Homework 2

1. In the following exercise $\forall$ sequence $\{x_n\}$ find an N such that $|x_n - l| < \varepsilon$ $\forall n \geq N$ where ε is given.

(i) $x_n = \frac{1}{n}$, $l = 0$, $\varepsilon = \frac{2}{3}$

<u>Ans.</u> $|x_n - l| = \frac{1}{n} < \varepsilon = 2/3$

$\iff n > 3/2 \iff n \geq 2$

Hence, $N \doteq 2$.

(ii) $x_n = \frac{n}{2n+1}$, $l = \frac{1}{2}$, $\varepsilon = 1, \frac{1}{2}, \frac{1}{10}$

(iii) $x_n = \frac{n-1}{n+1}$, $l = 1$, $\varepsilon = 1, \frac{2}{3}, \frac{3}{25}$

2. Using the definition of limit show that

(i) $\lim (-1)^n \frac{1}{n} = 0$

(ii) $\lim \frac{2n}{n+1} = 2$

3. Compute limits:

(i) $\lim \frac{n}{n^2+1}$ $\rightarrow$ Ans. $\lim \frac{n}{n^2+1} = \lim \frac{n/n^2}{(n^2+1)/n^2}$

$= \lim \frac{1/n}{1 + 1/n^2}$

Now, $\lim \frac{1}{n} = 0 \Rightarrow \lim \frac{1}{n^2}$

$= \lim \frac{1}{n} \cdot \frac{1}{n} = 0$ ($\because \lim x_n y_n = \lim x_n \cdot \lim y_n$ when they exist)

$\Rightarrow \lim (1 + 1/n^2) = 1$

$\Rightarrow \lim \frac{1/n}{1 + 1/n^2} = \frac{0}{1} = 0$ (Apply $\lim \frac{x_n}{y_n}$

(ii) $\lim \frac{n^2+n+1}{n^2+3n+5}$

(iii) $\lim \frac{n^3+n-1}{n^2(n+2)}$

$= \dfrac{\lim x_n}{\lim y_n}\Big)$

4. Explain why ~~the~~ limits do not exist for the following sequences

(i) $\{\sqrt{n}\}$ (ii) $\{2^n\}$ (iii) $\{(-1)^n \cdot n^2\}$

(iv) $\{1+(-1)^n\}$ (v) $\{(1.01)^n\}$

5. Do the following sequences converge? No need to find limits.

(i) $x_n = 1 + \frac{1}{2^1} + \frac{1}{2^2} + \cdots + \frac{1}{2^n}$

(Optional) for now (ii) $x_1 = 1$, $x_{n+1} = x_n + \frac{1}{n^n}$

6. If $\lim x_n = 0$ and $\{y_n\}$ is any bounded sequence then show that $\lim x_n y_n = 0$

7. If $\lim x_n = 0$ and $x_n \geq 0$ $\forall n$ then show that $\lim x_n^{1/p} = 0$ $\forall p \in \mathbb{N}$.