Basics

1 Statics

D'Alembert's principle states that for a body in static equillibrium,

$$\sum_i \mathbf{F}_i \cdot \delta \mathbf{r}_i = 0 \tag{1}$$

where the $\delta \mathbf{r}$ are virtual displacements, that is, displacements that satisfy the equations of constraint. D'Alembert's principle can be explained by noting that the LHS is a sum of virtual work, and if it were non-zero there would be a state of lower energy.

We now restrict to constraint forces that do no virtual work; this is satisfied by 1) rigid body constraints and 2) surface constraints.

For dynamics, the appropriate modification is

$$\sum_i (\mathbf{F}_i - \dot{\mathbf{p}}_i) \cdot \delta \mathbf{r}_i = 0 \tag{2}$$

Although the sum of the virtual work is 0, we cannot say anything about the individual terms because the $\delta \mathbf{r}$ are not independent. Here's what's so special about using generalized coordinates: **We can vary the q_j independantly**.

2 Generalized velocity

$$\mathbf{r_i} = \mathbf{r_i}(q_1, ...q_n, t) \tag{3}$$

If generalized coordinates are the q_j that describe the system, generalized velocities are the $\dot{q}_j$. If we change one of the q_j the change in the real position, $\mathbf{r}_i$ is

$$\begin{align} \dot{\mathbf{r}}_i &= \frac{d\mathbf{r}_i}{dt} \tag{4} \\ &= \sum_j \frac{\partial \mathbf{r}_i}{\partial q_j} \dot{q}_j + \frac{\partial \mathbf{r}_i}{\partial t} \tag{5} \end{align}$$

taking partial derivatives wrt $\dot{q}_j$ we can "cancel the dots"

$$\frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_j} = \frac{\partial \mathbf{r}_i}{\partial q_j} \tag{6}$$

Note that this is only true if the constraints are independent of the generalized velocities.

3 Derivation

Recalling

$$\mathbf{r_i} = \mathbf{r_i}(q_1, ...q_n, t) \tag{7}$$

first basic step of varying $\mathbf{r}_i$ and using chain rule:

$$\delta \mathbf{r_i} = \sum_j \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j \tag{8}$$

Let's consider the first term in DAL,

$$\sum_i \mathbf{F}_i \cdot \delta r_i = \sum_i \mathbf{F}_i \cdot (\sum_j \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j) \tag{9}$$

$$= \sum_{i,j} \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j \tag{10}$$

$$= \sum_j Q_j \delta q_j \tag{11}$$

where $Q_j = \sum_i \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j}$ is the generalized force.

4 Generalized force

If we write $F_{i,x} = -\frac{\partial V}{\partial x}$, then clearly

$$Q_j = -\frac{\partial V}{\partial q_j} \tag{12}$$

which is another way to derive that the virtual work done by varying the jth generalized cordinate is $Q_j \delta q_j$

Incidentally this is one very good reason to write

$$\mathbf{F} = -\nabla V = -\frac{\partial V}{\partial \mathbf{r}_i} \tag{13}$$

5 Momentum, Work, Kinetic Energy

Consider now the second term

$$\sum_i \dot{\mathbf{p}}_i \cdot \delta \mathbf{r}_i = \sum_i m_i \ddot{\mathbf{r}}_i \cdot \delta \mathbf{r}_i \tag{14}$$

$$= \sum_{i,j} m_i \ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j \tag{15}$$

the $\ddot{\mathbf{r}}_i \delta \mathbf{r}_i$ looks like it could be obtained from differentiating the kinetic energy.

$$T = \frac{1}{2} \sum_i m_i \dot{r}_i^2 \tag{16}$$

$$\tag{17}$$

there we differentiate wrt q_j and $\dot{q}_j$

$$\frac{\partial T}{\partial q_j} = \sum_i m_i \dot{\mathbf{r}}_i \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial q_j} \tag{18}$$

$$\frac{\partial T}{\partial \dot{q}_j} = \sum_i m_i \dot{\mathbf{r}}_i \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_j} \tag{19}$$

$$= \sum_i m_i \dot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \tag{20}$$

by dot-cancelling rule. Now take

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) = \sum_i m_i \ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} + \sum_i m_i \dot{\mathbf{r}}_i \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial q_j} \tag{21}$$

$$= Q_j + \frac{\partial T}{\partial q_j} \tag{22}$$

$$Q_j = \frac{d}{dt}\left(\frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} \tag{23}$$

from here the ELE are trivial.