Exercises

1. $(x^2-y^2)\,dx + 2xy\,dy = 0$.
Homogeneous separate vars. Soln: $x^2+y^2 = Cx$.
(Substitute: $y = vx$)

2. $(2y\,e^{y/x} - x)\,y' + 2x + y = 0$
This is deg 1 homogeneous in x, y since $e^{y/x}$ is of deg zero.
write $y = vx$, $y' = xv' + v$, cancel x, separate vars to get
$\log(v^2 + e^{-v}) + 2\log x = C$ or $y^2 + x^2 e^{-y/x} = C$.

PP. 9 3. $y' = \dfrac{4x - y + 7}{2x + y - 1}$

Lines $4x-y+7=0$ & $2x+y-1$ meet in $(-1,3)$. write $x = X-1$,
$y = Y+3$ to get $(2X+Y)\,dY = (4X-Y)\,dX$.
Now $Y = vX$, separate vars and get integrate: $(y-x-4)^3(y+4x+1)^2 =$

4. $(2x-4y+5)\,y' + x - 2y + 3 = 0$
$z = x-2y$, : $(2z+5)\,z' = 4z+11$, separate vals to get
$\left(1 - \frac{1}{4z+11}\right) dz = 2\,dx$ or $4x + 8y + \log|4x-8y+11| = C$.

5. $\dfrac{(1+y^2)\,y\,dx + (1+x^2)\,x\,dy}{(1+x^2+y^2)^{3/2}} = 0$.
Check that $\partial P/\partial y = \partial Q/\partial x$ So it's exact. Proceed to get
$\dfrac{xy}{\sqrt{1+x^2+y^2}} = C$.

6. $\log(y^2+1)\,dx + \dfrac{2y(x-1)}{y^2+1}\,dy = 0$
Check condition of integrability is satisfied. Soln. is $(x-1)\log(y^2+1) = C$

7. ~~6~~. $y\,dx - x\,dy = 0$.
This is not exact. I.F. are $\frac{1}{x^2}, \frac{1}{y^2}, \frac{1}{xy}$ etc. Soln. $\frac{x}{y} = \text{Const.}$

8. ~~7~~. $(1+y^2)\,y\,dx + (1+x^2)\,x\,dy = 0$
$(1+x^2+y^2)^{-3/2}$ is an I.F. Another one is $[xy(1+x^2)(1+y^2)]^{-1}$. This separates vars.

(9)

8. Let μ_1, μ_2 be two I.F of $Pdx + Qdy = 0$ (8.1) whose ratio is not a constant. Then the equation $\mu_2 = C\mu_1$ is an integral of the diff. eqn.

(This theorem is equiv. to the statement that if one I.F is known, many others are known)

Since $\mu_2 = C\mu_1$ is the primitive of the diff. eqn.

$$\left\{\mu_2 \frac{\partial \mu_1}{\partial x} - \mu_1 \frac{\partial \mu_2}{\partial x}\right\} dx + \left\{\mu_2 \frac{\partial \mu_1}{\partial y} - \mu_1 \frac{\partial \mu_2}{\partial y}\right\} dy = 0 \quad (8.3)$$

But since μ_1, μ_2 are I.F. of $Pdx + Qdy = 0$,

$$P \frac{\partial \mu_1}{\partial y} - Q \frac{\partial \mu_1}{\partial x} + \mu_1 \left\{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right\} = 0 \quad (8.4)$$

$$P \frac{\partial \mu_2}{\partial y} - Q \frac{\partial \mu_2}{\partial x} + \mu_2 \left\{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right\} = 0 \quad (8.5)$$

Hence $\mu_2 \left\{P \frac{\partial \mu_1}{\partial y} - Q \frac{\partial \mu_1}{\partial x}\right\} - \mu_1 \left\{P \frac{\partial \mu_2}{\partial y} - Q \frac{\partial \mu_2}{\partial x}\right\} = 0$

i.e. $\left\{\mu_2 \frac{\partial \mu_1}{\partial y} - \mu_1 \frac{\partial \mu_2}{\partial y}\right\} P = \left\{\mu_2 \frac{\partial \mu_1}{\partial x} - \mu_1 \frac{\partial \mu_2}{\partial x}\right\} Q$

which reduces (8.3) to (8.1).

Thus if μ_1 is known and we write $\mu_2 = v\mu_1$ we have from (8.5):

$$P \frac{\partial (v\mu_1)}{\partial y} - Q \frac{\partial (v\mu_1)}{\partial x} + v\mu_1 \left\{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right\} = 0$$

which by (8.4) reduces to $\mu_1 \left\{P \frac{\partial v}{\partial y} - Q \frac{\partial v}{\partial x}\right\} = 0$.

So that $v =$ Const is any general integral of (8.1). But if $u = C$ is one form of this integral we have $v = F(u)$

Thus $\mu_2 = \mu_1 F(u)$ where $F(u)$ is an arbitrary fn of u.

<u>Example</u>: $y\,dx - x\,dy = 0$ is not exact. I.F μ will satisfy $\frac{\partial (y\mu)}{\partial y} = -\frac{\partial (x\mu)}{\partial x}$ or $x \frac{\partial \mu}{\partial x} + y \frac{\partial \mu}{\partial y} + 2\mu = 0$

Check that possible μ are: $\frac{1}{x^2}, \frac{1}{y^2}, \frac{1}{xy}$. Taking $\mu = \frac{1}{x^2}$ soln is $\frac{y}{x} = C$.

By above, solns are $\frac{1}{x^2} : \frac{1}{y^2} =$ Const, $\frac{1}{x^2} : \frac{1}{xy} =$ Const, $\frac{1}{y^2} : \frac{1}{xy} =$ Const all equiv. to $\frac{y}{x} = C$. General $\mu = \frac{1}{xy} F\left(\frac{y}{x}\right)$

10. Special type of I.F. : It may happen that an I.F. can be found which depends on one var. only.
Say $Pdx + Qdy = 0$ admits an I.F. $\mu(x)$, depending on x

So $Q\frac{d\mu}{dx} = \mu\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$

or $\frac{d\mu}{dx}/\mu = \left\{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right\}/Q$.

Thus μ can be a fn. of x alone if RHS is independent of y.
μ is then determined by an integral.

Example : $(1-xy)dx + (xy - x^2)dy = 0$.

$\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)/Q = \frac{-x-(y-2x)}{xy - x^2} = -\frac{1}{x}$

Hence $\frac{d\mu}{\mu} = -\frac{dx}{x}$, $\log|\mu| = -\log|x|$ or $\mu = \frac{1}{x}$.

Eqn. becomes : $\left(\frac{1}{x} - y\right)dx + (y-x)dy = 0$

it is now exact & has the integral $\log|x| - xy + \frac{1}{2}y^2 = C$

• $|||^{ly}$ assuming existence of an ~~integral~~ I.F. which is a fn. of $x+y$, say $\mu = f(x+y) = f(z)$:

$$\frac{f'(z)}{f(z)} = -\left\{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right\}/(P-Q).$$

So necessary cond. for this is RHS is a fn. of $z = x+y$ alone

Ex.: $(5x^2 + 2xy + 3y^3)dx + 3(x^2 + xy^2 + 2y^3)dy = 0$.

$$\frac{f'(z)}{f(z)} = \frac{6y^2 - 4x}{2x^2 + 2xy - 3xy^2 - 3y^3} = \frac{2}{x+y} = \frac{2}{z} \text{ so } f(z) = z^2$$

So I.F is $(x+y)^2$

and Soln. is $(x^2+y^3)(x+y)^3 = C$.

• III^{ly} if $\mu = f(xy) = f(z)$, the cond. will become

$$\frac{f'(z)}{f(z)} = -\left\{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right\} / (Px - Qy).$$

where RHS is a fn. of $z = xy$ alone.

Ex: $(xy^3 + 2x^2y^2 - y^2)dx + (x^2y^2 + 2x^3y - 2x^2)dy = 0$

$$\frac{f'(z)}{f(z)} = 1 - \frac{2}{xy} = 1 - \frac{2}{z} \Rightarrow f(z) = e^{z} z^{-2}.$$

So I.F is $\mu = e^{xy} x^{-2} y^{-2}$.

Soln. is $e^{xy}\left(\frac{1}{x} + \frac{2}{y}\right) = C$.

11. Bernoulli Eqn.

Ex: $\frac{dy}{dx} - \frac{y}{2x} = 5x^2y^5$ i.e. $\frac{y'}{y^5} - \frac{1}{2xy^4} = 5x^2$

$v = y^{1-5} = y^{-4}$, $v' = -4y^{-5}y'$

$v' + \frac{2}{x}v = -20x^2$

This lin. eqn. has I.F. x^2.

Soln. is $y = \frac{1}{\sqrt[4]{Cx^{-2} - 4x^3}}$