1. The energy of a system of N localized magnetic spins, at temperature T and in the presence of magnetic field H is given by

$$\mathcal{H} = D\sum_{i=1}^{N} S_i^2 - \mu_0 H \sum_{i=1}^{N} S_i,$$

where the parameters D, μ_0, H are positive and spin variables S_j may assume values ± 1 or 0, for $i = 1, 2, 3.....$ This problem has been solved in the micro-canonical and canonical ensemble. We want to look at it from the grand-canonical perspective.

   a) Calculate the grand-canonical partition function.

   b) Obtain an expression for the internal energy, the entropy and the magnetization. How does it compare to the earlier results?

2. Consider a classical ultra-relativistic gas of particles, given by the Hamiltonian

$$\mathcal{H} = \sum_{i=1}^{N} c|\mathbf{p}_i|,$$

where c is a positive constant, inside a container of volume V, in contact with a reservoir of heat and particles.

   a) Obtain an expression for the grand-canonical partition function and the grand thermodynamic potential.

   b) Calculate the pressure of this system and compare it with the result that we derived in the class.

3. Obtain the grand partition function of a classical system of particles, inside a container of volume V, given by the Hamiltonian

$$\mathcal{H} = \sum_{i=1}^{N} \left[\frac{\mathbf{p}_i^2}{2m} + U(\mathbf{r}_i) \right].$$

Calculate the pressure and energy of the system. Does the system obey ideal gas laws?

4. The grand partition function for a simplified statistical model is given by the expression

$$\mathcal{Q}(z, V) = (1 + z)^V (1 + z^{\alpha V}),$$

where α is a positive constant and $z = e^{\beta\mu}$.

   a) Write the parametric forms of the equation of state.

   b) Show that this system displays a first-order phase transition.

   c) Calculate the zeros of the polynomial $\mathcal{Q}(z, V)$ in the complex z-plane, and show that there is a zero at $z = 1$, in the limit of $V \to \infty$.

 Instructor: Dipanjan Chakraborty