MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Spring 2006 | Physics 8.04 | Vuletic
Assignment #5 |
Due Thursday, March 23 at 4pm

1. **Continuity equation for probability density and probability current** (20 points)

A particle is in a state described by a wavefunction $\Psi(x,t)$.
a) (10 points) Using the Schroedinger equation, show that the probability density $P(x,t) = |\Psi(x,t)|^2$ obeys the continuity equation

$$\frac{\partial}{\partial t} P(x,t) + \frac{\partial}{\partial x} J(x,t) = 0,$$

if we define the probability current $J(x,t)$ by

$$J(x,t) = \frac{\hbar}{2im}\left(\Psi^*(x,t)\frac{\partial \Psi(x,t)}{\partial x} - \frac{\partial \Psi^*(x,t)}{\partial x}\Psi(x,t) \right).$$

b) (10 points) Write down an integral form of the continuity equation, i.e. a relation between the current J as defined above and the probability $P_{ab}(t)$ of finding a particle inside a finite interval $a \le x \le b$. Express the physical meaning of this equation in one sentence.

2. **Fictitious Bohr atom.** (20 points)

What would the Balmer formula look like for a fictitious Bohr atom where the electron is bound to the nucleus by a potential $V(r) = -\frac{C_6}{r^6}$? Use the Bohr quantization condition $L=n\hbar$ for the angular momentum for circular orbits to calculate the energy levels corresponding to different quantum number n, and remember which transitions the Balmer formula corresponds to. Find the quantity that would correspond to the Rydberg constant, and express it in terms of C_6, the electron mass m, and $\hbar$.

3. **Sommerfeld-Wilson quantization for linear potential in one dimension.** (25 points)

Consider a particle of mass m linear potential $V(x) = C|x|$, $C>0$. We want to determine the quantized energy levels in such a potential.

a) Assume $x(t=0) = A, A > 0,$ and $p(t=0) = 0$. Calculate $x(t)$ and $p(t)$ for one period T. How large is T?
b) Calculate $\oint p(x)dx$, i.e. the integral over one period, as a function of C, particle mass m, and amplitude of motion A. Note that $dx<0$ ($dx>0$) if the particle moves towards negative (positive) x values.
c) Now use the Sommerfeld-Wilson quantization condition $\oint p(x)dx = nh$ to determine a quantum mechanical condition on the amplitude A_n. What is the value of the amplitude for the ground state A_1, i.e. the amplitude for n=1?
d) Calculate the quantized energy levels E_n. Sketch the potential $V(x)$ and the quantized energy levels E_n Compare the dependence of the spacing between energy levels on quantum number n to the Bohr atom.

Plot the motion in phase space, i.e. in a momentum-versus-position diagram. What is the geometrical meaning of the Sommerfeld-Wilson quantization condition? In one sentence, how would you describe the stationary states in phase space?

4. **Momentum expectation values in terms of spatial wavefunctions.** (20 points)

We have defined the expectation value of a function $g(p)$ of momentum in terms of the probability density in wavevector (or momentum) space as

$$\langle g(p)\rangle = \int_{-\infty}^{\infty} dk\; g(\hbar k)\left|\tilde{\phi}(k)\right|^2 = \int_{-\infty}^{\infty} dp\; g(p)\left|\phi(p)\right|^2$$

For $g(p)=p$, this is simply $\langle p\rangle = \int_{-\infty}^{\infty} \hbar k\left|\tilde{\phi}(k)\right|^2 dk = \int_{-\infty}^{\infty} p\left|\phi(p)\right|^2 dp$

Show that we can instead calculate the expectation value of momentum directly from the spatial wavefunction as

$$\langle p\rangle = \int_{-\infty}^{\infty} dx\,\psi^*(x)\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)\psi(x) \quad ,$$

and in general

$$\langle p^n\rangle = \int_{-\infty}^{\infty} dx\,\psi^*(x)\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)^n \psi(x)\,.$$

This means that momentum p is represented in the spatial domain as an operator $\hat{p}' = \frac{\hbar}{i}\frac{\partial}{\partial x}$.

Show that similarly the expectation value of the position operator can be written as

$$\langle x\rangle = \int_{-\infty}^{\infty} dp\,\phi^*(p)\left(i\hbar\frac{\partial}{\partial p}\right)\phi(p) \qquad \text{and}$$

$$\langle x^n\rangle = \int_{-\infty}^{\infty} dp\,\phi^*(p)\left(i\hbar\frac{\partial}{\partial p}\right)^n \phi(p)\,.$$

What is the representation of the particle s position x in the momentum domain?

5. **Relations for probability current.** (15 points)

a) (5 points) Show that the probability current can be written as

$$J(x,t) = \frac{1}{2m}\left[\Psi^*(x,t)\,\hat{p}\Psi(x,t) + \left(\Psi^*(x,t)\,\hat{p}\Psi(x,t)\right)^*\right].$$

b) (10 points) Show that a complex potential $V(x)^* \neq V(x)$ contradicts the continuity equation.