Equations and Inequalities

1. Express $f(x) = |x-1| + |x+1|$ as a piecewise function. Sketch the graph of $y = f(x)$. Hence, or otherwise, solve the inequality

$$|x-1| + |x+1| \geq x + 2$$

2. The graph of $y = h(x)$ intersects the graph of $y = h^{-1}(x)$ at $x = 2$ and $x = 3$. Given that $h(x) = x^3 + ax + b$, find the values of a and b.

3. Find the value of p if the roots of the quadratic equation $4x^2 - x + 5p = 5px$ differ by 1, and p is a constant real number.

4. Given that the roots of the quadratic equation $2x^2 - x - 4$ are α and β, find a quadratic equation whose roots are $3\alpha + \beta$ and $\alpha + 3\beta$.

5. Show that $x = 1$ is a solution of $x^3 - 1 = 0$. Hence or otherwise, factorize $x^3 - 1$.

6. Find the term independent of x in $\left(x + \dfrac{1}{x}\right)^{2n}$ where n is an integer.

7. Show, without a calculator, that $2 < \sqrt{5} < 3$, and deduce that $0 < \sqrt{5} - 2 < 1$.

   Show that $(\sqrt{5} + 2)^4(\sqrt{5} - 2)^4 = 332$.

   Hence or otherwise, show that $321 < (\sqrt{5} + 2)^2 < 322$.

8. Express $\dfrac{x^2 - 1}{x^2 - 16}$ as a partial fraction.

9. Solve the simultaneous equation

$$\log_5 p + \log_5 q = 3$$
$$\log_p q = 2$$

10. Solve the simultaneous equation

$$16^x - 16^y = 64512 \tag{1}$$
$$4^x - 4^y = 224 \tag{2}$$

11. Given that the roots of $(\log 3x)(\log 7x) = 1$ are α and β, find $\alpha\beta$.

Trigonometry

12. Find the exact values of

    (a) $\sin\dfrac{\pi}{12}, \cos\dfrac{\pi}{12}$ and $\tan\dfrac{\pi}{12}$

    (b) $\cos^4\dfrac{\pi}{24} - \cos^4\dfrac{\pi}{24}$

    (c) $\tan^{-1}1 + \tan^{-1}2 + \tan^{-1}3$

13. Prove the following identities.

    (a) $\tan\theta + \cot\theta = 2\operatorname{cosec}2\theta$

    (b) $\tan 2A - 2\tan 2A\sin^2 A = \sin 2A$

Calculus

14. The equation of a curve is $y = e^{-ax^2}$, where $a > 0$ is a constant real number.

    (a) Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$

    (b) Find the stationary points of the curve. Over which interval is the function strictly increasing?

15. In an experiment, synthetic crude oil is piped into a silo at a rate of

$$R(t) = \frac{70}{\sqrt{t}}$$

tonnes per second, from $t = 1$ onwards. However, oil is leaking from the silo at a rate of

$$C(t) = \sqrt{t} + 9$$

tonnes per second. At $t = 1$ second, the amount of oil in the silo is 10,000 tonnes.

    (a) Find the net change in the amount of oil in the silo between $t = 4$ and $t = 16$ seconds.

    (b) Write down an expression involving an integral that gives the amount of synthetic crude oil inside the silo at time $t = 20$ second.

16. A cylinder has radius $\frac{t^{3/2}}{1+t^{3/2}}$ and height $\frac{t}{1+t}$, where t is the time in seconds. What is the rate of change of its volume? What is the rate of change of its surface area (including top and base)?

17. A chemist has a filter paper in the shape of a disc of radius R. The chemist cuts out a slice of the disc that subtends an angle θ and folds the remaining piece into a cone. Find the volume of the cone as a function of θ and R. What is the maximum volume of the cone? For what value of θ does the cone attain this volume?