CTCS

Sets

1

Let $h : W \to S$ be a function. Define a function $\text{Hom}(h, T) : \text{Hom}(S, T) \to \text{Hom}(W, T)$ by $\text{Hom}(h, T)(g) = g \circ h$. Show that if T has at least 2 elements, then h is surjective iif $\text{Hom}(h, T)$ is injective.

$\implies$: Let h be surjective. We wish to show that $\text{Hom}(h, T)$ is injective. Suppose $\text{Hom}(h, T)(f) = \text{Hom}(h, T)(g)$. Then

$$\begin{align} \text{Hom}(h, T)(f) &= \text{Hom}(h, T)(g) \tag{1} \\ f \circ h &= g \circ h \tag{2} \end{align}$$

Let $y \in S$ be arbitrary. Since h is onto S, there exists x such that $y = h(x)$. Then

$$\begin{align} f(h(x)) &= g(h(x)) \tag{3} \\ f(y) &= g(y) \tag{4} \end{align}$$

Since y was arbitrary, $f = g$. Hence $\text{Hom}(h, T)$ is injective.

$\impliedby$: Let h be not surjective. We wish to show that $\text{Hom}(h, T)$ is not injective. Since h is not surjective there exists $y \in S$ such that $y \neq h(x)$ for all x. Let f and g be functions both from XXX that agree on all values in their domain except that $f(y) \neq g(y)$. Note that they are different functions. However $\text{Hom}(h, T)(f) = \text{Hom}(h, T)(g)$ because $g \circ h = f \circ h$. Hence $\text{Hom}(h, T)$ is not injective.

2a

Show that the mapping that takes a pair $(f : X \to S, g : X \to T)$ of functions to the function $< f, g >: X \to S \times T$ defined by $< f, g > (x) =< f(x), g(x) >$ is a bijection from $\text{Hom}(X, S) \times \text{Hom}(X, T)$ to $\text{Hom}(X, S \times T)$.

Surjective: we show that the range of the mapping is equal to the codomain, $\text{Hom}(X, S \times T)$. Let $h \in \text{Hom}(X, S \times T)$ be given. Then $h : X \to S \times T$. We construct f and g as follows. Let x be arbitrary. Then $h(x) \in S \times T$, so $h(x) = (a, b)$. Then let $f(x) = a, g(x) = b$.

Injective: Suppose $< f, g >=< h, j >$. Then for all x we have

$$
\begin{aligned}
< f,g > (x) &=< h,j > (x) \\
(f(x),g(x)) &= (h(x),j(x)) \\
f(x) &= h(x) \\
g(x) &= j(x)
\end{aligned}
$$

Since this is true of all x we have $f = h, g = j$

2b

If you set $X = S \times T$ in (a) what does $id_{S\times T}$ correspond to under the bijection?

3a

Let S and T be disjoint sets. Let V be a set. Let $\phi : \mathrm{Hom}(S,V) \times \mathrm{Hom}(T,V) \to \mathrm{Hom}(S \cup T,V)$ be the mapping that takes a pair $(f : S \to V, g : T \to V)$ to the function $< f|g >: S \cup T \to V$ defined by $< f|g > (x) = f(x) if x \in S, g(x) if x \in T$. Show that ϕ is a bijection.

3b

If you set $V = S \cup T$ in (a), what is $\phi(id_{S\cup T})$?

4a

If $P(C)$ denotes the powerset of all subsets of C, then $Rel(A,B) = P(A \times b)$ denotes the set of relations from A to B. Let $\phi : Rel(A,B) \to \mathrm{Hom}(A,P(B))$ be defined by $\phi(\alpha)(a) = \{b \in B|(a,b) \in \alpha\}$. Show that ϕ is a bijection.

4b

Let $A = B$. What corresponds to Δ_A under this bijection?

4c

If we let $A = P(B)$ then $\phi^{-1} : \mathrm{Hom}(P(B),P(B)) \to Rel(P(B),B)$. What is $\phi^{-1}(id_{P(B)})$?