PHY304 - Statistical Mechanics

Spring 2021, IISER Mohali

Instructor: Dr. Anosh Joseph

PHY304: Homework 1 Solutions

Due: Thursday, January 21, 2021 at 11:00pm.

(Upload your solutions to Moodle as a single `.pdf` file.)

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1. Calculate the efficiency of a Carnot cycle for an engine using a magnetic substance. The magnetic substance has temperature T and magnetization M. The first law in this case takes the form

$$dU = TdS + HdM. \tag{1}$$

The internal energy is a function of T only: $U = f(T)$. The equation of state is

$$CH = MT, \tag{2}$$

where C is a constant proportional to the size of the system.

During the cycle the magnetic substance is doing work on the outside world by reducing its magnetization. Calculate all the contributions to the exchange of heat and work. Find the efficiency of this engine.

Solution:

1. The work done is by changing the magnetic field.

We have

$$U = f(T) = f(T(S, M)). \tag{3}$$

And also

$$M = \frac{CH}{T}. \tag{4}$$

We have

$$T = \left(\frac{\partial U}{\partial S}\right)\Bigg|_M = f'(T)\left(\frac{\partial T}{\partial S}\right)\Bigg|_M. \tag{5}$$

In the Carnot cycle we have

Step 1: Isothermal transition form (H_1, M_1) to (H_2, M_2) at T_h.

Step 2: Adiabatic transition form (H_2, M_2) to (H_3, M_3) with $T_h \to T_c$.

Step 3: Isothermal transition form (H_3, M_3) to (H_4, M_4) at T_c.

Step 4: Adiabatic transition form (H_4, M_4) to (H_1, M_1) with $T_c \to T_h$.

We assume that $M_2 < M_1$ and so in Step 1 we do work on the outside and heat flows in.

Process $1 \to 2$**:**

$$U_{1,2} = 0. \tag{6}$$

$$W_{1,2} = -\int_{M_1}^{M_2} H dM = -\frac{T_h}{C}\int_{M_1}^{M_2} M dM = -\frac{T_h}{2C}(M_2^2 - M_1^2). \tag{7}$$

$$Q_{1,2} = U_{1,2} + W_{1,2} = -\frac{T_h}{2C}(M_2^2 - M_1^2). \tag{8}$$

Process $2 \to 3$**:**

$$U_{2,3} = f(T_c) - f(T_h). \tag{9}$$

$$Q_{2,3} = 0. \tag{10}$$

$$W_{2,3} = -U_{2,3} + Q_{2,3} = f(T_h) - f(T_c). \tag{11}$$

Process $3 \to 4$**:**

$$U_{3,4} = 0. \tag{12}$$

$$W_{3,4} = -\int_{M_3}^{M_4} H dM = -\frac{T_c}{C}\int_{M_3}^{M_4} M dM = -\frac{T_c}{2C}(M_4^2 - M_3^2). \tag{13}$$

$$Q_{3,4} = U_{3,4} + W_{3,4} = -\frac{T_c}{2C}(M_4^2 - M_3^2). \tag{14}$$

Process $4 \to 1$**:**

$$U_{4,1} = f(T_h) - f(T_c). \tag{15}$$

$$Q_{4,1} = 0. \tag{16}$$

$$W_{4,1} = -U_{4,1} + Q_{4,1} = f(T_c) - f(T_h). \tag{17}$$

The total work is

$$W_{\text{total}} = -\frac{T_h}{2C}(M_2^2 - M_1^2) - \frac{T_c}{2C}(M_4^2 - M_3^2). \tag{18}$$

We also have

$$Q_{\text{in}} = -\frac{T_h}{2C}(M_2^2 - M_1^2). \tag{19}$$

The efficiency of the engine is

$$\eta = \frac{W_{\text{total}}}{Q_{\text{in}}} = 1 + \frac{T_c}{T_h}\frac{M_4^2 - M_3^2}{M_2^2 - M_1^2}. \tag{20}$$

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