Solution to HW 7

1) $g(x) = 1 - x^{2/3}$ is continuous on $[0, \infty)$
g in particular on $[0,1]$.
$g([0,1]) = [0,1] \subseteq [0,\infty)$
and $\sqrt{x}$ is continuous on $[0,\infty)$.
Thus their composition $\sqrt{g(x)}$ is continuous on $[0,1]$.

2) Suppose $|f(x)| \leq M$ for some $M > 0$.
$\Rightarrow \quad |x f(x)| \leq M|x|$
Hence $\forall\, \varepsilon > 0$ choose $\delta = \frac{\varepsilon}{M}$. Then
$0 \leq x < \delta \Rightarrow |x f(x)| < \varepsilon$
i.e. $0 \leq x - 0 < \delta \Rightarrow |g(x) - g(0)| < \varepsilon$
$\Rightarrow \lim\limits_{x \to 0+} g(x) = g(0)$.
Hence, g is continuous at $x = 0$.
Since, both $f(x)$ and x are continuous on $(0,1]$ so is $g(x)$.
$\Rightarrow g$ is continuous on $[0,1]$.

Note: 1 has no special role: The same is true if we replace $[0,1]$ by $[0,\infty)$ or $[0,b]$ $\forall\, b > 0$.

The 2nd part follows from this note.
since $\sin x$ is continuous, $|\sin x| \leq 1 \ \forall x \in \mathbb{R}$

4) If f is uniformly continuous on $[0,\infty)$ then it is uniformly continuous on $[k,\infty)$ $\forall k \in (0,\infty)$.

Conversely, suppose $k \in (0,\infty)$ and f is uniformly continuous on $[k,\infty)$.

We know that f is uniformly continuous on $[0,k]$ too. Now, given $\varepsilon>0$ let $\delta_1>0$, $\delta_2>0$ be such that

$$|f(x)-f(y)|<\varepsilon/2 \text{ for } |x-y|<\delta_1, \ x,y\in[0,k]$$

and $|f(z)-f(w)|<\varepsilon/2$ for $|z-w|<\delta_2$, $z,w\in[0,\infty)$

Let $\delta=\min(\delta_1,\delta_2)$.

<u>claim</u>: $\forall a,b \in [0,\infty)$ with $|a-b|<\delta$ we have

$$|f(a)-f(b)|<\varepsilon.$$

Note that if a,b are both in $[0,k]$ or $[0,\infty)$ then we are done.

Hence, without loss of generality, we may assume that $a\in[0,k)$ and $b\in(k,\infty)$.

Since $|a-b|<\delta$; $|a-k|<\delta$ and $|k-b|<\delta$.

Note: $k\in(a,b)$.

Hence, by the choice of δ;

$$|f(a)-f(k)|<\varepsilon/2 \text{ and } |f(k)-f(b)|<\varepsilon/2$$

$$\Rightarrow |f(a)-f(b)| \leqslant |(f(a)-f(k))+(f(k)-f(b))|$$
$$\leqslant |f(a)-f(k)|+|f(k)-f(b)| < \varepsilon.$$

This proves the claim and the solves the problem.

3) We will first show that $h(x)$ is uniformly continuous on $[0,\infty)$.

Since $\lim_{x\to 0} \frac{\sin x}{x} = 1$; $\lim_{x\to 0+} \frac{\sin x}{x} = 1$.

$$\Rightarrow \lim_{t\to\infty} \frac{\sin 1/t}{1/t} = \lim_{t\to\infty} t\sin\frac{1}{t}$$
$$= \lim_{t\to\infty} h(t) = 1$$

Hence, given $\varepsilon > 0$ $\exists M$ such that
$$|1-h(t)| < \varepsilon/4 \quad \forall t \geqslant M.$$

Hence, $\forall x, y \in [M,\infty)$
$$|1-h(x)| < \varepsilon/4$$
$$|1-h(y)| < \varepsilon/4$$
$$\Rightarrow |h(x)-h(y)| < \varepsilon/2$$

Since h is uniformly continuous on $[0,M]$ $\exists \delta > 0$ such that
$$|h(z)-h(w)| < \varepsilon/2 \quad \forall z, w \in [0,M]$$
with $|z-w| < \delta$.

Now, let $a, b \in [0,\infty)$ and $|a-b| < \delta$.

If a, b are both in $[0, M]$ or both in $[M, \infty)$ then $|h(a) - h(b)| < \varepsilon/2 < \varepsilon$.

If $a \in [0, M]$ and $b \in [M, \infty)$ then $|a - M| < \delta$.

$\Rightarrow |h(a) - h(M)| < \varepsilon/2$ and $|h(M) - h(b)| < \varepsilon/2$

$\Rightarrow |h(a) - h(b)| < \varepsilon/2 + \varepsilon/2 = \varepsilon$

Similarly, if $b \in [0, M]$, $a \in [M, \infty)$; $|a - b| < \delta$ then $|h(a) - h(b)| < \varepsilon$.

This proves that h is uniformly continuous on $[0, \infty)$.

Ex: Similarly check that h is uniformly continuous on $(-\infty, 0]$.

Then use the idea of (4) to show that h is uniformly continuous on $\mathbb{R}$.

5. (i) This similar to (3)

Note: $\lim_{x \to \infty} \frac{1}{x} = 0$. Check the rest and show uniform continuity

(ii) $\lim_{x \to \infty} \frac{x}{x+1} = 1$ and the use the proof of (3) to show uniform continuity

(iii) $f(x) = \sqrt{x}$ on $[0, \infty)$

This was done in class.

However $f(x) - f(y)$

$$= \sqrt{x} - \sqrt{y} = \frac{x - y}{\sqrt{x} + \sqrt{y}}$$

$$\Rightarrow \quad |f(x)-f(y)| = \frac{|x-y|}{\sqrt{x}+\sqrt{y}} \leq |x-y|$$

when $x \geq 1, y \geq 1$

Show that then f is uniformly continuous on $[1, \infty)$. Clearly f is uniformly continuous on $[0,1]$. Hence, by (4), f is uniformly continuous on $[0, \infty)$.

6. 19.12

(i) $f(x) = 3x + 11$ on $\mathbb{R}$

Then $|f(x)-f(y)| = 3|x-y| < \varepsilon$

for $|x-y| < \varepsilon/3 = \delta$.

(ii) $f(x) = x^2$ on $[0,3]$

$$|f(x)-f(y)| = |x^2-y^2|$$

$$= |x+y|\,|x-y| \leq 6|x-y|$$

Thus choose $\delta = \varepsilon/6$ in this case.

(iii) $f(x) = \frac{1}{x}$ on $[\frac{1}{2}, \infty)$

$$\Rightarrow |f(x)-f(y)| = \left|\frac{1}{x}-\frac{1}{y}\right|$$

$$= \frac{|x-y|}{xy} \leq \frac{|x-y|}{(\min(x,y))^2}$$

Since, $x, y \in [1/2, \infty)$, $\min(x,y) \geq 1/2$

$$\Rightarrow \quad \frac{1}{\min(x,y)} \leq 2$$

It follows that $|f(x) - f(y)| \leqslant 4|x-y|$.

Hence, we choose $\delta = \varepsilon/4$. $\forall \varepsilon > 0$.