PHY 310 - Mathematical Methods for Physicists I

Odd Term 2019, IISER Mohali

Instructor: Dr. Anosh Joseph

Homework 4 - Solutions

---

1. Show that $f(z) = z^*$ is not analytic in the complex plane.

Solution:

Writing the function in terms of real and imaginary parts

$$\begin{aligned} f(z) &= z^* \\ &= u(x,y) + iv(x,y) \\ &= x - iy. \end{aligned}$$

Taking the partial derivatives

$$\begin{aligned} \frac{\partial u}{\partial x} &= 1, \\ \frac{\partial v}{\partial y} &= -1. \end{aligned}$$

Applying Cauchy-Riemann conditions, we get

$$\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}.$$

Cauchy-Riemann conditions are not satisfied for $f(z)$ and thus $f(z) = z^*$ is not an analytic function.

■

---

2. Given $f(z) = z^*$ show that $f'(i)$ does not exist.

Solution:

Writing the function in terms of real and imaginary parts

$$\begin{aligned} f(z) &= z^* \\ &= u(x,y) + iv(x,y) \\ &= x - iy. \end{aligned}$$

Taking the partial derivatives

$$\begin{aligned} \frac{\partial u}{\partial x} &= 1, \\ \frac{\partial v}{\partial y} &= -1. \end{aligned}$$

Applying Cauchy-Riemann conditions, we get

$$\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}.$$

The Cauchy-Riemann conditions are not satisfied. This implies that $f(z) = z^*$ is not an analytic function and thus not differentiable in the complex plane. Thus $f'(i)$ does not exist.

■

3. Show that the function

$$u(x,y) = \frac{1}{2}\ln(x^2 + y^2)$$

is harmonic. Find the harmonic conjugate of this function.

Solution:

We have

$$u(x,y) = \frac{1}{2}\ln(x^2 + y^2).$$

Differentiating with respect to x we get

$$\frac{\partial u}{\partial x} = \frac{1}{2}\frac{1}{x^2+y^2} = \frac{x}{x^2+y^2}.$$

Differentiating with respect to y we get

$$\frac{\partial u}{\partial y} = \frac{y}{x^2+y^2}.$$

From the above two equations we have

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} &= \frac{y^2-x^2}{(x^2+y^2)^2},\\ \frac{\partial^2 u}{\partial y^2} &= \frac{x^2-y^2}{(x^2+y^2)^2}.\end{aligned}$$

Adding the above two equations

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Since u obeys the Laplace' equation, it is a harmonic function.

Let us take v as the harmonic conjugate of u. Then

$$\begin{aligned}dv &= \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy\\ &= -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy\\ &= -\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy.\end{aligned}$$

Thus

$$dv = \frac{xdy - ydx}{(x^2+y^2)} = d\left(\tan^{-1}\frac{y}{x}\right).$$

Integrating the above expression we get the harmonic conjugate as

$$v = \tan^{-1}\frac{y}{x} + C,$$

where C is the integration constant.

■

4. Evaluate the integral
$$I = \int_0^{1+i} dz(x^2 - iy)$$
along the path $y = x^2$.

Solution:

We have the path
$$y = x^2.$$

Thus
$$dy = 2xdx,$$

and
$$dz = dx + idy = dx + 2ixdx = (1 + 2ix)dx.$$

The integral is
$$\begin{aligned}
I &= \int_0^{1+i} dz(x^2 - iy) \\
&= \int_0^1 (x^2 - ix^2)(1 + 2ix)dx \\
&= \int_0^1 x^2(1 - i)(1 + 2ix)dx \\
&= (1 - i)\int_0^1 x^2(1 + 2ix)dx \\
&= (1 - i)\left[\frac{x^3}{3} + i\frac{x^4}{2}\right]\Bigg|_0^1 \\
&= (1 - i)\left[\frac{1}{3} + i\frac{1}{2}\right] = \frac{(1 - i)(2 + 3i)}{6} = \frac{5}{6} + i\frac{1}{6}.
\end{aligned}$$

■

5. Evaluate the integral
$$I = \int_C dz|z|,$$
where C is the left half of the unit circle $|z| = 1$ from $z = -i$ to $z = i$.

Solution:

For a point on the unit circle we have $|z| = 1$. Let us take

$$\begin{aligned} z &= e^{i\theta}, \\ dz &= ie^{i\theta}d\theta. \end{aligned}$$

We have the points $z = i \to \theta = 3\pi/2$ and $z = i \to \theta = \pi/2$.

Thus,

$$\begin{aligned} I &= \int_C dz|z| \\ &= \int_{3\pi/2}^{\pi/2} 1 \cdot e^{i\theta} i d\theta \\ &= e^{i\theta}\Big|_{3\pi/2}^{\pi/2} = e^{i\pi/2} - e^{3i\pi/2} \\ &= \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} - \cos\frac{3\pi}{2} - i\sin\frac{3\pi}{2} \\ &= 2i. \end{aligned}$$

■

6. Evaluate the integral

$$I = \oint \frac{e^{-z}}{(z+1)} dz,$$

where C is the circle $|z| = \frac{1}{2}$.

Solution:

Let us apply Cauchy's integral theorem

$$f(z_0) = \oint \frac{f(z)}{(z - z_0)} dz = \oint \frac{e^{-z}}{(z - (-1))} dz.$$

The point z_0 lies exterior to the contour. Also, the function is analytic within and on C.

Thus we have

$$I = \oint \frac{e^{-z}}{(z+1)} dz = 0.$$

■

---

7. Evaluate the integral

$$I = \oint \frac{e^{-z}}{(z+1)} dz,$$

where C is the circle $|z| = 2$.

Solution:

Let us apply Cauchy's integral theorem

$$\begin{aligned} f(z_0) &= \oint \frac{f(z)}{(z-z_0)} dz \\ &= \oint \frac{e^{-z}}{(z-(-1))} dz \\ &= 2\pi i \, e^{-z}\Big|_{z=-1} \\ &= 2\pi i e. \end{aligned}$$

■
