Lecture - 05

Saturday, 4 December 2021 8:18 AM

* Inner product for signals & orthogonality

* The set of signals $\{1, \cos(k\omega_0 t), \sin(k\omega_0 t)\}_{k=1,2,\dots\infty}$ forms a basis for space of periodic signals with time-period $T = \frac{1}{f_0} = \frac{2\pi}{\omega_0}$

* $$x(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega_0 t) + \sum_{k=1}^{\infty} b_k \sin(k\omega_0 t)$$

* Let $x(t)$ be periodic with $T = \frac{2\pi}{\omega_0}$, then its FS coefficients are given by

* $$a_k = \frac{2}{T}\int_{\langle T\rangle} x(t)\cos(k\omega_0 t)\,dt$$

$k = 1, 2, \dots \infty$

* $$b_k = \frac{2}{T}\int_{\langle T\rangle} x(t)\sin(k\omega_0 t)\,dt$$

and * $$a_0 = \frac{1}{T}\int_{\langle T\rangle} x(t)\,dt$$ ..... (average value of signal in one period)

we will use ω_0 $(= 2\pi f_0)$ more commonly

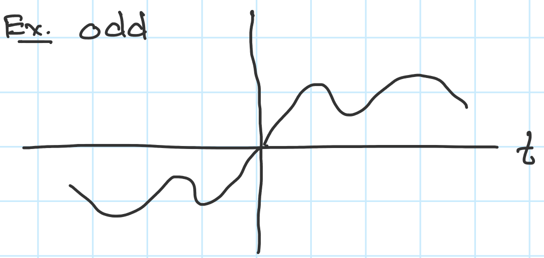
* For real-valued signals $x(t)$, the coefficients $\{a_k, b_k\}$ are real-valued

* Odd and Even signals

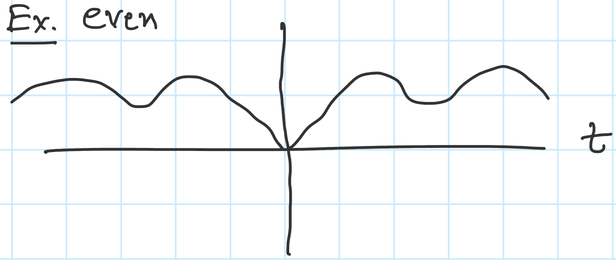
odd signal : $x(-t) = -x(t)$ Ex. $\sin(t)$

even signal : $x(-t) = x(t)$ Ex. $\cos(t)$

Ex. odd



Ex. even



★ decomposing a signal into odd & even components

$$x(t) = x_e(t) + x_o(t)$$

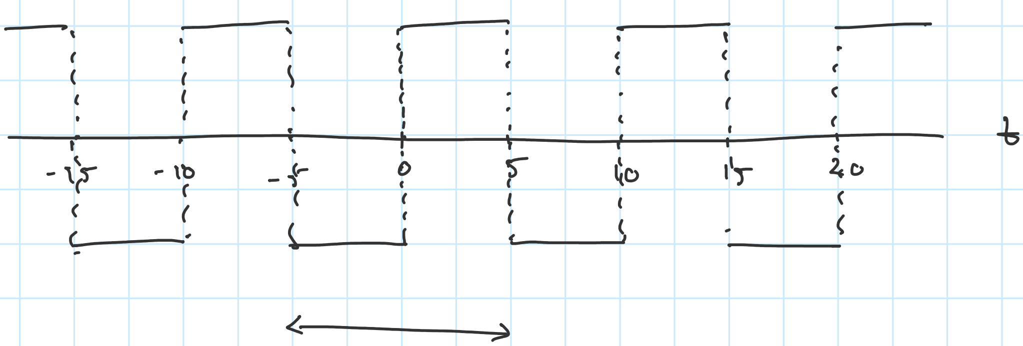
$$x_e(t) = \frac{x(t) + x(-t)}{2} \quad \& \quad x_o(t) = \frac{x(t) - x(-t)}{2}$$

★ Example: compute FS coefficients of the square wave below.

$$x(t) = \begin{cases} 3 & 0 < t < 5 \\ -3 & 5 < t < 10 \end{cases}$$ & period $T = 10$

Sketch :

odd signal



$x(t+T) = x(t)$ $T = 10$

T

$$a_0 = \frac{1}{T}\int_0 x(t)\,dt = 0$$

$$a_k = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos(k\omega_0 t)\,dt$$

$$= \frac{2}{10}\int_{-5}^{5} x(t)\cos(k\omega_0 t)\,dt = \frac{1}{5}\left[\int_{-5}^{0} x(t)\cos(k\omega_0 t)\,dt + \int_0^5 x(t)\cos(k\omega_0 t)\,dt\right]$$

$$= \frac{1}{5}\left[-3\int_{-5}^{0}\cos(k\omega_0 t)\,dt + 3\int_0^5 \cos(k\omega_0 t)\,dt\right]$$

$$\downarrow \tau = -t$$

$$= \frac{1}{5}\left[-3\int_0^5 \cos(k\omega_0 \tau)\,d\tau + 3\int_0^5 \cos(k\omega_0 t)\,dt\right] = 0$$

$$a_k = 0 \quad \forall \quad k = 1, 2, \cdots \infty$$

* Since $x(t)$ is an odd signal, we get $a_k = 0$

i.e there are no cos( ) terms in FS representation.

$$b_k = \frac{1}{5}\int_{-5}^{5} x(t)\sin(k\omega_0 t)\,dt \qquad \omega_0 = \frac{2\pi}{T} = \frac{2\pi}{10}$$

$$= \frac{1}{5}\left[\int_{-5}^{0} -3\sin(k\omega_0 t)\,dt + \int_0^5 3\sin(k\omega_0 t)\,dt\right]$$

$$b_k = \frac{3}{k\pi}\left[1 - 2\cos(k\pi) + \cos(2k\pi)\right]$$

$$b_k = \frac{6}{k\pi}\left[1 - \cos(k\pi)\right] = \frac{6}{k\pi}\left[1 - (-1)^k\right]$$

$$b_k = \begin{cases} \dfrac{12}{k\pi}, & k - \text{odd} \\ 0, & k - \text{even} \end{cases}$$

$$x(t) \equiv \sum_{k=2r-1}^{r=1,\dots\infty} \frac{12}{k\pi} \sin(k\omega_0 t) \qquad \text{FS representation.}$$

* For even $x(t)$, we get $b_k = 0$ $\forall$ $k = 1, 2, \dots \infty$

* Alternate FS representations :

  * Trigonometric FS representation

$$x(t) = a_0 + \sum a_k \cos(\ ) + \sum b_k \sin(\ )$$

* compact Trigonometric FS rep.

$$a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t) \equiv C_k \cos(k\omega_0 t + \theta_k)$$

$$\sqrt{a_k^2 + b_k^2} \left\{ \underbrace{\frac{a_k}{\sqrt{\ \ }}}_{\cos\theta_k} \cos(\ ) + \underbrace{\frac{b_k}{\sqrt{\ \ }}}_{-\sin\theta_k} \sin(\ ) \right\}$$

* $C_k = \sqrt{a_k^2 + b_k^2}$ & $\theta_k = \tan^{-1}\left(\dfrac{-b_k}{a_k}\right)$

$$x(t) \equiv C_0 + \sum_{k=1}^{\infty} C_k \cos(k\omega_0 t + \theta_k)$$

* Exponential FS representation