Centre of Mass

The COM frame is useful when systems have translational symmetry. This happens when V is a function of $\mathbf{r_1} - \mathbf{r_2}$ (relative position) instead of on the absolute position of the particles.

1 Classical

$$m_1 \frac{d^2\mathbf{r}_1}{dt^2} + \frac{\partial V}{\partial \mathbf{r}_1} = 0 \tag{1}$$
$$m_2 \frac{d^2\mathbf{r}_2}{dt^2} + \frac{\partial V}{\partial \mathbf{r}_2} = 0 \tag{2}$$

The problem with this is that the equations are coupled, because $V = V(\mathbf{r}_1 - \mathbf{r}_2)$.
Introduce $\mathbf{r} = \mathbf{r_1} - \mathbf{r_2}$ so that $V = V(\mathbf{r})$, and $(m_1 + m_2)\mathbf{R} = m_1\mathbf{r}_1 + m_2\mathbf{r}_2$

$$\frac{\partial V}{\partial \mathbf{r}_1} = \frac{\partial V}{\partial \mathbf{r}} \frac{\partial \mathbf{r}}{\partial \mathbf{r}_1} \tag{3}$$
$$= \frac{\partial V}{\partial \mathbf{r}} \tag{4}$$
$$\frac{\partial V}{\partial \mathbf{r}_2} = -\frac{\partial V}{\partial \mathbf{r}} \tag{5}$$

our equations are now

$$m_1 \frac{d^2\mathbf{r}_1}{dt^2} + \frac{\partial V}{\partial \mathbf{r}} = 0 \tag{6}$$
$$m_2 \frac{d^2\mathbf{r}_2}{dt^2} - \frac{\partial V}{\partial \mathbf{r}} = 0 \tag{7}$$

adding the equations, $M\frac{d^2\mathbf{R}}{dt^2} = 0$. Momentum is conserved if there are no external forces (or, by noether's theorem, if the system has translational symmetry)
now

$$m_1 m_2 \frac{d^2\mathbf{r}_1}{dt^2} + m_2 \frac{\partial V}{\partial \mathbf{r}} = 0 \tag{8}$$
$$m_2 m_1 \frac{d^2\mathbf{r}_2}{dt^2} - m_1 \frac{\partial V}{\partial \mathbf{r}} = 0 \tag{9}$$

subtract,

$$m_1 m_2 \frac{d^2\mathbf{r}}{dt^2} + (m_1 + m_2)\frac{\partial V}{\partial \mathbf{r}} = 0 \tag{10}$$
$$\mu \frac{d^2\mathbf{r}}{dt^2} + \frac{\partial V}{\partial \mathbf{r}} = 0 \tag{11}$$

2 Quantum