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Lista 2 - Integrais triplas

1) Calcule a integral iterada.

a)

$$\int_0^1 \int_0^z \left[\int_0^{x+z} 6xz \, dy\right] dx \, dz$$

$$\left[6xyz\right]_0^{x+z} = 6xz.(x+z) \Rightarrow \int_0^1 \left[\int_0^z 6xz(x+z) \, dx\right] dz$$

$$\left[6xz.x + 6xz.z\right] = \left[6x^2z + 6xz^2\right] \Rightarrow \left[\frac{\cancel{6}^{2}x^3z}{\cancel{3}} + \frac{\cancel{6}^{3}x^2z^2}{\cancel{2}}\right]_0^z$$

$$= 2z^3.z + 3z^2.z^2 - (0) = 2z^4 + 3z^4 = 5z^4$$

$$= \int_0^1 5z^4 \, dz = \left[\frac{\cancel{5}z^5}{\cancel{5}}\right]_0^1 = 1^5 - 0 = 1$$

b) $\int_0^3 \int_0^1 \left[\int_0^{\sqrt{1-z^2}} ze^y \, dx\right] dz \, dy$

$$\left[ze^y.x\right]_0^{\sqrt{1-z^2}} = ze^y.\sqrt{1-z^2} - 0 \Rightarrow \int_0^3 \left[\int_0^1 ze^y.\sqrt{1-z^2} \, dz\right] dy$$

$$\left[\frac{z^2e^y.\sqrt{1-z^2}}{2 \quad t}\right]_0^1 = \frac{e^y}{2}.\frac{2\sqrt{t}.t}{3} = \left[\frac{e^y}{2}.\frac{2\sqrt{1-z^2}.1-z^2}{3}\right]_0^1$$

$$= \left[\frac{e^y}{\cancel{2}}.\frac{\cancel{2}.\sqrt{1-1^2}}{3}.\cancel{1-1^2}\right] - \left(-\frac{e^y}{2}.\frac{2\sqrt{1}.1-0^2}{3}\right) = +\frac{e^y}{2}$$

$$\int_0^3 \frac{e^y}{2} \, dy = \frac{1}{2}\left[e^y\right]_0^3 = \frac{1}{2}(e^3 - 1)$$

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c) $\int_0^{\pi/2} \int_0^{y} \left[\int_0^{x} \cos(\underbrace{x+y+z}_{t})\, dz \right] dx\, dy$

$\int_0^{x} \cos(t)\, dz = \left[\sin(t) \right]_0^{x} = \sin(x+y+x) - \sin(x+y+0)$

$= \int_0^{y} \sin(\underbrace{2x+y}_{t}) - \sin(\underbrace{x+y}_{t})\, dx = \left[-\frac{\cos(2x+y)}{2} - \cos(x+y) \right]_0^{y}$

$= -\frac{\cos(2y+y+y)}{2} - \cos(y+y) - \left(-\frac{\cos(y)}{2} - \cos(y) \right)$

$= -\frac{\cos(4y)}{2} - \cos(2y) + \frac{\cos(y)}{2} + \cos(y)$

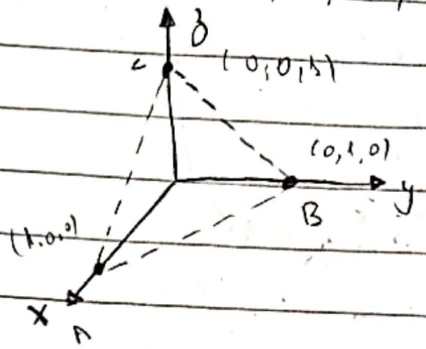
$= -\frac{\cos(3y) + \cos(y)}{2} + \cos(2y)$

$= \int_0^{\pi/2} -\frac{\cos(3y) + \cos(y)}{2} + \cos(2y)\, dy = \left[-\frac{\sin(3y)}{6} - \frac{\sin(y)}{2} + \frac{\sin(2y)}{2} \right]_0^{\pi/2}$

$\left[-\frac{\sin\left(3 \cdot \frac{\pi}{2}\right)}{6} - \cancel{\frac{\sin(\pi/2)}{2}} + \frac{\sin(2 \cdot \pi/2)}{2} - \left(-\cancel{\frac{\sin(0)}{6}}^{0} - \cancel{\frac{\sin(0)}{2}}^{0} + \cancel{\frac{\sin(0)}{2}}^{0} \right) \right]$

$\left[-\frac{\sin(3\pi/2)}{6} + \frac{\sin\left(\frac{\pi}{2}\right)}{2} \right] = -\frac{1}{3}$ //

d) $\iiint_T x^2 \, dV$, onde T é o tetraedro sólido com vértices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ e $(0,0,1)$.



$\vec{AC} = C - A = (0,0,1) - (1,0,0)$

$\vec{AC} = (-1,0,1)$

$\vec{AB} = B - A = (0,1,0) - (1,0,0)$

$\vec{AB} = (-1,1,0)$

$\vec{AX} = X - A = (x,y,z) - (1,0,0)$

$\vec{AX} = (x-1, y, z)$

$z = 0 \Rightarrow y = -x+1$

$$\det \begin{vmatrix} (x-1) & y & z \\ -1 & 0 & 1 \\ +1 & 1 & 0 \end{vmatrix} \Rightarrow z = -x-y+1$$

$z = 0 \Rightarrow y = -x+1$

$0 \leq z \leq -x-y+1$
$0 \leq y \leq -x+1$
$0 \leq x \leq 1$

$$\Rightarrow \int_0^1 \int_0^{-x+1} \left[\int_0^{-x-y+1} x^2 \, dz \right] dy \, dx$$

$$\left[x^2 z\right]_0^{-x-y+1} = x^2(-x-y+1)$$

$$\int_0^1 x^2 \int_0^{-x+1} [-x-y+1] \, dy \, dx = \int_0^1 x^2 \left[-xy - \frac{y^2}{2} + y\right]_0^{-x+1} dx$$

$$\int_0^1 x^2 \left[x^2 - x - \frac{(-x+1)^2}{2} - x + 1\right] dx$$

$$\int_0^1 \left[x^4 - x^3 - \frac{(x^4 - 2x^3 + x^2)}{2} - x^3 + x^2\right] dx$$

$$= \left[\frac{x^5}{5} - \frac{x^4}{4} - \frac{x^5}{10} + \frac{x^4}{4} - \frac{x^3}{6} - \frac{x^4}{4} + \frac{x^3}{3}\right]_0^1 = \frac{1^5}{5} - \frac{1^5}{10} - \frac{1^4}{4} - \frac{1^3}{6} + \frac{1^3}{3}$$

$$= \frac{12}{60} - \frac{6}{60} - \frac{15}{60} - \frac{10}{60} + \frac{20}{60} = \boxed{\frac{1}{60}}$$

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2) Calcule a integral tripla

a) $\iiint_E 2x\, dV$, onde $E = \{(x,y,z) \mid 0 \leq y \leq 2,\ 0 \leq x \leq \sqrt{4-y^2},\ 0 \leq z \leq y\}$

$\int_0^2 \int_0^{\sqrt{4-y^2}} \left[\int_0^y 2x\, dz\right] dx\, dy = \left[2xz\right]_0^y = 2xy - 0$

$\int_0^2 \left[\int_0^{\sqrt{4-y^2}} 2xy\, dx\right] dy = \left[\frac{\cancel{2}x^2 y}{\cancel{2}}\right]_0^{\sqrt{4-y^2}} = (\sqrt{4-y^2})^{\cancel{2}} \cdot y - 0$

$\int_0^2 4 - y^3\, dy = \left[4y - \frac{y^4}{4}\right] = 8 - \frac{16}{4} - 0 = 4$ //

c) $\iiint_E x^2 e^y\, dV$, onde E é delimitado pelo cilindro parabólico $z = 1 - y^2$ e pelos planos $z = 0$, $x = 1$ e $x = -1$.

$\int_{-1}^{1} \int_{-1}^{1} \left[\int_0^{1-y^2} x^2 e^y\, dz\right] dy\, dx = \left[x e^y z\right]_0^{1-y^2} = x^2 e^y \cdot (1-y^2) - (0)$

$\int_{-1}^{1} \left[\int_{-1}^{1} x^2 e^y \cdot 1 - x^2 e^y \cdot y^2\, dy\right] dx = \left[e^y - (y^2 - 2y + 2)e^y\right]_{-1}^{1} =$

$\int_{-1}^{1} x^2\, dx = \left[\frac{x^3}{3}\right]_{-1}^{1} \Rightarrow \left[\frac{x^3}{3}\right]_{-1}^{1} \cdot \left[e^y - (y^2 - 2y + 2) \cdot e^y\right]_{-1}^{1}$

$= \left[\frac{x^3}{3}\right]_{-1}^{1} \cdot \left[\cancel{e} - \cancel{e} + \cancel{2e} - \cancel{2e} - \cancel{e^{-1}} + \cancel{e^{-1}} + 2e^{-1} + 2e^{-1}\right]$

$= \left[\frac{1}{3} + \frac{1}{3}\right] \cdot \left[4e^{-1}\right] = \boxed{\frac{8}{3e}}$ //

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e) $\iiint_E x\, dV$, onde E é limitado pelo paraboloide $x = 4y^2 + 4z^2$ e pelo $x = 4$.

$4\pi \leq x \leq 4$ $\quad 4 = 4y^2 + 4z^2$

$0 \leq r \leq 1$ $\quad 1 = y^2 + z^2$

$0 \leq \theta \leq 2\pi$ $\quad r^2 = 1 \Rightarrow R = 1$

$$\int_0^{2\pi}\int_0^1\left[\int_{4\pi}^{4} 2xr\, dx\right] dr\, d\theta \Rightarrow \left[\frac{2x^2}{2}\right]_{4r^2}^{4} = \overset{\to 16}{4^2} - 4r^2$$

$$\int_0^{2\pi}\int_0^1 4^2 - 4r^2\, dr\, d\theta \Rightarrow \int_0^{2\pi} d\theta \cdot \int_0^1 4^2 - 4r\, dr$$

$$= 2\pi \cdot \left[16r - \frac{4r^3}{3}\right]_0^1 = 2\pi \cdot \left(16 - \frac{4}{3}\right) = \frac{16\pi}{3}$$

3) Volume do sólido

a) O tetraedro limitado pelos planos coordenados e o plano $2x + y + z = 4$.

$\hookrightarrow -2x - y + 4 = z$ $\quad \to z = 0 \Rightarrow y = 4 - 2x$

$0 \leq z \leq 4 - 2x - y$ $\quad y + z = 0 \Rightarrow x = \frac{-4}{-2} = 2$

$0 \leq y \leq 4 - 2x$

$0 \leq x \leq 2$

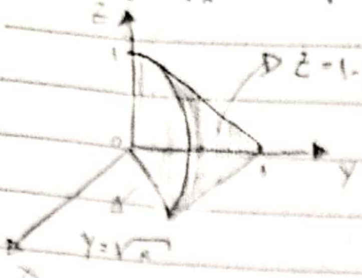
$$\int_0^2\int_0^{4-2x}\int_0^{4-2x-y} dz\, dy\, dx = \int_0^2\left[\int_0^{4-2x} 4 - 2x - y\, dy\right] dx$$

$$\left[4y - 2xy - \frac{y^2}{2}\right]_0^{4-2x} = 4\cdot(4-2x) - 2x\cdot(4-2x) - \frac{(4-2x)^2}{2}$$

$$\int_0^2 16 - 8x - 8x + 4x^2 - 8 + 8x - 2x^2\, dx = \int_0^2 (8 - 8x + 2x^2)\, dx$$

$$\left[8x - \frac{8x^2}{2} + \frac{2x^3}{3}\right]_0^2 = 16 - 16 + \frac{16}{3}$$

4) $\int_0^1\int_{\sqrt{x}}^1\int_0^{1-y} f(x,y,z)\,dz\,dy\,dx$



$\int_0^1\int_0^{y^2}\int_0^{1-y} f(x,y,z)\,dz\,dx\,dy$

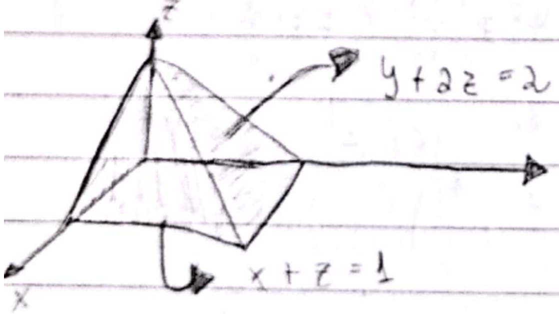
$\int_0^1\int_0^{1-\sqrt{x}}\int_{\sqrt{x}}^{1-z} f(x,y,z)\,dy\,dz\,dx$

$\int_0^1\int_0^{(1-z)^2}\int_{\sqrt{x}}^{1-z} f(x,y,z)\,dy\,dx\,dz$

$\int_0^1\int_0^{1-y}\int_0^{y^2} f(x,y,z)\,dx\,dz\,dy$

$\int_0^1\int_0^{1-z}\int_0^{y^2} f(x,y,z)\,dx\,dy\,dz$

5) Volume da região do primeiro octante limitada pelos planos coordenados e pelos planos $x+z=1$, $y+2z=2$



$z = 1-x$, $y = 2z-2$, $x = 1$

$\int_0^1\int_0^{1-x}\left[\int_0^{2z-2} dy\right]dz\,dx$

$\int_0^1\left[\int_0^{1-x} \frac{2z^2}{2}-2z\,dz\right]dx$

$\int_0^1 (1-x)^2+2(1-x)\,dx$

$\int_0^1 \cancel{2x}+2-1+\cancel{2x}-x^2\,dx = \left[1-x^2\right]_0^1 dx$

$\left[x-\frac{x^3}{3}\right]_0^1 = 1-\frac{1^3}{3} = 1-\frac{1}{3}$ // ou $\frac{2}{3}$ //

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6) Coordenadas cilíndricas

a) Calcule $\iiint_E \sqrt{x^2+y^2}\, dV$, onde E é a região que está dentro do cilindro $\underbrace{x^2+y^2}_{r^2}=16$ e entre os planos $z=-5$ e $z=4$

$r^2 = 16 \therefore r = 4$

$\theta = 2\pi$

$$\int_{-5}^{4}\int_{0}^{2\pi}\left[\int_{0}^{4} r^2\, dr\right]d\theta\, dz \Rightarrow \left[\frac{r^3}{3}\right]_0^4 = \frac{64}{3} - 0$$

$$\int_{-5}^{4}\left[\int_{0}^{2\pi} \frac{64}{3}\, d\theta\right]dz \Rightarrow \left[\frac{64\theta}{3}\right]_0^{2\pi} = \frac{64 \cdot 2\pi}{3}$$

$$\int_{-5}^{4} \frac{128\pi}{3}\, dz = \frac{128\pi \cdot z}{3} = \frac{512\pi}{3} + \frac{640\pi}{3} = \frac{1152\pi}{3} = 384\pi$$

b) Calcule $\iiint_E x^2\, dV$, onde E é limitado pelo plano xz e pelos hemisférios $y=\sqrt{9-x^2-z^2}$ e $y=\sqrt{16-x^2-z^2}$

$0 \leq z \leq 2r$

$0 \leq r \leq 1$

$0 \leq \theta \leq 2\pi$

$$\int_0^{2\pi}\int_0^1\left[\int_0^{4r} r^3 \cos^2\theta\, dz\right]dr\, d\theta$$

$$\left[z r^3 \cos^2\theta\right]_0^{4r} = 2r \cdot r^3 \cdot \cos^2\theta$$

$$\int_0^{2\pi}\left[\int_0^1 2r^4 \cdot \cos^2\theta\, dr\right]d\theta = \left[\frac{2r^5}{5} \cdot \cos^2\theta\right]_0^1$$

$$\int_0^{2\pi} \frac{1}{5}\cos^2\theta\, d\theta = \left[\frac{2}{5}\left(\theta + \frac{\operatorname{sen}\theta \cos\theta}{2}\right)\right]_0^{\pi}$$

$$= \frac{1}{5} \cdot 2\pi = \frac{2\pi}{5}$$

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d) Volume da região E limitada pelos parabolóides $z = x^2 + y^2$ e $z = 36 - 3x^2 - 3y^2$

$z = r^2$

$4r^2 = 36 \Rightarrow r^2 = 9 \Rightarrow r = 3$

$$\int_0^{2\pi}\int_0^3\left[\int_{r^2}^{36-r^2} r\, dz\right] dr\, d\theta = [rz]_{r^2}^{36-r^2} = r.(36-r^2) - r^3$$

$36r - r^3$

$$\int_0^{2\pi}\left[\int_0^3 36r - r^4\, dr\right] d\theta = \left[\frac{36r^2}{2} - \frac{r^4}{4}\right]_0^3 \quad \underbrace{18.9}_{162} - \frac{81}{4}\, d\theta$$

$$\int_0^{2\pi} 81\, d\theta = [81\theta]_0^{2\pi} = 81.2\pi - 0 = 162\pi$$

7) Calcule a integral $\int_{-2}^{2}\int_{\sqrt{4-y^2}}^{\sqrt{4-y^2}}\int_{\sqrt{x^2+y^2}}^{2} xz\, dz\, dx\, dy$

Transformando para coordenadas cilíndricas.

$r = 2, \quad \theta = 2\pi, \quad r \leq z \leq 2$

$$\int_0^2\int_0^{2\pi}\left[\int_r^2 r\cos\theta\, z\, dz\right] d\theta\, dr = \left[r\cos\theta\, \frac{z^2}{2}\right]_r^2 \quad r\cos\theta\, 2 - r\,\ldots$$

$$\int_0^2\left[\int_0^{2\pi} r\cos\theta\, 2 - \frac{r^3}{2}\cos\theta\, d\theta\right] dr = \left[+2r\, \text{sen}\,\theta - \frac{r^3}{3}\,\text{sen}\,\theta\right]_0^{2\pi}$$

$$= \int_0^2 0\, dr = 0$$

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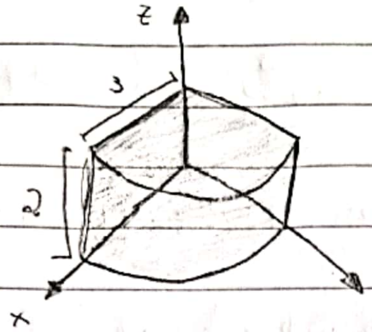
8) Escreva a integral Tripla de uma função $f(x,y,z)$ em coordenadas cilíndricas sobre o sólido mostrado.



$0 \leq r \leq 3$

$0 \leq z \leq 2$

$0 \leq \theta \leq \frac{\pi}{2}$

$$\int_0^{\pi/2}\int_0^3\int_0^2 f(r\cos\theta, r\,\text{sen}\,\theta, z)\, r\, dz\, dr\, d\theta$$

9) Coordenadas Esféricas

a) Calcule $\iiint_B (x^2+y^2+z^2)\, dV$, onde B é a Bola com centro na Origem e raio 5.

$0 \leq \theta \leq 2\pi$

$0 \leq \rho \leq 5$

$0 \leq \phi \leq \pi$

$$\int_0^{2\pi}\int_0^{\pi}\left[\int_0^5 \rho^3 \cdot \rho^2 \sin\phi\, d\rho\right] d\phi\, d\theta$$

$$\left[\frac{\rho^7}{7} \cdot \sin\phi\right]_0^5 = \frac{5^7}{7} \cdot \sin\phi$$

$$\int_0^{2\pi}\left[\int_0^{\pi} \frac{5^7}{7} \cdot \sin\phi\, d\phi\right] d\theta$$

$$\left[\frac{5^7}{7} \cdot (-\cos\phi)\right]_0^{\pi} = \frac{5^7}{7}(-\cos\pi) - \cos(0)$$

$$\int_0^{2\pi} \frac{5^7}{7}\, d\theta = \frac{5^7\theta}{7} = \frac{5^7 \cdot 2\pi \cdot 2}{7} = \frac{312.500\,\pi}{7}$$

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b) Calcule $\iiint_E z\, dV$, onde E está entre as esferas $x^2+y^2+z^2=1$ e $x^2+y^2+z^2=4$ no primeiro octante

$0 \leq \theta \leq \pi/2$

$0 \leq \phi \leq \pi/2$

$1 \leq \rho \leq 2$

$$\int_0^{\pi/2}\int_0^{\pi/2}\left[\int_1^2 \overbrace{\rho \cos\phi\, \rho^2}^{\rho^3} \operatorname{sen}\phi\, d\rho\right] d\phi\, d\theta$$

$$\left[\rho^3 \cos\phi \operatorname{sen}\phi\right] = \frac{\rho^4}{4} \cos\phi \operatorname{sen}\phi \Big|_1^2$$

$$= \int_0^{\pi/2}\left[\int_0^{\pi/2} 4\cos\phi \operatorname{sen}\phi - \frac{1}{4}\cos\phi \operatorname{sen}\phi\, d\phi\right] d\theta$$

$$= \frac{\pi}{\theta} \cdot \left[\frac{\operatorname{sen}^2\phi}{2}\right]_0^{\pi/2} \cdot \left(4 - \frac{1}{4}\right)$$

$$= \frac{\pi}{2} \cdot \frac{1}{2} \cdot \frac{15}{4} = \boxed{\frac{15\pi}{16}}$$

c) Determine o volume do sólido que está acima do cone $\phi = \pi/3$ e abaixo da esfera $\rho = 4\cos\phi$

$0 \leq \theta \leq 2\pi$

$0 \leq \phi \leq \pi/3$

$0 \leq \rho \leq 4\cos\phi$

$$\int_0^{2\pi}\int_0^{\pi/3}\left[\int_0^{4\cos\phi} \rho^2 \operatorname{sen}\phi\, d\rho\right] d\phi\, d\theta$$

$$\left[\frac{\rho^3}{3}\right]_0^{4\cos\phi} \operatorname{sen} = \frac{(4\cos\phi)^3}{3} \operatorname{sen}\phi$$

$$= \int_0^{2\pi} d\theta \cdot \int_0^{\pi/3} \frac{64}{3} \cos^3\phi \operatorname{sen}\phi\, d\phi$$

$$= 2\pi \cdot \frac{64}{3} \cdot \left[\frac{-\cos^4\phi}{4}\right]_0^{\pi/3}$$

$$= \frac{64}{3} \cdot 2\pi \cdot \left(\frac{-1}{64} + \frac{1}{4}\right) = \frac{64}{3} \cdot \frac{15}{64} \cdot 2\pi = \boxed{10\pi}$$

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9) d) Encontre o volume da parte da bola $\rho \leq a$ que está entre os cones $\phi = \frac{\pi}{6}$ e $\phi = \frac{\pi}{3}$

$0 \leq \theta \leq 2\pi$

$\pi/6 \leq \phi \leq \pi/3$

$0 \leq \rho \leq a$

$$\int_0^{2\pi}\int_{\pi/6}^{\pi/3}\int_0^{a} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_{\pi/6}^{\pi/3} \left[\rho^3/3 \sin\phi\right]_0^a d\phi \, d\theta$$

$$= \int_0^{2\pi} d\theta \int_{\pi/6}^{\pi/3} \sin\phi \, d\phi \cdot \frac{a^3}{3}$$

$$= 2\pi \cdot \frac{a^3}{3} \cdot \left[-\cos\phi\right]_{\pi/6}^{\pi/3} = 2\pi \cdot \frac{a^3}{3} \cdot \left[-\cos(\pi/3) + \cos(\pi/6)\right]$$

$$= 2\pi \cdot \frac{a^3}{3} \cdot \left(\frac{\sqrt{3}-1}{2}\right) = \frac{\pi a^3(\sqrt{3}-1)}{3}$$

10) Calcule a integral $\int_0^1\int_0^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} xy \, dz \, dy \, dx$

Transformando para coordenadas esféricas

$0 \leq \theta \leq \pi/2$

$0 \leq \phi \leq \pi/4$

$0 \leq \rho \leq \sqrt{2}$

$$\int_0^{\pi/2}\int_0^{\pi/4}\int_0^{\sqrt{2}} \rho^4 \sin^3\phi \sin\theta \cos\theta \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{\pi/2} \sin\theta \cos\theta \, d\theta \cdot \int_0^{\pi/4} \sin\phi \, d\phi \int_0^{\sqrt{2}} \rho^4 \, d\rho$$

$$= \left[\frac{\sin^2\theta}{2}\right]_0^{\pi/2} \cdot \left[-\cos\phi + \frac{\cos^3\phi}{3}\right]_0^{\pi/4} \cdot \left[\frac{\rho^5}{5}\right]_0^{\sqrt{2}}$$

$$= \frac{1}{2} \cdot \left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{12} + 1 - \frac{1}{3}\right) \cdot \frac{4\sqrt{2}}{5}$$

$$= \left(\frac{1}{2} \cdot \frac{-5\sqrt{2}}{12} + \frac{1}{2} \cdot \frac{2}{3}\right) \cdot \frac{4\sqrt{2}}{5} = -\frac{1}{3} + \frac{4\sqrt{2}}{15} = \frac{1}{15}(4\sqrt{2}-5)$$

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