PHY302: Quantum mechanics
Tutorial-1

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***Problem* 1** : A particle of mass m in a one-dimensional potential $V(x)$ has the wave function

$$\psi(x) = Nx \; exp\left(-\frac{1}{2}\alpha x^2\right), \qquad \alpha > 0.$$

(a) Normalize $\psi(x)$ to determine N. What is $\langle \hat{x} \rangle$ What is $\langle \hat{x}^2 \rangle$?

(b) What is $\langle \hat{p} \rangle$? What is $\langle \hat{p}^2 \rangle$?

(c) Is $\psi(x)$ a position eigenstate ? Is $\psi(x)$ a momentum eigenstate ? Explain.

(d) Suppose that $V(x) = 0$. What is $\langle \hat{H} \rangle$?

(e) Suppose that nothing is known about $V(x)$, but $\psi(x)$ is an energy eigenstate. Find the potential $V(x)$ and the energy eigenvalue E, assuming $V(0) = 0$. Could $\psi(x)$ be the ground state wavefunction for the particle?

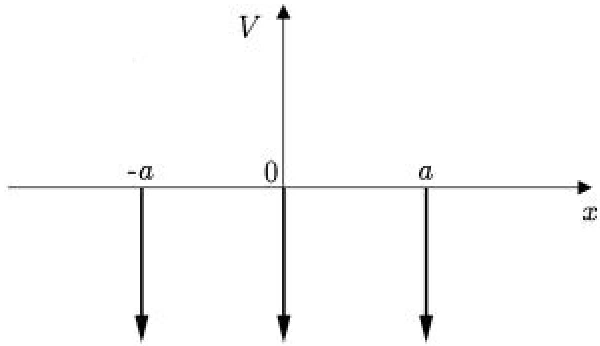
***Problem* 2** : A particle of mass m moves in one dimension, subject to a potential energy function $V(x)$ which is the sum of three evenly spaced attractive delta functions:

$$V(x) = -V_0 a \sum_{n=-1}^{1} \delta(x - na), \qquad \textit{where } V_0 > 0,\ a > 0 \textit{ are constants.}$$

1. Calculate the discontinuity in the first derivative of the wavefunction at $x = -a$, 0, and a.

2. Consider the possible number and locations of nodes in bound state wavefunctions for this system.

   (a) How many nodes are possible in the region $x > a$?



(b) How many nodes are possible in the region $0 < x < a$?

(c) Can there be a node at $x = a$?

(d) Can there be a node at $x = 0$?

***Problem* 3** : Consider the finite square well potential in section 2.6 of Griffiths:

$$V(x) = -V_0 \; for - a \leq x \leq a, \quad and \;\; V(x) = 0 \; for \; |x| > a.$$

(a) Number of bound states for deep well. Assume that the well is sufficiently deep and/or wide so that z_0, defined as

$$z_0 = \frac{a}{\hbar}\sqrt{2mV_0},$$

is a large number. Find an estimate for the number of bound states in this well using the result that the $(n+1)^{th}$ bound state has n nodes. Confirm that your result is a good approximation by comparing with Figure 2.18 in the book.

***Problem* 4** : Expectation value $\langle \hat{p} \rangle$ of the momentum.

(a) A particle's coordinate space wavefunction is square-integrable and real up to an arbitrary multiplicative phase:

$$\psi(x) = e^{i\alpha}\phi(x),$$

with α real and constant and $\phi(x)$ real. Prove that the expectation value of the momentum is zero.

(b) Consider instead the wavefunction

$$\psi(x) = \phi_1(x) + e^{i\alpha}\phi_2(x),$$

where $\phi_1(x)$ and $\phi_2(x)$ are each real and square-integrable. What is $\langle \hat{p} \rangle$? The answer can be expressed as a function of α times an integral that involves the functions ϕ_2 and $d\phi_1/dx$ (or ϕ_1 and $d\phi_2/dx$). For what values of α can we be sure that $\langle \hat{p} \rangle$ is zero without having further information about ϕ_1 and ϕ_2 ?

Problem* 5 : *Conserved probability current. Suppose $\Psi(x,t)$ obeys the one-dimensional *Schrödinger* equation,

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x,t) + V(x)\Psi(x,t) = i\hbar\frac{\partial}{\partial t}\Psi(x,t).$$

(a) Derive the conservation law for probability,

$$\frac{\partial \rho}{\partial t} + \frac{\partial J}{\partial x} = 0, \tag{1}$$

where $\rho(x,t)$ is the probability density and J(x,t) is the probability current density

$$\rho(x,t) = \Psi^*\Psi, \qquad\qquad J(x,t) = \frac{\hbar}{m} Im\left(\Psi^*\frac{\partial \Psi}{\partial x}\right)$$

What are the units of ρ and J?

(b) Explain why (1) is a conservation law for probability. In order to do so, define

$$P_{ab} = \int_a^b dx \rho(x,t),$$

evaluate $\frac{dP_{ab}}{dt}$ in terms of currents, and interpret your answer. Show then that a wavefunction $\Psi(x,t)$ that is normalized at time t remains normalized at later times.