PHY 310 - Mathematical Methods for Physicists I

Odd Term 2019, IISER Mohali

Instructor: Dr. Anosh Joseph

Homework 8 - Solutions

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1. Consider the set of functions $\{u_1, u_2, u_3\} = \{1, x, \sin x\}$. Also consider the inner product

$$\langle u_m | u_n \rangle = \int_{-\pi}^{\pi} dx u_m(x) u_n(x). \tag{1}$$

(i.) Are these functions orthogonal with respect to the inner product?

(ii.) If not, find the corresponding orthogonal functions using the Gram-Schmidt orthogonalization process.

Solution:

(i.) Let us consider

$$\begin{aligned} \langle u_2 | u_3 \rangle &= \int_{-\pi}^{\pi} x \sin x dx \\ &= -x \cos x \Big|_{-\pi}^{\pi} + \int_{-\pi}^{\pi} \cos x dx \\ &= (\sin x - x \cos x) \Big|_{-\pi}^{\pi} \\ &= -2\pi \cos \pi = 2\pi \neq 0. \end{aligned} \tag{2}$$

Thus the functions are not orthogonal.

(ii.) Let us apply the Gram-Schmidt orthogonalization process. We have

$$\psi_1 = 1, \tag{3}$$

$$\psi_2 = x - \frac{\int_{-\pi}^{\pi} x dx}{\int_{-\pi}^{\pi} dx} = x \tag{4}$$

$$\begin{aligned} \psi_3 &= \sin x - \frac{\int_{-\pi}^{\pi} \sin x dx}{\int_{-\pi}^{\pi} dx} - \frac{\int_{-\pi}^{\pi} x \sin x dx}{\int_{-\pi}^{\pi} x^2 dx} = \sin x - \frac{2\pi}{\frac{3}{2}\pi^3} x \\ &= \sin x - \frac{3x}{\pi^2}. \end{aligned} \tag{5}$$

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2. Use the Gram-Schmidt orthogonalization process to convert the set of polynomials $\{1, x, x^2\}$ to a set of orthogonal polynomials with respect to the inner product

$$\langle u_m | u_n \rangle = \int_0^\infty dx u_m(x) w(x) u_n(x), \tag{6}$$

where $w(x) = \exp(-ax)$ and $a > 0$.

Hint:

$$\int_0^\infty dx x^n e^{-ax} = \frac{n!}{a^{n+1}}. \tag{7}$$

Solution:

We have

$$\begin{aligned}
\psi_1(x) &= 1, && (8)\\
\psi_2(x) &= x - \frac{\int_0^\infty w(x) x dx}{\int_0^\infty dx} = x - \frac{1}{a} && (9)\\
\psi_3(x) &= x^2 - \frac{\int_0^\infty w(x) x^2 dx}{\int_0^\infty dx} - \frac{\int_0^\infty w(x) x^2 \left(x - \frac{1}{a}\right) dx}{\int_0^\infty \left(x - \frac{1}{a}\right)^2 dx} \left(x - \frac{1}{a}\right)\\
&= x^2 - \frac{2}{a^2} - \frac{4/a^4}{1/a^3} \left(x - \frac{1}{a}\right)\\
&= x^2 - \frac{4x}{a} + \frac{2}{a^2}. && (10)
\end{aligned}$$

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3. Consider the boundary value problem

$$y'' + 4y = x^2, \tag{11}$$

where $0 \leq x \leq 1$ and $y(0) = y(1) = 0$.

(i.) Construct the Greens function for this problem using the method of eigenfunction expansion.

(ii.) Find the solution $y(x)$ using the Green's function computed above.

Solution:

(i.) Writing this as the eigenvalue problem

$$\mathcal{L}\varphi = -\lambda\varphi, \tag{12}$$

with $\varphi(0) = 0$ and $\varphi(1) = 0$.

The general solution can be obtained from rewriting the equation as

$$\varphi''(x) + k^2\varphi(x) = 0, \tag{13}$$

where

$$k^2 = 4 + \lambda. \tag{14}$$

The solutions satisfying the boundary condition $x = 0$ are of the form

$$\varphi(x) = A \sin kx. \tag{15}$$

Imposing the boundary condition $\varphi(1) = 0$ gives

$$0 = A \sin k \implies k = n\pi, \quad k = 1, 2, 3, \cdots . \tag{16}$$

Thus the eigenvalues are

$$\lambda_n = n^2\pi^2 - 4, \quad n = 1, 2, \cdots . \tag{17}$$

The eigenfunctions are

$$\varphi_n = \sin n\pi x, \quad n = 1, 2, \cdots . \tag{18}$$

The normalization constant is

$$N_n = ||\varphi_n||^2 = \int_0^1 \sin^2 n\pi x = \frac{1}{2}. \tag{19}$$

Upon using the formula

$$G(x,z) = \sum_{n=1}^{\infty} \frac{1}{-\lambda_n N_n} \varphi_n(x)\varphi_n(z), \tag{20}$$

we have the Green's function

$$G(x,z) = 2\sum_{n=1}^{\infty} \frac{1}{(4-n^2\pi^2)} \sin(n\pi x)\sin(n\pi z). \tag{21}$$

(ii.) The solution is

$$\begin{aligned}
y(x) &= \int_0^1 G(x,z) f(z) dz \\
&= \int_0^1 \left(2\sum_{n=1}^{\infty} \frac{\sin n\pi x \sin n\pi z}{(4-n^2\pi^2} \right) z^2 dz \\
&= 2\sum_{n=1}^{\infty} \frac{\sin n\pi x}{(4-n^2\pi^2)} \int_0^1 z^2 \sin n\pi z dz \\
&= 2\sum_{n=1}^{\infty} \frac{\sin n\pi x}{(4-n^2\pi^2} \left[\frac{(2-n^2\pi^2)(-1)^n - 2}{n^3\pi^3} \right].
\end{aligned} \tag{22}$$

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4. Show that

$$\int_0^{\infty} dy e^{-ay} y^{n-1} = a^{-n}\Gamma(n). \tag{23}$$

Solution:

We have

$$\Gamma(n) = \int_0^{\infty} x^{n-1} e^{-x} dx. \tag{24}$$

Replacing x by ay we have

$$\begin{aligned}
\Gamma(n) &= \int_0^{\infty} (ay)^{n-1} e^{-ay} a dy \\
&= a^n \int_0^{\infty} e^{-ay} y^{n-1} dy.
\end{aligned} \tag{25}$$

Therefore

$$\int_0^\infty dy e^{-ay} y^{n-1} = a^{-n}\Gamma(n). \tag{26}$$

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5. Show that

$$B(m,n) = B(n,m). \tag{27}$$

Hint: Use $\int_0^a f(x)dx = \int_0^a f(a-x)dx$.

Solution:

We have

$$\begin{aligned} B(m,n) &= \int_0^1 x^{(m-1)}(1-x)^{n-1}dx \\ &= \int_0^1 (1-x)^{m-1}\left[1-(1-x)\right]^{n-1} dx \\ &= \int_0^1 (1-x)^{m-1}x^{n-1}dx \\ &= \int_0^1 x^{n-1}(1-x)^{m-1}dx \\ &= B(n,m). \end{aligned} \tag{28}$$

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6. Show that

$$\int_0^{\frac{\pi}{2}} d\theta \sin^p\theta \cos^q\theta = \frac{\Gamma(\frac{p+1}{2})\Gamma(\frac{q+1}{2})}{2\Gamma(\frac{p+q+2}{2})}. \tag{29}$$

Hint: Use $x = \sin^2\theta$ in the standard definition of the beta function.

Solution:

We have

$$B(m,n) = \int_0^1 dx x^{m-1}(1-x)^{n-1}. \tag{30}$$

Substituting

$$\begin{aligned} x &= \sin^2\theta, && (31)\\ dx &= 2\sin\theta\cos\theta d\theta, && (32)\\ 1-x &= 1-\sin^2\theta = \cos^2\theta, && (33)\end{aligned}$$

we have

$$B(m,n) = 2\int_0^{\frac{\pi}{2}} d\theta \sin^{2m-1}\cos^{2n-1}. \tag{34}$$

Upon using

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \tag{35}$$

and substituting

$$2m-1=p, \quad m=\frac{p+1}{2}, \quad 2n-1=q, \quad n=\frac{q+1}{2}, \tag{36}$$

we have

$$\frac{\Gamma(\frac{p+1}{2})\Gamma(\frac{q+1}{2})}{2\Gamma(\frac{p+q+2}{2})} = \int_0^{\frac{\pi}{2}} d\theta \sin^p\theta\cos^q\theta. \tag{37}$$

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