

# Catsters

## 1 Terminal and Initial objects

Let  $C$  be a category. A terminal object in  $C$  is an object  $T \in \text{Obj}(C)$  such that  $\forall X \in \text{Obj}(C) \exists!$  morphism  $X \rightarrow T$ .

Example: one-element sets are terminal in  $\text{Set}$ .

Example: the trivial group is terminal in  $\text{Grp}$ .

Example: any one-point space is terminal in  $\text{Top}$ .

It is pointless to think about “how many trivial groups there are” since they are all isomorphic (as groups), and the isomorphism is canonical.

Lemma: terminal objects are unique up to unique isomorphism.

Non-examples:

$\cdot \rightrightarrows \cdot$   
 $\cdot \longrightarrow \cdot \quad \cdot \longrightarrow \cdot$   
 $\cdot \longrightarrow \cdot \longrightarrow \cdot \longrightarrow \cdot \longrightarrow \dots$   
 Fields (assuming  $0 \neq 1$  in a field)

### 1.1 Initial 1

Let  $C$  be a category. An initial object in  $C$  is an object  $I \in \text{Obj}(C)$  such that  $\forall X \in \text{Obj}(C) \exists!$  morphism  $I \rightarrow X$ .

An initial object in  $C$  is a terminal object in  $C^{op}$ .

Example: the empty set is initial in  $\text{Set}$ .

Example: the one-point space is initial in  $\text{Top}$ .

Example: the trivial group is initial in  $\text{Grp}$ .

The trivial group is initial and terminal in  $\text{Grp}$ . We call such things null objects. Other examples of categories with null objects: pointed sets, based spaces.

In  $\text{Cat}$ , the empty category is initial and a category with one object and one morphism is terminal.

In  $\text{Field}$ , there is no initial object.

## 2 Products and Coproducts

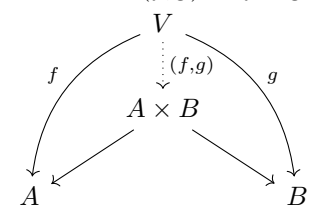
The cartesian product of  $\{1, 2\}$  and  $\{3, 4\}$  is any four element set, but the set must come with projection maps to  $\{1, 2\}$  and  $\{3, 4\}$ .

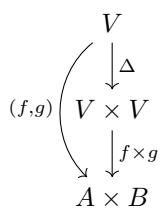
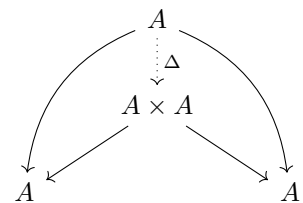
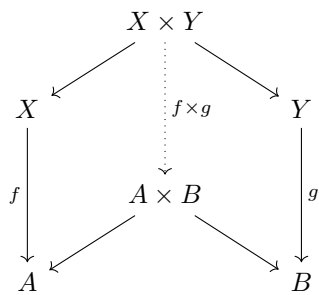
Examples: Product topology, group product, sum of vector spaces, product of categories, maximums in a poset.

Coproduct in group: you can try taking the disjoint union, but this is not a group. Hence you generate something freely. Free group. (similar intuition for vector space).

Example: coproduct in based topologies is disjoint union, but where the base points are identified.

Notations:  $(f, g)$  vs  $f \times g$ :





Let  $1$  be a terminal object. Then  $X \times 1 = X$ .

### 3 Pullbacks and Pushouts