→ ~~Axio~~ Linear Vector space

→ Linear operators

A linear operator maps the vectors of a vector space to vectors of another vector space

$$A|\psi\rangle = |\phi\rangle$$

→ linearity $A(c_1|\psi_1\rangle + c_2|\psi_2\rangle) = c_1 A|\psi_1\rangle + c_2 A|\psi_2\rangle$

→ Its sufficient to see the action of a linear operator on the basis vectors of a vector space in order to understand it completely.

→ $A = B \Rightarrow A|\psi\rangle = B|\psi\rangle \quad \forall\, |\psi\rangle \in V$

→ $A(B|\psi\rangle) = A|\chi\rangle = (AB)|\psi\rangle$

→ $\langle\phi| A|\psi\rangle = (\langle\phi|A)|\psi\rangle = (A^{+}|\phi\rangle, |\psi\rangle)$

A^{+} is called the Adjoint of the operator A.

→ Properties of the Adjoint operation

$$(CA)^{+} = C^{*}A^{+}$$

$$(A+B)^{+} = A^{+} + B^{+}$$

$$(AB)^{+} = B^{+}A^{+}$$

Think of the proof of these properties.

→ The outer product of two vectors $|\phi\rangle$ and $|\psi\rangle$ also result in an operator $|\phi\rangle\langle\psi|$

(what is the eigenvalue and eigenvector of this?)

→ $(|\phi\rangle\langle\psi|)^{\dagger} = |\psi\rangle\langle\phi|$

→ Trace of an operator

$$Tr(A) = \sum_i \langle u_i|A|u_i\rangle$$

$\{|u_i\rangle\}$ is an orthonormal and complete basis

→ Self-adjoint operators:

$$\langle\psi|A|\psi\rangle = (\langle\psi|A|\psi\rangle)^{*} \quad \forall\ |\psi\rangle$$

$$\langle\phi|A|\psi\rangle = (\langle\psi|A|\phi\rangle)^{*} \quad \forall\ |\psi\rangle, |\phi\rangle.$$

$$\Rightarrow \quad A = A^{\dagger}$$

Problem: Using this definition prove that $-i\hbar\vec{\nabla}$ is a hermitian operator.

→ Eigenvalues and eigenvectors of self-adjoint operator

$$A|\phi_n\rangle = a_n|\phi_n\rangle$$
$$A|\phi_r\rangle = a_r|\phi_r\rangle$$

$$\Rightarrow \underbrace{\left(\langle\phi_r|A|\phi_n\rangle - (\langle\phi_n|A|\phi_r\rangle)^{*}\right)}_{0} = a_n\langle\phi_r|\phi_n\rangle - a_r^{*}(\langle\phi_n|\phi_r\rangle)^{*}$$

$$= (a_n - a_r^{*})\langle\phi_r|\phi_n\rangle$$

$\Rightarrow$ For non-degenerate case $|a_n| \neq |a_r|$ if $n \neq r$

$\Rightarrow \langle\phi_r|\phi_n\rangle = 0$ for $r \neq n$

For $r = n$ $\quad a_n = a_n^{*} \Rightarrow$ Real.

$\Rightarrow$ The eigenvalues of the Hermitian operators is real and the eigenvectors are orthogonal.

(2)

→ For an $N \times N$ Hermitian operator we get N orthogonal vectors $\{|\phi_n\rangle\}$. The orthogonal vectors are $\underbrace{\text{linearly independent}}_{\text{prove it}}$. Therefore, the eigenstates of a Hermitian operator form an ortho-normal basis.

⇒ Any vector $|\chi\rangle$ can be decomposed in $\{|\phi_n\rangle\}$

$$|\chi\rangle = \sum C_n |\phi_n\rangle$$

where $C_n = \langle \phi_n | \chi \rangle$

$$\Rightarrow |\chi\rangle = \sum_n |\phi_n\rangle\langle\phi_n|\chi\rangle = \left(\sum_n |\phi_n\rangle\langle\phi_n|\right)|\chi\rangle$$

This equation is true for all the vectors $|\chi\rangle$

$$\Rightarrow \boxed{\sum_n |\phi_n\rangle\langle\phi_n| = \mathbb{1}}$$

→ $A \cdot |\phi_n\rangle = a_n |\phi_n\rangle$

$$A \cdot \mathbb{1} = A \sum_n |\phi_n\rangle\langle\phi_n| = \sum_n A|\phi_n\rangle\langle\phi_n| = \sum_n a_n |\phi_n\rangle\langle\phi_n|$$

$$\Rightarrow \boxed{A = \sum_n a_n |\phi_n\rangle\langle\phi_n|}$$ Spectral decomposition.

→ Skew adjoint operators : $A^\dagger = -A$

→ Unitary operators : $A^\dagger = A^{-1}$

→ Positive definite and +ve semidefinite operators

$\langle \psi | A | \psi \rangle \geq 0 \quad \forall |\psi\rangle$ +ve semidefinite

$\langle \psi | A | \psi \rangle > 0 \quad \forall |\psi\rangle$ +ve definite.

→ All the self-Adjoint operators with +ve spectrum are positive definite.

→ 〃 with non-negative spectrum are positive semi-definite.

Axiomatic approach to QM

→ States

→ ψ represents a state of a quantum system.

→ Superposition.

→ Overall phase does not matter.

→ ψ captures the entire information about a quantum system.

→ Operators (observables):

→ Any physical quantity is represented by an observable

→ Since physical, the average value and the experimental outcomes of the observable should be real ⇒ Hermitian

$$\Rightarrow \int \psi^*(x)\, O\, \psi(x)\, dx \in \mathbb{R}$$

→ Evolution:

→ Schrödinger eqn characterize the evolution of the quantum system.

→ All the evolutions are unitary.

→ Measurements :

→ Upon measurement of the observable $\hat{O}$ on the state ψ, the outcomes will be the eigenfunctions of $\hat{O}$, $\phi_n(x)$ with probability.

$$p_n = \int |\phi_n^*(x)\psi(x)|^2 dx$$

→ After measurement the state of the system will collapse to the outcome $\phi_n(x)$.

→ Hilbert space formulation of QM :

→ All the quantum states are vectors in the complex projective Hilbert space $P(\mathcal{H})$ or Ray space.

→ The projective Hilbert space $P(\mathcal{H})$ of a complex Hilbert space $\mathcal{H}$ is the set of equivalent classes of vectors v in $\mathcal{H}$, $v \neq 0$ such that

$v \sim w = \lambda v$; where λ is some non-zero complex number.

$P(\mathcal{H}_n) = \mathbb{C}P^{n-1}$ where n is the dimension of the Hilbert space.

→ Every vector in the complex projective space $P(\mathcal{H})$ represents a valid state.

→ The vector in $\mathbb{C}P^{n-1}$ are abstract vectors and their representation in different basis is what we are familiar with.

→ E.g. $\boxed{\psi(x) = \langle x|\psi\rangle}$

$\Rightarrow \quad |\psi\rangle = \int |x\rangle\langle x|\psi\rangle \, dx$

as $\int |x\rangle\langle x| \, dx = \mathbb{1}$

→ Similarly; the operators $\hat{x}$, $-i\hbar\frac{\partial}{\partial x}$ acting on the state $|\psi\rangle$

→ To understand that lets consider an operator M. Lets assume it is a differential operator and let $\psi_m(\vec{x})$ be its eigenfunction corresponding to d_m eigenvalue.

$\Rightarrow \quad M\psi_m(\vec{x}) = d_m \psi_m(\vec{x})$

→ Let the abstract vector corresponding to $\psi_m \to |m\rangle$ such that $\langle \vec{x}|m\rangle = \psi_m(\vec{x})$

$\Rightarrow \quad \hat{M}|m\rangle = d_m|m\rangle.$

→ To calculate the action of the operator $\hat{M}$ on any state $|\phi\rangle$ we can do the following

$|\phi\rangle = \sum_m a_m|m\rangle \quad ; \quad a_m = \langle m|\psi\rangle$

$M|\phi\rangle = \sum_m a_m M|m\rangle = \sum_m a_m d_m|m\rangle$

→ In the basis $\{|m\rangle\}$ $\quad |\phi\rangle = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix}$

$$\Rightarrow \quad M|\phi\rangle = \begin{bmatrix} d_1 a_1 \\ d_2 a_2 \\ \vdots \\ \vdots \\ d_N a_N \end{bmatrix} = \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix}$$

→ To see the action of an operator on a state $|\phi\rangle$ (4)
we need two things, the eigenvalues λ_m and
the projections $a_m = \langle m|\psi\rangle$. For example in L^2, L_x, L_y, L_z
we find the eigenvalues first using the commutation relation and then
construct the matrices.

→ Basis transformation:

Type I:

→ Consider an arbitrary basis $\{|\chi_n\rangle\}_{n=1}^{N}$ and $\{|m\rangle\}$

$$|\chi_n\rangle = \sum_m W_{nm} |m\rangle$$

→ What are the properties of W_{nm}?

$$\langle \chi_n|\chi_e\rangle = \sum_{jk} W^*_{nj} W_{ek} \underbrace{\langle j|k\rangle}_{\delta_{jk}}$$

$$= \sum_k (W)_{ek} (W^+)_{kn} = (WW^+)_{en}$$

we have $\langle \chi_n|\chi_e\rangle = \delta_{ne} = (WW^+)_{en} \Rightarrow \boxed{WW^+ = \mathbb{1}}$

$\underline{\text{Illy}}$ $\boxed{W^+W = \mathbb{1}}$ unitary matrix

Type II: $\{|\psi_i\rangle\} \rightarrow \{|\chi_i\rangle\}$

$$\Rightarrow U = \sum_i |\chi_i\rangle\langle\psi_i| \qquad \Big| \qquad UU^+ = U^+U = \mathbb{1}$$

$$\Rightarrow U|\psi_n\rangle = |\chi_n\rangle$$

→ Consider the operator H such that

$$H|\psi_n\rangle = \lambda_n|\psi_n\rangle$$

$$\Rightarrow \quad M = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_N \end{bmatrix} \quad \text{in } \{|\psi_m\rangle\} \text{ basis.}$$

→ How will M look like in $\{|\chi_m\rangle\}$ basis?

$$|\chi_m\rangle = U|\psi_m\rangle \Rightarrow |\psi_m\rangle = \sum W_{nm}|\chi_n\rangle$$

$$\Rightarrow M = \sum_m \lambda_m |\psi_m\rangle\langle\psi_m| = \sum_{m,n,n'} \lambda_m W_{nm} |\chi_n\rangle\langle\chi_{n'}| W^{*}_{n'm}$$

$$\Rightarrow \boxed{M = \sum_{nn'} \left(\sum_m W_{nm} \lambda_m W^{+}_{mn'}\right) |\chi_n\rangle\langle\chi_{n'}|}$$

→ What is the difference b/w basis transformation of the first type and IInd type?

→ Two Self-Adjoint operators with the same spectrum are related by unitary transformation

→ Two Self-Adjoint operation with different spectrum but same basis are commuting operators.

→ Eigenvalues and Eigenvectors of Unitary operator?

→ Consider a basis $\{|\eta_n\rangle\}$ such that

$$U|\eta_n\rangle = \alpha_n|\eta_n\rangle$$

→ What are the possible choices for α?

$$\Rightarrow \boxed{U = \sum_n \alpha_n |\eta_n\rangle\langle\eta_n|} \qquad \boxed{\alpha_n = e^{i\theta_n}}$$

→ The eigenvalues are phases and eigenvectors are orthonormal.

→ Consider

$$A|\psi_n\rangle = a_n|\psi_n\rangle$$

$$\Rightarrow A\left[|\psi_1\rangle \; |\psi_2\rangle \; \cdots \; |\psi_N\rangle\right] = \left[a_1|\psi_1\rangle \; a_2|\psi_2\rangle \; \cdots \; a_N|\psi_N\rangle\right]$$

Let $\quad S \equiv \left[|\psi_1\rangle \cdots |\psi_N\rangle\right]$

$$AS = SD \quad ; \quad D = \begin{bmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_N \end{bmatrix}$$

→ If $\{|\psi_1\rangle, \cdots |\psi_N\rangle\}$ are orthonormal

$$\Rightarrow \quad SS^{+} = \left[|\psi_1\rangle \cdots |\psi_N\rangle\right] \begin{bmatrix} \langle\psi_1| \\ \langle\psi_2| \\ \vdots \\ \langle\psi_N| \end{bmatrix}$$

$$= \sum_n |\psi_n\rangle\langle\psi_n| = \mathbb{1}$$

and $$S^{+}S = \begin{bmatrix} \langle\psi_1| \\ \langle\psi_2| \\ \vdots \\ \langle\psi_N| \end{bmatrix} \left[|\psi_1\rangle \cdots |\psi_N\rangle\right] = \begin{bmatrix} 1 & & \\ & 1 & \\ & & \ddots \; 1 \end{bmatrix}$$

⇒ S is Unitary

$$\Rightarrow \quad \boxed{A = SDS^{+}} \quad \text{or} \quad D = S^{+}AS \quad \text{diagonalisation.}$$

→ If S is not unitary

$$\Rightarrow \quad \boxed{A = S\,DS^{-1}} \quad \text{For non-normal matrices.}$$

→ <u>functions of operators</u> (Hermitian)

$$f(A) = f(UDU^{\dagger}) = U f(D)\, U^{\dagger}$$

$$= U \begin{bmatrix} f(d_1) & & & \\ & f(d_2) & & \\ & & \ddots & \\ & & & f(d_N) \end{bmatrix} U^{\dagger}$$

→ <u>Example:</u>

$$M = \exp[A] = \mathbb{1} + A + \frac{A^2}{2!} + \cdots$$

$$= \mathbb{1} + UDU^{\dagger} + \frac{UDU^{\dagger}UDU^{\dagger}}{2!} + \cdots$$

$$= U\left[\mathbb{1} + D + \frac{D^2}{2!} \cdots\right]U^{\dagger}$$

→ let $M^{\dagger} = M^{-1}$ → unitary

$$\Rightarrow \exp[A^{\dagger}] = \exp[-A] \Rightarrow \boxed{A^{\dagger} = -A}$$

Anti-Hermitian

$\Rightarrow$ $M = \exp[iH]$, i.e. any unitary operator can be written as the exponential of a Hermitian operator multiplied by i.

→ Therefore, for every Hermitian operator we have a unitary operator.

→ Despite this statement, the set of all the Hermitian matrices is unbounded whereas the set of all the unitary matrices is bounded. Explain why?

→ Consider two unitary operators U_1 and U_2 (6)

if $U_1 = \exp[iH_1]$ and $U_2 = \exp[iH_2]$ then H_1 and H_2 are Hermitian.

→ $U = U_1 \cdot U_2$ is also unitary

$= \exp[iH]$; H is Hermitian.

what is H in terms of H_1 and H_2.

→ Measurements:

→ Given state $|\psi\rangle$

→ observable $A = \sum_j a_j |x_j\rangle\langle x_j|$

→ Probability of jth outcome $p_j = |\langle\psi|x_j\rangle|^2$

⇒ Average value of the observable

$$\langle A\rangle = \sum_j a_j p_j = \sum_j a_j |\langle\psi|x_j\rangle|^2$$

$$= \sum_j a_j \langle\psi|x_j\rangle\langle x_j|\psi\rangle$$

$$\boxed{\langle A\rangle = \langle\psi|A|\psi\rangle}$$

→ Variance $(\Delta A)^2 = \langle (A - \langle A\rangle)^2\rangle$

$$= \langle A^2\rangle - \langle A\rangle^2$$

$$= \left[\sum_j a_j^2 p_j - \left(\sum_j a_j p_j\right)^2\right]$$

⇒ $\Delta A \to 0$ if $|\psi\rangle$ is an eigenstate of A.

→ Heisenberg uncertainity principle

→ Consider two observables A and B.

$$(\Delta A)^2 = \langle \psi | (A - \langle A \rangle)^2 | \psi \rangle = \langle \psi_A | \psi_A \rangle$$

$$|\psi_A\rangle = (A - \langle A \rangle)|\psi\rangle$$

$$(\Delta B)^2 = \langle \psi | (B - \langle B \rangle)^2 | \psi \rangle = \langle \psi_B | \psi_B \rangle$$

$$\Rightarrow (\Delta A)^2 (\Delta B)^2 = \underbrace{\langle \psi_A | \psi_A \rangle \langle \psi_B | \psi_B \rangle \geq |\langle \psi_A | \psi_B \rangle|^2}_{\text{Schwartz's inequality}}$$

$$\langle \psi_A | \psi_B \rangle = \langle \psi | (A - \langle A \rangle)(B - \langle B \rangle) | \psi \rangle$$

$$= \langle \psi | AB | \psi \rangle - \langle A \rangle \langle B \rangle$$

$$\rightarrow \langle AB \rangle - \langle A \rangle \langle B \rangle = \langle \psi | \left[\frac{\overset{\text{anti-Hermitian}}{[A,B]} + \overset{\text{Hermitian}}{\{A - \langle A \rangle, B - \langle B \rangle\}}}{2} \right] | \psi \rangle$$
$$= \langle \psi_A | \psi_B \rangle =$$

⇒ The second term $\langle \psi | \{A - \langle A \rangle, B - \langle B \rangle\} | \psi \rangle$ is the real component of the $\langle \psi_A | \psi_B \rangle$ and the first term is imaginary. Therefore, the magnitude is always greater that imy part.

$$\Rightarrow \boxed{(\Delta A)^2 (\Delta B)^2 \geq \left| \frac{\langle \psi | [A,B] | \psi \rangle}{2} \right|^2}$$

→ It means the error in two non-compatible observables is lower bounded by the expectation value of the commutator of the two observables.

→ Does it mean if we have two commuting observables then

→ Can the error in two non-commuting observables be zero??

Singular value decomposition:

→ $A \rightarrow$ arbitrary matrix.

$A^\dagger A \rightarrow$ Hermitian and +ve

$\Rightarrow \quad U^\dagger A^\dagger A U = D^2 \rightarrow$ Diagonal with +ve elements

$U \rightarrow$ unitary ; $D^2 = [d_i^2]$

$U \equiv [|\psi_1\rangle \, |\psi_2\rangle \; -- \, |\psi_N\rangle]$ orthonormal

→ $A|\psi_j\rangle = \gamma_j |\varphi_j\rangle$ $\quad \gamma_i \rightarrow$ some scalar

$|\varphi_i\rangle \rightarrow$ Normalized.

$\Rightarrow \quad \langle \varphi_i | \varphi_j \rangle = \frac{1}{\gamma_i \gamma_j} \langle \psi_i | A^\dagger A | \psi_j \rangle = \frac{d_i^2}{\gamma_i \gamma_j} \delta_{ij}$

$\Rightarrow \quad \{|\varphi_i\rangle\}$ are orthonormal

and $\gamma_j \equiv d_j$ (we can choose them to be +ve)

$$\Rightarrow \quad A = [|\varphi_1\rangle |\varphi_2\rangle -- |\varphi_M\rangle] \begin{bmatrix} d_1 & & \\ & d_2 & \\ & & \ddots \; d_{\min(M,N)} \end{bmatrix} \begin{bmatrix} \langle \psi_1 | \\ \langle \psi_2 | \\ \vdots \\ \langle \psi_N | \end{bmatrix}$$

$\boxed{A = W D U^\dagger}$ Spectral Decomposition

→ Here U is the matrix which diagonalise $A^{+}A$
and W is the " " AA^{+}.

⇒ if $AA^{+} = A^{+}A$

⇒ $\boxed{W \cong U}$

→ The $\cong$ sign is because we lose the freedom to choose d_i +ve.

→ <u>Polar Decomposition</u>:

$$A = W D U^{+} = W U^{+} U D U^{+}$$

$$= V H$$

→ V unitary $\equiv \exp[iH']$
→ H +ve Hermitian
⇒ Any arbitrary A can be written as unitary times +ve operator. This is called Polar Dec.

qubits:

- Any two-level quantum system is called a qubit.

<u>Examples:</u>
- Polarization states of a single photon
- Ground and excited states of an atom
- Quantum dots
- Superconducting charge and flux qubits
- Spin of an electron.
- Single photon travelling in two paths.

- State space of a qubit is $\mathbb{C}P^1$.

- <u>Hamiltonian</u>:

$$H = \sum_{\mu=0}^{3} a_\mu \sigma_\mu$$

$$= \begin{bmatrix} a_0+a_3 & a_1-ja_2 \\ a_1+ja_2 & a_0-a_3 \end{bmatrix}$$

- We dont care if we shift the energy of the energy levels by a constant $\Rightarrow$ $\boxed{a_0=0}$ chosen

$$\Rightarrow \quad H = \begin{bmatrix} a_3 & a_1-ja_2 \\ a_1+ja_2 & -a_3 \end{bmatrix}$$

- Eigenvalues $\pm\sqrt{a_1^2+a_2^2+a_3^2} \equiv E_\pm = \pm E$
- Eigenvectors $|\pm\rangle$ corresponding to $E_\pm$

$\Rightarrow \quad H = E_+ |+\rangle\langle+| + E_- |+\rangle\langle+|$

$= E\left[|+\rangle\langle+| - |-\rangle\langle-| \right]$

$\rightarrow$ If we choose $\{|+\rangle, |-\rangle\}$ as our computation basis then $\boxed{H \propto \sigma_z}$.

$H = \frac{\hbar\omega}{2}\sigma_z \quad ; \quad \boxed{E \equiv \frac{\hbar\omega}{2}}$

$\Rightarrow$ We can always choose σ_z as the Hamiltonian of a qubit system if other basis has not been specified.

$\Rightarrow \quad |+\rangle \equiv |0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \quad |-\rangle \equiv |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\Rightarrow \quad \boxed{\begin{aligned} H|0\rangle &= \frac{\hbar\omega}{2}|0\rangle \\ H|1\rangle &= -\frac{\hbar\omega}{2}|1\rangle \end{aligned}}$

$\rightarrow$ Most general state of qubit can be written as a linear superposition of $|0\rangle$ and $|1\rangle$

$\Rightarrow \quad |\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}; \quad |\alpha|^2 + |\beta|^2 = 1$

$\rightarrow \quad |\psi\rangle = \begin{bmatrix} \cos\theta/2 \\ \sin\theta/2 \, e^{i\phi} \end{bmatrix}$

$\rightarrow \quad p_0 = |\langle 0|\psi\rangle|^2 = \cos^2(\theta/2)$

$p_1 = |\langle 1|\psi\rangle|^2 = \sin^2(\theta/2)$

→ Time-evolution

$$|\psi(t)\rangle = \exp\left[-iHt/\hbar\right] |\psi(0)\rangle$$

$$= \begin{bmatrix} \cos\theta/2 \; e^{-i\frac{\omega t}{2}} \\ \sin(\theta/2)\, e^{i(\phi + \frac{\omega t}{2})} \end{bmatrix}$$

→ Measurement of σ_z gets us the expectation value.

$$\boxed{\langle \sigma_z \rangle = \cos^2\theta/2 - \sin^2\theta/2 = \cos\theta}$$

→ How do we estimate experimentally ϕ?

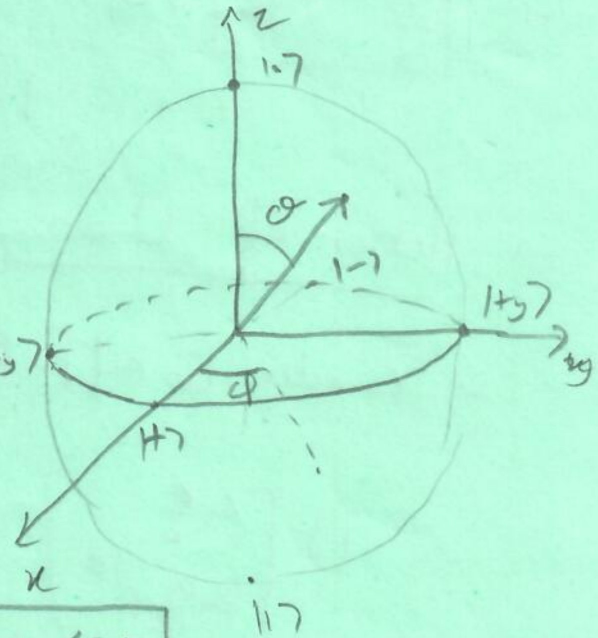
→ $\langle \sigma_x \rangle = \sin\theta \cos\phi$

→ $\langle \sigma_y \rangle = \sin\theta \sin\phi$

→ $\langle \sigma_z \rangle = \cos\theta$

Bloch Sphere

→ $\theta \to \theta + \pi$, $\phi \to \phi$ — orthogonal State



→ Expectation value of an arbitrary observable

$$A = \sum a_n \sigma_n \rightarrow \boxed{\langle A \rangle = \sum_n a_n \langle \sigma_n \rangle}$$

Unitary transformation on a qubit

$$U = \exp[iM] \quad ; \quad M = \sum_{\mu} a_\mu \sigma_\mu$$

→ a_0 contributes to an overall phase → neglect

→ What are the eigenvalues of U?

→ Eigenvectors are same as M.

→ let $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ is the first eigenvector, $|\alpha|^2 + |\beta|^2 = 1$

→ $\begin{bmatrix} -\beta^* \\ \alpha^* \end{bmatrix}$ is the IInd eigenvector.

→ $W = \begin{bmatrix} \alpha & -\beta^* \\ \beta & \alpha^* \end{bmatrix}$ is a unitary matrix

$W \in SU(2)$

→ $$\boxed{U = \exp[i\theta\, \hat{n}\cdot\vec{\sigma}] = \cos\theta\, \mathbb{1} + i\sin\theta\, \hat{n}\cdot\vec{\sigma}}$$

$$|\hat{n}| = 1$$

→ <u>Action of U on the states of a qubit.</u>

→ Consider $U = \exp\left[i\frac{\alpha}{2}\sigma_z\right] = \begin{bmatrix} e^{i\frac{\alpha}{2}} & 0 \\ 0 & e^{-i\frac{\alpha}{2}} \end{bmatrix}$

→ $$U\begin{bmatrix} \cos\frac{\theta}{2} \\ \sin\theta/_2\, e^{i\phi} \end{bmatrix} = \begin{bmatrix} e^{i\frac{\alpha}{2}}\cos\theta/_2 \\ \sin\theta/_2\, e^{i(\phi-\frac{\alpha}{2})} \end{bmatrix} \equiv e^{i\frac{\alpha}{2}}\begin{bmatrix} \cos\frac{\theta}{2} \\ \sin\theta/_2\, e^{i(\phi-\alpha)} \end{bmatrix}$$

⇒ Action of U is $\boxed{\begin{matrix} \theta \to \theta \\ \phi \to \phi - \alpha \end{matrix}}$

i.e. Rotation about z by angle α.

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→ let $U = \exp\left[-i\frac{\alpha}{2}\sigma_y\right] = \begin{bmatrix} \cos\frac{\alpha}{2} & +\sin\left(\frac{\alpha}{2}\right) \\ -\sin\left(\frac{\alpha}{2}\right) & \cos\frac{\alpha}{2} \end{bmatrix}$

→ let $|+\rangle = \begin{bmatrix} \cos\theta/2 \\ \sin\theta/2 \, e^{i\phi} \end{bmatrix}$ let $\phi = 0$ ⇒ x,z plane.

$$U|+\rangle = \begin{bmatrix} \cos\alpha/2 \cos\theta/2 + \sin\alpha/2 \sin\theta/2 \\ -\sin\alpha/2 \cos\theta/2 + \cos\alpha/2 \sin\theta/2 \end{bmatrix} = \begin{bmatrix} \cos\left(\frac{\theta-\alpha}{2}\right) \\ \sin\left(\frac{\theta-\alpha}{2}\right) \end{bmatrix}$$

⇒ $\theta \to \theta - \alpha$ ⇒ Rotation about y by an angle α.

→ $U = \exp\left[i\frac{\theta}{2}\hat{n}\cdot\vec{\sigma}\right]$ will give you a rotation about the $\hat{n}$ by an angle θ.

→ The arbitrary $SU(2)$

$$U = \begin{bmatrix} \alpha & -\beta^* \\ \beta & \alpha^* \end{bmatrix}; \quad |\alpha|^2 + |\beta|^2 = 1$$

$$U = \cos\left(\frac{\theta}{2}\right)\mathbb{1} + i\sin\left(\frac{\theta}{2}\right)\hat{n}\cdot\vec{\sigma}$$

$$\Rightarrow U = \begin{bmatrix} a e^{i\gamma} & -b e^{-i\delta} \\ b e^{i\delta} & a e^{-i\gamma} \end{bmatrix} \Rightarrow a^2 + b^2 = 1$$

$$e^{i\varepsilon\sigma_z} U e^{i\eta\sigma_z} = \begin{bmatrix} a e^{i(\gamma+\varepsilon+\eta)} & -b e^{-i(\delta-\varepsilon+\eta)} \\ b e^{i(\delta-\varepsilon+\eta)} & a e^{-i(\gamma+\varepsilon+\eta)} \end{bmatrix}$$

→ To make this matrix real we can choose

$\gamma + \varepsilon + \eta = 0$ and $\delta - \varepsilon + \eta = 0$

$$\Rightarrow \quad \eta = -\left(\frac{\delta+\gamma}{2}\right)$$

$$\varepsilon = \left(\frac{\delta+\gamma}{2}\right)$$

$$\Rightarrow \quad U = e^{-i\varepsilon\sigma_z} \begin{bmatrix} a & -b \\ b & a \end{bmatrix} e^{-i\eta\sigma_z}$$

→ A Real 2×2 unitary is a rotation about y.

⇒ An arbitrary 2×2 unitary matrix with $+1$ determinant can be decomposed as a product of 2 rotations about z axis and one rotation about y-axis.

→ The same can be done by taking two rotations about y and one about z.

→ This is called Euler angle representation.

<u>Exercise</u>: Prove that

$$U_N = O_1 D_U O_2^T$$

where $U_N \in SU(N)$

$O_1, O_2 \in SO(N)$

D_U is diagonal unitary operator.

Pure and Mixed States

(Density matrix formalism)

→ Consider a scenario where a ensemble of identical quantum systems is prepared which contain states $|\psi_1\rangle$ with probability q_1 and $|\psi_2\rangle$ with prob q_2. How will it affect the expectation values of the observables?

→ $\langle A \rangle = q_1 \langle \psi_1 | A | \psi_1 \rangle + q_2 \langle \psi_2 | A | \psi_2 \rangle$

$$= Tr\left[\{ q_1 |\psi_1\rangle\langle\psi_1| + q_2 |\psi_2\rangle\langle\psi_2| \} A \right]$$

→ let $\rho = q_1 |\psi_1\rangle\langle\psi_1| + q_2 |\psi_2\rangle\langle\psi_2|$

$\Rightarrow \boxed{\langle A \rangle = Tr[\rho A]}$

<u>IIIly</u>: If $\rho = \sum_{i=1}^{N} q_i |\psi_i\rangle\langle\psi_i|$

$$\boxed{\langle A \rangle = Tr[\rho A]}$$

→ ρ is called density operator (Matrix)

→ <u>Properties</u>:

- → Hermitian $\rho = \rho^\dagger$
- → positive $\Rightarrow$ $eig(\rho) \geq 0$
- → $Tr[\rho] = 1$

→ Any operator satisfying these properties can be a bonafide Density operator.

→ $\rho = |\psi\rangle\langle\psi|$ Represent a pure state

⇒ $\rho^2 = \rho$; $e_{ij}(\rho) = 1, 0, 0, \ldots$

→ $\rho^2 \neq \rho$ ⇒ Mixed state.

$\rho = \sum_i q_i |\psi_i\rangle\langle\psi_i|$ → Convex combination of Pure state.

→ Writing a Density operator ρ as a convex combination of pure states is called a preparation method or pure state Decomposition

→ A density operator ρ can be prepared in infinitely many ways and quantum mechanics does not allow us to Differentiate b/w different preparation methods.

→ Preparation method for Pure states is unique.

→ Recipe to get different preparation methods;

→ $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| = \sum_i |\tilde{\psi}_i\rangle\langle\tilde{\psi}_i|$

$|\tilde{\psi}_i\rangle = \sqrt{p_i}\, |\psi_i\rangle$

let $|\tilde{\varphi}_i\rangle = \sum_j U_{ji} |\tilde{\psi}_j\rangle$

→ $\sum_i |\tilde{\varphi}_i\rangle\langle\tilde{\varphi}_i| = \sum_i \sum_j \sum_k U_{ji} U^{*}_{ki} |\tilde{\psi}_j\rangle\langle\tilde{\psi}_k|$

$$= \sum_{jk} \left\{ \sum_{i} \left[U_{ji} (U^{\dagger})_{ik} \right] \right\} |\tilde{\psi}_j \rangle \langle \tilde{\psi}_k |$$

let $UU^{\dagger} = \mathbb{1} \Rightarrow \sum_i U_{ji} (U^{\dagger})_{ik} = \delta_{jk}$

$$\Rightarrow \boxed{\sum_i |\tilde{\varphi}_i \rangle \langle \tilde{\varphi}_i | = \sum_j |\tilde{\psi}_j \rangle \langle \tilde{\psi}_j | = \rho}$$

→ Geometric Representation of Density operators (qubit case)

$$\rho = \frac{1}{2} \left[\mathbb{1} + \vec{r} \cdot \vec{\sigma} \right] \quad ; \quad \vec{r} \in \mathbb{R}^3$$

$$\text{eig}(\rho) = \frac{1}{2} (1 \pm |\vec{r}|)$$

$\rho \geqslant \;\Rightarrow\; 1 - |\vec{r}| \geqslant 0 \Rightarrow |\vec{r}| \leq 1$

$\Rightarrow$ $|\vec{r}| \leq 1$ is the only condition on $\vec{r} \in \mathbb{R}^3$ to make ρ +ve and unit trace.

$\Rightarrow$ The set of all the qubit density operators is (geometrically) represented by a unit sphere.

→ $|\vec{r}| = 1 \Rightarrow$ One eigenvalue is zero

$\Rightarrow$ ρ rep. pure states.

$\Rightarrow$ The Surface of this sphere contains pure states.

$\rightarrow \quad \langle \sigma_x \rangle = \mathrm{Tr}[\rho\, \sigma_x] = r_x$

Let $\rho = p\,|\psi\rangle\langle\psi| + (1-p)\,|\bar\psi\rangle\langle\bar\psi|$ Spectral Decom

$\langle \psi | \bar\psi \rangle = 0$

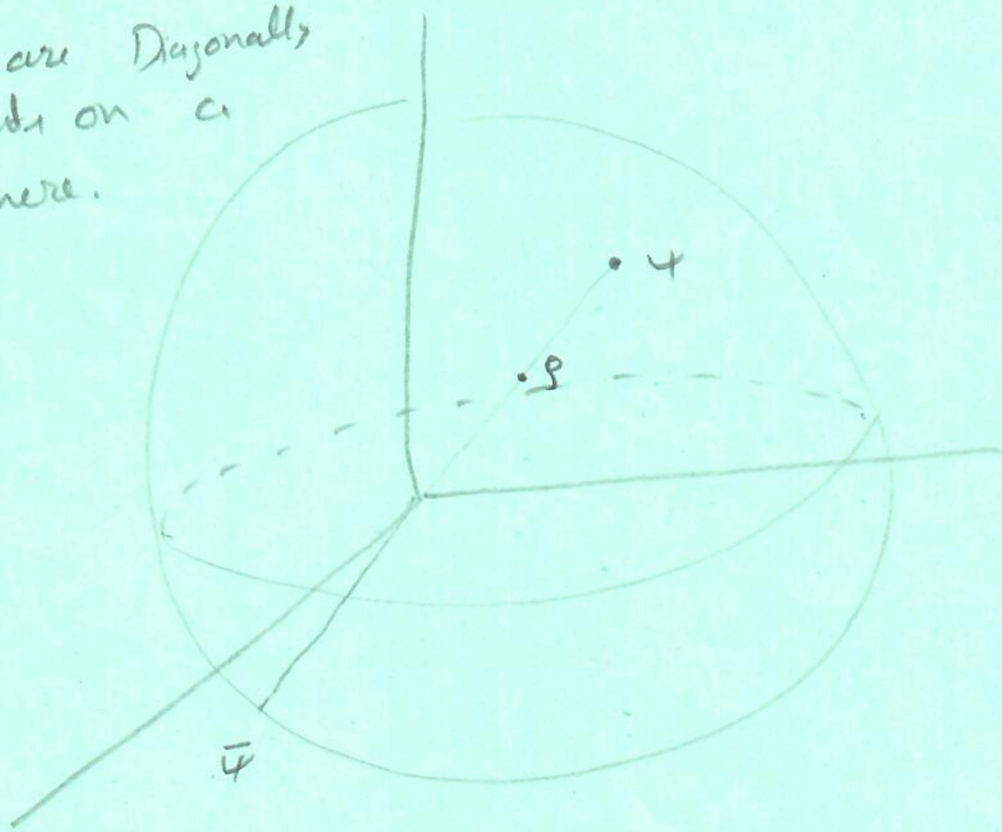
$\Rightarrow \quad \langle \sigma_x \rangle = p\,\langle \sigma_x \rangle_\psi + (1-p)\,\langle \sigma_x \rangle_{\bar\psi}$

III^{ly} $\langle \sigma_y \rangle = r_y = p\,\langle \sigma_y \rangle_\psi + (1-p)\,\langle \sigma_y \rangle_{\bar\psi}$

$\langle \sigma_z \rangle = r_z = p\,\langle \sigma_z \rangle_\psi + (1-p)\,\langle \sigma_z \rangle_{\bar\psi}$

$\rightarrow$ ψ and $\bar\psi$ are Diagonally opposit points on a Bloch sphere.

$\rightarrow$



$\rightarrow$ On the Bloch Sphere $\langle \vec\sigma \rangle_\psi$ is the vector rep of ψ

$\rightarrow$ $\bar\psi$ is Rep by $\langle \vec\sigma \rangle_{\bar\psi}$

$$\langle \vec\sigma \rangle_\rho = p\,\langle \vec\sigma \rangle_\psi + (1-p)\,\langle \vec\sigma \rangle_{\bar\psi}$$

$\Rightarrow$ The vector corresponding to ρ falls inside the Bloch sphere and is a convex combination of ψ and $\bar\psi$.

→ In a purely geometric terms we can see that the same point $\vec{r} = \langle \vec{\sigma} \rangle_\rho$ can be written as a convex combination of many other states.

→ Time evolution in terms of Density operators.

$$\rho(0) = \sum p_j |\varphi_j(0)\rangle\langle\varphi_j(0)|$$

$$|\varphi_j(t)\rangle = U(t,0)\,|\varphi_j(0)\rangle \qquad U = \exp\left[-iHt/\hbar\right]$$

$$\Rightarrow \rho(t) = U\rho(0)U^\dagger$$

$$\Rightarrow \frac{\partial}{\partial t}\rho(t) = -i\frac{H}{\hbar}U\rho(0)U^\dagger + U\rho(0)U^\dagger\left(\frac{iH}{\hbar}\right)$$

$$\Rightarrow \frac{\partial}{\partial t}\rho(t) = \frac{-i}{\hbar}\left[H, \rho(t)\right]$$

→ Von-Neumann eq^n or Liouville eq^n or

Liouville - von-Neumann eq^n.

→ Measurement on Density operator

$A,\ a_i,\ |a_i\rangle$

$p(a_i) = \langle a_i|\rho|a_i\rangle$

Polarization of classical Light

→ Jones vector

$$\vec{E} = E_x e^{i(\omega t - kz)}\hat{x} + E_y e^{i(\omega t - kz)}\hat{y}$$

Jones vector $E = \begin{bmatrix} E_x \\ E_y \end{bmatrix}$ $E_x, E_y \in \mathbb{C}$

→ Intensity $I = |E_x|^2 + |E_y|^2 = E^\dagger E$

→ Jones vector Represents the state of Polarization of classical light

→ An arbitrary optical setup which alter the state of polarization is written as a 2×2 complex matrix called Jones matrix.

→ x-polarizer → $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ Jones Matrix

→ y-polarizer → $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

→ Diagonal polarizer anti diagonal → $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & \pm 1 \\ \pm 1 & 1 \end{bmatrix}$

→ Quarter Wave plate
fast axis vertical $e^{i\pi/4}\begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix}$

QWP (fast axis Horizontal) $e^{i\pi/4}\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$

→ Half wave plate $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

→ Rotated Jones matrix $R(\theta)\, J\, R^T(\theta)$

→ Stokes Vector:

We the state of polarization is not pure we use Stokes vector instead of Jones vector

$$\vec{S} = \begin{bmatrix} s_0 \\ s_1 \\ s_2 \\ s_3 \end{bmatrix} \qquad \begin{aligned} s_0 &= \text{Total intensity} \\ s_1 &= p_H - p_V \\ s_2 &= p_D - p_A \\ s_3 &= p_R - p_L \end{aligned}$$

→ For a pure polarized light

$$s_0^2 = s_1^2 + s_2^2 + s_3^2$$

→ Degree of polarization for unpolarized light

$$\boxed{V = \frac{\left(s_1^2 + s_2^2 + s_3^2\right)^{1/2}}{s_0}}$$

→ Examples of Stokes vectors (Normalized intensity)

Horizontal Pol $\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ → Jones vector

Vertical $\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

45° Pol $\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

135° poly $\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$ $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

R-Circular $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix}$

L-Circular $\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$ $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix}$

→ The Stokes parameters are expectation values of $\sigma_x, \sigma_y, \sigma_z$

$$\sigma_z = |H\rangle\langle H| - |V\rangle\langle V|$$

$$s_1 = \langle \sigma_z \rangle$$

$$\sigma_x = |45^\circ\rangle\langle 45^\circ| - |135^\circ\rangle\langle 135^\circ|$$

$$s_2 = \langle \sigma_x \rangle$$

$$\sigma_y = |R\rangle\langle R| - |L\rangle\langle L|$$

$$s_3 = \langle \sigma_y \rangle$$

$$\Rightarrow \quad S = I\begin{bmatrix} 1 \\ \langle \sigma_z \rangle \\ \langle \sigma_x \rangle \\ \langle \sigma_y \rangle \end{bmatrix}$$

→ I is the intensity.

⇒ The Stokes vector S and the Density matrix for qubits is isomorphic.

Mueller matrices:

→ Any optical transformation on the state of polarization can be written as a real 4×4 matrix which maps a Stokes vector to a Stokes vector

$$\boxed{S' = MS}$$

M – is the Mueller matrix.

→ Write the Mueller matrix for Quarter wave-plate and Half wave-plate.

Exercise:

→ If $A \geq 0$; $B \geq 0$ $N \times N$ Matrices

Prove that $C = A + B \geq 0$

→ $A_{ij} = \dfrac{1}{d_i + d_j}$; $d_i, d_j > 0$

Prove that

(a) $A_{ij} = \int_0^\infty e^{-(d_i + d_j)t} \, dt$

(b) $A \geq 0$

Composite System:

→ Particle 1 $\mathcal{H}_{d_1}$, operator $T_1, T_2 \in \mathcal{B}(\mathcal{H}_{d_1})$

→ Particle 2 $\mathcal{H}_{d_2}$, " $S_1, S_2 \in \mathcal{B}(\mathcal{H}_{d_2})$

→ $|\psi\rangle \in \mathcal{H}_{d_1}$; $|\phi\rangle \in \mathcal{H}_{d_2}$

→ Then the total state can be represented by $(|\psi\rangle, |\phi\rangle)$, i.e., some ordered set.

→ A fancy way to write it is $|\psi\rangle \otimes |\phi\rangle$. We will see what are the properties of $\otimes$.

→ $|\psi\rangle \otimes |\phi\rangle \in \mathcal{H}_{d_1} \otimes \mathcal{H}_{d_2}$

→ $\alpha |\psi\rangle \otimes |\phi\rangle = |\psi\rangle \otimes \alpha |\phi\rangle$

→ $(\alpha |\psi_1\rangle + \beta |\psi_2\rangle) \otimes |\phi\rangle = \alpha |\psi_1\rangle \otimes |\phi\rangle + \beta |\psi_2\rangle \otimes |\phi\rangle$

→ $\{|\psi_i\rangle\}_{i=1}^{d_1}$ ONB for $\mathcal{H}_{d_1}$

→ $\{|\phi_j\rangle\}_{j=1}^{d_2}$ ONB for $\mathcal{H}_{d_2}$

⇒ $\{|\psi_i\rangle \otimes |\phi_j\rangle\}_{i,j=1}^{d_1 d_2}$ ONB $\mathcal{H}_{d_1} \otimes \mathcal{H}_{d_2}$

→ A general state $|\Psi\rangle \in \mathcal{H}_{d_1} \otimes \mathcal{H}_{d_2}$

⇒ $$\boxed{|\Psi\rangle = \sum_{ij} \alpha_{ij} |\psi_i\rangle \otimes |\phi_j\rangle}$$

→ **Important** Not all the state $|\Psi\rangle \in \mathcal{H}_{d_1} \otimes \mathcal{H}_{d_2}$ can be written as a tensor product if $|\psi\rangle \in \mathcal{H}_{d_1}$ and $|\varphi\rangle \in \mathcal{H}_{d_2}$

$$|\Psi\rangle \neq |\psi\rangle \otimes |\varphi\rangle \quad \text{in general.}$$

Proof: Let $|\Psi\rangle = \sum_{ij} \alpha_{ij} |i\rangle \otimes |j\rangle$

let we have $|\psi\rangle$ and $|\varphi\rangle$ such that

$|\Psi\rangle = |\psi\rangle \otimes |\varphi\rangle$

→ let $|\psi\rangle = \sum s_i |i\rangle$

$|\varphi\rangle = \sum r_j |j\rangle$

$\Rightarrow |\psi\rangle \otimes |\varphi\rangle = \sum_{ij} s_i r_j |i\rangle \otimes |j\rangle$

$\Rightarrow$ we must have s_i and r_j such that

$\alpha_{ij} = s_i r_j$

let $A_{ij} = \alpha_{ij}$; $|s\rangle = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_{d_1} \end{bmatrix}$ $|r\rangle = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_{d_2} \end{bmatrix}$

$\Rightarrow \boxed{A = |s\rangle\langle r|} \Rightarrow$ Rank 1.

A can be an arbitrary matrix $\Rightarrow$ Rank $= \min(d_1, d_2)$

→ The states $|\Psi\rangle$ which can not be written as a product of two states are called entangled states.

→ Operation acting on $\mathcal{H}_{d_1} \otimes \mathcal{H}_{d_2}$

→ Let T acts on $\mathcal{H}_{d_1} \Rightarrow \in \mathcal{B}(\mathcal{H}_{d_1})$

S 〃 $\mathcal{H}_{d_2} \Rightarrow \in \mathcal{B}(\mathcal{H}_{d_2})$

→ $T \otimes S \quad \in \mathcal{B}(\mathcal{H}_{d_1} \otimes \mathcal{H}_{d_2})$

→ $\boxed{(T \otimes S)\,(|\psi\rangle \otimes |\phi\rangle) = T|\psi\rangle \otimes S|\phi\rangle}$

→ $(T \otimes S)\,|\chi\rangle = T \otimes S \sum_{ij} \alpha_{ij}\, |i\rangle \otimes |j\rangle$

$$= \sum_{ij} \alpha_{ij}\, T|i\rangle \otimes S|j\rangle$$

→ $T \otimes \mathbb{1}$ and $\mathbb{1} \otimes S$ commutes.

→ $\boxed{(A \otimes B)\,(C \otimes D) = AC \otimes BD}$

→ If $|\psi\rangle = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_{d_1} \end{bmatrix}$ $\quad |\phi\rangle = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_{d_2} \end{bmatrix}$

$$|\psi\rangle \otimes |\phi\rangle = \begin{bmatrix} \psi_1\phi_1 \\ \psi_1\phi_2 \\ \vdots \\ \psi_{d_1}\phi_{d_2} \end{bmatrix} \equiv \begin{bmatrix} \psi_1|\phi\rangle \\ \psi_2|\phi\rangle \\ \psi_3|\phi\rangle \\ \vdots \\ \psi_{d_1}|\phi\rangle \end{bmatrix}$$

III(3) : $A \otimes B = \left[\begin{array}{c|c|c} a_{11}B & a_{12}B & a_{13}B \quad \cdots \\ \hline a_{21}B & - & - \\ & & \\ - & + & \cdot \end{array}\right]$

$$(A \otimes B)_{ij,kl} = A_{ik} \cdot B_{jl} .$$

→ <u>Exercise</u> : $H = H_1 \otimes I + I \otimes H_2$

→ Calculate $U = \exp[-i Ht/\hbar]$ in terms of $U_1 = \exp[-iH_1t/\hbar]$ and $U_2 = \exp[-iH_2t/\hbar]$

→ <u>Schmidt Decomposition</u>:

$$|\Psi\rangle = \sum \alpha_{ij} |i\rangle|j\rangle \equiv \sum \alpha_{ij} |ij\rangle$$

→ $A \equiv [\alpha_{ij}]$

→ Using SVD

$A = UDV^T$, U and V are unitary and D is Diagonal $D \geq 0$

$$\Rightarrow \quad A_{ij} = \alpha_{ij} = \sum_{kl} U_{ik} D_{kl} (V^T)_{lj}$$

$$= \sum_{k} U_{ik} d_k V_{jk}$$

$$\Rightarrow |\Psi\rangle = \sum_{ij} \sum_{k} U_{ik} V_{jk} d_k |i\rangle \otimes |j\rangle$$ (18)

let $|\psi_k\rangle = \sum_i U_{ik} |i\rangle$

$|\phi_k\rangle = \sum_j V_{jk} |j\rangle$

$$\Rightarrow \boxed{|\Psi\rangle = \sum_k d_k |\psi_k\rangle \otimes |\phi_k\rangle}$$ Schmidt Decomposition.

→ $\{|\psi_k\rangle\}$ ONB

$\{|\phi_k\rangle\}$ ONB

Density matrix

→ $\rho_{d_1 d_2 \times d_1 d_2}$ → Hermitian

→ Positive

→ Trace 1

→ $\rho \in \mathcal{B}(\mathcal{H}_{d_1} \otimes \mathcal{H}_{d_2})$

→ Two-qubit system: $\mathcal{H}_2 \otimes \mathcal{H}_2$

$(\mathbb{1}, \sigma_x, \sigma_y, \sigma_z)$ → Basis for $\mathcal{B}(\mathcal{H}_2)$

$\Rightarrow \{\mathbb{1}\otimes\mathbb{1}, \mathbb{1}\otimes\vec{\sigma}, \vec{\sigma}\otimes\mathbb{1}, \vec{\sigma}\otimes\vec{\sigma}\}$ → Basis for $\mathcal{B}(\mathcal{H}_2 \otimes \mathcal{H}_2)$

$$\rho = \frac{1}{4}\left[\mathbb{1} + \vec{r}\cdot\vec{\sigma} \otimes \mathbb{1} + \mathbb{1} \otimes \vec{s}\cdot\vec{\sigma} + \sum_{ij} t_{ij}\, \sigma_i \otimes \sigma_j\right]$$

Bloch-Representation

$\vec{r} = Tr[\vec{\sigma} \otimes \mathbb{1} \, \rho] = \langle \vec{\sigma} \otimes \mathbb{1} \rangle$

$\vec{s} = Tr[\mathbb{1} \otimes \vec{\sigma} \, \rho] = \langle \mathbb{1} \otimes \vec{\sigma} \rangle$

$t_{ij} = Tr[\sigma_i \otimes \sigma_j \, \rho] = \langle \sigma_i \otimes \sigma_j \rangle$

→ Normal form of density matrix $\rho \in B(\mathcal{H}_1 \otimes \mathcal{H}_2)$

$$\rho \rightarrow M = \left[\begin{array}{c|c} 1 & \vec{s}^T \\ \hline \vec{r} & t \end{array}\right]$$

→ Reduced Density operator

Density matrix of subsystem A ρ_A

$$\rho_A = \frac{1}{2}[\mathbb{1} + \vec{r} \cdot \vec{\sigma}]$$

$$\vec{r} = \langle \vec{\sigma} \otimes \mathbb{1} \rangle$$

||ly $\quad \rho_B = \frac{1}{2}[\mathbb{1} + \vec{s} \cdot \vec{\sigma}]$

$$\vec{s} = \langle \mathbb{1} \otimes \vec{\sigma} \rangle$$

ρ_A, ρ_B can also be calculated by tracing over the other subsystem, i.e,

$$\rho_A = Tr_B[\rho_{AB}] = Tr_B\left[\frac{1}{4}\left\{\mathbb{1} + \vec{r} \cdot \vec{\sigma} \otimes \mathbb{1} + \mathbb{1} \otimes \vec{s} \cdot \vec{\sigma} + \sum_{ij} t_{ij} \, \sigma_i \otimes \sigma_j\right\}\right]$$

$$= \frac{1}{4}\left[2\mathbb{1} + 2\vec{r} \cdot \vec{\sigma} + \mathbb{1} \otimes tr(\vec{s} \cdot \vec{\sigma}) + \sum_{ij} t_{ij} \, \sigma_i \otimes tr(\sigma_j)\right]$$

$$= \frac{1}{2}[\mathbb{1} + \vec{r} \cdot \vec{\sigma}]$$

$\Rightarrow \rho_A = Tr_B[\rho_{AB}]$
$\rho_B = Tr_A[\rho_{AB}]$ } This operation is called partial trace.

(41)

→ Give a ρ_{AB} we can calculate ρ_A and ρ_B.
However, given ρ_A and ρ_B we can not get ρ_{AB}.
From ρ_A and ρ_B we get only $\rho = \rho_A \otimes \rho_B$

$$\Rightarrow \rho = \frac{1}{4}\left[\mathbb{1} + \vec{r}\cdot\vec{\sigma} \otimes \mathbb{1} + \mathbb{1} \otimes \vec{s}\cdot\vec{\sigma} + \sum_{ij} r_i s_j \sigma_i \otimes \sigma_j \right]$$

$$\Rightarrow t_{ij} = r_i s_j$$

→ Purification:

$$|\Psi\rangle = \sum \alpha_{ij} |i\rangle|j\rangle$$

$$\Rightarrow \rho = |\Psi\rangle\langle\Psi| = \sum \alpha_{ij} \alpha^*_{kl} |ij\rangle\langle kl|$$

$$\rho_A = Tr_B[\rho] = \sum_{\substack{ij \\ kl}} \alpha_{ij} \alpha^*_{kl} |i\rangle\langle k| \delta_{jl}$$

$$= \sum_{ikl} \alpha_{il} \alpha^*_{kl} |i\rangle\langle k|$$

$\Rightarrow$ if $M = [\alpha_{ij}]$

$$\rho_A = M M^\dagger$$

$$\text{|||}^{ly} \quad \boxed{\rho_B = M^\dagger M \Rightarrow \rho_{B\,ij} = \sum_k \alpha^*_{ki} \alpha_{kj} |i\rangle\langle j|}$$

→ We can also invert the process

If we are given a mixed state ρ_A

$$\rho_A = \sum p_i |e_i\rangle\langle e_i| \; ; \quad |e_i\rangle \to \text{eigenvector}$$
$$p_i \to \text{eigenvalue}$$

Then ρ_A can be thought of as a reduced state of a bigger system in the pure-state.

Let $|\Psi\rangle = \sum_{ij} \alpha_{ij} |ij\rangle$

$$= \sum_k d_k |\psi_k \varphi_k\rangle \quad \to \text{Schmidt Deco.}$$

$$\Rightarrow \quad \rho_A = \sum_k d_k^2 |\psi_k\rangle\langle\psi_k|$$

So by choosing $d_k^2 = p_k$ and $|\psi_k\rangle = |e_k\rangle$

we can say $\rho_A = Tr_B[|\Psi\rangle\langle\Psi|]$

where $|\Psi\rangle = \sum_k \sqrt{p_k} |e_k \varphi_k\rangle$

where $\{|\varphi_k\rangle\}$ is an ONB which can be chosen as we wish.

→ Separable and entangled states

$$\rho = \sum_i p_i \rho_{Ai} \otimes \rho_{Bi} \quad \to \text{Separable}$$

$$\rho \neq \sum_i p_i \rho_{Ai} \otimes \rho_{Bi} \quad \to \text{entangled.}$$

Quantum operations and Maps :

→ Positive maps

→ Quantum operation are the maps which maps +ve operators to +ve operator.

→ $\rho \to \rho'$

→ Unitary operations are valid quantum map

→ Maps need not be unitary

→ $\Omega : \mathcal{B}_+(\mathcal{H}) \to \mathcal{B}_+(\mathcal{H})$

→ +ve maps satisfies the following conditions.

→ Hermiticity:

$$\Omega(A^\dagger) = (\Omega(A))^\dagger$$

→ Positivity:

$$\Omega(A) \geq 0 \quad \forall \ A \geq 0$$

→ (optional) trace preservation.

$$\boxed{tr(\Omega(A)) = tr(A)}$$

→ linearity:

$$\Omega(aA + bB) = a\Omega(A) + b\Omega(B)$$

→ Maps are operators acting upon the vector space of Hermitian operators, i.e, Maps are operators acting on the vector space of operators. Some times they are called superoperators.

→ Any linear transformation on a vector space can be represented by a matrix. Therefore, we must get a matrix rep. of the map Ω.

$$\boxed{\Omega(A)_{ij} = \sum_{kl} M_{ij,kl} A_{kl}}$$

→ The matrix M rep. the action of Ω on A.

→ Hermiticity:

$$\Omega(A^\dagger) = (\Omega(A))^\dagger \qquad \forall\ A = A^\dagger$$

$$\Rightarrow \quad \forall\ A_{ij} = A^*_{ji}$$

$$\Omega(A)_{ij} = \Omega(A)^*_{ji}$$

→ $$\Omega(A)_{ij} = \sum_{kl} M_{ij,kl} A_{kl}$$

$$\Omega(A)^*_{ji} = \sum_{kl} M^*_{ji,kl} A^*_{kl}$$

$$\Rightarrow \sum_{kl} M_{ij,kl} A_{kl} = \sum_{kl} M^*_{ji,kl} A^*_{kl}$$

$$\boxed{A^*_{kl} = A_{lk}}$$

$$\Rightarrow \boxed{\sum_{kl} M_{ij,kl} A_{kl} = \sum_{kl} M^*_{ji,lk} A_{kl}} \quad \forall\ A_{kl}$$

$$\Rightarrow \boxed{M_{ij,kl} = M^*_{ji,lk}}$$ For Hermiticity preserving maps.

→ Let us define a matrix H such that (21)

$$\boxed{H_{ij,kl} = M_{ik,jl}}$$

$$\Rightarrow \boxed{M_{ij,kl} = M^{*}_{ji,lk} \Rightarrow H_{ij,kl} = H^{*}_{kl,ij}}$$

$\Rightarrow$ If $H = H^{\dagger}$; M is Hermiticity preserving.

→ Positivity preserving maps:

$$\Omega(A) \geq 0 \quad \forall \; A \geq 0$$

$\Rightarrow$ if $\langle\psi|A|\psi\rangle \geq 0 \quad \forall \, |\psi\rangle \in \mathcal{H}$

then $\langle\phi|\Omega(A)|\phi\rangle \geq 0 \quad \forall \, |\phi\rangle \in \mathcal{H}$

$$\Omega(A)_{ij} = \sum_{kl} M_{ij,kl} A_{kl}$$

$$A \geq 0 \Rightarrow \sum_{kl} \phi^{*}_{k} A_{kl} \phi_{l} \geq 0$$

$$\sum_{ij} \phi^{*}_{i} \Omega(A)_{ij} \phi_{j} = \sum_{kl} \sum_{ij} \phi^{*}_{i} M_{ij,kl} \phi_{j} A_{kl}$$

$$A = \sum_{n} d_n |\psi_n\rangle\langle\psi_n|$$

$$\Rightarrow A_{kl} = \sum_{n} d_n \psi_{nk} \psi^{*}_{nl}$$

$$\Rightarrow \sum_{ij} \phi^{*}_{i} \Omega(A)_{ij} \phi_{j} = \sum_{kl} \sum_{ij} \sum_{n} d_n \phi^{*}_{i} M_{ij,kl} \phi_{j} \psi_{nk} \psi^{*}_{nl}$$

→ Here we can replace $M \to H$ matrix we get

$$\sum_{ij} \varphi_i^* \Omega(A)_{ij} \varphi_j = \sum_{kl} \sum_{ij} \sum_n d_n \varphi_i^* H_{ik,jl} \varphi_j \psi_{nk}^* \psi_{nl}$$

$$= \sum_n d_n \left[\sum_{ik} \sum_{jl} \varphi_i^* \psi_{nk} H_{ik,jl} \varphi_j \psi_{nl}^* \right]$$

$$= \sum_n d_n \left(\langle \varphi | \otimes \langle \psi_n^* | \; H \; | \varphi \rangle \otimes | \psi_n^* \rangle \right)$$

⇒ For Ω to be tvity preserving

$\langle \varphi | \otimes \langle \psi | H | \varphi \rangle \otimes | \psi \rangle \geq 0 \quad \forall \; |\varphi\rangle$ and $|\psi\rangle$

⇒ H must be positive for all product states.

→ Trace preservation:

$$Tr[\Omega(A)] = \sum_i \Omega(A)_{ii} = \sum_i \sum_{kl} M_{ii,kl} A_{kl}$$

$$= Tr[A] = \sum_k A_{kk}$$

$$\Rightarrow \boxed{\sum_i M_{ii,kl} = \delta_{kl}}$$

$$\Rightarrow \sum_i H_{ik,il} = \delta_{kl}$$

$$H = \left[\begin{array}{c|c} h_{11} & h_{12} \\ \hline h_{21} & h_{22} \end{array} \right] \Rightarrow \boxed{h_{11} + h_{22} = 1}$$

Block matrix

→ Interesting Rep.

→ $H = H^\dagger$ ⟶ Hermiticity preserving maps

$H \geq 0$ ∀ product states ⟶ +ve maps

→ Since H is Hermitian

$$H = \sum_m k_m |s^m\rangle\langle s^m| \qquad \text{Spectral Deco.}$$

$$\Rightarrow \quad H_{ik,jl} = \sum_m k_m S^m_{ik} S^{m*}_{jl} = M_{ij,kl}$$

$$\Rightarrow \quad \Omega(\rho)_{ij} = \sum_{kl} M_{ij,kl}\, \rho_{kl}$$

$$= \sum_m \sum_{kl} k_m S^m_{ik} S^{m*}_{jl} \rho_{kl}$$

$$= \sum_m k_m \left(S^m \rho S^{m\dagger}\right)_{ij}$$

$$\Rightarrow \quad \boxed{\Omega(\rho) = \sum_m k_m S^m \rho S^{m\dagger}} \quad \text{and} \quad \boxed{M = \sum_m k_m S \otimes S^*}$$

→ Not all the +ve maps represent physical process.

→ Example: Transposition

→ $\rho^T \geq 0 \;\; \forall \rho \geq 0$

But $(\mathbb{1} \otimes \Omega_T)(\rho_{AB})$ may not be +ve, i.e., partial transpose need not be +ve map.

$$(\mathbb{1}\otimes\Omega_T)(\rho_{AB}) = \left[\begin{array}{c|c} \rho_{11} & \rho_{12} \\ \hline \rho_{21} & \rho_{22} \end{array}\right]^{T_B} = \left[\begin{array}{c|c} \rho_{11}^T & \rho_{12}^T \\ \hline \rho_{21}^T & \rho_{22}^T \end{array}\right]$$

Block-Matrix

→ Consider $\rho_{AB} = \left(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\right)\left(\frac{\langle 00| + \langle 11|}{\sqrt{2}}\right)$

$$= \frac{1}{2}\left[\begin{array}{cc|cc} 1 & & & 1 \\ & 0 & & \\ \hline & & 0 & \\ 1 & & & 1 \end{array}\right]$$

$$\rho_{AB}^{T_B} = \frac{1}{2}\left[\begin{array}{cc|cc} 1 & & & \\ & & 1 & \\ \hline & 1 & & \\ & & & 1 \end{array}\right]$$

$eig(\rho_{AB}) = 1, 0, 0, 0$

$eig(\rho_{AB}^{T_B}) = \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}$

Hence $\rho_{AB}^{T_B}$ is not +ve

⇒ partial Transposition is not a +ve maps.

⇒ Transposition is +ve but not completely +ve.

Def: <u>m +ve maps</u>: A positive map Ω is m positive if $(\mathbb{1}_m \otimes \Omega)$ is +ve

<u>Completely +ve map</u>: A +ve map Ω which is ... for all m

Interesting:

→ A positive map is always +ve for separable state.

Proof: $\rho_{AB} = \sum_i p_i \rho_{Ai} \otimes \rho_{Bi}$ → Separable.

→ Ω → +ve

⇒ $\Omega(\rho_{AB}) \geq 0$

and $\Omega(\rho_A) \geq 0$

$\Omega(\rho_B) \geq 0$

$$\boxed{(\mathbb{1} \otimes \Omega)(\rho_{AB}) = \sum_i p_i \rho_{Ai} \otimes \Omega(\rho_{Bi}) \geq 0}$$

QED.

⇒ Maps which are +ve but not completely +ve may be helpful in detecting entanglement

→ Only completely +ve maps can represent physical processes.

Choi's theorem for completely +ve maps.

Theorem 1: A map $\Omega: \mathcal{B}(\mathcal{H}_n) \to \mathcal{B}(\mathcal{H}_m)$ given by $\Omega(\rho) = \sum_{j=1}^{l} K_j \rho K_j^{\dagger}$ is completely +ve.

→ Step 1: $K \rho K^{\dagger} \geq 0 \quad \forall \rho \geq 0$

→ Step 2: $(\mathbb{1} \otimes K) \rho (\mathbb{1} \otimes K)^{\dagger} \geq 0 \quad \forall \rho \geq 0$

→ Step 3: $\sum_i K_i \rho K_i^{\dagger} \geq 0$

$$\Rightarrow \sum_i (\mathbb{1}\otimes K_i)\, \rho_{AB} (\mathbb{1}\otimes K_i)^\dagger \geq 0$$

$\Rightarrow \Omega(\rho)$ is CP. $\square$ QED

<u>Theorem 2</u>: For any completely +ve map Ω, there exist $\{K_i\}$ such that $\boxed{\Omega(\rho) = \sum_i K_i \rho K_i^\dagger}$

<u>Proof</u>: Since Ω is completely +ve $\Rightarrow \mathbb{1}\otimes\Omega \geq 0$

→ Consider the computational basis $\{|i\rangle\}$

→ Consider a matrix $E = \sum_{i,j} |i\rangle\langle j| \otimes |i\rangle\langle j|$

$$E = \begin{bmatrix} |1\rangle\langle 1| & |1\rangle\langle 2| & |1\rangle\langle 3| & |1\rangle\langle 4| \\ |2\rangle\langle 1| & |2\rangle\langle 2| & |2\rangle\langle 3| & |2\rangle\langle 4| \\ |3\rangle\langle 1| & |3\rangle\langle 2| & |3\rangle\langle 3| & |3\rangle\langle 4| \\ |4\rangle\langle 1| & |4\rangle\langle 2| & |4\rangle\langle 3| & |4\rangle\langle 4| \end{bmatrix}$$ Block matrix

$$\boxed{E = \sum_{i,j} |ii\rangle\langle jj|} \Rightarrow E \geq 0$$

→ $E' = (\mathbb{1}\otimes\Omega)\, E = \sum_{i,j} |i\rangle\langle j| \otimes \Omega(|i\rangle\langle j|)$

→ $E' \geq 0$

$\Rightarrow E' = \sum_i d_i\, |\tilde{x}_i\rangle\langle\tilde{x}_i|$ → Spectral Deco.

$= \sum_i |x_i\rangle\langle x_i|$; $|x_i\rangle = \sqrt{d_i}\, |\tilde{x}_i\rangle$

if $|x_i\rangle = \begin{bmatrix} |x_{i1}\rangle \\ |x_{i2}\rangle \\ \vdots \\ |x_{in}\rangle \end{bmatrix} \Rightarrow E' = \left[\sum_i |x_{ij}\rangle\langle x_{ik}|\right]$ jk-th Block

Let us define $K_i = \left[|x_{i1}\rangle \; |x_{i2}\rangle \; \cdots \; |x_{in}\rangle\right]$

Then $\sum_i |x_{ij}\rangle\langle x_{ik}| = \sum_i K_i |j\rangle\langle k| K_i^+$

$K_i |j\rangle = |x_{ij}\rangle$

$$\Rightarrow \boxed{\Omega(|j\rangle\langle k|) = \sum_i K_i |j\rangle\langle k| K_i^+}$$

$$\Rightarrow \Omega(\rho) = \sum_{i,j,k} K_i \, \rho_{jk} |j\rangle\langle k| K_i^+$$

$$\boxed{\Omega(\rho) = \sum_i K_i \rho K_i^+}$$ □ QED.

$\Rightarrow$ All the completely +ve (CP) maps can be written as $\sum_i K_i \rho K_i^+$ and all the maps of the form $\sum_i K_i \rho K_i^+$ are CP.

→ A map is CP iff $\boxed{\Omega(\rho) = \sum_m K_m \rho K_m^+}$

→ local action of a map Ω on a pure state $|\Psi\rangle = \frac{1}{\sqrt{N}} \sum_i |ii\rangle$ will reveal whether the map Ω is CP or not.

→ $\{K_m\}$ are called Kraus-operators.

→ All the physical process are CP maps.

→ Are all the CP maps physical processes?

→ Given a map $\Omega(\rho) = \sum_j K_j \rho K_j^\dagger$ with trace preservation condition $\boxed{\sum_j K_j^\dagger K_j = \mathbb{1}}$, if there exist a unitary operator U acting on a bigger composite system such that it transform the state of the local system as $\rho \rightarrow \sum_j K_j \rho K_j^\dagger$, then the map Ω can be considered as CP.

→ Let the state of the system is ρ.

→ " " ancilla $|\psi\rangle\langle\psi|$

→ Let there exist U for a given Ω.

$$\Rightarrow \quad \Omega(\rho) = \sum_j K_j \rho K_j^\dagger = Tr_B\left[U(\rho \otimes |\psi\rangle\langle\psi|)U^\dagger\right]$$

$$= Tr \sum_n \sum_m \left[(\mathbb{1} \otimes |n\rangle\langle n|)\ U(\rho \otimes |\psi\rangle\langle\psi|)U^\dagger(\mathbb{1} \otimes |m\rangle\langle m|)\right]$$

→ Note that $\langle n|U|\psi\rangle$ is an operator of the system A.

$$\Rightarrow \quad \boxed{K_n = \langle n|U|\psi\rangle}$$

$$\Rightarrow \quad \boxed{\Omega(\rho) = \sum_n K_n \rho K_n^\dagger} = \sum_n \langle n|U|\psi\rangle \rho \langle\psi|U^\dagger|n\rangle$$

→ The same result holds if we interchange the position of system and ancilla.

→ In case $\mathcal{H}_a \otimes \mathcal{H}_s$

and choosing $|\psi\rangle = |1\rangle$

$$K_n = \langle n|U|\psi\rangle \equiv \left(\langle n| \otimes \mathbb{1}\right) U \left(|1\rangle \otimes \mathbb{1}\right)$$

$$= U_{n1} \text{ Block.}$$

$$\Rightarrow \quad U = \begin{bmatrix} K_1 & & & \\ K_2 & & & \\ K_3 & & & \\ K_4 & & & \\ \vdots & & & \end{bmatrix}$$

→ In this notation the first block column of the unitary matrix will be the Kraus operators.

→ M matrix for Given Kraus operators.

$$\Omega(\rho) = \rho' = \sum_i K_i \rho K_i^{\dagger}$$

$$\Rightarrow |\rho'\rangle = \left(\sum_i K_i \otimes K_i^{*}\right)|\rho\rangle$$

$$\Rightarrow \boxed{M = \sum_i K_i \otimes K_i^{*}}$$

and $H = \sum_i |K_i\rangle\langle K_i| = \sum_i d_i |\tilde{K}_i\rangle\langle \tilde{K}_i|$

$|\tilde{K}_i\rangle$ → Normalized

$\Rightarrow$ H is a +ve operator for CP maps.

$\rightarrow$ Trace preservation $\sum_i K_i^{\dagger} K_i = \mathbb{1} \Rightarrow \sum_i d_i = 1$

$\Rightarrow$ H Represents a Density operator.

$\Rightarrow$ A CP-map acting on a d-dimensional Hilbert space can be viewed as a State of a a d^2-dimensional Hilbert space. IIIly. a state in a d^2-dim HS can be viewed as a map on d-dim Hilbert space.

This relation b/w CP maps and quantum states is called Channel-State Duality.

$\rightarrow$ Since for CPTP maps H is a Density operator, we can have multiple decomposition of H.

, i.e,

$$H = \sum_i d_i |\tilde{K}_i\rangle\langle\tilde{K}_i|$$

$$= \sum_i p_i |\tilde{\Psi}_i\rangle\langle\tilde{\Psi}_i|$$

$$\Rightarrow \boxed{H = \sum_i d_i \tilde{K}_i \otimes \tilde{K}_i^{*} = \sum_i p_i \tilde{\Psi}_i \otimes \tilde{\Psi}_i^{*}}$$

$\Rightarrow$ Kraus operators are not unique.

Prove that