Assignments $\mathbf{G}_{\mu\nu}\mathbf{R}_{\mu\nu}$: $PHY635$

January-April 2025

Problem Sheet 1

1. If an inertial frame S' is related to another inertial frame S by boost (β) along the y-axis, while another frame $S^{''}$ is related to S' with a boost (β') in an arbitrary direction in the $y' - z'$ plane, what will be the transformation matrix taking one from frame S to S''?

2. How different is the transformation matrix in the previous problem from a general boost in $y - z$ plane starting from S in the leading order of (β') ? [Hint : Write the resulting matrix of the previous problem in leading order of (β') and compare it with a pure boost $(\beta'')$from S to S']

3. Show that the spacetime position co-ordinates (t, r, θ, ϕ) do not transform linearly under Lorentz transformation. How does $(dt, dr, d\theta, d\phi)$ transform under Lorentz transformation ? Is it a linear transformation ?

4. If charge is an Lorentz invariant quantity argue that the charge density ρ and the the current density $\mathbf{J} \equiv \rho\mathbf{v}$ constitute a Lorentz 4-vector J^μ.

5. In a general co-ordinate transformation of 3-space $x^i \equiv \{x^1, x^2, x^3\} \rightarrow x^{i'} \equiv \{x^{1'}(x^i), x^{2'}(x^i), x^{3'}(x^i)\}$, how does the volume element transform ? What transformation will keep the element invariant ?

6. If a frame S' is moving w.r.t. to an inertial frame S at a velocity $\mathbf{v}$ which is **not** necessarily along any axis (i.e. is in a general direction), then show that

$$\begin{aligned} ct' &= \gamma(ct - \beta \cdot \mathbf{r}); \\ \mathbf{r}' &= \mathbf{r} + \frac{(\beta \cdot \mathbf{r})\beta(\gamma - 1)}{\beta^2} - \beta\gamma ct. \end{aligned}$$

7. Find out the trajectory of a particle of mass m with angular momentum l moving in the gravitational potential of point mass M. Obtain the conditions in terms of l, m, M for the closed and open orbits. When will the trajectory turn parabolic in r, ϕ space ?

8. If there is an extra b/r^2 term added to the potential of the point mass M in the previous problem, **which is to be treated perturbatively**, set the new differential equation in terms of $u = 1/r$.
a.) Now if this extra term is perturbative, we can feed in the solution of the unperturbed part $u = (1 + e\cos\phi)l^2/m$ *for this term only* to obtain the new corrected differential equation.
b.) Noting that equations

$$f''(\phi) + f(\phi) = \{ \quad A, \ A\cos\phi, \ A\cos^2\phi$$

have following particular solutions

$$f(\phi) = \{ \quad A, \ \frac{A}{2}\phi\sin\phi, \ \frac{A}{2} - \frac{A}{6}\cos 2\phi$$

find out the perturbed solution of the orbit.
c.) What would be the criterion of identifying perihelion of the orbit $u(\phi)$? Find out the ϕ separation of two successive perihelion.

9. The Newtonian gravitational potential outside a mass distribution is written as

$$\phi(\mathbf{x}) = -G\int \frac{\rho(\mathbf{x})}{|\mathbf{x} - \mathbf{x}'|}.$$

If the potential is expanded in terms of multi-poles as

$$\phi(\mathbf{x}) = 4\pi \sum_{l,m} \frac{Q_{lm}}{2l+1}\frac{1}{r^{l+1}} Y_{lm}(\hat{\mathbf{x}}),$$

find out the expression for the multi-pole moments Q_{lm}, for few initial $l = 0, 1, 2$.

10. In the previous example for a density distribution which is symmetric under reflection in the $x - y$ plane and also in azimuthal direction then $Q_{lm} = 0$ for $m \neq 0$. Show that the dipole contribution vanishes. Show the corrected potential is

$$\phi = -\frac{GM}{r}\left[1 - \sum_{l=2}^{\infty} J_l \left(\frac{r'}{r}\right)^l P_l(\cos\theta)\right],$$

with

$$J_l = -\frac{1}{Mr'^l}\int \rho(\mathbf{y}) y^l P_l(\cos\theta) d^3\mathbf{y}$$