Q 1. Answer

$$H = a\left(|1\rangle\langle 1| - |2\rangle\langle 2| + |1\rangle\langle 2| + |2\rangle\langle 1|\right)$$

if $|1\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ $\quad |2\rangle = \begin{pmatrix}0\\1\end{pmatrix}$

Then,

$$H = a\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$$

To find Eigenvalues,

$$|A - \lambda I| = 0 \quad (C \cdot E)$$

$$\left| a\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} - \lambda\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right| = 0$$

$$\begin{vmatrix} a-\lambda & a \\ a & -a-\lambda \end{vmatrix} = 0 \qquad -(a-\lambda)(a+\lambda) - a^2 = 0$$

$$-a^2 + \lambda^2 - a^2 = 0$$

$$\lambda^2 = 2a^2$$

$$\boxed{\lambda = \pm\sqrt{2}\,a}$$

To find eigenvectors,

$$AX_1 = \lambda_1 X_1 \Rightarrow (A - \lambda_1 I)X_1 = 0$$

So, for $\lambda_1 = \sqrt{2}a$

$$a\begin{pmatrix} 1-\sqrt{2} & 1 \\ 1 & -1-\sqrt{2}\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} = 0$$

$$(1-\sqrt{2})x_1 + x_2 = 0$$

$$(\sqrt{2}-1)x_1 = x_2$$

$$X = \begin{pmatrix}x_1\\x_2\end{pmatrix} = N_1\begin{pmatrix}1\\ \sqrt{2}-1\end{pmatrix}.$$

for $\lambda_2 = -\sqrt{2}a$

$$AX_2 = \lambda_2 X_2 \Rightarrow (A - \lambda_2 I)X_2 = 0$$

$$\begin{pmatrix} 1+\sqrt{2} & 1 \\ 1 & -1+\sqrt{2}\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} = 0 \qquad X_2 = N_2\begin{pmatrix}-1\\ 1+\sqrt{2}\end{pmatrix}.$$

$$(1+\sqrt{2})x_1 + x_2 = 0$$

$$(1+\sqrt{2})x_1 = -x_2$$

For Normalization

$$x_1^* x_1 = 1$$

or

$$\langle x_1 | x_1 \rangle = 1.$$

$$|N_1|^2 \begin{pmatrix} 1 \cdot & \sqrt{2}-1 \end{pmatrix} \begin{pmatrix} 1 \\ \sqrt{2}-1 \end{pmatrix} = 1$$

$$|N_1|^2 \left\{ 1 + (\sqrt{2}-1)^2 \right\} = 1$$

$$|N_1|^2 \left(1 + 1 + 2 - 2\sqrt{2} \right) = 1$$

$$|N_1|^2 = \frac{1}{4 - 2\sqrt{2}}$$

$$(N_1) = \left(\frac{1}{4 - 2\sqrt{2}} \right)^{1/2}$$

$$X_1 = \left(\frac{1}{4 - 2\sqrt{2}} \right)^{1/2} \begin{pmatrix} 1 \\ \sqrt{2}-1 \end{pmatrix}$$

$$X_1 = \left(\frac{1}{4 - 2\sqrt{2}} \right)^{1/2} . |1\rangle + \left(\frac{(\sqrt{2}-1)^2}{4 - 2\sqrt{2}} \right)^{1/2} |2\rangle$$

for $X_2 =$

$$|N_2|^2 \langle x_2 | x_2 \rangle = 1$$

$$|N_2|^2 \begin{pmatrix} -1 & \sqrt{2}+1 \end{pmatrix} \begin{pmatrix} -1 \\ \sqrt{2}+1 \end{pmatrix} = 1$$

$$|N_2|^2 \left\{ 1 + 1 + 2 + 2\sqrt{2} \right\} = 1$$

$$N_2 \quad \cancel{|N_2|^2} = \frac{1}{(4 + 2\sqrt{2})^{1/2}}$$

$$X_2 = \frac{-1}{(4 + 2\sqrt{2})^{1/2}} |1\rangle + \left(\frac{(\sqrt{2}+1)^2}{4 + 2\sqrt{2}} \right)^{1/2} |2\rangle .$$

Q 2. Solutions

$$H = H_{11}|1\rangle\langle 1| + H_{22}|2\rangle\langle 2| + H_{12}\left[|1\rangle\langle 2| + |2\rangle\langle 1|\right]$$

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$H = \begin{pmatrix} H_{11} & H_{12} \\ H_{12} & H_{22} \end{pmatrix}.$$

For eigen values,

$$|H - \lambda I| = 0$$

$$(H_{11} - \lambda)(H_{22} - \lambda) - H_{12}^2 = 0$$

$$\lambda^2 - H_{22}\lambda - H_{11}\lambda + H_{22}H_{11} - H_{12}^2 = 0$$

$$\lambda^2 - (H_{22} + H_{11})\lambda + H_{22}H_{11} - H_{12}^2 = 0$$

$$ax^2 + bx + c = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\lambda = \frac{(H_{11} + H_{22}) \pm \sqrt{(H_{22} + H_{11})^2 + 4(H_{22}H_{11} - H_{12}^2)}}{2}$$

$$\lambda = \frac{(H_{11} + H_{22})}{2} \pm \sqrt{\left(\frac{H_{11} - H_{22}}{2}\right)^2 + H_{12}^2}$$

So, Let $H_{11} + H_{22}/_2 = A$

$H_{11} - H_{22}/_2 = B$

$H_{12} = C$

So. $\lambda = A \pm \sqrt{B^2 + C}$

Now,

$$H = A\mathbb{1} + B\sigma_z + C\sigma_x$$

$$\begin{pmatrix} A+B & C \\ C & A-B \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A + \sqrt{B^2 + C^2}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\begin{pmatrix} A+B-A-\sqrt{B^2+C^2} & C \\ C & A-B-A-\sqrt{B^2+C^2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$B-\sqrt{B^2+C^2}\, x_1 + C x_2 = 0$$

Let $B = \cos\theta \qquad C = \sin\theta$

$$\cos\theta - \sqrt{\cos^2\theta + \sin^2\theta}\, x_1 + \sin\theta\, x_2 = 0$$

$$\cos\theta - 1\, x_1 + \sin\theta\, x_2 = 0$$

$$(2\cos^2\theta/2 + 1 - 1).x_1 + 2\sin\theta/2 \cos\theta/2\, x_2 = 0$$

$$\cos\theta/2\, x_1 + \sin\theta/2\, x_2 = 0$$

$$|\lambda_+\rangle = \cos\theta/2\, |1\rangle + \sin\theta/2\, |2\rangle.$$

if $\theta \to \pi - \theta$

$$|\lambda_-\rangle = -\sin\theta/2\, |1\rangle + \cos\theta/2\, |2\rangle.$$

(3)

With all the specification given,

$$\vec{S}\cdot\hat{n} = \cos\phi \; \hat{S}_z + \sin\phi \; \hat{S}_x$$

The eignvector corresponding to $\frac{\hbar}{2}$

$$|\phi,+\rangle = \cos\frac{\phi}{2}|+\rangle + \sin\frac{\phi}{2}|-\rangle$$

$|+\rangle$ and $|-\rangle$ are eigenstates of $\hat{S}_z$.

The probability of obtaining the outcome $\frac{\hbar}{2}$ in a measurement of $\hat{S}_x$ on $|\phi,+\rangle$ is

$$P(+) = |\langle x,+|\phi,+\rangle|^2 \quad \text{where } |x,+\rangle = \frac{|+\rangle + |-\rangle}{\sqrt{2}}$$

$$\boxed{P(+) = \frac{1}{2}\left|\cos\frac{\phi}{2} + \sin\frac{\phi}{2}\right|^2 = \frac{1+\sin\phi}{2}}$$

Outcome probability for $\left(-\frac{\hbar}{2}\right)$ in the $\hat{S}_x$ measurement

$$\boxed{P(-) = 1 - P(+) = \frac{1-\sin\phi}{2}}$$

Expectation value of $\hat{S}_x$

$$\langle\hat{S}_x\rangle = \frac{\hbar}{2}P(+) - \hbar/2 \, P(-)$$

$$= \frac{\hbar}{2}\left(\frac{1+\sin\phi}{2}\right) - \frac{\hbar}{2}\left(\frac{1-\sin\phi}{2}\right)$$

$$\boxed{\langle\hat{S}_x\rangle = \frac{\hbar}{2}\sin\phi}$$

Dispersion of $\hat{S}_x$

$$Var(\hat{S}_x) = \langle (\hat{S}_x - \langle \hat{S}_x \rangle)^2 \rangle$$

$$= \langle \hat{S}_x^2 \rangle - \langle S_x \rangle^2$$

$$\hat{S}_x^2 = \frac{\hbar^2}{4} I$$

$$\boxed{Var(\hat{S}_x) = \frac{\hbar^2}{4} \cos^2 \phi}$$

r $\phi = 0, \pi/2, \pi$ the variance is $\frac{\hbar^2}{4}, 0, \frac{\hbar^2}{4}$ respectively. This agrees with the expectation since for $\phi = 0, \pi$ the measured state is an eigenstate of $\hat{S}_z$, ~~for ϕ~~. and for $\phi = \pi/2$ an eigenstate of $\hat{S}_x$.

(4)

observable

$$A = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \Rightarrow A = \frac{1}{\sqrt{2}} A'$$

Eigenvalue

$$\det\left[A' - \lambda I\right] = 0$$

$$\begin{vmatrix} -\lambda & 1 & 0 \\ 1 & 0-\lambda & 1 \\ 0 & 1 & 0-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^3 - 2\lambda = 0$$

$\boxed{\lambda = 0, \pm\sqrt{2}}$ Eigenvalues of A'

$\boxed{\text{Eigenvalues of } A, \ \lambda = 0, \pm 1}$

Eigenvectors are

$$\frac{1}{2}\begin{bmatrix} 1 \\ \sqrt{2} \\ -1 \end{bmatrix}, \quad \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \frac{1}{2}\begin{bmatrix} 1 \\ -\sqrt{2} \\ -1 \end{bmatrix}$$

$\boxed{\text{No. Degeneracy}}$.

Tutorial - 3

Problem 5.

Two observables A & B share simultaneous eigen kets $\{|a', b'\rangle\}$.

That means

$$\left.\begin{aligned} A|a', b'\rangle &= \lambda_{a'}|a', b'\rangle \\ B|a', b'\rangle &= \gamma_{b'}|a', b'\rangle \end{aligned}\right\} \forall a', b'$$

where $\lambda_{a'}$ & $\gamma_{b'}$ are some ~~oth~~ numbers.

Now we may check, for all a' & b',

$$\begin{aligned}[A, B]|a', b'\rangle &= (AB - BA)|a', b'\rangle \\ &= AB|a', b'\rangle - BA|a', b'\rangle \\ &= A\gamma_{b'}|a', b'\rangle - B\lambda_{a'}|a', b'\rangle \\ &= \gamma_{b'}\lambda_{a'}|a', b'\rangle - \lambda_{a'}\gamma_{b'}|a', b'\rangle \\ &= (\gamma_{b'}\lambda_{a'} - \lambda_{a'}\gamma_{b'})|a', b'\rangle \\ &= 0|a', b'\rangle.\end{aligned}$$

Therefore, $[A, B] = 0$.

Problem 6 : Let us assume that operators A & B share a common set of eigenkets $\{|a', b'\rangle\}$ with the eigenvalues $\{\lambda_{a'}\}$ & $\{\gamma_{b'}\}$ respectively.

Then $\{A, B\}|a', b'\rangle = (AB + BA)|a', b'\rangle$

$$= 2\lambda_{a'}\gamma_{b'}|a', b'\rangle, \ \forall a', b'.$$

The $\{A, B\} = 0$ then implies

$$\lambda_{a'}\gamma_{b'} = 0,$$ where the possiblities or conditions are–

① $\lambda_{a'} \neq 0, \ \lambda_{b'} = 0,$

② $\lambda_{a'} = 0, \ \lambda_{b'} \neq 0,$

③ $\lambda_{a'} = 0, \ \lambda_{b'} = 0.$

Therefore, $\{A, B\} = 0$ whenever either of the conditions ①, ② or ③ is satified.

In conclusion, $\{A, B\} = 0$ does not necessarily imply that the operators A & B will always have simultaneous eigenkets. Even when they share common set eigenkets, $\{A, B\}$ does not vanishes always.

⇒ For example, consider $A = \sigma_x$ & $B = \sigma_y$, the Pauli ~~spin~~ operators. You may check that $\{\sigma_x, \sigma_y\} = 0$ but they do not share a common set of eigenkets.