Assignment 8

PHY310: Mathematical Methods for Physicists I

Instructor: Dr. Prasenjit Das **No Need to Submit**

1. Write the Sturm-Liouville form and identify $p(x)$, $q(x)$, and $w(x)$ in each of the following equations:

   (a) $(1-x^2)y''(x) - xy'(x) + \lambda y(x) = 0;\ -1 \le x \le +1$

   (b) $xy''(x) + (1-x)y'(x) + \lambda y(x) = 0;\ -1 \le x < \infty$

   (c) $(1-x^2)y''(x) - 2xy'(x) + \left[\lambda - \frac{m^2}{1-x^2}\right] y(x) = 0;\ -1 \le x \le +1$

2. Show that the eigenvalues and the corresponding eigenfunctions of the Sturm-Liouville problems

   (a) with $u(1) = 0$ and $u(e^2) = 0$ in $1 \le x \le e^2$,
   $$x^2 u''(x) + xu'(x) + \lambda u(x) = 0$$
   are given by $\lambda_n = (n\pi/2)^2$ and $u_n(x) = C_n \sin(\frac{n\pi}{2} \ln x)$, $n = 1, 2, 3, \cdots$.

   (b) with $u(1) = 0$ and $u(2) = 0$ in $1 \le x \le 2$,
   $$x^4 u''(x) - 2x^3 u'(x) + \lambda u(x) = 0$$
   are given by $\lambda_n = (2n\pi)^2$ and $u_n(x) = C_n \sin(2n\pi/x)$, $n = 1, 2, 3, \cdots$.

   (c) with $u(0) = 0$ and $u(1) = 0$ in $0 \le x \le 1$,
   $$xu''(x) - u'(x) + 4x^3 \lambda u(x) = 0$$
   are given by $\lambda_n = n^2\pi^2$ and $u_n(x) = C_n \sin(n\pi x^2)$, $n = 1, 2, 3, \cdots$.

3. Using the Schmidt Orthogonalization procedure to obtain the first four normalized orthogonal polynomials $\psi_n(x)$ from a non-orthogonal set of linearly independent functions $u_n(x) = x^n$; $n = 0, 1, 2, 3, \cdots$ for

   (a) $w(x) = 1$ in $-1 \le x \le +1$.

   (b) $w(x) = 1$ in $0 \le x \le +1$.

   (c) $w(x) = 1$ in $0 \le x \le +2$.

   (d) $w(x) = \exp(-x)$ in $0 \le x < \infty$.

   (e) $w(x) = \exp(-x^2)$ in $-\infty < x < +\infty$.

4. The functions $u_1(x)$ and $u_2(x)$ are eigen functions of the same Hermitian operator but for distinct eigenvalues λ_1 and λ_2. Prove that $u_1(x)$ and $u_2(x)$ are linearly independent.

5. Use the Gram-Schmidt process to determine an orthonormal basis for the subspace of $\mathbb{R}^n$ spanned by the given set of vectors.

   (a) $\{(2, 1, -2), (1, 3, -1)\}$.
   (b) $\{(1, -1, -1), (2, 1, -1)\}$.
   (c) $\{(-1, 1, 1, 1), (1, 2, 1, 2)\}$.
   (d) $\{(1, 2, 0, 1), (2, 1, 1, 0), (1, 0, 2, 1)\}$.
   (e) $\{(1, 0, -1, 0), (1, 1, -1, 0), (-1, 1, 0, 1)\}$.
   (f) $\{(1, 1, -1, 0), (-1, 0, 1, 1), (2, -1, 2, 1)\}$.

6. Which of the following boundary conditions do not satisfy the orthonormality conditions?

   (a) $p(x) = 1$, $0 \leq x \leq 1$, $u(0) = u(1) = 2$, and $u'(0) = u'(1)$.
   (b) $p(x) = x^2$, $1 \leq x \leq 2$, $u(0) = u(2)$, and $u'(0) = u'(2)$.
   (c) $p(1) = p(2)$, $1 \leq x \leq 2$, $u(1) = u'(2)$, and $u'(1) = u(2)$.

7. Which of the following boundary conditions ensure that all the eigenvalues will be non-negative, if $q(x) \geq 0$?

   (a) $p(x) = 2$, $0 \leq x \leq 1$, $u(0) = 0$, and $u'(1) = 0$.
   (b) $p(x) = \sin x$, $0 \leq x \leq \pi$, $u(0) = u'(\pi)$, and $u'(0) = u(\pi)$.
   (c) $p(x) = e^{-x}$, $-10 \leq x \leq +10$, $u(-10) = u(10)$, and $u'(-10) = u'(10)$.