PHY 310 - Mathematical Methods for Physicists I

Odd Term 2019, IISER Mohali

Instructor: Dr. Anosh Joseph

Quiz 1

In class - Tuesday, August 27, 2019

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1. Find the first three coefficients in the expansion of the function

$$f(x) = \begin{cases} 0 & -1 \leq x \leq 0 \\ 1 & 0 \leq x \leq 1 \end{cases}$$

in a series of Legendre polynomials $P_l(x)$ over the interval $(-1, 1)$.

2. Express $\tan\theta$ using Legendre polynomials $P_0(\cos\theta)$ and $P_1(\cos\theta)$.

3. Express the function

$$f(x) = x^4$$

in terms of Legendre polynomials, $P_l(x)$.

4. Consider an electric charge q located at position $\mathbf{R}$ from the origin. We need to compute the electric potential due to this point charge at some other position $\mathbf{r}$. Let us take the polar angle θ to be the angle between $\mathbf{r}$ and $\mathbf{R}$.

We have, from Gauss' law in electromagnetism

$$\nabla^2 \Phi(r, \theta, \phi) = -\frac{\rho(r, \theta, \phi)}{\epsilon_0},$$

with $\Phi(r, \theta, \phi)$ denoting the electric potential and $\rho(r, \theta, \phi)$ the charge density.

For all $r < R$ the charge density is zero. This gives us

$$\nabla^2 \Phi = \left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial}{\partial \theta}\right)\right] \Phi(r, \theta) = 0.$$

We can find a solution to this using the separation of variables ansatz. We take $\Phi(r, \theta) = R_l(r) P_l(\theta)$. Then the general solution is

$$\Phi(r, \theta) = \sum_{l=0}^{\infty} a_l R_l(r) P_l(\cos\theta),$$

with $R_l(r) = Ar^l + Br^{-l-1}$ and $P_l(\cos\theta)$ is the l-th Legendre polynomial. A finite solution at $r = 0$ requires $B = 0$. Thus

$$\Phi(r,\theta) = \sum_{l=0}^{\infty} a_l r^l P_l(\cos\theta).$$

Determine the constants a_l using the boundary condition that when $\theta = 0$ we must recover the potential of a point charge

$$\Phi(r) = \frac{1}{4\pi\epsilon_0}\frac{q}{R-r}.$$

5. Express the Cartesian coordinates x, y and z in terms of spherical harmonics.

   Hint: We have

$$\begin{aligned} Y_1^{\pm 1} &= \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{\pm i\phi}, \\ Y_1^0 &= \sqrt{\frac{3}{4\pi}}\cos\theta. \end{aligned}$$

   and

$$\begin{aligned} x &= r\sin\theta\cos\phi, \\ y &= r\sin\theta\sin\phi, \\ z &= r\cos\theta. \end{aligned}$$