AT12 - Transformada de Laplace

* Propriedades

① Deslocamento no tempo

$$x(t) \longleftrightarrow X(s)$$

$$\underset{\uparrow \text{delay}}{x(t-t_0)} \longleftrightarrow X(s)\,\underbrace{e^{-st_0}}$$

② Deslocamento na frequência

$$x(t) \longleftrightarrow X(s)$$

$$x(t)\,e^{s_0 t} \longleftrightarrow X(s - s_0)$$

③ Diferenciação no tempo

$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s)$$

$$\frac{dx(t)}{dt} \overset{\mathcal{L}}{\longleftrightarrow} s\,\overset{\mathcal{L}\{x(t)\}}{X(s)} - \overset{x(t)|_{t=0}}{x(0^-)}$$

dom. freq.

$$\frac{d^2x(t)}{dt^2} \overset{\mathcal{L}}{\longleftrightarrow} s^2 X(s) - s\,\underbrace{x(0^-)}_{x(t)|_{t=0}} - \underbrace{\dot{x}(0^-)}_{\dot{x}(t)|_{t=0}}$$

$$\frac{d^3x(t)}{dt^3} \overset{\mathcal{L}}{\longleftrightarrow} s^3 X(s) - s^2 x(0^-) - s\,\dot{x}(0^-) - \ddot{x}(0^-)$$

④ Propriedade de integração no tempo

$$x(t) \overset{\mathcal{L}\{\}}{\longleftrightarrow} X(s)$$

↑ dom. Freq

$$\int_{0^-}^{t} x(\tau)\,d\tau \overset{\mathcal{L}\{\}}{\longleftrightarrow} \frac{X(s)}{s}$$

Prova: Seja $g(t) = \int_{0^-}^{t} x(\tau)\, d\tau$, $g(0^-) = 0$

$$\therefore \quad x(t) = \frac{d}{dt} g(t),$$

$$g(t) \xrightarrow{\mathcal{L}\{\}} G(s)$$

$$x(t) = \frac{d}{dt} g(t) \xrightarrow{\mathcal{L}} X(s) = s\,G(s) - \cancel{g(0^-)}^{\,0} = s\,G(s)$$

$$\therefore \boxed{X(s) = s\,G(s)}$$

$$x(t) \xleftrightarrow{\mathcal{L}\{\}} X(s)$$

$$\underbrace{\int_{0^-}^{t} x(\tau)\, d\tau}_{g(t)} \longleftrightarrow \underbrace{\frac{X(s)}{s}}_{G(s)}$$

$$g(t) \xleftrightarrow{\mathcal{L}\{\}} G(s)$$

⑤ Escalamento

$$x(t) \xleftrightarrow{\mathcal{L}\{\}} X(s)$$

$$x(at) \longleftrightarrow \frac{1}{a} X\left(\frac{s}{a}\right) \quad , \; a > 0$$

$a > 1 \rightsquigarrow$ compressão

$a < 1 \rightarrow$ expansão

Exemplo 1) Sabendo que

$$\cos(bt)\, u(t) \xleftrightarrow{\mathcal{L}\{\}} \frac{s}{s^2 + b^2}$$

Determine a transformada de Laplace p/ $e^{-at} \cdot \underbrace{\cos(bt)\, u(t)}$.

Solução: Propriedade do deslocamento na frequência

$$x(t)\, e^{s_0 t} \xleftrightarrow{\mathcal{L}\{\}} X(s - s_0)$$

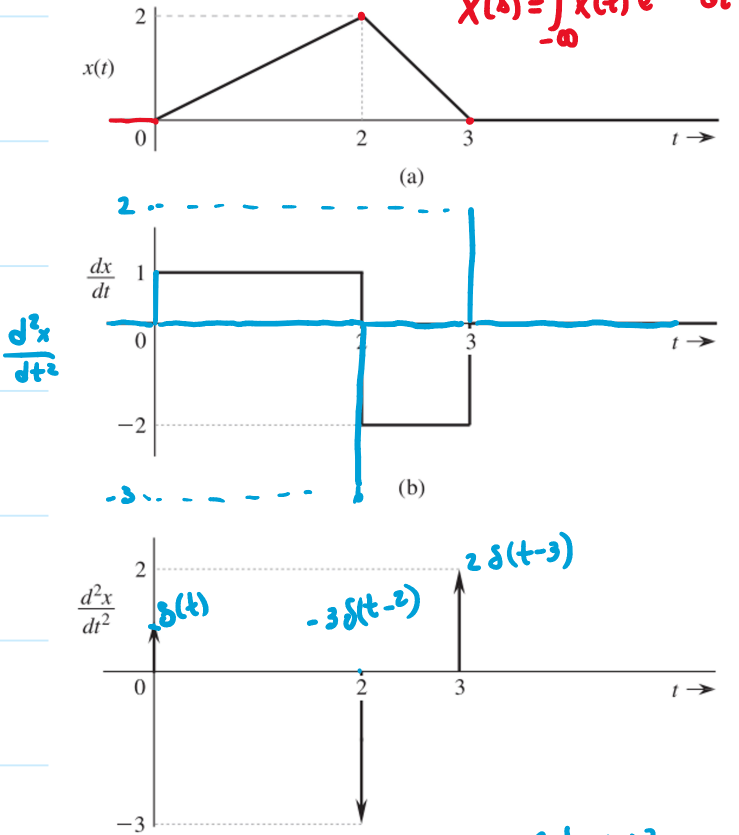
$s_0 = \underline{-a}$, $(s - s_0) = (s + a)$

$$\therefore \quad e^{-at} \cos(bt)\, u(t) \longleftrightarrow \frac{(s+a)}{(s+a)^2 + b^2}$$

$$(s+a)^2 + b^2$$

Exemplo 2) Determine $X(s) = \mathcal{L}\{x(t)\}$, por meio das propriedades de diferenciação e deslocamento no tempo.

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$



$$\mathcal{L}\{\delta(t)\} = 1$$

$$\frac{d^2 x(t)}{dt} = \delta(t) - 3\delta(t-2) + 2\delta(t-3)$$

$$\mathcal{L}\left\{\frac{d^2 x(t)}{dt}\right\} = \mathcal{L}\left\{\delta(t) - 3\delta(t-2) + 2\delta(t-3)\right\}$$

$$s^2 X(s) - s\,\underbrace{x(0^-)} - \underbrace{\dot{x}(0^-)} = 1 - 3e^{-2s} + 2e^{-3s} \rightarrow \text{dom. freq. } (s)$$

sendo $x(0^-) = \dot{x}(0^-) = 0$

$$s^2 X(s) = 1 - 3e^{-2s} + 2e^{-3s}$$

$$X(s) = \frac{1}{s^2}\left(1 - 3e^{-2s} + 2e^{-3s}\right)$$

$$X(s) = \mathcal{L}\{x(t)\}$$

⑥ Convolução no tempo e na frequência

$$x_1(t) \overset{\mathcal{L}}{\longleftrightarrow} X_1(s) \qquad x_2(t) \overset{\mathcal{L}}{\longleftrightarrow} X_2(s)$$

$$x_1(t) * x_2(t) \longleftrightarrow X_1(s) \cdot X_2(s)$$

$$x_1(t) \cdot x_2(t) \longleftrightarrow \frac{1}{2\pi j}\left[X_1(s) * X_2(s)\right]$$

→ Discussão e análise p/ sistemas

SLIT

$$\underset{e^{st}}{\cancel{x(t)}} \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

F.T.

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, \underbrace{x(t-\tau)}\, d\tau$$

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, \underset{e^{st} \cdot e^{-s\tau}}{e^{s(t-\tau)}}\, d\tau$$

$$y(t) = e^{st} \cdot \underbrace{\int_{-\infty}^{\infty} h(\tau)\, e^{-s\tau}\, d\tau}_{H(s)}$$

$$y(t) = H(s) \cdot e^{st}$$

↓

Função transferência

$$H(s) = \frac{\text{Sinal de saída}}{\text{Sinal de entrada}}$$

Ou

SLIT

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) \quad \xrightarrow{\mathcal{L}\{\}} \quad X(s) \longrightarrow \boxed{H(s)} \longrightarrow Y(s)$$

$$y(t) = h(t) * x(t) \quad \xrightarrow{\mathcal{L}\{\}} \quad Y(s) = H(s)\, X(s)$$

$$\therefore\ H(s) = \frac{Y(s) \to \text{saída}}{X(s) \to \text{entrada}}$$

$$H(s) = \frac{\mathcal{L}\{\text{Resp. de estado nulo}\}}{\mathcal{L}\{\text{Sinal de entrada}\}}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}$$

Exemplo 3) Determine $C(s) = \mathcal{L}\{c(t)\}$, $c(t) = \underbrace{e^{at}u(t)} * \underbrace{e^{bt}u(t)}$

Solução: $x_1(t) * x_2(t) \longleftrightarrow X_1(s)\,X_2(s)$

$$x_1(t) = e^{at}u(t) \longleftrightarrow \frac{1}{s-a}$$

$$x_2(t) = e^{bt}u(t) \longleftrightarrow \frac{1}{s-b}$$

$c(t) = x_1(t) * x_2(t)$

$$C(s) = X_1(s)\cdot X_2(s) = \frac{1}{(s-a)}\cdot\frac{1}{(s-b)} = \frac{k_1}{(s-a)} + \frac{k_2}{(s-b)}$$

$$k_1 = (s-a)\,C(s)\Big|_{s=a} = \frac{1}{a-b}$$

$$k_2 = (s-b)\,C(s)\Big|_{s=b} = \frac{1}{b-a} = -\frac{1}{(a-b)}$$

$$\therefore \quad C(s) = \frac{1}{a-b}\left[\frac{1}{(s-a)} - \frac{1}{(s-b)}\right]$$

$\mathcal{L}^{-1}\{\}$

$$c(t) = \frac{1}{a-b}\left(e^{at} - e^{bt}\right)u(t)$$

* Teorema de valor final

$$x(\infty) = \lim_{t\to\infty} x(t) \ ?$$

$$x(t)\Big|_{t\to\infty} = \lim_{s\to 0} s\,X(s) \quad , \quad \text{se } \exists\ \mathcal{L}\{x(t)\}, \mathcal{L}\{\dot{x}(t)\}$$

* Teorema de valor inicial

$$x(t)\Big| = \lim \ x(t) \ ?$$

* Teorema do valor inicial

$$x(t)\Big|_{t\to 0^+} = \lim_{t\to 0^+} x(t) \ ?$$

$$x(t)\Big|_{t\to 0^+} = \lim_{s\to\infty} s\,X(s)$$

Exemplo 5) Determine o valor inicial e final de y(t), sendo

$$Y(s) = \frac{10(2s+3)}{s(s^2+2s+5)}$$

Solução:

$$y(0^+) = \lim_{s\to\infty} s\,Y(s) = \lim_{s\to\infty} \cancel{s}\,\frac{10(2s+3)}{\cancel{s}(s^2+2s+5)} = 0$$

$$y(\infty) = \lim_{s\to 0} s\,Y(s) = \lim_{s\to 0} \cancel{s}\,\frac{10(2s+3)}{\cancel{s}(s^2+2s+5)} = \frac{30}{5} = 6$$

* Solução de equações diferenciais

$$\frac{d^k y(t)}{dt^k} \longleftrightarrow s^k Y(s)$$

$$\therefore \quad y(t) = \mathcal{L}^{-1}\{Y(s)\}$$

Exemplo 6) Resolva a equação diferencial

$$(D^2 + 5D + 6)\,y(t) = (D+1)\,x(t)$$

para condições iniciais $y(0^-) = 2$ e $\dot{y}(0^-) = 1$ e $x(t) = e^{-4t}u(t)$.

Solução)

$$\frac{d^2y(t)}{dt} + 5\frac{dy(t)}{dt} + 6y(t) = \frac{dx(t)}{dt} + x(t)$$

Se $\quad y(t) \longleftrightarrow Y(s)$

$$\frac{dy(t)}{dt} \longleftrightarrow sY(s) - y(0^-) = sY(s) - 2$$

$$\frac{d^2y(t)}{dt^2} \longleftrightarrow s^2Y(s) - s\,y(0^-) - \dot{y}(0) = s^2Y(s) - 2s - 1$$

E para

$$x(t) = e^{-4t}u(t) \longleftrightarrow X(s) = \frac{1}{s+4}$$

$$\frac{dx(t)}{dt} \longleftrightarrow sX(s) - x(0^-)$$

$$= \frac{s}{s+4} - 0$$

$$= \frac{s}{s+4}$$

Portanto,

$$\mathcal{L}\left\{\frac{d^2y(t)}{dt} + 5\frac{dy(t)}{dt} + 6y(t)\right\} = \mathcal{L}\left\{\frac{dx(t)}{dt} + x(t)\right\}$$

$$\left[s^2Y(s) - 2s - 1\right] + 5\left[sY(s) - 2\right] + 6Y(s) = \frac{s}{s+4} + \frac{1}{s+4}$$

$$(s^2 + 5s + 6)Y(s) - (2s + 11) = \frac{s+1}{s+4}$$

$$(s^2 + 5s + 6)Y(s) = (2s + 11) + \frac{s+1}{s+4}$$

COND. INI $\qquad$ ENTRADA

$$Y(s) \quad (2s+11) \qquad s+1$$

$$Y(s) = \frac{\overbrace{(2s+11)}^{\text{COND. INI}}}{\underbrace{(s^2+5s+6)}_{\text{Comp. Entrada nula}}} + \frac{\overbrace{s+1}^{\text{ENTRADA}}}{\underbrace{(s^2+5s+6)(s+4)}_{\text{Comp. Estado nulo}}}$$

$$Y(s) = \frac{(2s+11)(s+4) + s+1}{\underbrace{(s^2+5s+6)}\underbrace{(s+4)}}$$

$$Y(s) = \frac{2s^2+20s+45}{(s+2)(s+3)(s+4)} = \frac{K_1}{(s+2)} + \frac{K_2}{(s+3)} + \frac{K_3}{(s+4)}$$

$$Y(s) = \frac{13/2}{(s+2)} - \frac{3}{(s+3)} - \frac{3/2}{(s+4)}$$

$\mathcal{L}^{-1}\{\}$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \left(\tfrac{13}{2}e^{-2t} - 3e^{-3t} - \tfrac{3}{2}e^{-4t}\right)u(t)$$

↳ Comentários

* Comp. de ent. nula e estado nulo

$$Y(s) = \frac{(2s+11)}{\underbrace{(s^2+5s+6)}_{\text{Comp. Entrada nula}}} + \frac{s+1}{\underbrace{(s^2+5s+6)(s+4)}_{\text{Comp. Estado nulo}}}$$

$$Y(s) = \left[\frac{7}{s+2} - \frac{5}{s+3}\right] + \left[\frac{-1/2}{s+2} + \frac{2}{s+3} - \frac{3/2}{s+4}\right]$$

$\mathcal{L}^{-1}\{\}$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \underbrace{\left(7e^{-2t} - 5e^{-3t}\right)}_{\text{Resposta de entrada nula}} + \underbrace{\left(-\tfrac{1}{2}e^{-2t} + 2e^{-3t} - \tfrac{3}{2}e^{-4t}\right)}_{\text{Resposta de estado nulo}}$$

* Resposta de estado nulo

Dado um SLIT de ordem N

$$Q(D)\, y(t) = P(D)\, x(t)$$

$$(D^N + a_1 D^{N-1} + \cdots + a_N)\, y(t) = (b_0 D^N + b_1 D^{N-1} + \cdots + b_N)\, x(t)$$

A resposta de estado nulo considera condições iniciais nulas

$$y(0^-) = \dot{y}(0^-) = \cdots = y^{(n-1)}(0^{-1}) = 0$$

$$x(0^-) = \dot{x}(0^-) = \cdots = x^{(n-1)}(0^{-1}) = 0$$