

Diagonalization and Powers of A

Let A be a $n \times n$ matrix with n linearly independent eigenvectors. I put the eigenvectors in the columns of a matrix S , the eigenvector matrix. What is $S^{-1}AS$?

1 First way

First, what is AS ? Remember that $S_i = x_i$, the columns of S , are eigenvectors.

$$AS = A \begin{bmatrix} | & | & | & \dots \\ x_1 & x_2 & x_3 & \dots \\ | & | & | & \dots \end{bmatrix} \quad (1)$$

$$= \begin{bmatrix} | & | & | & \dots \\ Ax_1 & Ax_2 & Ax_3 & \dots \\ | & | & | & \dots \end{bmatrix} \quad (2)$$

$$= \begin{bmatrix} | & | & | & \dots \\ \lambda_1 x_1 & \lambda_2 x_2 & \lambda_3 x_3 & \dots \\ | & | & | & \dots \end{bmatrix} \quad (3)$$

Let's factor it out

$$AS = \begin{bmatrix} | & | & | & \dots \\ x_1 & x_2 & x_3 & \dots \\ | & | & | & \dots \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \dots & \end{bmatrix} \quad (4)$$

So

$$AS = S\Lambda \quad (5)$$

$$S^{-1}AS = \Lambda \quad (6)$$

2 Second way

S is a similarity transformation into the eigenbasis, so Λ is A in a new basis. This tells us something important: in the eigenbasis, A is diagonal, very simple. $A \sim \Lambda$. If we are given a vector in the eigenbasis it is very easy to apply Λ ; just multiply each basis vector by the appropriate λ .

3 Example

If

$$Ax = \lambda x \tag{7}$$

$$A^2x = \lambda Ax \tag{8}$$

$$= \lambda^2 x \tag{9}$$

$$A = S\lambda S^{-1} \tag{10}$$

$$A^2 = S\lambda S^{-1}S\lambda S^{-1} \tag{11}$$

$$= S\lambda^2 S^{-1} \tag{12}$$

$$A^k = S\lambda^k S^{-1} \tag{13}$$

This representation gives us a great way to calculate and understand the powers of a matrix.

For any A defined as above,

$$A^k \rightarrow 0 \text{ as } k \rightarrow \infty \tag{14}$$

$$\text{iff all } \|\lambda_i\| < 1 \tag{15}$$

4 Niceness

A is sure to have n independent eigenvectors (and be diagonalizable) if the λ 's are all different (no repeated eigenvalues)

If there are repeated eigenvectors, I have to look more closely. For instance for I we have only one eigenvalue, 1, but every nonzero vector is an eigenvector.

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \tag{16}$$

$\lambda = 2, 2$. However there is only one line of eigenvectors (the nullspace of $A - 2I$ has dimension 1).

Algebraic multiplicity = 2, but geometric multiplicity = 1.

5 $u_{k+1} = Au_k$

$u_{k+1} = Au_k$, start with u_0 . $u_k = A^k u_0$.

Solution: decompose u_0 into eigenbasis, then go!

Example: fibonacci.

$$u_k = [F_{k+1}, F_k] \quad u_{k+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} u_k$$

$\lambda_1 = \frac{1}{2}(1 + \sqrt{5})$ dominates as k increases. We see the Fibonacci series grows as 1.618^k