Non-linear Physics

PHYM038

University of Surrey

Department of Physics

Spring 2021

Coursework instructions

In non-linear systems, chaotic behaviour arises from the recursive stretching and overlapping of volumes in phase space. In the *Lorenz* system, this is attained by means of two symmetric spiral centres, known as the the butterfly wings. Trajectories on each wing spiral out, causing stretching, but the motion remains bound because the diverging trajectories are cut out of the attractor and re-injected into the centre of the opposite, overlapping, wing. The result is a *strange* or *chaotic attractor* with fractal geometry.

Another common mechanism that generates chaos is stretching and folding. Notable examples are the *Hénon map*, the *Rössler attractor* and the *Duffing oscillato*r. In this coursework you will study properties of the latter system and simulate the cascade mechanism that leads to chaos in unimodal maps.

One of the key features of chaotic systems is that, although their motion appears erratic and unpredictable, there is usually an underlying order such that some features of the motion exhibit stable and controlled behaviour. The final exercise is an example of how one might use this to distinguish chaos from random motion.

Your completed coursework should be submitted in electronic form through SurreyLearn by 16h on Friday 14th May 2021. Please submit *one file* in *PDF* format and name it `PHYM038_CW_<ID>.pdf` where `<ID>` is your student ID. You must also submit an honesty declaration into the appropriate folder on Surreylearn.
I recommend using LaTeX and an editor such as LyX or *Overleaf* to make your document. Graphics should be produced on computer, e.g. with a tool like *Gnuplot*, *pyplot*, *matplotlib* or *seaborn*. Coursework submitted late will be subject to penalties as per University Regulations.
Regarding formatting, please:

- Stick to the question numbering scheme and preserve the order of the questions: I will read from page one to the end, in that order.
- Put your URN at the top. Use at least 2cm margins. Use single-column, double-spaced text in 12-point, sans-serif font (e.g. Arial, Helvetica).
- Number equations appropriately (LaTeX does this for you) and learn about how equations should be formatted (roman font for constants, labels and expressions, italic font for variables, see e.g. https://academic.oup.com/mnras/pages/General_Instructions).
- Format graphs professionally: if the font is too small, the plot too small, the graph unreadable, too crowded, unlablled axes, you will be penalised. Your marker may well be colourblind.

Regarding science:

- Be quantitative! There are solutions to the problems.
- Check your work. Typos and algebraic errors are costly, but you have time to read through it many times.
- Be careful with computer codes: ensure you know what you are doing. Say which algorithm you use, give the code in the appendix.

More information...

- https://en.wikipedia.org/wiki/PDF
- https://www.latex-project.org/
- https://www.lyx.org/
- https://www.overleaf.com/
- http://gnuplot.info/
- https://matplotlib.org/tutorials/introductory/pyplot.html
- https://matplotlib.org/
- https://seaborn.pydata.org/
- Duffing equation https://www.youtube.com/watch?v=6lVLNAdF4pI
- Intro to LaTeX https://www.youtube.com/watch?v=Jp0lPj2-DQA
- Intro to LyX https://www.youtube.com/watch?v=Xv7rFWJ0taI
- Colourblind tips http://jfly.iam.u-tokyo.ac.jp/color/#assign

You MUST complete the assessment on your own and all the work submitted must be your own. Surrey University defines plagiarism as:

- inserting words, concepts, or images or other content from the work of someone else into work submitted for assessment without acknowledging the originator's contribution or,
- representing the work of another as one's own, whether purchased or not, or taken with or without permission.

1. DO NOT allow anyone else to complete the assessment on your behalf. If you do, you will be in breach of university regulations and could be expelled from the university.
2. DO NOT allow anyone else to help you directly with your assessment. You may discuss science and scientific methods in general, but submitted work MUST be your own.
3. DO NOT work together, or collude, with any other person.
4. DO NOT post questions, or question parts, on any online forum.
5. DO NOT copy and paste text from other sources into your work. This is plagiarism and will be classed as cheating.

I have been very good at spotting where people have copied in the past. Mostly this has been copying of code from textbooks or websites, which is highly illegal given that websites generally have licencing agreements for their code (which are obviously ignored when people pass it off as their own without citing a source). If you can find the code online, so can I, so please do not do it! Instead, learn from it and write your own code.

More information...

- https://exams.surrey.ac.uk/academic-integrity-and-misconduct
- https://www.surrey.ac.uk/office-student-complaints-appeals-and-regulation/academic-misconduct
- https://exams.surrey.ac.uk/academic-integrity-and-misconduct/plagiarism
- https://surrey-content.surrey.ac.uk/sites/default/files/2019-01/academic-integrity.pdf

1 *The Duffing equation*

At the beginning of the 20th century Georg Duffing[1] investigated the physics of periodically driven nonlinear oscillators with damping. What we now call the *Duffing equation* is a generalisation of the linear differential equation that describes a damped and forced harmonic motion. Consider the following specific form of the Duffing equation,

$$\ddot{x} + 2\gamma\dot{x} + \alpha x + \beta x^3 = F_\mathrm{e}\cos\left(\omega t\right) \ ,$$

with the following constant input parameters,

$$\begin{aligned} \alpha &= -1\,, \\ \beta &= +1\,, \\ \omega &= +1\,. \end{aligned}$$

1. Interpret the Duffing equation in terms of its mechanical analogue of a mass subject to a conservative force,

   $$F_\mathrm{cons} = -\frac{\partial V\left(x\right)}{\partial x} \ ,$$

   a non-conservative force and an external force. Plot the potential $V\left(x\right)$.

   [4 marks]

2. Consider this system in the absence of the external force, by setting $F_\mathrm{e} = 0$. Perform a full phase space analysis of the behaviour of the oscillator for all regimes of $\gamma < 0$, $\gamma = 0$ and $\gamma > 0$.
   You should discuss at least nullclines, fixed points, and sketch phase space diagrams for all relevant cases.

   [12 marks]

3. Rewrite the Duffing equation in the form of a three dimensional autonomous system in the variables x, $v \equiv \dot{x}$ and $\theta \equiv \omega t$, including the driving force term. Implement the solution of this system on a computer. Study the behaviour of the system with different values of γ and F_e. You should:

   (a) Plot the phase trajectory in the v–x plane.

   (b) Plot the solution $x\left(t\right)$.

   Five or six cases should suffice. However, make sure to include a couple of cases for $F_\mathrm{e} = 0$ to corroborate your results.

   [9 marks]

[1] https://de.wikipedia.org/wiki/Georg_Duffing

2 *Cascade route to chaos and folding of strange attractors*

In the following set $\gamma = 0.25$. Consider the *Poincaré section*[2] obtained by the intersection of the attractor with the plane $\theta = \omega t = 0$. For a given set of parameters (α, β, γ, F_e and ω) the Poincaré section is made of the set of points that is transformed into itself after one revolution. Thus, for example, if the attractor is a limit cycle with one orbit the Poincaré section will contain a single point. If the orbit closes after two oscillations the section will contain two separate points, and so on.

1. Compute the points of the Poincaré section in the chaotic regime given by $F_\mathrm{e} = 0.43$ when $\alpha = -1$, $\beta = 1$, $\gamma = 0.25$ and $\omega = 1$. Plot the projections of this section in a 2-dimensional graph where x components of each point are on the vertical axis and F_e (constant) is on the horizontal axis.

   [5 marks]

2. Compute the projected Poincaré sections using many values of F_e in the interval $0.1 \leq F_\mathrm{e} \leq 0.43$ while α, β, γ and ω remain constant, and obtain the bifurcation diagram in the (F_e, x) plane.

   [6 marks]

3. With $F_\mathrm{e} = 0.42$, plot Poincaré sections at different phases $0 \leq \theta = \omega t \leq 2\pi$ (that is, for one cycle of the driving force F) to show the evolution in time of the Poincaré section. You should obtain a plot of the section in the (x, v) plane for each θ. Interpret your results in terms of folding of the phase space and the fractal shape of these sections.

   [10 marks]

To obtain the Poincaré sections, it is suggested that you first let the system evolve for a certain time until the trajectory has approached the attractor. After this transient time, you can start storing the values of x and v every time the trajectory crosses the relevant θ plane. In the chaotic regime one needs to record at least 500 points or more to have an accurate description of the full Poincaré section. For the projected sections one only needs to retain the values of x.

[2] https://en.wikipedia.org/wiki/Poincaré_map

3 Chaos Recognition

An important task in many areas of science is to distinguish between random noise and deterministic chaos. Typically, the experimentalist has acquired data sampled over some time interval an would like to know if there exists some underlying chaotic attractor which would allow the data to be interpreted and perhaps future predictions made, or is one dealing with noise from which nothing can be inferred. The main idea is that if deterministic chaos occurs, then there should be some deterministic relations in the sample that link the data points.

Fig. 1 shows the time evolution of a single coordinate (one out of possibly many degrees of freedom) of three different oscillating systems. Some are actual nonlinear chaotic oscillators while others are random noise. The three data files used to plot these pictures are part of this assignment and can be downloaded from *SurreyLearn*. Only the initial part of the evolution is shown here but much longer time intervals are in included in the files.

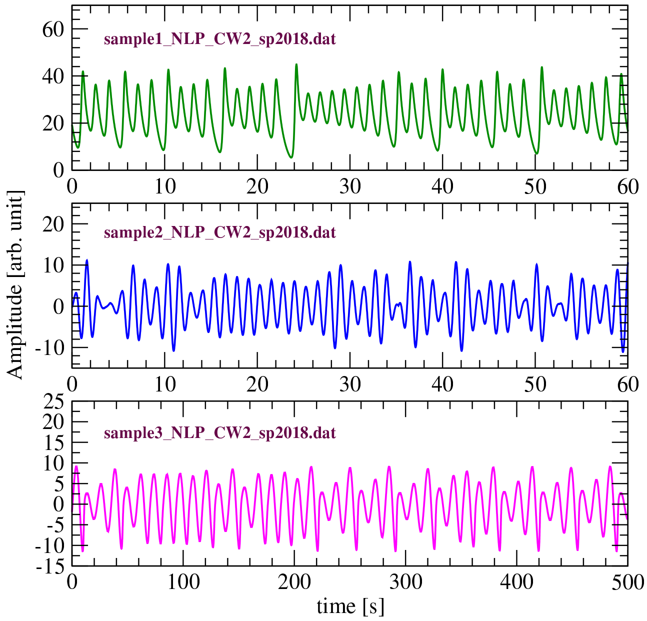

Figure 1: Time series used in the exercises.

Record the peak value of each oscillation and hence generate a sequence z_n of the maxima where n ranges from 1 to the total number of peaks in each data file. This will define a return map, $z_{n+1} = f(z_n)$, for each data sample.

1. Plot the map z_{n+1} vs z_n in each case and discuss which samples correspond to chaotic motion and which not. Justify your conclusions.

[9 marks]

2. With the sample(s) corresponding to chaotic motion, calculate numerically the Lyapunov exponent according to the definition studied in class[3]

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln |f'(z_i)| \ .$$

[3] See also https://en.wikipedia.org/wiki/Lyapunov_exponent.

You will need to evaluate the derivative $f'(z)$ numerically[4].

[5 marks]

Total marks: 60 /60

[4] https://en.wikipedia.org/wiki/Numerical_differentiation