§§ Linear equations with Constant Coefficients.

A linear DE of order n with Constant coeff. is an eqn. of the form

$$a_0 y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \cdots + a_n y = b(x)$$

Where $a_0 \neq 0, \ldots, a_n$ are Complex Constants and b is some Complex valued function on an interval I. Divide by a_0 to assume that eqn. is monic:

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = b(x).$$

Denote $L(y) = y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y$

Eqn becomes $L(y) = b(x)$.

If $b(x) \equiv 0$ the eqn. is called <u>homogeneous</u>
otherwise " " non-homogeneous.

L is a <u>differential operator</u> which operates on fns. which have n derivatives on I and transform a fn. ϕ to $L(\phi)$ whose value at x is

$$L(\phi)(x) = \phi^{(n)}(x) + a_1 \phi^{(n-1)}(x) + \cdots + a_n \phi(x).$$

Thus $L(\phi) = \phi^{(n)} + a_1 \phi^{(n-1)} + \cdots + a_n \phi$.

So a soln of $L(y) = b(x)$ is a fn ϕ with n derivates and S.T. $L(\phi) = b$.

▲ For $n = 1$ we found all solutions. Now we consider $n = 2$.

§ $L(y) = y'' + a_1 y' + a_2 y = 0$, a_1, a_2 Constants.

Recall that first order eqn with const. coeff. $y' + ay = 0$ has a soln. e^{-ax}. The Constant $-a$ is a soln of the eqn $r + a = 0$. Since differentiating e^{rx} (r = Const.) any no. of times yields a Constant times e^{rx}, it is reasonable to expect that e^{rx} with an appropriate Constant is a soln. to $L(y) = 0$.

$$L(e^{rx}) = (r^2 + a_1 r + a_2)\, e^{rx}$$

and e^{rx} will be a soln. of $L(y) = 0$ i.e. $L(e^{rx}) = 0$ if r satisfies $r^2 + a_1 r + a_2 = 0$.

We let $p(r) = r^2 + a_1 r + a_2$

and call p the <u>characteristic polynomial</u> of L.

$p(r)$ can be obtained from L by replacing $y^{(k)}$ everywhere by r^k.

By Fund. thm of algebra there are roots r_1, r_2 (possibly Complex) of $p(r)$.

If $r_1 \neq r_2$, we see that $e^{r_1 x}$ and $e^{r_2 x}$ are solns of $L(y) = 0$.

In Case $r_1 = r_2$ we have $L(e^{rx}) = p(r)\, e^{rx}$ $\forall r \,\forall x$.

Since we have repeated roots, $p'(r_1) = 0$ as well.

This suggests differentiating $L(e^{rx}) = p(r)\, e^{rx}$ w.r.t. r.

Since L involves only diff. w.r.t. x,

$$\frac{\partial}{\partial r} L(e^{rx}) = L\left(\frac{\partial}{\partial r} e^{rx}\right) = L(xe^{rx}).$$

$$\Rightarrow L(xe^{rx}) = [p'(r) + x\, p(r)]\, e^{rx}.$$

Setting $r = r_1$ we see that $L(xe^{r_1 x}) = 0$
Showing that $xe^{r_1 x}$ is another soln. in case $r_1 = r_2$.
We have proven:

<u>Theorem</u> Let a_1, a_2 be constants and consider
$$L(y) = y'' + a_1 y' + a_2 y = 0.$$
If r_1, r_2 are distinct roots of the char. poly of p:
$$p(r) = r^2 + a_1 r + a_2$$
then the fn. φ_1, φ_2 defined by
$$\varphi_1(x) = e^{r_1 x}, \quad \varphi_2(x) = e^{r_2 x}$$ are solns. of $L(y) = 0$.
If r_1 is a repeated root of p then the fns φ_1, φ_2 defined by $\varphi_1(x) = e^{r_1 x}$, $\varphi_2(x) = xe^{r_1 x}$ are solns. of $L(y) = 0$.

△ Note immediately that $\varphi = c_1\varphi_1 + c_2\varphi_2$ is also a solution of $L(y) = 0$ [linearity of L]. ; $c_1, c_2 \in \mathbb{C}$, constants
And 0 is also a solution of $L(y) = 0$.

<u>Example</u> [Harmonic oscillator] $y'' + \omega^2 y = 0$, ω is positive constant.

Char. poly is $p(r) = r^2 + \omega^2 = 0$, roots $i\omega$ & $-i\omega$.

So solns are $e^{i\omega x}$, $e^{-i\omega x}$ and more generally $c_1 e^{i\omega x} + c_2 e^{-i\omega x}$

Take $c_1 = c_2 = \frac{1}{2}$ to see $\cos\omega x$ is a soln.
$c_1 = \frac{1}{2i}$, $c_2 = -1/2i$ to see $\sin\omega x$ is a soln.

- We will now show that all solutions are obtained by the remark above: the linear combination of the two solutions we found.

§ Initial value problems for second order equations.

An I.V.P for $L(y)=0$ is a problem of finding a soln φ s.t. $\varphi(x_0)=\alpha$ and $\varphi'(x_0)=\beta$.
where x_0 is some real no. & α,β given constants.

Problem denoted: $L(y)=0$, $y(x_0)=\alpha$, $y'(x_0)=\beta$.

<u>Theorem</u> (Existence theorem) For any real x_0, and constants α,β, there exists a solution ~~of~~ φ of the I.V.P on $-\infty<x<\infty$.

<u>Proof:</u> ~~Let~~ We must have $c_1\varphi_1(x_0)+c_2\varphi_2(x_0)=\alpha$
We show $\exists!\, c_1,c_2$ s.t. $c_1\varphi_1'(x_0)+c_2\varphi_2'(x_0)=\beta$
$\varphi=c_1\varphi_1+c_2\varphi_2$ is a soln.

$\exists$ a ! soln c_1,c_2 if the determinant $\Delta=\begin{vmatrix}\varphi_1(x_0) & \varphi_2(x_0)\\ \varphi_1'(x_0) & \varphi_2'(x_0)\end{vmatrix}$
$=\varphi_1(x_0)\varphi_2'(x_0)-\varphi_1'(x_0)\varphi_2(x_0)\neq 0$.

In case $r_1\neq r_2$, $\varphi_1(x)=e^{r_1x}$, $\varphi_2(x)=e^{r_2x}$.

and $\Delta = r_2e^{r_1x_0}e^{r_2x_0}-r_1e^{r_1x_0}e^{r_2x_0}=(r_2-r_1)e^{(r_1+r_2)x_0}$
$\neq 0$ since $r_1\neq r_2$.

If $r_1=r_2$, $\varphi_1(x)=e^{r_1x}$, $\varphi_2(x)=xe^{r_1x}$.
and $\Delta=e^{r_1x_0}(e^{r_1x_0}+x_0r_1e^{r_1x_0})-r_1x_0e^{r_1x_0}e^{r_1x_0}$
$=e^{2r_1x_0}\neq 0$.

So $\Delta\neq 0$ satisfied in both cases and we get unique c_1,c_2 and hence a soln ~~of the~~ of the I.V.P.

Now we show that this solution is even unique.

~~Theorem~~

Lemma [Estimate on solutions $\varphi(x)$ of $L(y)=0$]. Let φ be soln. of $L(y)=0$ on $I \ni x_0$. $\forall\, x \in I$

$$\|\varphi(x_0)\| e^{-k|x-x_0|} \leq \|\varphi(x)\| \leq \|\varphi(x_0)\| e^{k|x-x_0|}$$

where $\|\varphi(x)\| = \left[|\varphi(x)|^2 + |\varphi'(x)|^2\right]^{1/2}$ (size of φ & φ')

& $k = 1 + |a_1| + |a_2|$ (size of L).

Proof: Let $u(x) = \|\varphi(x)\|^2 = \varphi\bar{\varphi} + \varphi'\bar{\varphi}'$

Then $u'(x) = \varphi'\bar{\varphi} + \varphi\bar{\varphi}' + \varphi''\bar{\varphi}' + \varphi'\bar{\varphi}''$

So $|u'(x)| \leq 2|\varphi(x)||\varphi'(x)| + 2|\varphi'(x)||\varphi''(x)|$.

Since $L(\varphi)=0$ we have $\varphi'' = -a_1\varphi' - a_2\varphi$.

So $|\varphi''(x)| \leq |a_1||\varphi'(x)| + |a_2||\varphi_2(x)|$

$\Rightarrow$ $|u'(x)| \leq 2(1+|a_2|)|\varphi(x)||\varphi'(x)| + 2|a_1||\varphi'(x)|^2$

Also $2|\varphi(x)||\varphi'(x)| \leq |\varphi(x)|^2 + |\varphi'(x)|^2$

So $|u'(x)| \leq (1+|a_2|)|\varphi(x)|^2 + (1+2|a_1|+|a_2|)|\varphi'(x)|^2$

$\leq 2(1+|a_1|+|a_2|)\left[|\varphi(x)|^2 + |\varphi'(x)|^2\right]$

or $|u'(x)| \leq 2k\,u(x)$.

$\Rightarrow$ $-2ku(x) \leq u'(x) \leq 2ku(x)$

Which directly gives the required inequalities by integration.

Theorem (Uniqueness of solutions): The IVP $L(y)=0, y(x_0)=\alpha, y'(x_0)=\beta$ has ~~at~~ ~~unique~~ atmost one solution on $I \ni x_0$.

Proof: Suppose φ, ψ are 2 solns. Let $\chi = \varphi - \psi$.

Then $L(\chi) = L(\varphi) - L(\psi) = 0$ & $\chi(x_0) = 0, \chi'(x_0) = 0$

So $\|\chi(x_0)\| = 0$. Hence by inequalities in lemma,
$\|\chi(x)\| = 0 \ \forall x \in I \Rightarrow \chi(x) = 0 \ \forall x \in I$.
i.e. $\varphi = \psi$.

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Hence we have the existence and uniqueness of soln.

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