Lecture 3.

Exercises contd.

4. Consider the equation $Ly' + Ry = E \sin \omega x$. where L, R, E, ω are >0 constants.

(a) Compute the solution φ satisfying $\varphi(0) = 0$.

(b) Show that this solution may be written in the form
$$\varphi(x) = \frac{E\omega L}{R^2+\omega^2L^2} e^{-(R/L)x} + \frac{E}{\sqrt{R^2+\omega^2L^2}} \sin(\omega x - \alpha).$$
where α is the angle satisfying $\cos\alpha = \frac{R}{\sqrt{R^2+\omega^2L^2}}$, $\sin\alpha = \frac{\omega L}{\sqrt{R^2+\omega^2L^2}}$

(c) Sketch the graph of the solution given in (b).

(6 inbk) 5. Let φ satisfy $y' + ay = b_1(x)$ & ψ satisfy $y' + ay = b_2(x)$. where b_1, b_2 are defined in the same interval I & a is constant.

(a) Show that $\chi = \varphi + \psi$ satisfies $y' + ay = b_1(x) + b_2(x)$.

(b) Apply (a) to find the solution of $y' + y = \sin x + 3\cos 2x$. whose graph passes through the origin.

(9 inbk) 6. Consider the eqn $y' + ay = b(x)$ where a is a constant s.t. $\operatorname{Re} a > 0$ and b is cts. on $0 \leq x < \infty$ which tends to β (a constant) as $x \to \infty$. Prove that every solution of this eqn. tends to β/a as $x \to \infty$.

§ The general linear equation of the first order.

Consider $y' + a(x)y = b(x)$. where a, b are cts fn. on some interval I. If we are given an equation of $\alpha(x)y' + \beta(x)y = \gamma(x)$ and $\alpha(x) \neq 0$ on I, we may divide by $\alpha(x)$ to obtain an equation of the earlier form.
The pts where $\alpha(x) = 0$, called singular pts. are troublesome. This will be dealt with later.

We apply same method. Try to find a fn. u s.t.

$$u(\varphi' + a\varphi) = (u\varphi)'$$

If A is fn. s.t. $A' = a$, fn ex. $A(x) = \int_{x_0}^{x} a(t)dt$. where $x_0 \in I$ is fixed. then such a fn. u is given by $u = e^A$.
Since $(e^A\varphi)' = e^A\varphi' + ae^A\varphi = e^A(\varphi' + a\varphi)$.

Therefore $\varphi' + a\varphi = b$ iff $(e^A\varphi)' = e^A b$. and this is valid iff

$e^A\varphi = B + C$. where c is a constant and B is a fn. whose der. is $e^A b$. Can choose B :

$$B(x) = \int_{x_0}^{x} e^{A(t)} b(t)\,dt.$$

So $e^A\varphi = B + C$ iff $\varphi(x) = e^{-A(x)} B(x) + ce^{-A(x)}$

Remark: $\varphi = e^{-A}B$ is a particular soln. (the case $c = 0$) and $\varphi_1 = e^{-A}$ is a solution of the homogeneous equation $y' + a(x)y = 0$.

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Theorem: Suppose a, b are cts. fns. on interval I. Let A be a fn. s.t. $A' = a$. Then the fn. ψ :

$$\psi(x) = e^{-A(x)} \int_{x_0}^{x} e^{A(t)} b(t)\, dt .$$

where $x_0 \in I$ is fixed, is a Solution of the equation $y' + a(x) y = b(x)$ on I.

The fn. $\varphi_1(x) = e^{-A(x)}$ is a Solution of the homogenous equation $y' + a(x) y = 0$.

If c is any constant, $\varphi = \psi + c\varphi_1$ is a Soln. and every Solution has this form.

Remark: $u' = g$ can be integrated and solved so the idea was to try to make one side a derivative of some function. Keeping this in mind will allow you to find Solutions.

Example: $y' + (\cos x)\, y = \sin x \cos x .$

Here $a(x) = \cos x$, $b(x) = \sin x \cos x$, and a choice for A is $A(x) = \sin x$. Thus if φ is any Solution

$$(e^{\sin x}\varphi)' = e^{\sin x} \sin x \cos x .$$

and integration gives

$$e^{\sin x}\varphi(x) = (\sin x - 1)\, e^{\sin x} + c .$$

or $\varphi(x) = (\sin x - 1) + c e^{-\sin x} .$