Exercises

1. Find all solutions of the following equations:

(a) $y' + 2xy = x$ (b) $xy' + y = 3x^3 - 1$ (for $x > 0$)

(c) $y' + e^x y = 3e^x$ (d) $y' - (\tan x)y = e^{\sin x}$ $(0 < x <$

(e) $y' + 2xy = x e^{-x^2}$.

2. Consider the equation $y' + (\cos x)y = e^{-\sin x}$.

(a) Find the solution φ which satisfies $\varphi(\pi) = \pi$.

(b) Show that any solution φ has the property that $\varphi(\pi k) - \varphi(0) = \pi k$ where k is any integer.

3. $x^2 y' + 2xy = 1$ on $0 < x < \infty$.

(a) Show that every soln. tends to zero as $x \to \infty$.

(b) Find that soln. φ which satisfies $\varphi(2) = 2\varphi(1)$.

4. ~~The~~ (Bernoulli's equation) $y' + \alpha(x)y = \beta(x)y^k$, k con

(a) Show that the formal substitution $z = y^{1-k}$ transforms this into the linear equation:

$$z' + (1-k)\alpha(x)z = (1-k)\beta(x)$$

(b) Find all solutions of $y' - 2xy = xy^2$

5. The Jacobi equation

$$(a_1 + b_1 x + c_1 y)(x\,dy - y\,dx) - (a_2 + b_2 x + c_2 y)\,dy + (a_3 + b_3 x + c_3 y)\,dx = 0.$$

Where a, b, c are constants is also closely related to Bernoulli. Make the substitution $x = X + \alpha$, $y = Y + \beta$. α, β are constants to be determined so as to make the coeff. of $X\,dY - Y\,dX$, dY and dX separately homogeneous in X, Y.

We get

$$(b_1 X + c_1 Y)(X dY - Y dX)$$
$$- \{A_2 + b_2 X + c_2 Y - \alpha(A_1 + b_1 X + c_1 Y) - A_1 X\} dY$$
$$+ \{A_3 + b_3 X + c_3 Y - \beta(A_1 + b_1 X + c_1 Y) - A_1 Y\} dX = 0$$

where $A_r = a_r + b_r \alpha + c_r \beta$, $r = 1, 2, 3$.

The coeff of dY and dX also become homogeneous if α, β are so chosen that $A_2 - \alpha A_1 = 0$, $A_3 - \beta A_1 = 0$.

or, more symmetrically, if $A_1 = \lambda$, $A_2 = \alpha\lambda$, $A_3 = \beta\lambda$.

i.e. if $a_1 - \lambda + b_1 \alpha + c_1 \beta = a_2 + (b_2 - \lambda)\alpha + c_2 \beta$
$= a_3 + b_3 \alpha + (c_3 - \lambda)\beta = 0.$

Thus λ is determined by the cubic equation.

$$\begin{vmatrix} a_1 - \lambda & b_1 & c_1 \\ a_2 & b_2 - \lambda & c_2 \\ a_3 & b_3 & c_3 - \lambda \end{vmatrix} = 0$$

and once λ is determined, α, β are the solutions of any two equations above.

The eqn is now written :

$$x\,dy - y\,dx - \Phi\left(\frac{y}{x}\right) dy - \Psi\left(\frac{y}{x}\right) dx = 0$$

The substitution $y = xu$ brings it to the form of a Bernoulli equation:

$$\frac{dx}{du} + U_1 x + U_2 x^2 = 0$$ where U_1, U_2

are fns of u alone. More will be discussed later.