Pushing charged particles around

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The goal is to simulate the motion of charged particles in a given electric field $\mathbf{E}(\mathbf{x}, t)$, and magnetic field $\mathbf{B}(\mathbf{x}, t)$. This is the test-particle approximation.

For each particle p of charge[1] q_p and mass m_p the equations to be solved (i.e. integrated in time) are

$$\frac{d\mathbf{x}_p}{dt} = \mathbf{v}_p \tag{1}$$

$$m\frac{d\mathbf{v}_p}{dt} = q_p \left[\mathbf{E}(\mathbf{x}_p, t) + \mathbf{v}_p \times \mathbf{B}(\mathbf{x}_p, t)\right] \tag{2}$$

where $\mathbf{v}_p(t)$ is the velocity and $\mathbf{x}_p(t)$ is its position. For simplicity we shall drop the subscript p.

Possibly, the simplest discritzation is that of Euler:

$$\frac{\mathbf{x}_{n+1} - \mathbf{x}_n}{\Delta t} = \mathbf{v}_n \tag{3}$$

$$\frac{\mathbf{v}_{n+1} - \mathbf{v}_n}{\Delta t} = \frac{q}{m}\left[\mathbf{E}_n(\mathbf{x}_n) + \mathbf{v}_n \times \mathbf{B}_n(\mathbf{x}_n)\right] \tag{4}$$

where the electric and magnetic fields are evaluated at the particle position $\mathbf{x}_{n+\frac{1}{2}}$. This scheme is however only first order in Δt and not very useful, but for better understanding the next method.

A much better numerical scheme is achived by staggering in time the position and velocity (leapfrog), so that

$$\frac{\mathbf{x}_{n+\frac{1}{2}} - \mathbf{x}_{n-\frac{1}{2}}}{\Delta t} = \mathbf{v}_n \tag{5}$$

$$\frac{\mathbf{v}_{n+1} - \mathbf{v}_n}{\Delta t} = \frac{q}{m}\left[\mathbf{E}_{n+\frac{1}{2}} + \frac{\mathbf{v}_n + \mathbf{v}_{n+1}}{2} \times \mathbf{B}_{n+\frac{1}{2}}(\mathbf{x}_{n+\frac{1}{2}})\right] \tag{6}$$

The electric and magnetic fields are evaluated at the particle position $\mathbf{x}_{n+\frac{1}{2}}$. In these equations, time has been discretized as $t = n\Delta t$ where the subscript n is an integer. The solution is then found at times $(1, 2, 3, \ldots)\Delta t$ for the velocity and at times $(-\frac{1}{2}, \frac{1}{2}, \frac{3}{2} \ldots)\Delta t$ for the position.

One of the most widely used, for its accuracy and simplicity, is the so-called Boris pusher[2]. The method is based on defining two new variables for the velocity

$$\mathbf{v}^- = \mathbf{v}_n + \frac{q}{m}\mathbf{E}_{n+\frac{1}{2}}\frac{\Delta t}{2} \tag{7}$$

$$\mathbf{v}^+ = \mathbf{v}_{n+1} - \frac{q}{m}\mathbf{E}_{n+\frac{1}{2}}\frac{\Delta t}{2} \tag{8}$$

[1] $q = -e$ for electrons and Ze for ions, where Z is the ion charge

[2] J. Boris, in Proceedings of the Fourth Conference on Numerical Simulation of Plasmas (Naval Research Laboratory, Washington DC, 1970), p. 3.

which when substituted into eq. 6 gives

$$\frac{\mathbf{v}^+ - \mathbf{v}^-}{\Delta t} = \frac{q}{m}\frac{\mathbf{v}^- + \mathbf{v}^+}{2} \times \mathbf{B}_{n+\frac{1}{2}} \tag{9}$$

The Boris algorithm results in the "separation" of the electric field and magnetic field forces acting on the particle.

The steps to update the velocity from $\mathbf{v}_n$ to $\mathbf{v}_{n+1}$ are: given by:

1. Add half of the electric impulse to $\mathbf{v}_n$ using eq.7 to obtain $\mathbf{v}^-$
2. Perform a rotation using eq. 9 to obtain $\mathbf{v}^+$. This is done by decomposing the rotation as

$$\mathbf{v}' = \mathbf{v}^- + \mathbf{v}^- \times \mathbf{t} \tag{10}$$

$$\mathbf{v}^+ = \mathbf{v}^- + \mathbf{v}' \times \mathbf{s} \tag{11}$$

   where

$$\mathbf{t} = \frac{q\Delta t}{2m}\mathbf{B}_{n+\frac{1}{2}} \tag{12}$$

$$\mathbf{s} = \frac{2}{1+t^2}\mathbf{t} \tag{13}$$

3. Add the remaining electric impulse using eq. 8 to obtain the updated velocity $\mathbf{v}_{n+1}$

$$\mathbf{v}_{n+1} = \mathbf{v}^+ + \frac{q}{m}\mathbf{E}_{n+\frac{1}{2}}\frac{\Delta t}{2} \tag{14}$$

Today's work

Write a code to solve the motion of a charged particle using the Boris particle pusher described above. Particles can be ions ($q = Ze$), electrons ($q = -e$) or positrons ($q = e$). For numerical stability and accuracy use $\omega_0 \Delta t = 0.1$ where $\omega_o = \frac{|q|B_0}{m}$. **Think carefully about initial conditions and the first time-step. Ideally the code should be able to deal with N particles.**

Like all numerical models, we **must** test the code against known analytical solutions. The tests **must** be quantitative. Examples of tests that you should consider are: simple gyro-motion: $\mathbf{B} = (0, 0, B_0)$ and $\mathbf{E} = (0, 0, 0)$; $\mathbf{E} \times \mathbf{B}$ drift ($\mathbf{E}$ and $\mathbf{B}$ are constant in time and do not vary in space); energy conservation when $\mathbf{E} = 0$; when possible, you should compare to the known analytical solutions.