Eigenvectors

We have a matrix A. Sometimes A behaves very specially on a special kind of vector; it simply lengthens or shortens it. x is parallel to Ax. In equations

$$Ax = \lambda x$$

x is the eigenvector and λ is the eigenvalue.

We can see that $x = 0$ always satisfies the equation for any λ. That's not nice, so we don't consider 0 to be an eigenvector.

Notice that if x is an eigenvector, so is kx for any k. Eigenvectors lie on (at least) a line.

For example, if P is the projection matrix, we can find its eigenvectors by observation: everything that lies on the projection plane will get projected to itself.

1 Finding eigenvectors

There is a special case we can consider: $\lambda = 0$, zero eigenvalues. Then the equation becomes

$$Ax = 0$$

so $\{x\} = Null(A)$

What about in general? The equation $Ax = \lambda x$ has two unknowns. Let us fix λ and rewrite

$$\begin{aligned} Ax &= \lambda x \\ &= \lambda I x \\ (A - \lambda I)x &= 0 \end{aligned}$$

so $\{x\} = Null(A - \lambda I)$. We see that eigenvectors belonging to the same eigenvalue form a vector space (eigenspace), which explains the kx observation above.

2 Permutation matrix

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

A permutes two elements by swapping. We can see one eigenvector by observation: $x = (1\ 1), \lambda = 1$. The other one is harder to spot; $x = (1\ \ -1), \lambda = -1$. Notice that $\lambda_1\lambda_2 = -1$ is the determinant of A while $\lambda_1 + \lambda_2 = 0$ is the trace.

3 Finding λ

Now that we know how to find x given λ, how do we find λ? Remember that $\{x\} = Null(A - \lambda I)$ and $x \neq 0$, we see that $A - \lambda I$ must be singular. So we have to solve the characteristic equation, a polynomial in λ, $det(A - \lambda I) = 0$. In general it is an n-th order equation for an n-dimensional matrix.

Another way to derive this is to see that if $A \to A + 3I$ the eigenvalues increase by 3 and the eigenvectors remain unchanged. Since we know how to find eigenvectors of a singular matrix we bring A to a singular matrix in order to find their eigenvectors.

4 Good Example

$$\begin{aligned}
A &= \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \\
A - \lambda I &= \begin{bmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{bmatrix} \\
det &= \lambda^2 - 6\lambda + 8 \\
&= (\lambda - 4)(\lambda - 2)
\end{aligned}$$

Notice that $Tr(A) = 6$, $det(A) = 8$ both appear in the characteristic polynomial.

$$\begin{aligned}
A - 2I &= \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \\
x &= \begin{bmatrix} 1 \\ -1 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
A - 4I &= \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \\
x &= \begin{bmatrix} 1 \\ 1 \end{bmatrix}
\end{aligned}$$