Algebra

1. Sets and Categories

1.1

Suppose that for any property, we can define a set whose members are items that satisfy that property. Then let $r = \{x : x \notin x\}$. Then $r \in r \iff r \notin r$.

In naive set theory and ZFC, this is avoided as the axiom schema of specification (for any property P and set S the set $\{s \in S : s \text{ satisfies } P\}$ exists) requires an existing set S. Also, a set of all sets cannot exist, as otherwise we can take S to be such a set.

1.2

Nonempty: any equivalence class $[a]$ contains a.

Disjoint: we wish to show that $[a] \neq [b] \implies [a] \cap [b] = \{\}$. Equivalently, $[a] \cap [b] \neq \{\} \implies [a] = [b]$. Suppose $c \in [a] \cap [b]$, and let $a' \in [a]$. Then $a' \sim a \sim c \sim b$, hence $[a] \subseteq [b]$. Similarly $[b] \subseteq [a]$.

Union is S: S contains the union as it contains every equivalence class by construction. Let $s \in S$. Then $s \in [s]$, hence the union contains S.

1.3

Define $a \sim b$ when there exists $p \in P$ such that $a \in p, b \in p$.

1.4

We count partitions. A partition will have 1, 2 or 3 parts. If it has 1 part it must be the whole set and if it has 3 parts it must be $\{[1], [2], [3]\}$. If it has 2 parts, exactly one part will have 2 items and the other 1 item, and we have 3 choices for the singleton part. Hence there are 5 equivalence relations.

1.5

For $a, b \in \mathbb{R}$ define $a \sim b$ when $|a - b| < 2$. Then $0 \sim 1$ and $1 \sim 2$ but $0 \not\sim 2$.

1.6

Reflexive: for $r \in \mathbb{R}$ we have $r \sim r$ since $r - r = 0 \in \mathbb{Z}$.

Symmetric: suppose $a \sim b$. Then $a - b \in \mathbb{Z}$. Then $b - a = -(b - a) \in \mathbb{Z}$, hence $b \sim a$.

Transitive: suppose $a \sim b, b \sim c$. Then $a - c = (a - b) + (b - c) \in \mathbb{Z}$.

Every class in $\mathbb{R}/\sim$ is represented by a unique real number $r \in [0, 1)$, which is equivalent to all real numbers with that fractional part.

Every class in $\mathbb{R}^2/\approx$ is represented by a unique pair of real numbers (r_1, r_2) with each $r_i \in [0, 1)$, which is equivalent to all pairs of real numbers whose fractional parts are the representative.

1. Functions between sets

2.1

$n!$

2.2

Let $f : A \to B$. We wish to show that f has a right-inverse iff it is surjective.

$\Longrightarrow$: let $b \in B$. Suppose the right inverse is g; then $f \circ g = id_B$ hence $b = f(g(b)) \in f(A)$.

$\Longleftarrow$: we define $g : B \to A$ as follows. Let $b \in B$. Choose $a \in f^{-1}(b)$, which is nonempty since f is surjective, then set $g(b) = a$. Then g is a right inverse. Proof: for all $b \in B$ we have $f(g(b)) = f(a) = b$.

2.3

Let $f : A \to B$ be a bijection and f^{-1} its inverse. Since f^{-1} is a right inverse of f, we have $f \circ f^{-1} = id_B$ hence f is a left inverse of f^{-1}. Similarly it is a right inverse as well; hence f^{-1} is a bijection.

Let $g : A \to B, f : B \to C$ be bijections. Then $(f \circ g) \circ (g^{-1} \circ f^{-1}) = id_C$, hence $f \circ g$ is surjective. Similarly, it is injective.

2.4

Reflexive: for any set $A, |A| = |A|$ since id_A is a bijection.

Symmetric: suppose $|A| = |B|$, that is, there is a bijection $f : A \to B$. Then $f^{-1} : B \to A$ is a bijection, hence $|B| = |A|$.

Transitive: suppose $|A| = |B|, |B| = |C|$ with bijections f and g. Then $f \circ g$ is a bijection between A and C.

2.5

A function $f : A \to B$ is an epimorphism (is epic) if for all sets Z and all functions $\alpha', \alpha'' : B \to Z$ we have $\alpha' \circ f = \alpha'' \circ f \implies \alpha' = \alpha$. A function is surjective iff it is epic.

$\Longrightarrow$: Let $f : A \to B$ be surjective and $\alpha', \alpha'' : B \to Z$ such that $\alpha' \circ f = \alpha'' \circ f$. Then f has a right inverse g and $\alpha' \circ f \circ g = \alpha'' \circ f \circ g$. Hence $\alpha' = \alpha''$.

$\Longleftarrow$: Let $f : A \to B$ be epic and let $b \in B$. We wish to show that $b \in f(A)$. Let $Z = \{0, 1\}$ and suppose α' and α'' disagree only on b, that is, $\alpha'(c) = \alpha''(c) \iff c \neq b$. Then $\alpha' \neq \alpha''$. Suppose $b \notin f(A)$. Then for all $a \in A \alpha' \circ f(a) = \alpha'' \circ f(a)$ (proof: $f(a) \neq b$, hence $\alpha'(f(a)) = \alpha''(f(a))$), which contradicts f is an epimorphism.

2.6

For any $f : A \to B$ define $\phi_f : A \to A \times B$ by $\phi_f(a) = (a, f(a))$. Then ϕ_f is a section (right inverse) of π_A. Proof: for all $a \in A$ we have $\pi_A \circ \phi_f(a) = \pi_A(a, f(a)) = a$. In fact every section of π_A corresponds to a unique function, since the $\pi_B(\pi_A^{-1}(x)) = B$.

2.7

For all $f : A \to B$ let $\phi_f : A \to \Gamma_f$ be given by $\phi(a) = (a, f(a))$. Then π_A is a left inverse of ϕ_f by exercise 2.6, hence ϕ_f is injective. Also ϕ_f is surjective. Proof: suppose $y = (a, b) \in \Gamma_f$. Then $b = f(a)$, hence $y = \phi_f(a)$.

2.8

TBD

2.9

TBD

2.10

Any function from A to B is defined by the value of $f(a)$ for all $a \in A$; the choice for each a is independent, there are $|A|$ choices and for each choice we may choose $|B|$ ways, so there are $|A| \times |A| \times \ldots |A|$ different functions.

For an explicit bijection, we can use the fact that any finite set can be well-ordered to order the elements of $|B^A|$ lexicographically on their valuations $(f(a_1), f(a_2) \ldots)$.

2.11

A subset $S \subseteq A$ is determined uniquely by its indicator function $f_S \in 2^A$ given by $f_S(a) = 0 \iff a \in S$.