Mid-Semester Exam I.

$$U = \frac{1}{2} k r^2$$

$$r^2 = x^2 + y^2$$

(a)

$$x = r\cos\phi$$
$$y = r\sin\phi$$

$$\Rightarrow \dot{x} = \dot{r}\cos\phi - \dot{\phi} r\sin\phi$$
$$\dot{y} = \dot{r}\sin\phi + \dot{\phi} r\cos\phi$$

$$\Rightarrow T = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\phi}^2)$$

$$L = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) - \frac{1}{2} k(x^2 + y^2)$$
$$= \frac{1}{2} m(\dot{r}^2 + r^2\dot{\phi}^2) - \frac{1}{2} k r^2$$

(b)

$$\frac{\partial L}{\partial \phi} = 0 \Rightarrow \phi : \text{Cyclic coordinate}$$

$$\Rightarrow \frac{\partial L}{\partial \dot{\phi}} = \boxed{m r^2 \dot{\phi} = J = \text{Const}}$$

$$\cancel{\frac{\partial L}{\partial r} = -kr, \quad \frac{\partial L}{\partial \dot{r}}}$$

$$\frac{\partial L}{\partial r} = mr\dot{\phi}^2 - kr$$

$$= \frac{J^2}{mr^3} - kr$$

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r}$$

$$\Rightarrow \boxed{m\ddot{r} = \frac{J^2}{mr^3} - kr}$$

Alternatively.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x} \Rightarrow \boxed{m\ddot{x} = -kx}$$

similarly for y

$$\boxed{m\ddot{y} = -ky}$$

(c) ϕ is the cyclic coordinate as $\frac{\partial L}{\partial \phi} = 0$

While there is no corresponding variable in Cartesian coordinates, it is seen that x and y motions decouple.

(d) ϕ cyclic $\Rightarrow$ $J = mr^2\dot{\phi}$ is conserved.

L: time independent, so E is conserved.

$$E = \frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} + \frac{1}{2}kr^2.$$

in Cartesian coordinates, we can show that

$$E_x = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$$

& $$E_y = \frac{1}{2}m\dot{y}^2 + \frac{1}{2}ky^2$$

are conserved separately.

$$m\ddot{x} = -kx.$$

$$\Rightarrow m\dot{x}\ddot{x} = -kx\dot{x}$$

$$\Rightarrow m\frac{d}{dt}\left(\frac{1}{2}\dot{x}^2\right) = -\frac{1}{2}k\frac{d}{dt}\left(x^2\right).$$

$$\Rightarrow \frac{d}{dt}\left(\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2\right) = 0.$$

similarly for y.

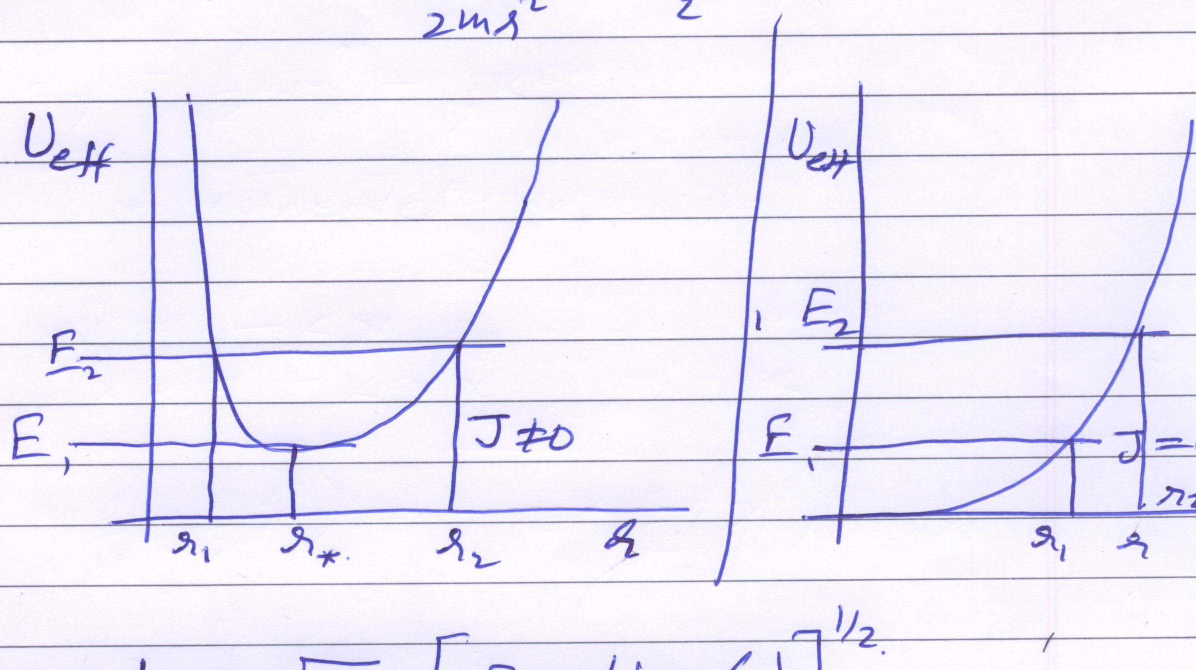
(e)

$$E = \frac{p_r^2}{2m} + \frac{J^2}{2mr^2} + \frac{1}{2}kr^2$$

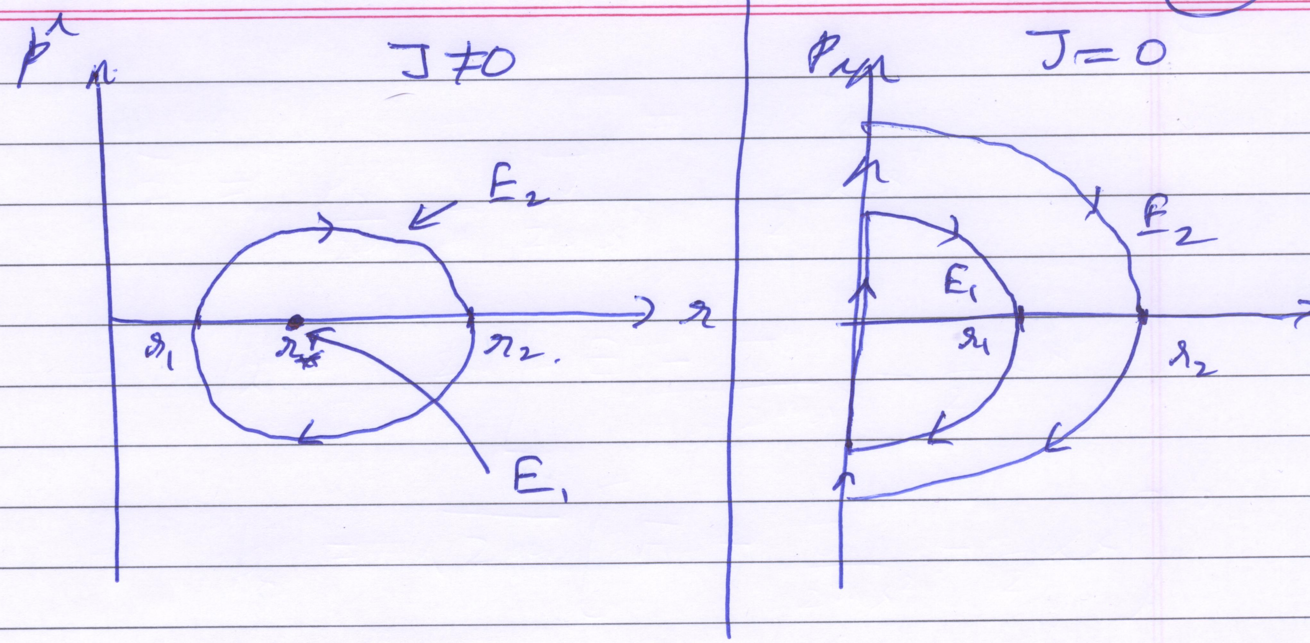
$$\Rightarrow p_r^2 = 2m\left[E - \frac{J^2}{2mr^2} - \frac{1}{2}kr^2\right]$$

$$\Rightarrow p_r = \sqrt{2m}\left[E - \frac{J^2}{2mr^2} - \frac{1}{2}kr^2\right]^{1/2}.$$

$$U_{eff} = \frac{J^2}{2mr^2} + \frac{1}{2}kr^2$$



$$p_r = \sqrt{2m}\left[E - U_{eff}(r)\right]^{1/2}.$$



In Cartesian coordinates, phase plots are ellipses for x & y.

(f) $$L = \frac{1}{2} m \dot{r}^2 + \underbrace{\frac{J}{2mr^2}}_{= \frac{1}{2} m r^2 \dot{\phi}^2} - \frac{1}{2} k r^2$$

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = m r^2 \dot{\phi}$$

$$H = p_r \dot{r} + p_\phi \dot{\phi} - L$$

$$= m\dot{r}^2 + m r^2 \dot{\phi}^2 - \left[\frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \dot{\phi}^2 - \frac{1}{2} k r^2 \right.$$

$$= \frac{p_r^2}{2m} + \frac{p_\phi^2}{2mr^2} + \frac{1}{2} k r^2$$

In Cartesian coordinates

$$p_x = m\dot{x}, \quad p_y = m\dot{y}$$

$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{1}{2}kx^2 + \frac{1}{2}ky^2$$

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$$\frac{\partial H}{\partial \phi} = 0 \Rightarrow \boxed{p_\phi = \text{Const} = mr^2\dot{\phi} = J}$$

as $\dot{p}_\phi = 0$

$$\boxed{\dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{mr^2} = \frac{J}{mr^2}}$$

$$\frac{\partial H}{\partial r} = -\frac{p_\phi^2}{mr^3} + kr$$

$$\boxed{\dot{p}_r = -\frac{\partial H}{\partial r} = \frac{p_\phi^2}{mr^3} - kr}$$

$$\boxed{\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m}}$$

In Cartesian Coordinates

$$\dot{p}_x = -\frac{\partial H}{\partial x} = -kx$$

$$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{p_x}{m}$$

$$\dot{p}_y = -ky$$

$$\dot{y} = \frac{p_y}{m}.$$