

Loaded Hoop on a conveyer belt

1 Preliminary calculations

1.1 Acceleration

Let v_r be the velocity of the load relative to the hoop, v_x be the velocity of the load in the x -axis in the lab frame and v_y be the velocity of the load in the y -axis in the lab frame. Then

$$v_r = R\dot{\theta}$$

$$\begin{aligned}v_x &= \dot{X} + v_r \cos \theta \\&= \dot{X} + R\dot{\theta} \cos \theta \\v_y &= v_r \sin \theta \\&= R\dot{\theta} \sin \theta\end{aligned}$$

$$\begin{aligned}a_x &= \dot{v}_x \\&= \ddot{X} + R\ddot{\theta} \cos \theta - R\dot{\theta}^2 \sin \theta \\a_y &= \dot{v}_y \\&= R\ddot{\theta} \sin \theta + R\dot{\theta}^2 \cos \theta\end{aligned}$$

1.2 Energy

Let K_m be the kinetic energy of the load, K_M be the kinetic energy of the hoop and U be the potential energy of the load. Then

$$\begin{aligned}K_m &= \frac{1}{2}m(v_x^2 + v_y^2) \\&= \frac{1}{2}m(R\dot{\theta} \sin \theta)^2 + \frac{1}{2}m(\dot{X} + R\dot{\theta} \cos \theta)^2 \\&= \frac{1}{2}mR^2\dot{\theta}^2 + \frac{1}{2}m\dot{X}^2 + mR\dot{X}\dot{\theta} \cos \theta\end{aligned}$$

$$K_M = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}MR^2\dot{\theta}^2$$

$$U = mgR(1 - \cos \theta)$$

$$\begin{aligned}E &= K_m + K_M + U \\&= \frac{1}{2}(M + m)R^2\dot{\theta}^2 + \frac{1}{2}(M + m)\dot{X}^2 + mR\dot{X}\dot{\theta} \cos \theta + mgR(1 - \cos \theta)\end{aligned}$$

2 Oscillations on a stationary belt

Let L be the lagrangian $K - U$, ignoring constant terms. Then

$$L = \frac{1}{2}(M + m)R^2\dot{\theta}^2 + \frac{1}{2}(M + m)\dot{X}^2 + mR\dot{\theta}\dot{X}\cos\theta + mgR\cos\theta$$

2.1 Period when friction is negligible

In this case X and θ are independent variables and we must use the Euler-Lagrange equation on each of them.

$$\frac{\partial L}{\partial \theta} = -mgR\sin\theta - mR\dot{\theta}\dot{X}\sin\theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = (M + m)R^2\dot{\theta} + mR\dot{X}\cos\theta$$

$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) &= (M + m)R^2\ddot{\theta} + mR\ddot{X}\cos\theta - mR\dot{X}\dot{\theta}\sin\theta \\ &= \frac{\partial L}{\partial \theta} \end{aligned}$$

$$\begin{aligned} -mgR\sin\theta &= (M + m)R^2\ddot{\theta} + mR\ddot{X}\cos\theta \\ mg\sin\theta + (M + m)R\ddot{\theta} + m\ddot{X}\cos\theta &= 0 \end{aligned}$$

We can also obtain this equation by considering $\tau = I\alpha$ about the center of the hoop in the (accelerating) frame of the hoop where τ is the torque, I the moment of inertia of the system about the center of the hoop and $\alpha = \ddot{\theta}$ the angular acceleration.

$$\begin{aligned} I &= (M + m)R^2 \\ \tau &= I\ddot{\theta} \\ &= -m\ddot{X}R\sin\left(\frac{\pi}{2} - \theta\right) - mgR\sin\theta \end{aligned}$$

For the X -coordinate,

$$\frac{\partial L}{\partial X} = 0$$

$$\frac{\partial L}{\partial \dot{X}} = (M + m)\dot{X} + mR\dot{\theta}\cos\theta$$

since $\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{X}} \right) = \frac{\partial L}{\partial X} = 0$, $\frac{\partial L}{\partial \dot{X}}$ is constant. Since $\dot{X} = \dot{\theta} = 0$ at the start, $\frac{\partial L}{\partial \dot{X}} = 0$ always.

$$(M + m)\dot{X} + mR\dot{\theta}\cos\theta = 0$$

We can also obtain this equation by using conservation of horizontal momentum, $M\dot{X} + mv_x = 0$

$$\begin{aligned} \dot{X} &= -\frac{m}{M + m}R\dot{\theta}\cos\theta \\ \ddot{X} &= \frac{m}{M + m}(-R\ddot{\theta}\cos\theta + R\dot{\theta}^2\sin\theta) \end{aligned}$$

combining the two we get

$$mg \sin \theta + (M + m)R\ddot{\theta} + \frac{m^2}{M + m}(R\dot{\theta}^2 \sin \theta \cos \theta - R\ddot{\theta} \cos^2 \theta) = 0$$

we now make the approximation that $\theta \ll 1$, so $\sin \theta = \theta$, $\dot{\theta}^2 = 0$, $\cos \theta = 1$

$$\begin{aligned} mg\theta + (M + m)R\ddot{\theta} - \frac{m^2}{M + m}R\ddot{\theta} &= 0 \\ \ddot{\theta} + \frac{m^2 + Mm}{M^2 + 2Mm} \frac{g}{R} + \ddot{\theta} &= 0 \\ \omega^2 &= \frac{m^2 + Mm}{M^2 + 2Mm} \frac{g}{R} \end{aligned}$$

2.2 Period when the hoop does not slip

The non-slip condition occurs when the velocity of the contact point between the hoop and the floor is 0.

$$\dot{X} + R\dot{\theta} = 0$$

We can simplify the Lagrangian with this

$$L = (M + m)R^2\dot{\theta}^2 - mR^2\dot{\theta}^2 \cos \theta + mgR \cos \theta$$

notice that horizontal momentum is not conserved since there is an external frictional force which has a horizontal component. However since this is static friction which does no work, energy is conserved and the ELE can still be used.

$$\frac{\partial L}{\partial \dot{\theta}} = 2(M + m)R^2\dot{\theta} - 2mR^2\dot{\theta} \cos \theta$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} = 2(M + m)R^2\ddot{\theta} - 2mR^2\ddot{\theta} \cos \theta + 2mR^2\dot{\theta}^2 \sin \theta$$

$$\frac{\partial L}{\partial \theta} = mR^2\dot{\theta}^2 \sin \theta - mgR \sin \theta$$

$$2MR\ddot{\theta} + mg\theta = 0$$

$$\ddot{\theta} + \frac{m}{2M} \frac{g}{R} = 0$$

Alternatively, we can also introduce f as the friction force.

$$\begin{aligned} f &= M\ddot{X} + ma_x \\ &= M\ddot{X} + m\ddot{X} + mR\ddot{\theta} \cos \theta - mR\dot{\theta}^2 \sin \theta \\ &\approx M\ddot{X} \end{aligned}$$

this makes sense as for small θ the load is close to the point of contact between the ground and the hoop, which is almost stationary in the non-slip condition. Using $\tau = I\ddot{\theta}$ again

$$\begin{aligned} (M + m)R^2\ddot{\theta} &= fR - m\ddot{X} \cos \theta R - mgR \sin \theta \\ (M + m)R\ddot{\theta} &= -MR\ddot{\theta} + mR\ddot{\theta} - mg \sin \theta \end{aligned}$$

$$2MR\ddot{\theta} + mg \sin \theta = 0$$

3 Stable Angular Orientations on an Accelerating Conveyer Belt

Let $a = \ddot{X}$ be the (constant) acceleration of the hoop and load.

In the frame of the accelerating system, we can balance torques about the center of the hoop:

$$f = mg \sin \theta + ma \cos \theta$$

Balancing forces

$$f = (M + m)a$$

equating the two

$$(M + m)a = mg \sin \theta + ma \cos \theta$$

now choose k and ϕ such that $k \sin \phi = a$ and $k \cos \phi = g$; k and ϕ will obviously always exist because we have not reduced the number of degrees of freedom (2) and k is unbounded. Then

$$\begin{aligned}\phi &= \tan^{-1} \frac{a}{g} \\ k &= \sqrt{a^2 + g^2}\end{aligned}$$

Now

$$\begin{aligned}(M + m)a &= mk \cos \phi \sin \theta + mk \sin \phi \cos \theta \\ &= mk \sin(\phi + \theta) \\ \frac{M + m}{m} \frac{a}{k} &= \sin(\phi + \theta)\end{aligned}$$

$$\begin{aligned}\phi + \theta &= \sin^{-1} \frac{M + m}{m} \frac{a}{k} \\ \theta &= \sin^{-1} \frac{M + m}{m} \frac{a}{k} - \phi \\ &= \sin^{-1} \frac{M + m}{m} \frac{a}{\sqrt{a^2 + g^2}} - \tan^{-1} \frac{a}{g}\end{aligned}$$

or, in terms of $\gamma = \frac{m}{M}$,

$$\theta = \sin^{-1} \left(\left(1 + \frac{1}{\gamma} \right) \frac{a}{\sqrt{a^2 + g^2}} \right) - \tan^{-1} \frac{a}{g}$$

3.1 $\mu_s = 1.0$

There are two cases we have to consider, static and kinetic friction. For static friction,

$$a = K = \sqrt{3}g$$

$$\begin{aligned}f &= (M + m)a \\ &= (M + m)\sqrt{3}g\end{aligned}$$

Let N be the normal frictional force. Balancing forces in the vertical direction,

$$N = (M + m)g$$

unfortunately, $f > \mu_s N$. Hence, the friction must be kinetic in nature.

$$\begin{aligned} f &= \mu_k N \\ &= \frac{\sqrt{3}}{3}(M + m)g \end{aligned}$$

and since $f = (M + m)a$,

$$a = \frac{\sqrt{3}}{3}g$$

and

$$\frac{a}{\sqrt{a^2 + g^2}} = \frac{1}{2}$$

and since $\frac{a}{g} = \frac{\sqrt{3}}{3}$ and $\tan^{-1} \frac{\sqrt{3}}{3} = \frac{\pi}{6}$, hence

$$\alpha_1 = \sin^{-1}\left(\left(1 + \frac{1}{\gamma}\right)\frac{1}{2}\right) - \frac{\pi}{6}$$

where $\gamma = \frac{m}{M}$

3.2 $\mu_s = 2.0$

In this case, for static friction, the working is the same; we still have

$$\begin{aligned} f &= (M + m)\sqrt{3}g \\ N &= (M + m)g \end{aligned}$$

but now $f < \mu_s N$ because $\sqrt{3} < 2$. Hence $a = \sqrt{3}g$ and

$$\frac{a}{\sqrt{a^2 + g^2}} = \frac{\sqrt{3}}{2}$$

and since $\frac{a}{g} = \sqrt{3}$ and $\tan^{-1} \sqrt{3} = \frac{\pi}{3}$, hence

$$\alpha_2 = \sin^{-1}\left(\left(1 + \frac{1}{\gamma}\right)\frac{\sqrt{3}}{2}\right) - \frac{\pi}{3}$$

where $\gamma = \frac{m}{M}$

4 Oscillation, Rotation, and Sliding on an Accelerating Conveyor Belt

4.1 Type A motion

Let x be the position of the center of the hoop relative to the point on the belt it was first placed on in a frame of reference moving at acceleration K (thus $X = \frac{1}{2}Kt^2 + x$). Then in this frame, the conveyor belt is stationary; hence friction does no work. The fictitious force does work on the system

$$W_f = MKx + mK(X + R \sin \theta)$$

by the work-energy theorem,

$$MKx + mK(x + R \sin \theta) + \frac{1}{2}(M + m)R^2\dot{\theta}^2 + \frac{1}{2}(M + m)\dot{x}^2 + mR\dot{\theta}\dot{x} \cos \theta + mgR(1 - \cos \theta) = 0$$

since it is constant and $x, \dot{x}, \theta, \dot{\theta}$ were all 0 initially. The non-slip condition is that

$$x + R\theta = 0$$

simplifying,

$$-MKR\theta + mKR(\sin \theta - \theta) + (M + m)R^2\dot{\theta}^2 - mR^2\dot{\theta}^2 \cos \theta + mgR(1 - \cos \theta) = 0$$

with $M = m$,

$$K(\sin \theta - 2\theta) + R\dot{\theta}^2(2 - \cos \theta) + g(1 - \cos \theta) = 0$$

$$\begin{aligned} R\dot{\theta}^2(2 - \cos \theta) &= K(2\theta - \sin \theta) + g(\cos \theta - 1) \\ R\dot{\theta}^2 &= \frac{K(2\theta - \sin \theta) + g(\cos \theta - 1)}{(2 - \cos \theta)} \\ \frac{1}{2}MR^2\dot{\theta}^2 &= \frac{1}{2}MR \frac{K(2\theta - \sin \theta) + g(\cos \theta - 1)}{(2 - \cos \theta)} \end{aligned}$$

This means that the system is equivalent to a point of mass M in a one-dimensional potential $V(\theta) \sim \frac{K(2\theta - \sin \theta) + g(\cos \theta - 1)}{(2 - \cos \theta)}$. Assuming that $V < 0$ when $0 < \theta < \beta$, when $\theta = \theta_{maximum}$, $\dot{\theta} = 0$ because otherwise we can increase θ by $d\theta$ by increasing or decreasing t by dt . Another way to see this is that in the equivalent system, $V = 0$ when $\theta = 0$; hence θ will oscillate between two roots of V . Hence

$$K(\sin \beta - 2\beta) + g(1 - \cos \beta) = 0$$

$$\begin{aligned} \frac{K}{g} &= \frac{1 - \cos \beta}{2\beta - \sin \beta} \\ &= 0.244837712 \end{aligned}$$

with $\beta = \frac{\pi}{6}$

4.2 Type B motion

If we try $\beta = \pi$,

$$\begin{aligned} \frac{K}{g} &= \frac{1 - \cos \beta}{2\beta - \sin \beta} \\ &= 0.318 \end{aligned}$$

we see that there is a hump where $V > 0$ as $0 < \theta < \pi$. Hence the condition that $\dot{\theta} = 0$ is insufficient; we must also make sure that $V < 0$ as $0 < \theta < \beta$. Since $2 - \cos \theta > 0$, this is equivalent to making $Z < 0$ where

$$Z = K(2\theta - \sin \theta) + g(\cos \theta - 1)$$

$$K/g = 0.47$$

the local maximum occurs when

$$\frac{dZ}{d\theta} = 0$$

$$K(2 - \cos \theta) - g \sin \theta = 0$$

$$2K - (K \cos \theta + g \sin \theta) = 0$$

$$2K - \sqrt{K^2 + g^2} \sin(\theta + \tan^{-1} \frac{K}{g}) = 0$$

$$\sin(\theta + \tan^{-1} \frac{K}{g}) = \frac{2K}{\sqrt{K^2 + g^2}}$$

$$\theta = \sin^{-1}(\frac{2K}{\sqrt{K^2 + g^2}}) - \tan^{-1} \frac{K}{g}$$

4.3 Sliding motion