Module
4

Analysis of Statically Indeterminate Structures by the Direct Stiffness Method

Lesson
28
The Direct Stiffness Method: Beams (Continued)

Instructional Objectives

After reading this chapter the student will be able to
1. Derive member stiffness matrix of a beam element.
2. Assemble member stiffness matrices to obtain the global stiffness matrix for a beam.
3. Write the global load-displacement relation for the beam.
4. Impose boundary conditions on the load-displacement relation of the beam.
5. Analyse continuous beams by the direct stiffness method.

28.1 Introduction

In the last lesson, the procedure to analyse beams by direct stiffness method has been discussed. No numerical problems are given in that lesson. In this lesson, few continuous beam problems are solved numerically by direct stiffness method.

Example 28.1

Analyse the continuous beam shown in Fig. 28.1a. Assume that the supports are unyielding. Also assume that EI is constant for all members.
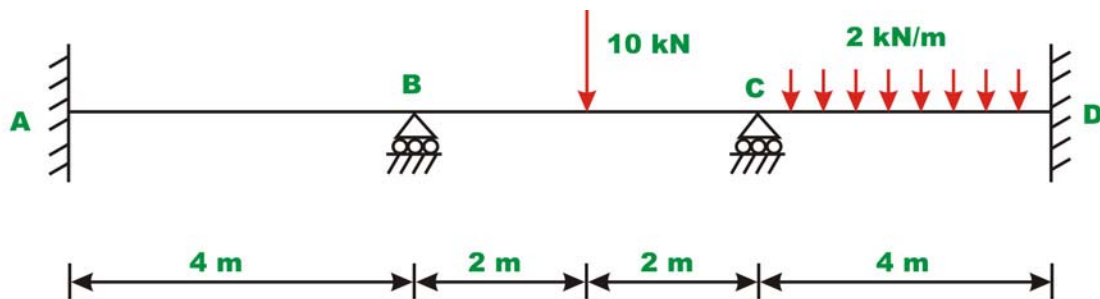

Fig. 28.1a

The numbering of joints and members are shown in Fig. 28.1b. The possible global degrees of freedom are shown in the figure. Numbers are put for the unconstrained degrees of freedom first and then that for constrained displacements.
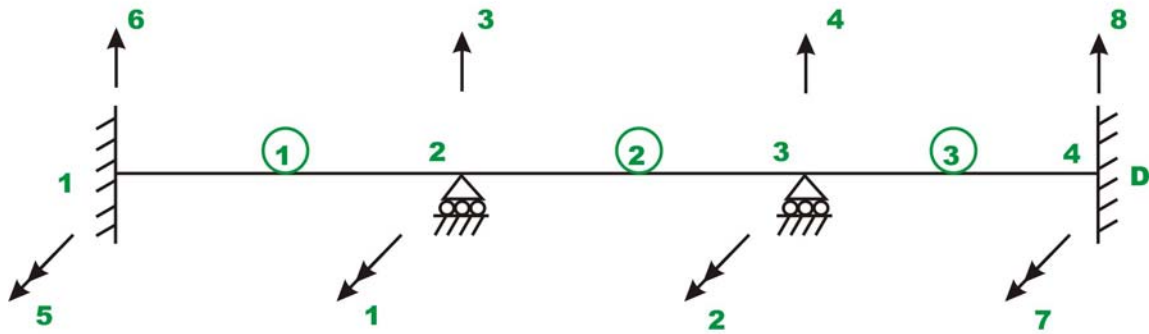
Figure 28.1b

The given continuous beam is divided into three beam elements Two degrees of freedom (one translation and one rotation) are considered at each end of the member. In the above figure, double headed arrows denote rotations and single headed arrow represents translations. In the given problem some displacements are zero, i.e., $u_3 = u_4 = u_5 = u_6 = u_7 = u_8 = 0$ from support conditions.

In the case of beams, it is not required to transform member stiffness matrix from local co-ordinate system to global co-ordinate system, as the two co-ordinate system are parallel to each other.

Figure 28.1c

First construct the member stiffness matrix for each member. This may be done from the fundamentals. However, one could use directly the equation (27.1) given in the previous lesson and reproduced below for the sake convenience.

$$[k] = \begin{bmatrix} \dfrac{12EI_z}{L^3} & \dfrac{6EI_z}{L^2} & -\dfrac{12EI_z}{L^3} & \dfrac{6EI_z}{L^2} \\ \dfrac{6EI_z}{L^2} & \dfrac{4EI_z}{L} & -\dfrac{6EI_z}{L^2} & \dfrac{2EI_z}{L} \\ -\dfrac{12EI_z}{L^3} & -\dfrac{6EI_z}{L^2} & \dfrac{12EI_z}{L^3} & -\dfrac{6EI_z}{L^2} \\ \dfrac{6EI_z}{L^2} & \dfrac{2EI_z}{L} & -\dfrac{6EI_z}{L^2} & \dfrac{4EI_z}{L} \end{bmatrix} \tag{1}$$

The degrees of freedom of a typical beam member are shown in Fig. 28.1c. Here equation (1) is used to generate element stiffness matrix.

Member 1: $L = 4\,m$, node points 1-2.

The member stiffness matrix for all the members are the same, as the length and flexural rigidity of all members is the same.

$$\begin{array}{l} Global\ d.o.f \\ \\ [k'] = EI_{zz} \end{array} \begin{array}{cccc} 6 & 5 & 3 & 1 \end{array}$$

$$[k'] = EI_{zz} \begin{bmatrix} 0.1875 & 0.375 & -0.1875 & 0.375 \\ 0.375 & 1.0 & -0.375 & 0.5 \\ -0.1875 & -0.375 & 0.1875 & -0.375 \\ 0.375 & 0.5 & -0.375 & 1.0 \end{bmatrix} \begin{array}{c} 6 \\ 5 \\ 3 \\ 1 \end{array} \tag{2}$$

On the member stiffness matrix, the corresponding global degrees of freedom are indicated to facilitate assembling.

Member 2: $L = 4\,m$, node points 2-3.

$$\begin{array}{l} Global\ d.o.f \end{array} \begin{array}{cccc} 3 & 1 & 4 & 2 \end{array}$$

$$[k^2] = EI_{zz} \begin{bmatrix} 0.1875 & 0.375 & -0.1875 & 0.375 \\ 0.375 & 1.0 & -0.375 & 0.5 \\ -0.1875 & -0.375 & 0.1875 & -0.375 \\ 0.375 & 0.5 & -0.375 & 1.0 \end{bmatrix} \begin{array}{c} 3 \\ 1 \\ 4 \\ 2 \end{array} \tag{3}$$

Member 3: $L = 4\,m$, node points 3-4.

$$
\begin{array}{c}
Global\; d.o.f \qquad\quad 4 \qquad\quad 2 \qquad\quad 8 \qquad\quad 7 \\[4pt]
\left[k^{3}\right]=EI_{zz}
\begin{bmatrix}
0.1875 & 0.375 & -0.1875 & 0.375 \\
0.375 & 1.0 & -0.375 & 0.5 \\
-0.1875 & -0.375 & 0.1875 & -0.375 \\
0.375 & 0.5 & -0.375 & 1.0
\end{bmatrix}
\begin{matrix}
4 \\ 2 \\ 8 \\ 7
\end{matrix}
\end{array}
\tag{4}
$$

The assembled global stiffness matrix of the continuous beam is of the order 8×8. The assembled global stiffness matrix may be written as,

$$
\left[K\right]=EI_{zz}
\begin{bmatrix}
2.0 & 0.5 & 0.0 & -0.375 & 0.5 & 0.375 & 0 & 0 \\
0.5 & 2.0 & 0.375 & 0 & 0 & 0 & 0.5 & -0.375 \\
0 & 0.375 & 0.375 & -0.1875 & -0.375 & -0.1875 & 0 & 0 \\
-0.375 & 0 & -0.1875 & 0.375 & 0 & 0 & 0.375 & -0.1875 \\
0.5 & 0 & -0.375 & 0 & 1.0 & 0.375 & 0 & 0 \\
0.375 & 0 & -0.1875 & 0 & 0.375 & 0.1875 & 0 & 0 \\
0 & 0.5 & 0 & 0.375 & 0 & 0 & 1.0 & -0.375 \\
0 & -0.375 & 0 & -0.1875 & 0 & 0 & -0.375 & 0.1875
\end{bmatrix}
\tag{5}
$$

Now it is required to replace the given members loads by equivalent joint loads. The equivalent loads for the present case is shown in Fig. 28.1d. The displacement degrees of freedom are also shown in Fig. 28.1d.
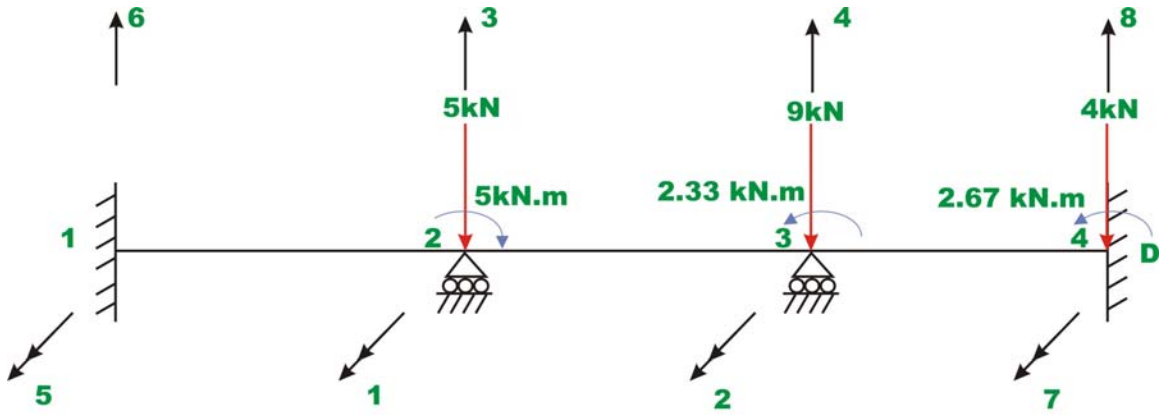
Fig. 28.1 (d) Equivalent joint loads

Thus the global load vector corresponding to unconstrained degree of freedom is,

$$\{p_k\} = \begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix} = \begin{Bmatrix} -5 \\ 2.33 \end{Bmatrix} \tag{6}$$

Writing the load displacement relation for the entire continuous beam,

$$\begin{Bmatrix} -5 \\ 2.33 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \\ p_7 \\ p_8 \end{Bmatrix} = EI_{zz} \begin{bmatrix} 2.0 & 0.5 & 0.0 & -0.375 & 0.5 & 0.375 & 0 & 0 \\ 0.5 & 2.0 & 0.375 & 0 & 0 & 0 & 0.5 & -0.375 \\ 0 & 0.375 & 0.375 & -0.187 & -0.375 & -0.187 & 0 & 0 \\ -0.375 & 0 & -0.187 & 0.375 & 0 & 0 & 0.375 & -0.187 \\ 0.5 & 0 & -0.375 & 0 & 1.0 & 0.375 & 0 & 0 \\ 0.375 & 0 & -0.187 & 0 & 0.375 & 0.187 & 0 & 0 \\ 0 & 0.5 & 0 & 0.375 & 0 & 0 & 1.0 & -0.375 \\ 0 & -0.375 & 0 & -0.187 & 0 & 0 & -0.375 & 0.187 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{Bmatrix} \tag{7}$$

where $\{p\}$ is the joint load vector, $\{u\}$ is displacement vector.

We know that $u_3 = u_4 = u_5 = u_6 = u_7 = u_8 = 0$. Thus solving for unknowns u_1 and u_2, yields

$$\left\{ \begin{array}{c} -5 \\ 2.33 \end{array} \right\} = EI_{zz} \begin{bmatrix} 2.0 & 0.5 \\ 0.5 & 2.0 \end{bmatrix} \left\{ \begin{array}{c} u_1 \\ u_2 \end{array} \right\} \tag{8}$$

$$\left\{ \begin{array}{c} u_1 \\ u_2 \end{array} \right\} = \frac{1}{3.75 EI_{zz}} \begin{bmatrix} 2.0 & -0.5 \\ -0.5 & 2.0 \end{bmatrix} \left\{ \begin{array}{c} -5 \\ 2.333 \end{array} \right\} \tag{9}$$

$$= \frac{1}{EI_{zz}} \left\{ \begin{array}{c} -2.977 \\ 1.909 \end{array} \right\}$$

Thus displacements are,

$$u_1 = \frac{-2.977}{EI_{zz}} \quad \text{and} \quad u_2 = \frac{1.909}{EI_{zz}} \tag{10}$$

The unknown joint loads are given by,

$$\left\{ \begin{array}{c} p_3 \\ p_4 \\ p_5 \\ p_6 \\ p_7 \\ p_8 \end{array} \right\} = EI_{zz} \begin{bmatrix} 0 & 0.375 \\ -0.375 & 0 \\ 0.5 & 0 \\ 0.375 & 0 \\ 0 & 0.5 \\ 0 & -0.375 \end{bmatrix} \frac{1}{EI_{zz}} \left\{ \begin{array}{c} -2.977 \\ 1.909 \end{array} \right\} \tag{11}$$

$$= \left\{ \begin{array}{c} 0.715 \\ 1.116 \\ -1.488 \\ -1.116 \\ 0.955 \\ -0.715 \end{array} \right\}$$

The actual reactions at the supports are calculated as,

$$
\begin{Bmatrix} R_3 \\ R_4 \\ R_5 \\ R_6 \\ R_7 \\ R_8 \end{Bmatrix} = \begin{Bmatrix} p_3^F \\ p_4^F \\ p_5^F \\ p_6^F \\ p_7^F \\ p_8^F \end{Bmatrix} + \begin{Bmatrix} p_3 \\ p_4 \\ p_5 \\ p_6 \\ p_7 \\ p_8 \end{Bmatrix} = \begin{Bmatrix} 5 \\ 9 \\ 0 \\ 0 \\ -2.67 \\ 4 \end{Bmatrix} + \begin{Bmatrix} 0.715 \\ 1.116 \\ -1.488 \\ -1.116 \\ 0.955 \\ -0.715 \end{Bmatrix} = \begin{Bmatrix} 5.716 \\ 10.116 \\ -1.489 \\ -1.116 \\ -1.715 \\ 3.284 \end{Bmatrix} \tag{12}
$$

Member end actions for element 1

$$
\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} + EI_{zz} \begin{bmatrix} 0.1875 & 0.375 & -0.1875 & 0.375 \\ 0.375 & 1.0 & -0.375 & 0.5 \\ -0.1875 & -0.375 & 0.1875 & -0.375 \\ 0.375 & 0.5 & -0.375 & 1.0 \end{bmatrix} \frac{1}{EI_{zz}} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -2.977 \end{Bmatrix}
$$

$$
= \begin{Bmatrix} -1.116 \\ -1.488 \\ 1.116 \\ -2.977 \end{Bmatrix} \tag{13}
$$

Member end actions for element 2

$$
\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} = + EI_{zz} \begin{bmatrix} 0.1875 & 0.375 & -0.1875 & 0.375 \\ 0.375 & 1.0 & -0.375 & 0.5 \\ -0.1875 & -0.375 & 0.1875 & -0.375 \\ 0.375 & 0.5 & -0.375 & 1.0 \end{bmatrix} \frac{1}{EI_{zz}} \begin{Bmatrix} 0 \\ -2.977 \\ 0 \\ 1.909 \end{Bmatrix}
$$

$$
= \begin{Bmatrix} 4.6 \\ 2.98 \\ 5.4 \\ -4.58 \end{Bmatrix} \tag{14}
$$

Member end actions for element 3

$$
\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} = \begin{Bmatrix} 4.0 \\ 2.67 \\ 4.0 \\ -2.67 \end{Bmatrix} + EI_{zz} \begin{bmatrix} 0.1875 & 0.375 & -0.1875 & 0.375 \\ 0.375 & 1.0 & -0.375 & 0.5 \\ -0.1875 & -0.375 & 0.1875 & -0.375 \\ 0.375 & 0.5 & -0.375 & 1.0 \end{bmatrix} \frac{1}{EI_{zz}} \begin{Bmatrix} 0 \\ 1.909 \\ 0 \\ 0 \end{Bmatrix}
$$

$$
= \begin{Bmatrix} 4.72 \\ 4.58 \\ 3.28 \\ -1.72 \end{Bmatrix} \tag{15}
$$

Example 28.2

Analyse the continuous beam shown in Fig. 28.2a. Assume that the supports are unyielding. Assume EI to be constant for all members.
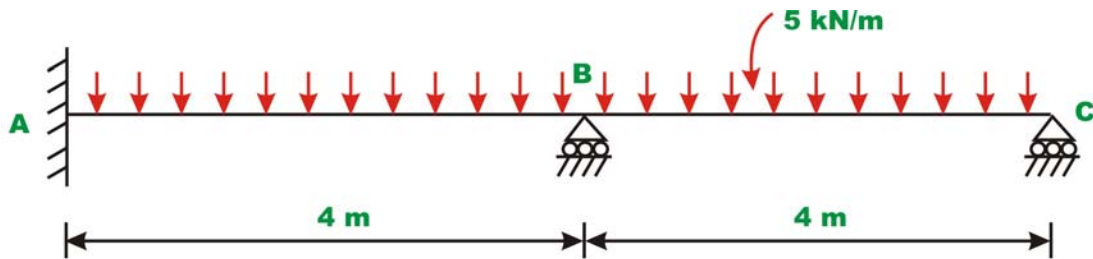

Fig. 28.2a

The numbering of joints and members are shown in Fig. 28.2b. The global degrees of freedom are also shown in the figure.

The given continuous beam is divided into two beam elements. Two degrees of freedom (one translation and one rotation) are considered at each end of the member. In the above figure, double headed arrows denote rotations and single headed arrow represents translations. Also it is observed that displacements $u_3 = u_4 = u_5 = u_6 = 0$ from support conditions.

First construct the member stiffness matrix for each member.

Member 1: $L = 4\,m$, node points 1-2.

The member stiffness matrix for all the members are the same, as the length and flexural rigidity of all members is the same.

$$\begin{array}{cccc} Global\ d.o.f & 6 & 5 & 3 & 1 \end{array}$$

$$[k'] = EI_{zz} \begin{bmatrix} 0.1875 & 0.375 & -0.1875 & 0.375 \\ 0.375 & 1.0 & -0.375 & 0.5 \\ -0.1875 & -0.375 & 0.1875 & -0.375 \\ 0.375 & 0.5 & -0.375 & 1.0 \end{bmatrix} \begin{array}{c} 6 \\ 5 \\ 3 \\ 1 \end{array} \qquad (1)$$

On the member stiffness matrix, the corresponding global degrees of freedom are indicated to facilitate assembling.

Member 2: $L = 4m$, node points 2-3.

$$\begin{array}{cccc} Global\ d.o.f & 3 & 1 & 4 & 2 \end{array}$$

$$[k^2] = EI_{zz} \begin{bmatrix} 0.1875 & 0.375 & -0.1875 & 0.375 \\ 0.375 & 1.0 & -0.375 & 0.5 \\ -0.1875 & -0.375 & 0.1875 & -0.375 \\ 0.375 & 0.5 & -0.375 & 1.0 \end{bmatrix} \begin{array}{c} 3 \\ 1 \\ 4 \\ 2 \end{array} \qquad (2)$$

The assembled global stiffness matrix of the continuous beam is of order 6×6. The assembled global stiffness matrix may be written as,

$$[K] = EI_{zz} \begin{bmatrix} 2 & 0.5 & 0 & -0.375 & 0.5 & 0.375 \\ 0.5 & 1.0 & 0.375 & -0.375 & 0 & 0 \\ 0 & 0.375 & 0.375 & -0.1875 & -0.375 & -0.1875 \\ -0.375 & -0.375 & -0.1875 & 0.1875 & 0 & 0 \\ 0.5 & 0 & -0.375 & 0 & 1.0 & 0.375 \\ 0.375 & 0 & -0.1875 & 0 & 0.375 & 0.1875 \end{bmatrix} \qquad (3)$$

Now it is required to replace the given members loads by equivalent joint loads. The equivalent loads for the present case is shown in Fig. 28.2c. The displacement degrees of freedom are also shown in figure.
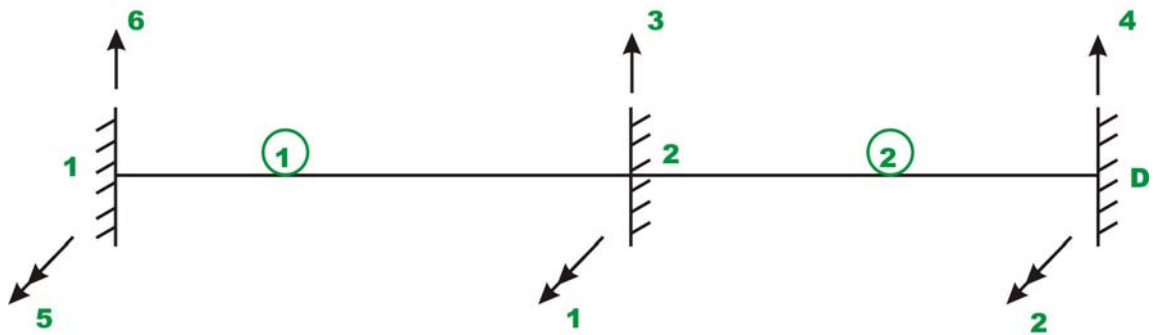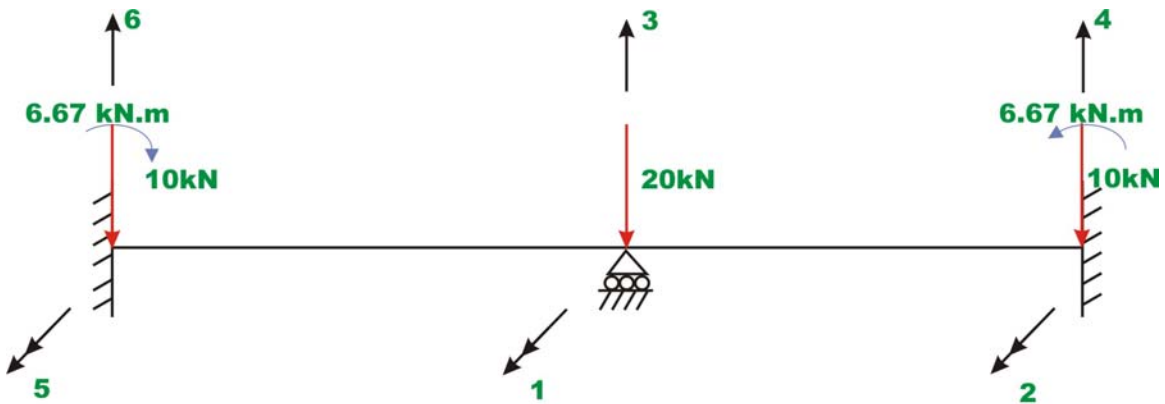
Fig. 28.2b Node and member

Fig. 28.2c Equivalent joint loads

Thus the global load vector corresponding to unconstrained degree of freedom is,

$$\{p_k\} = \begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 6.67 \end{Bmatrix} \qquad (4)$$

Writing the load displacement relation for the entire continuous beam,

$$
\left\{\begin{array}{c} 0 \\ 6.67 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \end{array}\right\} = EI_{zz} \begin{bmatrix} 2 & 0.5 & 0 & -0.375 & 0.5 & 0.375 \\ 0.5 & 1.0 & 0.375 & -0.375 & 0 & 0 \\ 0 & 0.375 & 0.375 & -0.1875 & -0.375 & -0.1875 \\ -0.375 & -0.375 & -0.1875 & 0.1875 & 0 & 0 \\ 0.5 & 0 & -0.375 & 0 & 1.0 & 0.375 \\ 0.375 & 0 & -0.1875 & 0 & 0.375 & 0.1875 \end{bmatrix} \left\{\begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array}\right\} \qquad (5)
$$

We know that $u_3 = u_4 = u_5 = u_6 = 0$. Thus solving for unknowns u_1 and u_2, yields

$$
\left\{\begin{array}{c} 0 \\ 6.67 \end{array}\right\} = EI_{zz} \begin{bmatrix} 2.0 & 0.5 \\ 0.5 & 1.0 \end{bmatrix} \left\{\begin{array}{c} u_1 \\ u_2 \end{array}\right\} \qquad (6)
$$

$$
\left\{\begin{array}{c} u_1 \\ u_2 \end{array}\right\} = \frac{1}{1.75 EI_{zz}} \begin{bmatrix} 1.0 & -0.5 \\ -0.5 & 2.0 \end{bmatrix} \left\{\begin{array}{c} 0 \\ 6.67 \end{array}\right\}
$$

$$
= \frac{1}{EI_{zz}} \left\{\begin{array}{c} -1.905 \\ 7.62 \end{array}\right\} \qquad (7)
$$

Thus displacements are,

$$
u_1 = \frac{-1.905}{EI_{zz}} \qquad \text{and} \qquad u_2 = \frac{7.62}{EI_{zz}}
$$

The unknown joint loads are given by,

$$
\left\{\begin{array}{c} p_3 \\ p_4 \\ p_5 \\ p_6 \end{array}\right\} = EI_{zz} \begin{bmatrix} 0 & 0.375 \\ -0.375 & -0.375 \\ 0.5 & 0 \\ 0.375 & 0 \end{bmatrix} \frac{1}{EI_{zz}} \left\{\begin{array}{c} -1.905 \\ 7.62 \end{array}\right\}
$$

$$= \begin{Bmatrix} 2.857 \\ -2.14 \\ -0.95 \\ -0.714 \end{Bmatrix} \tag{8}$$

The actual support reactions are,

$$\begin{Bmatrix} R_3 \\ R_4 \\ R_5 \\ R_6 \end{Bmatrix} = \begin{Bmatrix} 20 \\ 10 \\ -6.67 \\ 10 \end{Bmatrix} + \begin{Bmatrix} 2.857 \\ -2.14 \\ -0.95 \\ -0.714 \end{Bmatrix} = \begin{Bmatrix} 22.857 \\ 7.86 \\ -7.62 \\ 9.286 \end{Bmatrix} \tag{9}$$

Member end actions for element 1

$$\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} = \begin{Bmatrix} 10 \\ 6.66 \\ 10 \\ -6.66 \end{Bmatrix} + EI_{zz} \begin{bmatrix} 0.1875 & 0.375 & -0.1875 & 0.375 \\ 0.375 & 1.0 & -0.375 & 0.5 \\ -0.1875 & -0.375 & 0.1875 & -0.375 \\ 0.375 & 0.5 & -0.375 & 1.0 \end{bmatrix} \frac{1}{EI_{zz}} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -1.905 \end{Bmatrix}$$

$$= \begin{Bmatrix} 9.285 \\ 5.707 \\ 10.714 \\ -8.565 \end{Bmatrix} \tag{10}$$

Member end actions for element 2

$$\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} = \begin{Bmatrix} 10 \\ 6.66 \\ 10 \\ -6.66 \end{Bmatrix} + EI_{zz} \begin{bmatrix} 0.1875 & 0.375 & -0.1875 & 0.375 \\ 0.375 & 1.0 & -0.375 & 0.5 \\ -0.1875 & -0.375 & 0.1875 & -0.375 \\ 0.375 & 0.5 & -0.375 & 1.0 \end{bmatrix} \frac{1}{EI_{zz}} \begin{Bmatrix} 0 \\ -1.905 \\ 0 \\ 7.62 \end{Bmatrix}$$

$$= \begin{Bmatrix} 12.14 \\ 8.565 \\ 7.856 \\ 0 \end{Bmatrix}$$

Summary

In the previous lesson the beam element stiffness matrix is derived from fundamentals. Assembling member stiffness matrices, the global stiffness matrix is generated. The procedure to impose boundary conditions on the load-displacement relation is discussed. In this lesson, a few continuous beam problems are analysed by the direct stiffness method.