End-Sem Exam: Complex Analysis

All questions are worth 5 marks.

1. Let $\mathbb{S} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1^2 + x_2^2 + x_3^2 = 1\}$ be the Riemann sphere. We identify the XY-plane with $\mathbb{C}$ under $(x, y) \mapsto x + iy$. Consider the Stereographic projection

$$\pi : \mathbb{S} \setminus (0, 0, 1) \to \mathbb{C}.$$

   a) For $p \in \mathbb{S} \setminus \{\pm(0, 0, 1)\}$, write the image p and its antipodal point under π.

   b) Show that z and z' correspond to diametrically opposite points on $\mathbb{S}$ if and only if $z\overline{z'} = -1$, where $\overline{z'}$ denotes the complex conjugate of z'.

2. Find the number of zeros of $p(z) = z^6 + 9z^4 + z^3 + 2z + 4$ in the unit disc in $\mathbb{C}$.

3. Let $m, n \geq 1$. Compute the poles and residues of

$$\frac{1}{z^n(1-z)^m}.$$

4. Compute the real integral

$$\int_0^{2\pi} \frac{d\theta}{13 - 12cos(\theta)}.$$

5. Consider the rectangle $R = (\frac{-1}{2}\pi, \frac{5}{2}\pi) \times (-1, 2)$ in $\mathbb{C}$. On the oriented boundary of R, compute the contour integral

$$\int \frac{e^z dz}{sin(z)}.$$

6. Consider the fractional linear transformation from the extended complex plane $\mathbb{C} \cup \{\infty\}$ to itself given by

$$z \mapsto \frac{z}{z-4}.$$

   a) Show that the image of the circle $C = \{z | |z-2| = 2\}$ is a line. Hint: Consider the case $z = 4$.

   b) Determine the image of C.

c) Describe the image of the interior of C.

7. Consider the exponential map $\mathbb{C} \to \mathbb{C}$ given by $z \mapsto e^z$ i.e.

$$x + iy \mapsto e^x(cos(y) + isin(y)).$$

Let $a, b, c, d \in \mathbb{R}$ be constants with $a < b$ and $c < d$.

   a) Describe the image of the strip $\{z \in \mathbb{C} | a < Re(z) < b\}$. Hint: Describe first the image of the line $Re(z) = a$.

   b) Assuming $c - d < 2\pi$, describe the image of the strip $\{z \in \mathbb{C} | c < Img(z) < d\}$. Hint: Describe first the image of the line $Img(z) = c$.

   c) Describe the image of the Rectangle $R = \{a < Re(z) < b, -\pi < Img(z) < \pi\}$.

   d) At what angle do the images of $Re(z) = a$ and $Img(z) = b$ intersect and why?

8. Let a and b be complex numbers. Consider the rational function $(z - a)(z - b)$ as a function from the extended complex plane $\mathbb{C} \cup \{\infty\}$ to itself. Compute the order of pole and residue at ∞.