ROLL NO : | MS | | | | | | **NAME :**

PHY638 EndSem Part A   Date : May 5, 2025   Inst: Abhishek Chaudhuri

- **Time : 75 minutes**
- **Max Marks : 5×4 = 20**
- Attempt all questions.   No aids (Books/Notes/Gadgets).

Please give your answers in the space provided.

1. A long cylinder of radius R is immersed in a very viscous, incompressible fluid. At time $t = 0$, it begins to rotate with a constant angular velocity Ω about its axis. Assume steady, axisymmetric and unidirectional Stokes flow (only $u_\theta(r)$ is non-zero). Under these conditions, $u_\theta(r)$ obeys: $\mu\left(\frac{d^2u_\theta}{dr^2} + \frac{1}{r}\frac{du_\theta}{dr} - \frac{u_\theta}{r^2}\right) = 0$ with general solution $u_\theta(r) = Ar + B/r$. Using no-slip boundary conditions and assuming the fluid to be at rest far away, find $u_\theta(r)$. Hence determine the magnitude of torque per unit length on the cylinder given that the shear stress at the cylinder wall is: $\tau(R) = \mu\left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right)\Big|_{r=R}$.

2. A doublet of strength κ is placed in a uniform flow U. The potential and stream function for the doublet are given as $\phi_d = \kappa\cos\theta/r$ and $\psi_d = -\kappa\sin\theta/r$, respectively. Write down expressions for the streamlines, draw them and find the stagnation points.

3. For the velocity field $\mathbf{u} = (-y, x)$, compute the vorticity and stream function and hence classify the flow. Comment on the streamlines.

4. Consider the slow flow equations: $-\boldsymbol{\nabla} p + \mu \boldsymbol{\nabla}^2 \mathbf{u} = 0$; $\boldsymbol{\nabla} \cdot \mathbf{u} = 0$, where $\mathbf{u}$ is the flow velocity and p is the pressure. Let the fluid be in some region V which is bounded by a closed surface S. Let $\mathbf{u} = \mathbf{u_B}(\mathbf{x})$, say, on S. Then show that there is at most one solution of the slow flow equations which satisfies that boundary condition.

5. For a turbulent flow, the outer scale is where the fluid is being stirred, i.e. where energy is being injected with $Re = U\,L/\nu \gg 1$, where U is a velocity scale, L is a length scale and ν is the kinematic viscosity. The inner scale is where viscous dissipation occurs. The typical velocity v_d and lengthscale l_d are such that $v_d l_d/\nu \sim 1$. Noting that in a steady cascade, the energy transfer rate ε from scale to scale must be constant, how do v_d and l_d scale with ε and Re?

ROLL NO : | MS | | | | | | **NAME :**

PHY638 EndSem Part B   Date : May 5, 2025   Inst: Abhishek Chaudhuri

- **Time : 1 hr 45 minutes**
- **Max Marks : 9+8+8 = 25**
- Attempt all questions.   No aids (Books/Notes/Gadgets).

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1. Consider gravity-driven surface waves in a shallow layer of inviscid, incompressible fluid of depth H, with weak nonlinearity and weak dispersion. The governing equation is the following for the wave height $h(x,t)$:

$$\frac{\partial h}{\partial t} + c_0 \frac{\partial h}{\partial x} + \frac{3}{2}\frac{c_0}{H} h \frac{\partial h}{\partial x} + \frac{1}{6} c_0 H^2 \frac{\partial^3 h}{\partial x^3} = 0,$$

where $c_0 = \sqrt{gH}$. [2+3+4]

   (a) Non-dimensionalize the KdV equation using: $x = L\tilde{x}$, $t = \frac{L}{c_0}\tilde{t}$, $h = a\tilde{h}$. Show that the resulting equation involves the parameters $\epsilon = \frac{a}{H}$ and $\delta = \left(\frac{H}{L}\right)^2$, and interpret their physical meanings.

   (b) The KdV equation admits solitary wave solutions of the form: $h(x,t) = a\,\mathrm{sech}^2[\kappa(x - ct)]$. For what value of κ and c will this satisfy the KdV equation? Express in terms of a, H and c_0. Briefly discuss how the wave speed and width depend on amplitude.

   (c) Using the form of the KdV equation, estimate when nonlinearity dominates over dispersion, based on ϵ/δ. Linearize the KdV equation (drop the nonlinear term) and assume a wave solution of the form $h(x,t) = e^{i(kx-\omega t)}$. Derive the corresponding dispersion relation $\omega(k)$ and comment on how it compares with the dispersion relation for linear shallow water gravity waves.

2. Two immiscible, incompressible viscous fluids of equal density ρ flow steadily down a plane inclined at an angle α to the horizontal under the influence of gravity. The **lower fluid** has viscosity μ_1 and thickness h_1, while the **upper fluid** has viscosity μ_2 and thickness h_2. Let y be the coordinate perpendicular to the inclined plane, with $y = 0$ at the solid boundary. Assume no-slip at the wall and no shear stress at the free surface. [2+2+3+1]

   (a) Write down the governing differential equation for the velocity profile in each fluid layer, indicating any assumptions used to simplify the Navier–Stokes equations.

   (b) Solve the differential equation to obtain general expressions for the velocity profiles $u_1(y)$ in the lower layer ($0 \le y \le h_1$) and $u_2(y)$ in the upper layer ($h_1 \le y \le h_1 + h_2$), including constants of integration.

   (c) Apply appropriate boundary conditions to solve for $u_1(y)$ and $u_2(y)$.

   (d) Does the velocity profile in the lower fluid depend on the viscosity of the upper fluid? Explain.

3. One deep layer of inviscid fluid, density ρ_2, flows with uniform speed U_2 over another deep layer of density $\rho_1 > \rho_2$ which flows with uniform speed U_1 in the same direction. Initially, there is an interface between the two at $z = 0$. Let the interface be perturbed to $z = \eta(x,t) = \eta_0 \exp(i(kx - \omega t))$. [3+2+3]

   (a) Derive the dispersion relation for the case where the gravity $g \neq 0$ but with negligible surface tension $\gamma = 0$.

   (b) What is the condition for instability?

   (c) How would the perturbation of the interface evolve when gravity is ignored? Explain how instabilities may emerge in such a case.

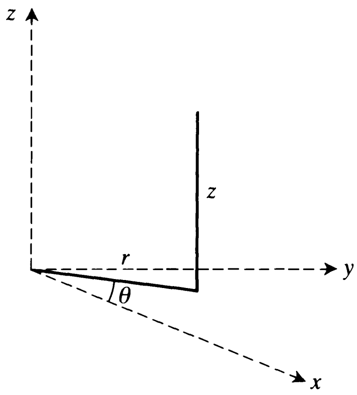

Fig. A.2 Cylindrical polar coordinates.

Also,

$$\nabla\phi = \frac{\partial\phi}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\boldsymbol{e}_\theta + \frac{\partial\phi}{\partial z}\boldsymbol{e}_z,$$

$$\nabla\cdot\boldsymbol{F} = \frac{1}{r}\frac{\partial}{\partial r}(rF_r) + \frac{1}{r}\frac{\partial F_\theta}{\partial\theta} + \frac{\partial F_z}{\partial z},$$

$$\nabla\wedge\boldsymbol{F} = \frac{1}{r}\begin{vmatrix} \boldsymbol{e}_r & r\boldsymbol{e}_\theta & \boldsymbol{e}_z \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial z} \\ F_r & rF_\theta & F_z \end{vmatrix},$$

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2},$$

$$\boldsymbol{u}\cdot\nabla = u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial\theta} + u_z\frac{\partial}{\partial z}.$$

$$\int_0^\infty x^n e^{-bx^2}\,dx = \frac{\Gamma(\frac{n+1}{2})}{2b^{\frac{n+1}{2}}}$$

The Navier–Stokes equations in cylindrical polar coordinates are:

$$\frac{\partial u_r}{\partial t} + (\boldsymbol{u}\cdot\nabla)u_r - \frac{u_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta}\right),$$

$$\frac{\partial u_\theta}{\partial t} + (\boldsymbol{u}\cdot\nabla)u_\theta + \frac{u_r u_\theta}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial\theta} + \nu\left(\nabla^2 u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial\theta} - \frac{u_\theta}{r^2}\right), \tag{A.35}$$

$$\frac{\partial u_z}{\partial t} + (\boldsymbol{u}\cdot\nabla)u_z = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\,\nabla^2 u_z,$$

$$\frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{\partial u_z}{\partial z} = 0.$$

The components of the rate-of-strain tensor are given by:

$$e_{rr} = \frac{\partial u_r}{\partial r}, \qquad e_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_r}{r}, \qquad e_{zz} = \frac{\partial u_z}{\partial z},$$

$$2e_{\theta z} = \frac{1}{r}\frac{\partial u_z}{\partial\theta} + \frac{\partial u_\theta}{\partial z}, \qquad 2e_{zr} = \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}, \tag{A.36}$$

$$2e_{r\theta} = r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial\theta}.$$

$$\frac{\partial\boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\boldsymbol{u} + \boldsymbol{g},$$

$$\nabla\cdot\boldsymbol{u} = 0.$$

$$\frac{\partial\boldsymbol{\omega}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{\omega} = (\boldsymbol{\omega}\cdot\nabla)\boldsymbol{u} + \nu\nabla^2\boldsymbol{\omega}.$$

For surface gravity waves of uniform depth H, we had the following conditions:

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial z^2} = 0.$$

subject to the conditions

$$\frac{\partial\phi}{\partial z} = 0 \quad \text{at} \quad z = -H,$$

$$\frac{\partial\phi}{\partial z} = \frac{\partial\eta}{\partial t} \quad \text{at} \quad z = 0,$$

$$\frac{\partial\phi}{\partial t} = -g\eta \quad \text{at} \quad z = 0.$$

$$\frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} = 0 \qquad \text{(continuity)},$$

$$\frac{1}{r}\frac{\partial}{\partial r}(ru_\theta) - \frac{1}{r}\frac{\partial u_r}{\partial\theta} = 0 \qquad \text{(irrotationality)},$$

$$u_r = \frac{\partial\phi}{\partial r} = \frac{1}{r}\frac{\partial\psi}{\partial\theta},$$

$$u_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -\frac{\partial\psi}{\partial r},$$

$$\nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0,$$

$$\nabla^2\psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} = 0,$$

Slow Flow Equations:

$$0 = -\nabla p + \mu\,\nabla^2\boldsymbol{u},$$

$$\nabla\cdot\boldsymbol{u} = 0$$