1 Parseval's Theorem

$$\int_{-\infty}^{\infty} |\psi(x)|^2 dx = \int_{-\infty}^{\infty} |\phi(k)|^2 dk$$

$$\begin{aligned}
\int_{-\infty}^{\infty} |\psi(x)|^2 dx &= \int \left(\int \frac{1}{\sqrt{2\pi}} \phi(k) e^{ikx} dk \right) \left(\int \frac{1}{\sqrt{2\pi}} \phi^*(k') e^{-ik'x} dk' \right) dx \\
&= \frac{1}{2\pi} \int \int \int \phi(k) \phi^*(k') e^{i(k-k')x} dk' dk dx \\
&= \frac{1}{2\pi} \int \int \phi(k) \phi^*(k') 2\pi \delta(k-k') dk dk' \\
&= \int \phi(k) \phi^*(k) dk
\end{aligned}$$

To evaluate the $\int e^{ikx} dx$ integral,

$$\int_{-L}^{L} e^{ikx} dx = \frac{2\sin(kL)}{L}$$

The central peak between the two roots closest to $x = 0$ is roughly triangular, height $2L$, width $\frac{2\pi}{L}$, so the area is

$$\frac{1}{2} 2L \frac{2\pi}{L} = 2\pi$$

letting $L \to \infty$,

$$\int_{-\infty}^{\infty} e^{ikx} dx = 2\pi\delta(k)$$

2 Fourier transform of a square wave packet

$$\begin{aligned}
\phi(k) &= \frac{1}{\sqrt{2\pi}} \int_{-d/2}^{d/2} \frac{1}{\sqrt{d}} e^{-ikx} dx \\
&= \frac{1}{\sqrt{2\pi d}} \int_{-d/2}^{d/2} e^{-ikx} dx \\
&= \frac{1}{\sqrt{2\pi d}} \frac{2\sin(\frac{dk}{2})}{k}
\end{aligned}$$

The width of $\psi(x)$ (which can also be defined as the distance between the first two zereos) is clearly

$$d$$

while the width of $\phi(k)$ is

$$\frac{4\pi}{d}$$

3 Momentum distribution due to slit and their diffraction patterns

Let $\psi(x,y) \sim \psi_x(x)\psi_y(y)$. Then

$$\psi_x(x) \sim [-d/2 < x < d/2]$$
$$\phi_x(k_x) \sim \frac{2\sin(\frac{dk_x}{2})}{k_x}$$

Where $\phi_x(k_x)$ is the probability distribution for the x component of the wavevector k_x. For the particle to land on x', the ratio momenta upon leaving the slit must be $x' : L$; this is the same as the ratio of wavenumber since $p \sim k$. Hence

$$k_x = \frac{k_0 x'}{L}$$

Also we have $k_0 = \frac{2\pi}{\lambda}$. The argument to the sin function in $\phi(k_x)$ is then

$$\begin{aligned} \frac{dk_x}{2} &= \frac{d(2\pi/\lambda)(x'/L)}{2} \\ &= \frac{d\pi x'}{\lambda L} \\ &= Ax' \end{aligned}$$

Which is the answer up to normalization.

4 de Broglie wavelength of macroscopic objects

5 Gaussian wavepacket in free space

a)

$$\begin{aligned} \int dx e^{-ikx}\psi(x) &\sim \int dx e^{-(ikx + \frac{x^2}{4w_0^2})} \\ &= \int dx e^{-(\frac{x}{2w} + ikw_0)^2} e^{-k^2 w_0^2} \\ &= 2\sqrt{\pi} w_0 e^{-k^2 w_0^2} \end{aligned}$$

Where

$$w_0 = \frac{1}{2k_0}$$

For the pile of multiplicative factors in front we can note that $\phi(k)$ must be normalized, ie have the same form as $\psi(x)$.

b)

$$w_0 k_0 = \frac{1}{2}$$