CTCS

Sets

1

Let $h : W \to S$ be a function. Define a function $\mathrm{Hom}(h, T) : \mathrm{Hom}(S, T) \to \mathrm{Hom}(W, T)$ by $\mathrm{Hom}(h, T)(g) = g \circ h$. Show that if T has at least 2 elements, then h is surjective iif $\mathrm{Hom}(h, T)$ is injective.

$\implies$: Let h be surjective. We wish to show that $\mathrm{Hom}(h, T)$ is injective. Suppose $\mathrm{Hom}(h, T)(f) = \mathrm{Hom}(h, T)(g)$. Then

$$\mathrm{Hom}(h, T)(f) = \mathrm{Hom}(h, T)(g) \tag{1}$$
$$f \circ h = g \circ h \tag{2}$$

Let $y \in S$ be arbitrary. Since h is onto S, there exists x such that $y = h(x)$. Then

$$f(h(x)) = g(h(x)) \tag{3}$$
$$f(y) = g(y) \tag{4}$$

Since y was arbitrary, $f = g$. Hence $\mathrm{Hom}(h, T)$ is injective.

$\impliedby$: Let h be not surjective. We wish to show that $\mathrm{Hom}(h, T)$ is not injective. Since h is not surjective there exists $y \in S$ such that $y \neq h(x)$ for all x. Let f and g be functions both from XXX that agree on all values in their domain except that $f(y) \neq g(y)$. Note that they are different functions. However $\mathrm{Hom}(h, T)(f) = \mathrm{Hom}(h, T)(g)$ because $g \circ h = f \circ h$. Hence $\mathrm{Hom}(h, T)$ is not injective.

2a

Show that the mapping that takes a pair $(f : X \to S, g : X \to T)$ of functions to the function $\langle f, g\rangle : X \to S \times T$ defined by $\langle f, g\rangle(x) = \langle f(x), g(x)\rangle$ is a bijection from $\mathrm{Hom}(X, S) \times \mathrm{Hom}(X, T)$ to $\mathrm{Hom}(X, S \times T)$.

Surjective: we show that the range of the mapping is equal to the codomain, $\mathrm{Hom}(X, S \times T)$. Let $h \in \mathrm{Hom}(X, S \times T)$ be given. Then $h : X \to S \times T$. We construct f and g as follows. Let x be arbitrary. Then $h(x) \in S \times T$, so $h(x) = (a, b)$. Then let $f(x) = a, g(x) = b$.

Injective: Suppose $\langle f, g\rangle = \langle h, j\rangle$. Then for all x we have

$$\begin{aligned}
\langle f, g\rangle(x) &= \langle h, j\rangle(x) \\
(f(x), g(x)) &= (h(x), j(x)) \\
f(x) &= h(x) \\
g(x) &= j(x)
\end{aligned}$$

Since this is true of all x we have $f = h, g = j$

2b

If you set $X = S \times T$ in (a) what does $id_{S\times T}$ correspond to under the bijection?

Let $id_{S\times T} = \langle f, g\rangle$. Then

$$\begin{aligned}
id_{S\times T}(s, t) &= (s, t) \\
\langle f, g\rangle(s, t) &= (f(s), g(t)) \\
f(s) &= s \\
g(t) &= t
\end{aligned}$$

It corresponds to (id_S, id_T)

3a

Let S and T be disjoint sets. Let V be a set. Let $\phi : \text{Hom}(S, V) \times \text{Hom}(T, V) \to \text{Hom}(S \cup T, V)$ be the mapping that takes a pair $(f : S \to V, g : T \to V)$ to the function $\langle f|g\rangle : S \cup T \to V$ defined by $\langle f|g\rangle(x) = f(x)$ if $x \in S, g(x)$ if $x \in T$. Show that ϕ is a bijection.

Surjective: Let S, T be disjoint sets. Let $h \in \text{Hom}(S \cup T, V)$ be given. Then $h : S \cup T \to V$. Construct $f : S \to V, g : T \to V$ as follow: for each $s \in S$ let $f(s) = h(s)$, and for each $v \in V$ let $g(v) = h(v)$. Then $\langle f|g\rangle = h$.

Injective: Suppose $\langle f|g\rangle = \langle h|j\rangle$. For all $s \in S$ we have

$$\begin{aligned}
\langle f|g\rangle(s) &= \langle h|j\rangle(s) \\
f(s) &= h(s)
\end{aligned}$$

Hence $f = h$. Similarly, $g = j$.

3b

If you set $V = S \cup T$ in (a), what is $\phi^{-1}(id_{S\cup T})$?

Let $s \in S, t \in T$. Then $\langle id_S|id_T\rangle(s) = s, \langle id_S|id_T\rangle(t) = t$. Hence $\langle id_S|id_T\rangle = id_{S\cup T}$.

4a

If $P(C)$ denotes the powerset of C (all subsets of C), then $Rel(A,B) = P(A \times B)$ denotes the set of relations from A to B. Let $\phi : Rel(A,B) \to \mathrm{Hom}(A, P(B))$ be defined by $\phi(\alpha)(a) = \{b \in B | (a,b) \in \alpha\}$. Show that ϕ is a bijection.

Surjective: Let $h \in \mathrm{Hom}(A, P(B))$ be given. Then $h : A \to P(B)$ and for all $a \in A$ we have $h(a) \subseteq B$. Construct α as follow: $\alpha = \{(a,b) | a \in A, b \in h(a)\}$. Then

$$\begin{aligned}
\phi(\alpha)(a) &= \{b \in B | (a,b) \in \alpha\} \\
&= \{b \in B | (a,b) \in \{(a,b) | a \in A, b \in h(a)\}\} \\
&= \{b \in B | b \in h(a)\} \\
&= h(a)
\end{aligned}$$

as required.

Injective: suppose $\phi(X) = \phi(Y)$. Then for all $a \in A$,

$$\begin{gathered}
\phi(X) = \phi(Y) \\
\phi(X)(a) = \phi(Y)(a) \\
\{b \in B | (a,b) \in X\} = \{b \in B | (a,b) \in Y\}
\end{gathered}$$

so for all $a \in A$, for all $b \in B$,

$$\begin{aligned}
b \in \{b \in B | (a,b) \in X\} &\iff b \in \{b \in B | (a,b) \in Y\} \\
(a,b) \in X &\iff (a,b) \in Y
\end{aligned}$$

Hence $X = Y$.

4b

Let $A = B$. What corresponds to Δ_A under this bijection?

$$\begin{aligned}
\phi(\Delta_A)(a) &= \{b | (a,b) \in \Delta_A\} \\
&= \{b | a = b\} \\
&= \{a\}
\end{aligned}$$

It corresponds to the singleton function $f(a) = \{a\}$

4c

If we let $A = P(B)$ then $\phi^{-1} : \mathrm{Hom}(P(B), P(B)) \to Rel(P(B), B)$. What is $\phi^{-1}(id_{P(B)})$?

Let $\phi^{-1}(id_{P(B)}) = \alpha$. Let $s \subseteq B$. Then

$$
\begin{aligned}
\phi(\alpha)(s) &= \{b \in B | (s, b) \in \alpha\} \\
&= id_{P(B)}(s) \\
&= s \\
&= \{b \in B | b \in s\} \\
&= \{b \in B | (s, b) \in \{(s, b) | b \in s\}\}
\end{aligned}
$$

Hence α is the subset relation.