# The rainbow

## 1 The maximal diameter of a spherical water droplet

We use spherical coordinates for this section, with the symbols $r$ for distance, $\theta$ for polar angle and $\phi$ for azimuthal angle. Since $r$ does not vary across our calculations, we set it to be $r = D/2$.

### 1.1 Air resistant force

The upward component of the force on a small section of area $dS$ is $f dS \sin\theta$. Hence the total upward force is

$$\begin{aligned} F_f &= \iint f dS \sin\theta \\ &= \iint f r^2 \sin\theta d\theta d\phi \sin\theta \\ &= r^2 f \int_0^{\pi} \sin^2\theta d\theta \int_0^{2\pi} d\phi \\ &= r^2 f \frac{\pi}{2} 2\pi \\ &= \pi^2 r^2 f \end{aligned}$$

For constant speed this must balance the gravitational force

$$F_g = \rho(\frac{4}{3}\pi r^3)g$$

Hence

$$\begin{aligned} \pi^2 r^2 f &= \rho(\frac{4}{3}\pi r^3)g \\ \pi f &= \rho(\frac{4}{3} r)g \\ f &= \frac{4\rho r g}{3\pi} \\ &= \frac{2\rho D g}{3\pi} \end{aligned}$$

### 1.2 Horizontal component of air resistant force

The horizontal component of the air resistant force is

$$
\begin{aligned}
F_a &= \iint f dS \cos\theta \sin\phi \\
&= \iint f r^2 \sin\theta d\theta d\phi \cos\theta \sin\phi \\
&= r^2 f \int_{\pi}^{\frac{\pi}{2}} \sin\theta\cos\theta d\theta \int_0^{\pi} \sin\phi d\phi \\
&= r^2 f \\
&= \frac{D^2}{4}\frac{2\rho D g}{3\pi} \\
&= \frac{\rho D^3 g}{6\pi}
\end{aligned}
$$

### 1.3 Surface tension

The area of contact has a length $L = \pi D/2$; hence $F_t = \sigma L$

$$F_t = \frac{\sigma\pi D}{2}$$

### 1.4 Spherical water droplet

$$
\begin{aligned}
F_t &= 100 F_a \\
\frac{\sigma\pi D}{2} &= 100\frac{\rho D^3 g}{6\pi} \\
\sigma\pi &= 100\frac{\rho D^2 g}{3\pi} \\
D &= \pi\sqrt{\frac{3\sigma}{100\rho g}} \\
&= 0.00277 \text{ m}
\end{aligned}
$$

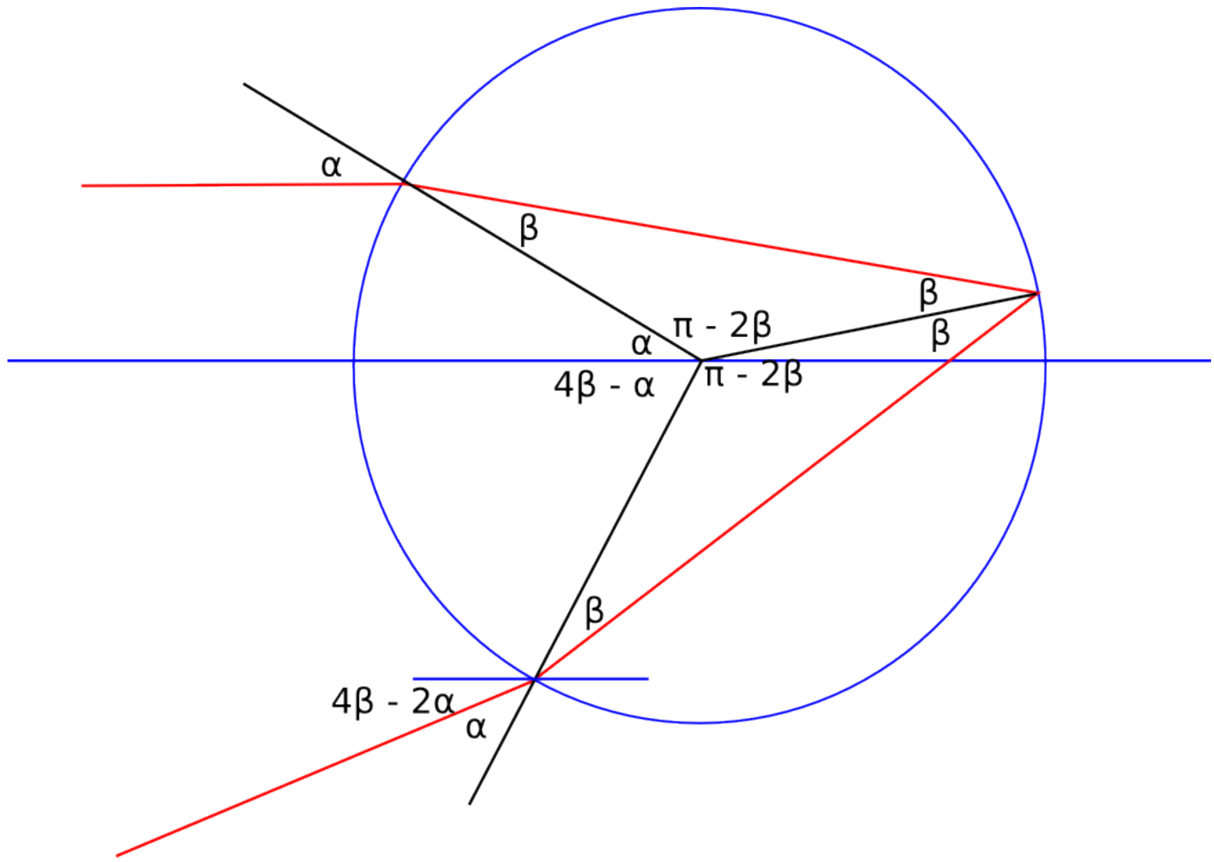
The maximum diameter is 0.00277 m.

## 2 Refraction and reflection of light rays in a spherical water droplet

### 2.1 Angle of reflected wave

Let $\beta$ be the refracted angle; then

$$
\begin{aligned}
\sin\alpha &= n\sin\beta \\
\beta &= \sin^{-1}(\frac{1}{n}\sin\alpha)
\end{aligned}
$$




$$\begin{aligned} \theta &= 4\beta - 2\alpha \\ &= 4\sin^{-1}(\frac{1}{n}\sin\alpha) - 2\alpha \end{aligned}$$

[plot 4 * asin(0.74951 * sin(x)) - 2*x]

## 2.2 Optical power distribution

Let $x$ and $y$ be the coordinates of the incident beam, as seen when staring into the beam. Then

$$\Delta P = I_0 \Delta x \Delta y$$

a fraction $T_1 T_2 R$ of the beam is transmitted to the outgoing wave; hence we must relate $\Delta x$ and $\Delta y$ to $\Delta\theta$ and $\Delta\phi$. Now

$$\theta = 4\sin^{-1}(\frac{1}{n}\sin\alpha) - 2\alpha$$

$$\Delta\theta = \left( \frac{4\cos\alpha}{n\sqrt{1 - (\frac{1}{n}\sin\alpha)^2}} - 2 \right) \Delta\alpha$$

and

$$\begin{aligned} r\sin\alpha &= y \\ r\cos\alpha\Delta\alpha &= \Delta y \\ \Delta\alpha &= \frac{\Delta y}{r\cos\alpha} \end{aligned}$$

so

$$\Delta\theta = \left( \frac{4}{rn\sqrt{1 - (\frac{1}{n}\sin\alpha)^2}} - \frac{2}{r\cos\alpha} \right) \Delta y$$

also, if we displace the beam a bit to the left or right (ie add $\Delta x$), $\alpha$ does not change to first order because the tangent to the circle of constant $\alpha$ is parallel to the direction of $\Delta x$ when $x$ is 0, as is the case here.

$$\Delta x = \frac{r\sin\theta\sin\alpha}{\sin(4\beta - \alpha)} \Delta\phi$$

Hence

$$\Delta P = I_0 \frac{\frac{r\sin\theta\sin\alpha}{\sin(4\beta-\alpha)}\Delta\phi}{\left( \frac{4}{rn\sqrt{1-(\frac{1}{n}\sin\alpha)^2}} - \frac{2}{r\cos\alpha} \right)} \Delta\theta$$

Therefore

$$\begin{aligned} J &= I_0 \frac{\frac{\sin\theta\sin\alpha}{\sin(4\beta-\alpha)}}{\left( \frac{4}{n\sqrt{1-(\frac{1}{n}\sin\alpha)^2}} - \frac{2}{\cos\alpha} \right)} T_1 T_2 R \\ &= I_0 \frac{\sin(4\sin^{-1}(\frac{1}{n}\sin\alpha) - 2\alpha)\sin\alpha}{\sin(4\sin^{-1}(\frac{1}{n}\sin\alpha) - 2\alpha)\left( \frac{4}{n\sqrt{1-(\frac{1}{n}\sin\alpha)^2}} - \frac{2}{\cos\alpha} \right)} T_1 T_2 R \end{aligned}$$

### 2.3 $\lambda = 550nm$

[plot (sin(4*asin(0.74951 * sin(x))-2*x) * sin(x))/(sin(4 * asin(0.74951 * sin(x)) - 2*x) * (4/(1.3342 * sqrt(1 - (0.74951 * sin(x))**2)) - 2/(cos(x))))]

When we plot it, we see that it diverges at $\alpha = 1.04$; indeed, for this angle $\Delta\theta = 0$.

### 2.4 Spectral Intensity

## 3 Basic characteristics of the rainbow