PHY 310 - Mathematical Methods for Physicists I

Odd Term 2019, IISER Mohali

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Homework 2 - Solutions

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1. Show that Bessel function $J_p(x)$ is an even function when p is even and is an odd function when p is odd.

Solution:

We have the series expansion of Bessel function

$$J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+1+p)} \left(\frac{x}{2}\right)^{2n+p}.$$

Replacing x by $-x$ it becomes

$$J_p(-x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+1+p)} \left(\frac{-x}{2}\right)^{2n+p}.$$

If p is even we know that $2n+p$ is even. Thus

$$\left(\frac{-x}{2}\right)^{2n+p} = \left(\frac{x}{2}\right)^{2n+p}.$$

The series expansion becomes

$$\begin{aligned} J_p(-x) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+1+p)} \left(\frac{x}{2}\right)^{2n+p} \\ &= J_p(x). \end{aligned}$$

If p is odd we know that $2n+p$ is odd. Thus

$$\left(\frac{-x}{2}\right)^{2n+p} = -\left(\frac{x}{2}\right)^{2n+p}.$$

The series expansion becomes

$$\begin{aligned} J_p(-x) &= -\sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+1+p)} \left(\frac{x}{2}\right)^{2n+p} \\ &= -J_p(x). \end{aligned}$$

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2. Show that, for a positive integer p, $J_{-p}(x) = (-1)^p J_p(x)$.

Solution:

We have the series expansion of the Bessel function

$$J_{-p}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+1-p)} \left(\frac{x}{2}\right)^{2n-p}.$$

Splitting the summation into two parts

$$\begin{aligned} J_{-p}(x) &= \sum_{n=0}^{n=p-1} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+1-p)} \left(\frac{x}{2}\right)^{2n-p} + \sum_{n=p}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+1-p)} \left(\frac{x}{2}\right)^{2n-p} \\ &= 0 + \sum_{n=p}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+1-p)} \left(\frac{x}{2}\right)^{2n-p}. \end{aligned}$$

Substituting $n = p + m$ we have

$$\begin{aligned} J_{-p}(x) &= \sum_{m=0}^{\infty} \frac{(-1)^{p+m}}{\Gamma(p+m+1)\Gamma(m+1)} \left(\frac{x}{2}\right)^{2m+p} \\ &= (-1)^p \sum_{m=0}^{\infty} \frac{(-1)^m}{\Gamma(m+1)\Gamma(p+m+1)} \left(\frac{x}{2}\right)^{2m+p} \\ &= (-1^p) J_p(x). \end{aligned}$$

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3. Show that

$$J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos x.$$

Solution:

We have the series expansion of the Bessel function

$$J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+1+p)} \left(\frac{x}{2}\right)^{2n+p}.$$

Substituting $p = -\frac{1}{2}$ it becomes

$$\begin{aligned}
J_{-\frac{1}{2}}(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+\frac{1}{2})} \left(\frac{x}{2}\right)^{2n-\frac{1}{2}} \\
&= \frac{x^{-\frac{1}{2}}}{2^{-\frac{1}{2}}} \frac{1}{\Gamma\left(\frac{1}{2}\right)} \left[1 - \frac{x^2}{2\cdot 2\cdot \left(1-\frac{1}{2}\right)} + \frac{x^4}{2\cdot 4\cdot 2^2\cdot \left(1-\frac{1}{2}\right)\left(2-\frac{1}{2}\right)} - \cdots\right] \\
&= \sqrt{\frac{2}{\pi x}} \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) \\
&= \sqrt{\frac{2}{\pi x}} \cos x.
\end{aligned}$$

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4. Express $J_5(x)$ in terms of $J_1(x)$ and $J_2(x)$.

Solution:

We have the recurrence relation

$$J_{p+1}(x) = \frac{2p}{x} J_p(x) - J_{p-1}(x).$$

For $J_3(x)$, $J_4(x)$ and $J_5(x)$ this gives

$$J_3(x) = \frac{4}{x} J_2(x) - J_1(x),$$

$$J_4(x) = \frac{6}{x} J_3(x) - J_2(x),$$

and

$$J_5(x) = \frac{8}{x} J_4(x) - J_3(x).$$

Substituting the expressions for $J_3(x)$ and $J_4(x)$ in the expression for $J_5(x)$

$$\begin{aligned}
J_5(x) &= \frac{8}{x} J_4(x) - J_3(x) \\
&= \frac{8}{x}\left[\frac{6}{x} J_3(x) - J_2(x)\right] - \frac{4}{x} J_2(x) + J_1(x) \\
&= \frac{48}{x^2} J_3(x) - \frac{8}{x} J_2(x) - \frac{4}{x} J_2(x) + J_1(x) \\
&= \frac{48}{x^2} J_3(x) - \frac{12}{x} J_2(x) + J_1(x).
\end{aligned}$$

Substituting the expression for $J_3(x)$ in the above expression

$$\begin{aligned}
J_5(x) &= \frac{48}{x^2} J_3(x) - \frac{12}{x} J_2(x) + J_1(x) \\
&= \frac{48}{x^2}\left[\frac{4}{x} J_2(x) - J_1(x)\right] - \frac{12}{x} J_2(x) + J_1(x) \\
&= \frac{4 \cdot 48}{x^3} J_2(x) - \frac{48}{x^2} J_1(x) - \frac{4 \cdot 3}{x} J_2(x) + J_1(x) \\
&= \frac{4}{x}\left[\frac{48}{x^2} - 3\right] J_2(x) + \left[1 - \frac{48}{x^2}\right] J_1(x).
\end{aligned}$$

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5. Show that

$$\frac{d}{dx}\left[x^{-p} J_p(x)\right] = -x^{-p} J_{p+1}(x).$$

Solution:

We have the recurrence relation

$$x J_p'(x) = p J_p(x) - x J_{p+1}(x).$$

Multiplying both sides by x^{-p-1} we get

$$x^{-p} J_p'(x) = p x^{-p-1} J_p(x) - x^{-p} J_{p+1}(x).$$

Simplifying

$$\begin{aligned} x^{-p}J_p'(x) - px^{-p-1}J_p(x) &= -x^{-p}J_{p+1}(x) \\ \frac{d}{dx}\left[x^{-p}J_p(x)\right] &= -x^{-p}J_{p+1}(x). \end{aligned}$$

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6. Prove that
$$4J_0'''(x) - 3J_1(x) + J_3(x) = 0.$$

Solution:

We have the recurrence relation

$$2J_p'(x) = J_{p-1}(x) - J_{p+1}(x).$$

Differentiating both sides and multiplying by 2 we get

$$2^2 J_p''(x) = 2J_{p-1}'(x) - 2J_{p+1}'(x).$$

Using the recurrence relation given above

$$\begin{aligned} 2^2 J_p''(x) &= (J_{p-2}(x) - J_p(x)) - (J_p(x) - J_{p+2}(x)) \\ &= J_{p-2}(x) - 2J_p(x) + J_{p+2}(x). \end{aligned}$$

Differentiating again and multiplying by 2

$$2^3 J_p'''(x) = 2J_{p-2}'(x) - 2^2 J_p'(x) + 2J_{p+2}'(x).$$

Upon using the same recurrence relation given above

$$\begin{aligned} 2^3 J_p'''(x) &= J_{p-3}(x) - J_{p-1}(x) - 2J_{p-1}(x) + 2J_{p+1}(x) + J_{p+1}(x) - J_{p+3}(x) \\ &= J_{p-3}(x) - 3J_{p-1}(x) + 3J_{p+1}(x) - J_{p+3}(x). \end{aligned}$$

Setting $p = 0$

$$\begin{aligned}
2^3 J_0'''(x) &= J_{-3}(x) - 3J_{-1}(x) + 3J_1(x) - J_3(x) \\
&= (-1)^3 J_3(x) - 3(-1)J_1(x) + 3J_1(x) - J_3(x) \\
&= -J_3(x) + 3J_1(x) + 3J_1(x) - J_3(x) \\
&= -2J_3(x) + 6J_1(x).
\end{aligned}$$

That is

$$4J_0'''(x) = -J_3(x) + 3J_1(x).$$

Rearranging the terms

$$4J_0'''(x) - 3J_1(x) + J_3(x) = 0.$$

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7. Prove that

$$\cos x = J_0(x) - 2J_2(x) + 2J_4(x) - \cdots$$

Solution:

Let us making use of the following expression

$$\cos(x\sin\theta) = J_0(x) + 2\sum_{p=1}^{\infty} J_{2p}(x)\cos 2p\theta.$$

Setting $\theta = \frac{\pi}{2}$ the above expression becomes

$$\begin{aligned}
\cos(x) &= J_0(x) + 2\sum_{p=1}^{\infty} J_{2p}(x)\cos p\pi \\
&= J_0(x) + 2J_2(x)\cos\pi + 2J_4(x)\cos 4\pi - \cdots \\
&= J_0(x) - 2J_2(x) + 2J_4(x) - \cdots
\end{aligned}$$

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8. Prove that
$$J_0(x)^2 + 2J_1(x)^2 + 2J_2(x)^2 + \cdots = 1.$$

Solution:

We have the trigonometric relations involving Bessel functions

$$\begin{aligned}\sin(x\sin\theta) &= 2J_1(x)\sin 2\theta + 2J_3(x)\sin 3\theta + 2J_5(x)\sin 5\theta + \cdots \\ \cos(x\sin\theta) &= J_0(x) + 2J_2(x)\cos 2\theta + 2J_4(x)\cos 4\theta + \cdots\end{aligned}$$

Squaring both sides of this expression and integrating from 0 to π we get

$$\begin{aligned}\int_0^\pi d\theta\sin^2(x\sin\theta) &= \left(2J_1(x)\sin 2\theta + 2J_3(x)\sin 3\theta + 2J_5(x)\sin 5\theta + \cdots\right)^2 \\ \int_0^\pi d\theta\cos^2(x\sin\theta) &= \left(J_0(x) + 2J_2(x)\cos 2\theta + 2J_4(x)\cos 4\theta + \cdots\right)^2.\end{aligned}$$

Adding the above two expression we get

$$\begin{aligned}&\int_0^\pi d\theta\left[\sin^2(x\sin\theta) + \cos^2(x\sin\theta)\right] \\ &\quad = \int_0^\pi d\theta\left(2J_1(x)\sin 2\theta + 2J_3(x)\sin 3\theta + 2J_5(x)\sin 5\theta + \cdots\right)^2 \\ &\qquad + \int_0^\pi d\theta\left(J_0(x) + 2J_2(x)\cos 2\theta + 2J_4(x)\cos 4\theta + \cdots\right)^2.\end{aligned}$$

Noting that

$$\begin{aligned}\int_0^\pi d\theta 2\sin^2 p\theta &= \pi, \\ \int_0^\pi d\theta 2\cos^2 p\theta &= \pi, \\ \int_0^\pi d\theta 2\sin p\theta\sin q\theta &= 0,\end{aligned}$$

and

$$\int_0^\pi d\theta 2\cos p\theta\cos q\theta = 0,$$

the above integral becomes

$$\pi = \pi\left(2J_1^2(x) + 2J_3^2(x) + 2J_5^2(x) + \cdots\right) + \pi\left(J_0^2(x) + 2J_2^2(x) + 2J_4^2(x) + \cdots\right).$$

This gives

$$J_0(x)^2 + 2J_1(x)^2 + 2J_2(x)^2 + \cdots = 1.$$

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