PHY302: Quantum mechanics
Tutorial-4

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Question no.1 Basis independent quantities

Consider a vector space V and a change of basis from $(v_1, v_2.....v_n)$ to $(u_1, u_2......u_n)$ defined by linear operator $A : v_k \to u_k$ for $k = 1, 2....n$. The operator is clearly *invertible* because, letting $B : u_k \to v_k$, we have $BA : v_k \to v_k$, showing that $BA = \mathbb{1}$ and $AB : u_k \to u_k$, showing that $AB = \mathbb{1}$. Thus B is the inverse of A.

(a) Consider the mapping equations

$$u_k = Av_k, \qquad \textit{and} \qquad v_k = Bu_k,$$

and write them explicitly using the matrix representation of A in the $v-$basis and the matrix representation of B in the $u-$basis. Show that these two matrices are inverse of each other.

Consider now the linear operator T in V. Let $T_{ij}(\{v\})$ denote its matrix representation in the v basis and $T_{ij}(\{u\})$ denote its matrix representation in the u basis.

(b) Find a matrix relation between $T_{ij}(\{v\})$ and $T_{ij}(\{u\})$, written in terms of the matrix representative of A and its inverse.

(c) Show that the trace of the matrix representation of T is basis independent.

(d) Show that the determinant of the matrix representation of T is basis independent.

Question no.2 Identities for commutators

In the following problem A,B and C are linear operators. So are q and p.

(a) Prove the following commutator identity:

$$[A, BC] = [A, B]C + B[A, C].$$

This is the derivation property of commutator: the commutator with A, that is the object $[A, \cdot]$, acts like a derivative on product BC. In the result the commutator is first taken with B and then taken with C while the operator that stays untouched is positioned at expected place.

(b) Prove the Jacobi identity:

$$[[A, B], C] + [[B, C], A] + [[C, A], B] = 0.$$

(c) Using $[q, p] = i\hbar$ and the result of (a), Show that

$$[q^n, p] = i\hbar n q^{n-1}.$$

(d) For any function $f(q)$ that can be expanded in a power series in q, use (c) to show

$$[f(q), p] = i\hbar f'(q).$$

(e) On the space of position-dependent function, the operator $f(x)$ acts multiplicatively and p acts $\frac{\hbar}{i}\frac{\partial}{\partial x}$. Calculate $[f(x), p]$ by letting this operator act on an arbitrary wave-function.

Question no.3 Useful operator identities and translations Suppose that A and B are two Operators that do not commute, $[A, B] \neq 0$.
(a) Let t be a formal variable. Show that

$$\frac{d}{dt}e^{(A+B)t} = (A + B)e^{(A+B)t} = e^{(A+B)t}(A + B).$$

(b) Now suppose $[A, B] = c$, where c is a c-number(a complex number times the identity operator). Prove that

$$e^A B e^{-A} = B + c \tag{1}$$

[Hint: Define an operator-valued function $F(t) = e^{At}Be^{-At}$. What is $F(0)$? Derive a differential equation for $F(t)$ and integrate it.]
Comment: Equation (1) is a special case of the Hadamard lemma, to be considered below.

(c) Let a be real number and $\hat{p}$ be the momentum operator. Show that the unitary **translation operator**

$$\hat{T}(a) = e^{-ia\hat{p}/\hbar}$$

translate the position operator:

$$\hat{T}^\dagger\,(a)\hat{x}\,\hat{T}(a) = \hat{x} + a$$

If a state $|\psi\rangle$ is described by the wave function $\langle x|\psi\rangle = \psi(x)$, show that the state $\hat{T}(a)\,|\psi\rangle$ is described by the wave function $\psi(x-a)$.

Question no.4 Bras and Kets.

Consider a three-dimensional Hilbert space with an orthonormal basis $|1\rangle\,, |2\rangle\,, |3\rangle\,.$ Using complex constant a and b define the kets

$$|\psi\rangle = a\,|1\rangle - b\,|2\rangle + a\,|3\rangle\,; \qquad\qquad |\phi\rangle = b\,|1\rangle + a\,|2\rangle\,.$$

(a) Write down $\langle\psi|$ and $\langle\phi|$. Calculate $\langle\psi|\phi\rangle$ and $\langle\phi|\psi\rangle$. Check that $\langle\psi|\phi\rangle = \langle\phi|\psi\rangle^*$

(b) Express $|\psi\rangle$ and $|\phi\rangle$ as column vector in the $|1\rangle\,, |2\rangle\,, |3\rangle$ basis and repeat (a).

(c) Let $A = |\phi\rangle\,\langle\psi|$. Find the 3 x 3 matrix that represents A in the given basis.

(d) Let $Q = |\psi\rangle\,\langle\psi| + |\phi\rangle\,\langle\phi|$. Is Q hermitian ? Give a simple argument (no computation) to show that Q has a zero eigenvalue.

Question no.5 Hermitian matrices and anticommutators

Consider Hermitian matrices M^1, M^2, M^3, M^4 that obey

$$M^iM^j + M^jM^i = 2\delta^{ij}\mathbb{1} \qquad\qquad i,j = 1,2,3,4.$$

(a) Show that the eigevales of M^i are ± 1.(Hint: go to the eigenbasis of M^i, and use the equation for i=j.)

(b) By considering the relation

$$M^iM^j = -M^jM^i \qquad for \qquad i \neq j,$$

show that M^i are trace less. [Hint: Tr(ABC)=Tr(BCA).]

(c) Show that they can not be odd-dimensional matrices.