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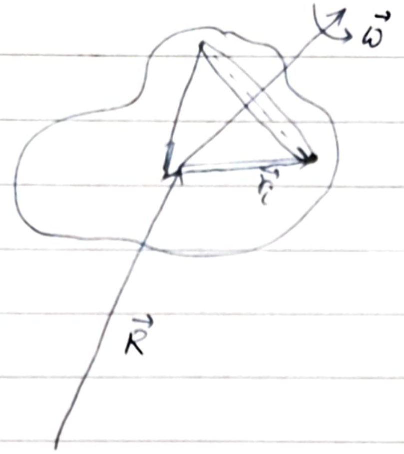

We can generalise this to an arbitrary rigid body which can be made up of single particles $\vec{r}_i$.

Then at each position of $\vec{r}_i$ the velocity of the point is

$$\vec{v}_i = \vec{\omega} \times \vec{r}_i .$$

Therefore, the total angular momentum is given by

$$\vec{L} = \sum_i \vec{r}_i \times \vec{p}_i = \sum_i \vec{r}_i \times (m_i \vec{v}_i)$$

$$= \sum_i m_i \, \vec{r}_i \times (\vec{\omega} \times \vec{r}_i)$$

$$= \sum_i m_i \left[(\vec{r}_i \cdot \vec{r}_i)\,\vec{\omega} - (\vec{r}_i \cdot \vec{\omega})\,\vec{r}_i \right]$$

$$= \sum_i \left[m_i r_i^2 \vec{\omega} - m_i (\vec{r}_i \cdot \vec{\omega})\,\vec{r}_i \right]$$

~~We will now change the indices from subscript to superscript to denote particle numbering and the subscript will denote components~~

$$= \sum_i \left[m_i \, r_i^2 \, \vec{\omega} - (x_i \omega_x + y_i \omega_y + z_i \omega_z)\,\vec{r}_i \right]$$

$$= \sum_i \left[m_i \, r_i^2 (\omega_x \hat{x} + \omega_y \hat{y} + \omega_z \hat{z}) - m_i (x_i \omega_x + y_i \omega_y + \omega_z z_i)(x_i \hat{x} + y_i \hat{y} + z_i \hat{z}) \right]$$

$$= \sum_i m_i \Big[(r_i^2 \omega_x - x_i^2 \omega_x - x_i y_i \omega_y - x_i z_i \omega_z)\,\hat{x}$$
$$+ \left[(r_i^2 - y_i^2)\,\omega_y - x_i y_i \omega_x - y_i z_i \omega_z \right] \hat{y}$$
$$+ \left[(r_i^2 - z_i^2)\,\omega_z - x_i z_i \omega_x - y_i z_i \omega_y \right] \hat{z} \Big]$$

$$\sum_i m_i \left[r_i^2 \vec{\omega} - (\vec{r}_i \cdot \vec{\omega})\, \vec{r}_i \right]$$

$$\sum_i m_i \left[r_i^2 \sum_k \omega_k \hat{e}_k - \sum_j r_j^i \omega_j) \sum_k r_k^{(i)} \hat{e}_k \right]$$

$$\sum_i m_i \left[\sum_k \left(\omega_k r_i^2 - \sum_j r_j^i \omega_j \; r_k \right) \hat{e}_k \right]$$

$$= \sum_i m_i \sum_k \left[\sum_k r^{(i)2} \omega_k - \sum_j r_j^{(i)} \omega_j r_k^{(i)} \right] \hat{e}_k$$

$$= \sum_i m_i \sum_k \left[\omega_k \sum_j r_j^{(i)2} - \sum_j r_j^{(i)} r_k^{(i)} \omega_j \right] \hat{e}_k$$

$$= \sum_i m_i \left[\sum \left(r^{(i)2} - x^{(i)2} \right) \omega_j \right.$$

$$L_x = \sum_i m_i \left(r_i^2 - x_i^2 \right) \omega_x - \sum_i m_i x_i y_i \omega_y - \sum_i m_i x_i z_i \omega_z .$$

$$L_y = \sum_i - m_i x_i y_i \omega_x + \sum_i m_i \left(r_i^2 - y_i^2 \right) \omega_y - \sum m_i y_i z_i \omega_z$$

$$L_z = - \sum_i m_i x_i z_i \omega_x + - \sum_i m_i y_i z_i \omega_y + \sum m_i \left(r_i^2 - z_i^2 \right) \omega_z$$

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

The off diagonal terms are known as products of inertia.

$$I_{xy} = - \sum_i m_i x_i y_i$$

$$= - \int d\vec{r}\, \rho(r)\, x\, y$$

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$$I_{xx} = \int d\vec{r}\ \rho(r)\,(r^2 - x^2)$$

and similarly the others follow. Note that $I_{xx} > 0$, $I_{yy} > 0$ $I_{zz} > 0$.

For ~~an~~ a single particle system

$$I_{xx} = m\,(y^2 + z^2) = m\,(r^2 - x^2) = m\,(r^2 - x_1^2)$$

$$I_{xy} = -\,m\,x\,y = -\,m\,x_1 x_2 .$$

Now $\vec{r}$ being a vector, we consider an orthogonal transformation so that $x \to x'$, $y \to y'$ and $z \to z'$ such that $\vec{r}\,' = \underline{\underline{R}}\,\vec{r}$

$T_{11} =$

$$x_i' = R_{ij}\, x_j$$

$$T_{11} \Rightarrow m\,(r^2 - x_1^2) \longrightarrow m\,(r'^2 - x_1'^2)$$

$$= m\,(r^2 - R_{1\ell}\, R_{1k}\, x_\ell x_k)$$

2 ~~$R_{i\ell}x_\ell R_{jk}x_k$~~

$\sum x_i\, x_j\, \delta_{ij} - x_1\, x_1$

$$= ~~m\,(x_i^2\,\delta_{ij}~~\ \ m\,\Big(\sum_{ij} x_i' x_j'\,\delta_{ij} - \sum_{\ell k} R_{1\ell}\, R_{1k}\, x_\ell x_k\Big).$$

$$= m\,\Big(\sum_{i,j}\sum_{\ell} R_{i\ell}\, x_\ell \sum_k R_{jk}\, x_k\, \delta_{ij} - \sum_{\ell k} R_{1\ell}\, x_\ell R_{1k}\, x_k$$

Stuck

Needs Careful Consideration.

$$= m \sum_\ell \sum_k \Big[\sum_{i,j} R_{i\ell}\, R_{jk}\, x_\ell\, x_k\, \delta_{ij} - R_{1\ell}\, R_{1k}\, x_\ell\, x_k \Big].$$

$$= \sum_{\ell, k}$$

The general form of the moment of inertia tensor is given (for a single particle)

$$I_{ij} = m\left(r^2 \delta_{ij} - x_i x_j\right)$$

Under a linear transformation a square T transform as

$$\underline{T}' = \underline{A}\,\underline{T}\,\underline{A}^{-1} \longrightarrow$$ This is called a similarity transformation.

For an orthogonal transformation such as rotation where $R^{-1} = R^T$, we get

$$\underline{\underline{T}}'' = \mathbb{R}\, T\, \mathbb{R}^T$$

which implies that

$$T'_{ij} = \sum_l \sum_k \mathbb{R}_{ik} \mathbb{R}_{jl} T_{kl}$$

so that the moment of inertia matrix must transform as

$$I'_{ij} = \sum_l \sum_k \mathbb{R}_{ik} \mathbb{R}_{jl} I_{kl}.$$

To see this in a easier way consider the kinetic energy of the system.

$$E_k^{Rot} = \frac{1}{2} \sum_{ij} I_{ij}\, \omega_i \omega_j$$

For an isotropic body $I_{ij} = I_0 \delta_{ij}$ so that we have $E_k^{Rot} = \frac{1}{2} I_0 \sum_i \omega_i^2$

If the direction of $\vec{\omega}$ is along one fixed axis we get

$$E_k^{Rot} = \frac{1}{2} I_0 \omega^2$$

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Inverse of the transformation means

$$\vec{\omega} = R^{-1}\vec{\omega}'$$

$$\omega_i = \sum_j R_{ji}\, \omega_j'$$

$$E_K^{rot} = \frac{1}{2} \sum_{ij} I_{ij} \sum_l R_{li}\, \omega_l' \sum_m R_{mj}\, \omega_m'$$

$$= \frac{1}{2} \sum_{l,m} \sum_{i,j} I_{ij} R_{li} R_{mj} \quad \omega_l' \omega_m'$$

$$= \frac{1}{2} \sum_{l,m} \underbrace{\sum_{ij} R_{li} R_{mj} I_{ij}}_{I'_{lm}} \omega_l' \omega_m'$$

$$I'_{lm} = \sum_{i,j} R_{li} R_{mj} I_{ij}$$

The principle axes of motion.

If we look at the expression of the angular momentum vector for a rigid body of arbitrary shape ~~thus we see~~ :

$$\vec{L} = \underline{\underline{I}}\,\vec{\omega}$$

Then it is clear that ~~we~~ unless that unless the MI matrix has the form $I_{ij} = I_0 \delta_{ij}$, the angular momentum vector is not ~~pa~~ parallel to the ~~$\vec{\omega}$~~ angular velocity. ~~Fortunately, we can rotate the system so that~~ For any arbitrary case, ~~is~~ the component of the angular momentum is given by

$$L_i = \sum I_{ij}\, \omega_j$$

So that $\dot{L}_i = \sum_j I_{ij} \dot{\omega}_j$ and therefore, the dynamical equation take an extremely complicated form

$$\tau_i = \sum_j I_{ij} \dot{\omega}_j$$

If on the other hand, the MI tensor is diagonal, the equations are ~~still complicated, they~~ no longer coupled differential equations. Fortunately, since we can rotate the coordinate system, we can transform the MI tensor to make it diagonal. This is an eigenvalue problem. Given a moment of inertia tensor, we want to solve the eigenvalue problem

$$\underline{\underline{I}}\,\vec{x} = \lambda \vec{x}$$

$$\begin{bmatrix} I_{xx}-\lambda & I_{xy} & I_{xz} \\ I_{yx} & I_{yy}-\lambda & I_{yz} \\ I_{zx} & I_{zy} & I_{zz}-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0.$$

The secular equation is given by $\mathrm{Det}\left[\underline{\underline{I}} - \lambda \mathbb{1}\right] = 0$. Which gives three roots $\lambda_1, \lambda_2, \lambda_3$. These are the eigenvalues of the equation. Corresponding to a given eigenvalue we can determine the corresponding eigenvector $\vec{x}$. The three such eigenvectors are known as the principle axes of inertia. When we rotate our coordinate system along these directions, the moment of inertia matrix becomes diagonal. We shall come back to this later.

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Constraints: In our forgoing discussion it appears that if we solve a set of differential equations, then the problem can be solved. Unfortunately, it is not so straight forward. Most often ~~those~~ the coordinates are restricted to move in a certain way. For example, a particle can move on a one dimensional wire embedded on a three dimensional surface. Therefore, it is obvious that motion of the particle is constrained. The constraints limit the motion of the system. Other examples include particle rolling down a sphere, motion of gas particles confined by the walls of the container.

Constraints can be classified according to the following schemes. If the conditions of the constraint can be expressed as an equation relating the coordinates of the particles having the form

$$f(\vec{r}_1, \vec{r}_2, \ldots \vec{r}_N, t) = 0.$$

Then the constraint is called a holonomic constraint. For example, the definition of a rigid body is taken as the one in which the separation vector remains constant, so that

$$(\vec{r}_i - \vec{r}_j)^2 - c_{ij}^2 = 0$$

Constraints which are not possible to ~~expressed~~ be written in the form of a ~~coordinate~~ equation relating the coordinates are called non-holonomic constraints. A particle rolling down a sphere and eventually falling off from the surface is an example of a non-holonomic constraint.

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If r is the distance of the particle from the center of the sphere, we have $r^2 - a^2 \geq 0$.

Particles of gas confined by a wall is also a non-holonomic constraint. $0 \leq x_i \leq L$, $0 \leq y_i \leq L$, $0 \leq z_i \leq L$.

Constraints are further classified according to whether the equations of the constraint contain time as an explicit variable or not. If the equation of the constraint contain time as an explicit variable it is called rheonomous constraint and if it does not contain time explicitly it is called scleronomous constraint.

Constraints introduce two types of difficulties in the solution of a mechanical ~~system~~ problem.

1. The coordinates $\vec{r}_i$ are no longer independent. The constraint equation restricts the ~~no~~ evolution of $\vec{r}_i$'s.

2. The forces of constraint is not known a priori and can be obtained only from the solutions we obtain. They are unknown quantities. Constraints are simply another ~~ment~~ method of stating that there are forces ~~are~~ ~~unknown.~~ present in the problem which are unknown.

Note that in certain cases the constraints can also be given in a differential form — for example if there is condition on the ~~velocity~~ velocities.

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The constraint then have the form

~~$\Sigma c a_k (x_1, x_2, \dots x_N)$~~

$$\sum_k a_k (x_1, x_2, \dots) \, dx_k = 0$$

where x_k represent the various coordinates and a_k are functions of these coordinates.

We now distinguish two cases:

If the ~~differ~~ differential is an exact (total) differential of a function U, then we immediately integrate it to obtain an equation involving the coordinates. Consequently, the constraint is a holonomic constraint.

If we can not do it, ~~then~~ ~~then~~ that is the differential is an inexact differential, the constraint is a non-holonomic constraint.

When the differential is an exact differential, that is

$$\Sigma a_k \, dx_k = dU.$$

$$\Rightarrow \quad a_k = \frac{\partial U}{\partial x_k}$$

$$\Rightarrow \quad \left.\frac{\partial a_i}{\partial x_k} = \frac{\partial a_k}{\partial x_i}\right\} \text{ Constraint is holonomic.}$$

~~Co~~ Coming back to the difficulties a constraint ~~possess~~ poses, ~~the fi~~ for a holonomic constraint the first difficulty is solved by introducing the generalised coordinates.

If we consider N particles free from any constraints then there are $3N$ degrees of freedom. Now suppose I have k constraint ~~co~~ equations

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of the form

$$f_1(x_1, x_2, \dots x_{3N}) = 0 \quad \Rightarrow \quad x_1 = g(x_2, \dots x_{3N})$$
$$f_2(x_1, x_2, \dots x_{3N}) = 0 \qquad x_2 = h(x_3, \dots x_{3N})$$
$$\vdots \qquad\qquad \vdots$$
$$f_k(x_1, x_2, \dots x_{3N}) = 0.$$

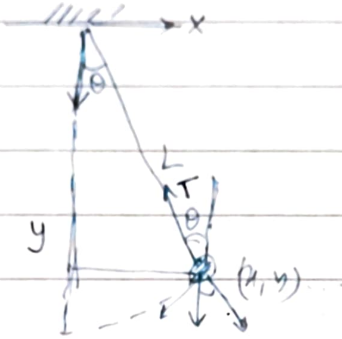
then it is obvious that I can eliminate k of the $3N$ coordinates, so that I am left out with $3N-k$ degrees of freedom. These $3N-k$ coordinates are independent of each other and for these we introduce $3N-k$ independent variables $(q_1, q_2, \dots q_{3N-k})$. These are the generalised coordinates. We can express the coordinates of the particles as

$$\vec{r}_1 = \vec{r}_1(q_1, q_2, \dots q_{3N-k}, t)$$
$$\vec{r}_2 = \vec{r}_2(q_1, q_2, \dots q_{3N-k}, t)$$
$$\vdots$$
$$\vec{r}_N = \vec{r}_N(q_1, q_2, \dots q_{3N-k}, t).$$

~~These are transf~~ The constraints are contained implicitly in these equations. These are transformations from the set $\{\vec{r}_i\}$ to the ~~ras~~ set $\{q_i\}$. The equations can also be viewed ~~in terms of the~~ as parametric representations of $\{\vec{r}_i\}$ variables. We can transform back from $\{q_i\}$ to $\{\vec{r}_i\}$ using the above set of equations together with the constraint equations.

Usually, the generalized coordinates q_i will divide into convinient groups of three which can together form a vector. All sorts of quantities can be used as a general coordinate. Often, they do not even have the dimension of the length. Generalised coordinates can not be viewed as a set orthogonal positional coordinates.

Simple Pendulum



$$\frac{y}{L} = \cos\theta$$

$$\frac{x}{L} = \sin\theta$$

Coordinate of the bob (x, y)

Constrain $\quad x^2 + y^2 - L^2 = 0.$

$$\vec{F} = mg\,\hat{y} - T\cos\hat{y} + T\sin\theta\,\hat{x}$$

$$= (mg - T\cos\theta)\,\hat{y} - T\sin\theta\,\hat{x}$$

$$m\frac{d^2x}{dt^2} = -T\sin\theta \qquad = -\frac{Tx}{L}$$

$$m\frac{d^2y}{dt^2} = mg - T\cos\theta = mg - T\frac{y}{L}$$

Here the tension is the unknown force ~~and~~ ~~acts as th~~ the manifestation of which is the constrain $x^2 + y^2 = L^2$. we ~~don~~ do not know what T is until we solve the equation.

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Now, in contrast we make the following observation.

1. The motion happens in a plane. ~~and this~~ Why?
~~Because the z angular momentum z pointing in z~~ ~~is conserved.~~ Consequently there are 2 Degrees of freedom with constraint. — ~~L = constant~~ $x^2+y^2=L^2$.

2. If we choose a spherical polar coordinate system then the constraint equation simply reads as
$r - L = 0$.
and we have only one degree of freedom defined by the angle θ.

The ~~torque~~ angular momentum on the system is given by $\vec{L} = L\hat{r} \times m v_\theta \hat{\theta}$

$= mL^2 \omega \hat{k}$

$\Rightarrow \quad \vec{L} = mL^2 \dot{\theta}\, \hat{k}$

$$\frac{d\vec{L}}{dt} = mL^2 \ddot{\theta}\, \hat{k}$$

$$\vec{\tau} = \vec{r} \times \vec{F} = L\hat{r} \times (mg\cos\theta\, \hat{r} - mg\sin\theta\, \hat{\theta})$$

$$= -mgL\sin\theta\, \hat{k}$$

$$\Rightarrow \quad mL^2\ddot{\theta} = -mgL\sin\theta$$

$$\ddot{\theta} + \frac{g}{L}\sin\theta = 0.$$

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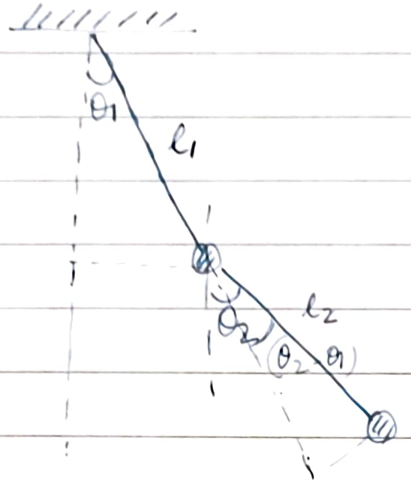
More complicated System: Double pendulum.



There are altogether 4 ~~deg~~ DOF.
And 2 Constraints - ~~l~~

$$(r_1 - l_1) = 0$$
$$(r_2 - l_2) = 0.$$

Therefore, the angles θ_1 and θ_2 serve as convinient generalised coordinates. Using ~~spherical~~ plane polar coordinates the kinetic energy of the system is

$$T = \frac{1}{2} m l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m l_2^2 \dot{\theta}_2^2 + m l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_2 - \theta_1)$$

$$P.E = mg\, l_1 \cos\theta_1 + mg\left[l_1 \cos\theta_1 + l_2 \cos(\theta_2 - \cancel{\theta_1})\right]$$
$$= 2mg\, l_1 \cos\theta_1 + mg\, l_2 \cos\theta_2$$

Total Energy: $$E = \frac{1}{2} m l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m l_2^2 \dot{\theta}_2^2 + m l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_2 - \theta_1)$$
$$+ 2mg l_1 \cos\theta_1 + mg\, l_2 \cos\theta_2 \; \cancel{mg\left[l_1 + l_2 \cos(\theta_2 - \theta_1)\right]}$$

$$E = \frac{1}{2} m l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m l_2^2 \dot{\theta}_2^2 + m l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)$$
$$+ 2mg l_1 \left(1 - \frac{\theta_1^2}{2}\right) + mg l_1 + mg l_2 \left(1 - \frac{(\theta_2 - \cancel{\theta_1})^2}{2}\right)$$

$$\cancel{= \frac{1}{2} m l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m l_2^2 \dot{\theta}_2^2}$$
$$\cancel{- \frac{mg(l_1 + l_2)\theta_1^2}{2} - \frac{mg l_2 \theta_2^2}{2} - mg l_2 \theta_1 \theta_2}$$

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To surmount the second difficulty, namely that the forces of constraint are unknown a-priori, we should like to formulate the mechanics that the forces of constraint disappear. We shall follow this up later.

Now consider the velocity of a mass point i according to the transformation equation

$$\vec{r}_i = \vec{r}_i(q_1, q_2, \dots q_f, t)$$

$$\frac{d\vec{r}_i}{dt} = \sum_{\alpha=1}^{f} \frac{\partial \vec{r}_i}{\partial q_\alpha} \dot{q}_\alpha + \frac{\partial \vec{r}_i}{\partial t}$$

When the constraint is time independent, we have

$$\dot{\vec{r}}_i = \sum_{\alpha=1}^{f} \frac{\partial \vec{r}_i}{\partial q_\alpha} \dot{q}_\alpha$$

where $\dot{q}_\alpha$ denote the generalised velocities. Consider the x component of the above equation.

$$\dot{x}_i = \sum_{\alpha=1}^{f} \frac{\partial x_i}{\partial q_\alpha} \dot{q}_\alpha$$

$$\ddot{x}_i = \sum_{\alpha=1}^{f} \frac{d}{dt}\left(\frac{\partial x_i}{\partial q_\alpha}\right) \dot{q}_\alpha + \sum_{\alpha=1}^{f} \frac{\partial x_i}{\partial q_\alpha} \ddot{q}_\alpha$$

$$\frac{d}{dt}\left(\frac{\partial x_i}{\partial q_\alpha}\right) = \sum_{\beta} \frac{\partial^2 x_i}{\partial q_\beta \partial q_\alpha} \dot{q}_\beta$$

so that

$$\ddot{x}_i = \sum_{\beta=1}^{f}\sum_{\alpha=1}^{f} \frac{\partial^2 x_i}{\partial q_\beta \partial q_\alpha} \dot{q}_\beta \dot{q}_\alpha + \sum_{\alpha=1}^{f} \frac{\partial x_i}{\partial q_\alpha} \ddot{q}_\alpha$$

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Now, let the generalised coordinates be changed q_α to $q_\alpha + dq_\alpha$. The work performed by the infinitesimal displacement is given by:

$$d\vec{r}_i = \sum_{\alpha=1}^{f} \frac{\partial \vec{r}_i}{\partial q_\alpha} dq_\alpha$$

$$\Rightarrow \quad dW = \sum_{i=1}^{N} \vec{F}_i \cdot d\vec{r}_i = \sum_{i=1}^{N} \sum_{\alpha=1}^{f} \vec{F}_i \cdot \frac{\partial \vec{r}_i}{dq_\alpha} dq_\alpha .$$

$$= \sum_{\alpha=1}^{f} \left(\sum_{i=1}^{N} \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_\alpha} \right) dq_\alpha$$

$$= \sum_{\alpha=1}^{f} Q_\alpha \, dq_\alpha$$

Where $$Q_\alpha = \sum_{i=1}^{N} \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_\alpha}$$

Q_α is called the generalised force. Since the generalised coordinate does not have the dimension of a length, Q_α may not have the dimension of force. However, $Q_\alpha \, dq_\alpha$ has the dimension of force.

Now, in a conservative system, W does not depend on time and therefore one has

$$dW = \sum \frac{\partial W}{\partial q_\alpha} dq_\alpha = \sum Q_\alpha dq_\alpha$$

$$\Rightarrow \quad dW - dW = \sum_\alpha \left(Q_\alpha - \frac{\partial W}{\partial q_\alpha} \right) dq_\alpha = 0$$

$$\Rightarrow \quad Q_\alpha = \frac{\partial W}{\partial q_\alpha}$$

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D'Alembert's Principle and Lagrange's Equation.

A virtual displacement of a system refers to a change in the configuration of the system as a result of any arbitrary ~~displacement~~ infinitesimal change of the coordinates $\delta \vec{r}_i$ that is consistent with the forces and constraints imposed on the system at the given ~~instant~~ instant t. The displacement is called virtual to distinguish it from an actual displacement of the system in a time interval dt, during which the forces and constraints may be changing.

Suppose the system is in equilibrium, that is the total force on each particle vanishes $\vec{F}_i = 0$. Then $\vec{F}_i \cdot \delta \vec{r}_i$ is also zero since $\delta \vec{r}_i$'s are ~~a~~ virtual displacement and not real displacements.
Which means:

$$\sum \vec{F}_i \cdot \delta \vec{r}_i = 0.$$

There is no new content here. We now break up the force as

$$\vec{F}_i = \vec{F}_i^{(a)} + \vec{f}_i^{\,c} \quad \text{so that}$$

$$\sum \vec{F}_i^{(a)} \cdot \delta \vec{r}_i + \sum \vec{f}_i^{\,c} \cdot \delta \vec{r}_i = 0.$$

We now restrict ourselves to systems for which the net virtual work done by the forces of constraint is zero. This condition holds for a large class of systems. Objects moving on a surface or line, the normal reaction exerted by the surface (the constraint) is perpendicular to the virtual displacement (which has to be consistent with the constraint) and therefore the virtual work done is zero.

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$$\vec{r}_1 = l_1 (\sin\theta_1, \cos\theta_1)$$

$$\frac{\partial \vec{r}_1}{\partial \theta_1} = l_1 (\cos\theta_1, -\sin\theta_1) = \hat{v}_1 l_1$$

$$\vec{r}_2 = \vec{r}_1 + l_2 (\sin\theta_2, \cos\theta_2)$$

$$\frac{\partial \vec{r}_2}{\partial \theta_1} = \frac{\partial \vec{r}_1}{\partial \theta_1} = l_1 (\cos\theta_1, -\sin\theta_1) = \hat{v}_1$$

$$Q_{\theta_1} = \left[\vec{F}_1 \cdot \frac{\partial \vec{r}_1}{\partial \theta_1} + \vec{F}_2 \cdot \frac{\partial \vec{r}_2}{\partial \theta_1} \right]$$

$$= T_1 (\hat{r}_1 \cdot \hat{v}_1) + T_2 (\hat{R} \cdot \hat{v}_1) + m(\vec{g} \cdot \hat{v}_1)$$
$$+ m(\vec{g} \cdot \hat{v}_1) - T_2 (\hat{R} \cdot \hat{v}_1)$$

$$= -2mg \sin\theta_1$$

$$Q_{\theta_2} = \left[\vec{F}_1 \cdot \frac{\partial \vec{r}_1}{\partial \theta_2} + \vec{F}_2 \cdot \frac{\partial \vec{r}_2}{\partial \theta_2} \right]$$

$$\frac{\partial \vec{r}_1}{\partial \theta_2} = 0. \qquad \frac{\partial \vec{r}_2}{\partial \theta_2} = l_2 (\cos\theta_2, -\sin\theta_2)$$

$$\vec{F}_2 = m\vec{g} - T_2 \hat{R} = m\vec{g} - T_2 (\sin\theta_2, \cos\theta_2).$$

$$\vec{F}_2 \cdot \frac{\partial \vec{r}_2}{\partial \theta_2} = -mg \sin\theta_2$$

$$Q_{\theta_2} = -mg \sin\theta_2 .$$

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For a sliding friction force, the displacement is not perpendicular to the constraint force and hence we must exclude them from the discussion. For a rolling friction though, this condition is not violated since the point of contact is at momentary rest and can do no work in an infinitesimal displacement consistent with the rolling constraint.

For holonomic constraints, the effect of the constraint reaction can be elucidated by the following: Consider the i^{th} constraint on the system

$$g_i(\vec{r}_1, \vec{r}_2, \dots \vec{r}_N, t) = 0.$$

The change of g_i with respect to a change of the position vector $\vec{r}_j$ must be the measure of the constraint reaction $\vec{f}^{\,c}_{ji}$ on the j^{th} particle due to the constraint $g_i = 0$.

$$\Rightarrow \quad \vec{f}^{\,c}_{ji} = \lambda_i \frac{\partial g_i}{\partial \vec{r}_j} \quad \lambda_i \vec{\nabla}_j \, g_i(\vec{r}_1, \vec{r}_2, \dots \vec{r}_N, t)$$

Here λ_i is an unknown factor, since the constraint's $g_i(\vec{r}_1, \vec{r}_2, \dots \vec{r}_N, t) = 0$ are known up to a non-vanishing factor. The total constraint force on the j^{th} particle is then the sum over all constraint reactions orginating from the individual k constraints.

$$\vec{f}^{\,C}_j = \sum_{i=1}^{k} \vec{f}^{\,c}_{ji} = \sum \lambda_i \frac{\partial g_i}{\partial \vec{r}_j}$$

Therefore, the virtual work done is given by

$$\delta W = \sum_{j=1}^{N} \vec{f}^{\,C}_j \cdot \delta \vec{r}_j = \sum_{i=1}^{K} \sum_{j=1}^{N} \lambda_i \frac{\partial g_i}{\partial \vec{r}_j} \cdot \delta \vec{r}_j$$

$$= \sum_{i=1}^{K} \lambda_i \, \delta g$$

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where $\delta g = \sum_{i=1}^{N} \frac{\partial g_i}{\partial \vec{r}_j} \cdot \delta \vec{r}_j$

But the virtual displacements are such that they are consistent with the constraints. That means, whatever $\{\delta \vec{r}_j\}$ you choose the constraints do not change., that is $\delta g = 0$.

$$\Rightarrow \quad \delta W = \sum_{j=1}^{N} \vec{f}_j^{\,c} \cdot \delta \vec{r}_j = 0.$$

Hence for holonomic constraints, the constraint reactions are perpendicular to the displacements that are compatible with the constraints.

The principle of virtual work only allows us to treat problems of statics. If we now introduce the inertial for $\dot{\vec{p}}_i$ and say that equilibrium is gained by adding a force $\dot{\vec{p}}_i$ in the opposite direction we get

$$\sum_i (\vec{F}_i - \dot{\vec{p}}_i) \cdot \delta \vec{r}_i = 0.$$

$$\Rightarrow \sum_i \left(\vec{F}_i^{\,a} - \dot{\vec{p}}_i\right) \cdot \delta \vec{r}_i = 0.$$

This is called D'Alembert's principle of virtual work.

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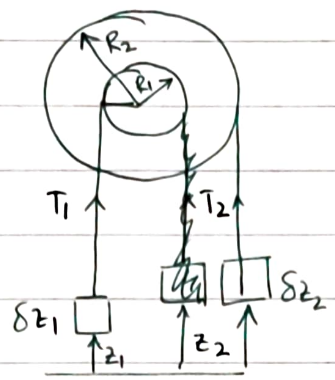


The constraint forces are T_1 and T_2.

$\sum \vec{f}_i^{\,c} \cdot \delta \vec{r}_i = 0.$

$\delta z_1 = R_1 \delta\phi_1$

$\delta z_2 = -R_2 \delta\phi_1.$

$$\Rightarrow \quad T_1 R_1 \delta\phi_1 - T_2 R_2 \delta\phi_1 = 0.$$

$$(T_1 R_1 - T_2 R_2)\,\delta\phi = 0.$$

$\Rightarrow T_1 R_1 = T_2 R_2 \quad \Rightarrow$ Torques must be the same.

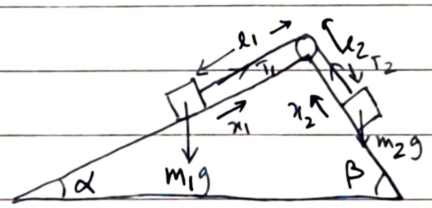
External force equilibrium:

$$\sum \vec{F}_i^{\,a} \cdot \delta \vec{r}_i = 0.$$

$$\Rightarrow \quad m_1 g R_1 \delta\phi_1 - m_2 g R_2 \delta\phi_1 = 0.$$

$$\Rightarrow \quad m_1 R_1 = m_2 R_2.$$

Ex 2



$x_1 + x_2 = l_1 + l_2 = $ Constant.

$\delta x_1 = -\delta x_2.$

Constraint forces are the tension T_1 and T_2.

$\therefore \sum_{i=1}^{2} \vec{f}_i^{\,c} \cdot \delta\vec{r}_i = 0 \quad \Rightarrow \quad T_1 \delta x - T_2 \delta x = 0.$

$\Rightarrow T_1 = T_2.$

External force balance :

$$\sum_{i=1}^{2} \vec{F}_i^{(a)} \cdot \delta \vec{r}_i = 0.$$

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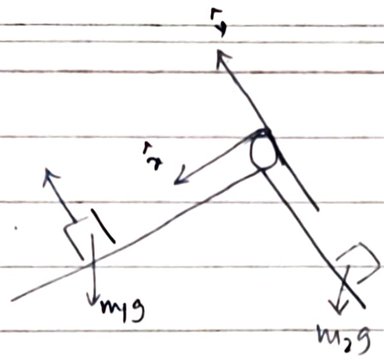
$\vec{E}_{ff}$



$$\vec{F}_1 = m_1 g \sin\alpha \, \hat{x} - m_1 g \cos\alpha \, \hat{y} + N \hat{y}$$

$$\vec{F}_2 = + m_2 g \cos\beta \, \hat{x} + N \hat{x} - m_2 g \sin\beta \, \hat{y}$$

$$\vec{r}_1 = x_1 \hat{x} \qquad \vec{r}_2 = -x_2 \hat{y} \qquad |\delta x_1| = |\delta x_2|$$

$$= x \hat{x} \qquad = -x \hat{y}$$

$$\vec{F}_1 \cdot \delta\vec{r}_1 + \vec{F}_2 \cdot \delta\vec{r}_2 = 0 \Rightarrow (m_1 g \sin\alpha - m_2 g \sin\beta)\delta x = 0.$$

$$\Rightarrow \boxed{\frac{m_1}{m_2} = \frac{\sin\beta}{\sin\alpha}}$$

What happens if the masses move?

$$\sum_{i=1}^{2} \left(\vec{F}_i^{(a)} - \dot{\vec{p}}_i \right) \cdot \delta\vec{r}_i = 0.$$

$$\left[(m_1 g \sin\alpha - m_1 \ddot{x}) - (m_2 g \sin\beta - m_2 \ddot{x}) \right] \delta x = 0.$$

$$(m_1 g \sin\alpha - m_2 g \sin\beta) - (m_1 + m_2)\ddot{x} = 0.$$

$$\Rightarrow \ddot{x} = \frac{(m_1 g \sin\alpha - m_2 g \sin\beta)}{(m_1 + m_2)}$$

Having introduced the generalised coordinates, we note that while we have prametrised the positions $\{\vec{r}_i\}$ in terms of $\{q_i\}$ going back and forth is ~~cub~~ cumbersome. Therefore, we shall try to express everything in terms of $\{q_i\}$.

$$\sum_{i=1}^{N} \vec{F}_i^{(a)} \cdot \delta \vec{r}_i \equiv \delta W = \sum_{\alpha=1}^{f} Q_\alpha \, \delta q_\alpha$$

Where $Q_\alpha = \cancel{\frac{\partial \vec{r}_i}{\partial r_i}} \sum_{i=1}^{N} \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_\alpha}$

Now consider the other term $\sum_{i=1}^{N} \dot{\vec{p}}_i \cdot \delta \vec{r}_i = \sum_i m_i \ddot{\vec{r}}_i \cdot \delta \vec{r}_i$

$$\cancel{\sum_i m_i \ddot{\vec{r}}_i} \qquad \delta \vec{r}_i = \sum_{\alpha=1}^{f} \frac{\partial \vec{r}_i}{\partial q_\alpha} \delta q_\alpha$$

$$\therefore \sum m_i \ddot{\vec{r}}_i \cdot \delta \vec{r}_i = \sum_{i=1}^{N} \sum_{\alpha=1}^{f} m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_\alpha} \delta q_\alpha$$

$$= \sum_{\alpha=1}^{f} \left(\sum_{i=1}^{N} m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_\alpha} \right) \delta q_\alpha$$

$$\sum_{i=1}^{N} m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_\alpha} = \sum_i \left[\frac{d}{dt}\left(m_i \dot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_\alpha} \right) - m_i \dot{\vec{r}}_i \cdot \frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_\alpha} \right) \right]$$

$$\frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_\alpha} \right) = \sum_\beta \frac{\partial^2 \vec{r}_i}{\partial q_\alpha \partial q_\beta} \dot{q}_\beta$$

$$= \frac{\partial}{\partial q_\alpha} \sum_\beta \frac{\partial \vec{r}_i}{\partial q_\beta} \dot{q}_\beta = \frac{\partial \vec{v}_i}{\partial q_\alpha}$$

$$\therefore \sum_{i=1}^{N} m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_\alpha} = \sum_i \left[\frac{d}{dt}\left(m_i \dot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_\alpha} \right) - m_i \vec{v}_i \frac{\partial \vec{v}_i}{\partial q_\alpha} \right]$$

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Now note that $\vec{v}_i = \sum_\beta \frac{\partial \vec{r}_i}{\partial q_\beta} \dot{q}_\beta$

$$\frac{\partial \vec{v}_i}{\partial \dot{q}_\beta} = \frac{\partial \vec{r}_i}{\partial q_\beta}$$

$$\Rightarrow \sum_i m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_\alpha} = \sum_i \left[\frac{d}{dt}\left(m_i \vec{v}_i \cdot \frac{\partial \vec{r}_i}{\partial q_\alpha}\right) - m_i \vec{v}_i \cdot \frac{\partial \vec{v}_i}{\partial q_\alpha}\right]$$

$$= \sum_i \left[\frac{d}{dt}\left(m_i \vec{v}_i \cdot \frac{\partial \vec{v}_i}{\partial \dot{q}_\alpha}\right) - \frac{\partial}{\partial q_\alpha} \frac{1}{2}(m_i \vec{v}_i^{\,2})\right].$$

$$= \frac{d}{dt} \sum_i \frac{\partial}{\partial \dot{q}_\alpha}\left(\frac{1}{2} m_i \vec{v}_i^{\,2}\right) - \sum_i \frac{\partial}{\partial q_\alpha}\left(\frac{1}{2} m_i \vec{v}_i^{\,2}\right)$$

$$= \frac{d}{dt} \frac{\partial}{\partial \dot{q}_\alpha}\left(\sum_i \frac{1}{2} m_i \vec{v}_i^{\,2}\right) - \frac{\partial}{\partial q_\alpha} \sum_i \frac{1}{2} m_i \vec{v}_i^{\,2}$$

$$= \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_\alpha} - \frac{\partial T}{\partial q_\alpha}$$

The D'Alembert's principle now looks

$$\sum \left(\vec{F}_i^{(a)} - \dot{\vec{p}}_i\right) \cdot \delta \vec{r}_i = 0.$$

$$\sum_{\alpha=1}^{f} Q_\alpha \delta q_\alpha - \left[\sum_{\alpha=1}^{f} \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right) - \left(\frac{\partial T}{\partial q_\alpha}\right)\right] \delta q_\alpha = 0.$$

$$\sum_{\alpha=1}^{f} \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right) - \frac{\partial T}{\partial q_\alpha} - Q_\alpha\right] \delta q_\alpha = 0.$$

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Since the q_α generalised coordinates are independent of each other, it follows

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right) - \frac{\partial T}{\partial q_\alpha} - Q_\alpha = 0.$$

Now, for a conservative force field we have

$$F_i = -\nabla_i U$$

$$Q_\alpha = \sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_\alpha} = -\sum_i \frac{\partial U}{\partial \vec{r}_i} \cdot \frac{\partial \vec{r}_i}{\partial q_\alpha} \quad \text{---}$$

If we expand the last term we have

$$\left(\frac{\partial U}{\partial x_i}\hat{x} + \frac{\partial U}{\partial y_i}\hat{y} + \frac{\partial U}{\partial z_i}\hat{z}\right)\cdot\left(\frac{\partial x_i}{\partial q_\alpha}\hat{x} + \frac{\partial y_i}{\partial q_\alpha}\hat{y} + \frac{\partial z_i}{\partial q_\alpha}\hat{z}\right)$$

$$= \frac{\partial U}{\partial x_i}\frac{\partial x_i}{\partial q_\alpha} + \frac{\partial U}{\partial y_i}\frac{\partial y_i}{\partial q_\alpha} + \frac{\partial U}{\partial z_i}\frac{\partial z_i}{\partial q_\alpha}$$

$$= \frac{\partial U}{\partial q_\alpha}$$

$$\Rightarrow Q_\alpha = -\frac{\partial U}{\partial q_\alpha}$$

$\Rightarrow$ The equation above takes the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right) - \frac{\partial (T-U)}{\partial q_\alpha} = 0.$$

Since $U(q_\alpha)$ does not depend on $\dot{q}_\alpha$, we have

$$\frac{d}{dt}\left(\frac{\partial (T-U)}{\partial \dot{q}_\alpha}\right) - \frac{\partial (T-U)}{\partial q_\alpha} = 0.$$

$$2ml_1^2\ddot{\theta}_1 + ml_1l_2\ddot{\theta}_2 = -2mgl_1\theta_1$$
$$ml_2^2\ddot{\theta}_2 + ml_1l_2\ddot{\theta}_1 = -mgl_2\theta_2$$

$l_1 = l_2$

$$ml^2\ddot{\theta}_1 + ml^2\ddot{\theta}_2 = -2mgl\theta_1$$

$$2l\ddot{\theta}_1 + l\ddot{\theta}_2 = -2g\theta_1$$
$$l\ddot{\theta}_1 + l\ddot{\theta}_2 = -g\theta_2$$

$\theta_1 = A_1e^{i\omega t}$ $\qquad$ $\theta_2 = A_2e^{i\omega t}$

$$-2l\omega^2A_1 + -l\omega^2A_2 = -2gA_1$$
$$-l\omega^2A_1 - l\omega^2A_2 = -gA_2$$

$$(2l\omega^2 - 2g)A_1 + l\omega^2A_2 = 0$$
$$l\omega^2A_1 + (l\omega^2 - g)A_2 = 0.$$

$$\underbrace{\begin{pmatrix} 2l\omega^2 - 2g & l\omega^2 \\ l\omega^2 & l\omega^2 - g \end{pmatrix}}_{\underline{M}} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0.$$

Scallar Equation: $\operatorname{Det}(\underline{M}) = 0.$

$$(2l\omega^2 - 2g)(l\omega^2 - g) - l^2\omega^4 = 0.$$

$$2l^2\omega^4 - 4gl\omega^2 + 2g^2 = 0. \qquad \omega^2 = (2 \pm \sqrt{2})\frac{g}{l}$$
$$l^2\omega^4 - 2gl\omega^2 + g^2 = 0.$$

$\omega_1^2 = (2+\sqrt{2})\dfrac{g}{l}$ $\qquad$ $\omega_2^2 = (2-\sqrt{2})\,g/l$

$\leftarrow A_2 = -\sqrt{2}A_1$ $\qquad$ $A_2 = \sqrt{2}A_1$ (in phase)