HW 6

0) Solve the exercises mentioned in class.

1) Draw the graphs of $f_1(x) = x$, $f_2(x) = x^2$, $f_3(x) = x^3$ and $f_4(x) = x^4$ and compare them on a single diagram.

2) Let
$$f(x) = \begin{cases} 2^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$
Show that f is continuous on $\mathbb{R}$.
Draw the graph of f.

3) Draw the graphs of $f(x) = x^2$ and $f(x) = 2^x$ and compare them.

4) Prove that $f(x) = \sqrt{x}$ is continuous on $[0, \infty)$.

5) (Dirichlet function)
Let
$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ 1 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$
Show that f is discontinuous at all points of $\mathbb{R}$.

6) Let
$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$
Show that f is continuous only at $x = 0$.

7) Use the idea of (6) to construct a function $f : \mathbb{R} \to \mathbb{R}$ such the f is continuous only at $x = 1, 2, \dots, n$ for any $n \in \mathbb{N}$.

8) i) If $f : \mathbb{R} \to \mathbb{R}$ is continuous and

$f(q)=0 \quad \forall q \in \mathbb{Q}$ the show that
$f(x)=0 \quad \forall x \in \mathbb{R}$

ii) If $f : \mathbb{R} \to \mathbb{R}$ is continuous and
$f(a+b) = f(a) + f(b) \quad \forall a, b \in \mathbb{R}$
then show that $f(x) = kx$ for some $k \in \mathbb{R}$.

9) Let $f : \mathbb{R} \to \mathbb{R}$ be defined as follows:

$$f(x) = \begin{cases} \frac{1}{q} & \text{if } x \in \mathbb{Q} \text{ and } x = \frac{p}{q} \text{ where } p \in \mathbb{Z}, q \in \mathbb{N} \text{ and } p, q \text{ co-prime} \\ 0 & \text{if } x \in \mathbb{R} - \mathbb{Q}. \end{cases}$$

Show that f is continuous at all points of $\mathbb{R} \setminus \mathbb{Q}$ but discontinuous at all points of $\mathbb{Q}$.

10) Construct a function $f : \mathbb{R} \to \mathbb{R}$ which is continuous on $\mathbb{Q}$ and discontinuous on $\mathbb{R} - \mathbb{Q}$.

11) i) If $\lim_{x \to a} f(x) = l$, $\lim_{x \to a} g(x) = l'$ then show that $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{l}{l'}$ under suitable restrictions on $g(x)$ and l'. Show by examples that this result fails if one is not careful.

ii) If $f, g : I \to \mathbb{R}$ are continuous the show that $\min(f, g)$, $\max(f, g)$ are continuous where

$$\min(f,g)(x) = \min(f(x), g(x))$$
$$\max(f,g)(x) = \max(f(x), g(x))$$

(P.T.O.)

11. Solve all the problems 18.1 – 18.12, Ross' book.

12. Let $f(x) = \frac{x}{|x|}$ $\quad x \neq 0$

Find $\lim_{x \to 0+} f(x)$ and $\lim_{x \to 0-} f(x)$

13. Consider the following functions and determine if they are continuous.

i) $f(x) = \begin{cases} x+1 & \text{for } x \geq 0 \\ 2^x & \text{for } x < 0 \end{cases}$

ii) $f(x) = \begin{cases} 2^{-1/x} & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases}$