PHY 635: Gravitation & Cosmology : Problem Sheet 6

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1. Modify the Friedmann equations in presence of a positive cosmological constant term starting from $G_{\mu\nu} = 8\pi G T_{\mu\nu} - \Lambda g_{\mu\nu}$. Find out the condition for the universe to become static. Show (i) In absence of Λ, the universe cannot be static if there is matter or radiation in the univere (ii) The static universe must be closed.

2. If the universe had only one component of vacuum energy, curvature, matter and radiation, find out the time taken by the universe to evolve into its being today starting from the big bang for $H_0 = 70.0 km/sec/Mpc$.

3. For an FRW universe dominated by a perfect fluid of equation of state ω, find out the Ricci scalar, Kretschmann scalar. If there is any condition for which either (or both) of them are zero ?

4. For nearby galaxies we approximated a relation $t_0 - t_e \approx a(t_0)\chi$, where t_0 is the time of observation of the galaxy on our galaxy, i.e. today and χ is the co-ordinate distance between that galaxy and us. Argue this can not be the relation for far away galaxies. If we define a new parameter (deceleration parameter) $q = -\frac{a\ddot{a}}{\dot{a}^2}$ show that

$$H_0(t_0 - t_e) = z - \left(1 + \frac{q_0}{2}\right) z^2 + \mathcal{O}(z^3),$$

   with H_0 being today's Hubble parameter value and z being the red-shift of the galaxy being observed.

5. **Proving Big Bang:** In a radiation dominated era [*Forget matter and dark energy contribution*] and starting with a boundary condition $a(t = 0) = 0$ for the second Friedmann equation, prove that

$$a(t) = \sqrt{2H_0\sqrt{\Omega_{R0}}t - \kappa t^2};$$

   with H_0, Ω_{R0} being values today, i.e. for $a(t_0) = 1$ and κ is the curvature of spatial section. From here show that the universe if born in radiation dominated era would have infinite initial speed of expansion.
   [Hint : Write down the two Friedmann equations in presence of curvature first and use radiation equation of state and energy density dilution in expansion.]