1 Arithmetic and Geometric Progressions

2 Summation of Series

2.1 Sigma Notation

$$\sum_{T=1}^{n} T_r = T_1 + T_2 + T_3 + \ldots + T_n$$

Number of terms $= m - n + 1$
Express the following in sigma notation

1. $1 - \dfrac{1}{2} + \dfrac{1}{3} - \dfrac{1}{4} + \ldots$
2. $3 + 5 + 7 + 9 + ... + 41$
3. $1 \times 3 - 2 \times 5 + 3 \times 7 - 4 \times 9 + 5 \times 11$

Write down the first 3 terms of the following sums

1. $\sum_{k=1}^{100}(3k-1)$
2. $\sum_{k=1}^{40}(2r^2)$
3. $\sum_{k=3}^{10}(r-1)(2r+1)$

2.2 Basic Properties of Sigma

$$\sum_{r=1}^{n} kT_r = k\sum_{r=1}^{n} T_r$$

$$k\sum_{r=1}^{n}(T_r + G_r) = \sum_{r=1}^{n} T_r + \sum_{r=1}^{n} G_r$$

$$\sum_{r=m}^{n} T_r = \sum_{r=1}^{n} T_r - \sum_{r=1}^{m-1} T_r$$

True or False?

1. $\sum_{r=1}^{100}(2r+1) = \sum_{r=1}^{100}(2r) + 1$
2. $\sum_{n=1}^{100}(2n+1) = \sum_{m=1}^{100}(2n+1)$
3. $\sum_{n=1}^{100} a_n = \sum_{n=0}^{99} T_{n+1}$
4. $\sum_{m=1}^{100} m^2 = \sum_{m=0}^{100} m^2$

5. $\sum_{r=1}^{100} a = 100a$
6. $\sum_{n=1}^{100}(2n+1) = \sum_{m=2}^{101}(2m-1)$
7. $\sum_{m=1}^{k}(m+2) = \sum_{m=0}^{k} m + \sum_{m=0}^{k} 2$

2.3 Basic Formulas

$$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$$
$$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(2n+1)$$
$$\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$$

1. Evaluate $\sum_{r=1}^{n}(r+2)(2r-1)$ in terms of n.
2. Find $\sum_{k=0}^{n}(2n+1-2k)$ in terms of n.
3. Find an expression, in simplified form, for $\sum_{r=n+1}^{2n}(2r-1)^2$.

2.4 Method of difference

If a general term T_r can be expressed as $G_{r+1} - G_r$, then

$$\begin{aligned}\sum_{r=1}^{n} T_r &= \sum_{r=1}^{n}(G_{r+1} - G_r) \\ &= G_n - G_0\end{aligned}$$

1. Evaluate $\sum_{r=1}^{100} \dfrac{1}{r(r+1)}$
2. Find the partial fractions of $\dfrac{1}{(2x-1)(2x+1))}$. Hence, find the sum of $\dfrac{1}{1\times 3} + \dfrac{1}{3\times 5} + \ldots + \dfrac{1}{(2n-1)(2n+1)}$
3. Express $\dfrac{2}{y(y+1)(y+2)}$ in partial fractions. Using this result, show that

$$\sum_{r=1}^{N} \frac{1}{r(r+1)(r+2)} = \frac{1}{4} - \frac{1}{2(N+1)(N+2)}$$

4. By considering $\dfrac{1}{1+a^{n-1}}-\dfrac{1}{1+a^n}$ or otherwise, show that $\sum_{n=1}^{N} \dfrac{a^{n-1}}{(1+a^{n-1})(1+a^n)} = \dfrac{a^N-1}{2(a-1)(a^N+1)}$, where a is positive and $a \neq 1$. Deduce that $\sum_{n=1}^{N} \dfrac{2^n}{(1+2^{n-1})(1+2^n)} < 1$.

5. Express $\dfrac{2}{n(n+1)(n+2)}$ in partial fractions. Hence, evaluate $\sum_{k=1}^{99} \dfrac{1}{k(k+1)(k+2)}$

6. Expand $n^2-(n-1)^2e$. Hence or otherwise, prove that $\sum_{n=1}^{N} e^n \left[(1-e)n^2+2ne-e\right] = N^2e^N$.

7. Find $\sum_{r=2}^{n} \left[r-2-2n\left(\dfrac{1}{r-1}-\dfrac{1}{r}\right)\right]$ in terms of n.

8. Show that $\dfrac{1}{n^2-n+2}-\dfrac{1}{n^2+n+2}=\dfrac{2n}{n^4+3n^2+4}$. Find an expression in terms of N for the sum S_N where $S_N = \sum_{n=1}^{N} \dfrac{n}{n^4+3n^2+4}$. Deduce that $S_N < \dfrac{1}{4}$.

3 Binomial Theorem

3.1 Permutation and Combination

Permutation: Arrangement o things. Combination: Selection of things (order not important)
Multiplication rule: If a work can be done in m ways and another in another n ways, both operations can be done $m \times n$ ways.

$$P_n = n!$$
$$^nC_r = \frac{n!}{r!(n-r)!}$$

1. How many ways can the 3 letters A, B and C be arranged in a row?

2. A committee of 5 is to be selected from 10 boys and 10 girls. How many different selections are there if the committee must contain a) boys only b) 2 girls

3.2 Binomial Theorem

$$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^r b^{n-r}$$

1. Expand $(2x+1)^5$

2. Expand $(2x^4-y^2)^3$

3. Find the 5th term of the expansion of $(3x-4)^{12}$

4. Find the coefficients of a^4b^3 and a^2b^5 in the expansion of $(2a - 3b)^7$

5. Find the coefficient of x^5 and the term independent of x in the binomial expansion $\left(\frac{x^2}{2} - \frac{3}{x^3}\right)$

3.3 Binomial Series

$$(1 + f(x))^n = [1 + nf(x) + \frac{n(n-1)}{2!}[f(x)]^2 + \frac{n(n-1)(n-2)}{3!}[f(x)]^3 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{r!}[f(x)]^r]$$

1. Expand $(3 - 4x)^{-2}$ as a series of ascending powers of x up to and including the term in x^3. State the range of values of x for which the expansion is valid.

2. Obtain the first 5 terms in the expansion of $(x^2 + 3x^3)^{\frac{1}{2}}$ in descending powers of x. State the range of values of x for which this expansion is valid.

3. Express $f(x) = \dfrac{x - 9}{(x-1)(x+3)}$ in partial fractions. Hence or otherwise, obtain $f(x)$ as a series expansion in asceding powers of x as far as the term in x^3, given that $-1 < x < 1$. Find also the coefficient of x^n in this expansion, where $n > 1$.

4. Find the possible values of a and b if the expansion in ascending powers of x up to the term in x^2 of $\dfrac{\sqrt{1 - ax}}{1 + bx}$ is $1 - \dfrac{9}{2}x^2$. With these values of a and b, state the set of values of x for which the expansion is valid.

5. Find the coefficient of x^3 in the expansion of $\left(1 - \dfrac{x}{2}\right)^{10}$.

6. Given that $x > \dfrac{1}{2}$, obtain the first 3 terms in th eseries expansion of $(x + 2x^2)^{\frac{1}{2}}$ in descending powers of x.

7. Given that $|x| > 1$, find the expansion of $\sqrt{x + x^2}$ up to and including the 4th nonzero term. Hence, by using a suitable substitution, find an approximation of $\sqrt{12}$ in the form $\dfrac{p}{q}$, where p and q are integers in its lowest terms.

8. Find the values of m if the constant term in the expansion of $\left(1 + 2x^2 - \dfrac{m}{x^4}\right)^6$ is 181.

9. The first three terms in the series expansion of $(a+x)^{-b}$, where a is a real constant and $b > 0$, in ascending powers of x are $\dfrac{1}{9} - \dfrac{2x}{27} + \dfrac{x^2}{27} + \ldots$. Find the values of a and b. State the range of values of x for which the expansion is valid.

4 Mathematical Induction

4.1 The Principle of Mathematical Induction

Let $P(n)$ be a statement about the positive integer n. Suppose that $P(1)$ is true and
For any natural number k, if $P(k)$ is true then $P(k+1)$ is also true, then $P(n)$ is true for all natural numbers n.

1. Prove by mathematical induction that $\frac{1}{1\times 2}+\frac{1}{2\times 3}+\ldots+\frac{1}{n\times(n+1)}=\frac{n}{n+1}$.

2. A sequence of positive integers $u_1, u_2, u_3 \ldots$ is defined by the relation

$$u_{n+1}=\frac{5u_n+4}{u_n+2}$$

   Where $u_1 = 1$. Prove by mathematical induction that $u_n < 4$ for all $n > 1$.

3. Prove that

$$\cos(a)\cos(2a)\cos(4a)\ldots\cos(2^n a)=\frac{\sin(2^{n+1}a)}{2^{n+1\sin(a)}}$$

4. Prove by mathematical induction that $3^{4n-2}+17^n+22$ is divisible by 16 for every positive integer n.

$$1^2=\frac{1\times 2\times 3}{6}$$
$$1^2+3^2=\frac{3\times 4\times 5}{6}$$
$$1^2+3^2+5^2=\frac{5\times 6\times 7}{6}$$

5. Write down the fourth row. Make a conjecture on a formula on the sum of squares of the first n od dpositive integers. Prove by mathematical induction that the formula is correct.

6. Suppose you have three posts and a stack of n disks, initially placed on one post with the largest disk on the bottom and each disk above it is smaller than the disk below. A legal move involves taking the top disk from one post and moving it so that it becomes the top disk on another post, but every move must place a disk either on an empty post, or on top of a disk larger than itself. Show that for every n, it is possible to move all the disks to a different post in finite steps. How many moves are required for an initial stack of n disks?

4.2 Fallacies

4.3 More examples

1. Prove by mathematical induction that 5 is a factor of $6^n - 1$ for all natural numbers n.

2. Let u_n denote the number of dots that make up the nth hexagon with side length n. Then $u_n = 3(n+1)^2 - 3(n+1) + 1$ for all n. Let $S_n = \sum_{r=1}^{n} u_r$. Find the values of S_1, S_2, S_3 and S_4. Make a conjecture for the formula S_n. Prove by induction your formula S_n.

3. Prove by induction that $\frac{2}{3!} + \frac{2 \times 2^2}{4!} + \frac{3 \times 2^3}{5!} + \ldots + \frac{n \times 2^n}{(n+2)!} = 1 - \frac{2^{n+1}}{(n+2)!}$ for all $n \in Z$.

   Use your result to find an expression in terms of n for $\sum_{r=n}^{2n} \frac{r2^r}{(r+2)!}$

4. The rth term of of a sequence is give by $u_r = \frac{1}{(2r)^2 - 1}$ for $r = 1, 2, 3\ldots$ Write down the first four terms of the sequence, and hence state the values of $\sum_{r=1}^{n} u_r$ for $n = 1, 2, 3, 4$. Make a conjecture for the formula for $\sum_{r=1}^{n} u_r$ in terms of n and prove the formula by induction.