

## PHY 101 : Problem Sheet 2

For the discussion on **cylindrical polar co-ordinate** please refer to the Fig. 1 on the next page. Any point in the 3-Dimensional space can either be written as  $(x, y, z)$  or a point on a cylinder as  $(r, \theta, z)$  where  $r$  is the magnitude of the projection ( $\mathbf{r}$ ) of the position vector  $\mathbf{R}$  into the  $x - y$  plane, and  $\theta$  is the angle the projected vector  $\mathbf{r}$  makes with the  $x -$  axis. Note that the unit vectors associated with  $\hat{\mathbf{r}}$  and  $\hat{\theta}$  is the same as we learnt for the polar co-ordinates in 2-dimensional case in the class. Thus,

$$\begin{aligned}\hat{\mathbf{r}} &= \cos \theta \hat{\mathbf{i}} + \sin \theta \hat{\mathbf{j}}; \\ \hat{\theta} &= -\sin \theta \hat{\mathbf{i}} + \cos \theta \hat{\mathbf{j}},\end{aligned}$$

while  $\hat{\mathbf{k}}$  keeps pointing along the  $z -$  axis

1. Find out  $\mathbf{v} = d\mathbf{R}/dt$  and  $\mathbf{a} = d\mathbf{v}/dt$  in  $\hat{\mathbf{r}}, \hat{\theta}, \hat{\mathbf{k}}$  basis.
2. Find out  $\hat{\mathbf{r}} \times \hat{\theta}$ ,  $\hat{\theta} \times \hat{\mathbf{k}}$  and  $\hat{\mathbf{k}} \times \hat{\mathbf{r}}$ .
3. Find out the expression of angular momentum  $\mathbf{L}$  of a particle at a location  $\mathbf{R}$  and moving with velocity  $\mathbf{v}$  in cylindrical polar co-ordinates.
4. In cylindrical polar co-ordinates write down the expression for torque  $\vec{\tau} = d\mathbf{L}/dt$ .

For the discussion on **spherical polar co-ordinate** please refer to the Fig. 2 on the next page. Any point  $P$  in the 3-Dimensional space can either be written as  $(x, y, z)$  or a point on a sphere as  $(r, \theta, \phi)$  where  $r$  is the magnitude of position vector  $\mathbf{r}$ ,  $\theta$  is the angle the vector  $\mathbf{r}$  makes with the  $z -$  axis and  $\phi$  is the angle the vector  $\mathbf{p}$ , which is projection of  $\mathbf{r}$  into the  $x - y$  plane, makes w.r.t. the  $x -$  axis. Note that the projection of  $\mathbf{r}$  along the  $x -$  axis (OA) can be calculated as  $OA = OB \cos \phi = r \sin \theta \cos \phi$ . Find out the projection of  $\mathbf{r}$  along the  $y -$  axis and show for a unit radius sphere we have

$$\hat{\mathbf{r}} = \sin \theta \cos \phi \hat{\mathbf{i}} + \sin \theta \sin \phi \hat{\mathbf{j}} + \cos \theta \hat{\mathbf{k}};$$

Also note that the unit vector  $\hat{\phi}$  plays the same role as the angular co-ordinate we learnt for the polar co-ordinates in 2-dimensional case in the class. Thus,

$$\hat{\phi} = -\sin \phi \hat{\mathbf{i}} + \cos \phi \hat{\mathbf{j}}.$$

Since  $\hat{\theta}$  is perpendicular to  $\hat{\mathbf{r}}$  [because  $\hat{\mathbf{r}}$  along the radial direction while  $\hat{\theta}$  is along the tangent to the sphere], show that

1. the projection of  $\hat{\theta}$  along the  $z -$  axis is “ $-\sin \theta$ ”.
2. If the projection of  $\hat{\theta}$  along the  $x -$  and  $y -$  axis are  $x$  and  $y$  respectively, i.e.

$$\hat{\theta} = x \hat{\mathbf{i}} + y \hat{\mathbf{j}} - \sin \theta \hat{\mathbf{k}},$$

then find out  $x$  and  $y$  from the conditions  $\hat{\mathbf{r}} \cdot \hat{\theta} = 0$  and  $\hat{\phi} \cdot \hat{\theta} = 0$ .

3. Find out the expression for  $d\hat{\mathbf{r}}/dt$  and  $d\hat{\phi}/dt$  in terms of  $\hat{\mathbf{r}}, \hat{\theta}, \hat{\phi}$ .
4. For a general position vector  $\mathbf{r} = r\hat{\mathbf{r}}$ , find the expression of velocity  $\hat{\mathbf{v}} = d\mathbf{r}/dt$ .

A hand-drawn diagram illustrating a cylinder in a 3D coordinate system. The cylinder is centered on the  $z$ -axis. A vector  $\vec{R}$  is drawn from the origin to a point on the top surface of the cylinder. A vector  $\vec{r}$  is drawn from the origin to a point on the bottom surface of the cylinder. A vertical vector  $\vec{z}$  is drawn from the bottom surface to the top surface. The vectors  $\vec{R}$ ,  $\vec{r}$ , and  $\vec{z}$  form a right-angled triangle. The axes are labeled  $x$ ,  $y$ , and  $z$ .

A diagram showing a sphere centered at the origin  $O$  of a Cartesian coordinate system with axes  $x$ ,  $y$ , and  $z$ . A point  $P$  is located on the surface of the sphere. The position vector from  $O$  to  $P$  is labeled  $\vec{r}$ . The angle between the  $z$ -axis and  $\vec{r}$  is  $\theta$ . The angle between the  $x$ -axis and the projection of  $\vec{r}$  onto the  $xy$ -plane is  $\phi$ . The distance from  $O$  to the projection of  $P$  on the  $xy$ -plane is labeled  $\rho$ . The radius of the sphere is labeled  $R$ . The coordinates of  $P$  are indicated as  $(x, y, z)$ .

## Worksheet - 2

1. The position vector

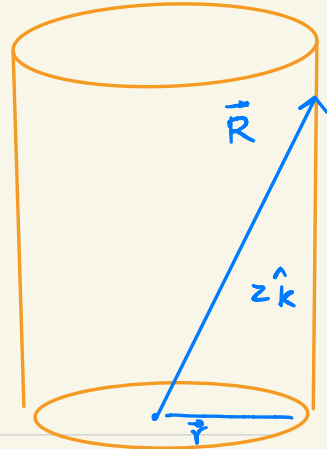
$$\vec{R} = \vec{r} + z \hat{k}$$

$$= r \hat{r} + z \hat{k}$$

$$\vec{V} = \frac{d\vec{R}}{dt} = \frac{d}{dt}(r \hat{r}) + \frac{d}{dt}(z \hat{k})$$

$$= \dot{r} \hat{r} + r \dot{\theta} \hat{\theta} + \dot{z} \hat{k}$$

because  $\dot{\hat{r}} = \dot{\theta} \hat{\theta}$  because  $\dot{\hat{k}} = 0$



Therefore,

$$\vec{a} = \frac{d\vec{V}}{dt} = \frac{d}{dt}(\dot{r} \hat{r}) + \frac{d}{dt}(r \dot{\theta} \hat{\theta}) + \frac{d}{dt}(\dot{z} \hat{k})$$

$$= \ddot{r} \hat{r} + \dot{r} \dot{\theta} \hat{\theta} + \dot{r} \dot{\theta} \hat{\theta} + r \ddot{\theta} \hat{\theta} - r \dot{\theta}^2 \hat{r} + \ddot{z} \hat{k}$$

$$= (\ddot{r} - r \dot{\theta}^2) \hat{r} + (r \ddot{\theta} + 2\dot{r} \dot{\theta}) \hat{\theta} + \ddot{z} \hat{k}$$

$$\begin{aligned}
 2. \quad \hat{i} &= \cos \theta \hat{i} + \sin \theta \hat{j} \\
 \hat{\theta} &= -\sin \theta \hat{i} + \cos \theta \hat{j} \\
 \hat{k} &= \hat{k}
 \end{aligned}$$

$$\begin{aligned}
 \hat{i} \times \hat{\theta} &= (\cos \theta \hat{i} + \sin \theta \hat{j}) \times (-\sin \theta \hat{i} + \cos \theta \hat{j}) \\
 &= (\cos^2 \theta + \sin^2 \theta) \hat{k} \\
 &= \hat{k}
 \end{aligned}$$

$$\begin{aligned}
 \hat{\theta} \times \hat{k} &= (-\sin \theta \hat{i} + \cos \theta \hat{j}) \times \hat{k} \\
 &= (\sin \theta \hat{j} + \cos \theta \hat{i}) = \hat{r}
 \end{aligned}$$

$$\begin{aligned}
 \hat{k} \times \hat{r} &= \hat{k} \times (\cos \theta \hat{i} + \sin \theta \hat{j}) \\
 &= (\cos \theta \hat{j} - \sin \theta \hat{i}) = \hat{\theta}
 \end{aligned}$$

$$\begin{aligned}
 3. \quad \vec{L} &= \vec{R} \times m \vec{V} \\
 &= (r \hat{r} + z \hat{k}) \times m (\dot{r} \hat{r} + r \dot{\theta} \hat{\theta} + \dot{z} \hat{k}) \\
 &= m r^2 \dot{\theta} \hat{k} + m r \dot{z} (-\hat{\theta})
 \end{aligned}$$

$$4. \quad \vec{\tau} = \frac{d\vec{L}}{dt} = \frac{d}{dt} (\vec{R} \times m\vec{V})$$

$$= \underbrace{\frac{d\vec{R}}{dt} \times m\vec{V}} + \vec{R} \times m \frac{d\vec{V}}{dt}$$

0 because  
 $\vec{V} \times \vec{V}$

$$= \vec{R} \times m\vec{a} = (r\hat{r} + z\hat{k}) \times$$

$$[ (\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta} + \ddot{z}\hat{k} ]$$

$$= (2r\dot{r}\dot{\theta} + r^2\ddot{\theta})\hat{k} - r\ddot{z}\hat{\theta} + (z\ddot{r} - zr\dot{\theta}^2)\hat{\theta} \\ + (2\dot{r}z\dot{\theta} + zr\ddot{\theta})\hat{r}$$

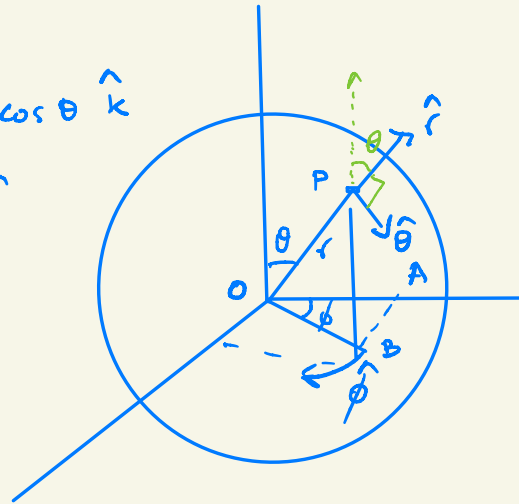
$$= - (2\dot{r}z\dot{\theta} + zr\ddot{\theta})\hat{r} + (z\ddot{r} - r\ddot{z} - zr\dot{\theta}^2)\hat{\theta} \\ + (2r\dot{r}\dot{\theta} + r^2\ddot{\theta})\hat{k}$$

Q. Set 2

$$1.) \hat{r} = \sin \theta \cos \phi \hat{i} + \sin \theta \sin \phi \hat{j} + \cos \theta \hat{k}$$

$$\hat{\phi} = -\sin \phi \hat{i} + \cos \phi \hat{j}$$

The vector  $\hat{\theta}$  makes  
an angle  $90^\circ + \theta$  with  
z-axis



$$\begin{aligned} \therefore \hat{\theta} &= x \hat{i} + y \hat{j} + \cos(90^\circ + \theta) \hat{k} \\ &= x \hat{i} + y \hat{j} - \sin \theta \hat{k} \end{aligned}$$

$$2.) \hat{r} \cdot \hat{\theta} = x \sin \theta \cos \phi + y \sin \theta \sin \phi - \sin \theta \cos \theta = 0 \quad \text{--- (1)}$$

$$\begin{aligned} \hat{\theta} \cdot \hat{\phi} &= -x \sin \phi + y \cos \phi = 0 \\ \Rightarrow x \sin \phi &= y \cos \phi \quad \text{--- (2)} \end{aligned}$$

$$\sin \phi \times (1) \Rightarrow x \sin \theta \sin \phi \cos \phi + y \sin \theta \sin^2 \phi = \sin \theta \cos \theta \sin \phi$$

$$\begin{aligned} \text{Using (2)} \Rightarrow y \sin \theta \cos^2 \phi + y \sin \theta \sin^2 \phi &= \sin \theta \cos \theta \sin \phi \\ &= \sin \theta \cos \theta \sin \phi \end{aligned}$$

$$\Rightarrow y \sin \theta = \sin \theta \cos \theta \sin \phi$$

$$y = \cos \theta \sin \phi$$

Using (2)  $x \sin \phi = y \cos \phi$   
 $= \cos \theta \sin \phi \cos \phi$

$$\therefore x = \cos \theta \cos \phi$$

$$\therefore \hat{\theta} = \cos \theta \cos \phi \hat{i} + \cos \theta \sin \phi \hat{j} - \sin \theta \hat{k}$$

3.)  $\hat{r} = \sin \theta \cos \phi \hat{i} + \sin \theta \sin \phi \hat{j} + \cos \theta \hat{k}$

$$\frac{d\hat{r}}{dt} = \frac{d}{dt} (\sin \theta \cos \phi) \hat{i} + \frac{d}{dt} (\sin \theta \sin \phi) \hat{j} + \frac{d}{dt} (\cos \theta) \hat{k}$$

$$= (\dot{\theta} \cos \theta \cos \phi - \dot{\phi} \sin \theta \sin \phi) \hat{i} + (\dot{\theta} \cos \theta \sin \phi + \dot{\phi} \sin \theta \cos \phi) \hat{j} - \dot{\theta} \sin \theta \hat{k}$$

$$= \dot{\theta} (\cos \theta \cos \phi \hat{i} + \cos \theta \sin \phi \hat{j} - \sin \theta \hat{k})$$

$$+ \dot{\phi} \sin \theta (-\sin \phi \hat{i} + \cos \phi \hat{j})$$

$$= \dot{\theta} \hat{\theta} + \dot{\phi} \sin \theta \hat{\phi}$$

$$\frac{d\hat{\phi}}{dt} = \frac{d}{dt} (-\sin \phi \hat{i} + \cos \phi \hat{j}) = -\dot{\phi} (\cos \phi \hat{i} + \sin \phi \hat{j})$$

$$= -\dot{\phi} (\sin \theta \hat{r} + \cos \theta \hat{\theta})$$

$$4. \quad \frac{d\vec{r}}{dt} = \frac{d}{dt} (r \hat{r}) = \dot{r} \hat{r} + r \frac{d}{dt} \hat{r}$$

$$= \dot{r} \hat{r} + r (\dot{\theta} \hat{\theta} + \sin \theta \dot{\phi} \hat{\phi})$$

$$= \dot{r} \hat{r} + r \dot{\theta} \hat{\theta} + r \sin \theta \dot{\phi} \hat{\phi}$$