PHY304: Statistical Mechanics

End Semester Examination 2024
April 23, 2024
(9:30 12:30 Hrs)

Instructor: R. Kapri **Max. Marks 50**

- All questions are compulsory.
- Some important results are given at the end.

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1. The enthalpy of a particular system is

$$H = A\frac{S^2}{N}\ln\left(\frac{P}{P_0}\right),$$

where A is a positive constant. Calculate the molar heat capacity at constant volume c_v as a function of T and P. [6]

2. Prove equipartition theorem i.e., show that every quadratic degree of freedom contributes $k_BT/2$ to the average energy and $k_B/2$ to the heat capacity. [6]

3. A lattice gas consists of N sites, each of which may be occupied by at most one atom. The energy of a site is ϵ if occupied, and 0 if empty.

   (a) Calculate the grand partition function $\mathcal{G}(z, T)$ at fugacity z and temperature T. [2]

   (b) What fraction of sites are occupied? [2]

   (c) Find the heat capacity as a function of T at fixed z. [2]

4. (a) Find the density matrix ρ of a partially polarized beam of spin-$\frac{1}{2}$ atoms containing a mixture of 75% of $|\uparrow_z\rangle$ and 25% of $|\uparrow_x\rangle$. [2]

   (b) Check whether it is a pure or a mixed state? [1]

   (c) Calculate $\langle S_x\rangle$, $\langle S_y\rangle$, and $\langle S_z\rangle$ for this system. [3]

5. Consider a free Fermi gas having N atoms in two dimensions, confined to a square area of $A = L^2$ at $T = 0$.

   (a) Calculate the density of single-particle states $D(\epsilon)d\epsilon$ lying between the energy ϵ and $\epsilon + d\epsilon$. [2]

   (b) Find the Fermi energy ϵ_F (in terms of N and A). [2]

   (c) Show that the average energy of the particles is $\epsilon_F/2$. [2]

6. The atom He^3 consists of three nucleons, has spin $\frac{1}{2}$ and is a fermion. The density of He^3 near absolute zero is 0.081 g/cm^3. Calculate:

(a) The Fermi energy ϵ_F (in meV). [3]

(b) The Fermi temperature T_F (in kelvin). [3]

7. Consider a *two dimensional solid* containing N atoms. Let $A = L^2$ be the area available to the system. In the Debye model, the atomic vibrations are treated as phonons in a box (the box being the solid). Obtain the *high* and the *low temperature* dependence of the heat capacity of the Debye solid. [8]

8. Consider a two dimensional ideal Bose gas. Let $V = L^2$ be the area available to the system. The number of particles (which is conserved) is given by

$$N = z\frac{\partial}{\partial z}\ln\left[\mathcal{Z}(z,V,T)\right] = \sum_p \left[z^{-1}\exp\left(\beta\epsilon_p\right) - 1\right]^{-1},$$

where $\mathcal{Z}$ is the grand partition function. Show that the chemical potential $\mu = 0$ is not possible for the above system. What does this signifies? [6]

Some useful results:

$$\hbar = 1.055\times10^{-27}\text{ergs/s}, \qquad m_{\text{necleon}} = 1.67\times10^{-27}\text{g}$$

$$k_B = 1.38\times10^{-16}\text{ergs K}^{-1} \qquad 1\text{eV} = 1.6022\times10^{-12}\text{ergs}$$

Binomial expansion:

$$(1+x)^N = \sum_{n=0}^{N}\binom{N}{n}x^n$$

Pauli's matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\zeta(s) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{x^{s-1}}{e^x - 1}dx; \qquad \zeta(1) = \infty$$

$$\int_{-\infty}^{\infty} e^{-ax^2}\,dx = \sqrt{\frac{\pi}{a}}$$