Tutorial 04 1.
01/02/22

Let us not forget the **elementary matrices**: $S_{p,q}$, $M_p(\lambda)$ and $L_{p,q}(\lambda)$. Write them once more for your recollection. Multiplying these matrices on the left of a matrix A is called an **elementary row operation** on A.

> Let us call a matrix to be a **row echelon matrix**[1] if it has the following three properties.
>
> I. First nonzero entry in each row is 1. This entry is to be called the **pivot** of the row.
>
> II. The pivot of a (not entirely 0) row is to the right of the pivot of the preceding row. If a row is entirely 0 then all the subsequent rows are also entirely 0.
>
> III. All entries above pivots are zero. (or equivalently, the pivot element of a row is the only nonzero element of the column it belongs to).
>
> [1] Different books will have a variation in this definition. We stick to the above definition in this course.

We may convert a matrix into a row echelon matrix through successive elementary row operations.

1. Which of the following are row echelon matrices?

III ✓ ✓

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 5 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \text{II}$$

III ✓ ✓ ✓! Vacuously!

$$\begin{pmatrix} 1 & 0 & 2 & 3 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

In each of the cases when matrix is not row echelon, list the condition(s) I, II, III of the definition that it fails to satisfy.

---

2. Using 0, 1 and 2 make as many 2×2 row echelon matrices as you can.

3. Using 0 and 1 how many 3×3 row echelon matrices can you make? List all of them.

2. $$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & * \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

There are six such matrices.

3. $$\begin{pmatrix} * & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} * & * & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 0 \end{pmatrix},$$

$* \; * = 1 \; 0$ or $0 \; 1$ or $1 \; 1$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$8 + 3 + 4 + 1 = 16$ matrices.

4. Is there a 3×3 rotation matrix which is a row echelon matrix?

Yes, $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = R_{x,0} = R_{y,0} = R_{z,0}$

There is no other rotation matrix that is a row echelon matrix. Why?

---

5. Convert the following matrices into a row echelon matrix by suitable sequence of elementary row operations.

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}, \quad \begin{pmatrix} 1 & -1 & 4 & 3 \\ 2 & 1 & 0 & 3 \\ 2 & 1 & 5 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \xrightarrow{L_{2,1}(-4)} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 7 & 8 & 9 \end{pmatrix} \xrightarrow{L_{3,1}(-7)}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{pmatrix} \xrightarrow{M_2(-\frac{1}{3})} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -6 & -12 \end{pmatrix} \xrightarrow{M_3(\frac{-1}{6})}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix} \xrightarrow{L_{3,2}(-1)} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{L_{1,2}(-2)}$$

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

This matrix is a row echelon matrix

Just to summarize

$$L_{1,2}(-2)\,L_{3,2}(-1)\,M_3\left(\tfrac{-1}{6}\right)M_2\left(-\tfrac{1}{3}\right)L_{3,1}(-7)\,L_{2,1}(-4)\begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix}$$

$$=\begin{pmatrix}1&0&-1\\0&1&2\\0&0&0\end{pmatrix}$$

For fun, let us calculate

$$L_{1,2}(-2)\,L_{3,2}(-1)\,M_3\left(\tfrac{-1}{6}\right)M_2\left(-\tfrac{1}{3}\right)L_{3,1}(-7)\,L_{2,1}(-4)$$

$$=L_{1,2}(-2)\,L_{3,2}(-1)\,M_3\left(\tfrac{-1}{6}\right)M_2\left(-\tfrac{1}{3}\right)L_{3,1}(-7)\begin{pmatrix}1&0&0\\-4&1&0\\0&0&1\end{pmatrix}$$

$$=L_{1,2}(-2)\,L_{3,2}(-1)\,M_3\left(\tfrac{-1}{6}\right)M_2\left(-\tfrac{1}{3}\right)\begin{pmatrix}1&0&0\\-4&1&0\\-7&0&1\end{pmatrix}$$

$$=L_{1,2}(-2)\,L_{3,2}(-1)\,M_3\left(\tfrac{-1}{6}\right)\begin{pmatrix}1&0&0\\4/3&-1/3&0\\-7&0&1\end{pmatrix}$$

$$=L_{1,2}(-2)\,L_{3,2}(-1)\begin{pmatrix}1&0&0\\4/3&-1/3&0\\7/6&0&-1/6\end{pmatrix}$$

$$=L_{1,2}(-2)\begin{pmatrix}1&0&0\\4/3&-1/3&0\\-1/6&1/3&-1/6\end{pmatrix}=\begin{pmatrix}-5/3&2/3&0\\4/3&-1/3&0\\-1/6&1/3&-1/6\end{pmatrix}$$

Check that

$$\begin{pmatrix} -5/3 & 2/3 & 0 \\ 4/3 & -1/3 & 0 \\ -1/6 & 1/3 & -1/6 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

and be sure that your calculations are correct!

$$\begin{pmatrix} 1 & -1 & 4 & 3 \\ 2 & 1 & 0 & 3 \\ 2 & 1 & 5 & 0 \end{pmatrix} \xrightarrow{L_{3,2}(-1)} \begin{pmatrix} 1 & -1 & 4 & 3 \\ 2 & 1 & 0 & 3 \\ 0 & 0 & 5 & -3 \end{pmatrix}$$

$$\xrightarrow{L_{2,1}(-2)} \begin{pmatrix} 1 & -1 & 4 & 3 \\ 0 & 3 & -8 & -3 \\ 0 & 0 & 5 & -3 \end{pmatrix} \xrightarrow{M_2\left(\frac{1}{3}\right)} \begin{pmatrix} 1 & -1 & 4 & 3 \\ 0 & 1 & -8/3 & -1 \\ 0 & 0 & 5 & -3 \end{pmatrix}$$

$$\xrightarrow{L_{1,2}(1)} \begin{pmatrix} 1 & 0 & 4/3 & 2 \\ 0 & 1 & -8/3 & -1 \\ 0 & 0 & 5 & -3 \end{pmatrix} \xrightarrow{M_3\left(\frac{1}{5}\right)} \begin{pmatrix} 1 & 0 & 4/3 & 2 \\ 0 & 1 & -8/3 & -1 \\ 0 & 0 & 1 & -3/5 \end{pmatrix}$$

$$\xrightarrow{L_{1,3}(-4/3)} \begin{pmatrix} 1 & 0 & 0 & 14/5 \\ 0 & 1 & -8/3 & -1 \\ 0 & 0 & 1 & -3/5 \end{pmatrix} \xrightarrow{L_{2,3}(8/3)} \begin{pmatrix} 1 & 0 & 0 & 14/5 \\ 0 & 1 & 0 & -13/5 \\ 0 & 0 & 1 & -3/5 \end{pmatrix}$$

6. Take a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ for which $ad - bc \neq 0$. Then multiply A by suitable elementary matrices to convert it to a row echelon matrix in each of the following cases.

(a) When $a \neq 0$. (b) When $a = 0$ but $b \neq 0$.

Keeping track of which elementary matrices were used in the process, find a 2×2 matrix B for which $AB = BA = I_2$? Can you write A itself as a product of elementary matrices?

(a) When $a \neq 0$, division by a is possible. Hence a^{-1} makes sense.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \xrightarrow{L_{2,1}(-a^{-1}c)} \begin{pmatrix} a & b \\ 0 & d - a^{-1}cb \end{pmatrix}$$

$$\xrightarrow{M_2\left(\frac{1}{d - a^{-1}cb}\right)} \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \xrightarrow{M_1(a^{-1})} \begin{pmatrix} 1 & ba^{-1} \\ 0 & 1 \end{pmatrix}$$

This is possible because $ad - bc \neq 0$

$$\begin{pmatrix} 1 & ba^{-1} \\ 0 & 1 \end{pmatrix} \xrightarrow{L_{1,2}(-ba^{-1})} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$$

Therefore

$$\underbrace{L_{1,2}(-ba^{-1})\, M_1(a^{-1})\, M_2\left(\frac{1}{d - a^{-1}cb}\right) L_{2,1}(-a^{-1}c)}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = I_2$$

This product must be the inverse of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

We calculate it.

$$L_{1,2}(-ba^{-1})\, M_1(a^{-1})\, M_2\left(\frac{1}{d - a^{-1}cb}\right) L_{2,1}(-a^{-1}c)$$

$$= L_{1,2}(-ba^{-1})\, M_1(a^{-1})\, M_2\left(\frac{1}{d - a^{-1}cb}\right) \begin{pmatrix} 1 & 0 \\ -a^{-1}c & 1 \end{pmatrix}$$

$$= L_{1,2}(-ba^{-1})\, M_1(a^{-1}) \begin{pmatrix} 1 & 0 \\ \dfrac{-a^{-1}c}{d-a^{-1}cb} & \dfrac{1}{d-a^{-1}cb} \end{pmatrix}$$

$$= L_{1,2}(-ba^{-1}) \begin{pmatrix} a^{-1} & 0 \\ \dfrac{-a^{-1}c}{d-a^{-1}cb} & \dfrac{1}{d-a^{-1}cb} \end{pmatrix}$$

$$= \begin{pmatrix} -a^{-1} + \dfrac{ba^{-1}a^{-1}c}{d-a^{-1}cb} & \dfrac{-ba^{-1}}{d-a^{-1}cb} \\ \dfrac{-a^{-1}c}{d-a^{-1}cb} & \dfrac{1}{d-a^{-1}cb} \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{d}{ad-bc} & \dfrac{-b}{ad-bc} \\ \dfrac{-c}{ad-bc} & \dfrac{a}{ad-bc} \end{pmatrix}$$

If you wish, you can verify your calculations for a particular matrix. My favorite is $\begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$, whose inverse, as per above calculation must be $\begin{pmatrix} 1 & -2 \\ -2 & 5 \end{pmatrix}$.

(b) When $a=0$, $b \neq 0$

$$\begin{tikzcd}
\begin{pmatrix} 0 & b \\ c & d \end{pmatrix} \arrow[r, "S_{1,2}"] & \begin{pmatrix} c & d \\ 0 & b \end{pmatrix} \arrow[r, "M_2(b^{-1})"] & \begin{pmatrix} c & d \\ 0 & 1 \end{pmatrix} \arrow[d, "L_{1,2}(-d)"] \\
& \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix} \arrow[l, "M_1(c^{-1})"']
\end{tikzcd}$$

Complete yourself rest of the calculation.

Hint:

Writing a matrix as product of elementary matrices.

" If $E_n \cdots E_3 E_2 E_1 A =$ Identity matrix

then $A = E_1^{-1} E_2^{-1} \cdots E_n^{-1}$ = a product of elementary matrices.