# HW 3

1) Use that $\lim \frac{1}{n^{1/p}} = 0$. $\forall p \in \mathbb{N}$ to show that $\lim \frac{1}{n^{\alpha}} = 0$ $\forall \alpha > 0$

[Help. Using $a^r \cdot a^s = a^{r+s}$ $\forall a > 0, r, s \in \mathbb{R}$ show that $t \mapsto a^t$ is an increasing function when $a > 1$.]

2) Suppose $\lim x_n = \lim y_n = \ell$.
Let $z_n = \max(x_n, y_n)$ and $w_n = \min(x_n, y_n)$
Show that $\lim z_n = \ell = \lim w_n$

3) Suppose $x_n > 0$ $\forall n$ and $\lim x_n = 1$.
$\forall k \in \mathbb{N}$ show that $\lim x_n^{1/k} = 1$.

[Help. Let $t_n = x_n^{1/k}$. Then $x_n - 1 = t_n^k - 1$
$= (t_n - 1)(t_n^{k-1} + t_n^{k-2} + \cdots + t_n^2 + t_n + 1)$
For all large $n$, $x_n$ is close to 1 and $t \mapsto t^{\alpha}$ $(\alpha > 0)$ is increasing. Thus $\sum_{i=0}^{k-1} t_n^i = \sum_{i=0}^{k-1} x_n^{i/k}$ is not small. Hence

$$\frac{x_n - 1}{\sum_{i=0}^{k-1} t_n^i} = t_n - 1$$ is close to zero. etc]

4) If $x_n > 0$ $\forall n$ and $x_n \to a$ then $\forall t > 0$ show that $\lim x_n^t = a^t$.
[Hint. Use (2), (3). ]

5) Suppose $a > 0$. If $x_n > 0$ $\forall n$ and $\lim x_n = 0$ then show that $\lim a^{x_n} = a^0 = 1$.
[Help. We know $\lim a^{1/n} = 1$. Also use that $t \mapsto a^t$

is increasing if $a \geq 1$ and decreasing otherwise.]

6) (i) If $a \in \mathbb{R}$ show that there is a sequence $\{x_n\}$ of rational no.s with $\lim x_n = a$.

(ii) If $S \subseteq \mathbb{R}$ is bounded above then show that there is a sequence of elements $\{x_n\}$ of $S$ with $\lim x_n = \sup S$.

7) i) Let $x_1 = 1$ and $x_{n+1} = \frac{n}{n+1} x_n^2 \;\forall n \geq 1$.
Show that $\lim x_n = 0$

ii) Let $y_1 = 1$, $y_{n+1} = \frac{1}{3}(y_n + 1) \;\forall n$
Show that $\lim y_n$ exists

8. Let $\{x_n\}$ be any sequence of real numbers. Show that there is a subsequence $\{x_{n_k}\}$ with $\lim x_{n_k} = l$ iff $\forall \varepsilon > 0$ the set

$$\{n \mid |x_n - l| < \varepsilon\}$$ is infinite.

9. Solve the problems and exercises mentioned in class. Complete the proofs of the theorems whose proofs were left as exercises.

10. Show that

(i) $\lim (\sqrt{n+1} - \sqrt{n}) = 0$

(ii) $\lim \frac{n!}{n^n} = 0$

(iii) $\lim \frac{n^k}{a^n} = 0 \quad \forall a > 1, k \in \mathbb{R}$

[Help. Let $t_n = \frac{n^k}{a^n}$. Then $\frac{t_n}{t_{n+1}} = a\left(\frac{n}{n+1}\right)^k$

$\Rightarrow t_{n+1} = t_n \cdot \frac{1}{a} \cdot \left(1 + \frac{1}{n}\right)^k$

By (4) $\lim \left(1 + \frac{1}{n}\right)^k = 1$. Since $a > 1$ for all large $n$ $\quad t_{n+1} \leq \varepsilon t_n$ for some $\varepsilon$, $0<\varepsilon<1$. Then $t_{n+2} \leq \varepsilon^2 t_n$

$\vdots$

$t_{n+k} \leq \varepsilon^k t_n$ etc. ]

(iv) If $x_n = 1+a+\frac{a^2}{2}+\cdots+\frac{a^n}{n!}$ $(a>0)$ then show that $\lim x_n$ exists.

11. $\{x_n\}$ is called a <u>Cauchy</u> <u>sequence</u> if $\forall \varepsilon>0$ $\exists N$ such that $|x_m - x_n| < \varepsilon$ $\forall m \geq n \geq N$.

(i) Show that a Cauchy sequence is bounded.
[Hint. $|x_m - x_n| < 1$ for all large $m, n$ ]

(ii) Show that a Cauchy sequence is convergent.
[Hint. Apply Bolzano-Weierstrass thm]

(iii) Show that convergent sequences are Cauchy. [<u>Hint</u>. $\forall \varepsilon>0$ $\exists N$ such that
$|x_n - l| < \varepsilon/2$ $\forall n \geq N$ where $l = \lim x_n$. ]