First player sealed bid auctions

1 Two-player, uniform valuation

Two players take part in this auction and bid for an item. Each player has a private valuation $v_i \in [0,1]$ of the item (how much utility he would get by winning the item) and bids b_i for it. For this model, v_i is uniformly distributed on [0,1]. If the bids are equal, a coin is tossed to determine the winner.

The payoff for each player is $u_i = \begin{cases} v_i - b_i & \text{if } b_i > b_j \\ \frac{v_i - b_i}{2} & \text{if } b_i = b_j \\ 0 & \text{if } b_i < b_j. \end{cases}$

Note that this is a function of v_i, b_i and b_j.

We want to find a function $b_i = b_i(v_i)$ (given how much I value the item, how much should I bid?) that maximises P_i.

Let's calculate the expected payoff; the expected payoff is

$$P_i = (v_i - b_i)Prob\{b_i > b_j\} + \frac{v_i - b_i}{2}Prob\{b_i = b_j\}$$

We now *guess* that $b_j = a_j + c_j v_j$ (b is a linear function of v). Of course this means we can't guarantee that we find all Nash Equilibria, but we are just trying to find one. Don't worry, later on we will strengthen our proof to avoid having to guess this.

Then $Prob\{b_i = b_j\} = 0$ and since v_i is uniformly distributed on [0,1],

$$\begin{aligned} Prob\{b_i > b_j\} &= Prob\{b_i > a_j + c_j v_j\} \\ &= Prob\{\frac{b_i - a_j}{c_j} > v_j\} \\ &= \frac{b_i - a_j}{c_j} \end{aligned}$$

To be a Nash Equilibrium b_i must maximise P_i for all v_i.

$$P_i = \frac{(b_i - a_j)(v_i - b_i)}{c_j}$$

and the maximum occurs when

$$\begin{aligned} b_i &= \frac{v_i + a_j}{2} \\ b_i &= a_i + c_i v_i \end{aligned}$$

Since this must be true for all v_i we compare coefficients of v_i to conclude that

$$b_i = \frac{v_i}{2}$$

Players will try to bid *half* of what they think the item is worth.

2 Uniqeness of Nash Equilibria

Is the NE we just found unique? Yes, we can prove it is. Recall that we had to assume that b_i was a linear function of v_i; now we relax this to require only that b_i be a continuous, differentiable and increasing function of v_i.

Because it's increasing we can define b^{-1} such that $b(b^{-1}(v_i)) = v_i$. Then the optimality condition implies that

$$\begin{aligned} 0 &= \frac{\partial P_i}{\partial b_i} \\ &= \frac{\partial}{\partial b_i}(v_i - b_i)Prob\{b_i > b_j\} \\ &= \frac{\partial}{\partial b_i}(v_i - b_i)Prob\{v_i > v_j\} \\ &= -v_i + (v_i - b_i)\frac{dv_i}{db_i} \end{aligned}$$

solving,

$$\begin{aligned} \frac{db}{dv}v + b &= v \\ d(bv) &= vdv \\ bv &= \frac{1}{2}v^2 + C \end{aligned}$$

together with the initial condition that b(0) = 0,

$$b_i = \frac{v_i}{2}$$

As before.

3 n player, arbitrary valuation

Now we generalize further: now we do not know for sure that the competitor's valuations are uniformly distributed; instead let them be distributed such that

$$Prob\{b_j > b_i\} = F(b_i)$$

with $n-1$ other players, the probability you will be beaten is just $F^{n-1}(b_i)$
applying the same procedure as before, the differential equation we obtain is

$$\frac{db}{dv} = (v - b)(n-1)\frac{f(v)}{F(v)}$$

where $f(v) = \frac{dF(v)}{dv}$. As before we use the boundary conditions that b(0) = 0, and hence F(0) = 0. The solution to the differential equation is

$$b = s - \frac{\int_0^v F^{n-1}(v)\mathrm{d}v.}{F^{n-1}(v)}$$