End-Semester Examination

Course : MTH301 (Real Analysis)
Duration of the Examination - Three Hours

Date : November 26, 2024
Maximum Marks - 50

Instructions : The question paper has TEN questions. Each question carries equal marks. Your answers should include justifications. You need to explain the relevant terms in the definitions and statement of theorems. Marks distribution is shown at the right side of the questions.

Notations: Let $a, b \in \mathbb{R}$ with $a < b$ and $[a, b]$ (or (a, b)) be a closed (resp. an open) interval. Let $\mathcal{R}[a, b]$ be the set of Riemann integrable functions on $[a, b]$. Let $\mathbb{R}^n$ be the n-dimensional Euclidean space. For a differentiable real-valued function f on an interval, f' denotes derivative of f. Let (X, d) (or X) be a metric space and $C(X)$ be the set of all real-valued continuous functions on X. If $f \in C(X)$ is bounded, then $||f||_\infty$ denotes the sup norm of f. If S is a subset of X, then $\overline{S}$ denotes the closure of S in X. Unexplained notations and terminologies are as discussed in the lectures.

1. Define an ordered field. Show that every ordered field $\mathbb{K}$ contains the field $\mathbb{Q}$ of rational numbers as a subfield. What do you mean by an Archimedean field? Explain. Show that if $\mathbb{Q}$ is dense in $\mathbb{K}$, then $\mathbb{K}$ is Archimedean. (1+1+1+2=5)

2. Define a homeomorphism between metric spaces. Construct explicit homeomorphisms $f: (0, 1) \to (a, b)$ and $g: (0, 1) \to \mathbb{R}$. Does there exist a homeomorphism $h : (0, 1) \longrightarrow [0, 1]$? Justify your answer. (1+1+1+2=5)

3. Define Cantor's ternary set. Show that no point of the Cantor's ternary set C is its interior point. Also show that C is a perfect set. (1+2+2=5)

   (**OR**) Let S be a nonempty subset of (X, d). Define $d_S(x) = \inf\{d(x, a) : a \in S\}$ for $x \in X$. Show that $d_S: X \to \mathbb{R}$ is uniformly continuous. Further, show that the closure $\overline{S} = \{x \in X : d_S(x) = 0\}$. (2+3=5)

4. Let $f: [a, b] \to \mathbb{R}$ be a continuous function. Show that $f \in \mathcal{R}[a, b]$. State the first and second fundamental theorem of calculus. (3+1+1=5)

   (**OR**) Let K be a (covering) compact subset of a metric space (X, d). Show that K is a closed subset. If $f: K \to \mathbb{R}$ is a continuous function, then show that f is uniformly continuous on K. (2+3=5)

5. State L'Hospital's rule. Show that $\lim_{y\to\infty} \frac{y}{e^y} = 0$. Let $f(x) = \begin{cases} e^{\frac{-1}{x}} & \text{for } x > 0, \\ 0 & \text{for } x \le 0. \end{cases}$ Show that f is continuously differentiable on $\mathbb{R}$. Also determine the derivative function f' of f. (1+1+2+1=5)

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6. Let (X, d) be a compact metric space. Define $\|f\|_\infty = \sup\{|f(x)| : x \in X\}$ for $f \in C(X)$. Show that $f \mapsto \|f\|_\infty$ defines a norm on $C(X)$. If $(f_n)_{n=1}^\infty$ is a sequence that converges uniformly to a function f, then show that $f \in C(X)$. Further, show that $C(X)$ is a complete metric space under the sup metric. (2+2+1=5)

7. Define an equicontinuous subset $\mathcal{E} \subseteq C[0,1]$. State Arzela-Ascoli theorem for $C[0,1]$. Let $f_n(x) = \frac{x}{n}$ and $g_n(x) = \frac{x}{x+(nx-1)^2}$ for $n \in \mathbb{N}$ and $x \in [0,1]$. Are $\mathcal{E}_1 = \{f_n : n \in \mathbb{N}\}$ and $\mathcal{E}_2 = \{g_n : n \in \mathbb{N}\}$ equicontinuous subsets of $C[0,1]$? Explain. (1+1+2+1=5)

(**OR**) State Weierstrass Approximation theorem. Let $f \in C[0,1]$. Suppose for $n = 0, 1, 2, \ldots$, we have $\int_0^1 f(x)x^n dx = 0$. Then show that $f(x) = 0$ for all $x \in [0,1]$. Define a sequence $(P_n(x))$ of polynomials recursively by

$$P_{n+1}(x) = P_n(x) + \frac{x^2 - (P_n(x))^2}{2} \quad \text{and} \quad P_0(x) = 0.$$

Show that $P_n(x) \to |x|$ uniformly on $[-1,1]$. (1+2+2=5)

8. Define a contraction $\varphi : X \to X$. Show that a contraction is continuous. If (X, d) is a complete metric space, then show that every contraction $\varphi : X \to X$ has a unique fixed point. (1+1+3=5)

9. Let $f(x,y) = \frac{xy^2}{x^2+y^4}$ for $(x,y) \neq (0,0)$ and $f(0,0) = 0$. Let $v = (v_1, v_2) \in \mathbb{R}^2$ be a unit vector. Determine the directional derivative $(D_v f)(0,0)$. Does the total derivative $f'(0,0)$ exist? Explain. Prove multivariate chain-rule of differentiation. (2+1+2=5)

10. State inverse function theorem. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$f(x_1, x_2) = (e^{x_1} \cos x_2,\ e^{x_1} \sin x_2) \quad \text{for } (x_1, x_2) \in \mathbb{R}^2.$$

Show that f is locally invertible at every $(x_1, x_2) \in \mathbb{R}^2$. Further, show that f is an *open mapping* (i.e. image of every open subset of $\mathbb{R}^2$ under f is an open subset of $\mathbb{R}^2$). Let $E = \{(x_1, x_2) \in \mathbb{R}^2 : 0 < x_2 < 2\pi\}$. Then show that $f : E \to f(E)$ has a global inverse $f^{-1} : f(E) \to E$. (1+2+1+1=5)

(**OR**) State implicit function theorem. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be the mapping given by

$$f(x,y,z) = x^2 y + e^x + z \quad \text{for } (x,y,z) \in \mathbb{R}^3.$$

Show that there is an open set $W \subseteq \mathbb{R}^2$ with $(1,-1) \in W$ and a C^1-mapping $g : W \to \mathbb{R}$ such that $g((1,-1)) = 0$ and $f(g(y,z), y, z) = 0$ for all $(y,z) \in W$. Determine the partial derivatives $\frac{\partial g}{\partial y}(1,-1)$ and $\frac{\partial g}{\partial z}(1,-1)$. (1+2+2=5)

(End of the Question Paper)