PROBLEM SET I
"TO DIE OF ANTICIPATION"

DUE FRIDAY, 16 SEPTEMBER

One of the first formal definitions of the notion of *finiteness* was offered by Dedekind.

Definition. We say that a set A is *Dedekind-infinite* if there exists a proper subset $A' \subsetneq A$ and a bijection $A \longrightarrow A'$. A set is said to be *Dedekind-finite* if it is not Dedekind-infinite.

It is true that for every Dedekind-finite set I, there exists a natural number n and a bijection between I and the set $I_n := \{i \in \mathbf{N} \mid i < n\}$. So Dedekind-finiteness is really the same as what we usually call finiteness. However, you *may not* use this fact to do the following three exercises.

Exercise 1. Show that any subset of a Dedekind-finite set is Dedekind-finite.

Exercise* 2. Show that the following are equivalent for a set A.
(1) Any injection $A \longrightarrow A$ is a surjection.
(2) The set A is Dedekind-finite.
(3) There is no injection $i \colon \mathbf{N} \longrightarrow A$.

Exercise** 3. Show that the union of two Dedekind-finite sets is Dedekind-finite. [Hint: Use characterization (3) from the previous exercise.]

Note that this exercise is *double-starred*. This is one of the very few double-starred problems of the semester. The claim *seems* obvious (After all, who is surprised to learn that the union of two finite sets is finite?), but the choice of *definition* of finiteness has made proving this fact a hair-raisingly difficult matter. This demonstrates the need to take care with the way we formalize our intuitions.

Exercise 4. Suppose a an integer. Show that if b is a rational number such that $b^2 = a$, then b is an integer.

Definition. For any real number x and any real number $s \geq 1$, a *Diophantine approximation of x to order s* is a pair of integers (m,n), with n positive, such that

$$0 < \left| x - \frac{m}{n} \right| < \frac{1}{n^s}.$$

So a Diophantine approximation of a real number x to order s is a rational number m/n, different from x, that is within a distance of $1/n^s$ to x.

Exercise 5. Find at least three Diophantine approximations of $\sqrt{2}$ to order 2. How many do you guess there are in all? Can you find any Diophantine approximations of $\sqrt{2}$ to order 3?

For any real number x and any real number $s \geq 1$, let $D(x,s)$ be the set of all Diophantine approximations of x of order s. That is,

$$D(x,s) = \left\{ (m,n) \in \mathbf{Z} \times \mathbf{Z} \;\middle|\; n > 0 \quad \text{and} \quad 0 < \left| x - \frac{m}{n} \right| < \frac{1}{n^s} \right\}$$

Exercise 6. Show that if s and t are real numbers such that $1 \leq s \leq t$, then $D(x,s) \supset D(x,t)$.

Exercise 7. Show that for any irrational number x, the set $D(x,1)$ is infinite.

The previous two exercises tell us something curious. As $s > 1$ increases, the sets $D(x,s)$ are getting smaller, and, for irrational numbers, $D(x,1)$ is infinite. If the sets $D(x,s)$ eventually become *finite* as s increases, then we can look for the precise moment this happens. This is called the *irrationality exponent* of x.

Definition. Now for any real number x, the *irrationality exponent* of x is the real number

$$\mu(x) := \inf\{s \in \mathbf{R} \mid \text{the set } D(x,s) \text{ is finite}\}.$$

We could have also defined $\mu(x)$ by

$$\mu(x) := \sup\{s \in \mathbf{R} \mid \text{the set } D(x,s) \text{ is infinite}\}.$$

(Why does this give us the same answer?) So the irrationality exponent $\mu(x)$ is the unique real number such that (1) for any $s > \mu(x)$, the set $D(x,s)$ of Diophantine approximations to x of order s is finite, and (2) for any $s < \mu(x)$, the set $D(x,s)$ of Diophantine approximations to x of order s is infinite.

Note that there's nothing saying that $\mu(x)$ should be finite. In fact, it can happen that $\mu(x)$ is infinite. (What would that say about the sets $D(x,s)$?) We'll get to that in a moment. For now, let's see that if x is *rational*, then not only is the irrationality exponent $\mu(x)$ finite, it must be precisely 1.

Exercise* 8. Show that the irrationality exponent of any rational number is precisely 1; that is, show that if x is a rational number, and $s > 1$, then $D(x,s)$ is finite.

So if x is rational, then $\mu(x)$ is as small as it can be. It turns out that by modifying the argument you gave for Ex. 7, you can show that the converse is also true: if x is *irrational*, then $\mu(x) > 1$. In fact, if x is irrational, then even $\mu(x) \geq 2$; this follows from a nifty theorem called the Dirichlet Approximation Theorem, but let's leave this aside for today.

Instead, let us see how the irrationality exponent $\mu(x)$ of a real number x can measure "how irrational" x is. To illustrate this principle, let's have a look at what sort of numbers have *infinite* irrationality exponent.

Theorem (Liouville's approximation theorem)**.** *Suppose n a positive integer. If x is a real, irrational solution to a polynomial equation*

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0,$$

where the coefficients a_i are all integers, then there exists a real number $c > 0$ such that for any integers p and q with $q > 0$,

$$\left| x - \frac{p}{q} \right| > \frac{c}{q^n}.$$

You will prove this theorem in a future homework, but for the purposes of the next problem, you may *assume* it.

Exercise 9. A real number with infinite irrationality exponent is called a *Liouville number*. Use Liouville's approximation theorem to conclude that any such number must be *transcendental*; i.e., it cannot be the solution to any polynomial equation with integer coefficients as above.

We will show precisely that lots such numbers actually exist in a future assignment, but for the sake of plausibility, let's write down a Liouville number:

$$\begin{array}{lll} 0 & \cdot & 1100010000000000000000010000000000000000000000000000000000000000000000000000000 \\ & & 000000000000000000000000000000000000000010000000000 \quad \cdots \end{array}$$

This is the number whose decimal expansion has a 1 in the $n!$-th place after the decimal point for each positive integer n and zeroes in all other places.