18.014 pset 1

A is Dedekind-infinite: $\exists\ A' \subset A$ such that $A \leftrightarrow A'$

1 Any subset of a Dedekind...

Given $A' \subseteq A$
Show that A is D-finite $\implies A'$ is D-finite
It suffices to show that A' is D-infinite $\implies A$ is D-infinite
Since A' is D-infinite $\exists A'' \subset A'$ and $A'' \leftrightarrow A'$
Call the bijection between A' and A'' f
Construct a bijection g such that
$g(x) = f(x)$ when $x \in A'$
$g(x) = x$ otherwise
g bijects A to $(A - A') \cup A''$
hence A is D-infinite

2 Show that the following are equivalent

1) Any injection $A \rightarrow A$ is a surjection
2) A is D-finite
3) There is no injection $i : N \rightarrow A$
First, $\neg 2 \implies \neg 1$. Also $\neg 1 \implies \neg 2$.
Proof: By definition
Next, $\neg 3 \implies \neg 2$
Let $R(N) \subseteq S$ be the range of i on N. i is now a bijection $N \leftrightarrow R(N)$
Since the given i is an injection, it has an inverse i^{-1}
Consider the injection: $x \in R(N) \rightarrow i(2i^{-1}(x))$
This injects $R(N)$ to $R(2N)$
And because $2N \subset N, R == N, R(2N) \subset R(N)$
Hence R is D-infinite and so is its parent set S.
Lastly, $\neg 2 \implies \neg 3$‘
Call the bijection p
Lemma: if X is D-infinite $p(X)$ is also D-infinite.
Proof: $p(X)$ bijects with $p(X') \subset p(X)$
So $p^n(A)$ is D-infinite for all n
Furthermore $p^n(A) - p^{n+1}(A)$ is nonempty because $p(p^n(A)) \subset p^n(A)$
Then map n to an element in $p^n(A) - p^{n+1}(A)$. Done!

3 Show that the union of two D-finite sets is D-finite

A finite and B finite $\implies A \cup B$ D-finite
We'll show $A \cup B$ D-infinite $\implies A$ D-infinite or B D-infinite
There is an injection i from $N \rightarrow A \cup B$
Consider the sets $N_A = i^{-1}(A)$ and $N_B = i^{-1}(B)$.
$N_A \cup N_B = N$
We will show that one of them must be bounded.
If N_A is bounded by m_A and N_B is bounded by m_B, N is bounded by $\max(m_A, m_B)$. Clearly wrong.
So S, which is A or B, is unbounded.

Consider the injection j defined by $0 \to$ smallest element of S
And $n \to$ smallest element of S larger than $j(n-1)$; this exists because S is unbounded
Hence $N \to S$, so S is infinite.

4 Roots

Use PNT

5 Find at least 3 diophantine approximants to $\sqrt{2}$

$1/1, 3/2, 7/5, 10/7, 17/12$

6 $1 < s < t \implies D(x,s) \supset D(x,t)$

$1/n^t < 1/n^s$

7 For any irrational x, $|D(x,1)| = \infty$

measure in units of $1/n$; then $m/n < x < (m+1)/n$.

8 If x is rational then $D(x,s) < \infty$ for $s > 1$

Let $s = 1+\epsilon$. $|\frac{a}{b} - \frac{m}{n}| = \frac{|an-bm|n^\epsilon}{n^s}$.

$$\frac{n^\epsilon}{bn^s} < \frac{|an-bm|n^\epsilon}{bn^s} < \frac{1}{n^s}$$

$$n^\epsilon < b$$

This fails when $n > b^{1/\epsilon}$; since there are a finite number of n and for each n a finite number of m, we are done

9 Liouville numbers are transcedental

We must prove that infinite irrationality exponent $\implies$ not a solution to polynomial equation
Solution to polynomial equation $\implies$ there exists n, c such that for all p, q, $|x - \frac{p}{q}| > \frac{c}{q^n}$. Let $c = 1/k$,

$$|x - \frac{p}{q}| > \frac{1}{kq^n} > \frac{1}{q^{n'}}$$

for some n' that depends on q. Hence for all q there exists n' such that $|x - \frac{p}{q}| > \frac{1}{q^{n'}}$ which means there are no n' or above approximants with denominator q. $n'(q) = n + 1/\ln_k q$ so the $q - n'$ plot cuts off everything to the right of some n'.