# Eigenvectors

We have a matrix A. Sometimes A behaves very specially on a special kind of vector; it simply lengthens or shortens it. $x$ is parallel to $Ax$. In equations

$$Ax = \lambda x \tag{1}$$

$x$ is the eigenvector and $\lambda$ is the eigenvalue.

We can see that $x = 0$ always satisfies the equation for any $\lambda$. That's not nice, so we don't consider 0 to be an eigenvector.

Notice that if $x$ is an eigenvector, so is $kx$ for any $k$. Eigenvectors lie on (at least) a line.

For example, if $P$ is the projection matrix, we can find its eigenvectors by observation: everything that lies on the projection plane will get projected to itself.

## 1 Finding eigenvectors

There is a special case we can consider: $\lambda = 0$, zero eigenvalues. Then the equation becomes

$$Ax = 0 \tag{2}$$

so $\{x\} = Null(A)$

What about in general? The equation $Ax = \lambda x$ has two unknowns. Let us fix $\lambda$ and rewrite

$$\begin{aligned} Ax &= \lambda x &(3)\\ &= \lambda I x &(4)\\ (A - \lambda I)x &= 0 &(5) \end{aligned}$$

so $\{x\} = Null(A - \lambda I)$. We see that eigenvectors belonging to the same eigenvalue form a vector space (eigenspace), which explains the $kx$ observation above.

## 2 Permutation matrix

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{6}$$

A permutes two elements by swapping. We can see one eigenvector by observation: $x = [1, 1]\lambda = 1$. The other one is harder to spot; $x = [1, -1]\lambda = -1$. Notice that $\lambda_1\lambda_2 = -1$ is the determinant of A while $\lambda_1 + \lambda_2 = 0$ is the trace.

## 3 Finding $\lambda$

Now that we know how to find $x$ given $\lambda$, how do we find $\lambda$? Remember that $\{x\} = Null(A - \lambda I)$ and $x \neq 0$, we see that $A - \lambda I$ must be singular. So we have to solve the characteristic equation, a polynomial in $\lambda$, $det(A - \lambda I) = 0$. In general it is an n-th order equation for an n-dimensional matrix.

Another way to derive this is to see that if $A \to A + 3I$ the eigenvalues increase by 3 and the eigenvectors remain unchanged. Since we know how to find eigenvectors of a singular matrix we bring $A$ to a singular matrix in order to find their eigenvectors.

## 4 Good Example

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \tag{7}$$

$$A - \lambda I = \begin{bmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{bmatrix} \tag{8}$$

$$det = \lambda^2 - 6\lambda + 8 \tag{9}$$

$$= (\lambda - 4)(\lambda - 2) \tag{10}$$

Notice that $Tr(A) = 6$, $det(A) = 8$ both appear in the characteristic polynomial.

$$A - 2I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \tag{11}$$

$$x = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \tag{12}$$

$$A - 4I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \tag{13}$$

$$x = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{14}$$