MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Spring 2006 | Physics 8.04 | Vuletic
Assignment #6 | page 1 of 2
Due Thursday, April 6 at 4pm

1. **Time evolution of wavefunction in box potential** (40 points).

Consider a particle of mass m inside a box of size a with infinite walls,

$$V(x) = \begin{cases} 0 & \text{for } 0 \le x \le a \\ \infty & \text{elsewhere} \end{cases}.$$

We want to determine the time evolution of a wavefunction that is specified at t=0.
Assume that $\psi(x,t=0) = C(2\sin kx + 3\sin 2kx + \sin 3kx)$, where $k = \pi / a$.

a) (5 points) Determine the normalization coefficient C so that $|\psi(x,t)|^2$ can be interpreted as a probability density.
b) (10 points) Expand the wavefunction at the initial time $\psi(x,t=0)$ in terms of the eigenfunctions $u_n(x)$ of the infinite box, i.e. determine the coefficients $c_n = \int_{-\infty}^{\infty} u_n^*(x)\psi(x,0)dx$, so that you can write $\psi(x,0)$ as a superposition of eigenstates of the infinite box.
c) (5 points) Using the known time evolution of eigenstates (negative phase factor evolving at angular frequency $\omega_n = E_n / \hbar$, where E_n is the eigenenergy of the state), find $\psi(x,t)$ an arbitrary later time t. [If you have Matlab or Mathematica at hand, you can make a movie of $|\psi(x,t)|^2$.]
d) (5 points) Is the motion periodic, is there a time T with $\psi(x,0) = \psi(x,T)$?
e) (5 points) If a measurement of the particle's energy is performed, what will be the outcome (outcomes) and with what probability will those values be measured?
f) (5 points) What is the average energy of the particle in the box? Does it change with time if no measurements are performed?
g) (5 points) Does the particle's energy change if an energy measurement is performed? Comment on energy conservation.

2. **Diabatic (sudden) expansion of infinite box.** (25 points).
A particle of mass m is prepared in the ground state of an infinite-potential box of size a extending from x=0 to x=a. Suddenly, the wall at x=a is moved to x=2a within a time Δt doubling the box size. You may assume that the wavefunction is the same immediately after the change, if the change happens fast enough.

a) (5 points) How fast is fast enough? ($\Delta t=0$ is a mathematician's answer, and not good enough for a physicist.)
b) (5 points) What is the probability that the particle is in the second (n = 2) state of the new well, immediately after the change? (Note that the wavelength within the well, and hence the energy, for this state is the same as for the initial state in the old well.) Make sure that you use properly normalized wavefunctions for your calculations.
c) (5 points) What is the probability that the particle would be found in the ground state of the new well?
d) (5 points) What is the expectation value of the energy of the particle before and after the sudden expansion?

e) (5 points) If you wanted to ensure that the particle has unity probability to be found in the ground state of the new potential, how would you change the potential (time scale)?

a) and e) do not require calculations, but a sentence explaining your answer.

3. **Hermitian operators** (15 points).
A Hermitian operator is an operator that has only real eigenvalues.

a) (5 points) Show that the Hamiltonian $H = \frac{p^2}{2m} + V(x)$ is a Hermitian operator.

b) (5 points) Show that the expectation value $\langle H \rangle$ of the Hamiltonian operator in an energy eigenstate is equal to the eigenenergy of that state.

c) (5 points) Show that the position operator in momentum space $x = i\hbar \frac{\partial}{\partial p}$ is a Hermitian operator.

2. **Square well centered at origin, parity** (20 points)
Solve the time-independent Schrödinger equation with appropriate boundary conditions for an infinite square well of width *a* centered at *x*=0, i.e.

$$V(x) = \begin{cases} 0 & \text{for } -a/2 \le x \le a/2 \\ \infty & \text{elsewhere} \end{cases}.$$

a) (10 points) Check that the allowed energies are consistent with those derived in lecture for an infinite well of width a centered at $x=a/2$. Confirm that the wave functions $\psi(x)$ can be obtained from those found in lecture if one uses the substitution $x \to x-a/2$.

b) b) (5 points) We define a parity operator P by $P\psi(x) = \psi(-x)$. Verify that the energy eigenstates for the above potential are also eigenstates of the parity operator. What are the corresponding eigenvalues? Is the same true for the eigenstates of the well centered at $x=a/2$?

c) (5 points) Find a general condition either on the potential $V(x)$ or on the Hamiltonian that ensures that the energy eigenstates are simultaneously eigenstates of the parity operator.