PROBLEM SET II
"HIC SUNT DRACONES."

DUE FRIDAY, 23 SEPTEMBER

Welcome to the wild world of functions. In this problem set, you'll meet some very strange and wonderful functions.

We'll begin with a little function that shows the immense descriptive power of recursion.

Definition. Define a function $h\colon \mathbf{N}\times\mathbf{N}\times\mathbf{N}\longrightarrow\mathbf{N}$ in the following manner:

$$h(n,p,q) := \begin{cases} q+1 & \text{if } n=0; \\ p & \text{if } n=1 \text{ and } q=0; \\ 0 & \text{if } n=2 \text{ and } q=0; \\ 1 & \text{if } n\geq 3 \text{ and } q=0; \\ h(n-1,p,h(n,p,q-1)) & \text{otherwise.} \end{cases}$$

Let's take a moment to unpack this. Let's look at $h(n,p,q)$ for fixed values of n.

Exercise 10. Show that we have the following equations for any $p,q\in\mathbf{N}$.

$$h(0,p,q)=q+1, \qquad h(1,p,q)=p+q, \qquad h(2,p,q)=pq, \quad \text{and} \quad h(3,p,q)=p^q.$$

Let's do some quick computations of some of the numbers $h(n,p,q)$ to get a feeling for the growth of this function.

Exercise 11. Compute explicitly the numbers $h(4,2,4)$ and $h(5,3,2)$. Show that for any positive integer n, one has $h(n,2,2)=4$.

Now let's assemble this function of three variables into a function of a single variable.

Definition. Define a function $a\colon \mathbf{N}\longrightarrow\mathbf{N}$ in the following manner

$$a(n) := h(n,n,n).$$

Thus

$$a(0)=1, \qquad a(1)=2, \qquad a(2)=4 \qquad a(3)=27, \quad \text{and} \quad a(4)=4^{4^{4^4}}=4^{4^{256}}.$$

The next term, $a(5)$, is a number so large that it cannot be written it down without new notation.

Exercise 12. Show that the function a is injective.

Exercise 13. For any natural number n, let B_n be the set of natural numbers m such that $a(m)\leq n$. Show that B_n is a finite set. Now for any natural number n, we write

$$b(n) := \begin{cases} 0 & \text{if } B_n=\varnothing; \\ \max(B_n) & \text{otherwise.} \end{cases}$$

This defines a function $b\colon \mathbf{N}\longrightarrow\mathbf{N}$. Compute $b(n)$ for $0\leq n\leq 2000$. Despite this, show that b *increases without bound*; that is, show that for any natural numbers $r\leq s$, one has $b(r)\leq b(s)$, and for any natural number N, there exists a natural number n such that $b(n)>N$.

Imagine making a table with the values of b. You'd see that the values were increasing unbelievably slowly. If you made an entry every nanosecond since the beginning of time, you *still* wouldn't have gotten to an n large enough to give you $b(n)=4$. You might even be tempted to conclude that the numbers would never get above 3. Nevertheless, the previous exercise shows you that the numbers are in fact increasing without bound.

Definition. The *Cantor set* $C \subset \mathbf{R}$ is defined in the following manner. Set $C_0 := [0,1]$. Proceed iteratively in the following manner. For every natural number $n > 0$, set

$$M_n := \bigcup_{k=0}^{3^{n-1}-1} \left(\frac{3k+1}{3^n}, \frac{3k+2}{3^n} \right) \qquad \text{and} \qquad C_n := C_{n-1} - M_n.$$

Finally, set

$$C := \bigcap_{n \in \mathbf{N}} C_n.$$

Now let $\mathrm{Map}(\mathbf{N}_+, \{0,1\})$ denote the set of all maps $\mathbf{N}_+ \longrightarrow \{0,1\}$, where $\mathbf{N}_+ = \{n \in \mathbf{N} \mid n \geq 1\}$. Let us define a map $f: C \longrightarrow \mathrm{Map}(\mathbf{N}_+, \{0,1\})$.

We define, recursively, a map $f_n: C_n \longrightarrow \mathrm{Map}(\{1,2,\dots,n\},\{0,1\})$ for every $n \in \mathbf{N}$. For $n = 0$, there's nothing to do, since there is only one map $\varnothing \longrightarrow \{0,1\}$. Now suppose the map f_{n-1} is defined; then set, for any point $x \in C_n$ and any integer $1 \leq m \leq n$,

$$f_n(x)(m) := \begin{cases} f_{n-1}(x)(m) & \text{if } 1 \leq m \leq n-1; \\ 0 & \text{if } m = n \text{ and if for some integer } k,\ x \in \left[\frac{3k}{3^n}, \frac{3k+1}{3^n}\right]; \\ 1 & \text{if } m = n \text{ and if for some integer } k,\ x \in \left[\frac{3k+2}{3^n}, \frac{3(k+1)}{3^n}\right]; \end{cases}$$

this defines our map $f_n: C_n \longrightarrow \mathrm{Map}(\{1,2,\dots,n\},\{0,1\})$. Now we define the map $f_n: C \longrightarrow \mathrm{Map}(\mathbf{N}_+, \{0,1\})$ by the formula $f(x)(m) := f_n(x)(m)$ for any integer $n \geq m$.

Exercise 14. Show that the function $f: C \longrightarrow \mathrm{Map}(\mathbf{N}_+, \{0,1\})$ is a bijection.

The so-called *indicator function* of the Cantor set is Riemann-integrable:

Exercise 15. Show that the function $f: [0,1] \longrightarrow [0,1]$ defined by the formula

$$f(x) := \begin{cases} 0 & \text{if } x \notin C; \\ 1 & \text{if } x \in C \end{cases}$$

is Riemann integrable. What is the integral $\int_0^1 f(x)\, dx$?

The rational numbers may seem like a tamer set than the Cantor set, but the *indicator function* of $\mathbf{Q}$ is *not* Riemann-integrable:

Exercise 16. Show that the function $d: \mathbf{R} \longrightarrow [0,1]$ defined by the formula

$$d(x) := \begin{cases} 0 & \text{if } x \notin \mathbf{Q}; \\ 1 & \text{if } x \in \mathbf{Q} \end{cases}$$

is not Riemann integrable on $[0,1]$.

Strangely, the even crazier-seeming popcorn function is Riemann-integrable:

Exercise 17. Show that the function $\theta: \mathbf{R} \longrightarrow [0,1]$ defined by the formula

$$\theta(x) := \begin{cases} 0 & \text{if } x \notin \mathbf{Q}; \\ 1/q & \text{if } x = p/q, \text{ where } p,q \in \mathbf{Z},\ \gcd(p,q) = 1 \text{ and } q \geq 1. \end{cases}$$

is Riemann integrable on $[0,1]$. What is the integral $\int_0^1 \theta(x)\, dx$?