Indian Institute of Science Education and Research Mohali

MTH101 : Linear Algebra (2023-24) **Tutorial 07 (November 09, 2023)**

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1. Let $(V, *, \bullet)$ be a vector space over real numbers. Let $n \in V$ be the neutral element of $(V, *)$. Show that the following properties hold.
   (a) $\alpha \bullet n = n$ for every $\alpha \in \mathbb{R}$.
   (b) $-1 \bullet v$ is equal to the inverse of v in $(V, *)$.
   (c) If $\alpha \bullet v = n$ for some $\alpha \in \mathbb{R}$ and some $v \in V$, then show that either $\alpha = 0$ or $v = n$.
   (d) Show that if $\alpha \neq 0$ and $v, w \in V$ are such that $\alpha v = \alpha w$, then $v = w$.

2. Let $(V, *, \bullet)$ be a vector space over reals. Let S_1 and S_2 be two subspaces of $(V, *, \bullet)$. Show that $S_1 \cap S_2$ is also subspace. What about $S_1 \cup S_2$?

3. Which of the following are subspaces of $(\mathbb{R}^3, +, \cdot)$? Here, $+$ is the addition of vectors in $\mathbb{R}^3$ and $\cdot$ is scaling of elements $\mathbb{R}^3$ by reals.
   (a) $S := \left\{ \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \in \mathbb{R}^3 : a_1 + a_2 = a_3 \right\}$.
   (b) $S := \left\{ \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \in \mathbb{R}^3 : \sin(a_1 + a_2) = \sin(a_1) + \sin(a_2) = \sin(a_3) \right\}$.
   (c) $S := \left\{ \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \in \mathbb{R}^3 : a_1 a_2 + a_2 a_3 + a_3 a_1 = 0 \right\}$.

4. Find all $\lambda \in \mathbb{R}$ such that $S := \left\{ \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \in \mathbb{R}^3 : a_1 + 2a_2 + 3a_3 + \lambda = 0 \right\}$ is a subspace of $(\mathbb{R}^3, +, \cdot)$.

5. Consider the vector space $(M_{n\times n}(\mathbb{R}), +, \cdot)$, where $+$ is the binary operation of matrix addition and $\cdot$ is the entrywise scaling of a matrix. Let $E := \{e_{ij} : 1 \leq i, j \leq n\}$ denote the set of matrix units in $M_{n\times n}(\mathbb{R})$. Show that $\mathrm{span}(E) = M_{n\times n}(\mathbb{R})$. Further, show that if $A, B \in E$ and $\alpha A + \beta B = 0$, then $\alpha = \beta = 0$.

6. Consider the vector space $(\mathbb{R}[x], +, \cdot)$. If $S := \{x^n + x^m : 1 \leq n, m \leq 2\} \subseteq \mathbb{R}[x]$, then
   (a) How many elements are there in the set S?
   (b) What is $\mathrm{span}(S)$?
   (c) If $S = \{x^n + x^m : n, m \text{ are non-negative integers}\}$, then is it true that $\mathrm{span}(S) = \mathbb{R}[x]$?
