PROBLEM SET IV
"BY EDWARD NASHTON"

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Prove or give counterexamples to justify your claims.

Exercise 29. Is there a continuous function $f:\mathbf{R}\longrightarrow\mathbf{R}$ such that the image of some closed interval is not also a closed interval? Is there a continuous function $f:\mathbf{R}\longrightarrow\mathbf{R}$ such that the image of some *open* interval is not also an open interval?

Exercise 30. Define the function $u:\mathbf{R}\longrightarrow\mathbf{R}$ by the formula

$$u(s):=\left|2\left(s-\left\lfloor\frac{1}{2}+s\right\rfloor\right)\right|.$$

Is u continuous? Consider the function $v:\mathbf{R}\longrightarrow\mathbf{R}$ defined by the formula

$$v(t):=\begin{cases}u(1/t) & \text{if } t\neq 0;\\ 0 & \text{if } t=0,\end{cases}$$

and the function $w:\mathbf{R}\longrightarrow\mathbf{R}$ defined by the formula

$$w(t):=\begin{cases}t\,u(1/t) & \text{if } t\neq 0;\\ 0 & \text{if } t=0,\end{cases}$$

Is v continuous? Is w?

Definition. A subset $E\subset\mathbf{R}$ is said to be *closed* if its complement $\mathbf{R}-E$ is open.

Exercise 31. Suppose $E\subset\mathbf{R}$ a closed set, and suppose $f:E\longrightarrow\mathbf{R}$ a continuous function. Must there be a function $F:\mathbf{R}\longrightarrow\mathbf{R}$ such that for any $x\in E$ one has $F(x)=f(x)$? What if E were only assumed open?

Exercise 32. Suppose $E\subset\mathbf{R}$. Is the function $d_E:\mathbf{R}\longrightarrow\mathbf{R}$ defined by the formula

$$d_E(x):=\inf\{|x-y|\mid y\in E\}$$

continuous?

Exercise 33. Suppose $E,E'\subset\mathbf{R}$ two disjoint closed subsets. Must there be a continuous function $f:\mathbf{R}\longrightarrow\mathbf{R}$ such that both $f^{-1}(0)=E$ and $f^{-1}(1)=E'$?

Exercise* 34. Let $a<b$ be two real numbers. Is there a discontinuous function $f:(a,b)\longrightarrow\mathbf{R}$ that is *convex*, in the sense that for any $x,y\in(a,b)$ and any $t\in[0,1]$, one has

$$f((1-t)x+ty)\leq(1-t)f(x)+tf(y)?$$

Exercise 35. Are there positive numbers $r,s,t\in\mathbf{R}$ with $r\geq t$ such that the limit

$$\lim_{x\to 0}\left(\frac{s^x+t}{r}\right)^{1/x}$$

does not exist? Are there positive numbers $r,s,t\in\mathbf{R}$ with $r\geq t$ such that this limit exists and is greater than 1?