Lecture - 04

Thursday, 2 December 2021 8:23 AM

* Classification of signals
* Periodic signals and sinusoids
* Fourier Series (FS) representation for periodic signals

period (T) signal : $x(t+T) = x(t) \quad \forall t$

Frequency : $f_0 = \frac{1}{T}$

for ANY* periodic signal $x(t)$ we have,

* $$x(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(2\pi k f_0 t) + \sum_{k=1}^{\infty} b_k \sin(2\pi k f_0 t)$$

k^{th} Harmonics (freq. $= kf_0$)

$\{a_k, b_k\} \rightarrow$ FS coefficients

$$x(t) \xleftrightarrow{FS} \{a_k, b_k\}$$

Q. $\cos(t) + \cos(2\pi t) = x(t) = x(t+T)$ if periodic for some T

$\cos(t+T) + \cos(2\pi(t+T))$

$T = 2\pi k$ & $2\pi T = 2\pi n$ } cannot be true simultaneously !

* FS Analysis : given periodic (T) $x(t)$, find its FS coefficients

$$x(t) \longrightarrow \{a_k, b_k\}$$

★ FS Synthesis : given FS coefficients, synthesize / reconstruct the periodic signal $x(t)$

★ partial reconstruction : use $\{a_k, b_k\}$, $k = 0, 1, 2, \dots K$.

$$\hat{x}(t) = a_0 + \sum_{k=1}^{K} a_k \cos(2\pi k f_0 t) + \sum_{k=1}^{K} b_k \sin(2\pi k f_0 t)$$

reconstruction error : $e(t) = x(t) - \hat{x}(t)$

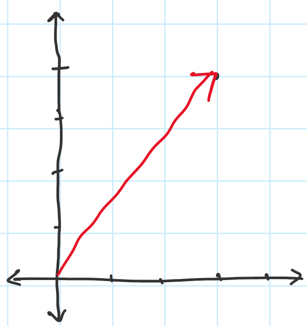
★ How to do FS analysis ?

★ detour into vector algebra

Ex. $\bar{v} = 3\hat{e}_1 + 4\hat{e}_2$

$3\hat{i} + 4\hat{j}$

same



$\hat{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ & $\hat{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\bar{v} \equiv (3, 4)$

$\uparrow$

$\{\hat{e}_1, \hat{e}_2\}$ basis

$\bar{v}$ using $\hat{f}_1 = 2\hat{e}_1$ & $\hat{f}_2 = 3\hat{e}_2$

$\bar{v} = \left(\frac{3}{2}\right)(2\hat{e}_1) + \left(\frac{4}{3}\right)(3\hat{e}_2)$

$\bar{v} \equiv \left(\frac{3}{2}, \frac{4}{3}\right)$

$\uparrow$

$\{\hat{f}_1, \hat{f}_2\}$ basis

$\bar{w} = \alpha_1\hat{e}_1 + \alpha_2\hat{e}_2$, find α_1 & α_2.

★ Inner product

$$\langle \bar{w}, \hat{e}_1 \rangle = (\alpha_1\hat{e}_1 + \alpha_2\hat{e}_2) \cdot \hat{e}_1$$

$$= \alpha_1 \langle \hat{e}_1, \hat{e}_1 \rangle + \alpha_2 \langle \hat{e}_2, \hat{e}_1 \rangle$$

orthogonal vectors

$$\Rightarrow \quad \alpha_1 = \frac{\langle \overline{w}, \hat{e}_1 \rangle}{\langle \hat{e}_1, \hat{e}_1 \rangle}$$

Similarly find α_2

★ Orthogonality $\quad \bar{a}$ & $\bar{b}$ are orthogonal if

$$\langle \bar{a}, \bar{b} \rangle = 0$$

★ Extending vector algebra to signals ... { Signals can be thought of as infinite dimensional vectors !

★ Inner product of signals : in interval $[a, b]$

$$\langle f(t), g(t) \rangle = \int_a^b f(t)\, g(t)\, dt \quad \text{... definition}$$

★ Orthogonal Signals $\Rightarrow \langle f, g \rangle = 0$

consider $\cos(2\pi k_1 f_0 t)$ & $\cos(2\pi k_2 f_0 t)$

Q. are they orthogonal in $\left[-\frac{T}{2}, \frac{T}{2}\right]$?

$$\Rightarrow \langle \cos(2\pi k_1 f_0 t), \cos(2\pi k_2 f_0 t) \rangle = \ ?$$

$$f_0 = \frac{1}{T}$$

$$k_1 \neq k_2 \quad \Rightarrow \quad \int^{T/2} \cos(2\pi k_1 f_0 t) \cos(2\pi k_2 f_0 t)\, dt$$

$k_1 \neq k_2 \Rightarrow \int_{-T/2}^{T/2} \cos(2\pi k_1 f_0 t) \cos(2\pi k_2 f_0 t)\, dt$

$$= \frac{1}{2} \int_{-T/2}^{T/2} \left(\cos\left[2\pi (k_1+k_2) \frac{1}{T} t\right] + \cos\left[2\pi (k_1-k_2) \frac{1}{T} t\right] \right) dt$$

$$= \begin{cases} 0 \quad , & k_1 \neq k_2 \quad \& \quad k_1 \,\&\, k_2 > 0 \\ T/2 \quad , & k_1 = k_2 \quad \& \quad k_1 > 0 \\ T \quad , & k_1 = k_2 = 0 \end{cases}$$

Similarly show for $\cos(2\pi k_1 f_0 t) \;\&\; \sin(2\pi k_2 f_0 t)$

$\sin(2\pi k_1 f_0 t) \;\&\; \sin(2\pi k_2 f_0 t)$

$\Rightarrow$ Each component in FS representation is orthogonal to every other component [except itself] *

$$FS: \quad x(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(2\pi k f_0 t) + \sum_{n=1}^{\infty} b_n \sin(2\pi n f_0 t)$$

$$\langle x(t), \cos(2\pi m f_0 t) \rangle = a_m \langle \cos(2\pi m f_0 t), \cos(2\pi m f_0 t) \rangle$$

$$\Rightarrow a_m = \frac{\langle x(t), \cos(2\pi m f_0 t) \rangle}{\langle \cos(2\pi m f_0 t), \cos(2\pi m f_0 t) \rangle}$$

$$a_m = \frac{2}{T} \int_{\langle T \rangle} x(t) \cos(2\pi m f_0 t)\, dt \quad , m = 1, 2, \cdots \infty$$

similarly we can show that

(H.W.) $$b_m = \frac{2}{T} \int_{\langle T \rangle} x(t) \sin(2\pi m f_0 t)\, dt \quad , m = 1, 2, \cdots \infty$$

notation : $\langle T \rangle$ — any region of length T

e.g. $\left[-\frac{T}{2}, \frac{T}{2}\right]$ or $[0, T]$ or $[-2T, -T]$, etc.

$$a_0 = \frac{1}{T} \int_{\langle T \rangle} x(t)\, dt$$