Consider a string fixed at both ends we write the verticle displacement $u = u(x,t)$; applying $F = ma$ yields

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0 \tag{1}$$

where $c^2 = T/\rho$

Subject to the boundary condition that $u(0,t) = u(L,t) = 0$. We want to solve the initial value problem; given $u(x,0)$ and $u_t(x,0)$ to find u. We will use two methods.

1 D'alembert's solution

We change variables to $\xi = x - ct$ and $\eta = x + ct$.

$$x = \xi + \eta \tag{2}$$
$$ct = \eta - \xi \tag{3}$$

using the chain rule,

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \tag{4}$$
$$\frac{1}{c}\frac{\partial}{\partial t} = \frac{\partial}{\partial \eta} - \frac{\partial}{\partial \xi} \tag{5}$$

Doing some manipulation we realise that

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = 0 \tag{6}$$

so obviously

$$u = f(x - ct) + g(x + ct) \tag{7}$$

where f and g are any nice (single-argument) functions. This corresponds to a wave propagating to the left and one propagating to the right.

2 Separation of variables

We assume that

$$u(x,t) = X(x)T(t) \tag{8}$$

We see that they satisfy ODEs

$$\frac{d^2X}{dx^2} + k^2X = 0 \tag{9}$$
$$\frac{d^2T}{dt^2} + (kc)^2T = 0 \tag{10}$$

the general solution for $X(x)$ is a sum of sines and cosines, but due to the boundary condition the only solutions are

$$X(x) = A\sin(k_n x) \tag{11}$$

where $k_n = \frac{n\pi}{L}, n = 1, 2, 3...$

There are no restrictions on $T(t)$ so a stationary solution is $u = A\sin(\frac{n\pi x}{L})\sin(\frac{n\pi vt}{L} + \phi)$

These are the familiar standing waves, with $n = 1$ being the fundamental, $n = 2$ first overtone (one octave higher), etc. The general solution