Loaded Hoop on a conveyer belt

1 Preliminary calculations

1.1 Acceleration

Let v_r be the velocity of the load relative to the hoop, v_x be the velocity of the load in the x-axis in the lab frame and v_y be the velocity of the load in the y-axis in the lab frame. Then

$$v_r = R\dot{\theta}$$

$$\begin{aligned}
v_x &= \dot{X} + v_r \cos\theta \\
&= \dot{X} + R\dot{\theta}\cos\theta \\
v_y &= v_r \sin\theta \\
&= R\dot{\theta}\sin\theta
\end{aligned}$$

$$\begin{aligned}
a_x &= \dot{v}_x \\
&= \ddot{X} + R\ddot{\theta}\cos\theta - R\dot{\theta}^2\sin\theta \\
a_y &= \dot{v}_y \\
&= R\ddot{\theta}\sin\theta + R\dot{\theta}^2\cos\theta
\end{aligned}$$

1.2 Energy

Let K_m be the kinetic energy of the load, K_M be the kinetic energy of the hoop and U be the potential energy of the load. Then

$$\begin{aligned}
K_m &= \frac{1}{2}m(v_x^2 + v_y^2) \\
&= \frac{1}{2}m(R\dot{\theta}\sin\theta)^2 + \frac{1}{2}m(\dot{X} + R\dot{\theta}\cos\theta)^2 \\
&= \frac{1}{2}mR^2\dot{\theta}^2 + \frac{1}{2}m\dot{X}^2 + mR\dot{X}\dot{\theta}\cos\theta
\end{aligned}$$

$$K_M = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}MR^2\dot{\theta}^2$$

$$U = mgR(1 - \cos\theta)$$

$$\begin{aligned}
E &= K_m + K_M + U \\
&= \frac{1}{2}(M+m)R^2\dot{\theta}^2 + \frac{1}{2}(M+m)\dot{X}^2 + mR\dot{\theta}\dot{X}\cos\theta + mgR(1-\cos\theta)
\end{aligned}$$

2 Oscillations on a stationary belt

Let L be the lagrangian $K - U$, ignoring constant terms. Then

$$L = \frac{1}{2}(M+m)R^2\dot{\theta}^2 + \frac{1}{2}(M+m)\dot{X}^2 + mR\dot{\theta}\dot{X}\cos\theta + mgR\cos\theta$$

2.1 Period when friction is negligible

In this case X and θ are independent variables and we must use the Euler-Lagrange equation on each of them.

$$\frac{\partial L}{\partial \theta} = -mgR\sin\theta - mR\dot{\theta}\dot{X}\sin\theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = (M+m)R^2\dot{\theta} + mR\dot{X}\cos\theta$$

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) &= (M+m)R^2\ddot{\theta} + mR\ddot{X}\cos\theta - mR\dot{X}\dot{\theta}\sin\theta \\
&= \frac{\partial L}{\partial \theta}
\end{aligned}$$

$$\begin{aligned}
-mgR\sin\theta &= (M+m)R^2\ddot{\theta} + mR\ddot{X}\cos\theta \\
mg\sin\theta + (M+m)R\ddot{\theta} + m\ddot{X}\cos\theta &= 0
\end{aligned}$$

We can also obtain this equation by considering $\tau = I\alpha$ about the center of the hoop in the (accelerating) frame of the hoop where τ is the torque, I the moment of inertia of the system about the center of the hoop and $\alpha = \ddot{\theta}$ the angular acceleration.

$$\begin{aligned}
I &= (M+m)R^2 \\
\tau &= I\ddot{\theta} \\
&= -m\ddot{X}R\sin(\frac{\pi}{2} - \theta) - mgR\sin\theta
\end{aligned}$$

For the X-coordinate,

$$\frac{\partial L}{\partial X} = 0$$

$$\frac{\partial L}{\partial \dot{X}} = (M+m)\dot{X} + mR\dot{\theta}\cos\theta$$

since $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{X}}\right) = \frac{\partial L}{\partial X} = 0$, $\frac{\partial L}{\partial \dot{X}}$ is constant. Since $\dot{X} = \dot{\theta} = 0$ at the start, $\frac{\partial L}{\partial \dot{X}} = 0$ always.

$$(M+m)\dot{X} + mR\dot{\theta}\cos\theta = 0$$

We can also obtain this equation by using conservation of horizontal momentum, $M\dot{X} + mv_x = 0$

$$\begin{aligned}
\dot{X} &= -\frac{m}{M+m}R\dot{\theta}\cos\theta \\
\ddot{X} &= \frac{m}{M+m}(-R\ddot{\theta}\cos\theta + R\dot{\theta}^2\sin\theta)
\end{aligned}$$

combining the two we get

$$mg\sin\theta + (M+m)R\ddot{\theta} + \frac{m^2}{M+m}(R\dot{\theta}^2\sin\theta\cos\theta - R\ddot{\theta}\cos^2\theta) = 0$$

we now make the approximation that $\theta << 1$, so $\sin\theta = \theta$, $\dot{\theta}^2 = 0, \cos\theta = 1$

$$\begin{aligned} mg\theta + (M+m)R\ddot{\theta} - \frac{m^2}{M+m}R\ddot{\theta} &= 0 \\ \ddot{\theta} + \frac{m^2+Mm}{M^2+2Mm}\frac{g}{R}\theta &= 0 \\ \omega^2 &= \frac{m^2+Mm}{M^2+2Mm}\frac{g}{R} \end{aligned}$$

where ω is the angular velocity. Hence the period $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{M^2+2Mm}{m^2+Mm}\frac{R}{g}}$

2.2 Period when the hoop does not slip

The non-slip condition occurs when the velocity of the contact point between the hoop and the floor is 0.

$$\dot{X} + R\dot{\theta} = 0$$

We can simplify the Lagrangian with this

$$L = (M+m)R^2\dot{\theta}^2 - mR^2\dot{\theta}^2\cos\theta + mgR\cos\theta$$

notice that horizontal momentum is not conserved since there is an external frictional force which has a horizontal component. However since this is static friction which does no work, energy is conserved and the ELE can still be used.

$$\frac{\partial L}{\partial\dot{\theta}} = 2(M+m)R^2\dot{\theta} \ - 2mR^2\dot{\theta}\cos\theta$$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} = 2(M+m)R^2\ddot{\theta} - 2mR^2\ddot{\theta}\cos\theta + 2mR^2\dot{\theta}^2\sin\theta$$

$$\frac{\partial L}{\partial\theta} = mR^2\dot{\theta}^2\sin\theta - mgR\sin\theta$$

$$\begin{aligned} 2MR\ddot{\theta} + mg\theta &= 0 \\ \ddot{\theta} + \frac{m}{2M}\frac{g}{R}\theta &= 0 \end{aligned}$$

Alternatively, we can also introduce f as the friction force.

$$\begin{aligned} f &= M\ddot{X} + ma_x \\ &= M\ddot{X} + m\ddot{X} + mR\ddot{\theta}\cos\theta - mR\ddot{\theta}^2\sin\theta \\ &\approx M\ddot{X} \end{aligned}$$

this makes sense as for small θ the load is close to the point of contact between the ground and the hoop, which is almost stationary in the non-slip condition. Using $\tau = I\ddot{\theta}$ again

$$(M+m)R^2\ddot{\theta} = fR - m\ddot{X}\cos\theta R - mgR\sin\theta$$
$$(M+m)R\ddot{\theta} = -MR\ddot{\theta} + mR\ddot{\theta} - mg\sin\theta$$

$$2MR\ddot{\theta} + mg\sin\theta = 0$$

hence,

$$\omega^2 = \frac{m}{2M}\frac{g}{R}$$

and the period $T = 2\pi\sqrt{\frac{2M}{m}\frac{R}{g}}$

3 Stable Angular Orientations on an Accelerating Conveyer Belt

Let $a = \ddot{X}$ be the (constant) acceleration of the hoop and load.

In the frame of the accelerating system, we can balance torques about the center of the hoop:

$$f = mg\sin\theta + ma\cos\theta$$

Balancing forces

$$f = (M+m)a$$

equating the two

$$(M+m)a = mg\sin\theta + ma\cos\theta$$

now choose k and ϕ such that $k\sin\phi = a$ and $k\cos\phi = g$; k and ϕ will obviously always exist because we have not reduced the number of degrees of freedom (2) and k is unbounded. Then

$$\phi = \tan^{-1}\frac{a}{g}$$
$$k = \sqrt{a^2+g^2}$$

Now

$$(M+m)a = mk\cos\phi\sin\theta + mk\sin\phi\cos\theta$$
$$= mk\sin(\phi+\theta)$$
$$\frac{M+m}{m}\frac{a}{k} = \sin(\phi+\theta)$$

$$\phi+\theta = \sin^{-1}\frac{M+m}{m}\frac{a}{k}$$
$$\theta = \sin^{-1}\frac{M+m}{m}\frac{a}{k} - \phi$$
$$= \sin^{-1}\frac{M+m}{m}\frac{a}{\sqrt{a^2+g^2}} - \tan^{-1}\frac{a}{g}$$

or, in terms of $\gamma = \frac{m}{M}$,

$$\theta = \sin^{-1}((1+\frac{1}{\gamma})\frac{a}{\sqrt{a^2+g^2}}) - \tan^{-1}\frac{a}{g}$$

3.1 $\mu_s = 1.0$

There are two cases we have to consider; either the friction is kinetic or static in nature. For static friction,

$$a = K = \sqrt{3}g$$

$$\begin{aligned} f &= (M+m)a \\ &= (M+m)\sqrt{3}g \end{aligned}$$

Let N be the normal frictional force. Balancing forces in the vertical direction,

$$N = (M+m)g$$

unfortunately, $f > \mu_s N$. Hence, the friction must be kinetic in nature.

$$\begin{aligned} f &= \mu_k N \\ &= \frac{\sqrt{3}}{3}(M+m)g \end{aligned}$$

and since $f = (M+m)a$,

$$a = \frac{\sqrt{3}}{3}g$$

and

$$\frac{a}{\sqrt{a^2+g^2}} = \frac{1}{2}$$

and since $\frac{a}{g} = \frac{\sqrt{3}}{3}$ and $\tan^{-1}\frac{\sqrt{3}}{3} = \frac{\pi}{6}$, hence

$$\alpha_1 = \sin^{-1}((1+(\frac{m}{M})^{-1})\frac{1}{2}) - \frac{\pi}{6}$$

3.2 $\mu_s = 2.0$

In this case, for static friction, the working is the same; we still have

$$\begin{aligned} f &= (M+m)\sqrt{3}g \\ N &= (M+m)g \end{aligned}$$

but now $f < \mu_s N$ because $\sqrt{3} < 2$. Hence $a = \sqrt{3}g$ and

$$\frac{a}{\sqrt{a^2+g^2}} = \frac{\sqrt{3}}{2}$$

and since $\frac{a}{g} = \sqrt{3}$ and $\tan^{-1}\sqrt{3} = \frac{\pi}{3}$, hence

$$\alpha_2 = \sin^{-1}((1+(\frac{m}{M})^{-1})\frac{\sqrt{3}}{2}) - \frac{\pi}{3}$$

4 Oscillation, Rotation, and Sliding on an Accelerating Conveyor Belt

4.1 Type A motion

Let x be the position of the center of the hoop relative to the point on the belt it was first placed on in a frame of reference moving at acceleration K (thus $X = \frac{1}{2}Kt^2 + x$). Then in this frame, the conveyor belt is stationary; hence friction does no work. The fictitious force does work W_{fic}on the system

$$W_{fic} = MKx + mK(x + R\sin\theta)$$

by the work-energy theorem,

$$MKx + mK(x + R\sin\theta) + \frac{1}{2}(M+m)R^2\dot{\theta}^2 + \frac{1}{2}(M+m)\dot{x}^2 + mR\dot{\theta}\dot{x}\cos\theta + mgR(1-\cos\theta) = 0$$

since it is constant and $x, \dot{x}, \theta, \dot{\theta}$ were all 0 initially. The non-slip condition is that

$$x + R\theta = 0$$

simplifying,

$$-MKR\theta + mKR(\sin\theta - \theta) + (M+m)R^2\dot{\theta}^2 - mR^2\dot{\theta}^2\cos\theta + mgR(1-\cos\theta) = 0$$

with $M = m$,

$$K(\sin\theta - 2\theta) + R\dot{\theta}^2(2-\cos\theta) + g(1-\cos\theta) = 0$$

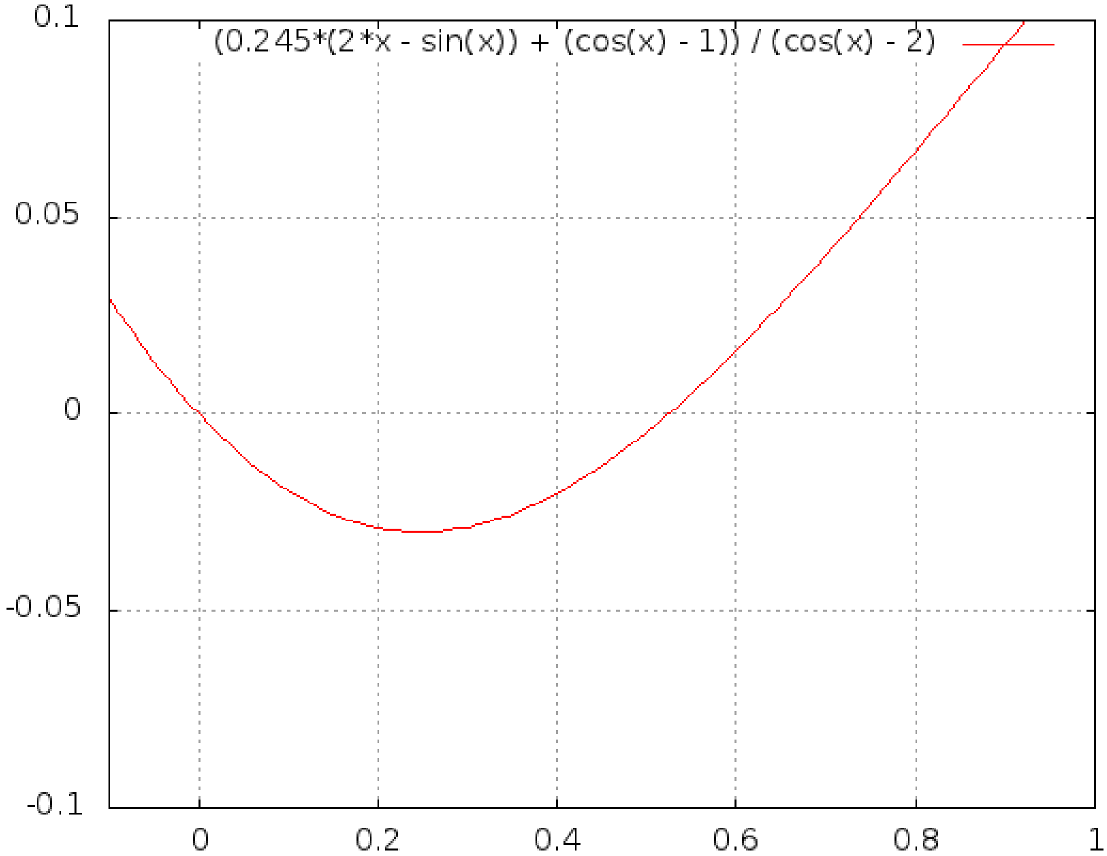
$$\begin{aligned}
R\dot{\theta}^2(2-\cos\theta) &= K(2\theta - \sin\theta) + g(\cos\theta - 1)\\
R\dot{\theta}^2 &= \frac{K(2\theta - \sin\theta) + g(\cos\theta - 1)}{(2-\cos\theta)}\\
\frac{1}{2}MR^2\dot{\theta}^2 &= \frac{1}{2}MR\frac{K(2\theta - \sin\theta) + g(\cos\theta - 1)}{(2-\cos\theta)}
\end{aligned}$$

This means that the system is equivalent to a point of mass M in a one-dimensional potential $V(\theta) \sim \frac{K(2\theta - \sin\theta) + g(\cos\theta - 1)}{(\cos\theta - 2)}$. Assuming that $V < 0$ when $0 < \theta < \beta$, when $\theta = \theta_{maximum}$, $\dot{\theta} = 0$ because otherwise we can increase θ by $d\theta$ by increasing or decreasing t by dt. Another way to see this is that in the equivalent system, $V = 0$ when $\theta = 0$; hence θ will oscillate between two roots of V. Hence

$$K(\sin\beta - 2\beta) + g(1-\cos\beta) = 0$$

$$\begin{aligned}
\frac{K}{g} &= \frac{1-\cos\beta}{2\beta - \sin\beta}\\
&= 0.244837712
\end{aligned}$$

with $\beta = \frac{\pi}{6}$. Now we plot $V(\theta)$ up to multiplicative constants for $\frac{K}{g} = 0.245$



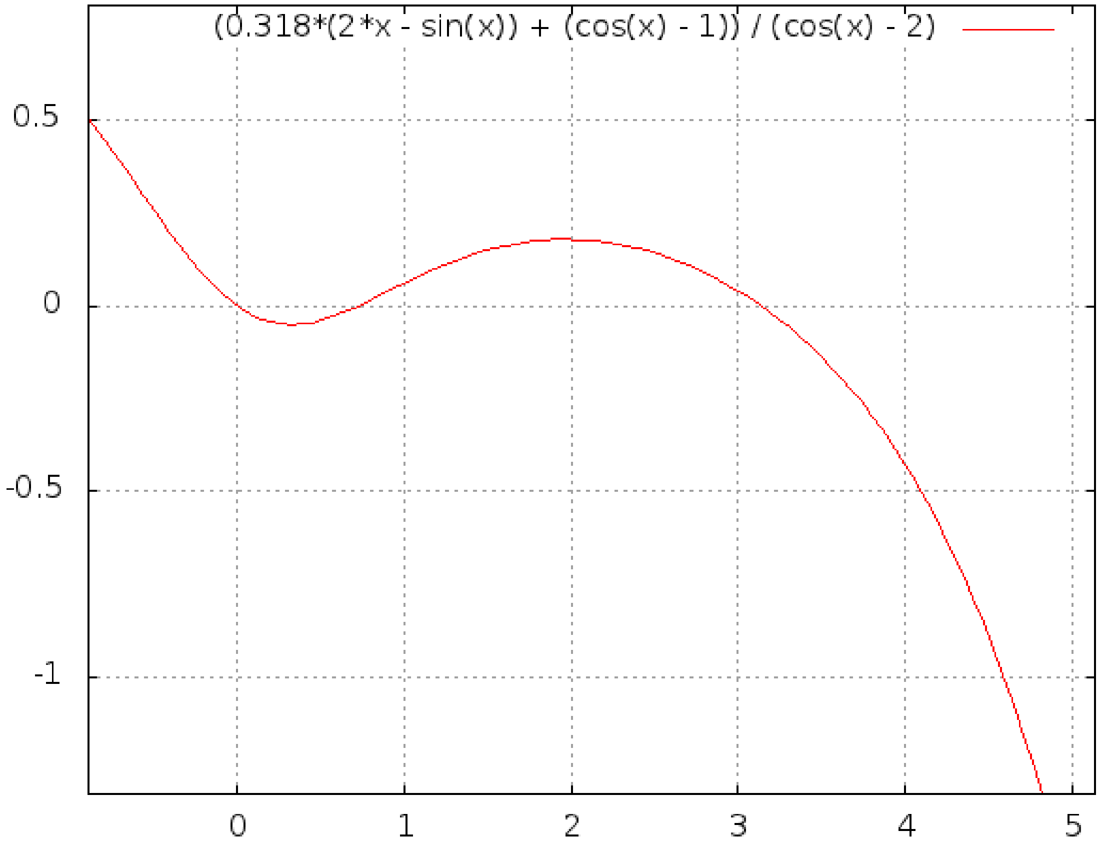
We see that the particle will exhibit Type A motion, oscillating between the two roots of $V(\theta)$

4.2 Type B motion

If we try $\beta = \pi$,

$$\begin{aligned}\frac{K}{g} &= \frac{1-\cos\beta}{2\beta-\sin\beta} \\ &= 0.318\end{aligned}$$

Now we plot $V(\theta)$ up to multiplicative constants for $\frac{K}{g} = 0.318$.



Although $V = 0$ at $\theta = \pi$, there is a positive-energy barrier (hump) where $V > 0$ as $0 < \theta < \pi$, preventing the particle from reaching $\theta = \pi$.

Hence the condition that $\dot{\theta} = 0$ is insufficient; we must also make sure that $V < 0$ as $0 < \theta < \pi$. Since $\cos\theta - 2 < 0$, this is equivalent to making $Z < 0$ where

$$Z = -K(2\theta - \sin\theta) - g(\cos\theta - 1)$$

the local maximum occurs when

$$\begin{aligned} \frac{dZ}{d\theta} &= 0 \\ K(2 - \cos\theta) - g\sin\theta &= 0 \\ 2K - (K\cos\theta + g\sin\theta) &= 0 \\ 2K - \sqrt{K^2 + g^2}\sin(\theta + \tan^{-1}\frac{K}{g}) &= 0 \\ \sin(\theta + \tan^{-1}\frac{K}{g}) &= \frac{2K}{\sqrt{K^2 + g^2}} \end{aligned}$$

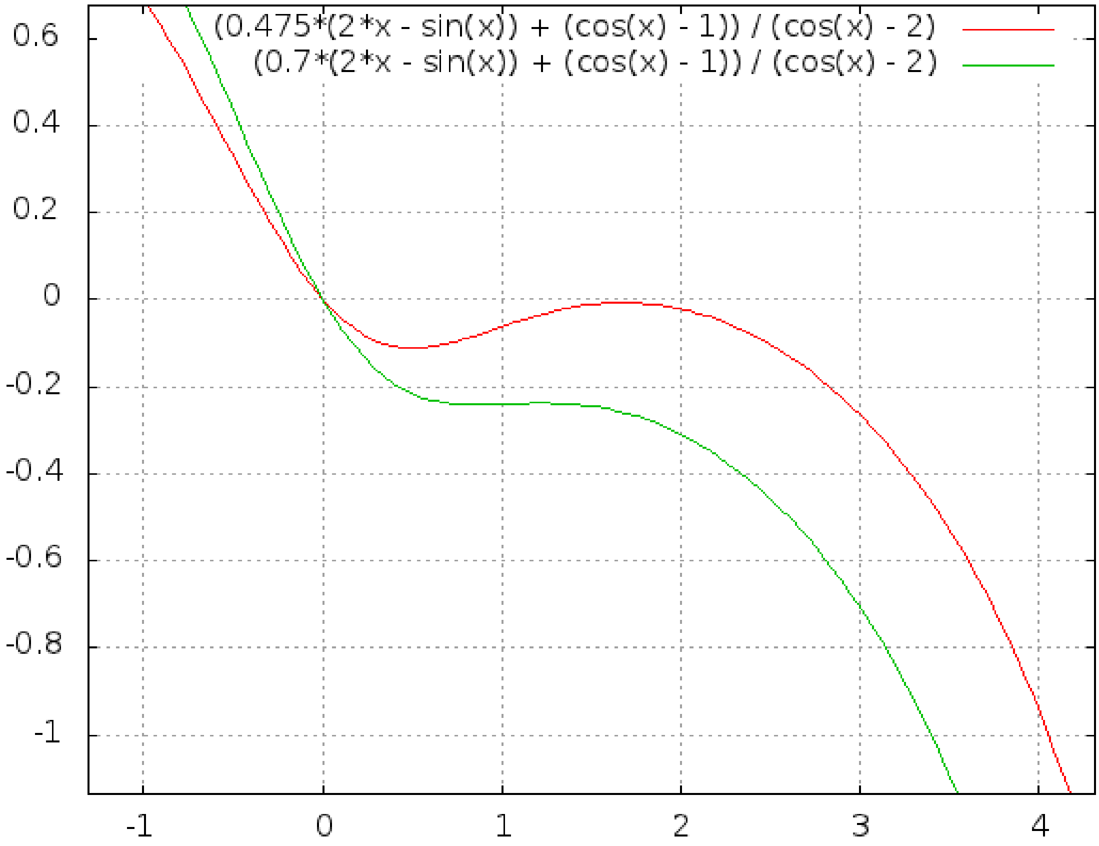
Let θ_{Zmax} be the positive θ such that Z is maximized. Then θ_{Zmax} is the second root because as seen from the graph, the first root corresponds to a local minimum while the second root corresponds to a local maximum.

$$\theta_{Zmax} = \pi - \sin^{-1}(\frac{2K}{\sqrt{K^2 + g^2}}) - \tan^{-1}\frac{K}{g}$$

We want to find $\frac{K}{g}$ such that $Z(\theta_{Zmax}) < 0$, so (substituting the expression for θ_{Zmax} into $Z(\theta)$) we must find the root of

$$-1-\cos\left(\sin^{-1}\frac{2\gamma}{(\sqrt{1+\gamma^2})}+\tan^{-1}\gamma\right)+\gamma\left(2\pi-2\sin^{-1}\frac{2\gamma}{\sqrt{1+\gamma^2}}-2\tan^{-1}\gamma-\sin(\sin^{-1}\frac{2\gamma}{\sqrt{1+\gamma^2}}+\tan^{-1}\gamma)\right)$$

with $\gamma = \frac{K}{g}$. Solving numerically, $\gamma = 0.4687 = 0.47$ (2 s.f.)
We now plot $V(\theta)$ up to multiplicative constants for two values of $\frac{K}{g} > \gamma$



Indeed, there are no longer any barriers.

4.3 Sliding motion

By force analysis in the horizontal direction,

$$\begin{aligned} f-(M+m)K &= M\ddot{x}+ma_x \\ &= -MR\ddot{\theta}-mR\ddot{\theta}(1-\cos\theta)-mR\dot{\theta}^2\sin\theta \end{aligned}$$

$$f=(M+m)K-MR\ddot{\theta}-mR\ddot{\theta}(1-\cos\theta)-mR\dot{\theta}^2\sin\theta$$

with $M=m$,

$$f=2MK-MR\ddot{\theta}(2-\cos\theta)-MR\dot{\theta}^2\sin\theta$$

Recall that

$$\frac{1}{2}MR^2\dot{\theta}^2=\frac{1}{2}MR\frac{K(2\theta-\sin\theta)+g(\cos\theta-1)}{(2-\cos\theta)}$$

differentiating with respect to θ,

$$MR^2\dot{\theta}\frac{d\dot{\theta}}{d\theta} = \frac{1}{2}MR\left(\frac{K(2\theta - \sin\theta) + g(\cos\theta - 1)}{(2 - \cos\theta)}\right)'$$

but since

$$\begin{aligned}\dot{\theta}\frac{d\dot{\theta}}{d\theta} &= \frac{d\theta}{dt}\frac{d\dot{\theta}}{d\theta} \\ &= \ddot{\theta}\end{aligned}$$

hence

$$R\ddot{\theta} = \frac{1}{2}\left(\frac{K(2\theta - \sin\theta) + g(\cos\theta - 1)}{(2 - \cos\theta)}\right)'$$

also

$$R\dot{\theta}^2 = \frac{K(2\theta - \sin\theta) + g(\cos\theta - 1)}{(2 - \cos\theta)}$$

substituting,

$$f = 2MK - \frac{1}{2}M\left(\frac{K(2\theta - \sin\theta) + g(\cos\theta - 1)}{(2 - \cos\theta)}\right)'(2 - \cos\theta) - M\frac{K(2\theta - \sin\theta) + g(\cos\theta - 1)}{(2 - \cos\theta)}\sin\theta$$

by a similar procedure but analyzing forces in the vertical direction,

$$N = 2Mg + \frac{1}{2}M\left(\frac{K(2\theta - \sin\theta) + g(\cos\theta - 1)}{(2 - \cos\theta)}\right)'\sin\theta + M\frac{K(2\theta - \sin\theta) + g(\cos\theta - 1)}{(2 - \cos\theta)}\cos\theta$$

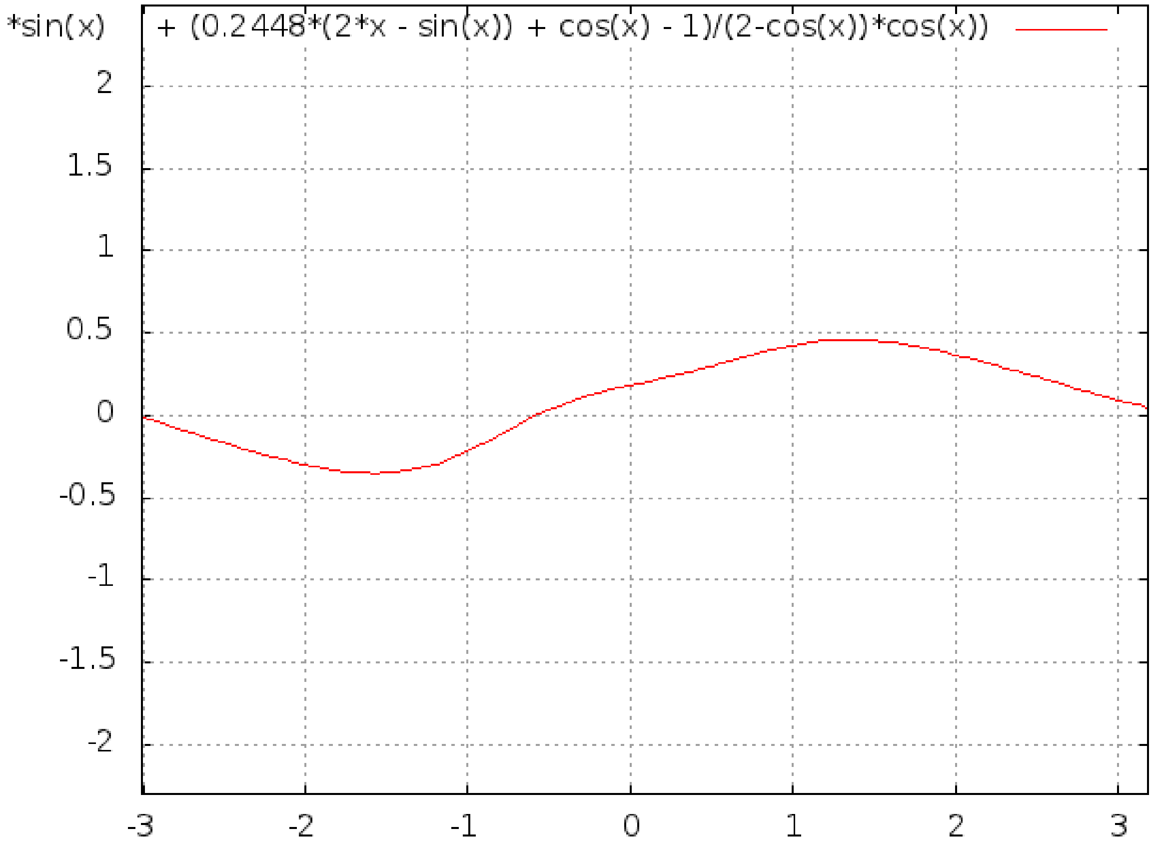
Let $\mu_e = \frac{f}{N}$. $\mu_{s,0}$ will be the maximum value of μ_e for $0 \le \theta \le \frac{\pi}{6}$ because if μ_s is less than that, then at some point in the system's motion (ie, for some θ) we will have $f > \mu_s N$.

$$\mu_e = \frac{2K - \frac{1}{2}\left(\frac{K(2\theta-\sin\theta)+g(\cos\theta-1)}{(2-\cos\theta)}\right)'(2 - \cos\theta) - \frac{K(2\theta-\sin\theta)+g(\cos\theta-1)}{(2-\cos\theta)}\sin\theta}{2g + \frac{1}{2}\left(\frac{K(2\theta-\sin\theta)+g(\cos\theta-1)}{(2-\cos\theta)}\right)'\sin\theta + \frac{K(2\theta-\sin\theta)+g(\cos\theta-1)}{(2-\cos\theta)}\cos\theta}$$

$$\left(\frac{K(2\theta - \sin\theta) + g(\cos\theta - 1)}{(2 - \cos\theta)}\right)' = \frac{5K - g\sin\theta - 4K\cos\theta - 2K\theta\sin\theta}{(2 - \cos\theta)^2}$$

$$\mu_e = \frac{2K - \frac{5K-g\sin\theta-4K\cos\theta-2K\theta\sin\theta}{4-2\cos\theta} - \frac{K(2\theta-\sin\theta)+g(\cos\theta-1)}{(2-\cos\theta)}\sin\theta}{2g + \frac{1}{2}\frac{5K-g\sin\theta-4K\cos\theta-2K\theta\sin\theta}{(2-\cos\theta)^2}\sin\theta + \frac{K(2\theta-\sin\theta)+g(\cos\theta-1)}{(2-\cos\theta)}\cos\theta}$$

plotting this



we see that the maximum value of μ_e for $0 \le \theta \le \frac{\pi}{6}$ is at $\theta = \frac{\pi}{6}$. Plugging in, $\mu_{s,0} = 0.307955 = 0.31$ (2 s.f.)