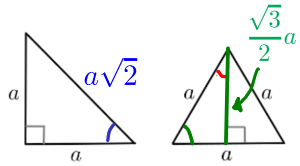
1.

Tutorial 02

18/01/22

1. Look at the triangles below and conclude the following (using Pythagoras theorem).



(a) $\sin\left(\frac{\pi}{6}\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$ (b) $\sin\left(\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$ (c) $\sin\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

Also, calculate $\sin\left(\frac{5\pi}{12}\right)$.

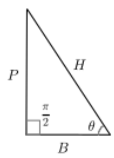
$$\frac{5\pi}{12} = \underset{\alpha}{\frac{\pi}{4}} + \underset{\beta}{\frac{\pi}{6}}$$

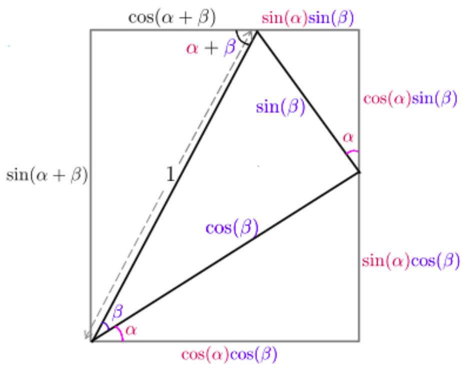
$\cos(\theta) = \frac{B}{H}$

$\sin(\theta) = \frac{P}{H}$

$B^2 + P^2 = H^2$

$\Rightarrow \cos(\theta)^2 + \sin(\theta)^2 = 1$





$$\Rightarrow \sin\left(\frac{\pi}{4} + \frac{\pi}{6}\right) = \cos\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{6}\right) + \sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{6}\right)$$

$$= \frac{1}{\sqrt{2}}\left(\frac{1+\sqrt{3}}{2}\right)$$

and

$$\cos\left(\frac{\pi}{4} + \frac{\pi}{6}\right) = \cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{6}\right) - \sin\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{6}\right)$$

$$= \frac{1}{\sqrt{2}}\left(\frac{\sqrt{3}-1}{2}\right)$$

3. A point $P = (1, 1)$ is rotated by an angle $\frac{5\pi}{12}$ in a plane so that it moves to point Q. Now the point Q is reflected about y-axis so that it moves to the point Q'. What are the coordinates of Q and Q'?

$P = (1,1) = (a, b)$.

$Q = (a', b')$. Then $a' = (\cos\theta, -\sin\theta).(1, 1)$

$= \cos\theta - \sin\theta$

$\theta = \frac{5\pi}{12}$ $= \frac{1}{\sqrt{2}}\left(\frac{\sqrt{3}-1}{2} - \frac{(\sqrt{3}+1)}{2}\right)$

$$= \frac{1}{\sqrt{2}}\left(\frac{-1-1}{2}\right) = \frac{-1}{\sqrt{2}}$$

and $b' = (\sin\theta, \cos\theta)\cdot(1,1)$

$$= \sin\theta + \cos\theta$$

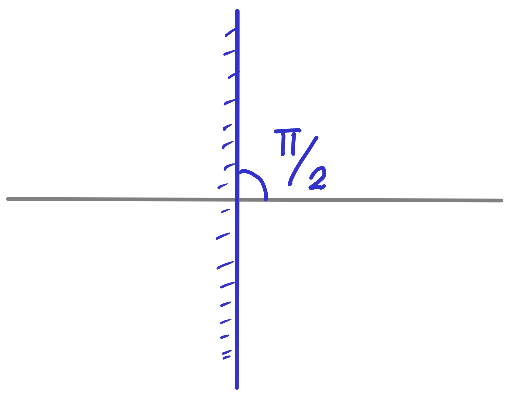
$\theta = \frac{5\pi}{12}$

$$= \frac{1}{\sqrt{2}}\left(\frac{\sqrt{3}+1}{2} + \frac{\sqrt{3}-1}{2}\right)$$

$$= \frac{\sqrt{3}}{\sqrt{2}}$$

So $Q' = \left(\frac{-1}{\sqrt{2}}, \frac{\sqrt{3}}{\sqrt{2}}\right)$

Reflection about Y-axis



$Q'' = (a'', b'')$

$\Rightarrow a'' = (\cos 2\theta, \sin 2\theta)\cdot(a', b')$

$= (\cos\pi, \sin\pi)\cdot(a', b')$

$= (-1, 0)\cdot(a', b') \Rightarrow a'' = -a' = \frac{1}{\sqrt{2}}$

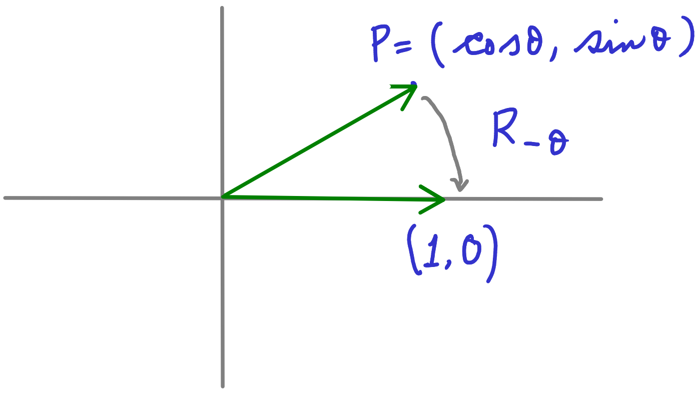
and $b'' = (\sin 2\theta, -\cos 2\theta)\cdot(a', b')$

$= (\sin\pi, -\cos\pi)\cdot(a', b')$

$= (0, 1)\cdot(a', b') \Rightarrow b'' = b' = \frac{\sqrt{3}}{\sqrt{2}}$

So $Q'' = \left(\frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{\sqrt{2}}\right)$.

2. Why is $\sin(-\theta) = -\sin(\theta)$ but $\cos(-\theta) = \cos(\theta)$? Substantiate your argument with the help of an example involving rotation in a plane.



Let $c = \cos(-\theta)$, $s = \sin(-\theta)$

Then

$$1 = (c, -s) \cdot (\cos\theta, \sin\theta)$$

$$0 = (s, c) \cdot (\cos\theta, \sin\theta)$$

$$\Rightarrow \quad \begin{aligned} \cos\theta\, c - \sin\theta\, s &= 1 \quad - \text{I} \\ \sin\theta\, c + \cos\theta\, s &= 0 \quad - \text{II} \end{aligned}$$

By $\sin\theta\,\text{I} - \cos\theta\,\text{II}$

$$-(\sin\theta)^2 s - (\cos\theta)^2 s = \sin\theta$$

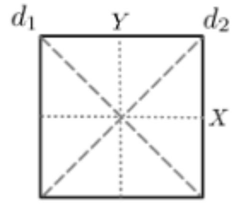
$$\Rightarrow \quad -s\left((\sin\theta)^2 + (\cos\theta)^2\right) = \sin\theta$$

$$\Rightarrow \quad s = -\sin\theta$$

Now do $\cos\theta\,\text{I} + \sin\theta\,\text{II}$ to complete the question.

4.

4. Match the following in connection with the symmetries of a square.



Use rotation matrix

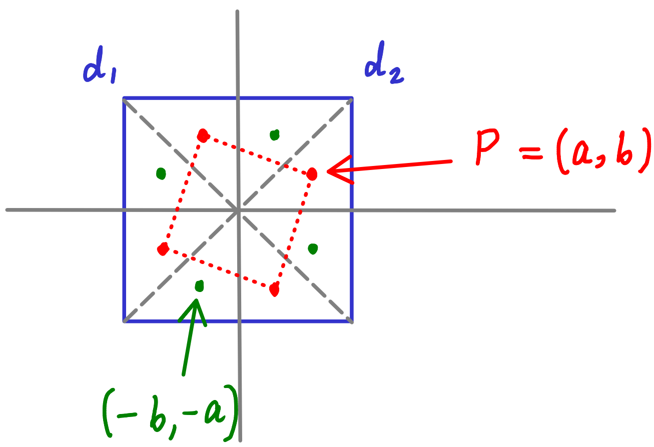
$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

and reflection matrix

$$\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$$

Matrix	**Symmetry**
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	f_{d_2}
$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	f_Y
$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$r_{3\pi/2}$
$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	$r_{\pi/2}$
$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	1
$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$	f_{d_1}
$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$	f_X
$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	r_π

Try making a similar exercise for symmetries of a triangle.



5. Write 3×3 rotation matrices $R_{x,\theta}$, $R_{y,\theta}$ and $R_{z,\theta}$. Consider a point $P = (a, b, c)$ and calculate the following.

   (a) $R_{x,\theta}(R_{y,\theta}(P))$ and $R_{y,\theta}(R_{x,\theta}(P))$ when $\theta = \pi$.

   (b) $R_{x,\theta}(R_{y,\theta}(P))$ and $R_{y,\theta}(R_{x,\theta}(P))$ when $\theta = \frac{\pi}{2}$.

   Do you connect these computations with an ongoing discussion over moodle forum?

$$R_{x,\theta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

$$R_{y,\theta} = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

$$P = (a, b, c)$$

$\theta = \pi$.

$$R_{y,\pi}(a,b,c) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$= \begin{pmatrix} (-1,0,0)\cdot(a,b,c) \\ (0,1,0)\cdot(a,b,c) \\ (0,0,1)\cdot(a,b,c) \end{pmatrix}$$

$$= \begin{pmatrix} -a \\ b \\ -c \end{pmatrix}$$

so $R_{y,\pi}(a,b,c) = (-a, b, -c)$

$$\Rightarrow R_{x,\pi}\left(R_{y,\pi}(a,b,c)\right) = R_{x,\pi}(-a,b,-c)$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} -a \\ b \\ -c \end{pmatrix}$$

$$= \begin{pmatrix} -a \\ -b \\ c \end{pmatrix}$$

use dot products

so $R_{x,\pi}\left(R_{y,\pi}(a,b,c)\right) = (-a,-b,c)$

Now $R_{x,\pi}(a,b,c) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$$= \begin{pmatrix} a \\ -b \\ -c \end{pmatrix}$$

so $R_{y,\pi}\left(R_{x,\pi}(a,b,c)\right) = R_{y,\pi}(a,-b,-c)$

$$= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} a \\ -b \\ -c \end{pmatrix}$$

$$= \begin{pmatrix} -a \\ -b \\ c \end{pmatrix}$$

so $R_{x,\pi}\left(R_{y,\pi}(P)\right) = R_{y,\pi}\left(R_{x,\pi}(P)\right)$ for all P.

Now for $\theta = \frac{\pi}{2}$, let us do a faster calculation.

$$R_{x,\frac{\pi}{2}}\left(R_{y,\pi/2}(a,b,c)\right)$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \left[\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right]$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} c \\ b \\ -a \end{pmatrix} = \begin{pmatrix} c \\ a \\ b \end{pmatrix}$$

And $R_{y,\pi/2}\left(R_{x,\pi/2}(a,b,c)\right)$

$$= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right]$$

$$= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ -c \\ b \end{pmatrix} = \begin{pmatrix} b \\ -c \\ -a \end{pmatrix}$$

so $R_{x,\frac{\pi}{2}}\left(R_{y,\frac{\pi}{2}}(a,b,c)\right) = (c, a, b)$

$$\neq (b, -c, -a)$$

$$= R_{y,\pi/2}\left(R_{x,\frac{\pi}{2}}(a,b,c)\right)$$