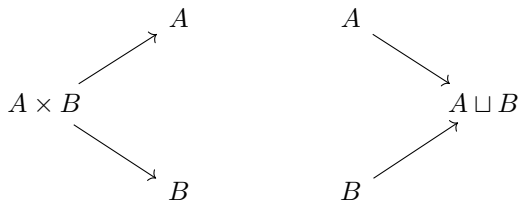


# Algebra

## Short Notes



## I. Set Theory and Categories

### 1. Sets and Categories

#### 1.1

Suppose that for any property, we can define a set whose members are items that satisfy that property. Then let  $r = \{x : x \notin x\}$ . Then  $r \in r \iff r \notin r$ .

In naive set theory and ZFC, this is avoided as the axiom schema of specification (for any property  $P$  and set  $S$  the set  $\{s \in S : s \text{ satisfies } P\}$  exists) requires an existing set  $S$ . Also, a set of all sets cannot exist, as otherwise we can take  $S$  to be such a set.

#### 1.2

Nonempty: any equivalence class  $[a]$  contains  $a$ .

Disjoint: we wish to show that  $[a] \neq [b] \implies [a] \cap [b] = \{\}$ . Equivalently,  $[a] \cap [b] \neq \{\} \implies [a] = [b]$ . Suppose  $c \in [a] \cap [b]$ , and let  $a' \in [a]$ . Then  $a' \sim a \sim c \sim b$ , hence  $[a] \subseteq [b]$ . Similarly  $[b] \subseteq [a]$ .

Union is  $S$ :  $S$  contains the union as it contains every equivalence class by construction. Let  $s \in S$ . Then  $s \in [s]$ , hence the union contains  $S$ .

#### 1.3

Define  $a \sim b$  when there exists  $p \in P$  such that  $a \in p, b \in p$ .

## 1.4

We count partitions. A partition will have 1, 2 or 3 parts. If it has 1 part it must be the whole set and if it has 3 parts it must be  $\{[1], [2], [3]\}$ . If it has 2 parts, exactly one part will have 2 items and the other 1 item, and we have 3 choices for the singleton part. Hence there are 5 equivalence relations.

## 1.5

For  $a, b \in \mathbb{R}$  define  $a \sim b$  when  $|a - b| < 2$ . Then  $0 \sim 1$  and  $1 \sim 2$  but  $0 \not\sim 2$ .

## 1.6

Reflexive: for  $r \in \mathbb{R}$  we have  $r \sim r$  since  $r - r = 0 \in \mathbb{Z}$ .

Symmetric: suppose  $a \sim b$ . Then  $a - b \in \mathbb{Z}$ . Then  $b - a = -(a - b) \in \mathbb{Z}$ , hence  $b \sim a$ .

Transitive: suppose  $a \sim b, b \sim c$ . Then  $a - c = (a - b) + (b - c) \in \mathbb{Z}$ .

Every class in  $\mathbb{R}/\sim$  is represented by a unique real number  $r \in [0, 1)$ , which is equivalent to all real numbers with that fractional part.

Every class in  $\mathbb{R}^2/\approx$  is represented by a unique pair of real numbers  $(r_1, r_2)$  with each  $r_i \in [0, 1)$ , which is equivalent to all pairs of real numbers whose fractional parts are the representative.

# 1. Functions between sets

## 2.1

$n!$

## 2.2

Let  $f : A \rightarrow B$ . We wish to show that  $f$  has a right-inverse iff it is surjective.

$\implies$  : let  $b \in B$ . Suppose the right inverse is  $g$ ; then  $f \circ g = id_B$  hence  $b = f(g(b)) \in f(A)$ .

$\impliedby$  : we define  $g : B \rightarrow A$  as follows. Let  $b \in B$ . Choose  $a \in f^{-1}(b)$ , which is nonempty since  $f$  is surjective, then set  $g(b) = a$ . Then  $g$  is a right inverse. Proof: for all  $b \in B$  we have  $f(g(b)) = f(a) = b$ .

## 2.3

Let  $f : A \rightarrow B$  be a bijection and  $f^{-1}$  its inverse. Since  $f^{-1}$  is a right inverse of  $f$ , we have  $f \circ f^{-1} = id_B$  hence  $f$  is a left inverse of  $f^{-1}$ . By a similar argument, it is a right inverse as well; hence  $f^{-1}$  is a bijection.

Let  $g : A \rightarrow B, f : B \rightarrow C$  be bijections. Then  $(f \circ g) \circ (g^{-1} \circ f^{-1}) = id_C$ , hence  $f \circ g$  is surjective. By a similar argument, it is injective.

## 2.4

Reflexive: for any set  $A$ ,  $|A| = |A|$  since  $id_A$  is a bijection.

Symmetric: suppose  $|A| = |B|$ , that is, there is a bijection  $f : A \rightarrow B$ . Then  $f^{-1} : B \rightarrow A$  is a bijection, hence  $|B| = |A|$ .

Transitive: suppose  $|A| = |B|$ ,  $|B| = |C|$  with bijections  $f$  and  $g$ . Then  $f \circ g$  is a bijection between  $A$  and  $C$ .

## 2.5

A function  $f : A \rightarrow B$  is an epimorphism (is epic) if for all sets  $Z$  and all functions  $\alpha', \alpha'' : B \rightarrow Z$  we have  $\alpha' \circ f = \alpha'' \circ f \implies \alpha' = \alpha''$ . A function is surjective iff it is epic.

$\implies$  : Let  $f : A \rightarrow B$  be surjective and  $\alpha', \alpha'' : B \rightarrow Z$  such that  $\alpha' \circ f = \alpha'' \circ f$ . Then  $f$  has a right inverse  $g$  and  $\alpha' \circ f \circ g = \alpha'' \circ f \circ g$ . Hence  $\alpha' = \alpha''$ .

$\impliedby$  : Let  $f : A \rightarrow B$  be epic and let  $b \in B$ . We wish to show that  $b \in f(A)$ . Let  $Z = \{0, 1\}$  and suppose  $\alpha'$  and  $\alpha''$  disagree only on  $b$ , that is,  $\alpha'(c) = \alpha''(c) \iff c \neq b$ . Then  $\alpha' \neq \alpha''$ . Suppose  $b \notin f(A)$ . Then for all  $a \in A$ ,  $\alpha' \circ f(a) = \alpha'' \circ f(a)$  (proof:  $f(a) \neq b$ , hence  $\alpha'(f(a)) = \alpha''(f(a))$ ), which contradicts  $f$  is an epimorphism.

## 2.6

For any  $f : A \rightarrow B$  define  $\phi_f : A \rightarrow A \times B$  by  $\phi_f(a) = (a, f(a))$ . Then  $\phi_f$  is a section (right inverse) of  $\pi_A$ . Proof: for all  $a \in A$  we have  $\pi_A \circ \phi_f(a) = \pi_A(a, f(a)) = a$ . In fact every section of  $\pi_A$  corresponds to a unique function, since the  $\pi_B(\pi_A^{-1}(x)) = B$ .

## 2.7

For all  $f : A \rightarrow B$  let  $\phi_f : A \rightarrow \Gamma_f$  be given by  $\phi(a) = (a, f(a))$ . Then  $\pi_A$  is a left inverse of  $\phi_f$  by exercise 2.6, hence  $\phi_f$  is injective. Also  $\phi_f$  is surjective. Proof: suppose  $y = (a, b) \in \Gamma_f$ . Then  $b = f(a)$ , hence  $y = \phi_f(a)$ .

## 2.8

The equivalence relation is:  $r_1 \sim r_2 \iff e^{2\pi i r_1} = e^{2\pi i r_2} \iff r_1 - r_2 \in \mathbb{Z}$ . Hence the surjection maps  $r$  to (the equivalence class represented by) its fractional part;  $\bar{f}$  maps  $[r]$  to  $e^{2\pi i r}$ , with the unit circle as codomain; the injection is the inclusion map (of the unit circle in the complex plane).

## 2.9

Let the bijection between  $A'$  and  $A''$  be  $f$  and the bijection between  $B'$  and  $B''$  be  $g$ . Define a bijection  $h : A' \cup B' \rightarrow A'' \cup B''$  as follows:  $h(x)$  is  $f(x)$  if  $x \in A'$  otherwise  $g(x)$ . This is well-defined as exactly one of  $x \in A'$ ,  $x \in B'$  is true. The inverse can be defined explicitly similarly.

To form the disjoint union of  $A$  and  $B$  we produce disjoint copies; suppose we do this in one way to get  $A'$  and  $B'$ , and another way to get  $A''$  and  $B''$ . Since they are copies,  $A' \cong A''$ ,  $B' \cong B''$ , hence the disjoint unions are isomorphic (as sets).

## 2.10

Any function from  $A$  to  $B$  is defined by the value of  $f(a)$  for all  $a \in A$ ; the choice for each  $a$  is independent, there are  $|A|$  choices and for each choice we may choose  $|B|$  ways, so there are  $|A| \times |A| \times \dots \times |A|$  different functions.

For an explicit bijection, we can use the fact that any finite set can be well-ordered to order the elements of  $|B^A|$  lexicographically on their valuations  $(f(a_1), f(a_2) \dots)$ .

## 2.11

A subset  $S \subseteq A$  is determined uniquely by its indicator function  $f_S \in 2^A$  given by  $f_S(a) = 0 \iff a \in S$ .

# 3. Categories

## 3.1

We compose two morphisms in  $C^{op}$  by composing the two underlying morphisms in  $C$  “the other way”. That is, suppose  $f \in \text{Hom}_{C^{op}}(A, B) = \text{Hom}_C(B, A)$  and  $g \in \text{Hom}_{C^{op}}(B, C) = \text{Hom}_C(C, B)$ . Let composition in  $C$  be denoted by  $\cdot$  and composition in  $C^{op}$  by  $\circ$ . Then define their composition  $g \circ f = f \cdot g$ .

1. The identity homomorphism exists because  $1_A \in \text{Hom}_C(A, A) = \text{Hom}_{C^{op}}(A, A)$ .
2. We explicitly constructed the composition in  $C^{op}$ , hence it exists.
3.  $(f \circ g) \circ h = h \cdot (g \cdot f) = (h \cdot g) \cdot f = f \circ (g \circ h)$ .
4.  $f \circ 1_A = 1_A \cdot f = f$ ; similar proof for left inverse.

## 3.2

$$|\text{End}(A)| = |\text{Hom}(A, A)| = |A^A| = |A|^{|A|}.$$

## 3.3

That means it is a left and right identity. Left identity means for all morphisms  $f : a \rightarrow b$ ,  $1_b f = f$ . Proof that  $1_b$  is a left identity: let  $f : a \rightarrow b$ . By our definition of composition,  $1_b f = (a, b)$  (let  $c = b$  on p21), which is identically  $f$ . The proof for right identity is similar.

## 3.4

No, since  $n \not\prec n$ , so  $\text{Hom}(n, n)$  is empty.

## 3.5

$\subseteq$  in 3.4 is the  $\sim$  relation in 3.3 (that is required to be reflexive and transitive).

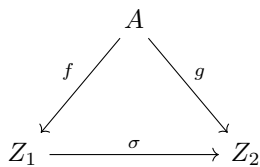
### 3.6

1. Let the identity in  $V$  be the identity matrix.
2. Let  $f : x \rightarrow y, g : y \rightarrow z$ . The composition  $gy$  is exactly the matrix product  $gy$ , which is the correct size ( $gy : x \rightarrow z$  is a  $z \times y$  matrix).
3. Associativity follows from associativity of matrix multiplication.
4. The identity matrix is an identity with respect to matrix multiplication.

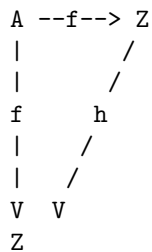
An object  $n$  of this category can be thought of as  $\mathbb{R}^n$ , and the morphisms  $\text{Hom}(m, n)$  as linear functions  $\mathbb{R}^m \rightarrow \mathbb{R}^n$ . (Eg use the standard basis on  $\mathbb{R}^m$  and  $\mathbb{R}^n$  to convert a linear function to an appropriately sized matrix).

### 3.7

Let us denote this category as  $C^A$ . Then the objects of  $C^A$  are morphisms of  $C$ . Given two objects in  $f, g \in \text{Obj}(C^A)$ , suppose  $f \in \text{Hom}_C(A, Z_1)$  and  $g \in \text{Hom}_C(A, Z_2)$ . Define a  $C^A$ -morphism from  $f$  to  $g$  to be a  $C$ -morphism  $\sigma$  such that the diagram commutes.

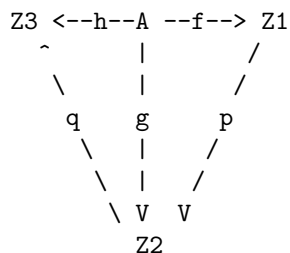


1. An identity  $C^A$ -morphism on the  $C^A$ -object  $f$  is a  $C$ -morphism  $h$  such that



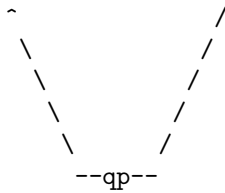
commutes;  $1_Z$  does the job.

2. Suppose  $f, g, h$  are  $C^A$ -objects and  $p$  is a  $C^A$ -morphism from  $f$  to  $g$  and  $q$  is a  $C^A$ -morphism from  $g$  to  $h$ . Then the following diagram commutes.



By removing the vertical line and composing  $p$  and  $q$ ,

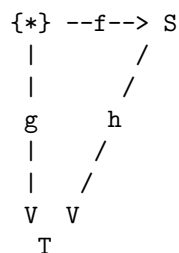




commutes; hence define composition in  $C^A$  as composition in  $C$ .

3. Associativity follows from associativity in  $C$ .

We will also verify the claim in example 3.8.



The requirement is that  $h$  commutes, that is, considered as set-functions,  $g = h \circ f$ . Since  $g$  is completely determined by its value on  $*$ , this means  $g(*) = h(f(*))$ , or  $t = h(s)$ .

### 3.8

The objects of  $C$  are infinite sets and the morphisms in  $C$  are set-functions between them (that is we define  $\text{Hom}_C(A, B) = \text{Hom}_{\text{Set}}(A, B)$ , forcing  $C$  to be a full subcategory of  $\text{Set}$ ). Identity, composition and associativity all follow from  $\text{Set}$ .

### 3.9

WIP

A function maps between two sets; we must generalize this concept to allow for maps between two multisets. How we do this determines the structure of morphisms in  $\text{MSet}$ . (In all cases, though, members of  $\text{Obj}(\text{MSet})$  will be pairs  $(S, \sim)$  where  $\sim$  is an equivalence relation on  $S$ ).

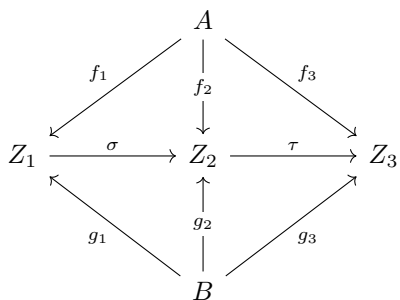
1. We can treat  $(S, \sim)$  as a partitioned set; then mfunctions between them will be functions between partitions.
2. We can simply inherit the morphisms from  $S$ , allowing morphisms to ignore  $\sim$  altogether. The downside is that mfunctions are now sensitive to which representative of a given element you feed it.
3. We can generalize functions to "multivalued functions" (eg the  $z \rightarrow z^{1/3}$  operator on  $\mathbb{C}$ ); TBD
4. We can generalize functions to "regular functions whose image contain elements with multiplicities"; for finite multiplicities this is like a regular function  $x \rightarrow (y, n)$  where  $x, y$  are our conceptual domain and codomain and  $n$  is the multiplicity. A contrived example is: given a fixed function  $f$  over  $\mathbb{R}$ , we could map  $x$  to  $(y, n)$  where  $y = f(x)$  and  $n$  is the algebraic multiplicity of the root of the function  $f'(t) = f(t) - f(x)$  at  $y$  (hence  $n$  would mostly be 1, and at stationary points it would increase).

### 3.10

The object  $\Omega = \{0, 1\}$  will do. A subobject (subset)  $A'$  of an object (set)  $A$  can be identified with a morphism (set-function)  $f : A \rightarrow \Omega$  by setting  $a \in A' \iff f(a) = 1$ .

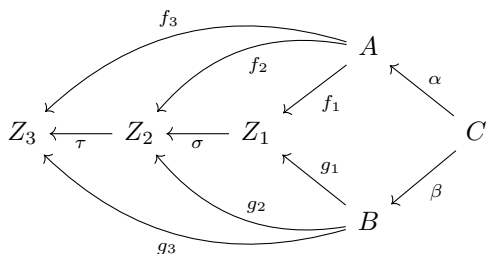
### 3.11 - $C^{A,B}$

A  $C^{A,B}$ -object is a pair of  $C$ -morphisms  $(f, g)$ . A  $C^{A,B}$ -morphism between  $(f_1, g_1)$  and  $(f_2, g_2)$  is a  $C$ -morphism  $\sigma$  such that  $f_2 = \sigma f_1$  and  $g_2 = \sigma g_1$ . Composition is inherited from  $C$ .



### 3.11 - $C^{\alpha,\beta}$

Let  $\alpha : C \rightarrow A$  and  $\beta : C \rightarrow B$ . A  $C^{\alpha,\beta}$ -object is a pair of  $C$ -morphisms  $(f, g)$  with a common target  $Z$  such that  $f\alpha = g\beta$ . A  $C^{\alpha,\beta}$ -morphism between objects  $(f_1, g_1)$  with common target  $Z_1$  and  $(f_2, g_2)$  with common target  $Z_2$  is a  $C$ -morphism  $\sigma : Z_1 \rightarrow Z_2$  such that  $f_2 = \sigma f_1$  and  $g_2 = \sigma g_1$ .



## 4. Morphisms

### 4.1

Proof omitted.

### 4.2

They are relations which are also symmetric. Proof: let  $C$  be a category of this kind and let  $A \sim B$ , that is,  $f \in \text{Hom}_C(A, B)$ . By the groupoid property there exists a left inverse  $g \in \text{Hom}_C(B, A)$ , hence  $\text{Hom}_C(B, A)$  is nonempty, hence  $B \sim A$ .

### 4.3

Let  $\beta' \circ f = \beta'' \circ f$ . Let  $g$  be a left inverse of  $f$ . Then  $\beta' \circ f \circ g = \beta'' \circ f \circ g$ , hence  $\beta' = \beta''$ .

TBD

#### 4.4

As long as  $Obj(C)$  is nonempty, the structure  $C_{nonmono}$  constructed as such cannot work since the identity function is a monomorphism, hence not a homomorphism in  $C_{nonmono}$ .

#### 4.5

TBD

### 5. Universal Properties

#### 5.1

Let  $Z$  be final in  $C$ . Consider  $Z$  as an element of  $C^{op}$  and let  $Z'$  be any element of  $C^{op}$ . There is a unique  $C$ -morphism between  $Z$  and  $Z'$ , hence there is a unique  $C^{op}$ -morphism between  $Z'$  and  $Z$ .

#### 5.2

$0$  is initial since for all sets  $S$  there is a unique function from  $0$  to  $S$ , the empty function. Let  $T \neq 0$  be a set. Then  $T$  must have at least one element, say  $t \in T$ . Consider  $Hom(T, P)$  where  $P = \{a, b\}$  where  $a = \{\}, b = \{a\}$  is a set with two elements. If  $Hom(T, P)$  is nonempty, say  $f \in Hom(T, P)$ , then either  $f(t) = a$  or  $f(t) = b$ ; either way we can construct  $f'$  which differs from  $f$  in its value of  $t$ .

#### 5.3

Let  $F$  be final. We show that there is a unique automorphism. This follows because  $F$  is final and an automorphism is  $F \rightarrow F$ .

Let  $F_1, F_2$  be final. We show that there is a unique isomorphism between  $F_1$  and  $F_2$ . Since  $F_1$  is final there is a unique morphism  $f : F_2 \rightarrow F_1$ . It suffices to show that  $f$  is an isomorphism. Since  $F_2$  is final there is a unique morphism  $g : F_1 \rightarrow F_2$ . Then  $gf = 1_{F_2}$  and  $fg = 1_{F_1}$ .

#### 5.4

Let  $(I, i)$  be an initial member of  $Set^*$ . Then there must be exactly one morphism from  $(I, i)$  to any other object  $(S, s)$ . All singleton sets (that is,  $(\{i\}, i)$ ) are initial, since given any  $(S, s)$  the unique morphism maps  $i$  to  $s$ . All non-singleton sets are not initial since there are at least two morphisms to  $(S, s)$  for  $|S| > 1$ .

Let  $(F, f)$  be a final member of  $Set^*$ . Then there must be exactly one morphism from any other object  $(S, s)$  to  $(F, f)$ . All singleton sets are final with the morphism mapping  $s$  to  $f$ . All non-singleton sets are not final (same argument as above).

#### 5.5

We define the category  $C$  explicitly. A  $C$ -object is a Set-morphism  $\sigma : A \rightarrow Z$  such that  $a \sim a' \implies \sigma(a) \sim \sigma(a')$ , and  $C$ -morphisms between  $\sigma_1$  and  $\sigma_2$  are Set-morphisms  $f$  such that  $\sigma_2 = f\sigma_1$ . The final



objects in this category are the Set-morphisms whose target are singleton sets.

## 5.6

Let  $m, n$  be objects of this category and let  $f$  be a final objects in  $C_{m,n}$ . Then  $f|m$  and  $f|n$ . Furthermore, for all  $f'$  such that  $f'|m$  and  $f'|n$  we must have  $f'|f$ . Hence  $f$  is the gcd of  $m$  and  $n$ .

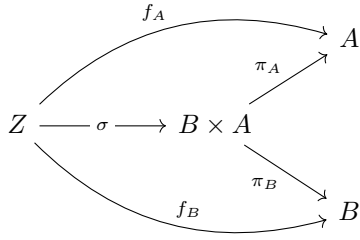
Let  $f$  be an initial object in  $C^{m,n}$ . Then  $m|f$  and  $n|f$ . Furthermore, for all  $f'$  such that  $m|f'$  and  $n|f'$ , we have  $f|f'$ . Hence  $f$  is the lcm of  $m$  and  $n$ .

## 5.7

The definition of  $A \sqcup B$  given makes it a coproduct in  $Set$ , since it satisfies the universal property of coproducts.

## 5.8

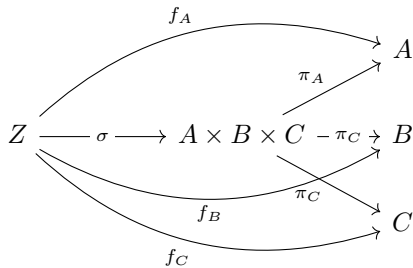
We need to show that a final object of  $C_{B,A}$  is final in  $C_{A,B}$ . Let  $f$  be any final object in  $C_{B,A}$ , that is,  $f$  consists of an element  $B \times A$  of  $C$  together with morphisms  $\pi_A : B \times A \rightarrow A$  and  $\pi_B : B \times A \rightarrow B$ . Then  $f$  is final in  $C_{A,B}$ . Proof: let  $Z$  be any element of  $C$  and  $f_A, f_B$  be morphisms from  $Z$  to  $A$  and  $B$  respectively. By finality of  $f$  in  $C_{B,A}$ , there exists a unique morphism  $\sigma$  such that the following diagram commutes.



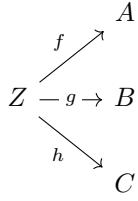
Hence  $f$  is final in  $C_{A,B}$ .

## 5.9

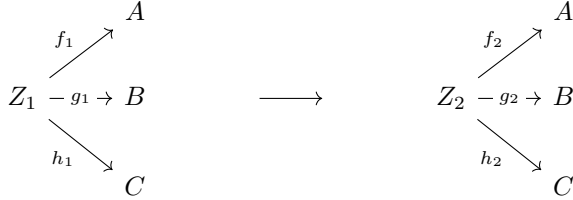
Let  $A, B, C$  be sets. Then the product  $A \times B \times C$  is universal with respect to following property: for any  $Y$  and morphisms  $f_A : Y \rightarrow A, f_B : Y \rightarrow B, f_C : Y \rightarrow C$  there is a unique morphism  $\sigma : Y \rightarrow A \times B \times C$  such that the following diagram commutes.



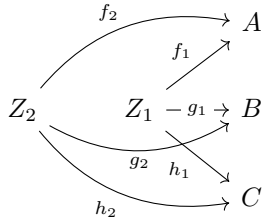
In other words,  $A \times B \times C$  is final in the category  $C_{A,B,C}$ . We now define this category. Objects in this category are diagrams



in  $C$ , and morphisms



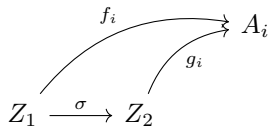
are commutative diagrams



We show that  $(A \times B) \times C$  satisfies this universal property. The function  $\sigma$  defined by  $\sigma(z) = ((f_A(z), f_B(z)), f_C(z))$  makes the diagram commute.

## 5.10

Let  $C$  be a category and  $S = \{A_i : i \in I\}$  be an indexed set of objects of  $C$ . Define the category  $C_S$  as follows. An object in  $C_S$  consists of an object  $Z$  in  $C$  together with indexed morphisms  $\{f_i : Z \rightarrow A_i\}$  where  $f_i : Z \rightarrow A_i$ . A morphism in  $C_S$  between  $(Z_1, \{f_i : i \in I\})$  and  $(Z_2, \{g_i : i \in I\})$  is a morphism  $\sigma : Z_1 \rightarrow Z_2$  in  $C$  such that for all  $i \in I$  the following diagram commutes.

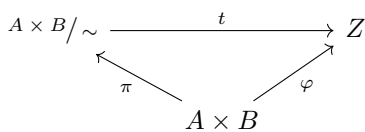


(If  $I$  is sufficiently small, we can convert this condition into the commutativity of a big diagram.)

### 5.11.1

check: <https://math.stackexchange.com/questions/242149/problem-1-5-11-in-aluffis-algebra-chapter-0>

The universal property says that for all  $\phi, Z$  for which  $(a_1, b_1) \sim (a_2, b_2) \implies \phi((a_1, b_1)) = \phi((a_2, b_2))$  there exists a unique  $t$  such that the following diagram commutes



Let  $Z = A / \sim_A$  and  $\varphi$  be defined by

$$\begin{array}{ccccc}
 & & \varphi & & \\
 & \nearrow & & \searrow & \\
 A \times B & \xrightarrow{\pi_A} & A & \xrightarrow{\pi} & A/\sim_A
 \end{array}$$

where  $\pi$  is any representative-choice function  $A \rightarrow A/\sim_A$  (such a function is included in the definition of  $A/\sim_A$ ). By the above, there exists a set-function  $A \times B/\sim \rightarrow A/\sim_A$ .

### 5.11.2

Lemma. The representative-choice function has an inverse. Proof: by using the universal property of quotients,

$$\begin{array}{ccc}
 A/\sim & \xrightarrow{\exists!} & A \\
 \nwarrow \pi & & \nearrow id \\
 & A &
 \end{array}$$

Let the functions be labelled as such:

$$\begin{array}{ccc}
 & & A/\sim_A \\
 & \nearrow \pi_a & \\
 A \times B/\sim & & \\
 & \searrow \pi_b & \\
 & & B/\sim_B
 \end{array}$$

Note that this is an element of  $Set_{A/\sim_A, B/\sim_B}$ . Let the following be another element.

$$\begin{array}{ccc}
 & \frac{A}{\sim_A} & \\
 f_A \nearrow & & \\
 Z & & \\
 f_B \searrow & & \\
 & \frac{B}{\sim_B} &
 \end{array}$$

We wish to show that there is a unique  $\sigma : Z \rightarrow A \times B/\sim$  such that the combined diagram commutes. Consider the following diagram.

$$\begin{array}{ccccc}
 & & f_A & \xrightarrow{\quad} & A/\sim_A \xrightarrow{\exists!} A \\
 & \nearrow & & \nearrow & \\
 & & \text{dotted line} & & \\
 Z & \xrightarrow{\quad} & A \times B/\sim & \xleftarrow{\pi_\sim} & A \times B \xrightarrow{\quad} B/\sim_B \xrightarrow{\exists!} B \\
 & \searrow & & \searrow & \\
 & & f_B & \xrightarrow{\quad} &
 \end{array}$$

By the universal property of  $A \times B$ , the dotted line is inhabited by a unique set-function. Its composition with  $\pi_\sim$  provides the required  $\sigma$ .

### 5.12

tbd. “pullback”.

## 2. Groups

### 1.4

It suffices to show that  $ghg^{-1}h^{-1} = 1$  for all  $g, h$ . This is true because it is  $(gh^{-1})^2$ .

### 1.6

Let  $G$  be a group. We know  $1 \in G$ . Hence if  $|G| = 1$  then  $G = \{e\}$ .

If  $|G| = 2$  then  $G = \{1, x\}$ . The multiplication table is

	1	x
1	1	x
x	x	

The last spot must be 1.

If  $|G| = 3$  then  $G = \{1, x, y\}$ . Suppose  $x^2 = 1$ . Then

	1	x	y
1	1	x	y
x	x	1	
y	y		

No choice is possible for the value of  $xy$ . Hence  $x^2 = y$ .

	1	x	y
1	1	x	y
x	x	y	1
y	y	1	x

Let  $|G| = 4, G = \{1, x, y, z\}$ . The table is

	1	x	y	z
1	1	x	y	z
x	x			
y	y			
z	z			

Either  $x^2 = 1, y$  or  $z$ . The later two cases are equivalent by relabeling. In the first case,

	1	x	y	z
1	1	x	y	z
x	x	1	z	y
y	y	z		
z	z	y		

Either the remaining diagonal entries are 1 (giving us the table for  $C_2 \times C_2$ ) or they are  $x$  (giving us the table for  $C_4$ ).

In the other case,

	1	x	y	z
1	1	x	y	z
x	x	y	z	1
y	y	z	1	x
z	z	1	x	y

which is the table for  $C_4$ .

### 1.8

Every element pairs up with its inverse except  $f$ .

### 1.9

The elements which are their own inverse are either 1 or have order 2. There are  $n - m$  of these and the non-1 elements come in pairs. If  $n$  is even, then  $m$  is odd, hence nonzero.

### 1.10

Let the order of  $g$  be  $2k + 1$ . Then the distinct powers of  $g$  are  $1, g, g^2, \dots, g^{2k}$  with  $g^{2k+1} = 1$ . The distinct powers of  $g^2$  are  $1, g^2, g^4, \dots, g^{2k}, g^{2k+2} = g, g^3, \dots, g^{2k-1}$ . Hence the order of  $g^2$  is the same as the order of  $g$ .

### 1.11

We prove the hint. We know that the distinct powers of  $g$  are  $1, g, g^2, \dots, g^{k-1}$ . It suffices to show that the powers of  $aga^{-1}$  which are  $1, aga^{-1}, ag^2a^{-1}, \dots, ag^{k-1}a^{-1}$  are distinct. These are distinct because otherwise, supposing  $ag^pa^{-1} = ag^qa^{-1}$  for some  $0 \leq p, q \leq k$ , by cancellation,  $g^p = g^q$ , a contradiction.

Then  $|gh| = |g^{-1}ghg| = |hg|$ .

### 3.1

By the universal property of products applied to  $H \times H$  there is a unique morphism  $q$  (which we denote by  $\varphi \times \varphi$ ) such that the following diagram commutes.

$$\begin{array}{ccccc}
 & & G & \xrightarrow{\varphi} & H \\
 \pi_{G,1} \nearrow & & & \searrow \pi_{H,1} & \\
 G \times G & \xrightarrow{q} & H \times H & & \\
 \pi_{G,2} \searrow & & & \nearrow \pi_{H,2} & \\
 & & G & \xrightarrow{\varphi} & H
 \end{array}$$

### 3.2

The following diagram commutes.

$$\begin{array}{ccccc}
& & G & \xrightarrow{\varphi} & H & \xrightarrow{\psi} & K \\
& \nearrow \pi_{G,1} & & \nearrow \pi_{H,1} & & \nearrow \pi_{K,1} & \\
G \times G & \xrightarrow{\varphi \times \varphi} & H \times H & \xrightarrow{\psi \times \psi} & K \times K & & \\
& \searrow \pi_{G,2} & & \searrow \pi_{H,2} & & \searrow \pi_{K,1} & \\
& & G & \xrightarrow{\varphi} & H & \xrightarrow{\psi} & K
\end{array}$$

By the universal property of products applied to  $K \times K$  as well as the outer edges of the diagram (i.e. the morphisms  $\psi\varphi\pi_{G,i}$ ), there is a unique isomorphism denoted by  $\psi\varphi \times \psi\varphi$  that makes the diagram commute. By its uniqueness, it must equal the composition  $\psi \times \psi$  and  $\varphi \times \varphi$

### 3.3

Let  $A, B, A \times B, Y$  be abelian groups, and let  $f_A : A \rightarrow Y$  and  $f_B : B \rightarrow Y$  be group homomorphisms. Define  $f : A \times B \rightarrow Y$  by  $f((g, h)) = f_A(g)f_B(h)$ . It suffices to show that  $f$  is the unique group homomorphism that makes the following diagram commute [diagram].

Proof of homomorphism: TBD. (Note that this is no longer true in the category of groups).

Commutativeness: for all  $a \in A$  we have  $f(i_A(a)) = f((a, 1)) = f_A(a)$ . The proof that  $f_B = i_B \circ f$  is similar.

Uniqueness: suppose  $f'$  is a group homomorphism that makes the diagram commute. TBD.

### 3.4

$\mathbb{Z}^+ \times \mathbb{Z}[x]^+ \simeq \mathbb{Z}[x]^+$  via the homomorphism  $(n, p) \rightarrow n + px$ .

### 3.5

Suppose  $\mathbb{Q} = G \times H$ . Then there are two subgroups of  $\mathbb{Q}$ , namely  $i_G(G) = G \times \{0_H\}$  and  $i_H(H)$ , which intersect at  $\{0\}$ . However any two nontrivial subgroups of  $\mathbb{Q}$  intersect at at least 2 points.

### 3.6

Let  $\phi_2 : C_2 \rightarrow S_3$  be given by  $\phi([0]) = (), \phi([1]) = (1, 2)$ .

Let  $\phi_3 : C_3 \rightarrow S_3$  be given by  $\phi([0]) = (), \phi([1]) = (1, 2, 3), \phi([2]) = (1, 3, 2)$ .

Suppose  $C_2 * C_3$  is a coproduct in Grp. Then there must be a (unique) group homomorphism  $f$  such that the following diagram commutes

$$\begin{array}{ccc}
C_2 & \xrightarrow{\phi_2} & S_3 \\
& \searrow & \nearrow \exists! f \\
& C_2 * C_3 & \\
& \nearrow & \searrow \\
C_3 & \xrightarrow{\phi_3} & S_3
\end{array}$$

The image of  $C_2$  in  $C_2 * C_3$  has size 2, and the image of  $C_3$  in  $C_2 * C_3$  has size 3. The image of  $f$  either has size 6, 3, 2 or 1. If it has size 1 or 2,  $\phi_3$  cannot have an image of size 3. If it has size 3,  $f$  maps a

subgroup of  $C_2 * C_3$  (namely the image of  $C_2$ ) to a subgroup of its image, but such subgroups must have size 1 or 3. Hence the image of  $f$  has size 6 and  $f$  is injective. However this means  $C_2 * C_3$  and  $S_3$  are isomorphic, which is impossible because one is abelian and the other is not.

### 3.7

Lemma: A group homomorphism is surjective iff it is an epimorphism in Grp. Proof: omitted.

We know that there is a unique arrow  $f$  such that the following commutes:

$$\begin{array}{ccccc}
 \mathbb{Z} & \xrightarrow{s_2} & C_2 & & \\
 & \searrow & & \searrow i_2 & \\
 & \mathbb{Z} * \mathbb{Z} & \xrightarrow{\exists! f} & C_2 * C_3 & \\
 & \nearrow & & \nearrow i_3 & \\
 \mathbb{Z} & \xrightarrow{s_3} & C_3 & & 
 \end{array}$$

We show that  $f$  is an epimorphism. Let  $g_1, g_2$  be two morphisms from  $C_2 * C_3$  to some group  $C$ , such that  $g_1 \circ f = g_2 \circ f$ . We wish to show that  $g_1 = g_2$ . We do this by exhibiting 5 diagrams; these are not commutative diagrams, but in each of them, the maximal paths (there are two) commute.

$$\mathbb{Z} * \mathbb{Z} \longrightarrow C_2 \times C_3 \rightrightarrows C$$

$$\begin{array}{c}
 \mathbb{Z} \\
 \searrow \\
 \mathbb{Z} * \mathbb{Z} \longrightarrow C_2 \times C_3 \rightrightarrows C
 \end{array}$$

$$\begin{array}{ccc}
 \mathbb{Z} & \longrightarrow & C_2 \\
 & & \searrow \\
 & & C_2 \times C_3 \rightrightarrows C
 \end{array}$$

$$\begin{array}{ccc}
 C_2 & & \\
 \searrow & & \\
 C_2 \times C_3 & \longrightarrow & C
 \end{array}$$

in the last step we made use of the fact that  $s_2$  is an epimorphism (since it is surjective).

A similar argument shows that the following diagram commutes:

$$\begin{array}{ccc}
 C_2 & \xrightarrow{g_1 \circ i_2 = g_2 \circ i_2} & C \\
 \searrow & & \nearrow \\
 & C_2 * C_3 & \\
 \nearrow & & \searrow \\
 C_3 & \xrightarrow{g_1 \circ i_3 = g_2 \circ i_3} & C
 \end{array}$$

By the universal property of coproduct, there is a unique morphism  $g : C_2 * C_3 \rightarrow C$  that factorizes the curved arrows, hence  $g = g_1 = g_2$ .

### 3.8

Let  $C_2$  be generated by  $x'$  and  $C_3$  be generated by  $y'$ . Let the group homomorphism from  $C_2$  to  $G$  be given by  $f_2(x') = x$  and from  $C_3$  to  $G$  be given by  $f_3(y') = y$ . Let  $g_2 : C_2 \rightarrow H$  and  $g_3 : C_3 \rightarrow H$  be group homomorphisms. We wish to show that there exists  $f : G \rightarrow H$  such that the following diagram commutes. TBD.

We define  $f$  as follows. Let  $f(x) = f_2(x')$  and  $f(y) = f_3(y')$ . TBD.