PROBLEM SET V
"NOWHERE NEAR THE FINAL FRONTIER"

DUE FRIDAY, 21 OCTOBER

Exercise 36. Suppose $s \in \mathbf{R}$. Show that the function $r_s \colon (0, \infty) \longrightarrow (0, \infty)$ given by $r_s(x) = x^s$ is continuous.

Exercise 37. Show that if $K \subset \mathbf{R}$ is compact, then there exists a real number $N > 0$ such that $K \subset [-N, N]$.

Exercise 38. Suppose $I_1 \supset I_2 \supset \cdots$ a sequence of closed (bounded) intervals. Show that the intersection $\bigcap_{k=1}^{\infty} I_k \neq \varnothing$.

Definition. The *diameter* of a bounded subset $E \subset \mathbf{R}$ is the real number

$$\operatorname{diam}(E) := \sup\{|x - y| \mid x, y \in E\}.$$

Exercise* 39. Suppose $K \subset \mathbf{R}$ a compact subset, and suppose $\{U_\alpha\}_{\alpha \in \Lambda}$ an open cover of K. Show that there exists a real number λ such that for any subset $V \subset K$ such that $\operatorname{diam}(V) < \lambda$, there exists an element $\alpha \in \Lambda$ such that $V \subset U_\alpha$.

Definition. Suppose $E \subset \mathbf{R}$. Then a map $f \colon E \longrightarrow \mathbf{R}$ is *uniformly continuous* if for any $\varepsilon > 0$, there exists a quantity $\delta > 0$ such that for any points $x_0, x_1 \in E$, if $|x_0 - x_1| < \delta$, then $|f(x_0) - f(x_1)| < \varepsilon$.

Exercise 40. Suppose $E \subset \mathbf{R}$. Show that a uniformly continuous map $E \longrightarrow \mathbf{R}$ is continuous, and show that the converse holds if E is compact.