REAL ANALYSIS

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1. Sequences

1.1. Cauchy's criterion of convergence.

Theorem 1.1. *Let $\{a_n\}$ be a sequence in $\mathbb{R}$. The following are equivalent:*

(1) The sequence $\{a_n\}$ converges in $\mathbb{R}$.
(2) $\{a_n\}$ is a Cauchy sequence.

Proof. (1) $\implies$ (2) Let the sequence $\{a_n\}$ converges to a. By definition, given $\epsilon > 0$ there exists N such that $|a_n - a| < \epsilon/2$ for all $n \geq N$. We choose $n, m > N$, by triangle inequality $|a_n - a_m| < \epsilon$ for all $n, m > N$. Hence $\{a_n\}$ is a Cauchy sequence.

(2) $\implies$ (1) Enough to show that $\{a_n\}$ is bounded. Then we know that there exists a subsequence of $\{a_n\}$ which is convergent. Call that subsequence $\{a_{n_k}\}_{k\in\mathbb{N}}$. Let $a_{n_k} \to a$ as $k \to \infty$. As $\{a_n\}$ is Cauchy, it is a simple exercise to show that $\lim_{n\to\infty} a_n = a$ as well. Let $\epsilon = 1$ then there exists N_1 such that $|a_n - a_{N_1}| < 1$ for $n \geq N_1$. Let $M = \sup_{1\leq n\leq N_1} |a_n|$. By triangle inequality we have $|a_n| \leq |a_n - a_{N_1}| + |a_{N_1}|$. By the previous inequalities we get that $|a_n| \leq M + 1$ for all $n \geq N_1$. Therefore we get that $\sup_{n\in\mathbb{N}} |a_n| \leq M + 1$. Hence $\{a_n\}$ is bounded.

□

Exercise 1 Let $\{a_n\}$ be a Cauchy sequence and there exists a convergent subsequence $\{a_{n_k}\}$ of $\{a_n\}$. Let $a = \lim_{k\to\infty} a_{n_k}$. Prove that $a_n \to a$ as $n \to \infty$.

Exercise 2 Let $\{a_n\}$ be a convergent sequence. P.T. every convergent subsequence of $\{a_n\}$ converges to the same limit as that of the sequence a_n.

Exercise 3 Let $A \neq \phi$ and $A \subset \mathbb{R}$ which is bounded above and below. Let L be the set of all upper bounds of A. Prove that $\sup A = \inf L$. What is the analogue for the $\inf A$?

1.2. Limsup and Liminf of a bounded sequence.

We have already seen that bounded monotonic sequences are always convergent. We will define a way of constructing monotonic sequences out of a given sequence. Let $\{a_n\}$ be a bounded sequence in $\mathbb{R}$. Define

$$A_n = \{a_k : k \geq n\}.$$

Clearly A_n is a bounded set. Let $b_n := \sup A_n$ and $c_n := \inf A_n$. As $A_{n+1} \subset A_n$ for every $n \geq 1$. It is easy to see that b_n is a decreasing sequence and is bounded below. Similarly c_n is an increasing sequence and bounded above. Hence $\lim_{n\to\infty} b_n = b$ exists and in fact is equal to $\inf_{n\in\mathbb{N}} b_n$. b is called the limit superior of the sequence $\{a_n\}$. We call

$$b = \limsup a_n = \inf_{n\geq 1} \left(\sup_{k\geq n} a_k \right).$$

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Let $c = \lim_{n\to\infty} c_n$ and again we have $c = \sup_{n\in\mathbb{N}} c_n$. c is called the limit inferior of the sequence $\{a_n\}$. We write

$$c = \liminf a_n = \sup_{n\geq 1}\left(\inf_{k\geq n} a_k\right).$$

Example Let $a_n = (-1)^n$. Then $b_n = 1$ and $c_n = -1$ for all $n \in \mathbb{N}$.

Note that in general $\liminf a_n \leq \limsup a_n$. Let $c = \liminf a_n$ and $b = \limsup a_n$. In the above notation $c = \lim_{n\to\infty} c_n$ and $b = \lim_{n\to\infty} b_n$, where b_n and c_n are as defined above. By definition $c_n \leq b_n$ for all $n \in \mathbb{N}$. As $c_n \uparrow c$ and $b_n \downarrow b$, given $\epsilon > 0$ there exists N_1, N_2 such that $c - \epsilon < c_n, \forall n \geq N_1$ and $b_n < b + \epsilon, \forall n \geq N_2$. Let $N = \max\{N_1, N_2\}$. We get that

$$c < c_n + \epsilon \leq b_n + \epsilon < b + 2\epsilon$$

for all $n \geq N$. Therefore we have $c < b + 2\epsilon$. As ϵ is any arbitrary positive real number we claim that $c \leq b$. Suppose not, i.e. $b < c$ this implies if $2\epsilon < c - b$ then $b + 2\epsilon < b + c - b = c$ which contradicts that $c < b + 2\epsilon$ for any arbitrary $\epsilon > 0$. Hence $c \leq b$.

Exercise 4 Let $a_n \to a$. Prove that if there exists $N, c, d \in \mathbb{R}$ s.t. $c < a_n < d$ for all $n \geq N$, then $c \leq a \leq d$.

We will present another criteria for the convergence of a sequence.

Theorem 1.2. *Let $\{a_n\}$ be a sequence in $\mathbb{R}$. the following are equivalent:*

(1) $\{a_n\}$ converges in $\mathbb{R}$.
(2) $\{a_n\}$ is bounded and $\limsup a_n = \liminf a_n$.

In fact for such a sequence

$$\lim_{n\to\infty} a_n = \limsup a_n = \liminf a_n.$$

Proof. (1) $\implies$ (2) Let $a = \lim_{n\to\infty} a_n$. We will show that $\limsup a_n = a$. The other part follows similarly. Given $\epsilon > 0$, there exists N such that $a_n \in (a - \epsilon, a + \epsilon)$ for all $n \geq N$. This implies that $A_n \subset (a - \epsilon, a + \epsilon)$ for all $n \geq N$ where $(u, v) := \{x \in \mathbb{R} : u < x < v\}$ for $u < v$. This shows that $a - \epsilon < b_n \leq a + \epsilon$ for all $n \geq N$. As $\lim_{n\to\infty} b_n = \limsup a_n$, as shown above we get that that $a - \epsilon \leq \limsup a_n \leq a + \epsilon$. As $\epsilon > 0$ is an arbitrary positive number, we get that $a = \limsup a_n$.

(2) $\implies$ (1) Let $\limsup a_n = \liminf a_n = a$. We will show that $lim_{n\to\infty} a_n = a$. We know that $\limsup a_n = a$. Given $\epsilon > 0$ there exists N such that $a \leq b_n < a + \epsilon$ for all $n \geq N$. As $b_n \geq a_k$ for all $k \geq n$ we have $a_n < a + \epsilon$ for all $n \geq N$. Similarly using the fact that $\liminf a_n = a$, given $\epsilon > 0$ there exists M such that $a - \epsilon < a_n$ for all $n \geq M$. Let $P = \max\{N, M\}$. Combining the above we get $|a_n - a| < \epsilon$ for all $n \geq P$. Hence proved. □

One can ask if any convergent subsequence of a bounded sequence $\{a_n\}$ also has a relation with $\limsup a_n$ and $\liminf a_n$. That indeed is true.

Theorem 1.3. *Let $\{a_n\}$ be a bounded sequence in $\mathbb{R}$. Let*

$$S = \{x \in \mathbb{R} : a_{n_k} \to x \quad \text{as } k \to \infty \text{ for some subsequence } \{a_{n_k}\}\}.$$

Let $b = \limsup a_n$ and $c = \liminf a_n$, then $b, c \in S$ and $c \leq x \leq b$ for all $x \in S$.

Proof. We will first show that $b \in S$, i.e. we will find a subsequence $\{a_{n_k}\}$ converging to b. The case of showing $c \in S$ follows similarly. Let b_n be as defined above. We know that $\inf_{n\in\mathbb{N}} b_n = b$ and $b_n \to b$ as $n \to \infty$. Therefore given $\epsilon > 0$ there exists N such that $b_n < b + \epsilon$. We also know that $a_k \leq b_n$ for all $k \geq n$, so we get that $a_k < b + \epsilon$ for all $k \geq N$.

Our claim is there exists $m_1 < m_2 < m_3...$ depending on ϵ such that $a_{m_k} > b - \epsilon$. If not this would imply that there exists some M such that $a_n \leq b - \epsilon$ for all $n \geq M$. This implies that $b-\epsilon$ is an upper bound of the set A_n for all $n \geq M$, i.e. $b_n \leq b-\epsilon$ for all $n \geq M$ which implies that $b \leq b-\epsilon$ which is a contradiction. Let us choose $\epsilon = 1$ we get a_{n_1} such that

$$b-1 < a_{n_1} < b+1.$$

For $\epsilon = 1/2$ choose $n_2 > n_1$ such that

$$b-\frac{1}{2} < a_{n_2} < b+\frac{1}{2}.$$

Going this way we get a subsequence $\{a_{n_k}\}$ such that

$$b-\frac{1}{k} < a_{n_k} < b+\frac{1}{k}.$$

It is easy to check that $\{a_{n_k}\}$ is Cauchy and converges to b.

We will show that $c \leq x \leq b$ for all $x \in S$. We will only show that $c \leq x$ for all $x \in S$. The other case follows similarly. As $x \in S$, there exists a subsequence $\{a_{n_k}\}$ such that $\lim_{k\to\infty} a_{n_k} = x$. We know that $c = \liminf a_n = sup_{n\in\mathbb{N}} c_n$. This implies that given $\epsilon > 0$ there exists N such that $c_n > c-\epsilon$ for all $n \geq N$ as c_n is an increasing sequence. As $c_n \leq a_k$ for all $k \geq n$ we get that $a_k > c-\epsilon$ for all $k \geq N$. In particular for $a_{n_k} > c-\epsilon$ for all $k \geq K$ such that $n_K > N$. this implies that $x \geq c-\epsilon$ for any $\epsilon > 0$. As discussed before this implies $x \geq c$. Hence proved.

□

Exercise 5 Find the lim sup and lim inf of the following:

(i) $a_n = \frac{1}{n} + \frac{(-1)^n}{n^2}$.
(ii) $a_n = (-1)^n + \frac{1}{n}$

Exercise 6 Let $\{a_n\}$ and $\{b_n\}$ be bounded sequences. Prove the following:

(a) $\limsup(a_n + b_n) \leq \limsup a_n + \limsup b_n$.
(b) $\liminf(a_n + b_n) \geq \liminf a_n + \liminf b_n$.
(c) Let $a_n \geq 0$ and $b_n \geq 0$ for all n. Show that $\limsup(a_n b_n) \leq \limsup a_n \limsup b_n$.
(d) Let $a_n \to a$. Show that

$$\limsup(a_n + b_n) = a + \limsup b_n,$$
$$\liminf(a_n + b_n) = a + \liminf b_n.$$

(e) Find examples where the inequalities above can be strict.