PHY 635: Gravitation & Cosmology : Problem Sheet 3

Submit before 05th March 2025

1. In the case if the spacetime has torsion, i.e. $T^{\lambda}_{\mu\nu} \equiv \Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\nu\mu} \neq 0$, a vector (or a co-vector) fails to comeback to itself when parallely translated along a closed loop, even in the globally flat spacetime (no curvature).

2. Using the relation $\log DetM = Tr(\log M)$ for a matrix M, find out the expression for $\delta(-g)^{-n/2}/\delta g^{\mu\nu}$, in terms of $\delta g_{\mu\nu}$ and $\delta g^{\mu\nu}$ with g being the determinant of matrix form of $g_{\mu\nu}$.

3. Show hat the Einstein's field equation $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ suggests that all vacuum spacetimes are *Ricci flat*, i.e. $R_{\mu\nu} = 0$. What is the condition if the Einstein's equation gets corrected by a presence of cosmological constant to $G_{\mu\nu} = 8\pi G T_{\mu\nu} + \Lambda g_{\mu\nu}$?

4. A scalar field has the Lagrangian density as

$$\mathcal{L} = \frac{1}{2\alpha} \sin\left[\alpha(\partial_\mu \phi)(\partial^\mu \phi)\right] - V(\phi).$$

   Obtain the expression of the stress energy tensor prescribed as

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S[\phi]}{\delta g^{\mu\nu}},$$

   where $S[\phi]$ is the action for the scalar field.

5. A simple spacetime has a certain quantity called geodesic deviation expansion parameter θ defined as

$$\frac{d\theta}{d\lambda} = -\alpha\theta^2 - R_{\mu\nu} u^\mu u^\nu,$$

   where $R_{\mu\nu}$ is the Ricci tensor, u^μ is tangent along the geodesic which is parameterized by λ and α is a constant. Find out for the geodesic expansion rate to become positive (i.e. for gravity to become repulsive), what is the necessary condition for the matter stress energy tensor $T_{\mu\nu}$, in Einstein's theory of relativity ?