Let X a set and $d : X \times X \to \mathbb{R}^+$

s.t. (i) $d(x,y) \geq 0$ (iv) $d(x,y) = d(y,x)$

(ii) $d(x,y) = 0 \Leftrightarrow x = y$

(iii) $d(x,y) \leq d(x,z) + d(z,y)$ $\forall x,y,z \in X$.

triangle inequality

we call 'd' a metric on X

(X,d) a metric sp.

Examples:- (i) $X = \mathbb{R}$, $d(x,y) = |x-y|$

then d satisfies (i) to (iv)

(ii) $X = \mathbb{C}$, $d(u,v) := |u-v|$

d is a metric on $\mathbb{C}$.

(iii) $X = \mathbb{R}^n = \{(x_1, x_2, \ldots x_n) : x_i \in \mathbb{R},\ 1 \leq i \leq n\}$

we define $d(x,y) = \left(\sum_{i=1}^{n} (x_i - y_i)^2\right)^{1/2}$ $n \in \mathbb{N}$

where $x = (x_1, \ldots x_n)$ and $y = (y_1, \ldots y_n)$

we will show that 'd' is a metric on $\mathbb{R}^n$.

(i) and (iv) trivially hold.

Let $d(x,y) = 0$ i.e. $\sum_{i=1}^{n} (x_i - y_i)^2 = 0$

As $(x_i - y_i)^2 \geq 0$ for all $i = 1, 2, \ldots n$.

$\ldots^2 = 0 \quad \forall i = 1$ to n

$\Rightarrow \quad (x_i - y_i)^2 = 0 \ \forall i = 1 \text{ to } n$

$\therefore \ x_i = y_i$ i.e. $x = y$

$i = 1 \text{ to } n$

For the triangle inequality

we need to show that

$$d(x,y) \leq d(x,z) + d(z,y) \quad \text{———} \ (*)$$

for $x = (x_1, \ldots x_n)$, $y = (y_1, \ldots y_n)$ and $z = (z_1, \ldots z_n)$

$$d(x,y) = \left(\sum_{i=1}^{n} (x_i - y_i)^2\right)^{1/2}$$

If we define $\|x\| = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}$

In order to prove $(*)$ E.T.S.T.

(1) — $\|x+y\| \leq \|x\| + \|y\|$ $\quad x+y = (x_1+y_1, \ldots x_n+y_n)$

as $d(x,y) = \|x-y\|$ where $x-y = (x_1-y_1, \ldots x_n-y_n)$

(1) holds $\Leftrightarrow$ $\|x+y\|^2 \leq (\|x\| + \|y\|)^2$

$$\|x+y\|^2 = \sum_{i=1}^{n} (x_i+y_i)^2 = \sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} y_i^2 + 2\sum_{i=1}^{n} x_i y_i$$

$$(\|x\| + \|y\|)^2 = \sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} y_i^2 + 2\|x\|\|y\|$$

(1) holds $\Leftrightarrow$ $(\|x\| + \|y\|)^2 - \|x+y\|^2 \geq 0$

i.e. $\displaystyle\sum_{i=1}^{n} x_i y_i \leq \|x\|\|y\|$

This is nothing but Cauchy Schwartz inequality.

Some definitions :-

Let X be a metric sp. with metric 'd'.

(a) Let $p \in X$. we define $(\varepsilon > 0)$

$$N_\varepsilon(p) = \{ x : d(x,p) < \varepsilon \}$$

$N_\varepsilon(p)$ is called a nbd of 'p' of radius 'ε'.

(b) A pt. $x \in X$ is a limit pt. of $E \subseteq X$ if for every nbd of x $\exists$ $q \neq p$ ($q \in$ nbd of p) s.t. $q \in E$.

(c) If $p \in E$ and 'p' is not a limit pt. of E, then p is an isolated pt. of E.

(d) We say $E \subseteq X$ is a closed set if every limit pt. of E belongs to E.

(e) A pt. $p \in X$ is an interior pt. of E if $\exists$ a nbd N of p s.t. $N \subset E$

(f) E is open if every pt. of E is an interior pt. of E.

(g) $E^c = \{ p \in X : p \notin E \}$

(h) E is perfect if E is closed and if every pt. of E is a limit pt. of E.

(i) E is bdd if $\exists$ $\cancel{M} \in \mathbb{R}$ and a pt. $q \in X$ s.t. $d(p,q) < N$ $\forall$ $p \in E$.

(j) $E \subseteq X$ is dense in X if every pt. of X is a limit pt. of E, or a pt. of E (or both).

Note that in $\mathbb{R}$ nbds are segments.

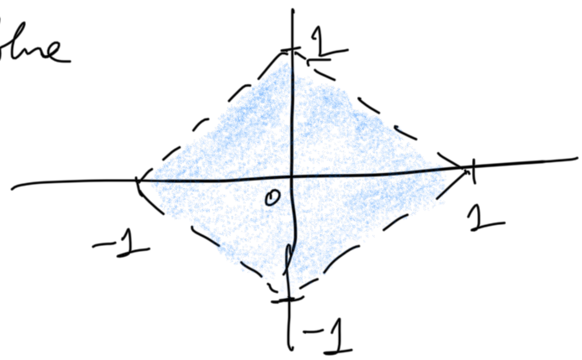
In the above defined metric on $\mathbb{R}^n$, $n=2$ nbds is ball without the bd.

In general the shapes of nbds depend on the metric we define.

For eg if we define

$$d(x,y) = \sum_{i=1}^{2} |x_i - y_i| \quad \text{on } \mathbb{R}^2$$

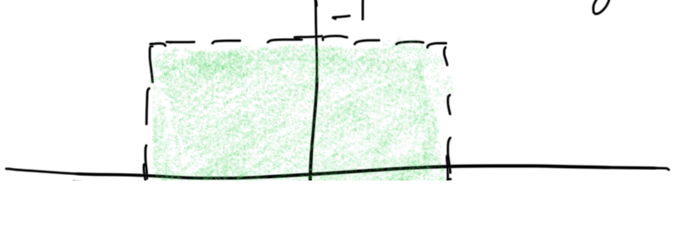
$N_1(0,0)$ ~~is~~ the blue shaded region

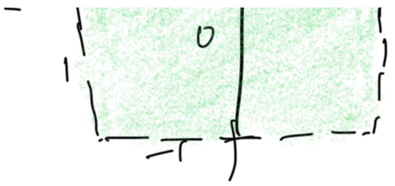


Similarly if we define the metric on $\mathbb{R}^2$ as

$$d(x,y) = \max_{i=1,2} |x_i - y_i|$$

Check that $N_1(0)$ is the green shaded region below.





Let us do some more properties of a metric space.

Thm 1:– Every nbd is an open set.

Note that nbd of a pt. is always non empty

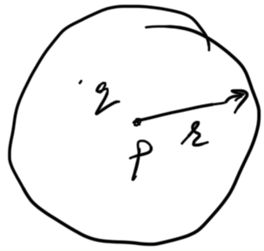
Pf:– Let E be a nbd i.e. $\exists\, r>0,\ p\in X$ s.t.

$$E = N_r(p)$$

We need to show that $\forall q\in E$, q is an int. pt.

Let $q\in E \Rightarrow d(p,q)<r$

We will show $\exists\, s>0$ (s depends on 'q') s.t. $N_s(q)\subseteq E$



Claim:– If we choose $s = r - d(p,q) > 0$ then $N_s(q)\subseteq E$

Let $x\in N_s(q)$ i.e. $d(x,q)<s$

We need to show that $x\in E$ which is equivalent to showing that $d(x,p)<r$.

Let us look at

$$d(x,p) \le d(x,q) + d(p,q)$$

(by triangle inequality

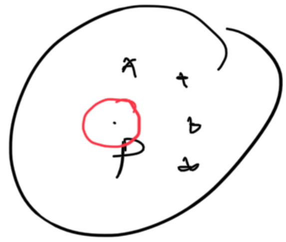
as $d(x,q) < r - d(p,q)$

we get $d(x,p) < r$. Hence proved. #

we get

Thm :- If p is a limit pt. of $E \subseteq X$, then every nbd of p contains inf many pts of E.

Pf :- Suppose $\exists$ a nbd of p which contains only finitely many pts. of E.



Let those pts be $x_1, \dots, x_N$

$x_i \neq p$

$$M = \min_{1 \le i \le N} d(x_i, p) > 0$$

Now if we look at $N_{\frac{M}{2}}(p)$. By the above assumpt we get ~~$N_{\frac{M}{2}}(p) \cap E = \phi$~~, $N_{M/2}(p) \cap E \setminus \{p\} = \phi$ which contradicts the fact that p is a limit pt. of E.

Corollary :- A finite pt. set has no limit pts.

Exercise :- Let (X, d) be a metric sp. and $E \subseteq X$. Then x_0 is a limit pt. of E

$\Leftrightarrow \exists\ x_n \in E$ s.t. $d(x, x_n) \to 0$ as $n \to \infty$

$(x_n \neq x, n \geq N)$

Examples :- (a) Let $A = \{ z \in \mathbb{C} : |z| < \}$

The set of limit pts is $\{ z \in \mathbb{C} : |z| \le 1 \}$

$X = \mathbb{C}$, standard metric

(b) $X = \mathbb{R}$ with the standard metric.

What is the set of limit pts ?

When what is ... of ... ?

$\mathbb{Z}$? Empty set.

(C) $\{1/n ; n \in \mathbb{N}\}$ it is limit pt. is '0'.

Ex: Determine which sets are closed/open in the above example.

Note: There may be subsets in a metric sp which are neither closed nor open.

$(a,b) = \{x : a < x < b\}$ → Open

$[a,b] = \{x : a \le x \le b\}$ → Closed

$[a,b) = \{x : a \le x < b\}$ → Neither open or closed

$(a,b] = \{x : a < x \le b\}$ → Neither open or closed

Thm: Let $\{E_\alpha\}$ be a (finite or infinite) collection of sets E_α. Then

$$\left(\bigcup_\alpha E_\alpha\right)^c = \bigcap E_\alpha^c$$

PLEASE READ THE PROOF FROM RUDIN OR DO IT YOURSELF. SIMPLE SET THEORETIC EXERCISE

Thm: A set E is open iff E^c is closed.

Proof: Let us first assume that E is open.

T.P. E^c is closed which is equivalent to ~~... all its limit~~

showing that E^c contains all its limit pts. Let x be a limit pt. of E^c.

$\Rightarrow N_\varepsilon(x) \cap E^c \neq \phi$

$\Rightarrow x$ is not an interior pt. of E

i.e. $x \notin E$ $\therefore x \in E^c$

Let E^c be closed. Need to show that E is open i.e. $\forall x \in E$, x is an int pt. of E.

If $x \in E \Rightarrow x$ is not a limit pt. of E^c. i.e. $\exists$ a nbd of x, call it $N_\varepsilon(x)$ s.t. $N_\varepsilon(x) \cap E^c = \phi$

$\Rightarrow N_\varepsilon(x) \subseteq E$

$\therefore E$ is open.

<u>Corollary</u>:- F is closed $\Leftrightarrow F^c$ is open.

<u>Thm</u>:- (a) For any collection $\{G_\alpha\}$ of open sets $\bigcup_\alpha G_\alpha$ is open

(b) For any collection $\{F_\alpha\}$ of closed sets, $\bigcap_\alpha F_\alpha$ is closed.

(c) Let $G_1, \ldots G_n$ be a finite collection of open sets. Then $\bigcap_{i=1}^{n} G_i$ is open.

(d) Let $F_1, F_2, \ldots F_n$ be a finite collection of

(d) Let $F_1, \ldots, F_n$ closed sets. Then $\bigcup_{i=1}^{n} F_i$ is a closed set.

Pf:- Clearly $a \Leftrightarrow b$ and $c \Leftrightarrow d$.

Let $\{G_\alpha\}$ be a collection of open sets

Let $x \in \bigcup G_\alpha$. Need to show it's an int pt.

$\exists$ some α st. $x \in G_\alpha \Rightarrow \exists\, \varepsilon > 0$ st.

$N_\varepsilon(x) \subseteq G_\alpha \subseteq \bigcup_\alpha G_\alpha$

$\therefore \bigcup_\alpha G_\alpha$ is open.

(c) Let $G_1, \ldots G_n$ be collection of open sets

$G = \bigcap_{i=1}^{n} G_i$. We will show G is open.

Let $x \in G \Rightarrow x \in G_i \ \forall i$

$\exists\, \varepsilon_i$ st. $N_{\varepsilon_i}(x) \subseteq G_i \ \forall i = 1, \ldots n$

Let $\varepsilon < \min_{1 \le i \le n} \varepsilon_i$

$\Rightarrow N_\varepsilon(x) \subseteq N_{\varepsilon_i}(x) \subseteq G_i \ \forall i = 1 \ldots n$

i.e. $N_\varepsilon(x) \subseteq \bigcap_{i=1}^{n} G_i$

$\therefore x$ is an int pt. of G.

#.

Examples :-

(I) Let $G_n = (-1/n, 1/n)$ $\bigcap_n G_n = \{0\}$ is a ~~closed~~ closed set

(i) ... set.

Def^n: Let (X, d) be a metric space. $E \subseteq X$. Let E' be the set of all limit pts of E in X, then closure of E is the set $\overline{E} = E \cup E'$

Thm: (X, d) a metric sp. $E \subseteq X$ then

(a) $\overline{E}$ is closed.

(b) $E = \overline{E} \iff E$ is closed

(c) $\overline{E} \subset F$ for every closed $F \subset X$ s.t. $E \subset F$.

Thm: Let $E \neq \phi$, $E \subseteq \mathbb{R}$ and E be bdd above. Let $y = \sup E$. Then $y \in \overline{E}$. If E is closed then $y \in E$.

Remark: Let (X, d) a metric sp.

$E \subseteq Y \subseteq X$

(Y, d) is also a metric sp.

We say E is open relative to Y for $E \subseteq Y$ if for any $q \in E$ $\exists$ $\varepsilon > 0$ s.t. $x \in E$ whenever $d(x, q) < \varepsilon$ and $x \in Y$

For eg :- $X = \mathbb{R}$

$Y = [0, 1)$

Then $E = [0, \tfrac{1}{2})$ is an open set relative to Y

Thm Suppose $Y \subseteq X$.

$A \subseteq Y$ is open relative to Y $\Leftrightarrow$ $A = Y \cap G$ for some open subset G of X.