1. **Gaussian wavepacket in free space.** (30 points).

A free particle of mass *m* has the initial wave function

$$\psi(x,t=0)=\frac{1}{(2\pi)^{1/4}w_0^{1/2}}e^{-\frac{x^2}{4w_0^2}},$$

where the initial width of the wavepacket w_0 is a real, positive constant. You have previously computed the Fourier transform of $\psi(x, 0)$ and shown that it equals:

$$\phi(k,t=0)=\frac{1}{(2\pi)^{1/4}k_0^{1/2}}e^{-\frac{k^2}{4k_0^2}},$$

with $k_0=1/(2w_0)$.
a) (15 points) Show that the wavefunction at a later time is given by:

$$\psi(x,t)=\frac{w_0^{1/2}}{(2\pi)^{1/4}\chi(t)}e^{-\frac{x^2}{4[\chi(t)]^2}},$$

where the complex quantity characterizing the wavepacket $\chi(t)$ at time *t* is given by

$$\chi(t)=w_0\sqrt{1+\frac{i\hbar t}{2mw_0^2}}.$$

(Use the fact that in free space the momentum eigenfunctions are also energy eigenfunctions.)
b) (5 points) Calculate $|\psi(x,t)|^2$, and show that it represents a Gaussian wavepacket. What is the time-dependent (real) width $w(t)$ of the wavepacket if we write the probability density in the form

$$|\psi(x,t)|^2=\frac{1}{(2\pi)^{1/2}w(t)}e^{-\frac{x^2}{2[w(t)]^2}}?$$

Does the width $w(t)$ at long times depend on the initial width w_0?
c) (10 points) Determine the time evolution of the wave function in the wavevector representation $\phi(k,t)$ (see above). Write down the probability density in wavevector space $|\phi(k,t)|^2$, and show that it does not spread in time. Plot $|\psi(x,t)|^2$ and $|\phi(k,t)|^2$, and explain in one or two sentences how it is possible that the wavefunction spreads in position space without spreading or shrinking in momentum space, if one is the Fourier transform of the other.

Useful formula:

$$\int_{-\infty}^{\infty}dx e^{-\alpha(x-\beta)^2}=\sqrt{\frac{\pi}{\alpha}}$$ for any complex α,β with $\mathrm{Re}(\alpha)>0$.

2. **Hermitian adjoint operators and Hermitian operators.** (25 points).
For an operator $\hat{A}$ we define the Hermitian adjoint operator $\hat{A}^{+}$ by the relation

$$\int_{-\infty}^{\infty}dx\left(\hat{A}^{+}\psi_1(x)\right)^*\psi_2(x)=\int_{-\infty}^{\infty}dx\psi_1^*(x)\hat{A}\psi_2(x),$$

for any two wavefunctions $\psi_1(x), \psi_2(x)$. Here the operator acts, as usual, on wavefunctions to its right, and the parentheses on the left-hand side of the equation indicate that the operator $\hat{A}^+$ acts only on $\psi_1(x)$.

a) (5 points) Show that the expectation value of the adjoint operator $\hat{A}^+$ in any state is related to the expectation value of $\hat{A}$ by $\langle \hat{A}^+ \rangle = \langle \hat{A} \rangle^*$.

b) (5 points) Show that for any two operators $\hat{A}, \hat{B}$, we have $(\hat{A}\hat{B})^+ = \hat{B}^+\hat{A}^+$.

c) (5 points) An operator is called self-adjoint or Hermitian if $\hat{A}^+ = \hat{A}$. Show that all eigenvalues of a Hermitian operator are real.

d) (5 points) Show that if $\hat{A}, \hat{B}$ are Hermitian operators, then the operator $\hat{A}\hat{B} + \hat{B}\hat{A}$ is also Hermitian.

e) (5 points) For a particle moving in one direction show that the operator $\hat{x}\hat{p}$ is not Hermitian. Construct an operator that corresponds to this physically observable product that is Hermitian.

3. **Commutators.** (20 points).

We define the commutator $[\hat{A}, \hat{B}]$ between two operators $\hat{A}, \hat{B}$ as the operator

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}.$$

a) (5 points) Show that for any three operators $\hat{A}, \hat{B}, \hat{C}$ the relations $[\hat{A} + \hat{B}, \hat{C}] = [\hat{A}, \hat{C}] + [\hat{B}, \hat{C}]$ and $[\hat{A}\hat{B}, \hat{C}] = \hat{A}[\hat{B}, \hat{C}] + [\hat{A}, \hat{C}]\hat{B}$ hold.

b) (5 points) Using either the position or the momentum representation, calculate the commutator $[\hat{x}, \hat{p}]$ between the position and momentum operators. (Remember that operators act on wavefunctions to the right.)

c) (5 points) Show that $[\hat{x}, \hat{p}^2] = 2i\hbar\hat{p}$.

d) (5 points) Two operators are said to commute if $[\hat{A}, \hat{B}] = 0$. Does the position operator commute with the Hamiltonian for a particle in free space? Does the momentum operator commute with the Hamiltonian for a particle in free space? We will show in class that if two operators commute, then we can construct a set of common eigenfunctions of both operators.

4. **Free particle.** (10 points)

A free particle moving in one dimension is in the state

$$\psi(x) = \int_{-\infty}^{\infty} dk \exp\left(-\frac{(ak)^2}{2} + ikx \right)(i \sin ak),$$ where a is a real constant.

a) (5points) What values of the momentum will not be found?

b) (5points) In which momentum state is the particle most likely to be found?

5. **Simultaneous eigenstates** (15 points).
Your friend from the previous year s 8.04 class argues the following: If a particle is in an eigenstate of a one-dimensional box of width a, then we know its energy exactly. We also know that the energy in the box is purely kinetic. Hence we know the particle s momentum exactly as well. This contradicts the Heisenberg uncertainty relation since the uncertainty in the particle position is finite ($\Delta x = a$). Punch a hole into your friend s argument.