Cauchy-Riemann equations

1 Complex differentiability

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \tag{1}$$

In general, the limit depends on which direction z approaches z_0 from. This is similar to defining the derivative for

$$|x| = \begin{cases} x & \text{if } x \geq 0; \\ -x & \text{if } x < 0. \end{cases}$$

Where the limit for $x = 0$ depends on which direction Δx approaches 0 from. For complex functions there are an infinite number of directions, not just 2, and we say the limit exists only if it is independant of the direction. [take average of all directions?]

For instance clearly if $f(z) = z^2$ then

$$\begin{aligned} \frac{(z + \Delta z)^2 - z^2}{\Delta z} &= \frac{2z\Delta z}{\Delta z} && (2) \\ &= 2z + \Delta z && (3) \\ &\to 2z && (4) \end{aligned}$$

$f'(z) = 2z$ independant of the direction. In contrast for $f(z) = z^*$

$$\begin{aligned} \frac{(z + \Delta z)^* - z^*}{\Delta z} &= \frac{\Delta z^*}{\Delta z} && (5) \\ &= \frac{\Delta r e^{-i\theta}}{\Delta r e^{i\theta}} && (6) \\ &= e^{-2i\theta} && (7) \end{aligned}$$

clearly dependant on the direction θ.

2 An Isomorphism

Consider a function $f : \mathbb{C} \to \mathbb{C}$. If we consider a complex number to be a pair of real numbers we can say recast f as $f_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$.

Obviously every f corresponds to exactly one f_2, but what if we restrict f to the class of functions that can be written without complex conjugation? For instance we allow $f(z)$ to be

$$sin(z)/z \tag{8}$$

$$log(z) \tag{9}$$

but not

$$|z|^2 = zz^* \tag{10}$$

$$\text{Re}(z) = \frac{1}{2}(z + z^*) \tag{11}$$

You should have an intuition that these *holomorphic* functions are precisely those for which a complex derivative exists.

What structure does this impose on f_2?

3 Two directions suffice

4 Derivation from definition of derivative

Let us rewrite the derivative as

$$f'(z) = \lim_{\substack{h \to 0 \\ h \in \mathbb{C}}} \frac{f(z+h) - f(z)}{h} \tag{12}$$

If this limit exists, then it may be computed by taking the limit as $h \to 0$ along the real axis or imaginary axis; in either case it should give the same result. Approaching along the real axis, one finds

$$\lim_{\substack{h \to 0 \\ h \in \mathbb{R}}} \frac{f(z_0+h) - f(z_0)}{h} = \frac{\partial f}{\partial x}(z) \tag{13}$$

On the other hand, approaching along the imaginary axis,

$$\lim_{\substack{h \to 0 \\ h \in \mathbb{R}}} \frac{f(z+ih) - f(z)}{ih} = \frac{1}{i}\frac{\partial f}{\partial y}(z) \tag{14}$$

we can write the CRE in one single equation more suggestively (and rather non-rigorously) as

$$\frac{df}{dz} = \frac{df}{d(x+yi)} = \frac{\partial f}{\partial x} = \frac{\partial f}{\partial iy} \tag{15}$$

5 Derivation from chain rule

This is a fast and standard way to derive it but I don't like it.

Let us first label $z = x + iy$ and $f(z) = u + iv$ for the real and imaginary coponents of the complex numbers. Then

$$\frac{\partial f}{\partial x} = \frac{df}{dz}\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \tag{16}$$
$$\frac{\partial f}{\partial y} = \frac{df}{dz}\frac{\partial f}{\partial y} = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y} \tag{17}$$

noting that $\frac{\partial z}{\partial x} = 1$, $\frac{\partial z}{\partial y} = 1$ we have

$$\frac{df}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \tag{18}$$
$$= \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \tag{19}$$

or, equating real and imaginary parts to obtain an equation in u, v, x, y,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \tag{20}$$
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{21}$$

(observe how the i swaps the real and imaginary components)

6 Wirtinger derivatives

7 Conformal mapping

8 Vector calculus

9 Analytic function, Liouville's theorem

An analytic function is a function that is locally given by a convergent power series. There is an important theorem that holomorphic functions are analytic.