0.5)^2}\right] \\
Y[z] &= \frac{1}{4}\left[\frac{4}{9}\frac{z}{z-1} - \frac{13}{9}\frac{z}{z+0.5} + \frac{5}{6}\frac{z}{(z+0.5)^2}\right] \\
y[n] &= \left[\frac{1}{9} - \frac{13}{36}(-0.5)^n - \frac{5}{12}n(-0.5)^n\right] u[n]
\end{aligned}
$$

(b) To find the zero-input and the zero-state components, we observe that the only term arising because of the initial conditions is 1 on the left-hand side of Eq. (1). Hence, we can rewrite Eq. (2) with explicit zero-input and zero-state components as

$$\frac{4z^2+4z+1}{z^2}Y[z] = -1 + \frac{1}{z-1}$$

Here the term -1 on the right-hand side represents the zero-input terms and the second term on the right-hand side represents the zero-state component. Rearranging the equation, we obtain

$$\frac{Y[z]}{z} = \frac{z}{4(z+0.5)^2}\left[-1 + \frac{1}{z-1}\right]$$

$$= \underbrace{\frac{-z}{4(z+0.5)^2}}_{\text{zero-input}} + \underbrace{\frac{z}{4(z-1)(z+0.5)^2}}_{\text{zero-state}}$$

$$= \underbrace{\frac{-1/4}{z+0.5} + \frac{1/8}{(z+0.5)^2}}_{\text{zero-input}} + \underbrace{\frac{1/9}{z-1} - \frac{1/9}{z+0.5} + \frac{1/12}{(z+0.5)^2}}_{\text{zero-state}}$$

Therefore

$$Y[z] = \underbrace{\frac{(-1/4)z}{z+0.5} + \frac{(1/8)z}{(z+0.5)^2}}_{\text{zero-input}} + \underbrace{\frac{(1/9)z}{z-1} - \frac{(1/9)z}{z+0.5} + \frac{(1/12)z}{(z+0.5)^2}}_{\text{zero-state}}$$

$$y[n] = \left\{ \underbrace{\frac{-1}{4}(-0.5)^n - \frac{n(-0.5)^n}{4}}_{\text{zero-input}} + \underbrace{\frac{1}{9} - \frac{(-0.5)^n}{9} - \frac{n(-0.5)^n}{6}}_{\text{zero-state}} \right\} u[n]$$

(c) The terms which vanish as $n \to \infty$ correspond to the transient component and the terms which do not vanish correspond to the steady-state component. Hence

$$y[n] = \underbrace{\left[-\frac{13}{36}(-0.5)^n - \frac{5}{12}n(-0.5)^n\right]u[n]}_{\text{transient}} + \underbrace{\frac{1}{9}u[n]}_{\text{steady-state}}$$

5.3-6. The system in delay form is

$$y[n] - 3y[n-1] + 2y[n-2] = x[n-1]$$

Also

$$y[n] \iff Y[z] \quad y[n-1] \iff \frac{1}{z}Y[z] + 2 \quad y[n-2] \iff \frac{1}{z^2}Y[z] + \frac{2}{z} + 3$$

$$x[n] \iff X[z] \quad x[n-1] \iff \frac{1}{z}X[z]$$

$$X[z] = \frac{z}{z-3}$$

The z-transform of the system equation is

$$\begin{aligned} Y[z] - 3\left[\frac{1}{z}Y[z] + 2\right] + 2\left[\frac{1}{z^2}Y[z] + \frac{2}{z} + 3\right] &= \frac{1}{z-3} \\ \left(1 - \frac{3}{z} + \frac{2}{z^2}\right)Y[z] = -\frac{4}{z} + \frac{1}{z-3} &= \frac{-3z+12}{z(z-3)} \end{aligned}$$

$$\frac{Y[z]}{z} = \frac{-3z+12}{(z^2-3z+2)(z-3)} = \frac{-3z+12}{(z-1)(z-2)(z-3)} = \frac{9/2}{z-1} - \frac{6}{z-2} + \frac{3/2}{z-3}$$

$$
\begin{aligned}
Y[z] &= \frac{9}{2}\frac{z}{z-1} - 6\frac{z}{z-2} + \frac{3}{2}\frac{z}{z-3} \\
y[n] &= \left[\frac{9}{2} - 6(2)^n + \frac{3}{2}(3)^n\right] u[n]
\end{aligned}
$$

5.3-7. The system equation in delay form is

$$
y[n] - 2y[n-1] + 2y[n-2] = x[n-2]
$$

$$
\begin{aligned}
y[n] &\iff Y[z] \quad y[n-1] \iff \frac{1}{z}Y[z] + 1 \quad y[n-2] \iff \frac{1}{z^2}Y[z] + \frac{1}{z} \\
x[n-2] &\iff \frac{1}{z^2}X[z] \quad \text{and} \quad X[z] = \frac{z}{z-1}
\end{aligned}
$$

The z-transform of the difference equation is

$$
\begin{aligned}
Y[z] - 2\left[\frac{1}{z}Y[z] + 1\right] + 2\left[\frac{1}{z^2}Y[z] + \frac{1}{z}\right] &= \frac{1}{z(z-1)} \\
\frac{(z^2 - 2z + 2)}{z^2}Y[z] &= \frac{2z^2 - 4z + 3}{z(z-1)}
\end{aligned}
$$

$$
\begin{aligned}
\frac{Y[z]}{z} &= \frac{2z^2 - 4z + 3}{(z-1)(z^2 - 2z + 2)} = \frac{1}{z-1} + \frac{z-1}{z^2 - 2z + 2} \\
Y[z] &= \frac{z}{z-1} + \frac{z(z-1)}{z^2 - 2z + 2}
\end{aligned}
$$

For the second fraction on the right-hand side, we use pair 12c with $A = 1$, $B = -1$, $a = -1$, $|\gamma|^2 = 2$. This yields $r = 1$, $\beta = \frac{\pi}{4}$, and $\theta = 0$. Therefore

$$
y[n] = \left[1 + (\sqrt{2})^n \cos(\frac{\pi}{4}n)\right] u[n]
$$

5.3-8. The equation in advance form is

$$
y[n+2] + 2y[n+1] + 2y[n] = x[n+1] + 2x[n]
$$

$$
\begin{aligned}
y[n] &\iff Y[z] \quad y[n+1] \iff zY[z] \quad y[n+2] \iff z^2Y[z] - z \\
x[n] &\iff X[z] \quad x[n+1] \iff zX[z] - z \quad \text{and} \quad X[z] = \frac{z}{z-e}
\end{aligned}
$$

The z-transform of the difference equation is

$$
\begin{aligned}
z^2Y[z] - z + 2zY[z] + 2Y[z] &= \frac{z^2}{z-e} - z + \frac{2z}{z-e} = \frac{z(e+2)}{z-e} \\
(z^2 + 2z + 2)Y[z] &= z + \frac{z(e+2)}{z-e} = \frac{z(z+2)}{z-e}
\end{aligned}
$$

Therefore

$$
\frac{Y[z]}{z} = \frac{z+2}{(z-e)(z^2 + 2z + 2)} = \frac{0.318}{z-e} + \frac{-0.318z - 0.502}{z^2 + 2z + 2}
$$

$$Y[z] = 0.318\frac{z}{z-e} - \frac{z(0.318z+0.502)}{z^2+2z+2}$$

For the second fraction on the right-hand side, we use pair 12c with $A = 0.318$, $B = 0.502$, $a = 1$, $|\gamma|^2 = 2$ and

$$r = 0.367 \qquad \beta = \cos^{-1}(\frac{-1}{\sqrt{2}}) = \frac{3\pi}{4} \qquad \theta = \tan^{-1}(\frac{-0.184}{0.318}) = -0.525$$

$$y[n] = \left[0.318(e)^n - 0.367(\sqrt{2})^n \cos(\frac{3\pi}{4}n - 0.525)\right] u[n]$$

5.3-9. In transform domain, $H(z) = z^{-1}\frac{2z/3}{z-1/3}$ and $Y(z) = z^{-1}\frac{-2z}{z+2}$. Since $Y(z) = H(z)X(z)$, we know $X(z) = Y(z)/H(z) = \frac{z^{-1}\frac{-2z}{z+2}}{z^{-1}\frac{2z/3}{z-1/3}}$. Thus, $X(z) = -3\frac{z-1/3}{z+2}$. Using tables, $x[n] = -3\left((-2)^n u[n] - \frac{1}{3}(-2)^{n-1}u[n-1]\right) = -3\left(-2(-2)^{n-1}u[n] - \frac{1}{3}(-2)^{n-1}u[n-1]\right)$ or

$$x[n] = -3\delta[n] + 7(-2)^{n-1}u[n-1].$$

5.3-10. Taking the z-transform of $b[m] = (1.01)b[m-1] + p[m]$ and solving for $B(z)$ yields $B(z) = P(z)\frac{1}{1-1.01z^{-1}}$. Thus, $P(z)$ is required to solve for $b[m]$. One way to represent Sally's deposit schedule is $p[m] = 100\left(u[m] - \sum_{k=0}^{\infty}\delta[m-(12k+11)]\right)$. Defined this way, Sally deposits one hundred dollars on the first day of every month m except for Decembers, ($m = 12k+11$ for $k = \{0, 1, 2, \ldots\}$).

Taking the z-transform yields $P(z) = 100\left(\frac{1}{1-z^{-1}} - \sum_{m=-\infty}^{\infty}\sum_{k=0}^{\infty}\delta[m-(12k+11)]z^{-m}\right) =$ $100\left(\frac{1}{1-z^{-1}} - \sum_{k=0}^{\infty}\sum_{m=-\infty}^{\infty}\delta[m-(12k+11)]z^{-m}\right) =$ $100\left(\frac{1}{1-z^{-1}} - \sum_{k=0}^{\infty}z^{-(12k+11)}\right)$. Substituting $P(z)$ into the expression for $B(z)$ yields $B(z) = 100\left(\frac{1}{1-z^{-1}} - \sum_{k=0}^{\infty}z^{-(12k+11)}\right)\frac{1}{1-1.01z^{-1}} =$ $100\left(\frac{1}{(1-1.01z^{-1})(1-z^{-1})} + \sum_{k=0}^{\infty}\frac{z^{-(12k+11)}}{1-1.01z^{-1}}\right) = 100\left(\frac{101}{1-1.01z^{-1}} + \frac{-100}{1-z^{-1}} + \sum_{k=0}^{\infty}\frac{z^{-(12k+11)}}{1-1.01z^{-1}}\right)$. The first two terms are easily inverted using a table of z-transform pairs, while the last sum is inverted using tables and the shifting property.

$$b[m] = 100\left(101(1.01)^m u[m] - 100u[m] - \sum_{k=0}^{\infty}(1.01)^{m-(12k+11)}u[m-(12k+11)]\right).$$

5.3-11. (a) Note, $h_1[n] = (-1 + (0.5)^n)\, u[n] = -(1)^n u[n] + (1/2)^n u[n]$. Thus, two real poles are evident at $z = 1$ and $z = 1/2$. Since $h[n]$ is not absolutely summable, the system is not BIBO stable. Thought of another way, the pole on the unit-circle makes the system marginally stable, at best. Marginally stable systems are not BIBO stable.

(b) Notice, $h_2[n] = (j)^n\,(u[n] - u[n-10])$ is a finite duration, causal signal. Thus, $h_2[n]$ has no poles (other than at zero). Since $h_2[n]$ is absolutely summable, the system is BIBO stable.

5.3-12. (a) Let

$$y[n] = \sum_{k=0}^{n} k$$

then

$$y[n] - y[n-1] = n \qquad \text{with} \qquad y[0] = 0$$

Setting $n = 0$ in this equation and $y[0] = 0$, yields

$$y[0] - y[-1] = 0 \Longrightarrow y[-1] = 0$$

The z-transform of the difference equation is

$$\left(1 - \frac{1}{z}\right) Y[z] = \frac{z}{(z-1)^2}$$

and

$$\frac{Y[z]}{z} = \frac{z}{(z-1)^3} = \frac{1}{(z-1)^2} + \frac{1}{(z-1)^3}$$

Hence

$$Y[z] = \frac{z}{(z-1)^2} + \frac{z}{(z-1)^3}$$

and

$$y[n] = n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2}$$

(b) Let

$$y[n] = \sum_{k=0}^{n} k^2$$

then

$$y[n] - y[n-1] = n^2 \qquad \text{with} \qquad y[0] = 0$$

Setting $n = 0$, we get

$$0 - y[-1] = 0 \Longrightarrow y[-1] = 0$$

The z-transform of the difference equation is

$$\frac{z-1}{z} Y[z] = \frac{z(z+1)}{(z-1)^3}$$

Hence

$$\begin{aligned}
\frac{Y[z]}{z} &= \frac{z(z+1)}{(z-1)^4} = \frac{1}{(z-1)^2} + \frac{3}{(z-1)^3} + \frac{2}{(z-1)^4} \\
Y[z] &= \frac{z}{(z-1)^2} + \frac{3z}{(z-1)^3} + \frac{2z}{(z-1)^4} \\
y[n] &= \left[n + \frac{3n(n-1)}{2} + \frac{n(n-1)(n-2)}{3}\right] u[n] \\
&= \frac{2n^3 + 3n^2 + n}{6} u[n] = \frac{n(n+1)(2n+1)}{6} u[n]
\end{aligned}$$

5.3-13. Let

$$y[n] = \sum_{k=0}^{n} k^3$$

then

$$y[n] - y[n-1] = n^3 \qquad \text{with} \qquad y[0] = 0$$

Setting $n = 0$ in this equation and using $y[0] = 0$, we get

$$0 - y[-1] = 0 \Longrightarrow y[-1] = 0$$

The z-transform of the difference equation is

$$\frac{z-1}{z}Y[z] = \frac{z(z^2+4z+1)}{(z-1)^4}$$

$$\begin{aligned}
\frac{Y[z]}{z} &= \frac{z(z^2+4z+1)}{(z-1)^5} = \frac{1}{(z-1)^2} + \frac{7}{(z-1)^3} + \frac{12}{(z-1)^4} + \frac{6}{(z-1)^5} \\
Y[z] &= \frac{z}{(z-1)^2} + 7\frac{z}{(z-1)^3} + 12\frac{z}{(z-1)^4} + 6\frac{z}{(z-1)^5} \\
y[n] &= n + \frac{7}{2}n(n+1) + 2n(n-1)(n-2) + \frac{1}{4}n(n-1)(n-2)(n-3) \\
&= \frac{n^4+2n^3+n^2}{4} = \frac{n^2(n+1)^2}{4}
\end{aligned}$$

5.3-14. Let

$$y[n] = \sum_{k=0}^{n} ka^k \qquad a \neq 1$$

then

$$y[n] - y[n-1] = na^k \qquad \text{with} \qquad y[0] = 0$$

Setting $n = 0$ and $y[0] = 0$ in this equation yields

$$0 - y[-1] = 0 \Longrightarrow y[-1] = 0$$

The z-transform of the difference equation is

$$\frac{z-1}{z}Y[z] = \frac{az}{(z-a)^2}$$

or

$$\frac{Y[z]}{z} = \frac{az}{(z-1)(z-a)^2} = \frac{\frac{a}{(a-1)^2}}{z-1} - \frac{\frac{a}{(a-1)^2}}{z-a} + \frac{\frac{a^2}{a-1}}{(z-a)^2}$$

and

$$\begin{aligned}
Y[z] &= \frac{a}{(a-1)^2}\left[\frac{z}{z-1} - \frac{z}{z-a} + a(a-1)\frac{z}{(z-a)^2}\right] \\
y[n] &= \frac{a}{(a-1)^2}\left[1 - a^n + (a-1)na^n\right]u[n] \\
&= \frac{a + a^{n+1}[n(a-1)-1]}{(a-1)^2} \qquad a \neq 1
\end{aligned}$$

5.3-15. (i) Let

$$x[n] = nu[n]$$

Then

$$X[z] = \frac{z}{(z-1)^2}$$

Use of the result in Prob. 5.2-8a yields

$$\sum_{k=0}^{n} k \Longleftrightarrow \frac{z^2}{(z-1)^3} = \frac{z(z+1)}{(z-1)^3} - \frac{z}{(z-1)^3}$$

Hence

$$\sum_{k=0}^{n} k = n^2 - \frac{n(n-1)}{2} = \frac{n(n+1)}{2}$$

(ii) Let

$$x[n] = n^2 u[n]$$

Then

$$X[z] = \frac{z(z+1)}{(z-1)^3}$$

Use of the result in Prob. 5.2-8a yields

$$\sum_{k=0}^{n} k^2 \Longleftrightarrow \frac{z^2(z+1)}{(z-1)^4} = \frac{z(z^2+4z+1)}{(z-1)^4} - \frac{3z(z-1)}{(z-1)^4} - \frac{4z}{(z-1)^4}$$

Hence

$$\sum_{k=0}^{n} k^2 = n^3 - \frac{3n(n-1)}{2} - \frac{2n(n-1)(n-2)}{3} = \frac{2n^3+3n^2+n}{6} = \frac{n(n+1)(2n+1)}{6}$$

5.3-16. Let

$$x[n] = n^3 u[n]$$

Then

$$X[z] = \frac{z(z^2+4z+1)}{(z-1)^4}$$

Use of the result in Prob. 5.2-8a yields

$$\sum_{k=0}^{n} k^3 \Longleftrightarrow \frac{z^2(z^2+4z+1)}{(z-1)^5} = \frac{z}{(z-1)^2} + \frac{7z}{(z-1)^3} + \frac{12z}{(z-1)^4} + \frac{6z}{(z-1)^5}$$

Hence

$$\sum_{k=0}^{n} k^3 = n + \frac{7}{2}n(n-1) + 2n(n-1)(n-2) + \frac{n(n-1)(n-2)(n-3)}{4} = \frac{n^4+2n^3+n^2}{4} = \frac{n^2(n+1)^2}{4}$$

5.3-17. Let

$$x[n] = na^n u[n]$$

Then

$$X[z] = \frac{az}{(z-a)^2}$$

Use of the result in Prob. 5.2-8a yields

$$\sum_{k=0}^{n} ka^k \Longleftrightarrow \frac{az^2}{(z-1)(z-a)^2} = \frac{a}{(a-1)^2}\left[\frac{z}{z-1} - \frac{z}{z-a} + a(a-1)\frac{z}{(z-a)^2}\right]$$

Hence

$$\sum_{k=0}^{n} ka^k = \frac{a}{(a-1)^2}\left[1 - a^n + (a-1)na^n\right]u[n] = \frac{a + a^{n+1}[n(a-1)-1]}{(a-1)^2} \qquad a \neq 1$$

5.3-18. (a)

$$\begin{aligned} x[n] &= ee^n u[n] \qquad X[z] = \frac{ez}{z-e} \\ Y[z] &= X[z]H[z] = \frac{ez^2}{(z-e)(z+0.2)(z-0.8)} \end{aligned}$$

Therefore

$$\begin{aligned} \frac{Y[z]}{z} &= \frac{ez}{(z-e)(z+0.2)(z-0.8)} = \frac{1.32}{z-e} - \frac{0.186}{z+0.2} - \frac{1.13}{z-0.8} \\ Y[z] &= 1.32\frac{z}{z-e} - 0.186\frac{z}{z+0.2} - 1.13\frac{z}{z-0.8} \\ y[n] &= \left[1.32(e)^n - 0.186(-0.2)^n - 1.13(0.8)^n\right]u[n] \end{aligned}$$

(b) From the given $H[z]$, we can write

$$(z^2 - 0.6z - 0.16)Y[z] = zX[z]$$

Hence, the corresponding difference equation of the system is

$$y[n+2] - 0.6y[n+1] - 0.16y[n] = x[n+1]$$

or

$$y[n] - 0.6y[n-1] - 0.16y[n-2] = x[n-1]$$

5.3-19. (a)

$$Y[z] = X[z]H[z] = \frac{z(2z+3)}{(z-1)(z-2)(z-3)}$$

Therefore

$$\begin{aligned} \frac{Y[z]}{z} &= \frac{2z+3}{(z-1)(z-2)(z-3)} = \frac{5/2}{z-1} - \frac{7}{z-2} + \frac{9/2}{z-3} \\ Y[z] &= \frac{5}{2}\frac{z}{z-1} - 7\frac{z}{z-2} + \frac{9}{2}\frac{z}{z-3} \\ y[n] &= \left[\frac{5}{2} - 7(2)^n + \frac{9}{2}(3)^n\right]u[n] \end{aligned}$$

(b) From the given $H[z]$, we can write

$$(z^2 - 5z + 6)Y[z] = (2z+3)X[z]$$

Hence, the corresponding difference equation of the system is

$$y[n+2] - 5y[n+1] + 6y[n] = 2x[n+1] + 3x[n]$$

or

$$y[n] - 5y[n-1] + 6y[n-2] = 2x[n-1] + 3x[n-2]$$

5.3-20. All cases use the same transfer function. From the given $H[z]$ (after dividing the numerator and the denominator by 6), we can write

$$\left(z^2 - \frac{5}{6}z + \frac{1}{6}\right) Y[z] = (5z-1)X[z]$$

Hence, the corresponding difference equation of the system is

$$y[n+2] - \frac{5}{6}y[n+1] + \frac{1}{6}y[n] = 5x[n+1] - x[n]$$

or

$$y[n] - \frac{5}{6}y[n-1] + \frac{1}{6}y[n-2] = 5x[n-1] - x[n-2]$$

(a) $x[n] = 4^{-n}u[n] = (\frac{1}{4})^n u[n]$ so that $X[z] = \frac{z}{z-\frac{1}{4}}$, and

$$Y[z] = X[z]H[z] = \frac{6z(5z-1)}{(z-\frac{1}{4})(6z^2-5z+1)} = \frac{z(5z-1)}{(z-\frac{1}{4})(z-\frac{1}{3})(z-\frac{1}{2})}$$

Therefore

$$\begin{aligned}
\frac{Y[z]}{z} &= \frac{5z-1}{(z-\frac{1}{4})(z-\frac{1}{3})(z-\frac{1}{2})} = \frac{12}{z-\frac{1}{4}} - \frac{48}{z-\frac{1}{3}} + \frac{36}{z-\frac{1}{2}} \\
Y[z] &= 12\frac{z}{z-\frac{1}{4}} - 48\frac{z}{z-\frac{1}{3}} + 36\frac{z}{z-\frac{1}{2}} \\
y[n] &= \left[12(\frac{1}{4})^n - 48(\frac{1}{3})^n + 36(\frac{1}{2})^n\right] u[n] \\
&= 12\left[4^{-n} - 4(3)^{-n} + 3(2)^{-n}\right] u[n]
\end{aligned}$$

(b) Here the input is $4^{-(n-2)}u[n-2]$ which is identical to the input in part (a) delayed by 2 units. Therefore the response will be the output in part (a) delayed by 2 units (time-invariance property). Therefore

$$y[n] = 12\left[4^{-(n-2)} - 4(3)^{-(n-2)} + 3(2)^{-(n-2)}\right] u[n-2]$$

(c) Here the input can be expressed as

$$x[n] = 4^{-(n-2)}u[n] = 16(4)^{-n}u[n]$$

This input is 16 times the input in part (a). Therefore the response will be 16 times the output in part (a) (linearity property). Therefore

$$y[n] = 192\left[4^{-n} - 4(3)^{-n} + 3(2)^{-n}\right] u[n]$$

(d) Here the input can be expressed as

$$x[n] = 4^{-n}u[n-2] = \frac{1}{16}(4)^{-(n-2)}u[n-2]$$

This input is $\frac{1}{16}$ times the input in part (b). Therefore the response will be $\frac{1}{16}$

times the output in part (b). Therefore

$$y[n] = \frac{3}{4}\left[4^{-(n-2)} - 4(3)^{-(n-2)} + 3(2)^{-(n-2)}\right] u[n-2]$$

5.3-21. (a)

$$Y[z] = X[z]\,H[z] = \frac{z(2z-1)}{(z-1)(z^2-1.6z+0.8)}$$

$$\begin{aligned}\frac{Y[z]}{z} &= \frac{2z-1}{(z-1)(z^2-1.6z+0.8)} = \frac{5}{z-1} - \frac{5(z-1)}{z^2-1.6z+0.8} \\ Y[z] &= 5\frac{z}{z-1} - 5\frac{z(z-1)}{z^2-1.6z+0.8}\end{aligned}$$

For the second fraction on the right-hand side, we use pair 12c with $A = 1$, $B = -1$, $a = -0.8$, $\gamma = \frac{2}{\sqrt{5}}$, $|\gamma|^2 = 0.8$. Therefore

$$r = 1.118 \qquad \beta = \cos^{-1}(\frac{0.8\sqrt{5}}{2}) = 0.464 \qquad \theta = \tan^{-1}(\frac{0.2}{0.4}) = 0.464$$

$$\begin{aligned}y[n] &= \left[5 - 5(1.118)\left(\frac{2}{\sqrt{5}}\right)^n \cos(0.464n + 0.464)\right] u[n] \\ &= \left[5 - 5.59\left(\frac{2}{\sqrt{5}}\right)^n \cos(0.464n + 0.464)\right] u[n]\end{aligned}$$

(b) From the given $H[z]$, we can write

$$\left(z^2 - 1.6z + 0.8\right) Y[z] = (2z-1)X[z]$$

Hence, the corresponding difference equation of the system is

$$y[n+2] - 1.6y[n+1] + 0.8y[n] = 2x[n+1] - x[n]$$

or

$$y[n] - 1.6y[n-1] + 0.8y[n-2] = 2x[n-1] - x[n-2]$$

5.3-22. (a) $\frac{z}{z+2}$

(b) $\frac{4z^2-3z}{2z^2-3z+1}$

(c) $\frac{z}{4z^2+4z+1}$

(d) We convert the equation to advanced operator form. This yields $(E^2 + 2E + 2)y[n] = (E+2)x[n]$. Hence, $H[z] = \frac{z+2}{z^2+2z+2}$

5.3-23. (a)

$$H[z] = \frac{z^2+3z+3}{z^2+3z+2} = \frac{z^2+3z+3}{(z+1)(z+2)}$$

Therefore

$$\frac{H[z]}{z} = \frac{z^2+3z+3}{z(z+1)(z+2)} = \frac{3/2}{z} - \frac{1}{z+1} + \frac{1/2}{z+2}$$

$$\begin{aligned} H[z] &= \frac{3}{2} - \frac{z}{z+1} + \frac{1}{2}\frac{z}{z+2} \\ h[n] &= \left[\frac{3}{2}\delta[n] - (-1)^n + \frac{1}{2}(-2)^n\right] u[n] \end{aligned}$$

(b)

$$H[z] = \frac{2z^2 - z}{z^2 + 2z + 1} = \frac{z(2z-1)}{(z+1)^2}$$

Therefore

$$\begin{aligned} \frac{H[z]}{z} &= \frac{2z-1}{(z+1)^2} = \frac{2}{z+1} - \frac{3}{(z+1)^2} \\ H[z] &= 2(\frac{z}{z+1}) - 3\frac{z}{(z+1)^2} \\ h[n] &= [2(-1)^n + 3n(-1)^n]\, u[n] = (2+3n)(-1)^n u[n] \end{aligned}$$

(c)

$$H[z] = \frac{z^2 + 2z}{z^2 - z + 0.5} = \frac{z(z+2)}{z^2 - z + 0.5}$$

Therefore

$$\frac{H[z]}{z} = \frac{z+2}{z^2 - z + 0.5}$$

We use pair 12c with $A = 1$, $B = 2$, $a = -0.5$, $|\gamma|^2 = 0.5$, $|\gamma| = \frac{1}{\sqrt{2}}$, and

$$r = 5.099 \qquad \beta = \cos^{-1}(0.5\sqrt{5}) = \frac{\pi}{4} \qquad \theta = \tan^{-1}(\frac{-2.5}{0.5}) = -1.373$$

$$h[n] = 5.099\left(\frac{1}{\sqrt{2}}\right)^n \cos(\frac{\pi}{4} - 1.373)u[n]$$

5.3-24. (a)

$$\begin{aligned} \frac{H[z]}{z} &= \frac{1}{(z+0.2)(z-0.8)} = \frac{-1}{z+0.2} + \frac{1}{z-0.8} \\ H[z] &= -\frac{z}{z+0.2} + \frac{z}{z-0.8} \\ h[n] &= [-(-0.2)^n + (0.8)^n]\, u[n] \end{aligned}$$

(b)

$$\begin{aligned} \frac{H[z]}{z} &= \frac{2z+3}{z(z-2)(z-3)} = \frac{1/2}{z} - \frac{7/2}{z-2} + \frac{3}{z-3} \\ H[z] &= \frac{1}{2} - \frac{7}{2}\frac{z}{z-2} + 3\frac{z}{z-3} \\ h[n] &= \left[\frac{1}{2}\delta[n] - \frac{7}{2}(2)^n + 3(3)^n\right] u[n] \end{aligned}$$

(c)

$$\frac{H[z]}{z} = \frac{2z-1}{z(z^2-1.6z+0.8)} = \frac{-1.25}{z} + \frac{1.25z}{z^2-1.6z+0.8}$$

For the second fraction on the right-hand side, $A = 1.25$, $B = 0$, $a = -0.8$, $|\gamma|^2 = 0.8$, $|\gamma| = \frac{2}{\sqrt{5}}$, and

$$r = 2.795 \qquad \beta = \cos^{-1}(\frac{0.8\sqrt{5}}{2}) = 0.464 \qquad \theta = \tan^{-1}(-2) = -1.107$$

$$h[n] = -1.25\delta[n] + 2.795(\frac{2}{\sqrt{5}})^n \cos(0.464n - 1.107)u[n]$$

5.3-25. (a) Noting that $H(z) = z^{-3}\frac{z}{z-1}$, $H^{-1}(z) = \frac{1}{H(z)} = \frac{z-1}{z^{-2}} = z^3 - z^2$. Thus,

$$h^{-1}[n] = \delta[n+3] - \delta[n+2].$$

(b) Since $h^{-1}[n]$ is absolutely summable, the system inverse is stable. However, $h^{-1}[n] \neq 0$ for $n < 0$ so the system is not causal.

(c) For systems that have time as the independent variable, it is only possible to realize causal systems. Shifting $h^{-1}[n]$ by three makes it causal and therefore realizable. That is, implement $h^{-1}_{\text{causal}}[n] = h^{-1}[n-3] = \delta[n] - \delta[n-1]$, as shown in S5.3-25c. Within a delay factor, this implementation functions as the system inverse.

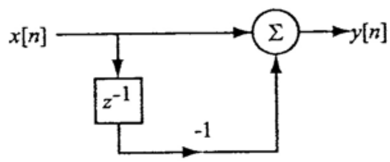


Figure S5.3-25c: Block realization of $h^{-1}_{\text{causal}}[n]$.

5.4-1. For convenience, let $\gamma = \frac{1+j}{\sqrt{8}}$ and rewrite $h[n] = (\gamma^n + (\gamma^*)^n)\,u[n]$.

(a) By inspection, the structure is a parallel implementation of two modes, which are easily identified in $h[n]$. The transfer function of the structure is $H(z) = \frac{1}{1+A_1z^{-1}} + \frac{1}{1+A_2z^{-1}}$. Taking the transform of $h[n]$, $H(z) = \frac{1}{1-\gamma z^{-1}} + \frac{1}{1-\gamma^* z^{-1}}$. Thus,

$$A_1 = -\gamma = -\frac{1+j}{\sqrt{8}} \text{ and } A_2 = -\gamma^* = -\frac{1-j}{\sqrt{8}}.$$

(b) $y_0[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] = \left\{\sum_{k=0}^{n+3} \gamma^n + (\gamma^*)^n\right\} u[n+3]$. Thus,

$$y_0[n] = \left\{\frac{1-\gamma^{n+4}}{1-\gamma} + \frac{1-(\gamma^*)^{n+4}}{1-\gamma^*}\right\} u[n+3], \text{ where } \gamma = \frac{1+j}{\sqrt{8}}.$$

Written another way, $y_0[n] = 2\mathcal{R}\left\{\frac{1-\gamma^{n+4}}{1-\gamma}\right\} u[n+3]$.

5.4-2. (a)

$$H[z] = \frac{3z^2 - 1.8z}{z^2 - z + 0.16} = \frac{3z(z-0.6)}{(z-0.2)(z-0.8)} = \left(\frac{3z}{z-0.2}\right)\left(\frac{z-0.6}{z-0.8}\right)$$

Parallel form: To realize parallel form, we could expand $H[z]$ or $H[z]/z$ into partial fractions. In our case:

$$H[z] = 3 + \frac{1.2z - 0.48}{(z-0.2)(z-0.8)}$$

$$H[z] = 3 + \frac{0.4}{(z-0.2)} + \frac{0.8}{(z-0.8)}$$

Alternatively we could expand $H[z]/z$ into partial fractions as:

$$\frac{H[z]}{z} = \frac{3(z-0.6)}{(z-0.2)(z-0.8)} = \frac{2}{z-0.2} + \frac{1}{z-0.8}$$

and

$$H[z] = 2\frac{z}{z-0.2} + \frac{z}{z-0.8}$$

The realizations are shown in Figure S5.4-2.

(b) Refer to Figure S5.4-2.

5.4-3. (a)

$$\begin{aligned} H[z] &= \frac{5z+2.2}{(z+0.2)(z+0.8)} = \frac{5z+2.2}{z^2+z+0.16} \\ &= \left(\frac{1}{z+0.2}\right)\left(\frac{5z+2.2}{z+0.8}\right) = \frac{2}{z+0.2} + \frac{3}{z+0.8} \end{aligned}$$

All the realizations are shown in Figure S5.4-3.

(b) Refer to Figure S5.4-3.

5.4-4. (a)

$$H[z] = \frac{3.8z - 1.1}{z^3 - 0.8z^2 + 0.37z - 0.05}$$

For a cascade form, we express $H[z]$ as:

$$H[z] = \left(\frac{1}{z-0.2}\right)\left(\frac{3.8z - 1.1}{z^2 - 0.6z + 0.25}\right)$$

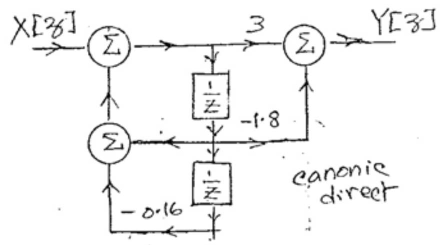
For a parallel form, we express $H[z]$ as:

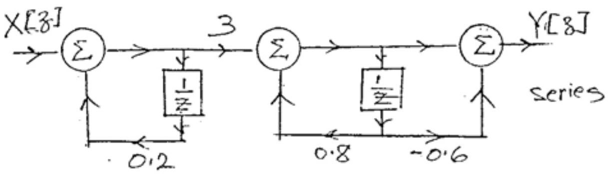
$$H[z] = \frac{-2}{z-0.2} + \frac{2z+3}{z^2 - 0.6z + 0.25}$$

All the realizations are shown in Figure S5.4-4.

(b) Refer to Figure S5.4-4.

5.4-5. (a) Note: the complex conjugate poles must be realized together as a second order factor


X[z]
3
Y[z]
-1.8
-0.16
canonic
direct


X[z]
3
Y[z]
series
0.2
0.8
-0.6

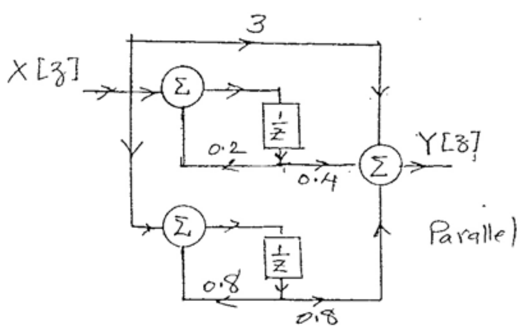
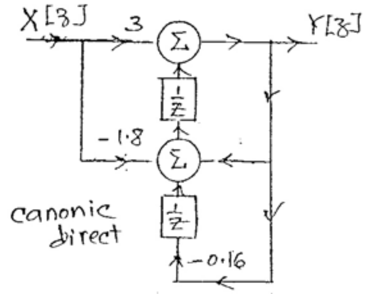
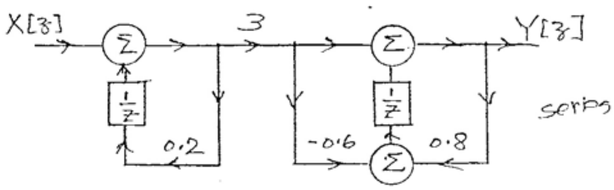

3
X[z]
0.2
Y[z]
0.4
Parallel
0.8
0.8

Fig. S5.4-2b


X[z]
3
Y[z]
-1.8
-0.16
canonic
direct


X[z]
3
Y[z]
0.2
-0.6
0.8
series

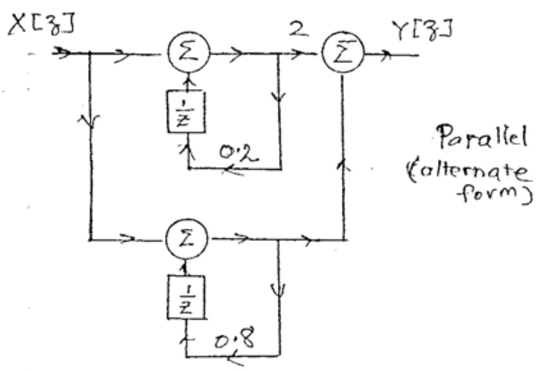
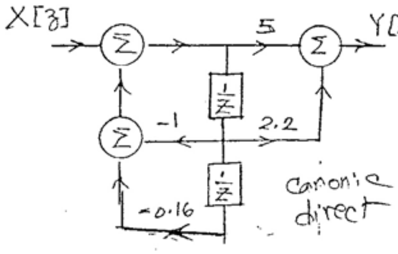
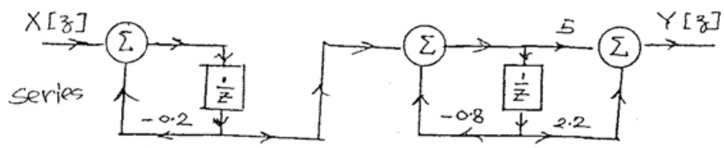

X[z]
2
Y[z]
0.2
0.8
Parallel
(alternate
form)

Fig. S5.4-3a


X[z]
5
Y[z]
-1
2.2
-0.16
canonic direct


X[z]
series
-0.2
5
Y[z]
-0.8
2.2

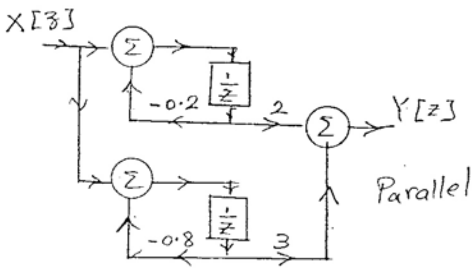
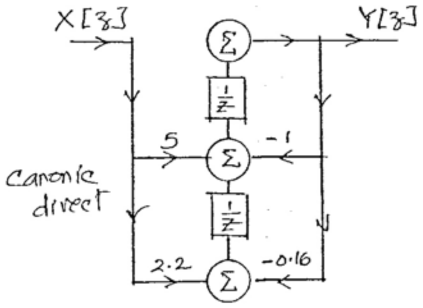
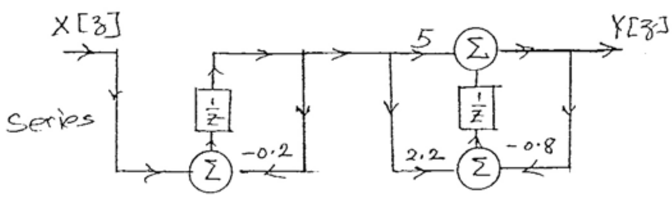

X[z]
-0.2
2
Y[z]
Parallel
-0.8
3

Fig. S5.4-3b


X[z]
Y[z]
$\frac{1}{z}$
5
-1
Canonic direct
$\frac{1}{z}$
2.2
-0.16


X[z]
Y[z]
Series
$\frac{1}{z}$
-0.2
5
$\frac{1}{z}$
2.2
-0.8

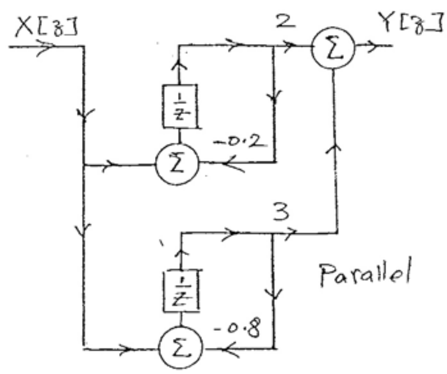
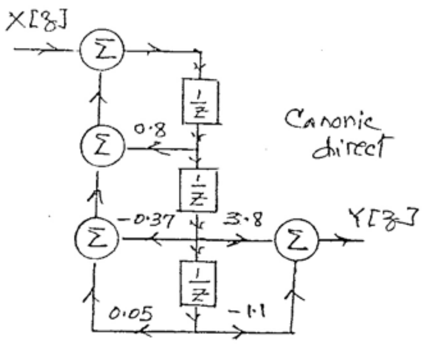
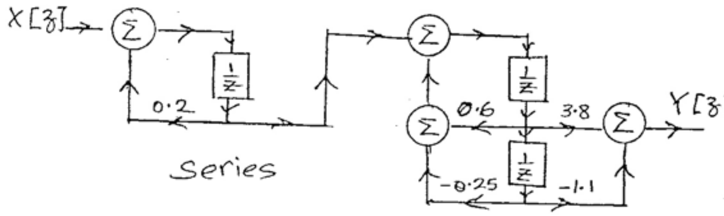

X[z]
2
Y[z]
$\frac{1}{z}$
-0.2
3
Parallel
$\frac{1}{z}$
-0.8

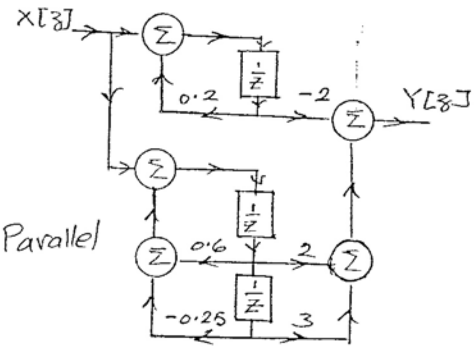
Fig. S5.4-4a



Canonic direct

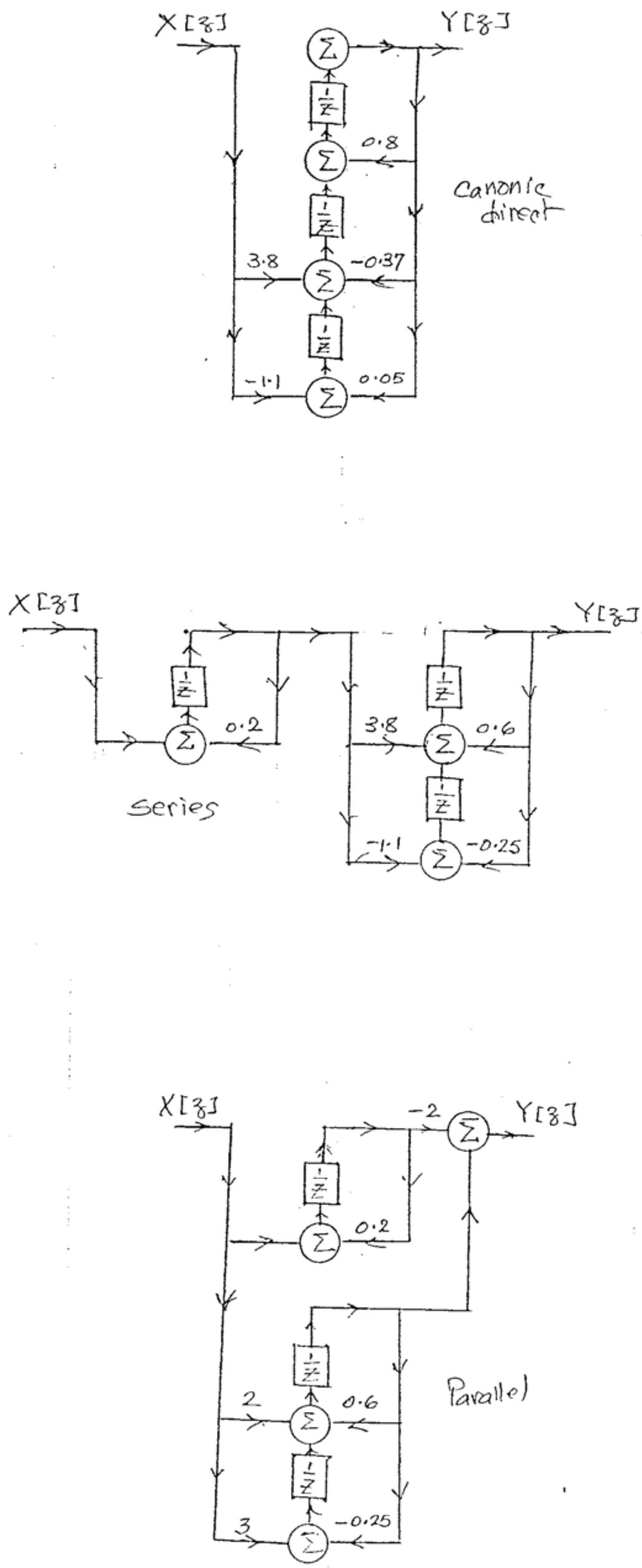


Series



Parallel

Fig. S 5.4-4b


X[z]
Y[z]
0.8
Canonic direct
3.8
-0.37
-1.1
0.05
X[z]
Y[z]
0.2
3.8
0.6
Series
-1.1
-0.25
X[z]
-2
Y[z]
0.2
Parallel
2
0.6
3
-0.25

Cascade form:

$$H[z] = \left(\frac{z}{z-0.2}\right)\left(\frac{1.6z-1.8}{z^2+z+0.5}\right)$$

Parallel form:

$$H[z] = \frac{-2z}{z-0.2} + \frac{2z^2+4z}{z^2+z+0.5}$$

All the realizations are shown in Figure S5.4-5.

(b) Refer to Figure S5.4-5.

5.4-6. (a) Cascade form:

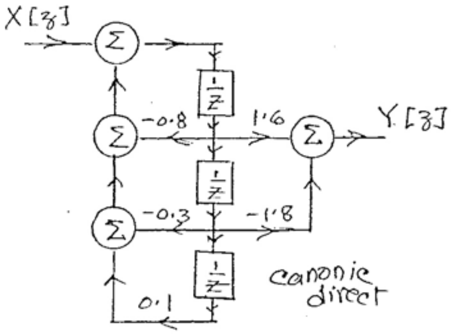
$$H[z] = \left(\frac{z}{z+0.5}\right)\left(\frac{2z^2+1.3z+0.96}{z^2-0.8z+0.16}\right)$$

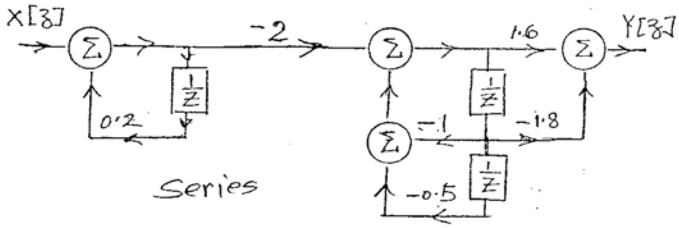
Parallel form:

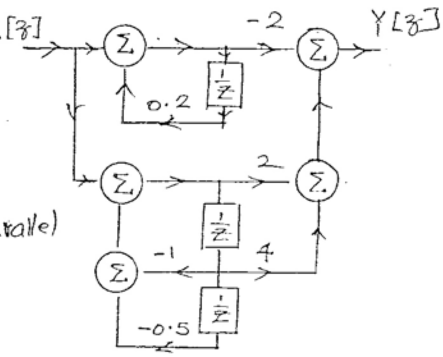
$$H[z] = \frac{z}{z+0.5} + \frac{z}{z-0.4} + \frac{2z}{(z-0.4)^2}$$

(b) Refer to Figure S5.4-6.

Fig S 5.4-5a

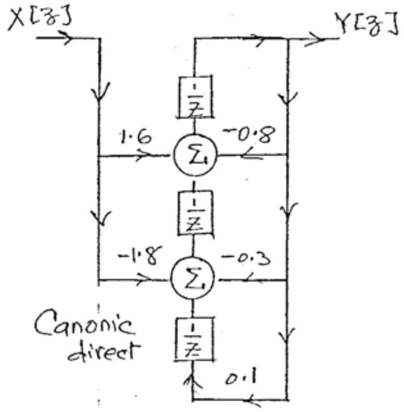


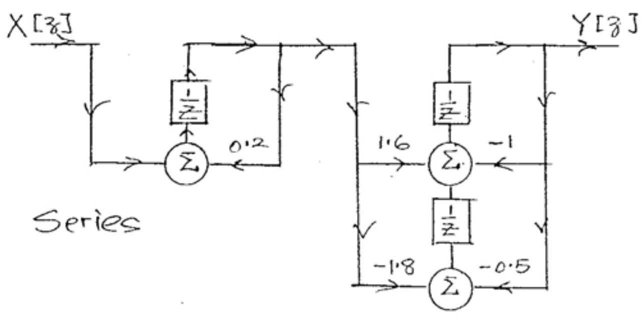




Parallel

fig S 5 4-5b


X[z]
Y[z]
1/z
1.6
-0.8
-1.8
-0.3
0.1
Canonic direct


X[z]
Y[z]
1/z
0.2
1.6
-1
-1.8
-0.5
Series

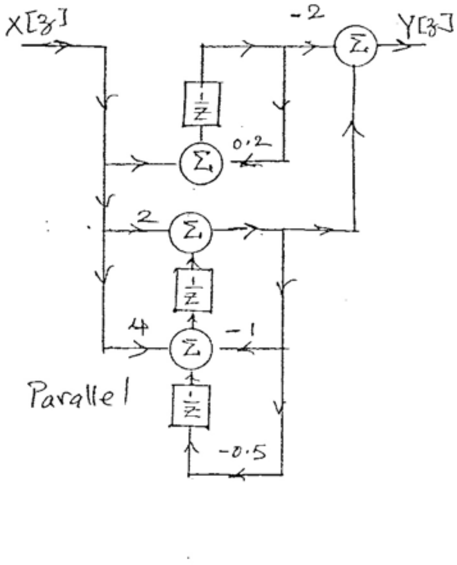
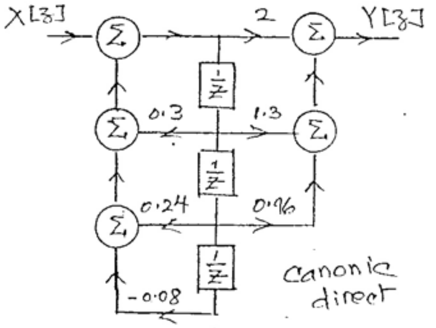
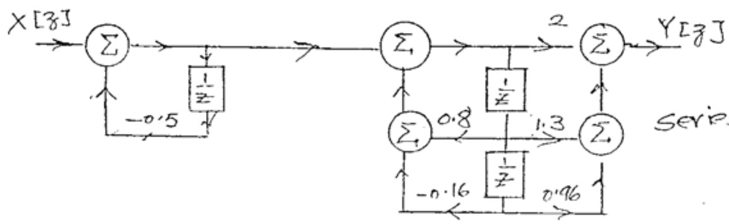

X[z]
Y[z]
-2
1/z
0.2
2
4
-1
-0.5
Parallel

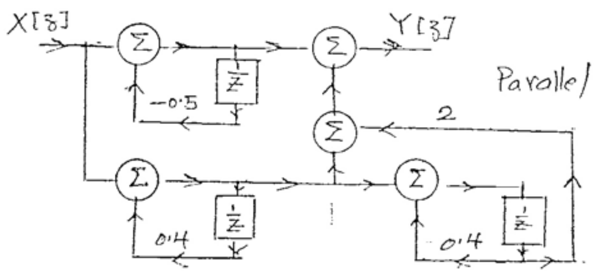
Fig. S5.4-6a



canonic direct

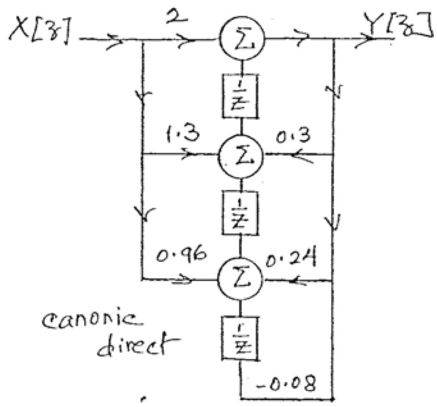


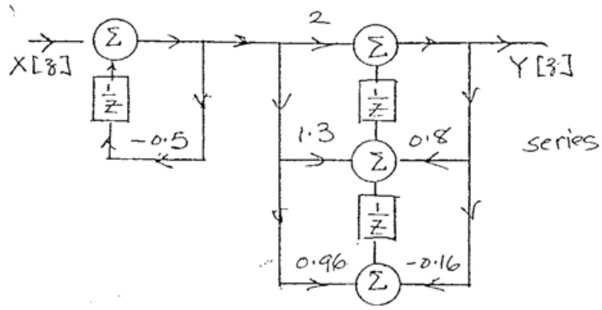
series

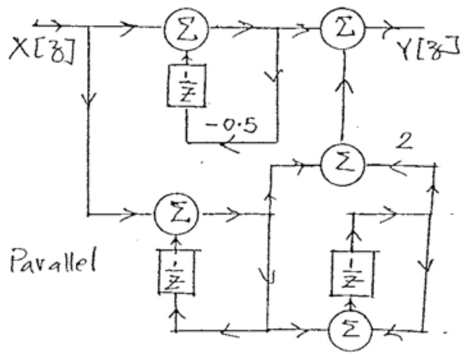


Parallel

Fig. S5 4-6 B


X[3]
2
Y[3]
1/z
1.3
0.3
1/z
0.96
0.24
canonic direct
1/z
-0.08


X[3]
1/z
-0.5
2
Y[3]
1/z
1.3
0.8
series
1/z
0.96
-0.16


X[3]
Y[3]
1/z
-0.5
2
Parallel
1/z
1/z

5.4-7.

$$H[z] = 2 + \frac{1}{z} + \frac{0.8}{z^2} + \frac{2}{z^3} + \frac{8}{z^4} = \frac{2z^2 + z^3 + 0.8z^2 + 2z + 8}{z^4}$$

The realization of this transfer function is shown in Figure S5.4-7. It can be explained in two ways. The realization has 5 paths in parallel, and each path represents one term in the transfer function. The first path (which bypasses all the delays) has transfer function 2. The second path (going through only one delay) has transfer function $1/z$, and so on. Alternately we observe that this transfer function has $a_4 = a_3 = a_2 = a_1 = 0$, and $b_4 = 8$, $b_3 = 2$, $b_2 = 0.8$, $b_1 = 1$, $b_0 = 2$. Therefore all the feedback coefficients are zero, and there are no feedback paths. There are 4 feed forward paths with gains 8, 2, 0.8, 1, and 2 as shown in the realization.

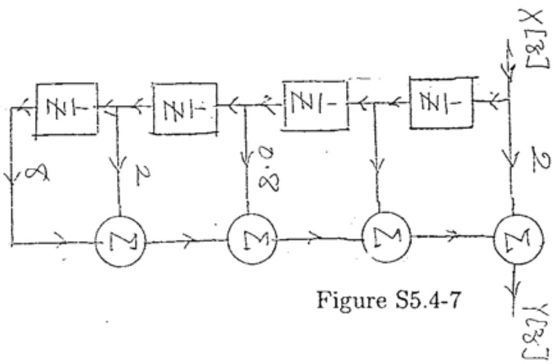


Figure S5.4-7

5.4-8.

$$H[z] = \frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3} + \frac{4}{z^4} + \frac{5}{z^5} + \frac{6}{z^6}$$

This transfer function is similar to that in Prob. 5.4-7. Its realization is shown in Figure S5.4-8.

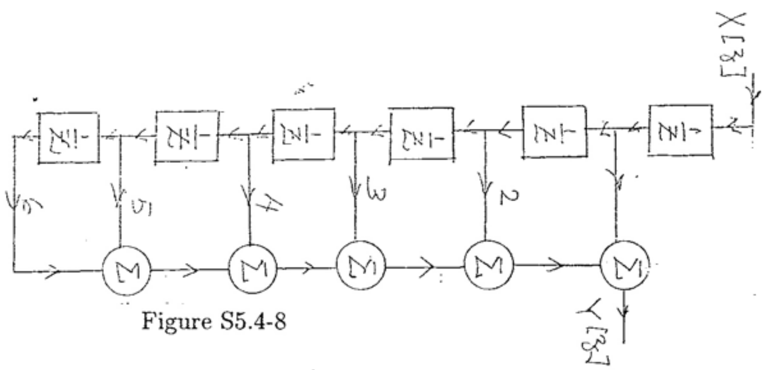


Figure S5.4-8

5.4-9. Consider a transfer function

$$H(s) = \frac{(s+a)(s+b)}{(s+c)(s+d)}$$

Let us use abbreviated notation:
A to denote $s + a$
B to denote $s + b$
C to denote $s + c$
D to denote $s + d$

A^T, B^T, C^T, D^T, denote transposed realization of A, B, C, and D.
The realization $\left(\frac{A}{C}\ \frac{B}{D}\right)$ denote a direct form canonic realization by cascading $(s+a)/(s+c)$ with $(s+b)/(s+d)$, and $\left(\frac{A^T}{C}\ \frac{B}{D}\right)$ represents a canonic realization using the cascade of the transpose of $(s+a)/(s+c)$ with direct form of $(s+b)/(s+d)$, etc ...
We can count at least the following realization.
1 Direct canonic
1 Transpose of the above
Cascade forms: The following 16 forms

$$\left(\frac{A}{C}\ \frac{B}{D}\right),\quad \left(\frac{A}{D}\ \frac{B}{C}\right),\quad \left(\frac{B}{D}\ \frac{A}{C}\right),\quad \left(\frac{B}{C}\ \frac{A}{D}\right),$$
$$\left(\frac{A^T}{C}\ \frac{B}{D}\right),\quad \left(\frac{A^T}{D}\ \frac{B}{C}\right),\quad \left(\frac{B^T}{D}\ \frac{A}{C}\right),\quad \left(\frac{B^T}{C}\ \frac{A}{D}\right),$$
$$\left(\frac{A}{C}\ \frac{B^T}{D}\right),\quad \left(\frac{A}{D}\ \frac{B^T}{C}\right),\quad \left(\frac{B}{D}\ \frac{A^T}{C}\right),\quad \left(\frac{B}{C}\ \frac{A^T}{D}\right),$$
$$\left(\frac{A^T}{C}\ \frac{B^T}{D}\right),\quad \left(\frac{A^T}{D}\ \frac{B^T}{C}\right),\quad \left(\frac{B^T}{D}\ \frac{A^T}{C}\right),\quad \left(\frac{B^T}{C}\ \frac{A^T}{D}\right),$$

Parallel forms: The following 4 forms

$$\left(\frac{k_1}{C}+\frac{k_2}{D}\right),\ \left(\frac{k_1}{C^T}+\frac{k_2}{D^T}\right),\ \left(\frac{k_1}{C^T}+\frac{k_2}{D}\right),\ \left(\frac{k_1}{C}+\frac{k_2}{D^T}\right)$$

This represent a total of 22 ways. We have not considered here the possibility of change of variables, which can yield unlimited number of realizations.

5.5-1. (a)

$$H[z]=\frac{1}{z-0.4} \quad \text{and} \quad H\left[e^{j\Omega}\right]=\frac{1}{e^{j\Omega}-0.4}=\frac{1}{\cos\Omega-0.4+j\sin\Omega}$$

$$\left|H\left[e^{j\Omega}\right]\right|^2=HH^*=\frac{1}{(e^{j\Omega}-0.4)(e^{-j\Omega}-0.4)}=\frac{1}{1.16-0.8\cos\Omega}$$

$$\left|H\left[e^{j\Omega}\right]\right|=\frac{1}{\sqrt{1.16-0.8\cos\Omega}}$$

and

$$\angle H\left[e^{j\Omega}\right]=-\tan^{-1}\frac{\sin\Omega}{\cos\Omega-0.4}$$

(b)

$$H[z]=\frac{z}{z-0.4}=\frac{1}{1-0.4z^{-1}}$$

$$\text{and} \qquad H\left[e^{j\Omega}\right]=\frac{1}{1-0.4e^{-j\Omega}}=\frac{1}{1-0.4\cos\Omega-j\sin\Omega}$$

Therefore

$$\left|H\left[e^{j\Omega}\right]\right|=\sqrt{HH^*}=\sqrt{\frac{1}{1-0.4e^{-j\Omega}}\frac{1}{1-0.4e^{j\Omega}}}=\frac{1}{\sqrt{1.16-0.8\cos\Omega}}$$

and

$$\angle H\left[e^{j\Omega}\right] = -\tan^{-1}\left(\frac{0.4\sin\Omega}{1-0.4\cos\Omega}\right)$$

(c)

$$H[z] = \frac{3z^2 - 1.8z}{z^2 - z + 0.16}$$

and $$H\left[e^{j\Omega}\right] = \frac{3e^{2j\Omega} - 1.8e^{j\Omega}}{e^{2j\Omega} - e^{j\Omega} + 0.16} = \frac{(3\cos 2\Omega - 1.8\cos\Omega) + j(3\sin 2\Omega - 1.8\sin\Omega)}{(\cos 2\Omega - \cos\Omega + 0.16) + j(\sin 2\Omega - \sin\Omega)}$$

$$\begin{aligned}\left|H\left[e^{j\Omega}\right]\right|^2 &= \left[\frac{3e^{2j\Omega} - 1.8e^{j\Omega}}{e^{2j\Omega} - e^{j\Omega} + 0.16}\right]\left[\frac{3e^{-2j\Omega} - 1.8e^{-j\Omega}}{e^{-2j\Omega} - e^{-j\Omega} + 0.16}\right] \\ &= \frac{12.24 - 10.8\cos\Omega}{2.0256 - 2.32\cos\Omega + 0.32\cos 2\Omega}\end{aligned}$$

Therefore $$\left|H\left[e^{j\Omega}\right]\right| = \left[\frac{12.24 - 10.8\cos\Omega}{2.0256 - 2.32\cos\Omega + 0.32\cos 2\Omega}\right]^{1/2}$$

and

$$\angle H\left[e^{j\Omega}\right] = \tan^{-1}\left(\frac{3\sin 2\Omega - 1.8\sin\Omega}{3\cos 2\Omega - 1.8\cos\Omega}\right) - \tan^{-1}\left(\frac{\sin 2\Omega - \sin\Omega}{\cos 2\Omega - \cos\Omega + 0.16}\right)$$

5.5-2. (a)

$$H[z] = 1 + \frac{0.5}{z} + \frac{2}{z^2} + \frac{2}{z^3} + \frac{0.5}{z^4} + \frac{1}{z^5}$$

$$\begin{aligned}H\left[e^{j\Omega}\right] &= 1 + 0.5e^{-j\Omega} + 2e^{-2j\Omega} + 2e^{-j3\Omega} + 0.5e^{-j4\Omega} + e^{-j5\Omega} \\ &= e^{-j2.5\Omega}\left[e^{j2.5\Omega} + 0.5e^{j1.5\Omega} + 2e^{j0.5\Omega} + 0.5e^{-j1.5\Omega} + e^{-j2.5\Omega}\right] \\ &= 2e^{-j2.5\Omega}\left[2\cos\frac{\Omega}{2} + \frac{1}{2}\cos\frac{3\Omega}{2} + \cos\frac{5\Omega}{2}\right]\end{aligned}$$

Therefore $$\left|H\left[e^{j\Omega}\right]\right| = \left|4\cos\frac{\Omega}{2} + \cos\frac{3\Omega}{2} + 2\cos\frac{5\Omega}{2}\right|$$

and $$\angle H\left[e^{j\Omega}\right] = -2.5\Omega$$

(b) Using the same procedure as in Prob5.45a, we obtain:

$$\left|H\left[e^{j\Omega}\right]\right| = \left|4\sin\frac{\Omega}{2} + \sin\frac{3\Omega}{2} + 2\sin\frac{5\Omega}{2}\right|$$

and $$\angle H\left[e^{j\Omega}\right] = -2.5\Omega - \pi/2.$$

5.5-3. The advance operator form of this equation is

$$E^4 y[n] = \frac{1}{5}\left[E^4 + E^3 + E^2 + E + 1\right]$$

and

$$H[z] = \frac{1}{5}\left[\frac{z^4+z^3+z^2+z+1}{z^4}\right]$$

$$\begin{aligned}
H\left[e^{j\Omega}\right] &= \frac{1}{5}\left[\frac{e^{j4\Omega}+e^{j3\Omega}+e^{j2\Omega}+e^{j\Omega}+1}{e^{j4\Omega}}\right] \\
&= \frac{1}{5}e^{-j2\Omega}\left[e^{j2\Omega}+e^{j\Omega}+1+e^{-j\Omega}+e^{-j2\Omega}\right] \\
&= \frac{1}{5}e^{-j2\Omega}\left[1+2\cos\Omega+2\cos 2\Omega\right]
\end{aligned}$$

5.5-4. (a) The z-transform of the two equations yield:

$$\text{(i)}.\ \left(1+\frac{0.9}{z}\right)Y[z] = X[z]$$

$$\text{(ii)}.\ \left(1-\frac{0.9}{z}\right)Y[z] = X[z]$$

Hence the transfer functions of these filters are

$$\text{(i)}.H[z] = \frac{z}{z+0.9} \qquad \text{(ii)}.H[z] = \frac{z}{z-0.9}$$

Consider the first system.

(i)

$$H\left[e^{j\Omega}\right] = \frac{e^{j\Omega}}{e^{j\Omega}+0.9} = \frac{1}{1+0.9e^{-j\Omega}} = \frac{1}{1+0.9\cos\Omega - j0.9\sin\Omega}$$

$$\left|H\left[e^{j\Omega}\right]\right| = \frac{1}{\sqrt{1.81+1.8\cos\Omega}},\ \angle H\left[e^{j\Omega}\right] = -\tan^{-1}\left[\frac{-0.9\sin\Omega}{1+0.9\cos\Omega}\right]$$

(ii)

$$H\left[e^{j\Omega}\right] = \frac{e^{j\Omega}}{e^{j\Omega}-0.9} = \frac{1}{1-0.9e^{-j\Omega}} = \frac{1}{1-0.9\cos\Omega + j0.9\sin\Omega}$$

$$\left|H\left[e^{j\Omega}\right]\right| = \frac{1}{\sqrt{1.81-1.8\cos\Omega}},\ \angle H\left[e^{j\Omega}\right] = -\tan^{-1}\left[\frac{0.9\sin\Omega}{1-0.9\cos\Omega}\right]$$

Filter(i) has a zero at the origin and a pole at -0.9. Because the only pole is near $\Omega = \pi(z=-1)$, this is a highpass filter, as verified from the frequency response shown in Figure S5.5-4a.
Filter(ii) has a zero at the origin and a pole at 0.9. Because the only pole is near $\Omega = 0(z=1)$, this is a lowpass filter, as verified from the frequency response shown in Figure S5.5-4b.

(b) (i) For $\Omega = 0.01\pi$

$$H\left[e^{j0.01\pi}\right] = \frac{1}{\sqrt{1.81+1.8\cos 0.01\pi}} = 0.5966$$

For $\Omega = 0.99\pi$

$$H\left[e^{j0.99\pi}\right] = \frac{1}{\sqrt{1.81 + 1.8\cos 0.99\pi}} = 9.58$$

(ii) For $\Omega = 0.01\pi$

$$H\left[e^{j0.01\pi}\right] = \frac{1}{\sqrt{1.81 - 1.8\cos 0.01\pi}} = 9.58$$

For $\Omega = 0.99\pi$

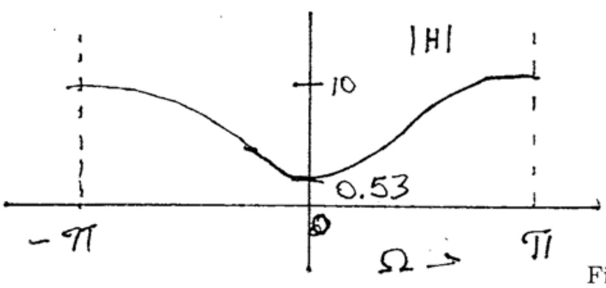
$$H\left[e^{j0.99\pi}\right] = \frac{1}{\sqrt{1.81 - 1.8\cos 0.99\pi}} = 0.5966$$

Filter(i) gain at Ω_0 is

$$|H| = \frac{1}{\sqrt{1.81 - 1.8\cos\Omega_0}}$$

Filter(ii) gain at $\pi - \Omega_0$ is

$$|H| = \frac{1}{\sqrt{1.81 - 1.8\cos(\pi - \Omega_0)}} = \frac{1}{\sqrt{1.81 + 1.8\cos(\Omega_0)}}$$



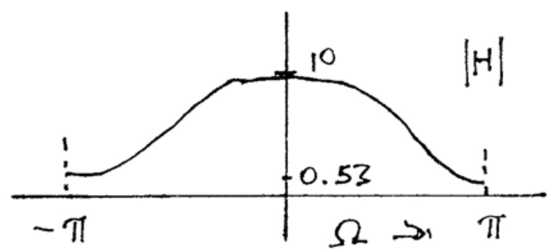


Figure S5.5-4

5.5-5.

$$H[z] = \frac{z + 0.8}{z - 0.5}$$

(a)

$$\begin{aligned}
H\left[e^{j\Omega}\right] &= \frac{e^{j\Omega} + 0.8}{e^{j\Omega} - 0.5} = \frac{(\cos\Omega + 0.8) + j\sin\Omega}{(\cos\Omega - 0.5) + j\sin\Omega} \\
\left|H\left[e^{j\Omega}\right]\right|^2 = H\left[e^{j\Omega}\right]H\left[e^{-j\Omega}\right] &= \frac{(e^{j\Omega} + 0.8)(e^{-j\Omega} + 0.8)}{(e^{j\Omega} - 0.5)(e^{-j\Omega} - 0.5)} = \frac{1.64 + 1.6\cos\Omega}{1.25 - \cos\Omega} \\
\angle H\left[e^{j\Omega}\right] &= \tan^{-1}\left(\frac{\sin\Omega}{\cos\Omega + 0.8}\right) - \tan^{-1}\left(\frac{\sin\Omega}{\cos\Omega - 0.5}\right)
\end{aligned}$$

(b) $\quad \Omega = 0.5$

$$\begin{aligned}
\left|H\left[e^{j0.5}\right]\right|^2 &= \frac{1.64 + 1.6\cos(0.5)}{1.25 - \cos(0.5)} = 8.174 \\
\left|H\left[e^{j0.5}\right]\right| &= 2.86
\end{aligned}$$

$$\angle H\left[e^{j0.5}\right] = 0.2784 - 0.9037 = -0.6253\,\text{rad}$$

Therefore

$$y[n] = 2.86\cos(0.5n - \frac{\pi}{3} - 0.6253) = 2.86\cos(0.5n - 1.6725)$$

5.5-6.

$$Y[z] = X[z]\,H[z]$$

For an input $x[n] = e^{j\Omega n}u[n]$, pair 6 in Table 5.1 yields

$$X[z] = \frac{z}{z - e^{j\Omega}}$$

and

$$Y[z] = \frac{zH[z]}{z - e^{j\Omega}}$$

Therefore if

$$H[z] = \frac{P[z]}{Q[z]} = \frac{P[z]}{(z-\gamma_1)(z-\gamma_2)\cdots(z-\gamma_N)}$$

then

$$Y[z] = \frac{zP[z]}{(z-\gamma_1)(z-\gamma_2)\cdots(z-\gamma_N)(z-e^{j\Omega})}$$

and

$$\begin{aligned}\frac{Y[z]}{z} &= \frac{P[z]}{(z-\gamma_1)(z-\gamma_2)\cdots(z-\gamma_N)(z-e^{j\Omega})}\\ &= \frac{c_1}{z-\gamma_1} + \frac{c_2}{z-\gamma_2} + \cdots + \frac{c_N}{z-\gamma_N} + \frac{A}{z-e^{j\Omega}}\end{aligned}$$

The coefficient A on the right-hand side is given by

$$\begin{aligned}A &= \left.\frac{P[z]}{(z-\gamma_1)(z-\gamma_2)\cdots(z-\gamma_N)(z-e^{j\Omega})}\right|_{z=e^{j\Omega}}\\ &= \left.H[z]\right|_{z=e^{j\Omega}}\\ &= H\left[e^{j\Omega}\right]\end{aligned}$$

Therefore

$$Y[z] = \sum_{i=1}^{N} c_i \frac{z}{z-\gamma_i} + H\left[e^{j\Omega}\right]\frac{z}{z-e^{j\Omega}}$$

and

$$y[n] = \left[\sum_{i=1}^{N} c_i {\gamma_i}^n + H\left[e^{j\Omega}\right]e^{j\Omega n}\right]u[n]$$

The sum on the right-hand side consists of N characteristic modes the system. For an asymptotically stable system $|\gamma_i| < 1$ ($i = 1, 2, \cdots, N$) and the sum on the right-hand side vanishes as $n \to \infty$. This sum is therefore the *transient* component of the response. The last terms $H\left[e^{j\Omega}\right]e^{j\Omega n}$, which does not vanish as $n \to \infty$, is the *steady-state* component of the response $y_{ss}[n]$:

$$y_{ss}[n] = H\left[e^{j\Omega}\right]e^{j\Omega n}$$

5.5-7. (a) $\Omega = 0.8\pi$ is in the fundamental range. Hence it is the apparent frequency.

(b) $1.2\pi = 2\pi - 0.8\pi$. Hence

$$\sin(1.2\pi n + \theta) = \cos\left(1.2\pi n - \frac{\pi}{2} + \theta\right) = \cos\left(0.8\pi + \frac{\pi}{2} - \theta\right) = -\sin(0.8\pi - \theta)$$

(c) $6.9 = 2\pi + 0.6168$. Hence

$$\cos(6.9n + \theta) = \cos(0.6168n + \theta)$$

The apparent frequency is 0.6168.

(d) $2.8\pi = 2\pi + 0.8\pi$ and $3.7\pi = 4\pi - 0.3\pi$.
Hence

$$\cos(2.8\pi n + \theta) = \cos(0.8\pi n + \theta)$$

and

$$\begin{aligned}\sin(3.7\pi n + \theta) &= \cos\left(3.7\pi n - \frac{\pi}{2} + \theta\right) = \cos\left(-0.3\pi n - \frac{\pi}{2} + \theta\right)\\ &= \cos\left(0.3\pi n + \frac{\pi}{2} - \theta\right) = -\sin(0.3\pi n - \theta)\end{aligned}$$

(e)

$$\text{sinc}\left(\frac{\pi n}{2}\right) = \frac{\sin(\pi n/2)}{(\pi n/2)}$$

$\frac{\pi}{2}$ is already in the fundamental range.

(f)

$$\begin{aligned}\text{sinc}\left(\frac{3\pi n}{2}\right) &= \frac{\sin 1.5\pi n}{1.5\pi n} = \frac{\cos\left(1.5\pi n - \frac{\pi}{2}\right)}{1.5\pi n} = \frac{\cos\left(0.5\pi n + \frac{\pi}{2}\right)}{1.5\pi n}\\ &= \frac{-\sin(0.5\pi n)}{1.5\pi n} = -\frac{1}{3}\text{sinc}(0.5\pi n)\end{aligned}$$

(g)

$$\text{sinc}(2\pi n) = \frac{\cos(2\pi n - \pi/2)}{2\pi n} = \frac{\cos(\pi/2)}{2\pi n} = 0$$

5.5-8. Because $1.4\pi = 2\pi - 0.6\pi$, $\cos(1.4\pi n + \frac{\pi}{3}) = \cos(-0.6\pi n + \frac{\pi}{3}) = \cos(0.6\pi n - \frac{\pi}{3})$. Also

$$\cos\left(0.6\pi n + \frac{\pi}{6}\right) = \cos 0.6\pi n \cos\frac{\pi}{6} - \sin 0.6\pi n \sin\frac{\pi}{6} = \frac{\sqrt{3}}{2}\cos 0.6\pi n - \frac{1}{2}\sin 0.6\pi n$$

Similarly

$$\cos\left(0.6\pi n - \frac{\pi}{3}\right) = \cos 0.6\pi n \cos\frac{\pi}{3} + \sin 0.6\pi n \sin\frac{\pi}{3} = \frac{1}{2}\cos 0.6\pi n + \frac{\sqrt{3}}{2}\sin 0.6\pi n$$

Therefore,

$$\cos\left(0.6\pi n + \frac{\pi}{6}\right) + \sqrt{3}\cos\left(1.4\pi n + \frac{\pi}{3}\right) = \sqrt{3}\cos 0.6\pi n + \sin 0.6\pi n = 2\cos\left(0.6\pi n - \frac{\pi}{6}\right)$$

5.5-9. (a) $f_h = \frac{1}{2T} = \frac{10^6}{50\times 2} = 10$ kHz.

(b) $f_s \geq 2f_h = 100$ kHz, $\qquad T \leq \frac{1}{f_s} = 10\,\mu$s.

5.5-10. Taking the z-transform of $y[n] = \sum_{k=0}^{\infty}(0.5)^k x[n-k]$ yields $Y(z) = \sum_{k=0}^{\infty}(0.5)^k X(z) z^{-k} = X(z)\sum_{k=0}^{\infty}(0.5/z)^k$. For $|z| > 1/2$, this becomes $Y(z) = X(z)\frac{1}{1-0.5z^{-1}}$. Thus, the system function is $H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1-0.5z^{-1}}$.

(a) Using $H(z)$ and letting $z = e^{j\Omega}$, the magnitude response is $\left|H(e^{j\Omega})\right| = \left|\frac{1}{1-0.5e^{-j\Omega}}\right| = \frac{1}{\sqrt{(1-0.5\cos(\Omega))^2+(-0.5\sin(\Omega))^2}} = \frac{1}{\sqrt{1-\cos(\Omega)+0.25(\cos^2(\Omega)+\sin^2(\Omega))}}$. Thus,

$$\left|H(e^{j\Omega})\right| = \frac{1}{\sqrt{5/4-\cos(\Omega)}}.$$

MATLAB is used to plot $\left|H(e^{j\Omega})\right|$.

```
>> Omega = linspace(-pi,pi,501);
>> Hmag = 1./sqrt(5/4-cos(Omega));
>> plot(Omega,Hmag,'k'); axis([-pi pi 0 2.5]); grid;
>> xlabel('\Omega'); ylabel('|H(e^{j\Omega})|');
>> set(gca,'xtick',[-pi:pi/4:pi],'xticklabel',['-p ';...
   '   ';'   ';'   ';' 0 ';'   ';'   ';'   ';' p '],...
   'fontname','symbol');
```

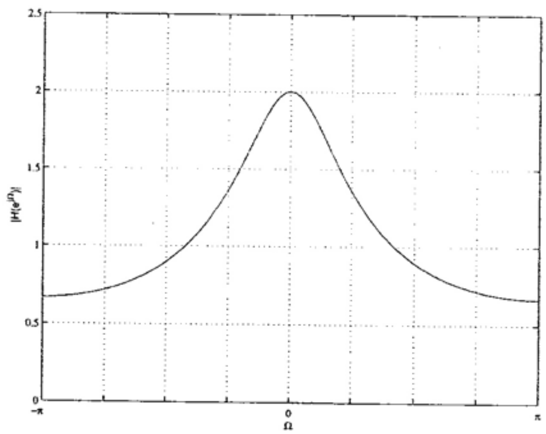


Figure S5.5-10a: $\left|H(e^{j\Omega})\right| = \frac{1}{\sqrt{5/4-\cos(\Omega)}}$.

(b) Using $H(z)$ and letting $z = e^{j\Omega}$, the phase response is $\angle H(e^{j\Omega}) = \angle\frac{1}{1-0.5e^{-j\Omega}} = -\angle(1-0.5\cos(\Omega)-0.5j\sin(\Omega))$. Thus,

$$\angle H(e^{j\Omega}) = -\arctan\left(\frac{-0.5\sin(\Omega)}{1-0.5\cos(\Omega)}\right).$$

MATLAB is used to plot $\angle H(e^{j\Omega})$.

```
>> Omega = linspace(-pi,pi,501);
>> Hang = -atan2(-0.5*sin(Omega),1-0.5*cos(Omega));
>> plot(Omega,Hang,'k');axis([-pi pi -0.6 0.6]); grid;
>> xlabel('\Omega'); ylabel('\angle H(e^{j\Omega}) [rad]');
>> set(gca,'xtick',[-pi:pi/4:pi],'xticklabel',['-p ';...
```

```
'   ';'   ';'   ';' 0 ';'   ';'   ';'   ';' p '],...
'fontname','symbol');
```

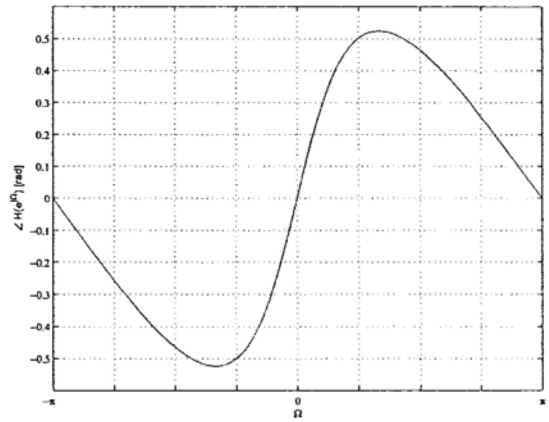


Figure S5.5-10b: $\angle H(e^{j\Omega}) = -\arctan\left(\frac{-0.5\sin(\Omega)}{1-0.5\cos(\Omega)}\right)$.

(c) Since $H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1-0.5z^{-1}}$, an equivalent difference equation description is $y[n] - 0.5y[n-1] = x[n]$. From this equation, an efficient block representation is found, as shown in Figure S5.5-10c.

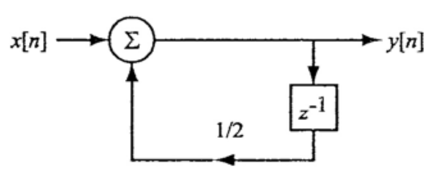


Figure S5.5-10c: Block representation of $y[n] - 0.5y[n-1] = x[n]$.

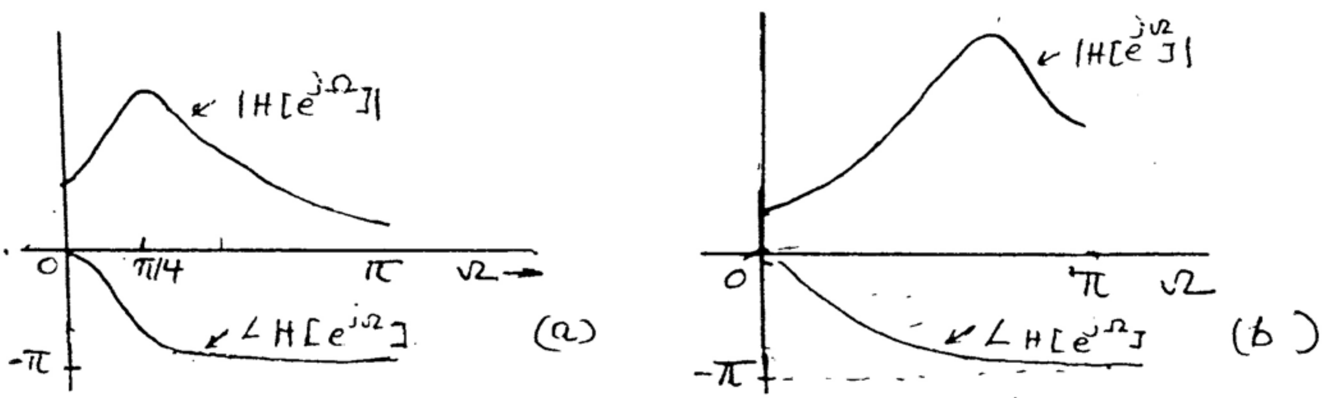


Figure S5.6-1

5.6-1. Figure S5.6-1 shows a rough sketch of the amplitude and phase response of this filter. For the case (a), the poles are in the vicinity of $\Omega = \frac{\pi}{4}$. Therefore, the gain $|H[e^{j\Omega}]|$ is high in the vicinity of $\Omega = \pi/4$. In the case (b), the poles are in the vicinity of $\Omega = \pi$. therefore, the gain $|H[e^{j\Omega}]|$ is high in the vicinity of $\Omega = \pi$. For case (a), the phases of the two poles are equal and opposite at $\Omega = 0$. Hence $\angle H[e^{j\Omega}]$ starts at 0 (for $\Omega = 0$). As Ω increases, the angle due to both poles increase. Hence, $\angle H[e^{j\Omega}]$ increases in negative direction until it reaches the value -2π at $\Omega = \pi$. For case (b), similar behavior is observed. Note that angle -2π is the same as 0.

5.6-2. The two systems are very similar and have identical steady-state characteristics. There is an important difference, however, between the two systems. The system $y[n] - y[n-1] = x[n] - x[n-1]$ is first-order and can support an initial condition; the system $y[n] = x[n]$ is zero-order and cannot support an initial condition. If the initial condition of the first system is non-zero, the output of the two systems can be quite different.

5.6-3. (a) From the magnitude response plot, it is clear that this is a lowpass filter. Low frequencies near $\Omega = 0$ are attenuated, and high frequencies near $\Omega = \pm\pi$ are passed with unity gain.

(b) From the magnitude and phase response plots, $H(e^{j\pi/2}) = \frac{1}{\sqrt{2}}e^{j3\pi/2}$. Thus, the output to $x_1[n] = 2\sin(\frac{\pi}{2}n + \frac{\pi}{4})$ is

$$y_1[n] = \sqrt{2}\sin(\frac{\pi}{2}n + \pi) = -\sqrt{2}\sin(\frac{\pi}{2}n).$$

(c) Notice, $H(e^{j7\pi/4}) = H(e^{-j\pi/4})$. From the magnitude and phase response plots, $H(e^{-j\pi/4}) \approx 0.071e^{j2.43}$. Thus, the output to $x_2[n] = \cos(\frac{7\pi}{4}n)$ is

$$y_2[n] = 0.071\cos(\frac{7\pi}{4}n + 2.43).$$

5.6-4. Refer to Figure S5.6-4 and the solution to 5.M-1.

5.6-5. Refer to Figure S5.6-5 and the solution to 5.M-4.

5.6-6.

$$H[z] = K\frac{z+1}{z-a}$$

Figure S5.6-6a shows the realization, Figure S5.6-6b shows the pole-zero configuration, and Figure S5.6-6c shows the amplitude response of the filter. Observe that the pole at a is close to $\Omega = 0$. Hence, there is the highest gain at dc. There is a zero at -1, which represents $\Omega = \pi$. Hence, the gain is zero at $\Omega = \pi$. This is a lowpass filter.

$$H\left[e^{j\Omega}\right] = K\left(\frac{e^{j\Omega}+1}{e^{j\Omega}-a}\right) = K\left(\frac{\cos\Omega + 1 + j\sin\Omega}{\cos\Omega - a + j\sin\Omega}\right)$$

$$\left|H\left[e^{j\Omega}\right]\right| = K\sqrt{\frac{2(1+\cos\Omega)}{1+a^2-2a\cos\Omega}}$$

For $a = 0.2$

$$\left|H\left[e^{j\Omega}\right]\right| = K\sqrt{\frac{2(1+\cos\Omega)}{1.04-0.4\cos\Omega}}$$

The dc gain is

$$|H[e^{j0}]| = 2.5K$$

For 3 dB bandwidth $|H[e^{j\Omega}]|^2 = \frac{1}{2}|H[e^{j0}]|^2 = 3.125K^2$. Hence

$$3.125K^2 = K^2\left[\frac{2(1+\cos\Omega)}{1.04-0.4\cos\Omega}\right] \Longrightarrow \Omega = 1.176$$

SEE SOLUTION FOR

5. M - 1

Figure S5.6-4

$$\text{Hence} \qquad B = \frac{\omega}{2\pi} = \frac{1.176}{2\pi T} = \frac{0.187}{T}\text{Hz}$$

5.6-7.

$$T \leq \frac{1}{2B_h} = \frac{1}{40000} = 25\mu\text{s}$$

Select $T = 25\mu$s Frequency 5000 Hz gives

$$\Omega = \omega T = 2\pi \times 5000 \times 25 \times 10^{-6} = \pi/4$$

Therefore frequency 5000 Hz corresponds to angles $\pm\pi/4$. We must place zeros at $e^{\pm j\pi/4}$. For fast recovery on either side of 5000 Hz, we read poles at $ae^{\pm j\pi/4}$ where $a < 1$ and $a \simeq 1$. The transfer function is

$$\begin{aligned} H[z] &= K\frac{(z - e^{j\pi/4})(z - e^{-j\pi/4})}{(z - ae^{j\pi/4})(z - ae^{-j\pi/4})} \\ &= \frac{K(z^2 - \sqrt{2}z + 1)}{z^2 - \sqrt{2}az + a^2} \end{aligned}$$

The constant K is chosen to have unity gain at $\omega = 0\,(\Omega = 0)$ or $z = e^{j\Omega} = 1$. $(H[1] = 1)$

$$H[1] = \frac{K(2 - \sqrt{2})}{1 + a^2 - \sqrt{2}a} = 1$$

SEE SOLUTION TO 5.M-4

Figure S5.6-5

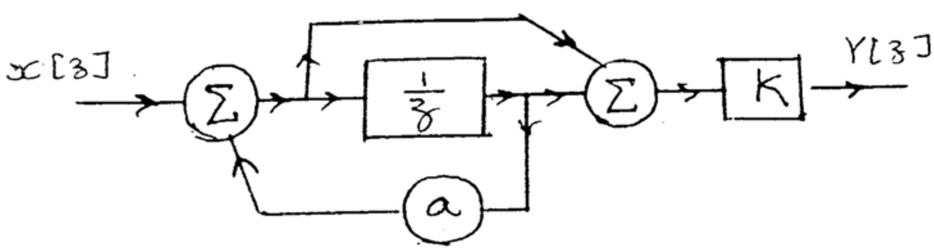


Figure S5.6-6

$$
\begin{aligned}
K &= \frac{1+a^2-\sqrt{2}a}{2-\sqrt{2}} = 1.707(1+a^2-\sqrt{2}a) \\
\left|H\left[e^{j\Omega}\right]\right|^2 &= K^2\frac{(e^{j2\Omega}-\sqrt{2}e^{j\Omega}+1)(e^{-j2\Omega}-\sqrt{2}e^{-j\Omega}+1)}{(e^{j2\Omega}-\sqrt{2}ae^{j\Omega}+a^2)(e^{-j2\Omega}-\sqrt{2}ae^{-j\Omega}+a^2)} \\
&= K^2\frac{4-4\sqrt{2}\cos\Omega+2\cos 2\Omega}{(1+a^2)^2-2\sqrt{2}a(1+a^2)\cos\Omega+2a^2\cos 2\Omega}
\end{aligned}
$$

5.6-8.

$$
\left|H\left[e^{j\Omega}\right]\right|^2 = H\left[e^{j\Omega}\right]H\left[e^{-j\Omega}\right] = \frac{(e^{j\Omega}-\frac{1}{r})(e^{-j\Omega}-\frac{1}{r})}{(e^{j\Omega}-r)(e^{-j\Omega}-r)} = \frac{1+\frac{1}{r^2}-\frac{2}{r}\cos\Omega}{1+r^2-2r\cos\Omega} = \frac{1}{r^2}
$$

This shows that the amplitude response is constant ($\left|H\left[e^{j\Omega}\right]\right| = \frac{1}{r}$) for all values of Ω. The filter is an allpass filter. This result can be generalized exactly the same way for complex poles and zeros.

5.6-9. (a)

$$\begin{aligned} h_2[n] &= (-1)^n h_1[n] \\ &= e^{\pm j\pi n} h_1[n] \end{aligned}$$

Use of the property in Eq. (5.20) with $\gamma = e^{\pm j\pi}$, we obtain $H_2[z] = H_1[e^{j\pm\pi}z]$. Hence

$$H_2[e^{j\Omega}] = H_1[e^{j(\Omega\pm\pi)}]$$

(b) Figure S5.6-9 shows the frequency response of an ideal lowpass filter with cutoff frequency Ω_c. Figure S5.6-9 also shows the same frequency response shifted by π (with 2π-periodicity). It is clear that the shifted response corresponds to an ideal high-pass filter with cutoff frequency $\pi - \Omega_c$.

5.6-10. (a) Let $H(z)$ represent the original filter, either highpass or lowpass. The transformed filter has system function $H_{\mathrm{T}}(z) = H(z)|_{z=-z} = H(-z)$. The basic character of the transformed filter can be assessed by its magnitude response, $|H_{\mathrm{T}}(e^{j\Omega})| = |H(-e^{j\Omega})| = |H(e^{j\pi}e^{j\Omega})| = |H(e^{j(\Omega+\pi)})|$. That is, the magnitude response of a transformed filter is just the magnitude response of the original filter shifted in frequency by π. If the magnitude response of a digital LPF is shifted in frequency by π, it becomes a highpass filter. Similarly, if the magnitude response of a digital HPF is shifted in frequency by π, it becomes a lowpass filter. Put another way, an original passband centered at $\Omega = 0$ (LPF) is shifted by the transformation to a passband centered at $\Omega = \pi$ (HPF), and vice-versa.

(b) If $H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n}$ is the original filter, $H_{\mathrm{T}}(z) = H(-z) = \sum_{n=-\infty}^{\infty} h[n](-z)^{-n} = \sum_{n=-\infty}^{\infty}(-1)^n h[n]z^{-n}$. Thus, the transformed filter has impulse response

$$h_{\mathrm{T}}[n] = (-1)^n h[n].$$

Put another way, the same transformation is accomplished by simply negating the values of $h[n]$ for every odd integer n.

5.6-11. The bilinear transformation states $s = \frac{2(1-z^{-1})}{T(1+z^{-1})}$. Rearranging yields $z = \frac{1+Ts/2}{1-Ts/2}$. Thus, $s = j\omega$ maps to $z = \frac{1+j\omega T/2}{1-j\omega T/2}$.

(a) The magnitude of this transformation is $|z| = \left|\frac{1+j\omega T/2}{1-j\omega T/2}\right| = \frac{|1+j\omega T/2|}{|1-j\omega T/2|} = 1$. Since only the unit circle has $|z| = 1$, the bilinear transform maps $s = j\omega$ onto the unit circle in the z-plane.

(b) Using the solution to 5.6-11a, we know that $z = \frac{1+j\omega T/2}{1-j\omega T/2}$ describes the unit circle in the complex plane. This allows us to write $z = e^{j\Omega} = \frac{1+j\omega T/2}{1-j\omega T/2}$. Thus, the bilinear transformation maps $(-\infty \le \omega \le \infty)$ to $(-\pi/2 \le \Omega\pi/2)$ in a monotonic, although non-linear, manner according to $\Omega = \angle z = \angle \frac{1+j\omega T/2}{1-j\omega T/2} = \arctan(\omega T/2) - \arctan(-\omega T/2) = 2\arctan(\omega T/2)$.

5.7-1. For the system in Eq. (3.15a), the transfer function $H(s)$ is given by

$$H(s) = \frac{1}{s+c}$$

and

$$h(t) = e^{-ct}u(t)$$

Using impulse invariance criterion, we obtain the equivalent digital filter transfer function from Table 5.3 corresponding to $H(s) = \frac{1}{s+c}$ as

$$H[z] = \frac{Tz}{z - e^{-ct}} \simeq \frac{Tz}{z-(1-cT)} \qquad \text{assuming} \quad T \to 0$$

The approximation used in Chapter 3, Eq. (3.15c), yields

$$\widehat{H}[z] = \frac{\beta z}{z+\alpha}$$

Substituting $\beta = \frac{T}{1+cT}$ and $\alpha = \frac{-1}{1+cT}$, we obtain

$$\widehat{H}[z] = \frac{\frac{Tz}{1+cT}}{z - \frac{1}{1+cT}}$$

As $T \to 0$, $\frac{1}{1+cT} \simeq 1 - cT$. Hence

$$\widehat{H}[z] \simeq \frac{T(1-cT)z}{z-(1-cT)} \simeq \frac{Tz}{z-(1-cT)}$$

which is same as that found by impulse-invariance method.

5.7-2. (a)

$$H_a(s) = \frac{7s+20}{2(s^2+7s+10)} = \frac{7s+20}{2(s+2)(s+5)} = \frac{1}{s+2} + \frac{5/2}{s+5}$$

Using Table 12.1, we get

$$H[z] = T\left[\frac{z}{z-e^{-2T}} + \frac{5}{2}\frac{z}{z-e^{-5T}}\right]$$

First we select T:

$$H_a(0) = 1 \quad \text{and for} \quad s \gg 5 \Longrightarrow H_a(s) \approx \frac{7}{2s}$$

$$\text{and} \qquad |H_a(j\omega)| \simeq \frac{7}{\omega} \qquad \omega \gg 5$$

(b) We shall choose the filter bandwidth to be that frequency where

$|H_a(j\omega_0)|$ is 1% of $|H_a(0)|$. Hence $\dfrac{7}{2\omega_0} = 0.01$ and $\omega_0 = 350$, $T = \dfrac{\pi}{350}$

Substituting this value of T in $H[z]$ yields

$$H[z] = 0.008976\left[\frac{z}{z-0.9822} + \frac{5}{2}\frac{z}{z-0.9561}\right] = 0.031416z\left[\frac{z-0.97474}{z^2-1.9383z+0.9391}\right]$$

1) Canonical realization:

$$H[z] = \frac{0.031416z^2 - 0.03062z}{z^2 - 1.9383z + 0.9391}$$

2) parallel realization:

$$H[z] = \frac{0.008976z}{z-0.9822} + \frac{0.02244z}{z-0.9561}$$

The canonical and the parallel realizations are shown in Figure S5.7-2.

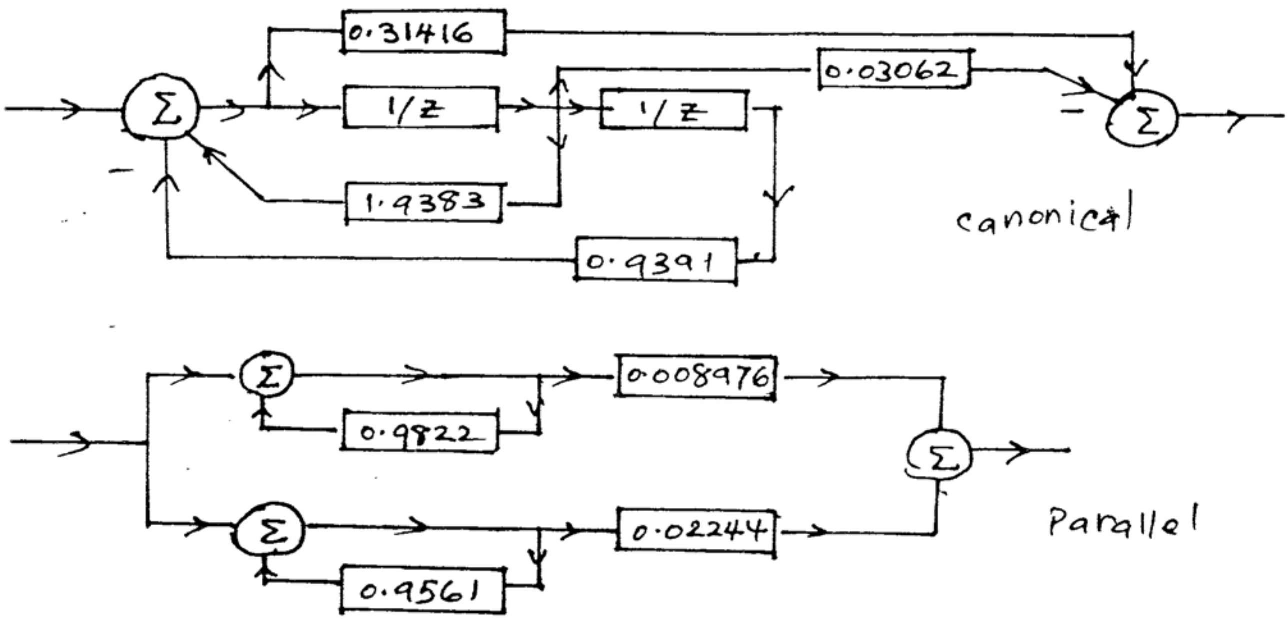


Figure S5.7-2

5.7-3. $H_a(s) = \frac{1}{s^2+\sqrt{2}s+1}$. Using Table 12.1, we get

$$H[z] = \left[\frac{\sqrt{2}Tze^{-T/\sqrt{2}}\sin\left(\frac{T}{\sqrt{2}}\right)}{z^2 - 2ze^{-T/\sqrt{2}}\cos\left(\frac{T}{\sqrt{2}}\right) + e^{-\sqrt{2}T}}\right]$$

We now select T:

$$H_a(0) = 1 \quad \text{and for high } s,\ H_a(s) \approx \frac{1}{s^2}$$

$$\text{and} \qquad |H_a(j\omega)| \simeq \frac{1}{\omega^2} \qquad \text{for high } \omega$$

For negligible aliasing, we select the frequency ω_0 to be that where

$$|H_a(j\omega_0)| \text{ is 1\% of } |H_a(0)|. \quad \text{Hence} \quad \frac{1}{\omega_0^2} = 0.01. \quad \text{and} \quad \omega_0 = 10$$

$$\text{and} \qquad T = \frac{\pi}{\omega_0} = \pi/10$$

Substitution of this value of T in $H[z]$ yields

$$H[z] = \frac{0.0784z}{z^2 - 1.5622z + 0.6413}$$

5.7-4. For an ideal integrator

$$H_a(s) = \frac{1}{s}$$

From Table 12.1, we find

$$H[z] = \frac{Tz}{z-1} \qquad \text{and} \qquad H[e^{j\omega T}] = \frac{Te^{j\omega T}}{e^{j\omega T} - 1}$$

Therefore,

$$\begin{aligned} |H[e^{j\omega T}]| &= \frac{T}{\left|\sqrt{(\cos\omega T - 1)^2 + \sin^2\omega T}\right|} \\ &= \frac{T}{|\sqrt{2(1-\cos\omega T)}|} \\ &= \frac{T}{2|\sin\frac{\omega T}{2}|} \quad |\omega| \leq \frac{\pi}{T} \end{aligned}$$

The ideal integrator amplitude response is

$$|H_a(j\omega)| = \frac{1}{\omega}$$

The two response characteristics are shown in Figure S5.7-4.

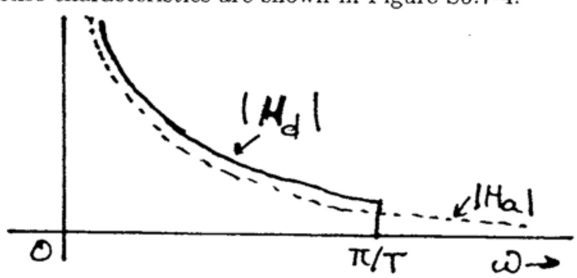


Figure S5.7-4

5.7-5. (a) Because an oscillator output is basically a system output with no input, a system with zero-input response of the form $\sin\Omega_0 n$ (or $\cos(\Omega_0 n + \theta)$ with any value of θ), where $\Omega_0 = \omega_0 T$ will serve as the desired oscillator. A marginally stable system with impulse response of the above form is a candidate. From Table 12.1, pair 11b, we see that a transfer function

$$H[z] = \frac{z \sin \omega_0 T}{z^2 - 2z \cos \omega_0 T + 1}$$

has an impulse response (or zero-input response) of the form $\sin \Omega_0 n$ ($\Omega_0 = \omega_0 T$). The period of the sinusoid is $T_0 = 2\pi/\Omega_0$, and there are 10 samples in each cycle. Therefore, the sampling interval $T = T_0/10 = \pi/5\Omega_0$, and $\Omega_0 T = \pi/5$. hence,

$$\begin{aligned} H[z] &= \frac{z \sin\left(\frac{\pi}{5}\right)}{z^2 - 2\cos\left(\frac{\pi}{5}\right) z + 1} \\ &= \frac{0.5878z}{z^2 - 1.618z + 1} \end{aligned}$$

This is one possible solution. By varying the phase in the impulse response, we could obtain variations of this transfer function.

(b) Another approach is to consider an analog system with transfer function $H_a(s)$ such that its impulse response (or zero-input response) is of the form $\sin \omega_0 t$ [or $\cos(\omega_0 t + \theta)$ for any value of θ]. From Table 6.1, pair 8b we find

$$H_a(s) = \frac{\omega_0}{s^2 + \omega_0^2}$$

Now using Table 12.1, we find the corresponding digital filter using impulse invariance method as

$$H[z] = \frac{Tz \sin \omega_0 T}{z^2 - 2z \cos \omega_0 T + 1}$$

Earlier we found that $\omega_0 T = \pi/5$. Because $\omega_0 = 2\pi(10,000) = 20,000\pi$, the period $T_0 = 10^{-4}$. There are 10 samples in each period. Hence the sampling interval $T = 10^{-5}$, and

$$H[z] = 10^{-5} \frac{0.5878z}{z^2 - 1.618z + 1}$$

This is identical to the answer in (a) except for an amplitude scaling by 10^{-5}. Because, we did not specify any amplitude requirement on the oscillator, different answers will differ by a constant multiplier. This is a marginally stable system and will oscillate without input with the response of the form

$$h[n] = 10^{-5} \sin(0.2\pi n)$$

This is a discrete sinusoid with 10 samples 1 cycle each. Sample is separated by 10^{-5} second. Hence the duration (period) of a cycle is $10 \times 10^{-5} = 10^{-4}$ and the frequency of oscillator is 10^4 Hz or 10 kHz as desired. The controller canonical form of realization is shown in Figure S5.7-5. Note that the multiplier 10^{-5} is not important in this realization, and hence is not shown in the figure. Also, there is no input to an oscillator. Hence no explicit input terminal is shown.

5.7-6. (a) If $g_a(t)$ is the unit step response of the system $H_a(s)$ in Figure 12.8b. Then $g_a(nT)$ should be the response of $H[z]$ to the input $u[n]$. We can use this criterion to design a digital filter to realize a given $H_a(s)$. Consider the filter

$$H_a(s) = \frac{\omega_c}{s + \omega_c}$$

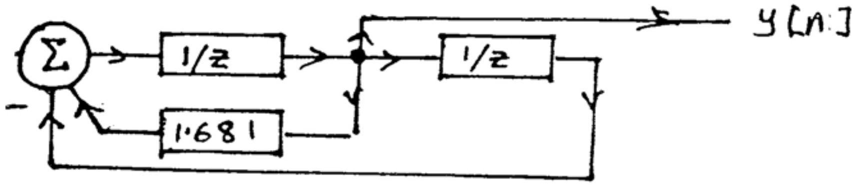


Figure S5.7-5

The unit step response $g_a(t)$ is given by:

$$g_a(t) = \mathcal{L}^{-1}\left[\frac{H_a(s)}{s}\right] = \mathcal{L}^{-1}\left[\frac{\omega_c}{s(s+\omega_c)}\right] = \mathcal{L}^{-1}\left[\frac{1}{s} - \frac{1}{s+\omega_c}\right]$$

$$\text{Therefore} \qquad g_a(t) = (1 - e^{-\omega_c t})u(t)$$

$$\text{and} \qquad g_a(nT) = (1 - e^{-\omega_c nT})u[n]$$

Also, $g[n]$, the response of $H[z]$ to $u[n]$ is given by:

$$g[n] = \mathcal{Z}^{-1}\left\{\frac{z}{z-1}H[z]\right\}$$

Since $g[n] = g_a(nT)$,

$$\begin{aligned}\frac{z}{z-1}H[z] &= \mathcal{Z}[(1 - e^{-\omega_c nT})u[n]] \\ &= \frac{z}{z-1} - \frac{z}{z - e^{-\omega_c T}} = \frac{z(1 - e^{-\omega_c T})}{(z-1)(z - e^{-\omega_c T})}\end{aligned}$$

$$\text{Therefore} \qquad H[z] = \frac{1 - e^{-\omega_c T}}{z - e^{-\omega_c T}}$$

Using the above argument, we can generalize:

$$H[z] = \frac{z-1}{z}\mathcal{Z}\left[\mathcal{L}^{-1}\left(\frac{H_a(s)}{s}\right)\right]_{t=nT}$$

(b)

$$\text{For} \qquad H_a(s) = \frac{\omega_c}{s+\omega_c}$$

the unit step invariance method gives

$$H[z] = \frac{1 - e^{-\omega_c T}}{z - e^{-\omega_c T}}$$

(c) For an integrator, $H_a(s) = 1/s$, and $\mathcal{L}^{-1}[H_s)/s = tu(t)$, and

$$H[z] = \frac{z-1}{z}\mathcal{Z}[nTu[n]] = \frac{T}{z-1}$$

and

$$H[e^{j\omega T}] = \frac{T}{e^{j\omega T} - 1}$$

Hence,

$$|H[e^{j\omega T}]| = \frac{T}{\left|\sqrt{(\cos\omega T - 1)^2 + \sin^2\omega T}\right|} = \frac{T}{|\sqrt{2(1-\cos\omega T)}|} = \frac{T}{2|\sin\frac{\omega T}{2}|} \quad |\omega| \le \frac{\pi}{T}$$

The ideal integrator amplitude response is

$$|H_a(j\omega)| = \frac{1}{\omega}$$

Observe that this amplitude response is identical to that found by the impulse invariance method in Prob. 5.7-4. Hence, this amplitude response and the ideal integrator amplitude response are the same as those in Figure S5.7-4. The only difference between the answers obtained by these methods is that the phase response of the step invariance response differs from that of the impulse invariance method by a constant $\frac{\pi}{2}$.

5.7-7. (a) For a differentiator

$$H_a(s) = s$$

The unit ramp response $r(t)$ is given by

$$r(t) = \mathcal{L}^{-1}F(s)H_a(s) = \mathcal{L}^{-1}\frac{1}{s^2}(s) = u(t)$$

Now we must design $H[z]$ such that its response to input $nTu[n]$ is $u[n]$, that is

$$\mathcal{Z}[u[n]] = H[z]\mathcal{Z}[nTu[n]]$$

$$\frac{z}{z-1} = \frac{Tz}{(z-1)^2}H[z] \quad \text{or} \quad H[z] = \frac{1}{T}(z-1)$$

(b) For an integrator

$$H(s) = \frac{1}{s}$$

The unit ramp response $r(t)$ is given by

$$r(t) = \mathcal{L}^{-1}\left(\frac{1}{s^2}\right)\frac{1}{s} = \frac{1}{2}t^2u(t)$$

Now we design $H[z]$ such that its response to $nTu[n]$ is $\frac{1}{2}n^2T^2u[n]$, that is

$$\mathcal{Z}\left\{\frac{1}{2}n^2T^2u[n]\right\} = H[z]\mathcal{Z}\{nTu[n]\}$$

or

$$\frac{T^2z(z+1)}{2(z-1)^3} = \frac{Tz}{(z-1)^2}H[z]$$

Hence

$$H[z] = \frac{T}{2}\left(\frac{z+1}{z-1}\right)$$

5.7-8.

$$\text{For} \qquad H_a(s) = \sum_i \frac{k_i}{s-\lambda_i}, \qquad H[z] = T\sum_i \frac{k_i z}{z - e^{\lambda_i T}}$$

If $\lambda_i = \alpha_i + j\beta_i$, then $e^{\lambda_i T} = e^{\alpha_i T}e^{j\beta_i T}$. When λ_i is in LHP, $\alpha_i < 0$ and $|e^{\lambda_i T}| = e^{\alpha_i T} < 1$. Hence if λ_i is in LHP, the corresponding pole of $H[z]$ is within the unit circle. Clearly if $H_a(s)$ is stable, the corresponding $H[z]$ is also stable.

5.7-9. (a) The ω-axis is given by $s = j\omega$. Rewriting the transformation $s = \frac{1-z^{-1}}{T}$ as $z = \frac{1}{1-sT}$ and substituting $s = j\omega$ yields $z = \frac{1}{1-j\omega T}$. Thus, we need to show that $z = \frac{1}{1-j\omega T}$ describes a circle centered at $(1/2, 0)$ with a radius of $1/2$.

For a complex variable z, the equation $|z - 1/2|^2 = (1/2)^2$ describes a circle centered at $(1/2, 0)$ with a radius of $1/2$. The transformation rule $z = \frac{1}{1-j\omega T}$ is substituted into this expression.

$$\begin{aligned}
|z-1/2|^2 &= (z-1/2)(z^*-1/2) \\
&= \left(\tfrac{1}{1-j\omega T} - \tfrac{1}{2}\right)\left(\tfrac{1}{1+j\omega T} - \tfrac{1}{2}\right) \\
&= \tfrac{1}{1+\omega^2T^2} + \tfrac{-1}{2(1+j\omega T)} + \tfrac{-1}{2(1-j\omega T)} + \tfrac{1}{4} \\
&= \tfrac{1}{1+\omega^2T^2} + \tfrac{-(1-j\omega T)-(1+j\omega T)}{2(1+j\omega T)(1-j\omega T)} + \tfrac{1}{4} \\
&= \tfrac{1}{1+\omega^2T^2} + \tfrac{-2}{2(1+\omega^2T^2)} + \tfrac{1}{4} \\
&= 1/4 = (1/2)^2
\end{aligned}$$

Since the equation is satisfied, the transformation rule $z = \frac{1}{1-j\omega T}$ maps the ω-axis to a circle centered at $(1/2, 0)$ with a radius of $1/2$.

Notice that different values of ω can map to the same value of z (aliasing), which makes an inverse transformation very problematic.

(b) First, rewrite the transformation $s = \frac{1-z^{-1}}{T}$ as $z = \frac{1}{1-sT}$ Next, notice

$$\begin{aligned}
|z|^2 &= zz^* \\
&= \tfrac{1}{1-sT}\tfrac{1}{1-s^*T} \\
&= \tfrac{1}{1-sT-s^*T+ss^*T^2} \\
&= \tfrac{1}{1-(\sigma+j\omega)T-(\sigma-j\omega)T+(\sigma^2+\omega^2)T^2} \\
&= \tfrac{1}{1-2\sigma T+(\sigma^2+\omega^2)T^2}
\end{aligned}.$$

For $\sigma < 0$, the denominator $1 - 2\sigma T + (\sigma^2 + \omega^2)T^2 > 1$ and $|z|^2 < 1$. That is, the left-half plane of s ($\sigma < 0$) is guaranteed to map to the interior of the unit circle in the z-plane.

5.9-1. (a)

$$x[n] = \underbrace{(0.8)^n u[n]}_{x_1[n]} + \underbrace{2^n u[-(n+1)]}_{x_2[n]}$$

$$\begin{aligned}
x_1[n] &\Longleftrightarrow \frac{z}{z-0.8} \qquad |z| > 0.8 \\
x_2[n] &\Longleftrightarrow \frac{-z}{z-2} \qquad |z| < 2
\end{aligned}$$

Hence

$$\begin{aligned} X[z] &= \frac{z}{z-0.8} - \frac{z}{z-2} \qquad 0.8 < |z| < 2 \\ &= \frac{-1.2z}{z^2 - 2.8z + 1.6} \qquad 0.8 < |z| < 2 \end{aligned}$$

(b)

$$\begin{aligned} X_1[z] &= \frac{z}{z-2} \qquad |z| > 2 \\ X_2[z] &= \frac{z}{z-3} \qquad |z| < 3 \end{aligned}$$

Hence

$$\begin{aligned} X[z] &= \frac{z}{z-2} + \frac{z}{z-3} \qquad 2 < |z| < 3 \\ &= \frac{z(2z-5)}{z^2 - 5z + 6} \qquad 2 < |z| < 3 \end{aligned}$$

(c)

$$\begin{aligned} X_1[z] &= \frac{z}{z-0.8} \qquad |z| > 0.8 \\ X_2[z] &= \frac{-z}{z-0.9} \qquad |z| < 0.9 \end{aligned}$$

Hence
$$\begin{aligned} X[z] &= \frac{z}{z-0.8} - \frac{z}{z-0.9} \\ &= \frac{-z}{10(z^2 - 1.7z + 0.72)} \qquad 0.8 < |z| < 0.9 \end{aligned}$$

(d)

$$\begin{aligned} [(0.8)^n + 3(0.4)^n]\, u[-(n+1)] &\Longleftrightarrow \left(\frac{-z}{z-0.8} - \frac{3z}{z-0.4}\right) \qquad |z| < 0.4 \\ &= \frac{-4z(z-0.7)}{(z-0.4)(z-0.8)} \qquad |z| < 0.4 \end{aligned}$$

(e)

$$\begin{aligned} [(0.8)^n + 3(0.4)^n]\, u[n] &\Longleftrightarrow \frac{z}{z-0.8} + \frac{3z}{z-0.4} \qquad |z| > 0.8 \\ &= \frac{4z(z-0.7)}{(z-0.4)(z-0.8)} \qquad |z| > 0.8 \end{aligned}$$

(f)

$$(0.8)^n u[n] + 3(0.4)^n u[-(n+1)]$$

The region of convergence for $(0.8)^n u[n]$ is $|z| > 0.8$. The region of convergence for $(0.4)^n u[-(n+1)]$ is $|z| < 0.4$. The common region does not exist. Hence the z-transform for this function does not exist.

(g)

$$\begin{aligned} x[n] = (0.5)^{|n|} &= 0.5^n u[n] + (0.5)^{-n} u[-n-1] \\ &= 0.5^n u[n] + 2^n u[-n-1] \\ 0.5^n u[n] &\Longleftrightarrow \frac{z}{z-0.5} \qquad |z| > 0.5 \\ 2^n u[-n-1] &\Longleftrightarrow \frac{-z}{z-2} \qquad |z| < 2 \end{aligned}$$

Hence

$$X[z] = \frac{z}{z-o.5} - \frac{z}{z-2} = \frac{-1.5z}{(z-0.5)(z-2)} \qquad 0.5 \leq |z| < 2$$

(h)

$$x[n] = nu[-(n+1)]$$

$$X[z] = \frac{-z}{(z-1)^2} \qquad |z| < 1$$

5.9-2.

$$\frac{X[z]}{z} = \frac{e^{-2}-2}{(z-e^{-2})(z-2)} = \frac{1}{z-e^{-2}} - \frac{1}{z-2}$$

$$\text{and} \qquad X[z] = \frac{z}{z-e^{-2}} - \frac{z}{z-2}$$

(a) The region of convergence is $|z| > 2$. Both terms are causal. And

$$x[n] = (e^{-2n} - 2^n)u[n]$$

(b) The region of convergence is $e^{-2} < |z| < 2$. In this case the 1st term is causal and the second is anticausal.

$$x[n] = e^{-2n}u[n] + 2^n u[-(n+1)]$$

(c) The region of convergence is $|z| < e^{-2}$. Both terms are anticausal in this case.

$$x[n] = (-e^{-2n} + 2^n)u[-(n+1)]$$

5.9-3. $X(z) = \frac{1}{(2z+1)(z+1)(z+\frac{1}{2})} = \frac{1/2}{(z+1/2)^2(z+1)} = \frac{1}{(z+1/2)^2} + \frac{-2}{(z+1/2)} + \frac{2}{(z+1)} = -2z^{-1}\frac{-z/2}{(z+1/2)^2} - 2z^{-1}\frac{z}{(z+1/2)} + 2z^{-1}\frac{z}{(z+1)}$. Since $(|z| < \frac{1}{2})$, the time-domain signal is left-sided. Thus, $x[n] = 2(n-1)(-1/2)^{n-1}u[-(n-1)-1] + 2(-1/2)^{n-1}u[-(n-1)-1] - 2(-1)^{n-1}u[-(n-1)-1] = -4(n-1)(-1/2)^n u[-n] - 4(-1/2)^n u[-n] + 2(-1)^n u[-n]$. Simplifying yields

$$x[n] = -4n(-1/2)^n u[-n] + 2(-1)^n u[-n].$$

5.9-4. (a) The three poles satisfy $z^3 = \frac{27}{8}$, or $z = 3/2e^{j2\pi k/3}$ for $k = (0,1,2)$. There are two finite zeros at $z = 0$ and $z = 1/2$ as well as a zero at infinity. MATLAB is used to create the corresponding pole-zero plot.

```
>> k = [0:2]; zp = 3/2*exp(j*2*pi*k/3); zz = [0,1/2];
```

```
>> plot(real(zz),imag(zz),'ko',real(zp),imag(zp),'kx');
>> xlabel('Re(z)'); ylabel('Im(z)');
>> axis([-1.5 1.5 -1.5 1.5]); axis equal; grid;
```

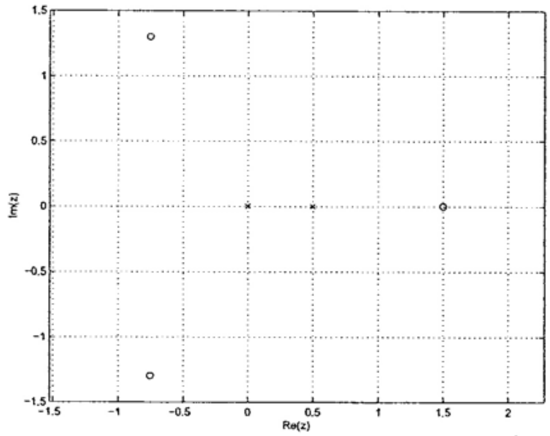


Figure S5.9-4a: Pole-zero plot for $H(z) = \frac{z(z-\frac{1}{2})}{(z^3-\frac{27}{8})}$.

There are two possible regions of convergence, both of which exclude the three system poles: $|z| < 3/2$ or $|z| > 3/2$.

(b) The poles and zeros of $H^{-1}(z)$ are just the zeros and poles, respectively, of $H(z)$. Thus, the three zeros of $H(z)$ satisfy $z^3 = \frac{27}{8}$, or $z = 3/2e^{j2\pi k/3}$ for $k = (0, 1, 2)$. There are two finite poles at $z = 0$ and $z = 1/2$ as well as a pole at infinity. MATLAB is used to create the corresponding pole-zero plot.

```
>> k = [0:2]; zz = 3/2*exp(j*2*pi*k/3); zp = [0,1/2];
>> plot(real(zz),imag(zz),'ko',real(zp),imag(zp),'kx');
>> xlabel('Re(z)'); ylabel('Im(z)');
>> axis([-1.5 1.5 -1.5 1.5]); axis equal; grid;
```

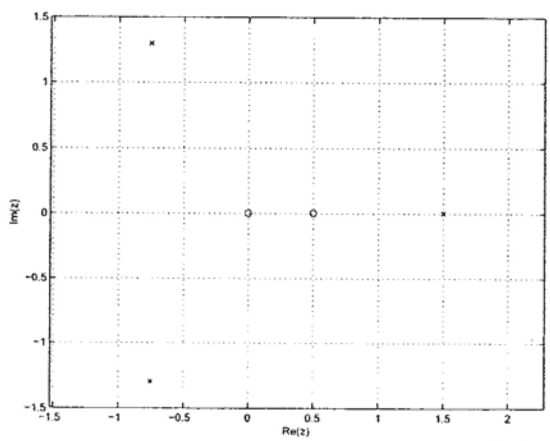


Figure S5.9-4b: Pole-zero plot for $H^{-1}(z) = \frac{(z^3-\frac{27}{8})}{z(z-\frac{1}{2})}$.

There are two possible regions of convergence, both of which exclude the three system poles: $0 < |z| < 1/2$ or $1/2 < |z| < \infty$.

5.9-5. It is known that $x[n]$ has a mode $(1/2)^n$ and that $x_1[n] = (1/3)^n x[n]$ is absolutely summable. For this to be true, the mode at $(1/2)^n$ must be right-sided. That is, $(1/2)^n(1/3)^n u[n]$ is absolutely summable but $(1/2)^n(1/3)^n u[-n]$ is not. It is also known that $x_2[n] = (1/4)^n x[n]$ is not absolutely summable. For this to be true, there must be a pole somewhere in the annulus $3 < |z| < 4$ that corresponds to a left-sided signal; such a mode when multiplied by $(1/3)^n$ is still absolutely summable but when multiplied by $(1/4)^n$ is not absolutely summable. Thus,

$$x[n] \text{ is a two-sided signal.}$$

5.9-6. In polar form, the known pole is at $z = \sqrt{18/16}e^{j\pi/4}$. To be absolutely summable, the signal's region of convergence must include the unit circle, $|z| = 1$.

(a) Yes, the signal can be left-sided. Since the known pole is outside the unit circle, a region of convergence that includes the unit circle (needed for absolute summability) implies that the known pole corresponds to a left-sided component of the signal.

(b) No, the signal cannot be right-sided. If the known pole outside the unit circle is right-sided, the region of convergence cannot include the unit circle and the signal cannot be absolutely summable as required.

(c) Yes, the signal can be two-sided. Let the known pole correspond to a left-sided component and let there be another pole within the unit circle that corresponds to a right-sided component. Such a signal is two-sided and has a region of convergence that includes the unit circle, which ensures the signal is absolutely summable as required.

(d) No, the signal cannot be finite duration. Finite duration signals cannot have poles in the region $0 < |z| < \infty$. Such poles, such as the pole known to exist, correspond to time-domain components with infinite duration.

5.9-7. (a) $X_1(z) = \frac{z}{z-3/4}$. Next, $Y_1(z) = H(z)X_1(z) = \frac{z-1/2}{z+1/2}\frac{z}{z-3/4}$. Thus, $Y_1(z)/z = \frac{z-1/2}{z+1/2}\frac{1}{z-3/4} = \frac{4/5}{z+1/2} + \frac{1/5}{z-3/4}$ or $Y_1(z) = \frac{4z/5}{z+1/2} + \frac{z/5}{z-3/4}$. Inverting yields

$$y_1[n] = \frac{4}{5}(-1/2)^n u[n] + \frac{1}{5}(3/4)^n u[n].$$

(b) The idea of frequency response is used to determine the output in response to the everlasting exponential input $x_2[n] = (3/4)^n$. That is, $H(z = 3/4) = \frac{3/4-1/2}{3/4+1/2} = \frac{1/4}{5/4} = \frac{1}{5}$. Thus,

$$y_2[n] = \frac{1}{5}(3/4)^n.$$

5.9-8. The given signal is $x[n] = (-1)^n u[n-n_0] + \alpha^n u[-n]$. If $|\alpha| = 2$, the z-transform $X(z) = (-1)^{n_0} z^{-n_0}\frac{z}{z+1} + \frac{-z}{z-\alpha}$ has region of convergence $1 < |z| < |\alpha| = 2$, as desired. Thus, the necessary constraint is

$$|\alpha| = 2.$$

There is no constraint on the integer n_0, other than it be finite.

5.9-9. (a) $X_1(z) = \sum_{n=-\infty}^{\infty} x_1[n]z^{-n} = \sum_{n=-\infty}^{\infty} \left((-j)^{-n}u[-n] + \delta[-n]\right) z^{-n} = \sum_{n=-\infty}^{0} \left(\frac{-1}{jz}\right)^n + \sum_{n=-\infty}^{\infty} \delta[-n]z^{-n} = \sum_{n=-\infty}^{0} \left(\frac{j}{z}\right)^n + 1$. For $|z| < 1$, this becomes

$$X_1(z) = 1 + \frac{0 - j/z}{1 - j/z} = 1 + \frac{-j}{z-j}; \text{ ROC } |z| < 1.$$

(b) $X_2(z) = \sum_{n=-\infty}^{\infty} x_2[n]z^{-n} = \sum_{n=-\infty}^{\infty} (\jmath)^n \cos(n+1)u[n] = \sum_{n=0}^{\infty} \jmath^n 0.5\left(e^{\jmath(n+1)} + e^{-jmath(n+1)}\right) z^{-n} = \sum_{n=0}^{\infty} 0.5e^{\jmath}\left(\frac{\jmath e^{\jmath}}{z}\right)^n + \sum_{n=0}^{\infty} 0.5e^{-\jmath}\left(\frac{\jmath e^{-\jmath}}{z}\right)^n$. For $|z| > 1$, this becomes

$$X_2(z) = \frac{0.5e^{\jmath}}{1-\jmath e^{\jmath}z^{-1}} + \frac{0.5e^{-\jmath}}{1-\jmath e^{-\jmath}z^{-1}};\ \text{ROC } |z| > 1.$$

(c) $X_3(z) = \sum_{n=-\infty}^{\infty} x_3[n]z^{-n} = \sum_{n=-\infty}^{\infty}\left(\jmath 0.5(e^n - e^{-n})u[-n+1]\right)z^{-n} = \sum_{n=-\infty}^{1} \jmath 0.5\left\{\left(\frac{e}{z}\right)^n - \left(\frac{1}{ez}\right)^n\right\}$. For $|z| < e$ and $|z| < e^{-1}$, this becomes $X_3(z) = \frac{\jmath}{2}\left\{\frac{0-(e/z)^2}{1-e/z} - \frac{0-(ez)^{-2}}{1-(ez)^{-1}}\right\}$. Thus,

$$X_3(z) = \frac{\jmath z^{-2}}{2}\left\{\frac{e^{-2}}{1-(ez)^{-1}} + \frac{e^2}{1-ez^{-1}}\right\};\ \text{ROC } |z| < e^{-1}.$$

(d) $X_4(z) = \sum_{n=-\infty}^{\infty} x_4[n]z^{-n} = \sum_{n=-\infty}^{\infty}\left(\sum_{k=-\infty}^{0}(2\jmath)^n\delta[n-2k]\right)z^{-n} = \sum_{k=-\infty}^{0}\sum_{n=-\infty}^{\infty}\left(\frac{2\jmath}{z}\right)^n\delta[n-2k] = \sum_{k=-\infty}^{0}\left(\frac{2\jmath}{z}\right)^{2k} = \sum_{k=-\infty}^{0}\left(\frac{-4}{z^2}\right)^k$. For $|z^2| < 4$, this becomes

$$X_4(z) = \frac{0-(-4z^{-2})}{1-(-4z^{-2})} = \frac{4z^{-2}}{1+4z^{-2}};\ \text{ROC } |z| < 2.$$

Note: $|z^2| = |z|^2$, so the region of convergence is $|z|^2 < 4$ or $|z| < 2$.

5.9-10. (a) The signal $X_1(z) = \frac{1}{1+\frac{13}{6}z^{-1}+\frac{1}{6}z^{-2}-\frac{1}{3}z^{-3}}$ has three finite poles, of which at least one must be real. Using the region of convergence, we know that a real root must be either 0.5, −0.5, 2, or −2. Trying these values, we find that $z = -0.5$ and $z = -2$ are both roots of the denominator. The remaining root can be found by noting that the denominator $1+\frac{13}{6}z^{-1}+\frac{1}{6}z^{-2}-\frac{1}{3}z^{-3}$ must be equal to $(1+0.5z^{-1})(1+2z^{-1})(1+Az^{-1})$. Equating the power of z^{-3} on each side yields $-1/3 = A$. Thus, $X_1(z) = \frac{1}{(1+0.5z^{-1})(1+2z^{-1})(1-z^{-1}/3)}$. Expanding yields $X_1(z) = \frac{-1/5}{1+0.5z^{-1}} + \frac{8/7}{1+2z^{-1}} + \frac{2/35}{1-z^{-1}/3}$. Using the region of convergence $(0.5 < |z| < 2)$ and tables, the inverse is

$$x_1[n] = -\frac{1}{5}(-1/2)^n u[n] - \frac{8}{7}(-2)^n u[-n-1] + \frac{2}{35}(1/3)^n u[n].$$

(b) $X_2(z) = \frac{1}{z^{-3}(2-z^{-1})(1+2z^{-1})} = z^3/2\frac{1}{(1-z^{-1}/2)(1+2z^{-1})} = z^3/2\left(\frac{1/5}{1-z^{-1}/2} + \frac{4/5}{1+2z^{-1}}\right)$. Using the region of convergence $(0.5 < |z| < 2)$ and tables, the inverse is

$$x_2[n] = \frac{1}{10}(1/2)^{n+3}u[n+3] - \frac{2}{5}(-2)^{n+3}u[-n-4].$$

5.9-11. (a) $H_1(z) = \frac{z^{-1}}{(z-\frac{1}{2})(1+\frac{1}{2}z^{-1})} = \frac{z^{-1}}{z^{-1}(z-\frac{1}{2})(z+\frac{1}{2})} = \left(\frac{1}{z-1/2} + \frac{-1}{z+1/2}\right) = z^{-1}\left(\frac{z}{z-1/2} + \frac{-z}{z+1/2}\right)$. Since $h_1[n]$ is stable, the region of convergence for $H_1(z)$ must be $|z| > 1/2$. Using this ROC and z-transform tables, the inverse transform is

$$h_1[n] = (1/2)^{n-1}u[n-1] - (-1/2)^{n-1}u[n-1].$$

(b) $H_2(z) = z^{-3}\left(\frac{z+1}{(z-2)(z+1/2)}\right) = z^{-3}\left(\frac{6/5}{z-2} + \frac{-1/5}{z+1/2}\right) = z^{-4}\left(\frac{6z/5}{z-2} + \frac{-z/5}{z+1/2}\right)$. Since $h_2[n]$ is stable, the region of convergence for $H_2(z)$ must be $1/2 < |z| < 2$. Using this ROC and z-transform tables, the inverse transform is

$$h_2[n] = \frac{-6}{5}(2)^{n-4}u[-n+3] + \frac{-1}{5}(-1/2)^{n-4}u[n-4].$$

5.9-12. $H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n} = \sum_{n=-\infty}^{\infty}\sum_{k=0}^{\infty}\delta[n-Nk]z^{-n} = \sum_{k=0}^{\infty}\sum_{n=-\infty}^{\infty}\delta[n-Nk]z^{-n} = \sum_{k=0}^{\infty} z^{-Nk} = \sum_{k=0}^{\infty}\left(\frac{1}{z^N}\right)^k$. For $|z| > 1$, this becomes

$$H(z) = \frac{1}{1-z^N};\ \text{ROC } |z| > 1.$$

Notice, $H(z)$ has N poles that correspond to the N roots of unity. That is, the poles of $H(z)$ satisfy $z = e^{j2\pi k/N}$ for $k = (0, 1, \ldots, N-1)$.

5.9-13. For causal signals, the region of convergence may be ignored. We shall consider it only for noncausal inputs

(a)

$$Y[z] = X[z]\,H[z] = \frac{z^2}{(z-e)(z+0.2)(z-0.8)}$$

Modified partial fraction expansion of $Y[z]$ yields

$$Y[z] = 0.477\frac{z}{z-e} - 0.068\frac{z}{z+0.2} - 0.412\frac{z}{z-0.8}$$

and

$$y[n] = [0.477e^n - 0.068(-0.2)^n - 0.412(0.8)^n]\,u[n]$$

(b)

$$\begin{aligned} X[z] &= \frac{-z}{z-2} \qquad |z| < 2 \\ H[z] &= \frac{z}{(z+0.2)(z-0.8)} \qquad |z| > 0.8 \end{aligned}$$

$$Y[z] = \frac{-z^2}{(z+0.2)(z-0.8)(z-2)} \qquad 0.8 < |z| < 2$$

and

$$\frac{Y[z]}{z} = \frac{-z}{(z+0.2)(z-0.8)(z-2)} = \frac{1/11}{z+0.2} + \frac{2/3}{z-0.8} - \frac{0.758}{z-2}$$

Therefore $\quad Y[z] = \frac{1}{11}\frac{z}{z+0.2} + \frac{2}{3}\frac{z}{z-0.8} - 0.758\frac{z}{z-2} \quad 0.8 < |z| < 2$

and $\quad y[n] = \left[\frac{1}{11}(-0.2)^n + \frac{2}{3}(0.8)^n\right]u[n] + 0.758(2)^n u[-(n+1)]$

(c) The input in this case is the sum of the inputs in parts a and b hence the response will be the sum of the responses in part a and b.

5.9-14.

$$x[n] = \underbrace{2^n u[n]}_{x_1[n]} + \underbrace{u[-(n+1)]}_{x_2[n]}$$

$$\begin{aligned} X_1[z] &= \frac{z}{z-2} \qquad |z| > 2 \\ X_2[z] &= \frac{-z}{z-1} \qquad |z| < 1 \end{aligned}$$

There is no region of convergence common to $X_1[z]$ and $X_2[z]$

$$H[z] = \frac{z}{(z+0.2)(z-0.8)}$$

The region of convergence of $H[z]$ is $|z| > 0.8$ (assuming a causal system). We should find the response to $x_1[n]$ and $x_2[n]$ separately.

$$Y_1[z] = \frac{z^2}{(z-2)(z+0.2)(z-0.8)} \qquad |z| > 2$$

The modified partial fractions of $Y[z]$ yield

$$Y_1[z] = -\frac{1}{11}\frac{z}{z+0.2} - \frac{2}{3}\frac{z}{z-0.8} + 0.758\frac{z}{z-2}$$

and

$$y_1[n] = \left[-\frac{1}{11}(-0.2)^n - \frac{2}{3}(0.8)^n + 0.758(2)^n\right]u[n]$$

Similarly

$$Y_2[z] = \frac{-25}{6}\frac{z}{z-1} + \frac{1}{6}\frac{z}{z+0.2} + 4\frac{z}{z-0.8} \quad 0.8 < |z| < 1$$

and

$$y_2[n] = \left[\frac{1}{6}(-0.2)^n + 4(0.8)^n\right]u[n] + \frac{25}{6}u[-(n+1)]$$

and

$$y[n] = y_1[n] + y_2[n] = \left[\frac{5}{66}(-0.2)^n + \frac{10}{3}(0.8)^n + 0.758(2)^n\right]u[n] + \frac{25}{6}u[-(n+1)]$$

5.9-15.

$$X[z] = \frac{-z}{z-e^2} \qquad |z| < e^2$$

and

$$H[z] = \frac{z}{(z+0.2)(z-0.8)} \qquad |z| > 0.8$$

No common region of convergence for $X[z]$ and $H[z]$ exists. Hence

$$y[n] = \infty$$

5.M-1. Taking the z-transform of $4y[n+2] - y[n] = x[n+2] + x[n]$ yields $Y(z)\left(4z^2-1\right) =$

$X(z)\,(z^2+1)$. Thus, the system function is $H(z) = \frac{Y(z)}{X(z)} = \frac{z^2+1}{4z^2-1} = 0.25\frac{1+z^{-2}}{1-z^{-2}/4}$.

(a) MATLAB is used to create the pole-zero diagram.

```
>> zz = roots([1 0 1]); zp = roots([4 0 -1]);
>> theta = linspace(0,2*pi,201);
>> plot(real(zz),imag(zz),'ko',real(zp),imag(zp),'kx',...
   cos(theta),sin(theta),'k');
>> xlabel('Re(z)'); ylabel('Im(z)'); grid;
>> axis([-1.1 1.1 -1.1 1.1]); axis equal;
```

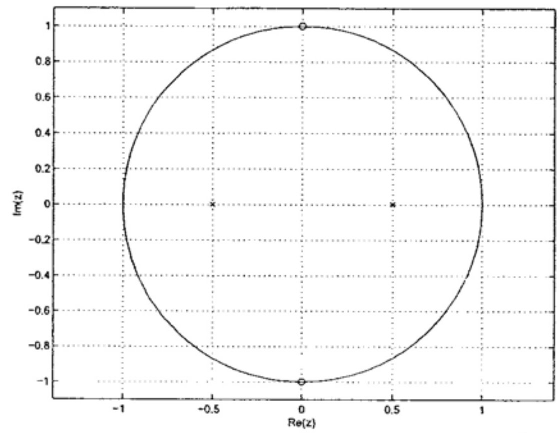


Figure S5.M-1a: Pole-zero diagram for $H(z) = \frac{z^2+1}{4z^2-1}$.

(b) $H(e^{j\Omega}) = \left.\frac{z^2+1}{4z^2-1}\right|_{z=e^{j\Omega}}$. MATLAB is used to plot the magnitude response $|H(e^{j\Omega})|$.

```
>> Omega = linspace(-pi,pi,501);
>> z = exp(j*Omega); H = (z.^2+1)./(4*z.^2-1);
>> plot(Omega,abs(H),'k'); axis([-pi pi 0 1]); grid;
>> xlabel('\Omega'); ylabel('|H(e^{j\Omega})|');
>> set(gca,'xtick',[-pi:pi/4:pi],'xticklabel',['-p ';...
   '   ';'   ';'   ';' 0 ';'   ';'   ';'   ';' p '],...
   'fontname','symbol');
```

(c) The pole-zero plot of Figure S5.M-1a and the magnitude response plot of Figure S5.M-1b confirm that this is a band-stop system.

(d) Yes, the system is asymptotically stable. Referring to Figure S5.M-1a, all the system poles are within the unit circle.

(e) Yes, the system is real. Since the system is expressed as a constant-coefficient linear difference equation with real coefficients, the impulse response $h[n]$ and system are both real.

(f) For an input of the form $x[n] = \cos(\Omega n)$, the greatest possible amplitude of the output corresponds to the greatest gain shown in the magnitude response plot of Figure S5.M-1b. Thus, 2/3 is the greatest output amplitude given an input of $x[n] = \cos(\Omega n)$. This output amplitude occurs when $\Omega = k\pi$, for any integer k.

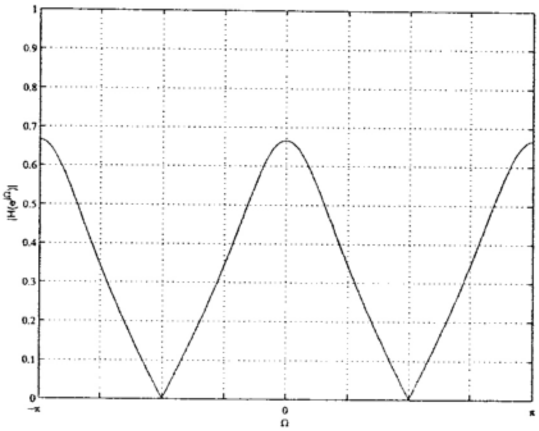


Figure S5.M-1b: Magnitude response plot for $4y[n+2] - y[n] = x[n+2] + x[n]$.

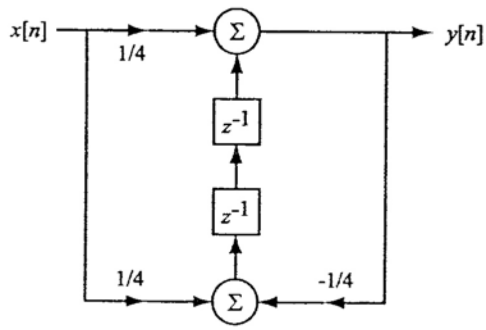


Figure S5.M-1g: TDFII implementation of $y[n] - 0.25y[n-2] = 0.25x[n] + 0.25x[n-2]$.

(g) Inverting $H(z) = \frac{Y(z)}{X(z)} = 0.25\frac{1+z^{-2}}{1-z^{-2}/4}$ provides $y[n] - 0.25y[n-2] = 0.25x[n] + 0.25x[n-2]$, which is a convenient form for implementation. Figure S5.M-1g illustrates a TDFII implementation of the system.

5.M-2. (a) $H(z) = \frac{z^2-j}{z-0.9e^{j3\pi/4}} = \frac{(z-e^{j\pi/4})(z-e^{-j3\pi/4})}{z-0.9e^{j3\pi/4}}$. Thus, there are zeros at $z = e^{j\pi/4}$ and $z = e^{-j3\pi/4}$, and there are poles at $z = 0.9e^{j3\pi/4}$ and $z = \infty$.

```
>> zz = roots([1 0 -j]); zp = 0.9*exp(j*3*pi/4);
>> theta = linspace(0,2*pi,201);
>> plot(real(zz),imag(zz),'ko',real(zp),imag(zp),'kx',...
   cos(theta),sin(theta),'k');
>> xlabel('Re(z)'); ylabel('Im(z)');
>> axis([-1.1 1.1 -1.1 1.1]); axis equal; grid;
```

Since this is a complex system, poles and zeros need not occur in complex conjugate pairs.

(b) Substituting $z = e^{j\Omega}$ into $H(z) = \frac{z^2-j}{z-0.9e^{j3\pi/4}}$, MATLAB is used to create the magnitude response plot.

```
>> Omega = linspace(-2*pi,2*pi,1001); z = exp(j*Omega);
>> H = (z.^2-j)./(z-0.9*exp(j*3*pi/4));
>> plot(Omega,abs(H),'k'); axis([-2*pi 2*pi 0 21]);
>> xlabel('\Omega'); ylabel('|H(e^{j\Omega})|'); grid;
```

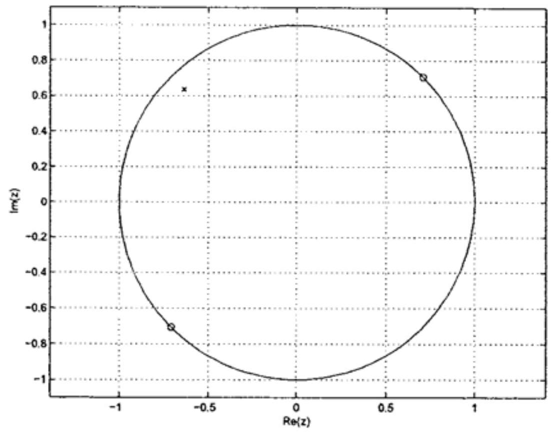


Figure S5.M-2a: Pole-zero diagram for $H(z) = \frac{z^2 - \jmath}{z - 0.9e^{\jmath 3\pi/4}}$.

```
>> set(gca,'xtick',[-2*pi:pi/4:2*pi],'xticklabel',['-2p';
   '   ';'   ';'   ';'-p ';'   ';'   ';'   ';' 0 ';...
   '   ';'   ';'   ';' p ';'   ';'   ';'   ';' 2p'],'fontname','symbol')
```

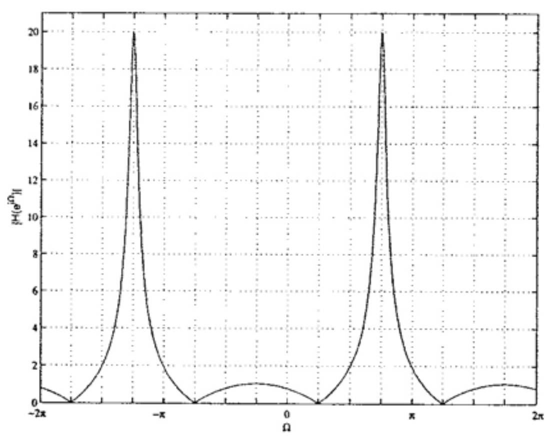


Figure S5.M-2b: Magnitude response plot for $H(z) = \frac{z^2 - \jmath}{z - 0.9e^{\jmath 3\pi/4}}$.

From Figure S5.M-2b, the system appears to be a type of bandpass filter. This particular filter tends to pass only positive frequency inputs near $\Omega = 3\pi/4$; the corresponding negative frequencies near $\Omega = -3\pi/4$ are significantly attenuated. Since the magnitude response is not an even function of Ω, the output of the filter will be complex-valued.

5.M-3. (a) $H(z) = \frac{z^4 - 1}{2(z^2 + 0.81\jmath)}$ has four finite zeros and two finite poles. The zeros are the four roots of unity, and the poles are at $z = 0.9e^{\jmath\pi/4}$ and $z = 0.9e^{-\jmath 3\pi/4}$. MATLAB is used to compute the pole-zero plot.

```
>> zz = roots([1 0 0 0 -1]); zp = roots([1 0 j*0.81]);
>> theta = linspace(0,2*pi,201);
>> plot(real(zz),imag(zz),'ko',real(zp),imag(zp),'kx',...
   cos(theta),sin(theta),'k');
```

```
>> xlabel('Re(z)'); ylabel('Im(z)');
>> axis([-1.1 1.1 -1.1 1.1]); axis equal; grid;
```

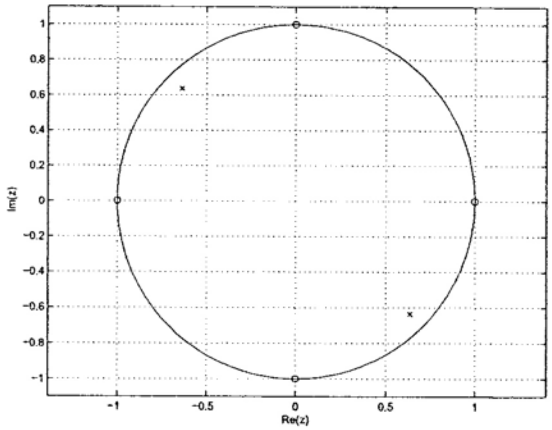


Figure S5.M-3a: Pole-zero diagram for $H(z) = \frac{z^4-1}{2(z^2+0.81j)}$.

(b) MATLAB is used to plot the magnitude response.

```
>> Omega = linspace(-pi,pi,1001); z = exp(j*Omega);
>> H = (z.^4-1)./(2*(z.^2+0.81*j));
>> plot(Omega,abs(H),'k'); axis([-pi pi 0 6]);
>> xlabel('\Omega'); ylabel('|H(e^{j\Omega})|'); grid;
>> set(gca,'xtick',[-pi:pi/4:pi],'xticklabel',['-p ';...
   '   ';'   ';'   ';' 0 ';'   ';'   ';'   ';' p '],...
   'fontname','symbol');
```

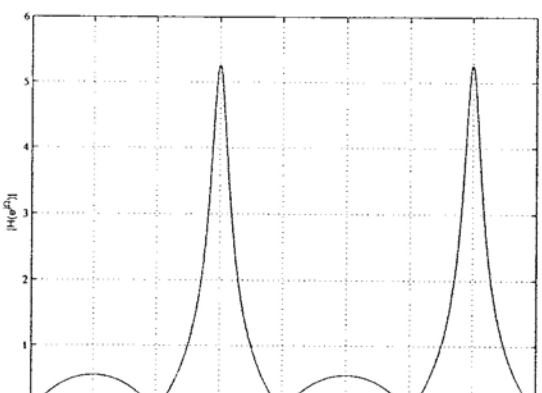


Figure S5.M-3b: Magnitude response plot for $H(z) = \frac{z^4-1}{2(z^2+0.81j)}$.

(c) Since $H(z)$ is an improper rational function and has poles at infinity, the system is non-causal. By dividing the numerator by the denominator (in such a way as to yield a right-sided impulse response), an improper rational function $H(z)$ will yield an impulse response $h[n]$ that is not zero for all negative n. Right-sided functions with poles at infinity are sometimes called "not-quite causal" since they are not causal only by a finite shift.

(d) Adding two poles at zero ($a = b = 0$) corresponds to a simple right-shift by two and does not affect the magnitude response. That is, $|H_{\text{causal}}(e^{j\Omega})| = \left|\frac{H(e^{j\Omega})}{(e^{j\Omega})^2}\right| = \frac{|H(e^{j\Omega})|}{|(e^{j\Omega})^2|} = |H(e^{j\Omega})|$. By adding two poles at zero to $H(z)$, $H_{\text{causal}}(z)$ becomes a proper rational function with only finite poles, and the corresponding impulse response function becomes causal.

(e) Noting that $H_{\text{causal}}(z) = \frac{Y(z)}{X(z)} = \frac{0.5z^4 - 0.5}{z^4 + 0.81jz^2} = \frac{0.5 - 0.5z^{-4}}{1 + 0.81jz^{-2}}$, the corresponding difference equation is $y[n] + 0.81jy[n-2] = 0.5x[n] - 0.5x[n-4]$. Figure S5.M-3e shows the corresponding TDFII implementation.

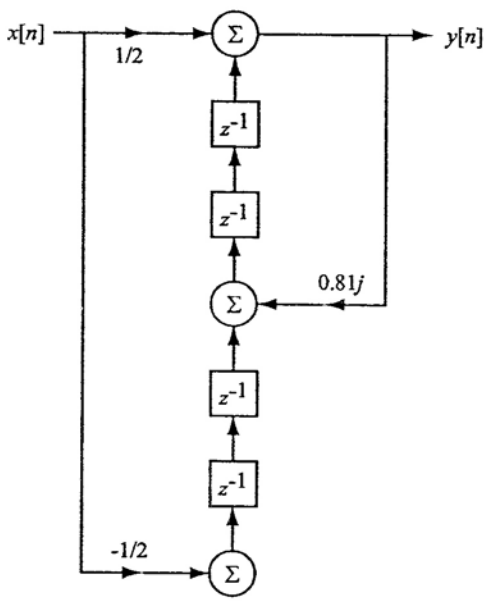


Figure S5.M-3e: TDFII implementation of $y[n] + 0.81jy[n-2] = 0.5x[n] - 0.5x[n-4]$.

5.M-4. (a) From the block diagram, the corresponding difference equation is written directly.

$$y[n] - 0.5y[n-2] = x[n].$$

(b) Taking the z-transform of $y[n] - 0.5y[n-2] = x[n]$ yields $Y(z)(1 - 0.5z^{-2}) = X(z)$. Thus, $H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1-0.5z^{-2}}$. Substituting $z = e^{j\Omega}$ into $H(z)$, the magnitude response is $\left|H(e^{j\Omega})\right| = \left|\frac{1}{1-0.5e^{-j2\Omega}}\right| = \frac{1}{\sqrt{(1-0.5\cos(2\Omega))^2 + (-0.5\sin(2\Omega))^2}} = \frac{1}{\sqrt{1-\cos(2\Omega)+0.25(\cos^2(2\Omega)+\sin^2(2\Omega))}}$. Thus,

$$\left|H(e^{j\Omega})\right| = \frac{1}{\sqrt{5/4 - \cos(2\Omega)}}.$$

MATLAB is used to plot $\left|H(e^{j\Omega})\right|$.

```
>> Omega = linspace(-pi,pi,501);
>> Hmag = 1./sqrt(5/4-cos(2*Omega));
```

```
>> plot(Omega,Hmag,'k'); axis([-pi pi 0 2.5]); grid;
>> xlabel('\Omega'); ylabel('|H(e^{j\Omega})|');
>> set(gca,'xtick',[-pi:pi/4:pi],'xticklabel',['-p ';...
     '   ';'   ';'   ';' 0 ';'   ';'   ';'   ';' p '],...
     'fontname','symbol');
```

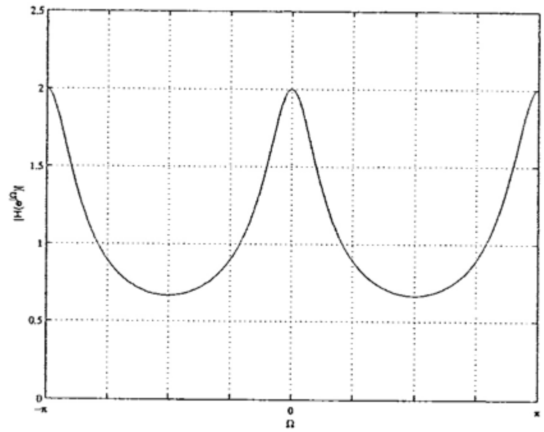


Figure S5.M-4b: $\left|H(e^{j\Omega})\right| = \frac{1}{\sqrt{5/4-\cos(2\Omega)}}$.

Standard filter types do not provide a good description of this filter. The system appears most like a bandstop filter, but its stopband attenuation is quite poor. The system boosts the gain of low and high frequencies more than it attenuates the middle frequencies.

(c) Inverting $H(z) = \frac{1}{1-0.5z^{-2}} = \frac{1/2}{1-z^{-1}/\sqrt{2}} + \frac{1/2}{1+z^{-1}/\sqrt{2}}$ yields

$$h[n] = 0.5\left((1/\sqrt{2})^n + (-1/\sqrt{2})^n\right)u[n].$$

5.M-5. Label the first summation block output as $v[n]$. Thus, $v[n] = x[n-1] + \frac{1}{4}y[n]$ or $V(z) = z^{-1}X(z) + \frac{1}{4}Y(z)$. The output of the second summation block is $y[n+1] = x[n]+v[n-1]$ or $zY(z) = X(z)+z^{-1}V(z)$. Combining the two equations yields $zY(z) = X(z) + z^{-1}\left(z^{-1}X(z) + \frac{1}{4}Y(z)\right)$ or $Y(z)\left(z - \frac{1}{4}z^{-1}\right) = X(z)\left(1+z^{-2}\right)$. Multiplying both sides by z^{-1} yields $Y(z)\left(1 - \frac{1}{4}z^{-2}\right) = X(z)\left(z^{-1}+z^{-3}\right)$. Thus, $H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-1}+z^{-3}}{1-\frac{1}{4}z^{-2}} = \frac{1+z^2}{z(z-1/2)(z+1/2)} = \frac{-4}{z} + \frac{5/2}{z-1/2} + \frac{5/2}{z+1/2} = \frac{-4}{z} + z^{-1}\frac{5z/2}{z-1/2} + z^{-1}\frac{5z/2}{z+1/2}$. Taking the inverse transform yields

$$h[n] = -4\delta[n-1] + \frac{5}{2}(1/2)^{n-1}u[n-1] + \frac{5}{2}(-1/2)^{n-1}u[n-1].$$

Since $h[n] = 0$ for $n < 0$, the system is causal.

The system has three poles located at $z = 0$, $z = 1/2$, and $z = -1/2$. Since all three poles are inside the unit circle, the system is stable.

5.M-6. Since $h[n] = \delta[n-1]+\delta[n+1]$, $H(z) = z^{-1}+z$ and $|H(e^{j\Omega})| = 2\left|\frac{e^{-j\Omega}+e^{j\Omega}}{2}\right| = 2|\cos(\Omega)|$.

```
>> Omega = linspace(-pi,pi,501);
```

```
>> Hmag = 2*abs(cos(Omega));
>> plot(Omega,Hmag,'k'); axis([-pi pi 0 2.5]); grid;
>> xlabel('\Omega'); ylabel('|H(e^{j\Omega})|');
>> set(gca,'xtick',[-pi:pi/4:pi],'xticklabel',['-p ';...
    '   ';'   ';'   ';' 0 ';'   ';'   ';'   ';' p '],...
    'fontname','symbol');
```

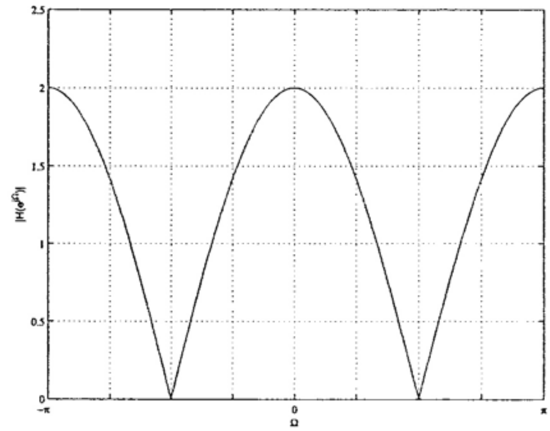


Figure S5.M-6: $|H(e^{j\Omega})| = 2|\cos(\Omega)|$.

Using Figure S5.M-6, this system is best described by a bandstop filter with gain. That is, low and high frequencies have a boosted gain of two and middle frequencies near $\Omega = \pi/2$ are attenuated.

5.M-7. (a)
```
>> Omega = linspace(-pi,pi,501); z = exp(j*Omega);
>> H = cos(1./z);
>> plot(Omega,abs(H),'k'); axis([-pi pi 0 2]); grid;
>> xlabel('\Omega'); ylabel('|H(e^{j\Omega})|');
>> set(gca,'xtick',[-pi:pi/4:pi],'xticklabel',['-p ';...
    '   ';'   ';'   ';' 0 ';'   ';'   ';'   ';' p '],...
    'fontname','symbol');
```

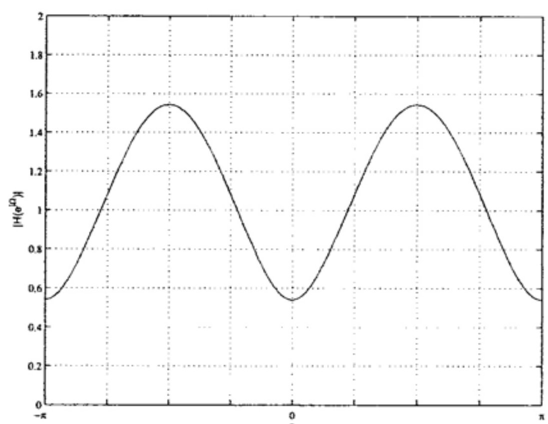


Figure S5.M-7a: Magnitude response for $H(z) = \cos(z^{-1})$.

From Figure S5.M-7a, it appears that this system behaves somewhat like a band-pass filter. Frequencies near $\Omega = \pi/2$ are boosted while frequencies near $\Omega = k\pi$ are somewhat attenuated.

(b) A Maclaurin series expansion of $\cos(x)$ is $\sum_{k=0}^{\infty} \frac{(-x^2)^k}{(2k)!}$. Substituting $n = z^{-1}$ yields $H(z) = \cos(z^{-1}) = \sum_{k=0}^{\infty} \frac{(-z^{-2})^k}{(2k)!}$. Inverting yields

$$h[n] = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\delta[n-2k] = \delta[n] - \frac{1}{2!}\delta[n-2] + \frac{1}{4!}\delta[n-4] - \frac{1}{6!}\delta[n-6] + \ldots.$$

```
>> n = [0:10]; h = zeros(size(n));
>> k = [0:5]; h(2*k+1) = (-1).^(k)./(gamma(2*k+1));
>> stem(n,h,'k'); xlabel('n'); ylabel('h[n]');
>> axis([-.5 10.5 -.6 1.1]);
```

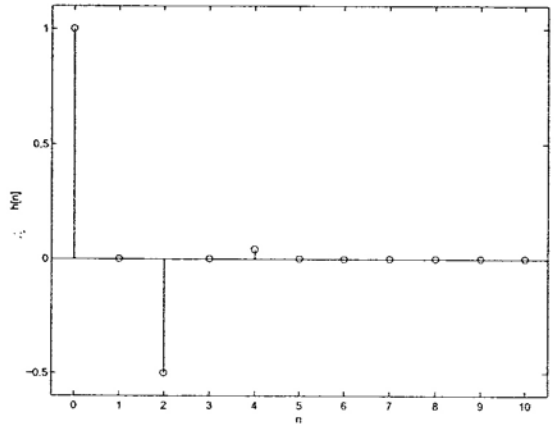


Figure S5.M-7b: Impulse response $h[n]$ for $H(z) = \cos(z^{-1})$.

(c) Using Figure S5.M-7b, it is clear that the impulse response quickly decays to zero. Thus, only the first few terms from $h[n]$ are needed for a good approximation. Note that $h[n] \approx \delta[n] - \delta[n-2]/2 + \delta[n-4]/24$. An FIR difference equation is found by letting $h[n] = y[n]$ and $\delta[n] = x[n]$. That is,

$$y[n] = x[n] - x[n-2]/2 + x[n-4]/24.$$

This is a fourth-order FIR filter with only three non-zero coefficients. The magnitude response is easily computed using MATLAB.

```
>> Omega = linspace(-pi,pi,501); z = exp(j*Omega);
>> H = 1 - z.^(-2)/2 + z.^(-4)/24;
>> plot(Omega,abs(H),'k'); axis([-pi pi 0 2]); grid;
>> xlabel('\Omega'); ylabel('|H(e^{j\Omega})|');
>> set(gca,'xtick',[-pi:pi/4:pi],'xticklabel',['-p ';...
     '   ';'   ';'   ';' 0 ';'   ';'   ';'   ';' p '],...
     'fontname','symbol');
```

Visually, Figure S5.M-7c is indistinguishable from Figure S5.M-7b. Thus, the FIR filter appears to closely approximate the original system.

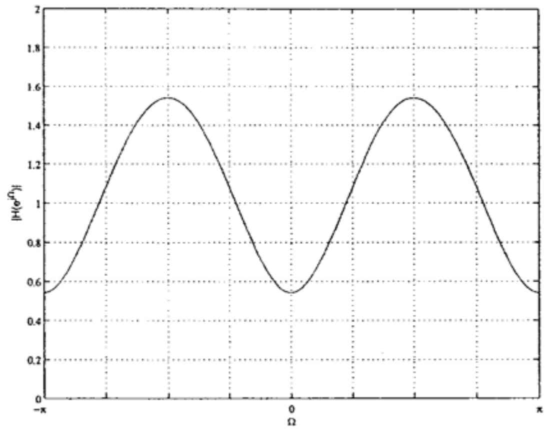


Figure S5.M-7c: Magnitude response for FIR approximation of $H(z) = \cos(z^{-1})$.

5.M-8. Factored form is used to plot roots, and standard transfer function form is used to compute magnitude response plots.

(a) Order-8 Butterworth LPF with $\Omega_c = \pi/3$.

```
>> Omega_c = pi/3; Omega = linspace(-pi,pi,1001);
>> [z,p,k] = butter(8,Omega_c/pi);
>> subplot(121),plot(real(p),imag(p),'kx',...
    real(z),imag(z),'ko',cos(Omega),sin(Omega),'k');
>> axis([-1.1 1.1 -1.1 1.1]); axis equal;
>> xlabel('Re(z)'); ylabel('Im(z)');
>> [B,A] = butter(8,Omega_c/pi);
>> HLP = polyval(B,exp(j*Omega))./polyval(A,exp(j*Omega));
>> subplot(122),plot(Omega,20*log10(abs(HLP)),'k');
>> axis([-pi pi -40 2]); grid;
>> xlabel('\Omega'); ylabel('|H_{LP}(e^{j\Omega})|');
>> set(gca,'xtick',[-pi:pi/3:pi],'xticklabel',['-p ';...
    '   ';'   ';' 0 ';'   ';'   ';' p '],...
    'fontname','symbol');
```

(b) Order-8 Butterworth HPF with $\Omega_c = \pi/3$.

```
>> Omega_c = pi/3; Omega = linspace(-pi,pi,1001);
>> [z,p,k] = butter(8,Omega_c/pi,'high');
>> subplot(121),plot(real(p),imag(p),'kx',...
    real(z),imag(z),'ko',cos(Omega),sin(Omega),'k');
>> axis([-1.1 1.1 -1.1 1.1]); axis equal;
>> xlabel('Re(z)'); ylabel('Im(z)');
>> [B,A] = butter(8,Omega_c/pi,'high');
>> HHP = polyval(B,exp(j*Omega))./polyval(A,exp(j*Omega));
>> subplot(122),plot(Omega,20*log10(abs(HHP)),'k');
>> axis([-pi pi -40 2]); grid;
>> xlabel('\Omega'); ylabel('|H_{HP}(e^{j\Omega})|');
>> set(gca,'xtick',[-pi:pi/3:pi],'xticklabel',['-p ';...
    '   ';'   ';' 0 ';'   ';'   ';' p '],...
    'fontname','symbol');
```

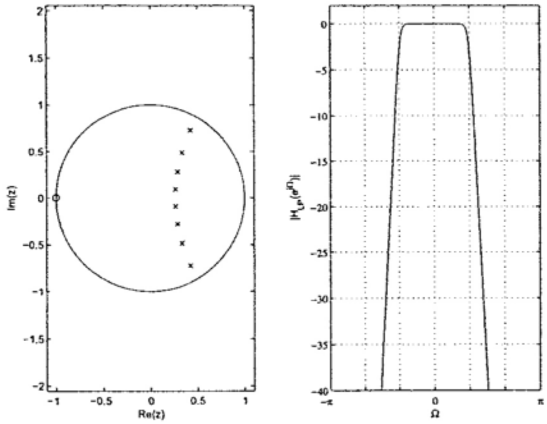


Figure S5.M-8a: Order-8 Butterworth LPF with $\Omega_c = \pi/3$.

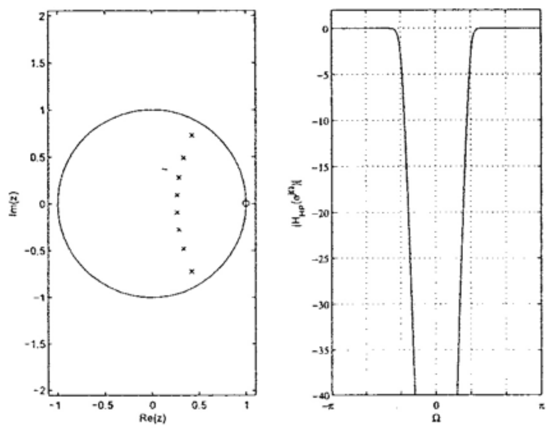


Figure S5.M-8b: Order-8 Butterworth HPF with $\Omega_c = \pi/3$.

(c) Order-8 Butterworth BPF with passband between $5\pi/24$ and $11\pi/24$. Notice that the command butter requires the parameter $N = 3$ to be used to obtain a $(2N = 8)$-order bandpass filter.

```
>> Omega_c = [5*pi/24,11*pi/24]; Omega = linspace(-pi,pi,1001);
>> [z,p,k] = butter(4,Omega_c/pi);
>> subplot(121),plot(real(p),imag(p),'kx',...
    real(z),imag(z),'ko',cos(Omega),sin(Omega),'k');
>> axis([-1.1 1.1 -1.1 1.1]); axis equal;
>> xlabel('Re(z)'); ylabel('Im(z)');
>> [B,A] = butter(4,Omega_c/pi);
>> HBP = polyval(B,exp(j*Omega))./polyval(A,exp(j*Omega));
>> subplot(122),plot(Omega,20*log10(abs(HBP)),'k');
>> axis([-pi pi -40 2]); grid;
>> xlabel('\Omega'); ylabel('|H_{BP}(e^{j\Omega})|');
>> set(gca,'xtick',[-pi:pi/3:pi],'xticklabel',['-p ';...
    '   ';'   ';' 0 ';'   ';'   ';' p '],...
    'fontname','symbol');
```

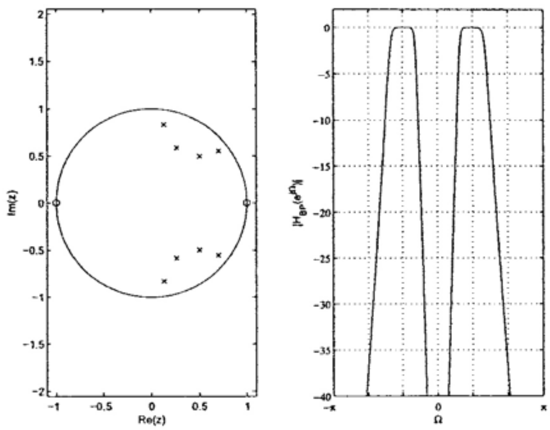


Figure S5.M-8c: Order-8 Butterworth BPF with passband between $5\pi/24$ and $11\pi/24$.

(d) Order-8 Butterworth BSF with stopband between $5\pi/24$ and $11\pi/24$. Notice that the command `butter` requires the parameter $N = 4$ to be used to obtain a $(2N = 8)$-order bandstop filter.

```
>> Omega_c = [5*pi/24,11*pi/24]; Omega = linspace(-pi,pi,1001);
>> [z,p,k] = butter(4,Omega_c/pi,'stop');
>> subplot(121),plot(real(p),imag(p),'kx',...
     real(z),imag(z),'ko',cos(Omega),sin(Omega),'k');
>> axis([-1.1 1.1 -1.1 1.1]); axis equal;
>> xlabel('Re(z)'); ylabel('Im(z)');
>> [B,A] = butter(4,Omega_c/pi,'stop');
>> HBS = polyval(B,exp(j*Omega))./polyval(A,exp(j*Omega));
>> subplot(122),plot(Omega,20*log10(abs(HBS)),'k');
>> axis([-pi pi -40 2]); grid;
>> xlabel('\Omega'); ylabel('|H_{BS}(e^{j\Omega})|');
>> set(gca,'xtick',[-pi:pi/3:pi],'xticklabel',['-p ';...
     '   ';'   ';' 0 ';'   ';'   ';' p '],...
     'fontname','symbol');
```

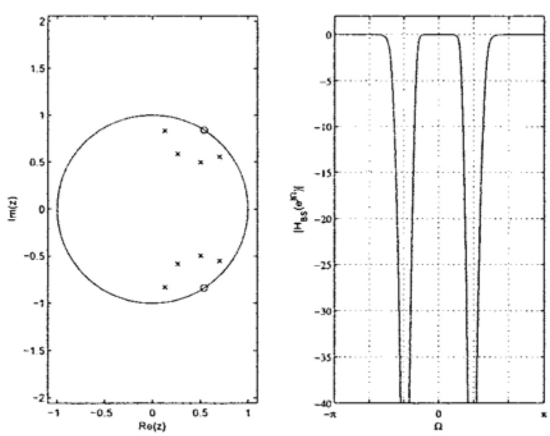


Figure S5.M-8d: Order-8 Butterworth BSF with stopband between $5\pi/24$ and $11\pi/24$.

5.M-9. Factored form is used to plot roots, and standard transfer function form is used to compute magnitude response plots.

(a) Order-8 Chebyshev Type I LPF with $\Omega_c = \pi/3$.

```
>> Omega_c = pi/3; Omega = linspace(-pi,pi,1001);
>> [z,p,k] = cheby1(8,3,Omega_c/pi);
>> subplot(121),plot(real(p),imag(p),'kx',...
   real(z),imag(z),'ko',cos(Omega),sin(Omega),'k');
>> axis([-1.1 1.1 -1.1 1.1]); axis equal;
>> xlabel('Re(z)'); ylabel('Im(z)');
>> [B,A] = cheby1(8,3,Omega_c/pi);
>> HLP = polyval(B,exp(j*Omega))./polyval(A,exp(j*Omega));
>> subplot(122),plot(Omega,20*log10(abs(HLP)),'k');
>> axis([-pi pi -40 2]); grid;
>> xlabel('\Omega'); ylabel('|H_{LP}(e^{j\Omega})|');
>> set(gca,'xtick',[-pi:pi/3:pi],'xticklabel',['-p ';...
   '   ';'   ';' 0 ';'   ';'   ';' p '],...
   'fontname','symbol');
```

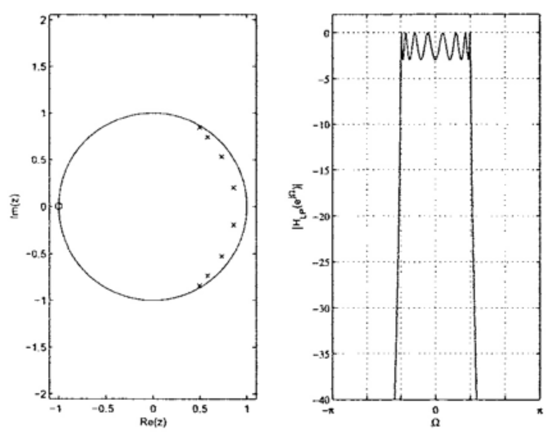


Figure S5.M-9a: Order-8 Chebyshev Type I LPF with $\Omega_c = \pi/3$.

(b) Order-8 Chebyshev Type I HPF with $\Omega_c = \pi/3$.

```
>> Omega_c = pi/3; Omega = linspace(-pi,pi,1001);
>> [z,p,k] = cheby1(8,3,Omega_c/pi,'high');
>> subplot(121),plot(real(p),imag(p),'kx',...
   real(z),imag(z),'ko',cos(Omega),sin(Omega),'k');
>> axis([-1.1 1.1 -1.1 1.1]); axis equal;
>> xlabel('Re(z)'); ylabel('Im(z)');
>> [B,A] = cheby1(8,3,Omega_c/pi,'high');
>> HHP = polyval(B,exp(j*Omega))./polyval(A,exp(j*Omega));
>> subplot(122),plot(Omega,20*log10(abs(HHP)),'k');
>> axis([-pi pi -40 2]); grid;
>> xlabel('\Omega'); ylabel('|H_{HP}(e^{j\Omega})|');
>> set(gca,'xtick',[-pi:pi/3:pi],'xticklabel',['-p ';...
   '   ';'   ';' 0 ';'   ';'   ';' p '],...
   'fontname','symbol');
```

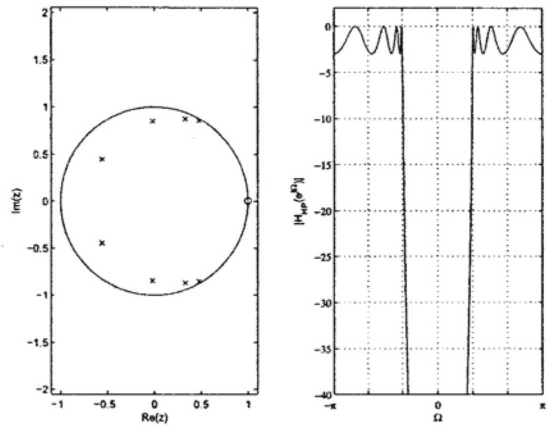


Figure S5.M-9b: Order-8 Chebyshev Type I HPF with $\Omega_c = \pi/3$.

(c) Order-8 Chebyshev Type I BPF with passband between $5\pi/24$ and $11\pi/24$. Notice that the command cheby1 requires the parameter $N = 4$ to be used to obtain a $(2N = 8)$-order bandpass filter.

```
>> Omega_c = [5*pi/24,11*pi/24]; Omega = linspace(-pi,pi,1001);
>> [z,p,k] = cheby1(4,3,Omega_c/pi);
>> subplot(121),plot(real(p),imag(p),'kx',...
   real(z),imag(z),'ko',cos(Omega),sin(Omega),'k');
>> axis([-1.1 1.1 -1.1 1.1]); axis equal;
>> xlabel('Re(z)'); ylabel('Im(z)');
>> [B,A] = cheby1(4,3,Omega_c/pi);
>> HBP = polyval(B,exp(j*Omega))./polyval(A,exp(j*Omega));
>> subplot(122),plot(Omega,20*log10(abs(HBP)),'k');
>> axis([-pi pi -40 2]); grid;
>> xlabel('\Omega'); ylabel('|H_{BP}(e^{j\Omega})|');
>> set(gca,'xtick',[-pi:pi/3:pi],'xticklabel',['-p ';...
   '   ';'   ';' 0 ';'   ';'   ';' p '],...
   'fontname','symbol');
```

(d) Order-8 Chebyshev Type I BSF with stopband between $5\pi/24$ and $11\pi/24$. Notice that the command cheby1 requires the parameter $N = 4$ to be used to obtain a $(2N = 8)$-order bandstop filter.

```
>> Omega_c = [5*pi/24,11*pi/24]; Omega = linspace(-pi,pi,1001);
>> [z,p,k] = cheby1(4,3,Omega_c/pi,'stop');
>> subplot(121),plot(real(p),imag(p),'kx',...
   real(z),imag(z),'ko',cos(Omega),sin(Omega),'k');
>> axis([-1.1 1.1 -1.1 1.1]); axis equal;
>> xlabel('Re(z)'); ylabel('Im(z)');
>> [B,A] = cheby1(4,3,Omega_c/pi,'stop');
>> HBS = polyval(B,exp(j*Omega))./polyval(A,exp(j*Omega));
>> subplot(122),plot(Omega,20*log10(abs(HBS)),'k');
>> axis([-pi pi -40 2]); grid;
>> xlabel('\Omega'); ylabel('|H_{BS}(e^{j\Omega})|');
>> set(gca,'xtick',[-pi:pi/3:pi],'xticklabel',['-p ';...
   '   ';'   ';' 0 ';'   ';'   ';' p '],...
```

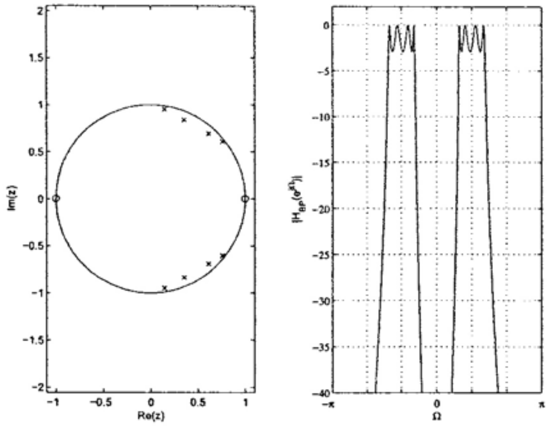


Figure S5.M-9c: Order-8 Chebyshev Type I BPF with passband between $5\pi/24$ and $11\pi/24$.

```
'fontname','symbol');
```

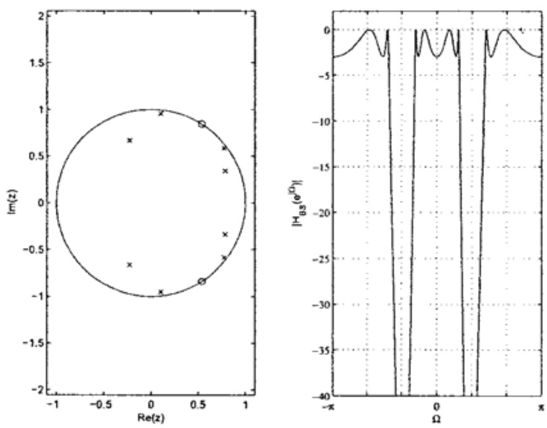


Figure S5.M-9d: Order-8 Chebyshev Type I BSF with stopband between $5\pi/24$ and $11\pi/24$.

To demonstrate the effect of decreasing the passband ripple, consider magnitude response plots for Chebyshev Type I LPFs with $R_p = \{0.1, 1.0, 3.0\}$.

```
>> Omega_c = pi/3; Omega = linspace(-pi,pi,1001);
>> [B,A] = cheby1(8,.1,Omega_c/pi);
>> HLP1 = polyval(B,exp(j*Omega))./polyval(A,exp(j*Omega));
>> [B,A] = cheby1(8,1,Omega_c/pi);
>> HLP2 = polyval(B,exp(j*Omega))./polyval(A,exp(j*Omega));
>> [B,A] = cheby1(8,3,Omega_c/pi);
>> HLP3 = polyval(B,exp(j*Omega))./polyval(A,exp(j*Omega));
>> plot(Omega,20*log10(abs(HLP1)),'k-',...
        Omega,20*log10(abs(HLP2)),'k--',...
        Omega,20*log10(abs(HLP3)),'k:');
>> axis([-pi pi -40 2]); grid;
>> xlabel('\Omega'); ylabel('|H_{LP}(e^{j\Omega})|');
```

```
>> legend('R_p = 0.1','R_p = 1.0','R_p = 3.0',0);
>> set(gca,'xtick',[-pi:pi/3:pi],'xticklabel',['-p ';...
    '   ';'   ';' 0 ';'   ';'   ';' p '],...
    'fontname','symbol');
```

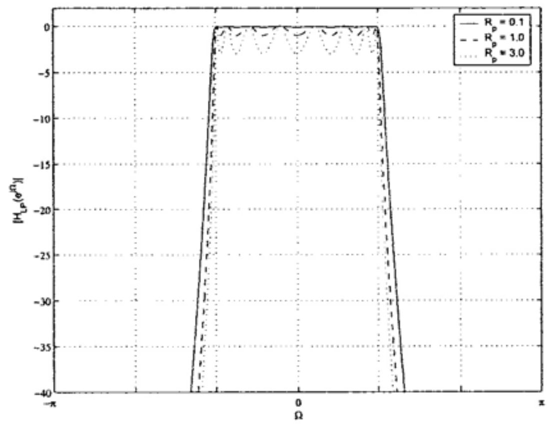


Figure S5.M-9d: Changing R_p for a digital Chebyshev Type I filter.

Thus, reducing the allowable passband ripple R_p tends to broaden the transition bands of the filter.

5.M-10. Factored form is used to plot roots, and standard transfer function form is used to compute magnitude response plots.

(a) Order-8 Chebyshev Type II LPF with $\Omega_c = \pi/3$.

```
>> Omega_c = pi/3; Omega = linspace(-pi,pi,1001);
>> [z,p,k] = cheby2(8,20,Omega_c/pi);
>> subplot(121),plot(real(p),imag(p),'kx',...
    real(z),imag(z),'ko',cos(Omega),sin(Omega),'k');
>> axis([-1.1 1.1 -1.1 1.1]); axis equal;
>> xlabel('Re(z)'); ylabel('Im(z)');
>> [B,A] = cheby2(8,20,Omega_c/pi);
>> HLP = polyval(B,exp(j*Omega))./polyval(A,exp(j*Omega));
>> subplot(122),plot(Omega,20*log10(abs(HLP)),'k');
>> axis([-pi pi -40 2]); grid;
>> xlabel('\Omega'); ylabel('|H_{LP}(e^{j\Omega})|');
>> set(gca,'xtick',[-pi:pi/3:pi],'xticklabel',['-p ';...
    '   ';'   ';' 0 ';'   ';'   ';' p '],...
    'fontname','symbol');
```

(b) Order-8 Chebyshev Type II HPF with $\Omega_c = \pi/3$.

```
>> Omega_c = pi/3; Omega = linspace(-pi,pi,1001);
>> [z,p,k] = cheby2(8,20,Omega_c/pi,'high');
>> subplot(121),plot(real(p),imag(p),'kx',...
    real(z),imag(z),'ko',cos(Omega),sin(Omega),'k');
>> axis([-1.1 1.1 -1.1 1.1]); axis equal;
>> xlabel('Re(z)'); ylabel('Im(z)');
>> [B,A] = cheby2(8,20,Omega_c/pi,'high');
```

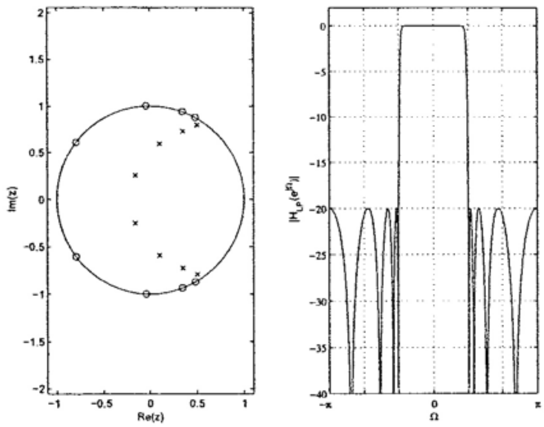


Figure S5.M-10a: Order-8 Chebyshev Type II LPF with $\Omega_c = \pi/3$.

```
>> HHP = polyval(B,exp(j*Omega))./polyval(A,exp(j*Omega));
>> subplot(122),plot(Omega,20*log10(abs(HHP)),'k');
>> axis([-pi pi -40 2]); grid;
>> xlabel('\Omega'); ylabel('|H_{HP}(e^{j\Omega})|');
>> set(gca,'xtick',[-pi:pi/3:pi],'xticklabel',['-p ';...
    '   ';'   ';' 0 ';'   ';'   ';' p '],...
    'fontname','symbol');
```

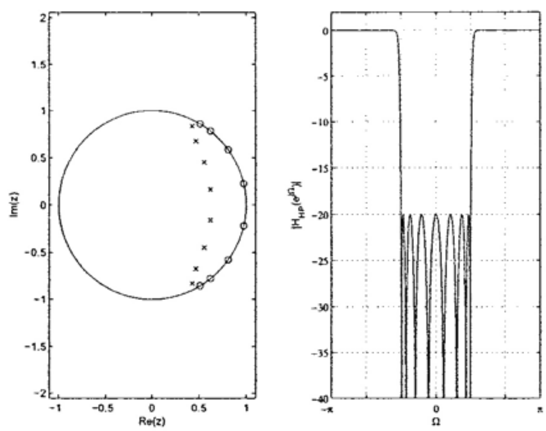


Figure S5.M-10b: Order-8 Chebyshev Type II HPF with $\Omega_c = \pi/3$.

(c) Order-8 Chebyshev Type II BPF with passband between $5\pi/24$ and $11\pi/24$. Notice that the command cheby2 requires the parameter $N = 4$ to be used to obtain a $(2N = 8)$-order bandpass filter.

```
>> Omega_c = [5*pi/24,11*pi/24]; Omega = linspace(-pi,pi,1001);
>> [z,p,k] = cheby2(4,20,Omega_c/pi);
>> subplot(121),plot(real(p),imag(p),'kx',...
    real(z),imag(z),'ko',cos(Omega),sin(Omega),'k');
>> axis([-1.1 1.1 -1.1 1.1]); axis equal;
>> xlabel('Re(z)'); ylabel('Im(z)');
>> [B,A] = cheby2(4,20,Omega_c/pi);
```

```
>> HBP = polyval(B,exp(j*Omega))./polyval(A,exp(j*Omega));
>> subplot(122),plot(Omega,20*log10(abs(HBP)),'k');
>> axis([-pi pi -40 2]); grid;
>> xlabel('\Omega'); ylabel('|H_{BP}(e^{j\Omega})|');
>> set(gca,'xtick',[-pi:pi/3:pi],'xticklabel',['-p ';...
    '   ';'   ';' 0 ';'   ';'   ';' p '],...
    'fontname','symbol');
```

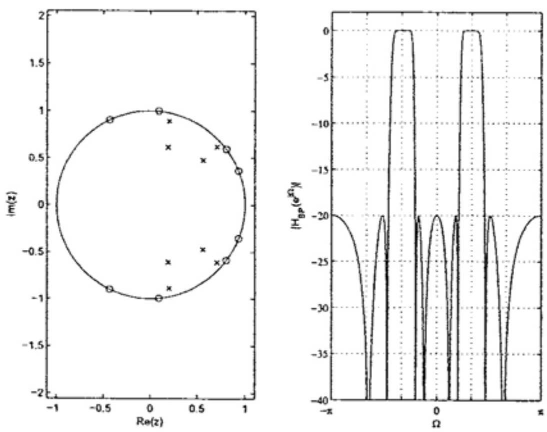


Figure S5.M-10c: Order-8 Chebyshev Type II BPF with passband between $5\pi/24$ and $11\pi/24$.

(d) Order-8 Chebyshev Type II BSF with stopband between $5\pi/24$ and $11\pi/24$. Notice that the command cheby2 requires the parameter $N = 4$ to be used to obtain a ($2N = 8$)-order bandstop filter.

```
>> Omega_c = [5*pi/24,11*pi/24]; Omega = linspace(-pi,pi,1001);
>> [z,p,k] = cheby2(4,20,Omega_c/pi,'stop');
>> subplot(121),plot(real(p),imag(p),'kx',...
    real(z),imag(z),'ko',cos(Omega),sin(Omega),'k');
>> axis([-1.1 1.1 -1.1 1.1]); axis equal;
>> xlabel('Re(z)'); ylabel('Im(z)');
>> [B,A] = cheby2(4,20,Omega_c/pi,'stop');
>> HBS = polyval(B,exp(j*Omega))./polyval(A,exp(j*Omega));
>> subplot(122),plot(Omega,20*log10(abs(HBS)),'k');
>> axis([-pi pi -40 2]); grid;
>> xlabel('\Omega'); ylabel('|H_{BS}(e^{j\Omega})|');
>> set(gca,'xtick',[-pi:pi/3:pi],'xticklabel',['-p ';...
    '   ';'   ';' 0 ';'   ';'   ';' p '],...
    'fontname','symbol');
```

To demonstrate the effect of increasing R_s, consider magnitude response plots for Chebyshev Type II LPFs with $R_s = \{10, 20, 30\}$.

```
>> Omega_c = pi/3; Omega = linspace(-pi,pi,1001);
>> [B,A] = cheby2(8,10,Omega_c/pi);
>> HLP1 = polyval(B,exp(j*Omega))./polyval(A,exp(j*Omega));
>> [B,A] = cheby2(8,20,Omega_c/pi);
>> HLP2 = polyval(B,exp(j*Omega))./polyval(A,exp(j*Omega));
>> [B,A] = cheby2(8,30,Omega_c/pi);
```

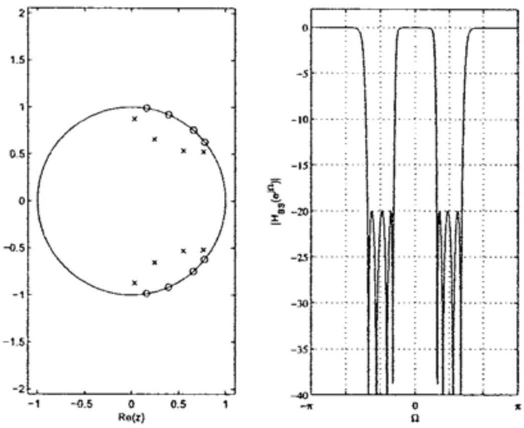


Figure S5.M-10d: Order-8 Chebyshev Type II BSF with stopband between $5\pi/24$ and $11\pi/24$.

```
>> HLP3 = polyval(B,exp(j*Omega))./polyval(A,exp(j*Omega));
>> plot(Omega,20*log10(abs(HLP1)),'k-',...
        Omega,20*log10(abs(HLP2)),'k--',...
        Omega,20*log10(abs(HLP3)),'k:');
>> axis([-pi pi -40 2]); grid;
>> xlabel('\Omega'); ylabel('|H_{LP}(e^{j\Omega})|');
>> legend('R_s = 10','R_s = 20','R_s = 30',0);
>> set(gca,'xtick',[-pi:pi/3:pi],'xticklabel',['-p ';...
    '   ';'   ';' 0 ';'   ';'   ';' p '],...
    'fontname','symbol');
```

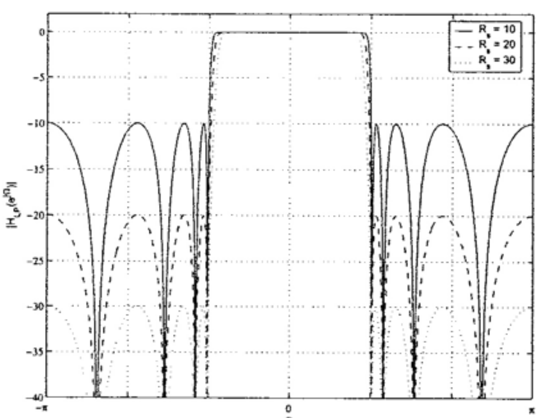


Figure S5.M-10d: Changing R_s for a digital Chebyshev Type II filter.

Thus, increasing R_s tends to broaden the transition bands of the filter.

5.M-11. Factored form is used to plot roots, and standard transfer function form is used to compute magnitude response plots.

(a) Order-8 Elliptic LPF with $\Omega_c = \pi/3$.

```
>> Omega_c = pi/3; Omega = linspace(-pi,pi,1001);
```

```
>> [z,p,k] = ellip(8,3,20,Omega_c/pi);
>> subplot(121),plot(real(p),imag(p),'kx',...
    real(z),imag(z),'ko',cos(Omega),sin(Omega),'k');
>> axis([-1.1 1.1 -1.1 1.1]); axis equal;
>> xlabel('Re(z)'); ylabel('Im(z)');
>> [B,A] = ellip(8,3,20,Omega_c/pi);
>> HLP = polyval(B,exp(j*Omega))./polyval(A,exp(j*Omega));
>> subplot(122),plot(Omega,20*log10(abs(HLP)),'k');
>> axis([-pi pi -40 2]); grid;
>> xlabel('\Omega'); ylabel('|H_{LP}(e^{j\Omega})|');
>> set(gca,'xtick',[-pi:pi/3:pi],'xticklabel',['-p ';...
    '   ';'   ';' 0 ';'   ';'   ';' p '],...
    'fontname','symbol');
```

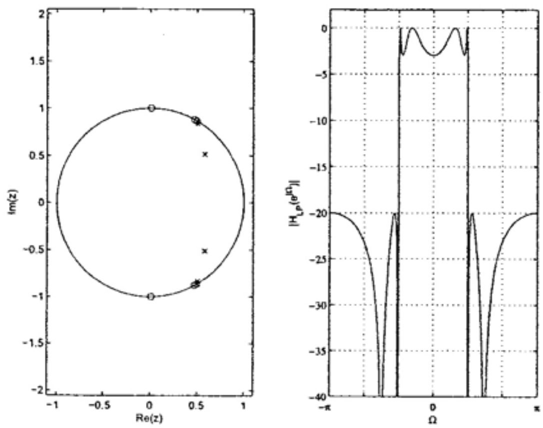


Figure S5.M-11a: Order-8 Elliptic LPF with $\Omega_c = \pi/3$.

(b) Order-8 Elliptic HPF with $\Omega_c = \pi/3$.

```
>> Omega_c = pi/3; Omega = linspace(-pi,pi,1001);
>> [z,p,k] = ellip(8,3,20,Omega_c/pi,'high');
>> subplot(121),plot(real(p),imag(p),'kx',...
    real(z),imag(z),'ko',cos(Omega),sin(Omega),'k');
>> axis([-1.1 1.1 -1.1 1.1]); axis equal;
>> xlabel('Re(z)'); ylabel('Im(z)');
>> [B,A] = ellip(8,3,20,Omega_c/pi,'high');
>> HHP = polyval(B,exp(j*Omega))./polyval(A,exp(j*Omega));
>> subplot(122),plot(Omega,20*log10(abs(HHP)),'k');
>> axis([-pi pi -40 2]); grid;
>> xlabel('\Omega'); ylabel('|H_{HP}(e^{j\Omega})|');
>> set(gca,'xtick',[-pi:pi/3:pi],'xticklabel',['-p ';...
    '   ';'   ';' 0 ';'   ';'   ';' p '],...
    'fontname','symbol');
```

(c) Order-8 Elliptic BPF with passband between $5\pi/24$ and $11\pi/24$. Notice that the command `ellip` requires the parameter $N = 4$ to be used to obtain a $(2N = 8)$-order bandpass filter.

```
>> Omega_c = [5*pi/24,11*pi/24]; Omega = linspace(-pi,pi,1001);
```

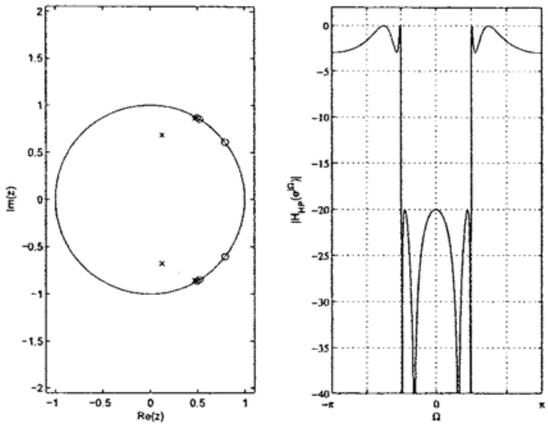


Figure S5.M-11b: Order-8 Elliptic HPF with $\Omega_c = \pi/3$.

```
>> [z,p,k] = ellip(4,3,20,Omega_c/pi);
>> subplot(121),plot(real(p),imag(p),'kx',...
    real(z),imag(z),'ko',cos(Omega),sin(Omega),'k');
>> axis([-1.1 1.1 -1.1 1.1]); axis equal;
>> xlabel('Re(z)'); ylabel('Im(z)');
>> [B,A] = ellip(4,3,20,Omega_c/pi);
>> HBP = polyval(B,exp(j*Omega))./polyval(A,exp(j*Omega));
>> subplot(122),plot(Omega,20*log10(abs(HBP)),'k');
>> axis([-pi pi -40 2]); grid;
>> xlabel('\Omega'); ylabel('|H_{BP}(e^{j\Omega})|');
>> set(gca,'xtick',[-pi:pi/3:pi],'xticklabel',['-p ';...
    '   ';'   ';' 0 ';'   ';'   ';' p '],...
    'fontname','symbol');
```

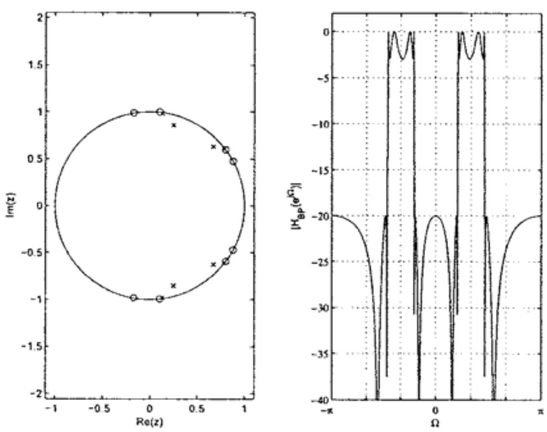


Figure S5.M-11c: Order-8 Elliptic BPF with passband between $5\pi/24$ and $11\pi/24$.

(d) Order-8 Elliptic BSF with stopband between $5\pi/24$ and $11\pi/24$. Notice that the command `ellip` requires the parameter $N = 4$ to be used to obtain a $(2N = 8)$-order bandstop filter.

```
>> Omega_c = [5*pi/24,11*pi/24]; Omega = linspace(-pi,pi,1001);
```

```
>> [z,p,k] = ellip(4,3,20,Omega_c/pi,'stop');
>> subplot(121),plot(real(p),imag(p),'kx',...
    real(z),imag(z),'ko',cos(Omega),sin(Omega),'k');
>> axis([-1.1 1.1 -1.1 1.1]); axis equal;
>> xlabel('Re(z)'); ylabel('Im(z)');
>> [B,A] = ellip(4,3,20,Omega_c/pi,'stop');
>> HBS = polyval(B,exp(j*Omega))./polyval(A,exp(j*Omega));
>> subplot(122),plot(Omega,20*log10(abs(HBS)),'k');
>> axis([-pi pi -40 2]); grid;
>> xlabel('\Omega'); ylabel('|H_{BS}(e^{j\Omega})|');
>> set(gca,'xtick',[-pi:pi/3:pi],'xticklabel',['-p ';...
    '   ';'   ';' 0 ';'   ';'   ';' p '],...
    'fontname','symbol');
```

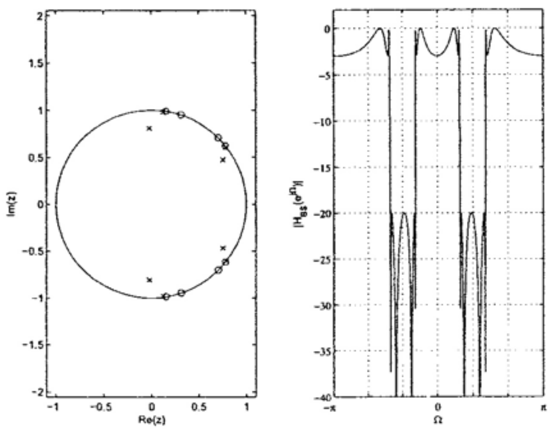


Figure S5.M-11d: Order-8 Elliptic BSF with stopband between $5\pi/24$ and $11\pi/24$.

Chapter 6 Solutions

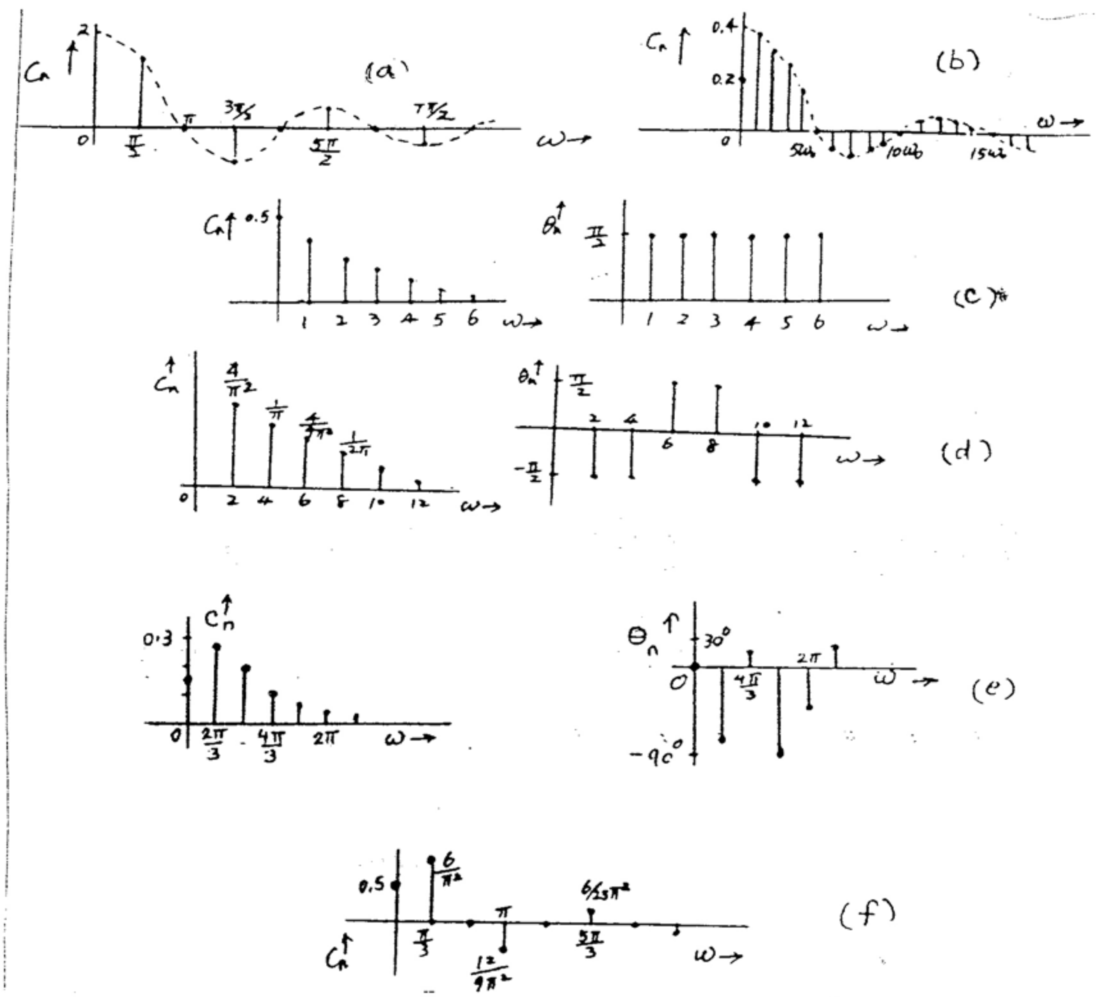


Figure S6.1-1

6.1-1. (a) $T_0 = 4$, $\omega_0 = \frac{2\pi}{T_0} = \frac{\pi}{2}$. Because of even symmetry, all sine terms are zero.

$$\begin{aligned} x(t) &= a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{2}t\right) \\ a_0 &= 0 \text{ (by inspection)} \\ a_n &= \frac{4}{4}\left[\int_0^1 \cos\left(\frac{n\pi}{2}t\right)\,dt - \int_1^2 \cos\left(\frac{n\pi}{2}t\right)\,dt\right] = \frac{4}{n\pi}\sin\frac{n\pi}{2} \end{aligned}$$

Therefore, the Fourier series for $x(t)$ is

$$x(t) = \frac{4}{\pi}\left(\cos\frac{\pi t}{2} - \frac{1}{3}\cos\frac{3\pi t}{2} + \frac{1}{5}\cos\frac{5\pi t}{2} - \frac{1}{7}\cos\frac{7\pi t}{2} + \cdots\right)$$

Here $b_n = 0$, and we allow C_n to take negative values. Figure S6.1-1a shows the plot of C_n.

(b) $T_0 = 10\pi$, $\omega_0 = \frac{2\pi}{T_0} = \frac{1}{5}$. Because of even symmetry, all the sine terms are zero.

$$\begin{aligned} x(t) &= a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n}{5}t\right) + b_n \sin\left(\frac{n}{5}t\right) \\ a_0 &= \frac{1}{5} \quad \text{(by inspection)} \\ a_n &= \frac{2}{10\pi}\int_{-\pi}^{\pi} \cos\left(\frac{n}{5}t\right)\,dt = \frac{1}{5\pi}\left(\frac{5}{n}\right)\sin\left(\frac{n}{5}t\right)\Big|_{-\pi}^{\pi} = \frac{2}{\pi n}\sin\left(\frac{n\pi}{5}\right) \\ b_n &= \frac{2}{10\pi}\int_{-\pi}^{\pi} \sin\left(\frac{n}{5}t\right)\,dt = 0 \qquad \text{(integrand is an odd function of } t) \end{aligned}$$

Here $b_n = 0$, and we allow C_n to take negative values. Note that $C_n = a_n$ for $n = 0, 1, 2, 3, \cdots$. Figure S6.1-1b shows the plot of C_n.

(c) $T_0 = 2\pi$, $\omega_0 = 1$.

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt \quad \text{with} \quad a_0 = 0.5 \quad \text{(by inspection)}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} \frac{t}{2\pi}\cos nt\,dt = 0, \qquad b_n = \frac{1}{\pi}\int_0^{2\pi} \frac{t}{2\pi}\sin nt\,dt = -\frac{1}{\pi n}$$

and

$$\begin{aligned} x(t) &= 0.5 - \frac{1}{\pi}\left(\sin t + \frac{1}{2}\sin 2t + \frac{1}{3}\sin 3t + \frac{1}{4}\sin 4t + \cdots\right) \\ &= 0.5 + \frac{1}{\pi}\left[\cos\left(t + \frac{\pi}{2}\right) + \frac{1}{2}\cos\left(2t + \frac{\pi}{2}\right) + \frac{1}{3}\cos\left(3t + \frac{\pi}{2}\right) + \cdots\right] \end{aligned}$$

The reason for vanishing of the cosines terms is that when 0.5 (the dc component) is subtracted from $x(t)$, the remaining function has odd symmetry. Hence, the Fourier series would contain dc and sine terms only. Figure S6.1-1c shows the plot of C_n and θ_n.

(d) $T_0 = \pi$, $\omega_0 = 2$ and $x(t) = \frac{4}{\pi}t$. $a_0 = 0$ (by inspection). $a_n = 0$ $(n >$

0) because of odd symmetry.

$$b_n = \frac{4}{\pi}\int_0^{\pi/4} \frac{4}{\pi} t \sin 2nt\, dt = \frac{2}{\pi n}\left(\frac{2}{\pi n}\sin\frac{\pi n}{2} - \cos\frac{\pi n}{2}\right)$$

$$\begin{aligned} x(t) &= \frac{4}{\pi^2}\sin 2t + \frac{1}{\pi}\sin 4t - \frac{4}{9\pi^2}\sin 6t - \frac{1}{2\pi}\sin 8t + \cdots \\ &= \frac{4}{\pi^2}\cos\left(2t - \frac{\pi}{2}\right) + \frac{1}{\pi}\cos\left(4t - \frac{\pi}{2}\right) + \frac{4}{9\pi^2}\cos\left(6t + \frac{\pi}{2}\right) + \frac{1}{\pi}\cos\left(8t + \frac{\pi}{2}\right) + \cdots \end{aligned}$$

Figure S6.1-1d shows the plot of C_n and θ_n.

(e) $T_0 = 3$, $\omega_0 = 2\pi/3$.

$$a_0 = \frac{1}{3}\int_0^1 t\, dt = \frac{1}{6}$$

$$a_n = \frac{2}{3}\int_0^1 t\cos\frac{2n\pi}{3} t dt = \frac{3}{2\pi^2 n^2}[\cos\frac{2\pi n}{3} + \frac{2\pi n}{3}\sin\frac{2\pi n}{3} - 1]$$

$$b_n = \frac{2}{3}\int_0^1 t\sin\frac{2n\pi}{3} t dt = \frac{3}{2\pi^2 n^2}[\sin\frac{2\pi n}{3} - \frac{2\pi n}{3}\cos\frac{2\pi n}{3}]$$

Therefore $C_0 = \frac{1}{6}$ and

$$C_n = \frac{3}{2\pi^2 n^2}\left[\sqrt{2 + \frac{4\pi^2 n^2}{9} - 2\cos\frac{2\pi n}{3} - \frac{4\pi n}{3}\sin\frac{2\pi n}{3}}\right]$$

and

$$\theta_n = \tan^{-1}\left(\frac{\frac{2\pi n}{3}\cos\frac{2\pi n}{3} - \sin\frac{2\pi n}{3}}{\cos\frac{2\pi n}{3} + \frac{2\pi n}{3}\sin\frac{2\pi n}{3} - 1}\right)$$

(f) $T_0 = 6$, $\omega_0 = \pi/3$, $a_0 = 5$ (by inspection). Even symmetry; $b_n = 0$.

$$\begin{aligned} a_n &= \frac{4}{6}\int_0^3 x(t)\cos\frac{n\pi}{3}\, dt \\ &= \frac{2}{3}\left[\int_0^1 \cos\frac{n\pi}{3}\, dt + \int_1^2 (2-t)\cos\frac{n\pi}{3} t\, dt\right] \\ &= \frac{6}{\pi^2 n^2}\left[\cos\frac{n\pi}{3} - \cos\frac{2n\pi}{3}\right] \end{aligned}$$

$$x(t) = 0.5 + \frac{6}{\pi^2}\left(\cos\frac{\pi}{3}t - \frac{2}{9}\cos\pi t + \frac{1}{25}\cos\frac{5\pi}{3}t + \frac{1}{49}\cos\frac{7\pi}{3}t + \cdots\right)$$

Observe that even harmonics vanish. The reason is that if the dc (0.5) is subtracted from $x(t)$, the resulting function has half-wave symmetry. (See Prob. 6.1-5). Figure S6.1-1f shows the plot of C_n.

6.1-2. (a) Here $T_0 = \pi$, and $\omega_0 = \frac{2\pi}{T_0} = 2$. Therefore

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos 2nt + b_n \sin 2nt$$

To compute the coefficients, we shall use the interval π to 0 for integration. Thus

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{0} e^{t/2}\,dt = 0.504 \\
a_n &= \frac{2}{\pi}\int_{-\pi}^{0} e^{t/2}\cos 2nt\,dt = 0.504\left(\frac{2}{1+16n^2}\right) \\
b_n &= \frac{2}{\pi}\int_{-\pi}^{0} e^{t/2}\sin 2nt\,dt = -0.504\left(\frac{8n}{1+16n^2}\right)
\end{aligned}$$

Therefore

$$\begin{aligned}
C_0 &= a_0 = 0.504 \\
C_n &= \sqrt{a_n^2+b_n^2} = 0.504\left(\frac{2}{\sqrt{1+16n^2}}\right) \\
\theta_n &= \tan^{-1}\left(\frac{-b_n}{a_n}\right) = \tan^{-1} 4n \\
x(t) &= 0.504 + 0.504\sum_{n=1}^{\infty}\frac{2}{\sqrt{1+16n^2}}\cos\left(2nt + tan^{-1}4n\right)
\end{aligned}$$

(b) This Fourier series is identical to that in Eq. (6.15a) with t replaced by $-t$.

(c) If $x(t) = C_0 + \sum C_n \cos(n\omega_0 t + \theta_n)$, then

$$x(-t) = C_0 + \sum C_n \cos(-n\omega_0 t + \theta_n) = C_0 + \sum C_n \cos(n\omega_0 t - \theta_n)$$

Thus, time inversion of a signal merely changes the sign of the phase θ_n. Everything else remains unchanged. Comparison of the above results in part (a) with those in Example 6.1 confirms this conclusion.

6.1-3. (a) Here $T_0 = \pi/2$, and $\omega_0 = \frac{2\pi}{T_0} = 4$. Therefore

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos 4nt + b_n \sin 4nt$$

where

$$\begin{aligned}
a_0 &= \frac{2}{\pi}\int_{0}^{\pi/2} e^{-t}\,dt = 0.504 \\
a_n &= \frac{4}{\pi}\int_{0}^{\pi/2} e^{-t}\cos 4nt\,dt = 0.504\left(\frac{2}{1+16n^2}\right)
\end{aligned}$$

and

$$b_n = \frac{4}{\pi}\int_{0}^{\pi/2} e^{-t}\sin 4nt\,dt = 0.504\left(\frac{8n}{1+16n^2}\right)$$

Therefore $C_0 = a_0 = 0.504$, $C_n = \sqrt{a_n^2+b_n^2} = 0.504\left(\frac{2}{\sqrt{1+16n^2}}\right)$, $\theta_n = -\tan^{-1} 4n$

(b) This Fourier series is identical to that in Eq. (6.15a) with t replaced by $2t$.

(c) If $x(t) = C_0 + \sum C_n \cos(n\omega_0 t + \theta_n)$, then

$$x(at) = C_0 + \sum C_n \cos(n(a\omega_0)t + \theta_n)$$

Thus, time scaling by a factor a merely scales the fundamental frequency by the same factor a. Everything else remains unchanged. If we time compress (or time expand) a periodic signal by a factor a, its fundamental frequency increases by the same factor a (or decreases by the same factor a). Comparison of the results in part **(a)** with those in Example 6.1 confirms this conclusion. This result applies equally well

6.1-4. (a) Here $T_0 = 2$, and $\omega_0 = \frac{2\pi}{T_0} = \pi$. Also $x(t)$ is an even function of t. Therefore

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi t$$

where, by inspection $a_0 = 0$ and from Eq. (6.18b)

$$a_n = \frac{4}{2}\int_0^1 A(-2t+1)\cos n\pi t\, dt = -\frac{4}{\pi^2 n^2}\,(\cos n\pi t - 1)\Big|_0^1 = \begin{cases} 0 & n \text{ even} \\ \frac{8A}{n^2\pi^2} & n \text{ odd} \end{cases}$$

Therefore

$$x(t) = \frac{8A}{\pi^2}\left[\cos \pi t + \frac{1}{9}\cos 3\pi t + \frac{1}{25}\cos 5\pi t + \frac{1}{49}\cos 7\pi t + \cdots\right]$$

(b) This Fourier series is identical to that in Eq. (6.16) with t replaced by $t + 0.5$.

(c) If $x(t) = C_0 + \sum C_n \cos(n\omega_0 t + \theta_n)$, then

$$x(t+T) = C_0 + \sum C_n \cos[n\omega_0(t+T)+\theta_n] = C_0 + \sum C_n \cos[n\omega_0 t + (\theta_n + n\omega_0 T)]$$

Thus, time shifting by T merely increases the phase of the nth harmonic by $n\omega_0 T$.

6.1-5. (a) For half wave symmetry

$$x(t) = -x\left(t \pm \frac{T_0}{2}\right)$$

and

$$a_n = \frac{2}{T_0}\int_0^{T_0} x(t)\cos n\omega_0 t\, dt = \frac{2}{T_0}\int_0^{T_0/2} x(t)\cos n\omega_0 t\, dt + \int_{T_0/2}^{T_0} x(t)\cos n\omega_0 t\, dt$$

Let $\tau = t - T_0/2$ in the second integral. This gives

$$\begin{aligned} a_n &= \frac{2}{T_0}\left[\int_0^{T_0/2} x(t)\cos n\omega_0 t\, dt + \int_0^{T_0/2} x\left(\tau + \frac{T_0}{2}\right)\cos n\omega_0\left(\tau + \frac{T_0}{2}\right) d\tau\right] \\ &= \frac{2}{T_0}\left[\int_0^{T_0/2} x(t)\cos n\omega_0 t\, dt + \int_0^{T_0/2} -x(\tau)[-\cos n\omega_0 \tau]\, d\tau\right] \\ &= \frac{4}{T_0}\left[\int_0^{T_0/2} x(t)\cos n\omega_0 t\, dt\right] \end{aligned}$$

In a similar way we can show that

$$b_n = \frac{4}{T_0}\int_0^{T_0/2} x(t)\sin n\omega_0 t\,dt$$

(b) (i) $T_0 = 8$, $\omega_0 = \frac{\pi}{4}$, $a_0 = 0$ (by inspection). Half wave symmetry. Hence

$$\begin{aligned} a_n &= \frac{4}{8}\left[\int_0^4 x(t)\cos\frac{n\pi}{4}t\,dt\right] = \frac{1}{2}\left[\int_0^2 \frac{t}{2}\cos\frac{n\pi}{4}t\,dt\right] \\ &= \frac{4}{n^2\pi^2}\left(\cos\frac{n\pi}{2} + \frac{n\pi}{2}\sin\frac{n\pi}{2} - 1\right) \quad (n \text{ odd}) \\ &= \frac{4}{n^2\pi^2}\left(\frac{n\pi}{2}\sin\frac{n\pi}{2} - 1\right) \quad (n \text{ odd}) \end{aligned}$$

Therefore

$$a_n = \begin{cases} \frac{4}{n^2\pi^2}\left(\frac{n\pi}{2} - 1\right) & n = 1, 5, 9, 13, \cdots \\ -\frac{4}{n^2\pi^2}\left(\frac{n\pi}{2} + 1\right) & n = 3, 7, 11, 15, \cdots \end{cases}$$

Similarly

$$b_n = \frac{1}{2}\int_0^2 \frac{t}{2}\sin\frac{n\pi}{4}t\,dt = \frac{4}{n^2\pi^2}\left(\sin\frac{n\pi}{2} - \frac{n\pi}{2}\cos\frac{n\pi}{2}\right) = \frac{4}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) \quad (n \text{ odd})$$

and

$$x(t) = \sum_{n=1,3,5,\cdots}^{\infty} a_n\cos\frac{n\pi}{4}t + b_n\sin\frac{n\pi}{4}t$$

(ii) $T_0 = 2\pi$, $\omega_0 = 1$, $a_0 = 0$ (by inspection). Half wave symmetry. Hence

$$x(t) = \sum_{n=1,3,5,\cdots}^{\infty} a_n\cos nt + b_n\sin nt$$

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^{\pi} e^{-t/10}\cos nt\,dt \\ &= \frac{2}{\pi}\left[\frac{e^{-t/10}}{n^2+0.01}(-0.1\cos nt + n\sin nt)\right]_0^{\pi} \quad (n \text{ odd}) \\ &= \frac{2}{\pi}\left[\frac{e^{-\pi/10}}{n^2+0.01}(0.1) - \frac{1}{n^2+0.01}(-0.1)\right] \\ &= \frac{2}{10\pi(n^2+0.01)}(e^{-\pi/10} - 1) = \frac{0.0465}{n^2+0.01} \end{aligned}$$

and

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^{\pi} e^{-t/10}\sin nt\,dt \\ &= \frac{2}{\pi}\left[\frac{e^{-t/10}}{n^2+0.01}(-0.1\sin nt - n\cos nt)\right]_0^{\pi} \quad (n \text{ odd}) \\ &= \frac{2n}{(n^2+0.01)}(e^{-\pi/10} - 1) = \frac{1.461n}{n^2+0.01} \end{aligned}$$

6.1-6. (a) Here, we need only cosine terms and $\omega_0 = \frac{\pi}{2}$. Hence, we must construct a pulse such that it is an even function of t, has a value t over the interval $0 \leq t \leq 1$, and repeats every 4 seconds as shown in Fig. S6.1-6a. We selected the pulse width $W = 2$ seconds. But it can be anywhere from 2 to 4, and still satisfy these conditions. Each value of W results in different series. Yet all of them converge to t over 0 to 1, and satisfy the other requirements. Clearly, there are infinite number of Fourier series that will satisfy the given requirements. The present choice yields

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{2}\right)t$$

By inspection, we find $a_0 = 1/4$. Because of symmetry $b_n = 0$ and

$$a_n = \frac{4}{4}\int_0^1 t\cos\frac{n\pi}{2}t\,dt = \frac{4}{n^2\pi^2}\left[\cos\left(\frac{n\pi}{2}\right) + \frac{n\pi}{2}\sin\left(\frac{n\pi}{2}\right) - 1\right]$$

(b) Here, we need only sine terms and $\omega_0 = 2$. Hence, we must construct a pulse with odd symmetry, which has a value t over the interval $0 \leq t \leq 1$, and repeats every π seconds as shown in Fig. S6.1-6b. As in the case (a), the pulse width can be anywhere from 1 to π. For the present case

$$x(t) = \sum_{n=1}^{\infty} b_n \sin 2nt$$

Because of odd symmetry, $a_n = 0$ and

$$b_n = \frac{4}{\pi}\int_0^1 t\sin 2nt\,dt = \frac{1}{\pi n^2}(\sin 2n - 2n\cos 2n)$$

(c) Here, we need both sine and cosine terms and $\omega_0 = \frac{\pi}{2}$. Hence, we must construct a pulse such that it has no symmetry of any kind, has a value t over the interval $0 \leq t \leq 1$, and repeats every 4 seconds as shown in Fig. S6.1-6c. As usual, the pulse width can be have any value in the range 1 to 4.

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{2}\right)t + b_n \sin\left(\frac{n\pi}{2}\right)t$$

By inspection, $a_0 = 1/8$ and

$$\begin{aligned} a_n &= \frac{2}{4}\int_0^1 t\cos\frac{n\pi}{2}t\,dt = \frac{2}{n^2\pi^2}\left[\cos\left(\frac{n\pi}{2}\right) + \frac{n\pi}{2}\sin\left(\frac{n\pi}{2}\right) - 1\right] \\ b_n &= \frac{2}{4}\int_0^1 t\sin\frac{n\pi}{2}t\,dt = \frac{2}{n^2\pi^2}\left[\sin\left(\frac{n\pi}{2}\right) - \frac{n\pi}{2}\cos\left(\frac{n\pi}{2}\right)\right] \end{aligned}$$

(d) Here, we need only cosine terms with $\omega_0 = 1$ and odd harmonics only. Hence, we must construct a pulse such that it is an even function of t, has a value t over the interval $0 \leq t \leq 1$, repeats every 2π seconds and has half-wave symmetry as shown in Fig. S6.1-6d. Observe that the first half cycle (from 0 to π) and the second half cycle (from π to 2π) are negatives of each other as required in half-wave symmetry. This will cause even harmonics to vanish. The pulse has an even and half-wave symmetry. This yields

$$x(t) = a_0 + \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} a_n \cos nt$$

By inspection, $a_0 = 0$. Because of even symmetry $b_n = 0$. Because of half-wave symmetry (see Prob. 6.1-5),

$$a_n = \frac{4}{2\pi}\left[\int_0^{\pi/2} t\cos nt\,dt - \int_{\pi/2}^{\pi}(t-\pi)\cos nt\,dt\right] = \frac{2}{\pi n^2}(\cos n\pi - 1) + \frac{2}{n}\sin\frac{n\pi}{2} \qquad n \text{ odd}$$

(e) Here, we need only sine terms with $\omega_0 = \pi$ and odd harmonics only. Hence, we must construct a pulse such that it is an odd function of t, has a value t over the interval $0 \le t \le 1$, repeats every 4 seconds and has half-wave symmetry as shown in Fig. S6.1-6e. Observe that the first half cycle (from 0 to 2) and the second half cycle (from 2 to 4) are negatives of each other as required in half-wave symmetry. This will cause even harmonics to vanish. The pulse has an odd and half-wave symmetry. This yields

$$x(t) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} b_n \sin\frac{n\pi}{2}t$$

By inspection, $a_0 = 0$. Because of odd symmetry $a_n = 0$. Because of half-wave symmetry (see Prob. 6.1-5),

$$b_n = \frac{4}{4}\int_0^1 t\sin\frac{n\pi}{2}t\,dt + \int_1^2(-t+2)\sin\frac{n\pi}{2}t\,dt = \frac{8}{n^2\pi^2}\sin\frac{n\pi}{2} \qquad n \text{ odd}$$

(f) Here, we need both sine and cosine terms with $\omega_0 = 1$ and odd harmonics only. Hence, we must construct a pulse such that it has half-wave symmetry, but neither odd nor even symmetry, has a value t over the interval $0 \le t \le 1$, and repeats every 2π seconds as shown in Fig. S6.1-6f. Observe that the first half cycle from 0 to π) and the second half cycle (from π to 2π) are negatives of each other as required in half-wave symmetry. By inspection, $a_0 = 0$. This yields

$$x(t) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} a_n \cos nt + b_n \sin nt$$

Because of half-wave symmetry (see Prob. 6.1-5),

$$a_n = \frac{4}{2\pi}\int_0^1 t\cos nt\,dt = \frac{2}{\pi n^2}(\cos n + n\sin n - 1)$$

$$b_n = \frac{4}{2\pi}\int_0^1 t\sin nt\,dt = \frac{2}{\pi n^2}(\sin n - n\cos n) \qquad n \text{ odd}$$

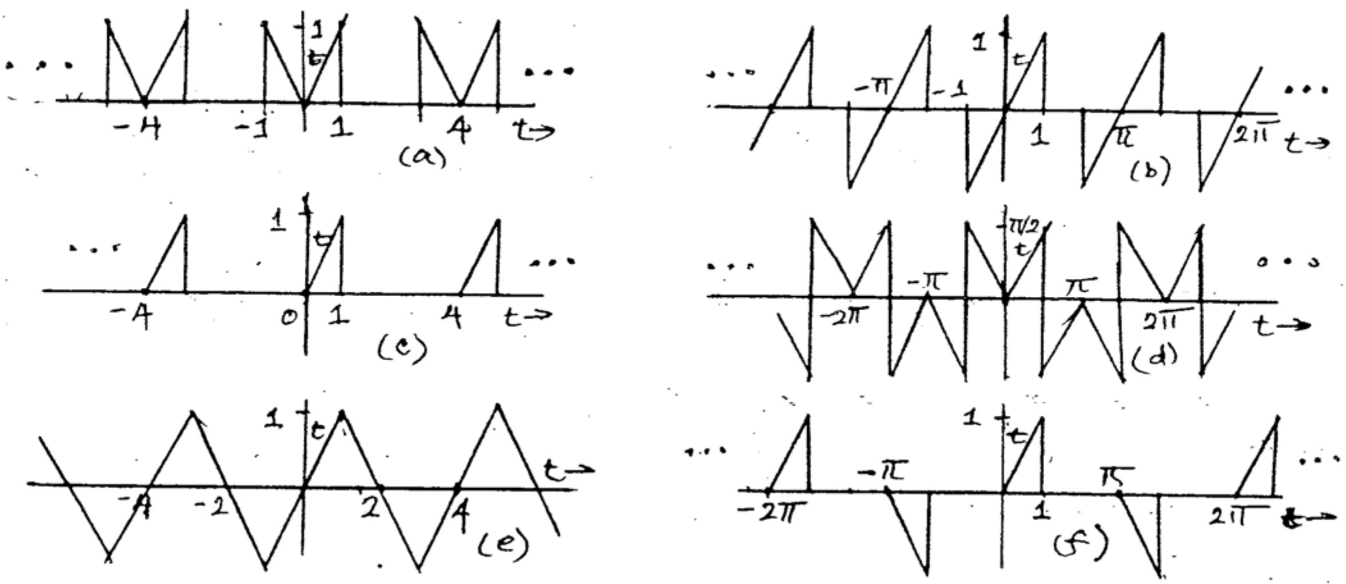


Figure S6.1-6

6.1-7.

	a	b	c	d	e	f	g	h	i
periodic ?	yes	yes	no	yes	no	yes	yes	yes	yes
ω_0	1	1		π		$\frac{1}{70}$	$\frac{3}{4}$	1	2
period	2π	2π		2		140π	$\frac{8\pi}{3}$	2π	π

6.3-1. (a) $T_0 = 4, \omega_0 = \pi/2$. Also $D_0 = 0$ (by inspection).

$$D_n = \frac{1}{2\pi}\int_{-1}^{1} e^{-j(n\pi/2)t}\,dt - \int_{1}^{3} e^{-j(n\pi/2)t}\,dt = \frac{2}{\pi n}\sin\frac{n\pi}{2} \qquad |n| \geq 1$$

(b) $T_0 = 10\pi$, $\omega_0 = 2\pi/10\pi = 1/5$

$$x(t) = \sum_{n=-\infty}^{\infty} D_n e^{j\frac{n}{5}t},$$

where $$D_n = \frac{1}{10\pi}\int_{\pi}^{\pi} e^{-j\frac{n}{5}t}\,dt = \frac{j}{2\pi n}\left(-2j\sin\frac{n\pi}{5}\right) = \frac{1}{\pi n}\sin\left(\frac{n\pi}{5}\right)$$

(c)

$$x(t) = D_0 + \sum_{n=-\infty}^{\infty} D_n e^{jnt}, \quad \text{where, by inspection} \qquad D_0 = 0.5$$

$$D_n = \frac{1}{2\pi}\int_{0}^{2\pi} \frac{t}{2\pi}e^{-jnt}\,dt = \frac{j}{2\pi n},$$

so that $|D_n| = \dfrac{1}{2\pi n}$, and $\angle D_n = \begin{cases} \frac{\pi}{2} & n > 0 \\ \frac{-\pi}{2} & n < 0 \end{cases}$

(d) $T_0 = \pi$, $\omega_0 = 2$ and $D_n = 0$

$$x(t) = \sum_{n=-\infty}^{\infty} D_n e^{j2nt},$$

$$\text{where} \qquad D_n = \frac{1}{\pi}\int_{-\pi/4}^{\pi/4} \frac{4t}{\pi} e^{-j2nt}\,dt = \frac{-j}{\pi n}\left(\frac{2}{\pi n}\sin\frac{\pi n}{2} - \cos\frac{\pi n}{2}\right)$$

(e) $T_0 = 3, \omega_0 = \frac{2\pi}{3}$.

$$x(t) = \sum_{n=\infty}^{\infty} D_n\, e^{j\frac{2\pi n}{3}t},$$

$$\text{where} \qquad D_n = \frac{1}{3}\int_0^1 t\, e^{-j\frac{2\pi n}{3}t}\,dt = \frac{3}{4\pi^2 n^2}\left[e^{-j\frac{2\pi n}{3}}\left(\frac{j2\pi n}{3} + 1\right) - 1\right]$$

Therefore

$$|D_n| = \frac{3}{4\pi^2 n^2}\left[\sqrt{2 + \frac{4\pi^2 n^2}{9} - 2\cos\frac{2\pi n}{3} - \frac{4\pi n}{3}\sin\frac{2\pi n}{3}}\right]$$

$$\text{and } \angle D_n = \tan^{-1}\left(\frac{\frac{2\pi n}{3}\cos\frac{2\pi n}{3} - \sin\frac{2\pi n}{3}}{\cos\frac{2\pi n}{3} + \frac{2\pi n}{3}\sin\frac{2\pi n}{3} - 1}\right)$$

(f) $T_0 = 6$, $\omega_0 = \pi/3$ $D_0 = 0.5$

$$x(t) = 0.5 + \sum_{n=-\infty}^{\infty} D_n e^{\frac{j\pi n t}{3}}$$

$$\begin{aligned} D_n &= \frac{1}{6}\left[\int_{-2}^{-1}(t+2)e^{-\frac{j\pi n t}{3}}\,dt + \int_{-1}^{1} e^{-\frac{j\pi n t}{3}}\,dt + \int_{1}^{2}(-t+2)e^{-\frac{j\pi n t}{3}}\,dt\right] \\ &= \frac{3}{\pi^2 n^2}\left(\cos\frac{n\pi}{3} - \cos\frac{2\pi n}{3}\right) \end{aligned}$$

6.3-2. Note that the signal $x(t)$ is defined as

$$x(t) = \begin{cases} \frac{1}{A}t & 0 \le t < A \\ 1 & A \le t < \pi \\ 0 & \pi \le t < 2\pi \\ x(t+2\pi) & \text{otherwise} \end{cases}$$

The exponential Fourier series coefficients are determined by

$$D_n = \frac{1}{T_0}\int_{T_0} x(t)e^{-jn\omega_0 t}dt$$

Since $T_0 = 2\pi$, $\omega_0 = \frac{2\pi}{T_0} = 1$. For $n = 0$,

$$D_0 = \frac{1}{T_0}\int_{T_0} x(t)dt$$

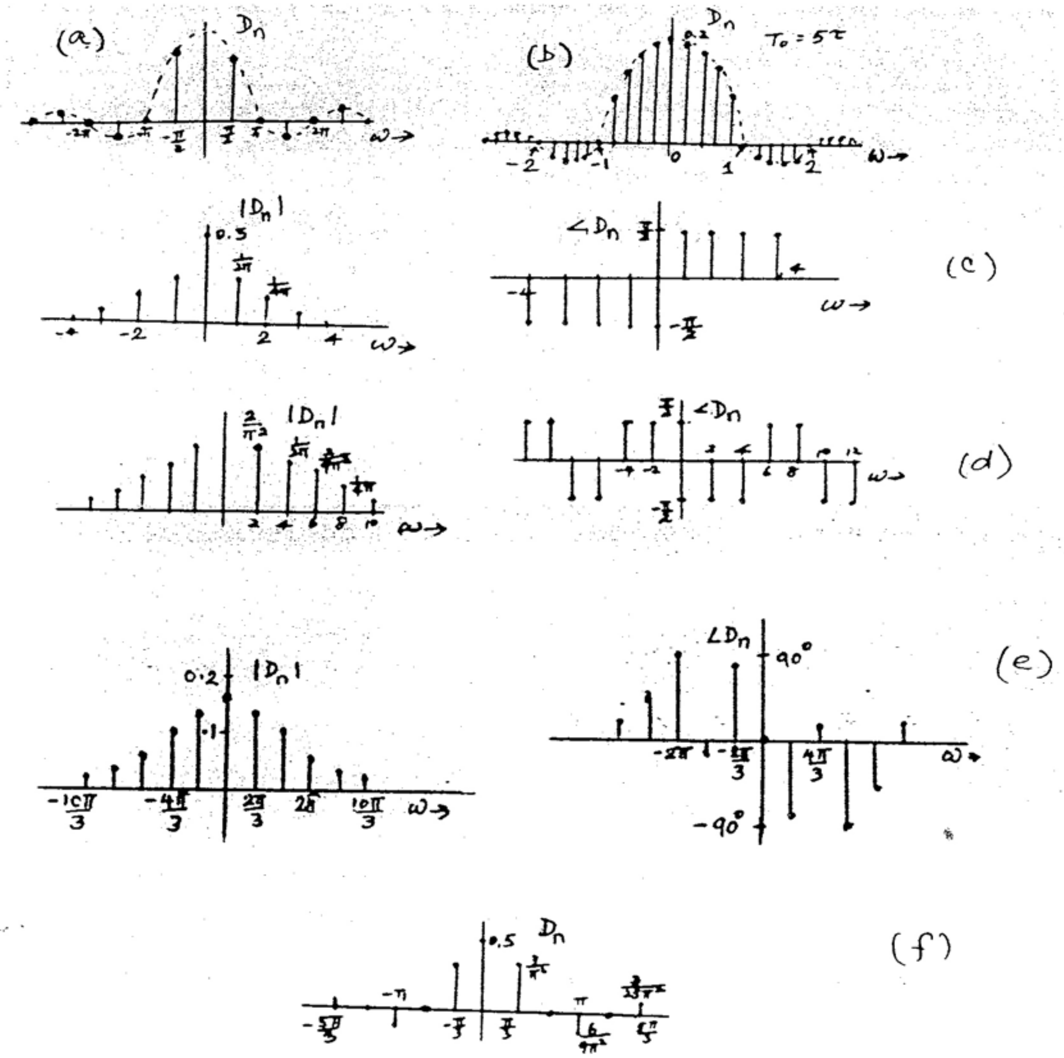


Figure S6.3-1

$$
\begin{aligned}
&= \frac{1}{2\pi}\left(\int_0^A \frac{t}{A}dt + \int_A^\pi dt\right) \\
&= \frac{1}{2\pi}\left(\left.\frac{t^2}{2A}\right|_{t=0}^{A} \left.t\right|_{t=A}^{\pi}\right) \\
&= \frac{1}{2\pi}\left(\frac{A}{2} + \pi - A\right) \\
&= \frac{2\pi - A}{4\pi}
\end{aligned}
$$

For $n \neq 0$,

$$
\begin{aligned}
D_n &= \frac{1}{T_0}\int_{T_0} e^{-jn\omega_0 t}x(t)dt \\
&= \frac{1}{2\pi}\left(\int_0^A \frac{t}{A}e^{-jn\omega_0 t}dt + \int_A^\pi e^{-jn\omega_0 t}dt\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2\pi}\left(\left.\frac{te^{-jnt}}{-jAn}\right|_{t=0}^{A} - \int_0^A \frac{e^{-jnt}}{jAn}dt + \left.\frac{e^{-jnt}}{-jn}\right|_{t=A}^{\pi}\right) \\
&= \frac{1}{2\pi}\left(\frac{e^{-jnA}}{-jn} - \left.\frac{e^{-jnt}}{-An^2}\right|_{t=0}^{A} + \frac{e^{-jn\pi} - e^{-jnA}}{-jn}\right) \\
&= \frac{1}{2\pi}\left(\frac{je^{-jn\pi}}{n} + \frac{e^{-jnA}-1}{An^2}\right) \\
&= \frac{1}{2\pi n}\left(\frac{e^{-jnA}-1}{An} + je^{-jn\pi}\right)
\end{aligned}
$$

Thus,

$$
D_n = \begin{cases} \frac{2\pi - A}{4\pi} & n = 0 \\ \frac{1}{2\pi n}\left(\frac{e^{-jnA}-1}{nA} + je^{-jn\pi}\right) & \text{otherwise} \end{cases}
$$

6.3-3. (a)

$$
x(t) = 3\cos t + \sin\left(5t - \frac{\pi}{6}\right) - 2\cos\left(8t - \frac{\pi}{3}\right)
$$

For a compact trigonometric form, all terms must have cosine form and amplitudes must be positive. For this reason, we rewrite $x(t)$ as

$$
\begin{aligned}
x(t) &= 3\cos t + \cos\left(5t - \frac{\pi}{6} - \frac{\pi}{2}\right) + 2\cos\left(8t - \frac{\pi}{3} - \pi\right) \\
&= 3\cos t + \cos\left(5t - \frac{2\pi}{3}\right) + 2\cos\left(8t - \frac{4\pi}{3}\right)
\end{aligned}
$$

In the preceding expression, we could have expressed the term $2\cos\left(8t - \frac{4\pi}{3}\right)$ as $2\cos\left(8t + \frac{2\pi}{3}\right)$. Figure S6.3-3a shows amplitude and phase spectra.

(b) By inspection of the trigonometric spectra in Fig. S6.3-3a, we plot the exponential spectra as shown in Fig. S6.3-3b.

(c) By inspection of exponential spectra in Fig. S6.3-3a, we obtain

$$
\begin{aligned}
x(t) &= \frac{3}{2}(e^{jt} + e^{-jt}) + \frac{1}{2}\left[e^{j(5t-\frac{2\pi}{3})} + e^{-j(5t-\frac{2\pi}{3})}\right] + \left[e^{j(8t-\frac{4\pi}{3})} + e^{-j(8t-\frac{4\pi}{3})}\right] \\
&= \frac{3}{2}e^{jt} + \left(\frac{1}{2}e^{-j\frac{2\pi}{3}}\right)e^{j5t} + \left(e^{-j\frac{4\pi}{3}}\right)e^{j8t} + \frac{3}{2}e^{-jt} + \left(\frac{1}{2}e^{j\frac{2\pi}{3}}\right)e^{-j5t} + \left(e^{j\frac{4\pi}{3}}\right)e^{-j8t}
\end{aligned}
$$

(d) By inspection of the first line in part (c), we can immediately write $x(t)$ in the trigonometric form as

$$
\begin{aligned}
x(t) &= 3\cos t + \cos\left(5t - \frac{2\pi}{3}\right) + 2\cos\left(8t - \frac{4\pi}{3}\right) \\
&= 3\cos t + \sin\left(5t - \frac{\pi}{6}\right) - 2\cos\left(8t - \frac{\pi}{3}\right)
\end{aligned}
$$

6.3-4. (a) In compact trigonometric form, all terms are of cosine form and amplitudes are positive. We can express $x(t)$ as

$$
x(t) = 3 + 2\cos\left(2t - \frac{\pi}{6}\right) + \cos\left(3t - \frac{\pi}{2}\right) + \frac{1}{2}\cos\left(5t + \frac{\pi}{3} - \pi\right)
$$

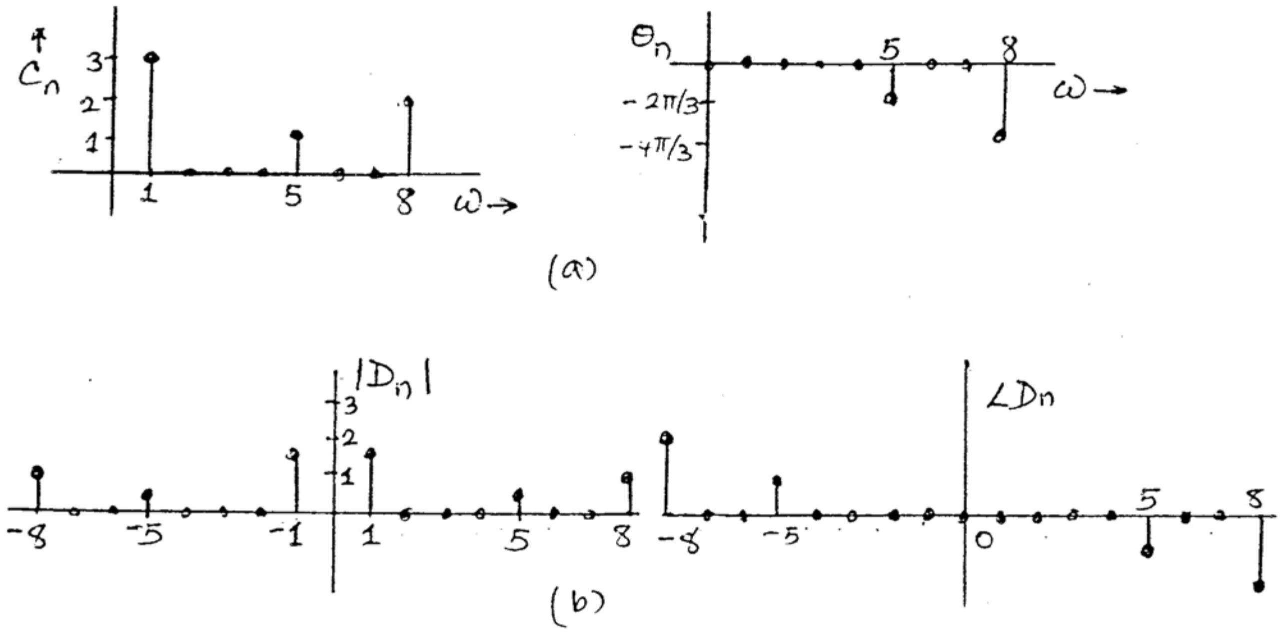


Figure S6.3-3

$$= \quad 3 + 2\cos\left(2t - \frac{\pi}{6}\right) + \cos\left(3t - \frac{\pi}{2}\right) + \frac{1}{2}\cos\left(5t - \frac{2\pi}{3}\right)$$

From this expression we sketch the trigonometric Fourier spectra as shown in Fig. S6.3-4a.

(b) By inspection of trigonometric spectra, we sketch the exponential Fourier spectra shown in Fig. S6.3-4b.

(c) From these exponential spectra, we can now write the exponential Fourier series as

$$x(t) = 3 + e^{j(2t-\frac{\pi}{6})} + e^{-j(2t-\frac{\pi}{6})} + \frac{1}{2}e^{j(3t-\frac{\pi}{2})} + \frac{1}{2}e^{-j(3t-\frac{\pi}{3})} + \frac{1}{4}e^{j(5t-\frac{2\pi}{3})} + \frac{1}{4}e^{-j(5t-\frac{2\pi}{3})}$$

(d) By inspection of the first line in part (c), we can immediately write $x(t)$ in the trigonometric form as

$$\begin{aligned} x(t) &= 3 + 2\cos\left(2t - \frac{\pi}{6}\right) + \cos\left(3t - \frac{\pi}{2}\right) + \frac{1}{2}\cos\left(5t - \frac{2\pi}{3}\right) \\ &= 3 + 2\cos\left(2t - \frac{\pi}{6}\right) + \sin 3t - \frac{1}{2}\cos\left(5t + \frac{\pi}{3}\right) \end{aligned}$$

6.3-5. (a) The exponential Fourier series can be expressed with coefficients in Polar form as

$$x(t) = (2\sqrt{2}e^{j\pi/4})e^{-j3t} + 2e^{j\pi/2}e^{-jt} + 3 + 2e^{-j\pi/2}e^{jt} + (2\sqrt{2}e^{-j\pi/4})e^{j3t}$$

From this expression the exponential Spectra are sketched as shown in Figure S6.3-5a.

(b) By inspection of the exponential spectra in Figure S6.3-5a, we sketch the trigonometric spectra as shown in Figure S6.3-5b. From these spectra, we can write the

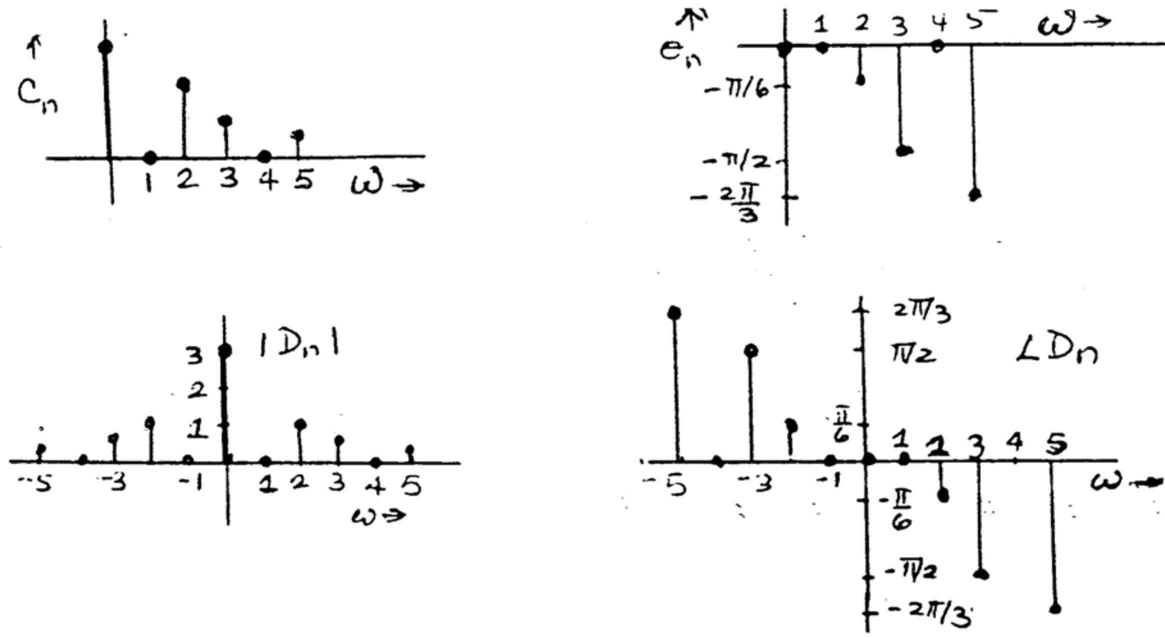


Figure S6.3-4

compact trigonometric Fourier series as

$$x(t) = 3 + 4\cos\left(t - \frac{\pi}{2}\right) + 4\sqrt{2}\cos\left(3t - \frac{\pi}{4}\right)$$

(c) Since, the trigonometric series in part (b) is obtained from the exponential series in part (a), the two series are equivalent.

(d) The lowest frequency in the spectrum is 0 and the highest frequency is 3. Therefore the bandwidth is 3 rad/s or $\frac{3}{2\pi}$ Hz.

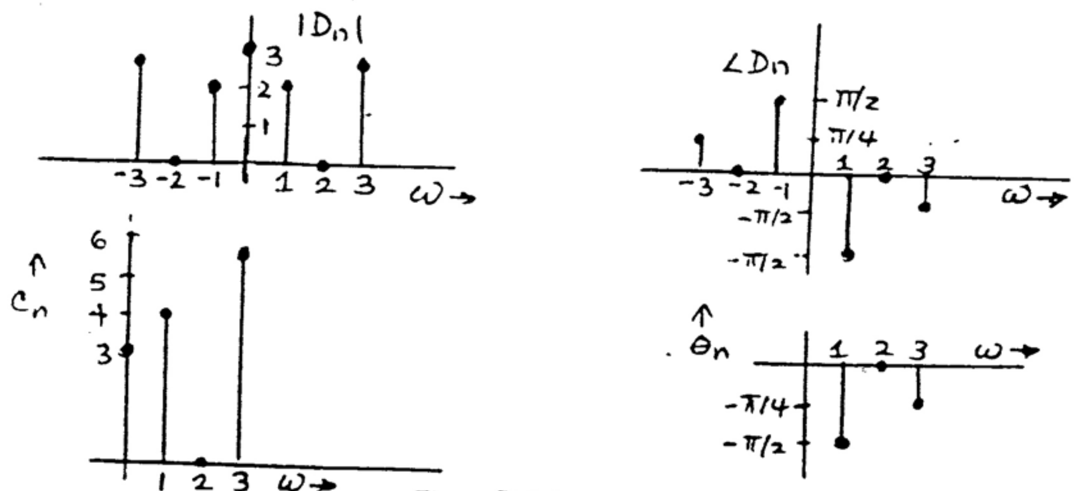


Figure S6.3-5

6.3-6. (a)

$$
\begin{aligned}
x(t) &= 2 + 2\cos(2t - \pi) + \cos(3t - \frac{\pi}{2}) \\
&= 2 - 2\cos 2t + \sin 3t
\end{aligned}
$$

(b) The exponential spectra are shown in Figure S6.3-6.

(c) By inspection of exponential spectra

$$
\begin{aligned}
x(t) &= 2 + \left[e^{(2t-\pi)} + e^{-j(2t-\pi)}\right] + \frac{1}{2}\left[e^{j(3t-\frac{\pi}{2})} + e^{-j(3t-\frac{\pi}{2})}\right] \\
&= 2 + 2\cos(2t - \pi) + \cos\left(3t - \frac{\pi}{2}\right)
\end{aligned}
$$

(d) Observe that the two expressions (trigonometric and exponential Fourier series) are equivalent.

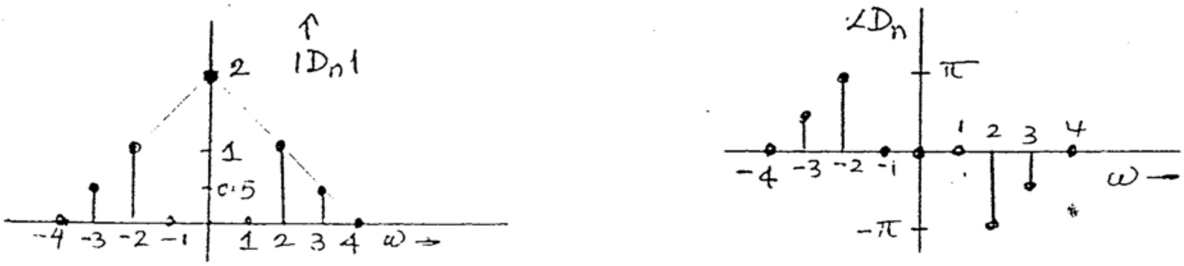


Figure S6.3-6

6.3-7. (a) The exponential Fourier series, as found by inspection of Figure P6.3-6, is

$$
x(t) = 2 + 2e^{j(t+\frac{2\pi}{3})} + 2e^{-j(t+\frac{2\pi}{3})} + e^{j(2t+\frac{\pi}{3})} + e^{-j(2t+\frac{\pi}{3})}
$$

(b) To find the corresponding trigonometric series, we consider only the positive frequency components, then double the exponential amplitudes (except for dc, which is kept the same), and maintain the same phase values to obtain the trigonometric spectrum, Figure S6.3-7.

(c) By inspection of the trigonometric spectra

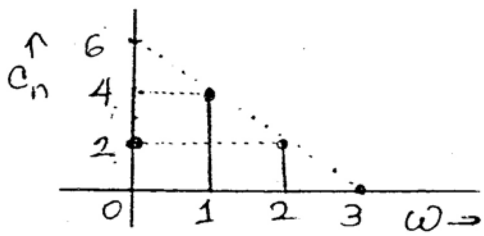
$$
x(t) = 2 + 4\cos\left(t + \frac{2\pi}{3}\right) + 2\cos\left(2t + \frac{\pi}{3}\right)
$$

(d)

$$
\begin{aligned}
x(t) &= 2 + 4\cos\left(t + \frac{2\pi}{3}\right) + 2\cos\left(2t + \frac{\pi}{3}\right) \\
&= 2 + 2e^{j(t+\frac{2\pi}{3})} + 2e^{-j(t+\frac{2\pi}{3})} + e^{j(2t+\frac{\pi}{3})} + e^{-j(2t+\frac{\pi}{3})}
\end{aligned}
$$

6.3-8. (a) The period is $T_0 = 8$ and $\omega_0 = \pi/4$. Also $D_0 = 0$ (by inspection), and

$$
x(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\frac{\pi}{4}t}
$$



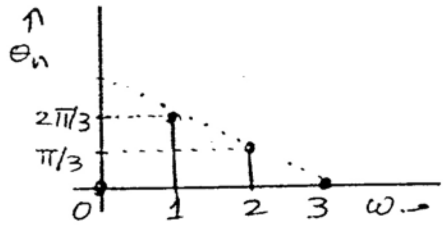


Figure S6.3-7

$$D_n = \frac{1}{8}\left[\int_{-4}^{0}\left(\frac{t}{2}+1\right)e^{-j2n(\pi/4)t}\,dt + \int_{0}^{4}\left(-\frac{t}{2}+1\right)e^{-j2n(\pi/4)t}\,dt\right] =$$

This yields

$$D_n = \begin{cases} \frac{4}{\pi^2 n^2} & n = \pm1, \pm3, \pm5, \pm7, \cdots \\ 0 & \text{otherwise} \end{cases}$$

Therefore

$$x(t) = \sum_{\substack{n=1 \\ n\ odd}}^{\infty} \frac{4}{\pi^2 n^2} e^{jn\frac{\pi}{4}t}$$

(b) Observe that $\hat{x}(t)$ is the same as $x(t)$ in Figure P6.3-8a delayed by 2 seconds. Therefore

$$\hat{x}(t) = x(t-2) = \sum_{\substack{n=1 \\ n\ odd}}^{\infty} D_n e^{jn\frac{\pi}{4}(t-2)} = \sum_{\substack{n=1 \\ n\ odd}}^{\infty} D_n e^{-jn\pi/2} e^{jn\frac{\pi}{4}t}$$

Therefore

$$\hat{x}(t) = \sum_{\substack{n=1 \\ n\ odd}}^{\infty} \hat{D}_n e^{jn\frac{\pi}{4}t}$$

where

$$\hat{D}_n = D_n e^{j\frac{n\pi}{2}} = \frac{4}{\pi^2 n^2} e^{-j\frac{n\pi}{2}}$$

(c) Observe that $\tilde{x}(t)$ is the same as $x(t)$ in Figure P6.2-8a time-compressed by a factor 2. Therefore

$$\tilde{x}(t) = x(2t) = \sum_{\substack{n=1 \\ n\ odd}}^{\infty} D_n e^{jn\frac{\pi}{4}(2t)} = \sum_{\substack{n=1 \\ n\ odd}}^{\infty} D_n e^{jn\frac{\pi}{2}t}$$

Therefore

$$\tilde{x}(t) = \sum_{\substack{n=1 \\ n\ odd}}^{\infty} \tilde{D}_n e^{jn\frac{\pi}{2}t}$$

where

$$\tilde{D}_n = D_n = \frac{4}{\pi^2 n^2}$$

6.3-9. (a)

$$\begin{aligned} x(t) &= \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t} \\ \hat{x}(t) &= x(t-T) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 (t-T)} = \sum_{n=-\infty}^{\infty} (D_n e^{-jn\omega_0 T}) e^{jn\omega_0 t} = \sum_{n=-\infty}^{\infty} \hat{D}_n e^{jn\omega_0 t} \end{aligned}$$

$$\hat{D}_n = D_n e^{-jn\omega_0 t} \quad \text{so that} \quad |\hat{D}_n| = |D_n|, \quad \text{and} \quad \angle\hat{D}_n = \angle D_n - jn\omega_0 T$$

(b)

$$\begin{aligned} x(t) &= \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t} \\ \hat{x}(t) &= x(at) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 (at)} \end{aligned}$$

6.3-10. (a) From Exercise E6.1a

$$x(t) = \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos n\pi t \qquad -1 \le t \le 1$$

The power of $x(t)$ is

$$P_x = \frac{1}{2}\int_{-1}^{1} t^4\, dt = \frac{1}{5}$$

Moreover, from Parseval's theorem Eq. (6.40)

$$P_x = C_0^2 + \sum_{1}^{\infty} \frac{C_n^2}{2} = \left(\frac{1}{3}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{4(-1)^n}{\pi^2 n^2}\right)^2 = \frac{1}{9} + \frac{8}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{1}{9} + \frac{8}{90} = \frac{1}{5}$$

(b) If the N-term Fourier series is denoted by $w(t)$, then

$$w(t) = \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{N-1} \frac{(-1)^n}{n^2} \cos n\pi t \qquad -1 \le t \le 1$$

The power P_x is required to be $99\% P_x = 0.198$. Therefore

$$P_x = \frac{1}{9} + \frac{8}{\pi^4}\sum_{n=1}^{N-1}\frac{1}{n^4} = 0.198$$

For $N = 1$, $P_x = 0.1111$; for $N = 2$, $P_x = 0.19323$, For $N = 3$, $P_x = 0.19837$, which is greater than 0.198. Thus, $N = 3$.

6.3-11. (a) From Exercise E6.1b

$$x(t) = \frac{2A}{\pi}(-1)^{n+1}\sum_{n=1}^{\infty}\frac{1}{n}\sin n\pi t \qquad -\pi \le t \le \pi$$

The power of $x(t)$ is

$$P_x = \frac{1}{2}\int_{-1}^{1} (At)^2\,dt = \frac{A^2}{3}$$

Moreover, from Parseval's theorem [Eq. (6.40)]

$$P_x = C_0^2 + \sum_1^\infty \frac{C_n^2}{2} = \frac{1}{2}\sum_1^\infty \frac{4A^2}{\pi^2 n^2} = \frac{2A^2}{\pi^2}\sum_1^\infty \frac{1}{n^2} = \frac{A^2}{3}$$

(b) If the N-term Fourier series is denoted by $w(t)$, then

$$w(t) = \frac{2A}{\pi}(-1)^{n+1}\sum_{n=1}^{N}\frac{1}{n}\sin n\pi t \qquad -\pi \le t \le \pi$$

The power P_w is required to be no less than $0.90\frac{A^2}{3} = 0.3A^2$. Therefore

$$P_w = \frac{1}{2}\sum_1^N \frac{4A^2}{\pi^2 n^2} \ge 0.3A^2$$

For $N=1$, $P_w = 0.2026A^2$; for $N=2$, $P_w = 0.2533A^2$, for $N=5$, $P_w = 0.29658A^2$, for $N=6$, $P_w = 0.30222A^2$, which is greater than $0.3A^2$. Thus, $N=6$.

6.3-12. The power of a rectified sine wave is the same as that of a sine wave, that is, 1/2. Thus $P_x = 0.5$. Let the $2N+1$ term truncated Fourier series be denoted by $\hat{x}(t)$. The power $P_{\hat{x}}$ is required to be no less than $0.9975P_x = 0.49875$. Using the Fourier series coefficients in Exercise E6.5, we have

$$P_{\hat{x}} = \sum_{n=-N}^{N} |D_n|^2 = \frac{4}{\pi^2}\sum_{n=-N}^{N}\frac{1}{(1-4n^2)^2} \ge 0.49875$$

Direct calculations using the above equation gives $P_{\hat{x}} = 4/\pi^2 = 0.4053$ for $N=0$ (only dc), $P_{\hat{x}} = 0.49535$ for $N=1$ (3 terms), and $P_{\hat{x}} = 0.49895$ for $N=2$ (5 terms). Thus, a 5-term Fourier series yields a signal whose power is 99.79% of the power of the rectified sine wave. The power of the error in the approximation of $x(t)$ by $\hat{x}(t)$ is only 0.21% of the signal power P_x.

6.4-1. Period $T_0 = \pi$, and $\omega_0 = 2$, and

$$H(j\omega) = \frac{j\omega}{(-\omega^2+3)+j2\omega}, \qquad \text{and from Eq. (6.30b)} \qquad D_n = \frac{0.504}{1+j4n}$$

Therefore, $$y(t) = \sum_{n=-\infty}^{\infty} D_n H(jn\omega_0)e^{jn\omega_0 t} = \sum_{n=-\infty}^{\infty}\frac{j1.08n}{(1+j4n)(-\omega^2+3+j2\omega)}e^{j2nt}$$

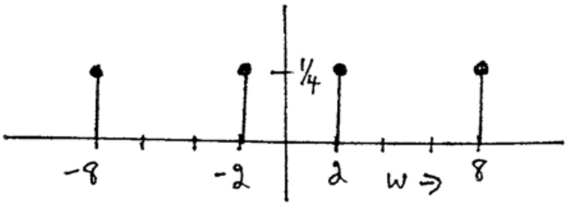
6.4-2. (a)

$$\begin{aligned}\cos 5t \sin 3t &= \frac{1}{2}[\sin 8t - \sin 2t] \\ &= \frac{1}{4j}\left[e^{j8t} - e^{-j8t} - e^{j2t} + e^{-j2t}\right]\end{aligned}$$

$$= \frac{1}{4j}\left[e^{j(8t-\frac{\pi}{2})} - e^{-j(8t-\frac{\pi}{2})} - e^{j(2t+\frac{\pi}{2})} + e^{-j(2t+\frac{\pi}{2})}\right]$$

This is the desired exponential Fourier series.

(b) There are four spectral components at $\omega = \pm 8$ and ± 2. The phases are either $\frac{\pi}{2}$ or $-\frac{\pi}{2}$, as shown in the spectrum in Figure S6.4-2b.



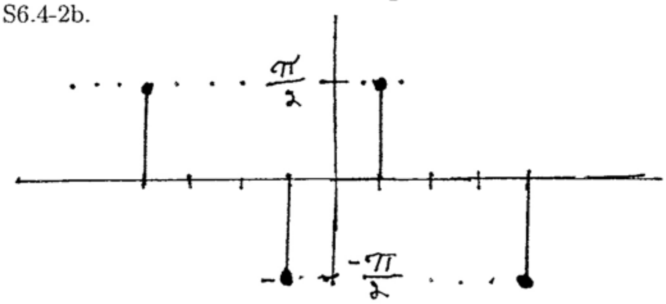


Figure S6.4-2b

(c) Since none of the spectral components of $x(t)$ appear in the pass-band of the filter, the output is $y(t) = 0$.

6.4-3.

$$D_n = \int_0^1 e^{-t}e^{-jn\omega_o t}dt = \frac{(e-1)(1-j2\pi n)}{e(1+4\pi^2n^2)}$$

The transfer function of the R-C circuit is

$$H(j\omega) = \frac{1}{1+(\frac{1}{j\omega})} = \frac{j\omega}{j\omega+1}$$

The input $x(t)$ can be expressed as a Fourier series

$$x(t) = \sum_{n=-\infty}^{\infty} \frac{(e-1)(1-j2\pi n)}{e(1+4\pi^2n^2)}e^{j2\pi nt}$$

Hence the output $y(t)$ is given by

$$\begin{aligned} y(t) &= \sum_{n=-\infty}^{\infty} D_n H(j2\pi n)e^{j2\pi nt} \\ &= \sum_{n=-\infty}^{\infty} \frac{(e-1)(1-j2\pi n)(j2\pi n)}{e(1+4\pi^2n^2)(j2\pi n+1)}e^{j2\pi nt} \\ &= \sum_{n=-\infty}^{\infty} \frac{2\pi n(e-1)(2\pi n+j)}{e(1+4\pi^2n^2)^2}e^{j2\pi nt} \end{aligned}$$

6.5-1. Equating the derivative (with respect to c) yields

$$2c|\mathbf{y}|^2 = 2\mathbf{x}.\mathbf{y}$$

which yields the desired result.

6.5-2. (a)

$$e(t) = x(t) - cx(t)$$

Also

$$\int_{t_1}^{t_2} x(t)[x(t) - cx(t)]\,dt = \int_{t_1}^{t_2} x(t)x(t)\,dt - c\int_{t_1}^{t_2} x^2(t)\,dt$$

But

$$c = \frac{\int_{t_1}^{t_2} x(t)x(t)\,dt}{\int_{t_1}^{t_2} x^2(t)\,dt}$$

Substitution of c in the earlier equation yields

$$\int_{t_1}^{t_2} x(t)[x(t) - cx(t)]\,dt = 0$$

Therefore $x(t)$ and $[x(t) - cx(t)]$ are orthogonal.

(b) We can readily see result from Figure 6.17. The error vector $\mathbf{e}$ is orthogonal to vector $\mathbf{x}$.

(c)

$$e(t) = \begin{cases} 1 - \frac{4}{\pi}\sin t & 0 \le t \le \pi \\ -1 - \frac{4}{\pi}\sin t & \pi \le t \le 2\pi \end{cases}$$

$$\begin{aligned}\int_0^{2\pi} e^2(t)\,dt &= \int_0^{\pi}\left(1 - \frac{4}{\pi}\sin t\right)^2 dt - \int_{\pi}^{2\pi}\left(1 + \frac{4}{\pi}\sin t\right)^2 dt \\ &= -\frac{8}{\pi}\left[\int_0^{\pi} \sin t\,dt + \int_{\pi}^{2\pi} \sin t\,dt\right] = -\frac{8}{\pi}\int_0^{2\pi} \sin t\,dt = 0\end{aligned}$$

6.5-3. (a) If $x(t)$ and $y(t)$ are orthogonal, then we showed [see Eq. (6.67)] the energy of $x(t) + y(t)$ is $E_x + E_y$. We now find the energy of $x(t) - y(t)$:

$$\begin{aligned}\int_{-\infty}^{\infty} |x(t) - y(t)|^2\,dt &= \int_{-\infty}^{\infty} |x(t)|^2\,dt + \int_{-\infty}^{\infty} |y(t)|^2\,dt \\ &\quad - \int_{-\infty}^{\infty} x(t)y^*(t)\,dt - \int_{-\infty}^{\infty} x^*(t)y(t)\,dt \\ &= \int_{-\infty}^{\infty} |x(t)|^2\,dt + \int_{-\infty}^{\infty} |y(t)|^2\,dt\end{aligned}$$

The last result follows from the fact that because of orthogonality, the two integrals of the cross products $x(t)y^*(t)$ and $x^*(t)y(t)$ are zero [see Eq. (6.80)]. Thus the energy of $x(t) + y(t)$ is equal to that of $x(t) - y(t)$ if $x(t)$ and $y(t)$ are orthogonal.

(b) Using similar argument, we can show that the energy of $c_1x(t) + c_2y(t)$ is equal to that of $c_1x(t) - c_2y(t)$ if $x(t)$ and $y(t)$ are orthogonal. This energy is given by $|c_1|^2E_x + |c_2|^2E_y$.

(c) If $z(t) = x(t) \pm y(t)$, then

$$\begin{aligned}\int_{-\infty}^{\infty} |x(t) \pm y(t)|^2\,dt &= \int_{-\infty}^{\infty} |x(t)|^2\,dt + \int_{-\infty}^{\infty} |y(t)|^2\,dt \\ &\quad \pm \int_{-\infty}^{\infty} x(t)y^*(t)\,dt \pm \int_{-\infty}^{\infty} x^*(t)y(t)\,dt\end{aligned}$$

$$= \quad E_x + E_y \pm (E_{xy} + E_{yx})$$

6.5-4. (a) In this case $E_x = \int_0^1 dt = 1$, and

$$c = \frac{1}{E_x}\int_0^1 x(t)y(t)\,dt = \frac{1}{1}\int_0^1 t\,dt = 0.5$$

(b) Thus, $x(t) \approx 0.5y(t)$, and the error $e(t) = t - 0.5$ over $(0 \le t \le 1)$, and zero outside this interval. Also E_x and E_e (the energy of the error) are

$$E_x = \int_0^1 x^2(t)\,dt = \int_0^1 t^2\,dt = 1/3 \quad \text{and} \quad E_e = \int_0^1 (t-0.5)^2\,dt = 1/12$$

The error $(t - 0.5)$ is orthogonal to $y(t)$ because

$$\int_0^1 (t-0.5)(1)\,dt = 0$$

By inspection of $y(t)$, we obtain $E_y = 1$. Note that $E_x = c^2E_y + E_e$. To explain these results in terms of vector concepts we observe from Figure 6.17 that the error vector $\mathbf{e}$ is orthogonal to the component $c\mathbf{x}$. Because of this orthogonality, the length-square of $\mathbf{x}$ [energy of $x(t)$] is equal to the sum of the square of the lengths of $c\mathbf{y}$ and $\mathbf{e}$ [sum of the energies of $cy(t)$ and $e(t)$].

6.5-5. In this case $E_x = \int_0^1 x^2(t)\,dt = \int_0^1 t^2\,dt = 1/3$, and

$$c = \frac{1}{E_x}\int_0^1 y(t)x(t)\,dt = 3\int_0^1 t\,dt = 1.5$$

Thus, $y(t) \approx 1.5x(t)$, and the error $e(t) = y(t) - 1.5x(t) = 1 - 1.5t$ over $(0 \le t \le 1)$, and zero outside this interval. Also E_e (the energy of the error) is $E_e = \int_0^1 (1-1.5t)^2\,dt = 1/4$.

6.5-6.

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos 2\pi nt + b_n \sin 2\pi nt \qquad \left(\omega_0 = \frac{2\pi}{1}\right)$$

$$a_0 = 1\int_0^1 x(t)\,dt = \int_0^1 t\,dt = \frac{1}{2}$$

$$a_n = 2\int_0^1 t\cos 2\pi nt\,dt = 0 \quad n \ge 1 \quad (n\,\text{integer})$$

$$b_n = 2\int_0^1 t\sin 2\pi nt\,dt = \frac{-1}{\pi n}$$

Hence

$$\begin{aligned} x(t) &= \frac{1}{2} - \frac{1}{\pi}\left(\sin 2\pi t + \frac{1}{2}\sin 4\pi t + \frac{1}{3}\sin 6\pi t + \cdots\right) \\ &= \frac{1}{2} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin 2\pi nt \end{aligned}$$

From Eq. (6.77)

$$\epsilon_k = 1\int_0^1 x^2(t)\,dt - \left[\left(\frac{1}{2}\right)^2 + \frac{1}{2}\left[\left(\frac{1}{\pi}\right)^2 + \left(\frac{1}{2\pi}\right)^2 + \cdots + \left(\frac{1}{(k-1)\pi}\right)^2\right]\right]$$

(Note that $K_j^2 = 1/2$ for $j = 1, 2, \ldots$ and $K_0^2 = 1$)

$$\begin{aligned}
\epsilon_1 &= \int_0^1 t^2\,dt - \frac{1}{4} = \frac{1}{3} - \frac{1}{4} = \frac{1}{12} \\
\epsilon_2 &= \frac{1}{3} - \frac{1}{4} - \frac{1}{2\pi^2} = 0.03267 \\
\epsilon_3 &= \frac{1}{3} - \frac{1}{4} - \frac{1}{2\pi^2} - \frac{1}{8\pi^2} = 0.02 \\
\epsilon_4 &= \frac{1}{3} - \frac{1}{4} - \frac{1}{2\pi^2} - \frac{1}{8\pi^2} - \frac{1}{18\pi^2} = 0.014378
\end{aligned}$$

6.5-7. (a) Figure S6.5-7a shows $x_1(t)$ that is a periodic extension of $x(t)$ to yield a series with $\omega_0 = 2\pi$ and only sine terms. This requires $T_0 = 2\pi/2\omega = 1$ and odd symmetry. From inspection, the dc component is 0.5. If we subtract dc (0.5) from $x_1(t)$, the remaining signal $x_1(t) - 0.5$ has odd symmetry (only sine terms). Therefore

$$\begin{aligned}
x_1(t) &= 0.5 + \sum_{n=1}^{\infty} b_n \sin 2\pi nt \\
b_n &= 2\int_0^1 t\,\sin 2\pi nt\,dt = -\frac{1}{\pi n} \\
x_1(t) &= \frac{1}{2} - \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{1}{n}\sin 2\pi nt
\end{aligned}$$

(b) $\omega_0 = \pi$ and $T_0 = 2\pi/\pi = 2$. For sine terms only, we need odd symmetry. Figure S6.5-7b shows a suitable function $x_2(t)$. It has no dc.

$$\begin{aligned}
x_2(t) &= \sum_{n=1}^{\infty} b_n \sin n\pi t \\
b_n &= \frac{4}{2}\int_0^1 t \sin n\pi t\,dt = (-1)^{n+1}\frac{2}{n\pi}
\end{aligned}$$

(c) $\omega_0 = \pi$, $T_0 = 2\pi/\omega_0 = 2$. For cosine terms only, we need an even function $x_3(t)$ as shown in Figure S6.5-7c. By inspection dc is 0.5. Therefore

$$\begin{aligned}
x_3(t) &= \frac{1}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi t \\
a_n &= \frac{4}{2}\int_0^1 t\cos n\pi t\,dt = -\frac{4}{\pi^2 n^2} \qquad n = 1, 3, 5, \ldots
\end{aligned}$$

6.5-8. (a) The signal $g(t)$ is the same as the signal $x(t)$ in Example 6.12 (Figure 6.23)

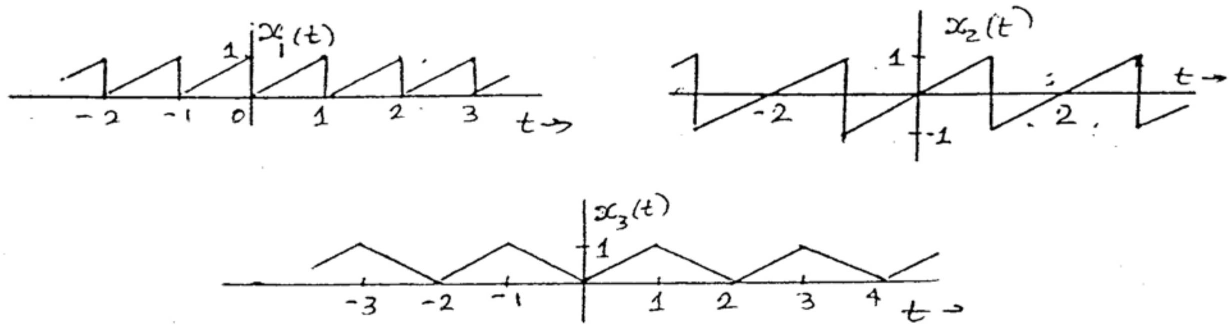


Figure S6.5-7

time-expanded by a factor π. Therefore from Eq. (6.90), we have

$$g(t) = x\left(\frac{t}{\pi}\right) = -\frac{3}{2}\left(\frac{t}{\pi}\right) + \frac{7}{8}\left[\frac{5}{2}\left(\frac{t}{\pi}\right)^3 - \frac{3}{2}\left(\frac{t}{\pi}\right)\right] + \cdots \tag{1}$$

For the representation in Eq. (6.90) for $x(t)$ in Figure 6.23

$$\int_{-1}^{1} x^2(t)\, dt = 2\pi \qquad \text{and}$$

Therefore from Eq. (6.77)

$$\begin{aligned} \epsilon_1 &= \int x^2(t)\, dt - \frac{1}{3}c_1^2 = 2 - \frac{3}{2} = 0.5 \\ \epsilon_2 &= \int x^2(t)\, dt - \frac{1}{3}c_1^2 - \frac{1}{7}c_3^2 = 0.28125 \end{aligned}$$

Since $g(t)$ is the same as $x(t)$ time-expanded by a factor π, all energies are increased by the same factor (π). Therefore

$$\begin{aligned} \epsilon_1 &= 0.5\pi \\ \epsilon_2 &= 0.28125\pi \end{aligned}$$

6.5-9.

$$x(t) = c_0 x_0(t) + c_1 x_1(t) + \cdots + c_7 x_7(t)$$

The energy E_n of $x_n(t)$, for all $n = 1, 2, 3, \ldots, 8$ is given by

$$E_n = \int_0^1 x_n^2(t)\, dt = 1$$

Hence

$$c_0 = \int_0^1 x(t) x_0(t)\, dt = \frac{1}{2}$$

$$\begin{aligned}
c_1 &= \int_0^1 x(t)x_1(t)\,dt = -\frac{1}{4} \\
c_2 &= c_4 = c_5 = c_6 = 0 \\
c_3 &= \int_0^1 x(t)x_3(t)\,dt = -\frac{1}{8} \\
c_7 &= \int_0^1 x(t)x_7(t)\,dt = -\frac{1}{16}
\end{aligned}$$

Hence

$$x(t) \simeq \frac{1}{2}x_0(t) - \frac{1}{4}x_1(t) - \frac{1}{8}x_3(t) - \frac{1}{16}x_7(t)$$

Also

$$\int_0^1 x^2(t)\,dt = \frac{1}{3} \quad \text{and} \quad E_n = 1$$

If $E_e(N)$ is the energy of the error signal in the approximation using first N terms, then From Eq. (6.77)

$$\begin{aligned}
E_e(1) &= \frac{1}{3} - c_0^2 = \frac{1}{12} = 0.0833 \\
E_e(2) &= \frac{1}{3} - c_0^2 - c_1^2 = \frac{1}{48} = 0.0204 \\
E_e(3) &= \frac{1}{3} - c_0^2 - c_1^2 - c_3^2 = \frac{1}{192} = 0.0052 \\
E_e(4) &= \frac{1}{3} - c_0^2 - c_1^2 - c_3^2 - c_7^2 = \frac{1}{768} = 0.001302
\end{aligned}$$

the corresponding trigonometric Fourier series found in Prob. 6.5-6 are 0.0833, 0.03267, 0.02, 0.014378. Clearly, the Walsh Fourier series gives smaller error than the corresponding trigonometric Fourier series for the same number of terms in the approximation.

6.M-1. (a) To determine a suitable set of $N = 10$ frequencies ω_n, we first determine ten points logarithmically spaced from 1 to 100.

```
>> f = logspace(0,2,10)
f =    1.0000    1.6681    2.7826    4.6416    7.7426
      12.9155   21.5443   35.9381   59.9484  100.0000
```

The problem with these points is that they are not all rational, and the resulting signal $m(t)$ is thus aperiodic. Truncating to the four decimal places shown makes the frequencies rational, but the resulting period T_0 is excessively long. An approximately logarithmic sequence that results in smaller T_0 is generated by rounding the logarithmic frequencies to the nearest tenths of a hertz.

```
>> f = round(10*logspace(0,2,10))/10
f =    1.0000    1.7000    2.8000    4.6000    7.7000
      12.9000   21.5000   35.9000   59.9000  100.0000
```

With these frequencies, the signal $m(t) = \sum_{n=1}^{N} \cos(\omega_n t + \theta_n)$ has period $T_0 = 10$. Thus, one reasonable choice of frequencies is

$\omega_n = 2\pi[1, 1.7, 2.8, 4.6, 7.7, 12.9, 21.5, 35.9, 59.9, 100]$ for which $m(t)$ has period $T_0 = 10$.

MATLAB is used to plot $m(t)$ when all θ_n are set to zero.

```
>> m = inline('sum(cos(omega*t+theta*ones(size(t))))',...
       'theta','t','omega');
>> omega = 2*pi*f'; theta = zeros(size(omega));
>> t = (-5:.01:5); plot(t,m(theta,t,omega),'k');
>> xlabel('t [sec]'); ylabel('m(t) [volts]');
```

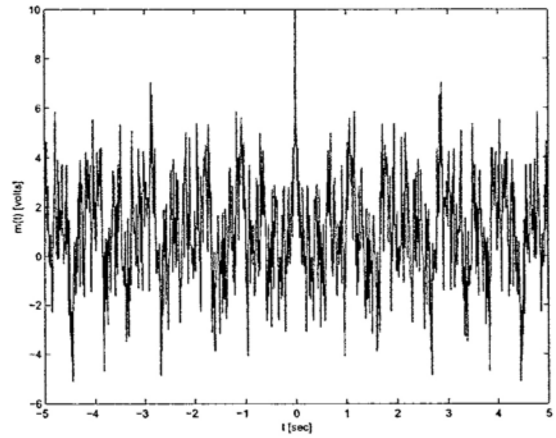


Figure S6.M-1a: Signal $m(t)$ with log-spaced ω_n and $\theta_n = 0$.

As expected, this worst-case version of $m(t)$ has a maximum amplitude of 10, which is also the number of sinusoids comprising the signal.

(b) MATLAB is used to try and find an optimal set of phases θ_n that minimizes the maximum amplitude of $m(t)$. The procedure followed is the same as that presented in MATLAB Session 6. To proceed, the code from part 6.M-1a needs to be first executed.

```
>> maxmagm = inline('max(abs(sum(cos(omega*t+theta*ones(size(t))))))',...
       'theta','t','omega');
>> t = [-5:.001 :5];
>> rand('state',0); theta_init = 2*pi*rand(N,1);
>> theta_opt = fminsearch(maxmagm,theta_init,[],t,omega);
>> mmag = max(abs(m(theta_opt,t,omega)))
mmag = 6.7711
```

The result of 6.7711 shows a reasonable reduction in maximum amplitude from the worst-case value of 10. Notice, a finely-spaced time vector `t` is required for the function `fminsearch` to determine a reliable result.

To make sure the result is good and not just a local minimum, the sequence is run again with a different initial guess for the phases.

```
>> theta_init = 2*pi*rand(N,1);
>> theta_opt = fminsearch(maxmagm,theta_init,[],t,omega);
>> mmag = max(abs(m(theta_opt,t,omega)))
mmag = 6.6854
```

Although the second result coincides well with the first, it is not exactly the same. To be safe, the sequence is therefore run several times, and the best solution is preserved.

```
>> mmag_opt = mmag; mmag = [mmag,zeros(1,9)];
>> for trial = 2:10;
>>     theta_init = 2*pi*rand(N,1);
>>     theta = fminsearch(maxmagm,theta_init,[],t,omega);
>>     mmag(trial) = max(abs(m(theta,t,omega)))
>>     if (mmag(trial)<mmag_opt),
>>         theta_opt = theta'; mmag_opt = mmag(trial);
>>     end
>> end
>> mmag, theta_opt
mmag =       6.6854  6.7069  6.6568  6.6421  6.7301
             6.4846  6.5906  6.5294  6.5292  6.5888
theta_opt = 2.7839  5.7162  5.1883  4.2464  5.1741
             4.0850  2.2366  1.9871  1.6831  3.7462
```

Thus, a good (but unlikely globally best) choice of phases is

$\theta_n = 2\pi[2.7839, 5.7162, 5.1883, 4.2464, 5.1741, 4.0850, 2.2366, 1.9871, 1.6831, 3.7462]$.

In this case, the maximum value of $m(t)$ is 6.4846, as shown in Figure 6.M-1b.

```
>> plot(t,m(theta_opt,t,omega),'k');
>> xlabel('t [sec]'); ylabel('m(t) [volts]');
```

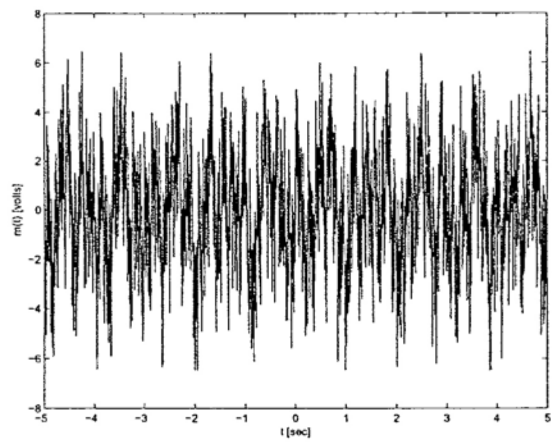


Figure S6.M-1b: Signal $m(t)$ with log-spaced ω_n and optimized phases.

(c) For environments with $1/f$ noise, it is appropriate to have lower frequency components have greater strength than higher frequency components. One simple possibility is to adjust the magnitude of each sinusoidal component to match the noise power at that frequency. In this way, the signal-to-noise ratio is kept constant for any frequency bin of the signal.

$$m(t) = \sum_{n=1}^{N} \frac{k}{\sqrt{\omega_n}} \cos(\omega_n t + \theta_n).$$

The constant k is selected to achieve the final desired signal power for the entire signal $m(t)$.

Chapter 7 Solutions

7.1-1.

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt = \int_{-\infty}^{\infty} x(t)\cos\omega t\,dt - j\int_{-\infty}^{\infty} x(t)\sin\omega t\,dt$$

If $x(t)$ is an even function of t, $x(t)\sin\omega t$ is an odd function of t, and the second integral vanishes. Moreover, $x(t)\cos\omega t$ is an even function of t, and the first integral is twice the integral over the interval 0 to ∞. Thus when $x(t)$ is even

$$X(\omega) = 2\int_{0}^{\infty} x(t)\cos\omega t\,dt \tag{1}$$

Similar argument shows that when $x(t)$ is odd

$$X(\omega) = -2j\int_{0}^{\infty} x(t)\sin\omega t\,dt \tag{2}$$

If $x(t)$ is also real (in addition to being even), the integral (1) is real. Moreover from (1)

$$X(-\omega) = 2\int_{0}^{\infty} x(t)\cos\omega t\,dt = X(\omega)$$

Hence $X(\omega)$ is real and even function of ω. Similar arguments can be used to prove the rest of the properties.

7.1-2.

$$\begin{aligned} x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|e^{j\angle X(\omega)}e^{j\omega t}\,d\omega \\ &= \frac{1}{2\pi}\left[\int_{-\infty}^{\infty} |X(\omega)|\cos[\omega t + \angle X(\omega)]\,d\omega + j\int_{-\infty}^{\infty} |X(\omega)|\sin[\omega t + \angle X(\omega)]\,d\omega\right] \end{aligned}$$

Since $|X(\omega)|$ is an even function and $\angle X(\omega)$ is an odd function of ω, the integrand in the second integral is an odd function of ω, and therefore vanishes. Moreover the integrand in the first integral is an even function of ω, and therefore

$$x(t) = \frac{1}{\pi}\int_{0}^{\infty} |X(\omega)|\cos[\omega t + \angle X(\omega)]\,d\omega$$

7.1-3. (a) Because $x(t) = x_o(t) + x_e(t)$ and $e^{-j\omega t} = \cos\,\omega t + j\sin\,\omega t$

$$X(\omega) = \int_{-\infty}^{\infty} [x_o(t) + x_e(t)]e^{-j\omega t}dt$$

$$= \int_{-\infty}^{\infty} [x_o(t) + x_e(t)] \cos \omega t dt - j \int_{-\infty}^{\infty} [x_o(t) + x_e(t)] \sin \omega t dt$$

Because $x_e(t) \cos \omega t$ and $x_o(t) \sin \omega t$ are even functions and $x_o(t) \cos \omega t$ and $x_e(t) \sin \omega t$ are odd functions of t, these integrals [properties in Eqs. (B.43), p. 38] reduce to

$$X(\omega) = 2\int_0^{\infty} x_e(t) \cos \omega t \, dt - 2j \int_0^{\infty} x_o(t) \sin \omega t \, dt \tag{1}$$

Also, from the results of Prob. 7.1-1, we have

$$\mathcal{F}\{x_e(t)\} = 2\int_0^{\infty} x_e(t) \cos \omega t \, dt \quad \text{and} \quad \mathcal{F}\{x_0(t)\} = -2j \int_0^{\infty} x_o(t) \sin \omega t \, dt \tag{2}$$

From Eqs. (1) and (2), the desired result follows.

(b) We can express $u(t)$ in terms of its even and odd components as follows

$$\begin{aligned} u(t) &= \frac{1}{2}[u(t) + u(-t)] + \frac{1}{2}[u(t) - u(-t)] \\ &= \underbrace{\frac{1}{2}}_{x_e(t)} + \underbrace{\frac{1}{2}\text{sgn}(t)}_{x_o(t)} \end{aligned}$$

and

$$X_e(\omega) = \pi\delta(\omega) \qquad \text{and} \qquad X_o(\omega) = \frac{1}{j\omega}$$

Clearly, $X_e(\omega)$ is the real part and $X_o(\omega)$ is the odd part of $X(\omega)$.
We follow the same procedure for $x(t) = e^{-at}u(t)$.

$$e^{-at}u(t) = \underbrace{\frac{1}{2}[e^{-at}u(t) + e^{-at}u(-t)]}_{x_e(t)} + \underbrace{\frac{1}{2}[e^{-at}u(t) - e^{-at}u(-t)]}_{x_o(t)}$$

Also

$$X_e(\omega) = \frac{1}{2}\left[\frac{1}{j\omega + a} - \frac{1}{j\omega - a}\right] = \frac{2a}{\omega^2 + a^2}$$

and

$$X_o(\omega) = \frac{1}{2}\left[\frac{1}{j\omega + a} + \frac{1}{j\omega - a}\right] = \frac{2j\omega}{\omega^2 + a^2}$$

Clearly, $X_e(\omega)$ is the real part and $X_o(\omega)$ is the odd part of $X(\omega)$.

7.1-4. (a)

$$X(\omega) = \int_0^T e^{-at}e^{-j\omega t} \, dt = \int_0^T e^{-(j\omega + a)t} \, dt = \frac{1 - e^{-(j\omega + a)T}}{j\omega + a}$$

(b)

$$X(\omega) = \int_0^T e^{at}e^{-j\omega t} \, dt = \int_0^T e^{-(j\omega - a)} \, dt = \frac{1 - e^{-(j\omega - a)T}}{j\omega - a}$$

7.1-5. (a)

$$X(\omega) = \int_0^1 4e^{-j\omega t}\,dt + \int_1^2 2e^{-j\omega t}\,dt = \frac{4 - 2e^{-j\omega} - 2e^{-j2\omega}}{j\omega}$$

(b)

$$X(\omega) = \int_{-\tau}^{0} -\frac{t}{\tau}e^{-j\omega t}\,dt + \int_0^{\tau} \frac{t}{\tau}e^{-j\omega t}\,dt = \frac{2}{\tau\omega^2}[\cos\omega\tau + \omega\tau\sin\omega\tau - 1]$$

This result could also be derived by observing that $x(t)$ is an even function. Therefore from the result in Prob. 7.1-1

$$X(\omega) = \frac{2}{\tau}\int_0^{\tau} t\cos\omega t\,dt = \frac{2}{\tau\omega^2}[\cos\omega\tau + \omega\tau\sin\omega\tau - 1]$$

7.1-6. (a)

$$\begin{aligned} x(t) &= \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0} \omega^2 e^{j\omega t}\,d\omega = \frac{1}{2\pi}\,\frac{e^{j\omega t}}{(jt)^3}\left[-\omega^2t^2 - 2j\omega t + 2\right]\Big|_{-\omega_0}^{\omega_0} \\ &= \frac{(\omega_0^2t^2 - 2)\sin\omega_0 t + 2\omega_0 t\cos\omega_0 t}{\pi t^3} \end{aligned}$$

(b) The derivation can be simplified by observing that $X(\omega)$ can be expressed as a sum of two gate functions $X_1(\omega)$ and $X_2(\omega)$ as shown in Figure S7.1-6. Therefore

$$x(t) = \frac{1}{2\pi}\int_{-2}^{2}[X_1(\omega)+X_2(\omega)]e^{j\omega t}\,d\omega = \frac{1}{2\pi}\left\{\int_{-2}^{2} e^{j\omega t}\,d\omega + \int_{-1}^{1} e^{j\omega t}\,d\omega\right\} = \frac{\sin 2t + \sin t}{\pi t}$$

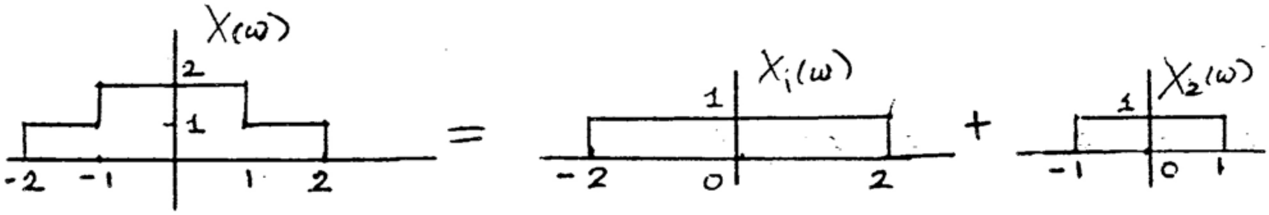


Figure S7.1-6

7.1-7. (a)

$$\begin{aligned} x(t) &= \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} \cos\omega\, e^{j\omega t}\,d\omega \\ &= \frac{e^{j\omega t}}{2\pi(1-t^2)}\left\{jt\cos\omega + \sin\omega\right\}_{-\pi/2}^{\pi/2} \\ &= \frac{1}{\pi(1-t^2)}\cos\left(\frac{\pi t}{2}\right) \end{aligned}$$

(b)

$$x(t) = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} X(\omega)e^{j\omega t}\,d\omega = \frac{1}{2\pi}\left[\int_{-\pi/2}^{\pi/2} X(\omega)\cos\omega t\,d\omega + j\int_{-\pi/2}^{\pi/2} X(\omega)\sin\omega t\,d\omega\right]$$

Because $X(\omega)$ is even function, the second integral on the right-hand side vanishes. Also the integrand of the first term is an even function. Therefore

$$\begin{aligned} x(t) &= \frac{1}{\pi}\int_0^{\pi/2} \frac{\omega}{\omega_0}\cos t\omega\, d\omega = \frac{1}{\pi\omega_0}\left[\frac{\cos t\omega + t\omega\sin t\omega}{t^2}\right]_0^{\omega_0} \\ &= \frac{1}{\pi\omega_0 t^2}[\cos\omega_0 t + \omega_0 t\sin\omega_0 t - 1] \end{aligned}$$

7.1-8.

$$\begin{aligned} X(\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \\ \text{Hence} \quad X(0) &= \int_{-\infty}^{\infty} x(t)dt \\ \text{Also} \quad x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega \\ \text{and} \quad x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)d\omega \end{aligned}$$

Because $\operatorname{sinc}(t) \leftrightarrow \pi\operatorname{rect}(\frac{\omega}{2})$

$$\pi\operatorname{rect}(0) = \pi = \int_{-\infty}^{\infty}\operatorname{sinc}(t)dt$$

Also $\operatorname{sinc}^2(t) \leftrightarrow \pi\Delta(\frac{\omega}{4})$

$$\pi\Delta(0) = \pi = \int_{-\infty}^{\infty}\operatorname{sinc}^2(t)dt$$

7.2-1. Figure S7.2-1 shows the plots of various functions. The function in part (a) is a gate function centered at the origin and of width 2. The function in part (b) can be expressed as $\Delta\left(\frac{\omega}{100/3}\right)$. This is a triangle pulse centered at the origin and of width 100/3. The function in part (c) is a gate function $\operatorname{rect}(\frac{t}{8})$ delayed by 10. In other words it is a gate pulse centered at $t = 10$ and of width 8. The function in part (d) is a sinc pulse centered at the origin and the first zero occurring at $\frac{\pi\omega}{5} = \pi$, that is at $\omega = 5$. The function in part (e) is a sinc pulse $\operatorname{sinc}(\frac{\omega}{5})$ delayed by 10π. For the sinc pulse $\operatorname{sinc}(\frac{\omega}{5})$, the first zero occurs at $\frac{\omega}{5} = \pi$, that is at $\omega = 5\pi$. Therefore the function is a sinc pulse centered at $\omega = 10\pi$ and its zeros spaced at intervals of 5π as shown in the figure S7.2-1e. The function in part (f) is a product of a gate pulse (centered at the origin) of width 10π and a sinc pulse (also centered at the origin) with zeros spaced at intervals of 5π. This results in the sinc pulse truncated beyond the interval $\pm 5\pi$ ($|t| \geq 5\pi$) as shown in Fig. f.

7.2-2.

$$\begin{aligned} X(\omega) &= \int_{4.5}^{5.5} e^{-j\omega t}\,dt = -\frac{1}{j\omega}e^{-j\omega t}\Big|_{4.5}^{5.5} = \frac{1}{j\omega}[e^{-j4.5\omega} - e^{-j5.5\omega}] \\ &= \frac{e^{-j5\omega}}{j\omega}[e^{j\omega/2} - e^{-j\omega/2}] = \frac{e^{-j5\omega}}{j\omega}\left[2j\sin\frac{\omega}{2}\right] \\ &= \operatorname{sinc}\left(\frac{\omega}{2}\right)e^{-j5\omega} \end{aligned}$$

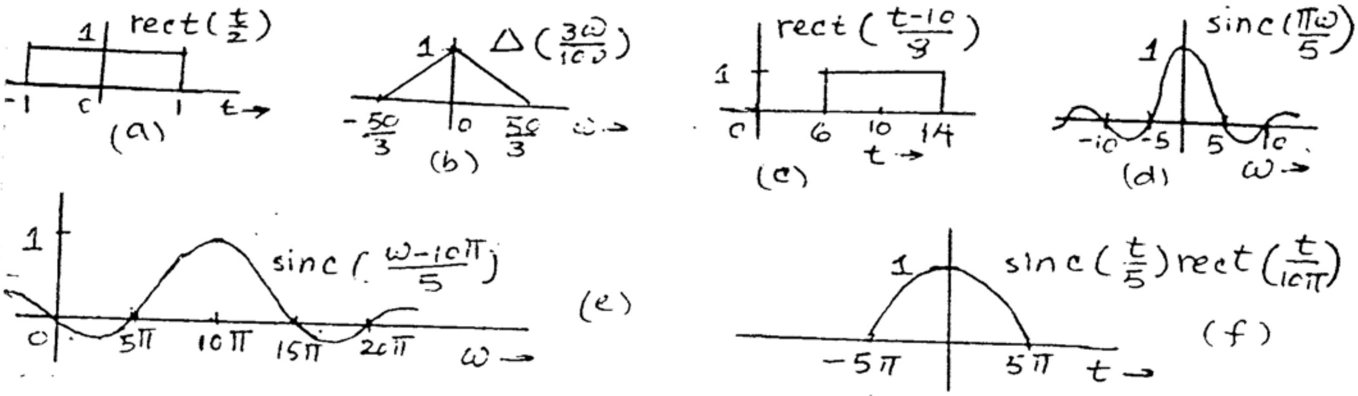


Figure S7.2-1

7.2-3.

$$
\begin{aligned}
x(t) &= \frac{1}{2\pi}\int_{10-\pi}^{10+\pi} e^{j\omega t}\,d\omega = \left.\frac{e^{j\omega t}}{2\pi(j\omega)}\right|_{10-\pi}^{10+\pi} = \frac{1}{j2\pi\omega}\left[e^{j(10+\pi)t} - e^{j(10-\pi)t}\right] \\
&= \frac{e^{j10t}}{j2\pi\omega}[2j\sin\pi t] = \text{sinc}(\pi t)e^{j10t}
\end{aligned}
$$

7.2-4. (a)

$$
\begin{aligned}
x(t) &= \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0} e^{-j\omega t_0}e^{j\omega t}\,d\omega = \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0} e^{j\omega(t-t_0)}\,d\omega \\
&= \left.\frac{1}{(2\pi)j(t-t_0)}e^{j\omega(t-t_0)}\right|_{-\omega_0}^{\omega_0} = \frac{\sin\omega_0(t-t_0)}{\pi(t-t_0)} = \frac{\omega_0}{\pi}\text{sinc}[\omega_0(t-t_0)]
\end{aligned}
$$

(b)

$$
\begin{aligned}
x(t) &= \frac{1}{2\pi}\left[\int_{-\omega_0}^{0} je^{j\omega t}\,d\omega + \int_0^{\omega_0} -je^{j\omega t}\,d\omega\right] \\
&= \left.\frac{1}{2\pi t}e^{j\omega t}\right|_{-\omega_0}^{0} - \left.\frac{1}{2\pi t}e^{j\omega t}\right|_0^{\omega_0} = \frac{1-\cos\omega_0 t}{\pi t}
\end{aligned}
$$

7.2-5. (a) When $a > 0$, we cannot find the Fourier transform of $e^{at}u(t)$ by setting $s = j\omega$ in the Laplace transform of $e^{at}u(t)$ because the ROC is $Re s > a$, which does not include the $j\omega$-axis.

(b) The Laplace transform of $x(t)$ is

$$
X(s) = \int_0^T e^{at}e^{-st}dt = \int_0^T e^{-(s-a)t}dt = \frac{1}{s-a}\left[1 - e^{-(s-a)T}\right]
$$

Interestingly, because $x(t)$ has a finite width, the ROC of its $X(s)$ is the entire s-plane, which includes $j\omega$-axis. Hence, the Fourier transform

$$
X(\omega) = X(s)\,|_{s=j\omega} = \frac{1}{s-a}\left[1 - e^{-(j\omega-a)T}\right]
$$

To verify this, we find the Fourier transform of $x(t)$

$$X(\omega) = \int_0^T e^{at}e^{-j\omega t}dt = \int_0^T e^{-(j\omega - a)t}dt = \frac{1}{j\omega - a}\left[1 - e^{-(j\omega-a)T}\right]$$

Which agrees with $X(j\omega)$

7.3-1. (a)

$$\underbrace{u(t)}_{x(t)} \Longleftrightarrow \underbrace{\pi\delta(\omega) + \frac{1}{j\omega}}_{X(\omega)}$$

Application of duality property yields

$$\underbrace{\pi\delta(t) + \frac{1}{jt}}_{X(t)} \Longleftrightarrow \underbrace{2\pi u(-\omega)}_{2\pi x(-\omega)}$$

or

$$\frac{1}{2}\left[\delta(t) + \frac{1}{j\pi t}\right] \Longleftrightarrow u(-\omega)$$

Application of Eq. (4.35) yields

$$\frac{1}{2}\left[\delta(-t) - \frac{1}{j\pi t}\right] \Longleftrightarrow u(\omega)$$

But $\delta(t)$ is an even function, that is $\delta(-t) = \delta(t)$, and

$$\frac{1}{2}[\delta(t) + \frac{j}{\pi t}] \Longleftrightarrow u(\omega)$$

(b)

$$\underbrace{\cos\,\omega_0 t}_{x(t)} \Longleftrightarrow \underbrace{\pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]}_{X(\omega)}$$

Application of duality property yields

$$\underbrace{\pi[\delta(t + \omega_0) + \delta(t - \omega_0)]}_{X(t)} \Longleftrightarrow \underbrace{2\pi\cos(-\omega_0\omega)}_{2\pi x(-\omega)} = 2\pi\cos(\omega_0\omega)$$

Setting $\omega_0 = T$ yields

$$\delta(t + T) + \delta(t - T) \Longleftrightarrow 2\cos\,T\omega$$

(c)

$$\underbrace{\sin\,\omega_0 t}_{x(t)} \Longleftrightarrow \underbrace{j\pi[\delta(\omega + \omega_0) - \delta(\omega - \omega_0)]}_{X(\omega)}$$

Application of duality property yields

$$\underbrace{j\pi[\delta(t + \omega_0) - \delta(t - \omega_0)]}_{X(t)} \Longleftrightarrow \underbrace{2\pi\sin(-\omega_0\omega)}_{2\pi x(-\omega)} = -2\pi\,\sin(\omega_0\omega)$$

Setting $\omega_0 = T$ yields

$$\delta(t+T) - \delta(t-T) \Longleftrightarrow 2j\sin T\omega$$

7.3-2. Refer to the solution of Prob. 1.2-3 for description of these signals.

(a)

$$x_1(t) = x(t+1) + x(1-t)$$

$$x(t+1) \Longleftrightarrow X(\omega)e^{j\omega}$$

Time inversion of $x(t+1)$ yields $x(-t+1)$, Hence

$$x(1-t) \Longleftrightarrow X(-\omega)e^{-j\omega}$$

$$\text{Hence} \qquad x_1(t) \Longleftrightarrow X(\omega)e^{j\omega} + X(-\omega)e^{-j\omega}$$

(b)

$$x_2(t) = x\left(\frac{t+1}{2}\right) + x\left(\frac{1-t}{2}\right)$$

$$x\left(t+\frac{1}{2}\right) \Longleftrightarrow X(\omega)e^{\frac{j\omega}{2}}$$

$$\text{and} \qquad x\left(\frac{t+1}{2}\right) = x\left(\frac{t}{2}+\frac{1}{2}\right) \Longleftrightarrow 2X(2\omega)e^{j\omega}$$

$$\text{Therefore} \qquad x\left(\frac{1-t}{2}\right) \Longleftrightarrow 2X(-2\omega)e^{-j\omega}$$

$$\text{Hence} \qquad x_2(t) \Longleftrightarrow 2\left[X(2\omega)e^{j\omega} + X(-2\omega)e^{-j\omega}\right]$$

(c)

$$x_3(t) = x\left(\frac{t+2}{4}\right) + x\left(\frac{2-t}{4}\right) + x\left(\frac{t}{2}\right) + x\left(\frac{-t}{2}\right)$$

$$x\left(\frac{t+2}{4}\right) \Longleftrightarrow 4X(4\omega)e^{j2\omega}$$

$$x\left(\frac{2-t}{4}\right) \Longleftrightarrow 4X(-4\omega)e^{-j2\omega}$$

$$x\left(\frac{t}{2}\right) \Longleftrightarrow 2X(2\omega) \qquad \text{and} \qquad x\left(-\frac{t}{2}\right) \Longleftrightarrow 2X(-2\omega)$$

$$\text{Hence} \qquad x_3(t) \Longleftrightarrow 4\left[X(4\omega)e^{j2\omega} + X(-4\omega)e^{-j2\omega} + 2X(2\omega) + 2X(-2\omega)\right]$$

(d)

$$x_4(t) = \frac{4}{3}x\left(\frac{t+2}{2}\right) + \frac{4}{3}x\left(\frac{2-t}{2}\right) - \frac{1}{3}x\left(\frac{t+2}{4}\right) - \frac{1}{3}x\left(\frac{2-t}{4}\right)$$

$$x\left(\frac{t+2}{2}\right) \Longleftrightarrow 2X(2\omega)e^{j2\omega} \qquad \text{and} \qquad x\left(\frac{2-t}{2}\right) \Longleftrightarrow 2X(-2\omega)e^{-j2\omega}$$

$$x\left(\frac{t+2}{4}\right) \Longleftrightarrow 4X(4\omega)e^{j2\omega} \qquad \text{and} \qquad x\left(\frac{2-t}{4}\right) \Longleftrightarrow 4X(-4\omega)e^{-j2\omega}$$

Hence $\quad x_4(t) \Longleftrightarrow \frac{8}{3}\left[X(2\omega)e^{j2\omega} + X(-2\omega)e^{-j2\omega}\right] - \frac{4}{3}\left[X(4\omega)e^{j2\omega} + X(-4\omega)e^{-j2\omega}\right]$

(e)

$$x_5(t)x(t+0.5) + x(0.5-t) + x(t+1.5) + x(1.5-t)$$

Hence $\quad x_5(t) \Longleftrightarrow X(\omega)e^{\frac{j\omega}{2}} + X(-\omega)e^{\frac{-j\omega}{2}} + X(\omega)e^{1.5j\omega} + X(-\omega)e^{-1.5j\omega}$

In all these expressions, we substitute

$$X(\omega) = \frac{1}{\omega^2}\left[ej\omega - j\omega e^{j\omega} - 1\right]$$

7.3-3. (a)

$$x(t) = \text{rect}\left(\frac{t+T/2}{T}\right) - \text{rect}\left(\frac{t-T/2}{T}\right)$$

$$\begin{aligned}
\text{rect}\left(\frac{t}{T}\right) &\Longleftrightarrow T\text{sinc}\left(\frac{\omega T}{2}\right) \\
\text{rect}\left(\frac{t \pm T/2}{T}\right) &\Longleftrightarrow T\text{sinc}\left(\frac{\omega T}{2}\right)e^{\pm j\omega T/2}
\end{aligned}$$

$$\begin{aligned}
X(\omega) &= T\text{sinc}\left(\frac{\omega T}{2}\right)[e^{j\omega T/2} - e^{-j\omega T/2}] \\
&= 2jT\text{sinc}\left(\frac{\omega T}{2}\right)\sin\frac{\omega T}{2} \\
&= \frac{j4}{\omega}\sin^2\left(\frac{\omega T}{2}\right)
\end{aligned}$$

(b) From Figure S7.3-3b we verify that

$$x(t) = \sin t\, u(t) + \sin(t-\pi)u(t-\pi)$$

Note that $\sin(t-\pi)u(t-\pi)$ is $\sin\, t\,u(t)$ delayed by π. Now, $\sin t\, u(t) \Longleftrightarrow \frac{\pi}{2j}[\delta(\omega-1) - \delta(\omega+1)] + \frac{1}{1-\omega^2}$ and

$$\sin(t-\pi)u(t-\pi) \Longleftrightarrow \left\{\frac{\pi}{2j}[\delta(\omega-1) - \delta(\omega+1)] + \frac{1}{1-\omega^2}\right\}e^{-j\pi\omega}$$

Therefore

$$X(\omega) = \left\{\frac{\pi}{2j}[\delta(\omega-1) - \delta(\omega+1)] + \frac{1}{1-\omega^2}\right\}(1+e^{-j\pi\omega})$$

Recall that $g(x)\delta(x-x_0) = g(x_0)\delta(x-x_0)$. Therefore $\delta(\omega \pm 1)(1+e^{-j\pi\omega}) = 0$, and

$$X(\omega) = \frac{1}{1-\omega^2}(1+e^{-j\pi\omega})$$

(c) From Figure S7.3-3c we verify that

$$x(t) = \cos t\left[u(t) - u\left(t - \frac{\pi}{2}\right)\right] = \cos t\, u(t) - \cos t\, u\left(t - \frac{\pi}{2}\right)$$

But $\sin(t - \frac{\pi}{2}) = -\cos t$. Therefore

$$\begin{aligned} x(t) &= \cos t\, u(t) + \sin\left(t - \frac{\pi}{2}\right) u\left(t - \frac{\pi}{2}\right) \\ X(\omega) &= \frac{\pi}{2}[\delta(\omega-1) + \delta(\omega+1)] + \frac{j\omega}{1-\omega^2} + \left\{\frac{\pi}{2j}[\delta(\omega-1) - \delta(\omega+1)] + \frac{1}{1-\omega^2}\right\} e^{-j\pi\omega/2} \end{aligned}$$

Also because $g(x)\delta(x - x_0) = g(x_0)\delta(x - x_0)$,

$$\delta(\omega \pm 1)e^{-j\pi\omega/2} = \delta(\omega \pm 1)e^{\pm j\pi/2} = \pm j\delta(\omega \pm 1)$$

Therefore

$$X(\omega) = \frac{j\omega}{1-\omega^2} + \frac{e^{-j\pi\omega/2}}{1-\omega^2} = \frac{1}{1-\omega^2}[j\omega + e^{-j\pi\omega/2}]$$

(d)

$$\begin{aligned} x(t) &= e^{-at}[u(t) - u(t-T)] = e^{-at}u(t) - e^{-at}u(t-T) \\ &= e^{-at}u(t) - e^{-aT}e^{-a(t-T)}u(t-T) \\ X(\omega) &= \frac{1}{j\omega + a} - \frac{e^{-aT}}{j\omega + a}e^{-j\omega T} = \frac{1}{j\omega + a}[1 - e^{-(a+j\omega)T}] \end{aligned}$$

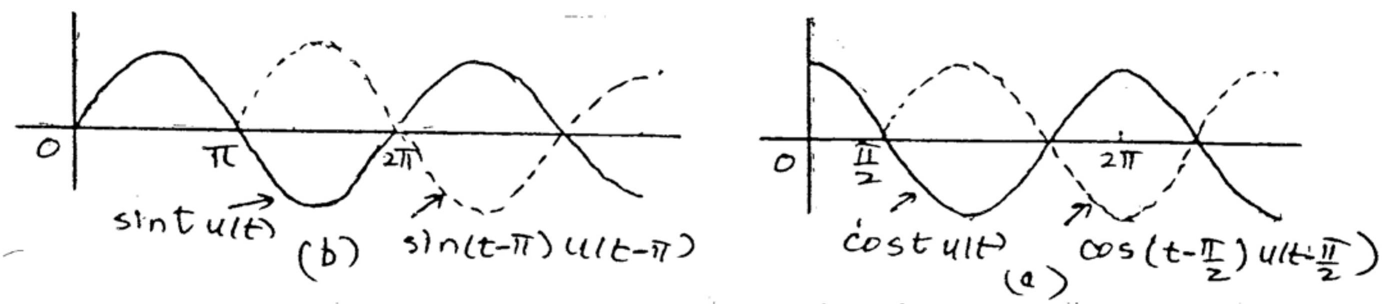


Figure S7.3-3

7.3-4. From time-shifting property

$$x(t \pm T) \Longleftrightarrow X(\omega)e^{\pm j\omega T}$$

Therefore

$$x(t+T) + x(t-T) \Longleftrightarrow X(\omega)e^{j\omega T} + X(\omega)e^{-j\omega T} = 2X(\omega)\cos\omega T$$

We can use this result to derive transforms of signals in Figure P7.3-4.

(a) Here $x(t)$ is a gate pulse as shown in Figure S7.3-4a.

$$x(t) = \text{rect}\left(\frac{t}{2}\right) \Longleftrightarrow 2\,\text{sinc}(\omega)$$

Also $T = 3$. The signal in Figure S7.3-4a is $x(t+3) + x(t-3)$, and

$$x(t+3) + x(t-3) \Longleftrightarrow 4\,\mathrm{sinc}(\omega)\cos 3\omega$$

(b) Here $x(t)$ is a triangular pulse shown in Figure S7.3-4b. From the Table 4.1 (pair 19)

$$x(t) = \Delta\left(\frac{t}{2}\right) \Longleftrightarrow \mathrm{sinc}^2\left(\frac{\omega}{2}\right)$$

Also $T = 3$. The signal in Figure P7.3-4b is $x(t+3) + x(t-3)$, and

$$x(t+3) + x(t-3) \Longleftrightarrow 2\,\mathrm{sinc}^2\left(\frac{\omega}{2}\right)\cos 3\omega$$

Figure S7.3-4

7.3-5. Frequency-shifting property states that

$$x(t)e^{\pm j\omega_0 t} \Longleftrightarrow X(\omega \mp \omega_0)$$

Therefore

$$x(t)\sin\omega_0 t = \frac{1}{2j}[x(t)e^{j\omega_0 t} + x(t)e^{-j\omega_0 t}] = \frac{1}{2j}[X(\omega - \omega_0) + X(\omega - \omega_0)]$$

Time-shifting property states that

$$x(t \pm T) \Longleftrightarrow X(\omega)e^{\pm j\omega T}$$

Therefore

$$x(t+T) - x(t-T) \Longleftrightarrow X(\omega)e^{j\omega T} - X(\omega)e^{-j\omega T} = 2jX(\omega)\sin\omega T$$

and

$$\frac{1}{2j}[x(t+T) - x(t-T)] \Longleftrightarrow X(\omega)\sin T\omega$$

The signal in Figure P7.3-5 is $x(t+3) - x(t-3)$ where

$$x(t) = \mathrm{rect}\left(\frac{t}{2}\right) \Longleftrightarrow 2\mathrm{sinc}(\omega)$$

Therefore

$$x(t+3) - x(t-3) \Longleftrightarrow 2j[2\mathrm{sinc}(\omega)\sin 3\omega] = 4j\,\mathrm{sinc}(\omega)\sin 3\omega$$

7.3-6. (a) The signal $x(t)$ in this case is a triangle pulse $\Delta(\frac{t}{2\pi})$ (Figure S7.3-6) multiplied by $\cos 10t$.

$$x(t) = \Delta\left(\frac{t}{2\pi}\right)\cos 10t$$

Also from Table 4.1 (pair 19) $\Delta(\frac{t}{2\pi}) \Longleftrightarrow \pi\,\text{sinc}^2(\frac{\pi\omega}{2})$ From the modulation property (4.41), it follows that

$$x(t) = \Delta\left(\frac{t}{2\pi}\right)\cos 10t \Longleftrightarrow \frac{\pi}{2}\left\{\text{sinc}^2\left[\frac{\pi(\omega-10)}{2}\right] + \text{sinc}^2\left[\frac{\pi(\omega+10)}{2}\right]\right\}$$

The Fourier transform in this case is a real function and we need only the amplitude spectrum in this case as shown in Figure S7.3-6a.

(b) The signal $x(t)$ here is the same as the signal in (a) delayed by 2π. From time shifting property, its Fourier transform is the same as in part (a) multiplied by $e^{-j\omega(2\pi)}$. Therefore

$$X(\omega) = \frac{\pi}{2}\left\{\text{sinc}^2\left[\frac{\pi(\omega-10)}{2}\right] + \text{sinc}^2\left[\frac{\pi(\omega+10)}{2}\right]\right\}e^{-j2\pi\omega}$$

The Fourier transform in this case is the same as that in part (a) multiplied by $e^{-j2\pi\omega}$. This multiplying factor represents a linear phase spectrum $-2\pi\omega$. Thus we have an amplitude spectrum [same as in part (a)] as well as a linear phase spectrum $\angle X(\omega) = -2\pi\omega$ as shown in Figure S7.3-6b. the amplitude spectrum in this case as shown in Figure S7.3-6b.
Note: In the above solution, we first multiplied the triangle pulse $\Delta(\frac{t}{2\pi})$ by $\cos 10t$ and then delayed the result by 2π. This means the signal in (b) is expressed as $\Delta(\frac{t-2\pi}{2\pi})\cos 10(t-2\pi)$.
We could have interchanged the operation in this particular case, that is, the triangle pulse $\Delta(\frac{t}{2\pi})$ is first delayed by 2π and then the result is multiplied by $\cos 10t$. In this alternate procedure, the signal in (b) is expressed as $\Delta(\frac{t-2\pi}{2\pi})\cos 10t$.
This interchange of operation is permissible here only because the sinusoid $\cos 10t$ executes integral number of cycles in the interval 2π. Because of this both the expressions are equivalent since $\cos 10(t-2\pi) = \cos 10t$.

(c) In this case the signal is identical to that in (b), except that the basic pulse is $\text{rect}(\frac{t}{2\pi})$ instead of a triangle pulse $\Delta(\frac{t}{2\pi})$. Now

$$\text{rect}\left(\frac{t}{2\pi}\right) \Longleftrightarrow 2\pi\,\text{sinc}(\pi\omega)$$

Using the same argument as for part (b), we obtain

$$X(\omega) = \pi\{\text{sinc}[\pi(\omega+10)] + \text{sinc}[\pi(\omega-10)]\}e^{-j2\pi\omega}$$

7.3-7. (a)

$$X(\omega) = \text{rect}\left(\frac{\omega-4}{2}\right) + \text{rect}\left(\frac{\omega+4}{2}\right)$$

Also

$$\frac{1}{\pi}\text{sinc}(t) \Longleftrightarrow \text{rect}\left(\frac{\omega}{2}\right)$$

Therefore

$$x(t) = \frac{2}{\pi}\text{sinc}(t)\cos 4t$$

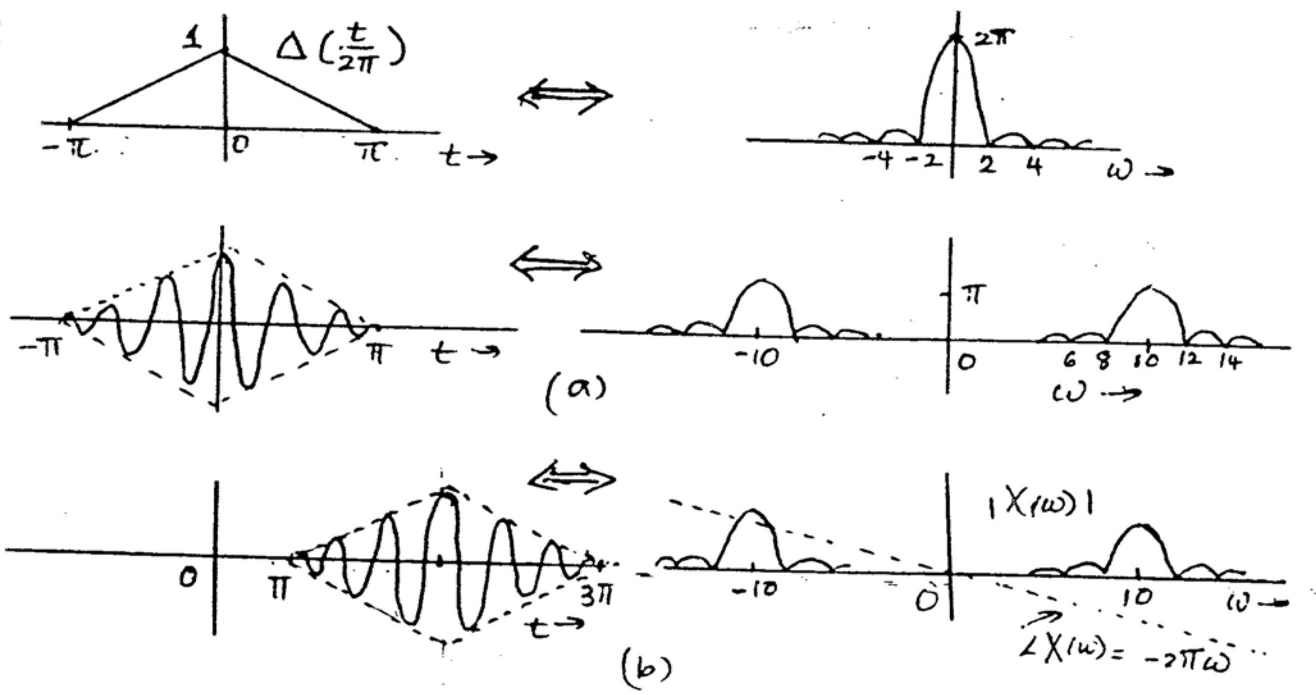


Figure S7.3-6

(b)

$$X(\omega) = \triangle\left(\frac{\omega+4}{4}\right) + \triangle\left(\frac{\omega-4}{4}\right)$$

Also

$$\frac{1}{\pi}\mathrm{sinc}^2(t) \Longleftrightarrow \triangle\left(\frac{\omega}{4}\right)$$

Therefore

$$x(t) = \frac{2}{\pi}\mathrm{sinc}^2(t)\cos 4t$$

7.3-8. (a)

$$e^{\lambda t}u(t) \Longleftrightarrow \frac{1}{j\omega - \lambda} \quad \text{and} \quad u(t) \Longleftrightarrow \pi\delta(\omega) + \frac{1}{j\omega}$$

If $x(t) = e^{\lambda t}u(t) * u(t)$, then

$$\begin{aligned} X(\omega) &= \left(\frac{1}{j\omega-\lambda}\right)\left(\pi\delta(\omega) + \frac{1}{j\omega}\right) \\ &= \frac{\pi\delta(\omega)}{j\omega-\lambda} + \left[\frac{1}{j\omega(j\omega-\lambda)}\right] \\ &= -\frac{\pi}{\lambda}\delta(\omega) + \left[\frac{-\frac{1}{\lambda}}{j\omega} + \frac{\frac{1}{\lambda}}{j\omega-\lambda}\right] \quad \text{because } g(x)\delta(x) = g(0)\delta(x) \\ &= \frac{1}{\lambda}\left[\frac{1}{j\omega-\lambda} - \left(\pi\delta(\omega) + \frac{1}{j\omega}\right)\right] \end{aligned}$$

Taking the inverse transform of this equation yields

$$x(t) = \frac{1}{\lambda}(e^{\lambda t} - 1)u(t)$$

(b)

$$e^{\lambda_1 t}u(t) \Longleftrightarrow \frac{1}{j\omega - \lambda_1} \quad \text{and} \quad e^{\lambda_2 t}u(t) \Longleftrightarrow \frac{1}{j\omega - \lambda_2}$$

If $x(t) = e^{\lambda_1 t}u(t) * e^{\lambda_2 t}u(t)$, then

$$X(\omega) = \frac{1}{(j\omega - \lambda_1)(j\omega - \lambda_2)} = \frac{\frac{1}{\lambda_1 - \lambda_2}}{j\omega - \lambda_1} - \frac{\frac{1}{\lambda_1 - \lambda_2}}{j\omega - \lambda_2}$$

Therefore

$$x(t) = \frac{1}{\lambda_1 - \lambda_2}(e^{\lambda_1 t} - e^{\lambda_2 t})u(t)$$

(c)

$$e^{\lambda_1 t}u(t) \Longleftrightarrow \frac{1}{j\omega - \lambda_1} \quad \text{and} \quad e^{\lambda_2 t}u(-t) \Longleftrightarrow -\frac{1}{j\omega - \lambda_2}$$

If $x(t) = e^{\lambda_1 t}u(t) * e^{\lambda_2 t}u(-t)$, then

$$X(\omega) = \frac{-1}{(j\omega - \lambda_1)(j\omega - \lambda_2)} = \frac{\frac{1}{\lambda_2 - \lambda_1}}{j\omega - \lambda_1} - \frac{\frac{1}{\lambda_2 - \lambda_1}}{j\omega - \lambda_2}$$

Therefore

$$x(t) = \frac{1}{\lambda_2 - \lambda_1}[e^{\lambda_1 t}u(t) + e^{\lambda_2 t}u(-t)]$$

Note that because $\lambda_2 > 0$, the inverse transform of $\frac{-1}{j\omega - \lambda_2}$ is $e^{\lambda_2 t}u(-t)$ and not $-e^{\lambda_2 t}u(t)$. The Fourier transform of the latter does not exist because $\lambda_2 > 0$.

(d)

$$e^{\lambda_1 t}u(-t) \Longleftrightarrow -\frac{1}{j\omega - \lambda_1} \quad \text{and} \quad e^{\lambda_2 t}u(-t) \Longleftrightarrow -\frac{1}{j\omega - \lambda_2}$$

If $x(t) = e^{\lambda_1 t}u(-t) * e^{\lambda_2 t}u(-t)$, then

$$X(\omega) = \frac{1}{(j\omega - \lambda_1)(j\omega - \lambda_2)} = \frac{\frac{-1}{\lambda_2 - \lambda_1}}{j\omega - \lambda_1} - \frac{\frac{-1}{\lambda_2 - \lambda_1}}{j\omega - \lambda_2}$$

Therefore

$$x(t) = \frac{1}{\lambda_2 - \lambda_1}(e^{\lambda_1 t} - e^{\lambda_2 t})u(-t)$$

The remarks at the end of part (c) apply here also.

7.3-9. From the frequency convolution property, we obtain

$$x^2(t) \Longleftrightarrow \frac{1}{2\pi}X(\omega) * X(\omega)$$

Because of the width property of the convolution, the width of $X(\omega) * X(\omega)$ is twice the width of $X(\omega)$. Repeated application of this argument shows that the bandwidth of $x^n(t)$ is nB Hz (n times the bandwidth of $x(t)$).

7.3-10. (a)

$$X(\omega) = \int_{-T}^{0} e^{-j\omega t}\,dt - \int_{0}^{T} e^{-j\omega t}\,dt = -\frac{2}{j\omega}[1 - \cos\omega T] = \frac{j4}{\omega}\sin^2\left(\frac{\omega T}{2}\right)$$

(b)

$$x(t) = \text{rect}\left(\frac{t + T/2}{T}\right) - \text{rect}\left(\frac{t - T/2}{T}\right)$$

$$\begin{aligned} \text{rect}\left(\frac{t}{T}\right) &\iff T\text{sinc}\left(\frac{\omega T}{2}\right) \\ \text{rect}\left(\frac{t \pm T/2}{T}\right) &\iff T\text{sinc}\left(\frac{\omega T}{2}\right) e^{\pm j\omega T/2} \end{aligned}$$

$$\begin{aligned} X(\omega) &= T\text{sinc}\left(\frac{\omega T}{2}\right)[e^{j\omega T/2} - e^{-j\omega T/2}] \\ &= 2jT\text{sinc}\left(\frac{\omega T}{2}\right)\sin\frac{\omega T}{2} \\ &= \frac{j4}{\omega}\sin^2\left(\frac{\omega T}{2}\right) \end{aligned}$$

(c)

$$\frac{df}{dt} = \delta(t + T) - 2\delta(t) + \delta(t - T)$$

The Fourier transform of this equation yields

$$j\omega X(\omega) = e^{j\omega T} - 2 + e^{-j\omega T} = -2[1 - \cos\omega T] = -4\sin^2\left(\frac{\omega T}{2}\right)$$

Therefore

$$X(\omega) = \frac{j4}{\omega}\sin^2\left(\frac{\omega T}{2}\right)$$

7.3-11. (a)

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt \quad \text{and} \quad \frac{dF}{d\omega} = \frac{d}{d\omega}\int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

Changing the order of differentiation and integration yields

$$\frac{dF}{d\omega} = \int_{-\infty}^{\infty} \frac{d}{d\omega}(x(t)e^{-j\omega t}) = \int_{-\infty}^{\infty} [-jtx(t)]e^{-j\omega t}\,dt$$

Therefore

$$-jtx(t) \iff \frac{dF}{d\omega}$$

(b)

$$
\begin{aligned}
e^{-at}u(t) &\iff \frac{1}{j\omega + a} \\
-jte^{-at}u(t) &\iff \frac{d}{d\omega}\left(\frac{1}{j\omega + a}\right) = \frac{-j}{(j\omega + a)^2}
\end{aligned}
$$

$$
te^{-at}u(t) \iff \frac{1}{(j\omega + a)^2}
$$

7.4-1.

$$
H(\omega) = \frac{1}{j\omega + 1}
$$

(a)

$$
\begin{aligned}
X(\omega) &= \frac{1}{j\omega + 2} \\
Y(\omega) &= \frac{1}{(j\omega + 1)(j\omega + 2)} = \frac{1}{j\omega + 1} - \frac{1}{j\omega + 2} \\
y(t) &= (e^{-t} - e^{-2t})u(t)
\end{aligned}
$$

(b)

$$
\begin{aligned}
X(\omega) &= \frac{1}{j\omega + 1} \\
Y(\omega) &= \frac{1}{(j\omega + 1)^2} \\
y(t) &= te^{-at}u(t)
\end{aligned}
$$

(c)

$$
\begin{aligned}
X(\omega) &= -\frac{1}{j\omega - 1} \\
Y(\omega) &= \frac{-1}{(j\omega + 1)(j\omega - 1)} = \frac{1/2}{j\omega + 1} - \frac{1/2}{j\omega - 1} \\
y(t) &= \frac{1}{2}e^{-t}u(t) + \frac{1}{2}e^{t}u(-t)
\end{aligned}
$$

(d)

$$
\begin{aligned}
X(\omega) &= \pi\delta(\omega) + \frac{1}{j\omega} \\
Y(\omega) &= \frac{1}{j\omega + 1}\left[\pi\delta(\omega) + \frac{1}{j\omega}\right] \\
&= \pi\delta(\omega) + \frac{1}{j\omega(j\omega + 1)} \qquad [\text{because} g(x)\delta(x) = g(0)\delta(x)] \\
&= \pi\delta(\omega) + \frac{1}{j\omega} - \frac{1}{j\omega + 1} \\
y(t) &= (1 - e^{-t})u(t)
\end{aligned}
$$

7.4-2. (a)

$$X(\omega) = \frac{1}{j\omega + 1} \qquad \text{and} \qquad H(\omega) = \frac{-1}{j\omega - 2}$$

and

$$Y(\omega) = \frac{-1}{(j\omega - 2)(j\omega + 1)} = \frac{1}{3}\left[\frac{1}{j\omega + 1} - \frac{1}{j\omega - 2}\right]$$

Therefore

$$y(t) = \frac{1}{3}\left[e^{-t}u(t) + e^{2t}u(-t)\right]$$

(b)

$$X(\omega) = \frac{-1}{j\omega - 1} \qquad \text{and} \qquad H(\omega) = \frac{-1}{j\omega - 2}$$

and

$$Y(\omega) = \frac{1}{(j\omega - 1)(j\omega - 2)} = \frac{-1}{j\omega - 1} - \frac{-1}{j\omega - 2}$$

Therefore

$$y(t) = [e^{t} - e^{2t}]u(-t)]$$

7.4-3.

$$X_1(\omega) = \text{sinc}\left(\tfrac{\omega}{20000}\right) \qquad \text{and} \quad X_2(\omega) = 1$$

Figure S7.4-3 shows $X_1(\omega)$, $X_2(\omega)$, $H_1(\omega)$ and $H_2(\omega)$. Now

$$\begin{aligned} Y_1(\omega) &= X_1(\omega)H_1(\omega) \\ Y_2(\omega) &= X_2(\omega)H_2(\omega) \end{aligned}$$

The spectra $Y_1(\omega)$ and $Y_2(\omega)$ are also shown in Figure S7.4-3. Because $y(t) = y_1(t)y_2(t)$, the frequency convolution property yields $Y(\omega) = Y_1(\omega) * Y_2(\omega)$. From the width property of convolution, it follows that the bandwidth of $Y(\omega)$ is the sum of bandwidths of $Y_1(\omega)$ and $Y_2(\omega)$. Because the bandwidths of $Y_1(\omega)$ and $Y_2(\omega)$ are 10 kHz, 5 kHz, respectively, the bandwidth of $Y(\omega)$ is 15 kHz.

7.4-4.

$$H(\omega) = 10^{-3}\,\text{sinc}\left(\tfrac{\omega}{2000}\right) \qquad \text{and} \quad P(\omega) = 0.5 \times 10^{-6}\,\text{sinc}^2\left(\tfrac{\omega}{4\times 10^6}\right)$$

The two spectra are sketched in Figure S7.4-4. It is clear that $H(\omega)$ is much narrower than $P(\omega)$, and we may consider $P(\omega)$ to be nearly constant of value $P(0) = 10^{-6}/2$ over the entire band of $H(\omega)$. Hence,

$$Y(\omega) = P(\omega)H(\omega) \approx P(0)H(\omega) = 0.5 \times 10^{-6}H(\omega) \quad \Longrightarrow \quad y(t) = 0.5 \times 10^{-6}h(t)$$

Recall that $h(t)$ is the unit impulse response of the system. Hence, the output $y(t)$ is equal to the system response to an input $0.5 \times 10^{-6}\delta(\omega) = A\delta(\omega)$.

7.4-5.

$$H(\omega) = 10^{-3}\,\text{sinc}\left(\tfrac{\omega}{2000}\right) \qquad \text{and} \quad P(\omega) = 0.5\,\text{sinc}^2\left(\tfrac{\omega}{4}\right)$$

The two spectra are sketched in Figure S7.4-5. It is clear that $P(\omega)$ is much narrower than $H(\omega)$, and we may consider $H(\omega)$ to be nearly constant of value $H(0) = 10^{-3}$

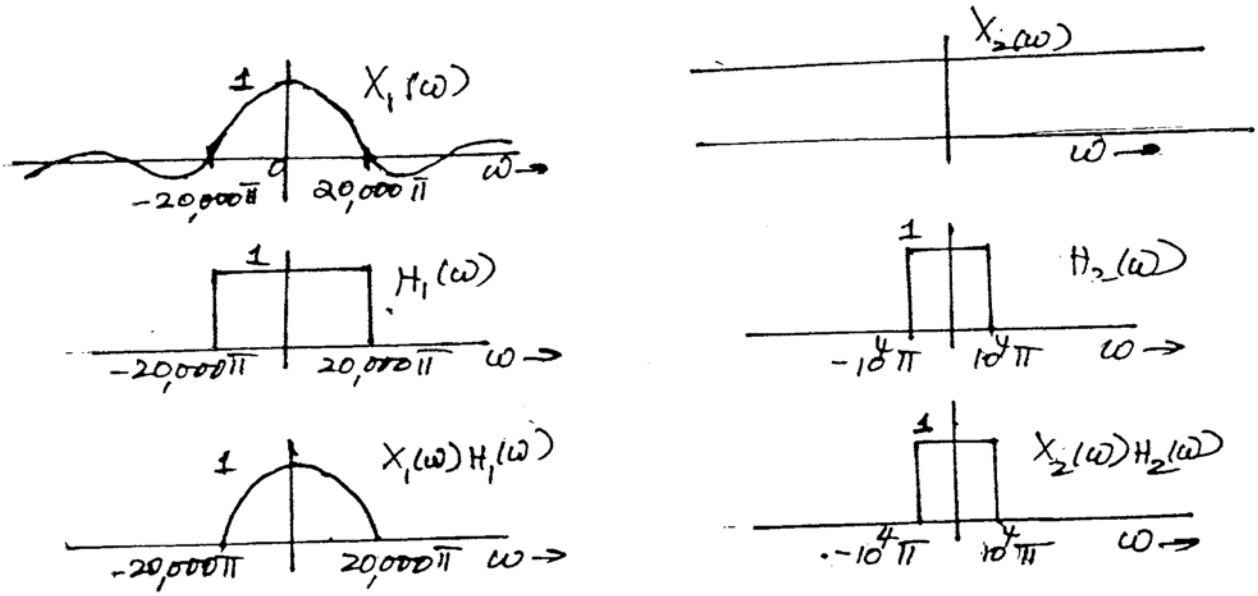


Figure S7.4-3

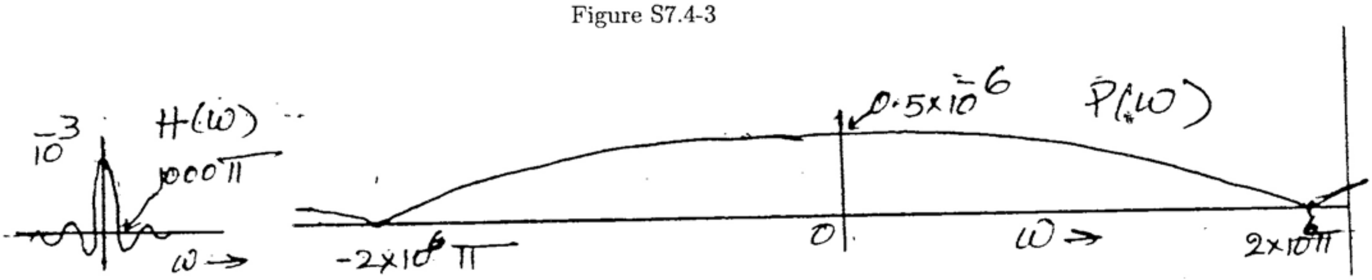


Figure S7.4-4

over the entire band of $P(\omega)$. Hence,

$$Y(\omega) = P(\omega)H(\omega) \approx P(\omega)H(0) = 10^{-3}P(\omega) \quad \Longrightarrow \quad y(t) = 10^{-3}p(t)$$

Note that the dc gain of the system is $k = H(0) = 10^{-3}$. Hence, the output is nearly $kP(t)$.

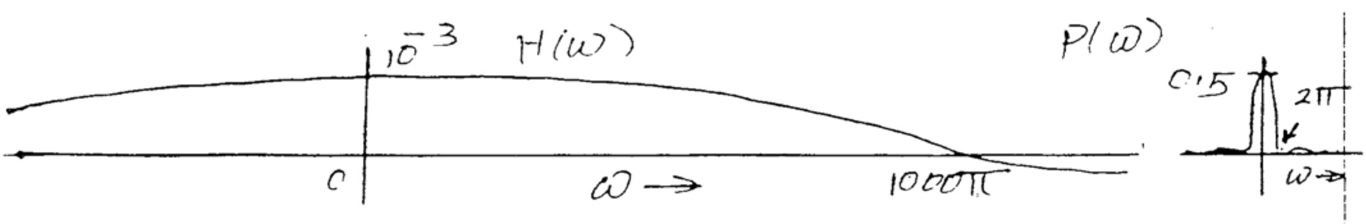


Figure S7.4-5

7.4-6. Every signal can be expressed as a sum of even and odd components (see Sec. 1.5-2). Hence, $h(t)$ can be expressed as a sum of its even and odd components a

$$h(t) = h_e(t) + h_o(t)$$

where $h_e(t) = \frac{1}{2}[h(t)u(t) + h(-t)u(-t)]$ and $h_o(t) = \frac{1}{2}[h(t)u(t) - h(-t)u(-t)]$. From these equations, we make an important observation that

$$h_e(t) = h_o(t)\,\text{sgn}(t) \qquad \text{and} \qquad h_o(t) = h_e(t)\,\text{sgn}(t) \tag{1}$$

provided that $h(t)$ has no impulse at the origin. This result applies only if $h(t)$ is causal. The graphical proof of this result may be seen in Figure 1.24.
Moreover, we have proved in Prob. 7.1-1 that the Fourier transform of a real and even signal is a real and even function of ω, and the Fourier transform of a real and odd signal is an imaginary odd function of ω. Therefore, if $X(\omega) = R(\omega) + jX(\omega)$, then

$$h_e(t) \Longleftrightarrow R(\omega) \qquad \text{and} \qquad h_o(t) \Longleftrightarrow jX(\omega) \tag{2}$$

Applying the convolution property to Eq. (1), we obtain

$$R(\omega) = \frac{1}{2\pi} jX(\omega) * \frac{2}{j\omega} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{X(y)}{\omega - y}\,dy$$

and

$$jX(\omega) = \frac{1}{2\pi} R(\omega) * \frac{2}{j\omega} = \frac{1}{j\pi}\int_{-\infty}^{\infty} \frac{R(y)}{\omega - y}\,dy$$

or

$$X(\omega) = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{R(y)}{\omega - y}\,dy$$

7.5-1.

$$H(\omega) = e^{-k\omega^2} e^{-j\omega t_0}$$

Using pair 22 (Table 4.1) and time-shifting property, we get

$$h(t) = \frac{1}{\sqrt{4\pi k}} e^{-(t-t_0)^2/4k}$$

This is noncausal. Hence the filter is unrealizable. Also

$$\int_{-\infty}^{\infty} \frac{|\ln|H(\omega)||}{\omega^2 + 1}\,d\omega = \int_{-\infty}^{\infty} \frac{k\omega^2}{\omega^2 + 1}\,d\omega = \infty$$

Hence the filter is noncausal and therefore unrealizable. Since $h(t)$ is a Gaussian function delayed by t_0, it looks as shown in the adjacent figure. Choosing $t_0 = 3\sqrt{2k}$, $h(0) = e^{-4.5} = 0.011$ or 1.1% of its peak value. Hence $t_0 = 3\sqrt{2k}$ is a reasonable choice to make the filter approximately realizable.

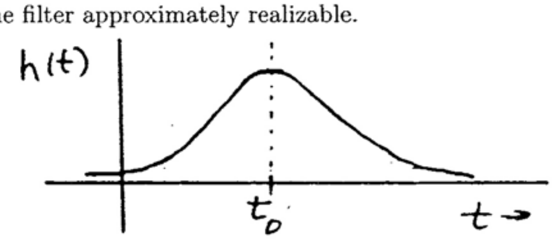


Figure S7.5-1

7.5-2.

$$H(\omega) = \frac{2 \times 10^5}{\omega^2 + 10^{10}} e^{-j\omega t_0}$$

From pair 3, Table 4.1 and time-shifting property, we get

$$h(t) = e^{-10^5|t-t_0|}$$

The impulse response is noncausal, and the filter is unrealizable.

The exponential delays to 1.8% at 4 times constants. Hence $t_0 = 4/a = 4 \times 10^{-5} = 40\mu s$ is a reasonable choice to make this filter approximately realizable.

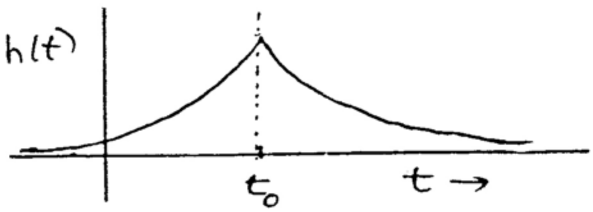


Figure S7.5-2

7.5-3. The unit impulse response is the inverse Fourier transform of $H(\omega)$. Hence, we have

$$h(t) = \text{(a)}\, 0.5\, \text{rect}\left(\frac{t}{2 \times 10^{-6}}\right) \quad \text{(b)}\, \text{sinc}^2(10{,}000\pi t) \quad \text{(c)}\, 1$$

All the three systems are noncausal (and, therefore, unrealizable) because all the three impulse responses start before $t = 0$.

For (a), the impulse response is a rectangular pulse starting at $t = -10^{-6}$. Hence, delaying the $h(t)$ by 1 μsecond will make it realizable. This will not change anything in the system behavior except the time delay of 1 μsecond in the system response.

For (b), the impulse response is a sinc square pulse, which extends all the way to $-\infty$. Clearly, this system cannot be made realizable with a finite time delay. The delay has to be infinite. However, because the sinc square pulse decays rapidly (see Figure 4.24d), we may truncate it at $t = 10^{-4}$, and then delay the resulting $h(t)$ by 10^{-4}. This makes the filter approximately realizable by allowing a time delay of 100 μseconds in the system response.

For (c), the impulse response is 1, which never decays. Consequently, this filter cannot be realized with any amount of delay.

7.6-1.

$$E_x = \int_{-\infty}^{\infty} x^2(t)\, dt = \frac{1}{2\pi\sigma^2} \int_{-\infty}^{\infty} e^{-t^2/\sigma^2}\, dt$$

Letting $\frac{t}{\sigma} = \frac{x}{\sqrt{2}}$ and consequently $dt = \frac{\sigma}{\sqrt{2}} dx$

$$E_x = \frac{1}{2\pi\sigma^2} \frac{\sigma}{\sqrt{2}} \int_{-\infty}^{\infty} e^{-x^2/2}\, dx = \frac{\sqrt{2\pi}}{2\sqrt{2}\pi\sigma} = \frac{1}{2\sigma\sqrt{\pi}}$$

Also from pair 22 (Table 4.1)

$$X(\omega) = e^{-\sigma^2\omega^2/2}$$

$$E_x = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\sigma^2\omega^2}\, d\omega$$

Letting $\sigma\omega = \frac{x}{\sqrt{2}}$ and consequently $d\omega = \frac{1}{\sigma\sqrt{2}}dx$

$$E_x = \frac{1}{2\pi}\frac{1}{\sigma\sqrt{2}}\int_{-\infty}^{\infty} e^{-x^2/2}\, dx = \frac{\sqrt{2\pi}}{2\pi\sigma\sqrt{2}} = \frac{1}{2\sigma\sqrt{\pi}}$$

7.6-2. Consider a signal

$$x(t) = \text{sinc}(kt) \quad \text{and} \quad X(\omega) = \frac{\pi}{k}\text{rect}(\frac{\omega}{2k})$$

$$\begin{aligned} E_x = \int_{-\infty}^{\infty} \text{sinc}^2(kt)\, dt &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\pi^2}{k^2}\left[\text{rect}\left(\frac{\omega}{2k}\right)\right]^2 d\omega \\ &= \frac{\pi}{2k^2}\int_{-k}^{k} d\omega = \frac{\pi}{k} \end{aligned}$$

7.6-3. If $x^2(t) \Longleftrightarrow A(\omega)$, then the output $Y(\omega) = A(\omega)H(\omega)$, where $H(\omega)$ is the lowpass filter transfer function (Figure S7.6-3). Because this filter band $\Delta f \to 0$, we may express it as an impulse function of area $4\pi\Delta f$. Thus,

$$H(\omega) \approx [4\pi\Delta f]\delta(\omega) \quad \text{and} \quad Y(\omega) \approx [4\pi A(\omega)\Delta f]\delta(\omega) = [4\pi A(0)\Delta f]\delta(\omega)$$

Here we used the property $g(x)\delta(x) = g(0)\delta(x)$ [Eq. (1.23a)]. This yields

$$y(t) = 2A(0)\Delta f$$

Next, because $x^2(t) \Longleftrightarrow A(\omega)$, we have

$$A(\omega) = \int_{-\infty}^{\infty} x^2(t)e^{-j\omega t}\, dt \quad \text{so that} \quad A(0) = \int_{-\infty}^{\infty} x^2(t)\, dt = E_x$$

Hence, $y(t) = 2E_x\Delta f$.

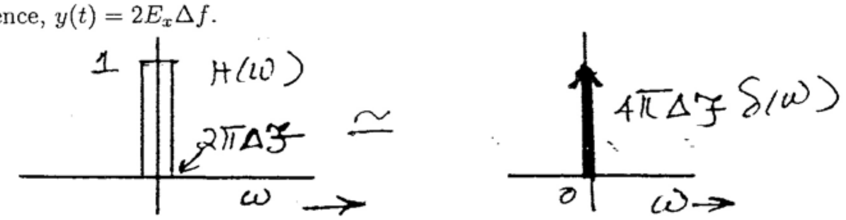


Figure S7.6-3

7.6-4. Recall that

$$x_2(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_2(\omega)e^{j\omega t}\, d\omega \quad \text{and} \quad \int_{-\infty}^{\infty} x_1(t)e^{j\omega t}\, dt = X_1(-\omega)$$

Therefore

$$\int_{-\infty}^{\infty} x_1(t)x_2(t)\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} x_1(t)[\int_{-\infty}^{\infty} X_2(\omega)e^{j\omega t}\, d\omega]\, dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X_2(\omega)[\int_{-\infty}^{\infty} x_1(t)e^{j\omega t}\,dt]\,d\omega = \frac{1}{2\pi}\int X_1(-\omega)X_2(\omega)\,d\omega$$

Interchanging the roles of $x_1(t)$ and $x_2(t)$ in the above development, we can show that

$$\int_{-\infty}^{\infty} x_1(t)x_2(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\omega)X_2(-\omega)\,d\omega$$

7.6-5. In the generalized Parseval's theorem in Prob. 7.6-4, if we identify $g_1(t) = \text{sinc}(Wt - m\pi)$ and $g_2(t) = \text{sinc}(Wt - n\pi)$, then

$$G_1(\omega) = \frac{\pi}{W}\,\text{rect}\left(\frac{\omega}{2W}\right)e^{\frac{jm\pi\omega}{W}}, \quad \text{and} \quad G_2(\omega) = \frac{\pi}{W}\,\text{rect}\left(\frac{\omega}{2W}\right)e^{\frac{jn\pi\omega}{W}}$$

Therefore

$$\int_{-\infty}^{\infty} g_1(t)g_2(t)\,dt = \frac{1}{2\pi}\left(\frac{\pi}{W}\right)^2\int_{-\infty}^{\infty}\left[\text{rect}\left(\frac{\omega}{2W}\right)\right]^2 e^{\frac{j(n-m)\pi\omega}{W}}\,d\omega$$

But $\text{rect}\left(\frac{\omega}{2W}\right) = 1$ for $|\omega| \le W$, and is 0 otherwise. Hence

$$\int_{-\infty}^{\infty} g_1(t)g_2(t)\,dt = \frac{\pi}{2W^2}\int_{-W}^{W} e^{\frac{j(n-m)\pi\omega}{W}}\,d\omega = \begin{cases} 0 & n \ne m \\ \frac{\pi}{W} & n = m \end{cases}$$

In evaluating the integral, we used the fact that $e^{\pm j2\pi k} = 1$ when k is an integer.

7.6-6. Application of duality property [Eq. (4.31)] to pair 3 (Table 4.1) yields

$$\frac{2a}{t^2 + a^2} \Longleftrightarrow 2\pi e^{-a|\omega|}$$

The signal energy is given by

$$E_x = \frac{1}{\pi}\int_0^{\infty} |2\pi e^{-a\omega}|^2\,d\omega = 4\pi\int_0^{\infty} e^{-2a\omega}\,d\omega = \frac{2\pi}{a}$$

The energy contained within the band (0 to W) is

$$E_W = 4\pi\int_0^{W} e^{-2a\omega}\,d\omega = \frac{2\pi}{a}[1 - e^{-2aW}]$$

If $E_W = 0.99E_x$, then

$$e^{-2aW} = 0.01 \quad \Longrightarrow W = \frac{2.3025}{a}\,\text{rad/s} = \frac{0.366}{a}\,\text{Hz}$$

7.7-1. (i) For $m(t) = \cos 1000t$

$$\begin{aligned} \varphi_{DSB-SC}(t) &= m(t)\cos 10,000t = \cos 1000t \cos 10,000t \\ &= \frac{1}{2}[\underbrace{\cos 9000t}_{\text{LSB}} + \underbrace{\cos 11,000t}_{\text{USB}}] \end{aligned}$$

(ii) For $m(t) = 2\cos 1000t + \cos 2000t$

$$\varphi_{DSB-SC}(t) = m(t)\cos 10,000t = [2\cos 1000t + \cos 2000t]\cos 10,000t$$

$$
\begin{aligned}
&= \cos 9000t + \cos 11,000t + \frac{1}{2}[\cos 8000t + \cos 12,000t] \\
&= \underbrace{[\cos 9000t + \frac{1}{2}\cos 8000t]}_{\text{LSB}} + \underbrace{[\cos 11,000t + \frac{1}{2}\cos 12,000t]}_{\text{USB}}
\end{aligned}
$$

(iii) For $m(t) = \cos 1000t \cos 3000t$

$$
\begin{aligned}
\varphi_{DSB-SC}(t) &= m(t)\cos 10,000t = \frac{1}{2}[\cos 2000t + \cos 4000t]\cos 10,000t \\
&= \frac{1}{2}[\cos 8000t + \cos 12,000t] + \frac{1}{2}[\cos 6000t + \cos 14,000t] \\
&= \frac{1}{2}\underbrace{[\cos 8000t + \cos 6000t]}_{\text{LSB}} + \frac{1}{2}\underbrace{[\cos 12,000t + \cos 14,000t]}_{\text{USB}}
\end{aligned}
$$

This information is summarized in a table below. Figure S7.7-1 shows various spectra.

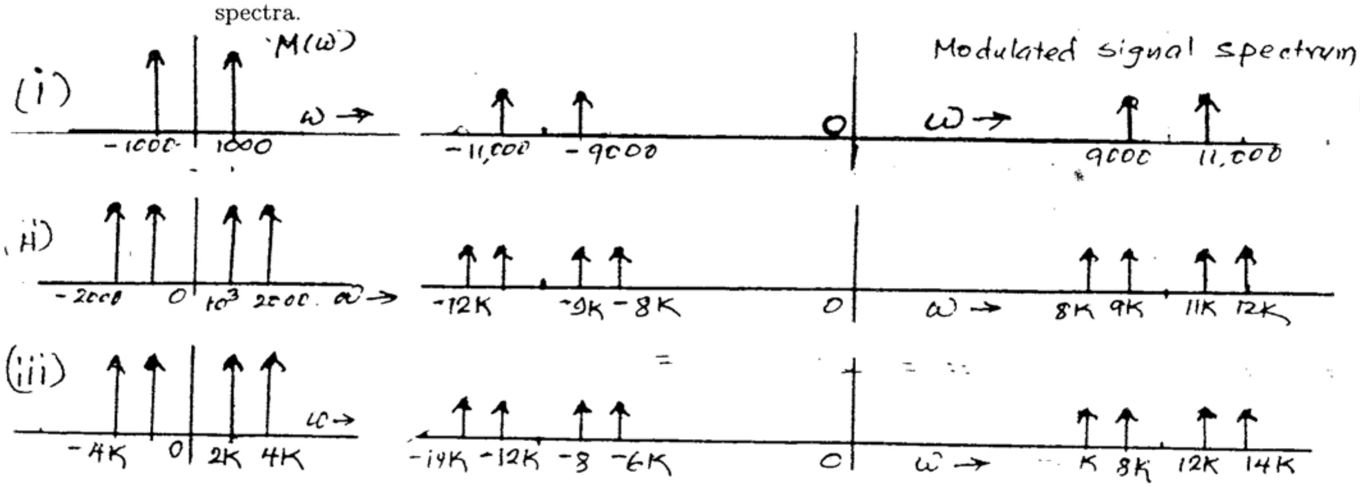


Figure S7.7-1

case	Baseband frequency	DSB frequency	LSB frequency	USB frequency
i	1000	9000 and 11,000	9000	11,000
ii	1000	9000 and 11,000	9000	11,000
	2000	8000 and 12,000	8000	12,000
iii	2000	8000 and 12,000	8000	12,000
	4000	6000 and 14,000	6000	14,000

7.7-2. (a) The signal at point b is

$$
\begin{aligned}
x_a(t) &= m(t)\cos^3 \omega_c t \\
&= m(t)\left[\frac{3}{4}\cos\omega_c t + \frac{1}{4}\cos 3\omega_c t\right]
\end{aligned}
$$

The term $\frac{3}{4}m(t)\cos\omega_c t$ is the desired modulated signal, whose spectrum is centered at $\pm\omega_c$. The remaining term $\frac{1}{4}m(t)\cos 3\omega_c t$ is the unwanted term, which represents the modulated signal with carrier frequency $3\omega_c$ with spectrum centered at $\pm 3\omega_c$ as shown in Figure S7.7-2. The bandpass filter centered at $\pm\omega_c$ allows to pass the desired term $\frac{3}{4}m(t)\cos\omega_c t$, but suppresses the unwanted term $\frac{1}{4}m(t)\cos 3\omega_c t$. Hence, this system works as desired with the output $\frac{3}{4}m(t)\cos\omega_c t$.

(b) Figure S7.7-2 shows the spectra at points b and c.

(c) The minimum usable value of ω_c is $2\pi B$ in order to avoid spectral folding at dc.

(d)

$$\begin{aligned} m(t)\cos^2\omega_c t &= \frac{m(t)}{2}\left[1+\cos 2\omega_c t\right] \\ &= \frac{1}{2}m(t)+\frac{1}{2}m(t)\cos 2\omega_c t \end{aligned}$$

The signal at point b consists of the baseband signal $\frac{1}{2}m(t)$ and a modulated signal $\frac{1}{2}m(t)\cos 2\omega_c t$, which has a carrier frequency $2\omega_c t$, not the desired value $\omega_c t$. Both the components will be suppressed by the filter, whose center center frequency is ω_c. Hence, this system will not do the desired job.

(e) The reader may verify that the identity for $\cos n\omega_c t$ contains a term $\cos\omega_c t$ when n is odd. This is not true when n is even. Hence, the system works for a carrier $\cos^n\omega_c t$ only when n is odd.

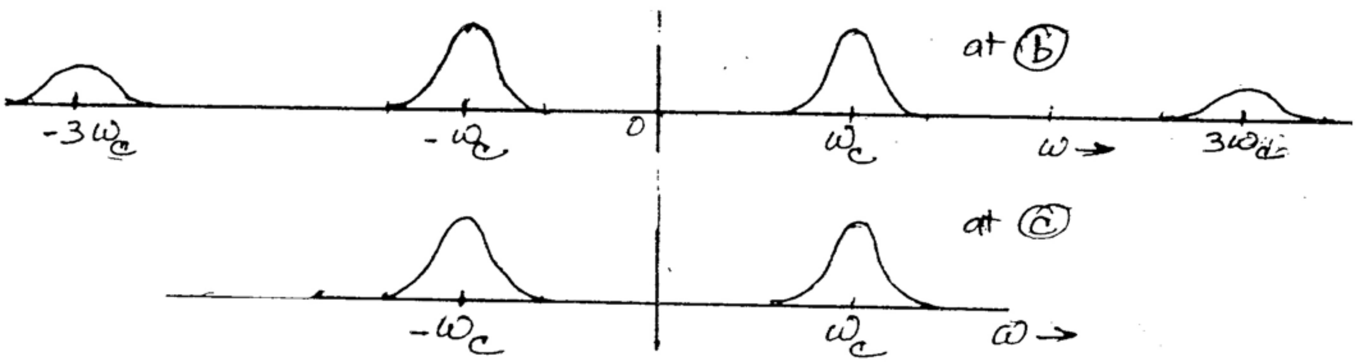


Figure S7.7-2

7.7-3. This signal is identical to that in Figure 3.8a with period T_0 (instead of 2π). We find the Fourier series for this signal as

$$x(t)=\frac{1}{2}+\frac{2}{\pi}\left[\cos\,\omega_c t-\frac{1}{3}\cos\,3\omega_c t+\frac{1}{5}\cos\,5\omega_c t+\cdots\right]$$

Hence, $y(t)$, the output of the multiplier is

$$y(t)=m(t)x(t)=m(t)\left[\frac{1}{2}+\frac{2}{\pi}\left(\cos\,\omega_c t-\frac{1}{3}\cos\,3\omega_c t+\frac{1}{5}\cos\,5\omega_c t+\cdots\right)\right]$$

The bandpass filter suppresses the signals $m(t)$ and $m(t)\cos\,n\omega_c t$ for all $n\neq 1$. Hence, the bandpass filter output is

$$km(t)\cos \omega_c t = \frac{2}{\pi}m(t)\cos \omega_c t$$

7.7-4. (a) Figure S7.7-4 shows the signals at points a, b, and c.

(b) From the spectrum at point c, it is clear that the channel bandwidth must be at least 30,000 rad/s (from 5000 to 35,000 rad/s.).

(c) Figure S7.7-4 shows the receiver to recover $m_1(t)$ and $m_2(t)$ from the received modulated signal.

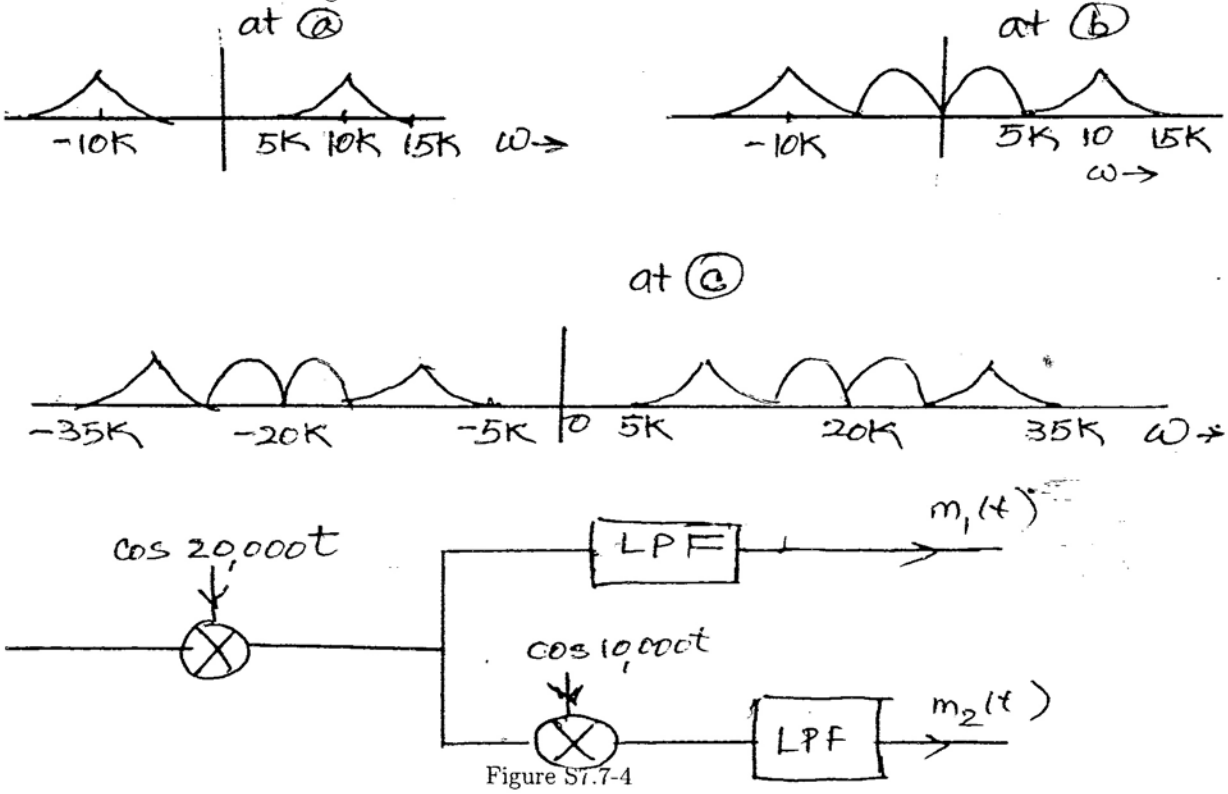


Figure S7.7-4

7.7-5. (a) Figure S7.7-5 shows the output signal spectrum $Y(\omega)$.

(b) Observe that $Y(\omega)$ is the same as $M(\omega)$ with the frequency spectrum inverted, that is, the high frequencies are shifted to lower frequencies and vice versa. Thus, the scrambler in Figure P7.7-5 inverts the frequency spectrum. To get back the original spectrum $M(\omega)$, we need to invert the spectrum $Y(\omega)$ once again. This can be done by passing the scrambled signal $y(t)$ through the same scrambler.

7.7-6. $x_a(t) = [A + m(t)]\cos\omega_c t$. Hence,

$$\begin{aligned} x_b(t) &= [A + m(t)]\cos^2 \omega_c t \\ &= \frac{1}{2}[A + m(t)] + \frac{1}{2}[A + m(t)]\cos 2\omega_c t \end{aligned}$$

The first term is a lowpass signal because its spectrum is centered at $\omega = 0$. The lowpass filter allows this term to pass, but suppresses the second term, whose spectrum

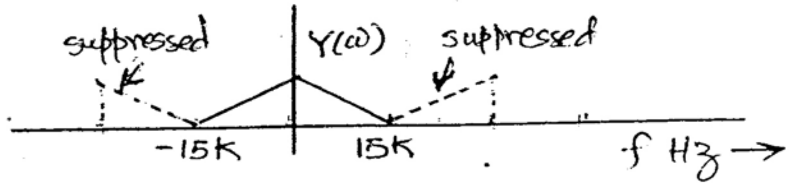


Figure S7.7-5

is centered at $\pm 2\omega_c$. Hence the output of the lowpass filter is

$$y(t) = A + m(t)$$

When this signal is passed through a dc block, the dc term A is suppressed yielding the output $m(t)$. This shows that the system can demodulate AM signal regardless of the value of A. This is a synchronous or coherent demodulation.

7.7-7. (a) $\mu = 0.5 = \frac{m_p}{A} = \frac{10}{A} \quad \Rightarrow A = 20$
(b) $\mu = 1.0 = \frac{m_p}{A} = \frac{10}{A} \quad \Rightarrow A = 10$
(c) $\mu = 2.0 = \frac{m_p}{A} = \frac{10}{A} \quad \Rightarrow A = 5$
(d) $\mu = \infty = \frac{m_p}{A} = \frac{10}{A} \quad \Rightarrow A = 0$

This means that $\mu = \infty$ represents the DSB-SC case. Figure S7.7-7 shows various waveforms.

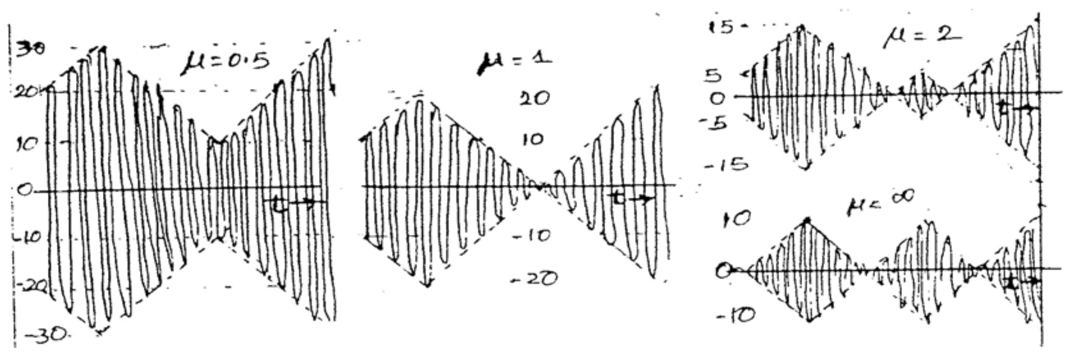


Figure S7.7-7

7.M-1. The signal $x(t) = e^{-at}u(t)$ has Fourier Transform given by $X(\omega) = \frac{1}{j\omega + a}$ and energy $E_x = \frac{1}{2a}$. Using this information, MATLAB program MS7P2 is modified.

```
function [W,E_W] = MS7P2mod1(a,beta,tol)
% MS7P2mod1.m
% Function M-file computes essential bandwidth W for exp(-at)u(t).
% INPUTS:   a = decay parameter of x(t)
%           beta = fraction of signal energy desired in W
%           tol = tolerance of relative energy error
% OUTPUTS:  W = essential bandwidth [rad/s]
```

```
%           E_W = Energy contained in bandwidth W
W = 0; step = a;         % Initial guess and step values
X_squared = inline('1./(omega.^2+a.^2)','omega','a');
E = beta/(2*a);          % Desired energy in W
relerr = (E-0)/E;        % Initial relative error is 100 percent
while(abs(relerr) > tol),
    if (relerr>0),       % W too small
        W=W+step;        % Increase W by step
    elseif (relerr<0),   % W too large
        step = step/2; W = W-step; % Decrease step size and then W.
    end
    E_W = 1/(2*pi)*quad(X_squared,-W,W,[],[],a);
    relerr = (E - E_W)/E;
end
```

(a) Setting $a = 1$ and using 95% signal energy results in

```
>> [W,E_W]=MS7P2mod1(1,.95,1e-9)
W = 12.7062
E_W = 0.4750
```

Thus,

$$W_1 = 12.7062.$$

From the text example, the essential bandwidth corresponding to 95% signal energy is derived as $W = 12.706a$ radians per second. For $a = 1$, this corresponds nicely with the computed value of $W_1 = 12.7062$.

(b) Setting $a = 2$ and using 90% signal energy results in

```
>> [W,E_W]=MS7P2mod1(2,.90,1e-9)
W = 12.6275
E_W = 0.2250
```

Thus,

$$W_2 = 12.6275.$$

(c) Setting $a = 3$ and using 75% signal energy results in

```
>> [W,E_W]=MS7P2mod1(3,.75,1e-9)
W = 7.2426
E_W = 0.1250
```

Thus,

$$W_3 = 7.2426.$$

7.M-2. To solve this problem, program MS7P2 is modified to solve for the pulse width to achieve a desired essential bandwidth, rather than solving for the essential bandwidth that corresponds to a desired pulse.

```
function [tau,E_W] =  MS7P2mod2(W,beta,tol)
% MS7P2mod2.m
% Function M-file computes essential bandwidth W for square pulse.
% INPUTS:   W = essential bandwidth [rad/s]
%           beta = fraction of signal energy desired in W
%           tol = tolerance of relative energy error
% OUTPUTS:  tau = pulse width
```

```
%           E_W = Energy contained in bandwidth W
tau = 1; step = 1;      % Initial guess and step values
X_squared = inline('(tau*MS7P1(omega*tau/2)).^2','omega','tau');
E_W = 1/(2*pi)*quad(X_squared,-W,W,[],[],tau);
E = beta*tau;           % Desired energy in W
relerr = (E - E_W)/E;
while(abs(relerr) > tol),
    if (relerr>0),      % tau too small
        tau=tau+step;   % Increase tau by step
    elseif (relerr<0),  % tau too large
        step = step/2;
        tau = tau-step; % Decrease step size and then tau.
    end
    E_W = 1/(2*pi)*quad(X_squared,-W,W,[],[],tau);
    E = beta*tau; % Desired energy in W
    relerr = (E - E_W)/E;
end
```

(a) Set $W = 2\pi 5$ and select 95% signal energy.

```
>> [tau,E_W] = MS7P2mod2(2*pi*5,.95,1e-9)
tau = 0.4146
E_W = 0.3939
```

Thus,

$$\tau_1 = 0.4146.$$

(b) Set $W = 2\pi 10$ and select 90% signal energy.

```
>> [tau,E_W] = MS7P2mod2(2*pi*10,.90,1e-9)
tau = 0.0849
E_W = 0.0764
```

Thus,

$$\tau_2 = 0.0849.$$

(c) Set $W = 2\pi 20$ and select 75% signal energy.

```
>> [tau,E_W] = MS7P2mod2(2*pi*20,.75,1e-9)
tau = 0.0236
E_W = 0.0177
```

Thus,

$$\tau_3 = 0.0236.$$

7.M-3. To solve this problem, program MS7P2 is modified to solve for the decay parameter a to achieve a desired essential bandwidth, rather than solving for the essential bandwidth that corresponds to a desired decay parameter.

```
function [a,E_W] =  MS7P2mod3(W,beta,tol)
% MS7P2mod3.m
% Function M-file computes decay parameter a needed to
%                   achieve a given essential bandwidth.
% INPUTS:   W = essential bandwidth [rad/s]
%           beta = fraction of signal energy desired in W
%           tol = tolerance of relative energy error
```

```
% OUTPUTS:  a = decay parameter
%           E_W = Energy contained in bandwidth W
a = 1; step = 1;         % Initial guess and step values
X_squared = inline('1./(omega.^2+a.^2)','omega','a');
E_W = 1/(2*pi)*quad(X_squared,-W,W,[],[],a);
E = beta/(2*a); % Desired energy in W
relerr = (E - E_W)/E;
while(abs(relerr) > tol),
    if (relerr<0),       % a too small
        a=a+step;        % Increase tau by step
    elseif (relerr>0),   % a too large
        step = step/2;
        a = a-step;      % Decrease step size and then tau.
    end
    E_W = 1/(2*pi)*quad(X_squared,-W,W,[],[],a);
    E = beta/(2*a); % Desired energy in W
    relerr = (E - E_W)/E;
end
```

(a) Set $W = 2\pi 5$ and select 95% signal energy.

```
>> [a,E_W] = MS7P2mod3(2*pi*5,.95,1e-9)
a = 2.4725
E_W = 0.1921
```

Thus,

$$a_1 = 2.4725.$$

(b) Set $W = 2\pi 10$ and select 90% signal energy.

```
>> [a,E_W] = MS7P2mod3(2*pi*10,.90,1e-9)
a = 9.9524
E_W = 0.0452
```

Thus,

$$a_2 = 9.9524.$$

(c) Set $W = 2\pi 20$ and select 75% signal energy.

```
>> [a,E_W] = MS7P2mod3(2*pi*20,.75,1e-9)
a = 52.0499
E_W = 0.0072
```

Thus,

$$a_3 = 52.0499.$$

7.M-4. Call the desired unit-amplitude, unit duration triangle function $x(t)$. First, notice that $x(t)$ can be constructed by convolving two rectangular pulses, each of width $\tau = 0.5$ and height $A = \sqrt{2}$. The energy of $x(t)$ is $E_x = 2\int_{t=0}^{0.5}(2t)^2 dt = 1/3$. Furthermore, using the convolution-in-time property and spectrum of a rectangular pulse, we know that $X(\omega) = \left(\frac{\sqrt{2}}{2}\text{sinc}(\omega/4)\right)^2$.

Next, program MS7P2 is modified to solve for the essential bandwidths of this signal for various signal energies.

```
function [W,E_W] = MS7P2mod4(beta,tol)
% MS7P2mod4.m
% Function M-file computes essential bandwidth W for a
%                  unit-amplitude, unit duration triangle function.
% INPUTS:   beta = fraction of signal energy desired in W
%           tol = tolerance of relative energy error
% OUTPUTS:  W = essential bandwidth [rad/s]
%           E_W = Energy contained in bandwidth W
W = 0; step = 1;        % Initial guess and step values
X_squared = inline('(sqrt(2)/2*MS7P1(omega/4)).^4','omega');
E = beta/3;             % Desired energy in W
relerr = (E-0)/E;       % Initial relative error is 100 percent
while(abs(relerr) > tol),
    if (relerr>0),      % W too small
        W=W+step;       % Increase W by step
    elseif (relerr<0),  % W too large
        step = step/2; W = W-step; % Decrease step size and then W.
    end
    E_W = 1/(2*pi)*quad(X_squared,-W,W);
    relerr = (E - E_W)/E;
end
```

Use 95% signal energy to compute the essential bandwidth:

```
>> [W,E_W] = MS7P2mod4(.95,1e-9)
W = 6.2877
E_W = 0.3167
```

Use 90% signal energy to compute the essential bandwidth:

```
>> [W,E_W] = MS7P2mod4(.9,1e-9)
W = 5.3350
E_W = 0.3000
```

Use 75% signal energy to compute the essential bandwidth:

```
>> [W,E_W] = MS7P2mod4(.75,1e-9)
W = 3.7872
E_W = 0.2500
```

Thus, the essential bandwidths are

$$W_{0.95} = 6.2877\text{rad/s},\ W_{0.90} = 5.3350\text{rad/s},\ W_{0.75} = 3.7872\text{rad/s}.$$

7.M-5. Following the example in MATLAB Session 7, the first 10 Fourier series coefficients of a 1/3 duty-cycle square wave are

$$D_n = \frac{\tau}{T_0}\text{sinc}\left(\frac{n\pi\tau}{T_0}\right).$$

(a) Setting $T_0 = 2\pi$ and $\tau = 2\pi/3$ yields

$$D_n = \frac{1}{3}\text{sinc}\left(\frac{n\pi}{3}\right).$$

MATLAB is used to evaluate and plot the first ten coefficients.

```
>> tau = 2*pi/3; T_0 = 2*pi; n = [0:10];
>> D_n = tau/T_0*MS7P1(n*pi*tau/T_0);
>> stem(n,D_n,'k'); xlabel('n'); ylabel('D_n');
>> axis([-0.5 10.5 -0.2 0.55]);
```

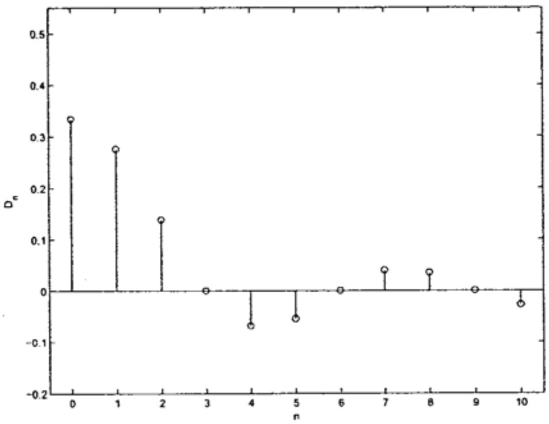


Figure S7.M-5a: Fourier series coefficients D_n for $x(t)$.

(b) Setting $T_0 = \pi$ and $\tau = \pi/3$ yields

$$D_n = \frac{1}{3}\text{sinc}\left(\frac{n\pi}{3}\right).$$

Notice, the coefficients D_n depend only on the duty-cycle of the signal, not the period. Since the duty cycle is fixed, the coefficients D_n are identical to those determined in 7.M-5a. Refer to solution 7.M-5a for the MATLAB code and plot.

7.M-6.

$$\begin{aligned} X(\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \\ &= \int_{-\infty}^{\infty} e^{-t^2}e^{-j\omega t}dt \\ &= \int_{-\infty}^{\infty} e^{-(t^2+j\omega t+(j\omega/2)^2-(j\omega/2)^2)}dt \\ &= e^{(j\omega/2)^2}\int_{-\infty}^{\infty} e^{-(t+j\omega/2)^2}dt \end{aligned}$$

Substituting $t'/\sqrt{2} = t$ and $dt'/\sqrt{2} = dt$ yields

$$X(\omega) = \frac{e^{-\omega^2/4}}{\sqrt{2}}\int_{-\infty}^{\infty} e^{-(t'+j\omega^2/\sqrt{2})/2}dt'.$$

However, $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\frac{-(t-a)^2}{2}}dt = 1$ for any a, so $\int_{-\infty}^{\infty} e^{\frac{-(t'+j\omega^2)^2}{2}}dt' = \sqrt{2\pi}$. Thus,

$$X(\omega) = \sqrt{\pi}e^{-\omega^2/4}.$$

MATLAB is used to plot $x(t)$ and $X(\omega)$.

```
>> t = linspace(-10,10,1001); x = exp(-t.^2);
>> omega = linspace(-10,10,1001); X = sqrt(pi)*exp(-omega.^2/4);
```

```
>> subplot(211); plot(t,x,'k');
>> xlabel('t'); ylabel('x(t)');
>> subplot(212); plot(t,X,'k');
>> xlabel('\omega'); ylabel('X(\omega)');
```

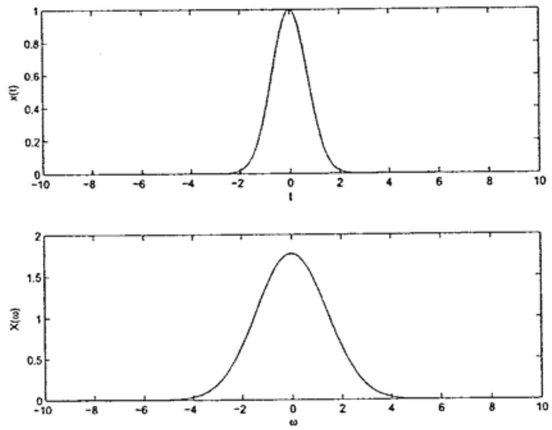


Figure S7.M-6: $x(t) = e^{-t^2}$ and $X(\omega) = \sqrt{\pi}e^{-\omega^2/4}$.

Figure S7.M-6 confirms that $X(\omega)$ is just a scaled and stretched version of $x(t)$. This is something remarkable; the Fourier Transform of a Gaussian pulse is itself a Gaussian pulse!

Chapter 8 Solutions

8.1-1. The bandwidths of $x_1(t)$ and $x_2(t)$ are 100 kHz and 150 kHz, respectively. Therefore the Nyquist sampling rates for $x_1(t)$ is 200 kHz and for $x_2(t)$ is 300 kHz. Also ${x_1}^2(t) \Longleftrightarrow \frac{1}{2\pi}F_1(\omega) * F_1(\omega)$, and from the width property of convolution the bandwidth of ${x_1}^2(t)$ is twice the bandwidth of $x_1(t)$ and that of ${x_2}^3(t)$ is three times the bandwidth of $x_2(t)$ (se also Prob. 4.3-10). Similarly the bandwidth of $x_1(t)x_2(t)$ is the sum of the bandwidth of $x_1(t)$ and $x_2(t)$. Therefore the Nyquist rate for ${x_1}^2(t)$ is 400 kHz, for ${x_2}^3(t)$ is 900 kHz, for $x_1(t)x_2(t)$ is 500 kHz.

8.1-2. (a)

$$\text{sinc}^2(100\pi t) \Longleftrightarrow 0.01\Delta(\frac{\omega}{400\pi})$$

The bandwidth of this signal is 200 π rad/s or 100 Hz. The Nyquist rate is 200 Hz (samples/sec).

(b) The Nyquist rate is 200 Hz, the same as in (a), because multiplication of a signal by a constant does not change its bandwidth.

(c)

$$\text{sinc}(100\pi t) + 3\,\text{sinc}^2(60\pi t) \Longleftrightarrow 0.01\,\text{rect}(\frac{\omega}{200\pi}) + \frac{1}{20}\Delta(\frac{\omega}{240\pi})$$

The bandwidth of $\text{rect}(\frac{\omega}{200\pi})$ is 50 Hz and that of $\Delta(\frac{\omega}{240\pi})$ is 60 Hz. The bandwidth of the sum is the higher of the two, that is, 60 Hz. The Nyquist sampling rate is 120 Hz.

(d)

$$\begin{aligned}\text{sinc}(50\pi t) &\Longleftrightarrow 0.02\,\text{rect}(\frac{\omega}{200\pi})\\ \text{sinc}(100\pi t) &\Longleftrightarrow 0.01\,\text{rect}(\frac{\omega}{400\pi})\end{aligned}$$

The two signals have bandwidths 25 Hz and 50 Hz respectively. The spectrum of the product of two signals is $1/2\pi$ times the convolution of their spectra. From width property of the convolution, the width of the convoluted signal is the sum of the widths of the signals convolved. Therefore, the bandwidth of $\text{sinc}(50\pi t)\text{sinc}(100\pi t)$ is $25 + 50 = 75$ Hz. The Nyquist rate is 150 Hz.

8.1-3. (a)

$$\begin{aligned}|X(\omega)| &= 3\pi\left[\delta(\omega + 6\pi) + \delta(\omega - 6\pi)\right]\\ &\quad +\pi\left[\delta(\omega + 18\pi) + \delta(\omega - 18\pi)\right]\\ &\quad +2\pi\left[\delta(\omega + [28 - \epsilon]\pi) + \delta(\omega - [28 - \epsilon]\pi)\right]\end{aligned}$$

Figure S8.1-3 shows the spectrum as a function of f in Hz. The spectrum in the range $|f| < 14$ Hz is the spectrum $X(\omega)$.

(b) The Nyquist rate is 28 Hz. Hence 25% above this results in the sampling rate $f_s = 35$ Hz. Figure S8.1-3 shows the sampled signal spectrum over the range $|f| < 50$ Hz. To reconstruct $x(t)$, we pass the sampled signal through an ideal lowpass filter of cutoff frequency anywhere between $(14+\epsilon)$ Hz to $(21-\epsilon)$ Hz where ϵ is a small positive number. The filter gain is $T = 1/f_s = 1/35$.

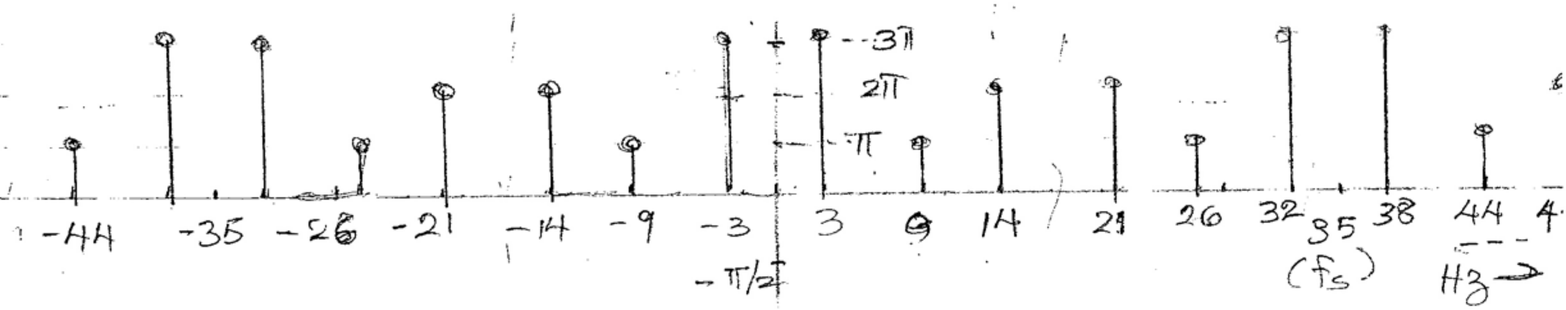


Figure S8.1-3

8.1-4. (a) From Eq. (7.27)

$$\delta_T(t) \Longleftrightarrow \omega_s \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_s) \qquad \omega_s = \frac{2\pi}{T}$$

Hence

$$\begin{aligned}\overline{x(t)} = x(t)\delta_T(t) \quad &\Longleftrightarrow \quad \omega_s X(\omega) * \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_s) \\ &= \quad \frac{\omega_s}{2\pi} \sum_{n=-\infty}^{\infty} X(\omega) * \delta(\omega - n\omega_s) \\ &= \quad \frac{1}{T} \sum_{n=-\infty}^{\infty} X(\omega - n\omega_s)\end{aligned}$$

(b) The sampling impulse train is given by

$$s(t) = \sum_n \delta(t - nT - \tau)$$

This is same as $\sum_n \delta(t - nT)$ right-shifted by τ.
Hence

$$\begin{aligned} s(t) \quad &\Longleftrightarrow \quad \omega_s \sum_n \delta(\omega - n\omega_s) e^{-jn\omega_s \tau} \\ x(t)s(t) \quad &\Longleftrightarrow \quad \frac{\omega_s}{2\pi} X(\omega) * \sum_n \delta(\omega - n\omega_s) e^{-jn\omega_s \tau} \\ &= \quad \frac{1}{T_s} \sum_{n=-\infty}^{\infty} X(\omega - n\omega_s) e^{-jn\omega_s \tau}\end{aligned}$$

8.1-5. If the corrupted spectrum is not filtered out, we need the minimum sampling rate 22 Hz. This is clarified by Figure S8.1-5a. For $f_s = 22$, the uncorrupted spectrum remains intact and can be recovered by a lowpass filter of cutoff frequency 10 Hz.

When the corrupted spectrum is suppressed, the resulting signal spectrum band is only 10Hz. Hence it is adequate to use $f_s = 20$ Hz, as shown in Figure S8.1-5b.

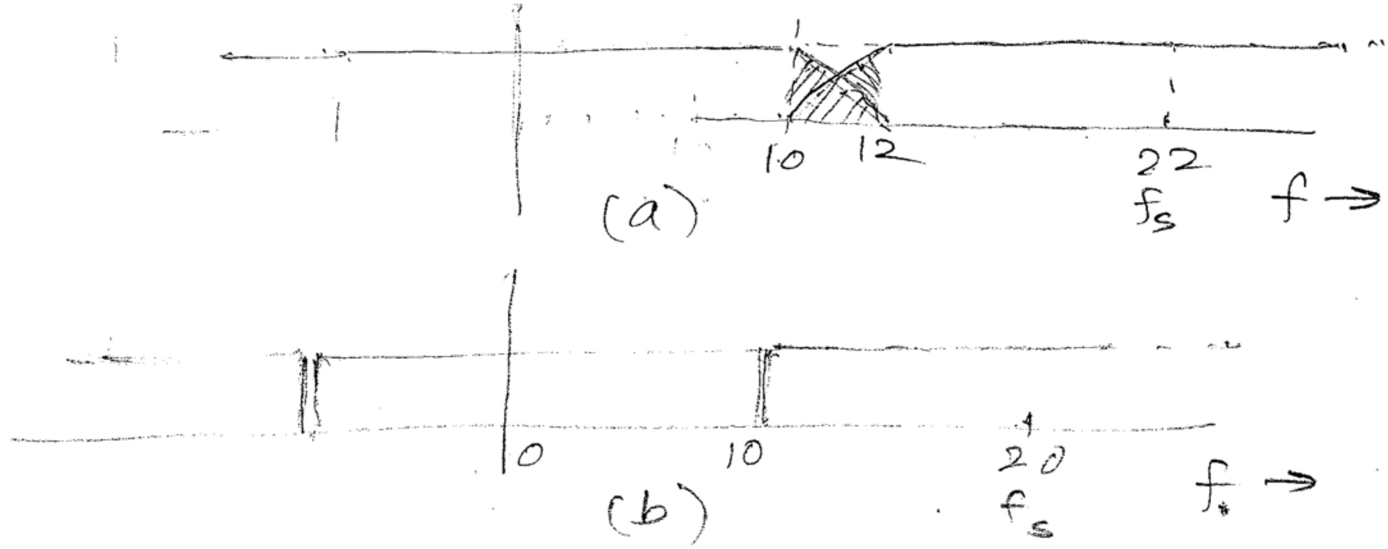


Figure S8.1-5

8.1-6.

$$\Delta\left(\frac{t-1}{2}\right) \Longleftrightarrow \text{sinc}^2\left(\frac{\omega}{2}\right) e^{-j\omega}$$

The spectrum $|X(\omega)|$ shows that most of the signal energy is concentrated within the band of 1 Hz. It can be shown that 90.28% energy is contained within the band of 1 Hz. If we use 90% energy criterion for bandwidth, sampling rate of 2 Hz is adequate. However, for a better approximation, (higher energy bandwidth criterion), we may go to $f_s = 4$ Hz. Theoretically, of course, $f_s = \infty$.

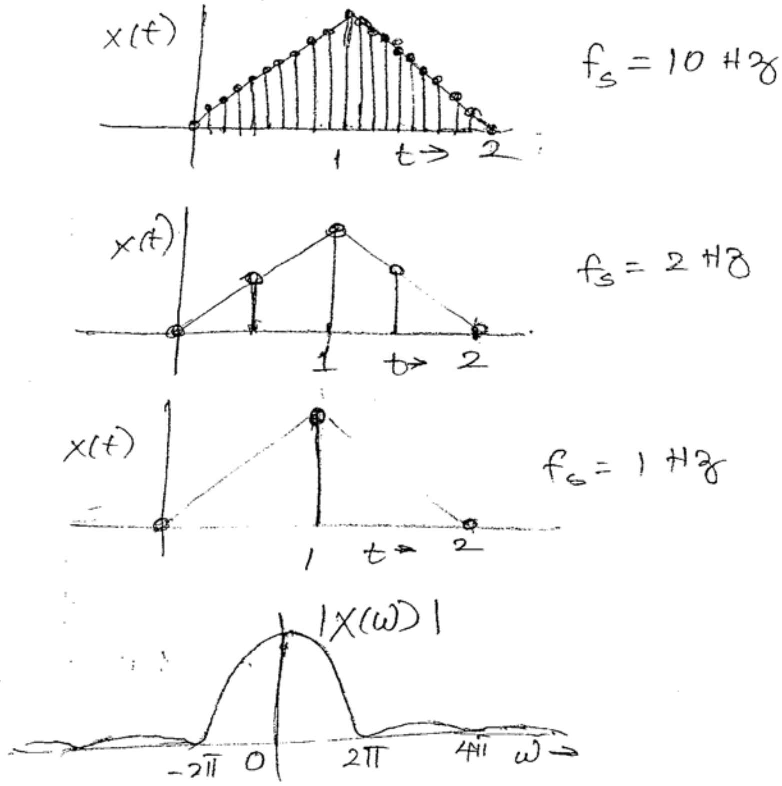
8.1-7. (a)

$$X(\omega) = \Delta\left(\frac{\omega}{20\pi}\right) + \pi[\delta(\omega + 20\pi) + \delta(\omega - 20\pi)]$$

The bandwidth is 10 Hz. There is an impulse at 10Hz, as seen from $X(\omega)$ shown in Figure S8.1-7a. The Nyquist rate is 20 Hz. Hence, $f_s = 10$Hz will not permit reconstruction of $x(t)$. This is verified from the sampled signal spectrum in Figure S8.1-7a, shown as a function of f in Hz.

(b) The Nyquist rate is 20 Hz. Hence the sampling rate $f_s = 20$ Hz is adequate despite the fact that $x(t)$ contains an impulse at the highest frequency 10 Hz. This is because, the impulse component is $\cos 20\pi t$.
To reconstruct $x(t)$ from the spectrum in Figure S8.1-7b, we need an ideal lowpass filter of cutoff frequency 10 Hz and gain $T = 1/20$. Because the rect function value is 0.5 at the edge (cutoff), the lowpass filter gain at the cutoff frequency 10 Hz is $0.5 \times 1/20 = 1/40$. Hence for the input of an impulse of strength 40π at

FIG. S8.1-6



± 10 Hz, the output will be an impulse of strength 40π at $f = \pm 10$ Hz. Hence, the filter output is the spectrum

$$X(\omega) = \Delta\left(\frac{\omega}{20\pi}\right) + \pi[\delta(\omega + 20\pi) + \delta(\omega - 20\pi)]$$

and

$$x(t) = 5\text{sinc}^2(5\pi t) + \cos 20\pi t$$

(c) In part (b), we found the cutoff $B_0 = 10$ Hz. If the filter bandwidth is $B_0 - \epsilon$, the filter will accept the entire impulse of strength 40π at $f = \pm 10$ Hz, and with gain 1/20, the reconstructed signal would be

$$\hat{x}(t) = 5\text{sinc}^2(5\pi t) + \cos 20\pi t$$

which is in error.
If the filter bandwidth is $B_0 + \epsilon$, it will miss the impulses at $f = \pm 10$ entirely; and the reconstructed signal would be

$$\tilde{x}(t) = 5\text{sinc}^2(5\pi t)$$

which is also in error.
In this case the impulses are because of a sine term. Hence, f_s =20 Hz is inadequate, even in theory. It is easily verified that the samples of $\sin 20\pi t$ at a rate 20 Hz ($T = 1/20$) are $\sin 20\pi nT = \sin \pi n = 0$, In this case the impulse at $\pm 10, \pm 30, \pm 50, \cdots$ cancel out because the two impulses have opposite phases (Figure S8.1-7c).

(d) Yes. In this case, the impulses do not overlap and there is no cancelation. Hence using a low pass filter of cutoff frequency 10.5 Hz, and gain $T = 1/21$, we can recover $x(t)$ from $\overline{x}(t)$ (see Figure S8.1-7d).

8.1-8. Figure S8.1-8a shows $X(\omega)$, which is a bandpass spectrum with bandwidth 10 Hz and band centered at 25 Hz. The highest frequency is 30 Hz. If we use 60 Hz sampling

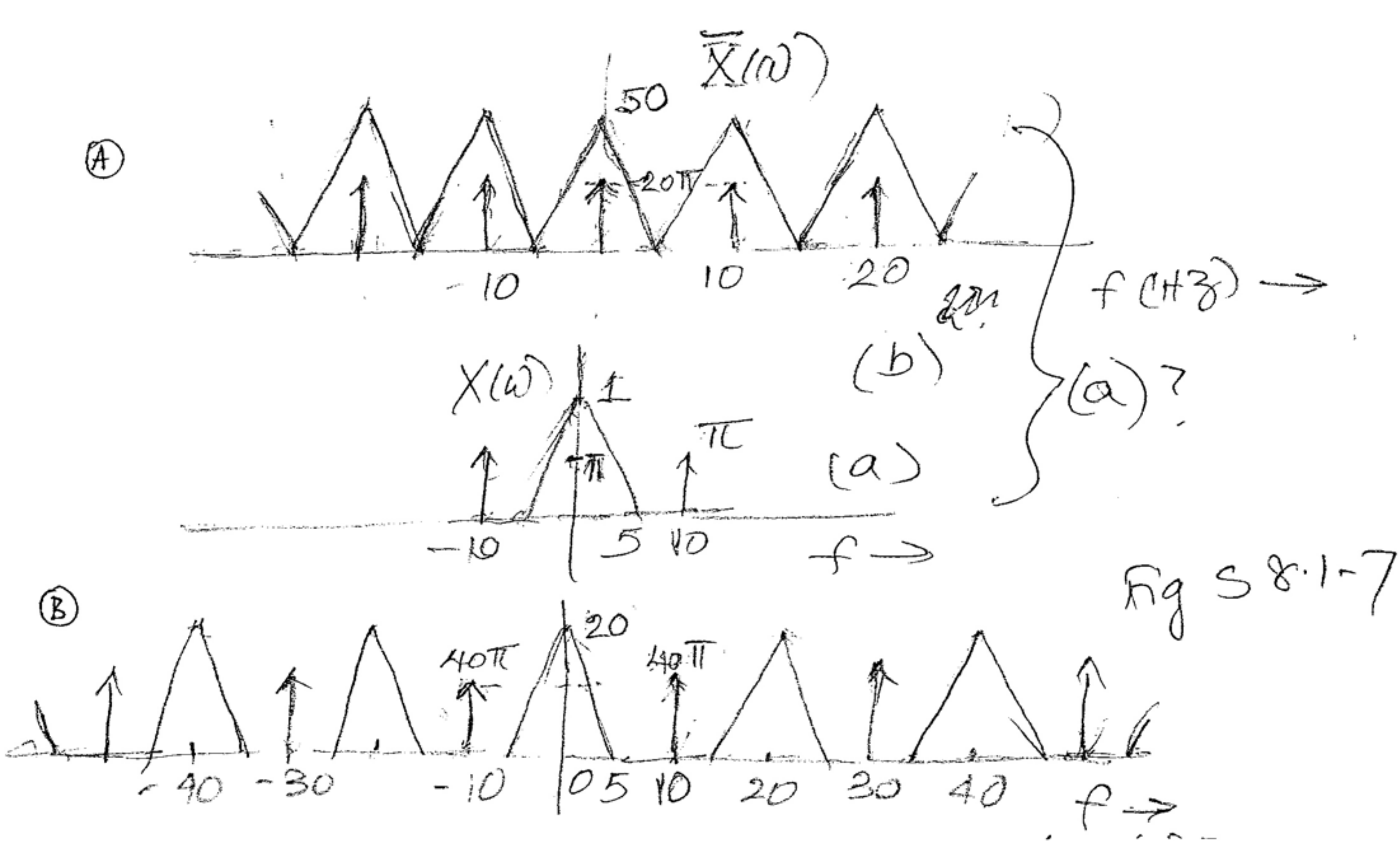


Fig S8.1-7

FIG. S8.1-7

frequency, the spectrum will shift by $\pm 60n (n = 1, 2, 3, \cdots)$ as shown in Figure S8.1-8b. The negative frequency spectrum is labeled P_0 and the positive frequency spectrum is labeled Q_0. Both these segments repeat with period 60 Hz. Let us label P_k as the segment P_0 shifted by 60 kHz and P_{-k} is P_0 shifted by -60 kHz. Similarly Q_k and Q_{-k} represent Q_0 shifted by ± 60 kHz. The sampled signal spectrum in Figure S8.1-8b shows the labels of various repeated segments. It is clear from Figure S8.1-8b that we can reconstruct the signal $x(t)$ from this spectrum by passing it through an ideal bandpass filter (shown dotted) centered at 25 Hz and having bandwidth 10Hz. The filter gain is $T = 1/60$.

Figure S8.1-8c shows the sampled signal spectrum with f_s =20 Hz (spectrum repeating at every 20 Hz). It is clear from this figure that we can reconstruct $x(t)$ from this spectrum also because the original segments P_0 and Q_0 are still intact without being overlapped by any other repeating segments.

Figure S8.1-8d shows the spectrum of the signal $y(t)$ sampled at a rate 20 Hz (spectrum repeating at every 20 Hz). The figure shows that P_0 overlaps with Q_{-2} and Q_0 overlaps with P_2. Hence it is impossible to reconstruct $y(t)$ from this spectrum.

8.1-9. This problem is trivial when worked out in the frequency-domain. The sampled signal spectrum is given by

$$\overline{X}(\omega) = \frac{1}{T} \sum_{n=-\infty}^{\infty} X(\omega - 2\pi n f_s)$$

We repeat the spectrum periodically with period $(f_1 + f_2)$ Hz, as shown in Figure S8.1-9. The amplitude at the origin is $1/T = f_1 + f_2$. From Figure S8.1-8a, it is obvious that the resulting spectrum $\overline{X}(\omega)$ is constant for all ω and has a value $f_1 + f_2$. Hence

$$\overline{X}(\omega) = f_1 + f_2$$

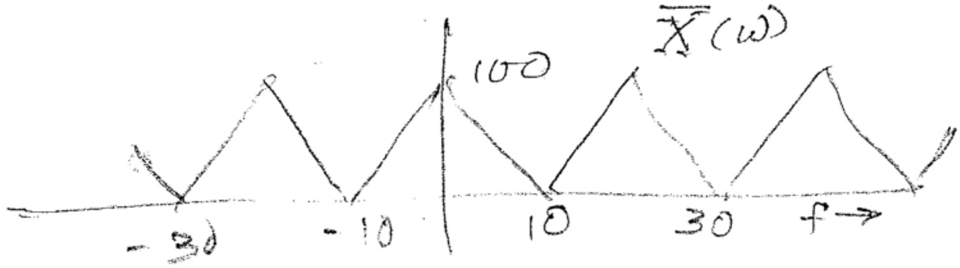
and

$$\overline{x}(t) = (f_1 + f_2)\delta(t)$$

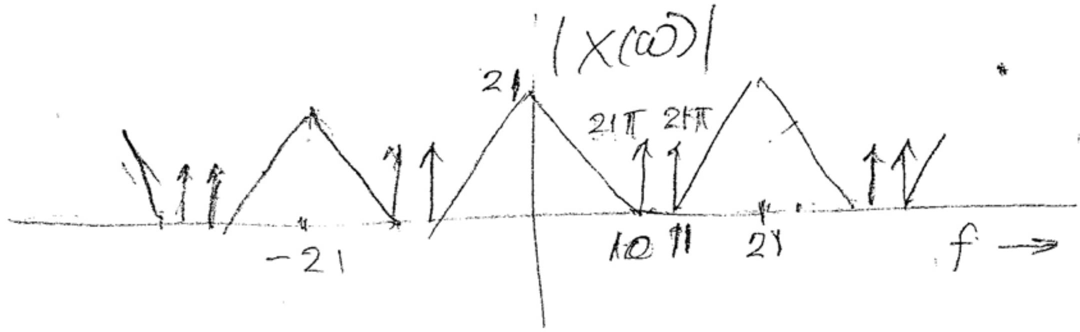
Hence all the samples of $x(t)$ at a rate $f_s = f_1 + f_2$ are zero except the sample at $t = 0$

FIG. S 8.1-7 (CONTINUED)

(c)



(D)



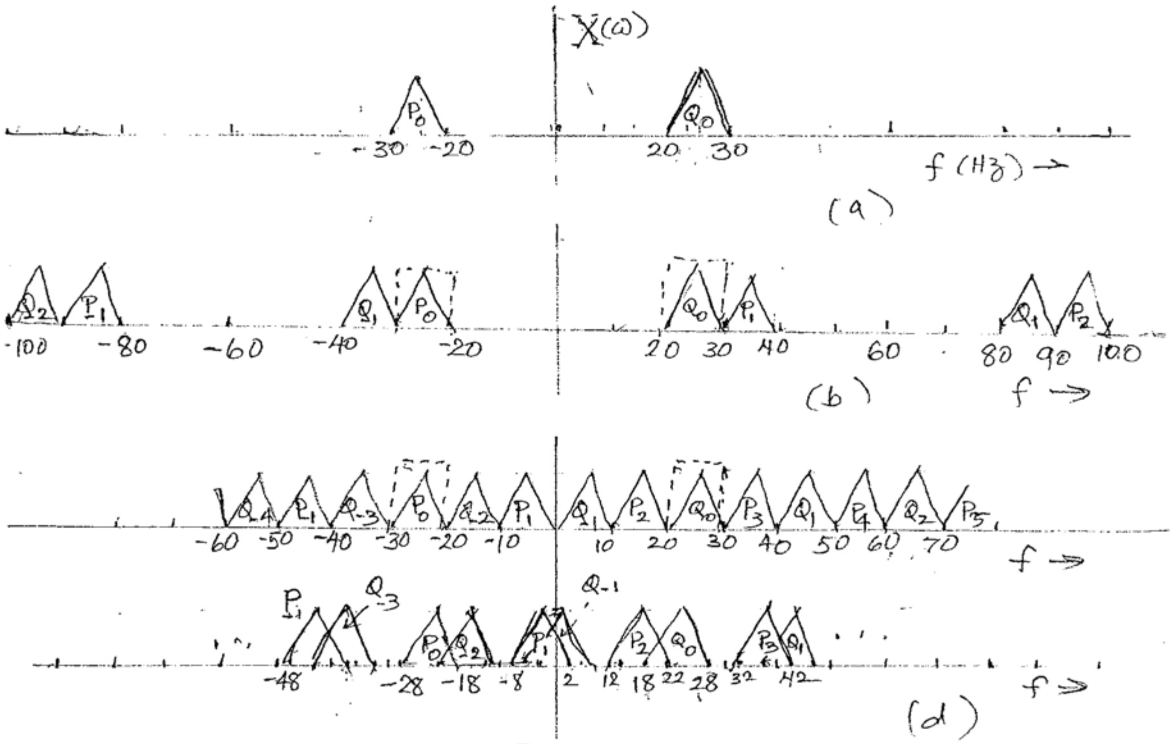


Figure S8.1-8

or $n = 0$.

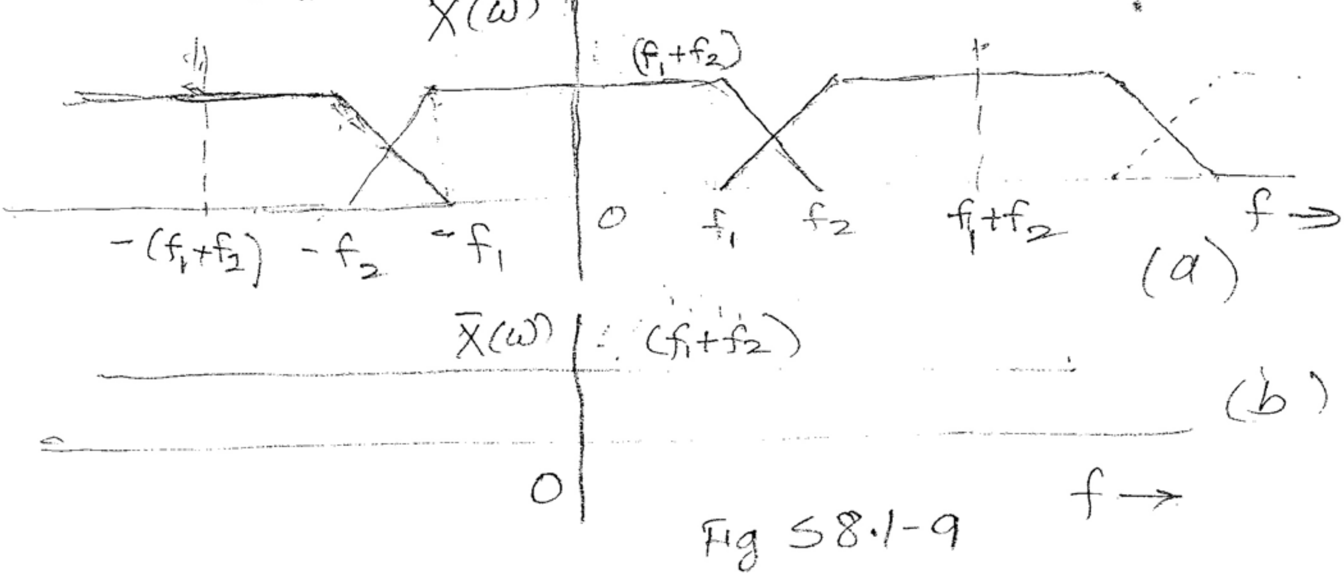


Figure S8.1-9

8.1-10. The Maximum Information Rate: 2 Pieces of Information per Second per Hertz.

A knowledge of the maximum rate of information that can be transmitted over a channel of bandwidth B Hz is of fundamental importance in digital communication. We can demonstrate a scheme which allows errorfree transmission of $2B$ independent pieces of information per second over a channel of bandwidth B Hz. Recall that a continuous-time signal $x(t)$ of bandwidth B Hz can be constructed from its Nyquist samples (which are at a rate of $2B$ Hz) using the interpolation formula (8.11). Letting the the kth piece of information equal to $x(kT)$, the kth Nyquist sample, and using the interpolation formula (8.11), we can construct a signal $x(t)$ that is bandlimited to B Hz. Clearly, this signal can be transmitted errorfree over the channel of bandwidth B Hz. Moreover, the $2B$ pieces of information are readily obtained (errorfree) from this signal by taking its Nyquist samples.

This theoretical rate of communication assumes a noisefree channel. In practice, channel noise is unavoidable, and consequently, this rate will cause some detection errors.

We shall prove the converse of this result using the method of *reductio ad absurdum*. Independent piece implies that each piece of information can be any one of the (uncountably) infinite number of amplitudes. To prove the converse, let us assume that a scheme exists that can transmit more than $2B$ independent pieces of information/second. If this were the case, the interpolation formula implies that we can transmit a signal of bandwidth higher than B Hz over a channel of bandwidth B Hz, which is not possible.

8.1-11. (a)

$$X(\omega) = \frac{1}{4}\text{rect}\left(\frac{\omega}{8\pi}\right)$$

This is a rect spectrum of band 4π rad/s or 2 Hz. To find the sampled signal spectrum $\overline{X}(\omega)$, we multiply $X(\omega)$ by $\frac{1}{T} = 4$ and repeat it periodically with period $f_s = 4$. the result is as shown in Figure S8.1-11a. Thus $\overline{X}(\omega) = 1$ for all ω.

(b) To reconstruct $x(t)$ from $\overline{X}(\omega)$, we pass $\overline{X}(\omega)$ through an ideal lowpass filter of gain $T = \frac{1}{4}$ and bandwidth $f_s/2 = 2$Hz (Figure S8.1-11b). The input is $\overline{X}(\omega) = 1$. Hence the output is $\frac{1}{4}\text{rect}\left(\frac{\omega}{8\pi}\right)$. The output is indeed $x(t)$, as expected.

(c) If we sample $x(t)$ at a rate 2 Hz $(T = 1/2)$,

$$\overline{X}(\omega) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X(\omega - 2\pi n f_s) = \frac{1}{2}\sum_{n=-\infty}^{\infty} \text{rect}\left(\frac{\omega - 4\pi n}{8\pi}\right)$$

Thus $\overline{X}(\omega)$ is obtained by repeating $\frac{1}{2}\text{rect}\left(\frac{\omega}{8\pi}\right)$ with period $\omega = 4\pi$ (or $f_s = 2$ Hz), as shown in Figure S8.1-11c. The two adjacent spectra, each of amplitude 1/2 overlap, yielding a constant value $\overline{X}(\omega) = 1$. When this $\overline{X}(\omega)$ is applied to the filter in Figure S8.1-11b, the output be again $x(t) = \frac{1}{4}\text{rect}(\omega/8\pi)$.

(d) We get identical result using f_s=1 Hz, where $X(\omega)$ repeats every 2π, but the amplitude is only 1/4. For any ω, we get 4 overlapping spectra, yielding $\overline{X}(\omega) = 1$.

(e) The signal $x(t) = \text{sinc}(4\pi t)$ is shown in Figure S8.1-11d. Observe that $x(t) = 0$ at $t = \frac{n}{4}$ for all positive and negative integer value of n. This means sampling $x(t)$ at a rate $\frac{4}{N}$ or $T = \frac{T}{4}$ yields all zero valued samples except at $t = 0 (N = 0)$.

(f) The signal $x(t) = \text{sinc}(4\pi t)$ is bandlimited to $B = 2$ Hz. Hence, its Nyquist rate is 4 Hz and the Nyquist interval is $T = 1/4$. As seen from Fig. S8.1-11d, the samples of this signal at $t = nT$ are zero at $t = M/4$ for any positive or negative integer M. In other words, if the sampling rate is $4M$ Hz, where M any arbitrarily large integer, the samples are given by

$$x(nT) = \begin{cases} 1 & n = 0 \\ 0 & n = \pm 1, \pm 2, \pm 3, \dots \end{cases}$$

Thus, we obtain the same set of samples if we sample the signal at a rate 4 Hz or 8 Hz, or 12 Hz, or $4M$ Hz, for any integer value of M. Therefore, regardless of the value of M, we can reconstruct the signal $x(t) = \text{sinc}(4\pi t)$ from these samples using the interpolator in Fig. S8.1-11b.

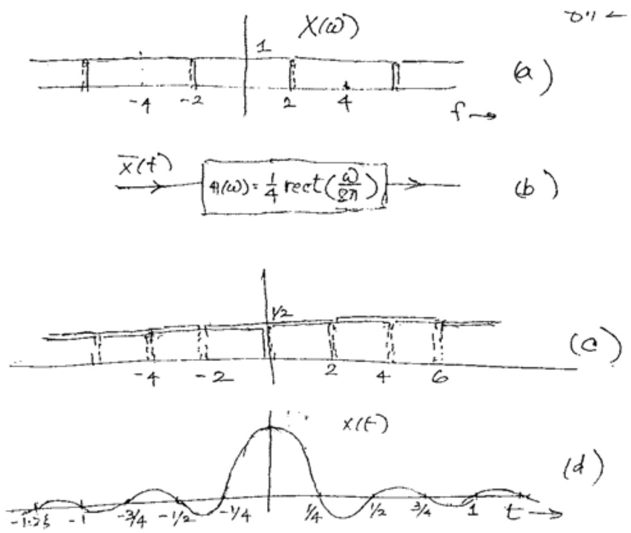


Figure S8.1-11

8.2-1. The signal $x(t) = \text{sinc}(200\pi t)$ is sampled by a rectangular pulse sequence $p_T(t)$ whose period is 4 ms so that the fundamental frequency (which is also the sampling frequency) is 250 Hz. Hence, $\omega_s = 500\pi$. The Fourier series for $p_T(t)$ is given by

$$p_T(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos n\omega_s t$$

Use of Eqs. (3.66) yields $C_0 = \frac{1}{5}, \quad C_n = \frac{2}{n\pi} \sin\left(\frac{n\pi}{5}\right)$, that is,

$$C_0 = 0.2, \quad C_1 = 0.374, \quad C_2 = 0.303, \quad C_3 = 0.202, \quad C_4 = 0.093, \quad C_5 = 0, \cdots$$

Consequently

$$\overline{x}(t) = x(t)p_T(t) = 0.2\,x(t) + 0.374x(t)\,\cos 500\pi t + 0.303\,x(t)\,\cos 1000\pi t + 0.202\,x(t)\,\cos 1500\pi t + \cdots$$

and

$$\begin{aligned} \overline{X}(\omega) &= 0.2\,X(\omega) + 0.187\,[X(\omega - 500\pi) + X(\omega + 500\pi)] \\ &\quad + 0.151\,[X(\omega - 1000\pi) + X(\omega + 1000\pi)] \\ &\quad + 0.101\,[X(\omega - 1500\pi) + X(\omega + 1500\pi)] + \cdots \end{aligned}$$

In the present case $X(\omega) = 0.005\,\text{rect}(\frac{\omega}{400\pi})$. The spectrum $\overline{X}(\omega)$ is shown in Figure S8.2-1. Observe that the spectrum consists of $X(\omega)$ repeating periodically at the interval of 500π rad/s (250 Hz). Hence, there is no overlap between cycles, and $X(\omega)$ can be recovered by using an ideal lowpass filter of bandwidth 100 Hz. An ideal lowpass filter of unit gain (and bandwidth 100 Hz) will allow the first term on the he right-side of the above equation to pass fully and suppress all the other terms. Hence the output $y(t)$ is

$$y(t) = 0.2\,x(t)$$

Because the spectrum $\overline{X}(\omega)$ has a zero value in the band from 100 to 150 Hz, we can use an ideal lowpass filter of bandwidth B Hz where $100 < B < 150$. But if $B > 150$ Hz, the filter will pick up the unwanted spectral components from the next cycle, and the output will be distorted.

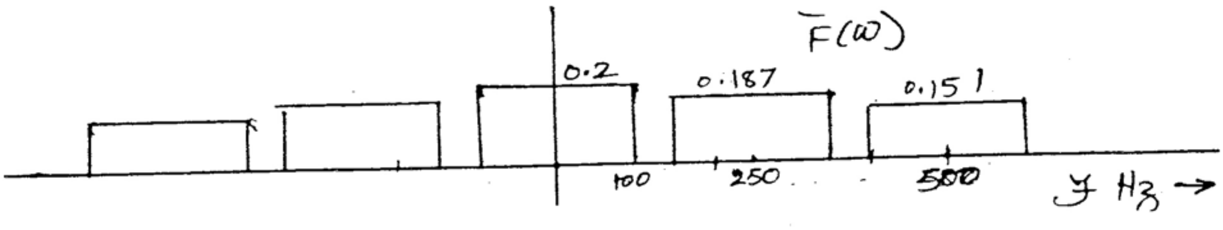


Figure S8.2-1

8.2-2. (a) When the input is $\delta(t)$, the input of the integrator is $[\delta(t) - \delta(t-T)]$. And, $h(t)$, the output of the integrator is:

$$h(t) = \int_0^t [\delta(\tau) - \delta(\tau - T)]\, d\tau = u(t) - u(t-T) = \text{rect}\left(\frac{t - \frac{T}{2}}{T}\right)$$

The impulse response $h(t)$ is shown in Figure S8.2-2a.

(b) The transfer function of this circuit is:

$$H(\omega) = T\text{sinc}\left(\frac{\omega T}{2}\right)e^{-j\omega T/2}$$

$$|H(\omega)| = T\left|\text{sinc}\left(\frac{\omega T}{2}\right)\right|$$

The amplitude response of the filter is shown in Figure S8.2-2b. Observe that the filter is a lowpass filter of bandwidth $2\pi/T$ rad/s or $1/T$ Hz. The impulse response of the circuit is a rectangular pulse. When a sampled signal is applied at the input, each sample generates a rectangular pulse at the output, proportional to the corresponding sample value. Hence the output is a staircase approximation of the input as shown in Figure S8.2-2c.

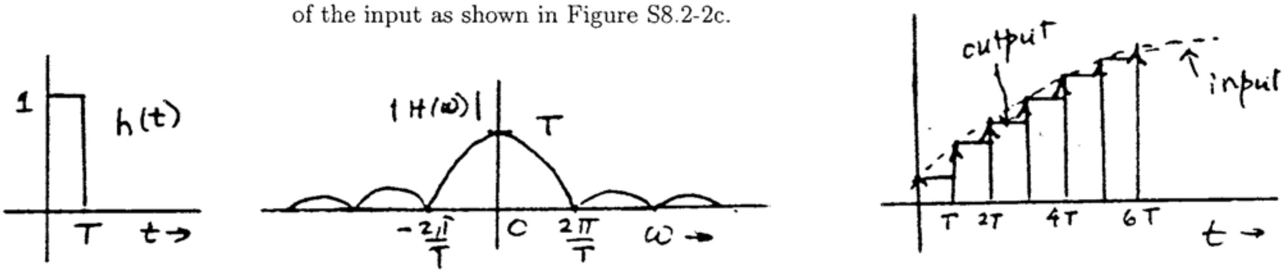


Figure S8.2-2

8.2-3. (a) Figure S8.2-3a shows the signal reconstruction from its samples using the first-order hold circuit. Each sample generates a triangle of width $2T$ and centered at the sampling instant. The height of the triangle is equal to the sample value. The resulting signal consists of straight line segments joining the sample tops.

(b) The frequency response of this circuit is:

$$H(\omega) = \mathcal{F}\{h(t)\} = \mathcal{F}\left\{\Delta\left(\frac{t}{2T}\right)\right\} = T\,\text{sinc}^2\left(\frac{\omega T}{2}\right)$$

Because $H(\omega)$ is positive for all ω, it also represents the amplitude response. Figure S8.2-3b shows this amplitude response and the ideal amplitude response (lowpass) required for signal reconstruction.

(c) A minimum of T secs delay is required to make $h(t)$ causal (realizable). Such a delay would cause the reconstructed signal in Figure S8.2-3a to be delayed by T secs.

(d) The impulse response and the frequency response of a ZOH circuit are

$$h(t) = \text{rect}\left(\frac{t - T/2}{T}\right)$$

and

$$H_{\text{zoh}}(\omega) = T\,\text{sinc}(\omega T/2)e^{-j\omega T/2}$$

The frequency response of the cascade two sections of the ZOH circuits is given by

$$H_{\text{cascade}}(\omega) = T^2\,\text{sinc}^2(\omega T/2)e^{-j\omega T}$$

In part (b), we found that the frequency response of an FOH circuit as

$$H_{\text{foh}}(\omega) = T\,\text{sinc}^2(\omega T/2)$$

This shows that the frequency response of the cascading two ZOH circuits is T times the frequency response of the FOH circuit with time delay T seconds. the time delay is a desirable feature as it makes the FOH circuit causal, and therefore, realizable. Thus, the cascade of two ZOH acts identical to an FOH circuit except for the amplification by factor T and delay by T seconds.

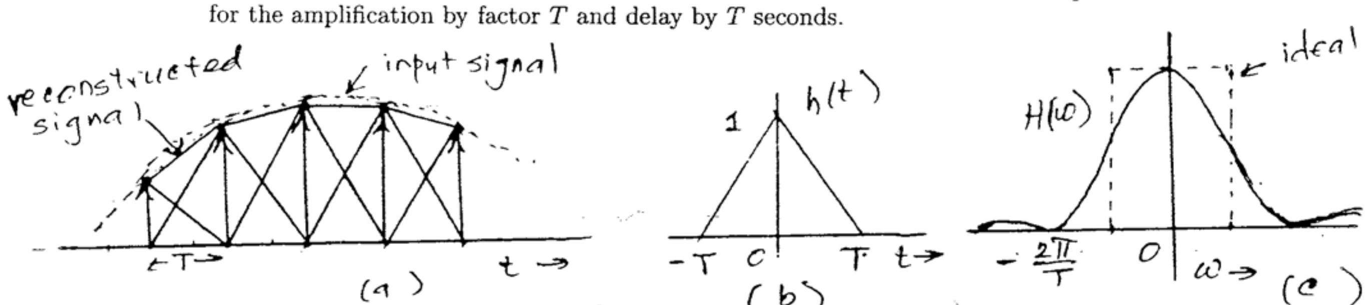


Figure S8.2-3

8.2-4. Consider a sampling pulse $p(t)$ of arbitrary width. The development in the text in Sec. 8.1-1 applies to pulse of arbitrary width. Hence, we can use the result in Eq. (8.7).

$$\overline{x}(t) = C_0 x(t) + \sum_{n=1}^{\infty} C_n x(t)\cos(n\omega_s t + \theta_n)$$

where C_i are the coefficients of the compact trigonometric Fourier series for $p_T(t)$ with $p(t) = e^{-at}u(t)$. Thus

$$p_T(t) = \sum_{k=-\infty}^{\infty} e^{-a(t-kT)}u(t-kT)$$

This is a periodic signal of period T and we can find the Fourier series for $p_T(t)$

$$p_T(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_s t + \theta_n)$$

Now, we use the arguments in Sec. 8.1-1 to show that $x(t)$ can be constructed from $\overline{x}(t) = x(t)p_T(t)$, provided the sampling rate is no less that $2B$ Hz or $T < 1/2B$.

8.2-5. The signal $x(t)$, when sampled by an impulse train, results in the sampled signal $x(t)\delta_T(t)$ (as shown in Figure 5.1d). If this signal is transmitted through a filter (Figure S8.2-5a) whose impulse response is $h(t) = p(t) = \text{rect}(\frac{t}{0.025})$, then each impulse in the input will generate a pulse $p(t)$, resulting in the desired sampled signal shown in Figure S8.2-5. Moreover, the sampled signal spectrum (impulse sampling) is $\frac{1}{T}\sum_{n=-\infty}^{\infty} X(\omega - n\omega_s)$. Hence, the output of the filter in Figure S8.2-5a is

$$\overline{X}(\omega) = H(\omega)\left[\frac{1}{T}\sum_{n=-\infty}^{\infty} X(\omega - n\omega_s)\right]$$

where $H(\omega) = P(\omega) = 0.025\text{sinc}(\frac{\omega}{80})$, the Fourier transform of $\text{rect}(\frac{t}{0.025})$. Figure S8.2-5b shows this spectrum consisting of the repeating spectrum $X(\omega)$ multiplied by $H(\omega) = 0.025\text{sinc}(\frac{\omega}{80})$. Thus, each cycle is somewhat distorted.
To recover the signal $x(t)$ from the flat top samples, we reverse the process in Figure S8.2-5a. First, we pass the sampled signal through a filter with transfer function $1/H(\omega)$. This will yield the signal sampled by impulse train. Now we pass this signal through an ideal lowpass filter of bandwidth B Hz to obtain $x(t)$.

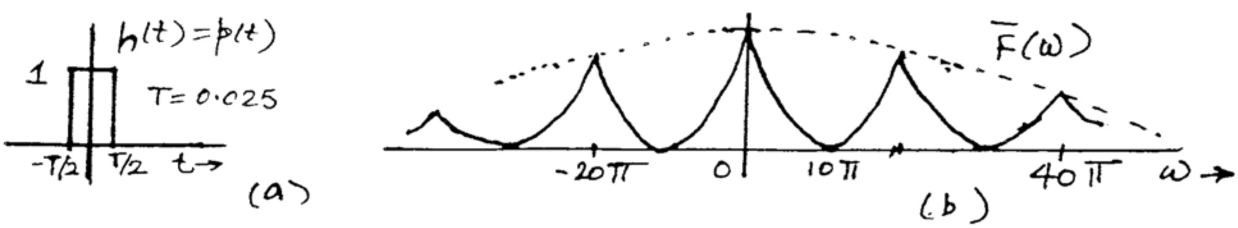


Figure S8.2-5

8.2-6.

$$f_s = 20\text{Hz}$$

$$|f_a| = |f - mf_s| \qquad |f_a| \le f_s/2 = 10$$

(a) $f = 8$ Hz is less that $f_s/2 = 10$ Hz.
Hence, this frequency is not aliased and $|f_a| = f = 8$ Hz

(b) $f = 12$ Hz
$|f_a| = |12 - 20| = 8$ Hz

(c) $f = 20$ Hz
$|f_a| = |20 - 20| = 0$ Hz

(d) $f = 22$ Hz
$|f_a| = |22 - 20| = 2$ Hz

(e) $f = 32$ Hz
$|f_a| = |32 - 40| = 8$ Hz

8.2-7. (a) $f_0 < 30$ Hz and $|f_a| = 20$ Hz.
Hence, $20 = |f_0 - 60m| \Longrightarrow m = 0$ and $f_0 = 20$Hz

(b) $20 = |f_0 - 60m| \Longrightarrow m = 1$ and $f_0 = 40$Hz

(c) $20 = |f_0 - 60m| \Longrightarrow m = 1$ and $f_0 = 80$Hz

(d) $20 = |f_0 - 60m| \Longrightarrow m = 2$ and $f_0 = 100$Hz

8.2-8.

$$X(\omega) = \pi\left[3\delta(\omega \pm 6\pi)) + \delta(\omega \pm 16\pi)) + 2\delta(\omega \pm 20\pi))\right]$$

The highest frequency is 10 Hz. The Nyquist rate is 20 Hz. 25% above this rate is 25 Hz is the actual sampling rate. Hence $T = 0.04$. Therefore $\overline{X}(\omega)$ consists of $25X(\omega)$ repeating periodically with period 25 Hz (or 50π rad/sec), as shown in Figure S8.2-8a. To reconstruct $x(t)$ from the sampled signal $\overline{x}(t)$, we pass $\overline{X}(\omega)$ through a lowpass filter of gain 1/25 and having a cutoff frequency anywhere between $(10 + \epsilon)$ Hz to $(15 - \epsilon)$ Hz where ϵ is an arbitrarily small number.
If the sampling rate is 25% below the NYquist rate, that is $f_s = 15$ Hz, the components of frequencies 8 Hz and 10 Hz will be aliased to other frequencies. The 8 Hz will appear as $|f_a| = |8 - 15| = 7$ Hz and 10 Hz will be aliased as $|10 - 15| = 5$ Hz. Hence the output contains frequencies 3 Hz, 5 Hz and 7 Hz.

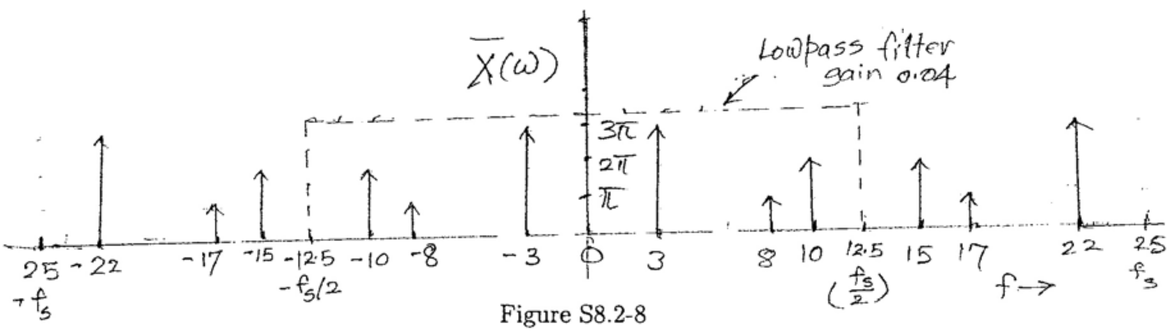


Figure S8.2-8

8.2-9. (a) The output in Eq. (8.11a) is the output of an ideal lowpass filter of bandwidth $B = 1/2T$ Hz. Clearly its bandwidth must be $\leq 1/2T$ Hz.

(b) Suppose there is another signal $\hat{x}(t)$ that passes through the samples $x(nT)$ and has bandwidth smaller than that of $x(t)$ obtained through Eq. (8-11a). Clearly, both $x(t)$ and $\hat{x}(t)$ have bandwidth $\leq 1/2T$ and the signal $x(t) - \hat{x}(t)$ also has bandwidth $\leq 1/2T$ Hz. Hence, we can reconstruct the signal $x(t) - \widehat{x}(t)$ from samples of $x(t) - \hat{x}(t)$ at a rate $1/T$ Hz and using these samples in Eq. (8.11a). But because both $x(t)$ and $\hat{x}(t)$ pass through sample $x(nT)$, the samples of $x(t)-\hat{x}(t)$ at a rate $1/T$ Hz are zero for all n. Clearly $x(t)-\hat{x}(t)=0$ and $\hat{x}(t)=x(t)$.

8.2-10. $T=1/R$ and the sample values are

$$p(0)=1, p(\pm T)=p(\pm 2T)=\cdots=p(\pm nT)=\cdots=0$$

We can use Eq. (8.11a) to reconstruct $p(t)$ from these sample values. There is only one nonzero-valued sample. Hence, we obtain

$$p(t)=p(0)\text{sinc}\left(\frac{\pi t}{T}\right)=\text{sinc}\,(\pi R t)$$

and

$$P(\omega)=\frac{1}{R}\text{rect}\left(\frac{\omega}{2\pi R}\right)$$

This is the only signal that has bandwidth πR rad/sec or $R/2$ Hz with samples $p(0)=1$ and $p(nT)=0$ for all $n\neq 0$.

8.2-11. Let us sample the pulse $p(t)$ at a rate R Hz or $T=1/R$ sec. The spectrum of the sampled signal consists of $TP(\omega)=RP(\omega)$ repeated periodically with period R Hz, as shown in Figure S8.2-11. Because of the odd symmetry of $P(\omega)$ about the dotted axis, the overlapping spectra add to a constant value 1 for all ω. Hence

$$\overline{P}(\omega)=1 \qquad \text{and} \qquad \overline{p}(t)=\delta(t)$$

This shows that

$$p(nT)=\begin{cases}1 & n=0\\ 0 & n\neq 0\end{cases} \qquad \text{and} \qquad T=1/R$$

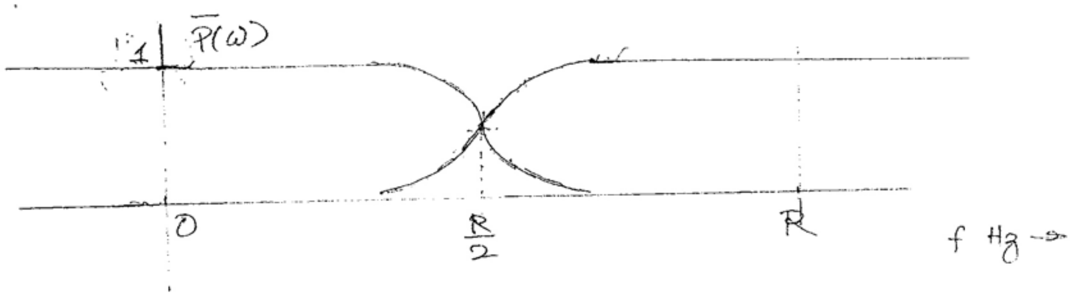


Figure S8.2-11

8.2-12. The Nyquist interval $T=1/2B$. The Nyquist samples are $x(\pm n/2B)$ for $(n=0,1,2,3,\ldots)$. We are given that $x(0)=x(1/2B)=1$ and $x(n/2B)=0$ for all other n. Hence, from Eq. (8.11b)

$$x(t)=\text{sinc}(2\pi Bt)+\text{sinc}(2\pi Bt-\pi)=\frac{\sin 2\pi Bt}{2\pi Bt}-\frac{\sin 2\pi Bt}{2\pi Bt-\pi}=\frac{\sin 2\pi Bt}{2\pi Bt(1-2Bt)}=\frac{\text{sinc}(2\pi Bt)}{1-2Bt}$$

8.2-13. (a) From Eq. (8.11a)

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT)\text{sinc}(Wt - n\pi) \qquad W = \frac{\pi}{T}$$

and

$$\begin{aligned}\int_{-\infty}^{\infty} x(t)dt &= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \text{sinc}(Wt - n\pi)dt \\ &= \sum_{n=-\infty}^{\infty} x(nT) \int_{-\infty}^{\infty} \text{sinc}(Wt - n\pi)dt \\ &= \frac{1}{W} \sum_{n=-\infty}^{\infty} x(nT) \int_{-\infty}^{\infty} \text{sinc}(t)dt\end{aligned}$$

Using the results in Prob. 7.1-9, we have

$$\int_{-\infty}^{\infty} x(t)dt = \frac{\pi}{W} \sum_{n=-\infty}^{\infty} x(nT) = T \sum_{n=-\infty}^{\infty} x(nT)$$

(b)

$$\begin{aligned}\int_{-\infty}^{\infty} |x(t)|^2 dt &= \int_{-\infty}^{\infty} x(t)x^*(t)dt \\ &= \int_{-\infty}^{\infty} \left[\sum_{n=-\infty}^{\infty} x(nT)\text{sinc}(Wt - nT)\right] \left[\sum_{m=-\infty}^{\infty} x^*(mT)\text{sinc}(Wt - mT)\right] dt\end{aligned}$$

Because of the orthogonality of sinc function, stated in Prob. 7.6-5, all the cross-product terms when $m \neq n$ vanish. Only the terms for $m = n$ survive. Using the results in Prob. 7.6-5, we obtain

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{\pi}{W} \sum_{n=-\infty}^{\infty} x(nT)x^*(nT) = T \sum_{n=-\infty}^{\infty} |x(nT)|^2$$

8.2-14. Assume a signal $x(t)$ that is simultaneously timelimited and bandlimited. Let $X(\omega) = 0$ for $|\omega| > 2\pi B$. Therefore $X(\omega)\text{rect}(\frac{\omega}{4\pi B'}) = X(\omega)$ for $B' > B$. Therefore from the time-convolution property (4.42)

$$\begin{aligned}x(t) &= x(t) * [2B'\text{sinc}(2\pi B't)] \\ &= 2B'x(t) * \text{sinc}(2\pi B't)\end{aligned}$$

Because $x(t)$ is timelimited. $x(t) = 0$ for $|t| > T$. But $x(t)$ is equal to convolution of $x(t)$ with $\text{sinc}(2\pi B't)$ which is not timelimited. It is impossible to obtain a time-limited signal from the convolution of a time-limited signal with a non-timelimited signal.

8.3-1. (a) The bandwidth is 15 kHz. The Nyquist rate is 30 kHz.

(b) $65536 = 2^{16}$, so that 16 binary digits are needed to encode each sample.

(c) $30000 \times 16 = 480000$ bits/s.

(d) $44100 \times 16 = 705600$ bits/s.

8.3-2. (a) The Nyquist rate is $2 \times 4.5 \times 10^6 = 9$ MHz. The actual sampling rate $= 1.2 \times 9 =$ 10.8 MHz.

(b) $1024 = 2^{10}$, so that 10 bits or binary pulses are needed to encode each sample.

(c) $10.8 \times 10^6 \times 10 = 108 \times 10^6$ or 108 Mbits/s.

8.3-3. (a) For $L = 16$, we need a 4-bit binary code because $16 = 2^4$. The natural binary code is given in the text. We give here two's complement code, which is used in signal processing.

Digit	binaty code	Digit	binary code
0	0111	8	1111
1	0110	9	1110
2	0101	10	1101
3	0100	11	1100
4	0011	12	1011
5	0010	13	1010
6	0001	14	1001
7	0000	15	1000

For a quaternary code, we use four symbols 0, 1, 2, and 3. For this code, we need only a group of 2 symbols to form 16 combinations $(4 \times 4) = 16$. One possible quaternary code is given below.

Digit	code	Digit	code	Digit	code	Digit	code
0	00	4	10	8	20	12	30
1	01	5	11	9	21	13	31
2	02	6	12	10	22	14	32
3	03	7	13	11	23	15	33

(b) Let a minimum of b_2 binary digits of b_4 quaternary digits be required to represent L level. Now b_2 binary digits can form at most 2^{b_2} distinct combinations. Similarly b_4 quaternary digits can form at most 4^{b_4} distinct combinations. Hence

$$L = 2^{b_2} = 4^{b_4}$$

and

$$b_2 \log 2 = b_4 \log 4 = 2b_4 \log 2$$

Hence

$$b_2/b_4 = 2$$

8.3-4. If V is the peak sample amplitude, then

$$\text{quantization error} \leq \frac{(0.2)(V)}{100} = \frac{V}{500}$$

Because the maximum quantization error is $\frac{\Delta}{2} = \frac{2V}{2L} = \frac{V}{L}$, it follows that

$$\frac{V}{L} = \frac{V}{500} \quad \Longrightarrow \quad L = 500$$

Because L should be a power of 2, we choose $L = 512 = 2^9$. This requires a 9-bit binary code per sample. The Nyquist rate is $2 \times 1000 = 2000$ Hz. 20% above this rate is $2000 \times 1.2 = 2400$ Hz. Thus, each signal has 2400 samples/second, and each sample is encoded by 9 bits. Therefore, each signal uses $9 \times 2400 = 21.6$ kbits/second. Five such signals are multiplexed. Hence, we need a total of $5 \times 21.6 = 108$ kBits/second data bits.

8.4-1. In section 8.4, we have shown that when a timelimited signal $x(t)$ is repeated periodically with a period $T_0 > \tau$ (the signal duration), the Fourier series coefficients for the resulting periodic signal $x_{T_0}(t)$ are proportional to the samples of $X(\omega)$, the Fourier transform of $x(t)$ at frequency interval of $f_0 = \frac{1}{T_0}$ Hz. This result is quite general and applied even if $x(t)$ is bandlimited, and therefore, nontimelimited. To show this we convolve $x(t)$ with unit impulse train $\delta_T(t)$. This will result in periodic repetition of $x(t)$ with period T.
Moreover

$$\begin{aligned} y(t) = x(t) * \delta_T(t) \quad &\Longleftrightarrow \quad \frac{1}{2\pi} X(\omega) \left[\frac{2\pi}{T}\delta_{\omega_s}(\omega)\right] \\ &= \quad \frac{1}{T} X(\omega)\delta_{\omega_s}(\omega) \qquad \omega_s = \frac{2\pi}{T} \end{aligned}$$

In the present case, $T = \frac{1.25}{B}$ and the fundamental frequency is $\omega = 2\pi/T$.
Therefore

$$Y(\omega) = \frac{1}{T} X(\omega) \sum_{n=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi n}{T}\right)$$

Hence $Y(\omega)$ represents the spectrum $\frac{1}{T}X(\omega)$ sampled at intervals of $\frac{1}{T}$ Hz. This means $y(t)$ is a periodic signal with fundamental frequency $f_0 = \frac{1}{T}$

$$y(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_s t}$$

where

$$D_n = \frac{1}{T} X(n\omega_s) = \frac{1}{T} X\left(\frac{2\pi n}{T}\right)$$

Moreover, $T = \frac{1.25}{B}$. Hence, the fundamental frequency $f_0 = \frac{B}{1.25} = 0.8B$. But $X(\omega)$ is bandlimited to B Hz. This means $Y(\omega)$ contains only the dc and the fundamental component. Frequencies of all the remaining components are beyond $1.6B$, where $X(\omega) = 0$ and hence $Y(\omega) = 0$. The nonzero component amplitudes are $D_0 = \frac{1}{T}X(0)$ and $D_1 = \frac{1}{T}X\left(\frac{2\pi}{T}\right)$. We can write $y(t)$ as a trigonometric Fourier series

$$y(t) = C_0 + C_1 \cos(1.6\pi B t + \theta_1)$$

where

$$C_0 = D_0 = \frac{1}{T} X(0)$$

and

$$C_1 = 2|D_1| = \frac{2}{T}\left|X\left(\frac{2\pi}{T}\right)\right|$$

and

$$\theta_1 = \angle D_1 = \angle X\left(\frac{2\pi}{T}\right)$$

8.5-1.

$$T_0 = \frac{1}{f_o} = \frac{1}{50} = 20\text{ms}$$

$$B = 10000 \quad \text{Hence} \quad f_s \geq 2B = 20000$$

$$T = \frac{1}{f_s} = \frac{1}{20000} = 50\mu s$$

$$N_0 = \frac{T_0}{T} = \frac{20 \times 10^{-3}}{50 \times 10^{-6}} = 400$$

Since N_0 must be a power of 2, we choose $N_0 = 512$. Also $T = 50\mu$s, and $T_0 = N_0 T = 512 \times 50\mu s = 25.6$ms, $f_o = 1/T_0 = 39.0625$ Hz. Since $x(t)$ is of 10 ms duration, we need zero padding over 15.6 ms. Alternatively, we could also have used

$$T = \frac{20 \times 10^{-3}}{512} = 39.0625 \ \mu\text{s}$$

This gives $T_0 = 20$ ms, $f_o = 50$ Hz. And

$$f_s = \frac{1}{T} = 25600\text{Hz}$$

There are also other possibilities of reducing T as well as increasing the frequency resolution.

8.5-2. For the signal $x(t)$,

$$T_0 \geq \frac{1}{0.25} = 4, \qquad T \leq \frac{1}{f_s} = \frac{1}{3 \times 2} = \frac{1}{6}$$

Let us choose $T = 1/8$. Also $T_0 = 4$. Therefore, $N_0 = T_0/T = 32$. The signal $x(t)$ repeats every 4 seconds with samples every 1/8 second. The samples are $Tx(kT) = (1/8)x(k/8)$. Thus, the first sample is (at $k = 0$) $1 \times (1/8) = 1/8$. The 32 samples are (starting at $k = 0$)

$$\frac{1}{8}, \frac{7}{64}, \frac{3}{32}, \frac{5}{64}, \frac{1}{16}, \frac{3}{64}, \frac{1}{32}, \frac{1}{64}, 0, 0, 0, 0, 0, 0, 0, 0,$$
$$0, 0, 0, 0, 0, 0, 0, 0, 0, \frac{1}{64}, \frac{1}{32}, \frac{3}{64}, \frac{1}{16}, \frac{5}{64}, \frac{3}{32}, \frac{7}{64}$$

The samples of $x(t)$ and $g(t)$ are shown in Figure S8.5-2.

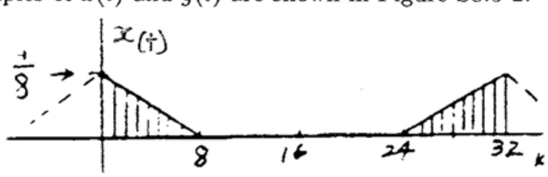


Figure S8.5-2

8.5-3.

$$x(t) = e^{-t}u(t) \qquad X(\omega) = \frac{1}{j\omega + 1}$$

(a) We take the folding frequency f_s to be the frequency where $|X(\omega)|$ is 1% of its peak value, which happens to be 1 (at $\omega = 0$). Hence,

$$|X(\omega)| \approx \frac{1}{\omega} = 0.01 \Rightarrow \omega = 2\pi B = 100$$

This yields $B = 50/\pi$, and $T \leq 1/2B = \pi/100$. Let us round T to 0.03125, resulting in 32 samples per second. The time constant of e^{-t} is 1. For T_0, a reasonable choice is 5 to 6 time constants or more. Value of $T_0 = 5$ or 6 results in $N_0 = 160$ or 192, neither of which is a power of 2. Hence, we choose $T_0 = 8$, resulting in $N_0 = 32 \times 8 = 256$, which is a power of 2.

(b)

$$|X(\omega)| = \frac{1}{\sqrt{\omega^2 + 1}} \simeq \frac{1}{\omega} \quad \omega \gg 1$$

We take the folding frequency f_s to be the 99% energy frequency as explained in Example 4.16. From the results in Example 4.16, (with $a = 1$) we have

$$\frac{0.99\pi}{2} = \tan^{-1}\frac{W}{a} \quad \Rightarrow W = 63.66a = 63.66 \text{ rad/sec.}$$

This yields $B = \frac{W}{2\pi} = 10.13$ Hz. Also $T \leq 1/2B = 0.04936$. This results in the sampling rate $\frac{1}{T} = 20.26$ Hz. Also $T_0 = 8$ as explained in part (a). This yields $N_0 = 20.26 \times 8 = 162.08$, which is not a power of 2. Hence, we choose the next higher value, that is $N_0 = 256$, which yields $T = 0.03125$ and $T_0 = 8$, the same as in part (a).

8.5-4. (a)

$$x(t) = \frac{2}{t^2 + 1}$$

Application of duality property to pair 3 (Table 4.1) yields

$$\frac{2}{t^2 + 1} \Longleftrightarrow 2\pi e^{-|\omega|}$$

Following the approach of Prob. 8.5-2, we find that the peak value of $|X(\omega)| = 2\pi e^{-|\omega|}$ is 2π (occurring at $\omega = 0$). Also, $2\pi e^{-|\omega|}$ becomes $0.01 \times 2\pi$ (1% of the peak value) at $\omega = \ln 100 = 4.605$. Hence, $B = 4.605/2\pi = 0.733$ Hz, and $T \leq 1/2B = 0.682$. Also,

$$x(0) = 2 \quad \text{and} \quad x(t) \simeq \frac{2}{t^2} \quad t \gg 1$$

Choose T_0 (the duration of $x(t)$) to be the instant where $x(t)$ is 1% of $x(0)$.

$$x(T_0) = \frac{2}{T_0^2 + 1} = \frac{2}{100} \Longrightarrow T_0 \approx 10$$

This results in $N_0 = T_0/T = 10/0.682 = 14.66$. We choose $N_0 = 16$, which is a power of 2. This yields $T = 0.625$ and $T_0 = 10$.

(b) The energy of this signal is

$$E_f = \frac{2}{2\pi}\int_0^\infty (2\pi)^2 e^{-2\omega}\, d\omega = 2\pi$$

The energy within the band from $\omega = 0$ to W is given by

$$E_W = \frac{8\pi^2}{2\pi}\int_0^W e^{-2\omega}\,d\omega = 2\pi(1 - e^{-2W})$$

But $E_W = 0.99E_f = 0.99 \times 2\pi$. Hence,

$$0.99(2\pi) = 2\pi(1 - e^{-2W}) \quad \Rightarrow W = 2.303$$

Hence, $B = W/2\pi = 0.366$ Hz. Thus, $T \leq 1/2B = 1.366$. Also, $T_0 = 10$ as found in part (a). Hence, $N_0 = T_0/T = 7.32$. We select $N_0 = 8$ (a power of 2), resulting in $N_0 = 8$ and $T = 1.25$.

8.5-5. The widths of $x(t)$ and $g(t)$ are 1 and 2 respectively. Hence the width of the convolved signal is $1 + 2 = 3$. This means we need to zero-pad $x(t)$ for 2 secs. and $g(t)$ for 1 sec., making $T_0 = 3$ for both signals. Since $T = 0.125$

$$N_0 = \frac{3}{0.125} = 24$$

N_0 must be a power of 2. Choose $N_0 = 32$. This permits us to adjust T_0 to 4. Hence the final values are $T = 0.125$ and $T_0 = 4$. The samples of $x(t)$ and $g(t)$ are shown in Figure S8.5-5.

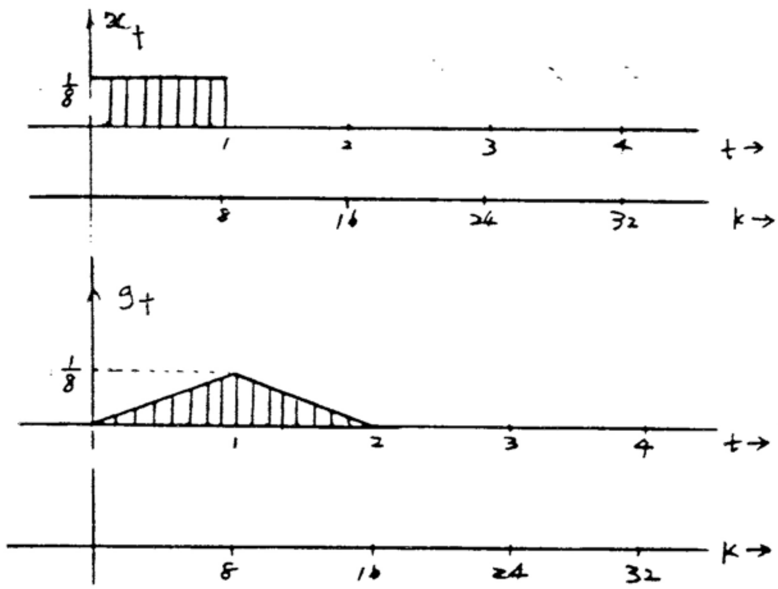


Figure S8.5-5

8.5-6. Valid N-point DFTs need to be N-periodic functions of frequency index r.

(a) Yes, $X(r)$ is a valid DFT. Since $X(r)$ is just a constant, it is periodic in any positive integer. Thus, of the DFT can be any positive integer $N = 1, 2, \ldots$. Since $X(r)$ is not conjugate symmetric, the time domain signal $x[n]$ is complex.

(b) No, $X_2(r)$ is not a valid DFT. Since r is constrained to be an integer, $X(r) = \sin(r/10)$ is not a periodic function in r.

(c) Yes, $X(r)$ is a valid DFT. Since $X(r)$ is periodic with fundamental period equal to 20, the DFT length can be any integer multiple of 20. That is, the DFT length is $N = 20, 40, 60, \ldots$. Since $X(r)$ is not conjugate symmetric, the time-domain signal $x[n]$ is complex.

(d) Yes, $X(r)$ is a valid DFT. Since $X(r)$ is periodic with fundamental period equal to eight, the DFT length can be any integer multiple of 8. That is, the DFT length is $N = 8, 16, 24, \ldots$. Since $X(r)$ is conjugate symmetric, the signal $x[n]$ is real.

(e) Yes, $X(r)$ is a valid DFT. Since $X(r)$ is periodic with fundamental period equal to ten, the DFT length can be any integer multiple of 10. That is, the DFT length is $N = 10, 20, 30, \ldots$. Since $X(r)$ is not conjugate symmetric, the time domain signal $x[n]$ is complex.

8.M-1. Given signal $x[n]$ with DFT $X_r = \sum_{n=0}^{N_0-1} x[n]e^{-jr\Omega_0 n}$, the time-shifting property tells us that $y[n] = x[n - n_0]$ has DFT $Y_r = e^{-jr\Omega_0 n_0} X_r$. Therefore, if MATLAB computes the DFT X_r for a signal $x[n]$ assuming that it starts at 0, then the DFT of the signal shifted to start at n_0 is found by scaling X_r by $e^{-jr\Omega_0 n_0}$.

In MATLAB, this is easy to accomplish. Assuming constants `n_0` and `Omega_0` are defined and DFT `X` is already computed, the corrected DFT `X_shift` is computed by `X_shift = exp(-j*([0:length(X)-1]')*Omega_0*n_0}.*X(:)`.

8.M-2. Ideally, $x_1[n] = e^{j2\pi n30/100} + e^{j2\pi n33/100}$ is characterized by two spikes of equal height located at $f_r = 0.30$ and $f_r = 0.33$. For most cases, two DFT magnitude plots are included: the first covers the entire range of digital frequencies and the second details the range near the true frequency content of $x_1[n]$.

(a)
```
>> n = (0:9); x1 = exp(j*2*pi*n*30/100)+exp(j*2*pi*n*33/100);
>> X1 = fft(x1); f_r = (0:length(x1)-1)/length(x1);
>> stem(f_r-0.5,fftshift(abs(X1)),'k.');
>> xlabel('f_r'); ylabel('|X_1(f_r)|');
>> axis([-0.5 0.5 0 20]);
```

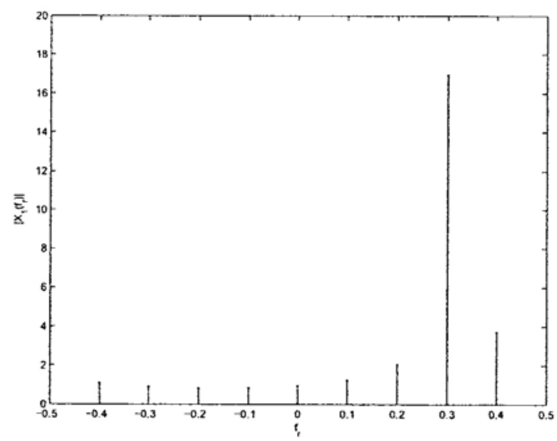


Figure S8.M-2a: DFT of 10-point $x_1[n]$.

For this DFT, only ten samples of $x_1[n]$ are used. As a result, the DFT has only 10 frequency bins uniformly spaced over the frequency interval $[-0.5, 0.5)$. As Figure S8.M-2a shows, there is insufficient frequency resolution to separately identify the two closely spaced exponentials at $f_r = 0.30$ and $f_r = 0.33$.

(b)
```
>> n = (0:9); x1 = exp(j*2*pi*n*30/100)+exp(j*2*pi*n*33/100);
>> x1 = [x1,zeros(1,490)];
>> X1 = fft(x1); f_r = (0:length(x1)-1)/length(x1);
>> subplot(211),stem(f_r-0.5,fftshift(abs(X1)),'k.');
>> xlabel('f_r'); ylabel('|X_1(f_r)|');
>> axis([-0.5 0.5 0 20]);
>> subplot(212),stem(f_r-0.5,fftshift(abs(X1)),'k.');
>> xlabel('f_r'); ylabel('|X_1(f_r)|');
>> axis([0.2 0.4 0 20]);
```

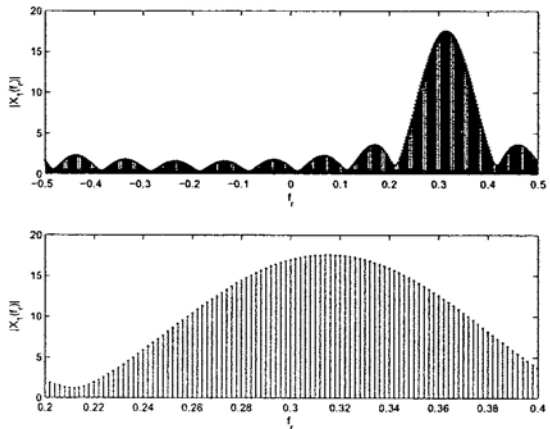


Figure S8.M-2b: DFT of 10-point $x_1[n]$ zero-padded to 500-points.

For this DFT, only ten samples of $x_1[n]$ are used but the sequence is zero-padded to a length of 500. Although the DFT has 500 frequency bins uniformly spaced over the frequency interval $[-0.5, 0.5)$, there is insufficient information about $x_1[n]$ (only 10 samples) to resolve the closely spaced exponentials at $f_r = 0.30$ and $f_r = 0.33$. Using the picket fence analogy, zero-padding increases the number of "pickets" in our DFT fence, but it does not change what lies behind the fence. Still, Figure S8.M-2b does show a concentration of signal energy centered at $f_r = 0.315$, the average of the two exponential frequencies.

(c)
```
>> n = (0:99); x1 = exp(j*2*pi*n*30/100)+exp(j*2*pi*n*33/100);
>> X1 = fft(x1); f_r = (0:length(x1)-1)/length(x1);
>> subplot(211),stem(f_r-0.5,fftshift(abs(X1)),'k.');
>> xlabel('f_r'); ylabel('|X_1(f_r)|');
>> axis([-0.5 0.5 0 110]);
>> subplot(212),stem(f_r-0.5,fftshift(abs(X1)),'k.');
>> xlabel('f_r'); ylabel('|X_1(f_r)|');
>> axis([0.2 0.4 0 110]);
```

For this DFT, 100 samples of $x_1[n]$ are used. As a result, the DFT has 100 frequency bins uniformly spaced over the frequency interval $[-0.5, 0.5)$. The set of DFT bins also happens to include both exponential frequencies $f_r = 0.30$ and $f_r = 0.33$. As Figure S8.M-2c shows, the two exponentials are each easily identified. It is rare, however, for the windowed data record $x_1[n]$ to contain an integer number of periods, as occurs in this case, so Figure S8.M-2c paints a somewhat optimistic picture of the performance of this 100-point DFT.

(d)
```
>> n = (0:99); x1 = exp(j*2*pi*n*30/100)+exp(j*2*pi*n*33/100);
```

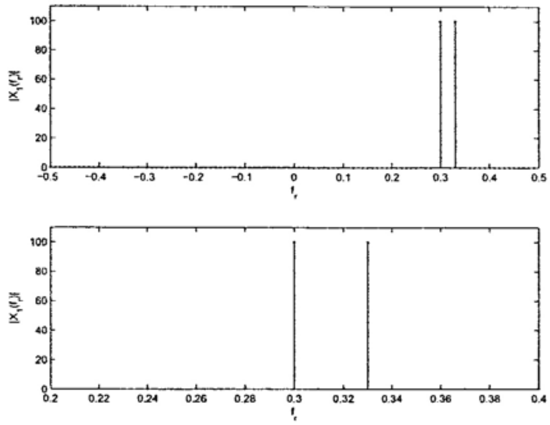


Figure S8.M-2c: DFT of 100-point $x_1[n]$.

```
>> x1 = [x1,zeros(1,400)];
>> X1 = fft(x1); f_r = (0:length(x1)-1)/length(x1);
>> subplot(211),stem(f_r-0.5,fftshift(abs(X1)),'k.');
>> xlabel('f_r'); ylabel('|X_1(f_r)|');
>> axis([-0.5 0.5 0 110]);
>> subplot(212),stem(f_r-0.5,fftshift(abs(X1)),'k.');
>> xlabel('f_r'); ylabel('|X_1(f_r)|');
>> axis([0.2 0.4 0 110]);
```

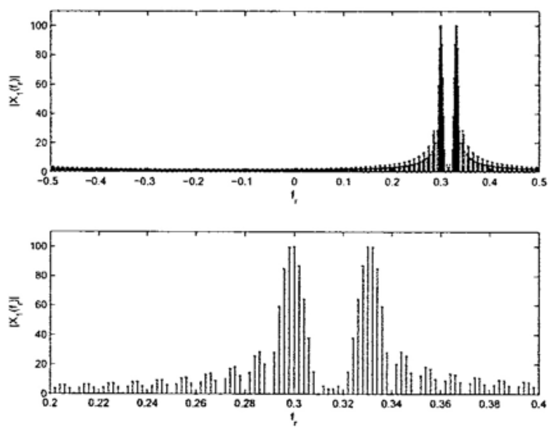


Figure S8.M-2d: DFT of 100-point $x_1[n]$ zero-padded to 500-points.

For this DFT, 100 samples of $x_1[n]$ are used, and the sequence is zero-padded to a length of 500. The DFT has 500 frequency bins uniformly spaced in the frequency interval $[-0.5, 0.5)$, and these bins include both exponential frequencies $f_r = 0.30$ and $f_r = 0.33$. As shown in Figure S8.M-2d, the two exponentials can be separately identified, but there is some added clutter throughout the frequency spectrum. This clutter is the result of applying a finite-length window to the signal $x_1[n]$. Although Figure S8.M-2d may appear less accurate than Figure S8.M-2d, both contain the same information about $x_1[n]$. Using the picket fence analogy, Figure S8.M-2d uses more pickets than Figure S8.M-2c, but the background behind both is the same. In many respects, Figure S8.M-2d paints a more honest picture of the data than Figure S8.M-2c; recall that Figure S8.M-2c looks uncommonly good since data record $x_1[n]$ includes an integer number of periods (a rare occurrence).

8.M-3. Ideally, $x_2[n] = e^{j2\pi n30/100} + e^{j2\pi n31.5/100}$ is characterized by two spikes of equal height located at $f_r = 0.30$ and $f_r = 0.315$. For most cases, two DFT magnitude plots are included: the first covers the entire range of digital frequencies and the second details the range near the true frequency content of $x_2[n]$.

(a)
```
>> n = (0:9); x2 = exp(j*2*pi*n*30/100)+exp(j*2*pi*n*31.5/100);
>> X2 = fft(x2); f_r = (0:length(x2)-1)/length(x2);
>> stem(f_r-0.5,fftshift(abs(X2)),'k.');
>> xlabel('f_r'); ylabel('|X_2(f_r)|');
>> axis([-0.5 0.5 0 20]);
```

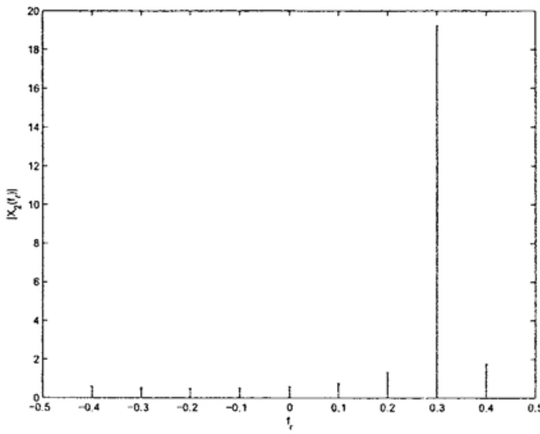


Figure S8.M-3a: DFT of 10-point $x_2[n]$.

For this DFT, only ten samples of $x_2[n]$ are used. As a result, the DFT has only 10 frequency bins uniformly spaced over the frequency interval $[-0.5, 0.5)$. As Figure S8.M-3a shows, there is insufficient frequency resolution to separately identify the two closely spaced exponentials at $f_r = 0.30$ and $f_r = 0.315$.

(b)
```
>> n = (0:9); x2 = exp(j*2*pi*n*30/100)+exp(j*2*pi*n*31.5/100);
>> x2 = [x2,zeros(1,490)];
>> X2 = fft(x2); f_r = (0:length(x2)-1)/length(x2);
>> subplot(211),stem(f_r-0.5,fftshift(abs(X2)),'k.');
>> xlabel('f_r'); ylabel('|X_2(f_r)|');
>> axis([-0.5 0.5 0 20]);
>> subplot(212),stem(f_r-0.5,fftshift(abs(X2)),'k.');
>> xlabel('f_r'); ylabel('|X_2(f_r)|');
>> axis([0.2 0.4 0 20]);
```

For this DFT, only ten samples of $x_2[n]$ are used but the sequence is zero-padded to a length of 500. Although the DFT has 500 frequency bins uniformly spaced over the frequency interval $[-0.5, 0.5)$, there is insufficient information about $x_2[n]$ (only 10 samples) to resolve the closely spaced exponentials at $f_r = 0.30$ and $f_r = 0.315$. Using the picket fence analogy, zero-padding increases the number of "pickets" in our DFT fence, but it does not change what lies behind the fence. Still, Figure S8.M-3b does show a concentration of signal energy centered around $f_r = 0.308$, the average of the two exponential frequencies.

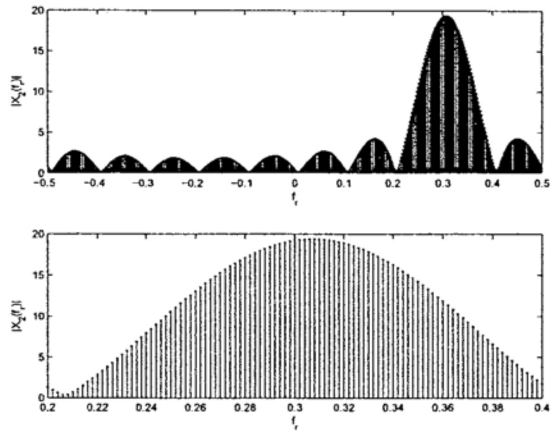


Figure S8.M-3b: DFT of 10-point $x_2[n]$ zero-padded to 500-points.

(c)
```
>> n = (0:99); x2 = exp(j*2*pi*n*30/100)+exp(j*2*pi*n*31.5/100);
>> X2 = fft(x2); f_r = (0:length(x2)-1)/length(x2);
>> subplot(211),stem(f_r-0.5,fftshift(abs(X2)),'k.');
>> xlabel('f_r'); ylabel('|X_2(f_r)|');
>> axis([-0.5 0.5 0 110]);
>> subplot(212),stem(f_r-0.5,fftshift(abs(X2)),'k.');
>> xlabel('f_r'); ylabel('|X_2(f_r)|');
>> axis([0.2 0.4 0 110]);
```

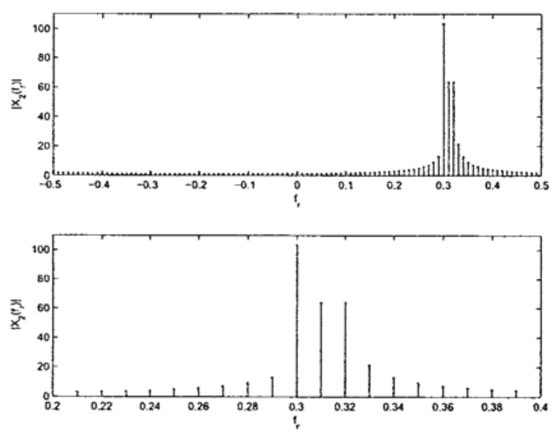


Figure S8.M-3c: DFT of 100-point $x_2[n]$.

For this DFT, 100 samples of $x_2[n]$ are used. As a result, the DFT has 100 frequency bins uniformly spaced over the frequency interval $[-0.5, 0.5)$. As Figure S8.M-3c shows, the two exponentials are not easily distinguished; even the number of dominant frequency components is difficult to identify. There difficulties partially occur because the exponentials are closely spaced and the data window is insufficiently large. Features are also obscured since the frequency $f_r = 0.315$ does not lie directly on a DFT frequency bin; therefore the effects of frequency leakage and smearing are pronounced.

(d)
```
>> n = (0:99); x2 = exp(j*2*pi*n*30/100)+exp(j*2*pi*n*31.5/100);
```

```
>> x2 = [x2,zeros(1,400)];
>> X2 = fft(x2); f_r = (0:length(x2)-1)/length(x2);
>> subplot(211),stem(f_r-0.5,fftshift(abs(X2)),'k.');
>> xlabel('f_r'); ylabel('|X_2(f_r)|');
>> axis([-0.5 0.5 0 110]);
>> subplot(212),stem(f_r-0.5,fftshift(abs(X2)),'k.');
>> xlabel('f_r'); ylabel('|X_2(f_r)|');
>> axis([0.2 0.4 0 110]);
```

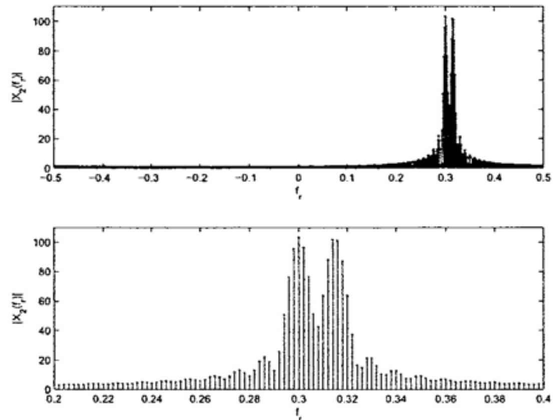


Figure S8.M-3d: DFT of 100-point $x_2[n]$ zero-padded to 500-points.

For this DFT, 100 samples of $x_2[n]$ are used, and the sequence is zero-padded to a length of 500. The DFT has 500 frequency bins uniformly spaced in the frequency interval $[-0.5, 0.5)$. As shown in Figure S8.M-3d, two dominant frequency components can be separately identified, but there is some added "clutter" throughout the frequency spectrum. This "clutter" is the result of applying a finite-length window to the signal $x_2[n]$. Comparing Figures S8.M-3c and S8.M-3d, it is clear that zero-padding assists in separating and locating the two dominant frequencies. That is, Figure S8.M-3d displays two discernable modes at the correct frequencies $f_r = 0.30$ and $f_r = 0.315$ while Figure S8.M-3c cannot distinguish these two features.

8.M-4. Ideally, $y_1[n] = 1 + e^{j2\pi n30/100} + 0.5 * e^{j2\pi n43/100}$ is characterized by three spikes located at $f_r = 0$, $f_r = 0.3$, and $f_r = 0.43$. The spikes at $f_r = 0$ and $f_r = 0.3$ should have equal height and the spike at $f_r = 0.43$ should have have a height that is half as high as the other two. For most cases, two DFT magnitude plots are included: the first covers the entire range of digital frequencies and the second details the range near the true frequency content of $x_2[n]$.

(a)
```
>> n = (0:19); y1 = 1+exp(j*2*pi*n*30/100)+0.5*exp(j*2*pi*n*43/100);
>> Y1 = fft(y1); f_r = (0:length(y1)-1)/length(y1);
>> stem(f_r-0.5,fftshift(abs(Y1)),'k.');
>> xlabel('f_r'); ylabel('|Y_1(f_r)|');
>> axis([-0.5 0.5 0 25]);
```

For this DFT, only 20 samples of $y_1[n]$ are used. As a result, the DFT has only 20 frequency bins uniformly spaced over the frequency interval $[-0.5, 0.5)$. As Figure S8.M-4a shows, strong content is seen at $f_r = 0$ and $f_r = 0.3$, but there

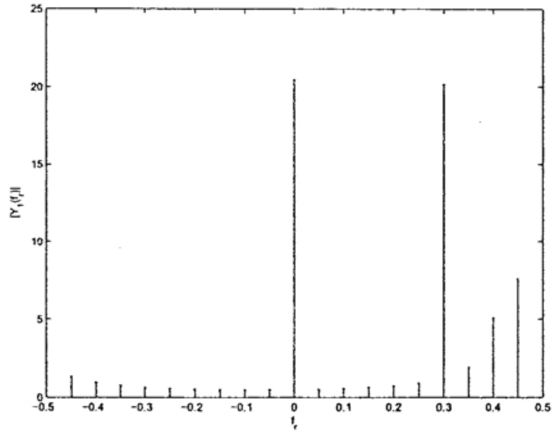


Figure S8.M-4a: DFT of 20-point $y_1[n]$.

is insufficient detail to identify the component at $f_r = 0.43$. As a result, it is not really possible to determine the relative strength of the two non-DC components.

(b)
```
>> n = (0:19); y1 = 1+exp(j*2*pi*n*30/100)+0.5*exp(j*2*pi*n*43/100);
>> y1 = [y1,zeros(1,480)];
>> Y1 = fft(y1); f_r = (0:length(y1)-1)/length(y1);
>> subplot(211),stem(f_r-0.5,fftshift(abs(Y1)),'k.');
>> xlabel('f_r'); ylabel('|Y_1(f_r)|');
>> axis([-0.5 0.5 0 25]);
>> subplot(212),stem(f_r-0.5,fftshift(abs(Y1)),'k.');
>> xlabel('f_r'); ylabel('|Y_1(f_r)|');
>> axis([0.25 0.45 0 25]);
```

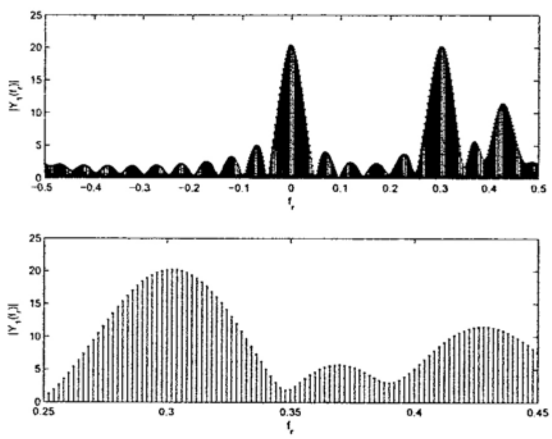


Figure S8.M-4b: DFT of 20-point $y_1[n]$ zero-padded to 500-points.

As shown in Figure S8.M-4b, the picture is improved by zero-padding the signal from 8.M-4a. In this case, each of the three signal components can be identified near their correct frequencies $f_r = 0$, $f_r = 0.3$, and $f_r = 0.43$. Interestingly, however, the peak amplitude near $f_r = 0.3$ is around 20.2 while the peak amplitude near $f_r = 0.43$ is around 11.5; the ratio of signal amplitudes appears to be $\frac{20.2}{11.5} = 1.7565$, which does not equal the true ratio of 2. The primary reason for this distortion is frequency leakage that results from applying a rectangular window to the signal $y_1[n]$.

(c)
```
>> n = (0:19); y1 = 1+exp(j*2*pi*n*30/100)+0.5*exp(j*2*pi*n*43/100);
>> y1 = y1.*window(@hanning,length(y1))';
>> Y1 = fft(y1); f_r = (0:length(y1)-1)/length(y1);
>> stem(f_r-0.5,fftshift(abs(Y1)),'k.');
>> xlabel('f_r'); ylabel('|Y_1(f_r)|');
>> axis([-0.5 0.5 0 12]);
```

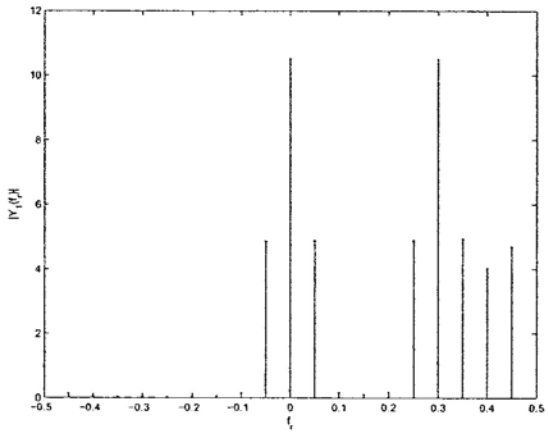


Figure S8.M-4c: DFT of 20-point Hanning-windowed $y_1[n]$.

Compared to a rectangular window, a Hanning window has a broader main lobe, which tends to broaden a signal's spectral features. This broadening, or frequency smearing as it is sometimes called, is evident when Figure S8.M-4c is compared to Figure S8.M-4a; the DFT of the Hanning-windowed signal has broader features than the DFT of the rectangular-windowed signal. As with Figure S8.M-4a, the component at $f_r = 0.43$ is not discernable in Figure S8.M-4c. Finally, notice that the peak amplitudes in Figure S8.M-4c are lower than the peak amplitudes in Figure S8.M-4a. This is primarily because the Hanning-window attenuates the edges of the original signal, resulting in a loss of signal energy (and thus smaller DFT coefficients).

```
>> n = (0:19); y1 = 1+exp(j*2*pi*n*30/100)+0.5*exp(j*2*pi*n*43/100);
>> y1 = [y1.*window(@hanning,length(y1))',zeros(1,480)];
>> Y1 = fft(y1); f_r = (0:length(y1)-1)/length(y1);
>> subplot(211),stem(f_r-0.5,fftshift(abs(Y1)),'k.');
>> xlabel('f_r'); ylabel('|Y_1(f_r)|');
>> axis([-0.5 0.5 0 12]);
>> subplot(212),stem(f_r-0.5,fftshift(abs(Y1)),'k.');
>> xlabel('f_r'); ylabel('|Y_1(f_r)|');
>> axis([0.25 0.45 0 12]);
```

By zero-padding the Hanning-windowed signal $y_1[n]$, the picture is again improved. Figure S8.M-4c shows that each of the three components of $y_1[n]$ are located at the correct frequencies $f_r = 0$, $f_r = 0.3$, and $f_r = 0.43$. Additionally, the peak amplitude at $f_r = 0.3$ is around 10.5 and the peak amplitude at $f_r = 0.43$ is around 5.24; the ratio of signal amplitudes is $\frac{10.5}{5.24} = 2.0038$, which is very close to the true ratio of 2. The computed ratio more accurate than that

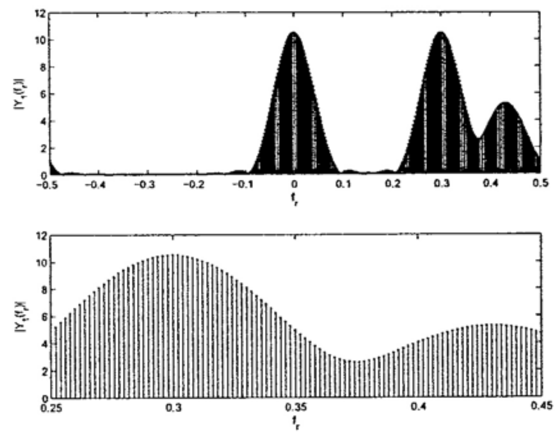


Figure S8.M-4c: DFT of 20-point Hanning-windowed $y_1[n]$ zero-padded to 500-points.

computed in 8.M-4b; the reason for this improvement is that the Hanning window has lower side lobes than the rectangular window, and is thus less susceptible to leakage from distantly spaced components.

In this particular case, the Hanning window improves the analysis of the signal. The lower side lobes of the Hanning window result in reduced leakage. Although the Hanning's broad main lobe results in increased smearing, signal components are spaced sufficiently far apart that each component can still be distinguished.

8.M-5. Ideally, $y_2[n] = 1+e^{j2\pi n30/100}+0.5*e^{j2\pi n38/100}$ is characterized by three spikes located at $f_r = 0$, $f_r = 0.3$, and $f_r = 0.38$. The spikes at $f_r = 0$ and $f_r = 0.3$ should have equal height and the spike at $f_r = 0.38$ should have have a height that is half as high as the other two. For most cases, two DFT magnitude plots are included: the first covers the entire range of digital frequencies and the second details the range near the true frequency content of $y_2[n]$.

(a)
```
>> n = (0:19); y2 = 1+exp(j*2*pi*n*30/100)+0.5*exp(j*2*pi*n*38/100);
>> Y2 = fft(y2); f_r = (0:length(y2)-1)/length(y2);
>> stem(f_r-0.5,fftshift(abs(Y2)),'k.');
>> xlabel('f_r'); ylabel('|Y_2(f_r)|');
>> axis([-0.5 0.5 0 25]);
```

For this DFT, only 20 samples of $y_2[n]$ are used. As a result, the DFT has only 20 frequency bins uniformly spaced over the frequency interval $[-0.5, 0.5)$. As Figure S8.M-5a shows, strong content is seen at $f_r = 0$ and $f_r = 0.3$, but there is insufficient detail to identify the component at $f_r = 0.38$. As a result, it is not really possible to determine the relative strength of the two non-DC components.

(b)
```
>> n = (0:19); y2 = 1+exp(j*2*pi*n*30/100)+0.5*exp(j*2*pi*n*38/100);
>> y2 = [y2,zeros(1,480)];
>> Y2 = fft(y2); f_r = (0:length(y2)-1)/length(y2);
>> subplot(211),stem(f_r-0.5,fftshift(abs(Y2)),'k.');
>> xlabel('f_r'); ylabel('|Y_2(f_r)|');
>> axis([-0.5 0.5 0 25]);
>> subplot(212),stem(f_r-0.5,fftshift(abs(Y2)),'k.');
>> xlabel('f_r'); ylabel('|Y_2(f_r)|');
>> axis([0.25 0.45 0 25]);
```

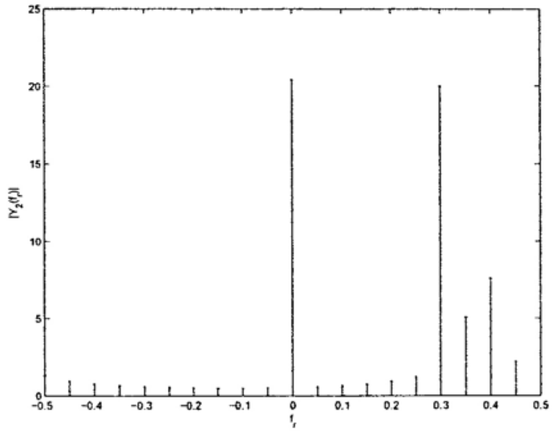


Figure S8.M-5a: DFT of 20-point $y_2[n]$.

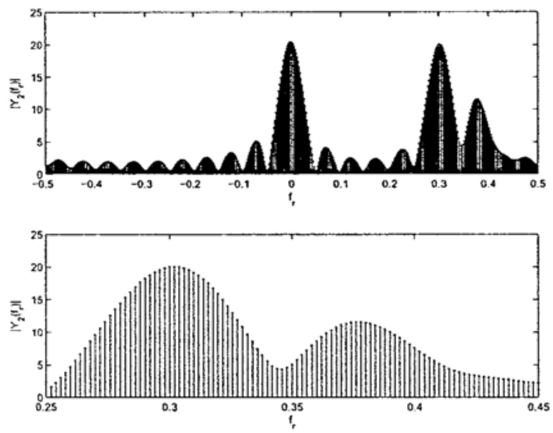


Figure S8.M-5b: DFT of 20-point $y_2[n]$ zero-padded to 500-points.

As shown in Figure S8.M-5b, the picture is greatly improved by zero-padding the signal from 8.M-5a. In this case, each of the three signal components can be identified near their correct frequencies $f_r = 0$, $f_r = 0.3$, and $f_r = 0.38$. Interestingly, however, the peak amplitude near $f_r = 0.3$ is around 20.0 while the peak amplitude near $f_r = 0.38$ is around 11.5; the ratio of signal amplitudes appears to be $\frac{20.0}{11.5} = 1.7391$, which does not equal the true ratio of 2. The primary reason for this distortion is frequency leakage that results from applying a rectangular window to the signal $y_2[n]$.

(c)
```
>> n = (0:19); y2 = 1+exp(j*2*pi*n*30/100)+0.5*exp(j*2*pi*n*38/100);
>> y2 = y2.*window(@hanning,length(y2))';
>> Y2 = fft(y2); f_r = (0:length(y2)-1)/length(y2);
>> stem(f_r-0.5,fftshift(abs(Y2)),'k.');
>> xlabel('f_r'); ylabel('|Y_2(f_r)|');
>> axis([-0.5 0.5 0 12]);
```

Compared to a rectangular window, a Hanning window has a broader main lobe, which tends to broaden a signal's spectral features. This broadening, or frequency smearing as it is sometimes called, is evident when Figure S8.M-5c is compared to

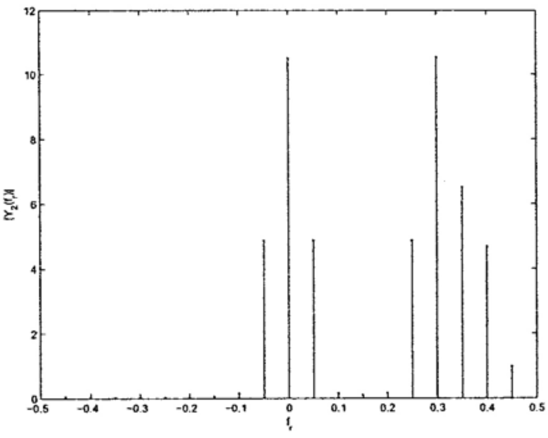


Figure S8.M-5c: DFT of 20-point Hanning-windowed $y_2[n]$.

Figure S8.M-5a; the DFT of the Hanning-windowed signal has broader features than the DFT of the rectangular-windowed signal. As with Figure S8.M-5a, the component at $f_r = 0.38$ is not discernable in Figure S8.M-5c. Finally, notice that the peak amplitudes in Figure S8.M-5c are lower than the peak amplitudes in Figure S8.M-5a. This is primarily because the Hanning-window attenuates the edges of the original signal, resulting in a loss of signal energy (and thus smaller DFT coefficients).

```
>> n = (0:19); y2 = 1+exp(j*2*pi*n*30/100)+0.5*exp(j*2*pi*n*38/100);
>> y2 = [y2.*window(@hanning,length(y2))',zeros(1,480)];
>> Y2 = fft(y2); f_r = (0:length(y2)-1)/length(y2);
>> subplot(211),stem(f_r-0.5,fftshift(abs(Y2)),'k.');
>> xlabel('f_r'); ylabel('|Y_2(f_r)|');
>> axis([-0.5 0.5 0 12]);
>> subplot(212),stem(f_r-0.5,fftshift(abs(Y2)),'k.');
>> xlabel('f_r'); ylabel('|Y_2(f_r)|');
>> axis([0.25 0.45 0 12]);
```

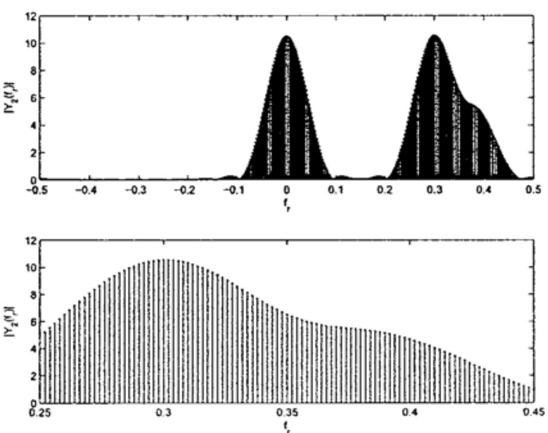


Figure S8.M-5c: DFT of 20-point Hanning-windowed $y_2[n]$ zero-padded to 500-points.

As shown in Figure S8.M-5c, zero-padding does not improve the picture of the Hanning-windowed signal $y_2[n]$. While the components near $f_r = 0$ and $f_r = 0.3$ can be identified, the component at $f_r = 0.38$ appears mostly lost! As such, the relative strengths of the components at $f_r = 0.3$ and $f_r = 0.38$ cannot be determined.

In this particular case, the Hanning window does not improve the analysis of the signal. While the lower side lobes of the Hanning window may reduce leakage, the Hanning's broad main lobe smears the components $f_r = 0.3$ and $f_r = 0.38$ to the point that the component $f_r = 0.38$ is completely obscured.

8.M-6. (a) MATLAB is used to plot the four samples corresponding to one period of the periodic signal $x[n] = \cos(n\pi/2)$.

```
>> N = 4; n = (0:N-1); x = cos(n*pi/2);
>> stem(n,x,'k.'); xlabel('n'); ylabel('x[n]');
>> axis([-.5 3.5 -1.1 1.1]);
```

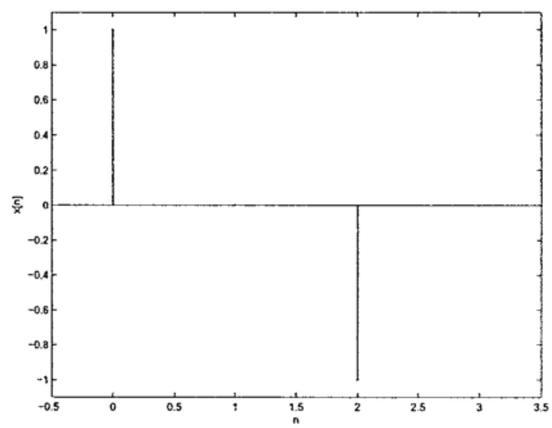


Figure S8.M-6a: $x[n] = \cos(n\pi/2)$.

Since the signal is so sparsely sampled, it doesn't much resemble a sinusoid.

(b)
```
>> N = 4; X = fft(x); r=(0:N-1); fr = r/N;
>> stem(fr-0.5,fftshift(abs(X)),'k.'); xlabel('f_r'); ylabel('|X(f_r)|');
>> axis([-.5 0.5 -0.1 2.1]);
```

The DFT shown in Figure S8.M-6b seems sensible. A pair of spikes appears that are consistent with the original sinusoid.

(c) Inserting zeros in the middle of the DFT X_r has the effect of increasing the sampling rate of $x[n]$. Zeros need to be placed in the middle to maintain the necessary symmetry of the DFT. Thought of another way, adding zeros to the middle of the DFT effectively specifies zero signal content for the newly added range of higher frequencies. The original signal content at lower frequencies is left unchanged.

```
>> Y = [X(1:3),zeros(1,100-length(X)),X(4)];
>> stem([0:99],real(ifft(Y)),'k.');
>> xlabel('n'); ylabel('y[n]');
```

As seen in Figure S8.M-6c, the signal $y[n]$ looks much more sinusoidal than $x[n]$. Both signals are plotted for one full period, but $y[n]$ has 25 times as many samples as $x[n]$. Notice also that the magnitude of $y[n]$ is 1/25 as great as $x[n]$.

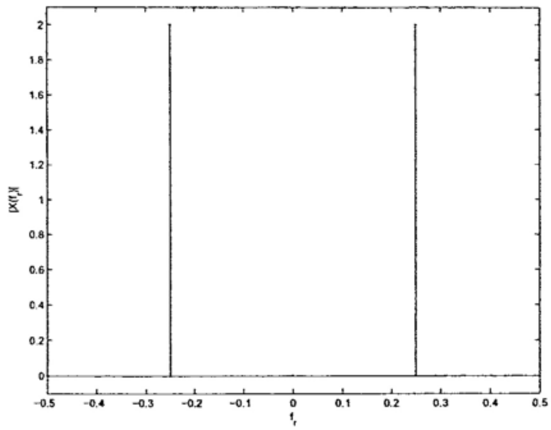


Figure S8.M-6b: $|X(f_r)|$ for $x[n] = \cos(n\pi/2)$.

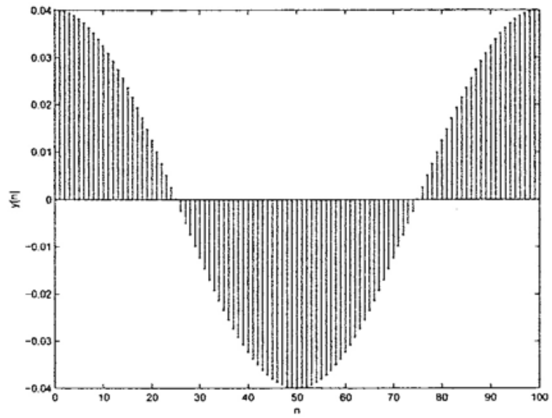


Figure S8.M-6c: $y[n]$ from zero-padded DFT.

Zero-padding in the frequency domain achieves a similar effect as zero-padding in the time domain. Using the picket fence analogy, zero-padding in frequency increases the number of pickets in the time-domain (increases the sampling rate), but it does not, other than a scale factor, change what is behind the pickets (in this case, one period of a sinusoid).

(d) As seen in the previous part, increasing the size of an N-point DFT by a factor K causes a reduction of the time-domain signal's amplitude by a factor $1/K$. To correct this reduction, scale the zero-padded DFT by the factor K. For example, a 4-point DFT zero-padded to a length of 100 would need to be scaled by $K = 100/4 = 25$.

(e)
```
>> temp = fft([1 1 1 1 -1 -1 -1 -1]);
>> S = (100/length(temp))*[temp(1:5),zeros(1,100-length(temp)),temp(6:8)];
>> stem([0:99],real(ifft(S)),'k.');
>> xlabel('n'); ylabel('s[n]');
```

As shown in Figure S8.M-6e, the reconstructed signal $s[n]$ has some appearance of a square wave, but lacks the sharp edges typical of a square wave. In fact, $s[n]$ might be best called a band-limited square wave. Although zero-padding in the frequency domain increases the sampling rate in the time-domain, zero-padding

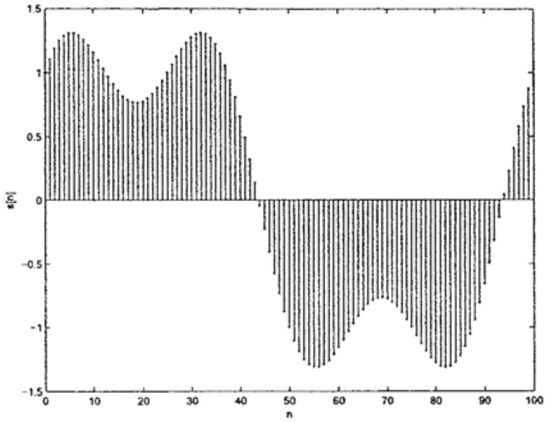


Figure S8.M-6e: $s[n]$ from zero-padded DFT.

cannot add the high-frequency harmonics needed to achieve a better square-wave approximation.

Chapter 9 Solutions

9.1-1.

$$\begin{aligned} x[n] &= 4\cos 2.4\pi n + 2\sin 3.2\pi n \\ &= 4\cos 0.4\pi n + 2\sin 1.2\pi n \\ &= 2[e^{j0.4\pi n} + e^{-j0.4\pi n}] + \frac{1}{j}[e^{j1.2\pi n} - e^{-j1.2\pi n}] \\ &= 2e^{j0.4\pi n} + 2e^{-j0.4\pi n} + e^{j(1.2\pi n - \pi/2)} + e^{-j(1.2\pi n - \pi/2)} \end{aligned}$$

The fundamental $\Omega_0 = 0.4\pi$ and $N_0 = \frac{2\pi}{\Omega_0} = 5$. Note also that.

$$e^{-j0.4\pi n} = e^{j1.6\pi n} \quad \text{and} \quad e^{-j1.2\pi n} = e^{j0.8\pi n}$$

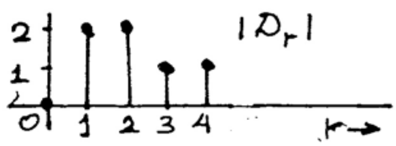
Therefore

$$x[n] = 2e^{j0.4\pi n} + 2e^{j1.6\pi n} + e^{j(1.2\pi n - \pi/2)} + e^{j(0.8\pi n + \pi/2)}$$

We have first, second, third and fourth harmonics with coefficients

$$\begin{aligned} &D_1 = D_2 = 2 \quad D_3 = -j \quad D_4 = j \\ &|D_1| = |D_2| = 2 \quad |D_3| = |D_4| = 1 \\ &\angle D_1 = \angle D_2 = 0 \quad \angle D_3 = -\frac{\pi}{2} \quad \text{and} \quad \angle D_4 = \frac{\pi}{2} \end{aligned}$$

The spectrum is shown in Figure S9.1-1.



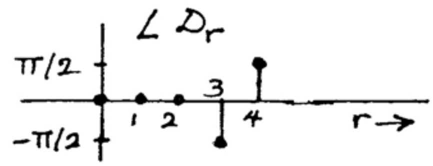


Figure S9.1-1

9.1-2.

$$\begin{aligned} x[n] = \cos 2.2\pi n \cos 3.3\pi n &= \frac{1}{2}[\cos 5.5\pi n + \cos 1.1\pi n] \\ &= \frac{1}{2}[\cos 1.5\pi n + \cos 1.1\pi n] \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}[e^{j1.5\pi n} + e^{-j1.5\pi n} + e^{j1.1\pi n} + e^{-j1.1\pi n}] \\
&= \frac{1}{2}[e^{j1.5\pi n} + e^{j0.5\pi n} + e^{j1.1\pi n} + e^{j0.9\pi n}]
\end{aligned}
$$

The fundamental frequency $\Omega_0 = 0.1$ and $N_0 = \frac{2\pi}{\Omega_0} = 20$. There are only 5th, 9th, 11th and 15th harmonics with coefficients

$$D_5 = D_9 = D_{11} = D_{15} = \frac{1}{2}$$

All the form coefficients are real (phases zero). The spectrum is shown in Figure S9.1-2.

Figure S9.1-2

9.1-3.

$$
\begin{aligned}
x[n] &= 2\cos 3.2\pi(n-3) = 2\cos(3.2\pi n - 9.6\pi) = 2\cos(1.2\pi n - 1.6\pi) \\
&= e^{j(1.2\pi n - 1.6\pi)} + e^{-j(1.2\pi n - 1.6\pi)} \\
&= e^{j(1.2\pi n - 1.6\pi)} + e^{j(0.8\pi n + 1.6\pi)}
\end{aligned}
$$

The fundamental frequency $\Omega_0 = 0.4\pi$ and $N_0 = \frac{2\pi}{\Omega_0} = 5$. Only 2nd, and 3rd harmonics are present.

$$|D_2| = |D_3| = 1 \quad \angle D_2 = 9.6\pi = 1.6\pi \quad \angle D_3 = -9.6\pi = -1.6\pi$$

The spectrum is shown in Figure S9.1-3.

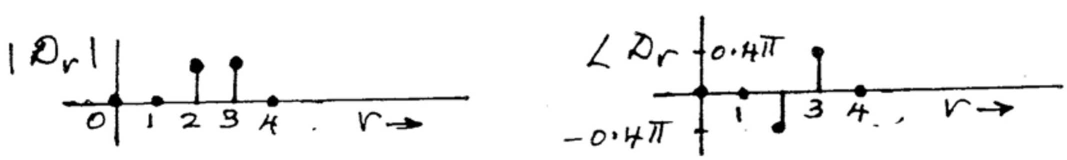


Figure S9.1-3

9.1-4. To compute coefficients D_r, we use Eq. (9.13) where summation is performed over any interval N_0. We choose this interval to be $-N_0/2$, $(N_0/2)-1$ (for even N_0). Therefore

$$D_r = \frac{1}{N_0} \sum_{n=-N_0/2}^{(N_0/2)-1} x[n] e^{-jr\Omega_0 n}$$

In the present case $N_0 = 6$, $\Omega_0 = \frac{2\pi}{N_0} = \frac{\pi}{3}$, and

$$D_r = \frac{1}{6}\sum_{n=-3}^{2} x[n]e^{-jr\frac{\pi}{3}n}$$

We have $x[0] = 3$, $x[\pm 1] = 2$, $x[\pm 2] = 1$, and $x[\pm 3] = 0$. Therefore

$$\begin{aligned} D_r &= \frac{1}{6}[3 + 2(e^{j\frac{\pi}{3}r} + e^{-j\frac{\pi}{3}r}) + (e^{j\frac{2\pi}{3}r} + e^{-j\frac{2\pi}{3}r})] \\ &= \frac{1}{6}[3 + 4\cos(\frac{\pi}{3}r) + 2\cos(\frac{2\pi}{3}r)] \end{aligned}$$

$$D_0 = \frac{3}{2} \quad D_1 = \frac{2}{3} \quad D_2 = 0 \quad D_3 = \frac{1}{6} \quad D_4 = 0 \quad D_5 = \frac{2}{3}$$

9.1-5. In this case $N_0 = 12$ and $\Omega_0 = \frac{\pi}{6}$.

$$\begin{aligned} &x[0] = 0 \quad x[1] = 1 \quad x[-1] = -1 \quad x[2] = 2 \quad x[-2] = -2 \\ &x[3] = 3 \quad x[-3] = -3 \quad x[\pm 4] = x[\pm 5] = x[\pm 6] = 0 \end{aligned}$$

Therefore

$$\begin{aligned} D_r &= \frac{1}{12}\sum_{n=-6}^{5} x[n]e^{-jr\frac{\pi}{6}n} \\ &= \frac{1}{12}[e^{-j\frac{\pi}{6}r} - e^{j\frac{\pi}{6}r} + 2(e^{-j\frac{2\pi}{6}r} - e^{j\frac{2\pi}{6}r}) + 3(e^{-j\frac{3\pi}{6}r} - e^{j\frac{3\pi}{6}r})] \\ &= \frac{-j}{12}[2\sin(\frac{\pi}{6}r) + 4\sin(\frac{\pi}{3}r) + 6\sin(\frac{\pi}{2}r)] \end{aligned}$$

9.1-6. Here, the period is N_0 and $\Omega_0 = 2\pi/N_0$. Using Eq. (9.9), we obtain

$$\mathcal{D}_r = \frac{1}{N_0}\sum_{n=0}^{N_0-1} a^n e^{-jr\Omega_0 n} = \frac{1}{N_0}\sum_{n=0}^{N_0-1}(ae^{-jr\Omega_0})^n$$

This is a geometric progression, whose sum is found from Sec. B.7-4 as

$$\mathcal{D}_r = \frac{1}{N_0}\frac{a^{N_0}e^{-jr\Omega_0 N_0} - 1}{ae^{-jr\Omega_0} - 1} = \frac{a^{N_0} - 1}{N_0(ae^{-jr\Omega_0} - 1)} \quad \text{because} \quad e^{-jr\Omega_0 N_0} = e^{-jr2\pi} = 1$$

Therefore

$$\begin{aligned} \frac{a^{N_0}}{N_0(ae^{-jr\Omega_0} - 1)} &= \frac{a^{N_0}}{N_0(a\cos r\Omega_0 - ja\sin\Omega_0 - 1)} \\ &= \underbrace{\frac{a^{N_0}}{N_0(\sqrt{a^2 - 2a\cos r\Omega_0 + 1}}}_{|\mathcal{D}_r|} \angle\underbrace{\{-\tan^{-1}\frac{-a\sin r\Omega_0}{a\cos r\Omega_0 - 1}\}}_{\angle\mathcal{D}_r} \end{aligned}$$

9.1-7. Because $|x[n]|^2 = x[n]x^*[n]$, using Eq. (9.8), we obtain

$$P_x = \frac{1}{N_0}\sum_{n=0}^{N_0-1}\left|\sum_{r=0}^{N_0-1} D_r e^{jr\Omega_0 n}\right|^2 = \frac{1}{N_0 - 1}\sum_{n=0}^{N_0-1}\left[\sum_{r=0}^{N_0-1} D_r e^{j\Omega_0 n}\sum_{m=0}^{N_0-1} D_m^* e^{-jm\Omega_0 n}\right]$$

Interchanging the order of summation yields

$$P_x = \frac{1}{N_0} \sum_{r=0}^{N_0-1} \sum_{m=0}^{N_0-1} D_r D_m^* \left[\sum_{n=0}^{N_0-1} e^{j(r-m)\Omega_0 n} \right]$$

From Eq. (5.43), in Appendix 5.1, the sum inside the parenthesis is N_0 when $r = m$, and is zero otherwise. Hence

$$P_x = \frac{1}{N_0} \sum_{n=0}^{N_0-1} |x[n]|^2 = \sum_{r=0}^{N_0-1} |D_r|^2$$

9.1-8. (a) Yes, the sum of aperiodic discrete-time sequences can be periodic. For example, consider two signals $x_1[n] = \sin(n)u[n]$ and $x_2[n]\sin(n)u[-n]$. The sum of these two aperiodic signals is the periodic function $x_1[n] + x_2[n] = sin(n)$.

(b) No, it is not possible for a sum of periodic discrete-time sequences to be aperiodic. Consider arbitrary periodic signals $x_1[n]$ and $x_2[n]$ with periods N_1 and N_2, respectively. Let $y[n] = x_1[n]+x_2[n]$. Notice that $y[n+N_1N_2] = x_1[n+N_1N_2]+x_2[n+N_1N_2]$. By periodicity, $x_1[n+kN_1] = x_1[n]$ and $x_2[n+kN_2] = x_2[n]$ for any k. Thus, $y[n+N_1N_2] = x_1[n]+x_2[n] = y[n]$. That is, the sum of two periodic signals must also be periodic.

9.2-1.

$$\begin{aligned} x[n] &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\Omega)e^{j\Omega n}\,d\Omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\Omega)|e^{j\angle X(\Omega)}e^{j\Omega n}\,d\Omega \\ &= \frac{1}{2\pi}\left[\int_{-\infty}^{\infty} |X(\Omega)|\cos[\Omega n + \angle X(\Omega)]\,d\Omega + j\int_{-\infty}^{\infty} |X(\Omega)|\sin[\Omega n + \angle X(\Omega)]\,d\Omega\right] \end{aligned}$$

Since $|X(\Omega)|$ is an even function and $\angle X(\Omega)$ is an odd function of Ω, the integrand in the second integral is an odd function of Ω, and therefore vanishes. Moreover the integrand in the first integral is an even function of Ω, and therefore

$$x[n] = \frac{1}{\pi}\int_{0}^{\infty} |X(\Omega)|\cos[\Omega n + \angle X(\Omega)]\,d\Omega$$

9.2-2. (a) Because $x[n] = x_o[n] + x_e[n]$ and $e^{-j\Omega n} = \cos\Omega n - j\sin\Omega n$

$$\begin{aligned} X(\Omega) &= \sum_{n=-\infty}^{\infty} \{x_o[n] + x_e[n]\}e^{-j\Omega n} \\ &= \sum_{n=-\infty}^{\infty} \{x_o[n] + x_e[n]\}\cos\Omega n - j\sum_{n=-\infty}^{\infty} \{x_o[n] + x_e[n]\}\sin\Omega n \end{aligned}$$

Because $x_e[n]\cos\Omega n$ and $x_o[n]\sin\Omega n$ are even functions and $x_o[n]\cos\Omega n$ and $x_e[n]\sin\Omega n$ are odd functions of n, these sums reduce to

$$X(\Omega) = 2\sum_{n=0}^{\infty} x_e[n]\cos\Omega n - 2j\sum_{n=0}^{\infty} x_o[n]\sin\Omega n \qquad (1)$$

Also, using the parallel development, we obtain the results similar to those of

Prob. 7.1-1, by which we have

$$\text{DTFT}\{x_e[n]\} = 2\sum_{n=0}^{\infty} x_e[n]\cos\Omega n \quad \text{and} \quad \text{DTFT}\{x_0[n]\} = -2j\sum_{n=0}^{\infty} x_o[n]\sin\Omega n \tag{2}$$

From Eqs. (1) and (2), the desired result follows.

(b) We shall prove the result for a general exponential $x[n] = a^n u[n]$. From Table 9.1, we obtain

$$X(\Omega) = \frac{1}{1-ae^{-j\Omega}} = \frac{1-a\cos\Omega}{1+a^2-2a\cos\Omega} + j\frac{-a\sin\Omega}{1+a^2-2a\cos\Omega}$$

The even and odd components of $x[n] = a^n u[n]$ are

$$x_e[n] = 0.5(a^n u[n] + a^{-n}u[-n]), \qquad \text{and} \qquad x_o[n] = 0.5(a^n u[n] - a^{-n}u[-n])$$

W e know that $a^n u[n] \Longleftrightarrow 1/(1-ae^{j\Omega})$. Moreover

$$a^{-n}u[-n] = \left(\frac{1}{a}\right)^n u[-(n+1)] + \delta[n]$$

Hence

$$a^{-n}u[-n] \Longleftrightarrow \frac{1}{\left(\frac{1}{a}\right)^n e^{-j\Omega} - 1} + 1 = \frac{1}{1-ae^{j\Omega}}$$

and

$$\text{DTFT}\{x_e[n]\} = 0.5\left(\frac{1}{1-ae^{-j\Omega}} + \frac{1}{1-ae^{\Omega}}\right) = \frac{1-a\cos\Omega}{1+a^2-2a\cos\Omega} = \text{Re } X(\Omega)$$

and

$$\text{DTFT}\{x_o[n]\} = 0.5\left(\frac{1}{1-ae^{-j\Omega}} - \frac{1}{1-ae^{\Omega}}\right) = \frac{-ja\sin\Omega}{1+a^2-2a\cos\Omega} = j\text{Im } X(\Omega)$$

9.2-3. (a)

$$X(\Omega) = \sum_{n=-\infty}^{\infty} \delta[n]e^{-j\Omega n} = 1$$

(b)

$$\begin{aligned} X(\Omega) &= \sum_{n=-\infty}^{\infty} \delta[n-k]e^{-j\Omega n} = e^{-j\Omega k} \\ |X(\Omega)| &= 1 \qquad \angle X(\Omega) = -\Omega k \end{aligned}$$

(c)

$$\begin{aligned} X(\Omega) &= \sum_{n=1}^{\infty} \gamma^n e^{-j\Omega n} = \sum_{n=1}^{\infty} (\gamma e^{-j\Omega})^n = \frac{(\gamma e^{-j\Omega})^{\infty} - (\gamma e^{-j\Omega})}{\gamma e^{-j\Omega} - 1} \\ &= \frac{0-\gamma e^{-j\Omega}}{\gamma e^{-j\Omega}-1} = \frac{\gamma}{e^{j\Omega}-\gamma} = \frac{\gamma}{(\cos\Omega - \gamma) + j\sin\Omega} \end{aligned}$$

$$|X(\Omega)| = \frac{\gamma}{\sqrt{(1+\gamma^2) - 2\gamma\cos\Omega}} \qquad \angle X(\omega) = -\tan^{-1}\left(\frac{\sin\Omega}{\cos\Omega - \gamma}\right)$$

Observe that

$$X(\Omega) = \frac{\gamma}{e^{j\Omega} - \gamma} = \frac{\gamma e^{-j\Omega}}{1 - \gamma e^{-j\Omega}}$$

Comparison of this equation with Eq. (9.34a) shows that $X(\Omega)$ in the present case is $\gamma e^{-j\Omega}$ times the $X(\Omega)$ for $\gamma^n u[n]$. Clearly, the amplitude spectrum in this case is γ times that in Figure 9.4b. Moreover, the angle spectrum in the present case is equal to $-\Omega$ plus that in Figure 9.4c. This is shown in Figure S9.2-3a.

(d)

$$X(\Omega) = \sum_{n=-1}^{\infty} (\gamma e^{-j\Omega})^n = \frac{(\gamma e^{-j\Omega})^\infty - (\gamma e^{-j\Omega})^{-1}}{\gamma e^{-j\Omega} - 1} = \frac{e^{j2\Omega}}{\gamma(e^{j\Omega} - \gamma)}$$

$$|X(\Omega)| = \frac{1}{\gamma\sqrt{1+\gamma^2 - 2\gamma\cos\Omega}} \qquad \angle X(\Omega) = 2\Omega - \tan^{-1}\left(\frac{\sin\Omega}{\cos\Omega - \gamma}\right)$$

Observe that

$$X(\Omega) = \frac{e^{j2\Omega}}{\gamma(e^{j\Omega} - \gamma)} = \frac{e^{j\Omega}/\gamma}{1 - \gamma e^{-j\Omega}}$$

Comparison of this equation with Eq. (9.34a) shows that $X(\Omega)$ in the present case is $\frac{1}{\gamma}e^{-j\Omega}$ times the $X(\Omega)$ for $\gamma^n u[n]$. Clearly, the amplitude spectrum in this case is $1/\gamma$ times that in Figure 9.4b. Moreover, the angle spectrum in the present case is equal to Ω plus that in Figure 9.4c. This is shown in Figure S9.2-3b.

(e)

$$\begin{aligned}
x[n] &= (-\gamma)^n u[n] \\
X(\Omega) &= \sum_{n=0}^{\infty} (-\gamma)^n e^{-j\Omega n} \\
&= \sum_{n=0}^{\infty} (-\gamma e^{-j\Omega})^n \\
&= \frac{1}{1 + \gamma e^{-j\Omega}} \\
&= \frac{1}{\sqrt{1+\gamma^2 + 2\gamma\cos\Omega}} \quad e^{-j\tan^{-1}\left[\frac{-\gamma\sin\Omega}{1+\gamma\cos\Omega}\right]}
\end{aligned}$$

(f)

$$\begin{aligned}
x[n] = \gamma^{|n|} &= \gamma^n u[n] + \gamma^{-n} u[-(n+1)] \\
&= \gamma^n u[n] + \left(\frac{1}{\gamma}\right)^n u[-(n+1)]
\end{aligned}$$

DTFT of both those components are found in the text Eq. (9.34) and Eq. (9.36). Hence

$$X(\Omega) = \frac{1}{1 - \gamma e^{-j\Omega}} + \frac{1}{\frac{1}{\gamma}e^{-j\Omega} - 1} = \frac{e^{-j\Omega}(1-\gamma^2)}{(1 - \gamma e^{-j\Omega})(e^{-j\Omega} - \gamma)}$$

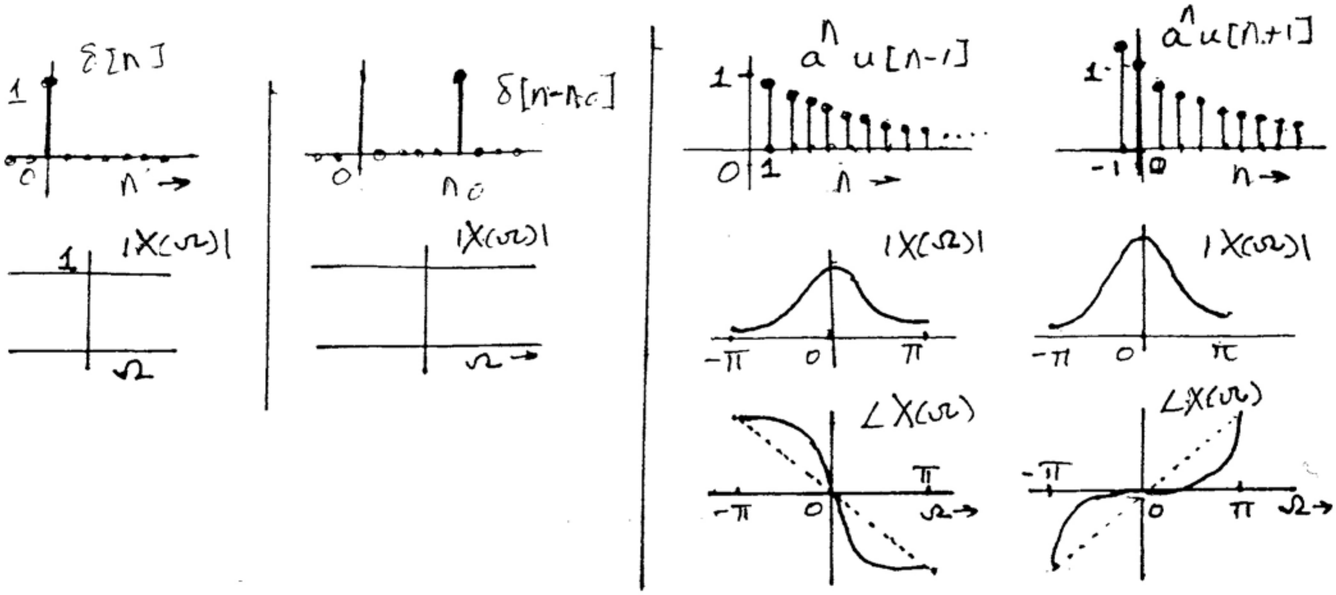


Figure S9.2-3

9.2-4. (a)

$$\begin{aligned} X(\Omega) &= e^{jk\Omega} \\ x[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{jk\Omega}e^{jn\Omega}d\Omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j(n+k)\Omega}d\Omega \\ &= \frac{1}{2\pi j}e^{j(n+k)\Omega}\Big|_{-\pi}^{\pi} \\ &= \text{sinc}[(n+k)\pi] \\ &= \delta[n+k] \end{aligned}$$

This follows from the fact that both n and k are integers and $\sin[(n+k)\pi] = 0$ for all $n \neq -k$. For $n = -k$, $\text{sinc}[(n+k)\pi] = 1$. Hence, the result.

(b)

$$X(\Omega) = \cos k\Omega = \frac{1}{2}[e^{jk\Omega} + e^{-jk\Omega}]$$

Hence, use of arguments in part (a) yields

$$x[n] = \frac{1}{2}(\delta[n+k] + \delta[n-k])$$

(c)

$$X(\Omega) = \cos^2(\frac{\Omega}{2}) = \frac{1}{2}(1+\cos\Omega) = \frac{1}{2} + \frac{1}{2}\cos\Omega$$

The IDTFT of $\frac{1}{2}$, denoted by $x_1[n]$, is given by

$$x_1[n] = \frac{1}{4\pi}\int_{-\pi}^{\pi} e^{j\Omega n}d\Omega = \frac{1}{2jn}e^{j\Omega n}\Big|_{-\pi}^{\pi} = \frac{1}{2}\text{sinc}(\pi n) = \frac{1}{2}\delta[n]$$

The DTFT if $x_2[n] = \frac{1}{2}\cos\Omega$ is found in part (b). Hence

$$x[n] = \frac{1}{2}\delta[n] + \frac{1}{4}\left(\delta[n+1] + \delta[n-1]\right)$$

(d)

$$\begin{aligned}
X(\Omega) &= \Delta\left(\frac{\Omega}{2\Omega_c}\right) = \begin{cases} 1 + \frac{\Omega}{\Omega_c} & \Omega < 0 \\ 1 - \frac{\Omega}{\Omega_c} & \Omega > 0 \end{cases} \\
x[n] &= \frac{1}{2\pi}\int_{-\pi}^{0}\left(1 + \frac{\Omega}{\Omega_c}\right)e^{j\Omega n}d\Omega + \frac{1}{2\pi}\int_{0}^{\pi}\left(1 - \frac{\Omega}{\Omega_c}\right)e^{j\Omega n}d\Omega \\
&= \frac{1}{2\pi}\left[\int_{-\Omega_c}^{\Omega_c} e^{j\Omega n}d\Omega + \frac{1}{\Omega_c}\int_{-\Omega_c}^{0}\Omega e^{j\Omega n}d\Omega - \frac{1}{\Omega_c}\int_{0}^{-\Omega_c}\Omega e^{j\Omega n}d\Omega\right]
\end{aligned}$$

The detailed derivation of these integrals yields

$$x[n] = \frac{1}{2\pi\Omega_c n^2}[1 - 2\cos\Omega_c n] = \frac{4}{2\pi\Omega_c n^2}\sin^2\left(\frac{\Omega_c n}{2}\right) = \frac{\Omega_c}{2\pi}\text{sinc}^2\left(\frac{\Omega_c}{2}\right)$$

(e)

$$X(\Omega) = 2\pi\delta(\Omega - \Omega_0)$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} 2\pi\delta(\Omega - \Omega_0)e^{j\Omega n}d\Omega$$

Use of the sampling property of the impulse in Eq. (1.24) yields

$$x[n] = e^{j\Omega_0 n}$$

(f)

$$\begin{aligned}
X(\Omega) &= \pi\left[\delta(\Omega - \Omega_0) + \delta(\Omega + \Omega_0)\right] \\
x[n] &= \frac{\pi}{2\pi}\int_{-\pi}^{\pi}\left[\delta(\Omega - \Omega_0) + \delta(\Omega + \Omega_0)\right]e^{j\Omega n}d\Omega \\
&= \frac{1}{2}\left[e^{j\Omega_0 n} + e^{-j\Omega_0 n}\right] = \cos\Omega_0 n
\end{aligned}$$

9.2-5.

$$\begin{aligned}
x[n] &= \frac{1}{2\pi}\int_{-\pi/4}^{3\pi/4} e^{j\Omega n}\,d\Omega = \left.\frac{e^{j\Omega n}}{2\pi jn}\right|_{-\pi/4}^{3\pi/4} = \frac{1}{j2\pi n}\left[e^{j(3\pi/4)n} - e^{-j(\pi/4)n}\right] \\
&= \frac{e^{j\pi n/4}}{j2\pi n}[2j\sin\pi n/2] = 0.5\,\text{sinc}(\pi n/2)e^{j\ pin/4}
\end{aligned}$$

9.2-6. (a) and (b)

$$X(\Omega) = \sum_{n=0}^{N_0} a^n e^{-j\Omega n} = \sum_{n=0}^{N_0}\left(ae^{-j\Omega}\right)^n = \frac{a^{N_0+1}e^{-j(N_0+1)\Omega} - 1}{ae^{-j\Omega} - 1}$$

The result applies to both parts (a) and (b). The only difference being in part (a),

$a < 1$ and in part (b), $a > 1$.

9.2-7. (a)

$$
\begin{aligned}
X(\Omega) &= 2\sum_{n=0}^{6} e^{-jn\Omega} + \sum_{n=7}^{12} e^{-jn\Omega} \\
&= 2\frac{e^{-j7\Omega}-1}{e^{-j\Omega}-1} + \frac{e^{-j13\Omega}-e^{-j7\Omega}}{e^{-j\Omega}-1} \\
&= \frac{e^{-j7\Omega}+e^{-j13\Omega}-2}{e^{-j\Omega}-1}
\end{aligned}
$$

(b)

$$
\begin{aligned}
X(\Omega) &= \sum_{n=-1}^{-(N_0-1)} \frac{-n}{N_0-1}e^{-jn\Omega} + \sum_{n=0}^{N_0-1} \frac{n}{N_0-1}e^{-jn\Omega} \\
&= \sum_{m=1}^{N_0-1} \frac{m}{N_0-1}e^{-jm\Omega} + \sum_{n=0}^{N_0-1} \frac{n}{N_0-1}e^{-jn\Omega} \\
&= \sum_{m=0}^{N_0-1} \frac{m}{N_0-1}e^{-jm\Omega} + \sum_{n=0}^{N_0-1} \frac{n}{N_0-1}e^{-jn\Omega} \\
&= \frac{1}{N_0-1}\left\{\frac{e^{j\Omega}+[(N_0-1)(e^{j\Omega}-1)-1]e^{jN_0\Omega}}{(e^{j\Omega}-1)^2}\right. \\
&\qquad \left. + \frac{e^{-j\Omega}+[(N_0-1)(e^{-j\Omega}-1)-1]e^{-jN_0\Omega}}{(e^{-j\Omega}-1)^2}\right\}
\end{aligned}
$$

9.2-8. (a)

$$
\begin{aligned}
x[n] &= \frac{1}{2\pi}\int_{-\Omega_0}^{\Omega_0} \Omega^2 e^{j\Omega n}\,d\Omega \\
&= \frac{1}{2\pi}\frac{e^{j\Omega n}}{(jn)^3}\left[-\Omega^2 n^2 - 2j\Omega n + 2\right]\Big|_{-\Omega_0}^{\Omega_0} \\
&= \frac{(\Omega_0^2 n^2 - 2)\sin\Omega_0 n + 2\Omega_0 n\cos\Omega_0 n}{\pi n^3}
\end{aligned}
$$

(b) The derivation can be simplified by observing that $X(\Omega)$ can be expressed as a sum of two gate functions $X_1(\Omega)$ and $X_2(\Omega)$ as shown in Figure S9.2-8. Therefore

$$
\begin{aligned}
x[n] &= \frac{1}{2\pi}\int_{-2}^{2}[X_1(\Omega)+X_2(\Omega)]e^{j\Omega n}\,d\Omega \\
&= \frac{1}{2\pi}\left\{\int_{-2}^{2} e^{j\Omega n}\,d\Omega + \int_{-1}^{1} e^{j\Omega n}\,d\Omega\right\} \\
&= \frac{\sin 2n + \sin n}{\pi n}
\end{aligned}
$$

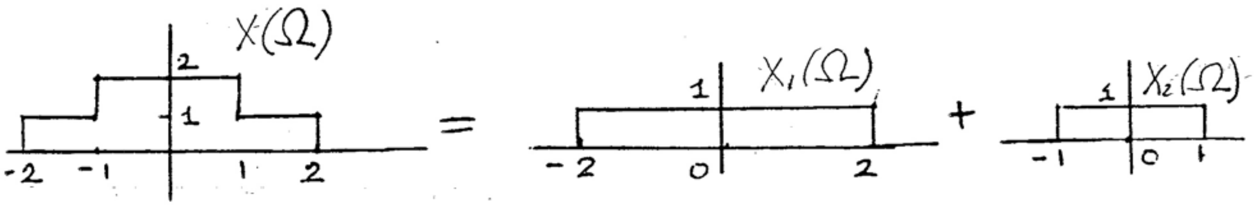


Figure S9.2-8b

9.2-9. (a)

$$
\begin{aligned}
x[n] &= \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} \cos\Omega\, e^{j\Omega n}\, d\Omega \\
&= \frac{e^{j\Omega n}}{2\pi(1-n^2)}\left\{jn\cos\Omega + \sin\Omega\right\}_{-\pi/2}^{\pi/2} \\
&= \frac{1}{\pi(1-n^2)}\cos\left(\frac{\pi n}{2}\right)
\end{aligned}
$$

(b)

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\Omega)e^{j\Omega n}\, d\Omega = \frac{1}{2\pi}\left[\int_{-\pi}^{\pi} X(\Omega)\cos\Omega n\, d\Omega + j\int_{-\pi}^{\pi} X(\Omega)\sin\Omega n\, d\Omega\right]$$

Because $X(\Omega)$ is even function, $X(\Omega)\,\sin\Omega n$ is an odd function of Ω. Hence, the second integral on the right-hand side vanishes. Also the integrand of the first term is an even function. Letting $\Omega_0 = \pi/4$, we obtain

$$
\begin{aligned}
x[n] &= \frac{1}{\pi}\int_0^{\Omega_0} \frac{\Omega}{\Omega_0}\cos n\Omega\, d\Omega = \frac{1}{\pi\Omega_0}\left[\frac{\cos n\Omega + n\Omega\sin n\Omega}{n^2}\right]\Bigg|_0^{\Omega_0} \\
&= \frac{1}{\pi\Omega_0 n^2}[\cos\Omega_0 n + \Omega_0 n\sin\Omega_0 n - 1] \\
&= \frac{4}{\pi^2 n^2}\left[\cos\frac{\pi n}{4} + \frac{\pi n}{4}\sin\frac{\pi n}{4} - 1\right]
\end{aligned}
$$

9.2-10. (a)

$$
\begin{aligned}
X(\Omega) = \sum_{n=-3}^{3} x[n]e^{-j\Omega n} &= 3 + 2(e^{-j\Omega} + e^{j\Omega}) + (e^{-j2\Omega} + e^{j2\Omega}) \\
&= 3 + 4\cos\Omega + 2\cos 2\Omega
\end{aligned}
$$

(b)

$$
\begin{aligned}
X(\Omega) = \sum_{n=0}^{6} x[n]e^{-jn\Omega} &= e^{-j\Omega} + 2e^{-j2\Omega} + 3e^{-j3\Omega} + 2e^{-j4\Omega} + e^{-j5\Omega} \\
&= e^{-j3\Omega}[(e^{j2\Omega} + e^{-j2\Omega}) + 2(e^{j\Omega} + e^{-j\Omega}) + 3] \\
&= e^{-j3\Omega}[3 + 4\cos\Omega + 2\cos 2\Omega]
\end{aligned}
$$

(c)

$$X(\Omega) = \sum_{n=-3}^{3} x[n]e^{-jn\Omega} = 3e^{-j\Omega} - 3e^{j\Omega} + 6e^{-j2\Omega} - 6e^{-j2\Omega} + 9e^{-j3\Omega} - 9e^{j3\Omega}$$
$$= 6j[\sin\Omega + 2\sin 2\Omega + 3\sin 3\Omega]$$

(d)

$$X(\Omega) = \sum_{n=-2}^{2} x[n]e^{-jn\Omega} = 2e^{-j\Omega} + 2e^{j\Omega} + 4e^{-j2\Omega} + 4e^{-j2\Omega}$$
$$= 4\cos\Omega + 8\cos 2\Omega$$

9.2-11. (a)

$$\begin{aligned} x[n] &= \frac{1}{2\pi}\int_{-\Omega_0}^{\Omega_0} e^{-j\Omega n_0}e^{j\Omega n}\,d\Omega = \frac{1}{2\pi}\int_{-\Omega_0}^{\Omega_0} e^{j\Omega(n-n_0)}\,d\Omega \\ &= \frac{1}{(2\pi)j(n-n_0)}e^{j\Omega(n-n_0)}\bigg|_{-\Omega_0}^{\Omega_0} = \frac{\sin\Omega_0(n-n_0)}{\pi(n-n_0)} = \frac{\Omega_0}{\pi}\mathrm{sinc}[\Omega_0(n-n_0)] \end{aligned}$$

(b)

$$\begin{aligned} x[n] &= \frac{1}{2\pi}\left[\int_{-\Omega_0}^{0} je^{j\Omega n}\,d\Omega + \int_{0}^{\Omega_0} -je^{j\Omega n}\,d\Omega\right] \\ &= \frac{1}{2\pi n}e^{j\Omega n}\bigg|_{-\Omega_0}^{0} - \frac{1}{2\pi n}e^{j\Omega n}\bigg|_{0}^{\Omega_0} = \frac{1-\cos\Omega_0 n}{\pi n} \end{aligned}$$

9.2-12. (a) We shall show that

$$\sum_{k=-\infty}^{\infty} x[k]\delta[n-Lk] = \begin{cases} x\left[\frac{n}{L}\right] & n = 0, \pm L, \pm 2L, \cdots \\ 0 & \text{otherwise} \end{cases}$$

When $n \neq mL$ where m is an integer, then for any integer values of k, $n - Lk$ cannot be zero and $\delta[n - Lk] = 0$ for all k and the sum on the left-hand side is zero for all $n \neq mL$ (m integer). When $n = mL$ (m integer), then $\delta[mL-Lk] = 1$ for $k = m$ and is zero for all $k \neq m$. Hence, the sum on the left-hand side has only one term $x[m]$ when $k = m$. Therefore

$$\begin{aligned} \sum_{k=-\infty}^{\infty} x[k]\delta[n-Lk] &= \begin{cases} x\left[\frac{m}{L}\right] & n = 0, \pm 1, \pm 2, \cdots \\ 0 & \text{otherwise} \end{cases} \\ &= x_e[n] \end{aligned}$$

(b)

$$X_e(\Omega) = \sum_{n=-\infty}^{\infty}\left(\sum_{k=-\infty}^{\infty} x[k]\delta[n-Lk]\right)e^{-j\Omega n}$$

Interchanging the order of the summation

$$\begin{aligned} X_e(\Omega) &= \sum_{n=-\infty}^{\infty} x[k]\left(\sum_{k=-\infty}^{\infty} \delta[n-Lk]e^{-j\Omega n}\right) \\ &= \sum_{k=-\infty}^{\infty} x[k]e^{-j\Omega Lk} = X(L\Omega) \end{aligned}$$

(c) $z[n]$ is the signal $x[n] = 1$ expanded by factor $L = 3$. Hence, from the above result and pair 11, we obtain

$$Z(\Omega) = 2\pi \sum_{k=-\infty}^{\infty} \delta(3\Omega - 2\pi k) = 2\pi \sum_{k=-\infty}^{\infty} \delta(\Omega - \frac{2\pi k}{3})$$

9.2-13. (a) We shall consider spectra within the band $|\Omega| \leq \pi$ only.
Pair 8: $\frac{\Omega_c}{\pi}\text{sinc}\,(\Omega_c n) \Longleftrightarrow \text{rect}\left(\frac{\Omega}{2\Omega_c}\right)$
This is identical to pair 18 in Table 7.1 with ω_c replaced by Ω_c and t replaced by n.
Pair 9: In the same way, we see that pair 9 is identical to pair 20 in Table 7.1.
Pair 11: is identical to pair 7 in Table 7.1.
Pair 12: is identical to pair 8 in Table 7.1.
Pair 13: is identical to pair 9 in Table 7.1.
Pair 14: is identical to pair 10 in Table 7.1.

(b) This method cannot be used for pairs 2, 3, 4, 5, 6, 7, 10, 15 and 16 because in all these cases $X(\Omega)$ is not bandlimited.

9.2-14. (a) Not valid because not 2π-periodic.
(b) Valid because it is a constant.
(c) Valid because it is $\frac{2\pi}{10}$-periodic, hence also 2π-periodic.
(d) Not valid because it is 20π-periodic and not 2π-periodic.
(e) Not valid because it is not 2π-periodic.

9.3-1. (a)

$$\begin{aligned} x[n] &= u[n] - u[n-9] \\ X(\Omega) &= \left[\frac{e^{j\Omega}}{e^{j\Omega}-1} + \pi\delta(\Omega)\right] - \left[\frac{e^{j\Omega}}{e^{j\Omega}-1} + \pi\delta(\Omega)\right] e^{-j9\Omega} \end{aligned}$$

Observe that $\delta(\Omega)e^{-j9\Omega} = \delta(\Omega)$ because $e^{-j9\Omega} = 1$ at $\Omega = 0$. Therefore

$$\begin{aligned} X(\Omega) &= \frac{e^{j\Omega}}{e^{j\Omega}-1}\left[1 - e^{-j9\Omega}\right] \\ &= \frac{e^{j\Omega}e^{-j4.5\Omega}\left[e^{j4.5\Omega} - e^{-j4.5\Omega}\right]}{e^{j\Omega/2}\left[e^{j\Omega/2} - e^{-j\Omega/2}\right]} \\ &= \frac{\sin 4.5\Omega}{\sin 0.5\Omega}e^{-j4\Omega} \end{aligned}$$

(b)

$$a^{n-m}u[n-m]$$

because

$$a^n u[n] \Longleftrightarrow \frac{e^{j\Omega}}{e^{j\Omega}-\gamma}$$

we obtain

$$a^{n-m}u[n-m] \Longleftrightarrow \frac{e^{j\Omega}e^{-jm\Omega}}{e^{j\Omega}-\gamma} = \frac{e^{j(1-m)\Omega}}{e^{j\Omega}-\gamma}$$

(c)

$$x[n] = a^{n-3}(u[n]-u[n-10]) = a^{-3}a^n u[n] - a^7 a^{n-10}u[n-10]$$

Hence

$$\begin{aligned} X(\Omega) &= a^{-3}\frac{e^{j\Omega}}{e^{j\Omega}-a} - a^7 \frac{e^{j\Omega}}{e^{j\Omega}-a}e^{-j10\Omega} \\ &= \frac{e^{j\Omega}\left(a^{-3}-a^7 e^{-10\Omega}\right)}{e^{j\Omega}-a} \end{aligned}$$

(d)

$$x[n] = a^{n-m}u[n] = a^{-m}a^n u[n]$$

Hence

$$X(\Omega) = \frac{a^{-m}e^{j\Omega}}{e^{j\Omega}-a}$$

(e)

$$x[n] = a^n u[n-m] = a^m a^{n-m}u[n-m]$$

Hence

$$X(\Omega) = \frac{a^m e^{j\Omega}e^{-jm\Omega}}{e^{j\Omega}-a} = a^m \frac{e^{j(1-m)\Omega}}{e^{j\Omega}-a}$$

(f)

$$x[n] = (n-m)a^{n-m}u[n-m]$$

Apply time-shift property to pair 5 to obtain

$$X(\Omega) = \frac{\gamma e^{j\Omega}e^{-jm\Omega}}{(e^{j\Omega}-\gamma)^2} = \frac{\gamma e^{j(1-m)\Omega}}{(e^{j\Omega}-\gamma)^2}$$

(g)

$$x[n] = (n-m)a^n u[n] = na^n u[n] - ma^n u[n]$$

Hence

$$\begin{aligned} X(\Omega) &= \frac{\gamma e^{j\Omega}}{(e^{j\Omega}-\gamma)^2} - \frac{me^{j\Omega}}{(e^{j\Omega}-\gamma)^2} \\ &= \frac{e^{j\Omega}(\gamma - me^{j\Omega}+m\gamma)}{(e^{j\Omega}-\gamma)^2} \end{aligned}$$

(h)

$$x[n] = na^{n-m}u[n-m] = (n-m)a^{n-m}u[n-m] + ma^{n-m}u[n-m]$$

Hence

$$\begin{aligned} X(\Omega) &= \frac{\gamma e^{j\Omega}}{(e^{j\Omega}-\gamma)^2}e^{-jm\Omega} + m\frac{e^{j\Omega}}{e^{j\Omega}-\gamma}e^{-jm\Omega} \\ &= \frac{\gamma + m(e^{j\Omega}-\gamma)}{(e^{j\Omega}-\gamma)^2}e^{j(1-m)\Omega} \end{aligned}$$

9.3-2.

$$x_1[n] = x[n-4] + x[-n-4] - 4\delta[n]$$

Note that at $n=0$, both $x[n-4]$ and $x[-n-4]$ have value 4. This duplication is corrected by the term $-4\delta[n]$. Thus

$$X_1(\Omega) = X(\Omega)e^{-j4\Omega} + X(-\Omega)e^{j4\Omega} - 4$$

$$\begin{aligned} x_2[n] &= x[n] + x[-n] \\ X_2(\Omega) &= X(\Omega) + X(-\Omega) \\ x_3[n] &= x[n-2] + x[-n-2] \\ X_3(\Omega) &= X(\Omega)e^{-j2\Omega} + X(-\Omega)e^{j2\Omega} \\ x_4[n] &= x[n-2] + x[-n-2] + x[n-7] + x[-n-7] \\ X_4(\Omega) &= X(\Omega)e^{-j2\Omega} + X(-\Omega)e^{j2\Omega} + X(\Omega)e^{-j7\Omega} + X(-\Omega)e^{j7\Omega} \end{aligned}$$

In all these expression, we substitute

$$X(\Omega) = \frac{4e^{j6\Omega} - 5e^{j5\Omega} + e^{j\Omega}}{(e^{j\Omega}-1)^2}$$

9.3-3. Let

$$Z(\Omega) = X(\Omega) \circledP Y(\Omega)$$

then

$$z[n] = 2\pi x[n]y[n]$$

$$x[n] = Z^{-1}\left[\sum_{k=0}^{4} a_k e^{-jk\Omega}\right] = a_0\delta[n] + a_1\delta[n-1] + a_2\delta[n-2] + a_3\delta[n-3] + a_4\delta[n-4]$$

and from pair 7 (Table 9.1)

$$y[n] = Z^{-1}\frac{\sin(5\Omega/2)}{\sin(\Omega/2)}e^{-j2\Omega} = \delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3] + \delta[n-4]$$

Both $x[n]$ and $y[n]$ are nonzero over $0 \le n \le 4$ and are zero outside this range clearly.

$$x[n]y[n] = x[n] \qquad \text{and} \qquad z[n] = 2\pi x[n]$$

$$Z(\Omega) = X(\Omega) \circledP Y(\Omega) = 2\pi X(\Omega)$$

9.3-4. (a) Apply modulation property [Eq. (9.55)] to pair 2 to obtain

$$\begin{aligned} a^n \cos\Omega_0 n u[n] &\Longleftrightarrow \frac{1}{2}\left[\frac{e^{j(\Omega-\Omega_0)}}{e^{j(\Omega-\Omega_0)}-a}+\frac{e^{j(\Omega+\Omega_0)}}{e^{j(\Omega+\Omega_0)}-a}\right] \\ &= \left[\frac{e^{j\Omega}-a\cos\Omega_0}{e^{j2\Omega}-2ae^{j\Omega}\cos\Omega_0+a^2}\right]e^{j\Omega} \end{aligned}$$

(b)

$$x[n] = n^2 a^n u[n]$$

Apply 'multiplication by n' property [Eq. (9.50)] to pair 2 to obtain

$$na^n u[n] \Longleftrightarrow j\frac{d}{d\Omega}\left[\frac{e^{j\Omega}}{e^{j\Omega}-a}\right] = \frac{ae^{j\Omega}}{(e^{j\Omega}-a)^2}$$

Apply the same property again to this result to obtain

$$n^2 a^n u[n] \Longleftrightarrow j\frac{d}{d\Omega}\left[\frac{ae^{j\Omega}}{(e^{j\Omega}-a)^2}\right] = \frac{ae^{j\Omega}(e^{j\Omega}+a)}{(e^{j\Omega}-a)^3}$$

(c)

$$x[n] = (n-k)a^{2n}u[n-m] = a^{2m}(n-m)a^{2(n-m)}u[n-m]+a^{2m}(m-k)a^{2(n-m)}u[n-m]$$

Application of 'multiplication by n' property to $na^n u[n]$ yields

$$na^{2n}u[n] \Longleftrightarrow \frac{a^2 e^{j\Omega}}{(e^{j\Omega}-a^2)^2}$$

Application of time-shift property now yields

$$\begin{aligned} X(\Omega) &= a^{2m}\frac{ae^{j\Omega}}{(e^{j\Omega}-a)^2}e^{-jm\Omega}+a^{2m}(m-k)\frac{e^{j\Omega}}{e^{j\Omega}-a^2}e^{-jm\Omega} \\ &= \frac{a^{2m}e^{-j(m-1)\Omega}}{e^{j\Omega}-a^2}\left[\frac{a}{e^{j\Omega}-a^2}+m-k\right] \end{aligned}$$

9.3-5. We shall consider spectra within the band $|\Omega| \le \pi$ only
Pair#11

$$1 = u[n] + u[-(n+1)]$$

But

$$u[n] \Longleftrightarrow \frac{e^{j\Omega}}{e^{j\Omega}-1}+\pi\delta(\Omega) \quad |\Omega| \le \pi$$

Hence

$$u[-n] \Longleftrightarrow \frac{e^{-j\Omega}}{e^{-j\Omega}-1}+\pi\delta(\Omega)$$

and

$$u[-(n+1)] = u[-n]-\delta[n] \Longleftrightarrow \frac{e^{-j\Omega}}{e^{-j\Omega}-1}+\pi\delta(\Omega)-1$$

and

$$1 \Longleftrightarrow \frac{e^{j\Omega}}{e^{j\Omega}-1}+\pi\delta(\Omega)+\frac{e^{-j\Omega}}{e^{-j\Omega}-1}+\pi\delta(\Omega)-1 = 2\pi\delta(\Omega)$$

Pair#12
Apply frequency-shift property [Eq. (9.54)] to the result for pair 11 to obtain

$$e^{j\Omega_0} \Longleftrightarrow 2\pi\delta(\Omega - \Omega_0) \qquad \Omega_0 \leq \pi$$

Pair#13
Apply modulation property [Eq. (9.55)] to the result for pair 11 to obtain

$$\cos\Omega_0 n \Longleftrightarrow \pi[\delta(\Omega - \Omega_0) + \delta(\Omega + \Omega_0)] \qquad \Omega_0 \leq \pi$$

Pair#14
Apply modulation property [Eq. (9.57)] with $\theta = -\pi/2$, to the result for pair 11 to obtain

$$\sin\Omega_0 n \Longleftrightarrow j\pi[\delta(\Omega + \Omega_0) - \delta(\Omega - \Omega_0)]$$

Pair#15

Apply modulation property [Eq. (9.55)] to the result for pair 10 to obtain

$$\begin{aligned}
\cos\Omega_0 n u[n] \quad &\Longleftrightarrow \quad \frac{1}{2}\left[\frac{e^{j(\Omega-\Omega_0)}}{e^{j(\Omega-\Omega_0)} - 1} + \frac{e^{j(\Omega+\Omega_0)}}{e^{j(\Omega+\Omega_0)} - 1}\right] + \frac{\pi}{2}[\delta(\Omega - \Omega_0) + \delta(\Omega + \Omega_0)] \\
&= \quad \frac{e^{j2\Omega} - e^{j\Omega}\cos\Omega_0}{e^{j2\Omega} - 2e^{j\Omega}\cos\Omega_0 + 1} + \frac{\pi}{2}[\delta(\Omega - \Omega_0) + \delta(\Omega + \Omega_0)]
\end{aligned}$$

Pair# 16
Apply modulation property [Eq. (9.57)] with $\theta = -\pi/2$ to pair 10 to obtain

$$\begin{aligned}
\sin\Omega_0 n u[n] \quad &\Longleftrightarrow \quad \frac{j}{2}\left[\frac{e^{j(\Omega+\Omega_0)}}{e^{j(\Omega+\Omega_0)} - 1} - \frac{e^{j(\Omega-\Omega_0)}}{e^{j(\Omega-\Omega_0)} - 1}\right] + \frac{j\pi}{2}[\delta(\Omega + \Omega_0) + \delta(\Omega - \Omega_0)] \\
&= \quad \frac{e^{j2\Omega}\sin\Omega_0}{e^{j2\Omega} - 2e^{j\Omega}\cos\Omega_0 + 1} + \frac{\pi}{2j}[\delta(\Omega - \Omega_0) - \delta(\Omega + \Omega_0)]
\end{aligned}$$

9.3-6.

$$\begin{aligned}
x[n + k] \quad &\Longleftrightarrow \quad X(\Omega)e^{jk\Omega} \\
x[n - k] \quad &\Longleftrightarrow \quad X(\Omega)e^{-jk\Omega}
\end{aligned}$$

Hence

$$x[n + k] + x[n - k] \Longleftrightarrow X(\Omega)\left[e^{jk\Omega} + e^{-jk\Omega}\right] = 2X(\Omega)\cos k\Omega$$

(a) Let $z[n] = u[n + 2] - u[n - 3]$ (see Figure S9.3-6a)
Then $x[n] = z[n - 4] + z[n + 4]$
Moreover $z[n]$ is $u[n] - u[n - 5]$ left-shifted by 2 units. Hence use of pair 7 and time-shifting property yields

$$Z(\Omega) = \frac{\sin 2.5\Omega}{\sin 0.5\Omega}$$

and

$$X(\Omega) = 2\frac{\sin 3\Omega}{\sin 0.5\Omega}\cos 4\Omega$$

(b) Figure S9.3-6b shows the signal $\Delta(\frac{n}{8})$. Hence,

$$x[n] = \Delta\left(\frac{n-8}{8}\right) + \Delta\left(\frac{n+8}{8}\right)$$

The DTFT of $\Delta\left(\frac{n}{8}\right)$ can be found in several way. Here we shall use the method of convolution. The reader can verify that the convolution $w[n]$ shown in Figure S9.3-6c with $w[-n]$ yields $\Delta\left(\frac{n}{8}\right)$.
From pair 7, we obtain

$$q[n] \Longleftrightarrow \frac{\sin 2\Omega}{\sin 0.5\Omega} e^{-j1.5\Omega}$$

Therefore

$$q[-n] \Longleftrightarrow \frac{\sin(-2\Omega)}{\sin(-0.5\Omega)} e^{j1.5\Omega} = \frac{\sin 2\Omega}{\sin 0.5\Omega} e^{j1.5\Omega}$$

Because

$$\Delta\left(\frac{n}{8}\right) = q[n] * q[-n]$$

$$\Delta\left(\frac{n}{8}\right) \Longleftrightarrow \frac{\sin^2(2\Omega)}{\sin^2(0.5\Omega)}$$

and

$$X(\Omega) = \frac{2\sin^2(2\Omega)}{\sin^2(0.5\Omega)} \cos 8\Omega$$

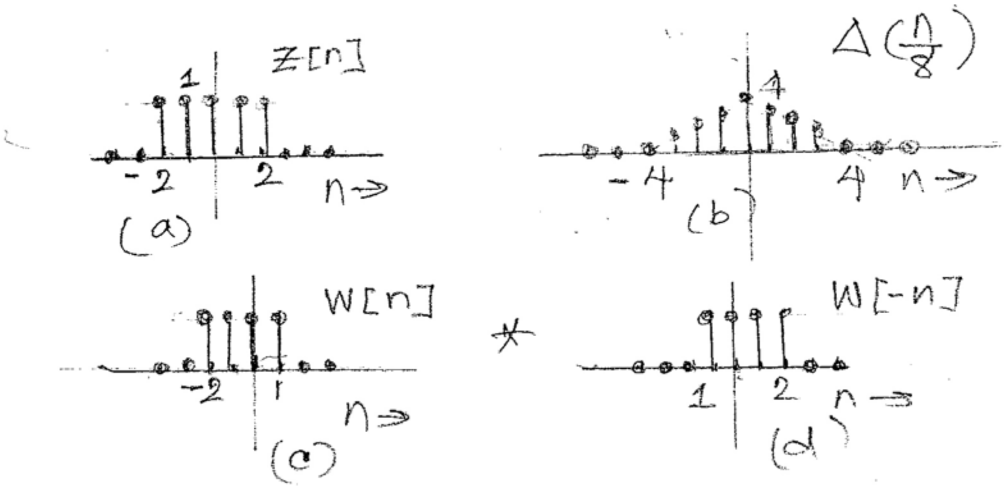


Figure S9.3-6

9.3-7. Use of time-shift property yields

$$x[n \pm k] \Longleftrightarrow X(\Omega) e^{\pm jk\Omega}$$

Therefore

$$x[n+k] - x[n-k] = X(\Omega)\left[e^{jk\Omega} - e^{-jk\Omega}\right] = 2X(\Omega)\sin k\Omega$$

From pair 7 (Table 9.1)

$$w[n] = u[n] - u[n-5] \Longleftrightarrow \frac{\sin 2.5\Omega}{\sin 0.5\Omega} e^{-j2\Omega}$$

and

$$x[n] = -\{w[n-2] - w[n+2]\} \Longleftrightarrow -\frac{\sin 2.5\Omega}{\sin 0.5\Omega} \sin 2\Omega e^{-j2\Omega}$$

9.3-8. If

$$W(\Omega) = \frac{e^{j\Omega}}{e^{j\Omega} - \gamma}$$

then

$$X(\Omega) = W(\Omega)W(\Omega)$$

and

$$\begin{aligned} x[n] &= \gamma^n u[n] * \gamma^n u[n] \\ &= \sum_{m=0}^{n} \gamma^m \gamma^{n-m} \qquad n \geq 0 \\ &= \gamma^n \sum_{m=0}^{n} 1 = \begin{cases} n\gamma^n & n \geq 0 \\ 0 & n < 0 \end{cases} \\ &= n\gamma^n u[n] \end{aligned}$$

9.3-9. Pair 2:

$$\gamma^n u[n] = \delta[n] + \gamma\delta[n-1] + \gamma^2\delta[n-2] + \ldots + \ldots$$

Therefore

$$\begin{aligned} \gamma^n u[n] &\Longleftrightarrow 1 + \gamma e^{-j\Omega} + \gamma^2 e^{-j2\Omega} + \gamma^3 e^{-j3\Omega} + \ldots + \ldots \\ &= \sum_{m=0}^{\infty} \left(\gamma e^{-j\Omega}\right)^m \\ &= \frac{1}{1 - \gamma e^{-j\Omega}} = \frac{e^{j\Omega}}{e^{j\Omega} - \gamma} \end{aligned}$$

Pair 3:
Use time inversion property to obtain

$$\lambda^{-n} u[-n] \Longleftrightarrow \frac{e^{-j\Omega}}{e^{-j\Omega} - \lambda} \qquad |\lambda| > 1$$

and

$$\lambda^{-n} u[-(n+1)] = \lambda^{-n} u[-n] - \delta[n]$$

Hence

$$-\lambda^{-n} u[-(n+1)] = -\lambda^{-n} u[-n] + \delta[n] \Longleftrightarrow \frac{-e^{-j\Omega}}{e^{-j\Omega} - \gamma} + 1 = \frac{-\lambda}{e^{-j\Omega} - \gamma} \qquad |\lambda| > 1$$

Letting $\lambda = 1/\gamma$, we obtain

$$-\gamma^n u[-(n+1)] \Longleftrightarrow \frac{-1/\gamma}{e^{j\Omega} - 1/\gamma} = \frac{e^{j\Omega}}{e^{j\Omega} - \gamma} \qquad |\gamma| < 1$$

Pair 4:

$$\gamma^{|n|} = \gamma^n u[n] - \left(\frac{1}{\gamma}\right)^n u[-(n+1)]$$

Hence

$$\gamma^{|n|} \Longleftrightarrow \frac{e^{j\Omega}}{e^{j\Omega}-\gamma} - \frac{e^{j\Omega}}{e^{j\Omega}-1/\gamma} = \frac{1-\gamma^2}{1-2\gamma\cos\Omega+\gamma^2}$$

Pair 5:
Apply 'multiplication by n' property [Eq. (9.50)] to pair 2 to obtain

$$n\gamma^n u[n] \Longleftrightarrow j\frac{d}{d\Omega}\left[\frac{e^{j\Omega}}{e^{j\Omega}-\gamma}\right] = \frac{\gamma e^{j\Omega}}{(e^{j\Omega}-\gamma)^2}$$

Pair 6:

$$\begin{aligned}
\gamma^n \cos[\Omega_0+\theta]u[n] &= \frac{1}{2}\left[\gamma^n e^{j(\Omega_0 n+\theta)} + \gamma^n e^{-j(\Omega_0 n+\theta)}\right]u[n] \\
&= \frac{1}{2}\left[e^{j\theta}\left(\gamma e^{j\Omega_0}\right)^n u[n] + e^{-j\theta}\left(\gamma e^{-j\Omega_0}\right)^n u[n]\right]
\end{aligned}$$

Hence

$$\begin{aligned}
X(\Omega) &= \frac{1}{2}\left[e^{j\theta}\frac{e^{j\Omega}}{e^{j\Omega}-\gamma e^{j\Omega_0}} + e^{-j\theta}\frac{e^{j\Omega}}{e^{j\Omega}-\gamma e^{-j\Omega_0}}\right] \\
&= \frac{e^{j\Omega}\left[e^{j\Omega}\cos\theta - \gamma\cos(\Omega_0-\theta)\right]}{e^{j2\Omega}-2\gamma\cos\Omega_0 e^{j\Omega}+\gamma^2}
\end{aligned}$$

Pair 7:

$$x[n] = \delta[n] + \delta[n-1] + \delta[n-2] + \cdots + \delta[n-M+1]$$

and

$$\begin{aligned}
X(\Omega) &= 1 + e^{-j\Omega} + e^{-j2\Omega} + \cdots + e^{-j(M-1)\Omega} \\
&= \sum_{k=0}^{M-1} e^{-jk\Omega} = \frac{e^{-jM\Omega}-1}{e^{-j\Omega}-1} = \frac{e^{-\frac{jM\Omega}{2}}\left(e^{-\frac{jM\Omega}{2}} - e^{\frac{jM\Omega}{2}}\right)}{e^{-\frac{j\Omega}{2}}\left(e^{-\frac{j\Omega}{2}} - e^{\frac{j\Omega}{2}}\right)} \\
&= \frac{\sin(M\Omega/2)}{\sin(\Omega/2)}e^{-j(M-1)\Omega/2}
\end{aligned}$$

9.3-10. We shall consider the spectrum only within the band $|\Omega| \le \pi$.

$$x[n] = e^{j\Omega n} = e^{j\frac{\Omega_0 n}{2}} \times e^{j\frac{\Omega_0 n}{2}}$$

Use of frequency convolution property yields

$$\begin{aligned}
X(\Omega) &= \frac{1}{2\pi}\left[2\pi\delta\left(\Omega-\frac{\Omega_0}{2}\right) * 2\pi\delta\left(\Omega-\frac{\Omega_0}{2}\right)\right] \qquad |\Omega| \le \pi \\
&= 2\pi\delta\left(\Omega-\frac{\Omega_0}{2}\right) * \delta\left(\Omega-\frac{\Omega_0}{2}\right) \qquad |\Omega| \le \pi \\
&= 2\pi\int_{-\infty}^{\infty}\delta\left(x-\frac{\Omega_0}{2}\right)\delta\left(\Omega-\Omega+\frac{\Omega_0}{2}\right)dx \qquad |\Omega| \le \pi
\end{aligned}$$

From the sampling property [Eq. (1.24)] of the impulse, we obtain

$$X(\Omega) = 2\pi\delta(\Omega - \Omega_0) \qquad |\Omega| \leq \pi$$

9.3-11. (a) Let

$$x[n] = \text{sinc}(\Omega_c n)$$

From pair 8, we have

$$X(\Omega) = \frac{\pi}{\Omega_c}\text{rect}\left(\frac{\Omega}{2\Omega_c}\right) \qquad |\Omega| \leq \pi$$

But

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$$

Hence

$$X(0) = \sum_{n=-\infty}^{\infty} x[n] = \sum_{n=-\infty}^{\infty} \text{sinc}(\Omega_c n)$$

But $X(0) = \frac{\pi}{\Omega_c}$, which proves the result.

(b) Let

$$x[n] = (-1)^n \text{sinc}(\Omega_c n)$$

First, we prove that if $x[n] \Longleftrightarrow X(\Omega)$, then $(-1)^n x[n] \Longleftrightarrow X(\Omega - \pi)$. This follows from the definition of the DTFT

$$\begin{aligned}
(-1)^n x[n] \quad &\Longleftrightarrow \quad \sum_{n=-\infty}^{\infty} (-1)^n x[n]e^{-j\Omega n} \\
&= \quad \sum_{n=-\infty}^{\infty} x[n]e^{-j\pi n}e^{j\Omega n} \\
&= \quad \sum_{n=-\infty}^{\infty} x[n]e^{-j(\Omega-\pi)n} \\
&= \quad X(\Omega - \pi)
\end{aligned}$$

Hence, we have

$$(-1)^n \text{sinc}(\Omega_c n) \Longleftrightarrow \frac{\pi}{\Omega_c}\text{rect}\left(\frac{\Omega - \pi}{2\Omega_c}\right) \qquad |\Omega| \leq \pi$$

Moreover

$$\text{rect}\left(\frac{\Omega - \pi}{2\Omega_c}\right) = 0 \qquad \text{at} \qquad \Omega = 0 \qquad \text{if} \qquad \Omega_c < \pi$$

Hence

$$X(0) = \sum_{n=-\infty}^{\infty} (-1)^n \text{sinc}(\Omega_c n) = 0$$

(c)

$$\text{sinc}^2(\Omega_c n) \Longleftrightarrow \frac{\pi}{\Omega_c}\Delta\left(\frac{\Omega}{4\Omega_c}\right) \qquad |\Omega| \leq \frac{\pi}{2}$$

Using the argument in part (a), we obtain the desired result.

(d) Using the argument in part (b) and (c), we obtain the desired result.

(e)

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\Omega)e^{j\Omega n}d\Omega$$

Hence

$$x[0] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\Omega)d\Omega \tag{1}$$

If we left-shift x[n] in pair 7 by $\frac{M-1}{2}$ units, we obtain

$$x[n] = u\left[n + \frac{M-1}{2}\right] - u\left[n + \frac{M+1}{2}\right] \Longleftrightarrow \frac{\sin(M\Omega/2)}{\sin(\Omega/2)}$$

Moreover $x[0] = 1$. Hence use of Eq. (1) above yields

$$2\pi = \int_{-\pi}^{\pi} \frac{\sin(M\Omega/2)}{\sin(\Omega/2)}d\Omega$$

(f) From pair 9, we have

$$\text{sinc}^2(\Omega_c n) \Longleftrightarrow \frac{\pi}{\Omega_c}\Delta\left(\frac{\omega}{4\Omega_c}\right) \qquad |\Omega| < \frac{\pi}{2}$$

Application of Parseval's theorem [Eq. (9.60)] yields

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} \left|\text{sinc}^2(\Omega_c n)\right|^2 &= \frac{2}{2\pi}\frac{\pi^2}{\Omega_c^2}\int_0^{2\Omega_c}\left|1 - \frac{\Omega}{\Omega_c}\right|^2 d\Omega \\
&= \frac{\pi}{\Omega_c^2}\int_0^{2\Omega_c}\left(1 - \frac{2\Omega}{\Omega_c} + \frac{\Omega^2}{\Omega_c^2}\right)d\Omega \\
&= \frac{2\pi}{3\Omega_c}
\end{aligned}$$

9.3-12.

$$\begin{aligned}
E_{x_c} &= \int_{-\infty}^{\infty} |x_c(t)|^2 dt = \int_{-\infty}^{\infty} x_c(t)x_c^*(t)dt \\
&= \int_{-\infty}^{\infty}\left[\sum_{n=-\infty}^{\infty} x_c(nT)\text{sinc}(2\pi Bt - n\pi)\right]\left[\sum_{m=-\infty}^{\infty} x_c^*(mT)\text{sinc}(2\pi Bt - m\pi)\right]dt
\end{aligned}$$

Because of orthogonality property of the sinc function, stated in the problem, all the cross-product terms, for $m \neq n$, vanish. Moreover when $m = n$, the integral is $1/2B$. Recall also that $x_c(kT) = x[k]$. Hence

$$E_{x_c} = \frac{1}{2B}\sum_{n=-\infty}^{\infty}|x[n]|^2 = T\sum_{n=-\infty}^{\infty}|x[n]|^2 = TE_x$$

9.4-1.

$$
\begin{aligned}
X(\Omega) &= \frac{1}{1+0.5e^{-j\Omega}} = \frac{e^{j\Omega}}{e^{j\Omega}+0.5} \\
Y(\Omega) &= X(\Omega)H(\Omega) = \frac{e^{j\Omega}\left(e^{j\Omega}+0.32\right)}{\left(e^{j\Omega}+0.5\right)\left(e^{j\Omega}+0.8\right)\left(e^{j\Omega}+0.2\right)} \\
\frac{Y(\Omega)}{e^{j\Omega}} &= \frac{e^{j\Omega}+0.32}{\left(e^{j\Omega}+0.5\right)\left(e^{j\Omega}+0.8\right)\left(e^{j\Omega}+0.2\right)} \\
&= \frac{2}{e^{j\Omega}+0.5} - \frac{8/3}{e^{j\Omega}+0.8} + \frac{2/3}{e^{j\Omega}+0.2} \\
Y(\Omega) &= 2\frac{e^{j\Omega}}{e^{j\Omega}+0.5} - \frac{8}{3}\frac{e^{j\Omega}}{e^{j\Omega}+0.8} + \frac{2}{3}\frac{e^{j\Omega}}{e^{j\Omega}+0.2} \\
y[n] &= \left[2\left(-0.5\right)^n - \frac{8}{3}\left(-0.8\right)^n + \frac{2}{3}\left(-0.2\right)^n\right]u[n]
\end{aligned}
$$

9.4-2.

$$
Y(\Omega) = \left[\frac{e^{j\Omega}+0.32}{(e^{j\Omega}+0.2)(e^{j\Omega}+0.8)}\right]\left[\pi\delta(\Omega) + \frac{e^{j\Omega}}{e^{j\Omega}-1}\right]
$$

We use the fact that $f(x)\delta(x) = f(0)\delta(x)$ to obtain

$$
Y(\Omega) = \frac{1.32\pi}{2.16}\delta(\Omega) + \frac{e^{j\Omega}(e^{j\Omega}+0.32)}{(e^{j\Omega}-1)(e^{j\Omega}+0.2)(e^{j\Omega}+0.8)}
$$

Using partial fraction expansion, we obtain

$$
Y(\Omega) = \frac{1.32\pi}{2.16}\delta(\Omega) + \frac{1.32}{2.16}\frac{e^{j\Omega}}{e^{j\Omega}-1} - \frac{1}{6}\frac{e^{j\Omega}}{e^{j\Omega}+0.2} - \frac{4}{9}\frac{e^{j\Omega}}{e^{j\Omega}+0.8}
$$

and

$$
y[n] = 0.611u[n] - \left[\frac{1}{6}(-0.2)^n + \frac{4}{9}(-0.8)^n\right]u[n]
$$

9.4-3.

$$
\begin{aligned}
X(\Omega) &= \frac{e^{j\Omega}}{e^{j\Omega}-0.8} - \frac{2e^{j\Omega}}{e^{j\Omega}-2} \\
Y(\Omega) &= X(\Omega)H(\Omega) = \frac{e^{j2\Omega}}{\left(e^{j\Omega}-0.5\right)\left(e^{j\Omega}-0.8\right)} - \frac{2e^{j2\Omega}}{\left(e^{j\Omega}-0.5\right)\left(e^{j\Omega}-2\right)} \\
&= \frac{-5/3}{e^{j\Omega}-0.5} + \frac{8/3}{e^{j\Omega}-0.8} + \frac{2/3}{e^{j\Omega}-0.5} - \frac{8/3}{e^{j\Omega}-2} \\
&= \frac{-1}{e^{j\Omega}-0.5} + \frac{8/3}{e^{j\Omega}-0.8} - \frac{8/3}{e^{j\Omega}-2} \\
Y(\Omega) &= -\frac{e^{j\Omega}}{e^{j\Omega}-0.5} + \frac{8}{3}\frac{e^{j\Omega}}{e^{j\Omega}-0.8} - \frac{8}{3}\frac{e^{j\Omega}}{e^{j\Omega}-2} \\
y[n] &= \left[-\left(0.5\right)^n + \frac{8}{3}\left(0.8\right)^n\right]u[n] + \frac{8}{3}\left(2\right)^n u\left[-\left(n+1\right)\right]
\end{aligned}
$$

9.4-4. (a) When $x[n] = \delta[n]$, the output is $h[n]$, given by (assuming causal accumulator)

$$h[n] = \sum_{k=0}^{n} \delta[k] = u[n]$$

and

$$H(\Omega) = \frac{e^{j\Omega}}{e^{j\Omega} - 1} + \pi\delta(\Omega)$$

(b) If this accumulator is used as a digital processor for the digital integrator, discussed in example 3.7, and the input is $x(t) = u(t)$, then the sampled $x(t)$ yields the digital input $x[n] = x(nT) = u[n]$

9.4-5. (a)

$$\text{sinc}\left(\frac{\pi n}{2}\right) \Longleftrightarrow 2\text{rect}\left(\frac{\Omega}{\pi}\right)$$

Hence the output corresponding to this input is

$$Y(\Omega) = 2\text{rect}^2\left(\frac{\Omega}{\pi}\right) e^{-j2\Omega} = 2\text{rect}\left(\frac{\Omega}{\pi}\right) e^{-j2\Omega}$$

Therefore

$$y[n] = \text{sinc}\left[\frac{\pi(n-2)}{2}\right]$$

(b)

$$\text{sinc}(\pi n) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases}$$

Hence

$$\text{sinc}(\pi n) = \delta[n] \Longleftrightarrow 1$$

The output corresponding to this input is

$$Y(\Omega) = \text{rect}\left(\frac{\Omega}{\pi}\right) e^{-j2\Omega}$$

and

$$y[n] = \frac{1}{2}\text{sinc}\left[\frac{\pi(n-2)}{2}\right]$$

(c)

$$\text{sinc}^2\left(\frac{\pi n}{4}\right) \Longleftrightarrow 4\Delta\left(\frac{\Omega}{\pi}\right)$$

The output corresponding to this input is

$$\begin{aligned} Y(\Omega) &= 4\Delta\left(\frac{\Omega}{\pi}\right) \text{rect}\left(\frac{\Omega}{\pi}\right) e^{-j2\Omega} \\ &= 4\Delta\left(\frac{\Omega}{\pi}\right) e^{-j2\Omega} \end{aligned}$$

and

$$y(n) = \text{sinc}^2\left[\frac{\pi(n-2)}{4}\right]$$

9.4-6. (a) Let

$$y[n] = (-1)^n x[n] = e^{-j\pi n}x[n]$$

Use of frequency-shifting property [Eq. (9.45)] yields

$$Y(\Omega) = X(\Omega - \pi)$$

(b) Figure S9.4-6a shows $\gamma^n u[n]$ and $(-1)\gamma^n u[n]$. The spectra for $(-1)^n\gamma^n u[n]$ are the same as those for $\gamma^n u[n]$ (Figure S9.4-6 b and c) but shifted by π, as shown in Figure S9.4-6 b over the fundamental band $|\Omega| \leq \pi$.

(c)

$$h_{LP}[n] = \frac{\Omega_c}{\pi}\text{sinc}(\Omega_c n)$$

The frequency response of $(-1)^n h_{LP}[n] = \frac{\Omega_c}{\pi}(-1)^n\text{sinc}(\Omega_c n)$ is $\text{rect}\left(\frac{\Omega}{2\Omega_c}\right)$ frequency-shifted by π is $\text{rect}\left(\frac{\Omega-\pi}{2\Omega_c}\right)$, as shown in Figure S9.4-6c. It is clear that this is a highpass filter.

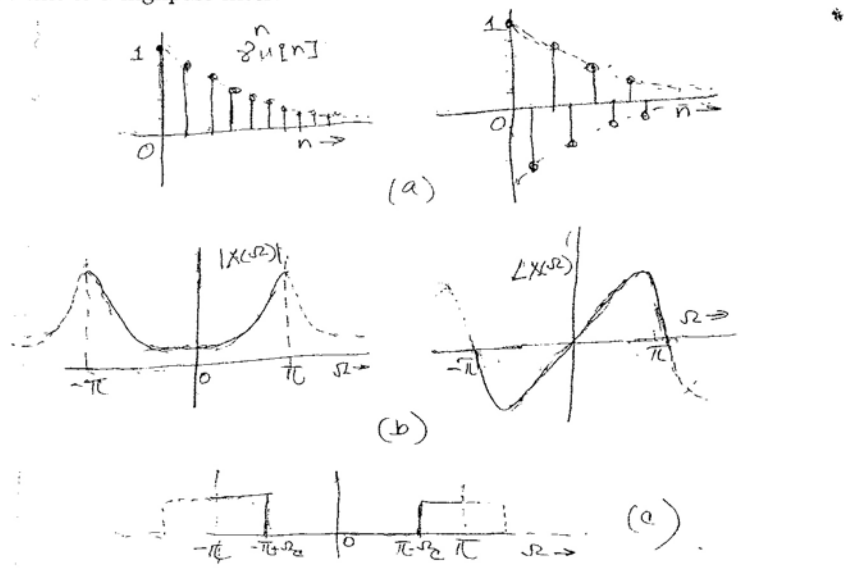


Figure S9.4-6

9.4-7.

$$w[n] = (-1)^n x[n]$$

Hence $W(\Omega) = X(\Omega - \pi)$, as shown in Problem 9.4-6a.

$$Q(\Omega) = X(\Omega - \pi)H(\Omega)$$

and because

$$y[n] = (-1)^n q[n]$$

$$
\begin{aligned}
Y(\Omega) = Q(\Omega - \pi) &= X(\Omega - 2\pi)H(\Omega - \pi) \\
&= X(\Omega)H(\Omega - \pi) \\
&= X(\Omega)H_1(\Omega)
\end{aligned}
$$

Therefore

$$H_1\Omega = H(\Omega - \pi)$$

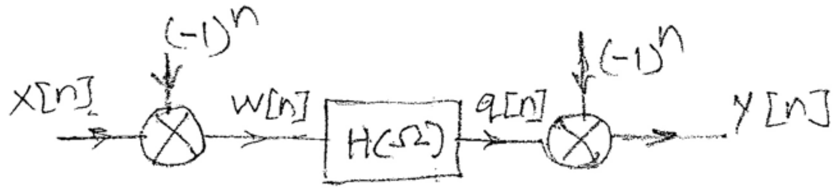


Figure S9.4-7

9.4-8. (a) System $\mathcal{S}_1$ can be described by Eq. (3.17)b as

$$y_1[n] + \sum_{k=1}^{N} a_k y_1[n-k] = \sum_{k=0}^{N} b_k x[n-k]$$

Take DTFT of this equation to obtain

$$Y_1(\Omega)\left[1 + \sum_{k=1}^{N} a_k e^{-jk\Omega}\right] = X(\Omega)\sum_{k=0}^{N} b_k e^{-jk\Omega}$$

and the frequency response of this system is given by

$$H_1(\Omega) = \frac{Y_1(\Omega)}{X(\Omega)} = \frac{\sum_{k=0}^{N} b_k e^{-jk\Omega}}{1 + \sum_{k=1}^{N} a_k e^{-jk\Omega}}$$

Now consider the system $\mathcal{S}_2$, whose a_i and b_i coefficients are $(-1)^k$ times the corresponding coefficients of $\mathcal{S}_1$. Hence

$$H_2(\Omega) = \frac{Y_2(\Omega)}{X(\Omega)} = \frac{\sum_{k=0}^{N}(-1)^k b_k e^{-jk\Omega}}{1 + \sum_{k=1}^{N}(-1)^k a_k e^{-jk\Omega}} = \frac{\sum_{k=0}^{N} b_k e^{-jk(\Omega-\pi)}}{1 + \sum_{k=1}^{N} a_k e^{-jk(\Omega-\pi)}} = H_1(\Omega-\pi)$$

(b) As shown in Prob. 9.4-6, if $H_1(\Omega)$ is a lowpass filter, then $H_2(\Omega) = H_1(\Omega - \pi)$ is a highpass filter.

(c) The DTFT of the system equation $y[n] - 0.8y[n-1] = x[n]$ is

$$Y_1(\Omega)\left[1 - 0.8e^{-j\Omega}\right] = X(\Omega)$$

and

$$H_1(\Omega) = \frac{Y_1(j\Omega)}{X(j\Omega)} = \frac{1}{1 - 0.8e^{-j\Omega}}$$

The frequency spectra for this system are shown in Figure 9.4, which shows that this is a lowpass system.
The DTFT of the system equation $y[n] + 0.8y[n-1] = x[n]$ is

$$Y_2(\Omega)\left[1 + 0.8e^{-j\Omega}\right] = X(\Omega)$$

and

$$H_2(\Omega) = \frac{Y_2(j\Omega)}{X(j\Omega)} = \frac{1}{1+0.8e^{-j\Omega}} = \frac{1}{1-0.8e^{-j(\Omega-\pi)}} = H_1(\Omega-\pi)$$

Hence the frequency response spectra for $H_2(\Omega)$ are same as those for $H_1(\Omega)$ frequency-shifted by π. Hence, $H_2(\Omega)$ is a highpass system.

9.4-9. (a) Let us compute the response $h[n]$ to the unit impulse input $\delta[n]$. Because the system contains time varying multipliers, however, we must also test whether it is a time varying or a time-invariant system. It is therefore appropriate to consider the system response to an input $\delta[n-k]$. This is an impulse at $n=k$. Using the fact that $x[n]\,\delta[n-k] = x[n]\delta[n-k]$, we can express the signals at various points as follows:

at a_1	$2\cos\,(\Omega_c k)\,\delta[n-k]$
a_2	$2\sin\,(\Omega_c k)\,\delta[n-k]$
b_1	$2\cos\,(\Omega_c k)h_o[n-k]$
b_2	$2\sin\,(\Omega_c k)h_o[n-k]$
c_1	$2\cos\,(\Omega_c k)\cos\,(\Omega_c n)h_o[n-k]$
c_2	$2\sin\,(\Omega_c k)\sin\,(\Omega_c n)h_o[n-k]$
d	$2h_o[n-k]\,[\cos\,(\Omega_c k)\cos\,(\Omega_c n)+\sin(\Omega_c k)\sin\,(\Omega_c n)]$
	$= 2h_o[n-k]\cos\,(\Omega_c[n-k])$

Thus, the system response to the input $\delta[n-k]$ is $2h_o[n-k]\cos[\Omega_c[n-k]]$. Clearly, the system is linear time-invariant, with impulse response

$$h[n] = 2h_o[n]\,\cos\Omega_c n$$

(b) From the modulation property, it follows that

$$H(\Omega) = H_0(\Omega-\Omega_c) + H_0(\Omega+\Omega_c)$$

If $H_0(\Omega) = (rect)(\Omega/2W)$, then

$$H(\omega) = \text{rect}\left(\frac{\Omega-\Omega_c}{2W}\right) + \text{rect}\left(\frac{\Omega+\Omega_c}{2W}\right)$$

The transfer function $H(\omega)$ [Figure S9.4-9b] represents an ideal bandpass filter when $\Omega_c + W \le \pi$.

9.M-1. (a) The inverse DTFS is given by $x[n] = \sum_{r=0}^{N_0-1}\mathcal{D}_r e^{jr\Omega_0 n}$. Like the DTFS, the IDTFS can be computed using a matrix based approach. First, define $W_{N_0} = e^{j\Omega_0}$, which is a constant for a given N_0. Substituting W_{N_0} into the IDTFS equation yields $x[n] = \sum_{n=0}^{N_0-1}\mathcal{D}_r W_{N_0}^{nr}$. An inner product of two vectors computes $x[n]$.

$$x[n] = \left[1, W_{N_0}^{n}, W_{N_0}^{2n}, \cdots, W_{N_0}^{(N_0-1)r}\right]\begin{bmatrix}\mathcal{D}_0\\ \mathcal{D}_1\\ \mathcal{D}_2\\ \vdots\\ \mathcal{D}_{N_0-1}\end{bmatrix}.$$

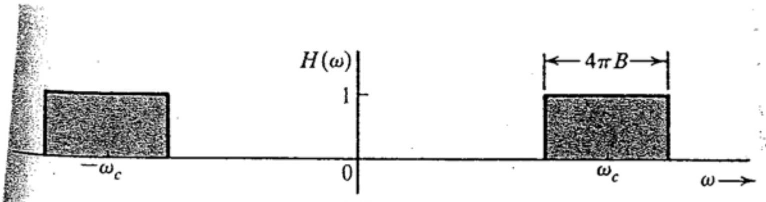


Figure S9.4-9b

Stacking the results for all n yields:

$$\begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ \vdots \\ x[N_0-1] \end{bmatrix} = \begin{bmatrix} 1, & 1, & 1, & \cdots, & 1 \\ 1, & W_{N_0}^1, & W_{N_0}^2, & \cdots, & W_{N_0}^{(N_0-1)} \\ 1, & W_{N_0}^2, & W_{N_0}^4, & \cdots, & W_{N_0}^{2(N_0-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1, & W_{N_0}^{(N_0-1)}, & W_{N_0}^{2(N_0-1)}, & \cdots, & W_{N_0}^{(N_0-1)^2} \end{bmatrix} \begin{bmatrix} \mathcal{D}_0 \\ \mathcal{D}_1 \\ \mathcal{D}_2 \\ \vdots \\ \mathcal{D}_{N_0-1} \end{bmatrix}.$$

In matrix notation, the IDTFS is compactly written as $\mathbf{x} = \mathbf{W}_{N_0}^* \boldsymbol{\mathcal{D}}$. Notice, $\mathbf{W}_{N_0}^*$ is just the conjugate of the DFT matrix $\mathbf{W}_{N_0}$.
In MATLAB, the N_0-by-N_0 IDTFS matrix is easily computed according to

```
>> Wconj= (exp(j*2*pi/N_0)).^((0:N_0-1)'*(0:N_0-1));
```

(b) MATLAB code, similar to that presented in MATLAB session 9, is used to test the execution speed of the matrix IDTFS approach to the inverse FFT approach. First, test vectors and IDTFS matrices are created.

```
>> X10 = fft(randn(10,1));
>> X100 = fft(randn(100,1));
>> X1000 = fft(randn(1000,1));
>> W10 = (exp(j*2*pi/10)).^((0:10-1)'*(0:10-1));
>> W100 = (exp(j*2*pi/100)).^((0:100-1)'*(0:100-1));
>> W1000 = (exp(j*2*pi/1000)).^((0:1000-1)'*(0:1000-1));
\end{verbatim}
Next, execution speeds are measured by repeating calculations within a loop.
Notice, output from MATLAB's {\tt ifft} command must be scaled by
$1/N_0$ to compute the IDTFS.
\begin{verbatim}
>> tic; for t=1:50000, ifft(X10)/10; end; T10ifft =toc;
>> tic; for t=1:50000, W10*X10; end; T10mat = toc;
>> tic; for t=1:5000, ifft(X100)/100; end; T100ifft =toc;
```

```
>> tic; for t=1:5000, W100*X100; end; T100mat = toc;
>> tic; for t=1:500, ifft(X1000)/1000; end; T1000ifft =toc;
>> tic; for t=1:500, W1000*X1000; end; T1000mat = toc;
>> [T10mat/T10ifft, T100mat/T100ifft, T1000mat/T1000ifft]
ans = 1.0323    3.5000   101.4754
```

For these trials, these results indicate that the IFFT approach is about as fast as the matrix approach for $N_0 = 10$, about an order of magnitude faster than the matrix approach for $N_0 = 100$, and about two orders of magnitude faster than the matrix approach for $N_0 = 1000$. While actual times will vary considerably from computer to computer and from trial to trial, the general trend is clear: the matrix based approach is less efficient than the inverse FFT approach, and this difference grows rapidly as N_0 increases.

(c) Substituting $\mathcal{D} = \frac{1}{N_0}\mathbf{W}_{N_0}\mathbf{x}$ into $\mathbf{x} = \mathbf{W}^*_{N_0}\mathcal{D}$ yields $\mathbf{x} = \mathbf{W}^*_{N_0}\frac{1}{N_0}\mathbf{W}_{N_0}\mathbf{x} = \frac{1}{N_0}\mathbf{W}^*_{N_0}\mathbf{W}_{N_0}\mathbf{x}$. For equality, $\frac{1}{N_0}\mathbf{W}^*_{N_0}\mathbf{W}_{N_0} = I_{N_0}$, where I_{N_0} is the N_0-by-N_0 identity matrix. Thus, $\mathbf{W}^*_{N_0}\mathbf{W}_{N_0} = N_0 I_{N_0} = \mathbf{W}_{N_0}\mathbf{W}^*_{N_0}$.

Thus, multiplying the DFT matrix $\mathbf{W}_{N_0}$ by the inverse DTFS matrix $\mathbf{W}^*_{N_0}$, or vice versa, yields the scaled identity matrix $N_0 I_{N_0}$:

$$\mathbf{W}^*_{N_0}\mathbf{W}_{N_0} = \mathbf{W}_{N_0}\mathbf{W}^*_{N_0} = N_0 I_{N_0}.$$

This result is consistent with the fact that the DTFS represents a signal using an orthogonal set of basis functions. Since the columns (or rows) of $\mathbf{W}_{N_0}$ are orthogonal, $\mathbf{W}^*_{N_0}\mathbf{W}_{N_0}$ must be a diagonal matrix. The scale factor of N_0 results from mixing matrices from the DFT and DTFS (recall, the DFT is N_0 times the DFTS).

9.M-2. (a) We know that $|H(e^{j\Omega_c})|^2 = \frac{1}{2} = \frac{(1+\alpha)^2}{4}\frac{|1-e^{-j\Omega}|^2}{|1-\alpha e^{-j\Omega}|^2} = \frac{1+2\alpha+\alpha^2}{4}\frac{(1-\cos(-\Omega_c))^2+(-\sin(-\Omega_c))^2}{(1-\alpha\cos(-\Omega_c))^2+(\alpha\sin(-\Omega_c))^2} = \frac{1+2\alpha+\alpha^2}{4}\frac{1-2\cos(\Omega_c)+\cos^2(\Omega_c)+\sin^2(\Omega_c)}{1-2\alpha\cos(\Omega_c)+\alpha^2\cos^2(\Omega_c)+\alpha^2\sin^2(\Omega_c)} = \frac{1+2\alpha+\alpha^2}{4}\frac{2-2\cos(\Omega_c)}{1+\alpha^2-2\alpha\cos(\Omega_c)} = \frac{1+2\alpha+\alpha^2}{4}\frac{2-2\cos(\Omega_c)}{1+\alpha^2-2\alpha\cos(\Omega_c)}$. Thus, $2(1+\alpha^2-2\alpha\cos(\Omega_c)) = (1+2\alpha+\alpha^2)(2-2\cos(\Omega_c))$ or $2+2\alpha^2-4\alpha\cos(\Omega_c) = 2+4\alpha+2\alpha^2-2\cos(\Omega_c)-4\alpha\cos(\Omega_c)-2\alpha^2\cos(\Omega_c)$. This simplifies to $0 = -2\alpha^2\cos(\Omega_c)+4\alpha-2\cos(\Omega_c)$ or $\cos(\Omega_c)\alpha^2-2\alpha+\cos(\Omega_c)$. Solving with the quadratic formula yields $\alpha = \frac{2\pm\sqrt{4-4\cos^2(\Omega_c)}}{2\cos(\Omega_c)} = \frac{1\pm\sin(\Omega_c)}{\cos(\Omega_c)}$. For $0 \le \Omega_c \le \pi$, $\left|\frac{1+\sin(\Omega_c)}{\cos(\Omega_c)}\right| \ge 1$ and $\left|\frac{1-\sin(\Omega_c)}{\cos(\Omega_c)}\right| \le 1$. Since a stable system is desired,

$$\alpha = \frac{1-\sin(\Omega_c)}{\cos(\Omega_c)}.$$

(b) For this part, $\Omega_c = \frac{2\pi}{5}$. Using MATLAB to solve:

```
>> Omega_c = 2*pi/5; alpha = (1-sin(Omega_c))/cos(Omega_c)
alpha = 0.1584
```

The corresponding difference equation is determined from $H(z) = \frac{Y(z)}{X(z)} = \frac{B(z)}{A(z)} = \left(\frac{1+\alpha}{2}\right)\left(\frac{1-z^{-1}}{1-\alpha z^{-1}}\right)$. Using MATLAB:

```
>> B = (1+alpha)/2*[1,-1], A = [1,-alpha]
B = 0.5792   -0.5792
A = 1.0000   -0.1584
```

Thus, the difference equation is

$$y[n] - 0.1584y[n-1] = 0.5792x[n] - 0.5792x[n-1].$$

This first order system has one pole at $z = \alpha = 0.1584$. Since this pole is inside the unit circle, the system is stable. Frequency response is computed using the signal processing toolbox function `freqz`:

```
>> Omega = linspace(0,pi,1001); H = freqz(B,A,Omega);
>> H3dB = freqz(B,A,[-Omega_c,Omega_c]);
>> plot(Omega,(abs(H)),'k',...
   [0,pi],[1/sqrt(2),1/sqrt(2)],'k:',...
   [Omega_c,Omega_c],[0,1],'k:');
>> axis([0,pi,0,1]);
>> xlabel('\Omega'); ylabel('|H(e^{j\Omega})|');
>> set(gca,'xtick',[0:pi/5:pi],'xticklabel',['  0 ';...
   ' p/5';'2p/5';'3p/5';'4p/5';'  p '],...
   'fontname','symbol');
```

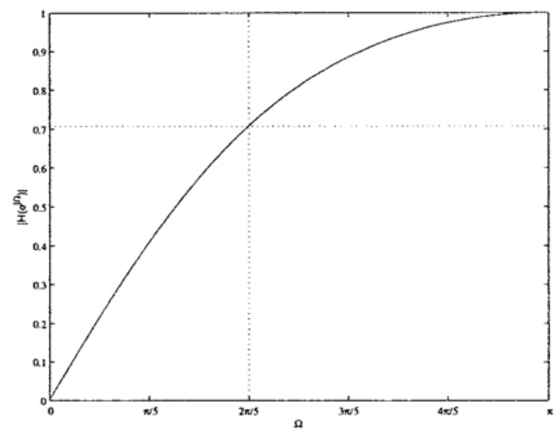


Figure S9.M-2b: $|H(e^{j\Omega})|$ for digital HPF with $\Omega_c = 2\pi/5$.

As seen if Figure S9.M-2b, the filter is highpass with a cutoff frequency $\Omega_c = 2\pi/5$, as desired.

(c) Since α, and therefore $H(z)$, is held constant, the cutoff frequency Ω_c remains constant as well. That is, changing the sampling frequency does not affect the digital cutoff frequency Ω_c of the filter. However, the cut-off frequency expressed in hertz scales directly with the sampling frequency. That is, as $\mathcal{F}_s$ is increased to $\mathcal{F}_s = 50\text{kHz}$, $f_c = 1\text{kHz}$ is increased to $f_c = 10\text{kHz}$.

(d) The inverse to $H(z) = \left(\frac{1+\alpha}{2}\right)\left(\frac{1-z^{-1}}{1-\alpha z^{-1}}\right)$ is $H^{-1}(z) = \left(\frac{2}{1+\alpha}\right)\left(\frac{1-\alpha z^{-1}}{1-z^{-1}}\right)$. Since the inverse has a root on the unit circle, it is not BIBO stable and therefore not well behaved.

(e) For $\Omega_c = \pi/2$, $\alpha = \frac{1-\sin(\Omega_c}{\cos(\Omega_c)} = \frac{1-1}{0}$, which is indeterminant. Using L'Hospital's rule, $\lim_{\Omega_c \to \pi/2} \frac{1-\sin(\Omega_c}{\cos(\Omega_c)} = \lim_{\Omega_c \to \pi/2} \frac{d}{dt}\left(\frac{1-\sin(\Omega_c}{\cos(\Omega_c)}\right) = \lim_{\Omega_c \to \pi/2} \frac{-\cos(\Omega_c}{-\sin(\Omega_c)} = \frac{0}{1} = 0$. Using $\alpha = 0$, the system function is $H(z) = 0.5(1 - z^{-1})$. What is particularly interesting is that when $\Omega_c = \pi/2$ this normally IIR filter $H(z)$ becomes an FIR filter! Notice that the impulse response is $h[n] = 0.5\delta[n] - 0.5\delta[n-1]$, which is a finite duration signal.

9.M-3. (a) It would be unlikely if not impossible to achieve this exact magnitude response with a practical FIR filter. Since the magnitude response has points of derivative discontinuities, an infinite length filter would be required, which is not practical.

(b) MATLAB is used to design a length-31 FIR filter that reasonably approximates the desired response. Different approximations are easily accomplished by changing N. To perform computations, program MS9P3 and MS5P1 are utilized. Notice, it is important to define Ω over $[0, 2\pi)$ not $[-\pi, \pi)$.

```
>> H_d = inline(['((mod(Omega,2*pi)>=0)&(mod(Omega,2*pi)<pi/4))',...
   '.*(4*mod(Omega,2*pi)/pi)+',...
   '((mod(Omega,2*pi)>=pi/4)&(mod(Omega,2*pi)<pi/2))',...
   '.*(2-4*mod(Omega,2*pi)/pi)+',...
   '((mod(Omega,2*pi)>7*pi/4)&(mod(Omega,2*pi)<=2*pi))',...
   '.*(-4*(mod(Omega,2*pi)-2*pi)/pi)+',...
   '((mod(Omega,2*pi)>3*pi/2)&(mod(Omega,2*pi)<=7*pi/4))',...
   '.*(2+4*(mod(Omega,2*pi)-2*pi)/pi)']);
>> N = 31; h = MS9P3(N,H_d);
>> Omega = linspace(0,2*pi,1001);
>> Omega_samples = linspace(0,2*pi*(1-1/N),N)';
>> H = MS5P1(h,1,Omega);
>> subplot(2,1,1); stem([0:N-1],h,'k');
>> xlabel('n'); ylabel('h[n]');
>> subplot(2,1,2); plot(Omega_samples,H_d(Omega_samples),'ko',...
   Omega,H_d(Omega),'k:',Omega,abs(H),'k');
>> axis([0 2*pi -0.1 1.3]); xlabel('\Omega'); ylabel('|H(\Omega)|');
>> legend('Samples','Desired','Actual',0);
>> set(gca,'xtick',[0:pi/2:2*pi],'xticklabel',['  0 ';...
   ' p/2';'';' p ';'3p/2';' 2p '],...
   'fontname','symbol');
```

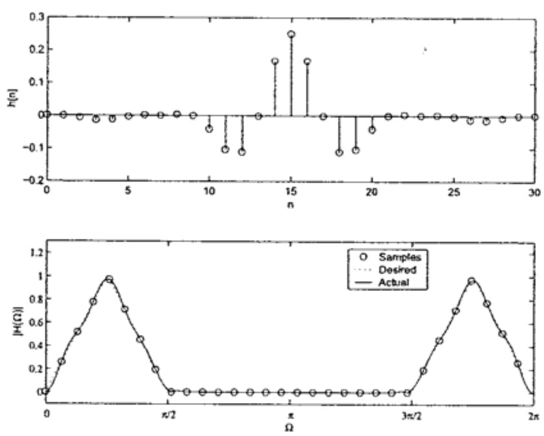


Figure S9.M-3b: Length-31 FIR "triangle" bandpass filter.

Repeating the above code using $N = 101$ yields a much closer approximation.

9.M-4. A simple first-order highpass filter is given by $H_{\text{HP}}(z) = k(1 - z^{-1})$. To achieve a gain of 3, solve $3 = k|1 - e^{-j\pi}| = 2k$. Thus, $k = 3/2$. To realize the desired comb filter, the

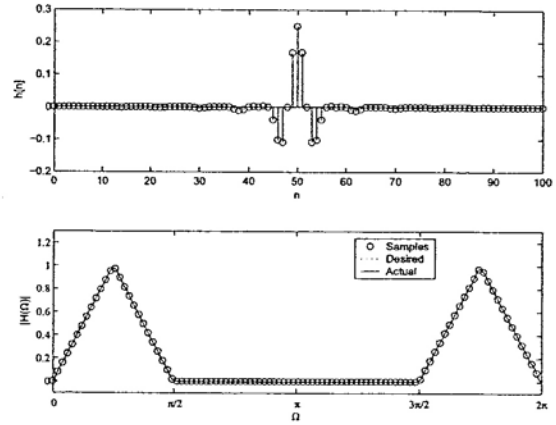


Figure S9.M-3b: Length-101 FIR "triangle" bandpass filter.

HPF response needs to be compressed by a factor of 4, which effectively replicates the original response four times over $[0, 2\pi)$. Compression is achieved by letting $z = z^4$. Thus,

$$H_{\text{comb}}(z) = 1.5(1 - z^{-4}).$$

The corresponding impulse response is

$$h_{\text{comb}} = 1.5\delta[n] - 1.5\delta[n-4].$$

MATLAB is used to verify operation:

```
>> Omega = linspace(0,2*pi,1001);
>> H = MS5P1([3/2 0 0 -3/2],1,Omega);
>> plot(Omega,abs(H),'k');
>> axis([0 2*pi -0.1 3.1]); xlabel('\Omega'); ylabel('|H(\Omega)|');
>> set(gca,'xtick',[0:pi/2:2*pi],'xticklabel',['  0 ';...
    ' p/2';'';'  p ';'3p/2';' 2p '],...
    'fontname','symbol');
```

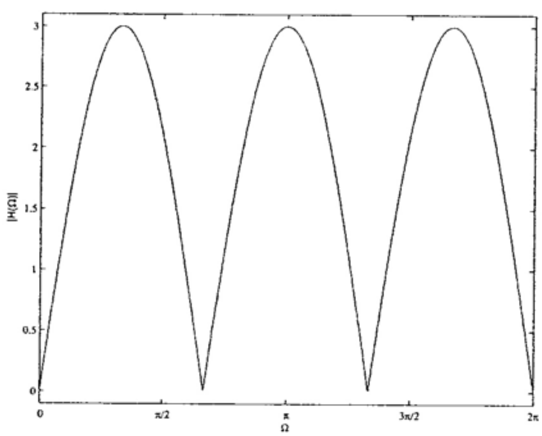


Figure S9.M-4: $|H(\Omega)|$ for FIR comb filter.

9.M-5. (a) Reversing the order of the elements of column vector $\mathbf{x}$ can be accomplished using a N_0-by-N_0 permutation matrix R_{N_0} that is simply a 90-degree rotated N_0-by-N_0 identity matrix. For example, R_5 is

$$R_5 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(b) Let integer $i \in \{0, 1, \ldots, N_0 - 1\}$ be used to designate row or column of $\mathbf{W}_{N_0}$. Row i of $\mathbf{W}_{N_0}$ is represented as $\mathbf{r}_i = e^{j2\pi[0,1,\ldots,N_0-1]i/N_0}$. Column i of $\mathbf{W}_{N_0}$ is represented as $\mathbf{c}_i = e^{j2\pi[0,1,\ldots,N_0-1]^T i/N_0}$. For $i \geq 1$, notice that column $N_0 - i$ is $\mathbf{c}_{N_0-i} = e^{j2\pi[0,1,\ldots,N_0-1]^T(N_0-i)/N_0} = e^{j2\pi[0,1,\ldots,N_0-1]^T} e^{-j2\pi[0,1,\ldots,N_0-1]^T i/N_0} = e^{-j2\pi[0,1,\ldots,N_0-1]^T i/N_0} = \mathbf{r}_i^H$. That is, for $i \geq 1$, column $(N_0 - i)$ is the complex-conjugate transpose of row i. Also notice that $\mathbf{W}_{N_0}$ is composed of orthogonal rows,

$$\mathbf{r}_i\mathbf{r}_k^H = \begin{cases} 0 & i \neq k \\ N_0 & i = k \end{cases}.$$

Combining these facts yields

$$\mathbf{W}_{N_0}^2 = N_0 \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & & & \\ \vdots & & R_{N_0-1} & \\ 0 & & & \end{bmatrix}.$$

For example, if $N_0 = 5$ then

$$\mathbf{W}_5^2 = 5 \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

By inspection, it is clear that $\mathbf{W}_{N_0}^2$ is a scaled permutation matrix. The operation $\mathbf{W}_{N_0}^2\mathbf{x}$ scales and reorders the vector $\mathbf{x}$: the first element of $\mathbf{x}$ is not moved, but the order of the remaining $N_0 - 1$ elements are reversed.

MATLAB is used to confirm these conclusions.

```
>> x = [1 2 3 4 5]';
>> W_5 = dftmtx(5);
>> real(W_5*W_5*x)'
ans = 5.0000   25.0000   20.0000   15.0000   10.0000
```

The last line includes the `real` command to remove minute imaginary components that result due to computer round-off. As expected, vector $\mathbf{x}$ is scaled by $N_0 = 5$ and the order of the last four elements is reversed.

(c) Using the previous result, $\mathbf{W}_{N_0}^4\mathbf{x} = (\mathbf{W}_{N_0}^2)(\mathbf{W}_{N_0}^2)\mathbf{x} = N_0^2\mathbf{x}$. The first multiplication by $(\mathbf{W}_{N_0}^2)$ scales $\mathbf{x}$ by N_0 and reverses the order of the last $N_0 - 1$ elements. The second multiplication again scales $\mathbf{x}$ by N_0 for a total of N_0^2 and reverses the previously reversed last $N_0 - 1$ elements, effectively leaving the order of $\mathbf{x}$ unchanged. MATLAB is used to confirm these conclusions.

```
>> x = [1 2 3 4 5]';
>> W_5 = dftmtx(5);
>> real(W_5*W_5*W_5*W_5*x)'
ans = 25.0000   50.0000   75.0000  100.0000  125.0000
```

The result is just x scaled by $N_0^2 = 25$.

Chapter 10 Solutions

10.1-1. (a)

$$\ddot{y} + 10\dot{y} + 2y = x$$

Choose: $q_1 = y \quad and \quad q_2 = \dot{y} = \dot{q}_1 \Longrightarrow \dot{q}_2 = \ddot{y}$

hence: $\begin{array}{l} \dot{q}_1 = q_2 \\ \dot{q}_2 = -2q_1 - 10q_2 + x \end{array}$

In matrix form we get:

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -10 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x$$

(b)

$$\ddot{y} + 2e^y\dot{y} + \log y = x$$

Choose $q_1 = y \quad and \quad q_2 = \dot{y} = \dot{q}_1$

hence: $\begin{array}{l} \dot{q}_1 = q_2 \\ \dot{q}_2 = -2e^{q_1}q_2 - \log q_1 + x \end{array}$

It is easy to see that this set is nonlinear.

(c)

$$\ddot{y} + \phi_1(y)\dot{y} + \phi_2(y)y = x$$

Choose $q_1 = y \quad and \quad q_2 = \dot{y}.$

hence: $\begin{array}{l} \dot{q}_1 = q_2 \\ \dot{q}_2 = -\phi_1(q_1)q_2 - \phi_2(q_1)q_1 + x \end{array}$

Also in this case we are dealing with a nonlinear set, since $\phi_2(q_1)$ and $\phi_1(q_1)$ are not constants.

10.2-1. Writing the loop equations we get:

$$x = q_1 + 2i + 3i_2 \qquad \text{where} \qquad i_2 = \frac{x - q_1 - \dot{q}_2}{2} - q_2$$

$$\text{and} \qquad i = \frac{x - q_1 - \dot{q}_2}{2}$$

$$\text{Also we have:} \qquad \frac{1}{2}\dot{q}_1 = \frac{x - q_1 - \dot{q}_2}{2} - q_1$$

$$\text{Therefore} \qquad \dot{q}_1 = x - q_1 - \dot{q}_2 - 2q_1 = -3q_1 - \dot{q}_2 + x \tag{1}$$

We can also write:

$$\dot{q}_2 = 3i_2 = 3\left[\frac{x - q_1 - \dot{q}_2}{2} - q_2\right] = \frac{3}{2}x - \frac{3}{2}q_1 - \frac{3}{2}\dot{q}_2 - 3q_2$$

$$\text{Hence} \qquad \frac{5}{2}\dot{q}_2 = -\frac{3}{2}q_1 - 3q_2 + \frac{3}{2}x$$

$$\text{or} \qquad \dot{q}_2 = -\frac{3}{5}q_1 - \frac{6}{5}q_2 + \frac{3}{5}x \tag{2}$$

Substituting equation (2) in equation (1) we obtain:

$$\dot{q}_1 = -3q_1 + x - \left[-\frac{3}{5}q_1 - \frac{6}{5}q_2 + \frac{3}{5}x\right] = -\frac{12}{5}q_1 + \frac{6}{5}q_2 + \frac{2}{5}x$$

Hence the state equations are:

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} -\frac{12}{5} & \frac{6}{5} \\ -\frac{3}{5} & -\frac{6}{5} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \begin{bmatrix} \frac{2}{5} \\ \frac{3}{5} \end{bmatrix} x(t)$$

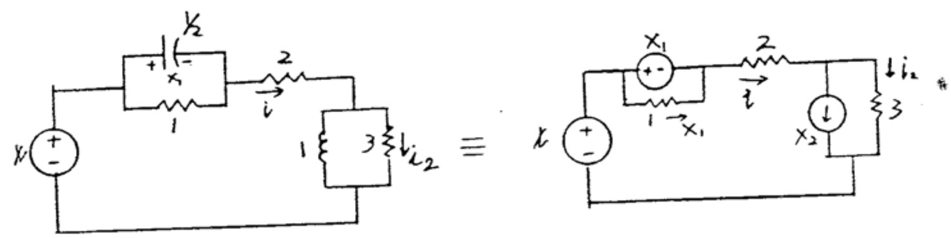


Figure S10.2-1

10.2-2. In the 1st loop, the current i_1 can be computed as:

$$x = \frac{1}{3}i_1 + q_1 \Longrightarrow i_1 = 3(x - q_1)$$

We also have: (using node equation)

$$\frac{1}{2}\dot{q}_1 = -2q_1 - q_2 - 3q_1 + 3x = -5q_1 - q_2 + 3x$$

$$\text{Hence} \qquad \dot{q}_1 = -10q_1 - 2q_2 + 6x \tag{1}$$

Writing the equations in the rightmost loop we get:

$$q_1 = q_2 + \dot{q}_2 \quad \text{and} \quad \dot{q}_2 = q_1 - q_2 \tag{2}$$

Hence from (1) and (2) the state equations are found as:

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} -10 & -2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \begin{bmatrix} 6 \\ 0 \end{bmatrix} x$$

The output equation is: $y = \dot{q}_2 = q_1 - q_2$

$$\text{or} \qquad y = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

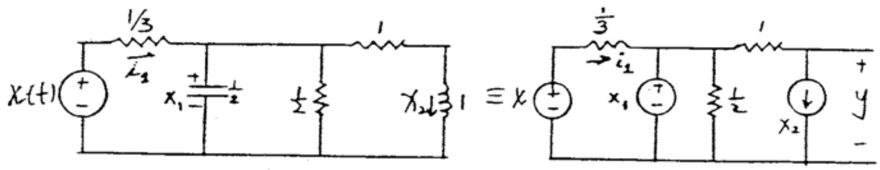


Figure S10.2-2

10.2-3. Let's choose the voltage across the capacitor and the current through the inductor as state variables q_1 and q_2, respectively. Writing the loop equations we get:

$$x_1 = q_1 + \frac{1}{5}[\dot{q}_1 - q_2]$$

Here we use the fact that: $\dot{q}_1 = i_1$ and $q_2 = i_2$.

$$x_2 = -\frac{1}{2}\dot{q}_2 - q_2 + \frac{1}{5}[\dot{q}_1 - q_2]$$

$$\text{And thus:} \qquad \begin{aligned} \dot{q}_1 &= -5q_1 + q_2 + 5x_1 \\ \dot{q}_2 &= -2q_1 - 2q_2 + 2x_1 - 2x_2 \end{aligned}$$

Hence the state equations are

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} -5 & 1 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

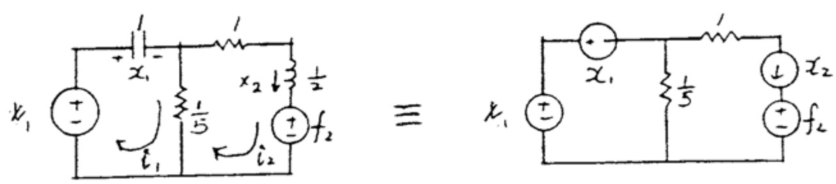


Figure S10.2-3

10.2-4. The loop equations yield:

$$\text{with} \qquad i_2 = \dot{q}_2 \qquad \text{and} \qquad i_1 = q_1 + i_2 = q_1 + \dot{q}_2$$

$$x = 2i_1 + q_1 + \dot{q}_1 = 2q_1 + 2\dot{q}_2 + q_1 + \dot{q}_1 = 3q_1 + \dot{q}_1 + 2\dot{q}_2 \tag{1}$$

$$x = 2i_1 + \dot{q}_2 + q_2 = 2q_1 + 2\dot{q}_2 + \dot{q}_2 + q_2 = 2q_1 + q_2 + 3\dot{q}_2 \tag{2}$$

The last equation gives:

$$\dot{q}_2 = -\frac{2}{3}q_1 - \frac{1}{3}q_2 + \frac{1}{3}x \tag{3}$$

Substituting $\dot{q}_2$ in the equation (1) we get:

$$\dot{q}_1 = -\frac{5}{3}q_1 + \frac{2}{3}q_2 + \frac{5}{3}x \tag{4}$$

From (3) and (4) the state equations are obtained as:

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} -\frac{5}{3} & \frac{2}{3} \\ -\frac{2}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} x(t)$$

And the output equations are: $y_1 = q_1$ and

$$y_2 = i_2 = \dot{q}_2 = -\frac{2}{3}q_1 - \frac{1}{3}q_2 + \frac{1}{3}x$$

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{2}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{3} \end{bmatrix} x(t)$$

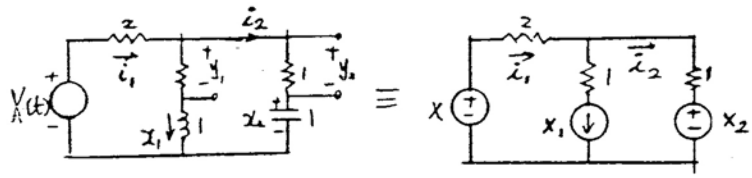


Figure S10.2-4

10.2-5. We have:

$$\begin{aligned} i &= q_1 + \dot{q}_1 \\ &= \frac{x - q_1}{2} + \frac{\dot{x} - \dot{q}_1}{2} \end{aligned}$$

Multiplying both sides of this equations by 2, we get:

$$\begin{aligned} 2q_1 + 2\dot{q}_1 &= x - q_1 + \dot{x} - \dot{q}_1 \\ \text{or} \qquad 3\dot{q}_1 &= -3q_1 + x + \dot{x} \\ \text{Hence} \qquad \dot{q}_1 &= -q_1 + \frac{x}{3} + \frac{\dot{x}}{3} \end{aligned}$$

Thus the only state equation is:

$$\dot{q}_1 = -q_1 + \frac{x}{3} + \frac{\dot{x}}{3}$$

The output equation is: $y = -q_1 + x$.
Note that although there are two capacitors, there is only one independent capacitor voltage, because the two capacitors form a loop with the voltage source. In such a case the state equation contains the terms x as well as $\dot{x}$. Similar situation exists when inductors along with current source(s) for a cut set.

10.2-6. Let us choose q_1, q_2 and q_3 as the outputs of the subsystem shown in the figure:

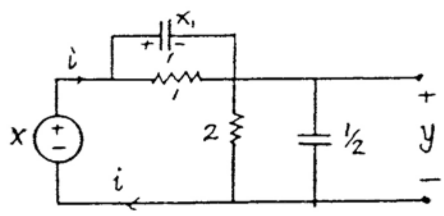


Figure S10.2-5

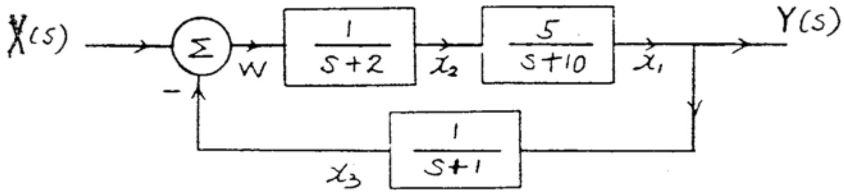


Figure S10.2-6

From the block diagram we obtain:

$$\begin{aligned} 5q_2 &= \dot{q}_1 + 10q_1 \Longrightarrow \dot{q}_1 = -10q_1 + 5q_2 && (1)\\ q_1 &= \dot{q}_3 + q_3 \Longrightarrow \dot{q}_3 = q_1 - q_3 && (2)\\ w &= \dot{q}_2 + 2q_2 \Longrightarrow \dot{q}_2 = w - 2q_2 && (3)\\ \dot{q}_2 &= -2q_2 - q_3 + x && (4) \end{aligned}$$

From (1), (2) and (3) the state equations can be written as:

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = \begin{bmatrix} -10 & 5 & 0 \\ 0 & -2 & -1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} x$$

And the output equation is:

$$y = q_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

10.2-7. From Figure P10.2-7., it is easy to write the state equations as:

$$\begin{aligned} \dot{q}_1 &= \lambda_1 q_1 \\ \dot{q}_2 &= \lambda_2 q_2 + x_1 \\ \dot{q}_3 &= \lambda_3 q_3 + x_2 \\ \dot{q}_4 &= \lambda_4 q_4 + x_2 \end{aligned}$$

or

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The output equation is:

$$\begin{array}{l} y_1 = q_1 + q_2 \\ y_2 = q_2 + q_3 \end{array} \Longrightarrow \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}$$

10.2-8.

$$H(s) = \frac{3s + 10}{s^2 + 7s + 12}$$

Direct form II:
We can write the state and output equations straightforward from the transfer function $H(s)$. Thus we get:

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -12 & -7 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x$$

$$y = \begin{bmatrix} 10 & 3 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

Transposed direct form II: In this case the block diagram can be drawn as shown in Figure S10.2-8a.

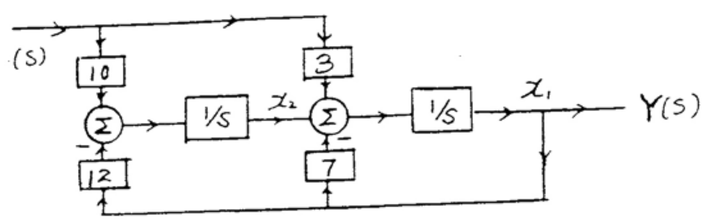


Figure S10.2-8a: transposed direct form II

hence:

$$\begin{aligned} \dot{q}_1 &= -7q_1 + q_2 + 3x \\ \dot{q}_2 &= -12q_1 + 10x \end{aligned}$$

or

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} -7 & 1 \\ -12 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \begin{bmatrix} 3 \\ 10 \end{bmatrix} x$$

The output equation is:

$$y = q_1 = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

The cascade form:

$$H(s) = \frac{3s+10}{s^2+7s+12} = \left(\frac{3s+10}{s+4}\right)\left(\frac{1}{s+3}\right)$$

Hence we can write:

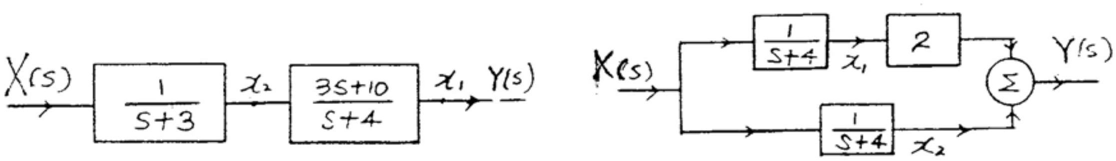


Figure S10.2-8b: cascade and parallel

$$\left.\begin{array}{r} \dot{q}_1 + 4q_1 = 3\dot{q}_2 + 10q_2 \\ \dot{q}_2 = -3q_2 + x \end{array}\right\} \Longrightarrow \begin{array}{l} \dot{q}_1 = -4q_1 - 9q_2 + 10q_2 + 3x \\ \dot{q}_2 = -3q_2 + x \end{array}$$

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} -4 & 1 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix} x$$

and

$$y = q_1 = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

Parallel form:

$$H(s) = \frac{2}{s+4} + \frac{1}{s+3}$$

$$\begin{array}{l} \dot{q}_1 = -4q_1 + x \\ \dot{q}_2 = -3q_1 + x \end{array} \Longrightarrow \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} -4 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} x$$

And the output equation is:

$$y = 2q_1 + q_2 = \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

10.2-9. (a)

$$H(s) = \frac{4s}{(s+1)(s+2)^2} = \frac{4s}{s^3+5s^2+8s+4}$$

Direct form II:

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -8 & -5 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} x$$

And

$$y = \begin{bmatrix} 0 & 4 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

Transposed direct form II:

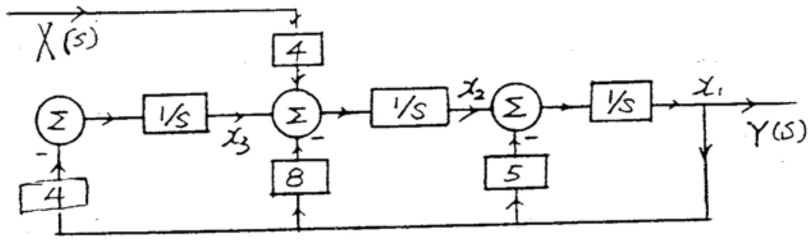


Figure S10.2-9a: transposed direct form II:

In this case:

$$\begin{aligned}\dot{q}_1 &= -5q_1 + q_2\\ \dot{q}_2 &= -8q_1 + q_3 + 4x\\ \dot{q}_3 &= -q_1\end{aligned}$$

$$\begin{bmatrix}\dot{q}_1\\ \dot{q}_2\\ \dot{q}_3\end{bmatrix} = \begin{bmatrix}-5 & 1 & 0\\ -8 & 0 & 1\\ -4 & 0 & 0\end{bmatrix}\begin{bmatrix}q_1\\ q_2\\ q_3\end{bmatrix} + \begin{bmatrix}0\\ 4\\ 0\end{bmatrix}x$$

And:

$$y = q_1 = \begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}q_1\\ q_2\\ q_3\end{bmatrix}$$

Cascade form:

$$H(s) = \left(\frac{1}{s+1}\right)\left(\frac{4s}{s+2}\right)\left(\frac{1}{s+2}\right)$$

From the block diagram we have:

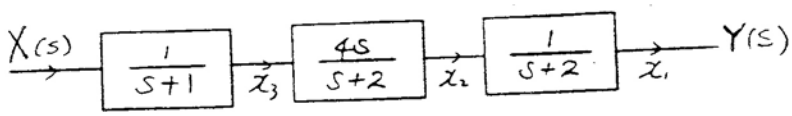


Figure S10.2-9a: cascade

$$\begin{aligned}\dot{q}_1 &= -2q_1 + q_2\\ \dot{q}_2 + 2q_2 &= 4\dot{q}_3\\ \dot{q}_3 &= -q_3 + x\end{aligned} \implies \begin{cases}\dot{q}_1 = -2q_1 + q_2\\ \dot{q}_2 = -4q_3 - 2q_2 + 4x\\ \dot{q}_3 = -q_3 + x\end{cases}$$

$$\begin{bmatrix}\dot{q}_1\\ \dot{q}_2\\ \dot{q}_3\end{bmatrix} = \begin{bmatrix}-2 & 1 & 0\\ 0 & -2 & -4\\ 0 & 0 & -1\end{bmatrix}\begin{bmatrix}q_1\\ q_2\\ q_3\end{bmatrix} + \begin{bmatrix}0\\ 4\\ 1\end{bmatrix}x$$

And the output:

$$y = q_1 = \begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}q_1\\ q_2\\ q_3\end{bmatrix}$$

Parallel form:

$$H(s) = \frac{-4}{s+1} + \frac{4}{s+2} + \frac{8}{(s+2)^2}$$

We have:

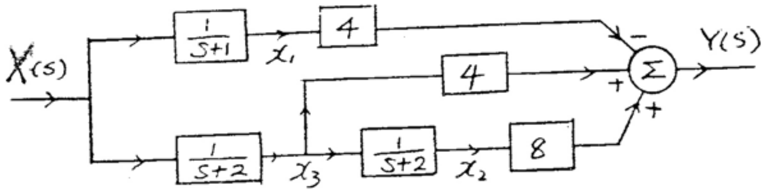


Figure S10.2-9a: parallel

$$\begin{aligned}\dot{q}_1 &= -q_1 + x\\ \dot{q}_2 &= -2q_2 + q_3\\ \dot{q}_3 &= -2q_3 + x\end{aligned}$$

$$\begin{bmatrix}\dot{q}_1\\ \dot{q}_2\\ \dot{q}_3\end{bmatrix} = \begin{bmatrix}-1 & 0 & 0\\ 0 & -2 & 1\\ 0 & 0 & -2\end{bmatrix}\begin{bmatrix}q_1\\ q_2\\ q_3\end{bmatrix} + \begin{bmatrix}1\\ 0\\ 1\end{bmatrix} x$$

And the output is:

$$y = -4q_1 + 8q_2 + 4q_3 = \begin{bmatrix}-4 & 8 & 4\end{bmatrix}\begin{bmatrix}q_1\\ q_2\\ q_3\end{bmatrix}$$

(b)

$$H(s) = \frac{s^3 + 7s^2 + 12s}{(s+1)^3(s+2)} = \frac{s^3 + 7s^2 + 12s}{s^4 + 5s^3 + 9s^2 + 7s + 2}$$

Direct form II:
Straightforward from $H(s)$, we have:

$$\begin{bmatrix}\dot{q}_1\\ \dot{q}_2\\ \dot{q}_3\\ \dot{q}_4\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -2 & -7 & -9 & -5\end{bmatrix}\begin{bmatrix}q_1\\ q_2\\ q_3\\ q_4\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 0\\ 1\end{bmatrix} x$$

And the output is:

$$y = \begin{bmatrix}0 & 12 & 7 & 1\end{bmatrix}\begin{bmatrix}q_1\\ q_2\\ q_3\\ q_4\end{bmatrix}$$

Transposed direct form II:
We can write the state equation directly from $H(s)$ as in the direct form II.

$$\begin{bmatrix}\dot{q}_1\\ \dot{q}_2\\ \dot{q}_3\\ \dot{q}_4\end{bmatrix} = \begin{bmatrix}-5 & 1 & 0 & 0\\ -9 & 0 & 1 & 0\\ -7 & 0 & 0 & 1\\ -2 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}q_1\\ q_2\\ q_3\\ q_4\end{bmatrix} + \begin{bmatrix}1\\ 7\\ 12\\ 0\end{bmatrix} x$$

And

$$y = q_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}$$

Cascade form:

$$H(s) = \frac{s(s+3)(s+4)}{(s+2)(s+1)^3} = \left(\frac{1}{s+2}\right)\left(\frac{s}{s+1}\right)\left(\frac{s+3}{s+1}\right)\left(\frac{s+4}{s+1}\right)$$

Cascade form: From the block diagram we obtain:

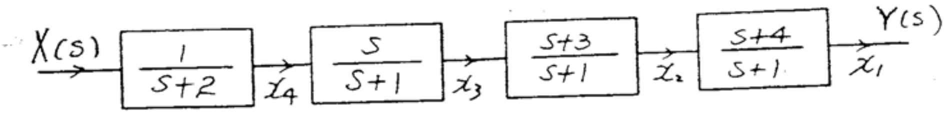


Figure S10.2-9b: cascade

$$\begin{array}{l} \dot{q}_1 + q_1 = \dot{q}_2 + 4q_2 \\ \dot{q}_2 + q_2 = \dot{q}_3 + 3q_3 \\ \dot{q}_3 = -q_3 + \dot{q}_4 \\ \dot{q}_4 = -2q_4 + x \end{array} \Longrightarrow \left\{ \begin{array}{l} \dot{q}_1 = -q_1 + 4q_2 - q_2 + 2q_3 - 2q_4 + x \\ \dot{q}_2 = -q_2 + 3q_3 - q_3 - 2q_4 + x \\ \dot{q}_3 = -q_3 - 2q_4 + x \\ \dot{q}_4 = -2q_4 + x \end{array} \right.$$

hence:

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \end{bmatrix} = \begin{bmatrix} -1 & 3 & 2 & -2 \\ 0 & -1 & 2 & -2 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} x$$

And

$$y = q_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}$$

Parallel form: we can rewrite $H(s)$ as (after partial fraction expansion)

$$H(s) = \frac{6}{s+2} + \frac{11}{s+1} + \frac{7}{(s+1)^2} - \frac{6}{(s+1)^3}$$

$$\begin{aligned} \dot{q}_1 &= -2q_1 + x \\ \dot{q}_2 &= -q_2 + q_3 \\ \dot{q}_3 &= -q_3 + q_4 \\ \dot{q}_4 &= -q_4 + x \end{aligned}$$

From the block diagram, we have

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} x$$

And the output can be written as:

$$y = 6q_1 - 6q_2 + 7q_3 + 11q_4$$

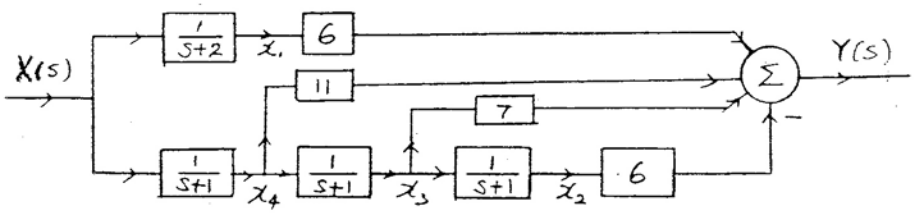


Figure S10.2-9b: parallel

or

$$y = \begin{bmatrix} 6 & -6 & 7 & 11 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}$$

10.3-1.

$$\dot{\mathbf{q}} = \mathbf{A}\mathbf{q} + \mathbf{B}\mathbf{x}$$

The solution of the state equation in the frequency domain is given by:

$$\mathbf{Q}(s) = \Phi(s)\mathbf{q}(0) + \Phi(s)\mathbf{B}\mathbf{X}(s)$$

but in this case $x(t) = 0 \Longrightarrow X(s) = 0$
hence: $\mathbf{Q}(s) = \Phi(s)\mathbf{q}(0)$ where $\Phi(s) = (s\mathbf{I} - \mathbf{A})^{-1}$

$$\Phi(s) = (s\mathbf{I} - \mathbf{A})^{-1} \qquad (s\mathbf{I} - \mathbf{A}) = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 2 \\ -1 & -3 \end{bmatrix}$$

$$s\mathbf{I} - \mathbf{A} = \begin{bmatrix} s & -2 \\ 1 & s+3 \end{bmatrix} \Longrightarrow \Phi(s) = (s\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} s+3 & 2 \\ -1 & s \end{bmatrix} \frac{1}{s^2+3s+2}$$

$$\Phi(s) = \begin{bmatrix} \frac{s+3}{s^2+3s+2} & \frac{2}{s^2+3s+2} \\ \frac{-1}{s^2+3s+2} & \frac{s}{s^2+3s+2} \end{bmatrix} = \begin{bmatrix} \frac{s+3}{(s+1)(s+2)} & \frac{2}{(s+1)(s+2)} \\ \frac{-1}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{bmatrix}$$

And hence: $\mathbf{Q}(s) = \Phi(s)\mathbf{q}(0)$

$$\mathbf{Q}(s) = \begin{bmatrix} \frac{2(s+3)+2}{(s+1)(s+2)} \\ \frac{-2+s}{(s+1)(s+2)} \end{bmatrix} = \begin{bmatrix} \frac{2s+8}{(s+1)(s+2)} \\ \frac{s-2}{(s+1)(s+2)} \end{bmatrix} = \begin{bmatrix} \frac{6}{s+1} - \frac{4}{s+2} \\ \frac{-3}{s+1} + \frac{4}{s+2} \end{bmatrix}$$

And finally:

$$\mathbf{q}(t) = \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} = \mathcal{L}^{-1}\left[\ \mathbf{Q}(s)\ \right] = \begin{bmatrix} (6e^{-t} - 4e^{-2t})u(t) \\ (-3e^{-t} + 4e^{-2t})u(t) \end{bmatrix}$$

10.3-2.

$$\mathbf{Q}(s) = \Phi(s)\mathbf{q}(0) + \Phi(s)\mathbf{B}\mathbf{X}(s) = \Phi(s)[\mathbf{q}(0) + \mathbf{B}\mathbf{X}(s)]$$

$$(s\mathbf{I} - \mathbf{A}) = \begin{bmatrix} s+5 & 6 \\ -1 & s \end{bmatrix} \quad \text{and} \quad \Phi(s) = (s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{s^2+5s+6} \begin{bmatrix} s & -6 \\ 1 & s+5 \end{bmatrix}$$

$$\Phi(s) = \begin{bmatrix} \frac{s}{(s+3)(s+2)} & \frac{-6}{(s+3)(s+2)} \\ \frac{1}{(s+3)(s+2)} & \frac{s+5}{(s+3)(s+2)} \end{bmatrix}$$

And hence:

$$\begin{aligned} \mathbf{Q}(s) = \Phi(s)\,[\mathbf{q}(0) + \mathbf{B}\mathbf{X}(s)] &= \begin{bmatrix} \frac{s}{(s+3)(s+2)} & \frac{-6}{(s+3)(s+2)} \\ \frac{1}{(s+3)(s+2)} & \frac{s+5}{(s+3)(s+2)} \end{bmatrix} \begin{bmatrix} 5 + \frac{100}{s^2+10^4} \\ 4 \end{bmatrix} \\ &= \begin{bmatrix} \frac{-34.02}{s+2} + \frac{39.03}{s+3} - \frac{10^{-2}s}{s^2+10^4} \\ \frac{17.01}{s+2} - \frac{10.01}{s+3} - \frac{0}{s^2+10^4} \end{bmatrix} \end{aligned}$$

hence: $\mathbf{q}(t) = \mathcal{L}^{-1}(\mathbf{Q}(s))$

$$\mathbf{Q}(s) = \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} = \begin{bmatrix} -34.02e^{-2t} + 39.03e^{-3t} - 0.01\cos 100t \\ 17.01e^{-2t} - 10.01e^{-3t} \end{bmatrix}$$

10.3-3.

$$\mathbf{Q}(s) = \Phi(s)[\mathbf{q}(0) + \mathbf{B}\mathbf{X}(s)]$$

$$\begin{aligned} (s\mathbf{I} - \mathbf{A}) = \begin{bmatrix} s+2 & 0 \\ -1 & s+1 \end{bmatrix} \text{ and } \Phi(s) = (s\mathbf{I} - \mathbf{A})^{-1} &= \frac{1}{(s+1)(s+2)} \begin{bmatrix} s+1 & 0 \\ 1 & s+2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{s+2} & 0 \\ \frac{1}{(s+1)(s+2)} & \frac{1}{s+1} \end{bmatrix} \end{aligned}$$

Also $x(t) = u(t) \Longrightarrow X(s) = \frac{1}{s}$

$$\text{Hence:} \quad \mathbf{B}\mathbf{X}(s) = \begin{bmatrix} \frac{1}{s} \\ 0 \end{bmatrix} \qquad \text{And} \qquad \mathbf{q}(0) + \mathbf{B}\mathbf{X}(s) = \begin{bmatrix} \frac{1}{s} \\ -1 \end{bmatrix}$$

And thus:

$$\begin{aligned} \mathbf{Q}(s) = \begin{bmatrix} \frac{1}{s+2} & 0 \\ \frac{1}{(s+1)(s+2)} & \frac{1}{s+1} \end{bmatrix} \begin{bmatrix} \frac{1}{s} \\ -1 \end{bmatrix} &= \begin{bmatrix} \frac{1}{s(s+2)} \\ \frac{1}{s(s+1)(s+2)} - \frac{1}{s+1} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2s} - \frac{1}{2(s+2)} \\ \frac{1}{2s} - \frac{2}{s+1} - \frac{1}{2(s+2)} \end{bmatrix} \end{aligned}$$

Hence:

$$\mathbf{q}(t) = \mathcal{L}^{-1}(\mathbf{Q}(s)) = \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} = \begin{bmatrix} (\frac{1}{2} - \frac{1}{2}e^{-2t})u(t) \\ (\frac{1}{2} - 2e^{-t} + \frac{1}{2}e^{-2t})u(t) \end{bmatrix}$$

10.3-4.

$$\mathbf{Q}(s) = \Phi(s)[\mathbf{q}(0) + \mathbf{B}\mathbf{X}(s)]$$

$$(s\mathbf{I} - \mathbf{A}) = \begin{bmatrix} s+1 & -1 \\ 0 & s+2 \end{bmatrix} \text{ and } \Phi(s) = (s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{(s+1)(s+2)} \begin{bmatrix} s+2 & 1 \\ 0 & s+1 \end{bmatrix}$$

$$\Phi(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{1}{(s+1)(s+2)} \\ 0 & \frac{1}{s+2} \end{bmatrix}$$

$$\text{and} \qquad x(t) = \begin{bmatrix} u(t) \\ \delta(t) \end{bmatrix} \Longrightarrow \mathbf{X}(s) = \begin{bmatrix} \frac{1}{s} \\ 1 \end{bmatrix}$$

$$\mathbf{BX}(s) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{s} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{s+1}{s} \\ 1 \end{bmatrix}$$

$$\text{and:} \qquad \mathbf{q}(0) + \mathbf{BX}(s) = \begin{bmatrix} \frac{s+1}{s} + 1 \\ 3 \end{bmatrix} \begin{bmatrix} \frac{2s+1}{s} + 1 \\ 3 \end{bmatrix}$$

$$\begin{aligned} \mathbf{Q}(s) = \Phi(s)[\mathbf{q}(0) + \mathbf{BX}(s)] &= \begin{bmatrix} \frac{1}{s+1} & \frac{1}{(s+1)(s+2)} \\ 0 & \frac{1}{s+2} \end{bmatrix} \begin{bmatrix} \frac{2s+1}{s} \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} \frac{(2s+1)(s+2)+3s}{s(s+1)(s+2)} \\ \frac{3}{s+2} \end{bmatrix} = \begin{bmatrix} \frac{1}{s} + \frac{4}{s+1} - \frac{3}{s+2} \\ \frac{3}{s+2} \end{bmatrix} \end{aligned}$$

And hence:

$$\mathbf{q}(t) = \mathcal{L}^{-1}(\mathbf{Q}(s)) = \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} = \begin{bmatrix} (1 + 4e^{-t} - 3e^{-2t})u(t) \\ 3e^{-2t}u(t) \end{bmatrix}$$

10.3-5.

$$\mathbf{Y}(s) = \mathbf{CQ}(s) + \mathbf{DX}(s) = \mathbf{C}\Phi(s)\mathbf{q}(0) + [\mathbf{C}\Phi(s)\mathbf{B} + \mathbf{D}]\mathbf{X}(s)]$$

$$(s\mathbf{I} - \mathbf{A}) = \begin{bmatrix} s+3 & -1 \\ 2 & s \end{bmatrix} \text{ and } \Phi(s) = (s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{(s+1)(s+2)} \begin{bmatrix} s & 1 \\ -2 & s+3 \end{bmatrix}$$

$$\Phi(s) = \begin{bmatrix} \frac{s}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} & \frac{s+3}{(s+1)(s+2)} \end{bmatrix} \qquad \text{and} \qquad \mathbf{BX}(s) = \begin{bmatrix} \frac{1}{s} \\ 1 \end{bmatrix}$$

Since $\mathbf{D} = 0 \Longrightarrow Y(s) = \mathbf{C}\Phi(s)[\mathbf{q}(0) + \mathbf{BX}(s)]$

$$\text{So} \qquad \mathbf{q}(0) + \mathbf{BX}(s) = \begin{bmatrix} 2 + \frac{1}{s} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{2s+1}{s} \\ 0 \end{bmatrix}$$

$$\text{and} \qquad \Phi(s)[\mathbf{q}(0)+\mathbf{BX}(s)] = \begin{bmatrix} \frac{s}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} & \frac{s+3}{(s+1)(s+2)} \end{bmatrix} \begin{bmatrix} \frac{2s+1}{s} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{2s+1}{(s+1)(s+2)} \\ \frac{-2(2s+1)}{s(s+1)(s+2)} \end{bmatrix}$$

$$\mathbf{Y}(s) = \mathbf{C}\Phi(s)[\mathbf{q}(0) + \mathbf{BX}(s)] = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{2s+1}{(s+1)(s+2)} \\ \frac{-2(2s+1)}{s(s+1)(s+2)} \end{bmatrix}$$

$$\mathbf{Y}(s) = \frac{-4s-2}{s(s+1)(s+2)} = \frac{-1}{s} - 2 \cdot \frac{1}{s+1} + \frac{3}{s+2}$$

$$y(t) = \mathcal{L}^{-1}[y(s)] = (-1 - 2e^{-t} + 3e^{-2t})u(t)$$

10.3-6.

$$\mathbf{y}(s) = \mathbf{CQ}(s) + \mathbf{DX}(s) = \mathbf{C}\Phi(s)\mathbf{q}(0) + [\mathbf{C}\Phi(s)\mathbf{B} + \mathbf{D}]\mathbf{X}(\mathrm{s})$$

$$= \mathbf{C}\{\Phi(s)[\mathbf{q}(0) + \mathbf{BX}(s)]\} + \mathbf{DX}(s)$$

$$(s\mathbf{I} - \mathbf{A}) = \begin{bmatrix} s+1 & -1 \\ 1 & s+1 \end{bmatrix} \text{ and } \Phi(s) = (s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{s^2+2s+2}\begin{bmatrix} s+1 & 1 \\ -1 & s+1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{s+1}{s^2+2s+2} & \frac{1}{s^2+2s+2} \\ \frac{-1}{s^2+2s+2} & \frac{s+1}{s^2+2s+2} \end{bmatrix}$$

$$\mathbf{BX}(s) = \begin{bmatrix} 0 \\ \frac{1}{s} \end{bmatrix} \qquad \text{and} \qquad \mathbf{q}(0) + \mathbf{BX}(s) = \begin{bmatrix} 2 \\ \frac{s+1}{s} \end{bmatrix}$$

Hence $$\Phi(s)[\mathbf{q}(0) + \mathbf{BX}(s)] = \begin{bmatrix} \frac{s+1}{(s+1)^2+1} & \frac{1}{(s+1)^2+1} \\ \frac{-1}{(s+1)^2+1} & \frac{s+1}{(s+1)^2+1} \end{bmatrix}\begin{bmatrix} 2 \\ \frac{s+1}{s} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2(s+1)}{(s+1)^2+1} + \frac{s+1}{s[(s+1)^2+1]} \\ \frac{-2}{(s+1)^2+1} + \frac{(s+1)^2}{s[(s+1)^2+1]} \end{bmatrix} = \begin{bmatrix} \frac{2s^2+3s+1}{s[(s+1)^2+1]} \\ \frac{s^2+1}{s[(s+1)^2+1]} \end{bmatrix}$$

$$\mathbf{C}\Phi(s)[\mathbf{q}(0) + \mathbf{BX}(s)] = \begin{bmatrix} 1 & 1 \end{bmatrix}\Phi(s)[\mathbf{q}(0) + \mathbf{BX}(s)] = \left[\frac{2s^2+3s+1+s^2+1}{s\{(s+1)^2+1\}}\right]$$

Also: $\mathbf{DX}(s) = \frac{1}{s}$

Hence $\mathbf{Y}(s) = \mathbf{C}\Phi(s)[\mathbf{q}(0) + \mathbf{BX}(s)] + \mathbf{DX}(s) = \frac{3s^2+3s+2}{s\{(s+1)^2+1\}} + \frac{1}{s} = \frac{4s^2+5s+4}{s\{(s+1)^2+1\}}$

$$Y(s) = \frac{4s^2+5s+4}{s(s^2+2s+2)} = \frac{\mathbf{C}}{s} + \frac{\mathbf{A}s + \mathbf{B}}{s^2+2s+2}$$

Using partial fractions and clearing fractions we get:

$$Y(s) = \frac{2}{s} + \frac{2s+1}{(s+1)^2+1^2} = \frac{2}{s} + 2\frac{(s+1)}{(s+1)^2+1^2} - \frac{1}{(s+1)^2+1^2}$$

$$\text{and} \qquad y(t) = \mathcal{L}^{-1}[Y(s)] = (2 + 2e^{-t}\cos t - e^{-t}\sin t)u(t)$$

10.3-7.

$$H(s) = \left(\frac{1}{s+3}\right)\left(\frac{3s+10}{s+4}\right) = \frac{3s+10}{s^2+7s+12}$$

This is the same transfer function as in Prob. 10.2-8, where the cascade form state equations were found to be

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} -4 & 1 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix}x$$

$$\text{And} \qquad y = q_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

In this case

$$s\mathbf{I} - \mathbf{A} = \begin{bmatrix} s+4 & -1 \\ 0 & s+3 \end{bmatrix} \text{ and } \Phi(s) = (s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{(s+3)(s+4)}\begin{bmatrix} s+3 & 1 \\ 0 & s+4 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{s+4} & \frac{1}{(s+3)(s+4)} \\ 0 & \frac{1}{s+3} \end{bmatrix}$$

Also in our case:

$$\mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad \text{and} \qquad \mathbf{B} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{D} = 0$$

Hence $$\Phi(s)\mathbf{B} = \begin{bmatrix} \frac{1}{s+4} & \frac{1}{(s+3)(s+4)} \\ 0 & \frac{1}{s+3} \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{3(s+3)+1}{(s+3)(s+4)} \\ \frac{1}{s+3} \end{bmatrix} = \begin{bmatrix} \frac{3s+10}{(s+3)(s+4)} \\ \frac{1}{s+3} \end{bmatrix}$$

And $$\mathbf{C}\phi(s)\mathbf{B} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{3s+10}{(s+3)(s+4)} \\ \frac{1}{s+3} \end{bmatrix} = \frac{3s+10}{(s+3)(s+4)}$$

Hence: $$\mathbf{C}\Phi(s)\mathbf{B} = \frac{3s+10}{s^2+7s+12} = H(s)$$

10.3-8.

$$H(s) = \mathbf{C}\Phi(s)\mathbf{B} + \mathbf{D}$$

in Prob. 10.3-5 we have found $\Phi(s)$. And

$$\Phi(s)\mathbf{B} = \begin{bmatrix} \frac{s}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} & \frac{s+3}{(s+1)(s+2)} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{s}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} \end{bmatrix}$$

Hence

$$\mathbf{C}\Phi(s)\mathbf{B} = \begin{bmatrix} 1 & 0 \end{bmatrix} \Phi(s)\mathbf{B} = \frac{-2}{(s+1)(s+2)} \qquad \text{and since} \qquad \mathbf{D} = 0$$

$$H(s) = \mathbf{C}\Phi(s)\mathbf{B} = \frac{-2}{s^2+3s+2}$$

10.3-9. From Prob. 10.3-6,

$$\Phi(s)\mathbf{B} = \begin{bmatrix} \frac{s+1}{(s+1)^2+1} & \frac{1}{(s+1)^2+1} \\ \frac{-1}{(s+1)^2+1} & \frac{s+1}{(s+1)^2+1} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{(s+1)^2+1} \\ \frac{s+1}{(s+1)^2+1} \end{bmatrix}$$

And:

$$\mathbf{C}\Phi(s)\mathbf{B} = \begin{bmatrix} 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{(s+1)^2+1} \\ \frac{s+1}{(s+1)^2+1} \end{bmatrix} = \frac{s+1+1}{(s+1)^2+1} = \frac{s+2}{(s+1)^2+1}$$

And
$$H(s) = \mathbf{C}\Phi(s)\mathbf{B} + \mathbf{D} = \frac{s+2}{(s+1)^2+1} + 1 = \frac{s^2+3s+4}{s^2+2s+2}$$

10.3-10. In this case:

$$s\mathbf{I} - \mathbf{A} = \begin{bmatrix} s & -1 \\ 1 & s+2 \end{bmatrix} \quad \text{and} \quad \Phi(s) = (s\mathbf{I} - \mathbf{A})^{-1} \quad = \quad \frac{1}{(s+1)^2} \begin{bmatrix} s+2 & 1 \\ -1 & s \end{bmatrix}$$

$$= \begin{bmatrix} \frac{s+2}{(s+1)^2} & \frac{1}{(s+1)^2} \\ \frac{-1}{(s+1)^2} & \frac{s}{(s+1)^2} \end{bmatrix}$$

And:

$$\Phi(s)\mathbf{B} = \begin{bmatrix} \frac{s+2}{(s+1)^2} & \frac{1}{(s+1)^2} \\ \frac{-1}{(s+1)^2} & \frac{s}{(s+1)^2} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{(s+1)^2} & \frac{s+2}{(s+1)^2} \\ \frac{s}{(s+1)^2} & \frac{-1}{(s+1)^2} \end{bmatrix}$$

$$\mathbf{C}\Phi(s)\mathbf{B} = \begin{bmatrix} 1 & 2 \\ 4 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{(s+1)^2} & \frac{s+2}{(s+1)^2} \\ \frac{s}{(s+1)^2} & \frac{-1}{(s+1)^2} \end{bmatrix}$$

$$\text{and} \quad H(s) = \mathbf{C}\Phi(s)\mathbf{B} + \mathbf{D} = \begin{bmatrix} \frac{2s+1}{(s+1)^2} & \frac{s}{(s+1)^2} \\ \frac{4+s}{(s+1)^2} & \frac{4s+7}{(s+1)^2} \\ \frac{1}{s+1} & \frac{1}{s+1} \end{bmatrix}$$

10.3-11. In the time domain, the solution $\mathbf{q}(t)$ is given by:

$$\mathbf{q}(t) = e^{\mathbf{A}t}\mathbf{q}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{x}(\tau)\, d\tau$$

$$\mathbf{q}(t) = e^{\mathbf{A}t}\mathbf{q}(0) + e^{\mathbf{A}t} * \mathbf{B}\mathbf{x}(t)$$

where:

$$e^{\mathbf{A}t} = \mathcal{L}^{-1}[(s\mathbf{I} - \mathbf{A})^{-1}] = \mathcal{L}^{-1}(\Phi(s))$$

From Prob. 10.3-1 we have found:

$$\Phi(s) = \begin{bmatrix} \frac{s+3}{(s+1)(s+2)} & \frac{2}{(s+1)(s+2)} \\ \frac{-1}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{bmatrix} = \begin{bmatrix} \frac{2}{s+1} - \frac{1}{s+2} & \frac{2}{s+1} - \frac{2}{s+2} \\ \frac{-1}{s+1} + \frac{1}{s+2} & \frac{-1}{s+1} + \frac{2}{s+2} \end{bmatrix}$$

$$e^{\mathbf{A}t} = \mathcal{L}^{-1}(\Phi(s)) = \begin{bmatrix} 2e^{-t} - e^{-2t} & 2e^{-t} - 2e^{-2t} \\ -e^{-t} + e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

$$e^{\mathbf{A}t}\mathbf{q}(0) = \begin{bmatrix} 4e^{-t} - 2e^{-2t} + 2e^{-t} - 2e^{-2t} \\ -2e^{-t} + 2e^{-2t} - e^{-t} + 2e^{-2t} \end{bmatrix} = \begin{bmatrix} 6e^{-t} - 4e^{-2t} \\ -3e^{-t} + 4e^{-2t} \end{bmatrix}$$

$$\text{Also:} \qquad \mathbf{B}\mathbf{x}(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \times 0 = 0$$

$$\text{hence:} \qquad \mathbf{q}(t) = \begin{bmatrix} (6e^{-t} - 4e^{-2t})u(t) \\ (-3e^{-t} + 4e^{-2t})u(t) \end{bmatrix}$$

which is the same thing as in Prob. 10.3-1.

10.3-12. From Prob. 10.3-2,

$$\Phi(s) = \begin{bmatrix} \frac{s}{(s+2)(s+3)} & \frac{-6}{(s+2)(s+3)} \\ \frac{1}{(s+2)(s+3)} & \frac{s+5}{(s+2)(s+3)} \end{bmatrix} = \begin{bmatrix} \frac{-2}{s+2} + \frac{3}{s+3} & \frac{-6}{s+2} + \frac{6}{s+3} \\ \frac{1}{s+2} - \frac{1}{s+3} & \frac{3}{s+2} - \frac{2}{s+3} \end{bmatrix}$$

$$\text{Hence:} \qquad e^{\mathbf{A}t} = \mathcal{L}^{-1}(\Phi(s)) = \begin{bmatrix} -2e^{-2t} + 3e^{-3t} & -6e^{-2t} + 6e^{-3t} \\ e^{-2t} - e^{-3t} & 3e^{-2t} - 2e^{-3t} \end{bmatrix}$$

And:

$$e^{\mathbf{A}t}\mathbf{q}(0) = \begin{bmatrix} -10e^{-2t} + 15e^{-3t} - 24e^{-2t} + 24e^{-3t} \\ 5e^{-2t} - 5e^{-3t} + 12e^{-2t} - 8e^{-3t} \end{bmatrix} = \begin{bmatrix} -34e^{-2t} + 39e^{-3t} \\ 17e^{-2t} - 13e^{-3t} \end{bmatrix}$$

Also: $$\mathbf{B}\mathbf{x}(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \sin 100t = \begin{bmatrix} \sin 100t \\ 0 \end{bmatrix}$$

And $$\begin{aligned} e^{\mathbf{A}t} * \mathbf{B}\mathbf{x}(t) &= \begin{bmatrix} -2e^{-2t} * \sin 100t + 3e^{-3t} * \sin 100t \\ e^{-2t} * \sin 100t - e^{-3t} * \sin 100t \end{bmatrix} \\ &= \begin{bmatrix} -\frac{2e^{-2t}}{100} + \frac{2\cos 100t}{100} + \frac{3e^{-3t}}{100} - \frac{3\cos 100t}{100} \\ +\frac{e^{-2t}}{100} - \frac{\cos 100t}{100} - \frac{e^{-3t}}{100} + \frac{\cos 100t}{100} \end{bmatrix} \\ &= \begin{bmatrix} -0.02e^{-2t} + 0.03e^{-3t} - 0.01\cos 100t \\ 0.01e^{-2t} - 0.01e^{-3t} \end{bmatrix} \end{aligned}$$

Hence:

$$\mathbf{q}(t) = e^{\mathbf{A}t}[\mathbf{q}(0)] + e^{\mathbf{A}t} * \mathbf{B}\mathbf{x}(t) = \begin{bmatrix} -34.02e^{-2t} + 39e^{-3t} + 0.01\cos 100t \\ 17.01e^{-2t} - 10.0e^{-3t} \end{bmatrix}$$

Hence

$$\mathbf{q}(t) = e^{\mathbf{A}t}\mathbf{q}(0) + e^{\mathbf{A}t} * \mathbf{B}\mathbf{x} = \begin{bmatrix} -34.02e^{-2t} + 39.03e^{-3t} + 0.01\cos 100t \\ 17.01e^{-2t} - 10.01e^{-3t} \end{bmatrix}$$

This is the same result as in Prob. 10.3-2.

10.3-13. From Prob. 10.3-3,

$$\Phi(s) = \begin{bmatrix} \frac{1}{s+2} & 0 \\ \frac{1}{(s+1)(s+2)} & \frac{1}{s+1} \end{bmatrix} = \begin{bmatrix} \frac{1}{s+2} & 0 \\ \frac{1}{s+1} - \frac{1}{s+2} & \frac{1}{s+1} \end{bmatrix}$$

Hence: $$e^{\mathbf{A}t} = \mathcal{L}^{-1}(\Phi(s)) = \begin{bmatrix} e^{-2t} & 0 \\ e^{-t} - e^{-2t} & e^{-t} \end{bmatrix}$$

And: $$e^{\mathbf{A}t}\mathbf{q}(0) = \begin{bmatrix} e^{-2t} & 0 \\ e^{-t} - e^{-2t} & e^{-t} \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -e^{-t} \end{bmatrix}$$

Also: $$\mathbf{B}\mathbf{x}(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) = \begin{bmatrix} u(t) \\ 0 \end{bmatrix}$$

And: $$e^{\mathbf{A}t} * \mathbf{B}\mathbf{x}(t) = \begin{bmatrix} e^{-2t} * u(t) \\ e^{-t} * u(t) - e^{-2t} * u(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(1 - e^{-2t})u(t) \\ (1 - e^{-t}) - \frac{1}{2}(1 - e^{-2t}) \end{bmatrix}$$

And hence:

$$\begin{aligned} \mathbf{q}(t) = e^{\mathbf{A}t}\mathbf{q}(0) + e^{\mathbf{A}t} * \mathbf{B}\mathbf{x}(t) &= \begin{bmatrix} 0 \\ -e^{-t} \end{bmatrix} + \begin{bmatrix} \frac{1}{2} - \frac{1}{2}e^{-2t} \\ \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2} - \frac{1}{2}e^{-2t} \\ \frac{1}{2} + \frac{1}{2}e^{-2t} - 2e^{-t} \end{bmatrix} \end{aligned}$$

10.3-14. From Prob. 10.3-4,

$$\Phi(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{1}{(s+1)(s+2)} \\ 0 & \frac{1}{s+2} \end{bmatrix} = \begin{bmatrix} \frac{1}{s+1} & \frac{1}{s+1} - \frac{1}{s+2} \\ 0 & \frac{1}{s+2} \end{bmatrix}$$

$$\text{Hence:} \qquad e^{\mathbf{A}t} = \mathcal{L}^{-1}(\Phi(s)) = \begin{bmatrix} e^{-t} & e^{-t} - e^{-2t} \\ 0 & e^{-2t} \end{bmatrix}$$

$$e^{\mathbf{A}t}\mathbf{q}(0) = \begin{bmatrix} e^{-t} & e^{-t} - e^{-2t} \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3e^{-t} - 2e^{-2t} \\ 2e^{-2t} \end{bmatrix}$$

$$\mathbf{B}\mathbf{x}(t) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u(t) \\ \delta(t) \end{bmatrix} = \begin{bmatrix} u(t) + \delta(t) \\ \delta(t) \end{bmatrix}$$

$$\text{And} \qquad e^{\mathbf{A}t} * \mathbf{B}\mathbf{x}(t) = \begin{bmatrix} e^{-t} * u(t) + e^{-t} * \delta(t) + e^{-t} * \delta(t) - e^{-2t} * \delta(t) \\ e^{-2t} * \delta(t) \end{bmatrix}$$

$$e^{\mathbf{A}t} * \mathbf{B}\mathbf{x}(t) = \begin{bmatrix} (1 - e^{-t}) + e^{-t} + e^{-t} - e^{-2t} \\ e^{-2t} \end{bmatrix} \begin{bmatrix} 1 + e^{-t} - e^{-2t} \\ e^{-2t} \end{bmatrix}$$

$$\text{And hence:} \quad \mathbf{q}(t) = e^{\mathbf{A}t}\mathbf{q}(0) + e^{\mathbf{A}t} * \mathbf{B}\mathbf{x}(t) = \begin{bmatrix} 3e^{-t} - 2e^{-2t} + 1 + e^{-t} - e^{-2t} \\ 2e^{-2t} + e^{-2t} \end{bmatrix} = \begin{bmatrix} 1 + 4e^{-t} - 3e^{-2t} \\ 3e^{-2t} \end{bmatrix}$$

10.3-15. From Prob. 10.3-5,

$$\Phi(s) = \begin{bmatrix} \frac{s}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} & \frac{s+3}{(s+1)(s+2)} \end{bmatrix} = \begin{bmatrix} \frac{-1}{s+1} + \frac{2}{s+2} & \frac{1}{s+1} - \frac{1}{s+2} \\ \frac{-2}{s+1} + \frac{2}{s+2} & \frac{2}{s+1} - \frac{1}{s+2} \end{bmatrix}$$

And $y(t)$ is given by: $\qquad y(t) = \mathbf{C}[e^{\mathbf{A}t}\mathbf{q}(0) + e^{\mathbf{A}t}\mathbf{B} * x(t)] + \mathbf{D}\mathbf{x}(t)$

$$\text{where:} \qquad e^{\mathbf{A}t} = L^{-1}(\Phi(s)) = \begin{bmatrix} -e^{-t} + 2e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & 2e^{-t} - e^{-2t} \end{bmatrix}$$

$$\text{And:} \qquad e^{\mathbf{A}t}\mathbf{q}(0) = e^{\mathbf{A}t} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -2e^{-t} + 4e^{-2t} \\ -2e^{-t} + 4e^{-2t} \end{bmatrix}$$

$$e^{\mathbf{A}t}\mathbf{B} = e^{\mathbf{A}t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -e^{-t} + 2e^{-2t} \\ -2e^{-t} + 2e^{-2t} \end{bmatrix}$$

$$e^{\mathbf{A}t} * \mathbf{B}\mathbf{x}(t) = \begin{bmatrix} -e^{-t} + 2e^{-2t} \\ -2e^{-t} + 2e^{-2t} \end{bmatrix} * u(t) = \begin{bmatrix} -e^{-t} * u(t) + e^{-2t} * u(t) \\ -2e^{-t} * u(t) + 2e^{-2t} * u(t) \end{bmatrix} = \begin{bmatrix} e^{-t} - e^{-2t} \\ -1 + 2e^{-t} - e^{-2t} \end{bmatrix}$$

$$\text{Since} \qquad \mathbf{D}=0 \Longrightarrow y(t)=\mathbf{C}[e^{\mathbf{A}t}\mathbf{q}(0)+e^{\mathbf{A}t}*\mathbf{B}\mathbf{x}(t)]$$

$$\begin{aligned}\text{And:} \quad e^{\mathbf{A}t}\mathbf{q}(0)+e^{\mathbf{A}t}*\mathbf{B}\mathbf{x}(t) &= \begin{bmatrix} -2e^{-t}+4e^{-2t} \\ -4e^{-t}+4e^{-2t} \end{bmatrix} + \begin{bmatrix} e^{-t}-e^{-2t} \\ -1+2e^{-t}-e^{-2t} \end{bmatrix} \\ &= \begin{bmatrix} -e^{-t}+3e^{-2t} \\ -1-2e^{-t}+3e^{-2t} \end{bmatrix}\end{aligned}$$

And hence:

$$y(t)=\begin{bmatrix} 0 & 1 \end{bmatrix} + \begin{bmatrix} -e^{-t}+3e^{-2t} \\ -1-2e^{-t}+3e^{-2t} \end{bmatrix} = (-1-2e^{-t}+3e^{-2t})u(t)$$

10.3-16.

$$y(t)=\mathbf{C}[e^{\mathbf{A}t}\mathbf{q}(0)+e^{\mathbf{A}t}*\mathbf{B}\mathbf{x}(t)]+\mathbf{D}\mathbf{x}(t)$$

From Prob. 10.3-6 we have obtained:

$$\Phi(s)=\begin{bmatrix} \frac{s+1}{(s+1)^2+1} & \frac{1}{(s+1)^2+1} \\ \frac{-1}{(s+1)^2+1} & \frac{s+1}{(s+1)^2+1} \end{bmatrix}$$

$$\text{Hence:} \qquad e^{\mathbf{A}t}=L^{-1}(\Phi(s))=\begin{bmatrix} e^{-t}\cos t & e^{-t}\sin t \\ -e^{-t}\sin t & e^{-t}\cos t \end{bmatrix}$$

$$e^{\mathbf{A}t}\mathbf{q}(0)=e^{\mathbf{A}t}\begin{bmatrix} 2 \\ 1 \end{bmatrix}=\begin{bmatrix} 2e^{-t}\cos t+e^{-t}\sin t \\ -2e^{-t}\sin t+e^{-t}\cos t \end{bmatrix}$$

$$\text{And:} \qquad e^{\mathbf{A}t}\mathbf{B}=e^{\mathbf{A}t}\begin{bmatrix} 0 \\ 1 \end{bmatrix}=\begin{bmatrix} e^{-t}\sin t \\ e^{-t}\cos t \end{bmatrix}$$

$$\text{And:} \quad e^{\mathbf{A}t}*\mathbf{B}\mathbf{x}(t)=\begin{bmatrix} e^{-t}\sin t*u(t) \\ e^{-t}\cos t*u(t) \end{bmatrix}=\begin{bmatrix} \frac{\cos(\frac{\pi}{2}-\phi)}{\sqrt{2}}-\frac{e^{-t}}{\sqrt{2}}\cos(t-\frac{\pi}{2}-\phi) \\ \frac{\cos(-\phi)}{\sqrt{2}}-\frac{e^{-t}}{\sqrt{2}}\cos(t-\phi) \end{bmatrix}$$

where: $\phi=\tan^{-1}\frac{-1}{1}=-\frac{\pi}{4}$.

$$\text{And hence:} \quad e^{\mathbf{A}t}\mathbf{q}(0)+e^{\mathbf{A}t}*\mathbf{B}\mathbf{x}(t)=\begin{bmatrix} \frac{1}{2}+\frac{3}{2}e^{-t}\cos t+\frac{1}{2}e^{-t}\sin t \\ \frac{1}{2}+\frac{1}{2}e^{-t}\cos t-\frac{3}{2}e^{-t}\sin t \end{bmatrix}$$

And

$$\begin{aligned} y(t) &= \mathbf{C}[e^{\mathbf{A}t}\mathbf{q}(0)+e^{\mathbf{A}t}*\mathbf{B}\mathbf{x}(t)]+\mathbf{D}\mathbf{x}(t) \\ &= \begin{bmatrix} 1 & 1 \end{bmatrix}[e^{\mathbf{A}t}\mathbf{q}(0)+e^{\mathbf{A}t}*\mathbf{B}\mathbf{x}(t)]+u(t) \\ &= [1+2e^{-t}\cos t-e^{-t}\sin t+1]u(t)=[2+2e^{-t}\cos t-e^{-t}\sin t]u(t) \end{aligned}$$

10.3-17.

$$H(s)=\frac{3s+10}{s^2+7s+12}$$

From Eq. (10.65) we have:

$$\mathbf{h}(t) = \mathbf{C}\phi(t)\mathbf{B} + \mathbf{D}\delta(t) \qquad \text{where} \qquad \phi(t) = e^{\mathbf{A}t}$$

From Prob. 10.3-7 we obtained $\Phi(s)$ as:

$$\Phi(s) = \begin{bmatrix} \frac{1}{s+4} & \frac{1}{(s+3)(s+4)} \\ 0 & \frac{1}{s+1} \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad \text{and} \qquad \mathbf{D} = 0$$

$$\text{hence:} \qquad e^{\mathbf{A}t} = L^{-1}(\Phi(s)) = \begin{bmatrix} e^{-4t} & e^{-3t} - e^{-4t} \\ 0 & e^{-3t} \end{bmatrix}$$

$$\text{And:} \qquad \phi(t)\mathbf{B} = \begin{bmatrix} 3e^{-4t} + e^{-3t} - e^{-4t} \\ e^{-3t} \end{bmatrix} = \begin{bmatrix} e^{-3t} + 2e^{-4t} \\ e^{-3t} \end{bmatrix}$$

$$\begin{aligned} \text{Since} \qquad \mathbf{D} = 0, \quad h(t) = \mathbf{C}\phi(t)\mathbf{B} &= [1 \quad 0]\phi(t)\mathbf{B} \\ &= (e^{-3t} + 2e^{-4t})u(t) \end{aligned}$$

10.3-18. From Prob. 10.3-6,

$$\Phi(s) = \begin{bmatrix} \frac{s+1}{(s+1)^2+1} & \frac{1}{(s+1)^2+1} \\ \frac{-1}{(s+1)^2+1} & \frac{s+1}{(s+1)^2+1} \end{bmatrix}$$

$$\text{hence:} \qquad \phi(t) = L^{-1}(\Phi(s)) = \begin{bmatrix} e^{-t}\cos t & e^{-t}\sin t \\ -e^{-t}\sin t & e^{-t}\cos t \end{bmatrix}$$

$$\phi(t)\mathbf{B} = \phi(t)\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} e^{-t}\sin t \\ e^{-t}\cos t \end{bmatrix}$$

$$\text{And:} \qquad \mathbf{C}\phi(t)\mathbf{B} = \begin{bmatrix} 1 & 1 \end{bmatrix}\phi(t)\mathbf{B} = (e^{-t}\sin t + e^{-t}\cos t)$$

And

$$h(t) = \mathbf{C}\phi(t)\mathbf{B} + \delta(t) = \delta(t) + (e^{-t}\sin t + e^{-t}\cos t)u(t)$$

10.3-19. From Prob. 10.3-10,

$$\phi(s) = \begin{bmatrix} \frac{2s+1}{(s+1)^2} & \frac{s}{(s+1)^2} \\ \frac{4+s}{(s+1)^2} & \frac{4s+7}{(s+1)^2} \\ \frac{s+2}{s+1} & \frac{1}{s+1} \end{bmatrix} = \begin{bmatrix} \frac{2}{s+1} - \frac{1}{(s+1)^2} & \frac{1}{s+1} - \frac{1}{(s+1)^2} \\ \frac{1}{s+1} + \frac{3}{(s+1)^2} & \frac{4}{s+1} + \frac{3}{(s+1)^2} \\ 1 + \frac{1}{s+1} & \frac{1}{s+1} \end{bmatrix}$$

And hence: the unit inputs response $\mathbf{h}(t)$ is given by:

$$\mathbf{h}(t) = \mathcal{L}^{-1}\{\mathbf{H}(s)\} = \begin{bmatrix} 2e^{-t} - te^{-t} & e^{-t} - te^{-t} \\ e^{-t} + 3te^{-t} & 4e^{-t} + 3te^{-t} \\ \delta(t) + e^{-t} & e^{-t} \end{bmatrix}$$

10.4-1.

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}x$$

$$\mathbf{w} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \mathbf{Pq}$$

The new state equation of the system is given by:

$$\dot{\mathbf{w}} = \mathbf{PAP}^{-1}\mathbf{w} + \mathbf{P}\dot{\mathbf{B}} = \hat{\mathbf{A}}\mathbf{w} + \hat{\mathbf{B}}x$$

$$\mathbf{P}^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \Longrightarrow \mathbf{PA} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ -1 & -2 \end{bmatrix}$$

$$\mathbf{PAP}^{-1} = \begin{bmatrix} -1 & -1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -3 & 1 \end{bmatrix}$$

And: $$\mathbf{PB} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}x$$

Hence $$\begin{bmatrix} \dot{w}_1 \\ \dot{w}_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix}x$$

Eigenvalues in the original system:
The eigenvalues are the roots of the characteristic equation, thus: in the original system:

$$|s\mathbf{I} - \mathbf{A}| = \begin{vmatrix} s & -1 \\ 1 & s+1 \end{vmatrix} = (s+1)s + 1 = s^2 + s + 1 = 0$$

The roots are given by: $$s_{1,2} = \frac{-1 \pm j\sqrt{3}}{2}$$

In the transformed system, the characteristic equation is given by:

$$|s\mathbf{I} - \hat{\mathbf{A}}| = \left|\begin{bmatrix} s+2 & -1 \\ 3 & s-1 \end{bmatrix}\right| = (s+2)(s-1) + 3 = s^2 - s + 2s - 2 + 3 = s^2 + s + 1$$

And the eigenvalues are given by:

$$s_{1,2} = \frac{-1 \pm j\sqrt{3}}{2}$$

which are the same as in the original system.

10.4-2.

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix}x(t)$$

(a) The characteristic equation is given by:

$$|s\mathbf{I} - \mathbf{A}| = 0 = \begin{vmatrix} s & -1 \\ 2 & s+3 \end{vmatrix} = s(s+3) - 2 = s^2 + 3s + 2 = (s+1)(s+2) = 0$$

$\lambda_1 = -1$ and $\lambda_2 = -2$ are the eigenvalues. And

$$\Lambda = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$

$\mathbf{w} = \mathbf{Pq}$ and $\dot{\mathbf{w}} = \mathbf{PAP}^{-1}\mathbf{w} + \mathbf{PB}x = \Lambda\mathbf{w} + \hat{\mathbf{B}}x$
Hence we have to find $\mathbf{P}$ such that: $\mathbf{PAP}^{-1} = \Lambda$ or $\Lambda\mathbf{P} = \mathbf{PA}$

$$\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

$$\left.\begin{array}{l} -p_{11} = 2p_{12} \Longrightarrow p_{11} = 2p_{12} \\ p_{12} = 3p_{12} - p_{11} \\ p_{21} = p_{22} \\ p_{22} = 3p_{22} - p_{21} \end{array}\right\} \Longrightarrow \begin{array}{l} p_{12} = 3p_{12} - 2p_{12} \\ \text{If we choose} \quad p_{11} = 2 \quad \text{then} \quad p_{12} = 1 \\ \text{And if} \quad p_{21} = 1 \quad \text{then} \quad p_{22} = 1 \end{array}$$

$$\text{Therefore} \qquad \mathbf{P} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\text{and hence} \qquad \mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 2q_1 + q_2 \\ q_1 + q_2 \end{bmatrix}$$

(b) $\mathbf{y} = \mathbf{Cq} + \mathbf{D}x$ where $\mathbf{D} = 0 \Longrightarrow \mathbf{y} = \mathbf{Cq}$.
we have $\mathbf{w} = \mathbf{Pq} \Longrightarrow \mathbf{P}^{-1}\mathbf{w} = \mathbf{q} \Longrightarrow \mathbf{y} = \mathbf{CP}^{-1}\mathbf{w}$. hence:

$$\mathbf{P}^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}, \quad \mathbf{CP}^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -3 & 5 \end{bmatrix}$$

$$\mathbf{y} = \begin{bmatrix} 0 & 1 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} w_2 \\ 5w_2 - 3w_1 \end{bmatrix}$$

10.4-3.

$$\dot{\mathbf{q}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix} \mathbf{q} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} x$$

The characteristic equation is given by:

$$\begin{aligned} |s\mathbf{I} - \mathbf{A}| &= \begin{vmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 0 & 2 & s+3 \end{vmatrix} \\ &= s\{(s)(s+3) + 2\} = s(s^2 + 3s + 2) = s(s+1)(s+2) = 0 \end{aligned}$$

Hence the eigenvalues are: $\lambda_1 = 0$, $\lambda_2 = -1$ and $\lambda_3 = -2$.

$$\text{And} \qquad \Lambda = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

In the transformed system we have: $\mathbf{w} = \mathbf{Pq}$ and $\dot{\mathbf{w}} = \mathbf{PAP}^{-1}\mathbf{w} + \mathbf{PB}x$
We have to find $\mathbf{P}$ such that: $\mathbf{PAP}^{-1} = \Lambda$ or $\Lambda\mathbf{P} = \mathbf{PA}$.

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix}$$

$$\Longrightarrow \left\{ \begin{array}{lll} p_{21} = 0 \quad p_{31} = 0 & \text{if} & p_{11} = 2 \text{ then } p_{13} = 1 \text{ and } p_{12} = 3 \\ p_{11} = p_{13} & & \\ p_{12} = 3p_{13} & & \\ p_{22} = 2p_{23} - p_{21} & \text{if} & p_{23} = 1, \text{ then } p_{22} = 2 \text{ and } p_{23} = 1 \\ p_{23} = 3p_{23} - p_{22} & \text{if} & p_{32} = 1 \text{ then } p_{33} = 1 \\ 2p_{32} = 2p_{33} - p_{31} & & \\ 2p_{33} = 3p_{33} - p_{32} \Longrightarrow p_{33} = p_{32} & & \end{array} \right.$$

$$\mathbf{w} = \mathbf{P}\mathbf{q} = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

10.4-4.

$$y(t) = \mathbf{C}[e^{\mathbf{A}t}\mathbf{q}(0) + e^{\mathbf{A}t} * \mathbf{B}\mathbf{x}(t)]$$

where: $$e^{\mathbf{A}t} = \mathcal{L}^{-1}(\phi(s))$$

$$(\phi(s))^{-1} = [s\mathbf{I} - \mathbf{A}] = \begin{bmatrix} s+1 & 0 & 0 \\ 0 & s+3 & 0 \\ 0 & 0 & s+2 \end{bmatrix}$$

$$\phi(t) = (s\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} \frac{1}{s+1} & 0 & 0 \\ 0 & \frac{1}{s+3} & 0 \\ 0 & 0 & \frac{1}{s+2} \end{bmatrix} \text{ and } e^{\mathbf{A}t} = \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-3t} & 0 \\ 0 & 0 & e^{-2t} \end{bmatrix}$$

And: $$e^{\mathbf{A}t}\mathbf{q}(0) = e^{\mathbf{A}t} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} e^{-t} \\ 2e^{-3t} \\ e^{-2t} \end{bmatrix}$$

$$e^{\mathbf{A}t}\mathbf{B} = e^{\mathbf{A}t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} e^{-t} \\ e^{-3t} \\ e^{-2t} \end{bmatrix}$$

and: $$e^{\mathbf{A}t} * \mathbf{B}\mathbf{x}(t) = e^{\mathbf{A}t} * \mathbf{B}u(t) = \begin{bmatrix} e^{-t} * u(t) \\ e^{-3t} * u(t) \\ e^{-2t} * u(t) \end{bmatrix} = \begin{bmatrix} (1 - e^{-t})u(t) \\ \frac{1}{3}(1 - e^{-3t})u(t) \\ \frac{1}{2}(1 - e^{-2t})u(t) \end{bmatrix}$$

Hence: $$e^{\mathbf{A}t}\mathbf{q}(0) + e^{\mathbf{A}t} * \mathbf{B}\mathbf{x}(t) = \begin{bmatrix} e^{-t} + 1 - e^{-t} \\ 2e^{-3t} + \frac{1}{3} - \frac{1}{3}e^{-3t} \\ e^{-2t} + \frac{1}{2} - \frac{1}{2}e^{-2t} \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{3} + \frac{5}{3}e^{-3t} \\ \frac{1}{2} + \frac{1}{2}e^{-2t} \end{bmatrix}$$

And finally: $$y(t) = \mathbf{C}[e^{\mathbf{A}t}\mathbf{q}(0) + e^{\mathbf{A}t} * \mathbf{B}\mathbf{x}(t)] \quad \text{with} \quad \mathbf{C} = \begin{bmatrix} 1 & 3 & 1 \end{bmatrix}$$

$$y(t) = \left(1 + 1 + 5e^{-3t} + \frac{1}{2} + \frac{1}{2}e^{-2t}\right) = \left(\frac{5}{2} + \frac{1}{2}e^{-2t} + 5e^{-3t}\right)$$

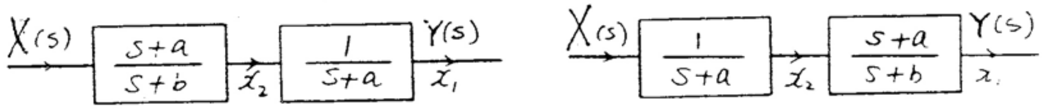


Figure S10.5-1: a and b

10.5-1. (a) state equations:

$$\begin{aligned} \dot{q}_2 + bq_2 &= (a-b)x \implies \dot{q}_2 = -bq_2 + (a-b)x \\ \dot{q}_1 + aq_1 &= q_2 + x \implies \dot{q}_1 = -aq_1 + q_2 + x \end{aligned}$$

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} -a & 1 \\ 0 & -b \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \begin{bmatrix} 1 \\ (a-b) \end{bmatrix} x$$

the output is: $$y = q_1 = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

The characteristic equation is

$$|s\mathbf{I}-\mathbf{A}| = 0 = \begin{vmatrix} s+a & -1 \\ 0 & s+b \end{vmatrix} = (s+a)(s+b) = 0$$

$\lambda_1 = -a$ and $\lambda_2 = -b$ are the eigenvalues.

$$\Lambda = \begin{bmatrix} -a & 0 \\ 0 & -b \end{bmatrix}$$

we also have: $\mathbf{w} = \mathbf{Pq}$ and $\dot{\mathbf{w}} = \mathbf{PAP}^{-1}\mathbf{w} + \mathbf{PB}x$.
We are looking for $\mathbf{P}$ such that: $\mathbf{PAP}^{-1} = \Lambda$ or $\Lambda\mathbf{P} = \mathbf{PA}$

$$\begin{bmatrix} -a & 0 \\ 0 & -b \end{bmatrix}\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}\begin{bmatrix} -a & 1 \\ 0 & -b \end{bmatrix}$$

$$\Longrightarrow \begin{cases} -ap_{11} = -ap_{11} \\ -bp_{21} = -ap_{21} \Longrightarrow p_{21} = 0 \\ -ap_{12} = p_{11} - bp_{12} = 0 \\ -bp_{22} = p_{21} - bp_{22} \Longrightarrow p_{21} = 0 \end{cases}$$

If $p_{11} = (b-a)$ then $p_{12} = 1, p_{21} = 0$ and p_{22} can be anything; let's take $p_{22} = 1$

$$\text{And thus:} \qquad \mathbf{w} = \mathbf{Pq} = \begin{bmatrix} b-a & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

Observability: the output in terms of $\mathbf{w}$ is: $y = \mathbf{Cq} = \mathbf{CP}^{-1}\mathbf{w} = \hat{\mathbf{C}}\mathbf{w}$.

$$\text{where:} \qquad \mathbf{P}^{-1} = \frac{1}{b-a}\begin{bmatrix} 1 & -1 \\ 0 & b-a \end{bmatrix} = \begin{bmatrix} \frac{1}{b-a} & \frac{-1}{b-a} \\ 0 & 1 \end{bmatrix}$$

$$\text{hence:} \qquad \hat{\mathbf{C}} = \mathbf{CP}^{-1} = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{b-a} & \frac{-1}{b-a} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{b-a} & \frac{-1}{b-a} \end{bmatrix}$$

We notice that in $\hat{\mathbf{C}}$, there is no column with all elements zeros, hence we conclude that the system is observable.
Controllability: In the new system (diagonalized form):

$$\hat{\mathbf{B}} = \mathbf{PB} = \begin{bmatrix} b-a & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ a-b \end{bmatrix} = \begin{bmatrix} 0 \\ a-b \end{bmatrix}$$

the 1st row in $\hat{\mathbf{B}}$ is zero. We affirm that this system is not controllable.

(b) State equations:

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} -b & 0 \\ 0 & -a \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}x$$

$$\text{and:} \qquad y = q_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

The matrix A is already in the diagonal form:

$$\mathbf{P} = \mathbf{A} = \begin{bmatrix} -b & 0 \\ 0 & -a \end{bmatrix} \Longrightarrow \mathbf{P}^{-1} = \frac{1}{ab} = \begin{bmatrix} -a & 0 \\ 0 & -b \end{bmatrix} = \begin{bmatrix} -\frac{1}{b} & 0 \\ 0 & -\frac{1}{a} \end{bmatrix}$$

In the transformed system: $\dot{\mathbf{w}} = \mathbf{PAP}^{-1}\mathbf{w} + \mathbf{PB}x = \mathbf{A}\mathbf{w} + \hat{\mathbf{B}}x$.

Observability:

$$\hat{\mathbf{C}} = \mathbf{CP}^{-1} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} -\frac{1}{b} & 0 \\ 0 & -\frac{1}{a} \end{bmatrix} = \begin{bmatrix} -\frac{1}{b} & 0 \end{bmatrix}$$

the second column in $\hat{\mathbf{C}}$ vanishes. This system is not observable.
Controllability:

$$\hat{\mathbf{B}} = \mathbf{PB} = \begin{bmatrix} -b & 0 \\ 0 & -a \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} = \begin{bmatrix} -b & -a \end{bmatrix}$$

in $\hat{\mathbf{B}}$, there is no row with all elements zeros; hence this system is controllable.

10.6-1. (a) Time-domain method: the output $y[n]$ is given by:

$$y[n] = \mathbf{CA}^n\mathbf{q}[0] + \mathbf{CA}^{n-1}u[n-1] * \mathbf{B}x[n] + \mathbf{D}x[n]$$

The characteristic equation of $\mathbf{A}$ is:

$$|\lambda\mathbf{I} - \mathbf{A}| = \begin{vmatrix} \lambda - 2 & 0 \\ -1 & \lambda - 1 \end{vmatrix} = (\lambda - 1)(\lambda - 2) = 0$$

$\lambda_1 = 1$ and $\lambda_2 = 2$ are the eigenvalues of $\mathbf{A}$. Also:

$$\mathbf{A}^n = \beta_0\mathbf{I} + \beta_1\mathbf{A} \qquad \text{where:} \qquad \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2^n \end{bmatrix}$$

$$\begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2^n \end{bmatrix} = \begin{bmatrix} 2 - 2^n \\ -1 + 2^n \end{bmatrix}$$

hence:

$$\mathbf{A}^n = \begin{bmatrix} \beta_0 & 0 \\ 0 & \beta_0 \end{bmatrix} + \begin{bmatrix} 2\beta_1 & 0 \\ \beta_1 & \beta_1 \end{bmatrix} = \begin{bmatrix} 2^n & 0 \\ 2^n - 1 & 1 \end{bmatrix}$$

$$\text{Hence:} \qquad \mathbf{CA}^n = \begin{bmatrix} 0 & 1 \end{bmatrix} \mathbf{A}^n = \begin{bmatrix} 2^n - 1 & 1 \end{bmatrix}$$

And:

$$y_x[n] = \mathbf{CA}^n\mathbf{q}(0) = \mathbf{CA}^n \begin{bmatrix} 2 \\ 1 \end{bmatrix} = (2^{n+1} - 1)u[n]$$

The zero-state component is given by:

$$y_x[n] = \mathbf{CA}^{n-1}u[n-1] * \mathbf{B}x[n] + \mathbf{D}x[n]$$

But $\quad \mathbf{CA}^n u[n] * \mathbf{B}x[n] = \begin{bmatrix} 2^n - 1 & 1 \end{bmatrix} u[n] * \begin{bmatrix} 0 \\ u[n] \end{bmatrix} = (n+1)u[n]$

Hence

$$y_x[n] = nu[n-1] + \mathbf{D}x[n] = nu[n-1] + u[n] = (n+1)u[n]$$

$$\text{and} \qquad y[n] = y_x[n] + y_x[n] = [2^{n+1} + n]u[n]$$

(b) Frequency-domain method: in this case:

$$\mathbf{Y}(z) = \mathbf{C}(\mathbf{I} - z^{-1}\mathbf{A})^{-1}\mathbf{q}[0] + [\mathbf{C}(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}]X[z]$$

$$
\begin{aligned}
(\mathbf{I}-z^{-1}\mathbf{A})^{-1} &= \begin{bmatrix} 1-2z^{-1} & 0 \\ -z^{-1} & 1-z^{-1} \end{bmatrix}^{-1} = \begin{bmatrix} 1-\frac{2}{z} & 0 \\ -\frac{1}{z} & 1-\frac{1}{z} \end{bmatrix}^{-1} = \begin{bmatrix} \frac{z-2}{z} & 0 \\ -\frac{1}{z} & \frac{z-1}{z} \end{bmatrix}^{-1} \\
&= \frac{z^2}{(z-1)(z-2)} \begin{bmatrix} \frac{z-1}{z} & 0 \\ \frac{1}{z} & \frac{z-2}{z} \end{bmatrix} = \begin{bmatrix} \frac{z}{z-2} & 0 \\ \frac{z}{(z-1)(z-2)} & \frac{z}{z-1} \end{bmatrix}
\end{aligned}
$$

Also:

$$
\begin{aligned}
(z\mathbf{I}-\mathbf{A})^{-1} &= \begin{bmatrix} z-2 & 0 \\ -1 & z-1 \end{bmatrix}^{-1} = \frac{1}{(z-1)(z-2)} \begin{bmatrix} z-1 & 0 \\ 1 & z-2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{z-2} & 0 \\ \frac{1}{(z-1)(z-2)} & \frac{1}{z-1} \end{bmatrix}
\end{aligned}
$$

$$
\text{and} \qquad \mathbf{C}(\mathbf{I}-z^{-1}\mathbf{A})^{-1} = \begin{bmatrix} \frac{z}{(z-1)(z-2)} & \frac{z}{z-1} \end{bmatrix}
$$

$$
\mathbf{C}(\mathbf{I}-z^{-1}\mathbf{A})^{-1}\mathbf{q}(0) = \begin{bmatrix} \frac{2z}{(z-1)(z-2)} + \frac{z}{z-1} \end{bmatrix} = \frac{z^2}{(z-1)(z-2)}
$$

Also

$$
\mathbf{C}(z\mathbf{I}-\mathbf{A})^{-1} = \begin{bmatrix} \frac{1}{(z-1)(z-2)} & \frac{1}{z-1} \end{bmatrix} \quad \text{and} \quad \mathbf{C}(z\mathbf{I}-\mathbf{A})^{-1}\mathbf{B} = \frac{1}{z-1}
$$

$$
\text{Hence:} \qquad \mathbf{C}(z\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}+\mathbf{D} = \frac{1}{z-1} + \mathbf{D} = \frac{1}{z-1} + 1 = \frac{z}{z-1}
$$

$$
x[n] = u[n] \qquad \text{and} \qquad X(z) = \frac{z}{z-1}
$$

$$
\text{And hence:} \qquad (\mathbf{C}(z\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}+\mathbf{D})X(z) = \left[\frac{z}{z-1}\right]^2 = \frac{z^2}{(z-1)^2}
$$

$$
\mathbf{Y}(z) = \mathbf{C}(\mathbf{I}-z^{-1}\mathbf{A})^{-1}\mathbf{q}(0) + [\mathbf{C}(z\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}+\mathbf{D}]X(z) = \frac{z^2}{(z-1)(z-2)} + \frac{z^2}{(z-1)^2}
$$

$$
\frac{\mathbf{Y}(z)}{z} = \frac{1}{z-2} + \frac{z}{(z-1)^2} = \frac{2}{z-2} + \frac{1}{(z-1)^2}
$$

$$
\mathbf{Y}(z) = \frac{2z}{z-2} + \frac{z}{(z-1)^2}
$$

$$
\begin{aligned}
\text{and} \qquad y[n] = z^{-1}[\mathbf{Y}(z)] &= [2^n+1]u[n] + (n+1)u[n] \\
&= [2^{n+1}+n]u[n]
\end{aligned}
$$

10.6-2.

$$
y[n] = \frac{E+0.32}{E^2+E+0.16}x[n]
$$

(a) In this case:

$$
\begin{aligned}
H(z) = \frac{Y(z)}{X(z)} &= \frac{z+0.32}{z^2+z+0.16} \\
&= \frac{z+0.32}{(z+0.2)(z+0.8)} = \frac{0.2}{z+0.2} + \frac{0.8}{z+0.8}
\end{aligned}
$$

(b) State and output equations for the direct form II: using the output of each delay as a state variable we get:

$$\begin{aligned} q_1[n+1] &= q_2[n] \\ q_2[n+1] &= -0.16q_1[n] - q_2[n] + x[n] \end{aligned}$$

Direct form II

$$\begin{bmatrix} q_1[n+1] \\ q_2[n+1] \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -0.16 & -1 \end{bmatrix} \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x[n]$$

output equation:

$$y[n] = 0.32q_1[n] + q_[n] = \begin{bmatrix} 0.32 & 1 \end{bmatrix} \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix}$$

State equations for the transposed direct form II:

$$\begin{aligned} q_1[n+1] &= -q_1[n] + q_2[n] + x[n] \\ q_2[n+1] &= -0.16q_1[n] + 0.32x[n] \end{aligned}$$

$$\begin{bmatrix} q_1[n+1] \\ q_2[n+1] \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -0.16 & 0 \end{bmatrix} \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix} + \begin{bmatrix} 1 \\ 0.32 \end{bmatrix} x[n]$$

The output equation is:

$$y[n] = q_1[n] = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix}$$

State equations for the cascade realization:

$$\begin{aligned} q_1[n+1] &= -0.8q_1[n] + q_2[n] \\ q_2[n+1] &= -0.2q_2[n] + x[n] \end{aligned}$$

$$\begin{bmatrix} q_1[n+1] \\ q_2[n+1] \end{bmatrix} = \begin{bmatrix} -0.8 & 1 \\ 0 & -0.2 \end{bmatrix} \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x[n]$$

The output equation is:

$$y[n] = 0.32q_1[n] - 0.8q_1[n] + q_2[n] = \begin{bmatrix} -0.48 & 1 \end{bmatrix} \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix}$$

State equations for the parallel realization:

$$\begin{aligned} q_1[n+1] &= -0.2q_1[n] + x[n] \\ q_2[n+1] &= -0.8q_2[n] + x[n] \end{aligned}$$

$$\begin{bmatrix} q_1[n+1] \\ q_2[n+1] \end{bmatrix} = \begin{bmatrix} -0.2 & 0 \\ 0 & -0.8 \end{bmatrix} \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} x[n]$$

The output equation is:

$$y[n] = 0.2q_1[n] + 0.8q_2[n] = \begin{bmatrix} 0.2 & 0.8 \end{bmatrix} \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix}$$

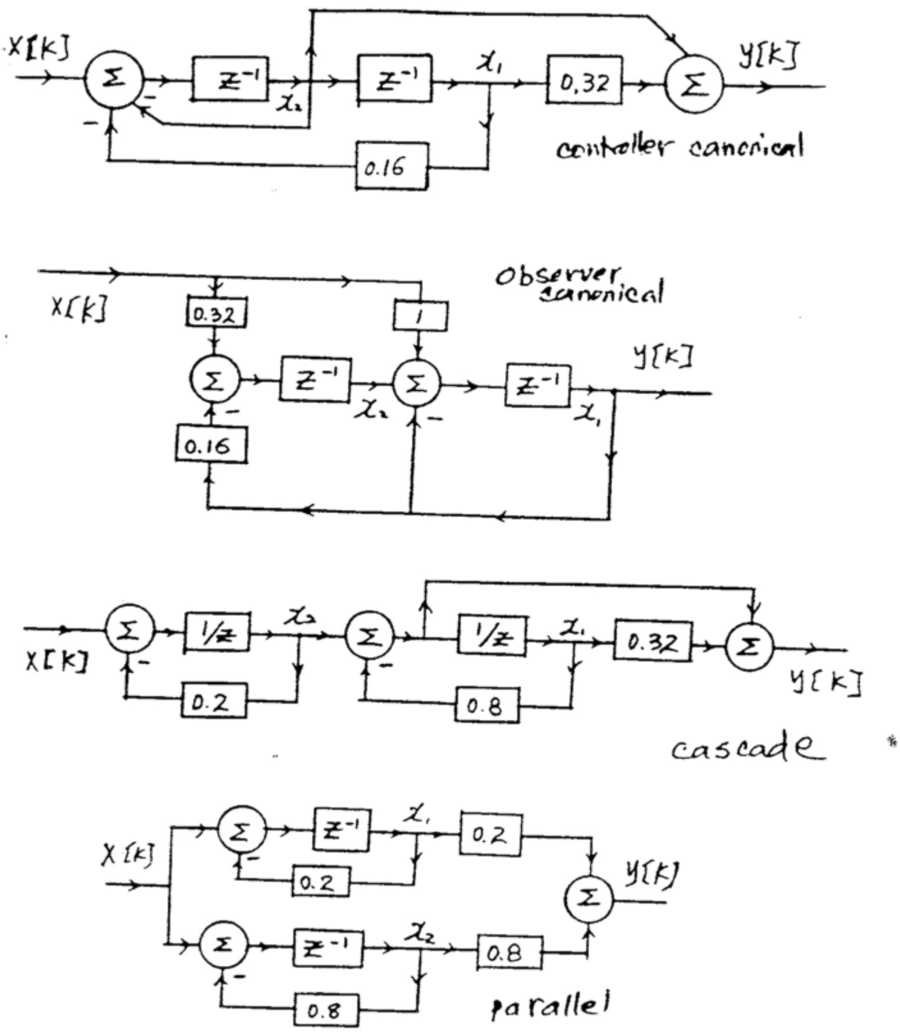


Figure S10.6-2:

10.6-3.

$$y[n] = \frac{E(2E+1)}{E^2+E-6}x[n]$$

(a)

$$\begin{aligned}
\frac{Y(z)}{X(z)} = H(z) &= \frac{z(2z+1)}{z^2+z-6} = \frac{2z^2+z}{z^2+z-6} \\
&= \frac{2z^2+z}{(z-2)(z+3)} = \left(\frac{z}{z-2}\right)\left(\frac{2z+1}{z+3}\right) \\
&= \frac{z}{z-2} + \frac{z}{z+3}
\end{aligned}$$

(b) State and output equations for the direct form II:

$$\begin{aligned} q_1[n+1] &= q_2[n] \\ q_2[n+1] &= 6q_1[n] - q_2[n] + x[n] \end{aligned}$$

and

$$\begin{bmatrix} q_1[n+1] \\ q_2[n+1] \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 6 & -1 \end{bmatrix} \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x[n]$$

The output equation is:

$$\begin{aligned} y[n] &= q_2[n] + 2[6q_1[n] - q_2[n] + x[n]] \\ &= 12q_1[n] - 2q_2[n] + 2x[n] \end{aligned}$$

Hence $\quad y[n] = \begin{bmatrix} 12 & -2 \end{bmatrix} \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix} + 2x[n]$

State equations for the transposed direct form II:

$$\begin{aligned} q_1[n+1] &= -q_1[n] + q_2[n] + x[n] \\ q_2[n+1] &= 6q_1[n] \end{aligned}$$

$$\begin{bmatrix} q_1[n+1] \\ q_2[n+1] \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 6 & 0 \end{bmatrix} \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} x[n]$$

The output equation is:

$$y[n] = q_1[n] + 2x[n] = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix} + 2x[n]$$

State equations for the cascade realization:

$$\begin{aligned} q_1[n+1] &= -0.3q_1[n] + 2q_2[n] + x[n] \\ q_2[n+1] &= 2q_2[n] + x[n] \end{aligned}$$

$$\begin{bmatrix} q_1[n+1] \\ q_2[n+1] \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} x[n]$$

The output equation is:

$$\begin{aligned} y[n] &= q_1[n] - 6q_1[n] + 4q_2[n] + 2x[n] \\ &= -5q_1[n] + 4q_2[n] + 2x[n] = \begin{bmatrix} -5 & 4 \end{bmatrix} \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix} + 2x[n] \end{aligned}$$

State equations for the parallel realization:

$$\begin{aligned} q_1[n+1] &= 2q_1[n] + x[n] \\ q_2[n+1] &= -3q_2[n] + x[n] \end{aligned}$$

$$\begin{bmatrix} q_1[n+1] \\ q_2[n+1] \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} x[n]$$

The output equation is:

$$y[n] = 2q_1[n] + x[n] + x[n] - 3q_2[n]$$

$$y[n] = \begin{bmatrix} 2 & -3 \end{bmatrix} \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix} + 2x[n]$$

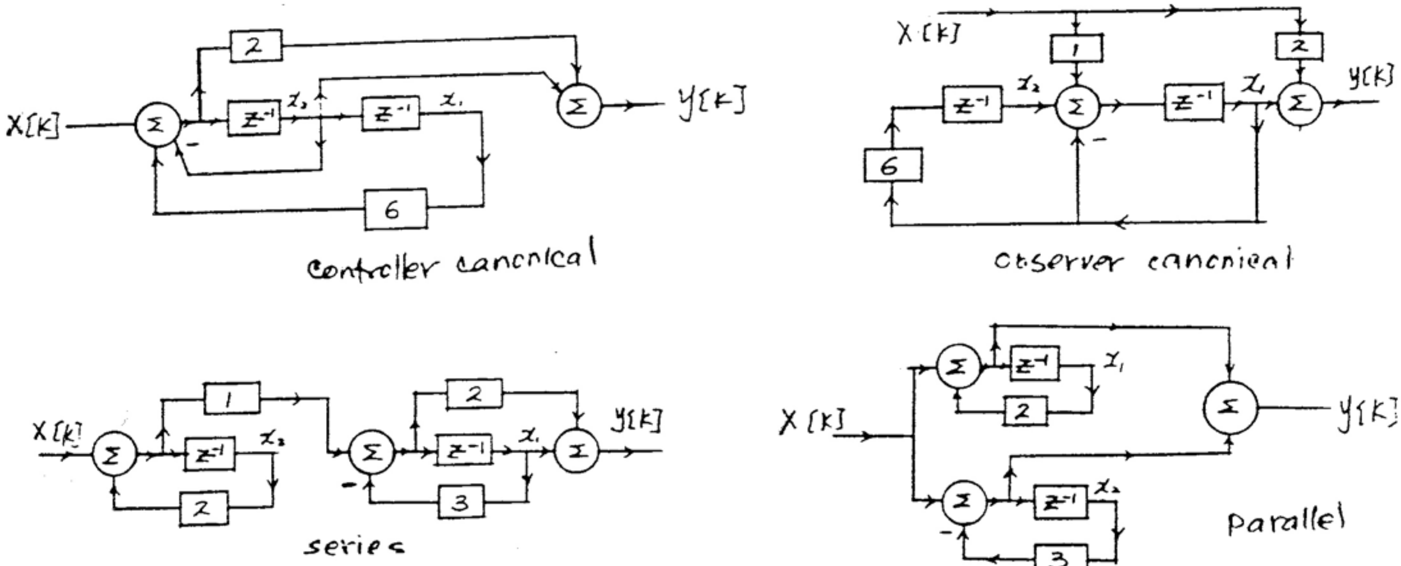


Figure S10.6-3

10.M-1. Figure 10.M-1 is used to help determine the state and output equations.

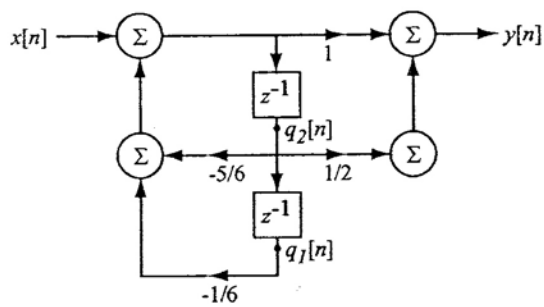


Figure S10.M-1: Direct form II realization of
$y[n] + \frac{5}{6}y[n-1] + \frac{1}{6}y[n-2] = x[n] + \frac{1}{2}x[n-1]$.

Directly from the diagram, note that $q_1[n+1] = q_2[n] + 0x[n]$ and $q_2[n+1] = -\frac{5}{6}q_2[n] - \frac{1}{6}q_1[n] + x[n]$. Taken together, these yield the state equation

$$\mathbf{Q}[n+1] = \begin{bmatrix} q_1[n+1] \\ q_2[n+1] \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{6} & -\frac{5}{6} \end{bmatrix} \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x[n] = \mathbf{A}\mathbf{Q}[n] + \mathbf{B}x[n]$$

The diagram is also used to write the output equation: $y[n] = \frac{1}{2}q_2[n] - \frac{5}{6}q_2[n] - \frac{1}{6}q_1[n] + x[n]$. Simplifying yields the output equation:

$$y[n] = \begin{bmatrix} -\frac{1}{6} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix} + 1x[n] = \mathbf{C}\mathbf{Q}[n] + \mathbf{D}x[n].$$

10.M-2. Figure 10.M-2 is used to help determine the state and output equations.

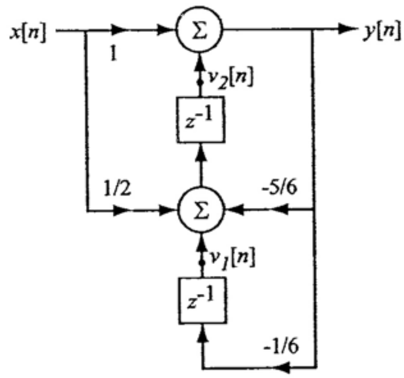


Figure S10.M-2: Transposed direct form II realization of
$y[n] + \frac{5}{6}y[n-1]\frac{1}{6}y[n-2] = x[n] + \frac{1}{2}x[n-1]$.

Directly from the diagram, note that $y[n] = v_2[n] + x[n]$. In standard form, the output equation is thus

$$y[n] = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} v_1[n] \\ v_2[n] \end{bmatrix} + 1x[n] = \mathbf{C}\mathbf{V}[n] + \mathbf{D}x[n].$$

Also using the diagram, note that $v_2[n+1] = v_1[n] + -\frac{5}{6}y[n] + \frac{1}{2}x[n]$ and $v_1[n+1] = -\frac{1}{6}y[n]$. Substituting $y[n] = v_2[n] + x[n]$ into each yields $v_2[n+1] = v_1[n] - \frac{5}{6}(v_2[n] + x[n]) + \frac{1}{2}x[n]$ and $v_1[n+1] = -\frac{1}{6}(v_2[n] + x[n])$. Simplifying to standard form, the state equations are represented in matrix form by

$$\mathbf{V}[n+1] = \begin{bmatrix} v_1[n+1] \\ v_2[n+1] \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{6} \\ 1 & -\frac{5}{6} \end{bmatrix} \begin{bmatrix} v_1[n] \\ v_2[n] \end{bmatrix} + \begin{bmatrix} -\frac{1}{6} \\ -\frac{1}{3} \end{bmatrix} x[n] = \mathbf{A}\mathbf{V}[n] + \mathbf{B}x[n].$$

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Contents

Chapter B Solutions

B.1. Given $w = re^{\jmath\theta} = r(\cos(\theta) + \jmath\sin(\theta)) = x + \jmath y$,

$$w^* = (x + \jmath y)^* = x - \jmath y = r(\cos(\theta) - \jmath\sin(\theta)) = re^{-\jmath\theta}.$$

B.2. (a) For $1 + \jmath$, $r = \sqrt{1^2 + 1^2} = \sqrt{2}$ and $\theta = \arctan\left(\frac{1}{1}\right) = \pi/4 = 0.584$. Thus,

$$1 + \jmath = \sqrt{2}e^{\jmath\pi/4} = 1.414e^{\jmath 0.584}.$$

(b) For $-4 + \jmath 3$, $r = \sqrt{(-4)^2 + 3^2} = 5$ and $\theta = \arctan\left(\frac{3}{-4}\right) = -0.643 + \pi = 2.498$. Thus,

$$-4 + \jmath 3 = 5e^{\jmath 2.498}.$$

(c) Using the results from B.2a and B.2b,

$$(1 + \jmath)(-4 + \jmath 3) = (\sqrt{2}e^{\jmath\pi/4})(5e^{\jmath 2.498}) = 7.071e^{\jmath 3.283}.$$

(d) $e^{j\pi/4} + 2e^{-j\pi/4} = \frac{1+\jmath}{\sqrt{2}} + \frac{2-\jmath 2}{\sqrt{2}} = \frac{3-\jmath}{\sqrt{2}}$. Thus, $r = \sqrt{\left(\frac{3}{\sqrt{2}}\right)^2 + \left(\frac{-1}{\sqrt{2}}\right)^2} = \sqrt{5} = 2.236$ and $\theta = \arctan\left(\frac{-1}{3}\right) = -0.322$, which yields

$$e^{j\pi/4} + 2e^{-j\pi/4} = 2.236e^{-\jmath 0.322}.$$

(e) $e^{\jmath} + 1 = \cos(1) + \jmath\sin(1) + 1$. Thus, $r = \sqrt{(\cos(1) + 1)^2 + (\sin(1))^2} = 1.755$ and $\theta = \arctan\left(\frac{\sin(1)}{\cos(1)+1}\right) = 0.500$, which yields

$$e^{\jmath} + 1 = 1.755e^{\jmath/2}.$$

(f) Using the results from B.2a and B.2b,

$$\frac{1 + \jmath}{-4 + \jmath 3} = \frac{\sqrt{2}e^{\jmath\pi/4}}{5e^{\jmath 2.498}} = 0.283e^{-\jmath 1.713}.$$

B.3. (a) Using Euler's identity,

$$3e^{\jmath\pi/4} = 3\cos(\pi/4) + \jmath 3\sin(\pi/4) = 2.121 + \jmath 2.121.$$

(b) Using Euler's identity,

$$\frac{1}{e^{\jmath}} = e^{-\jmath} = \cos(-1) + \jmath\sin(-1) = 0.540 - \jmath0.841.$$

(c) Expanding,

$$(1+\jmath)(-4+\jmath3) = (-4-3) + \jmath(-4+3) = -7-\jmath.$$

(d) Using Euler's identity,

$$e^{j\pi/4} + 2e^{-j\pi/4} = \frac{1+\jmath}{\sqrt{2}} + \frac{2-\jmath2}{\sqrt{2}} = \frac{3}{\sqrt{2}} + \jmath\frac{-1}{\sqrt{2}}.$$

(e) Using Euler's identity,

$$e^{\jmath} + 1 = \cos(1) + \jmath\sin(1) + 1 = (\cos(1)+1) + \jmath\sin(1).$$

(f) Start by expressing the denominator in standard polar form, $\frac{1}{2^{\jmath}} = \frac{1}{e^{\jmath\ln(2)}} = e^{-\jmath\ln(2)}$. Using Euler's identity,

$$\frac{1}{2^{\jmath}} = \cos(\ln(2)) - \jmath\sin(\ln(2)) = 0.769 - \jmath0.639.$$

B.4. For each proof, substitute the Cartesian form for w.

(a)

$$\frac{w+w^*}{2} = \frac{x+\jmath y+x-\jmath y}{2} = x = \mathrm{Re}\,(x+\jmath y) = \mathrm{Re}\,(w)\,.$$

(b)

$$\frac{w-w^*}{2\jmath} = \frac{x+\jmath y-x+\jmath y}{2\jmath} = y = \mathrm{Im}\,(x+\jmath y) = \mathrm{Im}\,(w)\,.$$

B.5. Using the results from B.4,

(a) $\mathrm{Re}\,(e^{w}) = \mathrm{Re}\,(e^{x-\jmath y}) = \frac{e^{x}e^{-\jmath y}+e^{x}e^{\jmath y}}{2} = e^{x}\frac{e^{-\jmath y}+e^{\jmath y}}{2}$. Using Euler's identity yields

$$\mathrm{Re}\,(e^{w}) = e^{x}\cos(y).$$

(b) $\mathrm{Im}\,(e^{w}) = \mathrm{Im}\,(e^{x-\jmath y}) = \frac{e^{x}e^{-\jmath y}-e^{x}e^{\jmath y}}{2\jmath} = e^{x}\frac{e^{-\jmath y}-e^{\jmath y}}{2\jmath}$. Using Euler's identity yields

$$\mathrm{Re}\,(e^{w}) = -e^{x}\sin(y).$$

B.6. For arbitrary complex constants w_1 and w_2,

(a) $\mathrm{Re}\,(\jmath w_1) = \mathrm{Re}\,(\jmath(x_1+\jmath y_1)) = \mathrm{Re}\,(-y_1+\jmath x_1) = -y_1$. Also, $-\mathrm{Im}\,(w_1) = -\mathrm{Im}\,(x_1+\jmath y_1) = -y_1$. Thus,

$$\text{True.} \qquad \mathrm{Re}\,(\jmath w_1) = -\mathrm{Im}\,(w_1)\,.$$

(b) $\mathrm{Im}\,(\jmath w_1) = \mathrm{Im}\,(\jmath(x_1+\jmath y_1)) = \mathrm{Im}\,(-y_1+\jmath x_1) = x_1$. Also, $\mathrm{Re}\,(w_1) = x_1$. Clearly,

$$\text{True.} \qquad \mathrm{Im}\,(\jmath w_1) = \mathrm{Re}\,(w_1)\,.$$

(c) $\mathrm{Re}(w_1)+\mathrm{Re}(w_2) = x_1+x_2$. Also, $\mathrm{Re}(w_1+w_2) = \mathrm{Re}(x_1+jy_1+x_2+jy_2) = x_1+x_2$. Thus,

$$\text{True.} \qquad \mathrm{Re}(w_1)+\mathrm{Re}(w_2) = \mathrm{Re}(w_1+w_2).$$

(d) $\mathrm{Im}(w_1)+\mathrm{Im}(w_2) = y_1+y_2$. Also, $\mathrm{Im}(w_1+w_2) = \mathrm{Im}(x_1+jy_1+x_2+jy_2) = y_1+y_2$. Thus,

$$\text{True.} \qquad \mathrm{Im}(w_1)+\mathrm{Im}(w_2) = \mathrm{Im}(w_1+w_2).$$

(e) $\mathrm{Re}(w_1)\mathrm{Re}(w_2) = x_1x_2$. Also, $\mathrm{Re}(w_1w_2) = \mathrm{Re}((x_1+jy_1)(x_2+jy_2)) = \mathrm{Re}(x_1x_2 - y_1y_2 + j(x_1y_2+x_2y_1)) = x_1x_2 - y_1y_2$. In general $x_1x_2 \neq x_1x_2 - y_1y_2$, so

$$\text{False.} \qquad \mathrm{Re}(w_1)\mathrm{Re}(w_2) \neq \mathrm{Re}(w_1w_2).$$

(f) $\mathrm{Im}(w_1)/\mathrm{Im}(w_2) = y_1/y_2$. Also, $\mathrm{Im}(w_1/w_2) = \mathrm{Im}\left(\frac{x_1+jy_1}{x_2+jy_2}\right) = \mathrm{Im}\left(\frac{x_1+jy_1}{x_2+jy_2}\frac{x_2-jy_2}{x_2-jy_2}\right) = \mathrm{Im}\left(\frac{x_1x_2-y_1y_2+j(x_1y_2+x_2y_1)}{x_2^2+y_2^2}\right) = \frac{x_1y_2+x_2y_1}{x_2^2+y_2^2}$. In general $y_1/y_2 \neq \frac{x_1y_2+x_2y_1}{x_2^2+y_2^2}$, so

$$\text{False.} \qquad \mathrm{Im}(w_1)/\mathrm{Im}(w_2) \neq \mathrm{Im}(w_1/w_2).$$

B.7. First, express w_1 in both rectangular and polar coordinates. By inspection, $w_1 = x_1 + jy_1 = 3 + j4$. Next, $r_1 = \sqrt{3^2+4^2} = 5$ and $\theta_1 = \arctan\left(\frac{4}{3}\right) = 0.927$ so $w_1 = r_1e^{j\theta_1} = 5e^{j0.927}$.

Second, express w_2 in both rectangular and polar coordinates. By inspection, $w_2 = r_2e^{j\theta_2} = 2e^{j\pi/4} = 2e^{j0.785}$. Next, $x_2 = r_2\cos(\theta_2) = 2\cos(\pi/4) = \sqrt{2} = 1.414$ and $y_2 = r_2\sin(\theta_2) = 2\sin(\pi/4) = \sqrt{2} = 1.414$. Thus, $w_2 = x_2 + jy_2 = 1.414 + j1.414$.

(a) From above,

$$w_1 = r_1e^{j\theta_1} = 5e^{j0.927}.$$

(b) From above,

$$w_2 = x_2 + jy_2 = 1.414 + j1.414.$$

(c)

$$|w_1|^2 = r_1^2 = 5^2 = 25.$$

Similarly,

$$|w_2|^2 = r_2^2 = 4.$$

(d)

$$w_1 + w_2 = (x_1+x_2) + j(y_1+y_2) = (3+1.414) + j(4+1.414) = 4.414 + j5.414.$$

(e) $w_1 - w_2 = (x_1+x_2) - j(y_1+y_2) = (3-1.414)+j(4-1.414) = 1.586+j2.586$. Converting to polar form, $r = \sqrt{(1.586)^2+(2.586)^2} = 3.033$ and $\theta = \arctan\left(\frac{2.586}{1.586}\right) = 1.021$. Thus,

$$w_1 - w_2 = re^{j\theta} = 3.033e^{j1.021}.$$

(f) $w_1w_2 = r_1e^{j\theta_1}r_2e^{j\theta_2} = 10e^{j1.713}$. Converting to Cartesian form, $x = 10\cos(1.713) = -1.414$ and $y = 10\sin(1.713) = 9.899$. Thus,

$$w_1w_2 = x + jy = -1.414 + j9.899.$$

(g)

$$\frac{w_1}{w_2} = \frac{r_1 e^{j\theta_1}}{r_2 e^{j\theta_2}} = \frac{r_1}{r_2} e^{j(\theta_1-\theta_2)} = 2.5e^{j0.142}.$$

B.8. First, express w_1 in both rectangular and polar coordinates. For rectangular form, $w_1 = (3+j4)^2 = 9-16+j(12+12) = -7+j24$. For polar form, $r_1 = \sqrt{(-7)^2+24^2} = 25$ and $\theta_1 = \arctan\left(\frac{24}{-7}\right) = -1.287+\pi = 1.855$. Thus, $w_1 = r_1 e^{j\theta_1} = 25e^{j1.855}$.

Second, express w_2 in both rectangular and polar coordinates. Since $j = e^{j\pi/2}$ and $e^{-j40\pi} = 1$, rectangular form is $w_2 = x_2 + jy_2 = j2.5$. For polar form, $w_2 = r_2 e^{j\theta_2} = 2.5e^{j\pi/2} = 2.5e^{j1.571}$.

(a) From above,

$$w_1 = r_1 e^{j\theta_1} = 25e^{j1.855}.$$

(b) From above,

$$w_2 = x_2 + jy_2 = j2.5.$$

(c)

$$|w_1|^2 = r_1^2 = 25^2 = 625.$$

Similarly,

$$|w_2|^2 = r_2^2 = 2.5^2 = 6.25.$$

(d)

$$w_1 + w_2 = (x_1 + x_2) + j(y_1 + y_2) = (-7+0) + j(24+2.5) = -7 + j26.5.$$

(e) $w_1 - w_2 = (x_1 + x_2) - j(y_1 + y_2) = (-7-0) + j(24-2.5) = -7 + j21.5$. Converting to polar form, $r = \sqrt{(-7)^2 + (21.5)^2} = 22.611$ and $\theta = \arctan\left(\frac{21.5}{-7}\right) = -1.256 + \pi = 1.886$. Thus,

$$w_1 - w_2 = re^{j\theta} = 22.611e^{j1.886}.$$

(f) $w_1 w_2 = r_1 e^{j\theta_1} r_2 e^{j\theta_2} = 62.5e^{j3.425}$. Converting to Cartesian form, $x = 62.5\cos(3.425) = -60$ and $y = 62.5\sin(3.425) = -17.5$. Thus,

$$w_1 w_2 = x + jy = -60 + j - 17.5.$$

(g)

$$\frac{w_1}{w_2} = \frac{r_1 e^{j\theta_1}}{r_2 e^{j\theta_2}} = \frac{r_1}{r_2} e^{j(\theta_1-\theta_2)} = 10e^{j0.284}.$$

B.9. First, express w_1 in both rectangular and polar coordinates. By inspection, $w_1 = x_1 + jy_1 = e^{\pi/4} + j = 2.193 + j$. Next, $r_1 = \sqrt{2.193^2 + 1^2} = 2.410$ and $\theta_1 = \arctan\left(\frac{1}{2.193}\right) = 0.428$ so $w_1 = r_1 e^{j\theta_1} = 2.410e^{j0.428}$.

Second, express w_2 in both rectangular and polar coordinates. Using Euler's identity, $w_2 = \cos(j) = \frac{e^{jj}+e^{-jj}}{2} = \frac{e^{-1}+e^{1}}{2} = \cosh(1) = 1.543$. Thus, $w_2 = x_2 + jy_2 = 1.543$. Polar form is $w_2 = r_2 e^{j\theta_2} = 1.543e^{j0}$.

(a) From above,

$$w_1 = r_1 e^{j\theta_1} = 2.410e^{j0.428}.$$

(b) From above,

$$w_2 = x_2 + jy_2 = 1.543.$$

(c)

$$|w_1|^2 = r_1^2 = 5.810.$$

Similarly,

$$|w_2|^2 = r_2^2 = 2.381.$$

(d)

$$w_1 + w_2 = (x_1 + x_2) + j(y_1 + y_2) = (2.193 + 1.543) + j(1 + 0) = 3.736 + j.$$

(e) $w_1 - w_2 = (x_1 + x_2) - j(y_1 + y_2) = (2.193 - 1.543) + j(1 - 0) = 0.650 + j$. Converting to polar form, $r = \sqrt{(0.650)^2 + (1)^2} = 1.193$ and $\theta = \arctan\left(\frac{1}{0.650}\right) = 0.994$. Thus,

$$w_1 - w_2 = re^{j\theta} = 1.193e^{j0.994}.$$

(f) $w_1 w_2 = r_1 e^{j\theta_1} r_2 e^{j\theta_2} = 3.720e^{j0.428}$. Converting to Cartesian form, $x = 3.720\cos(0.428) = 3.384$ and $y = 3.720\sin(0.428) = 1.543$. Thus,

$$w_1 w_2 = x + jy = 3.384 + j1.543.$$

(g)

$$\frac{w_1}{w_2} = \frac{r_1 e^{j\theta_1}}{r_2 e^{j\theta_2}} = \frac{r_1}{r_2} e^{j(\theta_1 - \theta_2)} = 1.562e^{j0.428}.$$

B.10. (a) Note, we can rewrite $2.5\cos(3t) - 1.5\sin(3t + \pi/3) = c\cos(3t + \phi)$ as $\mathrm{Re}\left(2.5e^{j3t} + j1.5e^{j(3t+\pi/3)}\right) = \mathrm{Re}\left(ce^{j(3t+\phi)}\right)$. Working with the left-hand side, $\mathrm{Re}\left(2.5e^{j3t} + j1.5e^{j(3t+\pi/3)}\right) = \mathrm{Re}\left(e^{j3t}(2.5 + 1.5e^{j(\pi/3+\pi/2)})\right)$. The unknown constants c and ϕ are determined by comparing the left- and right-hand sides.

$$c = |2.5 + 1.5e^{j(\pi/3+\pi/2)}| = \sqrt{(2.5 + 1.5\cos(5\pi/6))^2 + (1.5\sin(5\pi/6))^2} = 1.416$$

and

$$\phi = \angle\left(2.5 + 1.5e^{j(\pi/3+\pi/2)}\right) = \arctan\left(\frac{1.5\sin(5\pi/6)}{2.5 + 1.5\cos(5\pi/6)}\right) = 0.558.$$

(b) Note, $\cos(\theta \pm \phi) = \mathrm{Re}\left(e^{j(\theta\pm\phi)}\right) = \mathrm{Re}\left((\cos(\theta) + j\sin(\theta))(\cos(\phi) \pm j\sin(\phi))\right) = \mathrm{Re}\left((\cos(\theta)\cos(\phi) \mp \sin(\theta)\sin(\phi)) + j(\sin(\theta)\cos(\phi) \pm \cos(\theta)\sin(\phi))\right) = (\cos(\theta)\cos(\phi) \mp \sin(\theta)\sin(\phi))$. Thus,

$$\cos(\theta \pm \phi) = \cos(\theta)\cos(\phi) \mp \sin(\theta)\sin(\phi).$$

(c) Noting that $\sin(\alpha x) = \frac{e^{j\alpha x} - e^{-j\alpha x}}{2j}$, first solve the indefinite integral $\int e^{wx}\sin(\alpha x)dx = \int e^{wx}\frac{e^{j\alpha x} - e^{-j\alpha x}}{2j}dx = \int \frac{e^{x(w+j\alpha)} - e^{x(w-j\alpha)}}{2j} = \frac{1}{2j(w+j\alpha)}e^{x(w+j\alpha)} - \frac{1}{2j(w-j\alpha)}e^{x(w-j\alpha)}$. Substituting the limits of integration yields

$$\int_a^b e^{wx}\sin(\alpha x)dx =$$

$$\frac{1}{2j(w+j\alpha)}\left(e^{b(w+j\alpha)}-e^{a(w+j\alpha)}\right)-\frac{1}{2j(w-j\alpha)}\left(e^{b(w-j\alpha)}-e^{a(w-j\alpha)}\right).$$

B.11. Solutions to this problem are based on Euler's identity.

(a)

$$\begin{aligned}\cosh(w) &= \cosh(x+jy)=\frac{e^{x+jy}+e^{-x-jy}}{2}\\ &= 0.5\left((\cos(y)+j\sin(y))e^{x}+(\cos(y)-j\sin(y))e^{-x}\right)\\ &= 0.5\left(\cos(y)(e^{x}+e^{-x})+j\sin(y)(e^{x}-e^{-x})\right)\\ &= \cos(y)\cosh(x)+j\sin(y)\sinh(x)\end{aligned}$$

Thus,

$$\cosh(w)=\cosh(x+jy)=\cosh(x)\cos(y)+j\sinh(x)\sin(y).$$

(b)

$$\begin{aligned}\sinh(w) &= \sinh(x+jy)=\frac{e^{x+jy}-e^{-x-jy}}{2}\\ &= 0.5\left((\cos(y)+j\sin(y))e^{x}-(\cos(y)-j\sin(y))e^{-x}\right)\\ &= 0.5\left(\cos(y)(e^{x}-e^{-x})+j\sin(y)(e^{x}+e^{-x})\right)\\ &= \cos(y)\sinh(x)+j\sin(y)\cosh(x)\end{aligned}$$

Thus,

$$\sinh(w)=\sinh(x+jy)=\sinh(x)\cos(y)+j\cosh(x)\sin(y).$$

B.12. (a) $(w)^4=-1=e^{jj(\pi+2\pi k)}\Rightarrow w=(e^{j(\pi+2\pi k)})^{1/4}$. Thus,

$$w=e^{j\pi(1/4+k/2)}\quad\text{for}\quad k=[0,1,2,3].$$

```
>> k = [0:3]; w = exp(j*pi*(1/4+k/2)); t = linspace(0,2*pi,200);
>> h = plot(real(w),imag(w),'kx',cos(t),sin(t),'k:'); axis equal;
>> xlabel('Real'); ylabel('Imag'); grid;
>> set(h(1),'markersize',10,'linewidth',2);
```

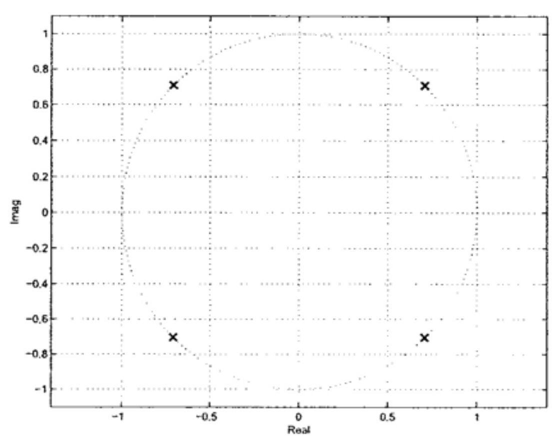

Figure SB.12a: Solutions to $(w)^4=-1$.

(b) Notice,

$$(w-(1+\jmath 2))^5 = \frac{32}{\sqrt{2}}(1+\jmath) = 32e^{\jmath(\pi/4+2\pi k)}.$$

This implies that

$$w-(1+\jmath 2) = \left(32e^{\jmath(\pi/4+2\pi k)}\right)^{1/5} = 2e^{\jmath(\pi/20+2\pi k/5)}.$$

Thus,

$$w = (1+\jmath 2) + 2e^{\jmath(\pi/20+2\pi k/5)} \quad \text{for} \quad k=[0,1,2,3,4].$$

```
>> k = [0:4]; w = (1+j*2)+2*exp(j*(pi/20+2*pi*k/5));
>> t = linspace(0,2*pi,200);
>> h = plot(real(w),imag(w),'kx',1+2*cos(t),2+2*sin(t),'k:');
>> axis equal; xlabel('Real'); ylabel('Imag'); grid;
>> set(h(1),'markersize',10,'linewidth',2);
```

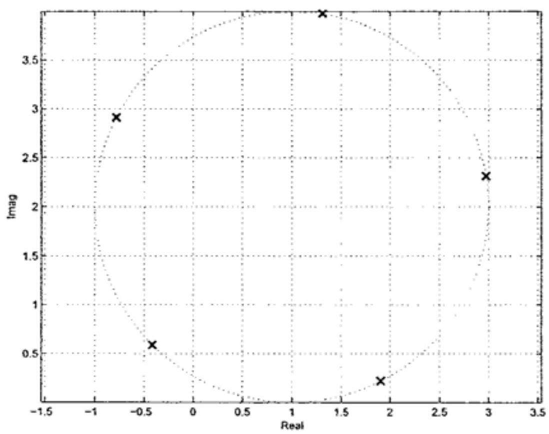

Figure SB.12b: Solutions to $(w-(1+\jmath 2))^5 = \frac{32}{\sqrt{2}}(1+\jmath)$.

(c) The solution set of $|\ w-2\jmath\ |=3$ describes a circle. To see this, note that $|\ w-2\jmath\ |^2 = (w-2\jmath)(w-2\jmath)^* = (x+\jmath(y-2))(x+\jmath(2-y)) = x^2+(y-2)^2 = 3^2 = 9$. The circle has center $(0,2)$ and radius $r=3$.

```
>> theta = linspace(0,2*pi,201); x = 3*cos(theta); y = 2+3*sin(theta);
>> plot(x,y,'k-'); axis equal; grid; xlabel('Real'); ylabel('Imag');
```

(d) Graph $w(t)=(1+t)e^{\jmath t}$ for $(-10 \le t \le 10)$.

```
>> t = [-10:.01:10]; w = (1+t).*exp(j*t);
>> im10 = find(t==-10); im = find(t<0);i0 = find(t==0);
>> ip = find(t>0); ip10 = find(t==10);
>> plot(real(w(im10)),imag(w(im10)),'vk',real(w(im)),imag(w(im)),'k-',...
>> real(w(i0)),imag(w(i0)),'ok',real(w(ip)),imag(w(ip)),'k:',...
>> real(w(ip10)),imag(w(ip10)),'k^'); axis equal; xlabel('Real');
>> ylabel('Imag'); legend('t=-10','t<0','t=0','t>0','t=10',0)
```

B.13. Since four distinct solutions are indicated, we know $n=4$. The solutions to $w^n = w_2 = r_2 e^{\jmath\theta_2}$ lie on a circle of radius $r_2^{1/n}$. The solutions to $(w-w_1)^n = w_2$ lie on the same circle shifted by w_1. To find w_1, drop perpendicular lines from the circle center to

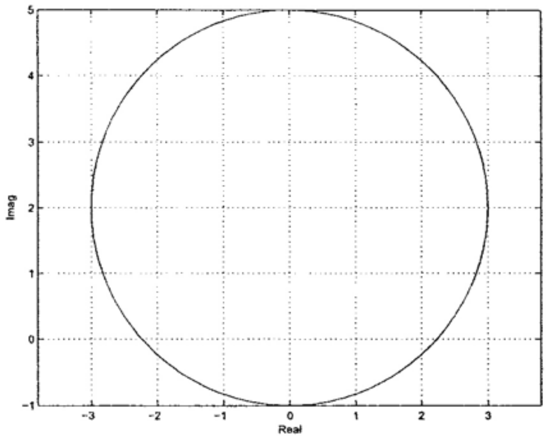

Figure SB.12c: Graph of $\mid w - 2\jmath \mid = 3$.

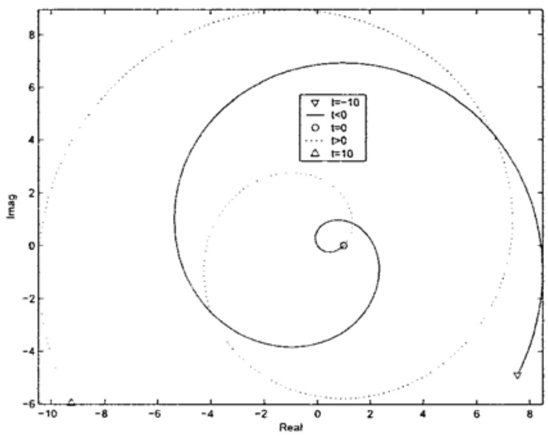

Figure SB.12d: Graph of $w(t) = (1+t)e^{\jmath t}$ for $(-10 \leq t \leq 10)$.

the real and imaginary axes, respectively. As shown, two similar triangles are formed. The circle center is $w_1 = A + \jmath A$. Furthermore, we know that $A + B = \sqrt{3} + 1$ and $A - B = \sqrt{3} - 1$. Clearly, $A = \sqrt{3}$ and $B = 1$. Thus, $w_1 = \sqrt{3} + \jmath\sqrt{3}$. The value of w_2 is now easily found by substitution: $w_2 = (\sqrt{3}+1-(\sqrt{3}+\jmath\sqrt{3}))^4 = (1+\jmath\sqrt{3})^4 = 16e^{\jmath 2\pi/3}$. Thus,

$$n = 4, w_1 = \sqrt{3} + \jmath\sqrt{3}, \text{ and } w_2 = 16e^{\jmath 2\pi/3}.$$

B.14. We can write $(\jmath - w)^{1.5} = (\jmath - w)^{3/2} = \sqrt{8}e^{\jmath\pi/4}$. Squaring both sides yields $(\jmath - w)^3 = 8e^{\jmath(\pi/2+2\pi k)}$. Taking the third root of each side yields $(\jmath - w) = 2e^{\jmath(\pi/6+2\pi k/3)}$. Rearranging yields three distinct solutions

$$w = \jmath - 2e^{\jmath(\pi/6+2\pi k/3)} \quad \text{for} \quad k = [0, 1, 2].$$

```
>> k = [0:2]; w = j-2*exp(j*(pi/6+2*pi*k/3)); t = linspace(0,2*pi,200);
>> h = plot(real(w),imag(w),'kx',...
      real(j-2*exp(j*t)),imag(j-2*exp(j*t)),'k-');
>> xlabel('Real'); ylabel('Imag'); grid;
```

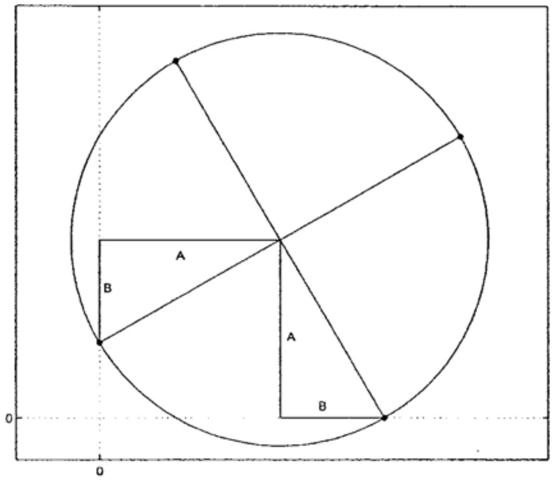

Figure SB.13: Distinct solutions to $(w - w_1)^n = w_2$.

```
>> axis equal; axis([-2.5 2.5 -1.5 3.5]);
>> set(h(1),'markersize',10,'linewidth',2);
```

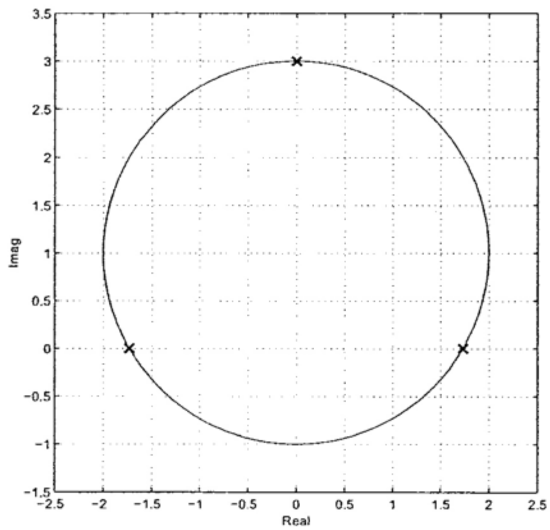

Figure SB.14: Distinct solutions to $(\jmath - w)^{1.5} = 2 + \jmath 2$.

B.15. $w = \sqrt{\jmath} = \left(e^{\jmath(\pi/2+2\pi k)}\right)^{1/2} = e^{\jmath(\pi/4+\pi k)}$. Thus, there are two distinct solutions

$$w = e^{\jmath(\pi/4+\pi k)} \quad \text{for} \quad k = [0, 1].$$

That is, $w = \pm(1+\jmath)/\sqrt{2}$.

B.16. $\ln(-e) = \ln(e^{1+\jmath(\pi+2\pi k)}) = 1 + \jmath(\pi + 2\pi k)$. Since k can be any integer, there are an infinite number of solutions

$$\ln(-e) = 1 + \jmath(\pi + 2\pi k) \quad \text{for integer } k.$$

MATLAB and other calculating devices generally give only the $k=0$ solution $1+\jmath\pi$.

B.17. $\log_{10}(-1) = \log_{10} e^{\jmath(\pi+2\pi k)} = \jmath(\pi+2\pi k)\log_{10}(e)$. Since k can be any integer, there are an infinite number of solutions

$$\log_{10}(-1) = 0 + \jmath(\pi+2\pi k)\log_{10}(e) \quad \text{for integer } k.$$

MATLAB and other calculating devices generally give only the $k=0$ solution $\jmath\pi\log_{10}(e)$.

B.18. (a) $\ln\left(\frac{1}{1+\jmath}\right) = \ln\left(\frac{1}{\sqrt{2}e^{\jmath\pi/4}}\right) = \ln\left((\sqrt{2})^{-1}e^{\jmath(-\pi/4+2\pi k)}\right) = -\ln(\sqrt{2}) + \jmath(-\pi/4+2\pi k)$. Since k can be any integer, there are an infinite number of solutions

$$\ln\left(\frac{1}{1+\jmath}\right) = -\ln(\sqrt{2}) + \jmath(-\pi/4+2\pi k) \quad \text{for integer } k.$$

MATLAB and other calculating devices generally give only the $k=0$ solution.

(b) $\cos(1+\jmath) = 0.5\left(e^{\jmath(1+\jmath)} + e^{-\jmath(1+\jmath)}\right) = 0.5\left(e^{-1}(\cos(1)+\jmath\sin(1)) + e^{1}(\cos(1)-\jmath\sin(1))\right) = \cos(1)\cosh(1) - \jmath\sin(1)\sinh(1)$. That is

$$\cos(1+\jmath) = \cos(1)\cosh(1) - \jmath\sin(1)\sinh(1).$$

(c) $(1-\jmath)^{\jmath} = \left(\sqrt{2}e^{-\jmath\pi/4}\right)^{\jmath} = \left(e^{\ln(\sqrt{2})}e^{-\jmath\pi/4}\right)^{\jmath} = e^{\jmath\ln(\sqrt{2})}e^{\pi/4} = e^{\pi/4}\left(\cos(\ln(\sqrt{2})) + \jmath\sin(\ln(\sqrt{2}))\right)$. Thus,

$$(1-\jmath)^{\jmath} = e^{\pi/4}\cos(\ln(\sqrt{2})) + \jmath e^{\pi/4}\sin(\ln(\sqrt{2})).$$

B.19. Letting $w=\jmath y$, $\cos(w) = \cos(\jmath y) = 0.5\left(e^{\jmath\jmath y} + e^{-\jmath\jmath y}\right) = 0.5\left(e^{-y}+e^{y}\right) = 2$. Multiplying both sides by $2e^y$ yields $1+(e^y)^2 - 4e^y = (e^y)^2 - 4e^y + 1 = 0$. This is a quadratic equation in e^y. Applying the quadratic formula yields $e^y = \frac{4\pm\sqrt{16-4}}{2} = 2\pm\sqrt{3}$. Solving for y gives $y = \ln\left(2\pm\sqrt{3}\right)$. Thus,

$$w = \jmath y = \jmath\ln\left(2\pm\sqrt{3}\right) = \pm\jmath 1.3170.$$

B.20. The general form is $x(t) = e^{-at}\cos(\omega t)$. At $t=0$, $e^{-at}=1$. Thus, a fifty percent decrease in two seconds requires $0.5 = e^{-a2}$, or $a = 0.5\ln(2)$. To oscillate three times per second requires $\omega = 6\pi$.

```
>> w = 3*2*pi; a = 0.5*log(2);
>> t = [-2:.01:2]; x = exp(-a*t).*cos(w*t);
>> plot(t,x,'k-'); xlabel('t'); ylabel('x(t)');
```

B.21. (a) $x_1(t) = \mathrm{Re}(2e^{(-1+\jmath 2\pi)t}) = 2e^{-t}\cos(2\pi t)$. This is 1Hz cosine wave that exponentially decays by a factor of $1-e^{-1} = 0.632$ every second. A signal peak is near $t=0$, where the signal has an amplitude of 2. See Figure SB.22a.

(b) $x_2(t) = \mathrm{Im}(3-e^{(1-\jmath 2\pi)t}) = e^{t}\sin(2\pi t)$. This is a 1Hz sine wave that exponentially grows by a factor of $e^1 = 2.718$ every second. A signal peak is near $t=1/4$, where the signal has an amplitude of 1.284. See Figure SB.22b.

(c) $x_3(t) = 3 - \mathrm{Im}(e^{(1-\jmath 2\pi)t}) = 3 + e^{t}\sin(2\pi t)$. This is a 1Hz sine wave that exponentially grows by a factor of $e^1 = 2.718$ every second and has an offset of 3.

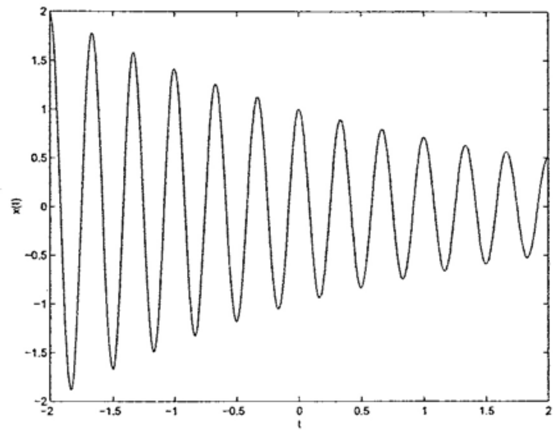

Figure SB.20: Plot of $x(t) = e^{-at}\cos(\omega t)$ for $\omega = 3(2\pi)$ and $a = 0.5\ln(2)$.

A signal peak is near $t = 1/4$, where the signal has an amplitude of 4.284. See Figure SB.22c.

B.22. (a)
```
>> t = [0:.001:3]; x_1 = 2*exp(-t).*cos(2*pi*t);
>> plot(t,x_1,'k-',t,2*exp(-t),'k:',t,-2*exp(-t),'k:');
>> xlabel('t'); ylabel('x_1(t)');
```

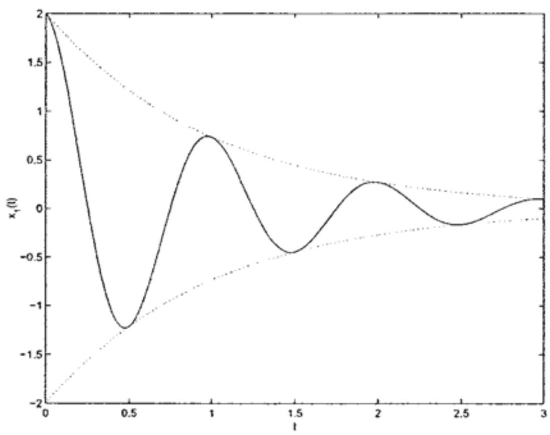

Figure SB.22a: Plot of $x_1(t) = 2e^{-t}\cos(2\pi t)$.

(b)
```
>> t = [0:.001:3]; x_2 = exp(t).*sin(2*pi*t);
>> plot(t,x_2,'k-',t,exp(t),'k:',t,-exp(t),'k:');
>> xlabel('t'); ylabel('x_2(t)');
```

(c)
```
>> t = [0:.001:3]; x_3 = 3+exp(t).*sin(2*pi*t);
>> plot(t,x_3,'k-',t,3+exp(t),'k:',t,3-exp(t),'k:');
>> xlabel('t'); ylabel('x_3(t)');
```

B.23. Since $\cos(t)$ oscillates at $\frac{1}{2\pi}$Hz, t should cover at least 2π seconds to span one period. Since $\sin(20t)$ has a period of $\frac{2\pi}{20} = 0.314$ seconds, the step size of t should be less than 0.0314 to ensure at least ten samples per period of this fastest component.

```
>> t = [0:.01:8]; x = cos(t).*sin(20*t);
```

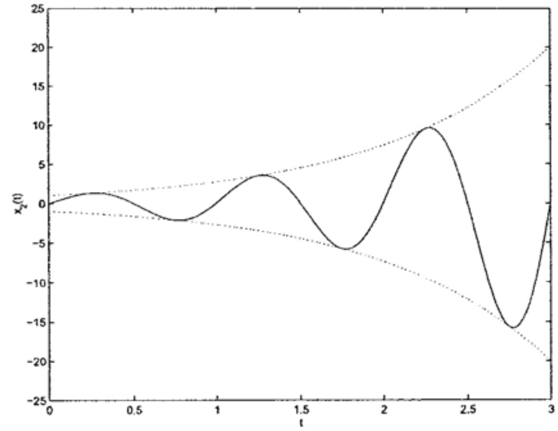

Figure SB.22b: Plot of $x_2(t) = e^t \sin(2\pi t)$.

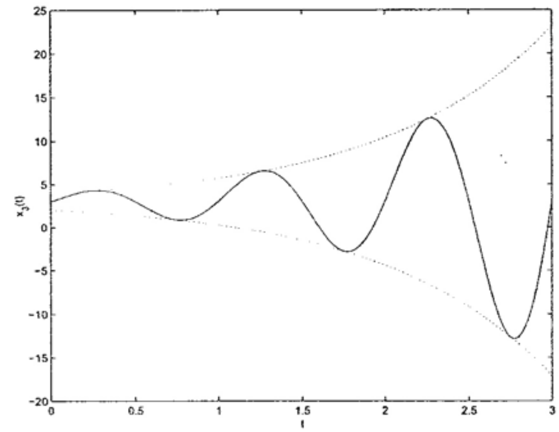

Figure SB.22c: Plot of $x_3(t) = 3 + e^t \sin(2\pi t)$.

```
>> plot(t,x,'k-'); xlabel('t'); ylabel('x(t)');
```

B.24. The highest frequency is 10Hz, so the step size of t should be 0.01 or less to provide ten samples per period of the fastest component. The lowest frequency is 1Hz, so t should span at least one second to cover one period of the slowest component.

```
>> t = [0:.005:2]; kt = (1:10)'*t; x = sum(cos(2*pi*kt));
>> plot(t,x,'k'); xlabel('t'); ylabel('x(t)');
```

B.25. There are many approximations possible for the sound of a bell. In the most simple case, we can model a bell as a decaying exponential. A small, light bell will have a high pitch and not sustain a sound for long. Thus, we might choose a base oscillation of 1kHz. A reasonably quick decay rate is obtained if the envelop decreases by 90% every second, or $e^{\ln(0.1)t}$. Thus, our bell model is $x(t) = e^{\ln(0.1)t}\cos(2\pi 1000t)$. The result, however, is somewhat "flat". Adding harmonics, such as $\cos(2\pi 2000t)$, adds richness to the sound. Furthermore, some low frequency modulation, perhaps as a result of a hand initially ringing the bell, improves the sound. For example,

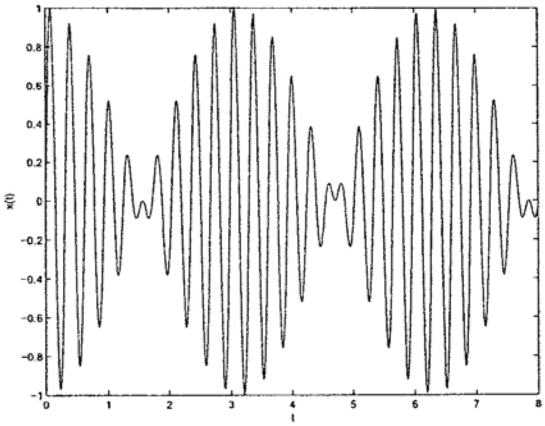

Figure SB.23: Plot of $x(t) = \cos(t)\sin(20t)$.

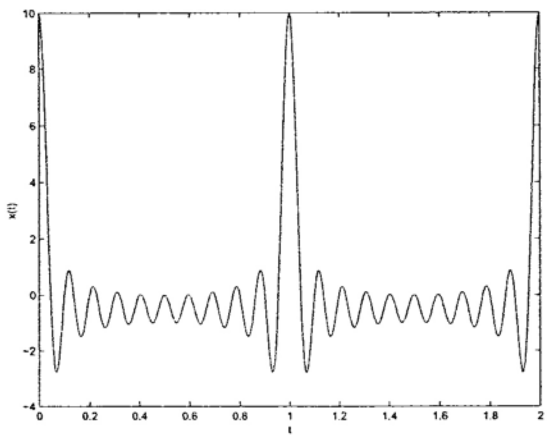

Figure SB.24: Plot of $x(t) = \sum_{k=1}^{10} \cos(2\pi kt)$.

$y(t) = e^{\ln(0.1)t}\cos(2\pi 3t)\left(\cos(2\pi 1000t) + 0.1\cos(2\pi 2000t)\right)$ sounds more natural than $x(t)$. The possibilities are endless.

```
>> t = [0:1/8000:3.5]; a = log(0.1); x = exp(a*t).*(cos(2*pi*1000*t));
>> y = exp(a*t).*(cos(2*pi*3*t).*(cos(2*pi*1000*t)+0.1*cos(2*pi*2000*t)));
>> sound([x,y],8000);
```

If a large, heavy bell is desired, the frequency and decay rates need to be reduced. For example, $z(t) = e^{\ln(0.5)t}\cos(2\pi 3t)\left(\cos(2\pi 100t) + 0.1\cos(2\pi 200t)\right)$.

```
>> t = [0:1/8000:5]; a = log(0.5);
>> z = exp(a*t).*(cos(2*pi*3*t).*(cos(2*pi*200*t)+0.1*cos(2*pi*400*t)));
>> sound(z,8000);
```

B.26. (a) To express e^{-x^2} as a Taylor series, recall that a Taylor series of e^u about zero is given by $e^u = \sum_{i=0}^{\infty} \frac{u^i}{i!}$. Substituting $-x^2$ for u yields

$$e^{-x^2} = \sum_{i=0}^{\infty} \frac{\left(-x^2\right)^i}{i!}.$$

(b) Integrating yields $\int e^{-x^2}dx = \int \sum_{i=0}^{\infty} \frac{(-1)^i x^{2i}}{i!} dx = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \int x^{2i} dx$ or

$$\int e^{-x^2} dx = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \frac{x^{2i+1}}{2i+1}.$$

(c) Since the lower limit of the definite integral is zero, it does not make any contribution. Thus $\int_0^1 e^{-x^2} dx = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \frac{x^{2i+1}}{2i+1}\Big|_{x=1} = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!(2i+1)}$. First, MATLAB is used to compute the first 10 terms of the sum.

```
>> i = 0:9; terms = (-1).^i./(gamma(i+1).*(2*i+1));
```

Next, one to ten term truncations are obtained using the MATLAB's cumulative sum command.

```
>> cumsum(terms)
```

The results are

$$(1.0000, 0.6667, 0.7667, 0.7429, 0.7475, 0.7467, 0.7468, 0.7468, 0.7468, 0.7468).$$

At a seven-term truncation, the result appears to converge to four digits.

B.27. (a) To express e^{-x^3} as a Taylor series, recall that a Taylor series of e^u about zero is given by $e^u = \sum_{i=0}^{\infty} \frac{u^i}{i!}$. Substituting $-x^3$ for u yields

$$e^{-x^3} = \sum_{i=0}^{\infty} \frac{\left(-x^3\right)^i}{i!}.$$

(b) Integrating yields $\int e^{-x^3}dx = \int \sum_{i=0}^{\infty} \frac{(-1)^i x^{3i}}{i!} dx = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \int x^{3i} dx$ or

$$\int e^{-x^3} dx = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \frac{x^{3i+1}}{3i+1}.$$

(c) Since the lower limit of the definite integral is zero, it does not make any contribution. Thus $\int_0^1 e^{-x^3} dx = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \frac{x^{3i+1}}{3i+1}\Big|_{x=1} = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!(3i+1)}$. First, MATLAB is used to compute the first 10 terms of the sum.

```
>> i = 0:9; terms = (-1).^i./(gamma(i+1).*(3*i+1));
```

Next, one to ten term truncations are obtained using the MATLAB's cumulative sum command.

```
>> cumsum(terms)
```

The results are

$$(1.0000, 0.7500, 0.8214, 0.8048, 0.8080, 0.8074, 0.8075, 0.8075, 0.8075, 0.8075).$$

At a seven-term truncation, the result appears to converge to four digits.

B.28. (a) To express $\cos(x^2) = 0.5\left(e^{jx^2} + e^{-jx^2}\right)$ as a Taylor series, recall that a Taylor series of e^u about zero is given by $e^u = \sum_{i=0}^{\infty} \frac{u^i}{i!}$. Substituting $\pm jx^2$ for u yields

$$\cos(x^2) = \sum_{i=0}^{\infty} 0.5 \left(\frac{\left(jx^2\right)^i}{i!} + \frac{\left(-jx^2\right)^i}{i!} \right).$$

(b) Integrating yields $\int \cos(x^2)dx = \int \sum_{i=0}^{\infty} 0.5\left(\frac{(\jmath x^2)^i}{i!} + \frac{(-\jmath x^2)^i}{i!}\right)dx = \sum_{i=0}^{\infty} 0.5\frac{(\jmath)^i(1+(-1)^i)}{i!}\int (x^2)^i dx$ or

$$\int \cos(x^2)dx = \sum_{i=0}^{\infty} 0.5\frac{(\jmath)^i(1+(-1)^i)}{i!}\frac{x^{2i+1}}{2i+1}.$$

(c) Since the lower limit of the definite integral is zero, it does not make any contribution. Thus $\int_0^1 \cos(x^2)dx = \sum_{i=0}^{\infty} 0.5\frac{(\jmath)^i(1+(-1)^i)}{i!}\frac{x^{2i+1}}{2i+1}\Big|_{x=1} = \sum_{i=0}^{\infty} 0.5\frac{(\jmath)^i(1+(-1)^i)}{i!(2i+1)}$. First, MATLAB is used to compute the first 10 terms of the sum.

```
>> i = 0:9; terms = 0.5*(j).^i.*(1+(-1).^i)./(gamma(i+1).*(2*i+1));
```

Next, one to ten term truncations are obtained using the MATLAB's cumulative sum command.

```
>> cumsum(terms)
```

The results are

$$(1.0000, 1.0000, 0.9000, 0.9000, 0.9046, 0.9046, 0.9045, 0.9045, 0.9045, 0.9045).$$

At a seven-term truncation, the result appears to converge to four digits.

B.29. (a) Using synthetic division, express $f_1(x) = \frac{1}{2-x^2} = \frac{1}{2} + \frac{1}{4}x^2 + \frac{1}{8}x^4 + \frac{1}{16}x^6 + \cdots$. Thus,

$$f_1(x) = \sum_{i=0}^{\infty}\left(\frac{1}{2}\right)^{i+1} x^{2i}.$$

(b) Rewrite as $f_2(x) = (0.5)^x = e^{-\ln(2)x}$. Recall that a Taylor series of e^u about zero is given by $e^u = \sum_{i=0}^{\infty}\frac{u^i}{i!}$. Substituting $-\ln(2)x$ for u yields

$$f_2(x) = \sum_{i=0}^{\infty}\frac{(-\ln(2)x)^i}{i!}.$$

B.30. (a) Begin by choosing a point on the unit circle, $w = e^{\jmath\Omega}$. Multiplying w by itself yields $ww = w^2 = e^{\jmath 2\Omega}$. Taking this result and again multiplying by w yields $ww^2 = w^3 = e^{\jmath 3\Omega}$. At step n, the result is $w^n = e^{\jmath n\Omega}$. From Euler's identity, we know $w^n = e^{\jmath n\Omega} = \cos(n\Omega) + \jmath\sin(n\Omega)$. The process does indeed provide the desired quadrature sinusoids: the real part provides the cosine term and the imaginary part yields the sine term.

(b) To produce a periodic signal, Ω needs to be a rational multiple of 2π. Most simply, choose $\Omega = 2\pi/N$, where N is the number of points computed per oscillation of each sinusoid. For reasonable quality sinusoids, N should be some moderately large integer, say 10 or 20. Although the quality of the sinusoids increases as N is increased, the required processing speed also increases with N. Thus, N represents a compromise between signal quality and processor speed. Taking $N = 20$, for example, yields $w = e^{\jmath\pi/10}$. In this case, only $\frac{1}{100000N} = 500e-9$ seconds (500ns) are available to process each sample. This is feasible with current processor technologies.

(c) Although not required by the procedure, a vector $x[n]$ is maintained so that the signal outputs can be plotted.

```
>> N = 20; w = exp(j*2*pi/N); w_n = w;
>> I = 40; x = zeros(1,I); x(1) = w_n;
>> for i = 1:I; x(i) = w_n; w_n = w_n*w; end
>> plot([1:I],real(x),'k-',[1:I],imag(x),'k:');
>> xlabel('n'); ylabel('Amplitude');
```

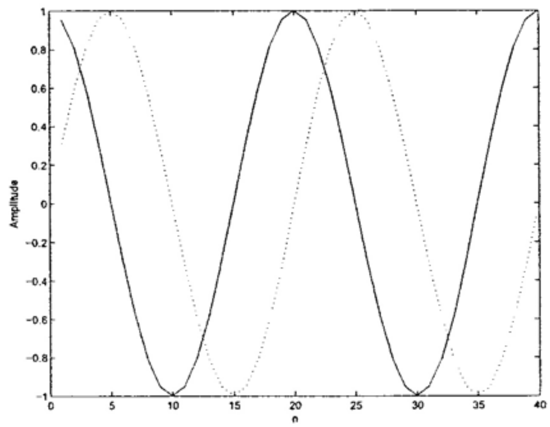

Figure SB.30c: Simulation to generate quadrature sinusoids.

(d) To work, this procedure requires several assumptions. First, we assume N is chosen large enough to provide good-quality sinusoids yet provide ample time during each step to compute the next value. For periodicity, the frequency Ω needs to be a rational multiple of 2π. Each of these assumptions can generally be met. However, there are at least two limitations that may affect the suitability of this procedure:

i. Since digital processors represent numbers with a finite number of bits, there is often an error associated with representing w. Instead of w, the computer stores $w + \Delta$. Due to the iterative nature of the procedure, the error grows with time. Generally, if $|w + \Delta| > 1$ then the signals will exponentially grow and if $|w + \delta| < 1$ the signals will exponentially decay. This limitation can prevent the procedure from working correctly over an indefinite time period.

ii. For the output signals to be truly periodic, the processor must take exactly the same amount of time between steps. This is impossible; timing errors are always present. Additionally, if the desired output frequency is not divisible by the processor clock speed, the resulting signals with either not be truly periodic, have slight frequency errors, or both.

B.31. First, the system of equations is written in matrix form.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = Ax = \begin{bmatrix} 1 \\ 3 \\ -3 \end{bmatrix}.$$

$|A| = 0 + 3 - 1 - (2 - 3 - 0) = 3.$

(a) $\begin{vmatrix} 1 & 1 & 1 \\ 3 & 2 & 3 \\ -3 & -1 & 0 \end{vmatrix} = 0 - 9 - 3 - (-6 - 3 - 0) = -3$. Thus,

$$x_1 = \frac{-3}{|A|} = \frac{-3}{3} = -1.$$

The same result is obtained in MATLAB by

```
>> A = [1 1 1;1 2 3;1 -1 0];
>> x_1 = det([[1;3;-3],A(:,2:3)])/det(A)
x_1 = -1
```

(b) $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 3 & 3 \\ 1 & -3 & 0 \end{vmatrix} = 0 + 3 - 3 - (3 - 9 - 0) = 6$. Thus,

$$x_2 = \frac{6}{|A|} = \frac{6}{3} = 2.$$

The same result is obtained in MATLAB by

```
>> A = [1 1 1;1 2 3;1 -1 0];
>> x_2 = det([A(:,1),[1;3;-3],A(:,3)])/det(A)
x_2 = 2
```

(c) $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & -1 & -3 \end{vmatrix} = -6 - 1 + 3 - (2 - 3 - 3) = 0$. Thus,

$$x_3 = \frac{0}{|A|} = \frac{0}{3} = 0.$$

The same result is obtained in MATLAB by

```
>> A = [1 1 1;1 2 3;1 -1 0];
>> x_3 = det([A(:,1:2),[1;3;-3]])/det(A)
x_3 = 0
```

B.32. (a) A matrix representation is

$$\begin{bmatrix} a & b \\ d & e \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \boldsymbol{Ax} = \boldsymbol{y} = \begin{bmatrix} c \\ f \end{bmatrix}.$$

(b) By inspection, $x_1 = 3$ and $x_2 = -2$ can be obtained by

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}.$$

Thus, $a = 1, b = 0, c = 3, d = 0, e = 1$, and $f = -2$ is one possible set of constants. These constants are not unique. Any linear combination of the rows yields the same solution set. For example, $a = 2, b = 0, c = 6, d = 1, e = 1$, and $f = 1$ also works.

To ensure unique values of x_1 and x_2, the matrix $\boldsymbol{A}$ must be full rank.

(c) For no solutions to exist, the matrix $\boldsymbol{A}$ must be rank deficient, and $[\boldsymbol{A}, \boldsymbol{y}]$ must increase the rank of $\boldsymbol{A}$ by one. For example,

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The rank of $\boldsymbol{A}$ is one and the rank of $[\boldsymbol{A}, \boldsymbol{y}]$ is two. Thus, there are no solutions. MATLAB verifies the desired ranks are obtained.

```
>> A = [1 1;2 2]; y = [1;1];
>> [rank(A), rank([A,y])]
ans =
     1     2
```

(d) For an infinite number of solutions to exist, the matrix $\boldsymbol{A}$ must be rank deficient, and $[\boldsymbol{A}, \boldsymbol{y}]$ must not increase the rank of $\boldsymbol{A}$. For example,

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

The rank of $\boldsymbol{A}$ is one and the rank of $[\boldsymbol{A}, \boldsymbol{y}]$ is also one. Thus, there are an infinite number of solutions. MATLAB verifies the desired ranks are obtained.

```
>> A = [1 1;2 2]; y = [1;2];
>> [rank(A), rank([A,y])]
ans =
     1     1
```

B.33. The system of equations is first written in matrix form.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \boldsymbol{A}\boldsymbol{x} = \begin{bmatrix} 4 \\ 2 \\ 0 \\ -2 \end{bmatrix}.$$

Next, the result is obtained using MATLAB.

```
>> A = [1 1 1 1;1 1 1 -1;1 1 -1 -1;1 -1 -1 -1];
>> x = A\[4;2;0;-2]
x =  1
     1
     1
     1
```

That is, $x_1 = 1$, $x_2 = 1$, $x_3 = 1$, and $x_4 = 1$.

B.34. First, the system of equations is written in matrix form.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -2 & 3 & 0 \\ 1 & 0 & -1 & 7 \\ 0 & -2 & 3 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \boldsymbol{A}\boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$

The result is obtained using MATLAB.

```
>> A = [1 1 1 1;1 -2 3 0;1 0 -1 7;0 -2 3 -4];
>> x = A\[1;2;3;4]
```

```
x = -30.0000
      8.0000
     16.0000
      7.0000
```

That is, $x_1 = -30$, $x_2 = 8$, $x_3 = 16$, and $x_4 = 7$.

B.35. (a) $H_1(s) = \frac{s^2+5s+6}{s^3+s^2+s+1} = \frac{s^2+5s+6}{(s-\jmath)(s+\jmath)(s+1)} = \frac{k_1}{s-\jmath} + \frac{k_2}{s+\jmath} + \frac{k_3}{s+1}$. Using the method of residues, $k_1 = \left.\frac{s^2+5s+6}{(s+\jmath)(s+1)}\right|_{s=\jmath} = \frac{5(1+\jmath)}{2\jmath(1+\jmath)} = -2.5\jmath$. Since the system is real, $k_2 = k_1^* = 2.5\jmath$. Lastly, $k_3 = \left.\frac{s^2+5s+6}{s^2+1}\right|_{s=-1} = 1$. Thus,

$$H_1(s) = \frac{1}{s+1} + \frac{-2.5\jmath}{s-\jmath} + \frac{2.5\jmath}{s+\jmath} = \frac{1}{s+1} + \frac{5}{s^2+1}.$$

(b) $H_2(s) = \frac{1}{H_1(s)} = \frac{s^3+s^2+s+1}{s^2+5s+6} = s - 4 + \frac{15s+25}{(s+2)(s+3)} = s - 4 + \frac{k_1}{s+2} + \frac{k_2}{s+3}$. Using the method of residues, $k_1 = \left.\frac{15s+25}{s+3}\right|_{s=-2} = -5$ and $k_2 = \left.\frac{15s+25}{s+2}\right|_{s=-3} = 20$. Thus,

$$H_2(s) = s - 4 + \frac{-5}{s+2} + \frac{20}{s+3}.$$

(c) $H_3(s) = \frac{1}{(s+1)^2(s^2+1)} = \frac{1}{(s+1)^2(s+\jmath)(s-\jmath)} = \frac{k_1}{s-\jmath} + \frac{k_2}{s+\jmath} + \frac{\tilde{a}_0}{(s+1)^2} + \frac{\tilde{a}_1}{(s+1)}$. Using the method of residues, $k_1 = \left.\frac{1}{(s+1)^2(s+\jmath)}\right|_{s=\jmath} = \frac{1}{(1+\jmath2-1)(\jmath2)} = -0.25$. Since the corresponding roots are complex conjugates, $k_2 = k_1^* = -0.25$. $\tilde{a}_0 = \left.\frac{1}{s^2+1}\right|_{s=-1} = 0.5$ and $\tilde{a}_1 = \frac{d}{ds}(s^2+1)^{-1}\big|_{s=-1} = -(s^2+1)^{-2}(2s)\big|_{s=-1} = \frac{-2}{-4} = 0.5$. Thus,

$$H_3(s) = \frac{-0.25}{s-\jmath} + \frac{-0.25}{s+\jmath} + \frac{0.5}{(s+1)^2} + \frac{0.5}{(s+1)}.$$

(d) $H_4(s) = \frac{s^2+5s+6}{3s^2+2s+1} = \frac{1}{3} + \frac{13s/3+17/3}{3s^2+2s+1} = \frac{13s/9+17/9}{s^2+2s/3+1/3}$. In some cases, this form is sufficient. A complete partial fraction expansion, however, requires the denominator roots $s = \frac{-2/3\pm\sqrt{4/9-4/3}}{2} = \frac{-1\pm\jmath\sqrt{2}}{3} = -0.3333 \pm 0.4714\jmath$. Thus, $H_4(s) = \frac{1}{3} + \frac{k_1}{s-(-1-\jmath\sqrt{2})/3} + \frac{k_2}{s-(-1+\jmath\sqrt{2})/3}$. Using the method of residues, $k_1 = \left.\frac{13s/9+17/9}{s-(-1+\jmath\sqrt{2})/3}\right|_{s=(-1-\jmath\sqrt{2})/3} = 0.7222 + 1.4928\jmath$. Since the system is real, $k_2 = k_1^* = 0.7222 - 1.4928\jmath$. Thus,

$$H_4(s) = \frac{1}{3} + \frac{0.7222 + 1.4928\jmath}{s + 0.3333 + 0.4714\jmath} + \frac{0.7222 - 1.4928\jmath}{s + 0.3333 - 0.4714\jmath}.$$

B.36. The MATLAB `residue` command computes the partial fraction expansion of a rational function by providing three quantities: the residues, the poles, and the direct terms.

(a)
```
>> [r,p,k] = residue([1 5 6],[1 1 1 1])
r =  1.0000
     0.0000 - 2.5000i
     0.0000 + 2.5000i
p = -1.0000
    -0.0000 + 1.0000i
```

```
   -0.0000 - 1.0000i
k = []
```

Thus,

$$H_1(s) = \frac{1}{s+1} + \frac{-2.5\jmath}{s-\jmath} + \frac{2.5\jmath}{s+\jmath} = \frac{1}{s+1} + \frac{5}{s^2+1}.$$

(b)
```
>> [r,p,k] = residue([1 1 1 1],[1 5 6])
r = 20.0000
    -5.0000
p = -3.0000
    -2.0000
k =  1    -4
```

Thus,

$$H_2(s) = \frac{20}{s+3} + \frac{-5}{s+2} + s - 4.$$

(c)
```
>> [r,p,k] = residue(1,poly([-1,-1,j,-j]))
r =   0.5000
      0.5000
     -0.2500 - 0.0000i
     -0.2500 + 0.0000i
p =  -1.0000
     -1.0000
      0.0000 + 1.0000i
      0.0000 - 1.0000i
k =  []
```

Thus,

$$H_3(s) = \frac{0.5}{(s+1)} + \frac{0.5}{(s+1)^2} + \frac{-0.25}{s-\jmath} + \frac{-0.25}{s+\jmath}.$$

(d)
```
>> [r,p,k] = residue([1 5 6],[3 2 1])
r =   0.7222 - 1.4928i
      0.7222 + 1.4928i
p =  -0.3333 + 0.4714i
     -0.3333 - 0.4714i
k =   0.3333
```

Thus,

$$H_4(s) = \frac{1}{3} + \frac{0.7222 - 1.4928\jmath}{s + 0.3333 - 0.4714\jmath} + \frac{0.7222 + 1.4928\jmath}{s + 0.3333 + 0.4714\jmath}.$$

B.37. First, express both sides of the expression with a common denominator $F(s) = \frac{s}{(s+1)^3} = \frac{a_0}{(s+1)^3} + \frac{a_1}{(s+1)^2} + \frac{a_2}{(s+1)} = \frac{a_0+a_1(s+1)+a_2(s+1)^2}{(s+1)^3} = \frac{a_2s^2+(a_1+2a_2)s+(a_0+a_1+a_2)}{(s+1)^3}$. Equating the coefficients of s^2 yields $a_2 = 0$. Thus $(a_1 + 2a_2) = a_1 = 1$. Finally $a_0 + a_1 + a_2 = a_0 + 1 + 0 = 0$ implies that $a_0 = -1$.

$$a_0 = -1, a_1 = 1, \text{ and } a_2 = 0.$$

B.38. Many solutions are possible to this problem, but the procedure is the same in each case. Consider a fictitious phone number 555-5555. Then, $H_N(s) = \frac{5s^2+5s+5+5s^{-1}}{5s^2+5s+5}\frac{s}{s} = \frac{5s^3+5s^2+5s+5}{5s^3+5s^2+5s+0}$. The partial fraction expansion of $H_N(s)$ is obtained using the MATLAB residue command.

```
>> [r,p,k] = residue([5 5 5 5],[5 5 5 0])
```

```
r = -0.5000 + 0.2887i
    -0.5000 - 0.2887i
     1.0000
p = -0.5000 + 0.8660i
    -0.5000 - 0.8660i
     0
k =  1
```

Thus,

$$H_N(s) = 1 + \frac{-0.5000 + 0.2887\jmath}{s + 0.5000 - 0.8660\jmath} + \frac{-0.5000 - 0.2887\jmath}{s + 0.5000 + 0.8660\jmath} + \frac{1}{s}.$$

B.39. (a)

```
>> omega = linspace(-pi,pi,201);
>> fr = cos(omega); fi = 0.1*sin(2*omega);
>> plot(fr,fi,'k-'); xlabel('Re(f)'); ylabel('Im(f)');
>> axis([-1.1 1.1 -1.1 1.1]); axis equal;
```

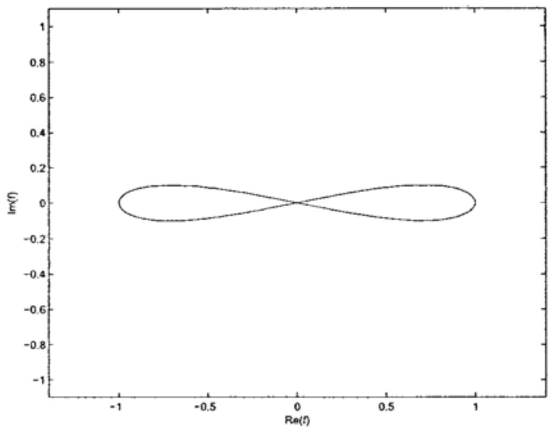

Figure SB.39a: Lissajous figure resembling horizontal propeller.

(b) Multiplying w by $e^{\jmath\theta}$ adds θ to the angle of w and thereby rotates w by θ. Also, $we^{\jmath\theta} = (x+\jmath y)(\cos(\theta)+\jmath\sin(\theta)) = (x\cos(\theta)-y\sin(\theta))+\jmath(x\sin(\theta)+y\cos(\theta))$. Furthermore, $\mathbf{R}\mathbf{w} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\cos(\theta) - y\sin(\theta) \\ x\sin(\theta) + y\cos(\theta) \end{bmatrix}$. Thus, $\mathbf{R}\mathbf{w}$ and $we^{\jmath\theta}$ are equivalent, and $\mathbf{R}\mathbf{w}$ rotates w by θ.

(c)

```
>> theta = 10*pi/180; R = [cos(theta) -sin(theta);sin(theta) cos(theta)];
>> f = [fr;fi]; f = R*f;
>> plot(f(1,:),f(2,:),'k-'); xlabel('Re(Rf)'); ylabel('Im(Rf)');
>> axis([-1.1 1.1 -1.1 1.1]); axis equal;
```

(d) If Rf rotates f by θ, then $RRf = R(Rf)$ rotates f by 2θ. Similarly, $RRRf$ rotates f by 3θ. In general, $(R^N)f$ rotates f by $N\theta$.

(e) As suggested in B.39b, multiplying $f(\omega)$ by the function $e^{\jmath\theta}$ simply rotates f by θ. For example, the previous plot is also obtained by

```
>> f = fr + j*fi; f = f*exp(j*theta);
>> plot(real(f),imag(f),'k-');
>> xlabel('Re(fe^{j\theta})'); ylabel('Im(fe^{j\theta})');
>> axis([-1.1 1.1 -1.1 1.1]); axis equal;
```

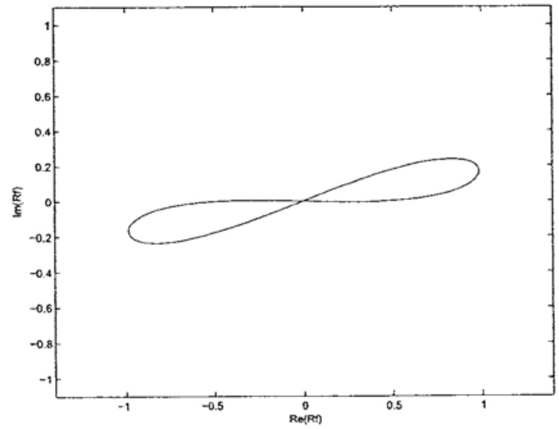

Figure SB.39c: Lissajous figure rotated 10 degrees CCW.

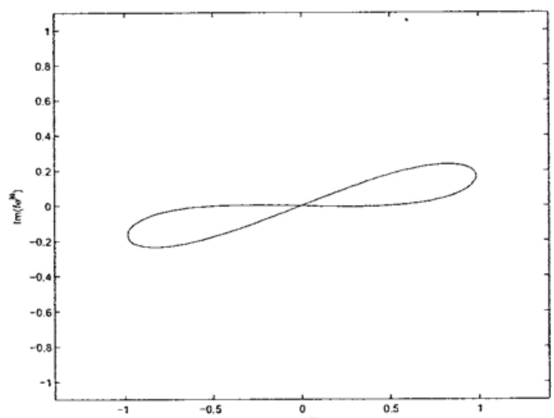

Figure SB.39e: Lissajous figure rotated by alternate method.