MTH302: INTEGERS, POLYNOMIALS AND MATRICES
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1. Ideals of a Matrix rings

Theorem 1.1. *Let R be a ring with 1.*

(1) If I is a left ideal of R then $M_n(I)$ is a left ideal of $M_n(R)$.
(2) If I is a right ideal of R then $M_n(I)$ is a right ideal of $M_n(R)$.
(3) If I is a two-sided ideal of R then $M_n(I)$ is a two-sided ideal of $M_n(R)$.
(4) If J is a two-sided ideal of $M_n(R)$ then $J = M_n(I)$ for some two-sided ideal of R.

Proof. (1), (2) and (3) are easy to prove. Let J be a two sided ideal of $M_n(R)$ and I be the set of all $(1,1)$ entries of the matrices of J. That is, $a \in I$ if and only if

$$\begin{pmatrix} a & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \in J$$

Since J is an ideal, it is easy to see that I is an ideal. Let $A = (a_{ij}) \in J$. Then $a_{ij}E_{11} = E_{1i} \cdot A \cdot E_{j1} \in J$ and thus $a_{ij} \in I$. This shows that $J \subset M_n(I)$. On the other hand, for any matrix $A = (a_{ij}) \in M_n(R)$, we have $E_{i1} \cdot A \cdot E_{1j} = a_{11}E_{ij}$. Now if $A = (a_{ij}) \in M_n(I)$ then there are matrices $N^{ij} \in J$ such that $(1,1)$ entry of N^{ij} is a_{ij}. Then

$$A = \sum_{i,j} a_{ij}E_{ij} = \sum_{i,j} E_{i1}N^{ij} \cdot E_{1j} \in J.$$

Thus $M_n(I) = J$. $\square$

Remark 1.2. If R is a division ring then $M_n(R)$ has no nontrivial two sided ideals; trivial ideals being (0) and $M_n(R)$.

Example 1.3. Consider the ring $R := M_2(2\mathbb{Z})$. Clearly R is without identity. Now let

$$J = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a \in 4\mathbb{Z} \subset 2\mathbb{Z}, b, c, d \in 2Z \right\}.$$

Verify that J is a two sided ideal of R: what is used here is the fact that $ab \in 4\mathbb{Z}$ for any $a, b \in 2\mathbb{Z}$. We claim that $J \neq M_2(I)$ for any ideal I of $2\mathbb{Z}$. Note that

$$\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \notin J$$

and therefore $J \neq M_2(2\mathbb{Z})$. Clearly, $M_2(4\mathbb{Z}) \subseteq J$ and

$$\begin{pmatrix} 4 & 2 \\ 0 & 0 \end{pmatrix} \notin M_2(4\mathbb{Z}).$$

Therefore $M_2(4\mathbb{Z}) \subsetneqq J \subsetneqq M_2(2\mathbb{Z})$. Since there are no ideals I of $2\mathbb{Z}$ such that $4\mathbb{Z} \subsetneqq I \subsetneqq 2\mathbb{Z}$, we see that $J \neq M_2(I)$ for any ideal of $2\mathbb{Z}$.

Simple Ring. A ring is *simple* if it has no nontrivial two-sided ideals.

Example 1.4. (1). Division rings are simple.
(2). Matrix rings over division rings are simple
(3). R is a simple ring with 1 then so is $M_n(R)$.

2. MINIMAL LEFT IDEALS OF MATRIX RINGS OVER DIVISION RINGS

Let R be a division ring. Consider the set

$$J_k = \left\{ \sum_{i=1}^{n} a_{ik} E_{ik} \mid a_{ik} \in R \right\}$$

of all matrices (a_{st}) such that $a_{st} = 0$ for all $t \neq k$. It is easy to see that J_k is a left ideal[1] of $M_n(R)$. Now we shall show that J_k is a minimal left ideal.

Let $A := (a_{st}) \in J_k$ and for a fixed i, k, let $a_{ik} \neq 0$. We claim that the left ideal generated by A is the ideal J_k and this would prove that J_k is a minimal left ideal. Verify that $E_{1k} = a_{ik}^{-1} E_{1i} \cdot A \in (A)_l$. Since, for each $s = 1, 2, \cdots, n$, $E_{sk} = E_{s1} \cdot E_{1k} \in (A)_l \subseteq J_k$ and R linear combinations of elements E_{sk} is the left ideal J_k, we have shown that the left ideal generated by the element A is the ideal J_k itself.

This show that J_k are minimal left ideals of R. Similarly, one can show that

$$I_k = \left\{ \sum_{i=1}^{n} a_{ki} E_{ki} \mid a_{ki} \in R \right\}$$

are minimal right ideals of $M_n(R)$.

[1] J_k is the left ideal containing matrices whose columns have zero entries except for the $k - th$ column.