PROBLEM SET III
"WHERE ONE STANDS AT TIMES OF CHALLENGE AND CONTROVERSY"

DUE FRIDAY, 30 SEPTEMBER

Exercise 18. For each of the following functions $f\colon [0,+\infty) \longrightarrow \mathbf{R}$, find the set of real numbers $t \geq 0$ such that f is Riemann-integrable on $[0,t]$, and compute $\int_0^t f(x)\,dx$:

$$f(x)=\sqrt{x};\qquad f(x)=\begin{cases}1/(x-1) & \text{if } x\neq 1,\\ 0 & \text{if } x=1;\end{cases}\qquad f(x)=\sum_{k=0}^{N}\frac{x^k}{k!}.$$

(In the last example, N is any natural number.)

Definition. We say a subset $E \subset \mathbf{R}$ is *bounded* if it is a subset $E \subset [a,b]$ of a closed interval. In this case, define the *indicator function of E* as the function $\chi_E\colon [a,b] \longrightarrow \mathbf{R}$ given by the formula

$$\chi_E(x) := \begin{cases}1 & \text{if } x\in E;\\ 0 & \text{if } x\notin E.\end{cases}$$

We say that E is *Jordan-measurable* if the indicator function χ_E is Riemann integrable on $[a,b]$. In this case, the *Jordan measure* of E is the value

$$\mu(E) := \int_a^b \chi_E.$$

Exercise 19. Show that a finite set $E \subset \mathbf{R}$ is Jordan-measurable, and that its Jordan measure is 0. Is it true that any bounded *countable* set is Jordan-measurable?

Exercise 20. Show that: (1) the union of two Jordan-measurable sets is Jordan-measurable, (2) the intersection of two Jordan-measurable sets is Jordan-measurable, and (3) the complement of a Jordan-measurable set is Jordan-measurable.

Exercise 21. Show that if $E \subset \mathbf{R}$ is a Jordan-measurable set, and if $f\colon \mathbf{R} \longrightarrow \mathbf{R}$ is a map given by $f(x)=ax+b$ for real numbers a and b, then the set

$$f(E) := \{x \in \mathbf{R} \mid \text{there exists } u \in E \text{ such that } x = f(u)\}$$

is also Jordan-measurable, and

$$\mu(f(E)) = a\,\mu(E).$$

Exercise 22. Define the *floor function* $\lfloor\cdot\rfloor\colon \mathbf{R} \longrightarrow \mathbf{R}$ in the following manner: for any real number x, let $\lfloor x\rfloor$ be the greatest integer that does not exceed x (if you like, $\lfloor x\rfloor := \sup\{n \in \mathbf{Z} \mid n \leq x\}$). At which real numbers x is $\lfloor\cdot\rfloor$ continuous?

Exercise 23. For any real number $t \geq 0$, compute the integral

$$\int_0^t (x - \lfloor x\rfloor)\,dx.$$

Exercise 24. Recall the function $\theta\colon \mathbf{R} \longrightarrow [0,1]$ defined by the formula

$$\theta(x) := \begin{cases}0 & \text{if } x \notin \mathbf{Q};\\ 1/q & \text{if } x = p/q, \text{ where } p,q\in\mathbf{Z},\ \gcd(p,q)=1 \text{ and } q\geq 1.\end{cases}$$

At which numbers $x \in [0,1]$ is θ continuous?

Exercise* 25. Show that if $f\colon [c,d] \longrightarrow \mathbf{R}$ is a continuous function, and if $g\colon [a,b] \longrightarrow [c,d]$ is a Riemann-integrable function, then the composite $f \circ g\colon [a,b] \longrightarrow \mathbf{R}$ is also Riemann-integrable. Give a counterexample to show that if we had assumed only that f and g are Riemann-integrable, this statement would be false.

Definition. Suppose $E \subset [a,b]$. Now for any function $f\colon [a,b] \longrightarrow \mathbf{R}$, we say that f is *Riemann-integrable on E* if the product $\chi_E \cdot f$ is Riemann integrable on $[a,b]$, and we write

$$\int_E f := \int_a^b \chi_E \cdot f.$$

Exercise* 26. Show that if $f, g\colon [a,b] \longrightarrow \mathbf{R}$ are Riemann-integrable functions, then the product $f \cdot g\colon [a,b] \longrightarrow \mathbf{R}$ is also Riemann-integrable. Deduce that if $E \subset [a,b]$ is Jordan measurable, and if $f\colon [a,b] \longrightarrow \mathbf{R}$ is Riemann integrable on $[a,b]$ then it is also Riemann-integable on E. [For the first part, observe that $4f \cdot g = (f+g)^2 - (f-g)^2$, and use the previous exercise.]

Exercise 27. Give an example to show that the converse above is false. That is, find two bounded functions $f, g\colon [a,b] \longrightarrow \mathbf{R}$, one of which is *not* Riemann integrable, such that the product $f \cdot g\colon [a,b] \longrightarrow \mathbf{R}$ *is* Riemann-integrable.

Exercise 28. Suppose E a Jordan-measurable set, and suppose f a Riemann-integrable function on E. Show that if $A \subset E$ is a Jordan-measurable set such that $\mu(E - A)$ is zero, then f is Riemann-integrable on A, and

$$\int_A f = \int_E f.$$