# MTH302: INTEGERS, POLYNOMIALS AND MATRICES
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*K*eywords: left ideal, maximal left ideal, partially ordered set, Zorn's lemma.

## 1. Notion of an Ideal

**Left ideal:** A subset $I$ of a ring $R$ is called a *left ideal* of $R$ if

LI-1. $I$ is a subgroup of $(R, +)$

LI-2. $r \cdot x \in I$ for all $x \in I$ and $r \in R$.

**Right ideal:** A subset $I$ of a ring $R$ is called a *right ideal* of $R$ if

RI-1. $I$ is a subgroup of $(R, +)$

RI-2. $x \cdot r \in I$ for all $x \in I$ and $r \in R$.

**Ideal:** A subset $I$ of a ring $R$ is called a *two-sided ideal* of $R$ or simply, an *ideal* of $R$ if $I$ is both a left ideal as well as a right ideal.

**Examples:**

(1) $(0) := \{0_R\}$ is an ideal of any ring $R$ called the *zero ideal*.
(2) $R$ is an ideal of $R$ (called the *unit ideal*)
(3) The subset $n\mathbb{Z} = \{kn \mid k \in \mathbb{Z}\}$ is an ideal of $\mathbb{Z}$.
(4) If $F$ is a field then $\{0\}$ and $F$ are the only ideals of $F$.
(5) Let $R$ be a ring. The sets

$$I_1 := \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \;\middle|\; a, b \in R \right\}, \qquad I_2 := \left\{ \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \;\middle|\; a, b \in R \right\}$$

are respectively, right and left ideals of $M_2(R)$.

**Remark 1.1.** Every two-sided ideal of a ring $R$ is a subring of $R$. However, subrings need not be ideals: $\mathbb{Z}$ is not an ideal of $\mathbb{R}$. A right ideal need not be a left ideal and vice-versa, see Example 5 above.

**Maximal left ideal:** A left ideal $I$ of $R$ is called a *maximal left ideal* of $R$ if $I \neq R$ and if $J$ is any ideal of $R$ such that $I \subseteq J$ then $I = J$ or $J = R$.

**Minimal left ideal:** A left ideal $I$ is called a *minimal left ideal* of $R$ if $I \neq (0)$ and if $J$ is any ideal of $R$ such that $J \subseteq I$ then $I = J$ or $J = (0)$.

**Maximal subring:** A subring $S$ of $R$ is called a *maximal subring* of $R$ if $S \neq R$ and if $T$ is any subring of $R$ such that $S \subseteq T$ then $T = S$ or $T = R$.

## 2. Partially ordered set and Zorn's Lemma

A nonempty set $X$ with a partial order $\leq$ is called a *poset* if $\leq$ satisfies the following conditions.

PO1. *Reflexivity:* $x \leq x$ for all $x \in X$.
PO2. *Anti-Symmetric:* If $x \leq y$ and $y \leq x$ then $x = y$ and
PO3. *Transitivity:* If $x \leq y$ and $y \leq z$ then $x \leq z$.

**Totally ordered set:** A poset $Y$ is called a *chain* or a *totally ordered set* if any two elements of $Y$ are comparable. That is, if $x, y \in Y$ then $x \leq y$ or $y \leq x$.

Let $X$ be a poset and $Y$ be a nonempty subset of $X$. The set $Y$ is a said to be *bounded above* (respectively, *bounded below*) if there is an element $x \in X$ such that $y \leq x$ (respectively, $x \leq y$) for all $y \in Y$. We say $Y$ is a *bounded* subset of $X$ if $X$ is both bounded above as well as bounded below. The set $Y$ is said to have a *maximal element* (respectively, a *minimal element*) if there is an element $a \in Y$ such that $a \nleq y$ (respectively, $y \nleq a$) for all $y \in Y$.

**Zorn's Lemma:** *Let $X$ be a nonempty poset. If every chain $Y$ of $X$ is bounded above (respectively, bounded below) then $X$ has a maximal (respectively, minimal) element.*

**Remark 2.1.** Zorn's Lemma is equivalent to the *Axiom of Choice*.

**Theorem 2.2.** *Let $R$ be a ring with $1$ and $I \neq R$ be a left (respectively, right/two-sided ) ideal of $R$. Then there is a maximal left (respectively, right/two-sided ) ideal $M$ of $R$ containing $I$.*

*Proof.* We shall only prove the theorem for left ideal $I$. The proof when $I$ is a right or a two sided ideal follows similarly. Consider the collection

$$\mathscr{C} := \{J \subseteq R \mid J \text{ is an left ideal of } R, \; I \subseteq J, \text{ and } J \neq R\}.$$

Since $I \in \mathscr{C}$, we see that $\mathscr{C} \neq \emptyset$. Clearly, $\mathscr{C}$ is a partially ordered set with respect to the partial order $\subseteq$.

Let $Y$ be a chain of $\mathscr{C}$ and let $P := \cup_{J \in Y} J$. We claim that $P$ is a left ideal, $P \in \mathscr{C}$ and that $Y$ is bounded above by $P$. For any $x, y \in P$ there are left ideals $J_1, J_2 \in Y$ such that $x \in J_1$ and $y \in J_2$. Since $Y$ is a chain, we have either $J_1 \subseteq J_2$ or $J_2 \subseteq J_1$. Assume that $J_1 \subseteq J_2$. Then $x, y \in J_2$ and we have $x - y \in J_2 \subseteq P$. For any $x \in P$, we know that $x \in J$ for some $J \in Y$. Since $J$ is a left ideal, $r \cdot x \in J \subseteq P$. Thus $P$ is also a left ideal. Similarly, when $J_2 \subseteq J_1$, one shows that $P$ is a left ideal.

For any $J \in Y$, we have $I \subseteq J$. Thus $I \subseteq P$. If $P = R$ then $1 \in P = \cup_{J \in Y} J$ and this implies $1 \in J \in Y \subseteq \mathscr{C}$, a contradiction. Thus $P \neq R$ and we have $P \in \mathscr{C}$. Finally, for any $J \in Y$, we have $J \subseteq P$ and thus $Y$ is bounded above by $P$. Hence the claim.

Now we shall apply Zorn's lemma and obtain a maximal element $M \in \mathscr{C}$. Clearly, $M$ is a left ideal and $M \neq R$. Now if $J$ is any ideal such that $J \neq R$ and $M \subseteq J$ then since $I \subseteq M \subseteq J$, we have $J \in \mathscr{C}$. Since $M$ is a maximal element of $\mathscr{C}$, we must have $M = J$. This implies $M$ is a maximal left ideal. $\square$

**Remark 2.3.** The ring $(\mathbb{Q}, +, *)$, where $+$ is the usual addition and $*$ is defined as $a * b = 0$ for all $a, b \in \mathbb{Q}$ has no unity. Also, the subgroups of $(\mathbb{Q}, +)$ are precisely the ideals of $(\mathbb{Q}, +, *)$ and there are no maximal subgroups (hence no maximal ideals).