Analysis in $\mathbb{R}^n$

Chapter 1. The Real and Complex Number Systems

§1.1 Real Numbers

Jotsaroop Kaur

Department of Mathematics
IISER Mohali

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Course Details

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Online Class Schedule Mon,Tue, Thu - 2 p.m to 3 p.m. & Tutorial on Fri- 2 p.m. to 3 p.m.

Course outline

1. Real number system as an Ordered field

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2. Metric spaces and topology on real number system

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7. Functions of severables variables

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Set of Reals $\mathbb{R}$ is the completion of $\mathbb{Q}$.

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Write $m = 2k$ for some integer k then $4k^2 = 2n^2$ which implies $2k^2 = n^2$, therefore n^2 is even and hence n is even.

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We examine the situation a little more closely:

Example 2

Let $A = \{p \in \mathbb{Q} \ : p > 0, p^2 < 2\}$ and $B = \{p \in \mathbb{Q} \ : p > 0, p^2 > 2\}$ then A contains no largest number and B contains no smallest.

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Proof: It is enough to prove that any for $p \in A$, there exists another rational $q \in A$ such that $p < q$ and for every $p \in B$ there exists another rational $q \in B$ such that $q < p$.

Existence of Irrationals

To do this, we associate with each rational $p > 0$ the number

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If p is in B then $p^2 - 2 > 0$ (0.1) shows that $0 < q < p$ and (0.2) show that $q^2 > 2$. Thus q is in B.

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In order to understand the structure of real number system we start with a brief discussion of the general concepts of ordered set and field.

Order

Let S be a set. An order on S is a relation, denoted by $<$, with the following two properties:

1. If $x \in S$ and $y \in S$ then one and only one of the statements
$$x < y, \quad x = y, \quad y < x$$
is true. (Law of trichotomy).
2. If $x, y, z \in S$, if $x < y$ and $y < z$ then $x < z$. (Transitive law)

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It is often convenient to write $y > x$ in place of $x < y$.

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For example, $\mathbb{Q}$ is an ordered set with the order defined by $r < s$ if $s - r$ is a positive rational number. This order is called standard order on $\mathbb{Q}$.

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Consider the set $\{\frac{1}{n} \ : n \in \mathbb{N}\}$, it is bounded in $\mathbb{Q}$.

Least upper bound (lub) or supremum

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Then l is called the least upper bound (lub) of E or the supremum of E and we write

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Examples:

Example 3: Consider the set $A = \{\frac{1}{n} \ : n \in \mathbb{N}\}$ in $\mathbb{Q}$, then the set A is bounded above and bounded below. 1 is an uppper bound and 0 is a lower bound. We can check that 0 is the infimum of A and 1 supremum of A.

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Example 5: Recall the sets A and B in Example 2:
$A = \{p \in \mathbb{Q} \ : p > 0, p^2 < 2\}$ and $B = \{p \in \mathbb{Q} \ : p > 0, p^2 > 2\}$. As a subset of $\mathbb{Q}$, A is bounded above. In fact upper bounds of A are exactly the members of B. Since B has no the smallest member, A has no least upper bound in $\mathbb{Q}$.

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Similarly, B is bounded below: The set of all lower bounds of B consists of A and of $r \in \mathbb{Q}$ with $r \leq 0$. Since A has no largest member B has no greatest lower bound in $\mathbb{Q}$.

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For the set $B = \{1 - \frac{1}{n^2} \ : \ n \in \mathbb{N}\} \subset \mathbb{Q}$, 1 is the supremum and 0 is the minimum of A.