HW 4

1) Find limsup and liminf for the following sequences:

(i) $\{1+(-1)^n\}$ (ii) $\{\sin n\pi/2\}$

(iii) $\{n(1+(-1)^n)\}$ (iv) $\{\frac{n}{n+1}\cdot(-1)^n\}$

2) If $\{x_n\}$, $\{y_n\}$ are sequences of positive numbers then show that

$$\limsup(x_n y_n) \leq (\limsup x_n)(\limsup y_n)$$

Find an example where the inequality is not an equality.

3) Let $\{a_n\}$ be a sequence of numbers with $a_n \geq 0$ $\forall n$. Let $s_n = (a_1 + \cdots + a_n)/n$

Show that $\liminf a_n \leq \liminf s_n \leq \limsup s_n \leq \limsup a_n$

Deduce that $\lim a_n$ exists $\Rightarrow \lim s_n$ exists.

Show by an example that $\lim s_n$ may exist even though $\lim a_n$ may not.

4) We have seen that for a sequence of nonzero numbers $\{x_n\}$

$$\liminf\left|\frac{x_{n+1}}{x_n}\right| \leq \liminf|x_n|^{1/n} \leq \limsup|x_n|^{1/n} \leq \limsup\left|\frac{x_{n+1}}{x_n}\right|$$

Hence, $\lim\left|\frac{x_{n+1}}{x_n}\right|$ exists $\Rightarrow \lim|x_n|^{1/n}$ exists.

☆ Contrary to what was said in class by mistake

The converse of the above is false in general.
Check that for the following sequence $\{x_n\}$ $\lim |x_n|^{1/n}$ exists but $\lim \left|\frac{x_{n+1}}{x_n}\right|$ does not:

$$x_n = \begin{cases} \frac{1}{2^n} & \text{for } n \text{ odd} \\ \frac{1}{2^{n-2}} & \text{for } n \text{ even} \end{cases}$$

5) Solve the exercises mentioned in class.