PHY638 MidSem II Date : March 7, 2025 Inst: Abhishek Chaudhuri

- **Time : 60 minutes, Max Marks : 20**
- **Attempt all questions.**

1. Consider the flow $\boldsymbol{u}(\boldsymbol{x},t) = -\frac{1}{2}\alpha r\boldsymbol{e}_r + u_\theta(r)\boldsymbol{e}_\theta + \alpha z\boldsymbol{e}_z$, where α is a positive constant. Given that the vorticity is $\boldsymbol{\omega} = \omega\boldsymbol{e}_z$ with $\omega = \frac{1}{r}\frac{d}{dr}(ru_\theta)$, answer the following:
   (a) What is $\boldsymbol{\nabla}\cdot\boldsymbol{u}$? [2]
   (b) Write down the vorticity equation for this flow in a steady state in terms of ω? (Hint: The equation can be reduced to a first-order differential equation). Your integration constant will need to be set by the condition that the circulation of the flow is given as: $\Gamma = \int_S \boldsymbol{\omega}\cdot d\boldsymbol{S} = 2\pi\int_0^\infty dr r\omega(r)$. [4]
   (c) Hence determine u_θ. [2]

2. Consider the following two-dimensional stream function composed of a uniform horizontal stream of speed U and two vortices of equal and opposite strength in (x,y)-Cartesian coordinates.

$$\psi(x,y) = Uy + (\Gamma/2\pi)\ln\sqrt{x^2+(y-b)^2} - (\Gamma/2\pi)\ln\sqrt{x^2+(y+b)^2}$$

   (a) Simplify this stream function for the combined limit of $b\to 0$ and $\Gamma\to\infty$ when $2b\Gamma = C$ (a constant) to find $\psi(x,y)$. (Hint: It may be useful to consider $r^2 = x^2+y^2$ while simplifying. [2]
   (b) Switch to (r,θ) polar coordinates and find both components of the velocity using the simplified stream function. [2]
   (c) Determine where $u_r = 0$ and $u_\theta = 0$ and hence sketch the streamlines for the flow. [2]

3. Consider stationary surface gravity waves in a rectangular container of length L and breadth b, containing water of undisturbed depth H. The velocity potential is given by

$$\phi = A\cos(m\pi x/L)\cos(n\pi y/b)\cosh[k(z+H)]e^{-i\omega t},$$

   where m and n are integers.
   (a) Do the velocities obey the boundary conditions at the wall? [2]
   (b) Under what conditions does the velocity potential satisfy the Laplace equation? [2]
   (c) What would be the dispersion relation to satisfy the linearised free surface boundary condition? [2]