PHY 310 - Mathematical Methods for Physicists I

Odd Term 2019, IISER Mohali

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Homework 3

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1. Evaluate the following integral involving Bessel function

$$I = \int_0^\infty dt\ e^{-at} J_0(bt), \quad a, b > 0$$

using the integral representation

$$J_0(x) = \frac{1}{\pi} \int_0^\pi d\theta\ \cos(x \sin\theta).$$

Hint: Use the expression

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \frac{a}{a^2 + b^2 \sin^2\theta} = \frac{1}{\sqrt{a^2+b^2}}.$$

2. The l-th spherical Bessel function is given by

$$j_l(x) = (-1)^l x^l \left(\frac{1}{x}\frac{d}{dx}\right)^l j_0(x).$$

Compute $j_1(x)$ and $j_2(x)$. Note that $j_0(x) = x^{-1}\sin x$.

3. Obtain the series expansion formula for the n-th Laguerre polynomial

$$L_n(x) = \sum_{m=0}^{n} (-1)^m \frac{n!}{(m!)^2 (n-m)!} x^m,$$

from the Rodrigues' formula

$$L_n(x) = \frac{e^x}{n!}\frac{d^n}{dx^n}\left(x^n e^{-x}\right).$$

4. Derive the following recurrence relation for Laguerre polynomials

$$(n+1)L_{n+1}(x) = (2n+1-x)L_n(x) - nL_{n-1}(x)$$

using the generating function

$$\phi(x,t) = \frac{e^{xt/(1-t)}}{(1-t)} = \sum_{n=0}^{\infty} L_n(x) t^n.$$

5. The generating function for associate Laguerre polynomials $L_n^m(x)$ is given by

$$\phi(x,t) = \frac{e^{-xt/(1-t)}}{(1-t)^{m+1}} = \sum_{n=0}^{\infty} L_n^m(x) t^n.$$

Use this generating function to find $L_n^m(0)$.

6. Evaluate

$$\int_{-\infty}^{\infty} dx\ e^{-x^2} \left[H_2(x)\right]^2,$$

where $H_2(x)$ is the Hermite polynomial of degree 2.