MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Spring 2006	Physics 8.04	Vuletic
Assignment #9	page 1 of 3	
Due Thursday, May 4 at 4pm		

1. **Transmission probability for a potential barrier and a potential well.** (20 points)

a) In lecture 16 we derived that the transmission probability $|t|^2$ for a particle incident on a barrier of width $2a$ and height V_0 for $E< V_0$ given by

$$|t|^2 = \frac{(2k\kappa)^2}{(2k\kappa)^2 + \left(k^2+\kappa^2\right)^2 \sinh^2 2\kappa a},$$

where $\frac{\hbar^2 k^2}{2m} = E, \quad \frac{\hbar^2\kappa^2}{2m} = V_0 - E$.

Plot or sketch $|t|^2$ as a function of κ for a very wide barrier (ka=10), a medium barrier width (ka=1), and a very thin barrier (ka=0.1). What is the limit of the transmission probability for a barrier height that approaches the energy of the particle ($V_0 \to E$, i.e. $\kappa \to 0$) in the three cases?

b) The transmission amplitude t for a potential well of the same width $2a$ and depth V_0 is given by

$$t = e^{-2ika} \frac{2kq}{2kq\cos 2qa - i(q^2+k^2)\sin 2qa},$$

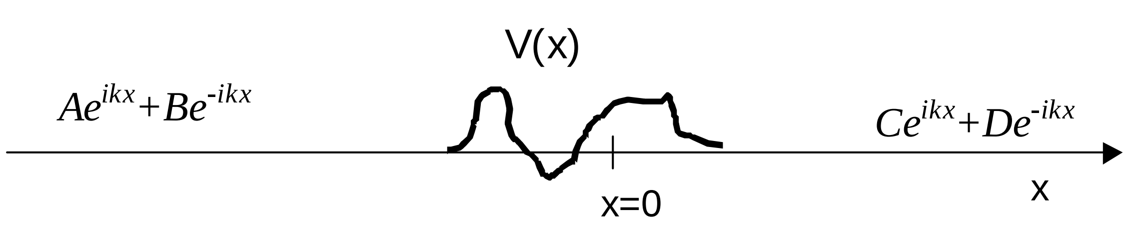
where $\frac{\hbar^2 k^2}{2m} = E, \quad \frac{\hbar^2 q^2}{2m} = V_0 + E$.

Calculate $|t|^2$ and plot or sketch it as a function of q for fixed barrier widths ka=10, ka=1, ka=0.1.

You can use a program of your choice to generate the curves, or sketch them by hand, indicating particular values.

2. **Scattering matrix**. (30 points)

Consider an arbitrary one-dimensional potential localized in a finite region near x=0, with V=0 outside that region. The most general solution of the Schroedinger equation outside the potential region is given by $Ae^{ikx}+Be^{-ikx}$, and $Ce^{ikx}+De^{-ikx}$ to the left and to the right of the potential, respectively.



a) (10 points) Show that if we write

$$\begin{aligned} B &= S_{11}A + S_{12}D \\ C &= S_{21}A + S_{22}D \end{aligned} \quad \text{, or} \quad \begin{pmatrix} B \\ C \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}\begin{pmatrix} A \\ D \end{pmatrix},$$

that the following relations for the matrix elements S_{ij} hold:

$$|S_{11}|^2 + |S_{21}|^2 = 1$$
$$|S_{12}|^2 + |S_{22}|^2 = 1$$
$$S_{11}S_{12}^* + S_{21}S_{22}^* = 0$$

b) (10 points)

$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}$ is called the scattering matrix. Use the above relations to show that the scattering matrix S and its transpose are unitary.
(Hint: Use flux conservation and the possibility that A and D are arbitrary complex numbers.)
What is the physical interpretation for each of the coefficients *A, B, C, D*?

c) (10 points) The scattering matrix S is a function of the wavenumber k (or momentum $\hbar k$).

Show that
$$\begin{aligned} S_{11}(-k) &= S_{11}^*(k) \\ S_{22}(-k) &= S_{22}^*(k), \\ S_{12}(-k) &= S_{21}^*(k) \end{aligned} \qquad \text{i.e.} \qquad S(-k) = S^+(k)$$

3. **Oscillating harmonic oscillator** (25 points)

A particle in a harmonic oscillator potential

$$V(x) = \frac{1}{2}m\omega^2 x^2$$

has an initial wave function

$$\Psi(x, t=0) = \frac{1}{\sqrt{2}}\left(\psi_0(x) + \psi_1(x)\right),$$

where ψ_0 and ψ_1 are the n=0 and n=1 normalized eigenstates for the harmonic oscillator.
a) (5 points) Write down $\Psi(x,t)$ and $|\Psi(x,t)|^2$. For this part, you may leave the expression in terms of $\psi_0(x)$ and $\psi_1(x)$.
b) (10 points) Find the expectation value of x as a function of time. Notice that it oscillates with time. What is the amplitude of the oscillation in terms of m, ω, and fundamental constants? What is its angular frequency?
c) (10 points) Find the expectation value of p as a function of time. Use your result from part b), and check if Ehrenfest's Theorem holds for this potential.

4. **Visual observation of a quantum harmonic oscillator** (25 points)

F&T 4-10

An experimenter asks for funds to observe visually through a microscope the quantum behavior of a small oscillator. According to his proposal, the oscillator consists of an object 10^{-4} cm in diameter and estimated mass of 10^{-12} g. It vibrates on the end of a thin fiber with a maximum amplitude of 10^{-3} cm and frequency 1000 Hz. You are referee for the proposal

a) (5 points) What is the approximate quantum number for the system in the state described?
b) (10 points) What would be its energy in eV if it were in its lowest-energy state? Compare with the average thermal energy (25 meV) of air molecules at room temperature.
c) (10 points) What would be its classical amplitude of vibration if it were in its lowest-energy state? Compare this with the wavelength of visible light (500 nm) by which it is presumably observed.
d) Would you, as referee of this proposal, recommend award of a grant to carry out this research?