Tutorial 05 1.

08/02/22

1. Convert the following matrices to a row echelon matrix and determine which of these are invertible.

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 4 & 9 \\ 1 & 8 & 27 \end{pmatrix}, \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}, \begin{pmatrix} 2 & 0 & -1 & 0 \\ 3 & 0 & 0 & -1 \\ 12 & 2 & -3 & -4 \\ 0 & 1 & 0 & -1 \\ 0 & 2 & -3 & 0 \end{pmatrix}$$

In each case where the matrix is invertible find the inverse.

We shall perform row operations on augmented matrix $(A|I_3)$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 1 & 4 & 9 & 0 & 1 & 0 \\ 1 & 8 & 27 & 0 & 0 & 1 \end{array}\right) \xrightarrow{L_{2,1}(-1)} \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 2 & 6 & -1 & 1 & 0 \\ 1 & 8 & 27 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{L_{3,1}(-1)} \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 2 & 6 & -1 & 1 & 0 \\ 0 & 6 & 24 & -1 & 0 & 1 \end{array}\right) \xrightarrow{M_2\left(\frac{1}{2}\right)} \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1/2 & 1/2 & 0 \\ 0 & 6 & 24 & -1 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{L_{1,2}(-2)} \left(\begin{array}{ccc|ccc} 1 & 0 & -3 & 2 & -1 & 0 \\ 0 & 1 & 3 & -1/2 & 1/2 & 0 \\ 0 & 6 & 24 & -1 & 0 & 1 \end{array}\right) \xrightarrow{L_{3,2}(-6)} \left(\begin{array}{ccc|ccc} 1 & 0 & -3 & 2 & -1 & 0 \\ 0 & 1 & 3 & -1/2 & 1/2 & 0 \\ 0 & 0 & 6 & 2 & -3 & 1 \end{array}\right)$$

$$\xrightarrow{M_3\left(\frac{1}{6}\right)} \left(\begin{array}{ccc|ccc} 1 & 0 & -3 & 2 & -1 & 0 \\ 0 & 1 & 3 & -1/2 & 1/2 & 0 \\ 0 & 0 & 1 & 1/3 & -1/2 & 1/6 \end{array}\right) \xrightarrow{L_{1,3}(3)} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -5/2 & 1/2 \\ 0 & 1 & 3 & -1/2 & 1/2 & 0 \\ 0 & 0 & 1 & 1/3 & -1/2 & 1/6 \end{array}\right)$$

$$\xrightarrow{L_{2,3}(-3)} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -5/2 & 1/2 \\ 0 & 1 & 0 & -3/2 & 2 & -1/2 \\ 0 & 0 & 1 & 1/3 & -1/2 & 1/6 \end{array}\right) = \left(I_3 \mid A^{-1} \right)$$

Hence $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 4 & 9 \\ 1 & 8 & 27 \end{pmatrix}^{-1} = \begin{pmatrix} 3 & -5/2 & 1/2 \\ -1/2 & 1/2 & 0 \\ 1/3 & -1/2 & 1/6 \end{pmatrix}$

$$(A \mid I_3) = \left(\begin{array}{ccc|ccc} \cos\theta & -\sin\theta & 0 & 1 & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{L_{2,1}\left(\frac{-\sin\theta}{\cos\theta}\right)} \left(\begin{array}{ccc|ccc} \cos\theta & -\sin\theta & 0 & 1 & 0 & 0 \\ 0 & 1/\cos\theta & 0 & \frac{-\sin\theta}{\cos\theta} & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{M_2(\cos\theta)} \left(\begin{array}{ccc|ccc} \cos\theta & -\sin\theta & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{M_1\left(\frac{1}{\cos\theta}\right)} \left(\begin{array}{ccc|ccc} 1 & \frac{-\sin\theta}{\cos\theta} & 0 & \frac{1}{\cos\theta} & 0 & 0 \\ 0 & 1 & 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{L_{1,2}\left(\frac{\sin\theta}{\cos\theta}\right)} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1-(\sin\theta)^2}{\cos\theta} & \sin\theta & 0 \\ 0 & 1 & 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$= \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \cos\theta & \sin\theta & 0 \\ 0 & 1 & 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

Therefore

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

No surprise! Why?

We had assumed $\cos\theta \neq 0$ in the above procedure. What if $\cos\theta = 0$?

$$(A|I_4) = \left(\begin{array}{cccc|cccc} 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

Try the following sequence of elementary row transformations.

$$S_{1,2} \longrightarrow L_{3,1}(-1) \longrightarrow L_{4,1}(-1) \longrightarrow L_{3,2}(-1) \longrightarrow L_{4,2}(-1)$$

$$\downarrow$$

$$M_4\left(\tfrac{1}{3}\right) \longleftarrow L_{4,3}(2) \longleftarrow L_{1,3}(-1) \longleftarrow M_3(-1) \longleftarrow S_{3,4}$$

$$\downarrow$$

$$L_{1,4}(1) \longrightarrow L_{2,4}(1) \longrightarrow L_{3,4}(-2)$$

to get

$$A^{-1} = \begin{pmatrix} -2/3 & 1/3 & 1/3 & 1/3 \\ 1/3 & -2/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & -2/3 & 1/3 \\ 1/3 & 1/3 & 1/3 & -2/3 \end{pmatrix}$$

$$(A|I_2) = \left(\begin{array}{cc|cc} \cos 2\theta & \sin 2\theta & 1 & 0 \\ \sin 2\theta & -\cos 2\theta & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{cc|cc} \cos 2\theta & \sin 2\theta & 1 & 0 \\ \sin 2\theta & -\cos 2\theta & 0 & 1 \end{array}\right) \qquad \left(\begin{array}{cc|cc} \sin 2\theta & -\cos 2\theta & 0 & 1 \\ \cos 2\theta & \sin 2\theta & 1 & 0 \end{array}\right)$$

$$\left(\begin{array}{cc|cc} 1 & \frac{-\cos 2\theta}{\sin 2\theta} & 0 & \frac{1}{\sin 2\theta} \\ \cos 2\theta & \sin 2\theta & 1 & 0 \end{array}\right) \qquad \left(\begin{array}{cc|cc} 1 & \frac{-\cos 2\theta}{\sin 2\theta} & 0 & \frac{1}{\sin 2\theta} \\ 0 & \frac{1}{\sin 2\theta} & 1 & \frac{-\cos 2\theta}{\sin 2\theta} \end{array}\right)$$

$$\left(\begin{array}{cc|cc} 1 & \frac{-\cos 2\theta}{\sin 2\theta} & 0 & \frac{1}{\sin 2\theta} \\ 0 & 1 & \sin 2\theta & -\cos 2\theta \end{array}\right)$$

$$\left(\begin{array}{cc|cc} 1 & 0 & \cos 2\theta & \frac{1-(\cos 2\theta)^2}{\sin 2\theta} \\ 0 & 1 & \sin 2\theta & -\cos 2\theta \end{array}\right) = \left(\begin{array}{cc|cc} 1 & 0 & \cos 2\theta & \sin 2\theta \\ 0 & 1 & \sin 2\theta & -\cos 2\theta \end{array}\right)$$

$$\Rightarrow \quad A^{-1} = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix} = A$$

What if $\sin 2\theta = 0$?

Again, not surprising! Why?

Extra There are infinitely many matrices whose square is identity matrix.

$$\begin{pmatrix} 2 & 0 & -1 & 0 \\ 3 & 0 & 0 & -1 \\ 12 & 2 & -3 & -4 \\ 0 & 1 & 0 & -1 \\ 0 & 2 & -3 & 0 \end{pmatrix}$$

This matrix is not even a square matrix, so it cannot be invertible.

Nevertheless, convert it to a row echelon matrix as it will help us in Q2. Row echelon matrix is

$$\begin{pmatrix} 1 & 0 & 0 & -1/3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2/3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

2. Use one of the matrices in Q1 to balance the following chemical reaction.

$$Al_2(SO_4)_3 + Ca(OH)_2 \rightarrow Al(OH)_3 + CaSO_4$$

Let x_1, x_2, x_3, x_4 be integers such that

$$x_1 Al_2(SO_4)_3 + x_2 Ca(OH)_2 \longrightarrow x_3 Al(OH)_3 + x_4 CaSO_4$$

Balancing elementwise, we get following 5 equations

Al	$2x_1 - x_3 = 0$
S	$3x_1 - x_4 = 0$
O	$12x_1 + 2x_2 - 3x_3 - 4x_4 = 0$
Ca	$x_2 - x_4 = 0$
H	$2x_2 - 3x_3 = 0$

We write this system of equations as

$$\begin{pmatrix} 2 & 0 & -1 & 0 \\ 3 & 0 & 0 & -1 \\ 12 & 2 & -3 & -4 \\ 0 & 1 & 0 & -1 \\ 0 & 2 & -3 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Augmentation matrix is

$$\left(\begin{array}{cccc|c} 2 & 0 & -1 & 0 & 0 \\ 3 & 0 & 0 & -1 & 0 \\ 12 & 2 & -3 & -4 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 2 & -3 & 0 & 0 \end{array}\right),$$ whose row echelon

matrix is

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & -1/3 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -2/3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Which amounts to

$$x_1 = x_4/3$$
$$x_2 = x_4$$
$$x_3 = 2x_4/3$$
$$x_4 = \text{free}$$

Smallest value that may be assigned to x_4, ensuring that x_1, x_2, x_3 are integers is : $x_4 = 3$.

In that case

$$x_1 = 1, \; x_2 = 3, \; x_3 = 2, \; x_4 = 3$$

Any other solution is a multiple of this solution.

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3. Write the rotation matrix in Q1 as a product of elementary matrices.

From the solution of Q1.

$$L_{1,2}\left(\frac{\sin\theta}{\cos\theta}\right) M_1\left(\frac{1}{\cos\theta}\right) M_2(\cos\theta)\, L_{2,1}\left(\frac{-\sin\theta}{\cos\theta}\right) R_{x,\theta} = I_3$$

$$\Rightarrow R_{x,\theta} = \left(L_{1,2}\left(\frac{\sin\theta}{\cos\theta}\right) M_1\left(\frac{1}{\cos\theta}\right) M_2(\cos\theta)\, L_{2,1}\left(\frac{-\sin\theta}{\cos\theta}\right)\right)^{-1}$$

$$= L_{2,1}\left(\frac{-\sin\theta}{\cos\theta}\right)^{-1} M_2(\cos\theta)^{-1} M_1\left(\frac{1}{\cos\theta}\right)^{-1} L_{1,2}\left(\frac{\sin\theta}{\cos\theta}\right)^{-1}$$

$$= L_{2,1}\left(\frac{\sin\theta}{\cos\theta}\right) M_2\left(\frac{1}{\cos\theta}\right) M_1(\cos\theta)\, L_{1,2}\left(\frac{-\sin\theta}{\cos\theta}\right)$$

4. Solve the following systems of linear equations.

(a)
$$\begin{aligned} 8x + y + 6z &= 20 \\ 3x + 5y + 7z &= 40 \\ 4x + 9y + 2z &= 60 \end{aligned}$$

(b)
$$\begin{aligned} 2x + 3y - z &= 2 \\ x - y + z &= 5 \\ x + 9y - 5z &= 10 \end{aligned}$$

(a) Augmentation matrix for this system of linear equations is

$$\left(\begin{array}{ccc|c} 8 & 1 & 6 & 20 \\ 3 & 5 & 7 & 40 \\ 4 & 9 & 2 & 60 \end{array}\right).$$

Magic square!

Upon converting it to a row echelon matrix, we get

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

Giving the unique solution $x_1 = 1$, $x_2 = 6$, $x_3 = 1$

(b) In this case, the augmented matrix is

$$\left(\begin{array}{ccc|c} 2 & 3 & -1 & 2 \\ 1 & -1 & 1 & 5 \\ 1 & 9 & -5 & 10 \end{array}\right).$$ On this matrix, we perform

$$S_{1,2} \longrightarrow L_{2,1}(-2) \longrightarrow L_{3,1}(-1) \longrightarrow M_2\left(\tfrac{1}{5}\right) \longrightarrow L_{1,2}(1) \longrightarrow L_{3,2}(-10)$$

and get

$$\left(\begin{array}{ccc|c} 1 & 0 & 2/5 & 17/5 \\ 0 & 1 & -3/5 & -8/5 \\ 0 & 0 & 0 & 21 \end{array}\right)$$

This is to give you an idea of solution. You are expected to provide complete solution, whenever asked.

This converts the given system of equations to

$$x_1 + \frac{2}{5}x_3 = 17/5, \quad x_2 - \frac{3}{5}x_3 = -8/5, \quad 0 = 21$$

The given system doesn't have a solution $\Leftarrow$ This is absurd!