General Relativity Fall 2019
Homework 3 solutions

Exercise 1

(*i*) *Prove that the 2-dimensional plane from which the origin was removed, $\mathbb{R}^2 \setminus \{(0,0)\}$, is an open set in $\mathbb{R}^2$.*

Let $X = (x, y) \in \mathbb{R}^2 \setminus \{(0,0)\}$. Let us define $r \equiv \sqrt{x^2 + y^2} > 0$. Then the open ball centered at X with radius r is included in $\mathbb{R}^2 \setminus \{(0,0)\}$. Indeed, this open ball includes all points whose distance from X is *strictly less* than r, thus it does not include the origin $(0,0)$, which by definition means that the open ball is in $\mathbb{R}^2 \setminus \{(0,0)\}$.

(*ii*) *Starting from their formal definition, show that a tangent vector applied to a constant function is zero, $\overline{V}$(constant) $= 0$.*

Let us define $\mathbf{1} : \mathcal{M} \to \mathbb{R}$ the constant function returning 1. It is clearly a smooth function. This function has the particularity that $\mathbf{1} \times \mathbf{1} = \mathbf{1}$. From Leibniz's rule, we thus have, for any vector $\overline{V}|_p$ at $p \in \mathcal{M}$,

$$\overline{V}|_p(\mathbf{1}) = \overline{V}|_p(\mathbf{1} \times \mathbf{1}) = \mathbf{1}|_p \times \overline{V}|_p(\mathbf{1}) + \mathbf{1}|_p \times \overline{V}|_p(\mathbf{1}) = 2\overline{V}|_p(\mathbf{1}). \tag{1}$$

This implies that $\overline{V}|_p(\mathbf{1}) = 0$. As a consequence of linearity, $\overline{V}|_p(\text{constant}) = \text{constant} \times \overline{V}|_p(\mathbf{1}) = 0$.

(*iii*) *Given a basis $\{e_{(\mu)}\}$ of a n-dimensional vector space $\mathcal{V}$, show that the dual vectors $\{e^{*(\mu)}\}$ defined by $e^{*(\mu)}(e_{(\nu)}) = \delta^\mu_\nu$ form a basis of the dual space $\mathcal{V}^*$.*

We must prove two properties: (*a*) the dual vectors $\{e^{*(\mu)}\}$ are linearly independent and (*b*) any dual vector $\underline{W}$ can be written as a linear combination of them.

(*a*) Suppose there exists a linear combination $\lambda_\mu e^{*(\mu)} = 0$. Apply this to any of the basis vectors $e_{(\nu)}$:

$$0 = \lambda_\mu e^{*(\mu)} \cdot e_{(\nu)} = \lambda_\nu. \tag{2}$$

Thus all the λ_ν's must be zero, i.e. there is no vanishing linear combination of the $\{e^{*(\mu)}\}$ with non-zero coefficients, which means they are linearly independent.

(*b*) Let $\underline{W} \in \mathcal{V}^*_p$. For any vector $\overline{V} = V^\mu e_{(\mu)}$, we have, by linearity

$$\underline{W} \cdot \overline{V} = V^\mu \underline{W} \cdot e_{(\mu)}. \tag{3}$$

But on the other hand $V^\mu = e^{*(\mu)} \cdot \overline{V}$. Thus

$$\underline{W} \cdot \overline{V} = (\underline{W} \cdot e_{(\mu)}) e^{*(\mu)} \cdot \overline{V} = (W_\mu e^{*(\mu)}) \cdot \overline{V}, \qquad W_\mu \equiv \underline{W} \cdot e_{(\mu)}. \tag{4}$$

This holds for any vector $\overline{V}$, thus it means that $\underline{W} = W_\mu e^{*(\mu)}$.

Exercise 2

Consider $\mathcal{M} = \mathbb{R}^3$ as a 3-dimensional manifold, and define $\{x, y, z\}$ to be the usual cartesian coordinates. The spherical polar coordinates $\{r, \theta, \varphi\}$ are related to the cartesian coordinates by

$$x = r \sin\theta \cos\varphi, \quad y = r \sin\theta \sin\varphi, \quad z = r\cos\theta. \tag{5}$$

(*i*) *Express the coordinate basis vectors $\{\partial_r, \partial_\theta, \partial_\varphi\}$ in terms of $\{\partial_{(x)}, \partial_y, \partial_z\}$, as well as the reciprocal relation. Also express the dual bases $\{dr, d\theta, d\varphi\}$ and $\{dx, dy, dz\}$ in terms of one another – note that if $x = x^1$, then by dx I mean $dx^{(1)}$, etc...*

First of all, from now on I will use ∂_x etc... instead of $\partial_{(x)}$, when the coordinates have names. The meaning should be unambiguous. I will keep the parentheses only when referring to generic coordinates, $\partial_{(\mu)}$.

We just us the chain rule $\partial_{(\mu')} = \frac{\partial x^\mu}{\partial x^{\mu'}} \partial_{(\mu)}$:

$$\partial_r = \frac{\partial x}{\partial r}\partial_x + \frac{\partial y}{\partial r}\partial_y + \frac{\partial z}{\partial r}\partial_z = \sin\theta\cos\varphi\ \partial_x + \sin\theta\sin\varphi\ \partial_y + \cos\theta\ \partial_z, \tag{6}$$
$$\partial_\theta = \frac{\partial x}{\partial \theta}\partial_x + \frac{\partial y}{\partial \theta}\partial_y + \frac{\partial z}{\partial \theta}\partial_z = r\cos\theta\cos\varphi\ \partial_x + r\cos\theta\sin\varphi\ \partial_y - r\sin\theta\ \partial_z, \tag{7}$$
$$\partial_\varphi = \frac{\partial x}{\partial \varphi}\partial_x + \frac{\partial y}{\partial \varphi}\partial_y + \frac{\partial z}{\partial \varphi}\partial_z = -r\sin\theta\sin\varphi\ \partial_x + r\sin\theta\cos\varphi\ \partial_y. \tag{8}$$

To obtain the reciprocal relation, we can either write explicit expressions for r, θ, φ in terms of x, y, z and then use the chain rule like above, or, equivalently, find the inverse of the relations above. This is relatively straightforward to do if we start from the more compact expressions

$$\partial_r = \sin\theta(\cos\varphi\ \partial_x + \sin\varphi\ \partial_y) + \cos\theta\ \partial_z, \tag{9}$$
$$\partial_\theta = r\cos\theta(\cos\varphi\ \partial_x + \sin\varphi\ \partial_y) - r\sin\theta\ \partial_z, \tag{10}$$
$$\partial_\varphi = r\sin\theta(-\sin\varphi\ \partial_x + \cos\varphi\ \partial_y). \tag{11}$$

These expressions are pretty easy to invert: the first step is to compute the following linear combinations:

$$\cos\varphi\ \partial_x + \sin\varphi\ \partial_y = \sin\theta\ \partial_r + \frac{1}{r}\cos\theta\ \partial_\theta, \tag{12}$$
$$-\sin\varphi\ \partial_x + \cos\varphi\ \partial_y = \frac{1}{r\sin\theta}\partial_\varphi. \tag{13}$$

Combining the two lines, we get

$$\partial_x = \cos\varphi\left(\sin\theta\ \partial_r + \frac{1}{r}\cos\theta\ \partial_\theta\right) - \frac{\sin\varphi}{r\sin\theta}\partial_\varphi, \tag{14}$$
$$\partial_y = \sin\varphi\left(\sin\theta\ \partial_r + \frac{1}{r}\cos\theta\ \partial_\theta\right) + \frac{\cos\varphi}{r\sin\theta}\partial_\varphi. \tag{15}$$

Lastly, we find

$$\partial_z = \cos\theta\ \partial_r - \frac{1}{r}\sin\theta\ \partial_\theta. \tag{16}$$

From these expressions, we can read off the partial derivatives of (r, θ, φ) with respect to (x, y, z):

$$\frac{\partial r}{\partial x} = \cos\varphi\sin\theta, \qquad \frac{\partial \theta}{\partial x} = \frac{1}{r}\cos\varphi\cos\theta, \qquad \frac{\partial \varphi}{\partial x} = -\frac{\sin\varphi}{r\sin\theta}, \tag{17}$$
$$\frac{\partial r}{\partial y} = \sin\varphi\sin\theta, \qquad \frac{\partial \theta}{\partial y} = \frac{1}{r}\sin\varphi\cos\theta, \qquad \frac{\partial \varphi}{\partial y} = \frac{\cos\varphi}{r\sin\theta}, \tag{18}$$
$$\frac{\partial r}{\partial z} = \cos\theta, \qquad \frac{\partial \theta}{\partial z} = -\frac{\sin\theta}{r}, \qquad \frac{\partial \varphi}{\partial z} = 0. \tag{19}$$

Now we can find the dual basis vectors:

$$dr = \frac{\partial r}{\partial x}\ dx + \frac{\partial r}{\partial y}\ dy + \frac{\partial r}{\partial z}\ dz = \sin\theta(\cos\varphi\ dx + \sin\varphi\ dy) + \cos\theta\ dz, \tag{20}$$
$$d\theta = \frac{\partial \theta}{\partial x}\ dx + \frac{\partial \theta}{\partial y}\ dy + \frac{\partial \theta}{\partial z}\ dz = \frac{\cos\theta}{r}(\cos\varphi\ dx + \sin\varphi\ dy) - \frac{\sin\theta}{r}\ dz, \tag{21}$$
$$d\varphi = \frac{\partial \varphi}{\partial x}\ dx + \frac{\partial \varphi}{\partial y}\ dy + \frac{\partial \varphi}{\partial z}\ dz = \frac{1}{r\sin\theta}(-\sin\varphi\ dx + \cos\varphi\ dy). \tag{22}$$

and, reciprocally,

$$dx = \frac{\partial x}{\partial r}\ dr + \frac{\partial x}{\partial \theta}\ d\theta + \frac{\partial x}{\partial \varphi}\ d\varphi = \cos\varphi(\sin\theta\ dr + r\cos\theta\ d\theta) - r\sin\theta\sin\varphi\ d\varphi, \tag{23}$$
$$dy = \frac{\partial y}{\partial r}\ dr + \frac{\partial y}{\partial \theta}\ d\theta + \frac{\partial y}{\partial \varphi}\ d\varphi = \sin\varphi(\sin\theta\ dr + r\cos\theta\ d\theta) + r\sin\theta\cos\varphi\ d\varphi, \tag{24}$$
$$dz = \frac{\partial z}{\partial r}\ dr + \frac{\partial z}{\partial \theta}\ d\theta + \frac{\partial z}{\partial \varphi}\ d\varphi = \cos\theta\ dr - r\sin\theta\ d\theta. \tag{25}$$

Important comments:

• Always check that your results have the correct dimensions. If r, x, y, z have dimensions of length and θ, φ are dimensionless, then dx, dy, dz, dr also have dimensions of length, and $d\theta, d\varphi$ are dimensionless, $\partial_r, \partial_x, \partial_y, \partial_z$ have dimensions of inverse length, and $\partial_\theta, \partial_\varphi$ are dimensionless.

• We know that if $e_{(\mu')} = M^{\mu}{}_{\mu'} e_{(\mu)}$, and the inverse relation is given by $e_{(\mu)} = M^{\mu'}{}_{\mu} e_{(\mu')}$, with $M^{\mu'}{}_{\mu} M^{\mu}{}_{\nu'} = \delta^{\mu'}_{\nu'}$, then $e^{*(\mu')} = M^{\mu'}{}_{\mu} e^{*(\mu)}$. Let us see what it means when written in matrix form. Let us define

$$A = \begin{pmatrix} M^{0}{}_{0'} & M^{0}{}_{1'} & \cdots \\ M^{1}{}_{0'} & M^{1}{}_{1'} & \cdots \\ \vdots & & \ddots \end{pmatrix}, \qquad B = \begin{pmatrix} M^{0'}{}_{0} & M^{0'}{}_{1} & \cdots \\ M^{1'}{}_{0} & M^{1'}{}_{1} & \cdots \\ \vdots & & \ddots \end{pmatrix}. \tag{26}$$

In other words, with the usual matrix convention (first index is row, second index is column), we have $A_{\mu\mu'} = M^{\mu}{}_{\mu'}$ and $B_{\mu'\mu} = M^{\mu'}{}_{\mu}$. Thus, $\delta^{\mu}_{\nu} = M^{\mu}{}_{\mu'} M^{\mu'}{}_{\nu} = A_{\mu\mu'} B_{\mu'\nu}$, which means that $B = A^{-1}$. Now let us further define

$$X = \begin{pmatrix} e_{(0)} \\ e_{(1)} \\ \vdots \end{pmatrix}, \quad X' = \begin{pmatrix} e_{(0')} \\ e_{(1')} \\ \vdots \end{pmatrix}, \quad Y = \begin{pmatrix} e^{*(0)} \\ e^{*(1)} \\ \vdots \end{pmatrix}, \quad Y' = \begin{pmatrix} e^{*(0')} \\ e^{*(1')} \\ \vdots \end{pmatrix}. \tag{27}$$

In matrix form, the equations $e_{(\mu')} = M^{\mu}{}_{\mu'} e_{(\mu)}$ and $e^{*(\mu')} = M^{\mu'}{}_{\mu} e^{*(\mu)}$ translate to

$$Y' = BY, \qquad X' = A^{\mathrm{T}} X = (B^{-1})^{\mathrm{T}} X. \tag{28}$$

We thus see that, if we wrote the coefficients for the change of basis in matrix form, the coefficients for the change of dual basis are obtained by taking the **transpose** of the inverse of this matrix.

A scalar product $\langle , \rangle$ on a vector space $\mathcal{V}$ is a bilinear symmetric operator on vectors, such that $\langle \overline{V}, \overline{W} \rangle \in \mathbb{R}$, and satisfying $\langle \overline{V}, \overline{V} \rangle > 0$ for $\overline{V} \neq 0$. The norm of a vector is then $||\overline{V}|| = \sqrt{\langle \overline{V}, \overline{V} \rangle}$. Two vectors $\overline{V}, \overline{W}$ are orthogonal if $\langle \overline{V}, \overline{W} \rangle = 0$.
(ii) Suppose we have defined a scalar product on the tangent space $\mathcal{V}_p$, for which the vectors $\{\partial_x, \partial_y, \partial_z\}$ are orthonormal, i.e. all have a unit norm, and are orthogonal to one another. Compute the norms of the vectors $\{\partial_r, \partial_\theta, \partial_\varphi\}$, as well as all the scalar products between them, $\langle \partial_r, \partial_\theta \rangle$, etc...

We use Eq. (8)-(8) above, to compute

$$\begin{pmatrix} \langle \partial_r, \partial_r \rangle & \langle \partial_r, \partial_\theta \rangle & \langle \partial_r, \partial_\varphi \rangle \\ & \langle \partial_\theta, \partial_\theta \rangle & \langle \partial_\theta, \partial_\varphi \rangle \\ & & \langle \partial_\varphi, \partial_\varphi \rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ & r^2 & 0 \\ & & r^2 \sin^2\theta \end{pmatrix}, \tag{29}$$

where I didn't write the lower part of the matrix of scalar products, as it is symmetric.

(iii) We define the vector

$$e_{\hat{r}} \equiv \frac{1}{||\partial_r||} \partial_r, \tag{30}$$

and similarly for $e_{\hat{\theta}}$ and $e_{\hat{\varphi}}$. From your answers to (ii), it should be clear that they form an orthonormal basis (which we symbolize by little hats on the indices). We denote he components of a vector $\overline{V}$ on this basis with little hats:

$$\overline{V} = V^{\hat{r}} e_{\hat{r}} + V^{\hat{\theta}} e_{\hat{\theta}} + V^{\hat{\varphi}} e_{\hat{\varphi}}. \tag{31}$$

Show rigorously *that, given a scalar function f,*

$$\overline{V} \cdot \nabla f = V^{\hat{r}} \frac{\partial f}{\partial r} + \frac{V^{\hat{\theta}}}{r} \frac{\partial f}{\partial \theta} + \frac{V^{\hat{\varphi}}}{r \sin\theta} \frac{\partial f}{\partial \varphi}. \tag{32}$$

Note that "$\cdot$" is not *the scalar product: it indicates the contraction of vectors and dual vectors.*

First, let us write the unit vectors explicitly:

$$e_{\hat{r}} = \partial_r, \quad e_{\hat{\theta}} = \frac{1}{r}\partial_\theta, \quad e_{\hat{\varphi}} = \frac{1}{r\sin\theta}\partial_\varphi. \tag{33}$$

This allows us to find the components of a vector on the coordinate basis:

$$\overline{V} = V^{\hat{r}} e_{\hat{r}} + V^{\hat{\theta}} e_{\hat{\theta}} + V^{\hat{\varphi}} e_{\hat{\varphi}} = V^{\hat{r}}\partial_r + \frac{V^{\hat{\theta}}}{r}\partial_\theta + \frac{V^{\hat{\varphi}}}{r\sin\theta}\partial_\varphi \equiv V^r\partial_r + V^\theta\partial_\theta + V^\varphi\partial_\varphi. \tag{34}$$

Now we know that *in a coordinate basis*, $\overline{V} \cdot \nabla f = V^\mu \partial_\mu f$. Thus we find

$$\overline{V} \cdot \nabla f = V^r\partial_r f + V^\theta\partial_\theta f + V^\varphi\partial_\varphi f = \left(V^{\hat{r}}\frac{\partial}{\partial r} + \frac{V^{\hat{\theta}}}{r}\frac{\partial}{\partial\theta} + \frac{V^{\hat{\varphi}}}{r\sin\theta}\frac{\partial}{\partial\varphi}\right) f. \tag{35}$$

Exercise 3

We will soon see that the inverse-square radius of curvature of spacetime is of order the tidal field, $\mathcal{R}^{-2} \sim \partial^2\phi$, where ϕ is the Newtonian potential. (i) Estimate the radius of curvature of spacetime near the center of the Sun and near the center of the Earth. (ii) Consider an object of mass M. What (approximate) conditions must be satisfied by its characteristic radius r for the radius of curvature of spacetime to be smaller than r? Numercially (in km), what is this condition on r if $M = M_\odot$?

(i) We were asked to assume that the radius of curvature is of order $\mathcal{R} \sim 1/\sqrt{\partial^2\phi}$. From Poisson's equation, $\nabla^2\phi = 4\pi G\rho$, we thus find,

$$\mathcal{R} \sim \frac{1}{\sqrt{4\pi G\rho}}, \tag{36}$$

where ρ is the density. We have to remember that this assumes $c = 1$, so, re-establishing c, we actually have

$$\mathcal{R} \sim \frac{c}{\sqrt{4\pi G\rho}}, \tag{37}$$

The characteristic density at the center of the Sun is of order $\rho_\odot \sim 10^2$ g/cm^3, and that at the center of Earth is of order $\rho_\oplus \approx 10$ g/cm^3. Thus we find

$$\mathcal{R}_\odot \sim 3 \times 10^{12}\text{cm}, \qquad \mathcal{R}_\oplus \sim 10^{13}\text{cm}. \tag{38}$$

These are comparable to one another, and of order an *astronomical unit*, i.e. the distance between the Sun and the Earth! So spacetime is only very gently curved by the Sun and the Earth.

(ii) The characteristic density of the object is $\rho \sim 3M/(4\pi r^3)$. Thus we find

$$\mathcal{R} \sim \frac{c}{\sqrt{4\pi G\rho}} \sim r\sqrt{rc^2/GM}. \tag{39}$$

Requiring $\mathcal{R} \lesssim r$ implies

$$r \lesssim GM/c^2. \tag{40}$$

For the Sun, we find

$$\boxed{GM_\odot/c^2 \approx 1.5 \ \ \text{km}}. \tag{41}$$

That's a very useful number to remember to make order of magnitude estimates. We will see later on that this would be (half) the Schwarzschild radius of a black hole with the mass of the Sun.