defining $\mathcal{L} = T - U$, we get

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_k}\right) - \frac{\partial \mathcal{L}}{\partial q_k} = 0.$$

$\mathcal{L}$ is called the Lagrangian of the system.

The ~~double~~ double pendulum

$$T = \frac{1}{2} m l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m l_2^2 \dot{\theta}_2^2 + m l_1 l_2 \dot{\theta}_1 \dot{\theta}_2$$

$$U = 2mg(l_1 + l_2) - 2mg\, l_1 \cos\theta_1 - mg\, l_2 \cos\theta_2 .$$

$$\mathcal{L} = T - U = \cancel{\frac{1}{2}} m l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m l_2^2 \dot{\theta}_2^2 + m l_1 l_2 \dot{\theta}_1 \dot{\theta}_2$$

$$- 2mg(l_1 + l_2) + 2mg\, l_1 \cancel{\theta} \left(1 - \frac{\theta_1^2}{2}\right) + mg\, l_2 \left(1 - \frac{\theta_2^2}{2}\right).$$

$$= \cancel{\frac{1}{2}} m l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m l_2^2 \dot{\theta}_2^2 + m l_1 l_2 \dot{\theta}_1 \dot{\theta}_2$$

$$- \cancel{+} 2mg\, l_1 \frac{\theta_1^2}{2} - mg\, l_2 \frac{\theta_2^2}{2}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1}\right) - \frac{\partial \mathcal{L}}{\partial \theta_1} = \frac{d}{dt}\left(2m l_1^2 \dot{\theta}_1 + m l_1 l_2 \dot{\theta}_2\right) + 2mg\, l_1 \theta_1 = 0.$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2}\right) - \frac{\partial \mathcal{L}}{\partial \theta_2} = \frac{d}{dt}\left(m l_2^2 \dot{\theta}_2 + m l_1 l_2 \dot{\theta}_1\right) + mg\, l_2 \theta_2 = 0.$$

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$$2ml_1^2\ddot{\theta}_1 + ml_1 l_2\ddot{\theta}_2 = -2mgl_1\theta_1$$
$$ml_2^2\ddot{\theta}_2 + ml_1 l_2\ddot{\theta}_1 = -mgl_2\theta_2$$

$l_1 = l_2$.

$$ml^2\ddot{\theta}_1 + ml^2\ddot{\theta}_2 = -2mgl\theta_1$$

$$2l\ddot{\theta}_1 + l\ddot{\theta}_2 = -2g\theta_1$$
$$l\ddot{\theta}_1 + l\ddot{\theta}_2 = -g\theta_2$$

$$\theta_1 = A_1 e^{i\omega t} \qquad \theta_2 = A_2 e^{i\omega t}$$

$$-2l\omega^2 A_1 - l\omega^2 A_2 = -2gA_1$$
$$-l\omega^2 A_1 - l\omega^2 A_2 = -gA_2$$

$$(2l\omega^2 - 2g)A_1 + l\omega^2 A_2 = 0$$
$$l\omega^2 A_1 + (l\omega^2 - g)A_2 = 0.$$

$$\underbrace{\begin{pmatrix} 2l\omega^2 - 2g & l\omega^2 \\ l\omega^2 & l\omega^2 - g \end{pmatrix}}_{\underline{M}} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0.$$

Secullar Equation: $\mathrm{Det}(\underline{M}) = 0$.

$$(2l\omega^2 - 2g)(l\omega^2 - g) - l^2\omega^4 = 0.$$

$$2l^2\omega^4 - 4gl\omega^2 + 2g^2 = 0. \qquad \omega^2 = (2 \pm \sqrt{2})\frac{g}{l}$$
$$l^2\omega^4 - 2gl\omega^2 + g^2 = 0.$$

$$\omega_1^2 = (2+\sqrt{2})\frac{g}{l} \qquad \omega_2^2 = (2-\sqrt{2})\,g/l$$

out of phase $\leftarrow A_2 = -\sqrt{2}A_1$ $\qquad A_2 = \sqrt{2}A_1$ (in phase)

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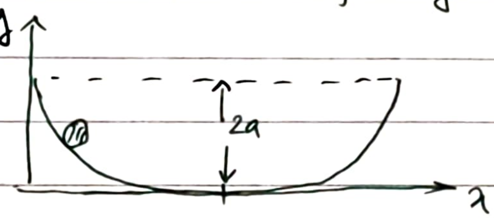
Ex

Mass on a cycloid trajectory.



$$x = a(\theta - \sin\theta) \qquad y = a(1+\cos\theta). \qquad 0 \le \theta \le 2\pi$$

$$T = \frac{1}{2}(m\dot{x}^2 + m\dot{y}^2) = \frac{m}{2}\left[(a\dot{\theta} - a\dot{\theta}\cos\theta)^2 + (-a\sin\theta\,\dot{\theta})^2\right]$$

$$= \frac{m}{2}\left[a^2\dot{\theta}^2(1 - 2\cos\theta + \cos^2\theta) + a^2\dot{\theta}^2\sin^2\theta\right].$$

$$= \frac{m}{2}\left[2a^2\dot{\theta}^2 - 2a^2\dot{\theta}^2\cos\theta\right] = ma^2\dot{\theta}^2(1-\cos\theta).$$

$$U = mgy = mga(1+\cos\theta)$$

$$\mathcal{L} = T - U = ma^2\dot{\theta}^2(1-\cos\theta) - mga(1+\cos\theta)$$

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) - \frac{\partial\mathcal{L}}{\partial\theta} = 0$$

$$\cancel{2ma^2\dot{\theta}\ddot{\theta}(1-\cos\theta) + ma^2\dot{\theta}^2(-\sin\theta\,\dot{\theta}) - (-mga(-\sin\theta))} = 0$$

$$\cancel{2ma^2\dot{\theta}\ddot{\theta}(1+\cos\theta) + ma^2\dot{\theta}^3\sin\theta - mga\sin\theta = 0}.$$

$$\cancel{\frac{d}{dt}\left(2ma^2\dot{\theta}(1-\cos\theta)\right) - \left(-mga(-\sin\theta)\right)} = 0.$$

$$\cancel{\frac{d}{dt}\left(\dot{\theta}(1-\cos\theta)\right) - \frac{g}{2a}\sin\theta = 0}.$$

$$\cancel{(1-\cos\theta)\ddot{\theta}}$$

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$$\frac{\partial L}{\partial \dot\theta} = 2ma^2\dot\theta\,(1-\cos\theta) \qquad \frac{\partial L}{\partial \theta} = ma^2\dot\theta^2 \sin\theta + mga\sin\theta$$

$$\frac{d}{dt}\left(\dot\theta(1-\cos\theta)\right) - \left[\frac{\dot\theta^2}{2}\sin\theta + \frac{g}{2a}\sin\theta\right] = 0.$$

$$\ddot\theta(1-\cos\theta) + \dot\theta^2\sin\theta - \frac{\dot\theta^2}{2}\sin\theta + \frac{g}{2a}\sin\theta = 0.$$

$$\ddot\theta(1-\cos\theta) + \frac{\dot\theta^2}{2}\sin\theta - \frac{g}{2a}\sin\theta = 0.$$

To solve : substitute $u = \cos\theta/2$.

$$\frac{d^2u}{dt^2} + \frac{g}{4a}u = 0.$$

$$u = \cos\theta/2 = A_1\cos\sqrt{\frac{g}{4a}}\,t + B_1\sin\sqrt{\frac{g}{4a}}\,t.$$