1. Wave is traveling in the $-\hat{z}$ direction
($\because$ given in $z + ct$).

$\vec{B}$ is $\perp^r$ to both the direction of propagation and $\vec{E}$. $\therefore \vec{B}$ must point in $\pm(\hat{x} - \hat{y})$ direction.

Since, $\vec{E} \times \vec{B} \propto$ direction of propagation which is $-\hat{z}$ direction.

$\therefore \vec{B}$ must take $+(\hat{x} - \hat{y})$ direction.

$|\vec{B}| = \frac{|E_0|}{c}$. $\therefore \vec{B} = \frac{E_0}{c}(\hat{x} - \hat{y}) \sin\left[\frac{2\pi}{\lambda}(z + ct)\right]$

$E_0 = 20 \frac{V}{m}$. $B_0 = \frac{E_0}{c} = (20)/(3 \times 10^8)$
$\sim 6.67 \times 10^{-8}$ T.

2.

$E_x = 0$, $E_y = E_0 \sin(kx + \omega t)$, $E_z = 0$

$B_x = 0$ $B_y = 0$, $B_z = -\frac{E_0}{c} \sin(kx + \omega t)$

$$\vec{\nabla} \cdot \vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0 .$$

Similarly, $\vec{\nabla} \cdot \vec{B} = 0$.

$$\vec{\nabla} \times \vec{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = \hat{z} E_0 k \cos(kx + \omega t)$$

$$-\frac{\partial \vec{B}}{\partial t} = \hat{z} \frac{E_0 \omega}{c} \cos(kx + \omega t)$$

To require, $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \Rightarrow k = \frac{\omega}{c} \Rightarrow \boxed{\omega = ck}$

Use this to show that, $\vec{\nabla} \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$.

3.

$$\vec{\nabla}\cdot\vec{E} = 0.$$

$$\vec{\nabla}\cdot\vec{B} = \left(-B_0 k \sin kx \sin ky + k B_0 \sin kx \cos ky\right) \sin\omega t$$

$$= 0.$$

$$\vec{\nabla}\times\vec{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & E_0 \cos kx \cos ky \cos\omega t \end{vmatrix}$$

$$= \hat{x}\left[\frac{\partial}{\partial y}\left(E_0 \cos kx \cos ky \cos\omega t\right) - \frac{\partial}{\partial z}(0)\right].$$

$$- \hat{y}\left[\frac{\partial}{\partial x}\left(E_0 \cos kx \cos ky \cos\omega t\right) - \frac{\partial}{\partial z}(0)\right].$$

$$+ \hat{z}\left[\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial y}(0)\right]$$

$$= -\hat{x} E_0 k \cos kx \sin ky \cos\omega t + \hat{y} E_0 k \sin kx \cos ky \cos\omega t$$

$$-\frac{\partial\vec{B}}{\partial t} = \hat{x} B_0 \omega \cos kx \sin ky \cos\omega t - \hat{y} B_0 \sin kx \cos kx \cos\omega t$$

Now for $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$ to hold,

(1) — $\boxed{E_0 k = B_0 \omega}$ (equating components).

Now, lets look at the other Maxwells eqn.

$$\vec{\nabla}\times\vec{B} = \mu_0 \epsilon_0 \frac{\partial\vec{E}}{\partial t}$$

$$\frac{\partial \vec{E}}{\partial t} = -\hat{z}\, \omega E_0 \cos kx \cos ky \sin \omega t.$$

$$\vec{\nabla} \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B_0 \cos kx \sin ky \sin \omega t & -B_0 \sin kx \cos ky \sin \omega t & 0 \end{vmatrix}$$

$$= \hat{x}(0-0) - \hat{y}(0-0) + \hat{z}\left(-B_0 k \cos kx \cos ky \sin \omega t - B_0 k \cos kx \cos ky \sin \omega t\right).$$

$$= -\hat{z}\, 2B_0 k \cos kx \cos ky \sin \omega t.$$

$\therefore$ For $\vec{\nabla} \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$ to hold,

$$2B_0 k = \mu_0 \epsilon_0 .\, \omega E_0 = \frac{\omega E_0}{c^2} \quad \left(\because c^2 = \frac{1}{\mu_0 \epsilon_0}\right).$$

$$\Rightarrow \boxed{2B_0 k = \frac{\omega E_0}{c^2}} \quad \text{—(2)}$$

$$(1) \div (2) \Rightarrow \frac{E_0}{2B_0} = \frac{B_0}{E_0/c^2} \Rightarrow E_0^2 = 2B_0^2 c^2$$

$$\therefore E_0 = \sqrt{2}\, c B_0.$$

Substituting for E_0 in (1) $\Rightarrow k\sqrt{2}\, c \cancel{B_0} = \cancel{B_0}\, \omega.$

$$\therefore \omega = \sqrt{2}\, ck.$$

4.

Proof of $\vec{E}'.\vec{B}' = \vec{E}.\vec{B}$ given in text.

Basically, for a frame F' moving with speed v in the $\hat{x}$ direction relative to F, the transformation equations are,

$$E_x' = E_x \; ; \; E_y' = \gamma(E_y - vB_z) \; ; \; E_z' = \gamma(E_z + vB_z)$$

$$B_x' = B_x \; ; \; B_y' = \gamma\left(B_z + \frac{v}{c^2} E_z\right) \; ; \; B_z' = \gamma\left(B_z - \frac{v}{c^2} E_y\right)$$

$$\vec{E}'.\vec{B}' = E_x' B_x' + E_y' B_y' + E_z' B_z'$$

Using the transformations given above it is easy to see that, $\vec{E}'.\vec{B}' = \vec{E}.\vec{B}$.

To show $E'^2 - cB'^2 = E^2 - cB^2$, we can use the same transformations as above & do it.

Or, as suggested by Purcell in prob 9.12, we can break $\vec{E}$ & $\vec{B}$ into $\parallel^{l}$ & $\perp^{r}$ vectors

$$\vec{E} = \vec{E}_{\parallel} + \vec{E}_{\perp} \quad \& \quad \vec{B} = \vec{B}_{\parallel} + \vec{B}_{\perp}$$

$$\therefore E'^2 - cB'^2 = \vec{E}'.\vec{E}' - c\vec{B}'.\vec{B}'$$

$$= (\vec{E}_{\parallel}' + \vec{E}_{\perp}').(\vec{E}_{\parallel}' + \vec{E}_{\perp}')$$

$$- c\,(\vec{B}_{\parallel}' + \vec{B}_{\perp}').(\vec{B}_{\parallel}' + \vec{B}_{\perp}')$$

Now, $\vec{E}_{\parallel}'\cdot\vec{E}_{\perp}' = 0$, $\vec{B}_{\parallel}'\cdot\vec{B}_{\perp}' = 0 = B_{\perp}'B_{\parallel}'$
$= \vec{E}_{\perp}'\cdot\vec{E}_{\parallel}'$

$\therefore \quad E'^2 - c^2B'^2 = \left(\vec{E}_{\parallel}'\cdot\vec{E}_{\parallel}' + \vec{E}_{\perp}'\cdot\vec{E}_{\perp}'\right) - c^2\left(\vec{B}_{\parallel}'\cdot\vec{B}_{\parallel}' + \vec{B}_{\perp}'\cdot\vec{B}_{\perp}'\right)$

Now, in vector form, the transformation eqs. are,

$$\vec{E}_{\parallel}' = \vec{E}_{\parallel} \quad ; \quad \vec{E}_{\perp}' = \gamma\left(\vec{E}_{\perp} + \vec{v}\times\vec{B}_{\perp}\right)$$
$$\vec{B}_{\parallel}' = \vec{B}_{\parallel} \quad ; \quad \vec{B}_{\perp}' = \gamma\left(\vec{B}_{\perp} - \frac{\vec{v}}{c^2}\times\vec{E}_{\perp}\right).$$

$$\therefore \quad \vec{E}_{\parallel}'\cdot\vec{E}_{\parallel}' - c^2\vec{B}_{\parallel}'\cdot\vec{B}_{\parallel}' = \vec{E}_{\parallel}\cdot\vec{E}_{\parallel} - c^2\vec{B}_{\perp}\cdot\vec{B}_{\perp}$$

$$\vec{E}_{\perp}'\cdot\vec{E}_{\perp}' - c^2\vec{B}_{\perp}'\cdot\vec{B}_{\perp}'$$
$$= \gamma^2\left(E_{\perp} + \vec{v}\times\vec{B}_{\perp}\right)\cdot\left(\vec{E}_{\perp} + \vec{v}\times\vec{B}_{\perp}\right)$$
$$- \gamma^2c^2\left(B_{\perp} - \frac{\vec{v}}{c^2}\times\vec{E}_{\perp}\right)\cdot\left(\vec{B}_{\perp} - \frac{\vec{v}}{c^2}\times\vec{E}_{\perp}\right).$$

$$= \gamma^2\Big[\vec{E}_{\perp}\cdot\vec{E}_{\perp} + \vec{E}_{\perp}\cdot\left(\vec{v}\times\vec{B}_{\perp}\right) + \left(\vec{v}\times\vec{B}_{\perp}\right)\cdot\vec{E}_{\perp}$$
$$+ \left(\vec{v}\times\vec{B}_{\perp}\right)\cdot\left(\vec{v}\times\vec{B}_{\perp}\right) - c^2\Big\{\vec{B}_{\perp}\cdot\vec{B}_{\perp}$$
$$- \frac{1}{c^2}\vec{B}_{\perp}\cdot\left(\vec{v}\times\vec{E}_{\perp}\right) - \frac{1}{c^2}\left(\vec{v}\times\vec{E}_{\perp}\right)\cdot\vec{B}_{\perp}$$
$$+ \frac{1}{c^4}\left(\vec{v}\times\vec{E}_{\perp}\right)\cdot\left(\vec{v}\times\vec{E}_{\perp}\right)\Big\}\Big].$$

Now, $\vec{E}_{\parallel}$ is parallel to $\vec{v}$ by definition.

$\therefore \vec{E}_\perp$ is $\perp^r$ to $\vec{v}$ $\quad \therefore \vec{v}\cdot\vec{E}_\perp = \vec{E}_\perp\cdot\vec{v} = 0$.

$\therefore$ ~~$\vec{E}_\perp\cdot(\vec{v}\times\vec{B}_\perp) = (\vec{v}\times\vec{B}_\perp)\cdot\vec{E}_\perp$~~ Similarly, $\vec{v}\cdot\vec{B}_\perp = \vec{B}_\perp\cdot\vec{v} = 0$

$$\therefore \underbrace{(\vec{v}\times\vec{E}_\perp)}_{\vec{A}}\cdot\underbrace{(\vec{v}}_{B}\times\underbrace{\vec{E}_\perp)}_{C} = \vec{v}\cdot\left[\vec{E}_\perp\times(\vec{v}\times\vec{E}_\perp)\right].$$

$$\left(\because \vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A})\right].$$

Now, $$\underbrace{\vec{E}_\perp}_{A}\times(\underbrace{\vec{v}}_{B}\times\underbrace{\vec{E}_\perp}_{C}) = \vec{v}(\vec{E}_\perp\cdot\vec{E}_\perp) - \vec{E}_\perp(\cancel{\vec{v}\cdot\vec{E}_\perp}^{\,0})$$

$$\left(\because \vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})\right].$$

$$= \vec{v}\,E_\perp^2$$

$$\therefore (\vec{v}\times\vec{E}_\perp)\cdot(\vec{v}\times\vec{E}_\perp) = (\vec{v}\cdot\vec{v})E_\perp^2 = v^2E_\perp^2.$$

$$\Rightarrow (\vec{v}\times\vec{E}_\perp)^2 = v^2E_\perp^2.$$

Similarly, $(\vec{v}\times\vec{B}_\perp)\cdot(\vec{v}\times\vec{B}_\perp) = v^2B_\perp^2$.

$$\therefore E_\perp'^2 - c^2B_\perp'^2 = \gamma^2\Big[E_\perp^2 + 2\vec{E}_\perp\cdot(\vec{v}\times\vec{B}_\perp) + v^2B_\perp^2 - c^2B_\perp^2$$
$$+ 2\vec{B}_\perp\cdot(\vec{v}\times\vec{E}_\perp) - \frac{1}{c^2}\cdot v^2E_\perp^2\Big].$$

But, $\vec{E}_\perp\cdot(\vec{v}\times\vec{B}_\perp) = -\vec{B}_\perp\cdot(\vec{v}\times\vec{E}_\perp)$ $\quad\left[\because \vec{A}\cdot(\vec{B}\times\vec{C}) = -\vec{C}\cdot(\vec{B}\times\vec{A}\right.$

$$\therefore E_\perp'^2 - c^2B_\perp'^2 = \gamma^2\Big[E_\perp^2\left(1-\frac{v^2}{c^2}\right) ~~+ v^2\left(B_\perp^2 - \frac{c^2}{v^2}\right.~~$$
$$- c^2B_\perp^2\left(1-\frac{v^2}{c^2}\right)\Big]$$

$$\therefore E_\perp'^2 - c^2B_\perp'^2 = \gamma^2\left[\frac{E_\perp^2}{\gamma^2} - \frac{c^2B_\perp^2}{\gamma^2}\right] = E_\perp^2 - c^2B_\perp^2$$

$$\therefore E'^2 - c^2B'^2 = E^2 - c^2B^2.$$ Hence proved.

5.

Q, $\vec{E}$, $\vec{J}$ are all functions of time.

Ampere & Biot-Savart law do not apply.

$$\vec{J}_d = \epsilon_0 \frac{\partial \vec{E}}{\partial t} = \frac{1}{4\pi} \frac{\dot{Q}}{r^2} \hat{r} = -\sigma \frac{Q}{4\pi \epsilon_0 r^2} \hat{r}.$$

This exactly cancels the conduction current & the magnetic field is indeed zero.