Symmetrey and conserevatin Laws:

conserevatin laws and symmetreyes have been always of considereable interest in science. Since the early stages of physics, symmetry have provided an extreemely powerful ~~too~~ & useful tool in our effort to understand nature. They gradually become backbone & important in formulati of many mathematical models.

It was belif preior to kepler, Galileo etc that
" God is perfect, therefore nature must be perfectly symmetric "

=> Planetary orbit must be perfect circles
→ kepler's planetary motion says its ellipses ← not a perfect circle.

=> celestial object must be perfect sphere
→ Galileo discovered, there are moutain on moon ← not a perfect sphere.

kepler established laws of planetary motion during 1609, about how planets move around the sun in ellipse. Next Newton in 1687 explained why they move in such a way. by formulatin law of gravity $F = G\frac{m_1 m_2}{r^2}$. Laplace showed that planets' movement along ellipse followed from the conserevatin law calculated by him.

$$A = v \times M + \mu \frac{\vec{x}}{r}$$

$\vec{v} \to$ velocity

$$A = \vec{v} \times \vec{L} - GMm\frac{\vec{r}}{r}$$

$L = m(\vec{r} \times \vec{v})$ angular momentum

A → area

$$\mu = -GMm$$

or $L = mr^2\theta$

$m \to$ mass of planet.

$\vec{r}, \vec{x} \to$ is position vector of planet

$r \to$ is magnitude of $\vec{x}$

Laplace used the formal definition of conservation law for calculation of this conserved vector. Later it was shown that using certain symmetry one can calculate the conservation of this vector. Kepler's 2nd Law, the conservation of area, follows from the conservation of angular momentum. Angular momentum corresponds to the central symmetry of Newton's gravitational field. Thus, the idea of symmetry and the corresponding conservation law helps to explain the movement of planets in ellipse.

(2) Newton's laws ~~are~~ implicitly assume that they are valid for all times in the past, present and future. Processes that we see occurring in these distant Galaxies actually happened billions of years ago. Thus this is kind of remaining same ~~of~~ over time translation, so we can say

" Newton's laws have time-translation symmetry"

(3) Newton's laws are supposed to apply equally well everywhere in universe. Newton realized that the same laws that cause apples to fall from trees here on earth apply to planets billion of kilometer way from earth

" Newton's laws have ~~time~~ space-translation symmetry"

(4) ~~Newton~~ Newton force is $\vec{F} = m\vec{a}$, if we change the direction of the motion the force doesn't change i.e. same rule for all direction, i.e. no preferred direction in space.

" Newton's laws have rotational symmetry"

So it's kind to established that :" symmetry resides in the laws of nature, not necessarily in the solutions to these laws.

What is the symmetrey ?

The mathematician Weyl gave a simple definitin of a symmetrey - " a symmetrey exists if one does somethin and it does not make a difference". Like a circular cylinder is an axially symmetric objects because if you rotate around its axis over some arbitrary angle, it looks exactly same. ~~State~~ Physics ~~it~~ remains same if a system of particles is placed under a different angle in otherwise empty space. There are no preferred directions in empty space. The angle that you place a system under does not make a difference. The correspondin conservatin law will turn out to be conservatin of angular momentum.

What it means in terms of physics that empty space has no prefered directions. According to QM, the schroedinger equatin or Dirac equatin for RQM describes the physics, it ~~it~~ says that the time derivatives of wave functin can be found as:

$$\frac{\partial \psi}{\partial t} = \frac{1}{i\hbar} H \psi$$

if space has no prefered directin, then Hamiltonian must be same ~~derivative of the wave functin~~ regardless of angular orientaten of coordinate system used. Consider an electronic structure of hydrogen atom, the electron is not in empty space, it is around a proton; but electric field of the proton has no preferred directin either. In terms of cartesian co-ordinate system. The H ~~for~~ & H' are

$$H = -\frac{\hbar^2}{2m}\left(\frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2} + \frac{\partial}{\partial z^2}\right) - \frac{e^2}{4\pi\epsilon_0}\frac{1}{\sqrt{x^2+y^2+z^2}}$$

$$H' = -\frac{\hbar^2}{2m}\left(\frac{\partial}{\partial x'^2} + \frac{\partial}{\partial y'^2} + \frac{\partial}{\partial z'^2}\right) - \frac{e^2}{4\pi\epsilon_0}\frac{1}{\sqrt{x^2+y^2+z^2}}$$

First term is kinetic operator, as you can see the laplacian operator is idependent of angular orientation & second term also independent of angular orientation as $\sqrt{x^2+y^2+z^2}$ under rotation remain same i.e. $\sqrt{x^2+y^2+z^2} = \sqrt{x'^2+y'^2+z'^2}$

Thus the expression for the Hamiltonian ~~does~~ doesnot depend on the angular orientation of coordinate system, you need to change primed system to unprimed only.

The equality of Hamiltonian in the original and rotated coordinate system has a consequence. This rotation operator must commute with the Hamiltonian. No matter whatever way you rotate the physics result still remain same. This observation can give a ~~an~~ indication that

"A symmetry of physics is described by unitary operator that commutes with the Hamiltonian"

we also can find that it is true to find a conserved quantity if there is a symmetry.

symmetry		conservation law
Rotation	$\longleftrightarrow$	Angular momentum.
space translation	$\longleftrightarrow$	Momentum
Time translation	$\longleftrightarrow$	Energy.

Noether, Emmy (1882-1935) discovered that "conservation laws are a consequence of the sime and elegent properties of the space and time." The content of Newton's laws is in their symmetry properties.

assume that the langrangian is symmetric under some transformation of variables.

$\Rightarrow$ that is all q's changes according to some rule.

$$q \to q(s)$$

but the Langrangian doesnot change, no matter what value of s is used

$$\frac{d}{ds} L\{q(s), \dot{q}(s); t\} = 0$$

Noether claimed that for any such symmetry, the quantity

$$C = P_q \frac{dq(s)}{ds}$$ must conserved.

Let's see what happens when we take the derivatives of c with respect to time (assume s has no time dipendence

$$\frac{dc}{dt} = \dot{P}_q \frac{dq(s)}{ds} + P_q \frac{d}{dt} \frac{dq(s)}{ds}$$

using definition of P_q this will become

$$\frac{dc}{dt} = \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i}\right) \frac{dq(s)}{ds} + \frac{\partial L}{\partial \dot{q}_i} \frac{d}{dt}\left(\frac{dq(s)}{ds}\right)$$

Now let's use ~~Langrean~~ Lagrange's equation of motion

$$\frac{dc}{dt} = \frac{\partial L}{dq_i} \frac{dq_s}{ds} + \frac{\partial L}{\partial \dot{q}_i} \frac{d\dot{q}(s)}{ds}$$

Right hand side $= \frac{d}{ds} L\{q(s), \dot{q}(s); t\}$.

$\Rightarrow \frac{dc}{dt} = 0$ That mean C is conserved.

This result is one of the most important theorems in physics. It holds not only for classical mechanics, but also for quantum mechanics, relativity and quantum field theories.

Symmetry considerations are a powerful tool to explore and understand the behaviour of elementary particles. Symmetry can be classified into two broad categories.

(1) Global symmetry: A Global Symmetry is one which is valid at all spacetime points. The existance of quantum numbers in a system always arise from the invarience of the system under a global geometrical transformation.

(2) Local symmetry: A local symmetry is one which has different symmetry transformation at different points in spacetime. Such symmetries play a pivotal role in physics as they form the basis of what are known as gauge theories.

Symmetries that are relevant to particle physics may be classified as follows.

(a) Permutation Symmetry: This is also called as the exchange symmetry, and deals with the symmetry of the system under permutation (or exchange) of identical particles. It results in Bose-Einstein statistics (for bosons) and Fermi-Dirac statistics (for fermions)

(b) Continuous Symmetry: This type of transformation deals with the symmetry of the system under infinitesimally small (therefore continuous) transformation. Translation in space and time, rotation in space

Lorentz tronsformatin are examples of such symmetry. The corresponding quentum numbers are additive.

(b) Descrete symmetry: This kind of symmetry is evidently not continuous. This system exhibits symmetries which are steps apart from each other. space ~~time~~ inversion time reversal, charge conjugation are example of descrete symmetry. The quentum number are multiplicative.

(c) Unitary Symmetry: Such symmetries arises from phase transformation of field, or from generalised rotations in internal space of the system. They are related to many generalised "charges" for example U(1) symmetry for electric charges, SU(2) for isospin, SU(3) for flavor or ~~etc~~ color symmetry. The associated quentum numbers are additive.

If the langrangian of the world would be fully known we could derive the equations of motion from it, and the symmetries of nature and conservation laws would automatically follow.

→ Maxwell Langrangian yeilds, via the Maxwell equation, all the symmetries and conservation laws of the electrodynamics.

- As we just gave the defnition descrete symetry gives multiplicative conserved quntum number and continuous symmetry lead to additive conserved quentum number.

when is an observable conserved?

The expectation value of a quantum mechanical operator F is $\langle F \rangle \equiv \langle \psi | F | \psi \rangle$

with $\langle F \rangle^* = \langle \psi | F^\dagger | \psi \rangle$ Hermitian conjugate.

The expectation value of an observable is a real number so that the operator of an observable should be Hermitian

$$F = F^\dagger \quad \text{if } \langle F \rangle \text{ is observable.}$$

Because energy is an observable the Hamiltonian H is Hermitian. We have for the schrodinger equation and its Hermition conjugate

$$i \frac{\partial |\psi\rangle}{\partial t} = H |\psi\rangle$$

$$-i \frac{\partial \langle \psi |}{\partial t} = \langle \psi | H^\dagger = \langle \psi | H$$

$$\Rightarrow \frac{\partial \langle F \rangle}{\partial t} = i \langle \psi | HF - FH | \psi \rangle = 0$$

$$\Rightarrow [HF - FH] = 0 \quad \Rightarrow [H, F] = 0$$

An observable constant of motion F is Hermitian and commutes with the Hamiltonian.

when H is known, we can find observable constants of motion by searching for Hermitian operators that commute with H. However, when H is not fully known, it is sufficient to establish (or postulate) the invariance of H or lagrangian, under a symmetry operation.

what is a symmetry operator:
A transformation operator U transform one wave function into another

$$|\psi'\rangle = U |\psi\rangle$$

wave functions are always normalised so that we must have $\langle \psi' | \psi' \rangle = \langle \psi | U^\dagger U | \psi \rangle = 1$

it follows the transformation operator must be unitary

$$U^{+}U = UU^{+} = I$$

we call U a symmetry operator when $|\psi'\rangle$ obeys the same Schrodinger equation $|\psi\rangle$. Then, with U time independent,

$$i\frac{\partial U}{\partial t}|\psi\rangle = HU|\psi\rangle \rightarrow i\frac{\partial|\psi\rangle}{\partial t} = U^{-1}HU|\psi\rangle$$

ie. $U^{-1}HU|\psi\rangle \xrightarrow[\text{it to be}]{\text{i want}} H|\psi\rangle$

This can happen only when

$$U^{-1}HU = H \quad \text{or} \quad [H, U] = 0$$

A symmetry operator U is unitary and commutes with the Hamiltonian. Thus U commuts with the Hamiltonian, as does a constant of motion. However, we canot identify U with an observable since it is unitary, and not necessarily Heremitian. Let's generalise some of the important features & basis of the formalism related to the Symmetry and conservation laws.

As we already saw, the symmetry and conservation laws are closely connected. Conservation laws are results of symmetries in the physical system. One can separate various symmetries into two categories:

(a) Space-time symmetries including space translation, time translation, rotation, Lorentz transformation, space inversion, time reversal, etc.

(b) other symmetries not related to space-time, such as isospin, permutation symmetry, charge-conjugation, gauge invariance, etc. These can be considered as internal symmetries.

we first consider examples of the space-time symmetries. suppose we describe a physical system with two different frames of references S and S'. S and S' are related by

$$(t, x, y, z) \longrightarrow (t', x', y', z')$$

The transformation can be ~~satisfie~~ specified by the number of parameters it contains. For example, a time translation $t \to t' = t + \tau$ is specified by a single parameter τ. Space translation, $\vec{v}' = \vec{v} + \vec{a}$, is specified by the three parameters (a_x, a_y, a_z). Rotation (re-orientation) of the coordinate system is also represented by three parameters.

In general, these transformations form families or groups of transformation and they have properties of group. what is group?

A group, usually denoted by G, is a set of elements with very specific mathematical properties. The elements can be labelled in a discrete index or by a continuous varying parameters. usually it is represented by.

$$\{g_i\} \in G$$

G would by called group if it satisfy to following basic properties:

(1) Closure Property: The product of two group elements should also be a group element.
Law of multiplication:

$\Rightarrow$ if $g_i \in G$, $g_j \in G$ $\Rightarrow$ $g_i g_j \in G$

(2) Associativity Property: In taking the product of three group elements, it doesn't matter if you first multiply the two left ones and then the right one or first the two right ones and then left one.
is also called law of combination

$\Rightarrow$ $g_i \in G$, $g_j \in G$ $g_k \in G$

$\Rightarrow$ $(g_i g_j) g_k = g_i (g_j g_k)$ Associativity

(3) Identity Properties: There exists a unique identity element, which is the same for both left and right multiplication

$e \in G$ & $g_i \in G$

such that all $eg = ge = g$ e usually denotes inverse by almost all books.

(4) Inverse: Each group element has its own unique inverse, which should also an element of group and is both left and a right inverse.

" ~~there is nothing~~ if $g \in G$ there must be $g^{-1} \in G$

so that $g g^{-1} = g^{-1} g = e$

(a) For a finite discrete group with n elements then $n = |G|$ is the order of the group.

(b) For any $g \in G$ the smallest integer m such that $g^m = e$ is the order of g.

(c) Two groups $G = \{g_i\}$, $G' = \{g'_i\}$ are isomorphic, $G \simeq G'$, if there is a one to one correspondence $\theta : g_i \Leftrightarrow g'_j$ between the elements consistent with the group multiplication rules. Even if $G \simeq G'$ there is not neccessarily a unique choice for θ but of course we must have $\theta : e \Leftrightarrow e'$

A crucial consequence of basic group axioms is

$$\{g_i g\} = \{g_i\} \text{ for any } g \text{ since } g_j g = g_i g$$
$$\Rightarrow g_j = g_i$$

which implies
$$\sum_i f(g_i) = \sum_i f(g_i g)$$

Sub group: For any group G a subgroup $H \subseteq G$ is naturally defined as a set of elements belonging to G which is also a group. A proper subgroup H is when $H \neq G$ and is denoted $H < G$. For any subgroup H there is an equivalance relation between $g_i g'_i \in G$

$$g_i \sim g'_i \Leftrightarrow g'_i = g_i h \quad \text{for } h \in H$$

that mean H must obey all the rules of group ~~and~~ ~~it is~~ of its own elements. i

- should follow associativity
- shoul follow closure propertis
- inverse
- identity

and H also a subset of $G \Rightarrow H \subseteq G$.

(3) $R_{2\pi}$ gives same orientation or it doesnot change the orientation thus it is an identity ~~an~~ element.

$$e R_{\pi} = R_{2\pi} R_{\pi} = R_{3\pi} = R_{\pi}$$

$$R_{\pi} e = R_{\pi 2\pi} = R_{3\pi} = R_{\pi}$$

$$\Rightarrow \quad e R_{\pi} = R_{\pi} e = R_{\pi}$$

Thus it has identity ~~mat~~ element.

(4) $R_{\pi/2} R_{3\pi/2} = R_{2\pi} = e. \Rightarrow R_{\pi/2} = (R_{3\pi/2})^{-1}$

$R_{\pi/2} R_{\pi} = R_{2\pi/2} = e \Rightarrow R_{\pi} = (R_{\pi})^{-1}$

Inverse is exists.

Since all the four required condition are satisfied for this rotation sets of operation, we can call this collection as a group.

$$\{e, R_{\pi/2}, R_{\pi}, R_{3\pi/2}\} \in G_R$$

G_R is a group of rotation.

Since this is not a group theory course we conclude ~~the~~ here about various groups. However we will pull up neccessary group mechanical tools wherever and whenever it is required. But lets also see some examples relavent to our context.

Consider space translation:

$$\vec{r}'' = \vec{r} + \vec{a}, \quad \vec{r}'' = \vec{r}' + b$$

(a) sequential $\vec{r} \rightarrow \vec{r}' \rightarrow \vec{r}''$ transformating gives $\vec{r}'' = \vec{r}' + (\vec{a} + \vec{b})$, which is also a space translation

(b) $\vec{a} = 0$ is the identity transformation

There are two observation we can make out of it.

(1) when we rotate the squares it forms a four set of fundamental rotation by which we can get any rotation

i.e. $R_{\pi/2}$ is equivalent to $R_{5\pi/2}$

or R_{π} is " " $R_{3\pi}$

Those basic rotation are $R_{\pi/2}, R_{\pi}, R_{3\pi/2}, R_{2\pi}$

(2) of all basic rotation $R_{2\pi}$ is a very special rotation as it give all vertices back to its original starting point. we can call is identity Rotation it doesnot chang. lets call is e.

Thus we can group these items to a set as

$$G = \{e, R_{\pi/2}\ R_{\pi}, R_{3\pi/2}\}.$$

(1) $R_{\pi/2} R_{\pi} = R_{3\pi/2}$

$R_{\pi/2} R_{\pi/2} = R_{\pi}$

⋮

Thus multiply any two elements gives ~~any~~ an element which also part of G. Thus this satisfy closure group.

(2) $(R_{\pi/2} R_{\pi}) R_{3\pi/2} = R_{3\pi/2} R_{3\pi/2} = R_{3\pi} = R_{\pi}$

$R_{\pi/2} (R_{\pi} R_{3\pi/2}) = R_{\pi/2} R_{5\pi/2} = R_{3\pi} = R_{\pi}$

$\Rightarrow (R_{\pi/2} R_{\pi}) R_{3\pi/2} = R_{\pi/2} (R_{\pi} R_{3\pi/2})$

Thus it satisfies associative properties.

G-7

(i) The simplest group consists of ~~two~~ one elements e with multiplication law $ee = e$.

(ii) Next simplest group consists of two element is e & a with law of multiplication

$$ee = e \qquad ea = ae = a \qquad aa = e$$

(iii) The groups whose element commutes are called abelian group.

$$g_i \in G \quad g_j \in G. \quad g_k \in G$$

$$g_i g_j = g_j g_i \Rightarrow g_i g_j - g_j g_i = 0 \Rightarrow [g_i, g_j] = 0$$

$$[g_k, g_j] = 0 \quad \text{or} \quad [g_i, g_k] = 0$$

(v) The smallest non abellian group is the group of permutation of three objects

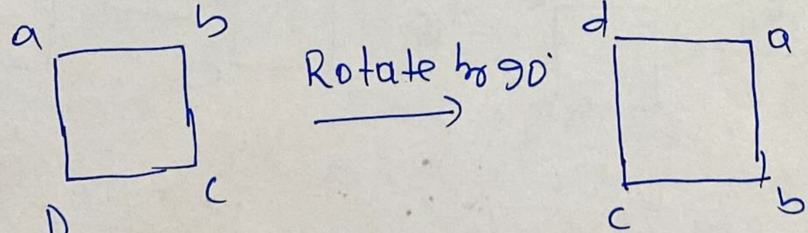
$$e = (1\,2\,3) \qquad a = (2\,1\,3) \qquad b = (1\,3\,2)$$

$$c = (3\,2\,1) \qquad d = (3\,1\,2) \qquad e = (2\,3\,1)$$

note that $\quad ab = (3\,1\,2)$ while $ba = (2\,3\,1)$

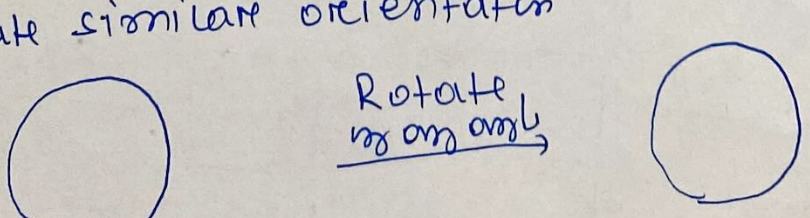
$$ab \neq ba$$

Now let's take a simple example of group: along with symmetrey through example. Taking an exmple of



symmetrey.

we can do the rotation in step of 90° ($\pi/2$) to generate similar orientation



Continous symmetrey.

Rotate by any angle it remain same.

Now the question is how many way we can rotate $\pi/2$, π, $3\pi/2$ and 2π. Now define this rotation operator. please note that although we can make rotation in any rotation but any rotation other than $\pi/2$, π, $3\pi/2$ and 2π would not make it look like as it was like the original ~~one~~ configuration. Original configuration has 4 vertices and when you make a rotation say $20°$ rotation the vertices points were all will be distorted. So the vertices will go into each other only when if the rotation is done by $\pi/2$ or ~~in~~ in integer multiple of $\pi/2$ like $\pi/2, \pi, 3\pi/2$ 2π ... ie..

let's define this Rotation as R thus the elements are.

$$\{ R_{\pi/2};\ R_{\pi},\ R_{3\pi/2},\ R_{2\pi},\ R_{5\pi/2},\ R_{3\pi},\ R_{7\pi/2},\ R_{4\pi},\ \ldots\ldots \}$$

- Let's see what happens in each rotation: