Assignment 7

PHY310: Mathematical Methods for Physicists I

Instructor:Dr. Prasenjit Das **No Need to Submit**

1. Show that
$$L_n(x) = \frac{e^x}{n!}\int_0^\infty e^{-t}t^n J_0\left[2(xt)^{1/2}\right]dt.$$

2. Show that
$$\int_0^\infty e^{-\alpha x}J_n(x)dx = \frac{1}{\sqrt{1+\alpha^2}}\left[\sqrt{1+\alpha^2}-\alpha\right]^n.$$

3. Show that
$$\int_0^a xJ_n^2(x)dx = \frac{1}{2}a^2J_n^2(a)\left[1-\frac{J_{n-1}(a)J_{n+1}(a)}{J_n^2(a)}\right].$$

4. If a_1, a_2, a_3, $\cdots\cdots$ are the roots of $J_0(x)$, show that
$$\sum_{i=1}^\infty \frac{2J_0(a_i x)}{a_i J_1(a_i)} = 1.$$

5. Prove that $x^2J_n''(x) = (n^2-n-x^2)J_n(x) + xJ_{n+1}(x)$.

6. Show that
$$\int x^2J_0(x)J_1(x)dx = \frac{x^2}{2}J_0'(x) + C.$$

7. If the r^{th} derivative of $J_n(x) = J_n^{(r)}(x)$, prove that
$$2^rJ_n^{(r)}(x) = J_{n-r}(x) - rJ_{n-r+2}(x) + \frac{r(r-1)}{2!}J_{n-r+4}(x) + \cdots\cdots + (-1)^rJ_{n+r}(x) + \cdots\cdots$$

8. Show that
$$\frac{x}{2}J_{n-1}(x) = nJ_n(x) - (n+2)J_{n+2}(x) + (n+4)J_{n+4}(x) + \cdots.$$

9. If $n > -1$, show that
$$\int_0^x x^{-n}J_{n+1}(x)dx = \frac{1}{2^n\Gamma(n+1)} - x^{-n}J_n(x).$$

10. Show that $[J_0(x)]^2 + 2[J_1(x)]^2 + 2[J_2(x)]^2 + \cdots = 1$.

11. Prove that the Neumann functions N_n (n is an integer) satisfy the recurrence relations:

(a) $N_{n-1}(x) + N_{n+1}(x) + = \frac{2n}{x} N_n(x)$.
(b) $N_{n-1}(x) - N_{n+1}(x) + = 2N_n'(x)$.

12. Show that

$$\sin x = 2 \sum_{n=0}^{\infty} (-1)^n J_{2n+1}(x).$$

13. Prove by mathematical induction that for n an arbitrary nonnegative integer,

$$j_n(x) = (-x)^n \left(\frac{1}{x} \frac{d}{dx} \right)^n \left(\frac{\sin x}{x} \right).$$

14. Show that

$$\int_{-\infty}^{+\infty} j_m(x) j_n(x) dx = \begin{cases} 0 & m \neq n \quad \text{and} \quad m, n \geq 0 \\ \frac{\pi}{2n+1} & m \neq n \quad \text{and} \quad m, n \geq 0 \end{cases}$$

15. Using a generating function $g(x,t) = g(u+v,t) = g(u,t)g(v,t)$, show that

(a)

$$J_n(u+v) = \sum_{s=-\infty}^{+\infty} J_s(u) \cdot J_{n-s}(v).$$

(b)

$$J_0(u+v) = J_0(u) \cdot J_0(v) + 2 \sum_{s=1}^{+\infty} J_s(u) \cdot J_{-s}(v).$$

16. For $x \to 0$, show that

(a)

$$j_n(x) \sim \frac{x^n}{(2n+1)!!}.$$

(b)

$$y_n(x) \sim -\frac{(2n-1)!!}{x^{n+1}}.$$

17. If $f(x)$ is defined in the region $0 \leq x \leq a$ and can be expanded in the form $\sum_{i=1}^{\infty} c_i J_n(\xi_i x)$ where ξ_i are roots of the equation $J_n(\xi a) = 0$, then show that

$$c_i = \frac{2 \int_0^a x f(x) J_n(\xi_i x) dx}{a^2 \left\{ J_{n+1}(\xi_i a) \right\}^2}.$$

18. If ξ_i are the solutions of the equation $J_0(\xi) = 0$, show that in $0 < x1$

$$\sum_{i=1}^{\infty} \frac{J_0(\xi_i x)}{\xi_i^2 \left\{ J_1(\xi_i) \right\}^2} = -\frac{1}{2} \ln x.$$