INDIAN INSTITUTE OF SCIENCE EDUCATION AND RESEARCH MOHALI

MTH101 (Symmetry)

Practice Problems Spring 2022

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1. Gallian: Problem 1.19.

2. Gallian: Problem 1.20.

3. Gallian: Problem 1.22.

4. Gallian: Problem 26.8.

5. Gallian: Problem 26.10.

6. A symmetry r of a shape is called **self inverse** if $rr = 1$. How many self inverse symmetries are there for the symmetries of a triangle? A square? A regular n-gon?

7. Label four vertices of a regular tetrahedron by $1, 2, 3, 4$.

8. **(Permutation Factory)** Take two permutations $a = (1\ 2\ 3\ 4)$ and $b = (1\ 4)$. Calculate a^2, b^2, ab and ba. Now you have permutations a, b, a^2, b^2, ab and ba. Multiply them with each other in all possible ways. Keep generating new permutations in this manner, until it is not possible to do so further. How many permutations do you have in the end?

9. Write all symmetries of a (regular) tetrahedron in terms of permutations. How many self inverse symmetries are there?

10. How many self inverse symmetries are there for a sphere? Write four of them.

11. Consider the rotations $R_{x,\alpha}$ and $R_{y,\beta}$ of a sphere. Can they ever be inverses of each other?

12. A matrix is called **unitriangular** if all its diagonal entries are 1, and all entries below the diagonal are 0. Thus a 3×3 unitriangular matrix looks like
$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$
   (a) Show that multiplication of two unitriangular matrices is a unitriangular matrix.
   (b) Perform row reduction on a unitriangular matrix and show that unitriangular matrices are invertible and the inverse is again unitriangular matrix.

13. Can you find two 3×3 matrices for which $AB - BA = I_3$, where I_3 denotes the 3×3 identity matrix.

14. Two distinct $n \times n$ swapper matrices A and B are picked at random. Determine, in terms of n, the probability that $(A + B)(A - B) = A^2 - B^2$.

15. Gallian: Understand examples of groups as given in Gallian, Chapter 2, Table 2.1.

16. Gallian: Problem 2.5.

17. Gallian: Problem 2.23

18. Gallian: Problem 2.29

19. Gallian: Problem 2.49

20. Take a group G and an element $g \in G$. The smallest positive integer n for which

$$g^n := \underbrace{g * g * g \cdots * g}_{n-\text{times}}$$

is equal to the identity element of G is called the *order* of g in G. Thus, identity is the only element in a group whose order is equal to 1.

If g^n is different from identity for every choice of $n \in \mathbb{N}$, then g is said to have *infinite order*.

Recall that $\mathcal{S}_n$ denotes the group of permutations of $1, 2, \cdots, n$. The group operation is the composition of permutations.

   (a) What is the order of (1 2) in then group $\mathcal{S}_3$?
   (b) How many elements of order 2 are there in Klein's 4-group?
   (c) How many elements of order 3 are there in the group of symmetries of a tetrahedron?
   (d) Find all elements of order 2 in $\mathcal{S}_6$.
   (e) Find an example of a group in which all elements, except identity, have infinite order.

21. **Conjugation Action**. Take a group G and $S = G$.

   (a) Show that $G \times S \to S$ defined by $g.s = gsg^{-1}$ is a group action.
   (b) Orbits of this action are called *conjugacy classes* of G. Find all conjugacy classes of the group $\mathcal{S}_3$.
   (c) For $s \in S$, consider the *stabilizer* defined by $\mathrm{stab}(s) := \{g \in G : g.s = s\}$. Show that the subset $\mathrm{stab}(s)$ of G is also a group, under the same group operation as that of G.

22. Let V be a vector space over $\mathbb{R}$. Show that scalar multiplication is an action of the multiplicative group of nonzero elements of $\mathbb{R}$ on V.

23. Which of the following is/are not a basis of the vector space $(\mathbb{R}^3, +, \cdot)$?

   (a) $S = \{(1, 2, 3), (4, 5, 6), (7, 8, 9)\}$.
   (b) $S = \{(1, 1, 2), (1, 1, 3), (1, 1, 4)\}$.
   (c) $S = \{(1, 1, 1), (1, 4, 9), (4, 7, 12)\}$.

24. Consider the vector space $(M_3(\mathbb{R}), +, \cdot)$ of 3×3 matrices. Show that $T : M_3(\mathbb{R}) \to \mathbb{R}$ defined by $T(A) = \det(A)$ is not a linear transformation. Here, $\det(A)$ stands for determinant of A.

25. Show, simply by matrix multiplication, that if A is a 3×3 rotation matrix then $A^t A$ is equal to the identity matirx I_3.

26. Consider the rotation $R_{x,\theta}$ in $\mathbb{R}^3$. Show that if v is an eigenvector of $R_{x,\theta}$, then $v = (\alpha, 0, 0)$ for some $\alpha \neq 0$.

27. Consider $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b \\ a + b \end{pmatrix}$. Show that T is a linear transformation. Does T have eigenvectors? What are their eigenvalues?

28. Let B be a 3×3 matrix. Show that if $v^t B w = v^t w$ for all column vectors $v, w \in \mathbb{R}^3$, then $B = I_3$.

29. Let $(V, +, \cdot)$ be a vector space over $\mathbb{R}$ and $S \subseteq V$ be a subset. Let us pick some vectors $v_1, v_2, \cdots, v_n \in V$ and some scalars $\alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbb{R}$. The vector $\alpha_1 v_1 + \alpha v_2 + \cdots + \alpha_n v_n \in V$ is called a *linear combination* originated in S. A subset $S \subseteq V$ is called *linearly independent* if the **only** way a linear combination originated in S is equal to $0 \in V$, is by through the choice $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$.

    Thus, $\{1, 1+x, 1+2x\}$ is not linearly independent in the vector space $(\mathbb{R}[x], +, \cdot)$, because for $\alpha_1 = 1, \alpha_2 = -2, \alpha = 1$, we have $\alpha_1 1 + \alpha_2(1+x) + \alpha_3(1+2x) = 1 + (-2-2x) + 1 + 2x = 0$.

    (a) Show that $S = \{(1,0,0), (0,1,0), (0,0,1)\}$ is linearly independent in $(\mathbb{R}^3, +, \cdot)$.

    (b) Determine, if $S = \{(1,2,3), (2,3,1), (3,1,2)\}$ is linearly independent in $(\mathbb{R}^3, +, \cdot)$.

    (c) Show that if $0 \in S$, then S is not linearly independent.

    (d) Show that a basis is linearly independent.
