Indian Institute of Science Education and Research, Mohali
Integrated MSc, Semester: IV
Probability and Statistics: MTH 202
Tutorial 7(March 01, 2023)

Summary:
Consider a continuous random variable $X : \Omega \to (a, b)$ with Probability Density Function f_X. The state space of X is $S = (a, b)$.
Theorem: For any continuous function $g : \mathbb{R} \to \mathbb{R}$ such that $\int_{-\infty}^{\infty} |g(t)| f_X(t) dt < \infty$,

$$\mathbb{E}(g(X)) = \int_{-\infty}^{\infty} g(t) f_X(t) dt.$$

The state space of $g(X)$ is $\{g(x) : x \in (a, b)\}$.

Theorem. For any continuously differentiable function $h : (a, b) \to (c, d)$ with $h'(x) \neq 0$ (h is either strictly increasing or decreasing), the Probability Density Function of the random variable $Z = h(X)$: $f_Z(t) = \chi_{(c,d)}(t) \;\; f_X(\alpha(t)) \;\; |\alpha'(t)|$. Here $\alpha = h^{-1}$ and $\chi_{(c,d)}$ is the indicator function of the interval (c, d).

Integration by parts: Let f, g be two differerntiable functions of an interval $[a, b]$ and $G(x) = \int_a^x g(t)dt$. Then $\int_a^b f(t)g(t)dt = [fG]_a^b - \int_a^b f'(t)G(t)dt$.

Conditional Probablity: $P(A|B) = \frac{P(A\cap B)}{P(B)}$.
Question

1. Suppose we pick a point x randomly from the interval $(0, 1)$. Let X be the first digit in the decimal expansion of x. Find Probability Mass Function P_X and the probability distribution function $F_X(t)$.

2. Let X denote the annual rainfall (in centimeters) here. Suppose X follows the NORMAL distribution with avarage $\mu = 110$ and standard Deviation $\sigma = 10$. What is the probability that rainfall in this year will be above 100 Centimeter? You may use the approximate value of $\Phi(1) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{1} e^{-\frac{1}{2}x^2} dx = 0.8413$. Find the Probability Density Function of random variable $Y = e^X$. Describe the state space $S \subseteq \mathbb{R}$ of possible values of Y.

3. Let X be a random variable with Cauchy distribution. Suppose X is having Probabilty Density Function $f_X(x) = \frac{1}{\pi(1+x^2)}, \forall x \in \mathbb{R}$. Find the probability $P(-1 \leq X \leq 1)$ and $P(X \geq 1)$.

4. Let X be the random variable which describes the life span (in years) of an electronic device. Suppose its probability density function $f_X(x) = 5e^{-5x}$ for $x \geq 0$ (zero for other $x \in \mathbb{R}$). Find the probability that the a 9 years old device will work for at least 2 more years. Also compute $P(X \geq 2)$.

5. Let X be uniformly distributed over $(-1, 1)$. Find $P(|X| \geq \frac{1}{2})$ and compute the density functions of the random variables $W = 2X + 5$, $Y = |X|$ and $Z = X^2$. For all these random variables describe the state space $S \subseteq \mathbb{R}$ of possible values.