PHY 310 - Mathematical Methods for Physicists I

Odd Term 2019, IISER Mohali

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Homework 1

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1. Find the expression for Legendre polynomial $P_2(x)$ from Rodrigues' formula

$$P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l} \left[(x^2 - 1)^l \right].$$

2. Obtain the Legendre polynomial $P_2(x)$ directly from Legendre's equation of order 2 by assuming a polynomial of degree 2

$$y(x) = ax^2 + bx + c.$$

3. Obtain the Legendre polynomial $P_4(x)$ by application of the recurrence formula

$$lP_l(x) = (2l - 1)xP_{l-1}(x) - (l - 1)P_{l-2}(x).$$

4. Find the first three coefficients in the expansion of the function

$$f(x) = \begin{cases} 0 & -1 \leq x \leq 0 \\ 1 & 0 \leq x \leq 1 \end{cases}$$

in a series of Legendre polynomials $P_l(x)$ over the interval $(-1, 1)$.

5. Find the first three coefficients in the expansion of the function

$$f(\theta) = \begin{cases} \cos\theta & 0 \leq \theta \leq \pi/2 \\ 0 & \pi/2 \leq \theta \leq \pi \end{cases}$$

in a series of the form

$$f(\theta) = \sum_{n=0}^{\infty} a_l P_l(\cos\theta), \quad 0 \leq \theta \leq \pi.$$

6. Obtain the associated Legendre functions $P_1^1(x)$, $P_1^2(x)$ and $P_1^{-1}(x)$.

7. Verify that the Legendre polynomials $P_1(x)$ and $P_2(x)$ are solutions of Legendre's equation for $l = 1, 2$.

8. Verify that the associated Legendre polynomial P_1^1 is a solution of associated Legendre's equation for $l = 1$ and $m = 1$.

9. Verify that the associated Legendre polynomial P_2^2 is a solution of associated Legendre's equation for $l = 2$ and $m = 2$.

10. Express x, x^2, x^3, x^4 using the set of Legendre polynomials

$$\{P_0(x), P_1(x), P_2(x), P_3(x), P_4(x)\}.$$

11 Express the following function using Legendre polynomials

$$f(x) = \sigma + \omega x^2 - \lambda x^4,$$

where σ, ω and λ are constants.

12. Express the function

$$f(x) = 30x^2 - 6$$

using Legendre polynomials. Use the method of solving algebraic equations to get the solution.

13. Express the function

$$f(x) = 30x^2 - 6$$

using Legendre polynomials. Use the orthogonality integral for Legendre polynomials to get the solution.

14. Obtain the first two Legendre coefficients of

$$f(x) = Ae^{-mx}.$$

15. Compute

$$\int_{-1}^{+1} dx\ P_1^1(x) P_2^1(x), \quad \text{and}$$
$$\int_{-1}^{+1} dx\ \left[P_2^1(x)\right]^2.$$

16. Using $P_0(x) = 1,\ P_1(x) = x$ and the recurrence relation

$$(2l+1)xP_l(x) = (l+1)P_{l+1}(x) + lP_{l-1}(x),$$

find $P_3(x)$.

17. Find P_0^0, P_1^1, P_1^0 and P_1^{-1} in terms of the angle variable $\cos\theta$.

18. Show that

$$\int_{-1}^{+1} dx\ P_l(x) = 0,\ \text{for } l > 0.$$

19. Let us consider the following potential

$$V = \frac{K}{r},$$

generated by two masses separated by a distance r

$$r = |\mathbf{A} - \mathbf{B}|.$$

The masses are located at the heads of the two vectors **A** and **B** originating from the origin of the coordinate system **O** and **r** is the distance vector starting at the head of vector **B** and ending at the head of vector **A**. The angle between the two vectors **A** and **B** is θ. Then from the low of cosines we have

$$\begin{aligned} r &= |\mathbf{A} - \mathbf{B}| \\ &= \sqrt{A^2 - 2AB\cos\theta + B^2}. \end{aligned}$$

Let us consider the case $|\mathbf{B}| \ll |\mathbf{A}|$. Then we can make the following change of variables

$$t = \frac{B}{A}, \quad x = \cos\theta.$$

In this situation

a.) Show that the gravitational potential is

$$V(r) = \frac{K}{A}\phi(x,t),$$

where $\phi(x,t)$ is the generating function for the Legendre polynomials.

b.) Expand the the potential using $P_l(\cos\theta)$.

20. Show from the generating function for Legendre polynomials

$$\phi(x,t) = \frac{1}{\sqrt{1-2xt+t^2}}, \quad |t| < 1,$$

that

$$(x-t)\frac{\partial \phi}{\partial x} = t\frac{\partial \phi}{\partial t}.$$