Lecture - 06

Tuesday, 7 December 2021 8:23 AM

* Joseph Fourier "An analytic theory of Heat" in 1822

* Odd and Even signals

* FS coefficients for odd / even periodic signals

  * odd signal : $a_k = 0 \quad \forall\, k = 0, 1, 2, \cdots \infty$

  * even signal : $b_k = 0 \quad \forall\, k = 1, 2, \cdots \infty$

* Square wave example :

$$x(t) = \begin{cases} 3, & 0 < t < 5 \\ -3, & -5 < t < 0 \end{cases} \quad \text{with period } T = 10$$

$$x(t) = \sum_{k=2r-1}^{r=1 \ldots \infty} \frac{12}{k\pi} \sin(k\omega_0 t) = \sum_{k\text{-odd}} \frac{12}{k\pi} \sin\left(\frac{2\pi k t}{10}\right)$$

Does this work ?

* visualization of partial FS synthesis / reconstruction

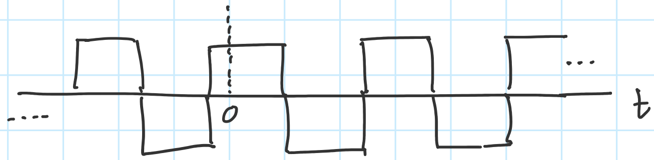
* Note on half-wave symmetry :

A periodic signal is said to have half-wave symmetry

* if $x(t) = -x\left(t + \frac{T}{2}\right)$, period of $x(t)$ is T.

Ex. $\sin(\omega_0 t)$, $\cos(\omega_0 t)$

Ex.



* half-wave symmetry independent of odd / even symmetry

* F S coefficients for half-wave symmetric periodic signals :

we can show that, (H. W.)

① $a_0 = 0$

② $a_k = 0$ for k - even

③ $b_k = 0$ for k - even

i.e. all even harmonics are absent *

* In Half-wave symmetric periodic signals, only odd harmonics are present

* Various forms of F S representation

* Trigonometric form

* compact trigonometric form

$$x(t) = c_0 + \sum_{k=1}^{\infty} c_k \cos(k\omega_0 t + \theta_k)$$

* Exponential form

$$x(t) = \sum_{k=-\infty}^{\infty} d_k e^{jk\omega_0 t} \quad ; \quad \omega_0 = \frac{2\pi}{T}$$

what is relation between d_k and a_k & b_k ?

$$x(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega_0 t) + \sum_{k=1}^{\infty} b_k \sin(k\omega_0 t)$$ ☆

Euler's formula: $e^{j\theta} = \cos\theta + j\sin\theta$

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2} \quad \& \quad \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

substitute

$$x(t) = a_0 + \sum_{k=1}^{\infty} a_k \frac{e^{jk\omega_0 t} + e^{-jk\omega_0 t}}{2} + \sum_{k} b_k \frac{e^{jk\omega_0 t} - e^{-jk\omega_0 t}}{2j}$$

$$= a_0 + \sum_{k=1}^{\infty} \underbrace{\left(\frac{a_k - jb_k}{2}\right)}_{d_k} e^{jk\omega_0 t} + \sum_{k=1}^{\infty} \underbrace{\left(\frac{a_k + jb_k}{2}\right)}_{f_k} e^{-jk\omega_0 t}$$

$k = -r$

$$= a_0 + \sum_{k=1}^{\infty} d_k e^{jk\omega_0 t} + \sum_{r=-\infty}^{-1} f_{-r} e^{jr\omega_0 t}$$

$a_0 = d_0$ $\qquad$ $f_{-r} = d_r$

$$x(t) = \sum_{k=-\infty}^{\infty} d_k e^{jk\omega_0 t}$$ ☆

$$* \quad d_k = \frac{1}{\;} \int x(t)\, e^{-jk\omega_0 t}\, dt$$

* $d_k = \frac{1}{T} \int_{\langle T \rangle} x(t)\, e^{-jk\omega_0 t}\, dt$

$$d_k = \begin{cases} \dfrac{a_k - jb_k}{2}, & k = 1, 2, \dots \infty \\ \dfrac{a_{-k} + jb_{-k}}{2}, & k = -1, -2, \dots \infty \\ a_0, & k = 0 \end{cases}$$

* orthogonality of complex sinusoids

$\{ e^{jk\omega_0 t} \}_k$ are orthogonal.

show $\langle e^{jk_1\omega_0 t}, e^{jk_2\omega_0 t} \rangle = 0$, $k_1 \neq k_2$

* Fourier Spectrum

$x(t) \overset{FS}{\longleftrightarrow}$ $\{a_k, b_k\} \rightarrow$ real-valued

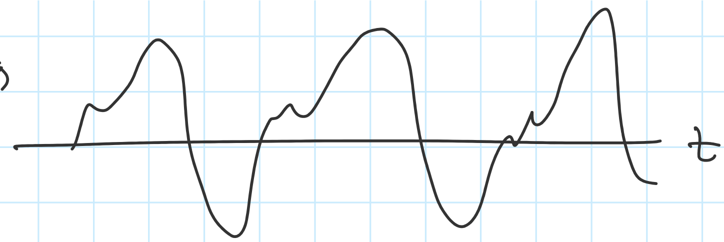
$\{C_k, \theta_k\} \rightarrow$ real-valued

$\{d_k\} \rightarrow$ complex-valued

$$\left. \begin{array}{l} |d_k| \text{ - magnitude spectrum} \\ \measuredangle d_k \text{ - phase spectrum} \end{array} \right\} \text{Fourier spectrum}$$

* Dual representations *

* <u>time-domain</u> $x(t)$



* <u>frequency-domain</u> $\{d_k\}$

freq.-domain version of $x(t)$

$d_k \longrightarrow k\omega_0$ in $x(t)$

freq. index

Ex. find d_k for square wave in prev. class.

plot mag. & phase spectrum.

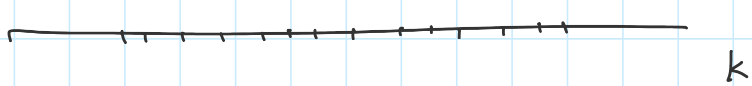
* For real signals:

$$d_k = d_{-k}^{*} \quad \cdots \text{ conjugate symmetry}$$

$$|d_k| = |d_{-k}| \quad \cdots \text{ even symmetry}$$

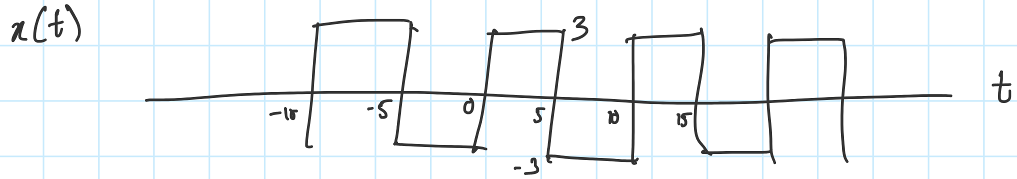
$$\measuredangle d_k = -\measuredangle d_{-k} \quad \cdots \text{ odd symmetry}$$

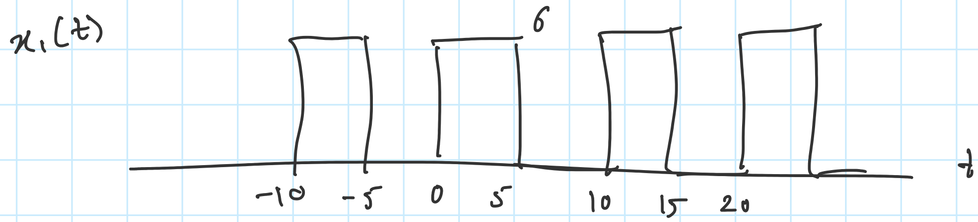
plot:

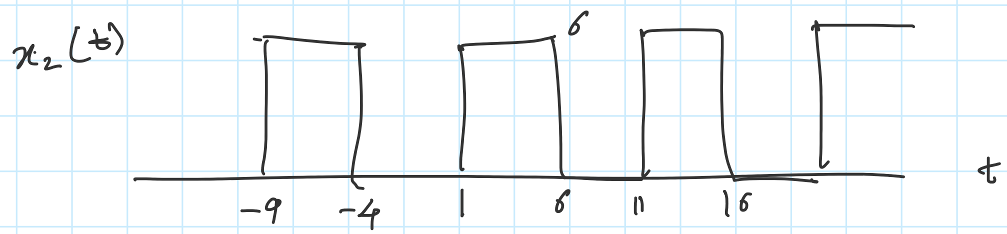


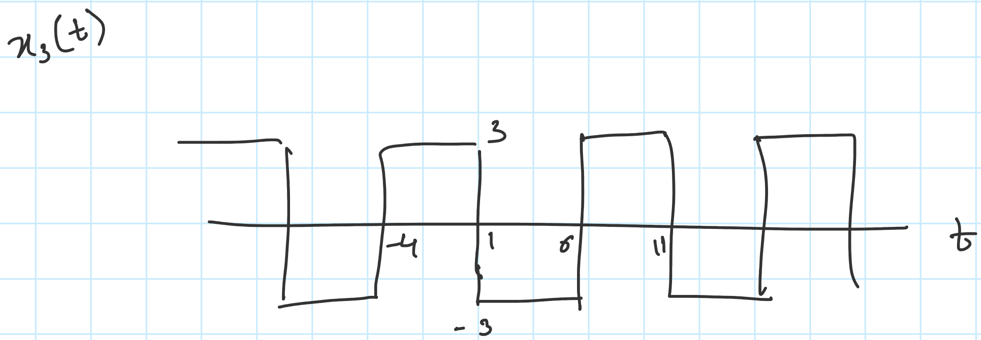
* <u>Properties</u> of Fourier series:

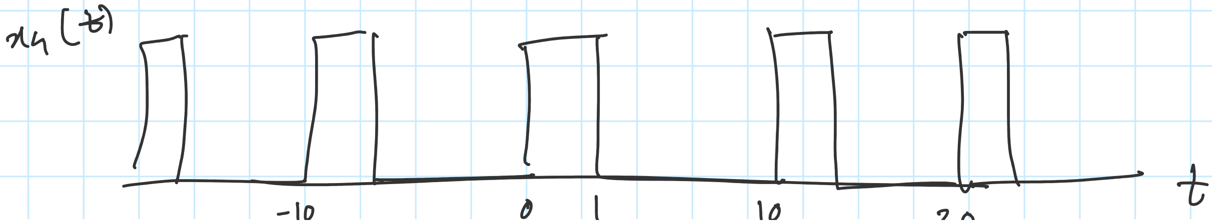
Q. given FS coefficients of $x(t)$, find FS coefficients of $x_1(t)$, $x_2(t)$, $x_3(t)$ & $x_4(t)$

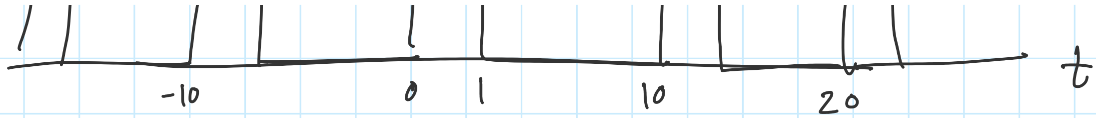












-10
0
1
10
20
t