Diagonalization and Powers of A

Let A be a $n \times n$ matrix with n linearly independent eigenvectors. I put the eigenvectors in the columns of a matrix S, the eigenvector matrix. What is $S^{-1}AS$?

1 First way

First, what is AS? Remember that $S_i = x_i$, the columns of S, are eigenvectors.

$$\begin{aligned} AS &= A \begin{bmatrix} | & | & | & \\ x_1 & x_2 & x_3 & ... \\ | & | & | & \end{bmatrix} && (1) \\ &= \begin{bmatrix} | & | & | & \\ Ax_1 & Ax_2 & Ax_3 & ... \\ | & | & | & \end{bmatrix} && (2) \\ &= \begin{bmatrix} | & | & | & \\ \lambda_1 x_1 & \lambda_2 x_2 & \lambda_3 x_3 & ... \\ | & | & | & \end{bmatrix} && (3) \end{aligned}$$

Let's factor it out

$$AS = \begin{bmatrix} | & | & | & \\ x_1 & x_2 & x_3 & ... \\ | & | & | & \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & ... \end{bmatrix} \tag{4}$$

So

$$\begin{aligned} AS &= S\Lambda && (5) \\ S^{-1}AS &= \Lambda && (6) \end{aligned}$$

2 Second way

S is a similarity transformation into the eigenbasis, so Λ is A in a new basis. This tells us something important: in the eigenbasis, A is diagonal, very simple. $A \sim \Lambda$. If we are given a vector in the eigenbasis it is very easy to apply Λ; just multiply each basis vector by the appropriate λ.

3 Example

If

$$\begin{aligned} Ax &= \lambda x && (7) \\ A^2 x &= \lambda A x && (8) \\ &= \lambda^2 x && (9) \end{aligned}$$

$$
\begin{aligned}
A &= S\lambda S^{-1} && (10)\\
A^2 &= S\lambda S^{-1}S\lambda S^{-1} && (11)\\
&= S\lambda^2 S^{-1} && (12)\\
A^k &= S\lambda^k S^{-1} && (13)
\end{aligned}
$$

This representation gives us a great way to calculate and understand the powers of a matrix. For any A defined as above,

$$
\begin{aligned}
k \to \infty \implies A^k \to 0 && (14)\\
iifall && (15)\\
\|\lambda_i\| < 1 && (16)
\end{aligned}
$$

4 Niceness

A is sure to have n independent eigenvectors (and be diagonalizable) if the $\lambda's$ are all different (no repeated eigenvalues)

If there are repeated eigenvectors, I have to look more closely. For instance for I we have only one eigenvalue, 1, but every nonzero vector is an eigenvector.

$$
A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \tag{17}
$$

$\lambda = 2, 2$. However there is only one line of eigenvectors (the nullspace of $A - 2I$ has dimension 1).

Algebraic multiplicity = 2, but geometric multiplicity = 1.

5 $u_{k+1} = Au_k$

$u_{k+1} = Au_k$, start with u_0. $u_k = A^k u_0$.

Solution: decompose u_0 into eigenbasis, then go!

Example: fibonacci.

$u_k = [F_{k+1}, F_k]$ $u_{k+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} u_k$

$\lambda_1 = \frac{1}{2}(1 + \sqrt{5})$ dominates as k increases. We see the Fibonacci series grows as 1.618^k