Jacobians

1 Introduction

To motivate the study of the Jacobian matrix and the Jacobian we begin by considering two operators that you should have learnt as a little child, the derivative and the gradient, and consider them as a way of answering this question: for a 'nice' function, how do you calculate a small change in the output given a small change in the input?

1.1 $f : \mathbb{R} \to \mathbb{R}$

Consider a function $f(x)$ from real numbers to real numbers.

If the function is differentiable and continuous we have

$$dx = (\frac{df}{dx})dx \tag{1}$$

At any given point $\frac{df}{dx}$ is a constant; hence small changes in f are approximately linearly related to small changes in f.

1.2 $f : \mathbb{R}^n \to \mathbb{R}$

Consider a function $f(\vec{r})$ from a vector (for our purposes, n real numbers) to real numbers.

for instance, the scalar electric potential as a function of position. We wish to know df as a linear function of $d\vec{r}$. We generalize the scalar derivitive to give a certain vector called the *gradient* such that

$$df = \vec{\nabla} f \cdot d\vec{r} \tag{2}$$

In 3-D cartesian coordinates $\vec{\nabla} f = (\frac{df}{dx}, \frac{df}{dy}, \frac{df}{dz})$.

1.3 $y : \mathbb{R}^m \to \mathbb{R}^n$

Consider a function $\vec{y}(\vec{x})$. The most common use for this is to to study transformations in $\mathbb{R}^n$, where we set $m = n$.

Notice that if $x \in \mathbb{R}^n$ then $dx \in \mathbb{R}^n$. to find dy as a function of dx we need a linear mapping between $\mathbb{R}^n$ and $\mathbb{R}^m$ that closely approximates the mapping at x. When $m = n = 1$ this mapping was *multiplication by a constant*; when $n = 1$ the mapping was *dot product with a constant vector*. In the general case we would probably need the mapping *multiplication by a constant m by n matrix*.

2 Construction

We now explicitly write out $\vec{y}$ as $(y_1, ...y_n)$. y_i is a function of $\vec{x}$, so can use our previous result to write $dy_i = \vec{\nabla} y_i \cdot dx$. Continuing we construct the Jacobian matrix J, a m by n matrix such that $d\vec{x} = Jd\vec{y}$ and

$$\begin{bmatrix} dy_1 \\ \vdots \\ \vdots \\ dy_m \end{bmatrix} = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial y_m}{\partial x_1} & \cdots & \frac{\partial y_m}{\partial x_n} \end{bmatrix} \begin{bmatrix} dx_1 \\ \vdots \\ \vdots \\ dx_m \end{bmatrix} \tag{3}$$

3 Geometrical meaning

linear approximation, stretching, determinant, volume, orientation of tangent plane

4 Application

Newton's method, stationary points in dynamical systems, classification of, change of variable