ESN and Cartesian-Tensors

1 Introduction

Consider vectors $\mathbf{a} = a_x\mathbf{x} + a_y\mathbf{y} + a_z\mathbf{z}$ and $\mathbf{b} = b_x\mathbf{x} + b_y\mathbf{y} + b_z\mathbf{z}$. If we wish to prove a simple theorem like the fact that $\frac{d}{dt}(\mathbf{a}\cdot\mathbf{b}) = \mathbf{a}\cdot\frac{d\mathbf{b}}{dt} + \mathbf{b}\cdot\frac{d\mathbf{a}}{dt}$ we'll have to write the following long proof

$$\begin{aligned}
\mathbf{a}\cdot\mathbf{b} &= a_xb_x + a_yb_y + a_zb_z && (1)\\
\frac{d}{dt}(\mathbf{a}\cdot\mathbf{b}) &= \frac{d}{dt}(a_xb_x + a_yb_y + a_zb_z) && (2)\\
&= \frac{d}{dt}(a_xb_x) + \frac{d}{dt}(a_yb_y) + \frac{d}{dt}(a_zb_z) && (3)\\
&= (\frac{d}{dt}a_x)b_x + a_x(\frac{d}{dt}b_x) + (\frac{d}{dt}a_y)b_y + a_y(\frac{d}{dt}b_y) + ... && (4)\\
&= a_x(\frac{d}{dt}b_x) + a_y(\frac{d}{dt}b_y) + ... && (5)\\
&+ (\frac{d}{dt}a_x)b_x + (\frac{d}{dt}a_y)b_y + ... && (6)\\
&= \mathbf{a}\cdot\frac{d\mathbf{b}}{dt} + \mathbf{b}\cdot\frac{d\mathbf{a}}{dt} && (7)
\end{aligned}$$

a tedious manipulation of components. We introduce ESN by the two rules:

1. indiced run over all possible values they can take
2. repeated indices are implicitly summed over

so in ESN

$$\begin{aligned}
\mathbf{a}\cdot\mathbf{b} &= a_ib_i && (8)\\
&= \sum_{i=1,2,3} a_ib_i && (9)
\end{aligned}$$

Normally we index our vectors by numbers instead of letters.

2 Orthogonal transformations preserve length

$$\begin{aligned}
a_i &= M_{ij}b_j && (10)\\
a_ia_i &= M_{ij}b_jM_{ij'}b_{j'} && (11)\\
&= M_{ij}M_{ij'}b_jb_{j'} && (12)\\
&= \delta_{jj'}b_jb_{j'} && (13)\\
&= b_jb_j && (14)
\end{aligned}$$

The first line may be writwen as $a = M_{\text{-}j}b_j$, showing that a is a linear combination of columns of $M(M_{\text{-}j})$ with the entries of $b(b_j)$ as coefficients.

3 Cartesian-tensors

$$
\begin{aligned}
*T' &= aTa^{-1} && (15)\\
[aT]_{il} &= a_{ik}T_{kl} && (16)\\
[aTa^{-1}]_{ij} &= [aT]_{il}a^{-1}_{lj} && (17)\\
&= a_{ik}T_{kl}a_{jl} && (18)
\end{aligned}
$$

$a_{ik}T_{kl} = a_{i\_}T_{\_l}$

4 Very useful identity

$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{jl}\delta_{im}$