ROLL NO : MS

NAME :

PHY638 EndSem Part A Date : May 5, 2025 Inst: Abhishek Chaudhuri

- Time : 75 minutes
- Max Marks : 5×4 = 20
- Attempt all questions. No aids (Books/Notes/Gadgets).

Please give your answers in the space provided.

1. A long cylinder of radius R is immersed in a very viscous, incompressible fluid. At time $t = 0$, it begins to rotate with a constant angular velocity Ω about its axis. Assume steady, axisymmetric and unidirectional Stokes flow (only $u_\theta(r)$ is non-zero). Under these conditions, $u_\theta(r)$ obeys: $\mu\left(\frac{d^2u_\theta}{dr^2} + \frac{1}{r}\frac{du_\theta}{dr} - \frac{u_\theta}{r^2}\right) = 0$ with general solution $u_\theta(r) = Ar + B/r$. Using no-slip boundary conditions and assuming the fluid to be at rest far away, find $u_\theta(r)$. Hence determine the magnitude of torque per unit length on the cylinder given that the shear stress at the cylinder wall is: $\tau(R) = \mu\left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right)\Big|_{r=R}$.

$u_\theta = Ar + \frac{B}{r}$. ; B.C.'s give, $u_\theta(r=R) = \Omega R$ —(a)

& $u_\theta(r\to\infty) = 0$. —(b)

(b) gives, $A = 0$; $\therefore$ From (a), $\Omega R = B/R \Rightarrow B = \Omega R^2$

$\therefore u_\theta = \frac{\Omega R^2}{r}$.

$\therefore \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} = -\frac{\Omega R^2}{r^2} - \frac{\Omega R^2}{r^2} = -2\frac{\Omega R^2}{r^2}$

$\tau(R) = \mu\left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right)\Big|_{r=R} = -2\mu\Omega\frac{R^2}{R^2} = -2\mu\Omega$

$\therefore$ Magnitude of torque/length $= |\tau(R) \times 2\pi R^2|$

$= 4\pi\mu\Omega R^2$

2. A doublet of strength κ is placed in a uniform flow U. The potential and stream function for the doublet are given as $\phi_d = \kappa\cos\theta/r$ and $\psi_d = -\kappa\sin\theta/r$, respectively. Write down expressions for the streamlines, draw them and find the stagnation points.

For Doublet of strength κ, $\phi_d = \kappa\frac{\cos\theta}{r}$, $\psi_d = -\kappa\frac{\sin\theta}{r}$

Uniform flow in x-direction : $\phi_u = Ur\cos\theta$

$\psi_u = Ur\sin\theta$.

$\therefore$ Total potential : $\phi = Ur\cos\theta + \frac{K\cos\theta}{r}$

Total stream function : $\psi = Ur\sin\theta - \frac{K\sin\theta}{r}$

For streamlines, consider $\psi = 0 \Rightarrow Ur^2 = K \Rightarrow r = \sqrt{\frac{K}{U}}$.

$\rightarrow$ closed streamline (circular in polar coordinates)

For stagnation points : velocity $= 0$. In polar coordinates,

$$u_r = \frac{\partial\phi}{\partial r} = U\cos\theta - \frac{K\cos\theta}{r^2}$$

$$u_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -U\sin\theta - \frac{K\sin\theta}{r^2}.$$

At stagnation, $u_r = 0 \Rightarrow U = \frac{K}{r^2} \Rightarrow r^2 = \frac{K}{U}$.

$u_\theta = 0 \Rightarrow \sin\theta = 0 \Rightarrow \theta = 0$ or π

$\therefore$ Stagnation pts : $r = \sqrt{\frac{K}{U}}$, $\theta = 0, \pi$.

$\Rightarrow x = \pm\sqrt{\frac{K}{U}}$, $y = 0$.

3. For the velocity field $\mathbf{u} = (-y, x)$, compute the vorticity and stream function and hence classify the flow. Comment on the streamlines.

$\underline{u}(x,y) = (-y, x)$.

In 2d, vorticity vector has only one component $\perp^r$ to the x-y plane (i.e. along $\hat{z}$) : $\omega_z = \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}$.

Now, $u_x = -y$, $u_y = x$. $\therefore \omega_z = 1 - (-1) = 2$

$\therefore$ Vorticity is constant & non-zero, indicating a uniformly rotating flow.

To get the stream function, $u_x = \frac{\partial \psi}{\partial y}$; $u_y = -\frac{\partial \psi}{\partial x}$.

$\therefore$ $-y = \frac{\partial \psi}{\partial y}$ $\Rightarrow$ Integrating, $\psi = -\frac{y^2}{2} + f(x)$

& $x = -\frac{\partial \psi}{\partial x}$ ~~Integrating~~

$\Rightarrow -x = -\frac{\partial f}{\partial x} \Rightarrow f(x) = -\frac{x^2}{2}$.

$\therefore \psi(x, y) = -\frac{1}{2}(x^2+y^2)$. $\Rightarrow$ circular streamlines

4. Consider the slow flow equations: $-\nabla p + \mu \nabla^2 \mathbf{u} = 0$; $\nabla \cdot \mathbf{u} = 0$, where $\mathbf{u}$ is the flow velocity and p is the pressure. Let the fluid be in some region V which is bounded by a closed surface S. Let $\mathbf{u} = \mathbf{u_B}(\mathbf{x})$, say, on S. Then show that there is at most one solution of the slow flow equations which satisfies that boundary condition.

Suppose there is another flow $\underline{u}^*$ which also satisfies the slow flow equations with pressure p^* and has $\underline{u}^* = \underline{u}_B(\underline{x})$ on S.

Let, $\underline{v} = \underline{u}^* - \underline{u}$ & $P = p^* - p$.

Slow flow equations are linear. $\therefore$ $0 = -\underline{\nabla} P + \mu \nabla^2 \underline{v}$ & $\underline{\nabla} \cdot \underline{v} = 0$.

with $\underline{v} = 0$ on S but $\underline{v} \neq 0$ in V by our hypothesis.

In component form, $-\frac{\partial P}{\partial x_i} + \mu \frac{\partial^2 v_i}{\partial x_j^2} = 0$ & $\frac{\partial v_i}{\partial x_i} = 0$

Multiplying the first equation by v_i,

$$-v_i \frac{\partial P}{\partial x_i} + \mu v_i \frac{\partial^2 v_i}{\partial x_j^2} = 0 \Rightarrow -\frac{\partial}{\partial x_i}(v_i P) + \mu v_i \frac{\partial^2 v_i}{\partial x_j^2} = 0$$

$$\left(\because \frac{\partial v_i}{\partial x_i} = 0\right)$$

Integrating over the volume & using divergence theorem,

$$-\int_S P v_i n_i \, dA + \mu \int_V v_i \frac{\partial^2 v_i}{\partial x_j^2} \, dV = 0.$$

Since $\underline{v} = 0$ on S, the first term $= 0$

Now, $\frac{\partial}{\partial x_j}\left(v_i \frac{\partial v_i}{\partial x_j}\right) = v_i \frac{\partial^2 v_i}{\partial x_j^2} + \left(\frac{\partial v_i}{\partial x_j}\right)^2$

$\Rightarrow v_i \frac{\partial^2 v_i}{\partial x_j^2} = \frac{\partial}{\partial x_j}\left(v_i \frac{\partial v_i}{\partial x_j}\right) - \left(\frac{\partial v_i}{\partial x_j}\right)^2$

$\therefore \mu \int_V \left[\frac{\partial}{\partial x_j}\left(v_i \frac{\partial v_i}{\partial x_j}\right) - \left(\frac{\partial v_i}{\partial x_j}\right)^2\right] dV = 0$

Again using divergence theorem, $\mu \int_S v_i \frac{\partial v_i}{\partial x_j} n_j \, dA - \mu \int \left(\frac{\partial v_i}{\partial x_j}\right)^2 dV = 0$

Since, $\underline{v} = 0$, 1st term $= 0$

$\Rightarrow \int_V \left(\frac{\partial v_i}{\partial x_j}\right)^2 dV = 0 \Rightarrow \frac{\partial v_i}{\partial x_j} = 0$ for all i & j $\Rightarrow \underline{v} = \text{const}$

But $\underline{v} = 0$ on S $\therefore \underline{v} = 0$ in V.

This is a contradiction.

5. For a turbulent flow, the outer scale is where the fluid is being stirred, i.e. where energy is being injected with $Re = U\,L/\nu \gg 1$, where U is a velocity scale, L is a length scale and ν is the kinematic viscosity. The inner scale is where viscous dissipation occurs. The typical velocity v_d and lengthscale l_d are such that $v_d l_d/\nu \sim 1$. Noting that in a steady cascade, the energy transfer rate ε from scale to scale must be constant, how do v_d and l_d scale with ε and Re?

Since ε from scale to scale is constant, hence from Dimensional arguments, $\varepsilon \sim \frac{U^3}{L} \sim \frac{v_d^3}{l_d}$

But, $\frac{v_d l_d}{\nu} \sim 1 \quad \therefore v_d^3 \sim \frac{\nu^3}{l_d^3} \quad \therefore \varepsilon \sim \frac{\nu^3}{l_d^4}$

$\Rightarrow l_d \sim \left(\frac{\nu^3}{\varepsilon}\right)^{1/4}$ & $v_d \sim (\nu\varepsilon)^{1/4}$.

Further, $\left(\frac{L}{l_d}\right)^4 = L^3 \times L \times l_d^{-4} \sim L^3 \times \frac{U^3}{\cancel{\varepsilon}} \times \frac{\cancel{\varepsilon}}{\nu^3}$

$\sim \left(\frac{LU}{\nu}\right)^3 = Re^3. \Rightarrow \left(\frac{L}{l_d}\right) \sim Re^{3/4}$

$\therefore l_d \sim \frac{L}{Re^{3/4}}$. $\therefore$ ~~velocity~~

$\frac{U^3}{L} \sim \frac{v_d^3}{l_d} \Rightarrow v_d \sim \left(\frac{l_d}{L}\right)^{1/3} U \sim \frac{U}{Re^{1/4}}$.

1. (a) $\frac{\partial h}{\partial t} + c_0 \frac{\partial h}{\partial x} + \frac{3}{2}\frac{c_0}{H} h \frac{\partial h}{\partial x} + \frac{1}{6} c_0 H^2 \frac{\partial^3 h}{\partial x^3} = 0.$

Let, $x = L\tilde{x}$, $t = \frac{L}{c_0}\tilde{t}$, $h = a\tilde{h}$

$$\frac{\partial h}{\partial t} = \frac{a}{L/c_0}\frac{\partial \tilde{h}}{\partial \tilde{t}} = \frac{a c_0}{L}\frac{\partial \tilde{h}}{\partial \tilde{t}}.$$

$$\frac{\partial h}{\partial x} = \frac{a}{L}\frac{\partial \tilde{h}}{\partial \tilde{x}} \; ; \; \frac{\partial^3 h}{\partial x^3} = \frac{a}{L^3}\frac{\partial^3 \tilde{h}}{\partial \tilde{x}^3}.$$

Substituting back in the KdV eqn:

$$\frac{a c_0}{L}\frac{\partial \tilde{h}}{\partial \tilde{t}} + c_0 \cdot \frac{a}{L}\frac{\partial \tilde{h}}{\partial \tilde{x}} + \frac{3}{2}\frac{c_0}{H}\frac{a^2}{L}\tilde{h}\frac{\partial \tilde{h}}{\partial \tilde{x}} + \frac{1}{6}c_0 H^2 \frac{a}{L^3}\frac{\partial^3 \tilde{h}}{\partial \tilde{x}^3} = 0$$

$$\Rightarrow \frac{\partial \tilde{h}}{\partial \tilde{t}} + \frac{\partial \tilde{h}}{\partial \tilde{x}} + \frac{3}{2}\frac{a}{H}\tilde{h}\frac{\partial \tilde{h}}{\partial \tilde{x}} + \frac{1}{6}\left(\frac{H}{L}\right)^2\frac{\partial^3 \tilde{h}}{\partial \tilde{x}^3} = 0$$

Define $\epsilon = \frac{a}{H}$; $\delta = \left(\frac{H}{L}\right)^2.$

Then, $\frac{\partial \tilde{h}}{\partial \tilde{t}} + \frac{\partial \tilde{h}}{\partial \tilde{x}} + \frac{3}{2}\epsilon \tilde{h}\frac{\partial \tilde{h}}{\partial \tilde{x}} + \frac{1}{6}\delta \frac{\partial^3 \tilde{h}}{\partial \tilde{x}^3} = 0.$

(b) $h(x,t) = a\,\text{sech}^2[k(x-ct)]$. ; Let $\xi = x - ct$.

$$\frac{\partial h}{\partial t} = -2ak(-c)\,\text{sech}(k\xi)\,\text{sech}(k\xi)\tanh(k\xi)$$

$$= 2akc\,\text{sech}^2(k\xi)\tanh(k\xi)$$

$$\frac{\partial h}{\partial x} = -2ak\,\text{sech}^2(k\xi)\tanh(k\xi)$$

$$\frac{\partial^2 h}{\partial x^2} = +4ak^2\,\text{sech}(k\xi)\,\text{sech}(k\xi)\tanh(k\xi)\tanh(k\xi)$$
$$- 2ak^2\,\text{sech}^2(k\xi)\,\text{sech}^2(k\xi).$$

$$= +4ak^2\,\text{sech}^2(k\xi)\left[\tanh^2(k\xi) - \frac{1}{2}\text{sech}^2(k\xi)\right]$$

$$= 4ak^2\,\text{sech}^2(k\xi)\left[1 - \frac{3}{2}\text{sech}^2(k\xi)\right].$$

Similarly,

$$\frac{\partial^3 h}{\partial x^3} = -8aK^3 \operatorname{Sech}^2(K\xi)\tanh(K\xi)\left[1-3\operatorname{Sech}^2(K\xi)\right]$$

Putting all of them back in the KdV eqn:

$$-2aK\operatorname{Sech}^2(K\xi)\tanh(K\xi)\Big[-c+c_0+\frac{3}{2}\frac{c_0 a}{H}\operatorname{Sech}^2(K\xi)\cdot$$

$$+\frac{4}{6}c_0 H^2K^2\left(1-3\operatorname{Sech}^2(K\xi)\right)\Big]=0.$$

For this expression to vanish for all ξ, and hence all values of $\operatorname{Sech}^2(K\xi)$, the coefficient of $\operatorname{Sech}^2(K\xi)\tanh(K\xi)$ must be identically zero.

$$\therefore\ (c_0-c)+\left(\frac{3}{2}\frac{c_0 a}{H}-2c_0H^2K^2\right)\operatorname{Sech}^2(K\xi)$$

$$+\frac{2}{3}c_0H^2K^2=0.$$

Collecting the constant term & the Sech² terms and noting that each needs to be zero to satisfy the above:

$$c_0-c+\frac{2}{3}c_0H^2K^2=0.$$

$$\Rightarrow\ c=c_0+\frac{2}{3}c_0H^2K^2=c_0\left(1+\frac{2}{3}H^2K^2\right)$$

$$\&\quad \frac{3}{2}\frac{c_0 a}{H}-2c_0H^2K^2=0.$$

$$\Rightarrow\ K^2=\frac{3a}{4H^3}\ \Rightarrow\ K=\sqrt{\frac{3a}{4H^3}}.$$

$$\therefore\ c=c_0\left(1+\frac{\cancel{2}}{\cancel{3}}\cdot H^2\cdot\frac{\cancel{3}a}{\cancel{4}2H^3}\right)$$

$$=c_0\left(1+\frac{a}{2H}\right).$$

$\rho_1 > \rho_2$.

(c) When $\epsilon >> \delta \Rightarrow$ nonlinearity dominates.

Dropping the nonlinear term:

$$\frac{\partial h}{\partial t} + c_0 \frac{\partial h}{\partial x} + \frac{1}{6} c_0 H^2 \frac{\partial^3 h}{\partial x^3} = 0.$$

Assume, $h(x,t) = e^{i(kx - \omega t)}$

Substituting,

$$-i\omega + ic_0k - i\frac{1}{6} c_0 H^2 k^3 = 0$$

$$\Rightarrow \omega = c_0 k \left(1 - \frac{1}{6} H^2 k^2\right)$$

Phase speed, $c = \frac{\omega}{k} = c_0\left(1 - \frac{1}{6} H^2 k^2\right)$

Group speed $c_g = \frac{d\omega}{dk} = c_0\left(1 - \frac{1}{2} H^2 k^2\right)$.

For linear shallow water waves ($kH << 1$),

$c \approx \sqrt{gH} \Rightarrow$ non-dispersive.

$\therefore$ KdV equation introduces dispersive correction to shallow water waves, with longer wavelengths travelling faster.

2. 

$\rho_1 > \rho_2$

Lower layer: Thickness h_1
viscosity μ_1
region $0 \le y \le h_1$

Upper layer: thickness h_2; viscosity μ_2
region $h_1 \le y \le h_1 + h_2$

No slip at bottom: $u = 0$ at $y = 0$

Shear free surface: $\frac{\partial u}{\partial y} = 0$ at $y = h_1 + h_2$

Continuity of velocity and shear stress at the interface $y = h_1$

Flow is steady, unidirectional $u(y)$ & driven by gravity.

Simplified N-S: $\mu \frac{d^2 u}{dy^2} = -\rho g \sin\alpha$.

Lower layer: $\frac{d^2 u_1}{dy^2} = -\frac{\rho g \sin\alpha}{\mu_1}$

Integrating twice: $u_1(y) = -\frac{\rho g \sin\alpha}{2\mu_1} y^2 + Ay + B$

A, B: constants of integration.

Upper layer: $\frac{d^2 u_2}{dy^2} = -\frac{\rho g \sin\alpha}{\mu_2}$

Integrating twice: $u_2(y) = -\frac{\rho g \sin\alpha}{2\mu_2} y^2 + Cy + D$

C, D: constants of integration

Boundary condition:

(i) No-slip at bottom wall: $u_1(0) = 0 \Rightarrow B = 0$

(ii) Shear free surface: $\left.\dfrac{du_2}{dy}\right|_{y=h_1+h_2} = 0.$

$$\Rightarrow -\frac{\rho g \sin\alpha}{\mu_2}(h_1+h_2) + C = 0$$

$$\Rightarrow C = \frac{\rho g \sin\alpha}{\mu_2}(h_1+h_2).$$

(iii) Continuity of shear stress at interface:

$$\mu_1 \left.\frac{du_1}{dy}\right|_{y=h_1} = \mu_2 \left.\frac{du_2}{dy}\right|_{y=h_1}$$

$$\Rightarrow \mu_1\left(-\frac{\rho g \sin\alpha}{\mu_1} h_1 + A\right) = \mu_2\left(-\frac{\rho g \sin\alpha}{\mu_2} h_1 + C\right).$$

$$\Rightarrow \cancel{-\rho g \sin\alpha\, h_1} + \mu_1 A = \cancel{-\rho g \sin\alpha\, h_1} + \mu_2 C.$$

$$\Rightarrow \mu_1 A = \mu_2 C$$

$$\Rightarrow A = \frac{\mu_2}{\mu_1} C = \frac{\rho g \sin\alpha}{\mu_1}(h_1+h_2)$$

(iv) Continuity of velocity at interface:

$$u_1(h_1) = u_2(h_1)$$

$$\Rightarrow -\frac{\rho g \sin\alpha}{2\mu_1} h_1^2 + A h_1 = -\frac{\rho g \sin\alpha}{2\mu_2} h_1^2 + C h_1 + D$$

$$\Rightarrow -\frac{\rho g \sin\alpha}{2\mu_1} h_1^2 + \frac{\rho g \sin\alpha}{\mu_1} h_1^2 + \frac{\rho g \sin\alpha}{\mu_1} \cancel{h_1} h_2$$

$$= -\frac{\rho g \sin\alpha}{2\mu_2} h_1^2 + \frac{\rho g \sin\alpha}{\mu_2} h_1^2 + \cancel{\frac{\rho g \sin\alpha}{\mu_2}} h_1 h_2 + D.$$

$$\Rightarrow \quad \frac{\rho g \sin\alpha}{2\mu_1} h_1^2 = \frac{\rho g \sin\alpha}{2\mu_2} h_1^2 + D$$

$$\Rightarrow \quad D = \frac{\rho g \sin\alpha}{2} h_1^2 \left(\frac{1}{\mu_1} - \frac{1}{\mu_2}\right).$$

$$\therefore \; u_2(y) = -\frac{\rho g \sin\alpha}{2\mu_2} y^2 + \frac{\rho g \sin\alpha}{\mu_2} y (h_1 + h_2) + \frac{\rho g \sin\alpha}{2} h_1^2 \left(\frac{1}{\mu_1} - \frac{1}{\mu_2}\right).$$

$$= \frac{\rho g \sin\alpha}{\mu_2} \left[\left(h_1 + h_2 - \frac{y}{2}\right) y + \frac{h_1^2}{2}\left(\frac{\mu_2}{\mu_1} - 1\right)\right]$$

The vel. profile of lower fluid does not depend on the viscosity of the upper fluid, although the velocity at the interface depends on matching condition.

3.

$\rho_1 > \rho_2$.

Inviscid fluid: Euler eqn: for each fluid:

$$\rho_i\left(\frac{\partial \underline{u}_i}{\partial t} + \underline{u}_i \cdot \underline{\nabla}\,\underline{u}_i\right) = -\underline{\nabla} P_i + \rho_i g \; ; \; i=1,2$$

$$\underline{u} = \underline{U}_1 + \underline{\nabla}\phi_1 \quad z < \eta \qquad z = \eta(x,t) = \eta_0 e^{i(kx - \omega t)}$$

$$= \underline{U}_2 + \underline{\nabla}\phi_2 \quad z > \eta \qquad \underline{U}_i = U_i \hat{x}$$

Fluid is incompressible: $\nabla^2\phi_1 = \nabla^2\phi_2 = 0$.

$\phi_1 \to 0$ as $z \to \infty$, $\phi_2 \to 0$ as $z \to \infty$

On the interface, $u_z = \frac{D\eta}{Dt}$.

$$\left.\frac{\partial \phi_i}{\partial z}\right|_{z=\eta} = \frac{\partial \eta}{\partial t} + \left(U_i + \left.\frac{\partial \phi_i}{\partial x}\right|_{z=\eta}\right)\frac{\partial \eta}{\partial x} \quad -(1).$$

$i=1,2$.

Pressure force must balance: $P_1(x,\eta) - P_2(x,\eta)$

γ: surface tension. $\qquad = -\gamma \frac{\partial^2 \eta}{\partial x^2}. \;(2)$

Bernoulli's principle:

$$\rho_i \frac{\partial \phi_i}{\partial t} + \frac{1}{2}\rho_i |\underline{U}_i + \underline{\nabla}\phi_i|^2 + P_i + \rho_i g z = f_i(t) \quad -(3)$$

From (2) above,

$$\rho_2\left(\frac{\partial \phi_2}{\partial t} + \frac{1}{2}|\underline{U}_2 + \underline{\nabla}\phi_2|^2 + gz\right)\Big|_{z=\eta}$$

$$-\rho_1\left(\frac{\partial \phi_1}{\partial t} + \frac{1}{2}|\underline{U}_1 + \underline{\nabla}\phi_1|^2 + gz\right)\Big|_{z=\eta}$$

$$= \tilde{f}(t) - \gamma\, \partial^2\eta/\partial x^2. \quad -(4)$$

Linearized Approximation: $k\eta_0 << 1$.

Then,

$$\frac{\partial \phi_i}{\partial z}\Big|_{z=0} = \frac{\partial \eta}{\partial t} + U_i \frac{\partial \eta}{\partial x}\Big|_{z=0} \quad i=1,2. \qquad -(5)$$

$$\rho_2\left(\frac{\partial \phi_2}{\partial t} + U_2\frac{\partial \phi_2}{\partial x}\right)\Big|_{z=0} - \rho_1\left(\frac{\partial \phi_1}{\partial t} + U_1\frac{\partial \phi_1}{\partial x}\right)\Big|_{z=0}$$

$$+ g(\rho_2 - \rho_1)\eta = f(t) - \gamma\frac{\partial^2 \eta}{\partial x^2} \qquad -(6)$$

For $\underline{\gamma = 0;\ g \neq 0}$.

$$\phi_i(x,z,t) = \hat{\phi}_i(z)e^{i(kx-\omega t)}$$

$$\eta(x,t) = \eta_0 e^{i(kx-\omega t)}$$

From Laplace eqn: $\frac{d^2\hat{\phi}_i}{dz^2} = k^2\hat{\phi}_i$

$$\Rightarrow \quad \hat{\phi}_1 = A_1 e^{+kz};\quad \hat{\phi}_2 = A_2 e^{-kz}$$

$\hat{\phi}_1 \to 0$ as $z \to -\infty$, $\hat{\phi}_2 \to 0$ as $z \to \infty$.

From (5): $kA_1 = (-i\omega + ikU_1)\eta_0$

& $-kA_2 = (-i\omega + ikU_2)\eta_0$

From (6) since $f(t)$ is independent of x & z, & $\gamma = 0$:

$$\rho_2(-i\omega + ikU_2)A_2 - \rho_1(-i\omega + ikU_1)A_1$$

$$+ g(\rho_2 - \rho_1)\eta_0 = 0.$$

$$\Rightarrow \ -\frac{\rho_2\eta_0}{k}(-i\omega + ikU_2)^2 - \frac{\rho_1 A_0}{k}(-i\omega + ikU_1)^2$$

$$+ g(\rho_2 - \rho_1)\eta_0 = 0$$

$$\Rightarrow \ + \rho_2(\omega^2 + k^2U_2^2 - 2\omega kU_2) + \rho_1(\omega^2 + k^2U_1^2 - 2\omega kU_1)$$

$$+ g(\rho_2 - \rho_1)k = 0.$$

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1.

$$\Rightarrow (\rho_1+\rho_2)\omega^2 - 2\omega k(\rho_1 U_1+\rho_2 U_2) + k^2(\rho_1 U_1^2+\rho_2 U_2^2) + gk(\rho_2-\rho_1) = 0.$$

$$\therefore \omega = \frac{1}{\rho_1+\rho_2}\left[k(\rho_1 U_1+\rho_2 U_2) \pm \sqrt{(\rho_1+\rho_2) gk(\rho_1-\rho_2) - \rho_1\rho_2 k^2 (U_1-U_2)^2}\right]$$

For instability;

$$\rho_1\rho_2 k^2 (U_1-U_2)^2 > gk(\rho_1^2-\rho_2^2).$$

$$\Rightarrow k > \frac{\rho_1^2-\rho_2^2}{\rho_1\rho_2}\frac{g}{(U_1-U_2)^2}.$$

Long wavelength (small k) modes are longer stable.

When $g=0$.

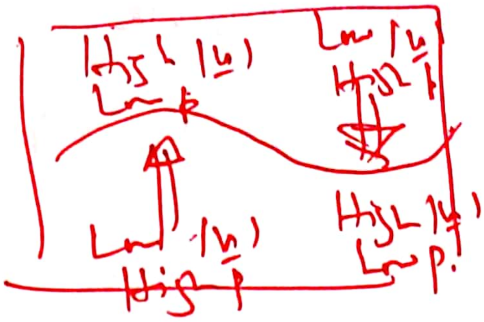
$$\omega = \frac{k}{\rho_1+\rho_2}\left[(\rho_1 U_1+\rho_2 U_2) \pm i\sqrt{\rho_1\rho_2}\,|U_1-U_2|\right].$$

$$z = \eta(x,t) = \eta_0 \exp\left[ik\left(x - \frac{\rho_1 U_1+\rho_2 U_2}{\rho_1+\rho_2}t\right) \pm \frac{\sqrt{\rho_1\rho_2}}{\rho_1+\rho_2}|U_1-U_2|kt\right]$$

Disturbance propagate with vel.

$$v = \frac{\rho_1 U_1+\rho_2 U_2}{\rho_1+\rho_2}.$$

Perturbation grows exponentially with time.


High (w)
Low p
Low (w)
High p
Low (w)
High p
High (w)
Low p