PHY306 Advanced Quantum Mechanics Jan-Apr 2025: EndSemester Exam

Max Marks=50 Time=03 Hours Dated May 6, 2025

1. (a) Consider an infinite 1D potential well of length L with walls at $x = 0$ and $x = L$ with the potential being given by

$$\begin{aligned} V(x) &= 0, \quad 0 < x < L \\ &= \infty, \quad \text{elsewhere} \end{aligned}$$

It is modified at the bottom by a perturbation $V_p(x)$

$$\begin{aligned} V_p(x) &= V_0, \quad 0 \leq x \leq L/2 \\ &= 0, \quad \text{elsewhere} \end{aligned}$$

where $V_0 << 1$. Calculate the energy E_n using first-order perturbation theory.

[Marks=02]

(b) Consider a particle in a 2D potential well of width a:

$$\begin{aligned} V(x, y) &= 0, \quad 0 \leq x \leq a \text{ and } 0 \leq y \leq a \\ &= \infty, \quad \text{elsewhere} \end{aligned}$$

A perturbation is added given by

$$\begin{aligned} W(x, y) &= w_0, \quad 0 \leq x \leq a/2 \text{ and } 0 \leq y \leq a/2 \\ &= \infty, \quad \text{elsewhere} \end{aligned}$$

Calculate to lowest non-vanishing order in w_0, the energy of the first excited state.

[Marks=03]

2. An electron is at rest at the origin, in the presence of a magnetic field whose magnitude B_0 is constant but whose direction rides around at a constant angular velocity ω on the lip of a cone of opening angle α such that

$$B(t) = B_0[\sin\alpha\cos\omega t\hat{i} + \sin\alpha\sin\omega t\hat{j} + \cos\alpha\hat{k}]$$

Use time-dependent perturbation theory (to first order) to calculate the probability of a transition from spin up (initial state) to spin down, as a function of time.

[Marks=05]

3. (a) Draw a diagram showing the splitting of the $np_{3/2}$ and $np_{1/2}$ levels of a hydrogen atom placed in a weak magnetic field (the anomalous Zeeman effect).

[Marks=02]

(b) Consider the first-order change in the energy levels of a hydrogen atom due to an external electric field of strength E along the z-axis (Stark effect). Calculate the effect for the four degenerate levels for $n = 2$.

[Marks=03]

4. (a) Calculate the first order correction to the energy E^1_{hf} (the hyperfine splitting) for the ground state of the hydrogen atom, due to the magnetic field arising from the proton's dipole moment (You need not put in any numerical values). How many levels does the unperturbed level split into?

[Marks=02]

(b) Write the Clebsch-Gordan coefficient matrix for the addition of two angular momenta $j_1 = 1$ and $j_2 = 1/2$.

[Marks=03]

5. A particle of mass m is in a potential well $V = (1/2)m\omega^2 x^2$. Use the variational method and a normalized trial wave function $(1/\sqrt{a})\cos(\pi x/2a)$ in the limits $-a < x < a$, to find the best value of a.

[Marks=05]

6. Consider bound states ($E < V(0)$) of a symmetrical double-well potential. Consider two turning points on the positive x-axis, x_1, x_2 where $x_1 < x_2$. In the region $0 < x < x_1$, the WKB wave function is given to you:

$$\begin{aligned}\psi(x) \;\approx\; & \frac{D}{\sqrt{|p(x)|}} \\ & [2\cos\theta e^{\frac{1}{\hbar}\int_x^{x_1}|p(x')|dx'} \\ + \; & \sin\theta e^{-\frac{1}{\hbar}\int_x^{x_1}|p(x')|dx'}]\end{aligned}$$

where

$$\theta \equiv \frac{1}{\hbar}\int_{x_1}^{x_2} p(x)dx$$

Consider only even (+) and odd (-) wave functions and impose the appropriate bounds on $\psi(0)$ and $\psi'(0)$ and find the quantization condition on θ in terms of ϕ where

$$\phi \equiv \frac{1}{\hbar}\int_{-x_1}^{x_1} p(x')dx'$$

[Marks=05]

7. (a) Use the one-dimensional Born approximation to compute the transmission coefficient ($T = 1 - R$) for scattering from a finite square well potential

$$\begin{aligned} V(x) &= -V_0, \quad -a < x < a \\ &= 0, \text{ otherwise}\end{aligned}$$

[Marks=05]

(b) For the case of low-energy scattering from a spherical delta-function shell, $V(r) = \alpha\delta(r-a)$, where α, a are constants. Assume $ka << 1$ so only $l = 0$ term contributes significantly. Use partial wave analysis to calculate the total cross-section σ and express the answer in terms of the dimensionless quantity $\phi = 2ma\alpha/\hbar^2$.

[Marks=05]

4. (a) For a nonrelativistic electron of velocity v, use the two-component spinor solutions of the Dirac equation u_A, u_B to show that u_A is larger than u_B by a factor of the order of v/c.

[Marks=05]

(b) Construct the normalized Dirac spinors u^+ and u^- representing an electron of momentum $\mathbf{p}$ and with helicity ± 1, where the helicity operator is given by $(\hat{p}.\Sigma)$.

[Marks=05]

Useful Formulae

$$R_{10}(r) = 2(\frac{1}{a_0})^{3/2} e^{-r/a_0};$$

$$R_{20}(r) = (2 - r/a_0)(\frac{1}{2a_0})^{3/2} e^{-r/2a_0};$$

$$R_{21}(r) = (\frac{1}{2a_0})^{3/2} \frac{r}{\sqrt{3}a_0} e^{-r/2a_0};$$

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

$$\int_0^{\infty} x^n e^{-ax} dx = \frac{n!}{a^{n+1}}$$

$$\int_0^{\infty} x^{2n+1} e^{-ax^2} dx = \frac{n!}{2a^{n+1}}$$

$$Y_0^0(\theta, \phi) = \sqrt{\frac{1}{4\pi}}; \quad Y_1^0(\theta, \phi) = \frac{1}{2}\sqrt{\frac{3}{\pi}} \cos\theta = \frac{1}{2}\sqrt{\frac{3}{\pi}} \frac{z}{r}$$

$$Y_1^{\pm 1}(\theta, \phi) = \mp\frac{1}{2}\sqrt{\frac{3}{2\pi}} \sin\theta e^{\pm i\phi} = \mp\frac{1}{2}\sqrt{\frac{3}{2\pi}} \frac{x \pm iy}{r};$$

$$\vec{L} = \frac{\hbar}{i}\left(\hat{\phi}\frac{\partial}{\partial\theta} - \hat{\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right)$$

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial f}{\partial r}) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial f}{\partial\theta}) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}$$