18.014 pset 2

10 Show that we have the following equations for any p, q

$h(0,p,q) = q+1$
$h(1,p,q) = h(0,p,h(1,p,q-1)) = h(1,p,q-1)+1 = ... = h(1,p,0)+q = p+q$
$h(2,p,q) = h(1,p,h(2,p,q-1)) = p + h(2,p,q-1) = ... = pq$
$h(3,p,q) = h(2,p,h(3,p,q-1)) = p * h(3,p,q-1) = ...p^q$
$h(4,p,q) = h(3,p,h(4,p,q-1)) = p^{h(4,p,q-1)} = p^{p^{p^{\cdots}}}$

11 Compute explicitly...

$h(4,2,4) = 2^{2^{2^2}} = 2^16 = 65536$
$h(5,3,2) = h(4,3,h(5,3,1)) = h(4,3,h(4,3,h(5,3,0))) = h(4,3,h(4,3,1)) = h(4,3,3) = 3^{3^3} = 3^27 \approx 8 \times 10^{12}$
For $n > 3$ we can prove
$h(n,2,1) = h(n-1,2,h(n,2,0)) = h(n-1,2,1) = 2$
and using this
$h(n,2,2) = h(n-1,2,h(n,2,1) = h(n-1,2,2)$

12 Show that a is injective

$h(n,n,n-1) > n\, h(n,n,n) = h(n-1,n,h(n,n,n-1)) > h(n-1,n-1,h(n,n,n-1)) > h(n-1,n-1,n-1)$

13 For any natural number n, let B_n be the set...

a is injective

14 Show that...

15 Show that the indicator function of the Cantor Set is Riemann integrable

16 Show that the function d is not Riemann integrable on $[0,1]$

In any section of finite width there will be both rational and irratoinal numbers.

17 Show that the function θ is Riemann integrable

For any partition, consider a rectangular block of height $1/q$. At most q rational numbers can be holding it up; then after $2q$ refinements, the block must decrease in area. Hence $\int = 0$.

Proof with fixed-width partitions

If all partitions have the same witdth $1/N$ where N is the number of partitions, the area is $\frac{1}{N}\sum h$ where h is the height. Obviously we choose the tallest values first; there are at most q numbers for which $h = 1/q$, and the sum of heights is 1; hence $\sum h < \ln N$. So $\overline{\int} < \frac{1}{N}\ln N = 0$ as $N \to \infty$