1. Si $f(n) \notin \mathcal{O}(g(n))$, entonces $g(n) \in \mathcal{O}(f(n))$.

$f(n) = n^3 \qquad g(n) = \begin{cases} n^2 & \text{si } n \text{ es par} \\ n^4 & \text{si } n \text{ es impar} \end{cases}$

PD: $f(n) \notin \mathcal{O}(g(n))$

Sup. $f(n) \in \mathcal{O}(g(n)) \longrightarrow \exists c \in \mathbb{R}^+ \; \exists n_0 \in \mathbb{N} . \; \forall n \geq n_0 . \; f(n) \leq c \cdot g(n)$

$n_1 = \max(\lceil c \rceil, n_0) + 1 \longrightarrow n_1 > c$

si n_1 es par:

$f(n_1) = n_1^3 > c \cdot n_1^2 = c \cdot g(n) \quad \rightarrow\leftarrow$

si n_1 es impar:

sea $n_2 = n_1 + 1 \rightarrow n_2$ es par

$f(n_2) = n_2^3 > c \cdot n_2^2 = c \cdot g(n_2) \quad \rightarrow\leftarrow$

$\Rightarrow f(n) \notin \mathcal{O}(g(n))$

PD: $g(n) \notin \mathcal{O}(f(n))$

Sup. que $g(n) \in \mathcal{O}(f(n)) \rightarrow \exists c, n_0 . \; \forall n \geq n_0 . \; g(n) \leq c \cdot f(n)$

$n_1 = \max(\lceil c \rceil, n_0) + 1 \longrightarrow n_1 > c \wedge n_1 > n_0$

Si n_1 es impar:

$g(n_1) = n_1^4 > c \cdot n_1^3 = c \cdot f(n_1) \quad \rightarrow\leftarrow$

Si n_1 es par:

$n_2 = n_1 + 1$

$g(n_2) = n_2^4 > c \cdot n_2^3 = c \cdot f(n_2) \quad \rightarrow\leftarrow$

$\Rightarrow g(n) \notin \mathcal{O}(f(n))$

$\therefore$ La afirmación es falsa.

2. Si $f(n) \in \mathcal{O}(g(n))$, entonces $2^{f(n)} \in \mathcal{O}(2^{g(n)})$.

$\exists c, n_0 .\ \forall n > n_0 .\ f(n) \leq c \cdot g(n)$

$2^{f(n)} \leq 2^{c \cdot g(n)}$

$f(n) = 2n \quad g(n) = n \longrightarrow n_0 = 0 \wedge c = 2$

Sup que $f'(n) = 2^{2n} \in \mathcal{O}(g'(n)) \quad g'(n) = 2^n$

$\exists c, n_0 .\ \forall n > n_0,\ 2^{2n} \leq c \cdot 2^n$

Sea $n_1 = \max(n_0, \lceil \log_2(c) \rceil) + 1 \longrightarrow n_1 > n_0 \wedge 2^{n_1} > c$

$f'(n_1) = 2^{n_1} \cdot 2^{n_1} > c \cdot 2^{n_1} = c \cdot g(n) \longrightarrow$

$2^{2n} = (2^n)^2 = 2^n \cdot 2^n \leq c \cdot 2^n$

$2^n > c$

3. $n^{\log(n)} \in \mathcal{O}(\log(n)^n)$.

$n = e^{\log(n)} \longrightarrow n^{\log(n)} = \left(e^{\log(n)}\right)^{\log(n)} = e^{\log^2(n)} = e^{(\log(n))^2}$

$\left.\begin{array}{l} \log(n) \geq e \longrightarrow n = e^e \leq 3^3 = 27 \\ n \geq \log^2(n) \longrightarrow n \geq 1 \end{array}\right\} n_0 = 27,\ c = 1$

Demuestre usando inducción constructiva que $T(n) \in \mathcal{O}(n \cdot \log_2(n))$, con $T(n)$ dado por:

$$T(n) = \begin{cases} 1 & n = 0 \\ T(\lfloor \frac{n}{3} \rfloor) + T(\lfloor \frac{2n}{3} \rfloor) + n & n > 0 \end{cases}$$

PD: $\exists c, n_0.\ \forall n \geq n_0.\quad T(n) \leq c \cdot n \cdot \log_2(n)$

Sup que $\forall k < n.\ \exists c.\ T(k) \leq c \cdot k \log_2(k)$

PD2: $T(n) \leq c \cdot n \log_2(n)$

$$T(n) = T\left(\left\lfloor \frac{n}{3} \right\rfloor\right) + T\left(\left\lfloor \frac{2n}{3} \right\rfloor\right) + n$$

$$\leq c \cdot \left\lfloor \frac{n}{3} \right\rfloor \log_2\left(\left\lfloor \frac{n}{3} \right\rfloor\right) + c \cdot \left\lfloor \frac{2n}{3} \right\rfloor \cdot \log_2\left(\left\lfloor \frac{2n}{3} \right\rfloor\right) + n$$

$$\leq c \cdot \frac{n}{3} \cdot \log_2\left(\frac{n}{3}\right) + c \cdot \frac{2n}{3} \cdot \log_2\left(\frac{2n}{3}\right) + n$$

$$= \frac{cn}{3} \cdot \left(\log_2(n) - \log_2(3)\right) + \frac{2nc}{3}\left(\log_2(2) + \log_2(n) - \log_2(3)\right) + n$$

$$= \cancel{\frac{cn}{3} \log(n)} - \underline{\frac{cn}{3} \log(3)} + 2\frac{cn}{3} \log(2) + \cancel{2\frac{cn}{3} \log(n)} - \underline{2\frac{cn}{3} \log(3)} + n$$

$$= cn \log(n) - cn \log(3) + \frac{2cn}{3} + n$$

$$= cn \log(n) - n\left(c \log(3) - \frac{2c}{3} - 1\right)$$

$$\overset{*}{\leq} cn \log(n)$$

$*\quad c \log(3) - \frac{2c}{3} - 1 > 0$

$$c\left(\log(3) - \frac{2}{3}\right) > 1$$

$$c > \frac{1}{\log(3) - 2/3}$$

Si $c > \frac{1}{\log(3) - 2/3} \longrightarrow T(n) \leq c \cdot n \log(n) \longrightarrow \exists c = 3.\ T(n) \leq 3 \cdot n \log(n)$

CB: $c = 3$

$T(0) = 1 \leq 3 \cdot 0 \log(0) = 0$ ✗

$T(1) = T(0) + T(0) + 1 = 3 \leq 3 \cdot 1 \log(1) = 0$ ✗

$T(2) = T(0) + T(1) + 2 = 6 \leq 3 \cdot 2 \log_2(2) = 6$ ✓

$T(3) = T(1) + T(2) + 3 = 12 \leq 3 \cdot 3 \log(3) \approx 14.3$ ✓

$T(4) = T(1) + T(2) + 4 = 13 \leq 3 \cdot 4 \log(4) = 24$ ✓

$T(5) = T(1) + T(3) + 5 = 20 \leq 3 \cdot 5 \log(5) \approx 34.8$ ✓

$T(6) = T(2) + T(4) + 6$ ✓✓

$\exists.\ \underline{n_0 = 2},\ c = 3 \longrightarrow T(n) \in \mathcal{O}(n \log_2(n))$

Dado un polinomio:

$$p(x) = \sum_{i=0}^{n-1} a_i x^i$$

Representado por la tupla de coeficientes $\overline{a} = (a_0, ..., a_{n-1})$, definimos la **Transformada Alternativa de Fourier** como:

$$\mathbf{TAF}(\overline{a}) = [p(\omega_{2n}^1, \omega_{2n}^3, ..., \omega_{2n}^{2n-1})]$$

Demuestre que las mismas ideas utilizadas en el algoritmo de **FFT** puede usarse para calcular $\mathbf{TAF}(\overline{a})$ en tiempo $\mathcal{O}(n\log(n))$, considerando la suma y multiplicación de números complejos como operación básica a contar.

$p(x) \rightarrow$ grado $n-1$ $\quad [(x_1, p(x_1)), (x_2, p(x_2)), \ldots, (x_n, p(x_n))]$

$x^n - 1 = 0 \quad \omega_n^k = e^{\frac{2\pi i}{n}\cdot k} \quad [p(\omega_n^0), p(\omega_n^1), \ldots, p(\omega_n^{n-1})]$

$(\omega_n^k)^n - 1 = 0 \quad (e^{\frac{2\pi i}{n}\cdot k})^n = e^{2\pi i \cdot k} = (1)^k = 1$

$$p(x) = \sum_{k=0}^{n-1} a_k x^k = \boxed{\underbrace{\sum_{k=0}^{\frac{n}{2}-1} a_{2k} x^{2k}}_{q(x^2)} + x \underbrace{\sum_{k=0}^{\frac{n}{2}-1} a_{2k+1} x^{2k}}_{r(x^2)} = p(x) = q(x^2) + x \cdot r(x^2)}$$

$[q((\omega_n^0)^2), \ldots, q((\omega_n^{n-1})^2)] \quad (\omega_n^{\frac{n}{2}+k})^2 = \omega_n^{n+2k} = \omega^n \cdot \omega^{2k} = \omega_n^{2k} = (\omega_n^k)^2$

$\omega_{m\cdot\alpha}^{k\cdot\alpha} = \omega_m^k \longrightarrow (\omega_n^k)^2 = \omega_n^{k\cdot 2} = \omega_{\frac{n}{2}\cdot 2}^{k\cdot 2} = \omega_{\frac{n}{2}}^k$

$p(\omega_n^k) = q((\omega_n^k)^2) \pm \omega_n^k \cdot r((\omega_n^k)^2)$

$\quad = q(\omega_{n/2}^k) \pm \omega_n^k \cdot r(\omega_{n/2}^k)$

$\mathrm{DFT}(q) = [q(\omega_{n/2}^0), q(\omega_{n/2}^1), \ldots, q(\omega_{n/2}^{\frac{n}{2}-1})]$

$\mathrm{DFT}(r)$

CB: $p(x) = a_0 + a_1 \cdot x \quad \begin{cases} p(\omega_2^0) = a_0 + a_1 \cdot 1 = a_0 + a_1 \\ p(\omega_2^1) = a_0 + a_1 \cdot (-1) = a_0 - a_1 \end{cases}$

$\mathrm{TAF} = p(\omega_{2n}^1), \ldots, p(\omega_{2n}^{2n-1})$

$p(x) = q(x^2) + x \cdot r(x^2) \longrightarrow [q((\omega_{2n}^1)^2), \ldots, q((\omega_{2n}^{2n-1})^2)]$

$[r((\omega_{2n}^1)^2), \ldots, r((\omega_{2n}^{2n-1})^2)]$

$m = 2n$

$(\omega_m^{\frac{m}{2}+k})^2 = (\omega_m^k)^2 \longrightarrow (\omega_{2n}^{n+k})^2 = (\omega_{2n}^k)^2$ ✓

$\omega_{m\cdot\alpha}^{k\cdot\alpha} = \omega_m^k \longrightarrow (\omega_{2n}^k)^2 = \omega_n^k$ ✓

$\mathrm{TAF}(q)$

$\frac{n}{2} \longrightarrow q(\omega_{\frac{n}{2}\cdot 2}^k) \rightarrow q(\omega_n^k)$

$$p(\omega_{2n}^k) = \underbrace{q(\omega_n^k)}_{\mathrm{TAF}(q)} + \omega_{2n}^k \cdot \underbrace{r(\omega_n^k)}_{\mathrm{TAF}(r)}$$

FFT$(a_0, \dots, a_{n-1})$

  if $n = 2$ **then**

    $y_0 = a_0 + a_1$

    $y_1 = a_0 - a_1$

    return $[y_0, y_1]$

  else

    $[u_0, \dots, u_{\frac{n}{2}-1}] :=$ **FFT**$(a_0, \dots, a_{n-2})$

    $[v_0, \dots, v_{\frac{n}{2}-1}] :=$ **FFT**$(a_1, \dots, a_{n-1})$

    $\omega_n := e^{\frac{2\pi i}{n}}$

    $\alpha := 1$

    for $k := 0$ **to** $\frac{n}{2} - 1$ **do**

      $y_k := u_k + \alpha \cdot v_k$

      $y_{\frac{n}{2}+k} := u_k - \alpha \cdot v_k$

      $\alpha := \alpha \cdot \omega_n$

    return $[y_0, \dots, y_{n-1}]$

$$\omega_n^k = e^{\frac{2\pi i}{n} \cdot k}$$

$k = 0$

$$e^{(\,)\cdot 0} = e^0 = 1$$

$k = 0 \quad - 1$

$k = 1 \quad \alpha \cdot e^{\frac{2\pi i}{n}} \qquad \omega_n^1$

$k = 2 \quad e^{\frac{2\pi i}{n}} \cdot e^{\frac{2\pi i}{n}} \qquad \omega_n^2$

$k = 3 \quad e^{\frac{2\pi i}{n} \cdot 2} \cdot e^{\frac{2\pi i}{n}} \qquad \omega_n^3$

$k = 4 \quad e^{\frac{2\pi i}{n} \cdot 3} \cdot e^{\frac{2\pi i}{n}} = e^{\frac{2\pi i}{n} \cdot 4} \qquad \omega_n^4$