# Differential Equations and exp(At)

Consider $\frac{du}{dt} = Au$

$A = \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix}$

$\lambda = 0, -3$

so for certain eigenvectors the equation becomes $\frac{du}{dt} = 0u$ and $\frac{du}{dt} = -3u$. The 0 solution is the steady state, the $-3$ one dies out.

$u(t) = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t}$

compare to the formula for difference equations

## 1 Cases

Stability, $u(\infty) = 0$ when all $\operatorname{Re}\lambda < 0$

Steady state, $\lambda_1 = 0$ and other eigenvectors have $\operatorname{Re}\lambda < 0$

Blowup if any $\operatorname{Re}\lambda > 0$

## 2 Special case

For $2 \times 2$ matrix:

$\mathrm{Tr} = a + d = \lambda_1 + \lambda_2 < 0$
$\mathrm{Det} = ad - bc = \lambda_1 \lambda_2 > 0$

positive determinant, negative trace.

## 3 S

Let's write the solution down in terms of $S$ and $\Lambda$. Eigenvalues uncouple them.

$\frac{du}{dt} = Au$

set $u = Av$

$S\frac{dv}{dt} = ASv$
$\frac{dv}{dt} = \Lambda v$

no coupling! Just a diagonal matrix.

$v(t) = e^{\Lambda t v(0)}$
$u(t) = Se^{\Lambda t}S^{-1}u(0) = e^{At}u(0)$

what do the matrix exponentials mean?

# 4 Power series

$e^{At} = I + At + \frac{(At)^2}{2!} + ...$
$(I - At)^{-1} = I + At + (At)^2 + ...$

the exponential always converges but if $A$ is too big (eigenvalue $> 1$) the second series will blow up.

now just write $A = S\Lambda S^{-1}$, we get $e^{At} = Se^{\Lambda t}S^{-1}$

# 5 Exponential of a diagonal matrix

$$e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & & \\ & e^{\lambda_2 t} & \\ & & ... \end{bmatrix}$$

this goes to zero when all $Re\lambda < 0$

stability for differential equation: $\lambda$ in left half plane. Powers of the matrix go to zero if $\lambda$ is contained in the unit disc.

# 6 Cheapskate trick

$y'' + by' + ky = 0$ $u = [y'y], u' = \begin{bmatrix} -b & -k \\ 1 & 0 \end{bmatrix} u$

in general, coefficients in he first row, and other rows have just one 1 among 0's.