PHY 310 - Mathematical Methods for Physicists I

Odd Term 2019, IISER Mohali

Instructor: Dr. Anosh Joseph

First Mid-Semester Examination - Solutions

13th September, 2019 from 8:00 AM - 9:55 AM in LH3/LH4

Maximum Marks 100

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1. [**4 + 4 + 2 = 10 Marks**] Consider the following function of complex variable $z = x+iy$

$$f(z) = \begin{cases} x^3y(y-ix)/(x^6+y^2) & \text{when } z \neq 0, \\ 0 & \text{when } z = 0. \end{cases}$$

(1a.) Find the derivative of the function $f'(z) = df/dz$, at $z = 0$, using the approximation

$$f'(z)\big|_{z=0} = \left.\frac{df}{dz}\right|_{z=0} = \lim_{z\to 0}\left[\frac{f(z)-f(0)}{z}\right],$$

along the path $y = Ax$.

(1b.) Find the derivative of the function using the same approximation given above along the path $y = x^3$.

(1c.) Is the function $f(z)$ differentiable at $z = 0$?

Solution:

We have the function

$$f(z) = \begin{cases} x^3y(y-ix)/(x^6+y^2) & \text{when } z \neq 0, \\ 0 & \text{when } z = 0. \end{cases}$$

(1a.) Taking $z \to 0$ along the path $y = Ax$

$$\begin{aligned}
\lim_{z\to 0}\left[\frac{f(z)-f(0)}{z}\right] &= \lim_{z\to 0}\left[\frac{\frac{x^3y(y-ix)}{(x^6+y^2)}-0}{(x+iy)}\right] \\
&= \lim_{z\to 0}\frac{-ix^3y(x+iy)}{(x^6+y^2)(x+iy)} \\
&= \lim_{z\to 0}\frac{-ix^3y}{(x^6+y^2)} = \lim_{z\to 0}\frac{-ix^3(Ax)}{(x^6+A^2x^2)} \\
&= \lim_{x\to 0}\frac{-iAx^2}{(x^4+A^2)} = 0.
\end{aligned}$$

(1b.) Taking $z \to 0$ along the path $y = x^3$

$$\begin{aligned}
\lim_{z\to 0}\left[\frac{f(z)-f(0)}{z}\right] &= \lim_{z\to 0}\left[\frac{-ix^3y}{(x^6+y^2)}\right] \\
&= \lim_{x\to 0}\frac{-ix^3(x^3)}{(x^6+(x^3)^2} \\
&= -\frac{i}{2}.
\end{aligned}$$

(1c.) The function $f(z)$ is not differentiable at $z = 0$ since we get different values of df/dz for different $z \to 0$ paths, as shown above.

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2. [**2 + 2 + 2 + 2 + 2 = 10 Marks**] Are the following statements true or false?

(2a.) A function, which is analytic everywhere (for all z in the complex plane) is known as an entire function. (True/False)

(2b.) Bessel function $J_p(x)$ is an odd function when p is even and is an even function p is odd. (True/False)

(2c.) A branch point of a multi-valued function is a point such that the function becomes discontinuous once we go around an arbitrarily small closed path around this point. (True/False)

(2d.) $P_n^0(x) = P_n(x)$. (True/False)

(2e.) Helmholtz' equation in spherical polar coordinates leads to Hermite polynomials. (True/False)

Solution:

(2a.) True.

(2b.) False.

(2c.) True.

(2d.) True.

(2e.) False.

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3. [**5 + 5 = 10 Marks**] The function $f(z) = 1/z$ is analytic everywhere in the complex plane.

(3a.) Is the above statement **true** or **false**?

(3b.) Provide the justification for your answer.

Solution:

(3a.) False

(3b.) The function is

$$
\begin{aligned}
f(z) &= \frac{1}{z} \\
&= u(x,y) + iv(x,y) \\
&= \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2}.
\end{aligned}
$$

Thus gives

$$
\begin{aligned}
u &= \frac{x}{x^2+y^2}, \\
v &= \frac{-y}{x^2+y^2}.
\end{aligned}
$$

We have

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{(x^2+y^2)\cdot 1 - x\cdot 2x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}, \\
\frac{\partial u}{\partial y} &= \frac{-2xy}{(x^2+y^2)^2}, \\
\frac{\partial v}{\partial x} &= \frac{2xy}{(x^2+y^2)^2}, \\
\frac{\partial v}{\partial y} &= \frac{y^2-x^2}{(x^2+y^2)^2}.
\end{aligned}$$

We have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Thus CR relations are satisfied. The partial derivatives are also continuous except at $(x, y) = (0, 0)$.

Therefore $1/z$ is analytic everywhere except at $z = 0$.

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4. [**10 Marks**] For a function $f(z) = u + iv$, the Cauchy-Riemann conditions take the following form

$$\begin{aligned}
\frac{\partial u}{\partial r} &= A\frac{\partial v}{\partial \theta}, \\
\frac{\partial u}{\partial \theta} &= B\frac{\partial v}{\partial r},
\end{aligned}$$

in polar coordinates, $(x, y) = (r\cos\theta, r\sin\theta)$. Find A and B.

Solution:

We have

$$z = re^{i\theta} = r\cos\theta + ir\sin\theta.$$

The function is

$$f(z) = u + iv = f(re^{i\theta}).$$

Differentiating partially with respect to r we get

$$\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = f'(re^{i\theta})e^{i\theta}.$$

Differentiating partially with respect to θ we get

$$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = f'(re^{i\theta})re^{i\theta}i.$$

Thus we have, from the above two equations

$$\begin{aligned} f'(re^{i\theta})e^{i\theta} &= \frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}, \\ f'(re^{i\theta})e^{i\theta} &= \frac{-i}{r}\frac{\partial u}{\partial \theta} + \frac{1}{r}\frac{\partial v}{\partial \theta}. \end{aligned}$$

That is

$$\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = \frac{-i}{r}\frac{\partial u}{\partial \theta} + \frac{1}{r}\frac{\partial v}{\partial \theta}.$$

Equating real and imaginary parts

$$\begin{aligned} \frac{\partial u}{\partial r} &= \frac{1}{r}\frac{\partial v}{\partial \theta}, \\ \frac{\partial v}{\partial r} &= -\frac{1}{r}\frac{\partial u}{\partial \theta}. \end{aligned}$$

This gives

$$\begin{aligned} \frac{\partial u}{\partial r} &= \frac{1}{r}\frac{\partial v}{\partial \theta}, \\ \frac{\partial u}{\partial \theta} &= -r\frac{\partial v}{\partial r}. \end{aligned}$$

Thus we have

$$A = 1/r, \quad \text{and} \quad B = -r.$$

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5. [**10 Marks**] Evaluate the value of the integral

$$I = \oint_C dz(z-a)^n,$$

where the contour C is a circle with center a and radius r and the direction of the contour is counterclockwise. Find the value of integral when $n \neq -1$ and when $n = -1$.

Solution:

We have

$$z - a = ee^{i\theta},$$

and θ varies from 0 to 2π.

We have

$$dz = ire^{i\theta}d\theta.$$

The value of the integral, when $n \neq -1$ is

$$\begin{aligned}
I &= \oint_C dz(z-a)^n \\
&= \int_0^{2\pi} r^n e^{in\theta} ire^{i\theta} d\theta \\
&= ir^{n+1} \int_0^{2\pi} d\theta e^{i(n+1)\theta} \\
&= ir^{n+1} \left[\frac{e^{i(n+1)\theta}}{i(n+1)} \right] \Big|_0^{2\pi} \\
&= \frac{r^{n+1}}{(n+1)} [e^{i2(n+1)\pi} - 1] \\
&= \frac{r^{n+1}}{(n+1)} [\cos 2(n+1)\pi + i \sin 2(n+1)\pi - 1] \\
&= \frac{r^{n+1}}{(n+1)} [1 + oi - 1] \\
&= 0.
\end{aligned}$$

When $n = -1$ we have

$$\begin{aligned}
I &= \oint_C \frac{dz}{(z-a)} \\
&= \int_0^{2\pi} \frac{ire^{i\theta}d\theta}{re^{i\theta}} \\
&= i \int_0^{2\pi} d\theta = 2\pi i.
\end{aligned}$$

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6. [**10 Marks**] Show that

$$\sqrt{\frac{1}{2}\pi x}\; J_{\frac{3}{2}}(x) = \frac{\sin x}{x} - \cos x.$$

Solution:

We have

$$J_p(x) = \frac{x^p}{2^p\Gamma(p+1)}\left[1 - \frac{x^2}{2\times 2\times (p+1)} + \frac{x^4}{2\times 4\times 2^2\times (p+1)(p+2)} - \cdots\right]$$

For $p = \frac{3}{2}$ we have

$$\begin{aligned}
J_{\frac{3}{2}}(x) &= \frac{x^{\frac{3}{2}}}{2^{\frac{3}{2}}\Gamma(\frac{3}{2}+1)}\left[1 - \frac{x^2}{2\times 2\times (\frac{3}{2}+1)} + \frac{x^4}{2\times 4\times 2^2\times (\frac{3}{2}+1)(\frac{3}{2}+2)} - \cdots\right] \\
&= \frac{x^{-\frac{1}{2}}}{2\sqrt{2}\frac{3}{2}\frac{1}{2}\sqrt{\pi}}\left[x^2 - \frac{x^4}{2\cdot 5} + \frac{x^6}{2\cdot 4\cdot 5\cdot 7} - \cdots\right] \\
&= \sqrt{\frac{2}{\pi x}}\left[\frac{x^2}{2!} - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \frac{x^6}{7!} + \cdots\right] \\
&= \sqrt{\frac{2}{\pi x}}\left[\frac{1}{x}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) - \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)\right] \\
&= \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right).
\end{aligned}$$

Thus

$$\sqrt{\frac{\pi x}{2}} J_{\frac{3}{2}}(x) = \frac{\sin x}{x} - \cos x$$

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7. [**10 Marks**] The Ber and Bei functions are defined as

$$J_0(i^{\frac{3}{2}}x) = \text{Ber } x + i\text{Bei } x,$$

where

$$\begin{aligned}\text{Ber } x &= 1+\sum_{p=1}^{\infty}(-1)^p\frac{x^{4p}}{2^2\times 4^2\times 6^2\times\cdots\times(4p)^2},\\ \text{Bei } x &= -\sum_{p=1}^{\infty}(-1)^p\frac{x^{4p-2}}{2^2\times 4^2\times 6^2\times\cdots\times(4p-2)^2}.\end{aligned}$$

Show that

$$\frac{d}{dx}\left(x\ \text{Ber}'\ x\right) = -x\text{Bei}\ x.$$

Solution:

We have

$$\text{Ber } x = 1-\frac{x^4}{2^2\times 4^2}+\frac{x^8}{2^2\times 4^2\times 6^2\times 8^2}+\cdots$$

Differentiating this with respect to x we get

$$\text{Ber}'\ x = -4\frac{x^3}{2^2\times 4^2}+8\frac{x^7}{2^2\times 4^2\times 6^2\times 8^2}+\cdots$$

Thus

$$x\text{Ber}'\ x = -\frac{x^4}{2^2\times 4}+\frac{x^8}{2^2\times 4^2\times 6^2\times 8}+\cdots$$

Now

$$\begin{aligned}\frac{d}{dx}\left(x\text{Ber}'\ x\right) &= -\frac{x^3}{2^2}+\frac{x^7}{2^2\times 4^2\times 6^2}+\cdots\\ &= -x\left[\frac{x^2}{2^2}-\frac{x^6}{2^2\times 4^2\times 6^2}+\cdots\right]\\ &= -x\ \text{Bei}\ x.\end{aligned}$$

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8. [**10 Marks**] Consider the integral

$$I=\int_0^{\pi}d\theta\sin 2\theta P_n(\cos\theta),$$

with $n>1$ and $P_n(x)$ is the Legendre polynomial of degree n. Evaluate I.

Solution:

We have

$$\begin{aligned}
I &= \int_0^\pi d\theta \sin 2\theta P_n(\cos\theta) \\
&= \int_0^\pi d\theta \sin 2\theta P_0(x) P_n(\cos\theta) \\
&= \int_0^\pi d\theta 2\sin\theta\cos\theta P_n(\cos\theta) \\
&= -2\int_0^\pi d\cos\theta \cos\theta P_n(\cos\theta) \\
&= -2\int_0^\pi d\cos\theta P_1(\cos\theta) P_n(\cos\theta), \quad n > 1 \\
&= 0,
\end{aligned}$$

using the orthogonality relation for Legendre polynomials.

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9. [**10 Marks**] Consider a particle of mass m that moves vertically under the influence of the Earth's gravitational field. It bounces elastically off some hard surface, which occupies the plane $x = 0$. The potential is given by

$$V(x) = \begin{cases} mgx & \text{when } x > 0, \\ \infty & \text{when } x < 0. \end{cases}$$

The Schroedinger equation for the particle under this potential is

$$-\frac{\hbar^2}{2m}\psi''(x) + V(x)\psi(x) = E\psi(x).$$

Use the substitution

$$z = \alpha\left(x - \frac{E}{mg}\right),$$

with

$$\alpha = \left(\frac{2m^2 g}{\hbar^2}\right)^{\frac{1}{3}},$$

to show that the Schroedinger's equation reduces to Airy differential equation.

Solution:

We have

$$-\frac{\hbar^2}{2m}\psi''(x) + V(x)\psi(x) = E\psi(x).$$

That is

$$\begin{aligned}\psi''(x) &= \frac{2m^2 g}{\hbar^2}\left(x - \frac{E}{mg}\right)\psi(x) \\ &= \alpha^3\left(x - \frac{E}{mg}\right)\psi(x).\end{aligned}$$

Using the substitution

$$z = \alpha\left(x - \frac{E}{mg}\right),$$

we have

$$\begin{aligned}\frac{\partial^2\psi}{\partial x^2} &= \alpha^2\frac{\partial^2\psi}{\partial z^2} \\ &= \alpha^3\frac{z}{\alpha}\psi.\end{aligned}$$

We have $dz = \alpha dx$. Thus the differential equation becomes

$$\frac{\partial^2\psi}{\partial z^2} - z\psi = 0,$$

which is the Airy differential equation.

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10. [**10 Marks**] Let Π be the parity operator. A function $f(\theta,\phi)$ transforms the following way under parity

$$\Pi f(\theta,\phi) \to f(\pi-\theta, \pi+\phi).$$

Show that $Y_l^l(\theta,\phi)$ is an eigenfunction of Π with eigenvalue $(-1)^l$.

Solution:

We have

$$Y_l^m(\theta,\phi) = (-1)^m \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}} P_l^m(\cos\theta)e^{im\phi}.$$

Setting $m = l$,

$$Y_l^l(\theta,\phi) = (-1)^l \sqrt{\frac{(2l+1)}{4\pi(2l)!}} P_l^l(\cos\theta)e^{il\phi}.$$

Upon using

$$\begin{aligned} P_l^l(x) &= (1-x^2)^{l/2} \sum_{k=0}^{0} \frac{(2l-2k)!}{2^l(l-k)!k!(2k)!}(-1)^k x^{-2k} \\ &= (1-x^2)^{l/2} \frac{(2l)!}{2^l l!}. \end{aligned}$$

Thus

$$P_l^l(\cos\theta) = \frac{\sin^l\theta}{2^l l!}(2l)!.$$

Using the above equation we have

$$Y_l^l(\theta,\phi) = (-1)^l \sqrt{\frac{(2l+1)}{4\pi(2l)!}} \frac{\sin^l\theta}{2^l l!}(2l)! e^{il\phi}.$$

Applying Π on $Y_l^l(\theta,\phi)$,

$$\Pi Y_l^l(\theta,\phi) = (-1)^l \sqrt{\frac{(2l+1)}{4\pi(2l)!}} \frac{\sin^l(\pi-\theta)}{2^l l!}(2l)! e^{il(\phi+\pi)}$$

Upon using $\sin(\pi-\theta) = \sin\theta$ and $e^{il\pi} = (-1)^l$,

$$\begin{aligned} \Pi Y_l^l(\theta,\phi) &= (-1)^l \left[(-1)^l \sqrt{\frac{(2l+1)}{4\pi(2l)!}} \frac{\sin^l\theta}{2^l l!}(2l)! e^{il\phi}\right] \\ &= (-1)^l \; Y_l^l(\theta,\phi). \end{aligned}$$

Thus $Y_l^l(\theta,\phi)$ is an eigenfunction of Π with eigenvalue $(-1)^l$.

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