REAL ANALYSIS

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Contents

1. Real numbers

In this section we define real number informally and study field and ordered structure of real number system. We know that following number systems:

- Set of Natural numbers $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$
- Set of Integers $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$
- Set of Rationals $\mathbb{Q} = \{\frac{m}{n} \ : \ m, n \in \mathbb{Z}, n \neq 0\}$
- Set of Reals $\mathbb{R}$ is the completion of $\mathbb{Q}$.

The set of rationals number system has certain gap, by filling them we get the real number system. To realize the gap in rational number system we will see the following:

Example 1 There is no rational p such that $p^2 = 2$.

Proof: Suppose there exists rational p such that $p^2 = 2$. Then p can be written as $p = \frac{m}{n}$ where m, n are integers which has no common factor. We see that $p^2 = 2$ implies $m^2 = 2n^2$ which implies m^2 is even. Hence m is even (if m were odd, m^2 would be odd).

Write $m = 2k$ for some integer k then $4k^2 = 2n^2$ which implies $2k^2 = n^2$, therefore n^2 is even and hence n is even.

This leads to the conclusion both m and n are even, contrary to our choice of m and n.

Hence the there is no rational p such that $p^2 = 2$.

We examine the situation a little more closely:

Example 2 Let $A = \{p \in \mathbb{Q} \ : p > 0, p^2 < 2\}$ and $B = \{p \in \mathbb{Q} \ : p > 0, p^2 > 2\}$ then A contains no largest number and B contains no smallest. Proof: It is enough to prove that any for $p \in A$, there exists another rational $q \in A$ such that $p < q$ and for every $p \in B$ there exists another rational $q \in B$ such that $q < p$.

To do this, we associate with each rational $p > 0$ the number

$$q = p - \frac{p^2 - 2}{p + 2} = \frac{2p + 2}{p + 2} \tag{1.1}$$

Then

$$q^2 - 2 = \frac{2(p^2 - 2)}{(p + 2)^2} \tag{1.2}$$

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If p is in A then $p^2 - 2 < 0$ (1.1) shows that $q > p$ and (1.2) show that $q^2 < 2$. Thus q is in A. If p is in B then $p^2 - 2 > 0$, (1.1) shows that $0 < q < p$ and (1.2) shows that $q^2 > 2$. Thus q is in B.

In view of Example 1 and Example 2, the rational number system has certain gaps. The real number system fills these gaps. This is the principal reason for the fundamental role it plays in analysis.

In order to understand the structure of real number system we start with a brief discussion of the general concepts of ordered set and field.

1.1. **Ordered Set.** Let S be a set. An order on S is a relation, denoted by $<$, with the following two properties:

(1) If $x \in S$ and $y \in S$ then one and only one of the statements

$$x < y, \quad x = y, \quad y < x$$

is true. (Law of trichotomy).

(2) If $x, y, z \in S$, if $x < y$ and $y < z$ then $x < z$. (Transitive law)

The statement "$x < y$" may be read as "x is less than y" or "x is smaller than y" or "x precedes y". It is often convenient to write $y > x$ in place of $x < y$. The notation $x \leq y$ indicates that $x < y$ or $x = y$, without specifying which of these two holds. In other words, $x \leq y$ is the negation of $x > y$.

Definition 1.1. *An ordered set is a set in which an order is defined.*

For example, $\mathbb{Q}$ is an ordered set with the order defined by $r < s$ if $s - r$ is a positive rational number. This order is called standard order on $\mathbb{Q}$.

1.2. **Bounded above and bounded below.** Suppose S is an ordered set and $E \subset S$. If there exists a $\beta \in S$ such that $x \leq \beta$ for every $x \in E$, we say that E is *bounded above* and call β an *upper bound* of E.

If there exists an $\alpha \in S$ such that $\alpha \leq x$ for every $x \in E$, we say that E is *bounded below* and call α a *lower bound* of E. We say that the set E is bounded if it is bounded above and bounded below.

Example: The set $\mathbb{N}$ of natural numbers is bounded below in $\mathbb{Q}$ with standard order and 0 is a lower bound for $\mathbb{N}$ but it is not bounded above.

Consider the set $\{\frac{1}{n} : n \in \mathbb{N}\}$, it is bounded in $\mathbb{Q}$.

Definition of lub (supremum) and glb (infimum)

Suppose S is an ordered set, $E \subset S$ and E is bounded above. Suppose there exists an $l \in S$ with the following properties:

(1) l is an upper bound of E.

(2) If $\gamma < l$ then γ is not an upper bound of E.

Then l is called the *least upper bound* (lub) of E or the ısupremum of E and we write

$$l = \sup E$$

Suppose S is an ordered set, $E \subset S$ and E is bounded below. Suppose there exists a $g \in S$ with the following properties:

(1) g is a lower bound of E.

(2) If $g < \gamma$ then γ is not an lower bound of E.

Then g is called the *greatest lower bound* (glb) of E or the *infimum* of E and we write

$$g = \inf E$$

Example 3: Consider the set $A = \{\frac{1}{n} \ : n \in \mathbb{N}\}$ in $\mathbb{Q}$, then the set A is bounded above and bounded below. 1 is an uppper bound and 0 is a lower bound. We can check that 0 is the infimum of A and 1 supremum of A.

Example 4: Let $B = \{1 - \frac{1}{n^2} \ : \ n \in \mathbb{N}\} \subset \mathbb{Q}$ with standard order. We can check that lub $B = 1$ and glb $B = 0$.

Example 5: Recall the sets A and B in Example 2: $A = \{p \in \mathbb{Q} \ : p > 0, p^2 < 2\}$ and $B = \{p \in \mathbb{Q} \ : p > 0, p^2 > 2\}$. As a subset of $\mathbb{Q}$, A is bounded above. In fact upper bounds of A are exactly the members of B. Since B has no the smallest member, A has no least upper bound in $\mathbb{Q}$. Similarly, B is bounded below: The set of all lower bounds of B consists of A and of $r \in \mathbb{Q}$ with $r \leq 0$. Since A has no largest member, B has no greatest lower bound in $\mathbb{Q}$.

Note that if $l = \sup E$ exists, then l may or may not be a member E. If $l = \sup E$ is the member of E, then l is called maximum of E.

Similarly $g = \inf E$ (if it exists) may or may not be a member of E. If $g = \inf E$ is the member of E then g is called minimum of E.

For the set $A = \{\frac{1}{n} \ : n \in \mathbb{N}\}$ in $\mathbb{Q}$, 1 is the maximum and 0 is the infimum of A. For the set $B = \{1 - \frac{1}{n^2} \ : \ n \in \mathbb{N}\} \subset \mathbb{Q}$, 1 is the supremum and 0 is the minimum of A.

1.3. **Least upper bound property.**

Definition 1.2. *An ordered set S is said to have the least upper bound property if the following is true: If $E \subset S, E \neq \phi$ and E is bounded above, then $\sup E$ exists in S.*

Example $\mathbb{Q}$ doesnot have the lub property.

Ques: What can one say about the set $\{p \in \mathbb{Z} : p^2 < 5\}$? Does this set has lub in $\mathbb{Z}$?

Theorem 1.3. *Suppose S is an ordered set with the lub property, $B \subset S, B \neq \phi$ and B is bounded below. Let L be the set of all lower bounds of B. Then*

$$\alpha = \sup L$$

exists in S and $\alpha = \inf B$. In particular, $\inf B$ exists in S.

Proof. Since B is bounded below, so $L \neq \phi$.By definition if $x \in L$, then $x \leq y$ for all $y \in B$. Therefore L is bounded above. By lub property of S, there exists $\alpha \in S$ such that $\alpha = \sup L$. Our claim is $\alpha = \inf B$. First we need to show that α is a lower bound. More precisely $\alpha \leq x$ for all $x \in B$. If not then there exists $z \in B$ such that $z < \alpha$. Since $\alpha = \sup L$ this implies that z is not an upper bound of L, i.e. there exists $x \in L$ such that $z < x$. This cannot hold as x is a lower bound of B. Therefore α is a lower bound of B

Secondly we need to show if $\gamma \in S$ such that $\gamma > \alpha$ then γ is not a lower bound of B. That follows from the fact that $\alpha = \sup L$. □

2. Fields

In this section we will first recall the definition of a Field.

Definition: A field is a set F with two operations , called addition and multiplication, satisfy the following axioms:

Axioms for addition:

a: If $x, y \in F$, then $x + y \in F$.
b: For all $x, y \in F$, $x + y = y + x$.
c: $(x + y) + z = x + (y + z)$ for all $x, y, z \in F$.
d: There exists $0 \in F$ (called additive identity) such that $x + 0 = x = 0 + x$ for all $x \in F$.

e: To every $x \in F$, there exists $-x \in F$ (called additive inverse) such that

$$x + (-x) = (-x) + x = 0$$

.

F also satisfies the same axioms for the multiplication operation. We denote multiplicative identity by $1 \neq 0$. The point e) holds for every $x \neq 0$ in F for the multiplication operation. We denote the multiplicative inverse of $x \neq 0$ by $\frac{1}{x}$.

F also satisfies the distributive law.

$$x(y + z) = xy + xz.$$

Example 1 The set $\mathbb{Q}$ is a Field under standard addition and multiplication operation

Definition 2.1. *An ordered field is a field F which is also an ordered set, such that*

- *if $y < z$ then $x + y < x + z$ for all $x, y, z \in F$,*
- *$xy > 0$ if $x, y > 0$.*

The following properties hold in every ordered field:

- If $x > 0$, then $-x < 0$ and vice versa.
- If $x > 0$ and $y < z$, then $xy < xz$.
- If $x < 0$ and $y < z$, then $xy > xz$.
- If $x \neq 0$ then $x^2 > 0$. In particular $1 > 0$.
- If $0 < x < y$ then $0 < \frac{1}{y} < \frac{1}{x}$.

Theorem 2.2. *There exists an ordered field $\mathbb{R}$ which has the lub property. Moreover $\mathbb{R}$ contains $\mathbb{Q}$ as a subfield.*

$\mathbb{Q}$ is a subfield of $\mathbb{R}$ means that $\mathbb{Q} \subset \mathbb{R}$ and the operations of addition and multiplication in $\mathbb{R}$ when applied to $\mathbb{Q}$ coincide with the usual addition and multiplication in $\mathbb{Q}$. The members of $\mathbb{R}$ are called the *real numbers*. We will assume the above theorem for the time being.

3. Archimedean Property

Theorem 3.1. (i): *(Archimedean Property)If $x \in \mathbb{R}$, $y \in \mathbb{R}$ and $x > 0$, then there exists a $n \in \mathbb{N}$ such that $nx > y$ where $nx = x + x...(ntimes) + x$.*

(ii): *If $x, y \in \mathbb{R}$ and $x < y$, then there exists a $p \in \mathbb{Q}$ such that $x < p < y$.*

Proof. (i) WLOG we can assume $y > 0$ and $y > x$. Suppose it doesn't hold. Then y is an upper bound of the set $A = \{nx : n \in \mathbb{N}\}$. $A \neq \phi$ and is bounded above. Therefore by lub property of $\mathbb{R}$ there exists $\alpha \in \mathbb{R}$ such that $\alpha = \sup A$. As $x > 0$ we have that $\alpha - x < \alpha$. There fore $\alpha - x$ is not an upper bound of A. So there exists $m \in \mathbb{N}$ such that $\alpha - x < mx$, i.e. $\alpha < (m + 1)x$.

(ii) According to given $y - x > 0$. By (1) we know that given 1 and $y - x > 0$ there exists $n \in \mathbb{N}$ such that $n(y - x) > 1$, i.e. $y - x > \frac{1}{n}$. Fix nx. We apply (1) again to obtain m_1 and m_2 such that $nx < m_1$ (for $1, nx$) and $-nx < m_2$ (for $1, -nx$). This implies that

$$-m_2 < nx < m_1.$$

We can find an integer m such that $m - 1 \leq nx < m$. Since $ny > nx + 1$ we get that $m < 1 + nx < ny$. Combining the above relations we get

$$x < \frac{m}{n} < y.$$

□

We observed in the beginning that solution of the equation $x^2 = 2$ in $\mathbb{Q}$ does not exist. However for $\mathbb{R}$ this is not the case.

Theorem 3.2. *For every $x > 0$ and $n \in \mathbb{N}$ there is one and only one real number $y > 0$ such that*
$$y^n = x.$$

We will denote y by $x^{1/n}$.

Proof. Let us prove the uniqueness part first. Let there exist $0 < y_1 < y_2$ such that both satisfy $y_i^n = x$ for $i = 1, 2$. As $\mathbb{R}$ is an ordered field, we get that $y_1^n < y_2^n$ which gives a contradiction.

Let us now prove the existence part. Let
$$E = \{t \in \mathbb{R} : t > 0 \text{ and } t^n < x\}.$$
We will first show that $E \neq \phi$. Define $y = \frac{x}{1+x}$. Clearly $0 < y < 1$ and $y < x$. Therefore $y^n \leq y < x$. This implies that $y \in E$.

Now we will show that E is bounded above. If $z > 1 + x$, this implies that $z^n \geq z > x$. So $z \notin E$. This implies that for all $t \in E$, $t \leq 1 + x$. Therefore E is bounded above. By lub property of $\mathbb{R}$, we know that there exists $y \in \mathbb{R}$ such that $y = \sup E$.

Claim: $y^n = x$.

If not then $y^n < x$ or $y^n > x$. We will show that both these inequalities lead to a contradiction. Let $b > a > 0$. It is easy to prove that
$$b^n - a^n = (b - a)\left(\sum_{j=0}^{n-1} b^{n-1-j}a^j\right).$$
The above equality gives the inequality
$$b^n - a^n < nb^{n-1}(b - a). \tag{3.1}$$
Assume that $y^n < x$. Choose $0 < h < 1$ (use part (*ii*) of Theorem 3.1) such that
$$h < \frac{x - y^n}{n(y + 1)^{n-1}}.$$
Put $a = y$ and $b = y + h$ and apply (3.1), we get
$$(y + h)^n - y^n < hn(y + h)^{n-1}.$$
By our choice of h we have
$$(y + h)^n - y^n < \frac{x - y^n}{(y + 1)^{n-1}}(y + h)^{n-1}.$$
Finally we get $(y + h)^n - y^n < x - y^n$, which implies that $(y + h)^n < x$. This means that $y + h \in E$. But y is an upper bound of E and $y + h > y$ which gives a contradiction.

Let us now assume $x < y^n$. Put
$$k = \frac{y^n - x}{ny^{n-1}}. \tag{3.2}$$

It is easy to check that $0 < k < y$. If $t \geq y - k$, we get
$$y^n - t^n \leq y^n - (y - k)^n < kny^{n-1}.$$
By our choice of k we get $y^n - t^n < y^n - x$, which implies $t^n > x$. Therefore $y - k$ is an upper bound of E. We know that y is lub of E and $y - k < y$ which is a contradiction.

□

3.1. **Complex Field.**

Definition 3.3. *A complex number is the ordered pair (a, b) of real numbers. Call this set $\mathbb{C}$. Here ordered means that (a, b) and (b, a) are distinct.*

We define the addition as

$$(a, b) + (c, d) = (a + b, c + d)$$

and multiplication as

$$(a, b)(c, d) = (ac - bd, bc + ad).$$

Exercise 1 Check that the above operations of addition and multiplication make $\mathbb{C}$ into a field. The additive identity is given by $(0, 0)$ and multiplicative identity is given by $(1, 0)$. Find the multiplicative inverse of a non-zero complex number (non-zero means $\mathbb{C} \setminus \{0\}$).
Exercise 2 $\mathbb{C}$ cannot be made into an ordered field.

Define $i = (0, 1)$ and $1 = (1, 0)$. It is easy to check that $i^2 = -1$ in the above notation. We can write every element of $\mathbb{C}$ as $a + ib$ for $a, b \in \mathbb{R}$.

Definition 3.4. *Let $z = a + ib, a, b \in \mathbb{R}$. We call $a = Re(z)$ and $b = Im(z)$. We define conjugate of z as $\bar{z} = a - ib$.*

Let $z, w \in \mathbb{C}$. Let 0 be the additive identity of $\mathbb{C}$.
By using the above definitions we have

- $\overline{z + w} = \bar{z} + \bar{w}$.
- $\overline{zw} = \bar{z}\bar{w}$.
- $z + \bar{z} = 2a$ and $z - \bar{z} = 2ib$.
- $z\bar{z} = a^2 + b^2$ which is strictly positive real number if and only if $z \neq 0$.

Definition 3.5. *For $z \in \mathbb{C}$, we define its absolute value $|z|$ as the positive square root of $z\bar{z}$, i.e. $|z| = (z\bar{z})^{1/2}$.*

Note that when x is real, then $= \bar{x}$ and $|x| = \sqrt{x^2}$. Thus $|x| = x$ when $x \geq 0$ and $|x| = -x$ when $x < 0$.

Theorem 3.6. *Let $z, w \in \mathbb{C}$. Then*
a: $|z| > 0$ *unless* $z = 0$,
b: $|\bar{z}| = |z|$,
c: $|zw| = |z||w|$,
d: $|Re(z)| \leq |z|$,
e: $|z + w| \leq |z| + |w|$.

4. Sequences

We will recall and study some properties of sequences of real numbers. A sequence of real numbers is a countable collection of real numbers denoted by $\{a_n\}_{n \in \mathbb{N}}, a_n \in \mathbb{R}$.

- We say that $\{a_n\}$ is a bounded sequence if there exists some $M > 0$ such that $|a_n| \leq M$ for all $n \in \mathbb{N}$.
- $\{a_n\}$ is a monotonically increasing (decreasing) sequence if $a_1 \leq a_2 \leq a_3 \leq ...$ $(a_1 \geq a_2 \geq a_3 \geq ...)$.
- We say $\{a_n\}$ is convergent in $\mathbb{R}$ if there exists $a \in \mathbb{R}$ and given $\epsilon > 0$ there exists N such that $|a_n - a| < \epsilon$ for all $n \geq N$. We also write $\lim_{n \to \infty} a_n = a$.
- $\{a_n\}$ is Cauchy if given $\epsilon > 0$ there exists $N > 0$ such that $|a_n - a_m| < \epsilon$ for all $n, m \geq N$.

- We say $\{a_n\}$ diverges to ∞ if given any $M > 0$ there exists N such that $a_n \geq M$ for all $n \geq N$.

Example 1 The sequence $a_n = (-1)^n$ is not Cauchy.

Example 2 Define $a_n = 1 + \frac{(-1)^n}{n}$. $\lim_{n\to\infty} a_n = 1$. **Example 3** The sequence $a_n = 2^n$ diverges to ∞.

Exercise 3 Show that if a bounded sequence $\{a_n\}$ is monotonically increasing (decreasing) then $\lim_{n\infty} a_n = \sup_{n\in\mathbb{N}} a_n (\inf_{n\in\mathbb{N}} a_n)$.

Let n_k be a strictly increasing sequence of natural numbers. Given a sequence $\{a_n\}_{n\in\mathbb{N}}$ and $\{n_k\}_{k\in\mathbb{N}}$ we say $\{a_{n_k}\}_{k\in\mathbb{N}}$ is a subsequence of $\{a_n\}_{n\in\mathbb{N}}$

Theorem 4.1. *Every bounded sequence in $\mathbb{R}$ has a convergent subsequence in $\mathbb{R}$.*

Proof. From Exercise 3 above it is enough to show that any sequence in $\mathbb{R}$ has a monotonic subsequence. Call a natural number n a peak point if $a_n \geq a_k$ for all $n \leq k$. There are two possibilities:

(1) Suppose first that n is frequently a peak point, i.e. there exists $n_1 < n_2 < n_3$... such that $a_{n_1} \geq a_{n_2} \geq a_{n_3} \geq$ .... Hence we get a monotonically decreasing subsequence.

(2) There is some N such that $n \geq N$ is not a peak point, i.e. when $n_1 = N$ there exists $n_2 > n_1$ such that $a_{n_2} > a_{n_1}$. Similarly for n_2 there exists $n_3 > n_2$ such that $a_{n_3} > a_{n_2}$. Proceeding in this manner we get an increasing subsequence.

□