MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Spring 2006 Physics 8.04 Vuletic

Assignment #3

Due Thursday, March 2 at 4pm

1. **Compton effect and electron recoil.** (25 points)
   a) Show that it is impossible for a free electron to absorb all of the energy of a single photon which collides with it.
   b) Derive the Compton wavelength shift for a photon scattered from a free, initially stationary electron
   $$\Delta\lambda = \lambda_1 - \lambda_0 = \lambda_c(1-\cos\theta),$$
   where θ is the photon scattering angle, λ_0 is the wavelength of the incident, λ_1 the wavelength of the scattered photon, and λ_c the Compton wavelength. Calculate the numerical value of the Compton wavelength.

   (c) The Compton shift in wavelength, $\Delta\lambda$, is independent of the incident photon energy $E_0=h\nu_0$. However, the Compton shift in energy, $\Delta E=E_1-E_0$, is strongly dependent on E_0. Find the expression for the Compton energy shift ΔE. (Does the photon gain or lose energy in the collision?) Compute the numerical value of the fractional shift in energy for a 10 keV photon and a 10 MeV photon, assuming $\theta=\pi/2$.

   (d) Is it easier to observe the Compton effect with visible light or with X-rays? Why?

   (e) Show that the relation between the directions of motion of the scattered photon and the recoiling electron is
   $$\cot\frac{\theta}{2} = \left(1+\frac{h\nu_0}{m_e c^2}\right)\tan\phi,$$
   where the angle ϕ specifies the direction of the recoiling electron.

2. **Practice with delta functions** (10 points).

The Dirac delta function may be defined as

$$\delta(x) = \begin{cases} \infty & \text{for } x=0 \\ 0 & \text{for } x\neq 0 \end{cases}, \text{ such that } \int_{-\infty}^{\infty} dx\,\delta(x) f(x) = f(0)$$

for any function $f(x)$. Evaluate the following integrals:

a) $\int_{-3}^{1} dx\left(x^4-4x^3+2x\right)\delta(x+2)$,

b) $\int_{0}^{\infty} dx\left[\cos(3x)+2e^{ix}\right]\left(\delta(x-\pi)+\delta(x)\right)$.

3. **Gaussian wave packet and Heisenberg uncertainty relation.** (50 points).

Consider the Gaussian wave function from assignment #2, problem 3,

$$\psi(x) = \frac{1}{(2\pi)^{1/4} w_0^{1/2}} e^{-\frac{x^2}{4w_0^2}},$$

and its Fourier transform $\phi(k) = \frac{1}{(2\pi)^{1/4} k_0^{1/2}} e^{-\frac{k^2}{4k_0^2}}$.

a) (5 points) Show that $\psi(x)$ is normalized, i.e. $\int_{-\infty}^{\infty} |\psi(x)|^2 dx = 1$. From Parseval's theorem we know that the Fourier transform is then automatically also normalized, $\int_{-\infty}^{\infty} |\phi(k)|^2 dk = 1$.

b) (5 points) The expectation values of position $\langle x \rangle$ and momentum $\langle p \rangle$ are given by

$$\langle x \rangle = \int_{-\infty}^{\infty} x |\psi(x)|^2 dx \quad \text{and} \quad \langle p \rangle = \int_{-\infty}^{\infty} \hbar k |\phi(k)|^2 dk .$$

Calculate $\langle x \rangle$ and $\langle p \rangle$. (Hint: Take advantage of symmetry.)

If you assume that the formula for $\langle p \rangle$ is correct, how would you (in one sentence) interpret the quantity $|\phi(k)|^2$?

c) (20 points) In general the expectation values $\langle f(x) \rangle$, $\langle g(p) \rangle$ for any function $f(x)$ of the position coordinate x or function $g(p)$ of the momentum coordinate p are given by

$$\langle f(x) \rangle = \int_{-\infty}^{\infty} f(x) |\psi(x)|^2 dx \quad \text{and} \quad \langle g(p) \rangle = \int_{-\infty}^{\infty} g(\hbar k) |\phi(k)|^2 dk .$$

Calculate $\langle x^2 \rangle$ and $\langle p^2 \rangle$.

d) (10 points) The uncertainties Δx in position x and Δp in position p can be formally defined as $(\Delta x)^2 = \langle x^2 \rangle - \langle x \rangle^2$ and $(\Delta p)^2 = \langle p^2 \rangle - \langle p \rangle^2$.

Calculate Δx and Δp. Compare the product $\Delta x \Delta p$ to the minimum value allowed by the Heisenberg relation.

e) (10 points) Now assume a Gaussian wavepacket of the same width displaced in space by an amount x_0, $\overline{\psi}(x) = \frac{1}{(2\pi)^{1/4} w_0^{1/2}} e^{-\frac{(x-x_0)^2}{4w_0^2}}$.

Calculate the Fourier transform $\bar{\phi}(k)$. Will a measurement of the momentum distribution yield a different result than a momentum measurement for the original wave packet $\psi(x)$ prepared at x_0=0? Interpret your finding in one sentence.

4. **Childish precision experiment.** (15 points)
A child on top of a ladder of height H is dropping marbles of mass m to the floor and trying to hit a crack in the floor. To aim, the child is using equipment of the highest possible precision. Assume that the effects of air resistance and breezes are entirely negligible. Show that the marbles will miss the crack by a typical distance of order $(\hbar/m)^{1/2}\,(2H/g)^{1/4}$, where g is the acceleration due to gravity. How large is this distance for H=3m, $m=10^{-2}$ kg? An experimentalist decides to perform the same experiment with ^{133}Cs atoms and a drop height H=0.2m. Will she be able to observe the effect?