Differential Equations and exp(At)

Consider $\frac{du}{dt} = Au$

$A = \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix}$

$\lambda = 0, -3$

so for certain eigenvectors the equation becomes $\frac{du}{dt} = 0u$ and $\frac{du}{dt} = -3u$. The 0 solution is the steady state, the -3 one dies out.

$u(t) = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t}$

compare to the formula for difference equations

1 Cases

Stability, $u(\infty) = 0$ when all $Re\lambda < 0$

Steady state, $\lambda_1 = 0$ and other eigenvectors have $Re\lambda < 0$

Blowup if any $Re\lambda > 0$

2 Special case

For 2×2 matrix:

$a + d = \lambda_1 + \lambda_2 < 0$ $det > 0$

positive determinant, negative trace.

3 S

Let's write the solution down in terms of S and Λ. Eigenvalues uncouple them.

$\frac{du}{dt} = Au$

set $u = Av$

$S\frac{dv}{dt} = ASv$

$\frac{dv}{dt} = \Lambda v$

no coupling! Just a diagonal matrix.

$v(t) = e^{\Lambda t v(0)}$

$u(t) = Se^{\Lambda t}S^{-1}u(0) = e^{At}u(0)$

what do the matrix exponentials mean?

4 Power series

$e^{At} = I + At + \frac{(At)^2}{2!} + ...$

$(I - At)^{-1} = I + At + (At)^2 + ...$

the exponential always converges but if A is too big (eigenvalue ¿ 1) the second series will blow up.

now just write $A = S\Lambda S^{-1}$, we get $e^{At} = Se^{\Lambda t}S^{-1}$

5 Exponential of a diagonal matrix

$$e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & & \\ & e^{\lambda_2 t} & \\ & & ... \end{bmatrix}$$

this goes to zero when all $Re\lambda < 0$

stability for differential equation: λ in left half plane. Powers of the matrix go to zero if λ is contained in the unit disc.

6 Cheapskate trick

$y'' + by' + ky = 0$ $u = [y'y], u' = \begin{bmatrix} -b & -k \\ 1 & 0 \end{bmatrix} u$