INDIAN INSTITUTE OF SCIENCE EDUCATION AND RESEARCH MOHALI

FINAL EXAM

MTH 102 - ANALYSIS IN ONE VARIABLE

Date: Aug 02, 2022

Time: 3 hours

Important Instructions.

(1) **Answer all the questions. Total points = 40. Numbers in the brackets indicate points for the questions.**
(2) **Write answers to all the parts of a question at the same place.**
(3) **Cancel any multiple attempts/answers to solving any question (or its part). Only one attempt/answer to a question will be graded.**
(4) **All answers must be supported by explanations. Only correct answers would not bring any points.**
(5) *Use blue or black ink pen only to write you answers.*
(6) *Draw figures if needed to answer any question.*
(7) *Students are suggested not to cross or erase any answer which may be potentially wrong. There are partial credits for most of the questions. Writing some rough ideas is better than writing nothing.*

Questions:

(1) (a) Suppose $S \subset \mathbb{R}$ is a bounded set. Write down the definitions of $\sup S$ and $\inf S$. [**1**]

(b) Suppose $S \subset \mathbb{R}$ is bounded below. If $\inf S \notin S$ then show that there is a sequence of numbers $\{x_n\}$ in S such that $\lim_{n\to\infty} x_n = \inf S$. [**1**]

(c) Show that $\lim_{n\to\infty} \frac{n^2}{2^n} = 0$. [**3**]

(2) (a) Suppose $\sum a_n$ is a convergent series. Show that $\lim a_n = 0$. [**1**]

(b) Check if the series $\sum(-1)^n n$ is convergent. [**1**]

(c) Show that the series $\sum \frac{1}{n!}$ is convergent. [**3**]

(3) (a) Suppose $P(x) = (x-a_1)(x-a_2)\cdots(x-a_n)$ where $a_i \in \mathbb{R}$. Suppose $f : \mathbb{R} \to \mathbb{R}$ is defined as follows: $f(x) = P(x)$ for $x \neq a_i$, $1 \leq i \leq n$ and $f(a_1) = f(a_2) = \cdots = f(a_n) = 1$. Show that f is discontinuous at all a_i and continuous everywhere else. [**3**]

(b) Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous functions such that $f(x) = x$ for all $x \in \mathbb{Q}$. Find $f(\sqrt{2})$ with proper justification. [**2**]

(4) (a) Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $f(x+1) = f(x)$ for all $x \in \mathbb{R}$. Show that f is uniformly continuous on $\mathbb{R}$. [3]

(b) Show that $2 + x = 2^x$ has at least two solutions. [2]

(5) (a) Suppose $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) - f(y) \leq (x-y)^2$ for all $x, y \in \mathbb{R}$. Find f with proper justification. [3]

(b) Suppose f is a differentiable function on $\mathbb{R}$ such that $f(0) = 0$ and $f(1) = 1$. Show that there exists $x_0 \in (0,1)$ such that $f'(x_0) = 2x_0$. [2]

(6) (a) Use the power series expansion of a simpler function to find a power series expansion in x (centered at 0) that has the function $f(x) = \frac{4}{(2-x)^2}$ as the sum. Give the exact interval where f equals the power series with proper justification. [3]

(b) Using Taylor's theorem show that the Taylor series of $f(x) = \ln(1+x)$ centered at 0 converges to $\ln 2$ at $x = 1$. Deduce a series expansion for $\ln 2$. You may use: $f^{(n)}(x) = \frac{(-1)^{n+1}(n-1)!}{(1+x)^n}$ for $n \geq 1$. [2]

(7) (a) Suppose f and g are continuous functions on $[a,b]$ such that $\int_a^b f = \int_a^b g$. Prove that there exists $c \in (a,b)$ such that $f(c) = g(c)$. [3]

(b) Find the limit with justification: $\lim_{x\to 0} \frac{1}{1-\cos(x)} \int_0^{\tan(x^2)} e^{(1+t^2)}\, dt$. [2]

(8) (a) Suppose f is an integrable function on $[0,1]$. Prove that the sequence $a_n = \sum_{k=1}^n f(\frac{k^2}{n^2})(\frac{2k-1}{n^2})$ converges as $n \to \infty$, and determine this limit. [3]

(b) Without using the Fundamental Theorem of Calculus, find $\int_0^1 \sqrt{x}\, dx$. [2]