Que 1 we know that,

$$[x_i, p_j] = i\hbar\,\delta_{ij}$$

$$\vec{L} = \vec{r} \times \vec{p} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix}$$

$$L_x = y p_z - z p_y$$

$$L_y = x p_z - z p_x$$

$$L_z = x p_y - y p_x$$

So,

(a) (i) $[L_z, x] = [x p_y - y p_x, x]$

$$= [x p_y, x] - [y p_x, x]$$

$$\Rightarrow x\cancel{[p_y, x]}^{\to 0} + \cancel{[x, x]}^{\to 0} p_y - y[p_x, x] - \cancel{[y, x]}^{\to 0} p_x$$

$$= -y[p_x, x]$$

$$= y[x, p_x] = i\hbar y$$

(ii) $[L_z, y] = -i\hbar x$

Sol^n

$$[xp_y - yp_x, y]$$

$$= [xp_y, y] - [yp_x, y]$$

$$= x[p_y, y] + \cancelto{0}{[x, y]}\, p_y - y\,\cancelto{0}{[p_x, y]} - \cancelto{0}{[y, y]}\, p_x$$

$$= -i\hbar x$$

(iii) $[L_z, z] = 0$

$$[xp_y - yp_x, z]$$

$$= [xp_y, z] - [y p_x, z]$$

$$= x\,\cancelto{0}{[p_y, z]} + \cancelto{0}{[x, z]}\, p_y - y\,\cancelto{0}{[p_x, z]} - \cancelto{0}{[y, z]}\, p_x$$

$$= 0$$

similarly for $[L_z, p_i]$

So,

(iv) $[L_z, p_x] = [xp_y - yp_x, p_x]$

$$= [xp_y, p_x] - [yp_x, p_x]$$

$$= x\,\cancelto{0}{[p_y, p_x]} + [x, p_x]\, p_y - y\,\cancelto{0}{[p_x, p_x]} + \cancelto{0}{[y, p_x]}\, p_x$$

$$= +i\hbar p_y$$

(v) $[L_z, p_y] = -i\hbar p_y$

$= [xp_y - yp_x, p_y]$

$= [xp_y, p_y] - [yp_x, p_y]$

$= x[p_y p_y] + [x, p_y]p_y - y[p_x, p_y] - [y, p_y]p_x$

$= -i\hbar p_x.$

(vi) $[L_z, p_z] = 0$

$= [xp_y - yp_x, p_z]$

$= [xp_y, p_z] - [yp_x, p_z]$

$= x[p_y, p_z] + [x, p_z]p_y - y[p_x, p_z] - [y, p_z]p_x$

$= 0$

(b) (i) $[L_z, r^2]$

$= [L_z, x^2 + y^2 + z^2]$

$= [L_z, x^2] + [L_z, y^2] + [L_z, z^2]$

$= [L_z, xx] + [L_z, yy] + [L_z, zz]$

$= [L_z, x]x + x[L_z, x] + [L_z, y]y + y[L_z, y] + \cancel{[L_z, z]}z + z\cancel{[L_z, z]}^{=0}$

$=$

$\Rightarrow i\hbar\{yx + xy - xy - yx\}$

$[L_z, r^2] = 0$

(ii) $[L_z, p^2]$ $\qquad p^2 = p_x^2 + p_y^2 + p_z^2$

$$\Rightarrow [L_z, p_x^2] + [L_z, p_y^2] + \cancel{[L_z, p_z^2]}^{\,0}$$

$$\Rightarrow p_x[L_z, p_x] + [L_z, p_x]p_x + p_y[L_z\, p_y] + [L_z, p_y]p_y$$

$$= i\hbar\{p_x p_y + p_y p_x - p_y p_x - p_x p_y\}$$

$$= 0.$$

Q2. L_x, L_y and L_z, L^2 in spherical coordinate.

$$\vec{L} = \vec{r} \times \vec{p} \qquad p = -i\hbar\nabla$$

$$\vec{L} = -i\hbar(\vec{r} \times \vec{\nabla})$$

$$\vec{\nabla} = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}$$

$$\vec{L} = -i\hbar\left[r\hat{r} \times \left\{\hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \frac{\hat{\phi}}{r\sin\theta}\frac{\partial}{\partial\phi}\right\}\right]$$

$$= -i\hbar\left[r\,\cancel{(\hat{r}\times\hat{r})}^{\,0}\frac{\partial}{\partial r} + r\frac{1}{r}\frac{\partial}{\partial\theta}(\hat{r}\times\hat{\theta}) + \frac{1}{r\sin\theta}\, r\,\frac{\partial}{\partial\phi}(\hat{r}\times\hat{\phi})\right]$$

$$\vec{L} = -i\hbar\left[\hat{\phi}\frac{\partial}{\partial\theta} - \hat{\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right].$$

Now, we can write $\hat{\phi}$ and $\hat{\theta}$ in terms of $\hat{i}$, $\hat{j}$ and $\hat{k}$.

$$\hat{r} = \hat{i}\sin\theta\cos\phi + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}$$

$$\hat{\theta} = \cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,\hat{k}$$

$$\hat{\phi} = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$$

$$\begin{pmatrix} r \\ \theta \\ \phi \end{pmatrix} = \begin{pmatrix} \cos\phi \sin\theta & \sin\theta \sin\phi & \cos\theta \\ \cos\theta \cos\phi & \cos\theta \sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 1 \end{pmatrix} \begin{pmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{pmatrix}$$

Using this Conversion matrix:

$$\vec{L} = -i\hbar \left[(-\sin\phi \hat{i} + \cos\phi \hat{j}) \frac{\partial}{\partial\theta} - (\cos\theta \cos\phi \hat{i} + \cos\theta \sin\phi \hat{j} - \sin\theta \hat{k}) \frac{1}{\sin\theta} \frac{\partial}{\partial\phi} \right]$$

$$\vec{L} = L_x \hat{i} + L_y \hat{j} + L_z \hat{k}$$

$$\boxed{L_x = -i\hbar \left[-\sin\phi \frac{\partial}{\partial\theta} - \cos\phi \cot\theta \frac{\partial}{\partial\phi} \right]}$$

$$\boxed{L_y = -i\hbar \left[\cos\phi \frac{\partial}{\partial\theta} - \sin\phi \cot\theta \frac{\partial}{\partial\phi} \right]}$$

$$L_z = -i\hbar \left[\sin\theta \frac{1}{\sin\theta} \frac{\partial}{\partial\phi} \right]$$

$$\boxed{L_z = -i\hbar \frac{\partial}{\partial\phi}}$$

and.

$$L^2 = L_x^2 + L_y^2 + L_z^2.$$

$$\vec{L} = \vec{R} \times \vec{P} = -i\hbar r(\hat{r} \times \vec{\nabla})$$

$$= (-i\hbar r)\, \hat{r} \times \left[\hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial \theta} + \frac{\hat{\phi}}{r\sin\theta}\frac{\partial}{\partial \phi}\right]$$

$$\vec{L} = -i\hbar\left(\hat{\phi}\frac{\partial}{\partial\theta} - \frac{\hat{\theta}}{\sin\theta}\frac{\partial}{\partial\phi}\right)$$

$$L_x = i\hbar\left(\sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi}\right).$$

$$L_y = i\hbar\left(-\cos\phi\frac{\partial}{\partial\theta} + \cot\theta\sin\phi\frac{\partial}{\partial\phi}\right).$$

$$L_z = -i\hbar\frac{\partial}{\partial\phi}.$$

$$\boxed{\cancel{L_+ = L_x + iL_y}}$$

$$L^2 = -\hbar^2 r^2(\hat{r}\times\vec{\nabla})\cdot(\hat{r}\times\vec{\nabla})$$

$$= -\hbar^2 r^2\left[\frac{\hat{\phi}}{r}\frac{\partial}{\partial\theta} - \frac{\hat{\theta}}{r\sin\theta}\frac{\partial}{\partial\phi}\right]\left[\frac{\hat{\phi}}{r}\frac{\partial}{\partial\theta} - \frac{\hat{\theta}}{r\sin\theta}\frac{\partial}{\partial\phi}\right]$$

$$= -\hbar^2\left[\left\{\hat{\phi}\frac{\partial}{\partial\theta} - \frac{\hat{\theta}}{\sin\theta}\frac{\partial}{\partial\phi}\right\}\left\{\hat{\phi}\frac{\partial}{\partial\theta} - \frac{\hat{\theta}}{\sin\theta}\frac{\partial}{\partial\phi}\right\}\right]$$

$$= -\hbar^2\left[\hat{\phi}\cdot\frac{\partial}{\partial\theta}\left(\hat{\phi}\frac{\partial}{\partial\theta}\right) - \hat{\phi}\frac{\partial}{\partial\theta}\left(\frac{\hat{\theta}}{\sin\theta}\frac{\partial}{\partial\phi}\right) - \frac{\hat{\theta}}{\sin\theta}\frac{\partial}{\partial\phi}\left(\hat{\phi}\frac{\partial}{\partial\theta}\right) + \frac{\hat{\theta}}{\sin\theta}\cdot\frac{\partial}{\partial\phi}\left(\frac{\hat{\theta}}{\sin\theta}\frac{\partial}{\partial\phi}\right)\right]$$

$$= -\hbar^2\left[\frac{\partial^2}{\partial\theta^2} - 0 - \frac{\hat{\theta}}{\sin\theta}\cdot\left(\frac{\partial\hat{\phi}}{\partial\phi}\frac{\partial}{\partial\theta} + \hat{\phi}\frac{\partial^2}{\partial\phi\partial\theta}\right) + \frac{\hat{\theta}}{\sin^2\theta}\cdot\left(\frac{\partial\hat{\theta}}{\partial\phi}\frac{\partial}{\partial\phi} + \hat{\theta}\frac{\partial^2}{\partial\phi^2}\right)\right]$$

$$= -\hbar^2 \left[\frac{\partial^2}{\partial \theta^2} - \frac{\hat{\theta}}{\sin\theta} \left(-\hat{r}\sin\theta - \hat{\theta}\cos\theta \right) \frac{\partial}{\partial \theta} + \hat{\phi} \frac{\partial^2}{\partial\theta\partial\phi} \right) + \right.$$

$$\left. \frac{\hat{\theta}}{\sin^2\theta} \left(\hat{\phi}\cos\theta \frac{\partial}{\partial\phi} + \hat{\theta}\frac{\partial^2}{\partial\phi^2} \right) \right]$$

using $\frac{\partial\hat{\phi}}{\partial\phi} = -\hat{r}\sin\theta - \hat{\theta}\cos\theta$

$\frac{\partial\hat{\theta}}{\partial\phi} = \hat{\phi}\cos\theta$

$$= -\hbar^2 \left[\frac{\partial^2}{\partial\theta^2} + \cot\theta \frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} \right]$$

(zettille page 658)

$$= -\hbar^2 \left[\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} \right]$$

From "Quantum mechanics, concepts and applications" by Noureddine Zettili, Wiley

B

Angular Momentum in Spherical Coordinates

In this appendix, we will show how to derive the expressions of the gradient $\vec{\nabla}$, the Laplacian ∇^2, and the components of the orbital angular momentum in spherical coordinates.

B.1 Derivation of Some General Relations

The Cartesian coordinates (x, y, z) of a vector $\vec{r}$ are related to its spherical polar coordinates (r, θ, φ) by

$$x = r\sin\theta\cos\varphi, \qquad y = r\sin\theta\sin\varphi, \qquad z = r\cos\theta \tag{B.1}$$

The orthonormal Cartesian basis $(\hat{x}, \hat{y}, \hat{z})$ is related to its spherical counterpart $(\hat{r}, \hat{\theta}, \hat{\varphi})$ by

$$\hat{x} = \hat{r}\sin\theta\cos\varphi + \hat{\theta}\cos\theta\cos\varphi - \hat{\varphi}\sin\varphi \tag{B.2}$$
$$\hat{y} = \hat{r}\sin\theta\sin\varphi + \hat{\theta}\cos\theta\sin\varphi + \hat{\varphi}\cos\varphi, \tag{B.3}$$
$$\hat{z} = \hat{r}\cos\theta - \hat{\theta}\sin\theta. \tag{B.4}$$

Differentiating (B.1), we obtain

$$dx = \sin\theta\cos\varphi\,dr + r\cos\theta\cos\varphi\,d\theta - r\sin\theta\sin\varphi\,d\varphi \tag{B.5}$$
$$dy = \sin\theta\sin\varphi\,dr + r\cos\theta\sin\varphi\,d\theta + r\cos\varphi\,d\varphi, \tag{B.6}$$
$$dz = \cos\theta\,dr - r\sin\theta\,d\theta. \tag{B.7}$$

Solving these equations for dr, $d\theta$ and $d\varphi$, we obtain

$$dr = \sin\theta\cos\varphi\,dx + \sin\theta\sin\varphi\,dy + \cos\theta\,dz \tag{B.8}$$
$$d\theta = \frac{1}{r}\cos\theta\cos\varphi\,dx + \frac{1}{r}\cos\theta\sin\varphi\,dy - \frac{1}{r}\sin\theta\,dz, \tag{B.9}$$
$$d\varphi = -\frac{\sin\varphi}{r\sin\theta}\,dx + \frac{\cos\varphi}{r\sin\theta}\,dy. \tag{B.10}$$

We can verify that (B.5) to (B.10) lead to

$$\frac{\partial r}{\partial x} = \sin\theta\cos\varphi, \qquad \frac{\partial \theta}{\partial x} = \frac{1}{r}\cos\varphi\cos\theta, \qquad \frac{\partial \varphi}{\partial x} = -\frac{\sin\varphi}{r\sin\theta}, \tag{B.11}$$

$$\frac{\partial r}{\partial y} = \sin\theta\sin\varphi, \qquad \frac{\partial \theta}{\partial y} = \frac{1}{r}\sin\varphi\cos\theta, \qquad \frac{\partial \varphi}{\partial y} = \frac{\cos\varphi}{r\sin\theta}, \tag{B.12}$$

$$\frac{\partial r}{\partial z} = \cos\theta, \qquad \frac{\partial \theta}{\partial z} = -\frac{1}{r}\sin\theta, \qquad \frac{\partial \varphi}{\partial z} = 0, \tag{B.13}$$

which, in turn, yield

$$\begin{aligned}\frac{\partial}{\partial x} &= \frac{\partial}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial x} + \frac{\partial}{\partial \varphi}\frac{\partial \varphi}{\partial x} \\ &= \sin\theta\cos\varphi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\cos\varphi\frac{\partial}{\partial \theta} - \frac{\sin\varphi}{r\sin\theta}\frac{\partial}{\partial \theta},\end{aligned} \tag{B.14}$$

$$\begin{aligned}\frac{\partial}{\partial y} &= \frac{\partial}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial y} + \frac{\partial}{\partial \varphi}\frac{\partial \varphi}{\partial y} \\ &= \sin\theta\sin\varphi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\sin\varphi\frac{\partial}{\partial \theta} + \frac{\cos\varphi}{r\sin\theta}\frac{\partial}{\partial \varphi},\end{aligned} \tag{B.15}$$

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial z} + \frac{\partial}{\partial \varphi}\frac{\partial \varphi}{\partial z} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{\partial r}\frac{\partial}{\partial \theta}. \tag{B.16}$$

B.2 Gradient and Laplacian in Spherical Coordinates

We can show that a combination of (B.14) to (B.16) allows us to express the operator $\vec{\nabla}$ in spherical coordinates:

$$\vec{\nabla} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z} = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\varphi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}, \tag{B.17}$$

and also the Laplacian operator ∇^2

$$\nabla^2 = \vec{\nabla}.\vec{\nabla} = \left(\hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial \theta} + \frac{\hat{\varphi}}{r\sin\varphi}\frac{\partial}{\partial \varphi}\right)\cdot\left(\hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial \theta} + \frac{\hat{\varphi}}{r\sin\theta}\frac{\partial}{\partial \varphi}\right). \tag{B.18}$$

Now, using the relations

$$\frac{\partial \hat{r}}{\partial r} = 0, \qquad \frac{\partial \hat{\theta}}{\partial r} = 0, \qquad \frac{\partial \hat{\varphi}}{\partial r} = 0, \tag{B.19}$$

$$\frac{\partial \hat{r}}{\partial \theta} = \hat{\theta}, \qquad \frac{\partial \hat{\theta}}{\partial \theta} = -\hat{r}, \qquad \frac{\partial \hat{\varphi}}{\partial \theta} = 0, \tag{B.20}$$

$$\frac{\partial \hat{r}}{\partial \varphi} = \hat{\varphi}\sin\theta, \qquad \frac{\partial \hat{\theta}}{\partial \varphi} = \hat{\varphi}\cos\theta, \qquad \frac{\partial \hat{\varphi}}{\partial \varphi} = -\hat{r}\sin\theta - \hat{\theta}\cos\theta, \tag{B.21}$$

we can show that the Laplacian operator reduces to

$$\nabla^2 = \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial}{\partial \theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial \varphi^2}\right]. \tag{B.22}$$

B.3 Angular Momentum in Spherical Coordinates

The orbital angular momentum operator $\vec{L}$ can be expressed in spherical coordinates as:

$$\hat{\vec{L}} = \hat{\vec{R}} \times \hat{\vec{P}} = (-i\hbar r)\hat{r} \times \vec{\nabla} = (-i\hbar r)\hat{r} \times \left[\hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial \theta} + \frac{\hat{\varphi}}{r\sin\theta}\frac{\partial}{\partial\varphi}\right], \tag{B.23}$$

or as

$$\hat{\vec{L}} = -i\hbar\left(\hat{\varphi}\frac{\partial}{\partial\theta} - \frac{\hat{\theta}}{\sin\theta}\frac{\partial}{\partial\varphi}\right). \tag{B.24}$$

Using (B.24) along with (B.2) to (B.4), we express the components $\hat{L}_x, \hat{L}_y, \hat{L}_z$ within the context of the spherical coordinates. For instance, the expression for $\hat{L}_x$ can be written as follows

$$\begin{aligned}\hat{L}_x &= \hat{x}.\vec{L} = -i\hbar\left(\hat{r}\sin\theta\cos\varphi + \hat{\theta}\cos\theta\cos\varphi - \hat{\varphi}\sin\varphi\right)\cdot\left(\hat{\varphi}\frac{\partial}{\partial\theta} - \frac{\hat{\theta}}{\sin\theta}\frac{\partial}{\partial\varphi}\right) \\ &= i\hbar\left(\sin\varphi\frac{\partial}{\partial\theta} + \cot\theta\cos\varphi\frac{\partial}{\partial\varphi}\right). \end{aligned} \tag{B.25}$$

Similarly, we can easily obtain

$$\hat{L}_y = i\hbar\left(-\cos\varphi\frac{\partial}{\partial\theta} + \cot\theta\sin\varphi\frac{\partial}{\partial\varphi}\right) \tag{B.26}$$

$$\hat{L}_z = -i\hbar\frac{\partial}{\partial\varphi}. \tag{B.27}$$

From the expressions (B.25) and (B.26) for $\hat{L}_x$ and $\hat{L}_y$, we infer that

$$\hat{L}_+ = \hat{L}_x + i\hat{L}_y = \hbar e^{i\varphi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right), \tag{B.28}$$

$$\hat{L}_- = \hat{L}_x - i\hat{L}_y = \hbar e^{-i\varphi}\left(\frac{\partial}{\partial\theta} - i\cot\theta\frac{\partial}{\partial\varphi}\right). \tag{B.29}$$

The expression for $\vec{L}^2$ is

$$\vec{L}^2 = -\hbar^2 r^2(\hat{r}\times\vec{\nabla})\cdot(\hat{r}\times\vec{\nabla}) = -\hbar^2 r^2\left[\nabla^2 - \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)\right]; \tag{B.30}$$

it can be easily written in terms of the spherical coordinates as

$$\vec{L}^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]; \tag{B.31}$$

this expression was derived by substituting (B.22) into (B.30).

Note that, using the expression (B.30) for $\vec{L}^{\,2}$, we can rewrite ∇^2 as

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{1}{\hbar^2 r^2}\vec{L}^{\,2} = \frac{1}{r}\frac{\partial^2}{\partial r^2}r - \frac{1}{\hbar^2 r^2}\vec{L}^{\,2}. \tag{B.32}$$

Q4.

$$\Psi = K(x+y+2z)\, e^{-\alpha r}, \text{ where}$$

$$r = \sqrt{x^2+y^2+z^2}.$$

In spherical coordinate (r, θ, φ),

$$\Psi = K\, r\, e^{-\alpha r}\left(\sin\theta\cos\varphi + \sin\theta\sin\varphi + 2\cos\theta\right).$$

The angular part is —

$$\Psi(\theta,\varphi) = K'\left(\sin\theta\cos\varphi + \sin\theta\sin\varphi + 2\cos\theta\right)$$

$$= K'\left[\frac{1}{2}\left(e^{i\varphi}+e^{-i\varphi}\right)\sin\theta + \frac{1}{2i}\left(e^{i\varphi}-e^{-i\varphi}\right)\sin\theta - 2\cos\theta\right]$$

$$\boxed{\Psi(\theta,\varphi) = K'\left[-\frac{1}{2}(1+i)\sqrt{\frac{8\pi}{3}}\, Y_1^{1} + \frac{1}{2}(1+i)\sqrt{\frac{8\pi}{3}}\, Y_1^{-1} + 2\sqrt{\frac{4\pi}{3}}\, Y_1^{0}\right]}$$

We may find out K' using the normalization condition of Y_l^m. $\Rightarrow$

$$(K')^2\left[\frac{1}{2}\cdot\frac{8\pi}{3}+\frac{1}{2}\,\frac{8\pi}{3}+4\cdot\frac{4\pi}{3}\right]=1$$

$$\Rightarrow \boxed{K'=\sqrt{\frac{1}{8\pi}}}$$

$$\Rightarrow \boxed{\psi(\theta,\varphi)=\frac{1}{\sqrt{8\pi}}\left[\sqrt{\frac{8\pi}{3}}\left\{-\frac{1}{2}(1-i)\,Y_1^{1}+\frac{1}{2}(1+i)\,Y_1^{-1}\right\}+2\sqrt{\frac{4\pi}{3}}\,Y_1^{0}\right]}$$

a) The total angular momentum is

$$\sqrt{\langle L^2\rangle}=\sqrt{l(l+1)}\,\hbar=\sqrt{2}\,\hbar, \quad \text{as } l=1.$$

b) The z-component of the angular momentum

$$\langle\psi(\theta,\varphi)|L_x|\psi(\theta,\varphi)\rangle=0$$

c) Probability of finding $L_z = +\hbar$ is

$$P = |\langle L_z = +\hbar | \psi(\theta, \varphi) \rangle|^2 = \frac{1}{8\pi} \cdot \frac{1}{2} \cdot \frac{8\pi}{3} = \frac{1}{6}.$$