MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Spring 2006 Physics 8.04 Vuletic
Assignment #2
Due Thursday, February 23 at 4pm

1. **Parseval's theorem. (**10 points)

Prove the following theorem: A function *ψ(x)* and its Fourier transform *ϕ(k)* have the same normalization, i.e.

$$\int_{-\infty}^{\infty} |\psi(x)|^2 dx = \int_{-\infty}^{\infty} |\phi(k)|^2 dk .$$

2. **Fourier transform of a square wave packet. (**20 points)

Calculate the Fourier transform

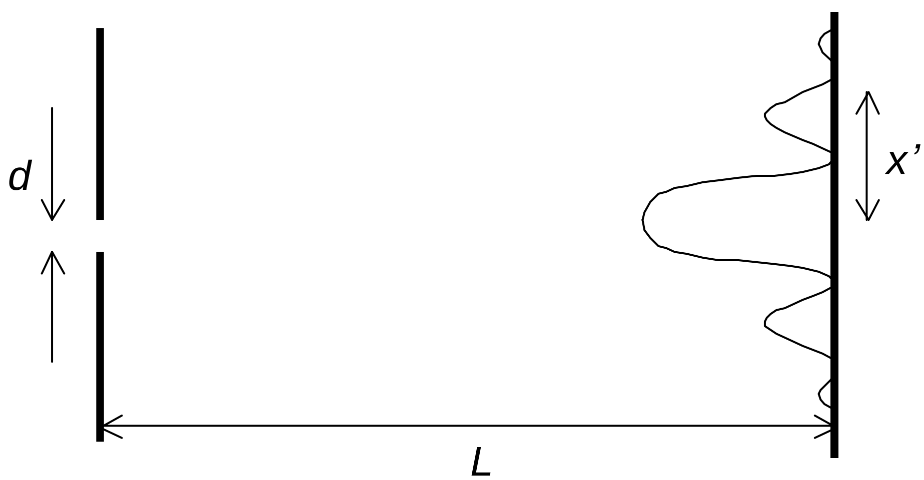
$$\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(x) e^{-ikx} dx$$

of the square wave packet

$$\psi(x) = \begin{cases} \frac{1}{\sqrt{d}} & \text{for } -d/2 \le x \le d/2 \\ 0 & \text{elsewhere} \end{cases}$$

Draw qualitative graphs of *ψ(x)* and *ϕ(k)*. What is the approximate "width of each function? (For *ϕ(k)*, one might define a suitable width as the spacing between its first two zeros.)

3. **Momentum distribution due to slit and the diffraction pattern. (**30 points)



Assume that the square wave packet

$$\psi(x,y) = \begin{cases} \frac{1}{\sqrt{d}} \exp(ik_0 y) & \text{for } -d/2 \le x \le d/2 \\ 0 & \text{elsewhere} \end{cases}$$

describes the wavefunction of the particle at the moment when the particle passes through the slit. This wavefunction describes a particle with a sharply defined *y*-component of momentum that is localized in the *x*-direction. Assume that the momentum of the particle after it passes through the slit is the same as when it is within the slit, and that it is conserved until the particle hits the screen. Using the relation between the Fourier transform and the probability that the particle has a certain value of momentum p_x along the x-direction, calculate the probability $I(x')$ of observing the particle at location x' on a screen that has been placed at a large distance $L \gg d,x'$ from the slit. Show that this reproduces the result obtained from a diffraction analysis

$$I(x') = C\left(\frac{\sin(Ax')}{Ax'}\right)^2$$

where $A=\pi d/(\lambda L)$, and C is a normalization constant.

4. **The deBroglie wavelength of macroscopic objects** (10 points)

What is the deBroglie wavelength of an automobile (2000kg) traveling at 25 miles per hour? A dust particle of radius 1μm and density 200 kg/m^3 being jostled around by air molecules at room temperature (T=300K)? An ^{87}Rb atom that has been laser cooled to a temperature of T=100μK? Assume that the kinetic energy of the particle is given by (3/2)k_BT.

3. **Gaussian wavepacket in free space.** (30 points).

A particle of mass m has the initial wave function

$$\psi(x) = \frac{1}{(2\pi)^{1/4} w_0^{1/2}} e^{-\frac{x^2}{4w_0^2}},$$

where the width of the wavepacket w_0 is a real, positive constant.
a) (15 points) Compute the Fourier transform of $\psi(x, 0)$, show that it also a Gaussian wavepacket:

$$\phi(k) = \frac{1}{(2\pi)^{1/4} k_0^{1/2}} e^{-\frac{k^2}{4k_0^2}}.$$

Determine the width k_0 of the wavepacket in momentum space.
(Hint: You should complete the square in the exponent in order to perform the integration using the formula for Gaussian integrals below.)
b) (15 points) Assuming that $\Delta x=w_0$ and $\Delta k= k_0$, (you will verify that in a later problem set), show that the Gaussian wavepacket has an uncertainty product given by the lower bound of the Heisenberg uncertainty relation.

The following formula is useful to calculate Gaussian integrals:

$$\int_{-\infty}^{\infty} dx e^{-\alpha(x-\beta)^2} = \sqrt{\frac{\pi}{\alpha}} \quad \text{for any complex } \alpha,\beta \text{ with } \mathrm{Re}(\alpha)>0.$$