MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Due Thursday, May 11 at 4pm

1. **Operators and the HO** (30 points)

Let $|0\rangle$ be the normalized ground state of the harmonic oscillator, defined by $\hat{a}|0\rangle = 0$ and $\langle 0|0\rangle = 1$, where $\hat{a}$ is the lowering (or annihilation) operator and $\hat{a}^+$ is the raising (or creation) operator.

a) (5 points) Show that the length of the unnormalized *n*-th eigenstate $|\tilde{n}\rangle$, defined by $|\tilde{n}\rangle = (a^+)^n |0\rangle$, is given by $\sqrt{n!}$.

b) (5 points) As a consequence of a), the normalized *n*-th eigenstate $|n\rangle$ is given by $|n\rangle = \frac{1}{\sqrt{n!}}(a^+)^n|0\rangle$. Verify that the eigenstates are orthonormal, i.e. show that $\langle n|m\rangle = \delta_{nm}$.

c) (5 points) Show that $\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$ and $\hat{a}^+|n\rangle = \sqrt{n+1}|n+1\rangle$.

d) (5 points) Define an operator $\hat{n}$ by $\hat{n} \equiv \hat{a}^+\hat{a}$. Show that $\hat{n}$ is a Hermitian operator. Consequently, $\hat{n}$ represents a measurable quantity. Calculate $\hat{n}|n\rangle$. What is the physical quantity that $\hat{n}$ represents? What are the eigenvalues and eigenstates of $\hat{n}$?

e) (5 points) Express $\hat{x}$ and $\hat{p}$ in terms of $\hat{a}$ and $\hat{a}^+$. Calculate $\langle m|\hat{x}|n\rangle$ and $\langle m|\hat{p}|n\rangle$, and show that they vanish unless $m = n \pm 1$.

f) (5 points) Express $\hat{x}^2$ and $\hat{p}^2$ in terms of $\hat{a}$ and $\hat{a}^+$. Show that in the *n*-th eigenstate of the harmonic oscillator the uncertainty product of *x* and *p* is given by $\Delta x \Delta p = \frac{\hbar}{2}(2n+1)$.

2. **Coherent states of the HO** (40 points)

A state $|\alpha\rangle$ that obeys the eigenequation $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$ with an arbitrary complex number α, is called a coherent state.

a) (5 points) Show that $\hat{a}(\hat{a}^+)^n|0\rangle = n(\hat{a}^+)^{n-1}|0\rangle$.

b) (10 points) Show that the coherent state may be written in the form $|\alpha\rangle = C\exp(\alpha\hat{a}^+)|0\rangle$, where *C* is a normalization constant. The operator $\exp(\alpha\hat{a}^+)$ is defined in terms of its Taylor expansion.

c) (5 points) Calculate the normalization constant *C*.

d) (10 points) Expand the coherent state in terms of the normalized eigenstates $|n\rangle$, and calculate the probability of finding the HO in the *n*-th eigenstate (or equivalently, of finding *n* quanta in the system). You should obtain the Poisson distribution.

e) (10 points) Calculate the average number of quanta $\langle n\rangle = \langle\alpha|\hat{n}|\alpha\rangle$ in the coherent state.

The coherent states are the closest quantum mechanical analog to classical states with a well-defined amplitude and phase. They are used, e.g., in the quantum mechanical description of the light field emitted by a laser or radiofrequency oscillator. The coherent states are also the appropriate states for describing a classical harmonic oscillator with a large average quantum number $\langle n \rangle$.

3. **Projection operator** (10 points)

Let $|n\rangle$ denote the eigenstate of a Hermitian operator. The operator $\hat{P}_n = |n\rangle\langle n|$ is called the projection operator. Show that $\hat{P}_m \hat{P}_n = \delta_{mn} \hat{P}_n$. Using the expansion of an arbitrary state into eigenstates $|n\rangle$ of a Hermitian operator, show that $\sum_n \hat{P}_n = \hat{1}$, where $\hat{1}$ is the unity operator.

4. **Particle in angular momentum eigenstate** (20 points)

A particle is in an eigenstate $|l,m\rangle$ of L^2 and L_z.
a) (10 points) Show that in this case $\langle L_x \rangle = \langle L_y \rangle = 0$.
b) (10 points) Show that $\langle L_x^2 \rangle = \langle L_y^2 \rangle = \frac{\hbar^2 l(l+1) - \hbar^2 m^2}{2}$.
Hints: For part a), use L_+ and L_-. For part b), use $\mathbf{L}^2 = L_x^2 + L_y^2 + L_z^2$.