INDIAN INSTITUTE OF SCIENCE EDUCATION AND RESEARCH MOHALI

MTH101 (Symmetry)

Tutorial Sheet 09 / March 29, 2022 Spring 2022

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1. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two distinct lines in $\mathbb{R}^2$ which pass through the origin $(0,0)$. Show that $\mathcal{L}_1 \cup \mathcal{L}_2$ is not a vector space under vector addition of vectors and usual scaling of vectors in $\mathbb{R}^2$. What if we take the union of finitely many distinct lines $\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_n$ for $n \geq 3$?

2. Consider the vector space $(\mathbb{R}[x], +.)$. If $S = \{x^n + x^m : 1 \leq n, m \leq 2\} \subseteq \mathbb{R}[x]$, then

   (a) How many elements are there in the set S?

   (b) What is span(S)?

   (c) What is the dimension of the vector space span(S)?

   (d) If $S = \{x^n + x^m : n, m$ are non-negative integers$\}$, then is is true that span$(S) = \mathbb{R}[x]$?

3. Consider the vector space $(M_n(\mathbb{R}), +, .)$, for $n \geq 2$. Let S be the set of swapper matrices in $M_n(\mathbb{R})$. Is it true that S is a basis of $M_n(\mathbb{R})$?

4. For $v = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3$, the positive real number $\ell(v) := (a^2 + b^2 + c^2)^{1/2}$ is called the *length* of v. A linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ is called *rigid*, if $\ell(T(v)) = \ell(v)$ for all $v \in \mathbb{R}^3$.

   (a) Show that the transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by $T(v) = R_{x,\theta}v$ is rigid, where $R_{x,\theta}$ is the rotation matrix about x-axis by angle θ.

   (b) For $v, w \in \mathbb{R}^3$, the quantity
   $$\beta(v, w) := \frac{v.w}{\ell(v)\ell(w)},$$
   where $v.w$ is the dot product of v and w, is called the *angle cosine* of v and w. Show that if $T : \mathbb{R}^3 \to \mathbb{R}^3$ is a rigid linear transformation then it preserves angle cosine, *i.e.*, for $v, w \in \mathbb{R}^3$, we have $\beta(v, w) = \beta(T(v), T(w))$.
