4.1.

$$\mu = \left(\frac{\partial U}{\partial N}\right)_{S V}$$

at $T=0$: $\mu = U_N - U_{N-1}$

For bosons : all the particles are in the g.s.

$$\Rightarrow \mu = N(\epsilon + mc^2) - (N-1)(\epsilon + mc^2)$$

$$\boxed{\mu = \epsilon + mc^2}$$

For fermions: (assuming no degeneracy of energy levels) each particle occupies one energy level.

$$\therefore \boxed{\mu = \epsilon_N = N\epsilon + mc^2}$$

4.2

$$N \simeq \int_0^\infty g(\epsilon)\, d\epsilon \, \langle n_{\vec{k}} \rangle = \sum_{\vec{k}} \langle n_{\vec{k}} \rangle$$

$\langle n_{\vec{k}} \rangle = \theta(\epsilon_F - \epsilon_{\vec{k}})$ at $T=0$.

$$= \int_0^{\epsilon_F} g(\epsilon)\, d\epsilon$$

In 2-dim, $g(\epsilon)\, d\epsilon = g_s \frac{Am}{2\pi \hbar^2} d\epsilon$

For e^- : $g_s = (2s+1) = 2$

$$\therefore N = \frac{Am}{\pi \hbar^2} \epsilon_F$$

or $\boxed{\epsilon_F = \frac{\pi \hbar^2 N}{mA}}$

4.3. $$A(T) = \int_0^\infty d^3r \, \{ e^{-\beta U(r)} - 1 \}$$

$$= \int_0^{r_0} d^3r \, (-1) + \int_{r_0}^\infty d^3r \left[e^{\beta U_0 (r_0/r)^n} - 1 \right]$$

$$e^{\beta U_0 (r_0/r)^n} \simeq 1 + \beta U_0 \left(\frac{r_0}{r}\right)^n$$

$\because \; r_0/r < 1$ & $\beta U_0 < 1$

$$\therefore \; A(T) = -\frac{4}{3}\pi r_0^3 + \cancel{4\pi r_0^n} \int_{r_0}^\infty 4\pi r^2 dr \left[\beta U_0 \left(\frac{r_0}{r}\right)^n \right]$$

$$= -\frac{4}{3}\pi r_0^3 + 4\pi r_0^n \beta \int_{r_0}^\infty \frac{dr}{r^{n-2}}$$

for convergence of the integral : $\boxed{n > 3}$

~~4.4~~ $$\langle n_i \rangle = \frac{1}{e^{\beta(\epsilon_i - \mu)} - 1}$$

In the limit $e^{\beta(\epsilon_i - \mu)} \gg 1$, $e^{-\beta(\epsilon_i - \mu)} \ll 1$.

$\langle n_i \rangle \sim e$

$\Rightarrow$ low # density for given V $\left(\sum_i \langle n_i \rangle = N \right)$

$$\underline{\beta(\epsilon-\mu) >> 1}$$

$$\Rightarrow \quad e^{\beta(\epsilon-\mu)} >> 1$$

~~or~~ $\quad e^{\beta\mu} << e^{\beta\epsilon}$

$$\Rightarrow \quad \boxed{e^{\beta\mu} << 1}$$ for $\epsilon < k_B T$ (classical limit)

$$\langle n_i \rangle = \frac{1}{e^{\beta(\epsilon_i-\mu)} \mp \eta} \quad -(1)$$

$\sim e^{-\beta(\epsilon_i-\mu)} << 1 \Rightarrow$ low # density for given V.

$$\Rightarrow \quad N = \sum_i n_i = e^{\beta\mu} \underbrace{\sum_i e^{-\beta\epsilon_i}}_{\text{single particle partition fn}}$$

$$\Rightarrow \quad N = e^{\beta\mu} \cdot \left(\frac{V}{\lambda^3}\right)$$

$$\Rightarrow \quad \boxed{e^{\beta\mu} = \frac{N}{V}\lambda^3} \quad -(2)$$

$\star\ \mu = \mu(T)$.

from 1 & (2) : $\beta(\epsilon-\mu) >> 1$ is ~~consistent~~ consistent with $\frac{N}{V}\lambda^3 << 1$.

4.5 Eqn of state for a quantum ideal gas.

classical limit: $PV = Nk_BT \left\{ 1 + \underbrace{\eta \frac{\lambda^3 N}{4\sqrt{2}\, V}}_{\text{The effect of quantum statistics.}} \right\}$

$\Rightarrow$ Pressure of a bose/fermi gas is lower/higher than that of a classical ideal gas ($PV = Nk_BT$).