Fourier series

Suppose

$$f(t) = \sum_{k=-n}^{n} c_k e^{2\pi ikt} \tag{1}$$

is a function of period 1 and we want to find c_m.

$$c_m e^{2\pi imt} = f(t) - \sum_{k\neq m} c_k e^{2\pi ikt} \tag{2}$$

multiply by $e^{-2\pi imt}$

$$c_m = f(t)e^{-2\pi imt} - \sum_{k\neq m} c_k e^{2\pi i(k-m)t} \tag{3}$$

integrating from 0 to 1,

$$c_m = \int_0^1 f(t)e^{-2\pi imt}dt \tag{4}$$

this trick works because the complex exponentials form an orthonormal set under the inner product of $\int_0^1$.

So given any $f(t)$, define

$$\hat{f}(k) = \int_0^1 f(t)e^{-2\pi ikt} \tag{5}$$

Any discontinuity in any derivative precludes writing

$$f(t) = \sum_k \hat{f}(k)e^{2\pi ikt} \tag{6}$$

for a finite sum.

It takes high frequencies to make sharp corners.

1 Results

Continuous case: converges! $\sum_k \hat{f}(k)e^{2\pi ikt}$ converges to $f(t)$ for each t.

Pointwise convergence

Smooth case: uniform convergence

Jump discontinuity: converges to the point in the middle of the jump.

General case: finite energy (square-integrable) mean convergence

2 Heated ring

$$u_t = \frac{1}{2}u_{xx} \tag{7}$$

write u as a fourier series and put the time dependence on the fourier exponents.

$$u(x,t) = \sum c_n(t)e^{2\pi inx} \tag{8}$$

$$u_t = \sum c_n'(t)e^{2\pi inx} \tag{9}$$

$$u_{xx} = \sum(-4\pi^2n^2)c_n(t)e^{2\pi inx} \tag{10}$$

comparing coefficients,

$$c'(t) = -2\pi^2n^2c(t) \tag{11}$$

with solution

$$c_n(t) = c_n(0)e^{-2\pi^2n^2t} \tag{12}$$

3 Convolution

$$c_n(0) = \hat{f}(n) = \int_0^1 f(y)e^{-2\pi iny}dy \tag{13}$$

$$u(x,t) = \sum e^{-2\pi^2n^2t}e^{2\pi inx}\int_0^1 f(y)e^{-2\pi iny}dy \tag{14}$$

$$= \int_0^1 \sum e^{-2\pi^2n^2t}e^{2\pi in(x-y)}f(y)dy \tag{15}$$

convolution with a function g puts g's fourier coefficients onto f's.