Algebra

I. Set Theory and Categories

1. Sets and Categories

1.1

Suppose that for any property, we can define a set whose members are items that satisfy that property. Then let $r = \{x : x \notin x\}$. Then $r \in r \iff r \notin r$.

In naive set theory and ZFC, this is avoided as the axiom schema of specification (for any property P and set S the set $\{s \in S : s \text{ satisfies } P\}$ exists) requires an existing set S. Also, a set of all sets cannot exist, as otherwise we can take S to be such a set.

1.2

Nonempty: any equivalence class $[a]$ contains a.

Disjoint: we wish to show that $[a] \neq [b] \implies [a] \cap [b] = \{\}$. Equivalently, $[a] \cap [b] \neq \{\} \implies [a] = [b]$. Suppose $c \in [a] \cap [b]$, and let $a' \in [a]$. Then $a' \sim a \sim c \sim b$, hence $[a] \subseteq [b]$. Similarly $[b] \subseteq [a]$.

Union is S: S contains the union as it contains every equivalence class by construction. Let $s \in S$. Then $s \in [s]$, hence the union contains S.

1.3

Define $a \sim b$ when there exists $p \in P$ such that $a \in p, b \in p$.

1.4

We count partitions. A partition will have 1, 2 or 3 parts. If it has 1 part it must be the whole set and if it has 3 parts it must be $\{[1], [2], [3]\}$. If it has 2 parts, exactly one part will have 2 items and the other 1 item, and we have 3 choices for the singleton part. Hence there are 5 equivalence relations.

1.5

For $a, b \in \mathbb{R}$ define $a \sim b$ when $|a - b| < 2$. Then $0 \sim 1$ and $1 \sim 2$ but $0 \not\sim 2$.

1.6

Reflexive: for $r \in \mathbb{R}$ we have $r \sim r$ since $r - r = 0 \in \mathbb{Z}$.

Symmetric: suppose $a \sim b$. Then $a - b \in \mathbb{Z}$. Then $b - a = -(b - a) \in \mathbb{Z}$, hence $b \sim a$.

Transitive: suppose $a \sim b, b \sim c$. Then $a - c = (a - b) + (b - c) \in \mathbb{Z}$.

Every class in $\mathbb{R}/\sim$ is represented by a unique real number $r \in [0, 1)$, which is equivalent to all real numbers with that fractional part.

Every class in $\mathbb{R}^2/\approx$ is represented by a unique pair of real numbers (r_1, r_2) with each $r_i \in [0, 1)$, which is equivalent to all pairs of real numbers whose fractional parts are the representative.

1. Functions between sets

2.1

$n!$

2.2

Let $f : A \to B$. We wish to show that f has a right-inverse iff it is surjective.

$\implies$: let $b \in B$. Suppose the right inverse is g; then $f \circ g = id_B$ hence $b = f(g(b)) \in f(A)$.

$\impliedby$: we define $g : B \to A$ as follows. Let $b \in B$. Choose $a \in f^{-1}(b)$, which is nonempty since f is surjective, then set $g(b) = a$. Then g is a right inverse. Proof: for all $b \in B$ we have $f(g(b)) = f(a) = b$.

2.3

Let $f : A \to B$ be a bijection and f^{-1} its inverse. Since f^{-1} is a right inverse of f, we have $f \circ f^{-1} = id_B$ hence f is a left inverse of f^{-1}. By a similar argument, it is a right inverse as well; hence f^{-1} is a bijection.

Let $g : A \to B, f : B \to C$ be bijections. Then $(f \circ g) \circ (g^{-1} \circ f^{-1}) = id_C$, hence $f \circ g$ is surjective. By a similar argument, it is injective.

2.4

Reflexive: for any set $A, |A| = |A|$ since id_A is a bijection.

Symmetric: suppose $|A| = |B|$, that is, there is a bijection $f : A \to B$. Then $f^{-1} : B \to A$ is a bijection, hence $|B| = |A|$.

Transitive: suppose $|A| = |B|, |B| = |C|$ with bijections f and g. Then $f \circ g$ is a bijection between A and C.

2.5

A function $f : A \to B$ is an epimorphism (is epic) if for all sets Z and all functions $\alpha', \alpha'' : B \to Z$ we have $\alpha' \circ f = \alpha'' \circ f \implies \alpha' = \alpha$. A function is surjective iff it is epic.

$\implies$: Let $f : A \to B$ be surjective and $\alpha', \alpha'' : B \to Z$ such that $\alpha' \circ f = \alpha'' \circ f$. Then f has a right inverse g and $\alpha' \circ f \circ g = \alpha'' \circ f \circ g$. Hence $\alpha' = \alpha''$.

$\impliedby$: Let $f : A \to B$ be epic and let $b \in B$. We wish to show that $b \in f(A)$. Let $Z = \{0, 1\}$ and suppose α' and α'' disagree only on b, that is, $\alpha'(c) = \alpha''(c) \iff c \neq b$. Then $\alpha' \neq \alpha''$. Suppose $b \notin f(A)$. Then for all $a \in A \alpha' \circ f(a) = \alpha'' \circ f(a)$ (proof: $f(a) \neq b$, hence $\alpha'(f(a)) = \alpha''(f(a))$), which contradicts f is an epimorphism.

2.6

For any $f : A \to B$ define $\phi_f : A \to A \times B$ by $\phi_f(a) = (a, f(a))$. Then ϕ_f is a section (right inverse) of π_A. Proof: for all $a \in A$ we have $\pi_A \circ \phi_f(a) = \pi_A(a, f(a)) = a$. In fact every section of π_A corresponds to a unique function, since the $\pi_B(\pi_A^{-1}(x)) = B$.

2.7

For all $f : A \to B$ let $\phi_f : A \to \Gamma_f$ be given by $\phi(a) = (a, f(a))$. Then π_A is a left inverse of ϕ_f by exercise 2.6, hence ϕ_f is injective. Also ϕ_f is surjective. Proof: suppose $y = (a, b) \in \Gamma_f$. Then $b = f(a)$, hence $y = \phi_f(a)$.

2.8

The equivalence relation is: $r_1 \sim r_2 \iff e^{2\pi i r_1} = e^{2\pi i r_2} \iff r_1 - r_2 \in \mathbb{Z}$. Hence the surjection maps r to (the equivalence class represented by) its fractional part; $\bar{f}$ maps $[r]$ to $e^{2\pi i r}$, with the unit circle as codomain; the injection is the inclusion map (of the unit circle in the complex plane).

2.9

Let the bijection between $A' and A''$ be f and the bijection between B' and B'' be g. Define a bijection $h : A' \cup B' \to A'' \cup B''$ as follows: $h(x)$ is $f(x)$ if $x \in A'$ otherwise $g(x)$. This is well-defined as exactly one of $x \in A', x \in B'$ is true. The inverse can be defined explicitly similarly.

To form the disjoint union of A and B be produce disjoint copies; suppose we do this in one way to get A' and B', and another way to get A'' and B''. Since they are copies, $A' \cong A'', B' \cong B''$, hence the disjoint unions are isomorphic (as sets).

2.10

Any function from A to B is defined by the value of $f(a)$ for all $a \in A$; the choice for each a is independent, there are $|A|$ choices and for each choice we may choose $|B|$ ways, so there are $|A| \times |A| \times \ldots |A|$ different functions.

For an explicit bijection, we can use the fact that any finite set can be well-ordered to order the elements of $|B^A|$ lexicographically on their valuations $(f(a_1), f(a_2) \ldots)$.

2.11

A subset $S \subseteq A$ is determined uniquely by its indicator function $f_S \in 2^A$ given by $f_S(a) = 0 \iff a \in S$.

3. Categories

3.1

We compose two morphisms in C^{op} by composing the two underlying morphisms in C "the other way". That is, suppose $f \in \mathrm{Hom}_{C^{op}}(A, B) = \mathrm{Hom}_C(B, A)$ and $g \in \mathrm{Hom}_{C^{op}}(B, C) = \mathrm{Hom}_C(C, B)$. Let composition in C be denoted by $\cdot$ and composition in C^{op} by $\circ$. Then define their composition $g \circ f = f \cdot g$.

1. The identity homorphism exists because $1_A \in \mathrm{Hom}_C(A, A) = \mathrm{Hom}_{C^{op}}(A, A)$.

2. We explicitly constructed the composition in C^{op}, hence it exists.

3. $(f \circ g) \circ h = h \cdot (g \cdot f) = (h \cdot g) \cdot f = f \circ (g \circ h)$.

4. $f \circ 1_A = 1_A \cdot f = f$; similar proof for left inverse.

3.2

$|\operatorname{End}(A)| = |\operatorname{Hom}(A, A)| = |A^A| = |A|^{|A|}$.

3.3

That means it is a left and right identity. Left identity means for all morphisms $f : a \to b, 1_b f = f$. Proof that 1_b is a left identity: let $f : a \to b$. By our definition of composition, $1_b f = (a, b)$ (let $c = b$ on p21), which is identically f. The proof for right identity is similar.

3.4

No, since $n \not< n$, so $\mathrm{Hom}(n, n)$ is empty.

3.5

$\subseteq$ in 3.4 is the $\sim$ relation in 3.3 (that is required to be reflexive and transitive).

3.6

1. Let the identity in V be the identity matrix.

2. Let $f : x \to y, g : y \to z$. The composition gy is exactly the matrix product gy, which is the correct size ($gy : x \to z$ is a $z \times y$ matrix).

3. Associativity follows from associativity of matrix multiplication.

4. The identity matrix is an identity with respect to matrix multiplication.

An object n of this category can be thought of as $\mathbb{R}^n$, and the morphisms $\mathrm{Hom}(m,n)$ as linear functions $\mathbb{R}^m \to \mathbb{R}^n$. (Eg use the standard basis on $\mathbb{R}^m$ and $\mathbb{R}^n$ to convert a linear function to an appropriately sized matrix).

3.7

Let us denote this category as C^A. Then the objects of C^A are morphisms of C. Given two objects in $f, g \in Obj(C^A)$, suppose $f \in \mathrm{Hom}_C(A, Z_1)$ and $g \in \mathrm{Hom}_C(A, Z_2)$. Define a C^A-morphism from f to g to be a C-morphism σ such that the diagram commutes.

$$\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[dr, "g"] & \\ Z_1 \arrow[rr, "\sigma"] & & Z_2 \end{tikzcd}$$

1. An identity C^A-morphism on the C^A-object f is a C-morphism h such that

```
A --f--> Z
|       /
|      /
f     h
|    /
|   /
V  V
Z
```

commutes; 1_Z does the job.

2. Suppose f, g, h are C^A-objects and p is a C^A-morphism from f to g and q is a C^A-morphism from g to h. Then the following diagram commutes.

```
Z3 <--h--A --f--> Z1
  ^      |       /
   \     |      /
    q    g     p
     \   |    /
      \  |   /
       \ V  V
         Z2
```

By removing the vertical line and composing p and q,

```
Z3 <--h--A --f--> Z1
  ^              /
   \            /
    \          /
     \        /
      \      /
       --qp--
```

commutes; hence define composition in C^A as composition in C.

3. Associativity follows from associativity in C.

We will also verify the claim in example 3.8.

```
{*} --f--> S
 |       /
 |      /
 g     h
 |    /
 |   /
 V  V
  T
```

The requirement is that h commutes, that is, considered as set-functions, $g = h \circ f$. Since g is completely determined by its value on $*$, this means $g(*) = h(f(*))$, or $t = h(s)$.

3.8

The objects of C are infinite sets and the morphisms in C are set-functions between them (that is we define $\mathrm{Hom}_C(A, B) = \mathrm{Hom}_{Set}(A, B)$, forcing C to be a full subcategory of Set). Identity, composition and associativity all follow from Set.

3.9

WIP

A function maps between two sets; we must generalize this concept to allow for maps between two multisets. How we do this determines the structure of morphisms in MSet. (In all cases, though, members of Obj(MSet) will be pairs $(S, \sim)$ where $\sim$ is an equivalence relation on S).

1. We can treat $(S, \sim)$ as a partitioned set; then mfunctions between them will be functions between partitions.

2. We can simply inherit the morphisms from S, allowing morphisms to ignore $\sim$ altogether. The downside is that mfunctions are now sensitive to which representative of a given element you feed it.

3. We can generalize functions to "multivalued functions" (eg the $z \to z^{1/3}$ operator on $\mathbb{C}$); TBD

4. We can generalize functions to "regular functions whose image contain elements with multiplicities"; for finite multiplicities this is like a regular function $x \to (y, n)$ where x, y are our conceptual domain and codomain and n is the multiplicity. A contrived example is: given a fixed function f over $\mathbb{R}$, we could map x to (y, n) where $y = f(x)$ and n is the algebraic multiplicity of the root of the function $f'(t) = f(t) - f(x)$ at y (hence n would mostly be 1, and at stationary points it would increase).

3.10

The object $\Omega = \{0, 1\}$ will do. A subobject (subset) A' of an object (set) A can be identified with a morphism (set-function) $f : A \to \Omega$ by setting $a \in A' \iff f(a) = 1$.

3.11 - $C^{A,B}$

A $C^{A,B}$-object is a pair of C-morphisms (f, g). A $C^{A,B}$-morphism between (f_1, g_1) and (f_2, g_2) is a C-morphisms σ such that $f_2 = \sigma f_1$ and $g_2 = \sigma g_1$. Composition is inherited from C.

$$
\begin{tikzcd}
 & A \arrow[dl, "f_1"'] \arrow[d, "f_2"] \arrow[dr, "f_3"] & \\
Z_1 \arrow[r, "\sigma"] & Z_2 \arrow[r, "\tau"] & Z_3 \\
 & B \arrow[ul, "g_1"] \arrow[u, "g_2"] \arrow[ur, "g_3"'] &
\end{tikzcd}
$$

3.11 - $C^{\alpha,\beta}$

Let $\alpha : C \to A$ and $\beta : C \to B$. A $C^{\alpha,\beta}$-object is a pair of C-morphisms (f, g) with a common target Z such that $f\alpha = g\beta$. A $C^{\alpha,\beta}$-morphism between objects (f_1, g_1) with common target Z_1 and (f_2, g_2) with common target Z_2 is a C-morphism $\sigma : Z_1 \to Z_2$ such that $f_2 = \sigma f_1$ and $g_2 = \sigma g_1$.

$$
\begin{tikzcd}
 & & & A \arrow[dl, "f_1"] \arrow[dll, bend right, "f_2"'] \arrow[dlll, bend right, "f_3"'] & \\
Z_3 & Z_2 \arrow[l, "\tau"] & Z_1 \arrow[l, "\sigma"] & & C \arrow[ul, "\alpha"'] \arrow[dl, "\beta"] \\
 & & & B \arrow[ul, "g_1"'] \arrow[ull, bend left, "g_2"] \arrow[ulll, bend left, "g_3"] &
\end{tikzcd}
$$

4. Morphisms

4.1

Proof ommitted.

4.2

The are relations which are also symmetric. Proof: let C be a category of this kind and let $A \sim B$, that is, $f \in \mathrm{Hom}_C(A, B)$. By the groupoid property there exists a left inverse $g \in \mathrm{Hom}_C(B, A)$, hence $\mathrm{Hom}_C(B, A)$ is nonempty, hence $B \sim A$.

4.3

Let $\beta' \circ f = \beta'' \circ f$. Let g be a left inverse of f. Then $\beta' \circ f \circ g = \beta'' \circ f \circ g$, hence $\beta' = \beta''$.

TBD

4.4

As long as $Obj(C)$ is nonempty, the structure $C_{nonmono}$ constructed as such cannot work since the identity function is a monomorphism, hence not a homomorphism in $C_{nonmono}$.

4.5

TBD

5. Universal Properties

5.1

Let Z be final in C. Consider Z as an element of C^{op} and let Z' be any element of C^{op}. There is a unique C-morphism between Z and Z', hence there is a unique C^{op}-morphism between Z' and Z.

5.2

0 is inital since for all sets S there is a unique function from 0 to S, the empty function. Let $T \neq 0$ be a set. Then T must have at least one element, say $t \in T$. Consider $Hom(T, P)$ where $P = \{a, b\}$ where $a = \{\}, b = \{a\}$ is a set with two elements. If $Hom(T, P)$ is nonempty, say $f \in Hom(T, P)$, then either $f(t) = a$ or $f(t) = b$; either way we can construct f' which differs from f in its value of t.

5.3

Let F be final. We show that there is a unique automorphism. This follows because F is final and an automorphism is $F \to F$.

Let F_1, F_2 be final. We show that there is a unique isomorphism between F_1 and F_2. Since F_1 is final there is a unique morphism $f : F_2 \to F_1$. It suffices to show that f is an isomorphism. Since F_2 is final there is a unique morphism $g : F_1 \to F_2$. Then $gf = 1_{F_2}$ and $fg = 1_{F_1}$.

5.4

Let (I, i) be an initial member of Set^*. Then there must be exactly one morphism from (I, i) to any other object (S, s). All singleton sets (that is, $(\{i\}, i)$) are initial, since given any (S, s) the unique morphism maps i to s. All non-singleton sets are not initial since there are at least two morphisms to (S, s) for $|S| > 1$.

Let (F, f) be a final member of Set^*. Then there must be exactly one morphism from any other object (S, s) to (I, i). All singleton sets are final with the morphism mapping s to i. All non-singleton sets are not final (same argument as above).

5.5

We define the category C explicitly. A C-object is a Set-morphism $\sigma : A \to Z$ such that $a \sim a' \implies \sigma(a) \sim \sigma(a')$, and C-morphisms between σ_1 and σ_2 are Set-morphisms f such that $\sigma_2 = f\sigma_1$. The final objects in this category are the Set-morphisms whose target are singleton sets.

5.6

Let m, n be objects of this category and let f be a final objects in $C_{m,n}$. Then $f|m$ and $f|n$. Furthermore, for all f' such that $f'|m$ and $f'|n$ we must have $f'|f$. Hence f is the gcd of m and n.

Let f be an initial object in $C^{m,n}$. Then $m|f$ and $n|f$. Furthermore, for all f' such that $m|f'$ and $n|f'$, we have $f|f'$. Hence f is the lcm of m and n.

5.7

The definition of $A \sqcup B$ given makes it a coproduct in Set, since it satisfies the universal property of coproducts.

5.8

We need to show that a final object of $C_{B,A}$ is final in $C_{A,B}$. Let f be any final object in $C_{B,A}$, that is, f consists of an element $B \times A$ of C together with morphisms $\pi_A : B \times A \to A$ and $\pi_B : B \times A \to B$. Then f is final in $C_{A,B}$. Proof: let Z be any element of C and f_A, f_B be morphisms from Z to A and B respectively. By finality of f in $C_{B,A}$, there exists a unique morphism σ such that the following diagram commutes.

$$\begin{tikzcd}
& & A \\
Z \arrow[r, "\sigma"] \arrow[urr, bend left, "f_A"] \arrow[drr, bend right, "f_B"'] & B \times A \arrow[ur, "\pi_A"] \arrow[dr, "\pi_B"'] & \\
& & B
\end{tikzcd}$$

Hence f is final in $C_{A,B}$.

5.9

Let A, B, C be sets. Then the product $A \times B \times C$ is universal with respect to following property: for any Y and morphisms $f_A : Y \to A, f_B : Y \to B, f_C : Y \to C$ there is a unique morphism $\sigma : Y \to A \times B \times C$ such that the following diagram commutes.

$$\begin{tikzcd}
& & A \\
Z \arrow[r, "\sigma"] \arrow[urr, bend left, "f_A"] \arrow[rr, bend right, "f_B"'] \arrow[drr, bend right, "f_C"'] & A \times B \times C \arrow[ur, "\pi_A"] \arrow[r, dashed, "\pi_C"] \arrow[dr, "\pi_C"'] & B \\
& & C
\end{tikzcd}$$

In other words, $A \times B \times C$ is final in the category $C_{A,B,C}$. We now define this category. Objects in this category are diagrams

$$\begin{tikzcd}
& A \\
Z \arrow[ur, "f"] \arrow[r, "g"] \arrow[dr, "h"'] & B \\
& C
\end{tikzcd}$$

in C, and morphisms

$$\begin{tikzcd} & A \\ Z_1 \arrow[ur, "f_1"] \arrow[r, "g_1"] \arrow[dr, "h_1"'] & B \\ & C \end{tikzcd} \qquad \longrightarrow \qquad \begin{tikzcd} & A \\ Z_2 \arrow[ur, "f_2"] \arrow[r, "g_2"] \arrow[dr, "h_2"'] & B \\ & C \end{tikzcd}$$

are commutative diagrams

$$\begin{tikzcd} & & A \\ Z_2 \arrow[urr, bend left, "f_2"] \arrow[rr, bend right, "g_2"'] \arrow[drr, bend right, "h_2"'] & Z_1 \arrow[ur, "f_1"] \arrow[r, "g_1"] \arrow[dr, "h_1"'] & B \\ & & C \end{tikzcd}$$

We show that $(A \times B) \times C$ satisfies this universal property. The function σ defined by $\sigma(z) = ((f_A(z), f_B(z)), f_C(z))$ makes the diagram commute.

5.10

Let C be a category and $S = \{A_i : i \in I\}$ be an indexed set of objects of C. Define the category C_S as follows. An object in C_S consists of an object Z in C together with indexed morphisms $\{f_i : i \in I\}$ where $f_i : Z \to A_i$. A morphism in C_S between $(Z_1, \{f_i : i \in I\})$ and $(Z_2, \{g_i : i \in I\})$ is a morphism $\sigma : Z_1 \to Z_2$ in C such that for all $i \in I$ the following diagram commutes.

$$\begin{tikzcd} & & A_i \\ Z_1 \arrow[urr, bend left, "f_i"] \arrow[r, "\sigma"] & Z_2 \arrow[ur, bend left, "g_i"'] & \end{tikzcd}$$

(If I is sufficiently small, we can convert this condition into the commutativeness of a big diagram.)

5.11.1

The universal property says that for all ϕ, Z for which $(a_1, b_1) \sim (a_2, b_2) \implies \phi((a_1, b_1)) = \phi((a_2, b_2))$ there exists a unique t such that the following diagram commutes

$$\begin{tikzcd} A \times B/\sim \arrow[rr, "t"] & & Z \\ & A \times B \arrow[ul, "\pi"] \arrow[ur, "\varphi"'] & \end{tikzcd}$$

Let $Z = A/\sim_A$ and φ be defined by

$$\begin{tikzcd} A \times B \arrow[r, "\pi_A"'] \arrow[rr, bend left, "\varphi"] & A \arrow[r, "\pi"'] & A/\sim_A \end{tikzcd}$$

where π is any representative-choice function $A \to A/\sim_A$ (such a function is included in the definition of $A/\sim_A$). By the above, there exists a set-function $A \times B/\sim \to A/\sim_A$.

5.11.2

Lemma. The representative-choice function has an inverse. Proof: by using the universal property of quotients,

$$\begin{tikzcd}
A/\sim \arrow[rr, "\exists!"] & & A \\
& A \arrow[ul, "\pi"] \arrow[ur, "id"'] &
\end{tikzcd}$$

Let the functions be labelled as such:

$$\begin{tikzcd}
& {}^{A}/{\sim_A} \\
A \times B/\sim \arrow[ur, "\pi_a"] \arrow[dr, "\pi_b"'] & \\
& {}^{B}/{\sim_B}
\end{tikzcd}$$

Note that this is an element of $Set_{A/\sim_A, B/\sim_B}$. Let the following be another element.

$$\begin{tikzcd}
& \frac{A}{\sim_A} \\
Z \arrow[ur, "f_A"] \arrow[dr, "f_B"'] & \\
& \frac{B}{\sim_B}
\end{tikzcd}$$

We wish to show that there is a unique $\sigma : Z \to A \times B/\sim$ such that the combined diagram commutes. Consider the following diagram.

$$\begin{tikzcd}
& & A/\sim_A \arrow[r, "\exists!"] & A \\
Z \arrow[urr, bend left, "f_A"] \arrow[drr, bend right, "f_B"'] \arrow[rr, bend left, dotted] & A \times B/\sim \arrow[ur] \arrow[dr] & A \times B \arrow[l, "\pi_\sim"] \arrow[ur] \arrow[dr] & \\
& & B/\sim_B \arrow[r, "\exists!"'] & B
\end{tikzcd}$$

By the universal property of $A \times B$, the dotted line is inhabited by a unique set-function. Its composition with $\pi_\sim$ provides the required σ.