Module
4

Analysis of Statically Indeterminate Structures by the Direct Stiffness Method

Lesson
24
The Direct Stiffness Method: Truss Analysis

Instructional Objectives

After reading this chapter the student will be able to
1. Derive member stiffness matrix of a truss member.
2. Define local and global co-ordinate system.
3. Transform displacements from local co-ordinate system to global co-ordinate system.
4. Transform forces from local to global co-ordinate system.
5. Transform member stiffness matrix from local to global co-ordinate system.
6. Assemble member stiffness matrices to obtain the global stiffness matrix.
7. Analyse plane truss by the direct stiffness matrix.

24.1 Introduction

An introduction to the stiffness method was given in the previous chapter. The basic principles involved in the analysis of beams, trusses were discussed. The problems were solved with hand computation by the direct application of the basic principles. The procedure discussed in the previous chapter though enlightening are not suitable for computer programming. It is necessary to keep hand computation to a minimum while implementing this procedure on the computer. In this chapter a formal approach has been discussed which may be readily programmed on a computer. In this lesson the direct stiffness method as applied to planar truss structure is discussed.

Plane trusses are made up of short thin members interconnected at hinges to form triangulated patterns. A hinge connection can only transmit forces from one member to another member but not the moment. For analysis purpose, the truss is loaded at the joints. Hence, a truss member is subjected to only axial forces and the forces remain constant along the length of the member. The forces in the member at its two ends must be of the same magnitude but act in the opposite directions for equilibrium as shown in Fig. 24.1.
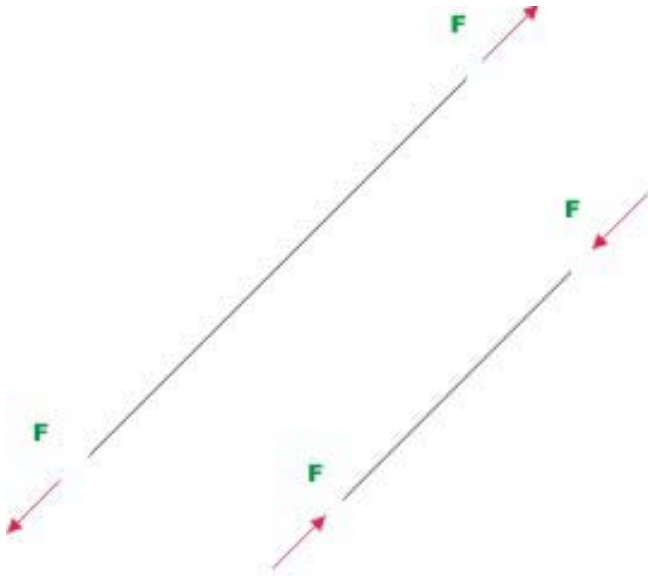
Fig 24.1 Truss member in equilibrium

Now consider a truss member having cross sectional area A, Young's modulus of material E, and length of the member L. Let the member be subjected to axial tensile force F as shown in Fig. 24.2. Under the action of constant axial force F, applied at each end, the member gets elongated by u as shown in Fig. 24.2.

Fig 24.2 Force - displacement relationship

The elongation u may be calculated by (vide lesson 2, module 1).

$$u = \frac{FL}{AE} \qquad (24.1)$$

Now the force-displacement relation for the truss member may be written as,

$$F = \frac{AE}{L} u \qquad (24.2)$$

$$F = ku \qquad\qquad (24.3)$$

where $k = \dfrac{AE}{L}$ is the stiffness of the truss member and is defined as the force required for unit deformation of the structure. The above relation (24.3) is true along the centroidal axis of the truss member. But in reality there are many members in a truss. For example consider a planer truss shown in Fig. 24.3. For each member of the truss we could write one equation of the type $F = ku$ along its axial direction (which is called as local co-ordinate system). Each member has different local co ordinate system. To analyse the planer truss shown in Fig. 24.3, it is required to write force-displacement relation for the complete truss in a co ordinate system common to all members. Such a co-ordinate system is referred to as global co ordinate system.
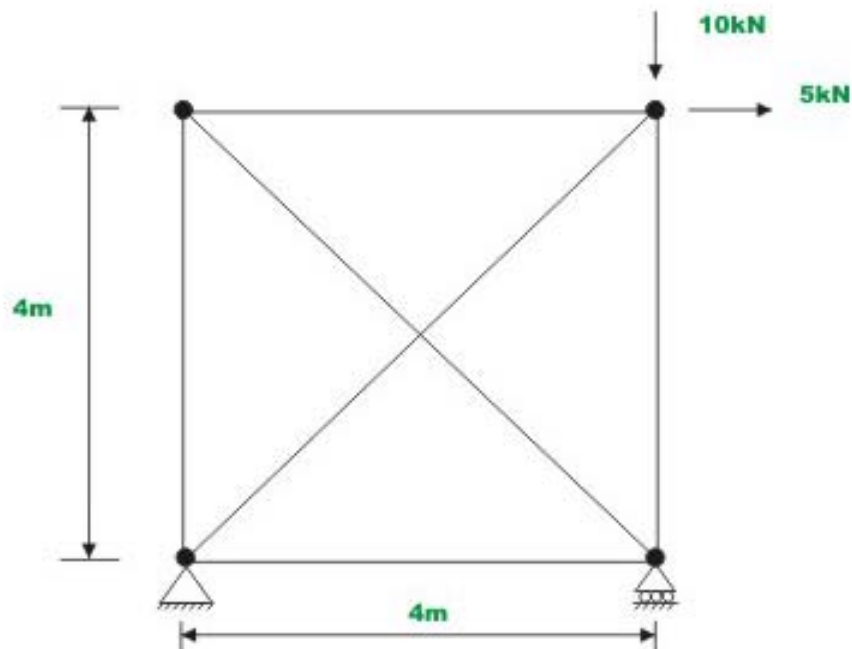

Fig 24.3 Plane truss

24.2 Local and Global Co-ordinate System

Loads and displacements are vector quantities and hence a proper coordinate system is required to specify their correct sense of direction. Consider a planar truss as shown in Fig. 24.4. In this truss each node is identified by a number and each member is identified by a number enclosed in a circle. The displacements and loads acting on the truss are defined with respect to global co-ordinate system xyz. The same co ordinate system is used to define each of the loads and displacements of all loads. In a global co-ordinate system, each node of a planer truss can have only two displacements: one along x-axis and another along y-axis. The truss shown in figure has eight displacements. Each displacement

(degree of freedom) in a truss is shown by a number in the figure at the joint. The direction of the displacements is shown by an arrow at the node. However out of eight displacements, five are unknown. The displacements indicated by numbers 6,7 and 8 are zero due to support conditions. The displacements denoted by numbers 1-5 are known as unconstrained degrees of freedom of the truss and displacements denoted by 6-8 represent constrained degrees of freedom. In this course, unknown displacements are denoted by lower numbers and the known displacements are denoted by higher code numbers.
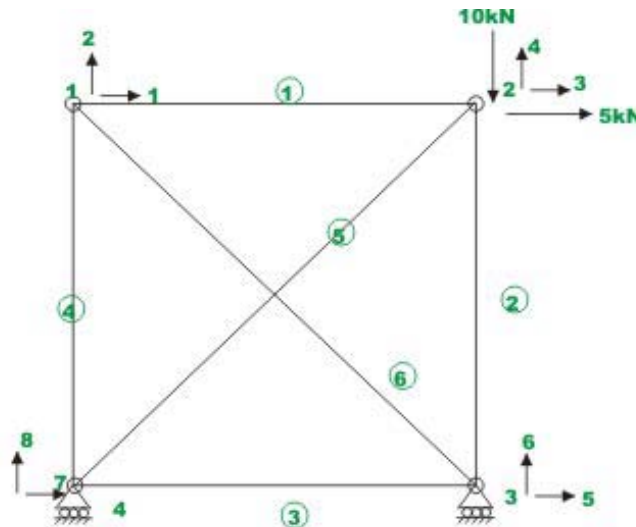

Fig 24.4 Node and members numbering

To analyse the truss shown in Fig. 24.4, the structural stiffness matrix K need to be evaluated for the given truss. This may be achieved by suitably adding all the member stiffness matrices k', which is used to express the force-displacement relation of the member in local co-ordinate system. Since all members are oriented at different directions, it is required to transform member displacements and forces from the local co-ordinate system to global co-ordinate system so that a global load-displacement relation may be written for the complete truss.

24.3 Member Stiffness Matrix

Consider a member of the truss as shown in Fig. 24.5a in local co-ordinate system $x'y'$. As the loads are applied along the centroidal axis, only possible displacements will be along x'-axis. Let the u'_1 and u'_2 be the displacements of truss members in local co-ordinate system $i.e.$ along x'-axis. Here subscript 1 refers to node 1 of the truss member and subscript 2 refers to node 2 of the truss member. Give displacement u'_1 at node 1 of the member in the positive x' direction, keeping all other displacements to zero. This displacement in turn

induces a compressive force of magnitude $\dfrac{EA}{L}u'_1$ in the member. Thus,

$q'_1 = \dfrac{EA}{L}u'_1$ and $q'_2 = -\dfrac{EA}{L}u'_1$ (24.4a) ($-ve$ as it acts in the $-ve$ direction for equilibrium). Similarly by giving positive displacements of u'_2 at end 2 of the member, tensile force of magnitude $\dfrac{EA}{L}u'_2$ is induced in the member. Thus,

$$q''_1 = -\dfrac{EA}{L}u'_2 \text{ and } q''_2 = \dfrac{EA}{L}u'_2 \qquad\qquad (24.4b)$$

Now the forces developed at the ends of the member when both the displacements are imposed at nodes 1 and 2 respectively may be obtained by method of superposition. Thus (vide Fig. 24.5d)
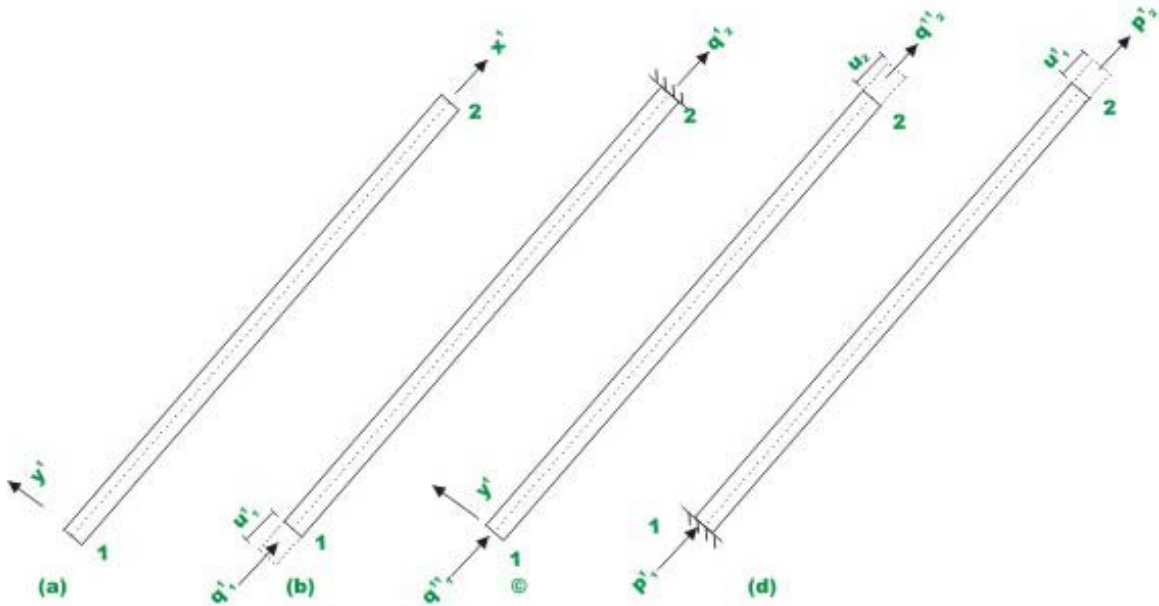

Fig 24.5 Force displacement reaction in load co-ordinate

$$p'_1 = \dfrac{EA}{L}u'_1 - \dfrac{EA}{L}u'_2 \qquad\qquad (24.5a)$$

$$p'_2 = \dfrac{EA}{L}u'_2 - \dfrac{EA}{L}u'_1 \qquad\qquad (24.5b)$$

Or we can write

$$\left\{ \begin{array}{c} p'_1 \\ p'_2 \end{array} \right\} = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \left\{ \begin{array}{c} u'_1 \\ u'_2 \end{array} \right\}$$ (24.6a)

$$\{p'\} = [k']\{u'\}$$ (24.6b)

Thus the member stiffness matrix is

$$k' = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$ (24.7)

This may also be obtained by giving unit displacement at node 1 and holding displacement at node 2 to zero and calculating forces developed at two ends. This will generate the first column of stiffness matrix. Similarly the second column of stiffness matrix is obtained by giving unit displacement at 2 and holding displacement at node 1 to zero and calculating the forces developed at both ends.

24.4 Transformation from Local to Global Co-ordinate System.

Displacement Transformation Matrix

A truss member is shown in local and global co ordinate system in Fig. 24.6. Let $x'y'z'$ be in local co ordinate system and xyz be the global co ordinate system.
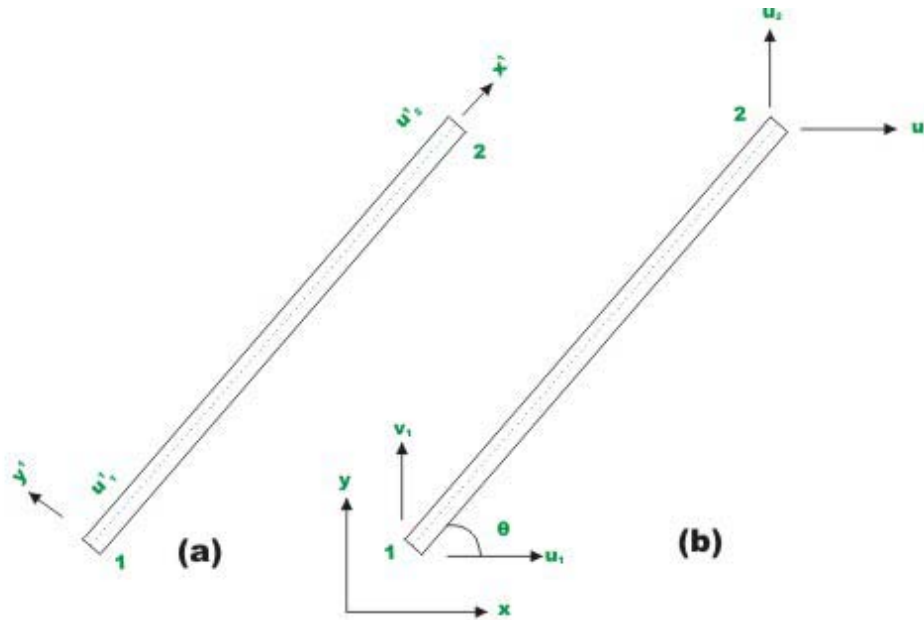
Fig 24.6 Truss element (a) local co-ordinate system
(b) global co-ordinate system

The nodes of the truss member be identified by 1 and 2. Let u'_1 and u'_2 be the displacement of nodes 1 and 2 in local co ordinate system. In global co ordinate system, each node has two degrees of freedom. Thus, u_1, v_1 and u_2, v_2 are the nodal displacements at nodes 1 and 2 respectively along x- and y- directions. Let the truss member be inclined to x axis by θ as shown in figure. It is observed from the figure that u'_1 is equal to the projection of u_1 on x' axis plus projection of v_1 on x'-axis. Thus, (vide Fig. 24.7)

$$u'_1 = u_1 \cos\theta + v_1 \sin\theta \qquad (24.8a)$$

$$u'_2 = u_2 \cos\theta + v_2 \sin\theta \qquad (24.8b)$$

This may be written as

$$\begin{Bmatrix} u'_1 \\ u'_2 \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix} \qquad (24.9)$$

Introducing direction cosines $l = \cos\theta$; $m = \sin\theta$; the above equation is written as

$$\begin{Bmatrix} u'_1 \\ u'_2 \end{Bmatrix} = \begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}$$ (24.10a)

Or, $\{u'\} = [T]\{u\}$ (24.10b)

In the above equation $[T]$ is the displacement transformation matrix which transforms the four global displacement components to two displacement component in local coordinate system.
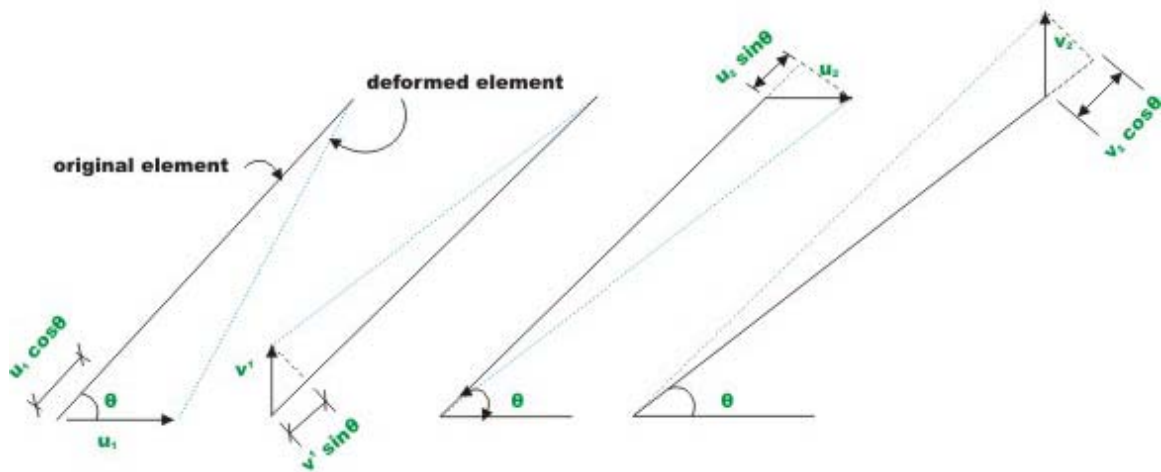

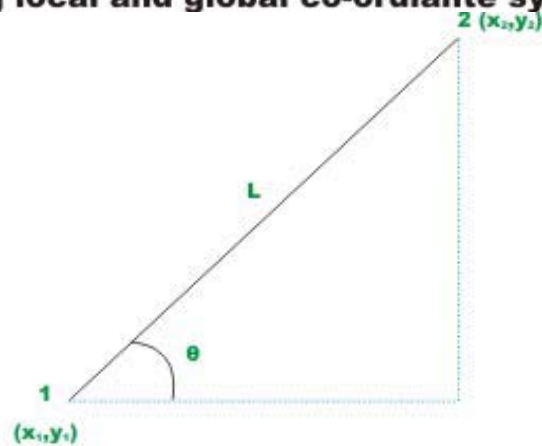**Fig 24.7 Generalized displacement along local and global co-ordiante system**

Fig 24.8 A typical truss member

Let co-ordinates of node 1 be (x_1, y_1) and node 2 be (x_2, y_2). Now from Fig. 24.8,

$$l = \cos\theta = \frac{x_2 - x_1}{L} \qquad\qquad (24.11a)$$

$$m = \sin\theta = \frac{y_2 - y_1}{L} \qquad\qquad (24.11b)$$

$$\text{and } L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \qquad\qquad (24.11c)$$

Force transformation matrix

Let p'_1, p'_2 be the forces in a truss member at node 1 and 2 respectively producing displacements u'_1 and u'_2 in the local co-ordinate system and p_1, p_2, p_3, p_4 be the force in global co-ordinate system at node 1 and 2 respectively producing displacements u_1, v_1 and u_2, v_2 (refer Fig. 24.9a-d).
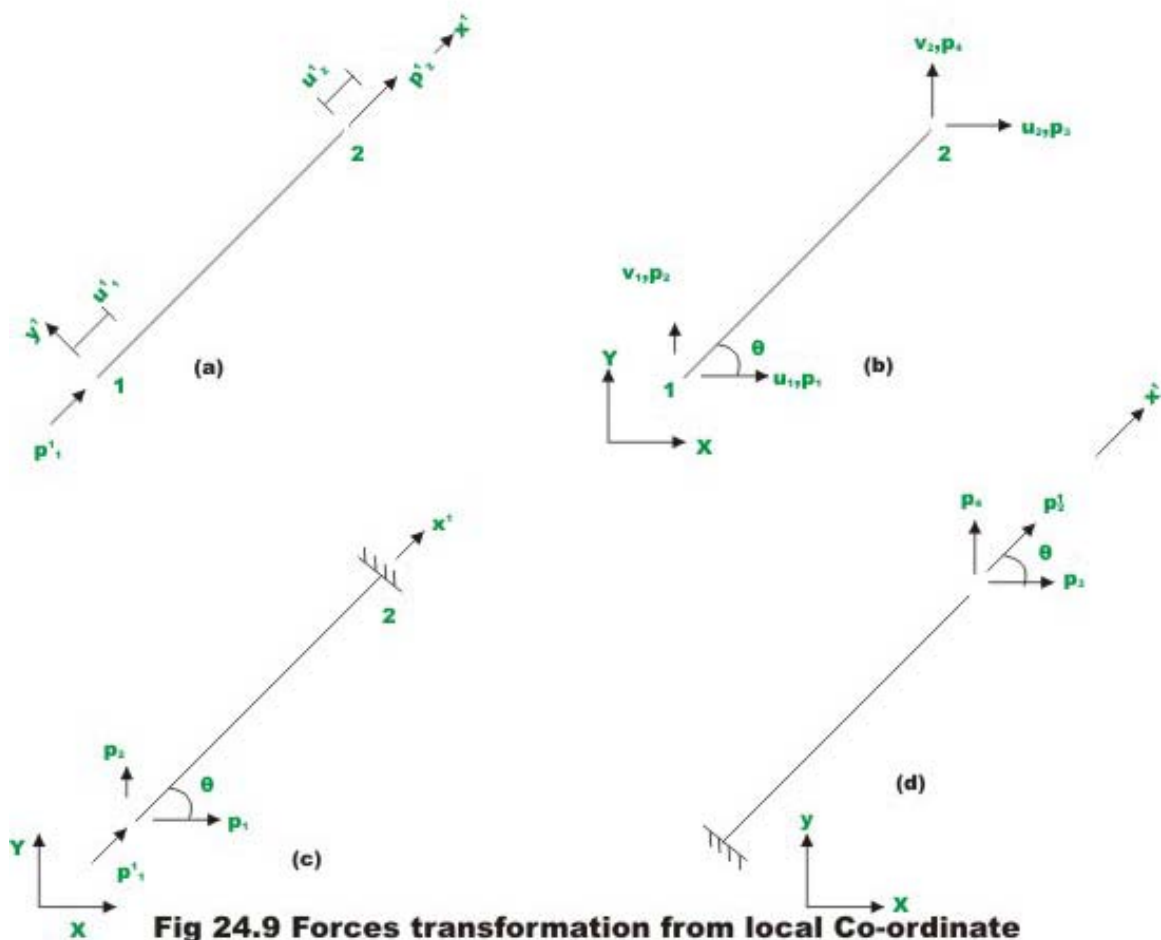

Fig 24.9 Forces transformation from local Co-ordinate system to global Co-ordinate system

Referring to fig. 24.9c, the relation between p'_1 and p_1, may be written as,

$$p_1 = p'_1 \cos\theta \qquad\qquad (24.12a)$$

$$p_2 = p'_1 \sin\theta \qquad\qquad (24.12b)$$

Similarly referring to Fig. 24.9d, yields

$$p_3 = p'_2 \cos\theta \qquad\qquad (24.12c)$$

$$p_4 = p'_2 \sin\theta \qquad\qquad (24.12d)$$

Now the relation between forces in the global and local co-ordinate system may be written as

$$\begin{Bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{Bmatrix} = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & \cos\theta \\ 0 & \sin\theta \end{bmatrix} \begin{Bmatrix} p'_1 \\ p'_2 \end{Bmatrix} \qquad\qquad (24.13)$$

$$\{p\} = [T]^T \{p'\} \qquad\qquad (24.14)$$

where matrix $\{p\}$ stands for global components of force and matrix $\{p'\}$ are the components of forces in the local co-ordinate system. The superscript T stands for the transpose of the matrix. The equation (24.14) transforms the forces in the local co-ordinate system to the forces in global co-ordinate system. This is accomplished by force transformation matrix $[T]^T$. Force transformation matrix is the transpose of displacement transformation matrix.

Member Global Stiffness Matrix

From equation (24.6b) we have,

$$\{p'\} = [k'] \{u'\}$$

Substituting for $\{p'\}$ in equation (24.14), we get

$$\{p\} = [T]^T [k'] \{u'\} \qquad\qquad (24.15)$$

Making use of the equation (24.10b), the above equation may be written as

$$\{p\} = [T]^T [k'][T]\{u\} \tag{24.16}$$

$$\{p\} = [k]\{u\} \tag{24.17}$$

Equation (24.17) represents the member load displacement relation in global co-ordinates and thus $[k]$ is the member global stiffness matrix. Thus,

$$\{k\} = [T]^T [k'][T] \tag{24.18}$$

$$[k] = \frac{EA}{L} \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta & -\cos^2\theta & -\cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta & -\cos\theta\sin\theta & -\sin^2\theta \\ -\cos^2\theta & -\cos\theta\sin\theta & \cos^2\theta & \cos\theta\sin\theta \\ -\cos\theta\sin\theta & -\sin^2\theta & \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$$

$$[k] = \frac{EA}{L} \begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix} \tag{24.19}$$

Each component k_{ij} of the member stiffness matrix $[k]$ in global co-ordinates represents the force in x-or y-directions at the end i required to cause a unit displacement along $x-$ or $y-$directions at end j.

We obtained the member stiffness matrix in the global co-ordinates by transforming the member stiffness matrix in the local co-ordinates. The member stiffness matrix in global co-ordinates can also be derived from basic principles in a direct method. Now give a unit displacement along x-direction at node 1 of the truss member. Due to this unit displacement (see Fig. 24.10) the member length gets changed in the axial direction by an amount equal to $\Delta l_1 = \cos\theta$. This axial change in length is related to the force in the member in two axial directions by
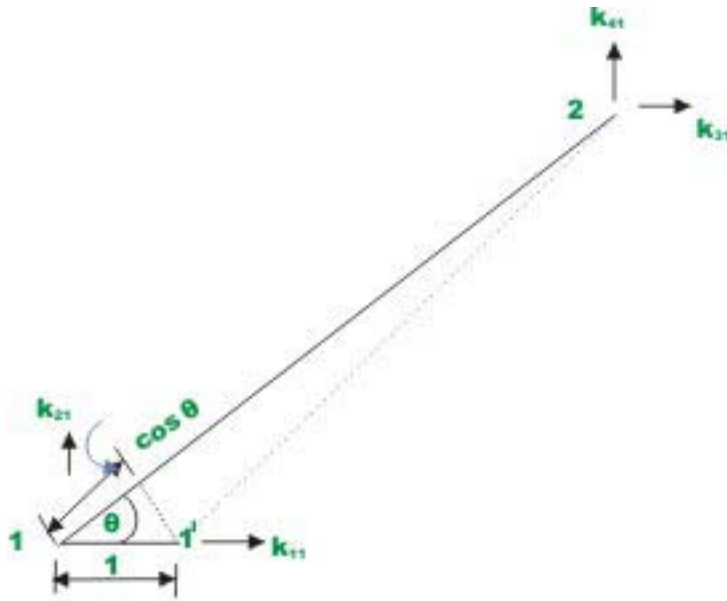
$$F_{1'2'} = \frac{EA}{L}\cos\theta \tag{24.20a}$$

Fig 24.10

This force may be resolved along u_1 and v_1 directions. Thus horizontal component of force $F_{1'2'}$ is $k_{11} = \dfrac{EA}{L}\cos^2\theta$ (24.20b)

Vertical component of force $F_{1'2'}$ is $k_{21} = \dfrac{EA}{L}\cos\theta\sin\theta$ (24.20c)

The forces at the node 2 are readily found from static equilibrium. Thus,

$$k_{31} = -k_{11} = -\frac{EA}{L}\cos^2\theta \tag{24.20d}$$

$$k_{41} = -k_{21} = \frac{EA}{L}\cos\theta\sin\theta \tag{24.20e}$$

The above four stiffness coefficients constitute the first column of a stiffness matrix in the global co-ordinate system. Similarly, remaining columns of the stiffness matrix may be obtained.

24.5 Analysis of plane truss.

Number all the joints and members of a plane truss. Also indicate the degrees of freedom at each node. In a plane truss at each node, we can have two displacements. Denote unknown displacements by lower numbers and known displacements by higher numbers as shown in Fig. 24.4. In the next step evaluate member stiffness matrix of all the members in the global co ordinate

system. Assemble all the stiffness matrices in a particular order, the stiffness matrix K for the entire truss is found. The assembling procedure is best explained by considering a simple example. For this purpose consider a two member truss as shown in Fig. 24.11. In the figure, joint numbers, member numbers and possible displacements of the joints are shown.

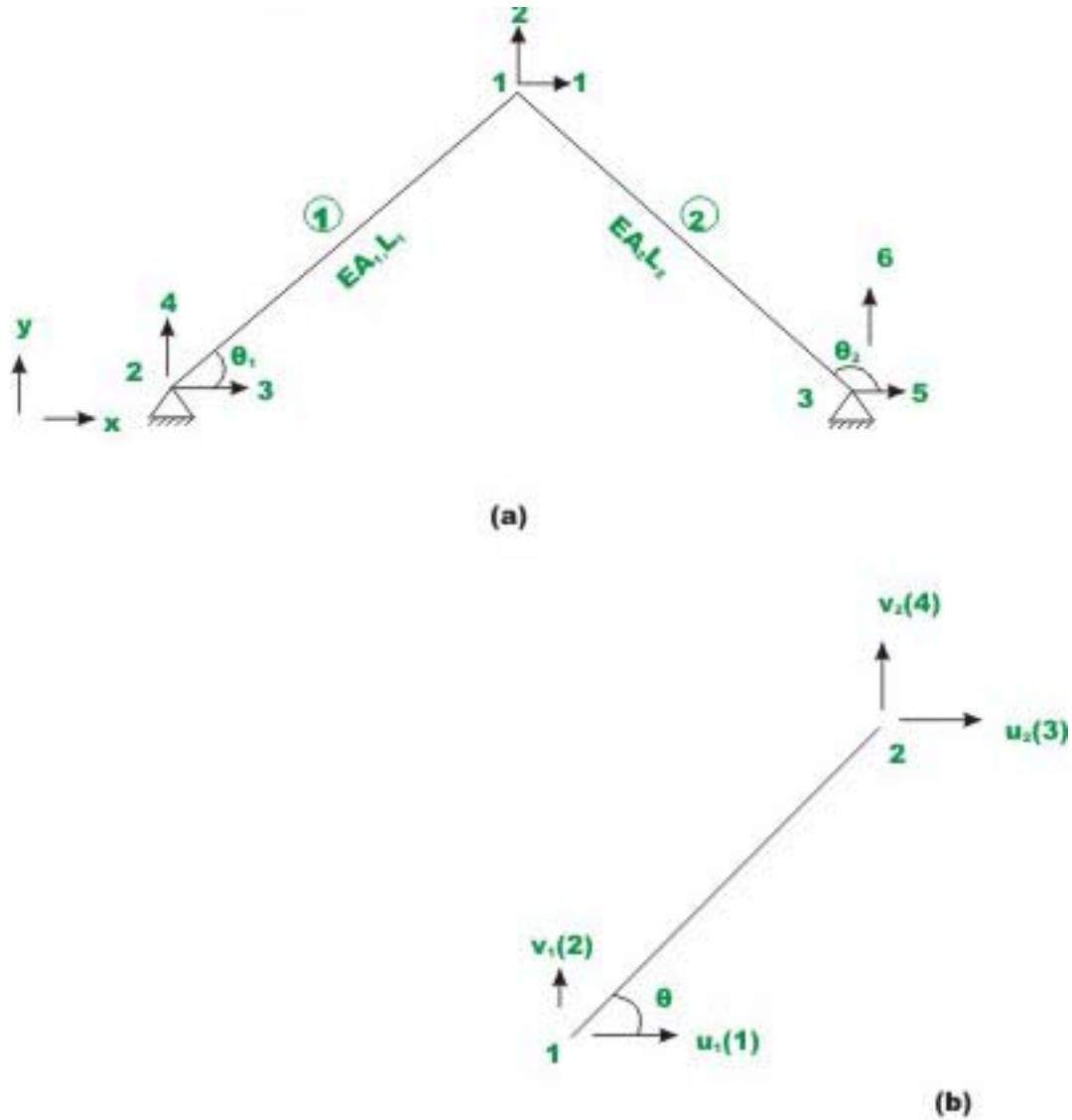**Fig 24.11 Analysis of plane - truss**

The area of cross-section of the members, its length and its inclination with the x - axis are also shown. Now the member stiffness matrix in the global co-ordinate system for both the members are given by

$$
\begin{array}{cccc}
\textit{Global} & 3 & 4 & 1 & 2 \\
\textit{Memb} & 1 & 2 & 3 & 4
\end{array}
$$

$$
\left[k^1 \right] = \frac{EA_1}{L_1}
\begin{bmatrix}
l_1^{\,2} & l_1 m_1 & -l_1^{\,2} & -l_1 m_1 \\
l_1 m_1 & m_1^{\,2} & -l_1 m_1 & -m_1^{\,2} \\
-l_1^{\,2} & -l_1 m_1 & l_1^{\,2} & l_1 m_1 \\
-l_1 m_1 & -m_1^{\,2} & l_1 m_1 & m_1^{\,2}
\end{bmatrix}
\tag{24.21a}
$$

On the member stiffness matrix the corresponding member degrees of freedom and global degrees of freedom are also shown.

$$
\left[k^2 \right] = \frac{EA_2}{L_2}
\begin{bmatrix}
l_2^{\,2} & l_2 m_2 & -l_2^{\,2} & -l_2 m_2 \\
l_2 m_2 & m_2^{\,2} & -l_2 m_2 & -m_2^{\,2} \\
-l_2^{\,2} & -l_2 m_2 & l_2^{\,2} & l_2 m_2 \\
-l_2 m_2 & -m_2^{\,2} & l_2 m_2 & m_2^{\,2}
\end{bmatrix}
\tag{24.21b}
$$

Note that the member stiffness matrix in global co-ordinate system is derived referring to Fig. 24.11b. The node 1 and node 2 remain same for all the members. However in the truss, for member 1, the same node (*i.e.* node 1 and 2 in Fig. 24.11b) are referred by 2 and 1 respectively. Similarly for member 2, the nodes 1 and 2 are referred by nodes 3 and 4 in the truss. The member stiffness matrix is of the order 4×4. However the truss has six possible displacements and hence truss stiffness matrix is of the order 6×6. Now it is required to put elements of the member stiffness matrix of the entire truss. The stiffness matrix of the entire truss is known as assembled stiffness matrix. It is also known as structure stiffness matrix; as overall stiffness matrix. Thus, it is clear that by algebraically adding the above two stiffness matrix we get global stiffness matrix. For example the element k^1_{11} of the member stiffness matrix of member 1 must go to location $(3,3)$ in the global stiffness matrix. Similarly k^2_{11} must go to location $(3,3)$ in the global stiffness matrix. The above procedure may be symbolically written as,

$$
K = \sum_{i=0}^{n} k^i
\tag{24.22}
$$

$$
= \frac{EA_1}{L_1}
\begin{bmatrix}
l_1^{\,2} & l_1 m_1 & -l_1^{\,2} & -l_1 m_1 \\
l_1 m_1 & m_1^{\,2} & -l_1 m_1 & -m_1^{\,2} \\
-l_1^{\,2} & -l_1 m_1 & l_1^{\,2} & l_1 m_1 \\
-l_1 m_1 & -m_1^{\,2} & l_1 m_1 & m_1^{\,2}
\end{bmatrix}
+ \frac{EA_2}{L_2}
\begin{bmatrix}
l_2^{\,2} & l_2 m_2 & -l_2^{\,2} & -l_2 m_2 \\
l_2 m_2 & m_2^{\,2} & -l_2 m_2 & -m_2^{\,2} \\
-l_2^{\,2} & -l_2 m_2 & l_2^{\,2} & l_2 m_2 \\
-l_2 m_2 & -m_2^{\,2} & l_2 m_2 & m_2^{\,2}
\end{bmatrix}
$$

$$
\tag{24.23a}
$$

The assembled stiffness matrix is of the order 6×6. Hence, it is easy to visualize assembly if we expand the member stiffness matrix to 6×6 size. The missing columns and rows in matrices k^1 and k^2 are filled with zeroes. Thus,

$$K = \frac{EA_1}{L_1}
\begin{bmatrix}
l_1^{\,2} & l_1 m_1 & -l_1^{\,2} & -l_1 m_1 & 0 & 0 \\
l_1 m_1 & m_1^{\,2} & -l_1 m_1 & -m_1^{\,2} & 0 & 0 \\
-l_1^{\,2} & -l_1 m_1 & l_1^{\,2} & l_1 m_1 & 0 & 0 \\
-l_1 m_1 & -m_1^{\,2} & l_1 m_1 & m_1^{\,2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

$$+ \frac{EA_2}{L_2}
\begin{bmatrix}
l_2^{\,2} & l_2 m_2 & -l_2^{\,2} & -l_2 m_2 & 0 & 0 \\
l_2 m_2 & m_2^{\,2} & -l_2 m_2 & -m_2^{\,2} & 0 & 0 \\
-l_2^{\,2} & -l_2 m_2 & l_2^{\,2} & l_2 m_2 & 0 & 0 \\
-l_2 m_2 & -m_2^{\,2} & l_2 m_2 & m_2^{\,2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

(24.24)

Adding appropriate elements of first matrix with the appropriate elements of the second matrix,

$$K = \begin{bmatrix}
\frac{EA_1}{L_1}l_1^2 + \frac{EA_2}{L_2}l_2^2 & \frac{EA_1}{L_1}l_1 m_1 + \frac{EA_2}{L_2}l_2 m_2 & -\frac{EA_1}{L_1}l_1^2 & -\frac{EA_1}{L_1}l_1 m_1 & -\frac{EA_2}{L_2}l_2^2 & -\frac{EA_2}{L_2}l_2 m_2 \\[2ex]
\frac{EA_1}{L_1}l_1 m_1 + \frac{EA_2}{L_2}l_2 m_2 & \frac{EA_1}{L_1}m_1^2 + \frac{EA_2}{L_2}m_2^2 & -\frac{EA_1}{L_1}l_1 m_1 & -\frac{EA_1}{L_1}m_1^2 & -\frac{EA_2}{L_2}l_2 m_2 & -\frac{EA_2}{L_2}m_2^2 \\[2ex]
-\frac{EA_1}{L_1}l_1^2 & -\frac{EA_1}{L_1}l_1 m_1 & \frac{EA_1}{L_1}l_1^2 & \frac{EA_1}{L_1}l_1 m_1 & 0 & 0 \\[2ex]
-\frac{EA_1}{L_1}l_1 m_1 & -\frac{EA_1}{L_1}m_1^2 & \frac{EA_1}{L_1}l_1 m_1 & \frac{EA_1}{L_1}m_1^2 & 0 & 0 \\[2ex]
-\frac{EA_2}{L_2}l_2^2 & -\frac{EA_2}{L_2}l_2 m_2 & 0 & 0 & \frac{EA_2}{L_2}l_2^2 & \frac{EA_2}{L_2}l_2 m_2 \\[2ex]
-\frac{EA_2}{L_2}l_2 m_2 & -\frac{EA_2}{L_2}m_2^2 & 0 & 0 & \frac{EA_2}{L_2}l_2 m_2 & \frac{EA_2}{L_2}m_2^2
\end{bmatrix}$$

If more than one member meet at a joint then the stiffness coefficients of member stiffness matrix corresponding to that joint are added.

After evaluating global stiffness matrix of the truss, the load displacement equation for the truss is written as,

$$\{p\} = [K]\ \{u\} \tag{24.26}$$

where $\{p\}$ is the vector of joint loads acting on the truss, $\{u\}$ is the vector of joint displacements and $[K]$ is the global stiffness matrix. The above equation is known as the equilibrium equation. It is observed that some joint loads are known and some are unknown. Also some displacements are known due to support conditions and some displacements are unknown. Hence the above equation may be partitioned and written as,

$$\begin{Bmatrix} \{p_k\} \\ \{p_u\} \end{Bmatrix} = \begin{bmatrix} [k_{11}] & [k_{12}] \\ [k_{21}] & [k_{22}] \end{bmatrix} \begin{Bmatrix} \{u_u\} \\ \{u_k\} \end{Bmatrix} \tag{24.27}$$

where $\{p_k\}, \{u_k\}$ denote vector of known forces and known displacements respectively. And $\{p_u\}, \{u_u\}$ denote vector of unknown forces and unknown displacements respectively.

Expanding equation 24.27,

$$\{p_k\} = [k_{11}]\{u_u\} + [k_{12}]\{u_k\} \tag{24.28a}$$

$$\{p_u\} = [k_{21}]\{u_u\} + [k_{22}]\{u_k\} \tag{24.28b}$$

In the present case (vide Fig. 24.11a) the known displacements are u_3, u_4, u_5 and u_6. The known displacements are zero due to boundary conditions. Thus,

$\{u_k\} = \{0\}$. And from equation (24.28a),

$$\{p_k\} = [k_{11}]\{u_u\} \tag{24.29}$$

Solving $\{u_u\} = [k_{11}]^{-1}\{p_k\}$

where $[k_{11}]$ corresponding to stiffness matrix of the truss corresponding to unconstrained degrees of freedom. Now the support reactions are evaluated from equation (24.28b).

$$\{p_u\} = [k_{21}]\{u_u\} \tag{24.30}$$

The member forces are evaluated as follows. Substituting equation (24.10b) $\{u'\} = [T]\{u\}$ in equation (24.6b) $\{p'\} = [k']\{u'\}$, one obtains

$$\{p'\} = [k'][T]\{u\} \tag{24.31}$$

Expanding this equation,

$$\begin{Bmatrix} p'_1 \\ p'_2 \end{Bmatrix} = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix} \tag{24.32}$$

Example 24.1

Analyse the two member truss shown in Fig. 24.12a. Assume EA to be constant for all members. The length of each member is $5\,m$.
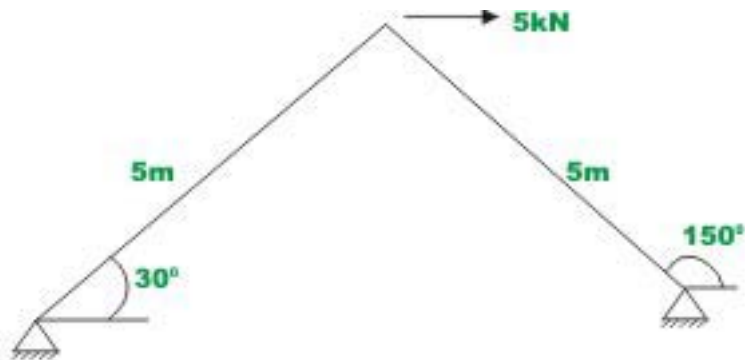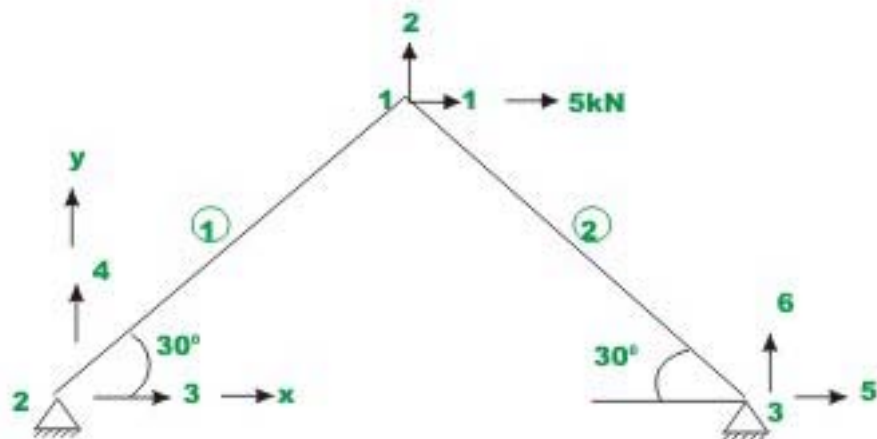

Fig 24.12(a) Example 24.1

Fig 24.12(b) Members and node numbering

The co-ordinate axes, the number of nodes and members are shown in Fig.24.12b. The degrees of freedom at each node are also shown. By inspection it is clear that the displacement $u_3 = u_4 = u_5 = u_6 = 0$. Also the external loads are

$$p_1 = 5 \quad \text{kN} \quad ; p_2 = 0 \quad \text{kN}. \tag{1}$$

Now member stiffness matrix for each member in global co-ordinate system is $(\theta_1 = 30°)$.

$$[k^1] = \frac{EA}{5} \begin{bmatrix} 0.75 & 0.433 & -0.75 & -0.433 \\ 0.433 & 0.25 & -0.433 & -0.25 \\ -0.75 & -0.433 & 0.75 & 0.433 \\ -0.433 & -0.25 & 0.433 & 0.25 \end{bmatrix} \tag{2}$$

$$[k^2] = \frac{EA}{5} \begin{bmatrix} 0.75 & -0.433 & -0.75 & 0.433 \\ -0.433 & 0.25 & 0.433 & -0.25 \\ -0.75 & 0.433 & 0.75 & -0.433 \\ 0.433 & -0.25 & -0.433 & 0.25 \end{bmatrix} \tag{3}$$

The global stiffness matrix of the truss can be obtained by assembling the two member stiffness matrices. Thus,

$$[K] = \frac{EA}{5} \begin{bmatrix} 1.5 & 0 & -0.75 & -0.433 & -0.75 & 0.433 \\ 0 & 0.5 & -0.433 & -0.25 & 0.433 & -0.25 \\ -0.75 & -0.433 & 0.75 & 0.433 & 0 & 0 \\ -0.433 & -0.25 & 0.433 & 0.25 & 0 & 0 \\ -0.75 & 0.433 & 0 & 0 & 0.75 & -0.433 \\ 0.433 & -0.25 & 0 & 0 & -0.433 & 0.25 \end{bmatrix} \tag{4}$$

Again stiffness matrix for the unconstrained degrees of freedom is,

$$[K] = \frac{EA}{5} \begin{bmatrix} 1.5 & 0 \\ 0 & 0.5 \end{bmatrix} \tag{5}$$

Writing the load displacement-relation for the truss for the unconstrained degrees of freedom

$$\{p_k\} = [k_{11}]\{u_u\} \tag{6}$$

$$\left\{ \begin{array}{c} p_1 \\ p_2 \end{array} \right\} = \frac{EA}{5} \begin{bmatrix} 1.5 & 0 \\ 0 & 0.5 \end{bmatrix} \left\{ \begin{array}{c} u_1 \\ u_2 \end{array} \right\} \tag{7}$$

$$\left\{ \begin{array}{c} 5 \\ 0 \end{array} \right\} = \frac{EA}{5} \begin{bmatrix} 1.5 & 0 \\ 0 & 0.5 \end{bmatrix} \left\{ \begin{array}{c} u_1 \\ u_2 \end{array} \right\}$$

$$u_1 = \frac{16.667}{EA} \quad ; u_2 = 0 \tag{8}$$

Support reactions are evaluated using equation (24.30).

$$\{p_u\} = [k_{21}]\{u_u\} \tag{9}$$

Substituting appropriate values in equation (9),

$$\{p_u\} = \frac{EA}{5} \begin{bmatrix} -0.75 & -0.433 \\ -0.433 & -0.25 \\ -0.75 & 0.433 \\ 0.433 & -0.25 \end{bmatrix} \frac{1}{AE} \left\{ \begin{array}{c} 16.667 \\ 0 \end{array} \right\} \tag{10}$$

$$\left\{ \begin{array}{c} p_3 \\ p_4 \\ p_5 \\ p_6 \end{array} \right\} = \left(\begin{array}{c} -2.5 \\ -1.443 \\ -2.5 \\ 1.443 \end{array} \right) \tag{11}$$

The answer can be verified by equilibrium of joint 1. Also,

$$p_3 + p_5 + 5 = 0$$

Now force in each member is calculated as follows,

Member 1: $l = 0.866$; $m = 0.5$; $L = 5m$.

$$\{p'\} = [k']\{u'\}$$

$$= [k'][T]\{u\}$$

$$\begin{Bmatrix} p'_1 \\ p'_2 \end{Bmatrix} = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix} \begin{Bmatrix} u_3 \\ v_4 \\ u_1 \\ v_2 \end{Bmatrix}$$

$$\{p'_1\} = \frac{AE}{L} \begin{bmatrix} l & m & -l & -m \end{bmatrix} \begin{Bmatrix} u_3 \\ v_4 \\ u_1 \\ v_2 \end{Bmatrix}$$

$$\{p'_1\} = \frac{AE}{L} \begin{bmatrix} -0.866 \end{bmatrix} \begin{Bmatrix} \dfrac{16.667}{AE} \end{Bmatrix} = -2.88 \quad \text{kN}$$

Member 2: $l = -0.866$; $m = 0.5$; $L = 5m$.

$$\begin{Bmatrix} p'_1 \\ p'_2 \end{Bmatrix} = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix} \begin{Bmatrix} u_5 \\ v_6 \\ u_1 \\ v_2 \end{Bmatrix}$$

$$\{p'_1\} = \frac{AE}{L} \begin{bmatrix} l & m & -l & -m \end{bmatrix} \begin{Bmatrix} u_3 \\ v_4 \\ u_1 \\ v_2 \end{Bmatrix}$$

$$\{p'_1\} = \frac{AE}{5} \begin{bmatrix} -0.866 \end{bmatrix} \begin{Bmatrix} \dfrac{16.667}{AE} \end{Bmatrix} = -2.88 \quad \text{kN}$$

Summary

The member stiffness matrix of a truss member in local co-ordinate system is defined. Suitable transformation matrices are derived to transform displacements and forces from the local to global co-ordinate system. The member stiffness matrix of truss member is obtained in global co-ordinate system by suitable transformation. The system stiffness matrix of a plane truss is obtained by assembling member matrices of individual members in global co-ordinate system. In the end, a few plane truss problems are solved using the direct stiffness matrix approach.