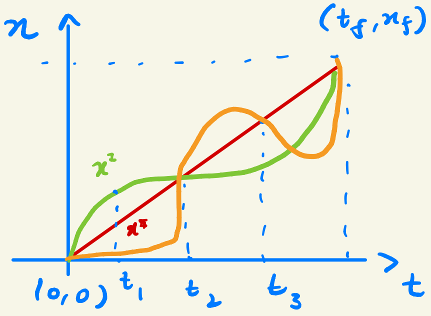
* Need of calculus

(Newton, Lebinitz)

- To identify the local, instantaneous information

• For both the curves, the average speed is same



$$V_{avg} = \frac{x_f - 0}{t_f - 0} = \frac{x_f}{t_f}$$

Still the green curve travels differently compared to the red curves.

$\Rightarrow$ This will show up if we ask questions related to smaller duration. For instance at $t = t_1$, the red curve and the green curve are at different points

$$V_{avg}(t_1) = \frac{x_{II} - 0}{t_1 - 0} = \frac{x_{II}}{t_1}$$

$$V_{avg}(t_1) = \frac{x_I - 0}{t_1 - 0} = \frac{x_1}{t_1}$$

Still for $t = t_2$, again the averages will be the same.

$\Rightarrow$ If we make the duration smaller and smaller, the difference becomes more illustrated.

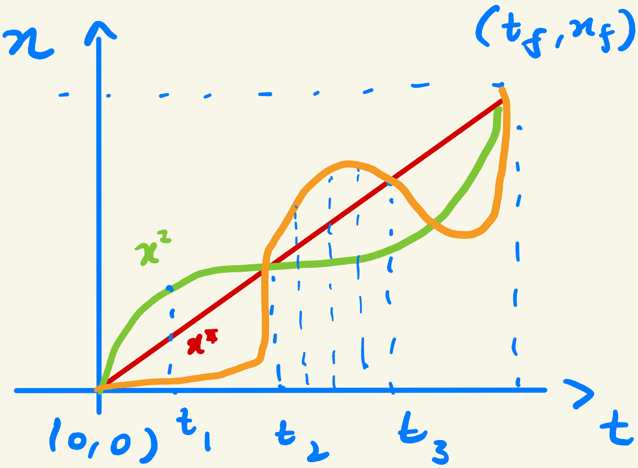
- Suppose we focus on the trajectory post t_2, the red and green curves initially behave differently but eventually end up on the same final point

$$\vec{V}_{avg}(t_3) = \frac{x(t_3) - x(t_2)}{t_3 - t_2}$$

Thus $\vec{V}_{avg}(t_3)$ will be able to differentiate the two curves.

Yet $\vec{V}_{avg}(t_3)$ will not be able to distinguish the red and the orange curves.

If we further break the time steps between t_2 and t_3, different curves can be identified.



Ultimately the local information is captured in the finest time step.

$$V(t) = \lim_{\Delta t \to 0} \frac{X(t+\Delta t) - X(t)}{\Delta t}$$

$$= \frac{dX}{dt}$$

This is called differentiation !!

This gives instantaneous speed at $\underline{t}$

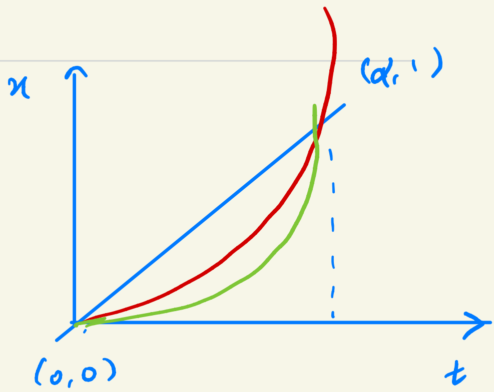
Example

The trajectories

$x = \alpha t$

$x = \alpha t^2$

$x = \alpha t^3$



all begin at $(0,0)$ and at $t = 1\,s$. are at $x = \alpha$.

Yet their local properties are different

$$v(t) = \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\alpha(t+\Delta t) - \alpha t}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\alpha \cancel{\Delta t}}{\cancel{\Delta t}} = \lim_{\Delta t \to} \alpha = \alpha$$

$$v(t) = \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\alpha (t+\Delta t)^2 - \alpha t}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\cancel{\alpha t^2} + \alpha \Delta t^2 + 2\alpha t \Delta t - \cancel{\alpha t^2}}{\Delta t} =$$

$$= \lim_{\Delta t \to 0} \frac{\alpha \Delta t^2 + 2\alpha t \Delta t}{\Delta t} = \lim_{\Delta t \to 0} (2\alpha t + \alpha \Delta t)$$

$$= 2\alpha t$$

$$v(t) = \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\alpha(t+\Delta t)^3 - \alpha t^3}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\cancel{\alpha t^3} + \alpha \Delta t^3 + 3\alpha t^2 \Delta t + 3\alpha t \Delta t^2 - \cancel{\alpha t^3}}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \left(3\alpha t^2 + 3\alpha t \Delta t + \alpha \Delta t^2 \right) = 3\alpha t^2$$

Exercise: Find out the derivative of

$$x(t) = \alpha t^n$$

Take Home Exercise 1:

The cost price of a stock is given as

$$C(t) = 3\alpha t^2 - 4\beta t^3.$$

Find out the local rate of increase of the stock price.

- Average information does not give the full picture
  - With the information $v_{avg} = x_f/t_f$, any of the curve is a possible trajectory
  - With local / instantaneous data it is possible to reconstruct the curve uniquely

- Distance = Velocity × time

In time t_1 the distance traveled

$$x = \frac{x_f}{t_f} t_1$$

$$x = \frac{x_f}{t_f} t_1$$

But the red curve and the green curve have different instantaneous velocity

$v(t)$ and $v(t)$

The velocity keeps changing at all times

$v(t_1), v(t_2), v(t_3) \ldots \ldots \quad v(t_f)$

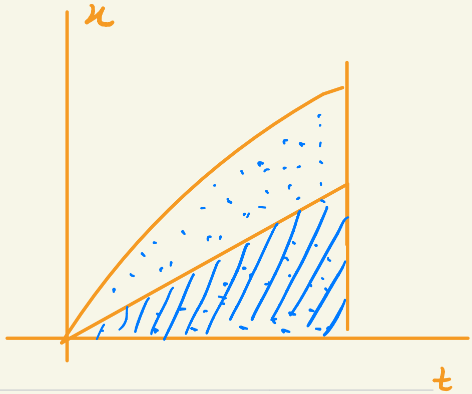
$v(t_1), v(t_2), v(t_3), \ldots \ldots \quad v(t_f)$

Then which of the $v(t)$ will sit in the formula $x = v \times t$

This formula works well for constant velocity

$v(t_1) = v(t_2) = v(t_3) \ldots$

$\ldots v(t_f)$

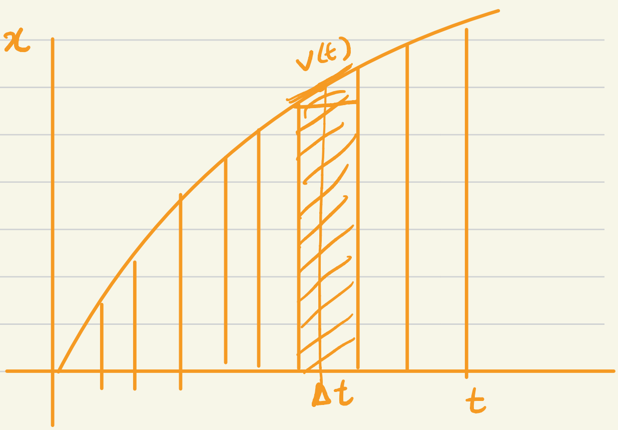


Then the question of

Which velocity does not arise!

Let us divide the step 0 to t into small segments of size Δt.
At the center of a segment let the local speed be

$v(t)$

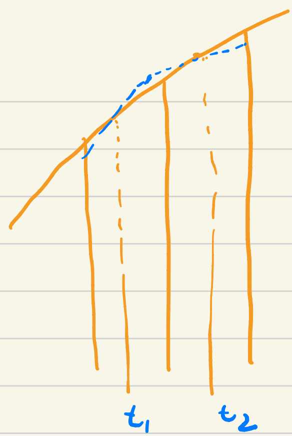


For a small Δt, let us assume $v(t)$ does not change 'much'.

Then for the small Δt we can use $x(\Delta t) = V(t) \times \Delta t$

-- Of course, this is not completely correct but a mindful approximation.

For two consecutive segments the distances travelled



$$x\left(t_1 - \frac{\Delta t}{2}, t_1 + \frac{\Delta t}{2}\right) + x\left(t_2 - \frac{\Delta t}{2}, t_2 + \frac{\Delta t}{2}\right)$$

$$\approx V(t_1)\,\Delta t + V(t_2)\,\Delta t$$

If we carry on this process we get

$$\sum_i V(t_i)\,\Delta t$$

$$= \sum_{i=1}^{n} V(t_0 + (i-1)\Delta t)\,\frac{t_f}{n}$$

$$\left\{ \begin{array}{l} t_0 \\ t_1 \end{array} \right\} \Delta t \quad \left. t_2 \right\} \Delta t \quad \left. t_3 \right\} \Delta t \quad \vdots$$

Ultimately we have to take $n \to \infty$

For $v(t) = \alpha$

$$x(t) = \lim_{n\to\infty} \sum_{i=1}^{n} (\alpha)\, \frac{t}{n} = \lim_{n\to\infty} \alpha t \sum_{i=1}^{n} \frac{1}{n}$$

$$= \lim_{n\to\infty} \alpha t \cdot \cancel{n} \times \frac{1}{\cancel{n}} = \alpha t \cdot \lim_{n\to\infty} 1 = \alpha t$$

For $v(t) = 2\alpha t$

$$x(t) = \lim_{n\to\infty} \sum_{i=1}^{n} 2\alpha\left(t_0 + i\,\Delta t\right)\frac{(t-t_0)}{n}$$

$$= \lim_{n\to\infty} 2\alpha \cdot \frac{\sum_{i=1}^{n}\left(t_0 + i\frac{(t-t_0)}{n}\right)(t-t_0)}{n}$$

$$= \lim_{n\to\infty} 2\alpha \cdot \left[\frac{t_0 n}{n} + \frac{n(n+1)(t-t_0)}{2n^2}\right](t-t_0)$$

$$= \lim_{n\to\infty} 2\alpha(t-t_0)\left[t_0 + \frac{(t-t_0)}{2} + \frac{(t-t_0)}{2n}\right]$$

$$= 2\alpha(t-t_0)\left[\frac{t+t_0}{2}\right] = \alpha\left(t^2 - t_0^2\right)$$

For $v(t) = 3\alpha t^2$

$$x(t) = \lim_{n\to\infty} \sum_{i=1}^{n} 3\alpha \left(t_0 + i\,\Delta t\right)^2 \Delta t$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} 3\alpha \left(t_0 + i\,\frac{(t-t_0)}{n}\right)^2 \left(\frac{t-t_0}{n}\right)$$

$$= \lim_{n\to\infty} 3\alpha(t-t_0) \sum_{i=1}^{n} \frac{\left(t_0 + i\,\frac{(t-t_0)}{n}\right)^2}{n}$$

$$= \lim_{n\to\infty} 3\alpha(t-t_0) \left[\sum_{i=1} \frac{1}{n} \left(t_0^2 + i^2 \frac{(t-t_0)^2}{n^2} + 2t_0\, i\, \frac{(t-t_0)}{n} \right) \right]$$

$$= \lim_{n\to\infty} 3\alpha(t-t_0) \left[t_0^2 + \frac{n(n+1)(2n+1)}{6n^3} (t-t_0)^2 + t_0 \frac{n(n+1)}{n} (t-t_0) \right]$$

$$= 3\alpha(t-t_0) \left[t_0^2 + \frac{(t-t_0)^2}{3} + t_0 (t-t_0) \right]$$

$$= 3\alpha(t-t_0) \left[\frac{3t_0 t + t^2 + t_0^2 - 2tt_0}{3} \right]$$

$$= \alpha(t-t_0)\left(t^2 + t_0^2 + tt_0\right)$$

$$= \alpha\left(t^3 - t_0^3\right)$$

Such sum of limit is called INTEGRATION

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f\left(x_0 + i\left(\frac{x - x_0}{n}\right)\right)\left(\frac{x - x_0}{n}\right) = \int_{x_0}^{x} f(x')\,dx'$$

Exercise : Find out integration of

$$v(t) = n\,\alpha\,t^n$$

Take Home Exercise : A stock price keeps changing with rate $R(t) = 2\alpha t - 3\beta t^2$

Find out how much money one would make during one day ? Half day ? Quarter day ?

Rules of derivative

$$\frac{d}{dt} c = 0 = \lim_{\Delta t \to 0} \frac{c - c}{\Delta t} = 0$$

$$\frac{d}{dt} f(t)\, g(t) = \lim_{\Delta t \to 0} \frac{f(t+\Delta t)\, g(t+\Delta t) - f(t)\, g(t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{f(t+\Delta t)\, g(t+\Delta t) - f(t)\, g(t+\Delta t) + f(t)\, g(t+\Delta t) - f(t)\, g(t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \left(\frac{f(t+\Delta t) - f(t))}{\Delta t} \, g(t+\Delta t) \right. + \lim_{\Delta t \to 0} f(t) \left\{ \frac{g(t+\Delta t) - g(t)}{\Delta t} \right\}$$

$$= \frac{df}{dt} g(t) + f(t) \frac{dg(t)}{dt}$$

$$\frac{d f(y)}{dt} = \frac{dy}{dt} \frac{df}{dy} = \lim_{\Delta t \to 0} \frac{y + \Delta y - y}{\Delta t} \frac{f(y+\Delta y) - f(y)}{\Delta y}$$

$$= \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}$$

Since $y(t) = y$; $y(t+\Delta t) = y + \Delta y$

Quantities in motion

Position $\qquad \vec{x}(t) \equiv (x(t), y(t), z(t))$

$$= x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k}$$

Velocity $\qquad \vec{V}(t) \equiv (V_x(t), V_y(t), V_z(t))$

$$= V_x(t)\,\hat{i} + V_y(t)\,\hat{j} + V_z(t)\,\hat{k}$$

$$\vec{V}(t) = \frac{d\vec{x}}{dt} = \frac{dx(t)}{dt}\,\hat{i} + \frac{dy(t)}{dt}\,\hat{j} + \frac{dz(t)}{dt}\,\hat{k}$$

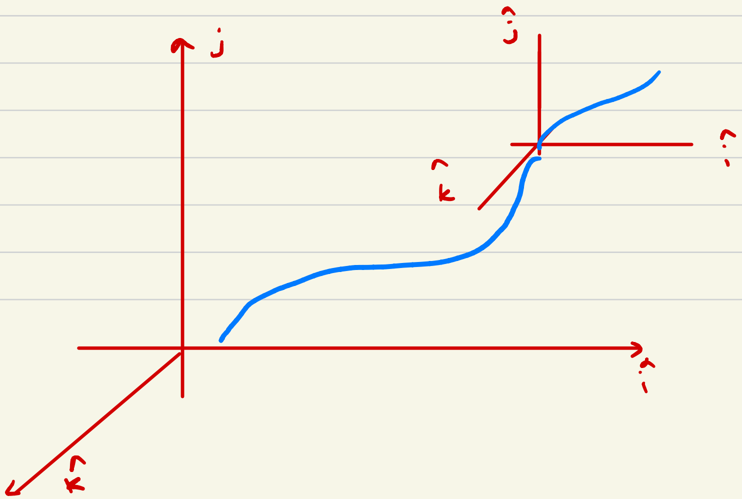
Because $\hat{i}, \hat{j}, \hat{k}$ do not change in t

Acceleration $\qquad \vec{a}(t) \equiv (a_x(t), a_y(t), a_z(t))$

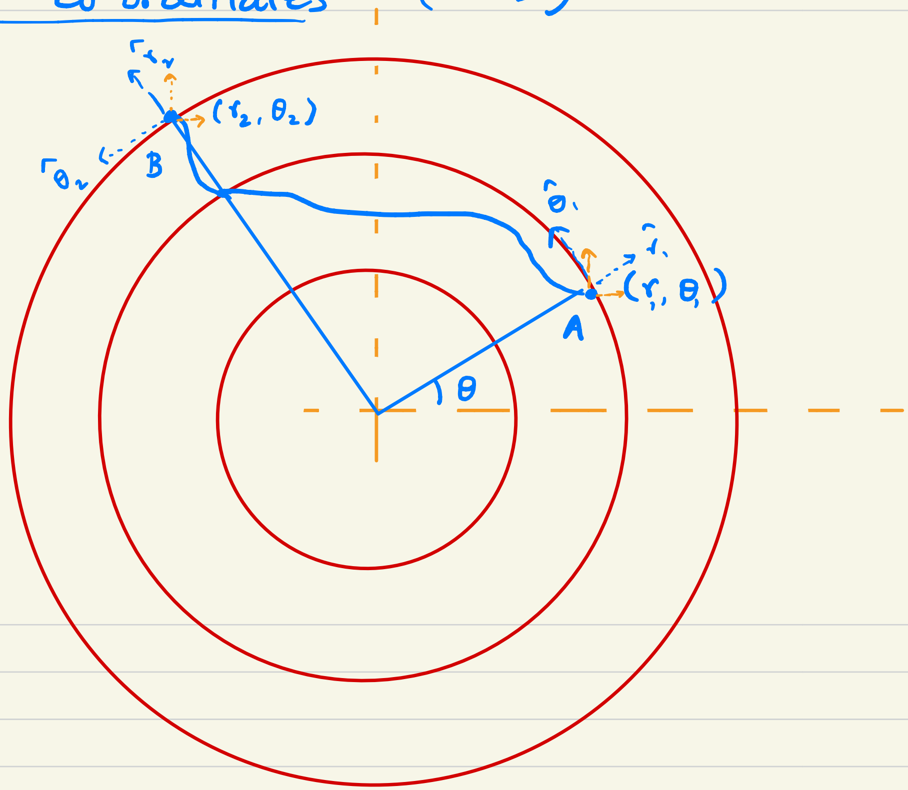
$$= a_x\,\hat{i} + a_y\,\hat{j} + a_z\,\hat{k}$$

$$\vec{a}(t) = \frac{d\vec{V}}{dt} = \frac{dV_x}{dt}\,\hat{i} + \frac{dV_y}{dt}\,\hat{j} + \frac{dV_z}{dt}\,\hat{k}$$

Because $\hat{i}, \hat{j}, \hat{k}$, do not change in t



Polar co-ordinates (2-D)



When a particle/system/person moves from A to B along the curve shown, its position at

location : A (x_1, y_1) or (r_1, θ_1)

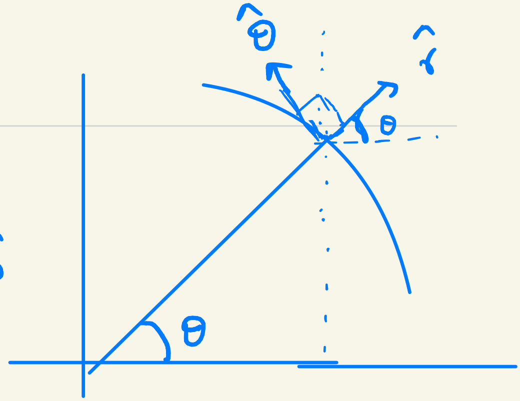
$$\vec{r}_1 = x_1\hat{i} + y_1\hat{j} \quad \text{or} \quad r_1\hat{r} + \theta_1\hat{\theta}_1$$

location B (x_2, y_2) or (r_2, θ_2)

$$\vec{r}_2 = x_2\hat{i} + y_2\hat{j} \quad \text{or} \quad r_2\hat{r}_2 + \theta_2\hat{\theta}_2$$

★ $\hat{i}, \hat{j}$ do not change along the trajectory

$\hat{r}, \hat{\theta}$ do !!



$$\hat{r} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$$

$$\hat{\theta} = \cos\left(\tfrac{\pi}{2}+\theta\right)\hat{i} + \sin\left(\tfrac{\pi}{2}+\theta\right)\hat{j}$$

$$= -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$$

Since along different points θ changes, hence $\hat{r}$ and $\hat{\theta}$ keep changing. In a dynamical case r and θ change with time

Then $\vec{r} = r\,\hat{r}$

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{dr}{dt}\hat{r} + r\frac{d\hat{r}}{dt}$$

$$= \dot{r}\,\hat{r} + r\left[-\sin\theta\,\dot{\theta}\,\hat{i} + \cos\theta\,\dot{\theta}\,\hat{j}\right]$$

$$= \dot{r}\,\hat{r} + r\dot{\theta}\,\hat{\theta}$$

Similarly $\dfrac{d\hat{\theta}}{dt} = -\dot{\theta}\cos\theta\,\hat{i} - \dot{\theta}\sin\theta\,\hat{j}$

$$= -\dot{\theta}\left(\cos\theta\,\hat{i} + \sin\theta\,\hat{j}\right)$$

$$= -\dot{\theta}\,\hat{r}$$

The expression for acceleration

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\dot{r}\,\hat{r} + r\,\dot{\theta}\,\hat{\theta}\right)$$

$$= \left(\ddot{r}\,\hat{r} + \dot{r}\,\frac{d\hat{r}}{dt}\right) + \left(\dot{r}\,\dot{\theta}\,\hat{\theta} + r\,\ddot{\theta}\,\hat{\theta} + r\,\dot{\theta}\,\frac{d\hat{\theta}}{dt}\right)$$

$$= \left(\ddot{r}\,\hat{r} + \dot{r}\,\dot{\theta}\,\hat{\theta}\right) + \left(\dot{r}\,\dot{\theta}\,\hat{\theta} + r\,\ddot{\theta}\,\hat{\theta} - r\,\dot{\theta}^2\,\hat{r}\right)$$

$$= \left(\ddot{r} - r\,\dot{\theta}^2\right)\hat{r} + \left(r\,\ddot{\theta} + 2\,\dot{r}\,\dot{\theta}\right)\hat{\theta}$$

H.W.: Deduce the meaning of

$$\frac{d\hat{r}}{dt} = \dot{\theta}\,\hat{\theta}$$

$$\frac{d\hat{\theta}}{dt} = -\,\dot{\theta}\,\hat{r}$$