Cauchy-Riemann equations

1 Introduction

Consider a function $f : \mathbb{C} \to \mathbb{C}$. If we consider a complex number to be a pair of real numbers we can say recast f as $f_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$.

Obviously every f corresponds to exactly one f_2, but what if we restrict f to the class of functions that can be written as without using "complex operations"? For instance we allow $f(z)$ to be

$$z^2 \tag{1}$$
$$sin(z)/z \tag{2}$$
$$log(z) \tag{3}$$

but not

$$z^* \tag{4}$$
$$|z|^2 = zz^* \tag{5}$$
$$\mathrm{Re}(z) = \frac{1}{2}(z + z^*) \tag{6}$$

we will call such functions *holomorphic* functions. What structure does this impose on f_2?

2 Complex differentiability

Surprisingly it turns out that the structure imposed is best phrased in terms of certain differential equations. To see this we define the complex derivative

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \tag{7}$$

In general, the limit depends on which direction z approaches z_0 from. There is a similar situation in real analysis where we have to consider left limits and right limits when defining a derivative. For complex functions there are an infinite number of directions, not just 2, and we say the limit exists only if it is independant of the direction. [take average of all directions?]

For instance clearly if $f(z) = z^2$ then $f'(z) = 2z$ independant of the direction (which can be verified by binomial theorem or taking a directional derivative).

We will now derive the abovementioned differential equations, known as the Cauchy-Riemann equations or CRE.

3 Derivation from chain rule

Let us first label $z = x + iy$ and $f(z) = u + iv$ for the real and imaginary coponents of the complex numbers. Then

$$\frac{\partial f}{\partial x} = \frac{df}{dz}\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \tag{8}$$
$$\frac{\partial f}{\partial y} = \frac{df}{dz}\frac{\partial f}{\partial y} = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y} \tag{9}$$

noting that $\frac{\partial z}{\partial x} = 1, \frac{\partial z}{\partial y} = 1$ we have

$$\frac{df}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \tag{10}$$
$$= \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \tag{11}$$

or, equating real and imaginary parts to obtain an equation in u, v, x, y,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \tag{12}$$
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{13}$$

(observe how the i swaps the real and imaginary components)

4 Derivation from definition of derivative

Let us rewrite the derivative as

$$f'(z) = \lim_{\substack{h \to 0 \\ h \in \mathbb{C}}} \frac{f(z+h) - f(z)}{h} \tag{14}$$

If this limit exists, then it may be computed by taking the limit as $h \to 0$ along the real axis or imaginary axis; in either case it should give the same result. Approaching along the real axis, one finds

$$\lim_{\substack{h \to 0 \\ h \in \mathbb{R}}} \frac{f(z_0+h) - f(z_0)}{h} = \frac{\partial f}{\partial x}(z) \tag{15}$$

On the other hand, approaching along the imaginary axis,

$$\lim_{\substack{h \to 0 \\ h \in \mathbb{R}}} \frac{f(z+ih) - f(z)}{ih} = \frac{1}{i}\frac{\partial f}{\partial y}(z) \tag{16}$$

we can write the CRE in one single equation more suggestively (and rather non-rigorously) as

$$\frac{df}{dz} = \frac{df}{d(x+yi)} = \frac{\partial f}{\partial x} = \frac{\partial f}{\partial iy} \tag{17}$$

5 Wirtinger derivatives

6 Conformal mapping, amplitwist

7 Vector calculus

8 Analytic function