P1

2

$$|\phi\rangle = N\left[a|100\rangle + b|200\rangle + c|322\rangle\right]$$

a) using $\pi|n\ell m\rangle = (-1)^{\ell}|n\ell m\rangle$

$$\pi|\phi\rangle = N\left[a(-1)^0|100\rangle + b(-1)^0|200\rangle + c(-1)^2|322\rangle\right]$$

$$= +|\phi\rangle$$

$\Rightarrow$ $|\phi\rangle$ is an even eigenstate of parity operator

A.

$\boxed{0.5}$

b) using $\langle n\ell m|H_0|n\ell m\rangle = \frac{-13.6}{n^2}\,eV = E_n$

& orthogonality of $|n\ell m\rangle$ states.

$$E = \langle \phi | H_0 | \phi \rangle = N^2 \{ a^2 E_1 + b^2 E_2 + c^2 E_3 \}$$

$$= -N^2 \, 13.6 \left( a^2 + \frac{b^2}{4} + \frac{c^2}{9} \right) eV \quad \boxed{0.5}$$

c) using $L^2 |n \ell m\rangle = \ell(\ell+1) \hbar^2 |n \ell m\rangle$

& $L_z |n \ell m\rangle = m\hbar \, |n \ell m\rangle$

$$\langle \phi | L^2 | \phi \rangle = N^2 \hbar^2 \{ a^2 \cdot 0 + b^2 \cdot 0 + c^2 \cdot 6 \}$$

$$= 6 c^2 N^2 \hbar^2 \quad \boxed{0.5}$$

& $\langle \phi | L_z | \phi \rangle = 2 c^2 N^2 \hbar \quad \boxed{0.5}$

$P_2$

**(2)**

For H-atom,

$$\Delta E = \frac{4}{3} g \left(\frac{m_e}{m_p}\right) \alpha^4 m_e c^2$$

$$\simeq 5.9 \times 10^{-6}$$

for muonic atom. $m_e \to m_\mu \simeq 200\, m_e$

$$\therefore \Delta E \simeq (200)^4 \times 5.9 \times 10^{-6} \text{ eV}$$

$$\simeq 0.2 \text{ eV} \quad \boxed{2}$$

$P_3$

**(4)**

$$S^2 = S_1^2 + S_2^2 + 2\vec{S_1}\cdot\vec{S_2}$$

$$S^2 \chi_i = \underbrace{S_1^2 \chi_i}_{3/4\hbar^2} + \underbrace{S_2^2 \chi_i}_{3/4\hbar^2} + 2\vec{S_1}\cdot\vec{S_2}\,\chi_i$$

$$\therefore \quad S^2 \chi_i = \frac{3}{2}\hbar^2 \chi_i + 2\,\underbrace{\vec{S_1}\cdot\vec{S_2}}\,\chi_i$$

$$S_{1x}S_{2x} + S_{1y}S_{2y} + S_{1z}S_{2z} \qquad \boxed{1}$$

$$* \; S_{1z}S_{2z}\,\chi_i = \frac{\hbar^2}{4}\begin{cases} +\chi_i & \text{for } i=1,4 \\ -\chi_i & \text{for } i=2,3 \end{cases}$$

$$* \quad S_{1x}S_{2x}\begin{cases} |\uparrow\uparrow\rangle = \frac{\hbar^2}{4}|\downarrow\downarrow\rangle \\ |\uparrow\downarrow\rangle = \frac{\hbar^2}{4}|\downarrow\uparrow\rangle \\ |\downarrow\uparrow\rangle = \hbar^2/4\,|\uparrow\downarrow\rangle \\ |\downarrow\downarrow\rangle = \hbar^2/4\,|\uparrow\uparrow\rangle \end{cases} \qquad S_{1y}S_{2y}\begin{cases} |\uparrow\uparrow\rangle = -\frac{\hbar^2}{4}|\downarrow\downarrow\rangle \\ |\uparrow\downarrow\rangle = \hbar^2/4\,|\downarrow\uparrow\rangle \\ |\downarrow\uparrow\rangle = \hbar^2/4\,|\uparrow\downarrow\rangle \\ |\downarrow\downarrow\rangle = -\hbar^2/4\,|\uparrow\uparrow\rangle \end{cases}$$

$$\boxed{1}$$

$$\Rightarrow \quad S^2 \begin{cases} \chi_1 = \frac{3}{2}\hbar^2 \chi_1 + 2\left(\cancel{\hbar^2/4}\,\chi_4 - \cancel{\hbar^2/4}\,\chi_4 + \hbar^2/4\, \chi_1\right) \\ \quad = 2\hbar^2 \chi_1 . \\ \chi_2 = \hbar^2 (\chi_2 + \chi_3) \\ \chi_3 = \hbar^2 (\chi_2 + \chi_3) \\ \chi_4 = 2\hbar^2 \chi_4 \end{cases}$$

$\Rightarrow$ $\chi_1$ and $\chi_4$ are the eigenstates of $S^2$. $\boxed{1}$

From the above result,

$\chi_2 + \chi_3$ & $\chi_2 - \chi_3$ are also eigenstates of $S^2$. $\boxed{1}$

* $\chi_1$, $\chi_2 + \chi_3$ & $\chi_4$ correspond to $S=1$ while $\chi_2 - \chi_3$ to $S=0$.

P4

5

In the ground state of He atom, the space part of the wave fn is symmetric. Therefore, the spin part has to be anti-symmetric. There is only one antisymmetric state which can be constructed out of two spin-$\frac{1}{2}$ particles.

$\Rightarrow$ the g.s. of He atom is non-degenerate $\boxed{1}$

For Hypothetical He atom, the total w.f. has be symmetric. Since the space part is still symmetric the spin part should also be symmetric. $\boxed{1}$

From addition of angular momenta,

$$1 \otimes 1 = 2 \oplus 1 \oplus 0 \quad \boxed{1}$$

$|1\,1\rangle \quad |1\,0\rangle \quad |1\,{-1}\rangle \;:\; |0\,1\rangle \quad |0\,0\rangle \quad |0\,{-1}\rangle \;:\; |{-1}\,1\rangle \quad |{-1}\,0\rangle \quad |{-1}\,{-1}\rangle$

$S_z: \quad 2 \quad 1 \quad 0 \qquad 1 \quad 0 \quad -1 \qquad 0 \quad -1 \quad -2$

$|1\,0\rangle \pm |0\,1\rangle$

$|0\,{-1}\rangle \pm |{-1}\,0\rangle$

$|1\,{-1}\rangle \pm |{-1}\,1\rangle$

Via inspection: $S=2$ &

$S=0$ are symmetric states $\boxed{1}$

$\Rightarrow$ total $5+1=6$ spin states $\therefore$ 6-fold degeneracy. $\boxed{1}$

P5

7

$$H' = e\,\mathcal{E}_0\, x$$
$\hookrightarrow r\sin\theta\cos\phi$

Since $n=1$ state is non-degenerate, the first order correction is given by

$$E^1_{n=1} = \langle 100|H'|100\rangle$$
$$= e\,\mathcal{E}_0 \langle 100|x|100\rangle$$
$$= e\,\mathcal{E}_0 \langle 100|\cdot\pi^2 x\,|100\rangle$$
$$= e\,\mathcal{E}_0 \langle 100|\pi(-x)\pi|100\rangle$$
$$= -e\,\mathcal{E}_0 \langle 100|x|100\rangle$$
$$\Rightarrow \boxed{E^1_{n=1} = 0}$$

$\boxed{1}$

$\underline{n=2}$ state has four fold degeneracy: $|200\rangle$ $|210\rangle$ $|21\pm1\rangle$

This requires computation of $\langle 2\,l\,m|x|2\,l'\,m'\rangle$

Parity $\Rightarrow$ $\Delta l = l'-l = \pm 1$ : selection rule for $l$

$\Rightarrow$ $\langle 21\pm1|x|21\mp\rangle = 0.$ $\boxed{1}$

& $\langle 210|x|21\pm1\rangle = 0.$

Using $[L_z, x] = i\hbar y$ ; $[L_z, y] = -i\hbar x$

$\langle 2lm|[L_z, y]|2l'm'\rangle = -i\hbar\langle 2lm|x|2l'm'\rangle$

$\underline{\text{or}}$ $(m-m')\langle 2lm|y|2l'm'\rangle = -i\langle 2lm|x|2l'm'\rangle$

$\hookrightarrow$ $[L_z, x]/i\hbar$

$$\Rightarrow \frac{(m-m')}{i\hbar} \underbrace{\langle 2lm|\,[L_z, x]\,|2l'm'\rangle} = i\,\langle 2lm|\,x\,|2l'm'\rangle$$

$$(m-m')\hbar\,\langle 2lm|\,x\,|2l'm'\rangle = i \quad "$$

$$\Rightarrow \quad [(m-m')^2 - 1]\,\langle 2lm|\,x\,|2l'm'\rangle = 0$$

$$\Rightarrow \quad \Delta m = \pm 1 \quad \Rightarrow \quad \langle 200|\,x\,|210\rangle = 0 \qquad \boxed{2}$$

& $\langle 200|x|21\pm1\rangle$ are non-zero matrix elements.

Let $\langle 200|\,x\,|21\,+1\rangle = a$

& $\langle 200|\,x\,|21\,-1\rangle = b$

Note:

$$\langle 2\,0\,0 | x | 2\,1 \pm 1 \rangle =$$

$$\int d^3r \; \Psi^*_{200} \; x \; \Psi_{21\pm1} = \int r^2 dr \; d\Omega . \; R^*_{20} \; Y^*_{00} \; r \sin\theta \cos\phi \; R_{21} \; Y_{1\pm1}$$

$$= \int r^3 dr \; R^*_{20} \; R_{21} \int d\Omega \; \frac{1}{\sqrt{4\pi}} \left( \mp \sqrt{\frac{3}{8\pi}} \right) \sin^2\theta \, e^{\pm i\phi} . \cos\phi$$

$$= \mp \sqrt{\frac{3}{2}} \; \frac{1}{4\pi} \int_0^\infty r^3 dr \, R^*_{20} R_{21} \int_{-1}^{1} d\cos\theta \; \sin^2\theta \quad \times \int_0^{2\pi} d\phi \; \underbrace{\left(\cos\phi \; e^{\pm i\phi}\right)}_{\cos^2\phi \,\pm\, i \cos\phi \sin\phi}$$

since $\int_0^{2\pi} d\phi \; \cos\phi \sin\phi = 0$

$\Rightarrow \; \langle 200 | x | 21\pm1 \rangle$ are real, equal and opposite

i.e. $b = -a$.

From degenerate P.T., we need to solve for states $|2\,0\,0\rangle$ & $|2\,1\,\pm1\rangle$

$$E^1 \cdot (1)_{3\times3} \cdot \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = (H')_{3\times3} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \quad \text{at first order.}$$

$$= \begin{pmatrix} 0 & a & -a \\ a & 0 & 0 \\ -a & 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \quad \boxed{1}$$

$\Rightarrow$ Eigen values are given by.

$$\begin{vmatrix} -E^1 & a & -a \\ a & -E^1 & 0 \\ -a & 0 & -E^1 \end{vmatrix} = 0 \Rightarrow -a(-E^1 a) - E^1((E^1)^2 - a^2) = 0$$

$$\Rightarrow E^1(2a^2 - (E^1)^2) = 0$$

$E^1 = 0 \ , \ \pm a\sqrt{2}$ $\boxed{1}$

for $E^1 = 0$, $a c_2 - a c_3 = 0 \Rightarrow c_3 = c_2 = 1$

$\& \ c_1 = 0$

$\Rightarrow$ corresponding state : $\frac{1}{\sqrt{2}} \left( |2\,1\,+1\rangle + |2\,1\,-1\rangle \right) = |\phi\rangle_0$

for $E^1 = \pm a\sqrt{2}$ $\quad a c_1 = \pm a\sqrt{2} c_2 \ \& \ a c_2 - a c_3 = \pm a\sqrt{2} c_1$

$$\underbrace{-a c_1 = \pm a\sqrt{2} c_3}$$

$\Rightarrow$ for $c_1 = 1$ ; $c_2 = \mp 1/\sqrt{2}$ ; $c_3 = \pm 1/\sqrt{2}$

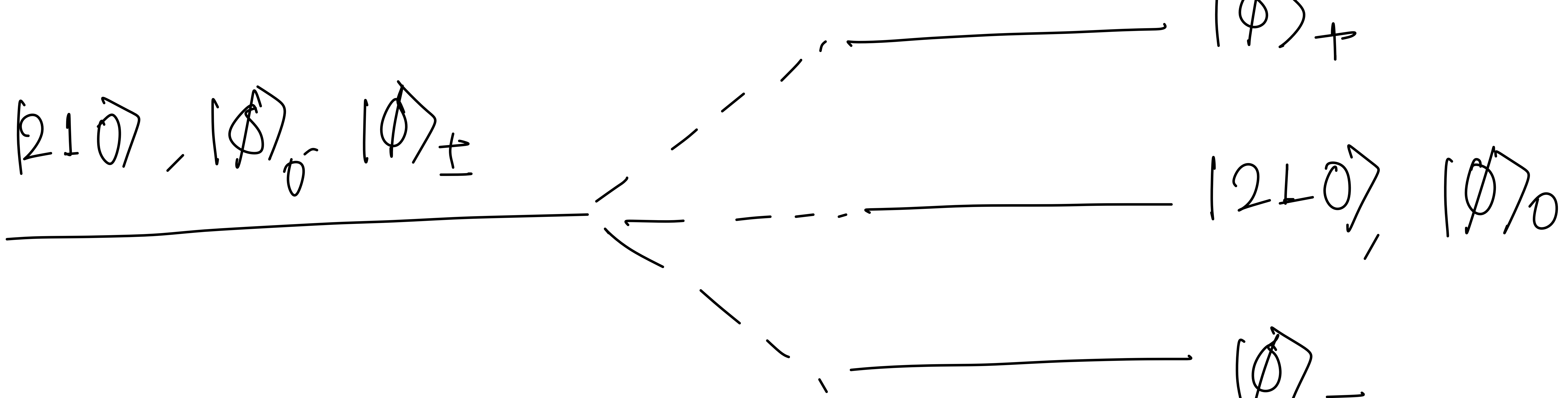
$\Rightarrow$ corresponding state : $\frac{1}{\sqrt{2}} \left( |2\,0\,0\rangle \mp \frac{1}{\sqrt{2}} |2\,1\,1\rangle \pm \frac{1}{\sqrt{2}} |2\,1\,-1\rangle \right)$

$|\phi\rangle_{\pm}$

$|210\rangle$ : $|\phi\rangle_0$ $|\phi\rangle_\pm$ are good states as they are orthogonal and off-diagonal matrix elements of $H'$ in this space is zero.

as $\langle 210|x|\phi\rangle_{0,\pm} = 0 = {}_0\langle\phi|x|\phi_\pm\rangle = {}_+\langle\phi|x|\phi\rangle_-$ . $\boxed{1}$

Final splitting:




$\varepsilon_0 = 0$ $\varepsilon_0 \neq 0$