PHY 101 : Problem Sheet 4

Look at the Figures 1(a) and 1(b) on the next page, where both the systems are put in earth's gravity.

1. In Fig. (1a), the wedge of mass M moves back by amount X as the ball of mass m slides down withput friction from the top of the wedge by amount x. Using the constraint that the movement of ball is constrained on a surface of angle θ relate the accelerations $\ddot{X}$ and $\ddot{x}$ and write down their equations of motion.
2. In Fig. (1b), relate the acceleration of the of the rightmost block m_2 to that of the left block m_1 if the thread connecting them is inextendible and massless. Write down their individual equations of motion and employ the constraint obtained.

A particle of mass m is moving in a straight line as seen by an inertial observer O who denotes the positon of the particle by $\mathbf{r}(\mathbf{t})$. Another non-inertial observer O' moves around the inertial observer in a circle of radius R as shown in Fig 2, with an angular velocity $\theta(t)$.

1. Find out the velocity and accleration of the particle P as seen by O'.
2. What are the possible accleration values of P as seen by O' when the points O', O and P are collinear ?

An observer finds a particle of unit mass moving along a trajectory $\mathbf{r} = \alpha\left(t^4/12 - 2\beta^2 t^2\right)\hat{\imath} + \beta(t^3/6)\hat{\jmath}$ under the action of a force $\mathbf{F} = \alpha(t^2 - 2\beta t)\hat{\imath} + \beta(t - 2\beta)\hat{\jmath}$, for constant α, β

1. Find out if the observer is inertial.
2. If the observer is non-inertial, find out the trajectory of this observer and that of the particle as seen by an inertial observer who was at the same location as that of this given observer at $t = 0$.
3. Find out the work done by the force according to the inertial observer during $t = 0$ and $t = 1$.
4. Find out the work done by the force according to the non-inertial observer during $t = 0$ and $t = 1$.

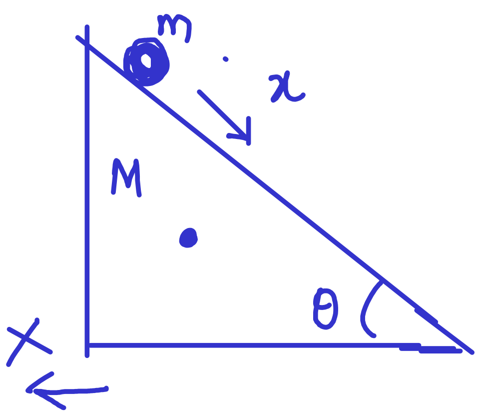


Fig 1 a

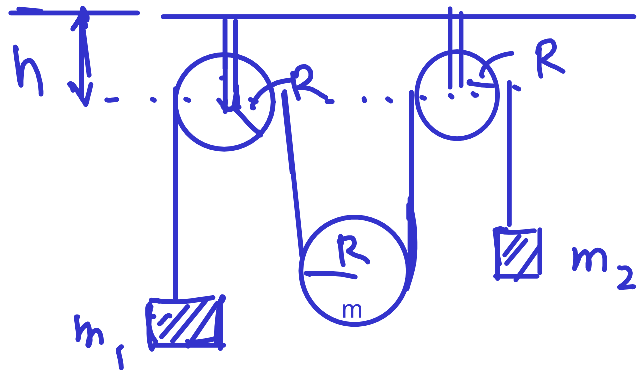


Fig 1 b

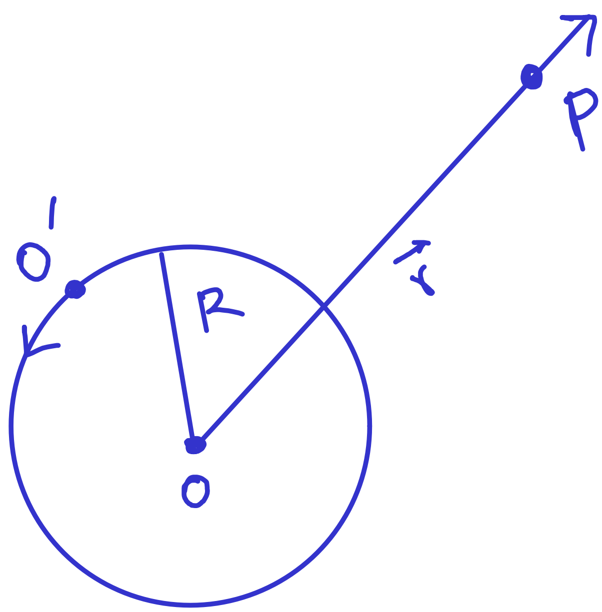
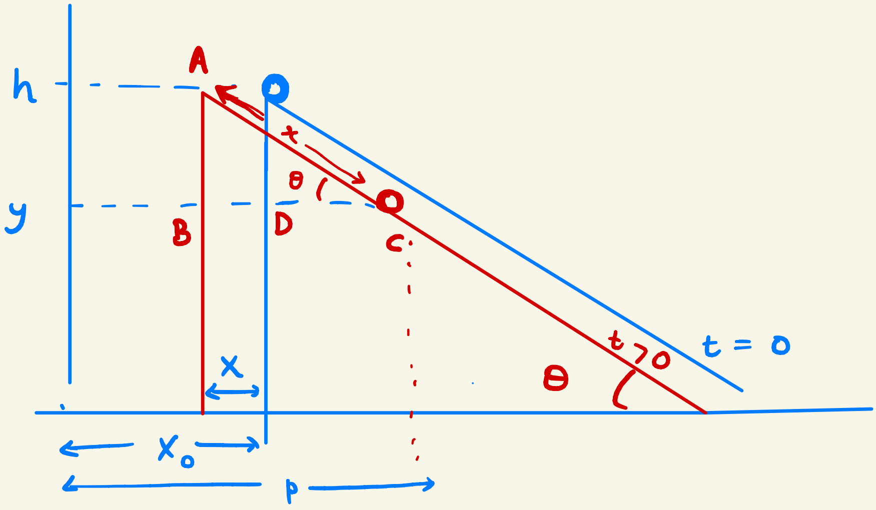


Fig 2

Problem Sheet 4

1.



At time $t > 0$, particle is at C. $\frac{AB}{BC} = \tan\theta$

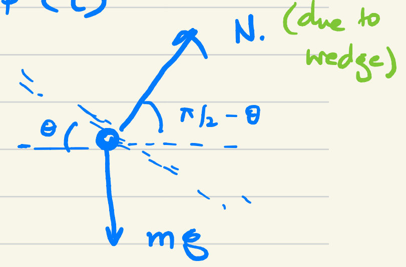
$AB = h - y$, $BC = AC\cos\theta = x\cos\theta$

Also, $BC = BD + DC = X + (p - X_0)$

$$\therefore \quad x(t)\cos\theta = X(t) + p(t) - X_0$$

$$\Rightarrow \quad \ddot{x}(t)\cos\theta = \ddot{X}(t) + \ddot{p}(t)$$

For the particle

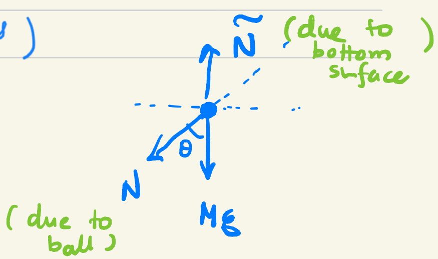


$$m\ddot{p} = N\cos\left(\tfrac{\pi}{2} - \theta\right) = N\sin\theta$$

$$\therefore \quad \ddot{x}(t)\cos\theta = \ddot{X}(t) + \frac{N}{m}\sin\theta$$

$$m\ddot{y} = -mg + N\cos\theta \quad \left(\text{y increases upwards}\right)$$

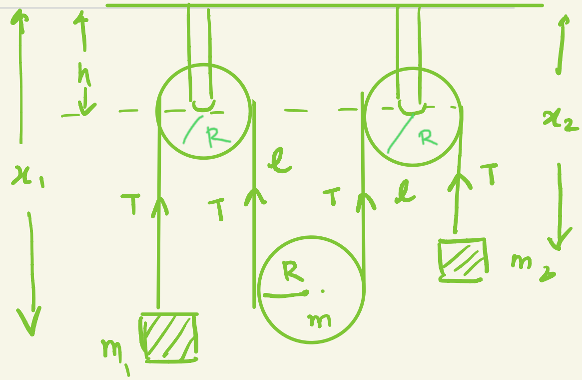
For the wedge



$$M\ddot{Y} = 0 = Mg - \tilde{N} + N\cos\theta$$

$$M\ddot{X} = N\sin\theta$$

2. At a time t the locations of m_1 and m_2 be $x_1(t)$ and $x_2(t)$ respectively



Then total string length

$$x_1(t) - h + \underset{\text{(Pulley 1)}}{\pi R} + \ell(t) + \underset{\text{(middle pulley \# 2)}}{\pi R} + \ell(t)$$

$$+ \underset{\text{(Pulley 3)}}{\pi R} + x_2(t) - h$$

$$= L \text{ (fixed)}$$

$$\Rightarrow x_1(t) + 2\ell(t) + x_2(t) + 3\pi R - 2h = L$$

$$\ddot{x}_1(t) + 2\ddot{\ell}(t) + \ddot{x}_2(t) = 0 \quad - \text{(1)}$$

For left mass $\quad m_1\ddot{x}_1 = m_1 g - T \quad$ – (2)

For middle pulley $\quad m\dfrac{d^2}{dt^2}(\ell+h) = m\ddot{\ell} = 2T - mg \quad$ – (3)

For right mass $\quad m_2\ddot{x}_2 = m_2 g - T \quad$ – (4)

(2) + (4) $\Rightarrow \quad m_1\ddot{x}_1 + m_2\ddot{x} = (m_1+m_2)g - 2T$

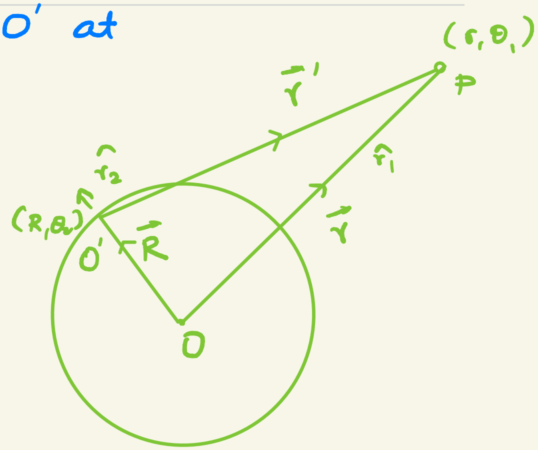
From (1) and (3) $\quad \dfrac{m}{2}(\ddot{x}_1 + \ddot{x}_2) = mg - 2T$

3. Let the location of O' at time t be

$$\vec{R}(t) = R\,\hat{r}_2(t)$$

That of the particle p

$$\vec{r}(t) = r\,\hat{r}_1(t)$$



The $\hat{r}_2$ is rotating about an axis $\odot$ to plane of paper (let us call it $\hat{k} \Rightarrow$ Z-direction)

$$\therefore \quad \frac{d\hat{r}_2}{dt} = \vec{\Omega} \times \hat{r}_2 = \Omega(\hat{k} \times \hat{r}_2) = \Omega\,\hat{\theta}_2$$

$$\frac{d\hat{\theta}_2}{dt} = \vec{\Omega} \times \hat{\theta}_2 = \Omega(\hat{k} \times \hat{\theta}_2) = -\Omega\,\hat{r}_2$$

$$\vec{V}_{in} = \vec{V}_{rot} + \vec{\Omega} \times \vec{r}$$

$$\vec{V}_{rot} = \vec{V}_{in} - \vec{\Omega} \times \vec{r} = \dot{r}\,\hat{r}_1 - \Omega r\,\hat{\theta}_1$$

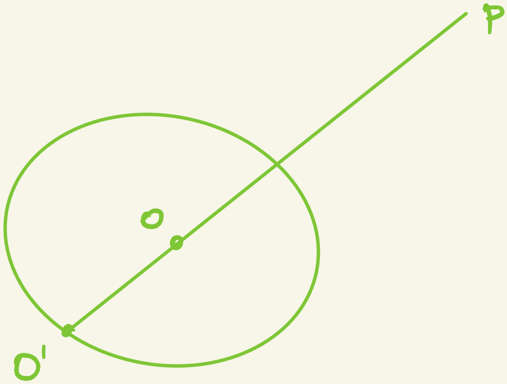
$$\vec{a}_{rot} = \vec{a}_{in} - 2(\vec{\Omega} \times \vec{V}_{rot}) - \vec{\Omega} \times (\vec{\Omega} \times \vec{r})$$

$$= \vec{a}_{in} - 2\left(\Omega\hat{k} \times \{\dot{r}\hat{r}_1 - \Omega r\hat{\theta}_1\}\right) - \Omega\hat{k} \times (\Omega\hat{k} \times r\hat{r}_1)$$

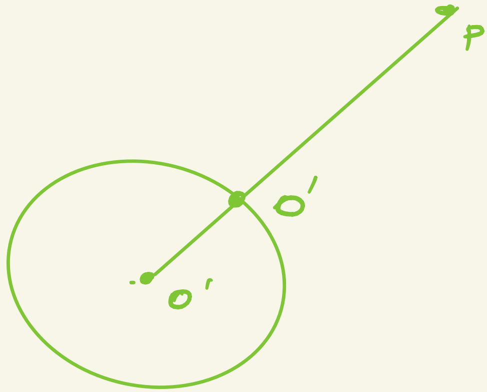
$$= \ddot{r}\,\hat{r}_1 - 2\left[\Omega\dot{r}\,\hat{\theta}_1 + \Omega^2 r\,\hat{r}_1\right] + \Omega^2 r\,\hat{r}_1$$

$$= \ddot{r}\,\hat{r}_1 - 2\Omega\dot{r}\,\hat{\theta}_1 - \Omega^2 r\,\hat{r}_1$$

4. When P, O and O′ are collinear



1 :



2 :

For 1 :

$\hat{r}_1 = -\hat{r}_2$

$\theta_2 = \pi + \theta_1$

$\hat{\theta}_2 = -\sin(\pi+\theta_1)\,\hat{i} + \cos(\pi+\theta_1)\,\hat{j} =$

$= +\sin\theta_1\,\hat{i} - \cos\theta_1\,\hat{j} = -\hat{\theta}_1$

$\vec{a}_{rot} = -\ddot{r}\,\hat{r}_2 + \Omega^2 r\,\hat{r}_2 + 2\Omega\dot{r}\,\hat{\theta}_2$

For 2 :

$\hat{r}_1 = \hat{r}_2$, $\hat{\theta}_1 = \hat{\theta}_2$

$\theta_1 = \theta_2$

$\vec{a}_{rot} = \ddot{r}\,\hat{r}_2 - \Omega^2 \hat{r}_2 - 2\Omega\dot{r}\,\hat{\theta}_2$

5. $\vec{r} = \alpha\left(t^4/12 - 2\beta^2 t^2\right)\hat{i} + \beta\left(t^3/6\right)\hat{j}$

$\ddot{\vec{r}} = \alpha\left(t^2 - 4\beta^2\right)\hat{i} + \beta t\,\hat{j}$

$\vec{F} = \alpha\left(t^2 - 2\beta t\right)\hat{i} + \beta\left(t - 2\beta\right)\hat{j}$

For inertial obsercverers

(i) No force $\Rightarrow$ No acceleration

Fails ! as at $t = 2\beta$, $\vec{F} = 0$, but $\ddot{\vec{r}} = 2\beta^2\,\hat{j}$

(ii) $\vec{F} = m\vec{a}$ (clearly fails for all time)

Thus the observer is not inertial

6. $\vec{r}_{ni} = \vec{r}_{in} - \vec{R}_f$

($\rightarrow$ location of frame)

$\therefore\ m\ddot{\vec{r}}_{ni} = m\ddot{\vec{r}}_{in} - m\ddot{\vec{R}}_f$

$= \vec{F} - m\ddot{\vec{R}}_f$

$\Rightarrow m\ddot{\vec{R}}_f = \vec{F} - m\ddot{\vec{r}}_{ni} = \vec{F} - \ddot{\vec{r}}_{ni}$ for unit mass

$\Rightarrow \ddot{\vec{R}}_f = \vec{F} - \ddot{\vec{r}}_{ni} = \left(4\alpha\beta^2 - 2\alpha\beta t\right)\hat{i} - 2\beta^2\hat{j}$

$\dot{\vec{R}}_f = \left(4\alpha\beta^2 t - \alpha\beta t^2\right)\hat{i} - 2\beta^2 t\,\hat{j} + \underbrace{\vec{C}}_{\text{can be zero}}$

put to $\dot{\vec{R}}_f(0) = 0$

$\therefore\ \vec{R}_f = \left(2\alpha\beta^2 t^2 - \frac{1}{3}\alpha\beta t^3\right)\hat{i} - \beta^2 t^2\,\hat{j}$

$\vec{r}_{in} = \vec{r}_{ni} + \vec{R}_f = \alpha\left(\frac{t^4}{12} - \frac{\beta t^3}{3}\right)\hat{i} + \beta t^2\left(\frac{t}{6} - \beta\right)\hat{j}$

$\dot{\vec{r}}_{in} = \alpha\left(\frac{t^3}{3} - \beta t^2\right)\hat{i} + \beta\left(\frac{t^2}{2} - 2\beta t\right)\hat{j}$

7. Work done as per inertial observer

$$W_{in} \equiv \int dW = \int \vec{F} \cdot d\vec{r} = \int_0^1 \vec{F} \cdot \left(\frac{d\vec{r}}{dt}\right)_{in} dt$$

$$W_{in} = \int_0^1 \left[\alpha^2 (t^2 - 2\beta t)\left(\frac{t^3}{3} - \beta t^2\right) + \beta^2 (t - 2\beta)\left(\frac{t^2}{2} - 2\beta t\right)\right] dt$$

Evaluate this

8. Work done as per non-inertial observer

$$\int dW = \int_0^1 \vec{F} \cdot \left(\frac{d\vec{r}}{dt}\right)_{ni} dt$$

$$= \int_0^1 \vec{F} \cdot \left[\left(\frac{d\vec{r}}{dt}\right)_{in} - \dot{\vec{R}}_f\right] dt$$

$$= W_{in} - \underbrace{\int_0^1 \vec{F} \cdot \dot{\vec{R}}_f \, dt}$$

$$\int_0^1 \vec{F} \cdot \dot{\vec{R}}_f \, dt = \int_0^1 \left[\alpha^2 \beta \left(t^2 - 2\beta t\right)\left(4\beta t - t^2\right) - 2\beta^2 t^2 (t - 2\beta)\right] dt$$

Evaluate this