http://web.mit.edu/jlee08/Public/7.91/8.04/sol4.pdf

1 Continuity equation for probability density and probability current

a)

$$\begin{aligned}
\psi_t &= i\frac{\hbar}{2m}\psi_x - i\frac{V}{\hbar}\psi \\
\frac{\partial}{\partial t}\psi\psi^* &= \psi\psi_t^* + \psi^*\psi_t \\
&= \psi(-i\frac{\hbar}{2m}\psi_{xx}^* + i\frac{V}{\hbar}\psi^*) + \psi^*(i\frac{\hbar}{2m}\psi_x - i\frac{V}{\hbar}\psi) \\
&= \psi(-i\frac{\hbar}{2m}\psi_{xx}^*) + \psi^*(i\frac{\hbar}{2m}\psi_{xx}) \\
&= i\frac{\hbar}{2m}(\psi^*\psi_{xx} - \psi\psi_{xx}^*)
\end{aligned}$$

$$\begin{aligned}
J_x &= \frac{\hbar}{2im}(\psi_x^*\psi_x + \psi^*\psi_{xx} - \psi_x^*\psi_x - \psi_{xx}^*\psi_x) \\
&= -i\frac{\hbar}{2m}(\psi^*\psi_{xx} - \psi_{xx}^*\psi_x)
\end{aligned}$$

b) $P_t(a,b) = J(a) - J(b)$

2 Fictitious Bohr Atom

Use the quantization condition $L = n\hbar$

$$\begin{aligned}
V &= -C_6 r^{-6} \\
F_r &= 6C_6 r^{-7} = \frac{m_e v^2}{r} \\
v &= \sqrt{\frac{6C_6}{m_e}}r^{-3} \\
L &= mvr = \sqrt{6m_eC_6}r^{-2} = n\hbar \\
r^2 &= \frac{\sqrt{6m_eC_6}}{n\hbar} \\
v^2 &= \frac{6C_6}{m_e}(\frac{n\hbar}{\sqrt{6m_eC_6}})^3 \\
\frac{1}{2}m_ev^2 &= 3C_6(\frac{n\hbar}{\sqrt{6m_eC_6}})^3 = \frac{n^3\hbar^3}{2\sqrt{6C_6}m_e^{3/2}} \\
\frac{C_6}{r^6} &= C_6(\frac{n\hbar}{\sqrt{6m_eC_6}})^3 = \frac{n^3\hbar^3}{6\sqrt{6C_6}m_e^{3/2}} \\
E_n &= \frac{1}{2}m_ev^2 - \frac{C_6}{r^6} = \frac{n^3\hbar^3}{3\sqrt{6C_6}m_e^{3/2}}
\end{aligned}$$

3 Sommerfeld-Wilson quantization for linear potential in one dimension

We will work over $T/4$.

a) $V = C|x|, p(t) = -Ct, x(t) = A - \frac{C}{2m}t^2$

b) $CA = p^2/2m + Cx$

$p = \sqrt{2mc(A-x)}$

$1/4 \int pdx = \int_0^A \sqrt{2mc(A-x)}dx = \sqrt{2mCA^3}2/3$

$\int = \frac{8}{3}\sqrt{2mCA^3}$

c) $\frac{8}{3}\sqrt{2mCA^3} = nh$

$A = (\frac{9n^2h^2}{128mC})^{1/3}$

d) $E_n = CA = (\frac{9n^2h^2C^2}{128m})^{1/3}$

4 Momentum expectation values in terms of spatial wavefunctions

5 Relations for probability current