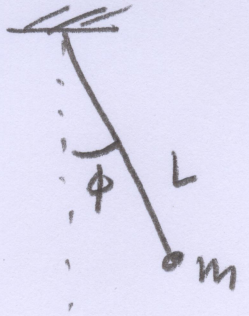
① ⓐ



$$U = mgh$$

$$= mgL\,(1-\cos\phi)$$

ignoring constant

$$U = -mgL\cos\phi$$

$$T = \frac{1}{2} m L^2 \dot{\phi}^2$$

$$\Rightarrow L = T - U = \frac{1}{2} m L^2 \dot{\phi}^2 + mgL\cos\phi$$

ⓑ

$$E = \frac{\partial L}{\partial \dot{q}}\dot{q} - L = \frac{\partial L}{\partial \dot{\phi}}\dot{\phi} - L$$

$$= mL^2\dot{\phi}^2 - \left[\frac{1}{2} m L^2 \dot{\phi}^2 + mgL\cos\phi\right]$$

$$= \frac{1}{2} m L^2 \dot{\phi}^2 - mgL\cos\phi$$

ⓒ

At maximum displacement, $\dot{\phi} = 0$

$$\Rightarrow E = -mgL\cos\phi_0$$

$$\Rightarrow -\cancel{m}g\cancel{L}\cos\phi_0 = \frac{1}{2}\cancel{m}L^{\cancel{2}}\dot{\phi}^2 - \cancel{m}g\cancel{L}\cos\phi$$

$$\Rightarrow \dot{\phi}^2 = \frac{2g}{L}\left(\cos\phi - \cos\phi_0\right)$$

(d)

$$dt = \left(\frac{L}{2g}\right)^{1/2} \frac{d\phi}{[\cos\phi - \cos\phi_0]^{1/2}}$$

$$\tau = \left(\frac{L}{2g}\right)^{1/2} \times 4 \times \int_0^{\phi_0} \frac{d\phi}{[\cos\phi - \cos\phi_0]^{1/2}}$$

$$= 2\left(\frac{2L}{g}\right)^{1/2} \int_0^{\phi_0} \frac{d\phi}{[\cos\phi - \cos\phi_0]^{1/2}}$$

(e)

$$\cos\phi = 1 - 2\sin^2\frac{\phi}{2}$$

$$\Rightarrow \cos\phi - \cos\phi_0 = 2\left(\sin^2\frac{\phi_0}{2} - \sin^2\frac{\phi}{2}\right)$$

$$= 2\sin^2\frac{\phi_0}{2}\left[1 - \frac{\sin^2\frac{\phi}{2}}{\sin^2\frac{\phi_0}{2}}\right]$$

$$\tau = 2\left(\frac{L}{g}\right)^{1/2} \int_0^{\phi_0} \frac{d\phi}{\sin\frac{\phi_0}{2}\left[1 - \frac{\sin^2\phi/2}{\sin^2\phi_0/2}\right]^{1/2}}$$

$$\sin\xi = \frac{\sin\phi/2}{\sin\phi_0/2}$$

$$\cos\xi \, d\xi = \frac{1}{2}\frac{\cos\frac{\phi}{2}\, d\phi}{\sin\frac{\phi_0}{2}}$$

$$\Rightarrow \frac{d\phi}{\sin\frac{\phi_0}{2}} \frac{1}{\left[1-\frac{\sin^2\phi/2}{\sin^2\phi_0/2}\right]^{1/2}} = 2\frac{\cancel{\cos\xi}\, d\xi}{\cos\frac{\phi}{2}\,\cancel{\left[1-\sin^2\xi\right]^{1/2}}}$$

$$\Rightarrow \; = 2\frac{d\xi}{\left[1-\sin^2\frac{\phi}{2}\right]^{1/2}} = 2\frac{d\xi}{\left[1-\sin^2\frac{\phi_0}{2}\sin^2\xi\right]^{1/2}}$$

$$\Rightarrow \tau = 4\left(\frac{L}{g}\right)^{1/2}\int_0^{\pi/2}\frac{d\xi}{\left[1-\sin^2\frac{\phi_0}{2}\sin^2\xi\right]^{1/2}}$$

$$\phi_0 < 1 \Rightarrow \sin\frac{\phi_0}{2} \ll 1$$

$$\Rightarrow \tau \simeq 4\left(\frac{L}{g}\right)^{1/2}\int_0^{\pi/2} d\xi\left[1+\frac{1}{2}\sin^2\frac{\phi_0}{2}\sin^2\xi + \frac{3}{8}\sin^4\frac{\phi_0}{2}\sin^4\xi + \cdots\right]$$

$$= 2\pi\left(\frac{L}{g}\right)^{1/2} + 2\left(\frac{L}{g}\right)^{1/2}\sin^2\frac{\phi_0}{2}\int_0^{\pi/2}\sin^2\xi\, d\xi + \frac{3}{2}\left(\frac{L}{g}\right)^{1/2}\sin^4\frac{\phi_0}{2}\int_0^{\pi/2}\sin^4\xi\, d\xi$$

$$= 2\pi\left(\frac{L}{g}\right)^{1/2} + 2\left(\frac{L}{g}\right)^{1/2}\sin^2\frac{\phi_0}{2}\int_0^{\pi/2}\frac{[1-\cos 2\xi]\,d\xi}{2}$$

$$+ \frac{3}{2}\left(\frac{L}{g}\right)^{1/2}\sin^4\frac{\phi_0}{2}\int_0^{\pi/2}\frac{1}{4}(1-\cos 2\xi)^2\,d\xi$$

$$\int_0^{\pi/2}(1-\cos 2\xi)\,d\xi = \frac{\pi}{2} - \left.\frac{\sin 2\xi}{2}\right|_0^{\pi/2} = \frac{\pi}{2}$$

$$(1-\cos 2\xi)^2 = 1 + \cos^2 2\xi - 2\cos 2\xi$$

$$= 1 - 2\cos 2\xi + \left(\frac{1+\cos 4\xi}{2}\right)$$

$$= \frac{3}{2} - 2\cos 2\xi + \frac{1}{2}\cos 4\xi$$

$$\Rightarrow \int_0^{\pi/2}(1-\cos 2\xi)^2\,d\xi = \frac{3\pi}{4}$$

$$\Rightarrow \tau = 2\pi\left(\frac{L}{g}\right)^{1/2}\left[1 + \frac{1}{4}\sin^2\frac{\phi_0}{2} + \frac{9}{16}\sin^4\frac{\phi_0}{2} + \cdots\right]$$

Power series in ϕ_0

$$\sin\frac{\phi_0}{2} = \frac{\phi_0}{2} - \frac{\phi_0^3}{32} + \frac{\phi_0^5}{3840} - \cdots$$

$$\sin^2\frac{\phi_0}{2} = \frac{\phi_0^2}{4} - \frac{\phi_0^4}{32}$$

$$\sin^4\frac{\phi_0}{2} = \frac{\phi_0^4}{16}$$

$$\Rightarrow \tau = 2\pi\left(\frac{L}{g}\right)^{1/2}\left[1 + \frac{\phi_0^2}{16} - \frac{\phi_0^4}{128} + \frac{9}{16}\cdot\frac{\phi_0^4}{16} + \cdots\right]$$

$$= 2\pi\left(\frac{L}{g}\right)^{1/2}\left[1 + \frac{\phi_0^2}{16} + \frac{7}{256}\phi_0^4 + \cdots\right]$$

$0.0385 + 0.0104$

0.0489

$\boxed{0.049}$

4.9%