Name ....
Regn. No. ....
Course Number ....
Subject ....
Instructor ....

IN PURSUIT OF KNOWLEDGE
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1.1.
$$dU = TdS - PdV$$

$$\Rightarrow \left(\frac{\partial U}{\partial V}\right)_T = T\left(\frac{\partial S}{\partial V}\right)_T - P$$

(Maxwell relation: From the exactness of F. F = U − TS) $\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial P}{\partial T}\right)_V$

$$\therefore \left(\frac{\partial U}{\partial V}\right)_T = T\left(\frac{\partial P}{\partial T}\right)_V - P.$$

1.2 From Gibbs – Duhem relation:

$$SdT - VdP + Nd\mu = 0$$

$$\Rightarrow N\left(\frac{\partial \mu}{\partial N}\right)_{V,T} = V\left(\frac{\partial P}{\partial N}\right)_{V,T}$$

$$* \left(\frac{\partial P}{\partial N}\right)_{V,T} = \frac{-1}{\left(\frac{\partial N}{\partial V}\right)_{P,T} \cdot \left(\frac{\partial V}{\partial P}\right)_{N,T}}$$ Chain rule.

$V(N, P, T)$

$$\Rightarrow N\left(\frac{\partial \mu}{\partial N}\right)_{V,T} = \frac{(\partial V/\partial N)_{P,T}}{K_T}$$

$$* \left(\frac{\partial V}{\partial N}\right)_{P,T} = \frac{V}{N}$$ since $V(N, P, T) = N f(P, T)$

from the extensivity of V

$$\Rightarrow \quad N\left(\frac{\partial \mu}{\partial N}\right)_{V,T} = \frac{V}{N K_T}$$

$$\underline{\text{or}} \quad \left(\frac{\partial N}{\partial \mu}\right)_{V,T} = \frac{N^2 K_T}{V}.$$

1.3. $$C_p\left(\frac{\partial T}{\partial P}\right)_H = \left(\frac{\partial H}{\partial T}\right)_P \left(\frac{\partial T}{\partial P}\right)_H$$

$$= -\left(\frac{\partial H}{\partial P}\right)_T \quad \text{using chain rule} \quad -(1)$$

From $dH = T\,dS + V\,dP$

$$\left(\frac{\partial H}{\partial P}\right)_T = T\left(\frac{\partial S}{\partial P}\right)_T + V \quad -(2)$$

~~From~~ Maxwell's relation from the exactness of

$G = U - TS + PV \Rightarrow$

$$\left(\frac{\partial S}{\partial P}\right)_T = -\left(\frac{\partial V}{\partial T}\right)_P \quad -(3)$$

Combining (1) (2) + (3) $\Rightarrow$

$$C_p\left(\frac{\partial T}{\partial P}\right)_H = T\left(\frac{\partial V}{\partial T}\right)_P - V.$$

2.

$$Z_G = \sum_{E,N} e^{-\beta(E - \mu N)}$$

$$= \sum_{N=1}^{\infty} e^{+\beta\mu N} \cdot \underbrace{\sum_{E} e^{-\beta E(N)}}_{Z(T,V,N)}$$

$$= \sum_{N=0}^{\infty} e^{\beta\mu N} \frac{[Z_1(T,V)]^N}{N!} = \exp\{e^{\beta\mu} Z_1\}$$

$$Z_1(T,V) = \int \frac{d^3\vec{q}\, d^3\vec{p}}{h^3} e^{-\beta E(\vec{p},\vec{q})}$$

for relativistic gas $E \simeq pc$

$$\Rightarrow Z_1 = \frac{V}{h^3} \int d^3\vec{p}\, e^{-\beta pc}$$

$$\simeq \frac{V}{h^3} \int_0^{\infty} 4\pi p^2 dp\, e^{-\beta pc}$$

$$\therefore Z_1 = \frac{V}{\Lambda^3} \; ; \; \Lambda = \frac{\hbar c}{k_B T} \pi^{2/3} .$$

$$\therefore Z_G(T,V,\mu) = \exp\left\{\frac{e^{\beta\mu} V}{\Lambda^3}\right\}$$

$$PV = k_B T \ln Z_G$$

$$PV = e^{\beta\mu}\left(\frac{V}{\Lambda^3}\right) \cdot k_B T$$

$$\Rightarrow \quad e^{\beta\mu} = \frac{P\Lambda^3}{k_B T}$$

$$= \frac{P}{k_B T} \cdot \frac{\hbar^3 c^3 \pi^2}{(k_B T)^3}$$

$$= \frac{(\hbar c)^3 P \pi^2}{(k_B T)^4}$$

or $$\boxed{\mu = k_B T \ln\left\{\frac{\hbar^3 c^3 \pi^2 P}{(k_B T)^4}\right\}}$$

$$= \mu(P, T).$$

$*$ $\mu = 0 \Rightarrow T = 0$

or $$\frac{\hbar^3 c^3 \pi^2 P}{(k_B T)^4} = 1$$

or $$\boxed{T = \left(\frac{\hbar^3 c^3 \pi^2 P}{k_B^4}\right)^{1/4}}$$

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3. $N = \sum_i \langle n_i \rangle$

$$\simeq \int_0^\infty g(\epsilon)\, d\epsilon \cdot \underbrace{\langle n(\epsilon) \rangle}_{\frac{1}{e^{\beta(\epsilon-\mu)}+1}}$$

In 2 dim.

$$g(\epsilon)\, d\epsilon = g_s \frac{A \times m}{2\pi\hbar^2} \; ; \; g_s = (2s+1)$$
$$= 2 \text{ for } e^-.$$

$$\therefore \; N = \frac{g_s A m}{2\pi\hbar^2} \int_0^\infty \frac{d\epsilon}{e^{\beta(\epsilon-\mu)}+1} \quad \text{— (1)}$$

$$* \quad \int_0^\infty \frac{d\epsilon}{e^{\beta(\epsilon-\mu)}+1} = \frac{1}{\beta} \int_0^\infty \frac{dx}{e^{x-x_0}+1} \; ;$$

where $x_0 = \beta\mu$

$$* \quad \frac{1}{e^{x-x_0}+1} = \frac{e^{-x}}{e^{-x_0}+e^{-x}}$$
$$= -\frac{d}{dx} \ln\{e^{-x}+e^{-x_0}\}$$

$$\therefore \int_0^\infty \frac{d\epsilon}{e^{\beta(\epsilon-\mu)}+1} = -\frac{1}{\beta} \ln\left(e^{-x}+e^{-x_0}\right)\Big|_0^\infty$$

or $\displaystyle\int_0^\infty \frac{d\epsilon}{e^{\beta(\epsilon-\mu)}+1} = \frac{-1}{\beta}\left\{ \ln\left[e^{-x_0}\right] - \ln\left[1+e^{-x_0}\right] \right\}$

$$= \frac{1}{\beta} \ln\left(1+e^{x_0}\right)$$

$$= \frac{1}{\beta} \ln\left[1+e^{\beta\mu}\right] \quad \text{— (2)}$$

from (1) & (2)

$\therefore \quad N = \dfrac{g_s A m}{2\pi\hbar^2} \dfrac{1}{\beta} \ln\left(1+e^{\beta\mu}\right) \quad g_s = 2$

Recall $\quad \epsilon_F = \dfrac{\pi\hbar^2 N}{mA}$. in 2 dim

$$\Rightarrow \quad \beta\epsilon_F = \ln\left(1+e^{\beta\mu}\right)$$

or $\quad \boxed{\mu = k_B T \ln\left[e^{\epsilon_F/k_B T} - 1\right]}$

4. $\chi = \lim_{H\to 0} \frac{\partial M}{\partial H}$ / or $M \propto H$
$= \chi H.$

* $M = N_+ \times (-\mu_B) + N_- (+\mu_B)$; $N \equiv N_+ + N_-$

$$= (N_- - N_+) \times \mu_B \quad -(1)$$

* $$N_\pm = \int_0^\infty \frac{g(\epsilon)\, d\epsilon}{e^{\beta(\epsilon \pm \mu_B H - \mu)} + 1}$$

$$= \cancel{\int_0^\infty g(\epsilon)} \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \int_0 \frac{d\epsilon \sqrt{\epsilon}}{e^{(\beta\epsilon - \alpha_\mp)} + 1}$$

where $\alpha_\mp = \beta(\mu \mp \mu_B H)$

$$N_\pm = \underbrace{\frac{V}{(2\pi)^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}}_{\alpha} \frac{2}{\beta^{3/2}} \int_0^\infty \frac{dy \sqrt{y}}{e^{y - \alpha_\mp} + 1} \quad -(2)$$

* Low Tem expansion

$$\int_0^\infty dy \frac{\sqrt{y}}{e^{y-\alpha}+1} \simeq \frac{2}{3}\alpha^{3/2} + \frac{\pi^2}{12}\alpha^{-1/2}. \quad -(3)$$

$$M = \frac{\mu_B \alpha}{\beta^{3/2}} \left\{ \frac{2}{3} \left(x_+^{3/2} - x_-^{3/2} \right) + \frac{\pi^2}{12} \left(x_+^{-1/2} - x_-^{-1/2} \right) \right\}$$

$$* \quad x_\pm = \beta(\mu \pm \mu_B H)$$

$$= \beta\mu \left\{ 1 \pm \frac{\mu_B H}{\mu} \right\}$$

$$\rightarrow (x_\pm)^n \simeq (\beta\mu)^n \left\{ 1 \pm n \frac{\mu_B H}{\mu} \right\}$$

$$\Rightarrow \boxed{(x_+)^n - (x_-)^n \simeq 2n \, \mu_B H \times \beta^n \mu^{n-1}} \quad (4)$$

$$M = \frac{\mu_B \alpha}{\beta^{3/2}} \times \left\{ \frac{2}{3} \times 2 \times \frac{3}{2} \beta^{3/2} \mu^{1/2} + \frac{\pi^2}{12} \times 2 \times \left(-\frac{1}{2}\right) \cdot \beta^{-1/2} \mu^{-3/2} \right\} \mu_B H$$

$$\frac{M}{H} = \frac{\beta \alpha \mu_B^2}{\beta^{3/2}} \left\{ 2 (\mu\beta)^{1/2} - \frac{\pi^2}{12} (\mu\beta)^{-3/2} \right\}$$

$$= 2\beta^{-3/2} \alpha \mu_B^2 \sqrt{\mu} \left\{ 1 - \frac{\pi^2}{24} \frac{1}{(\mu\beta)^2} \right\}$$

Recall: $\mu = \epsilon_F \left\{ 1 - \frac{\pi^2}{12} \frac{1}{(\beta \epsilon_F)^2} \right\}$.

$$\Rightarrow \frac{M}{H} \simeq 2 \alpha \mu_B^2 \cdot \beta^{3/2} \sqrt{\epsilon_F} \left\{ 1 - \frac{\pi^2}{12} \frac{1}{(\beta \epsilon_F)^2} \right\}$$

also. $N = \frac{4}{3} \alpha \beta^{3/2} (\epsilon_F)^{3/2}$.

$$\Rightarrow \boxed{\chi = \frac{M}{H} \simeq \frac{3}{2} \mu_B^2 \frac{N}{\epsilon_F} \left\{ 1 - \frac{\pi^2}{12} \frac{1}{(\beta \epsilon_F)^2} \right\}}$$