

1. The spin Hamiltonian of a system of N localized magnetic ions is given by

$$\mathcal{H} = D \sum_{j=1}^N S_j^2,$$

where $D > 0$ and spin variables S_j may assume values ± 1 or 0 , for $j = 1, 2, 3, \dots$. This spin Hamiltonian describes the effects of the electrostatic environment on spin-1 ions. An ion in state ± 1 has energy $D > 0$ and in state 0 has zero energy.

- a) Show that the number of accessible microstates of the system is given by

$$\Omega(U, N) = \frac{N!}{(N - U/D)!} \sum_{N_-} \frac{1}{(U/D - N_-)! N_-!}$$

- b) Calculate the binomial sum and obtain an exact result for $\Omega(U, N)$
- c) Now use Sterling's approximation and obtain the entropy of the system. Is it extensive?
- d) Calculate the specific heat of the system as a function of temperature.
2. Consider N particles distributed in a volume V . Now divide the volume into cell of size b , with $N \leq V/b$. Suppose that each cell may be either empty or can be occupied by a single particle.
- a) calculate the number of microstates accessible to the system.
- b) from the above result, calculate the entropy of the system and hence the quantity P/T , where the symbols have their usual meaning.
- c) do you see any difference with an ideal gas? If yes what do think is the reason behind this difference?

3. In the class we have worked out the problem of N quantum harmonic oscillators in the microcanonical ensemble. Now assume that the fundamental frequency has volume dependence given by:

$$\omega = \omega(v) = \omega_0 - A \ln \left(\frac{v}{v_0} \right),$$

where $v = V/N$, and ω_0, A , and v_0 are positive constants. Calculate the expansion coefficient and the compressibility of the system.

4. Consider system of ideal gas of N particles which can be in discrete energy states ϵ_j ; $j = 0, 1, 2, 3, \dots$
- a) how would you specify the microscopic state of the system?
- b) calculate the number of accessible microstates for the system.
- c) assuming that the total particle number and total energy of the system is fixed, write down the constraint equations.
- d) can you now extremize the system and find out probability of finding N_i particle in an energy state ϵ_i ? Do you recognize the distribution.
5. Consider a magnetic system with a total energy E and having N spins. The Hamiltonian for the system is $\mathcal{H} = -\mu H \sum_i \sigma_i$, with $\sigma_i = \pm 1$. In a microcanonical ensemble, we want to calculate the total number of accessible microstates.

- a) Assume that there are N_+ up spins and N_- down spins. Express the N_+ and N_- in terms of E and N .
 - b) From this calculate the total number of microstates accessible to the system for $E, N \rightarrow \infty$ and $E/N = u$ fixed.
 - c) Hence calculate the entropy per spin and derive an expression for the energy per spin of the system.
 - d) Using all the above informations (not all are required though) derive an expression for the magnetization of the system.
6. We want to look at the above problem using a different approach.
- a) The total number of accessible microstates is given by $\Omega = \sum_{\{\sigma_i = \pm 1\}} \delta(E + h \sum_i \sigma_i)$. Using the integral representation of the delta function $\delta(x - a) = \int_{-\infty}^{\infty} e^{ik(x-a)} \frac{dk}{2\pi}$, rewrite the expression for Ω .
 - b) Now do the sum over the states of the individual spins σ_i to express Ω as $\Omega = 2^N \int_{-\infty}^{\infty} e^{f(k)} \frac{dk}{2\pi}$. Determine $f(k)$.
 - c) Verify that the argument of the exponential goes as $\mathcal{O}(N)$. We can therefore use a saddle point approximation. To do this find the minimum k_0 of the function $f(k)$ and verify that it is indeed a minimum (i.e. $f''(k_0) < 0$). You need the following identity : $\arctan(ix) = \frac{1}{2} \log \left(\frac{1+x}{1-x} \right)$
 - d) Once you have the value of k_0 , you want to re-express the integral in Ω . Recall how we did it while establishing the connection between the canonical partition function and the free energy F . You have to approximate the function $f(k)$ by a Taylor series around k_0 and retain the second order term. Once you have done that carry out the integral over and determine Ω as function of $e = E/Nh$.