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1. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two distinct lines in $\mathbb{R}^2$ which pass through the origin $(0,0)$. Show that $\mathcal{L}_1 \cup \mathcal{L}_2$ is not a vector space under vector addition of vectors and usual scaling of vectors in $\mathbb{R}^2$. What if we take the union of finitely many distinct lines $\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_n$ for $n \geq 3$?

$\mathcal{L}_i$ = a line in $\mathbb{R}^2$, passing through

Let m_i be its slope

Then,

$$\mathcal{L}_i = \{(\alpha, m_i\alpha) : \alpha \in \mathbb{R}\}, \text{ if } m_i \neq \infty$$
$$\text{or } \{(0, \alpha) : \alpha \in \mathbb{R}\}, \text{ if } m_i = \infty$$

If $m_1 \neq \infty$ and $m_2 \neq \infty$, then

$$(1, m_1), (1, m_2) \in \mathcal{L}_1 \cup \mathcal{L}_2$$

but $(1, m_1) + (1, m_2) = \underbrace{(2, m_1 + m_2)}_{\text{slope is } \frac{m_1+m_2}{2},\ \text{and } m_1 \neq m_2} \notin \mathcal{L}_1 \cup \mathcal{L}_2$

If $0 \neq m_1 < m_2 = \infty$, then

$$\left(\frac{1}{m_1}, 1\right), (0, 1) \in \mathcal{L}_1 \cup \mathcal{L}_2$$

but $\left(\frac{1}{m_1}, 1\right) + (0, 1) = \underbrace{\left(\frac{1}{m_1}, 2\right)}_{\text{slope is } 2m_1} \notin \mathcal{L}_1 \cup \mathcal{L}_2$

If $m_1 = 0$, $m_2 = \infty$, then

$$(1, 0), (0, 1) \in \mathcal{L}_1 \cup \mathcal{L}_2$$

but $(1, 0) + (0, 1) = \underbrace{(1, 1)}_{\text{slope is } 1} \notin \mathcal{L}_1 \cup \mathcal{L}_2$

Thus $\mathcal{L}_1 \cup \mathcal{L}_2$ is not a vector space.

The case $n \geq 3$ is similar. We may assume $m_1 < m_2 < \cdots < m_n$ and argue using m_1 and m_2, as in $n = 2$ case.

2. Consider the vector space $(\mathbb{R}[x], +, \cdot)$. If $S = \{x^n + x^m : 1 \le n, m \le 2\} \subseteq \mathbb{R}[x]$, then

   (a) How many elements are there in the set S?

   (b) What is span(S)?

   (c) What is the dimension of the vector space span(S)?

   (d) If $S = \{x^n + x^m : n, m$ are non-negative integers$\}$, then is it true that span$(S) = \mathbb{R}[x]$?

(a) $S = \{2x,\ x+x^2,\ 2x^2\}$. So $\# S = 3$.

(b) A typical element of span(S) is

$$\alpha_1(2x) + \alpha_2(x+x^2) + \alpha_3(2x^2), \text{ where } \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$$

$$= (2\alpha_1 + \alpha_2)\,x + (2\alpha_3 + \alpha_2)\,x^2$$

As $\alpha_1, \alpha_2, \alpha_3$ vary over elements of $\mathbb{R}$, $(2\alpha_1+\alpha_2)\,x + (2\alpha_3+\alpha_2)\,x^2$ varies over <u>all</u> quadratic polynomials without constant term.

Thus, $\text{span}(S) = \{\beta_1 x + \beta_2 x^2 : \beta_1, \beta_2 \in \mathbb{R}\}$

(c) Note that span(S) is a vector space.

A minimal spanning set of span(S) is $\{x, x^2\}$.

So, dimension of span(S) is 2.

(d) Observe that : $2 = x^0 + x^0 \in \text{span}(S)$.

Thus $1 = \frac{1}{2}(2) \in \text{span}(S)$

Also $$-\frac{1}{2}\underbrace{(x^0+x^0)}_{\in S} + \underbrace{(x^m + x^0)}_{\in S} \in \text{span}(S)$$

$$= x^m \in \text{span}(S), \quad \text{for all integers } m \geq 0.$$

Hence $\{1, x, x^2, x^3, \ldots\ldots\} \subseteq \text{span}(S)$.

and therefore, $\mathbb{R}[x] \subseteq \text{span}(S) \subseteq \mathbb{R}[x]$.

i.e. $\text{span}(S) = \mathbb{R}[x]$.

3. Consider the vector space $(M_n(\mathbb{R}), +, \cdot)$, for $n \geq 2$. Let S be the set of swapper matrices in $M_n(\mathbb{R})$. Is it true that S is a basis of $M_n(\mathbb{R})$?

Let E_{ij} = matrix whose $(i,j)^{th}$ entry is 1 and other entries are 0.

Observe that

$S_1 = \{ E_{ij} : 1 \leq i, j \leq n \}$ is a spanning set of $M_n(\mathbb{R})$.

In fact, it is a minimal spanning set.

So, dimension of $M_n(\mathbb{R}) = \# S_1 = n^2$.

Now, if S = set of swapper matrices, then

$$\# S = \frac{n(n-1)}{2} \quad (\text{why?})$$

Since any two basis consist of equal no. of elements, S can not be a basis of $(M_n(\mathbb{R}), +, \cdot)$.

4. For $v = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3$, the positive real number $\ell(v) := (a^2 + b^2 + c^2)^{1/2}$ is called the *length* of v. A linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ is called *rigid*, if $\ell(T(v)) = \ell(v)$ for all $v \in \mathbb{R}^3$.

(a) Show that the transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by $T(v) = R_{x,\theta}v$ is rigid, where $R_{x,\theta}$ is the rotation matrix about x-axis by angle θ.

(b) For $v, w \in \mathbb{R}^3$, the quantity

$$\beta(v, w) := \frac{v.w}{\ell(v)\ell(w)},$$

where $v.w$ is the dot product of v and w, is called the *angle cosine* of v and w. Show that if $T : \mathbb{R}^3 \to \mathbb{R}^3$ is a rigid linear transformation then it preserves angle cosine, *i.e.*, for $v, w \in \mathbb{R}^3$, we have $\beta(v, w) = \beta(T(v), T(w))$.

(a) Note that $T(v) = R_{x,\theta} v$ is a linear transformation.

Now,

$$T(v) = R_{x,\theta} v = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$= \begin{pmatrix} a \\ b\cos\theta - c\sin\theta \\ b\sin\theta + c\cos\theta \end{pmatrix}$$

$$\Rightarrow \ell(T(v))^2 = a^2 + (b\cos\theta - c\sin\theta)^2 + (b\sin\theta + c\cos\theta)^2$$

$$= a^2 + b^2\cos^2\theta + c^2\sin^2\theta - 2bc\cos\theta\sin\theta$$
$$+ b^2\sin^2\theta + c^2\cos^2\theta + 2bc\sin\theta\cos\theta$$

$$= a^2 + b^2 + c^2$$

$$\Rightarrow \ell(T(v)) = T(v).$$

Thus T is rigid linear transformation.

(b) It is enough to show that if T is rigid, then

$v \cdot w = T(v) \cdot T(w) \quad \forall\, v, w \in \mathbb{R}^3$. - why?

Since T is rigid

$$\ell(v+w) = \ell(T(v+w)) \quad \forall\, v, w \in \mathbb{R}^3. \quad \text{—} (*)$$

observe that $l(v+w)^2 = (v+w)\cdot(v+w)$

$$= v\cdot v + v\cdot w + w\cdot v + w\cdot w$$

Thus, from (*),

$$v\cdot v + 2v\cdot w + w\cdot w = T(v)\cdot T(v) + 2T(v)\cdot T(w) + T(w)\cdot T(w)$$

$$\Rightarrow l(v)^2 + 2v\cdot w + l(w)^2 = l(T(v))^2 + 2T(v)\cdot T(w) + l(T(w))^2$$

since T is rigid, $l(v) = l(T(v))$
and $l(w) = l(T(w))$.

Thus, $v\cdot w = T(v)\cdot T(w) \quad \forall\, v, w \in \mathbb{R}^3$.