Module
5
Cables and Arches

Lesson
34
Symmetrical Hingeless Arch

Instructional Objectives:

After reading this chapter the student will be able to
1. Analyse hingeless arch by the method of least work.
2. Analyse the fixed-fixed arch by the elastic-centre method.
3. Compute reactions and stresses in hingeless arch due to temperature change.

34.1 Introduction

As stated in the previous lesson, two-hinged and three-hinged arches are commonly used in practice. The deflection and the moment at the center of the hingeless arch are somewhat smaller than that of the two-hinged arch. However, the hingeless arch has to be designed for support moment. A hingeless arch (fixed–fixed arch) is a statically redundant structure having three redundant reactions. In the case of fixed–fixed arch there are six reaction components; three at each fixed end. Apart from three equilibrium equations three more equations are required to calculate bending moment, shear force and horizontal thrust at any cross section of the arch. These three extra equations may be set up from the geometry deformation of the arch.
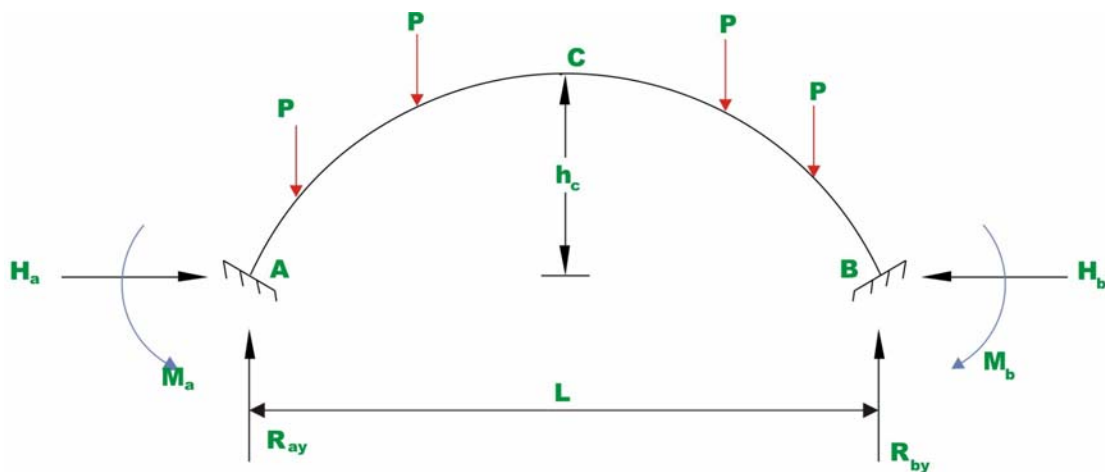
34.2 Analysis of Symmetrical Hingeless Arch

Fig. 34.1 Hingeless Arch

Consider a symmetrical arch of span L and central rise of h_c Let the loading on the arch is also symmetrical as shown in Fig 34.1. Consider reaction components

at the left support A i.e., bending moment M_a, vertical reaction R_{ay} and horizontal thrust H_a as redundants.

Considering only the strain energy due to axial compression and bending, the strain energy U of the arch may be written as

$$U = \int_0^s \frac{M^2 ds}{2EI} + \int_0^s \frac{N^2 ds}{2EA} \qquad (34.1)$$

where M and N are respectively the bending moment and axial force of the arch rib. Since the support A is fixed, one could write following three equations at that point.

$$\frac{\partial U}{\partial M_a} = 0 \qquad (34.2a)$$

$$\frac{\partial U}{\partial H_a} = 0 \qquad (34.2b)$$

$$\frac{\partial U}{\partial R_{ay}} = 0 \qquad (34.2c)$$

Knowing dimensions of the arch and loading, using the above three equations, the unknown redundant reactions M_a, H_a and R_{ay} may be evaluated.

Since the arch and the loading are symmetrical, the shear force at the crown is zero. Hence, at the crown we have only two unknowns. Hence, if we take the internal forces at the crown as the redundant, the problem gets simplified.
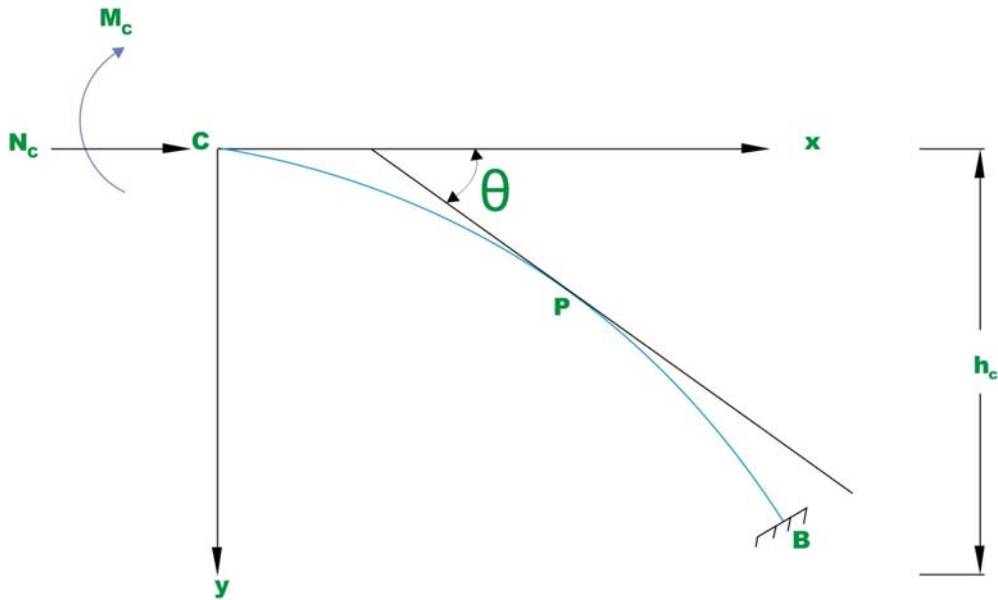
Fig. 34.2

Hence, consider bending moment M_c and the axial force N_c at the crown as the redundant. Since the arch and the loading is symmetrical, we can write from the principle of least work

$$\frac{\partial U}{\partial M_c} = 0 \tag{34.3a}$$

$$\frac{\partial U}{\partial N_c} = 0 \tag{34.3b}$$

$$\frac{\partial U}{\partial M_c} = \int_0^s \frac{M}{EI}\frac{\partial M}{\partial M_c}\,ds + \int_0^s \frac{N}{EA}\frac{\partial N}{\partial M_c}\,ds = 0 \tag{34.4a}$$

$$\frac{\partial U}{\partial N_c} = \int_0^s \frac{M}{EI}\frac{\partial M}{\partial N_c}\,ds + \int_0^s \frac{N}{EA}\frac{\partial N}{\partial N_c}\,ds = 0 \tag{34.4b}$$

Where, s is the length of centerline of the arch, I is the moment of inertia of the cross section and A is the area of the cross section of the arch. Let M_0 and N_0 be the bending moment and the axial force at any cross section due to external loading. Now the bending moment and the axial force at any section is given by

$$M = M_c + N_c y + M_0 \tag{34.5a}$$

$$N = N_c \cos\theta + N_0 \tag{34.5b}$$

$$\frac{\partial M}{\partial M_c} = 1 \ ; \quad \frac{\partial M}{\partial N_c} = y \ ; \quad \frac{\partial N}{\partial N_c} = \cos\theta \ ; \quad \frac{\partial N}{\partial M_c} = 0 \ . \tag{34.6}$$

Equation (34.4a) and (34.4b) may be simplified as,

$$\int_0^s \frac{M}{EI}(1)ds + \int_0^s \frac{N}{EA}(0)ds = 0$$

$$M_c \int_0^s \frac{ds}{EI} + N_c \int_0^s \frac{y\,ds}{EI} = -\int_0^s \frac{M_0}{EI}ds \tag{34.7a}$$

$$\int_0^s \frac{M}{EI}yds + \int_0^s \frac{N}{EA}\cos\theta\,ds = 0$$

$$\int_0^s \frac{M_c y}{EI}ds + \int_0^s \frac{N_c y^2}{EI}ds + \int_0^s \frac{N_c}{EA}\cos^2\theta \ \ ds = -\int_0^s \frac{M_0 y}{EI}ds - \int_0^s \frac{N_0}{EA}\cos\theta \ \ ds \tag{34.7b}$$

From equations 34.7a and 34.7b, the redundant M_c and N_c may be calculated provided arch geometry and loading are defined. If the loading is unsymmetrical or the arch is unsymmetrical, then the problem becomes more complex. For such problems either column analogy or elastic center method must be adopted. However, one could still get the answer from the method of least work with little more effort.
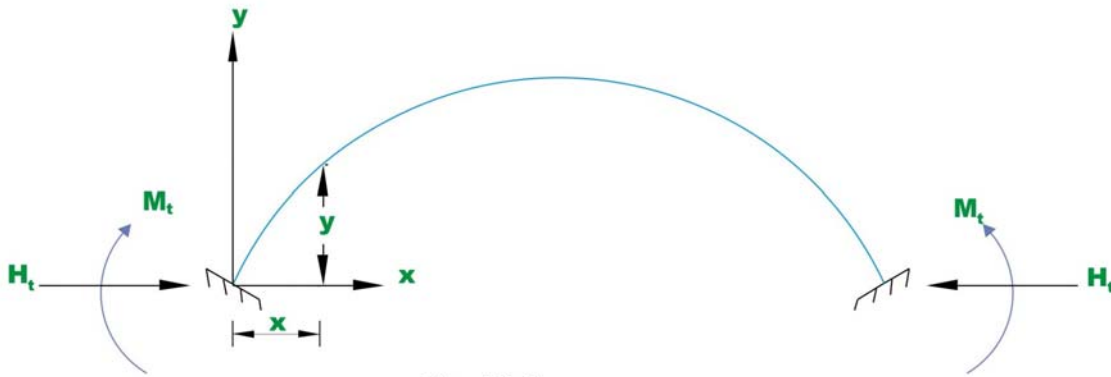
34.3 Temperature stresses

Fig. 34.3

Consider an unloaded fixed-fixed arch of span L. The rise in temperature, would introduce a horizontal thrust H_t and a moment M_t at the supports. Now due to rise in temperature, the moment at any cross-section of the arch

$$M = M_t - H_t t \qquad (34.8)$$

Now strain energy stored in the arch

$$U = \int_0^s \frac{M^2 ds}{2EI}$$

Now applying the Castigliano's first theorem,

$$\frac{\partial U}{\partial H_t} = \alpha \ L \ T = \int_0^s \frac{M}{EI} \frac{\partial M}{\partial H_t} ds$$

$$\alpha LT = = \int_0^s \frac{M_t y}{EI} ds - H_t \int_0^s \frac{y^2}{EI} ds \qquad (34.9)$$

Also,

$$\frac{\partial U}{\partial M_t} = 0 = \int_0^s \frac{M}{EI} \frac{\partial M}{\partial M_t} ds$$

$$\int_0^s \frac{(M_t - H_t y)}{EI} ds = 0$$

$$M_t \int_0^s \frac{ds}{EI} - H_t \int_0^s \frac{y ds}{EI} = 0 \qquad (34.10)$$

Solving equations 34.9 and 34.10, M_t and H_t may be calculated.

Example 34.1

A semicircular fixed-fixed arch of constant cross section is subjected to symmetrical concentrated load as shown in Fig 34.4. Determine the reactions of the arch.
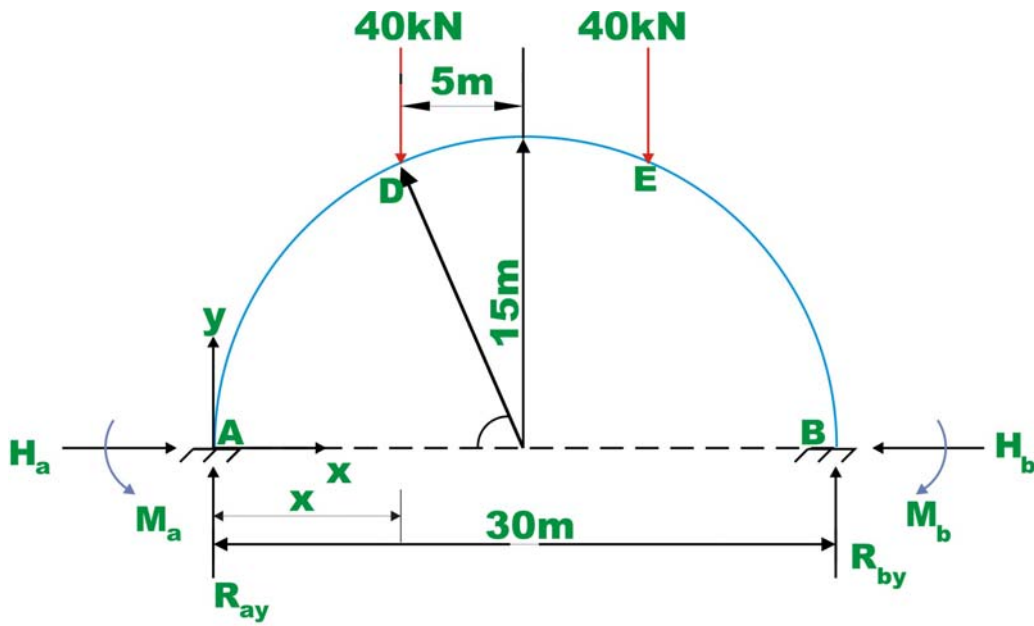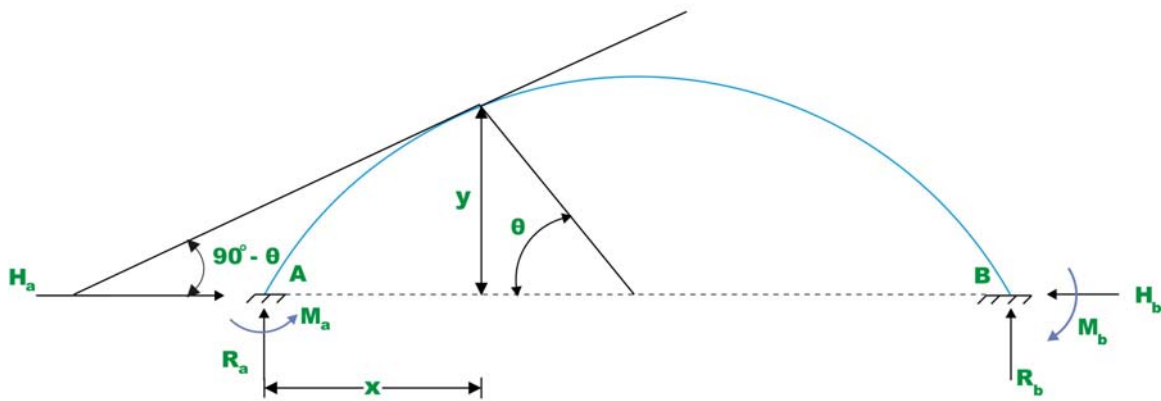
Fig. 34.4 Example 34.1

Fig. 34.5

Solution:

Since, the arch is symmetrical and the loading is also symmetrical,

$$R_{ay} = R_{by} = 40\text{kN} \qquad (1)$$

Now the strain energy of the arch is given by,

$$U = \int_0^s \frac{M^2 ds}{2EI} + \int_0^s \frac{N^2 ds}{2EA} \qquad (2)$$

Let us choose H_a and M_a as redundants. Then we have,

$$\frac{\partial U}{\partial M_a} = 0 \text{ and } \frac{\partial U}{\partial H_a} = 0 \qquad (3)$$

The bending moment at any cross section is given by,

$$M = R_{ay} x - M_a - H_a y \qquad\qquad 0 \le \theta \le \theta_D \qquad (4)$$

$$M = R_{ay} x - M_a - H_a y - 40(x - 10) \qquad\qquad \theta_D \le \theta \le \pi/2$$

$$N = H_a \cos(90 - \theta) + R_a \cos\theta$$

$$N = H_a \sin\theta + R_a \cos\theta \qquad\qquad 0 \le \theta \le \theta_D \qquad (5)$$

$$N = H_a \sin\theta + (R_a - 40)\cos\theta \qquad\qquad \theta \le \theta \le \pi/2 \qquad (6)$$

$$y = R\sin\theta$$

$$x = R(1 - \cos\theta)$$

And $ds = Rd\theta$

See Fig 34.5.

$$\frac{\partial U}{\partial M_a} = \int_0^s \frac{M}{EI}(-1)ds + \int_0^s \frac{N}{EA}(0)ds = 0$$

$\int_0^s \dfrac{M}{EI} ds = 0$ Since the arch is symmetrical, integration need to be carried out

between limits $0 \text{ to } \pi/2$ and the result is multiplied by two.

$$2\int_0^{\pi/2} \frac{M}{EI} ds = 0$$

$$\int_0^{\pi/2} 40R(1-\cos\theta)Rd\theta - \int_0^{\pi/2} M_a Rd\theta - H_a \int_0^{\pi/2} R\sin\theta\, Rd\theta - \int_{\pi/2.552}^{\pi/2} 40[R(1-\cos\theta)-10]Rd\theta = 0$$

$$22.8310R^2 - 1.571M_aR - H_aR^2 - 41.304R^2 + 135.92R = 0$$

$$342.477 - 1.571M_a - 15H_a - 169.56 + 135.92 = 0$$

$$1.571M_a + 15H_a - 308.837 = 0$$

$$\frac{\partial U}{\partial H_a} = \int_0^s \frac{M}{EI}(-y)ds + \int_0^s \frac{N}{EA}(\sin\theta)ds = 0$$

$$\frac{1}{EI}\int_0^{\pi/2}(-R\sin\theta)\{[40R(1-\cos\theta)] - M_a - H_a(R\sin\theta)\}Rd\theta - \frac{1}{EI}\int_{\pi/2.552}^{\pi/2}(-R\sin\theta)\{[40[R(1-\cos\theta)-10]]\}Rd\theta +$$

$$\int_0^{\pi/2}\frac{(H_a\sin\theta + R_a\cos\theta)}{EA}(\sin\theta)Rd\theta - \frac{1}{EA}\int_{\pi/2.552}^{\pi/2}(\sin\theta)40\cos\theta\,Rd\theta = 0$$

$$\int_0^{\pi/2}\{-\frac{40R^3}{EI}\sin\theta + \frac{40R^3}{EI}\sin\theta\cos\theta + \frac{M_aR^2}{EI}\sin\theta + \frac{H_aR^3}{EI}\sin^2\theta + \frac{H_aR}{EA}\sin^2\theta - \frac{R(R_{ay})\sin\theta\cos\theta}{EA}\}d\theta +$$

$$\int_{\pi/2.552}^{\pi/2}\{\frac{40R^3}{EI}\sin\theta - \frac{40R^3}{EI}\sin\theta\cos\theta - \frac{400R^2}{EI}\sin\theta - \frac{40R}{EA}\sin\theta\cos\theta\}d\theta = 0$$

$$\frac{-40}{I}(1) + \frac{40}{I}(\frac{1}{2}) + \frac{M_a}{IR}(1) + \frac{H_a}{I}(0.785) + \frac{H_a}{R^2A}(0.785) - \frac{40}{R^2A}(\frac{1}{2}) +$$

$$\frac{40}{I}(0.333) - \frac{40}{I}(0.0554) - \frac{400}{RI}(0.333) - \frac{40}{R^2A}(0.0555) = 0$$

$$-266 + 23.58H_a + 2M_a = 0 \tag{7}$$

Solving equations (6) and (7), H_a and M_a are evaluated. Thus,

$$H_a = 28.28\text{ kN}$$

$$M_a = -466.42\text{ kN} \tag{8}$$
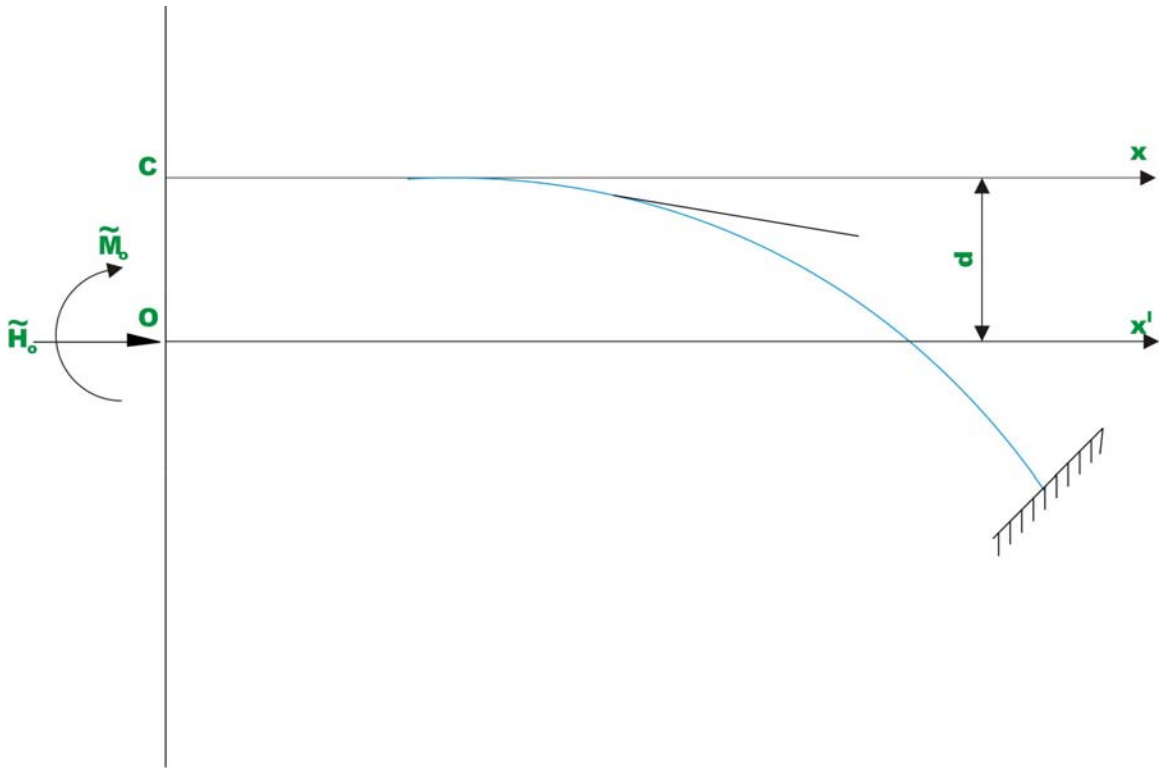
34.4 Elastic centre method

Fig. 34.6 Elastic centre

Equations (34.7a) and (34.7b) are quite difficult to solve. However, they can be further simplified if the origin of co-ordinates is moved from C to O in Fig. 34.3. The distance d is chosen such that y_1 $(= y - d)$ satisfies the following condition.

$$\int_0^s \frac{y_1}{EI} ds = \int_0^s \frac{(y-d)}{EI} ds = 0 \qquad (34.10a)$$

Solving which, the distance d may be computed as

$$d = \frac{\displaystyle\int_0^s \frac{y}{EI} ds}{\displaystyle\int_0^s \frac{ds}{EI}} \qquad 34.10b)$$

The point O is known as the elastic centre of the arch. Now equation (34.7a) can be written with respect to new origin O. Towards this, substitute $y = y_1 + d$ in equation (34.7a).

$$M_c \int_0^s \frac{ds}{EI} + N_c \int_0^s \frac{(y_1 + d)}{EI} ds = -\int_0^s \frac{M_0}{EI} ds \qquad (34.11)$$

In the above equation, $\int_0^s \frac{y_1}{EI} ds$ is zero. Hence the above equation is rewritten as

$$M_c + N_c d = -\frac{\int_0^s \frac{M_0}{EI} ds}{\int_0^s \frac{ds}{EI}} \qquad (34.12)$$

Now, $(M_c + N_c d)$ is the moment $\tilde{M}_0$ at O (see Fig. 34.3). Similarly the equation (34.7b) is also simplified. Thus we obtain,

$$\tilde{M}_0 = M_c + N_c d = -\frac{\int_0^s \frac{M_0}{EI} ds}{\int_0^s \frac{ds}{EI}} \qquad (34.13)$$

and,

$$\tilde{H}_0 = N_c = -\frac{\int_0^s \frac{M_0 y_1}{EI} ds + \int_0^s \frac{N_0 \cos\theta}{EA} ds}{\int_0^s \frac{y_1^2}{EI} ds + \int_0^s \frac{\cos^2\theta}{EA} ds} \qquad (34.14)$$

34.4.1 Temperature stresses

Consider a symmetrical hinge less arch of span L, subjected to a temperature rise of $T^\circ C$. Let elastic centre O be the origin of co-ordinates and $\tilde{H}_0, \tilde{M}_0$ be the redundants. The magnitude of horizontal force $\tilde{H}_0$ be such as to counteract the increase in the span $\frac{\alpha LT}{2}$ due to rise in temperature T. Also from Symmetry, there must not be any rotation at the crown. Hence,

$$\frac{\partial U}{\partial M_O} = 0 = \int_0^s \frac{M}{EI} \frac{\partial M}{\partial \tilde{M}_O} ds = 0 \qquad (34.15)$$

$$\frac{\partial U}{\partial \tilde{H}_O} = \int_0^s \frac{M}{EI} \frac{\partial M}{\partial \tilde{H}_O} ds + \int_0^s \frac{N}{EA} \frac{\partial N}{\partial \tilde{H}_O} ds = \frac{\alpha LT}{2} \qquad (34.16)$$

Moment at any section is calculated by,

$$M = \tilde{M}_O + \tilde{H}_O y$$

$$N = \tilde{H}_O \cos\theta$$

$$\int_0^s \frac{\tilde{M}_O}{EI} ds = 0$$

$$\tilde{M}_O = 0 \qquad\qquad (34.17)$$

and

$$\int_0^s \left(\frac{\tilde{H}_O y_1}{EI} \right) y_1 ds + \int_0^s \left(\frac{\tilde{H}_O \cos\theta}{EA} \right) \cos\theta ds = \frac{\alpha LT}{2}$$

Simplifying the above equation,

$$\tilde{H}_O = \frac{\dfrac{\alpha LT}{2}}{\displaystyle\int_0^s \left(\frac{y_1^2}{EI} \right) ds + \int_0^s \left(\frac{\cos^2\theta}{EA} \right) ds} \qquad\qquad (34.18)$$

Using equation (34.18), the horizontal thrust $\tilde{H}_O$ due to uniform temperature rise in the arch can be easily calculated provided the dimensions of the arch are known. Usually the area of the cross section and moment of inertia of the arch vary along the arch axis.

Example 30.2

A symmetrical hinge less circular arch of constant cross section is subjected to a uniformly distributed load of 10 kN/m. The arch dimensions are shown in Fig. 34.7a. Calculate the horizontal thrust and moment at A.
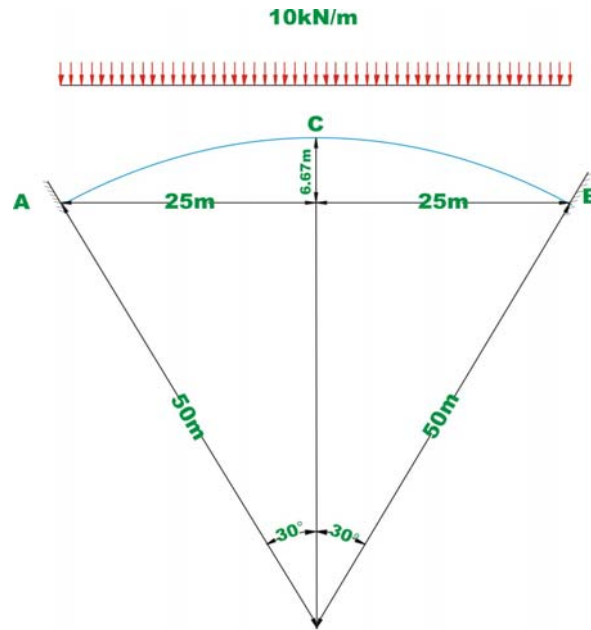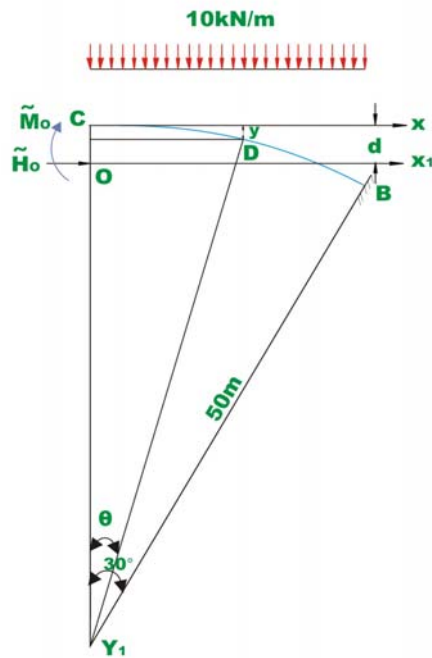
Fig. 34 .7a Example 34.2

Fig. 34.7b

The distance d of the elastic centre from the crown C is calculated by equation,

$$d = \dfrac{\displaystyle\int_0^s \dfrac{y}{EI}\,ds}{\displaystyle\int_0^s \dfrac{ds}{EI}} \qquad (1)$$

From Fig.34.7b, the ordinate at d, is given by

$$y = 50(1 - \cos\theta)$$

$$d = \dfrac{\displaystyle\int_0^{\pi/6} \dfrac{50(1-\cos\theta)}{EI}\,50\,d\theta}{\displaystyle\int_0^{\pi/6} \dfrac{50\,d\theta}{EI}}$$

$$d = \dfrac{50\left(\dfrac{\pi}{6} - \dfrac{1}{2}\right)}{\dfrac{\pi}{6}} = 2.2535 \quad \text{m.} \qquad (2)$$

The elastic centre O lies at a distance of $2.2535\,m$ from the crown. The moment at the elastic centre may be calculated by equation (34.12). Now the bending moment at any section of the arch due to applied loading at a distance x from elastic centre is

$$\tilde{M}_O = \dfrac{-\displaystyle\int_0^s \dfrac{5x^2}{EI}\,ds}{\displaystyle\int_0^s \dfrac{ds}{EI}} \qquad (3)$$

In the present case, $x = 50\sin\theta$ and $ds = 50\,d\theta$, $EI = \text{constant}$

$$\tilde{M}_O = \dfrac{-5 \times 50^3 \displaystyle\int_0^{\pi/6} \sin^2\theta\;d\theta}{50 \displaystyle\int_0^{\pi/6} d\theta}$$

$$M_C + N_C d = -\dfrac{5 \times 50^2}{2}\dfrac{\left(\dfrac{\pi}{6} - \dfrac{1}{2}\sin\left(\dfrac{\pi}{3}\right)\right)}{\dfrac{\pi}{6}} = -1081.29 \quad \text{kN.m} \qquad (4)$$
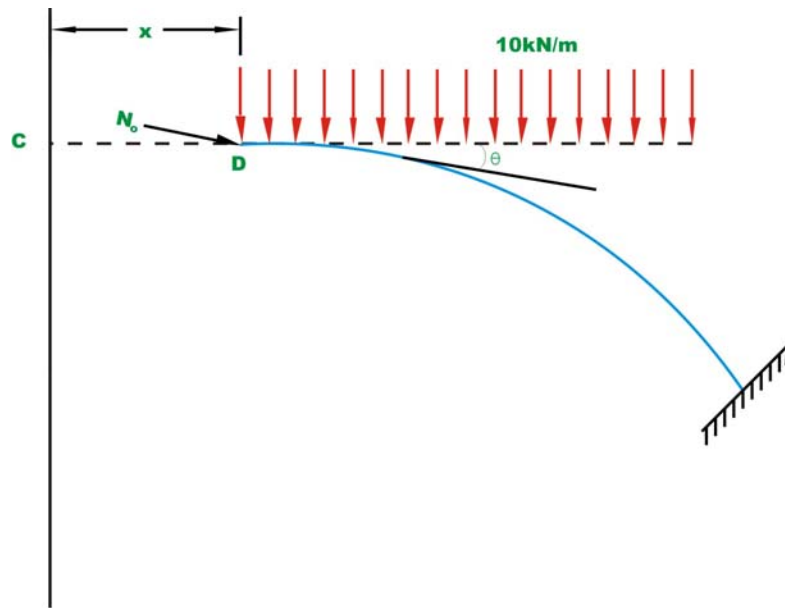
$$N_O = 10\left(\frac{L}{2} - x\right)\cos\theta$$
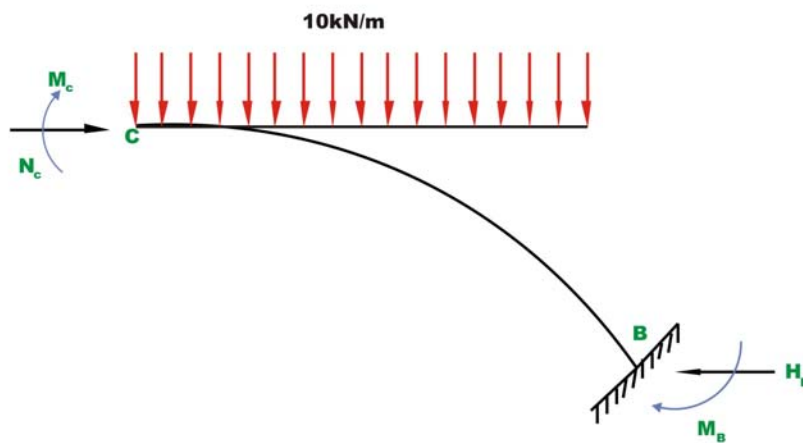

Fig. 34.7c

Fig. 34.7d

And.

$$y_1 = y - d$$

$$y_1 = 50(1 - \cos\theta) - 2.25$$

$$y_1 = 47.75 - 50\cos\theta$$

Now $\tilde{H}_O$ is given by equation (34.14). Thus

$$\tilde{H}_0 = N_c = -\frac{\displaystyle\int_0^s \frac{M_0 y_1}{EI}\,ds + \int_0^s \frac{N_0 \cos\theta}{EA}\,ds}{\displaystyle\int_0^s \frac{y^2_1}{EI}\,ds + \int_0^s \frac{\cos^2\theta}{EA}\,ds} \qquad (5)$$

$$\int_0^s \frac{M_0 y_1}{EI}\,ds = \frac{1}{EI}\int_0^{\pi/6} 5x^2 (47.75 - 50\cos\theta)50 \;\; d\theta$$

$$= \frac{250}{EI}\int_0^{\pi/6} (50\sin\theta)^2 (47.75 - 50\cos\theta) \;\; d\theta$$

$$= \frac{625000}{EI}\int_0^{\pi/6} \left(23.875(1 - \cos 2\theta) - 50\cos\theta\sin^2\theta\right) \;\; d\theta$$

$$= \frac{625000}{EI}\int_0^{\pi/6} \left(23.875(1 - \cos 2\theta) - 25\left(\cos\theta - \frac{1}{2}(\cos 3\theta + \cos\theta)\right)\right) \;\; d\theta$$

$$= \frac{49630.735}{EI} \qquad (6)$$

$$\int_0^s \frac{N_0 \cos\theta}{EA}\,ds = \frac{1}{EA}\int_0^{\pi/6} 10(25 - x)\cos^2\theta \;\; d\theta$$

$$= \frac{10}{EA}\int_0^{\pi/6} \left(25\left(\frac{1 + \cos 2\theta}{2}\right) - 50\sin\theta\cos^2\theta\right) \;\; d\theta$$

$$= \frac{10}{EA}\int_0^{\pi/6} \left(12.5(1 + \cos 2\theta) - 25(\sin\theta + \sin\theta\cos 2\theta)\right) \;\; d\theta$$

$$= \frac{10}{EA}\left(12.5(\theta + \sin 2\theta)\right)_0^{\pi/6} - 25\left(-(\cos\theta)_0^{\pi/6} + \frac{1}{2}\left(-\frac{1}{3}\cos 3\theta - \cos\theta\right)_0^{\pi/6}\right)$$

$$= \frac{81.795}{EA} \tag{7}$$

$$\int_0^s \frac{y^2_1}{EI} ds = \frac{1}{EI} \int_0^{\pi/6} (47.75 - 50\cos\theta)^2 \, 50 \; d\theta$$

$$= \frac{50}{EI} \int_0^{\pi/6} \left(2280.06 + 2500\cos^2\theta - 4775\cos\theta\right) \; d\theta$$

$$= \frac{50}{EI} \left(2280.06\left(\frac{\pi}{6}\right) + 1250\left(\frac{\pi}{6} + \frac{1}{2}\sin\frac{\pi}{3}\right) - 4775\sin\frac{\pi}{6}\right)$$

$$= \frac{105.046}{EI} \tag{8}$$

$$\int_0^s \frac{\cos^2\theta}{EA} ds = \frac{50}{2EA} \int_0^{\pi/6} (1 + \cos 2\theta) d\theta$$

$$= \frac{25}{EA}\left(\frac{\pi}{6} + \frac{1}{2}\sin\frac{\pi}{3}\right) = 23.915 \tag{9}$$

$$\tilde{H}_0 = -\frac{-\left(\dfrac{49630.735}{EI} + \dfrac{81.795}{EA}\right)}{\left(\dfrac{105.046}{EI} + \dfrac{23.915}{EA}\right)} \tag{10}$$

Consider an arch cross section of 300×500 mm ; and $I = 3.125 \times 10^{-3} \, \text{m}^4$
$A = 0.15$ m^2. Then,

$$\tilde{H}_0 = -\frac{-(15881835.2 + 545.3)}{(33614.72 + 159.43)} = -470.25 \quad \text{kN} \tag{11}$$

In equation (5), if the second term in the numerator and the second term in the denominator were neglected then, we get,

$$\tilde{H}_0 = -\frac{-\left(\dfrac{49630.735}{EI}\right)}{\left(\dfrac{105.046}{EI}\right)} = -472.67 \quad \text{kN} \tag{12}$$

Thus calculating $\tilde{H}_O$ by neglecting second term in the numerator and denominator induces an error which is less than 0.5%. Hence for all practical purposes one could simplify the expression for $\tilde{H}_O$ as,

$$\tilde{H}_0 = -\frac{\displaystyle\int_0^s \frac{M_0 y_1}{EI}\,ds}{\displaystyle\int_0^s \frac{y_1^2}{EI}\,ds} \tag{13}$$

Now we have,

$$M_C + N_C d = -1081.29$$

$$N_C = -470.25$$

$$M_C = -23.22 \text{ kN.m} \tag{14}$$

Moment at B, $M_B = M_C + 10 \times 25 \times \dfrac{25}{2}$

$$= -23.22 + 10 \times 25 \times \frac{25}{2}$$

$$= 3101.78 \quad \text{kN.m} \tag{15}$$

Also $H_B = N_C$.

Since the arch and the loading are symmetrical, $M_A = M_B$ and $H_A = H_B$.

Summary

In this lesson, hingeless arches are considered. The analysis of hingeless arch by the method of least work is given in the beginning. This is followed by the analysis of hingeless arch by the elastic centre method. The procedure to compute stresses developed in the hingeless arch due to temperature change is discussed. A few problems are solved illustrate the various issues involved in the analysis of hingeless arches.