Quantum Measurements:

→ Von-Neuman Measurements:

Projective measurement

$|\psi\rangle \in \mathcal{H}_S$

$A \in \mathcal{B}(\mathcal{H}_S)$

$A = A^\dagger = \sum_j a_j |\psi_j\rangle\langle\psi_j|$

$|\phi\rangle = \sum_n c_n |\psi_n\rangle$

$p_n = |\langle\psi_n|\phi\rangle|^2 = \mathrm{Tr}[|\psi_n\rangle\langle\psi_n| \, |\phi\rangle\langle\phi|] = |c_n|^2$

$\equiv$ Probability of getting n^{th} outcome.

→ Define projectors $P_n = |\psi_n\rangle\langle\psi_n|$

→ $P_n^2 = P_n$

→ $\mathrm{Eig}(P_n) = 1, 0$

→ Probability p_n for a Density matrix ρ

$p_n = \mathrm{Tr}[P_n \rho]$

→ After the measurement the state of the system is $\tilde{\rho} = \dfrac{P_n \rho P_n}{\mathrm{Tr}[P_n \rho]} = \dfrac{1}{p_n} |\psi_n\rangle\langle\psi_n| \rho |\psi_n\rangle\langle\psi_n|$

$$\boxed{\rho_n = |\psi_n\rangle\langle\psi_n|}$$

→ After repeating the measurements, we get

$$\boxed{\rho_{total} = \sum_n p_n \rho_n = \sum_n p_n |\psi_n\rangle\langle\psi_n|}$$

→ Projector $P^{(i)}$ can be $P^{(i)} = \sum_{n_i} |\psi_{n_i}\rangle\langle\psi_{n_i}|$

Still $P^{(i)2} = P^{(i)}$, $P^{(i)} = P^{(i)\dagger}$

$eig(P^{(i)}) = 0, 1$

→ If we perform measurements in the projector $P^{(i)}$ such that $P^{(i)}P^{(j)} = \delta_{ij}$ and $\sum_i P^{(i)} = \mathbb{1}$

Then the measurement is complete and the outcome will be $\rho^{(i)} = \frac{P^{(i)} \rho P^{(i)}}{Tr[P^{(i)} \rho]}$

→ $\rho_{final} = \sum_i p_i \rho^{(i)} = \sum_i P^{(i)} \rho P^{(i)}$

→ For composite systems:

$$|\Psi\rangle = \sum_{ij} \alpha_{ij} |i\rangle \otimes |j\rangle$$

→ Performing measurement on subsystem B in basis $\{|\psi_n\rangle\}$ will result in the B system collapsing to state $|\psi_n\rangle$ and corresponding projected state of A

$\Rightarrow$ $|\Psi\rangle \rightarrow (\mathbb{1} \otimes |\psi_n\rangle\langle\psi_n|)|\Psi\rangle$

or $\rho_{AB} = |\Psi\rangle\langle\Psi| \rightarrow \rho_n = \frac{(\mathbb{1} \otimes |\psi_n\rangle\langle\psi_n|)\rho_{AB}(\mathbb{1} \otimes |\psi_n\rangle\langle\psi_n|)}{Tr[\mathbb{1} \otimes |\psi_n\rangle\langle\psi_n| \rho_{AB}]}$

$$\boxed{\rho_{final} = \sum_n (\mathbb{1} \otimes |\psi_n\rangle\langle\psi_n|)\rho_{AB}(\mathbb{1} \otimes |\psi_n\rangle\langle\psi_n|)}$$

Using a Probe to make a measurement.

→ System + (ancilla = probe)

$\rho_{AS} = \rho_A \otimes \rho_S$ → initial state of the system + ancilla

→ Interaction b/w S+A → U

$\rho_{AS} \to U \rho_{AS} U^\dagger = U(\rho_A \otimes \rho_S) U^\dagger$

→ Projective measurements on the probe.

$\tilde{\rho}_n = (P_n \otimes \mathbb{1}) U \rho_{AS} U^\dagger (P_n \otimes \mathbb{1})$ → Un Normalized

$\equiv \tilde{\rho}_n = |n\rangle\langle n| \otimes A_n \rho_S A_n^\dagger$; $p_n = \mathrm{Tr}[A_n^\dagger A_n \rho_S]$

$A_n = (\langle n| \otimes \mathbb{1} \cup |+\rangle \otimes \mathbb{1})$

where $\rho_A = |+\rangle\langle +|$

$|n\rangle\langle n| = P_n$

→ $\{A_n\}$ are the measurement operators

Prove it. $\sum_n A_n^\dagger A_n = \mathbb{1}$

→ State after measurement

$\rho_{\text{fin}} = \sum_n |n\rangle\langle n| \otimes A_n \rho_S A_n^\dagger$

$\rho_{S,\text{fin}} = \sum_n A_n \rho_S A_n^\dagger$

→ In the measurement, the $|n\rangle$ clicks with probability $p_n = \mathrm{Tr}[A_n^\dagger A_n \rho_S]$

→ Interestingly, $p_n = Tr[A_n^\dagger A_n \rho_s]$ remains same for any set of Measurement operators

$\{B_n = W_n A_n\}$ where $W_n \in$ Unitary.

→ Therefore, for the same experimental outcome p_n, we have a freedom in choosing the measurement operators.

⇒ We need to have a formalism which is free of this problem.

→ Positive operator valued measurements (POVM):

→ A POVM is a set of +ve semidefinite operators $\{E_i\}$ called effect, acting on the Hilbert space $\mathcal{H}_d$ that sum to the identity operator

⇒ $\boxed{\sum_i E_i = \mathbb{1} \text{ and } E_i \geq 0}$

→ The measurement outcome is the probability $p_n = \langle E_n \rangle = Tr[E_n \rho]$

→ For every effect E_i we can choose a measurement operator A_i such that

$E_i = A_i^\dagger A_i$

→ The state after the measurement is $\boxed{\frac{A_i \rho A_i^\dagger}{\langle E_i \rangle}}$

→ Neumark's Dilation theorem:

Neumark's Dilation theorem states that a POVM can be lifted to a projective-measurements on an extended Hilbert space.

(The proof is similar to the Kraus-operators - Unitary link).

→ Expectation values of observables using POVM:

(For qubits only)

→ let the observable is σ_z or and $\hat{n}.\vec{\sigma}$ with $|\hat{n}|=1$

we can define the effects.

$$E_{\pm} = \frac{1}{2}\left[\mathbb{1} \pm d\,\hat{n}.\vec{\sigma}\right] \qquad 0 \le d \le 1$$

→ $d = 1$ implies projective measurements.

→ $d < 1$ // unsharp (weak) measurement.

$$\boxed{\langle \hat{n}.\vec{\sigma} \rangle = \frac{\langle E_+ \rangle - \langle E_- \rangle}{d}}$$

→ Quantum state tomography using POVM.

→ For state tomography we need a POVM with d^2 effects $\{E_i\}_{i=1}^{d^2}$

$$\langle E_j \rangle = \mathrm{Tr}\left[\rho E_i\right] = \langle E_j | \rho \rangle = p_j$$

→ Define a matrix $F = \begin{bmatrix} \langle E_1| \\ \langle E_2| \\ \vdots \\ \langle E_{d^2}| \end{bmatrix}$

$\Rightarrow \quad F|\rho\rangle = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_{d^2} \end{bmatrix}$

→ If E_i are linearly independent

$\Rightarrow$ F^{-1} exists.

$\Rightarrow \boxed{|\rho\rangle = F^{-1} p}$ From here we can find the density matrix

→ One example of POVM for qubits

$$\frac{|0\rangle\langle 0|}{2}, \frac{|+\rangle\langle +|}{2}, \frac{|+_y\rangle\langle +_y|}{2}, \mathbb{1} - \frac{|0\rangle\langle 0|}{2} - \frac{|+\rangle\langle +|}{2} - \frac{|+_y\rangle\langle +_y|}{2}$$

→ Symmetric informationally complete POVM

SIC-POVM

A set of d^2 Rank one POVM $\{E_i = |\psi_i\rangle\langle\psi_i|\}$ which satisfy the relation.

$$\mathrm{Tr}[E_i E_j] = |\langle\psi_i|\psi_j\rangle|^2 = \frac{d\delta_{ij} + 1}{d+1}$$

→ SIC-POVM gives you the best estimate of the density operators.

→ Optical examples of generalized measurement

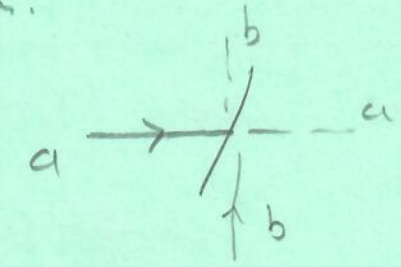
→ Projective measurement ($d \to 1$ limit)

→ Consider a photon in the polarization state

$|\psi\rangle = \alpha|H\rangle + \beta|V\rangle$

→ A Beam-Splitter as two input optical modes and two output modes.

→ $|a\rangle, |b\rangle$ are the two states of a beam splitter.



→ photon incoming in mode a $\Rightarrow$ state $|a\rangle$

// b $\Rightarrow$ $|b\rangle$

→ For a balanced (50:50) beam splitter

$$|a\rangle \to \frac{|a\rangle + |b\rangle}{\sqrt{2}}$$

$$|b\rangle \to \frac{|a\rangle - |b\rangle}{\sqrt{2}} \quad \Rightarrow \quad BS = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Beam-Splitter matrix.

→ Twin Beam splitter is independent of Polarization.

→ Polarizing beam splitter:

$$|H a\rangle \to |H a\rangle$$
$$|V a\rangle \to |V b\rangle$$
$$|H b\rangle \to |H b\rangle$$
$$|V b\rangle \to |V a\rangle$$

$\Rightarrow$

	H_a	V_a	H_b	V_b
H_a	1	0	0	0
V_a	0	0	0	1
H_b	0	0	1	0
V_b	0	1	0	0

=> A polarizing beam splitter will take the input state $(\alpha|H\rangle + \beta|V\rangle)\otimes|a\rangle \rightarrow \alpha|Ha\rangle + \beta|Vb\rangle$

=> Placing the detectors in $|a\rangle$ and $|b\rangle$ outputs will cause the state to collapse to $|H\rangle$ and $|V\rangle$.

=> Projective measurement.

→ Weak (Unsharp) measurement.

Consider a beam splitter with transmission t^2 and reflection r^2 => $t^2 + r^2 = 1$; let $t, r \in \mathbb{R}$

→ $BS = \begin{bmatrix} t & r \\ r & -t \end{bmatrix}$

=> $|a\rangle \rightarrow t|a\rangle + r|b\rangle$

$|b\rangle \rightarrow r|a\rangle - t|b\rangle$

→ Consider $|\psi\rangle = (\alpha|H\rangle + \beta|V\rangle)\otimes|a\rangle$.

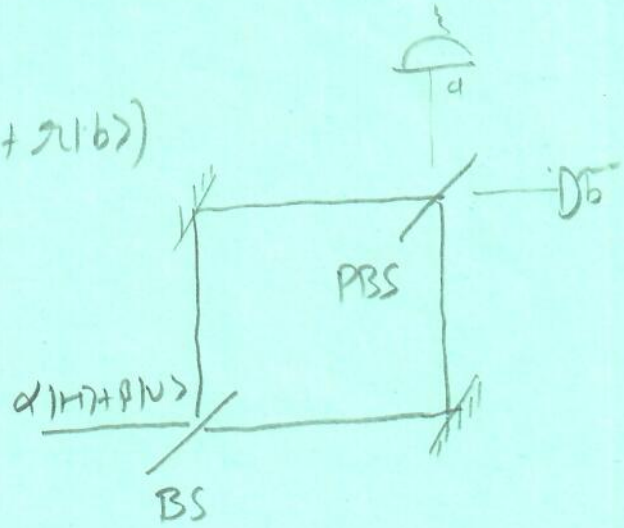
→ After BS

$|\psi\rangle \rightarrow (\alpha|H\rangle + \beta|V\rangle)\otimes(t|a\rangle + r|b\rangle)$

→ After PBS

$(t\alpha|H\rangle + \beta r|V\rangle)\otimes|a\rangle$

$+ (\beta t|V\rangle + \alpha r|H\rangle)\otimes|b\rangle$



=> Detecting photon in $|a\rangle$ => $(t\alpha|H\rangle + \beta r|V\rangle)$

$|b\rangle$ => $\beta t|V\rangle + \alpha r|H\rangle$

⇒ Measurement operator

$$K_1 = \begin{bmatrix} t & \\ & r \end{bmatrix} ; \quad K_2 = \begin{bmatrix} r & \\ & t \end{bmatrix}$$

$$\boxed{K_1^\dagger K_1 + K_2^\dagger K_2 = \mathbb{1}}$$

→ The probability of click in $|a\rangle \equiv p_a = |t\alpha|^2 + |\beta r|^2$

$$= \langle \psi | K_1^\dagger K_1 | \psi \rangle$$

// $\quad |b\rangle \equiv p_b = |\beta t|^2 + |\alpha r|^2$

$$= \langle \psi | K_2^\dagger K_2 | \psi \rangle$$

→ if $t = r = \frac{1}{\sqrt{2}}$ ⇒ Weakest measurement

if $t = 1, r = 0$ or $r = 1, t = 0$ ⇒ Projective measurements

→ Distinguishing non-orthogonal states:

→ Given two non-orthogonal states $|\psi\rangle, |\phi\rangle$ $\langle \psi | \phi \rangle \neq 0$, with equal probability, can we design an experimental scheme which distinguishes b/w $|\psi\rangle$ and $|\phi\rangle$.

→ The discrimination should by unambiguous.

→ We can choose the basis $\{|\psi\rangle, |\psi^\perp\rangle\}$ for the measurements. So that if we get a click in $|\psi^\perp\rangle$ then the state was $|\phi\rangle$. whereas if we get a click in $|\psi\rangle$ then we do not know if it was $|\psi\rangle$ or $|\phi\rangle$.

→ Probability of success is $p_s = \langle +_1 | \rho_s | +_1 \rangle$

where $\boxed{\rho_s = \frac{|+\rangle\langle+| + |\phi\rangle\langle\phi|}{2}}$

$$\Rightarrow \quad p_s = \frac{1}{2} |\langle +_1 | \phi \rangle|^2$$

→ Similarly, we can choose $\{|\phi\rangle, |\phi_1\rangle\}$ basis.

→ It seems like the best we can do.

→ For simplicity, let's consider

$$|+\rangle = \begin{bmatrix} \cos\theta/2 \\ \sin\theta/2 \end{bmatrix}; \quad |\phi\rangle = \begin{bmatrix} \cos\theta/2 \\ -\sin\theta/2 \end{bmatrix} \quad 0 \le \theta \le \pi/2$$

$$\Rightarrow \quad \boxed{p_s = \frac{1}{2} \sin^2\theta}$$

→ We now try generalized measurement to distinguish them.

→ Let the state of ancilla be $|0\rangle$.

→ Total State of system + ancilla

$$|\chi\rangle = \cos\frac{\theta}{2} |00\rangle \pm \sin\frac{\theta}{2} |10\rangle$$

→ Consider a unitary U such that

$U(\cos\frac{\theta}{2}|00\rangle) \rightarrow \sin\frac{\theta}{2}|00\rangle + \sqrt{\cos\theta}\,|11\rangle$

and $U|01\rangle = |01\rangle$ and $U|10\rangle = |10\rangle$

$$\Rightarrow U|\chi\rangle = \left(\sin\frac{\theta}{2}|00\rangle \pm \sin\frac{\theta}{2}|10\rangle\right) + \sqrt{\cos\theta}\,|11\rangle$$

$$= \sqrt{2}\sin\frac{\theta}{2}\left[\frac{|0\rangle \pm |1\rangle}{\sqrt{2}}\right]|0\rangle + \sqrt{\cos\theta}\ |11\rangle$$

⇒ if upon measurement on the ancilla we get $|1\rangle$ then the measurement result is inconclusive

→ if the outcome is $|0\rangle$ then the system will collapse to $\frac{|0\rangle \pm |1\rangle}{\sqrt{2}}$. Therefore, performing measurement on the system will result in the discrimination b/w $|+\rangle$ and $|-\rangle$

→ Success probability $p_s = 2\sin^2\frac{\theta}{2}$

$$\boxed{p_{ns} = 1 - |\langle +|-\rangle|^2}$$

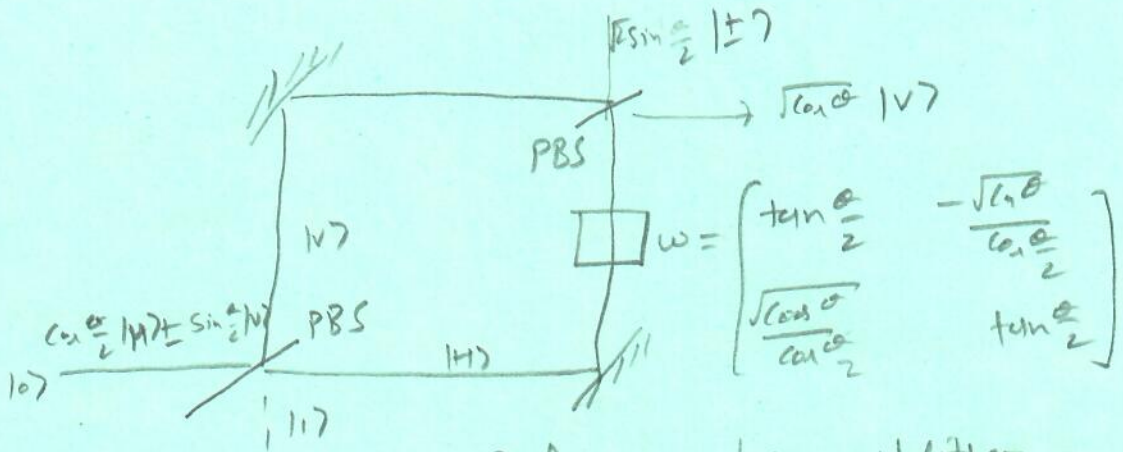
→ Surely this probability is more than what we got earlier

$$U = \left[\begin{array}{cc|cc} \tan\frac{\theta}{2} & 0 & 0 & -\frac{\sqrt{\cos\theta}}{\cos\theta/2} \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ \frac{\sqrt{\cos\theta}}{\cos\theta/2} & 0 & 0 & \tan\frac{\theta}{2} \end{array}\right]$$

→ Optical implementation.

→ $|+\rangle, |-\rangle$ are the polarization states of a single photon. Two spatial modes of a beam-splitter forms the ancilla.

→ Initially the photon is in the mode $|0\rangle$; therefore, the total state is $\left(\cos\frac{\theta}{2}|H\rangle \pm \sin\frac{\theta}{2}|V\rangle\right)\otimes|0\rangle$



→ Pass it through the Polarizing beam-splitter.

$$|X\rangle \rightarrow \cos\frac{\alpha}{2}|H0\rangle \pm \sin\frac{\alpha}{2}|V1\rangle$$

→ Apply Rotation on $|H\rangle$ in the $|0\rangle$ such that

$$|H\rangle \rightarrow \tan\frac{\alpha}{2}|H\rangle + \frac{\sqrt{\cos\alpha}}{\cos\frac{\alpha}{2}}|V\rangle$$

$$|V\rangle \rightarrow -\frac{\sqrt{\cos\alpha}}{\cos\frac{\alpha}{2}}|H\rangle + \tan\frac{\alpha}{2}|V\rangle$$

→ Let it interfere at another PBS.

→ In the language of POVM:

$E_1 = a|\psi_1\rangle\langle\psi_1|$

$E_2 = a|\phi_1\rangle\langle\phi_1|$; a is chosen optimally so that $E_3 \geq 0$ and Rank 1.

$E_3 = \mathbb{1} - E_1 - E_2$

$\Rightarrow \quad \langle E_1\rangle = \langle E_2\rangle = p_s$

$\langle E_3\rangle \rightarrow$ inconclusive.