Lagrangian Mechanics

Definition of particle - internal motion disregarded.

q and $\dot{q}$ - generalized coordinates and velocities.

If q and $\dot{q}$ are given at one time - motion for subsequent time determined. Mathematically, $\ddot{q} = \ddot{q}(q, \dot{q})$ these are the *equations of motion*.

1 Principle of Least Action

Motion of a system is completely characterised by a function $L(q, \dot{q}, t)$.

$$S = \int L(q, \dot{q}, t)\, dt \tag{1}$$

action is minimized. Derive! Replace $q(t)$ by

$$q(t) + \delta q(t) \tag{2}$$

such that $\delta q(t) = 0$ at endpoints.

$$\begin{aligned} \delta S &= \delta \int L(q, \dot{q}, t)\, dt \qquad (3) \\ &= \int \left(\frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right) dt \qquad (4) \end{aligned}$$

since $\delta \dot{q} = \frac{d \delta q}{dt}$,

$$dS = \left[\frac{\partial L}{\partial \dot{q}} \right]_a^b + \int_a^b \left(\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \right) dt = 0 \tag{5}$$

Hence

$$\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = 0 \tag{6}$$

2 Properties of L

Two non-interacting systems A and B have langrangian

$$L = L_A + L_B \tag{7}$$

multiplication of a lagrangian by arbitrary factor doesn't change equations of motion.

Let

$$L' = L + \frac{d}{dt} f(q, t) \tag{8}$$

Then

$$S' = S + \int \frac{df}{dt} dt \tag{9}$$
$$dS' = dS \tag{10}$$

3 Galilean relativity

Random frames of reference are bad. Experimentally, however, there exist a set of inertial reference frames; space is homogenous, anisotropic. In such a frame L cannot depend explicity on $\mathbf{r}$ or t, but only on $\mathbf{v} = \dot{\mathbf{r}}$. Anisotropy implies that $L = L(v^2)$ Lagrange's equation reads

$$\frac{d}{dt}\frac{\partial L}{\partial \mathbf{v}} = 0 \tag{11}$$
$$\frac{\partial L}{\partial \mathbf{v}} = const. \tag{12}$$

since $\frac{\partial L}{\partial \mathbf{v}}$ is a function of only v, $\mathbf{v}$ is constant.
Galilean relativity. Physical principle, cannot be derived.

4 Lagrangian for a free particle

We know that L depends only on v^2. Derive! Consider two frames K and K' with velocities $\mathbf{v}$ and

$$\mathbf{v}' = \mathbf{v} + \epsilon \tag{13}$$

They must differ by only a total derivative of a function of coordinates and time.

$$L(v'^2) = L(v^2 + 2\mathbf{v} \cdot \epsilon) \tag{14}$$
$$= L(v^2) + \frac{\partial L}{\partial v^2} 2\mathbf{v} \cdot \epsilon \tag{15}$$
$$= L(v^2) + 2\frac{\partial L}{\partial v^2}\frac{d}{dt}(\mathbf{r} \cdot \epsilon) \tag{16}$$

hence $\frac{\partial L}{\partial v^2}$ is independent of v and

$$L = \frac{1}{2}mv^2 \tag{17}$$

now we see that this holds true for finite ϵ;

$$L' = L + \frac{d}{dt}\left(m\mathbf{r} \cdot V + \frac{1}{2}mV^2 t\right) \tag{18}$$

5 Field

Consider a free particle with lagrangian $L = \frac{1}{2}mv^2$ interacting with another system. In general the interaction adds another term we call $-U$. Now writing the eqns,

$$L = \frac{1}{2}mv^2 - U \tag{19}$$
$$m\frac{d\mathbf{v}}{dt} = -\frac{\partial U}{\partial \mathbf{r}} - \frac{d}{dt}\left(\frac{\partial U}{\partial \mathbf{v}}\right) \tag{20}$$

we define the LHS as the force $m\mathbf{a}$. Introduce the idea of conservative force,

$$U = U(\mathbf{r}_a, \mathbf{r}_b, \mathbf{r}_c ...) \tag{21}$$

Now switch to generalized coordinates q;

$$L = \frac{1}{2} \sum a_{ik} \dot{q}_i \dot{q}_k - U \tag{22}$$

where $a_{ik} = a_{ik}(q)$

6 Conservation Laws

With s generalized coordinates there are $2s$ variables that define the state of the system. Howerever there are some functions of these variables that remain constant through the motion of the system known as *integrals* of motion. Some of them are additive.

Lagrangian of a closed system does not depend explicitly on time

$$\begin{aligned}
\frac{dL}{dt} &= \sum \frac{\partial L}{\partial q} \dot{q} + \sum \frac{\partial L}{\partial \dot{q}} \ddot{q} && (23)\\
&= \sum \dot{q}_i \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) + \sum \frac{\partial L}{\partial \dot{q}} \ddot{q} && (24)\\
&= \sum \frac{d}{dt} \left(\dot{q}_i \frac{\partial L}{\partial \dot{q}} \right) && (25)\\
E &= \sum \dot{q} \frac{\partial L}{\partial \dot{q}} - L && (26)
\end{aligned}$$

E is conserved for closed and conservative systems.

$$\begin{aligned}
\sum \dot{q} \frac{\partial L}{\partial \dot{q}} &= \sum \dot{q} \frac{\partial T}{\partial \dot{q}} && (27)\\
&= 2T && (28)
\end{aligned}$$

evidently $E = T + V$

7 Momentum

Homogenity of space; translate space by $\delta\mathbf{r}$. Then

$$\delta L = \sum \frac{\partial L}{\partial \mathbf{r}} \cdot \delta \mathbf{r} \tag{29}$$

so [use LE once] $\sum \frac{\partial L}{\partial \mathbf{r}} = m\mathbf{v}$ is conserved. $\sum F = 0$.
[generalized momenta and force, COM, p43] [angular momentum] [mechanical similarity]