Catsters

1 Terminal and Initial objects

Let C be a category. A terminal object in C is an object $T \in Obj(C)$ such that $\forall X \in Obj(C) \exists!$ morphism $X \to T$.

Example: one-element sets are terminal in Set.

Example: the trivial group is terminal in Grp.

Example: any one-point space is terminal in Top.

It is pointless to think about "how many trivial groups there are" since they are all isomorphic (as groups), and the isomorphism is canonical.

Lemma: terminal objects are unique up to unique isomorphism.

Non-examples:

$\cdot \rightrightarrows \cdot$

$\cdot \longrightarrow \cdot \qquad \cdot \longrightarrow \cdot$

$\cdot \longrightarrow \cdot \longrightarrow \cdot \longrightarrow \cdot \longrightarrow \dots$

Fields (assuming $0 \neq 1$ in a field)

1.1 Initial 1

Let C be a category. An initial object in C is an object $I \in Obj(C)$ such that $\forall X \in Obj(C) \exists!$ morphism $I \to X$.

An initial object in C is a terminal object in C^{op}.

Example: the empty set is initial in Set.

Example: the one-point space is initial in Top.

Example: the trivial group is initial in Grp.

The trivial group is initial and terminal in Grp. We call such things null objects. Other examples of categories with null objects: pointed sets, based spaces.

In Cat, the empty category is initial and a category with one object and one morphism is terminal.

In Field, there is no initial object.

2 Products and Coproducts

The cartesian product of $\{1, 2\}$ and $\{3, 4\}$ is any four element set, but the set must come with projection maps to $\{1, 2\}$ and $\{3, 4\}$.

Examples: Product topology, group product, sum of vector spaces, product of categories, maximums in a poset.

Coproduct in group: you can try taking the disjoint union, but this is not a group. Hence you generate something freely. Free group. (similar intuition for vector space).

Example: coproduct in based topologies is disjoint union, but where the base points are identified.

Notations: (f, g) vs $f \times g$:

$$\begin{tikzcd}
 & V \arrow[d, dotted, "(f,g)"] \arrow[ddl, bend right, "f"'] \arrow[ddr, bend left, "g"] & \\
 & A \times B \arrow[dl] \arrow[dr] & \\
A & & B
\end{tikzcd}$$

$$
\begin{tikzcd}
 & X \times Y \arrow[dl] \arrow[dr] \arrow[dd, dotted, "f \times g"] & \\
X \arrow[dd, "f"'] & & Y \arrow[dd, "g"] \\
 & A \times B \arrow[dl] \arrow[dr] & \\
A & & B
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & A \arrow[d, dotted, "\Delta"] \arrow[ddl, bend right] \arrow[ddr, bend left] & \\
 & A \times A \arrow[dl] \arrow[dr] & \\
A & & A
\end{tikzcd}
$$

$$
\begin{tikzcd}
V \arrow[d, "\Delta"] \arrow[dd, bend right, "{(f,g)}"'] \\
V \times V \arrow[d, "f \times g"] \\
A \times B
\end{tikzcd}
$$

Let 1 be a terminal object. Then $X \times 1 = X$.

3 Pullbacks and Pushouts