Module
3

Analysis of Statically Indeterminate Structures by the Displacement Method

Lesson
16
The Slope-Deflection Method: Frames Without Sidesway

Instructional Objectives

After reading this chapter the student will be able to
1. State whether plane frames are restrained against sidesway or not.
2. Able to analyse plane frames restrained against sidesway by slope-deflection equations.
3. Draw bending moment and shear force diagrams for the plane frame.
4. Sketch the deflected shape of the plane frame.

16.1 Introduction

In this lesson, slope deflection equations are applied to solve the statically indeterminate frames without sidesway. In frames axial deformations are much smaller than the bending deformations and are neglected in the analysis. With this assumption the frames shown in Fig 16.1 will not sidesway. i.e. the frames will not be displaced to the right or left. The frames shown in Fig 16.1(a) and Fig 16.1(b) are properly restrained against sidesway. For example in Fig 16.1(a) the joint can't move to the right or left without support A also moving .This is true also for joint D. Frames shown in Fig 16.1 (c) and (d) are not restrained against sidesway. However the frames are symmetrical in geometry and in loading and hence these will not sidesway. In general, frames do not sidesway if

1) They are restrained against sidesway.
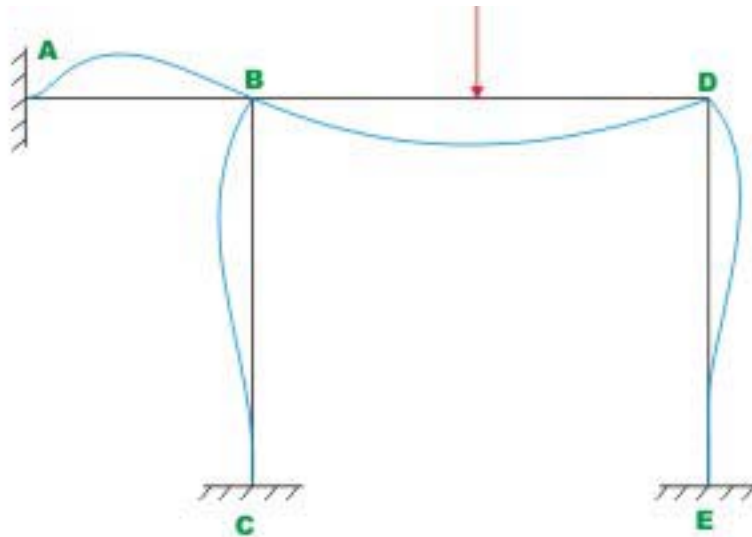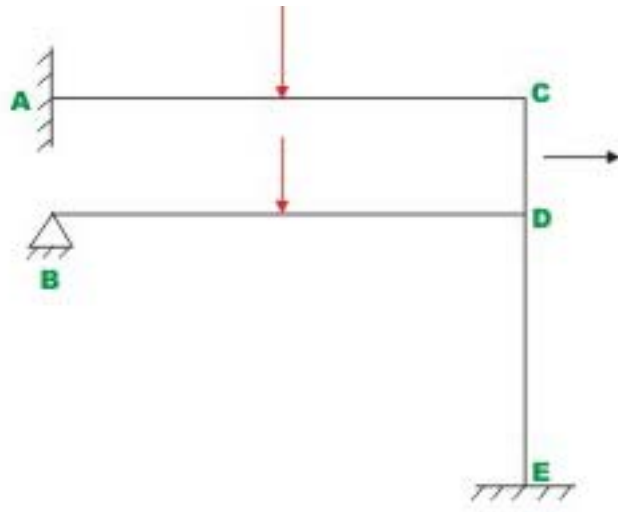2) The frame geometry and loading is symmetrical
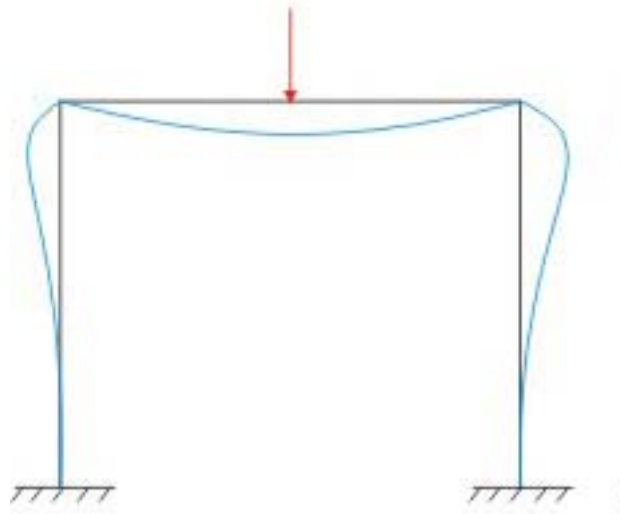

Fig- 16.1(a)
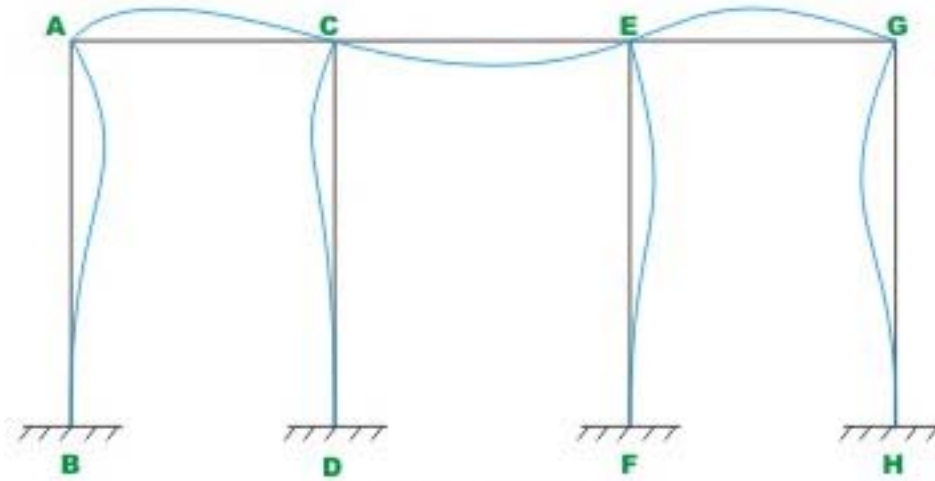
Fig- 16.1(b)

Fig- 16.1(c)

Fig- 16.1(d)

For the frames shown in Fig 16.1, the angle ψ in slope-deflection equation is zero. Hence the analysis of such rigid frames by slope deflection equation essentially follows the same steps as that of continuous beams without support settlements. However, there is a small difference. In the case of continuous beam, at a joint only two members meet. Whereas in the case of rigid frames two or more than two members meet at a joint. At joint C in the frame shown in Fig 16.1(d) three members meet. Now consider the free body diagram of joint C as shown in fig 16.2 .The equilibrium equation at joint C is
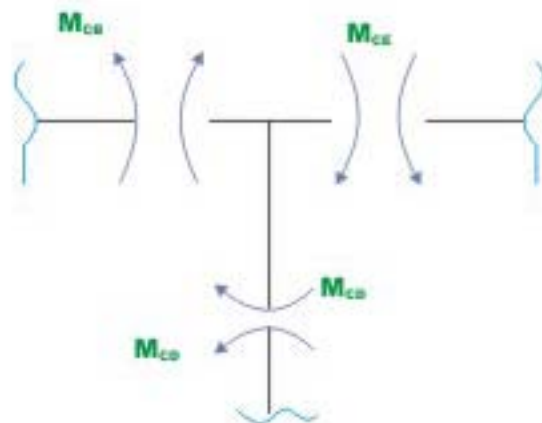

Fig- 16.2

$$\sum M_C = 0 \Rightarrow \qquad M_{CB} + M_{CE} + M_{CD} = 0$$

At each joint there is only one unknown as all the ends of members meeting at a joint rotate by the same amount. One would write as many equilibrium equations as the no of unknowns, and solving these equations joint rotations are evaluated. Substituting joint rotations in the slope–deflection equations member end moments are calculated. The whole procedure is illustrated by few examples. Frames undergoing sidesway will be considered in next lesson.

Example 16.1

Analyse the rigid frame shown in Fig 16.3 (a). Assume EI to be constant for all the members. Draw bending moment diagram and also sketch the elastic curve.

Solution

In this problem only one rotation needs to be determined i. e. θ_B. Thus the required equations to evaluate θ_B is obtained by considering the equilibrium of joint B. The moment in the cantilever portion is known. Hence this moment is applied on frame as shown in Fig 16.3 (b). Now, calculate the fixed-end moments by fixing the support B (vide Fig 16.3 c). Thus
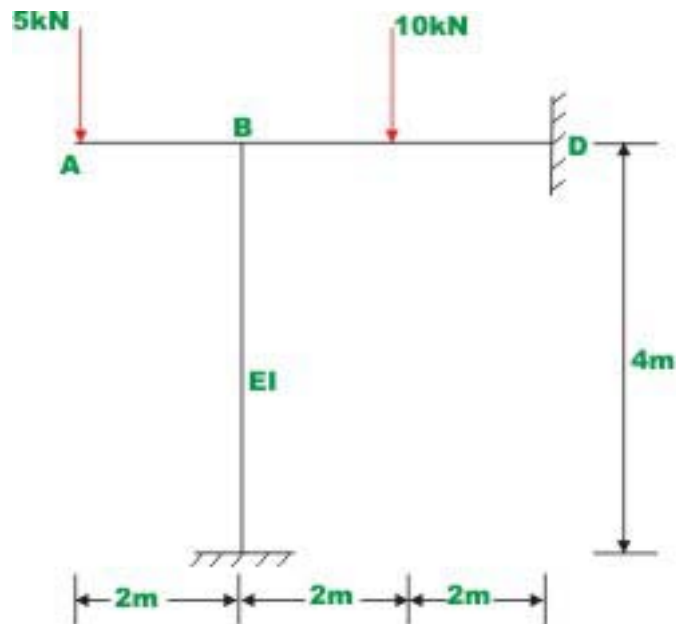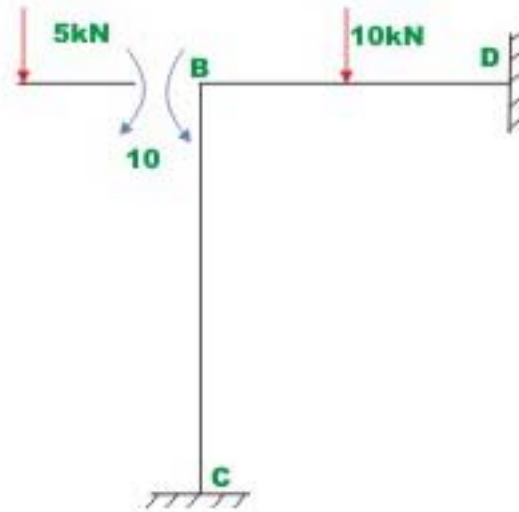

Fig- 16.3 a Example 16.1
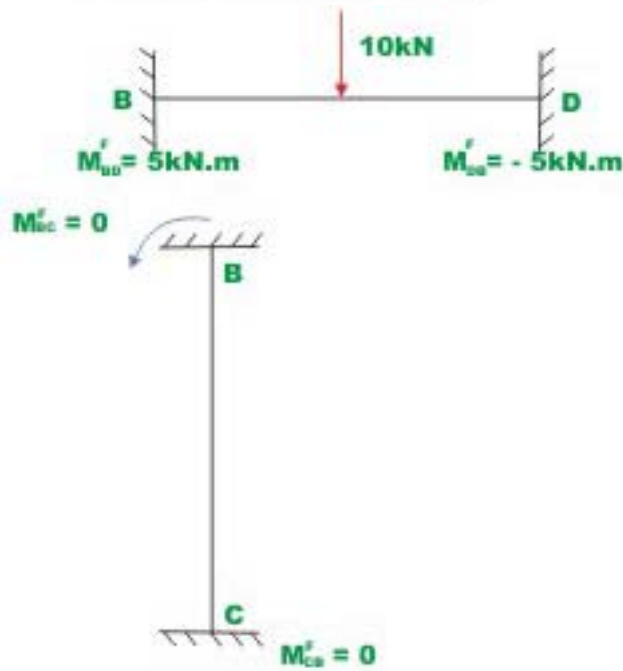
Fig- 16.3 b Moment at joint B due to overhang

Fig- 16.3 © Kinematically restrained structure

$$M^F_{BD} = +5 \text{ kNm}$$

$$M^F_{DB} = -5 \text{ kNm}$$

$$M^F_{BC} = 0 \text{ kNm}$$

$$M^F_{BC} = 0 \text{ kNm}$$

For writing slope–deflection equations two spans must be considered, BC and BD. Since supports C and D are fixed $\theta_C = \theta_D = 0$. Also the frame is restrained against sidesway.

$$M_{BD} = 5 + \frac{2EI}{4}\left[2\theta_B\right] = 5 + EI\theta_B$$
$$M_{DB} = 5 + \frac{2EI}{4}\left[\theta_B\right] = -5 + 0.5EI\theta_B$$
$$M_{BC} = EI\theta_B$$
$$M_{CB} = 0.5EI\theta_B \hspace{3cm} (2)$$

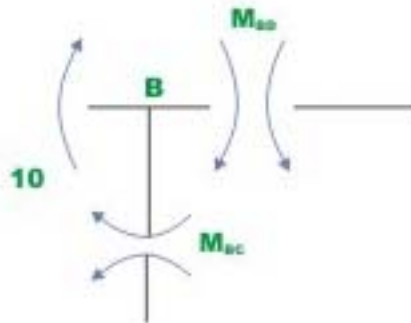Now consider the joint equilibrium of support B, (see Fig 16.3 d)

Fig- 16.3 (d) Free - body diagram of joint B

$$\sum M_B = 0 \quad \Rightarrow \quad M_{BD} + M_{BC} - 10 = 0 \tag{3}$$

Substituting the value of M_{BD} and M_{BC} and from equation (2) in the above equation

$$5 + EI\theta_B + EI\theta_B - 10 = 0$$

$$\theta_B = \frac{2.5}{EI} \tag{4}$$

Substituting the values of θ_B in equation (2), the beam end moments are calculated

$$M_{BD} = +7.5 \, \text{kN} \cdot \text{m}$$

$$M_{DB} = -3.75 \, \text{kN} \cdot \text{m}$$

$$M_{BC} = +2.5 \, \text{kN} \cdot \text{m}$$

$$M_{CB} = +1.25 \, \text{kN} \cdot \text{m} \tag{5}$$

The reactions are evaluated from static equations of equilibrium. The free body diagram of each member of the frame with external load and end moments are shown in Fig 16.3 (e)
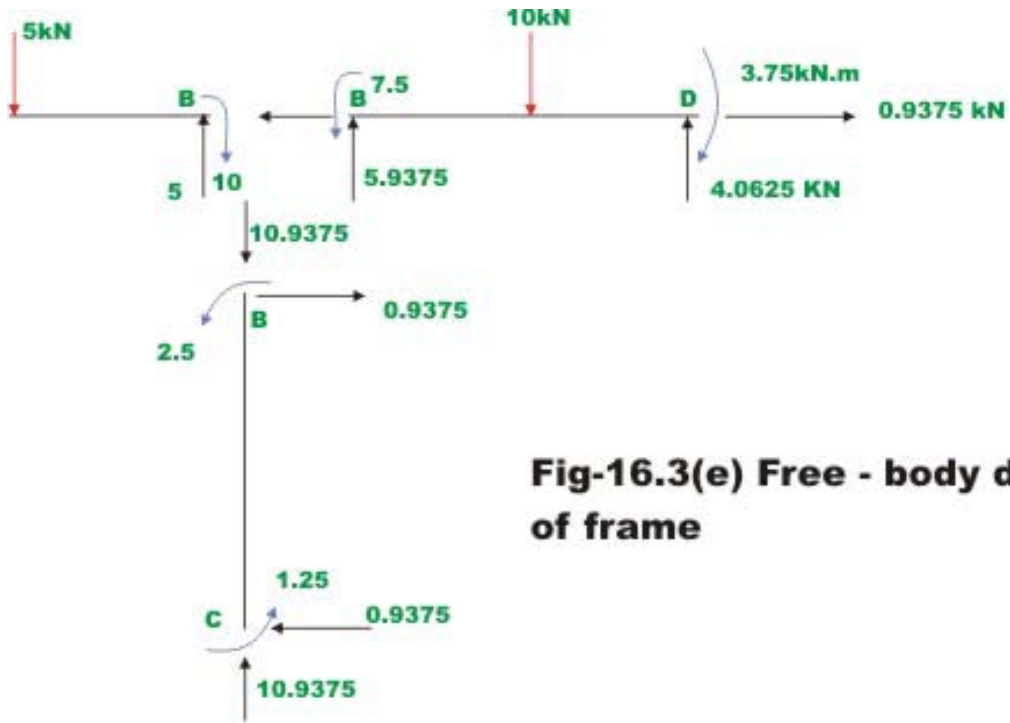
Fig-16.3(e) Free - body diagram of frame

$$R_{Cy} = 10.9375 \text{ kN}(\uparrow)$$

$$R_{Cx} = -0.9375 \text{ kN}(\leftarrow)$$

$$R_{Dy} = 4.0625 \text{ kN}(\uparrow)$$

$$R_{Dx} = 0.9375 \text{ kN}(\rightarrow) \qquad\qquad (6)$$
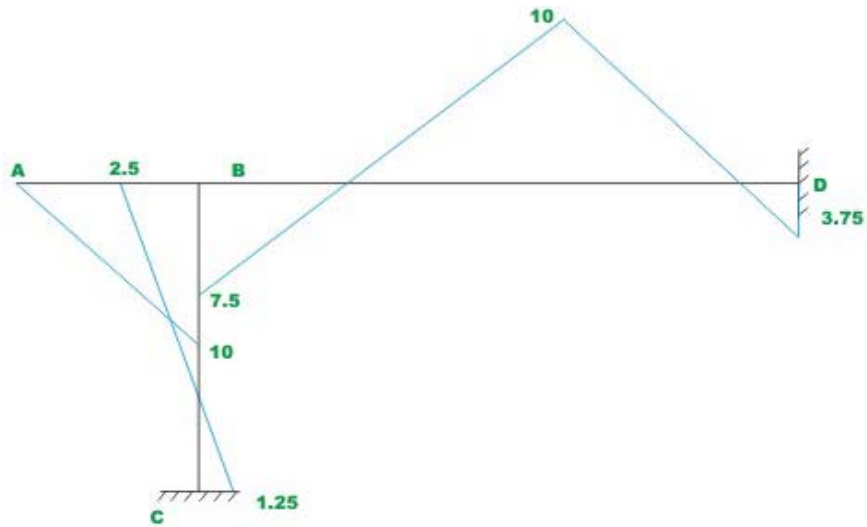
Bending moment diagram is shown in Fig 16.3(f)

Fig-16.3(f) Bending moment diagram plotted on compression side

The vertical hatching is use to represent the bending moment diagram for the horizontal member (beams) and horizontal hatching is used for bending moment diagram for the vertical members.

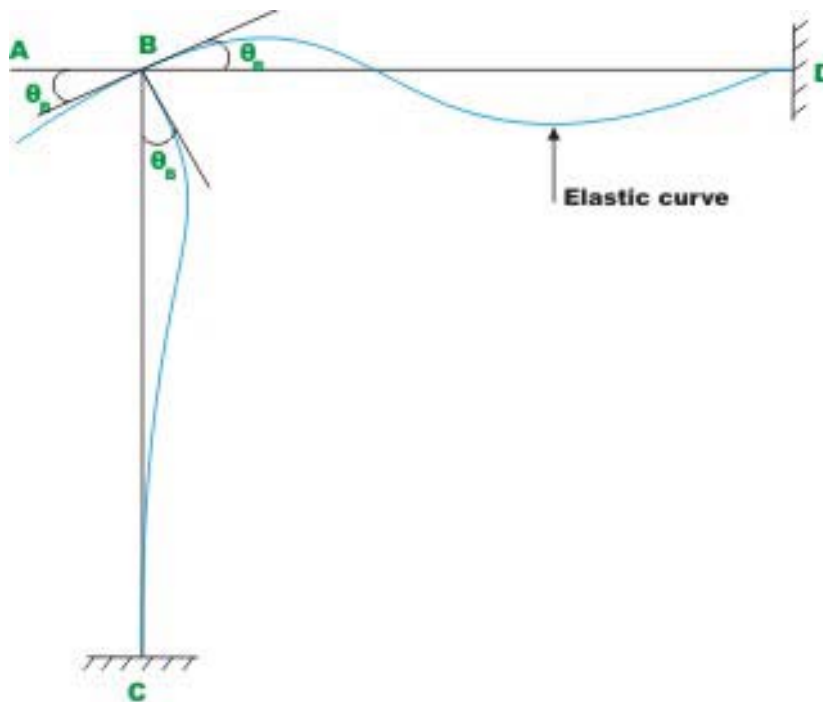The qualitative elastic curve is shown in Fig 16.3 (g).

Fig-16.3(g) Elastic curve

Example 16.2

Compute reactions and beam end moments for the rigid frame shown in Fig 16.4 (a). Draw bending moment and shear force diagram for the frame and also sketch qualitative elastic curve.
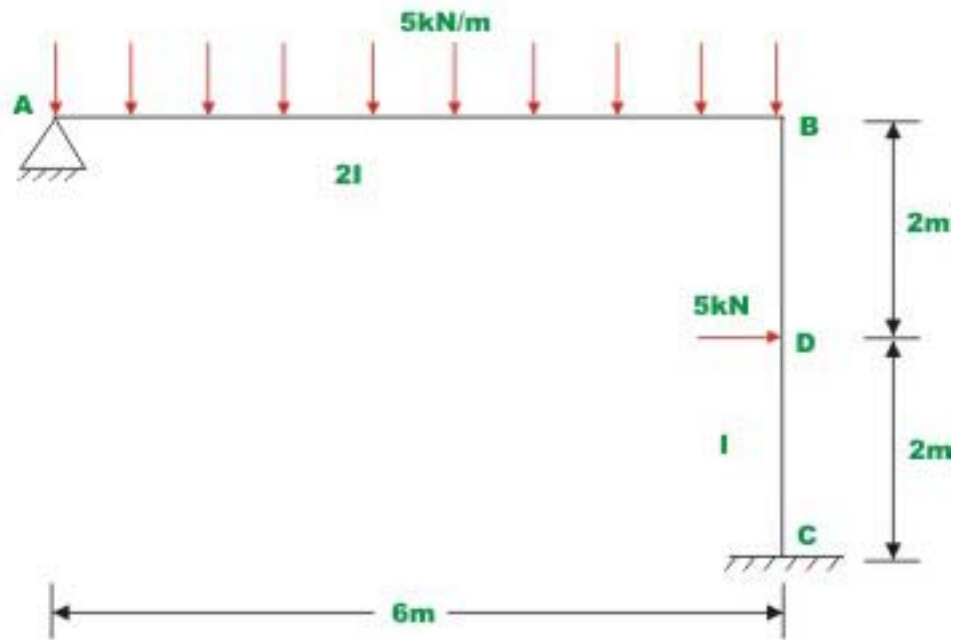
Solution

Fig-16.4(a) Example 16.2

In this frame rotations θ_A and θ_B are evaluated by considering the equilibrium of joint A and B. The given frame is kinematically indeterminate to second degree. Evaluate fixed end moments. This is done by considering the kinematically determinate structure. (Fig 16.4 b)
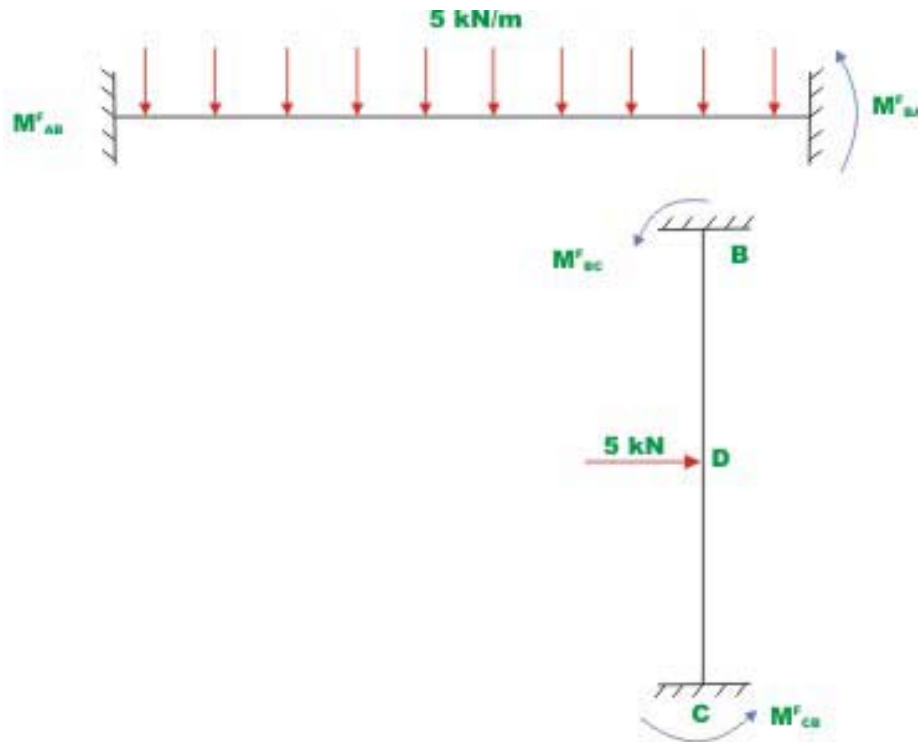
Fig-16.4(b) Kinematically restrained structure

$$M^F_{DB} = \frac{5 \times 6^2}{12} = 15 \text{ kN.m}$$

$$M^F_{BA} = \frac{-5 \times 6^2}{12} = -15 \text{ kN.m}$$

$$M^F_{BC} = \frac{5 \times 2 \times 2^2}{4^2} = 2.5 \text{ kN.m}$$

$$M^F_{CD} = \frac{-5 \times 2 \times 2^2}{4^2} = -2.5 \text{ kN.m} \tag{1}$$

Note that the frame is restrained against sidesway. The spans must be considered for writing slope-deflection equations viz, A, B and AC. The beam end moments are related to unknown rotations θ_A and θ_B by following slope-deflection equations. (Force deflection equations). Support C is fixed and hence $\theta_C = 0$.

$$M_{AB} = M^F_{ABL} + \frac{2E(2I)}{L_{AB}} (2\theta_A + \theta_B)$$

$$M_{AB} = 15 - + 1.333 EI\theta_A + 0.667 EI\theta_B$$

$$M_{BA} = -15 + 0.667 EI\theta_A + 1.333 EI\theta_B$$

$$M_{BC} = 2.5 + EI\theta_B + 0.5 EI\theta_C$$

$$M_{CB} = -2.5 + 0.5 EI\theta_B \qquad\qquad (2)$$

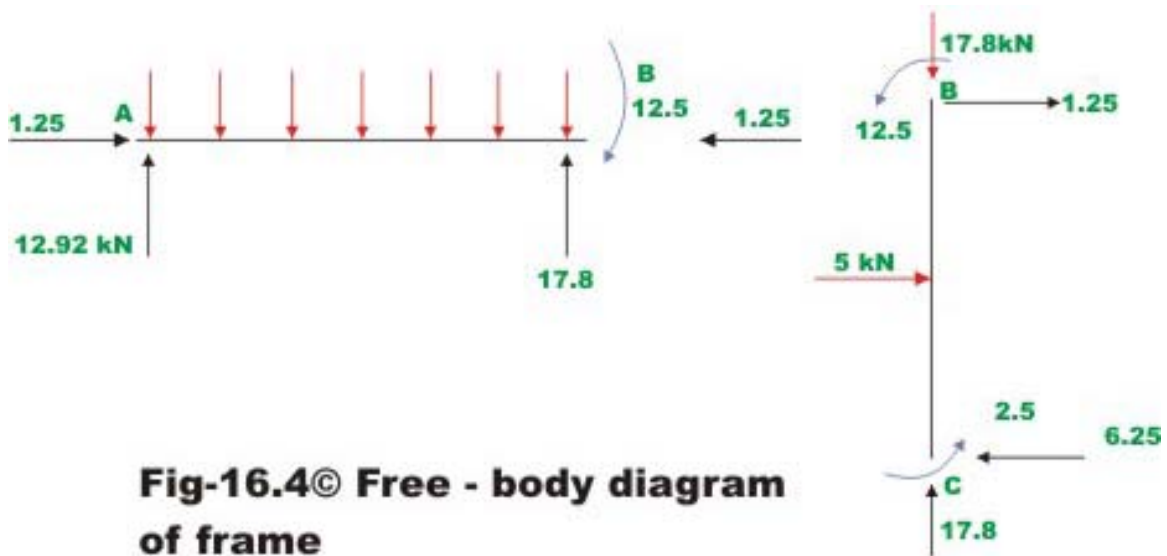Consider the joint equilibrium of support *A* (See Fig 16.4 (c))

$$\sum M_A = 0$$

$$M_{AB} = 0 = 15 + 1.333 EI\theta_A + 0.667 EI\theta_B \qquad\qquad (3)$$

$$1.333 EI\theta_A + + 0.667 EI\theta_B = -15$$

Or, $2\theta_A + \theta_B = \dfrac{-22.489}{EI}$

Equilibrium of joint *B* (Fig 16.4(d))

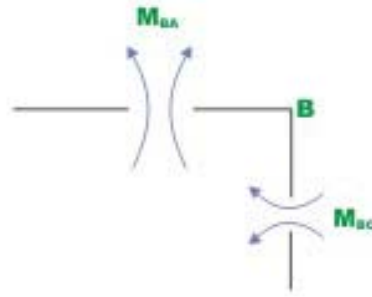**Fig-16.4© Free - body diagram of frame**

Fig-16.4(d) Free - body diagram of joint B

$$\sum M_B = 0 \quad \Rightarrow \quad M_{BC} + M_{BA} = 0 \tag{4}$$

Substituting the value of M_{BC} and M_{BA} in the above equation,

$$2.333EI\theta_B + 0.667EI\theta_A = 12.5 \tag{5}$$

Or,
$$3.498\theta_B + \theta_A = \frac{18.741}{EI}$$

Solving equation (3) and (4)

$$\theta_B = \frac{10.002}{EI}(counterclockwise)$$

$$\theta_B = \frac{-16.245}{EI}(clockwise) \tag{6}$$

Substituting the value of θ_A and θ_B in equation (2) beam end moments are evaluated.

$$M_{AB} = 15 + 1.333EI\left(\frac{-16.245}{EI}\right) + 0.667EI\left(\frac{10.002}{EI}\right) = 0$$

$$M_{BA} = -15 + 0.667EI\left(\frac{-16.245}{EI}\right) + .1.33EI\left(\frac{10.002}{EI}\right) = -1$$

$$M_{BC} = 2.5 + EI\left(\frac{10.002}{EI}\right) = 12.5 \text{ kN.m}$$

$$M_{CB} = -2.5 + 0.5EI\left(\frac{10.002}{EI}\right) = 2.5 \text{ kN.m} \tag{7}$$

Using these results, reactions are evaluated from equilibrium equations as shown in Fig 16.4 (e)
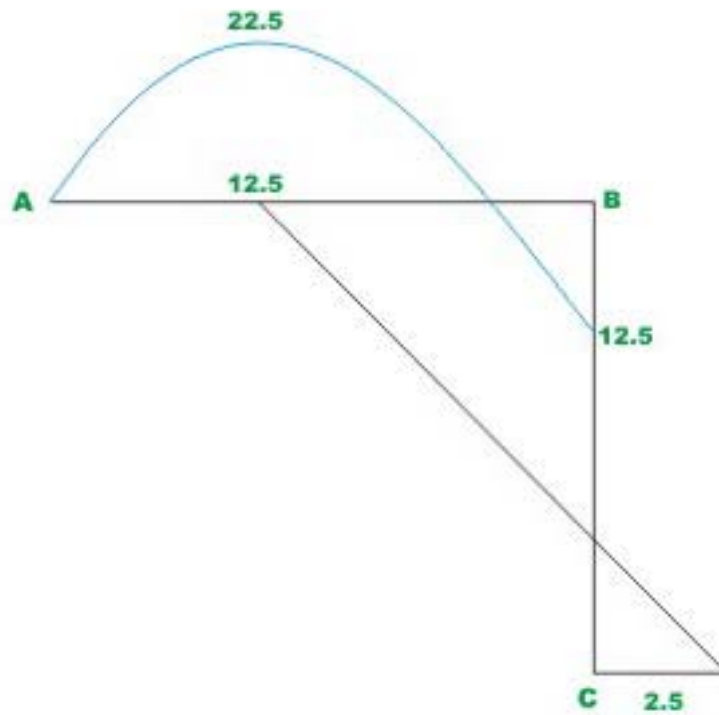

Fig-16.4(e) B.M.D

The shear force and bending moment diagrams are shown in Fig 16.4(g) and 16.4 h respectively. The qualitative elastic curve is shown in Fig 16.4 (h).
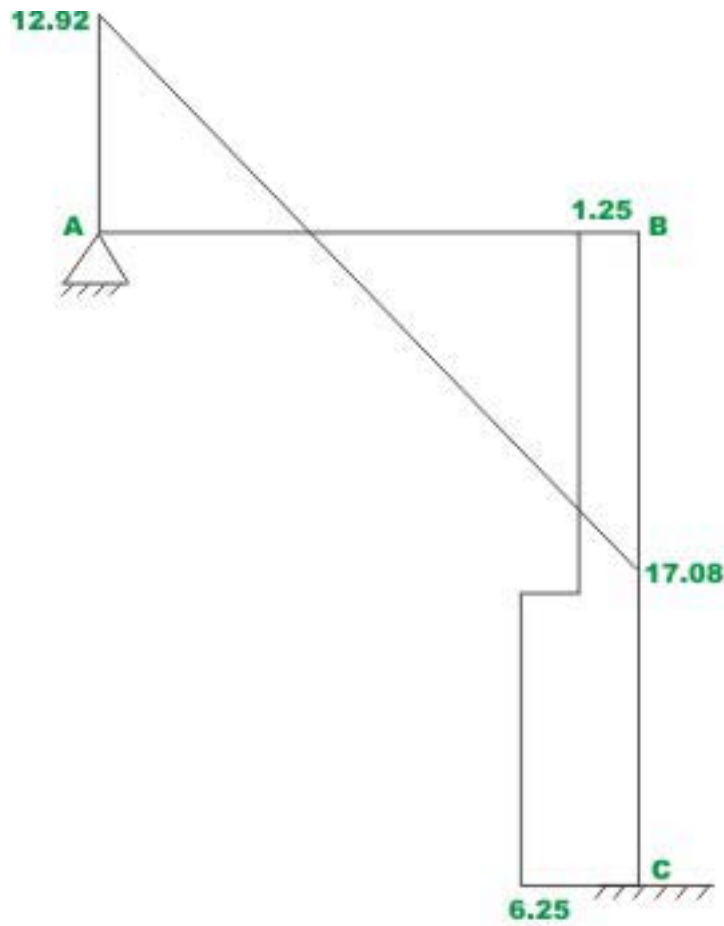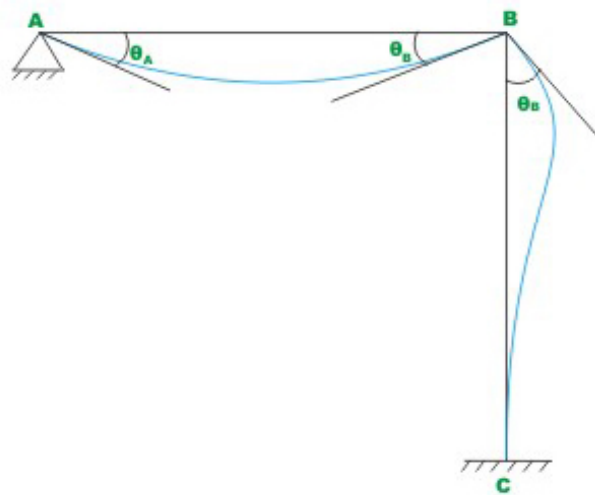
Fig-16.4(f) S.F.D
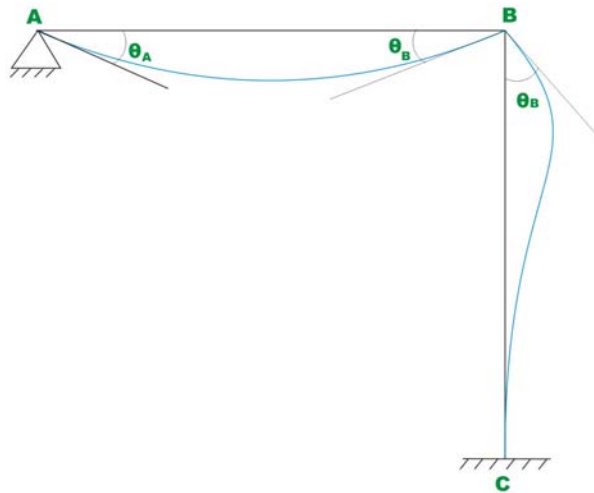

Fig.16.4 (g)Elastic Curve

Fig- 16.4(h) Elastic Curve

Example 16.3

Compute reactions and beam end moments for the rigid frame shown in Fig 16.5(a). Draw bending moment diagram and sketch the elastic curve for the frame.
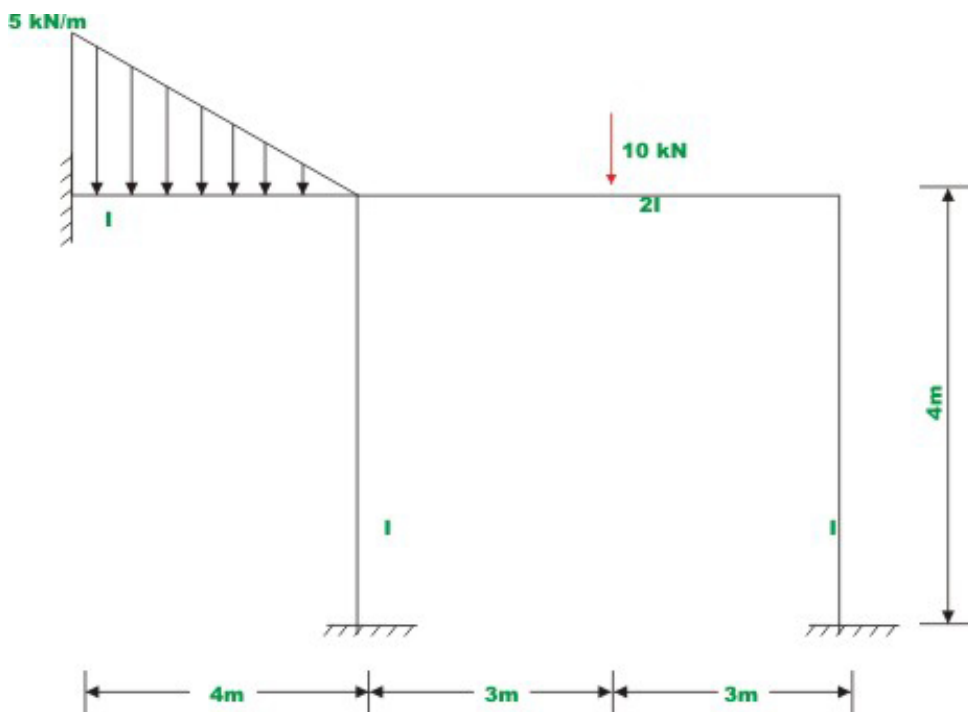
Solution

Fig-16.5(a) Example 16.3

The given frame is kinematically indeterminate to third degree so three rotations are to be calculated, θ_B, θ_C and θ_D. First calculate the fixed end moments (see Fig 16.5 b).
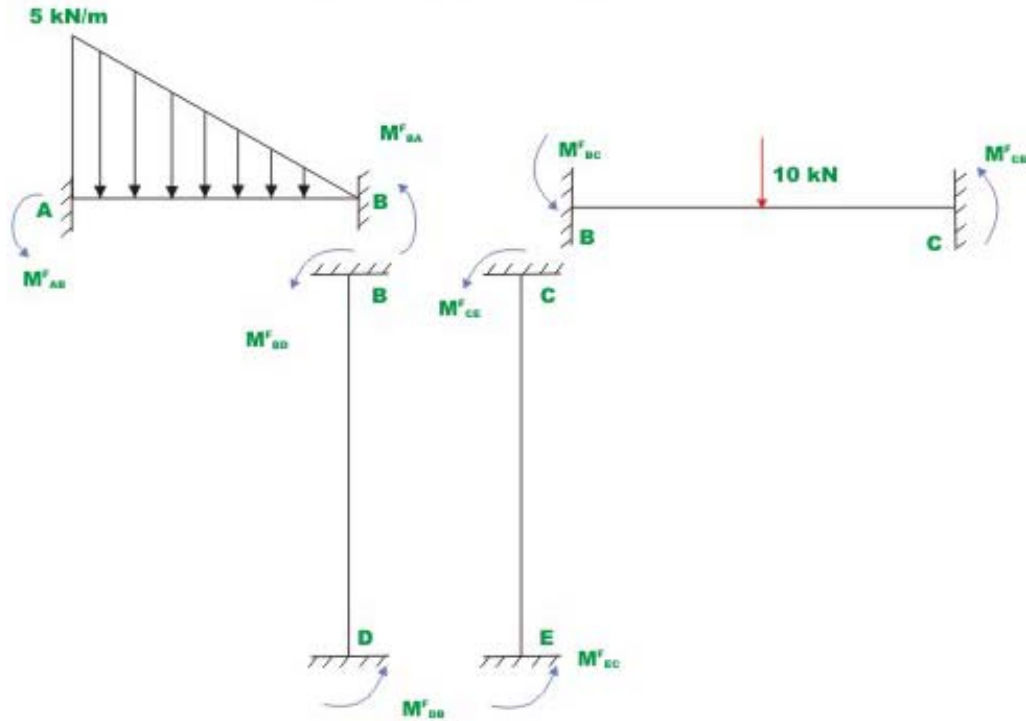

Fig.16.5b Kinematically restrained structure

$$M_{AB}^F = \frac{5 \times 4^2}{20} = 4 \text{ kN.m}$$

$$M_{BA}^F = \frac{-5 \times 4^2}{30} = -2.667 \text{ kN.m}$$

$$M_{BC}^F = \frac{10 \times 3 \times 3^2}{6^2} = 7.5 \text{ kN.m}$$

$$M_{CB}^F = \frac{-10 \times 3 \times 3^2}{6^2} = -7.5 \text{ kN.m}$$

$$M_{BD}^F = M_{DB}^F = M_{CE}^F = M_{EC}^F = 0 \tag{1}$$

The frame is restrained against sidesway. Four spans must be considered for rotating slope – deflection equation: AB, BD, BC and CE. The beam end

moments are related to unknown rotation at B, C, and D. Since the supports A and E are fixed. $\theta_A = \theta_E = 0$.

$$M_{AB} = 4 + \frac{2EI}{4}\left[2\theta_A + \theta_B\right]$$

$$M_{AB} = 4 + EI\theta_A + 0.5EI\theta_B = 4 + 0.5EI\theta_B$$

$$M_{BA} = -2.667EI\theta_A + EI\theta_B = -2.667 + EI\theta_B$$

$$M_{BD} = EI\theta_B + 0.5EI\theta_D$$

$$M_{DB} = 0.5EI\theta_B + EI\theta_D$$

$$M_{BC} = 7.5 + \frac{2E(2I)}{6}\left[2\theta_B + \theta_C\right] = 7.5 + 1.333EI\theta_B + 0.667EI\theta_C$$

$$M_{CB} = -7.5 + .667EI\theta_B + 1.333EI\theta_C$$

$$M_{CE} = EI\theta_C + 0.5EI\theta_E = EI\theta_C$$

$$M_{EC} = 0.5EI\theta_C + 0.5EI\theta_E = 0.5EI\theta_C \qquad\qquad (2)$$

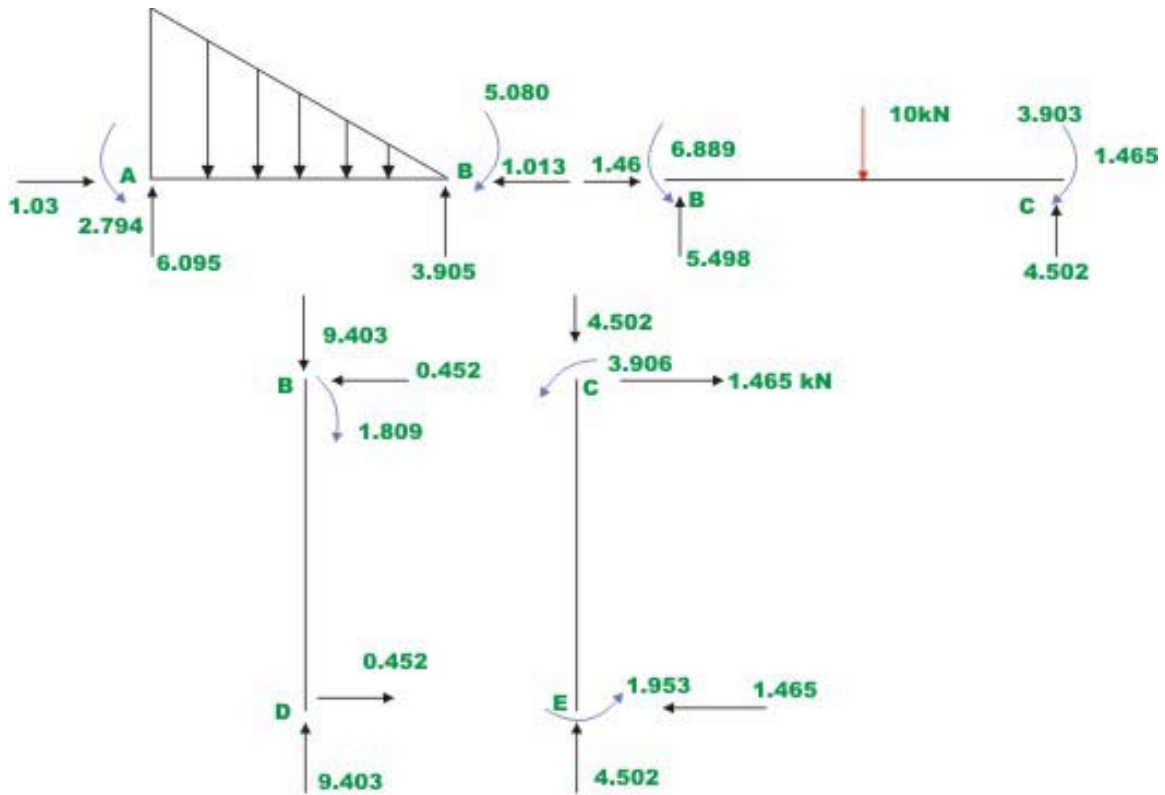Consider the equilibrium of joints B, D, C (vide Fig. 16.5(c))

Fig-16.5 (c) Free - body diagram

$$\sum M_B = 0 \quad \Rightarrow \quad M_{BA} + M_{BC} + M_{BD} = 0 \tag{3}$$

$$\sum M_D = 0 \quad \Rightarrow \quad M_{DB} = 0 \tag{4}$$

$$\sum M_C = 0 \quad \Rightarrow \quad M_{CB} + M_{CE} = 0 \tag{5}$$

Substituting the values of $M_{BA}, M_{BC}, M_{BD}, M_{DB}, M_{CB}$ and M_{CE} in the equations (3), (4), and (5)

$$3.333EI\theta_B + 0.667EI\theta_C + 0.5EI\theta_D = -4.833$$

$$0.5EI\theta_B + EI\theta_D = 0$$

$$2.333EI\theta_C + 0.667EI\theta_B = 7.5 \tag{6}$$

Solving the above set of simultaneous equations, θ_B, θ_C and θ_D are evaluated.

$$EI\theta_B = -2.4125$$

$$EI\theta_C = 3.9057$$

$$EI\theta_D = 1.2063 \tag{7}$$

Substituting the values of θ_B, θ_C and θ_D in (2), beam end moments are computed.

$$M_{AB} = 2.794 \text{ kN.m}$$

$$M_{BA} = -5.080 \text{ kN.m}$$

$$M_{BD} = -1.8094 \text{ kN.m}$$

$$M_{DB} = 0$$

$$M_{BC} = 6.859 \text{ kN.m}$$

$$M_{CB} = -3.9028 \text{ kN.m}$$

$$M_{CE} = 3.9057 \text{ kN.m}$$

$$M_{EC} = 1.953 \text{ kN.m} \tag{8}$$

The reactions are computed in Fig 16.5(d), using equilibrium equations known beam-end moments and given loading.
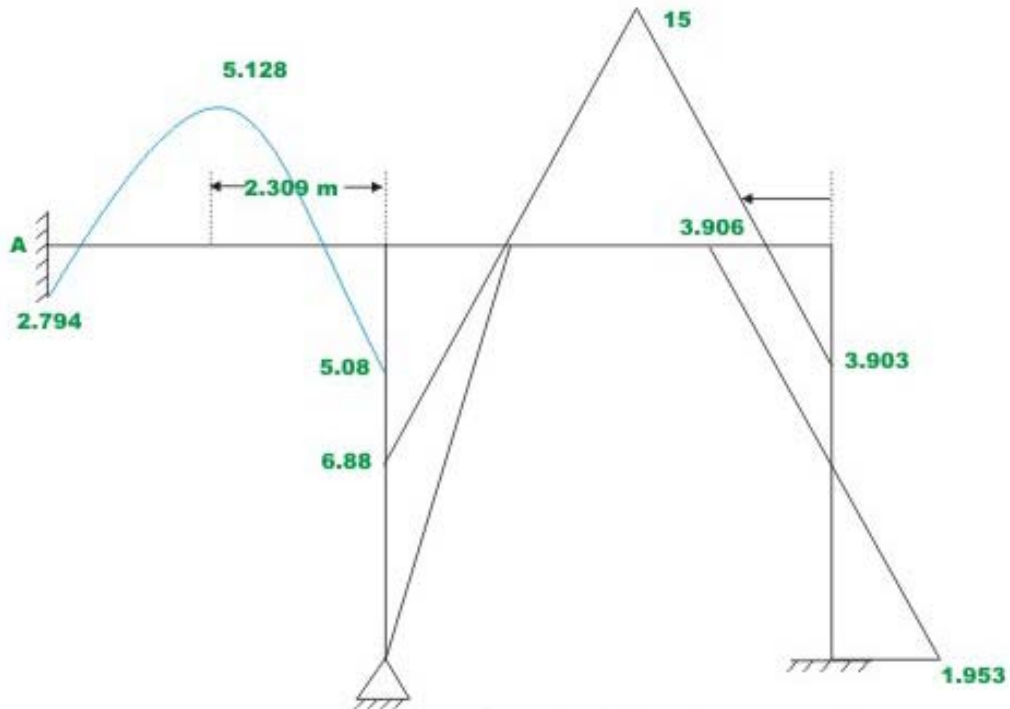
Fig-16.5(d) Bending moment diagram

$$R_{Ay} = 6.095 \text{ kN} (\uparrow)$$

$$R_{Dy} = 9.403 \text{ kN} (\uparrow)$$

$$R_{Ey} = 4.502 \text{ kN} (\uparrow)$$

$$R_{Ax} = 1.013 \text{ kN} (\rightarrow)$$

$$R_{Dx} = 0.542 \text{ kN} (\rightarrow)$$

$$R_{Ex} = -1.465 \text{ kN} (\leftarrow) \tag{9}$$

The bending moment diagram is shown in Fig 16.5.(e) and the elastic curve is shown in Fig 16.5(f).
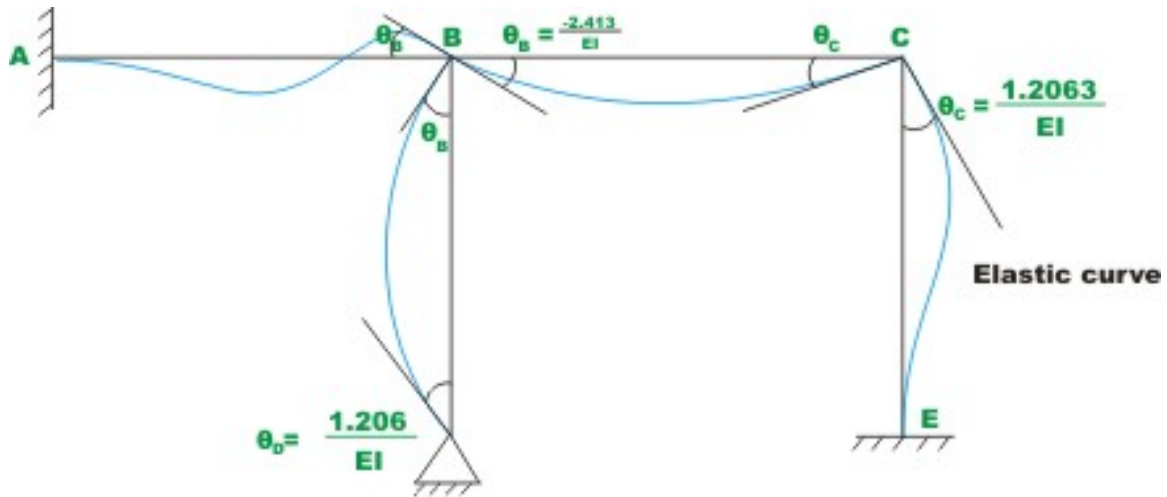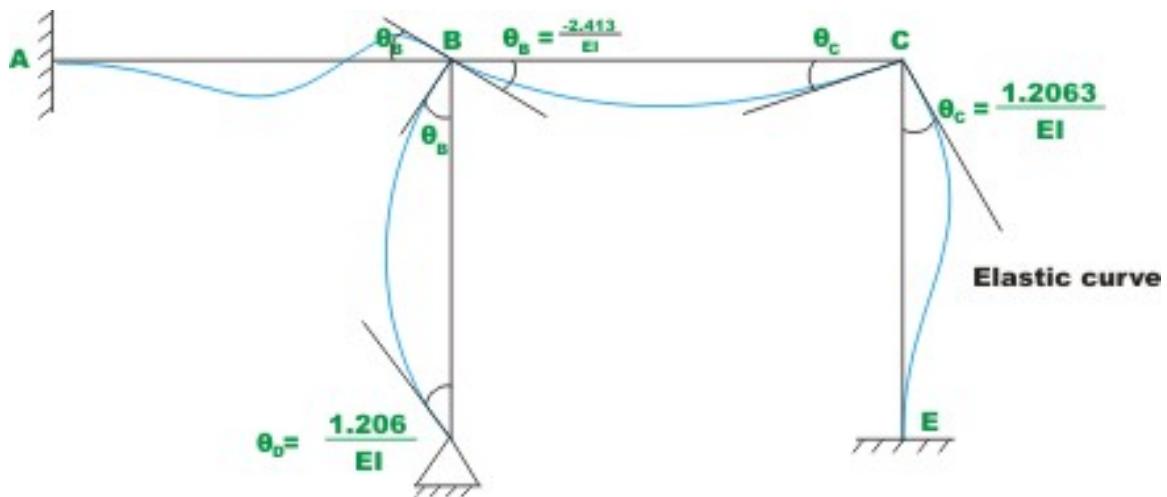
Fig-16.5(e)

Fig-16.5(f)

Summary

In this lesson plane frames restrained against sidesway are analysed using slope-deflection equations. Equilibrium equations are written at each rigid joint of the frame and also at the support. Few problems are solved to illustrate the procedure. The shear force and bending moment diagrams are drawn for the plane frames.