PHY 101 : End Semester Exam
28^{th} November 2023, IISER Mohali

This question paper has two parts. Questions in Part A are for 3 marks each; Questions in Part B carry 5 marks each. Symbols have their usual meanings.

Part A : 3 marks each

1. A fast moving charged particle in a room is moving along a circle with speed v w.r.t. a person in the room. If it is due to the Coriolis force and the radius of the circle is R, obtain the relation of the velocity with the radius of the circle. Find out the speed of the particle if it has to make a circle of radius $5m$ at an altitude of $\lambda = 30^o$, if no ohter forces are present. [2 + 1]

2. On a sphere of radius R, a particle is moving under Newton's law on the trajectory $\theta = \alpha t^3$, for some constant α. Find out the acceleration of the particle. What is the work done by the force in moving the particle from the north pole to the south ? [1 + 2]

3. For the Hamiltonian of a 1-D system given a $\mathcal{H} = \alpha p^2 + \beta x^2$ find out the rate of change of the momentum w.r.t. the position, i.e. dp/dx. From that expression find p as a function of x if at $x = 0$ the momentum was p_0. [2 + 1]

4. A particle has its location r, ϕ in an orbit of eccentricity e around a massive star such that it is at its maximum distance when $\phi = 0$ and at minimum at $\phi = \pi$. Prove that $dr/d\phi = -er\sin\phi/(1 - e\cos\phi)$. If the trajectory is parabolic one, what is its value when $\phi = \pi/2$ if its minimum distance is b? [1 + 2]

5. A test particle of mass m is moving in an elliptical orbit around a star of mass M with eccentricity e. Find out the relation of the angular momentum of the particle with the area of the orbit if the semi major axis is a. How does the time period of the orbit scale with a/m ? [1 + 2]

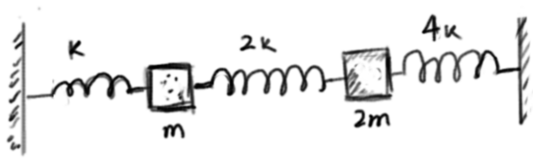
Part B : 5 marks each.
Attempt any 5 problems

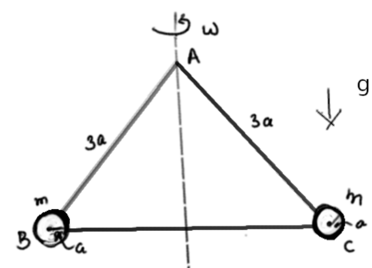
1. An observer finds a particle of mass m moving on a trajectory $\mathbf{r(t)} = \gamma[(t^3/6 - \alpha t^2/2)\hat{i} + (t^3 - 9\beta t^2)\hat{j}$ under the action of a force $2m\gamma[(t-3)\hat{i} + 2(t-3)\hat{j}]$, for constants α, β, γ. Is the observer inertial for all α, β, γ ? If not, for what choices of these parameters the observer be inertial ? When the observer is inertial, is the Newton's second law holding ? [2 + 2 + 1]

2. A Foucault pendulum is oscillating at a latitude of 45^o which has a time period t_{osc} and plane shift period T. Obtain the expression for t_{osc}/T, if Ω is the Earth's rotational speed and R its radius. If the Earth were to suddenly collapse to 8 times its present density how would the ratio change ? [One can treat Earth as a uniform sphere for this problem.] [2 + 3]

3. A unit mass forced and damped oscillator with parameters (β, ω_0^2) is driven with a force $f_1 \cos \omega t + f_2 \sin \omega t$. If in the steady state, the oscillator oscillates with amplitude A and phase difference ϕ w.r.t. to $f_2 \sin \omega t$, obtain the relations connecting f_1, f_2 to other parameters. Find out the expression for the amplitude and the phase. [1 + 2 + 2]

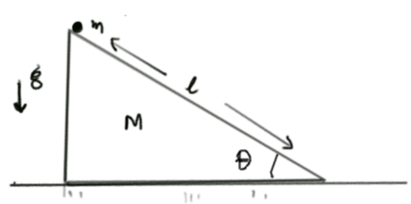
4. A system of springs $k, 2k, 4k$ and masses m and $2m$ are connected as shown in the diagram. Obtain the equations of motion for the two blocks. Obtain the condition for parameter α such that $x_1 + \alpha x_2$ oscillates as a normal mode, where x_1, x_2 are the displacements of the blocks m and $2m$ from their equilibrium positions. Find the normal mode frequencies. [1 + 2 + 2]





5. Two uniform spheres of radius a and mass m are put on two corners of a massless equilateral triangle of arm $3a$ as shown in the figure. If the triangular frame is rotated about an axis passing through the third corner with angular speed ω as shown. Find the angular momentum of the system along the axis of rotation. If the frame is hinged from the corner A (in the figure) and displaced a bit in the plane of the paper, find out the frequency of small oscillations due to gravitational force g acting vertically downwards. [2 + 3]

6. A particle of mass m is moving down the slope of a wedge of angle θ and slant length ℓ from rest at the top, under the action of gravity g vertically downwards. Once the particle reaches the bottom, how much velocity the wedge has obtained ? If the particle was replaced with a coin of same mass and radius a which rolls with a condition $v = a\omega$ where v, ω are its linear and angular speeds respectively at any instant, how would the answers change? What if the coin was replaced with a ring of same mass and radius ? [Assume friction to be negligible everywhere] [1 + 2 + 2]



Part A

Q. 1 Coriolis force $\vec{F} = -2m(\vec{\Omega} \times \vec{v})$

$$= -2m\Omega \sin\lambda \, v \; \hat{n}$$

where $\hat{n} \perp \vec{v}$

For circular trajectory this should provide the centripetal force (as follows)

For a person in the room $\vec{r} = R\hat{r}$

$\vec{v} = \dot{\vec{r}} = R\dot{\theta}\,\hat{\theta} \Rightarrow \vec{a} = \ddot{\vec{r}} = -R\dot{\theta}^2\,\hat{r} \Rightarrow \vec{a} \perp \vec{v}$

$$\therefore \quad \frac{mv^2}{R} = 2m\Omega\sin\lambda\, v \Rightarrow \boxed{v = 2\Omega R \sin\lambda}$$

For 30° $\Rightarrow$ $v = \Omega R = (5\,m)\,\Omega$

Q. 2. $\theta = \alpha t^3 \Rightarrow \theta = 0$ at $t = 0$, $\theta = \pi$ at $t = \left(\frac{\pi}{\alpha}\right)^{1/3}$

$\vec{r} = R\hat{r} \Rightarrow \dot{\vec{r}} = R\dot{\theta}\,\hat{\theta} = 3R\alpha t^2\,\hat{\theta}$

$$\ddot{\vec{r}} = (-R\dot{\theta}^2)\,\hat{r} + (R\ddot{\theta})\,\hat{\theta}$$

$$= -R(3\alpha t^2)^2\,\hat{r} + R(6\alpha t)\,\hat{\theta}$$

$$= R\left[-(3\alpha t^2)^2\hat{r} + (6\alpha t)\,\hat{\theta}\right]$$

$$\vec{F} = m\ddot{\vec{r}} = mR\left[-(3\alpha t^2)^2\hat{r} + (6\alpha t)\hat{\theta}\right]$$

$$W = \int \vec{F}\cdot d\vec{r} = m\int_0^{\left(\frac{\pi}{\alpha}\right)^{1/3}} R(6\alpha t)\,(3R\alpha t^2)\,dt$$

$$= 18\,mR^2\alpha^2 \int_0^{\left(\frac{\pi}{\alpha}\right)^{1/3}} t^3\,dt = \frac{9}{2}\,mR^2\alpha^2\left(\frac{\pi}{\alpha}\right)^{4/3}$$

Q.3 $\mathcal{H} = \alpha p^2 + \beta x^2$

$$\frac{dx}{dt} = \frac{\partial \mathcal{H}}{\partial p} = 2\alpha p \qquad \frac{dp}{dt} = -\frac{\partial \mathcal{H}}{\partial x} = -2\beta x$$

$$\frac{dp}{dx} = \frac{dp}{dt} \cdot \frac{dt}{dx} = -\frac{\beta x}{\alpha p}$$

$$\alpha p \, dp = -\beta x \, dx$$

$$\alpha \int_{p_0}^{p} p \, dp = -\beta \int_0^x x \, dx \Rightarrow \frac{\alpha}{2}(p^2 - p_0^2) = -\frac{\beta x^2}{2}$$

Q.4 $r = \dfrac{r_0}{1 + e\cos(\phi - \phi_0)}$

For r to be maximum for $\phi = 0$, $\phi_0 = \pi$

then $r = \dfrac{r_0}{1 + e\cos(\phi - \pi)} = \dfrac{r_0}{1 - e\cos\phi}$

$$\frac{dr}{d\phi} = -\frac{r_0}{(1 - e\cos\phi)^2} e\sin\phi = \frac{-e\, r \sin\phi}{(1 - e\cos\phi)}$$

For parabolic trajectory $e = 1$, $\Rightarrow r = \dfrac{r_0}{1 - \cos\phi}$

Minimum distance at $\phi = \pi$ $b = \dfrac{r_0}{2} \Rightarrow r_0 = 2b$

$$\frac{dr}{d\phi} = \frac{-r_0 \sin\phi}{(1 - \cos\phi)^2}$$

$$\left.\frac{dr}{d\phi}\right|_{\pi/2} = -r_0 = -2b$$

Q. 5 $\quad \frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{L}{2m}$

$$\tau = \frac{A}{\dot{A}} = \frac{2m A}{L}$$

$$L^2 = (1-e^2)\, m k a \Rightarrow \frac{a^3 L^2}{mk} = (1-e^2) a^4 = A^2$$

$$L = \sqrt{mk}\, A / a^{3/2}$$

$$\tau = \frac{2 m A\, a^{3/2}}{\sqrt{mk}\; A} = 2 \frac{m^2}{\sqrt{k}} \left(\frac{a}{m}\right)^{3/2}$$

PART B.

1. $\vec{r}(t) = \gamma \left[\left(\frac{t^3}{6} - \frac{\alpha t^2}{2}\right)\hat{i} + (t^3 - 9\beta t^2)\hat{j} \right]$

$$\dot{\vec{r}} = \gamma \left[\left(\frac{t^2}{2} - \alpha t\right)\hat{i} + (3t^2 - 18\beta t)\hat{j} \right]$$

$$\ddot{\vec{r}} = \gamma \left[(t-\alpha)\hat{i} + (6t - 18\beta)\hat{j} \right]$$

$$\vec{F} = 2m\gamma \left[(t-3)\,\hat{i} + 2(t-3)\,\hat{j} \right]$$

Inertial frame means if $\vec{F} = 0 \Rightarrow \vec{a} = 0$

$\vec{F} = 0$ for $\gamma = 0$ or $t = 3$

$\vec{a} = 0$ for $\gamma = 0$ or $t = \alpha = 3\beta$

Thus, at $t = 3$, $\vec{F} = 0$ but $\vec{a} \neq 0$ for all α, β

Therefore it is not inertial for all α, β

It will be inertial if at $t = 3$, $\vec{a} = 0$

Thus $\alpha = 3$, $\beta = 1$.

Then $m\vec{a} = m\gamma \left[(t-3)\,\hat{i} + 6(t-3)\,\hat{j} \right]$

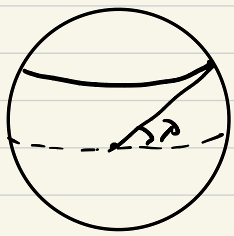
$\vec{F} \neq m\vec{a}$

Newton's second law does not hold.

2. $$\frac{t_{osc}}{T} = \sqrt{\ell}\,\frac{\Omega \sin\lambda}{\sqrt{g - \Omega^2 R \cos^2\lambda}}$$

$$= \sqrt{\frac{\ell}{2}}\,\frac{\Omega}{\sqrt{g - \frac{\Omega^2 R}{2}}}$$

$$= \sqrt{\ell}\,\frac{\Omega}{\sqrt{2g - \Omega^2 R}}$$



When it collapses to 8 times the density

$$8\,\frac{M}{\frac{4}{3}\pi R^3} = \frac{M}{\frac{4}{3} M R'^3} \Rightarrow R' = R/2$$

Then $I = \frac{2}{5} MR^2 \longrightarrow I' = \frac{2}{5} MR'^2 = \frac{I}{4}$

But without any torque $I'\Omega' = I\Omega$

$$\Rightarrow \Omega' = \Omega/4$$

$$\therefore \quad \frac{t'_{osc}}{T'} = \sqrt{e}\ \frac{\Omega'}{\sqrt{2g' - \Omega'^2 R'}}$$

$$g' = \frac{GM}{R'^2} = 4g$$

$$\frac{t'_{osc}}{T'} = \sqrt{e}\ \frac{4\Omega}{\sqrt{4g - 8\Omega^2 R}} = \frac{\sqrt{e}\ 2\Omega}{\sqrt{g - 2\Omega^2 R}}$$

3. Driving term : $f_1 \cos \omega t + f_2 \sin \omega t$

Equation of motion

$$\ddot{x} + \beta \dot{x} + \omega_0^2 x = f_1 \cos \omega t + f_2 \sin \omega t$$

$$x = A \sin(\omega t + \phi)$$

$$\dot{x} = \omega A \cos(\omega t + \phi) \quad , \quad \ddot{x} = -\omega^2 A \sin(\omega t + \phi)$$

$$\Rightarrow A(\omega_0^2 - \omega^2)(\sin \omega t \cos \phi + \cos \omega t \sin \phi) + \beta \omega A (\cos \omega t \cos \phi - \sin \omega t \sin \phi)$$
$$= f_1 \cos \omega t + f_2 \sin \omega t$$

$$\Rightarrow \quad \sin \omega t \left[A(\omega_0^2 - \omega^2)\cos\phi - \beta\omega A \sin\phi - f_2 \right]$$
$$+ \cos \omega t \left[A(\omega_0^2 - \omega^2)\sin\phi + \beta\omega A \cos\phi - f_1 \right] = 0$$

$$A(\omega_0^2 - \omega^2)\cos\phi - \beta\omega A \sin\phi = f_2$$
$$A(\omega_0^2 - \omega^2)\sin\phi + \beta\omega A \cos\phi = f_1$$

$$\Rightarrow \quad A(\omega_0^2-\omega^2) = f_1 \sin\phi + f_2 \cos\phi$$
$$\beta\omega A = f_1 \cos\phi - f_2 \sin\phi$$

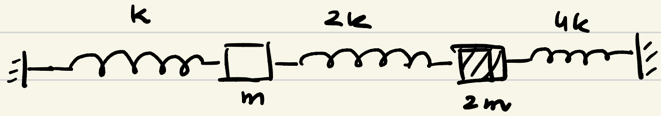
$$\Rightarrow \quad A^2 \left[(\omega_0^2-\omega^2)^2 + \beta^2\omega^2 \right] = f_1^2 + f_2^2$$

$$\Rightarrow \quad A = \frac{f_1^2 + f_2^2}{\left[(\omega_0^2-\omega^2)^2 + \beta^2\omega^2\right]^{1/2}}$$

$$\frac{(\omega_0^2-\omega^2)}{\beta\omega} = \frac{f_1 \sin\phi + f_2 \cos\phi}{f_1 \cos\phi - f_2 \sin\phi} = \frac{f_1 \tan\phi + f_2}{f_1 - f_2 \tan\phi}$$

$$\Rightarrow \quad (\omega_0^2-\omega^2) f_1 - (\omega_0^2-\omega^2) f_2 \tan\phi = \beta\omega f_1 \tan\phi + f_2 \beta\omega$$
$$\Rightarrow \quad \left[\beta\omega f_1 + (\omega_0^2-\omega^2) f_2\right] \tan\phi = (\omega_0^2-\omega^2) f_1 - f_2 \beta\omega$$
$$\Rightarrow \quad \tan\phi = \frac{(\omega_0^2-\omega^2) f_1 - \beta\omega f_2}{\left[\beta\omega f_1 + (\omega_0^2-\omega^2) f_2\right]}$$

4.



$$m\ddot{x}_1 = -kx_1 - 2k(x_1-x_2) = -3kx_1 + 2kx_2$$
$$2m\ddot{x}_2 = -4kx_2 + 2k(x_1-x_2) = 2kx_1 - 6kx_2$$
$$m\ddot{x}_2 = kx_1 - 3kx_2$$

$$m(\ddot{x}_1 + \alpha\ddot{x}_2) = -3kx_1 + 2kx_2 + \alpha(2kx_1 - 6kx_2)$$
$$= -(3k - 2\alpha k)x_1 - (6k\alpha - 2k)x_2$$
$$= -(3k - 2\alpha k)x_1 - \frac{(6k\alpha - 2k)}{\alpha}\alpha x_2$$

For normal mode

$$m(\ddot{x}_1 + \alpha\ddot{x}_2) = -m\omega^2(x_1 + \alpha x_2)$$

Therefore, $3k - 2\alpha k = \dfrac{6\alpha k - 2k}{\alpha}$

$$\Rightarrow \quad 3\alpha - 2\alpha^2 = 6\alpha - 2$$

$$\Rightarrow \quad 2\alpha^2 + 3\alpha - 2 = 0$$

$$\Rightarrow \quad (2\alpha - 1)(\alpha + 2) = 0$$

$$\alpha = +1/2 \quad \text{or} \quad \alpha = -2$$

For $\alpha = 1/2$

$$m\left(\ddot{x}_1 + \frac{\ddot{x}_2}{2}\right) = -(3k - k)x_1 - 2(3k - 2k)\frac{x_2}{2}$$

$$= -2k\left(x_1 + \frac{x_2}{2}\right)$$

$$\Rightarrow \omega^2 = \frac{2k}{m}$$

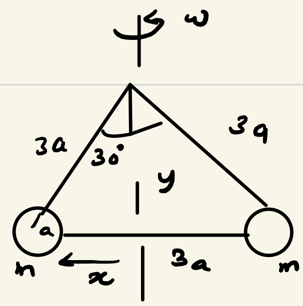
For $\alpha = -2$

$$m(\ddot{x}_1 - 2\ddot{x}_2) = -(3k + 4k)x_1 - \frac{(-12k - 2k)}{-2}(-2)x_2$$

$$= -7kx_1 + 14kx_2 = -7k(x_1 - 2x_2)$$

$$\omega^2 = \frac{7k}{m}$$

5. For rotation, perpendicular distance from the axis



$x : 3a \sin 30^\circ = \frac{3a}{2}$

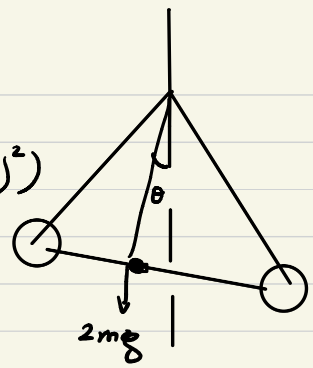
$$I = 2\left(\frac{2}{5}ma^2 + m\left(\frac{3a}{2}\right)^2\right)$$

↑ for both spheres

$$= 2\left(\frac{2}{5}ma^2 + \frac{9}{4}ma^2\right) = \frac{53}{10}ma^2$$

$$L = I\omega = \frac{53}{10}ma^2\omega$$

For oscillation



$$I_O = 2\left(I_{CM} + m\left(\frac{3\sqrt{3}}{2}a\right)^2\right)$$

$$= 2\left(\frac{2}{5}ma^2 + m\left(\frac{3a}{2}\right)^2 + m\left(\frac{3\sqrt{3}}{2}a\right)^2\right)$$

$$= 2\left(\frac{2}{5}ma^2 + m\,9a^2\right)$$

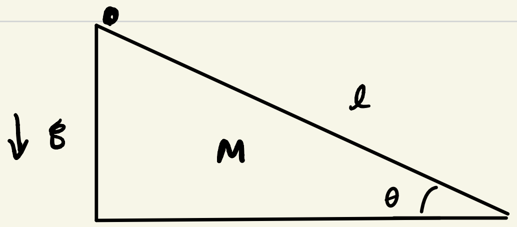
$$= \frac{94}{5}ma^2$$

$$I_O \frac{d\omega}{dt} = -2Mg\,\frac{3\sqrt{3}}{2}a\,\sin\theta \approx -3\sqrt{3}\,mga\,\theta$$

$$\frac{d^2\theta}{dt^2} = -\frac{3\sqrt{3}\,mga}{\frac{94}{5}ma^2}\theta = -\frac{15\sqrt{3}}{94}\frac{g}{a}\theta$$

$$\omega^2 = \frac{15\sqrt{3}}{94}\frac{g}{a}$$

6.



Since there is only gravity in y - direction the net momentum in x - direction will be zero. As the particle reaches bottom

$$m v = - M \bar{V}$$

$$\bar{V} = -\frac{m}{M} v$$

For friction less surfaces

$$mgh = \frac{1}{2} m v^2 + \frac{1}{2} M \bar{V}^2 = \frac{1}{2}\left(m + \frac{m^2}{M}\right) v^2 \Rightarrow v = \alpha \sqrt{2gh}$$

$$\Rightarrow \bar{V} = -\frac{m}{M}\alpha \sqrt{2gh}$$

with $\alpha = \left(\sqrt{m+M}\right)^{-1}$

For a coin which rolls down

$$KE = \frac{1}{2} m v^2 + \frac{1}{2} I \omega^2 + \frac{1}{2} M \bar{V}^2, \quad I_{coin} = \frac{1}{2} m R^2$$

$$= \frac{1}{2} m v^2 + \frac{1}{2} \frac{mR^2}{2} \frac{v^2}{R^2} = \frac{3}{4} m v^2 + \frac{1}{2} M \bar{V}^2$$

$$mgh = \beta m v^2 = v = \sqrt{\frac{1}{\beta} gh}$$

$$\bar{V} = -\frac{m}{M}\sqrt{\frac{1}{\beta} gh} \qquad \left\{ \beta = \frac{3}{4} + \frac{m}{2M} \right.$$

For ring $I = mR^2$

$$mgh = \frac{1}{2} m v^2 + \frac{1}{2} m R^2 \frac{v^2}{R^2} + \frac{1}{2} M \bar{V}^2 = m v^2 + \frac{1}{2} M \bar{V}^2$$

$$v = \sqrt{gh/\delta}, \quad \bar{V} = -\frac{m}{M}\sqrt{\frac{gh}{\delta}}; \quad \delta = \left(1 + \frac{m}{2M}\right)$$