Lemma: In a group $\langle G, \cdot \rangle$, then
$(a \cdot b)^{-1} = b^{-1} \cdot a^{-1}$, $\forall a, b \in G$.

Proof. We have:
$(a \cdot b) \cdot (a \cdot b)^{-1} = e_G$, the identity in G

$$\Rightarrow a^{-1} \cdot (a \cdot b) \cdot (a \cdot b)^{-1} = a^{-1} \cdot e_G = a^{-1}$$
$$\Rightarrow \underline{(a^{-1} \cdot a)} \cdot b \cdot (a \cdot b)^{-1} = a^{-1}$$
$$\Rightarrow \underline{e_G \cdot b} \cdot (a \cdot b)^{-1} = a^{-1}$$
$$\Rightarrow b \cdot (a \cdot b)^{-1} = a^{-1}$$
$$\Rightarrow \underline{(b^{-1} \cdot b)} \cdot (a \cdot b)^{-1} = b^{-1} \cdot a^{-1}$$
$$\Rightarrow e_G \cdot (a \cdot b)^{-1} = b^{-1} \cdot a^{-1}$$
$$\Rightarrow \boxed{(a \cdot b)^{-1} = b^{-1} \cdot a^{-1}}.$$

Problem: A group $\langle G, \cdot \rangle$ is abelian, iff $(a \cdot b)^{-1} = a^{-1} \cdot b^{-1}$, $\forall a, b \in G$.

Proof.

$(\Rightarrow)$: Given G is an abelian group

i.e, $a, b \in G$, $a \cdot b = b \cdot a$.

R.T.P. $(a \cdot b)^{-1} = a^{-1} \cdot b^{-1}$

We have: $(a \cdot b)^{-1} = b^{-1} a^{-1}$ in a group G, $\forall a, b \in G$.

Now, $(a \cdot b)^{-1} = (b \cdot a)^{-1}$
$= \underline{a^{-1} \cdot b^{-1}}$.

$(\Leftarrow)$: Given $(a\cdot b)^{-1} = a^{-1}\cdot b^{-1}$, $\forall a,b \in G$.

R.T.P. G is abelian ie, $a\cdot b = b\cdot a$, $\forall a,b \in G$.

We have:

$$(a\cdot b)^{-1} = a^{-1}\cdot b^{-1}$$

$$\Rightarrow \underline{(a\cdot b)\cdot(a\cdot b)^{-1}} = (a\cdot b)\cdot a^{-1}\cdot b^{-1}$$

$$\Rightarrow e_G = a\cdot(b\cdot a^{-1})\cdot b^{-1}, \text{ where } e_G \in G \text{ is the identity.}$$

$$\Rightarrow a\cdot(b\cdot a^{-1})\cdot \underbrace{b^{-1}\cdot b}_{e_G} = e_G\cdot b = b$$

$$\Rightarrow (a\cdot b)\cdot a^{-1} = b$$

$$\Rightarrow (a\cdot b)\cdot(\underbrace{a^{-1}\cdot a}_{e_G}) = b\cdot a$$

$$\Rightarrow \boxed{a\cdot b = b\cdot a}$$

Problem [left and right cosets]

Given $S_3 = \{e, (1\ 2), (1\ 3), (2\ 3), (1\ 2\ 3), (1\ 3\ 2)\}$ is a group

where $e = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$ is identity element,

$$(1\ 2) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix},\quad (1\ 3) = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix},$$

$$(2\ 3) = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix},\quad (1\ 2\ 3) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix},$$

$$(1\ 3\ 2) = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}.$$

Let $H = \{e, (1\ 2)\} \subseteq S_3$.

H is a subgroup, because

(i) $e \in H$

(ii) $\forall h_1, h_2 \in H, \quad h_1 \cdot h_2^{-1} \in H$

for example, $h_1 = e$, $h_2 = (1\ 2)$

$$h_2^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

Then, $h_1 \cdot h_2^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$

$$= \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \in H$$

$= (1\ 2) \in H$

↳ The left cosets of $G = S_3$ relative to subgroup H are:

$e.H = e.\{e, (1\ 2)\} = \{e, (1\ 2)\}$

$(1\ 2).H = \{(1\ 2), e\}$

$(1\ 3).H = \{(1\ 3), (1\ 2\ 3)\}$

$(2\ 3).H = \{(2\ 3), (1\ 3\ 2)\}$

$(1\ 2\ 3).H = \{(1\ 2\ 3), (1\ 3)\}$

$(1\ 3\ 2).H = \{(1\ 3\ 2), (2\ 3)\}$

$\therefore$ The distinct left cosets are: $e.H$, $(1\ 3).H$, and $\underline{(2\ 3).H}$.

↳ The right cosets of S_3 relative to H are:

$H.e = \{e, (1\ 2)\} = H$

$H.(1\ 2) = \{(1\ 2), e\} = H$

$H.(1\ 3) = \{(1\ 3), (1\ 3\ 2)\}$

$H.(2\ 3) = \{(2\ 3), (1\ 2\ 3)\}$

$H \cdot (1\ 2\ 3) = \{(1\ 2\ 3), (2, 3)\} = H \cdot (2\ 3)$

$H \cdot (1\ 3\ 2) = \{(1\ 3\ 2), (1\ 3)\} = H \cdot (1\ 3)$

$\therefore$ The distinct right cosets are:
$H \cdot e$, $H \cdot (1\ 3)$, and $H \cdot (2\ 3)$. $\blacksquare$

Problem: If H be a subgroup of a group $\langle G, \cdot \rangle$ and $h \in H$, then $h \cdot H = H \cdot h = H$.

Proof. Given H is a subgroup of a group G
$\therefore \forall h_1, h_2 \in H$, $h_1 \cdot h_2^{-1} \in H$

Given $h \in H$.

R.T.P. (i) $h \cdot H = H$ ie, $h \cdot H \subseteq H$ and $H \subseteq h \cdot H$

(ii) $H \cdot h = H$ ie, $H \cdot h \subseteq H$ and $H \subseteq H \cdot h$

$h \cdot H =$ left coset of H in G
$= \{h \cdot h' \mid h' \in H\}$

$H \cdot h =$ right coset of H in G
$= \{h' \cdot h \mid h' \in H\}$.

(i) Let $h' \in H$.
Then, $h \cdot h' \in H$, by the closure of H

Also, $\boxed{h \cdot h' \in h \cdot H}$

we have: $h \cdot h' \in h \cdot H$
and $h \cdot h' \in H$

$\Rightarrow \boxed{h \cdot H \subseteq H}$ ... (1)

Again, let $h' \in H$

Then, $h' = e \cdot h'$, where $e \in H$ is the identity

$= (h \cdot h^{-1}) \cdot h'$

$= h \cdot (\underline{h^{-1} \cdot h'})$, $h^{-1} \in H$

$\in h \cdot \underline{\underline{H}}$

Since $h^{-1} \in H$ and $h' \in H \Rightarrow h^{-1} \cdot h' \in H$

$\therefore \quad H \subseteq h \cdot H \quad \ldots (2)$

from (1) & (2): $\boxed{h \cdot H = H}$

(ii) ~~Let~~ $h' \in H$.

R.T.P (a) $H \cdot h \subseteq H$, where $H \cdot h = \{h' \cdot h \mid h' \in H\}$

$\therefore h \in H,\ h' \in H$

$\Rightarrow \boxed{h' \cdot h \in H \cdot h}$, by defn.

Since H is a subgroup, so

$h' \in H$ and $h \in H \Rightarrow \boxed{h' \cdot h \in H}$

$\therefore \quad H \cdot h \subseteq H \quad \ldots (3)$

(b) Again, let $h' \in H$

Then, $h' = h' \cdot (\underbrace{h^{-1} \cdot h}_{e})$

$= (h' \cdot h^{-1}) \cdot h$

$\in H \cdot h$

Since $h' \in H$, $h^{-1} \in H \Rightarrow h \cdot h^{-1} \in H$.

$\therefore \quad H \subseteq H \cdot h \quad \ldots (4)$

from (3) & (4): $\boxed{H \cdot h = H}$

Problem:

Given $\forall g \in G$,

$N_g = \{ h \mid h \cdot g \cdot h^{-1} = g \}$ which is the "normalizer" of g.

R.T.P. N_g is a subgroup of G, $\forall g \in G$.

i.e., R.T.P.: $\forall h_1, h_2 \in N_g$, $\underline{h_1 \cdot h_2^{-1}} \in N_g$.

i.e., R.T.P.

$\forall h_1, h_2 \in N_g$, $(h_1 \cdot h_2^{-1}) \cdot g \cdot (h_1 \cdot h_2^{-1})^{-1} = g$

Now, $(h_1 \cdot h_2^{-1}) \cdot g \cdot (\underline{h_1 \cdot h_2^{-1}})^{-1}$

$= h_1 \cdot h_2^{-1} \cdot g \cdot (h_2^{-1})^{-1} \cdot h_1^{-1}$

$= h_1 \cdot (h_2^{-1} \cdot g \cdot h_2) \cdot h_1^{-1}$

$\therefore$ R.T.P. $\boxed{\forall h_1, h_2 \in N_g, \quad h_1 \cdot (h_2^{-1} \cdot g \cdot h_2) \cdot h_1^{-1} = g}$.