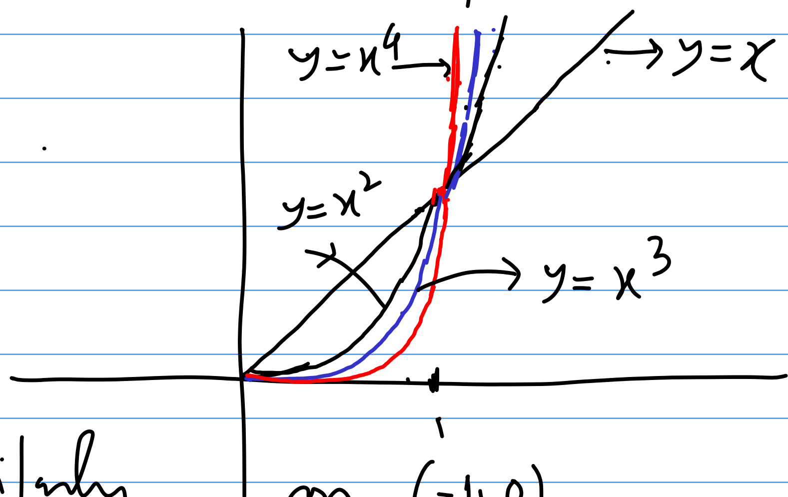
# Solution to HW6

1) On $(0,1)$

$$x > x^2 > x^3 > x^4$$

On $(1,\infty)$ $x < x^2 < x^3 < x^4$

Hence the graphs for $x \geq 0$ look as below.




Similarly on $(-1,0)$

$$x < x^3 < x^4 < x^2$$

on $(-\infty,-1)$

$$x^3 < x < x^2 < x^4$$

and we use them to draw the graph for $x \leq 0$.

Note: All the four curves go through $(0,0)$, $(1,1)$.

2. Note: For any sequence $x_n \to 0$

$$1/x_n^2 \to \infty \Rightarrow 2^{1/x_n^2} \to \infty$$

$$\Rightarrow \quad 2^{-1/x_n^2} = 1/2^{1/x_n^2} \to 0 = f(0)$$

Recall: If $\forall$ sequence $\{x_n\}$ in $I$,
$x_n \to a \Rightarrow f(x_n) \to f(a)$ where $f: I \to \mathbb{R}$
then $f$ is continuous at $a$.

Thus the given function is continuous at $x=0$.

If $x \neq 0$ then $2^{1/x^2}$ is the composition of $g(x) = 2^x$ and $h(x) = 1/x^2$ which are both continuous on $(-\infty, 0)$ and on $(0, \infty)$. Thus $2^{1/x^2}$ is continuous on both $(0, \infty)$ and $(-\infty, 0)$.

Thus the statement follows.

3. Let $g(x) = 2^x$, $h(x) = x^2$

You may ignore this!

Note that ① if $x_n \to -\infty$ then $x_n^2 \to \infty$ but $2^{x_n} = \frac{1}{2^{-x_n}} \to 0$

② $x^2, 2^x$ are both increasing on $[0, \infty)$

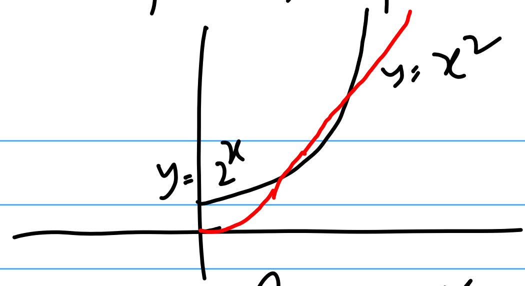
③ $\lim \frac{n^2}{2^n} = 0$

Ex. Using this one can check $\lim_{x \to \infty} \frac{x^2}{2^x} = 0$

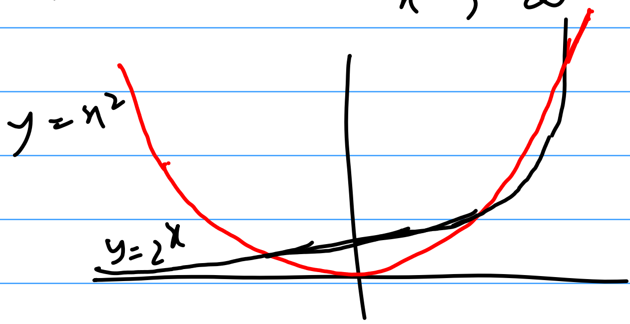
i.e. $\forall \varepsilon > 0$ $\exists N$ such that $\frac{x^2}{2^x} < \varepsilon$ $\forall x \geq N$.

In particular for large $x$, $x^2 < 2^x$

One requires derivative to show that $x^2 < 2^x$ $\forall x > 4$. Also $x^2 = 2^x$ at $x = 2, 4$. The graphs are follows for $x \geq 0$




For $x \leq 0$ : $\lim_{x \to -\infty} 2^x = 0$




4) $f(x) = \sqrt{x}$ Follow the proof done in class that $x^a$ is continuous $\forall a > 0$

Note: i) $x^a$ is defined on $[0, \infty)$

ii) To check continuity at $x = 0$ one needs to verify $\lim_{x \to 0} \sqrt{x} = 0$

However, in this case the proof is much simpler.

To show $\lim_{x \to a} f(x) = f(a)$ , $a > 0$:

This is equivalent to $\lim_{h \to 0} [f(a+h) - f(a)] = 0$

But $|f(a+h) - f(a)|$

$$= |\sqrt{a+h} - \sqrt{a}|$$

$$= \left|\frac{h}{\sqrt{a} + \sqrt{a+h}}\right| < \frac{h}{\sqrt{a}}$$

Hence, given $\varepsilon > 0$ choose $\delta = \sqrt{a}\,\varepsilon$.

T.S. $\lim_{h \to 0} \sqrt{h} = 0$. Given $\varepsilon > 0$ choose $\delta = \varepsilon^2$.

5. $f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ 1 & \text{if } x \notin \mathbb{Q} \end{cases}$

Given any $a \in \mathbb{R}$ $\exists$ a sequence $\{x_n\}$ with $x_n \to a$ & $x_n \in \mathbb{Q}$ $\forall n$

Thus if $f$ were continuous then $f(a) = \lim f(x_n) = 0$.

Also $\exists$ a sequence $\{y_n\}$ of irrational no.s with $\lim y_n = a$ whence

$$f(a) = \lim f(y_n) = 1$$

This shows that $f$ is discontinuous at $a$.

6. Use the sequential criteria of continuity as before. for $a \neq 0$. Use $\varepsilon$-$\delta$ definition for $a = 0$.

7. Let $f(x) = \begin{cases} (x-1)\cdots(x-n) & x \in \mathbb{Q} \\ 0 & \text{else} \end{cases}$

Check that $f(x)$ does the job.

8. (i) As in (5) use sequential criteria and that $\forall a \in \mathbb{R}$ $\exists$ a sequence of rational no.s $\{x_n\}$ with $x_n \to a$.

8. (ii) See below.

9. Claim 1. $\forall a \in \mathbb{R} \setminus \mathbb{Q}$, $\lim_{x \to a} f(x) = 0$

(i) Suppose $\{x_n\}$ is a sequence of irrational no.s and $x_n \to a$. Then $f(x_n) = 0 = f(a)$ $\forall n$. So $f(x_n) \to f(a)$.

(ii) If $\left\{\frac{p_n}{q_n}\right\}$ is a sequence of rational no.s where $q_n \in \mathbb{N}$, $p_n \in \mathbb{Z}$ are coprime (unless $p_n = 0$) and $p_n/q_n \to a$ then we claim that $\frac{1}{q_n} \to f(a) = 0$.

Given $\varepsilon > 0$ consider the set $S_\varepsilon$ of rational no.s of the form $\frac{p}{q}$ where $p \in \mathbb{Z}$, $q \in \mathbb{N}$, $p, q$ coprime

Note: $f(0) = f\left(\frac{0}{1}\right) = 1$. In the same way $f(n) = 1 \ \forall n \in \mathbb{Z}$

and $\frac{1}{q} \geq \varepsilon$.

Given $x \neq y \in S$, $|x-y| \geq \frac{1}{q^2}$ (check)

Hence in the sequence $\left\{\frac{p_n}{q_n}\right\}$ only finitely many terms are from $S_\varepsilon$. (check)

Hence, $\exists N$ such that $\forall n \geq N$, $\frac{p_n}{q_n} \notin S_\varepsilon$

$\Rightarrow \quad \frac{1}{q_n} < \varepsilon$

$$\Rightarrow \quad f\left(\frac{p_n}{q_n}\right) = \left|f\left(\frac{p_n}{q_n}\right) - f(a)\right| < \varepsilon$$

(iii) Using (i) and (ii) we claim that for any sequence $a_n \to a$, $f(a_n) \to f(a)$

Let $A = \{n \in \mathbb{N} : a_n \in \mathbb{Q}\}$

and $B = \{n \in \mathbb{N} : a_n \notin \mathbb{Q}\}$

Now, the elements of $A$ and those of $B$ form two subsequences $\{n_k\}$ and $\{n'_k\}$ of

~~elements~~ elements in $\mathbb{N}$.

Let $x_k = a_{n_k}$ $\forall k \in \mathbb{N}$.
$y_k = a_{n'_k}$

Then $x_k \to a$, $y_k \to a$; $x_k \in \mathbb{Q}$, $y_k \notin \mathbb{Q}$

Ex: Complete the proof.

8.(ii) $f(a+b) = f(a)+f(b)$

check: $f(na) = n f(a)$ $\forall n \in \mathbb{N}$, $a \in \mathbb{R}$

Also $f(a+(-a)) = f(a) + f(-a)$

$\Rightarrow f(0) = f(a) + f(-a)$ $\forall a \in \mathbb{R}$

In particular, letting $a=0$

$f(0) = f(0) + f(0) \Rightarrow f(0) = 0$

and $f(-a) = -f(a)$ $\forall a \in \mathbb{R}$

Thus $f(-na) = -f(na) = -n f(a)$ $\forall n \in \mathbb{N}$

Hence, $\boxed{f(na) = n f(a) \quad \forall n \in \mathbb{Z}}$ $(*)$

claim: $f\left(\frac{p}{q} a\right) = \frac{p}{q} f(a)$ $\forall p \in \mathbb{Z}$, $q \in \mathbb{N}$.

$q f\left(\frac{p}{q} a\right) = f\left(q \cdot \frac{p}{q} a\right)$ by $(*)$

$= f(pa)$ by $(*)$

$= p f(a)$ by $(*)$

$\Rightarrow \boxed{f\left(\frac{p}{q} a\right) = \frac{p}{q} f(a)}$ $(**)$

Let $f(1) = k$.

Then by $(**)$ $f\left(\frac{p}{q}\right) = \frac{p}{q} f(1) = \frac{p}{q} k$.

Hence $\forall x \in \mathbb{Q}$ $f(x) = kx$.

Now, $f$ is continuous, ~~so~~ and so is $kx$

$\Rightarrow g(x) = f(x) - kx$ is continuous and

$g(x) = 0 \;\; \forall x \in \mathbb{Q}$.

$\Rightarrow g(x) = 0 \;\; \forall x \in \mathbb{R}$ by 8(i)

$\Rightarrow f(x) = kx \;\; \forall x \in \mathbb{R}$.

Qn. Can you think of discontinuous function $f: \mathbb{R} \to \mathbb{R}$ such that $f(a) + f(b) = f(a+b)$ $\forall a, b \in \mathbb{R}$?

11) (i) Look at the textbook

(ii) $h(x) = \max(f(x), g(x))$

Let $a, b \in \mathbb{R}$ we claim that

$$\max(a, b) = \frac{a + b + |a-b|}{2}$$

Case 1: $a \geq b$

Then $|a-b| = a - b$

$$\Rightarrow \frac{a+b+|a-b|}{2} = a$$

Case 2: $a < b$

Then $|a-b| = b - a$

$$\Rightarrow \frac{a+b+|a-b|}{2} = b.$$

This proves the claim.

Thus $h(x) = \dfrac{f(x)+g(x)}{2} + \left|\frac{1}{2}(f(x)-g(x))\right|$

Since linear combination of continuous fn is continuous $\frac{1}{2}(f(x)+g(x))$ is continuous and so is $\frac{1}{2}(f(x)-g(x))$.

Now, $\left|\frac{1}{2}(f(x)-g(x))\right|$ is the composition of the functions $\frac{1}{2}(f(x)-g(x))$ and $|x|$. and it is continuous.

It follows that $\max(f,g)$ is continuous.

Finally, $\min(f,g) = f+g-\max(f,g)$.

12. For $x>0$ $\quad \dfrac{x}{|x|} = \dfrac{x}{x} = 1$

For $x<0$ $\quad \dfrac{x}{|x|} = \dfrac{x}{-x} = -1$

Hence, $\displaystyle\lim_{x\to 0+} \frac{x}{|x|} = 1, \quad \lim_{x\to 0-} \frac{x}{|x|} = -1.$

13. The given functions are continuous everywhere except at $x=0$ where the definition changes.

It is enough to check that

$$\lim_{x\to 0} f(x) = f(0).$$

To this end we show that

$$\lim_{x\to 0+} f(x) = \lim_{x\to 0-} f(x) = f(0)$$

(i) $\lim_{x\to 0+} f(x) = \lim_{x\to 0+} (x+1) = 1 = f(0)$

$\lim_{x\to 0-} f(x) = \lim_{x\to 0-} 2^x = 2^0 = 1 = f(0)$

(ii) $\lim_{x\to 0-} f(x) = 0 = f(0)$

$\lim_{x\to 0+} f(x) = \lim_{x\to 0} 2^{-1/x} = 0$

as in Ex(2) of this HW.

11 Ross

18.1 Check yourself

18.2 For instance $x_n \to a$ or $x_n \to b$ is possible.

18.3, 18.4 : Ignore them.

18.5. Let $h(x) = f(x) - g(x)$.

Then $h$ is continuous, $h(a) \geq 0$, $h(b) \leq 0$.

Now, use IVT to show $h(c) = 0$ for some $c \in [a,b]$:

If $h(a) = 0$ or $h(b) = 0$ then we are done. Suppose $h(a) > 0$, $h(b) < 0$ etc.

18.6. Let $f(x) = x - \cos x$. $f$ continuous on $\mathbb{R}$.

Then $f(0) = -\cos 0 = -1 < 0$

$f(\pi/2) = \pi/2 > 0$

Use IVT

18.7. Similar to 18.6

18.8 $f(a)f(b) < 0$

$\Rightarrow f(a), f(b)$ have opposite signs.

So we can apply IVT.

18.9. Done in class.

18.10 Use the hint.

18.11 Ex

18.12 (i) Choose different sequences $x_n \to 0$, $y_n \to 0$ where

$\lim f(x_n) \neq \lim f(y_n)$

(ii) $f$ is continuous on $(-\infty, 0)$ and on $(0, \infty)$.

Hence, if $a, b \in (0, \infty)$, $a < b$, then $\forall\, l$ between $f(a), f(b)$ $\exists\, c \in (a, b)$ with $f(c) = l$.

Similarly for $a, b \in (-\infty, 0)$.

Suppose $a \in (-\infty, 0)$, $b \in (0, \infty)$

Then (Ex) pick $a' \in (0, b)$ with $f(a) = f(a')$.

Then $\forall\, l$ between $f(a) = f(a')$ and $f(b)$ $\exists\, c \in (a', b) \subseteq (a, b)$ such that $f(c) = l$. (PTO)

How to choose $a'$?

Note $f(a) \in [-1, 1]$.

Choose $n$ such that $\frac{1}{n} < b$.

Then $f\left( \left[ \frac{1}{(n+2)\pi}, \frac{1}{n\pi} \right] \right) = [-1, 1]$ check

Hence, $\exists$ $a' \in \left[ \frac{1}{(n+2)\pi}, \frac{1}{n\pi} \right]$ with

$$f(a') = f(a).$$