On HW3 solutions

1) The function $t \mapsto a^t$ $(a>1)$ is increasing since $\forall s<t$ we have

$$a^t = a^{s+(t-s)} = a^s \cdot a^{(t-s)}$$

Since $a>1$ and $t-s>0$, $a^{t-s} > 1$. (check)

<u>Recall</u>: $a^{t-s} = \sup\{a^q : q \in \mathbb{Q}, q \le t-s\}$

$$\Rightarrow \quad a^t = a^s \cdot a^{t-s} > a^s.$$

Now, choose $p \in \mathbb{N}$ so that $p\alpha > 1$.

$$\Rightarrow \quad \frac{1}{p} < \alpha \Rightarrow n^{1/p} < n^{\alpha} \quad \forall n>1$$

$$\Rightarrow \quad \frac{1}{n^{\alpha}} < \frac{1}{n^{1/p}}$$

Since $\lim \frac{1}{n^{1/p}} = 0$, by sandwich theorem

$$\lim \frac{1}{n^{\alpha}} = 0$$

2. $z_n = \max(x_n, y_n)$, $w_n = \min(x_n, y_n)$

Given $\varepsilon>0$ $\exists N_1, N_2 \in \mathbb{N}$ such that

$$|x_n - l| < \varepsilon \quad \forall n \geq N_1$$

and $|y_n - l| < \varepsilon \quad \forall n \geq N_2$

Let $N = \max(N_1, N_2)$. Then $\forall n \geq N$

$$|x_n - l| < \varepsilon \Rightarrow l-\varepsilon < x_n < l+\varepsilon$$

and $|y_n - l| < \varepsilon \Rightarrow l-\varepsilon < y_n < l+\varepsilon$

It follows that $\forall n \geq N$

$$l-\varepsilon < z_n < l+\varepsilon$$

$$\& \quad l-\varepsilon < w_n < l+\varepsilon$$

i.e. $$|z_n - l| < \varepsilon$$

$$\& \quad |w_n - l| < \varepsilon.$$

Hence, we are done.

3. Choose $N \in \mathbb{N}$ such that

$$|x_n - 1| < 1/2 \quad \forall n \geq N$$

i.e. $$1/2 < x_n < 3/2 \quad \forall n \geq N.$$

$$\Rightarrow \quad 1/2^{1/k} < x_n^{1/k} < (3/2)^{1/k}$$ (since $t \mapsto t^{1/k}$ is increasing) check

Let $a = 1/2^{1/k}$

Thus $$1 + t_n + \cdots + t_n^{k-1} \geq 1 + a + \cdots + a^{k-1}$$

$$\Rightarrow \quad |t_n - 1| = \frac{|x_n - 1|}{1 + t_n + \cdots + t_n^{k-1}} \leq \frac{|x_n - 1|}{1 + a + \cdots + a^{k-1}}$$

Let $L = 1 + a + \cdots + a^{k-1}$.

$$\Rightarrow \quad |t_n - 1| \leq \frac{1}{L} |x_n - 1| \quad \forall n \geq N.$$

Now, $$\lim |x_n - 1| = 0$$

Hence, by sandwich thm $\lim |t_n - 1| = 0$

$$\Rightarrow \quad \lim (t_n - 1) = 0 \Rightarrow \lim t_n = 1.$$

(a) Choose $k \in \mathbb{N}$ such that $\frac{1}{k} < t$

Choose $L \in \mathbb{N}$ such that $t < L$

If $x \geq 1$ then $x^{1/k} \leq x^t \leq x^L$

as $s \mapsto x^s$ is increasing.

If $0<x<1$ then $x^{1/k} \geq x^t \geq x^L$.

Thus $\forall x \in \mathbb{R}, x>0$ we have

$\min\{x^{1/k}, x^L\} \leq x^t \leq \max\{x^{1/k}, x^L\} \longrightarrow (*)$

Now, clearly $a \geq 0$.

<u>Case 1</u>: Suppose $a>0$.

$$x_n \to a \iff \frac{x_n}{a} \to 1$$

$$\Rightarrow \quad \left(\frac{x_n}{a}\right)^{1/k} \to 1 \quad \text{by } (3)$$

and $\left(\frac{x_n}{a}\right)^L \to 1$ by the product rule.

$$\Rightarrow \min\left\{\left(\frac{x_n}{a}\right)^{1/k}, \left(\frac{x_n}{a}\right)^L\right\} \to 1$$

and $\max\left\{\left(\frac{x_n}{a}\right)^{1/k}, \left(\frac{x_n}{a}\right)^L\right\} \to 1$ by (2)

Using $(*)$ and the sandwich theorem

$$\left(\frac{x_n}{a}\right)^t \to 1 \Rightarrow x_n^t \to a^t.$$

(5) Note $\forall n \in \mathbb{N}\ \exists N$ such that

$0 < x_k < 1/n \quad \forall k \geq N$. etc

First do it for $a>1$.

(6) Let $\alpha \in \mathbb{R}$. $\forall n \in \mathbb{N}$ by the density of

(i) $\mathbb{Q}$ in $\mathbb{R}$ $\exists$ a rational number x_n in $(\alpha - 1/n, \alpha)$.

check that $x_n \to \alpha$

(ii) Let $a = \sup S$. $\forall n \in \mathbb{N}$, $a - 1/n$ is not the supremum of S. Hence, $\exists\, s_n \in S$ with $a - \frac{1}{n} < s_n \leq a$

check $s_n \to a$.

7) i) By induction show that $\forall n \geq 2$

$$x_n \leq \left(\frac{1}{2}\right)^{2^{n-2}}$$

ii) Check that

$$y_n = \frac{1}{3} + \frac{1}{3^2} + \cdots + \frac{1}{3^{n-2}} + \frac{1}{3^{n-1}} + \frac{1}{3^{n-1}}$$

$$= \frac{1}{3^{n-1}} + \frac{1}{3}\left(1 + \frac{1}{3} + \cdots + \frac{1}{3^{n-2}}\right)$$

$$= \frac{1}{3^{n-1}} + \frac{1}{3}\,\frac{1 - 1/3^{n-1}}{1 - \frac{1}{3}}$$

$$= \frac{1}{3^{n-1}} + \frac{1}{3}\cdot\frac{3}{2}\left(1 - \frac{1}{3^{n-1}}\right)$$

$$= \frac{1}{2}\cdot\frac{1}{3^{n-1}} + \frac{1}{2}$$

$$\Rightarrow \lim y_n = 1/2$$

8, 9, 10 (i) – (iii) are left to the students.

10) iv) Show that it is a Cauchy sequence.

See (11)

(11) Ex.