(1). If the function $\psi(x)$ and its second derivative $\psi''(x)$ had the same sign, then what would happen at a critical point x_0 where first derivative is zero.

$$\psi'(x)\Big|_{at\ x=x_0} = \psi'(x_0) = 0$$

If $\psi(x)$ were positive at the critical point, $\psi''(x)$ would have to be as well. This can be understood from

$$\frac{d^2\psi}{dx^2} = \frac{2m}{\hbar^2}\left(V(x)-E\right)\psi$$

$$\text{if } E < V(x)\Big|_{x=x_{min}} \Rightarrow E < V_{min} \Rightarrow (V_{min}-E) > 0$$

Then positive sign of ψ refer positive $\frac{d^2\psi}{dx^2}$

Negative sign of ψ refer negative $\frac{d^2\psi}{dx^2}$

Sign of $\frac{d^2\psi}{dx^2}$ also defines curvature. Thus for positive $\frac{d^2\psi}{dx^2}$, the wave function would be concave up. Since $\psi''(x_0) > 0$ the values of $\psi(x)$ near x_0 would be larger than $\psi(x_0)$ and $\psi''(x)$ would never change sign.

As a result, the wavefunction would just get larger and larger, getting away from the critical point (at x_0). Since the wave function couldn't reach zero for extreme values of x, the wave function wouldn't be normalizable.

(2) Expectation Value.

$$\langle \hat{p} \rangle = \int \psi^* \hat{p} \psi \, dx = -i\hbar \int \psi^* \frac{\partial \psi}{\partial x} \, dx.$$

Now, $\psi(x) = e^{ikx} \phi(x) \quad \Rightarrow \quad \phi(x)$ is real $\Rightarrow \phi^*(x) =$

$$\frac{\partial \psi}{\partial x} = ik\, e^{ikx} \phi(x) + e^{ikx} \phi'(x)$$

$$\langle \hat{p} \rangle = -i\hbar \int e^{-ikx} \phi^*(x) \left[ik\, e^{ikx} \phi(x) + e^{ikx} \phi'(x) \right] dx$$

$$= -i\hbar \int ik\, \phi^2(x) \, dx - i\hbar \int \phi \phi' \, dx$$

$$= \hbar k \int \phi^2(x) \, dx - \frac{i\hbar}{2} \int \frac{d}{dx}\left(\phi^2\right) dx$$

$$= \hbar k \int \phi^2(x) \, dx - \frac{i\hbar}{2} \, \phi^2 \Big|_{-\infty}^{\infty}$$

$$\boxed{\langle \hat{p} \rangle = \hbar k \int \phi^2(x) \, dx}$$

3 (a) Given wave function is

$$\psi(n) = e^{i\alpha(n)} \phi(n)$$

where $\alpha(n)$ and $\phi(n)$ are Real.

So, we know that

$$\rho = \psi^* \psi$$

$$\rho = e^{-i\alpha(n)} \phi(n) . e^{i\alpha(n)} \phi(n) = \phi^2(n) ..$$

or

$$\rho = |\phi(n)|^2$$

$$J(n) = \frac{-i\hbar}{2m} \left[\psi^* \nabla \psi - \psi \nabla \psi^* \right] \quad \text{when} \quad V = V^*$$

$$J(x) = \frac{-i\hbar}{2m} \left[e^{-i\alpha(n)} \phi(n) \frac{\partial}{\partial x} \left\{ e^{i\alpha(n)} \phi(n) \right\} - e^{i\alpha(n)} \phi(n) \frac{\partial}{\partial x} \left\{ e^{-i\alpha(n)} \phi(n) \right\} \right]$$

$$\Rightarrow \frac{-i\hbar}{2m} \left[e^{-i\alpha(n)} \phi(n) \left\{ e^{i\alpha(n)} \phi(n)\, i\alpha'(n) + \phi'(n)\, e^{i\alpha(n)} \right\} \right.$$

$$\left. - e^{i\alpha(n)} \phi(n) \left\{ e^{-i\alpha(n)} \phi(n) (-i\alpha'(n)) + e^{-i\alpha(n)} \phi'(n) \right\} \right]$$

$$= \frac{-i\hbar}{2m} \left[\phi^2(n)\, i\alpha'(n) + \cancel{\phi(n) \phi'(n)} + i\alpha'(n) \phi^2(n) - \cancel{\phi(n) \phi'(n)} \right]$$

$$= \frac{-i^2 \hbar}{\cancel{2}m} . \cancel{2} \alpha'(n) \phi^2(n) = \frac{\hbar}{m} \alpha'(n) \phi^2(n)$$

$$\boxed{\frac{J(n)}{\rho(n)} = \frac{\hbar}{m} \alpha'(n)}$$

(b) $\psi(x) = A e^{ipx/\hbar} + B e^{-ipx/\hbar}$

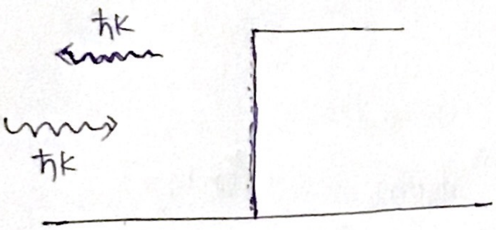
$\lambda = \dfrac{2\pi}{k}$

we can Re-write it,

$\hbar = \dfrac{h}{2\pi}$

$\psi(x) = A e^{ikx} + B e^{-ikx}$

de broglie wavelength.



$\lambda = \dfrac{h}{p}$

$\boxed{p = \hbar k}$

So, current density for this wave.

$$J = -\frac{i\hbar}{2m}\left[\psi^* \nabla \psi - \psi \nabla \psi^*\right] \qquad \text{for } V = V^*$$

$$\nabla \psi = \frac{\partial}{\partial x}\left\{A e^{ikx} + B e^{-ikx}\right\}$$

$$= A e^{ikx}(ik) - B e^{-ikx}(ik)$$

$$= (ik)\left\{A e^{ikx} - B e^{-ikx}\right\}$$

$$\psi^* \nabla \psi = (A^* e^{-ikx} + B^* e^{ikx})\, ik\, (A e^{ikx} - B e^{-ikx})$$

$$= ik\left(|A|^2 - A^* B e^{-i2kx} + B^* A e^{i2kx} - |B|^2\right)$$

$$\psi \nabla \psi^* = (ik)\left\{-|A|^2 + AB^* e^{i2kx} - BA^* e^{-i2kx} + |B|^2\right\}$$

$$J = -\frac{i\hbar}{2m}\left[\psi^* \nabla \psi - \psi \nabla \psi^*\right]$$

$$= -\frac{i\hbar\, ik}{2m}\left[|A|^2 - \cancel{A^* B e^{-i2kx}} + \cancel{B^* A e^{i2kx}} - |B|^2 + |A|^2 - \cancel{AB^* e^{i2kx}} + \cancel{BA^* e^{-i2kx}} - |B|^2\right]$$

$$J \Rightarrow \frac{\hbar k}{\cancel{2} m} \cdot \cancel{2}\left[|A|^2 - |B|^2\right] = \frac{\hbar k}{m}\left(|A|^2 - |B|^2\right)$$

Problem 4. $M = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ $f(z) = \sum_{n=0}^{\infty} f_n z^n.$

similar for Matrix

$f(M) = f_n M^n.$

Solⁿ

we know that

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} \cdots\cdots = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!}$$

Standard form of Tylor series Expansion:-

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \frac{f''(z_0)(z - z_0)^2}{2!} + \cdots\cdots$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} \cdots\cdots = \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{(2n+1)}}{(2n+1)!}$$

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \frac{\theta^8}{8!} \cdots\cdots = \sum_{n=0}^{\infty} (-1)^n \frac{(\theta)^{2n}}{2n!}$$

as we know Tylor series expansion also posible for Matrix.

$$e^{iM\theta} = 1 + iM\theta + \frac{(iM\theta)^2}{2!} + \frac{(iM\theta)^3}{3!} + \frac{(iM\theta)^4}{4!} \cdots\cdots$$

$$= 1 + iM\theta - \frac{M^2\theta^2}{2!} - \frac{iM^3\theta^3}{3!} + \frac{M^4\theta^4}{4!} + \frac{iM^5\theta^5}{5!} \cdots\cdots$$

$$\Rightarrow \left(1 - \frac{M^2\theta^2}{2!} + \frac{M^4\theta^4}{4!} - \cdots\cdots\right) + iM\left(\theta - \frac{M^2\theta^3}{3!} + \frac{M^4\theta^5}{5!} - \cdots\cdots\right)$$

So, if $M^2 = \mathbb{1}$ or $M^{2n} = \mathbb{1}$.

Then,

$$e^{iM\theta} = \mathbb{1}\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\cdots\right) + iM\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\cdots\right)$$

$$e^{iM\theta} = \mathbb{1}\cos\theta + iM\sin\theta$$

$A(\theta) = \cos\theta$

$B(\theta) = i\sin\theta$

Tutorial - 2

Prob. 5

a) The unnormalized state is

$$|\psi\rangle = (1+i)|z,+\rangle - (1+i\sqrt{3})\,|z,-\rangle .$$

To make it ~~no~~ normalized let ~~a~~ us take the inner-product.

$$\langle\psi|\psi\rangle = |1+i|^2 + |1+i\sqrt{3}|^2$$

Note, while deriving the inner-product we have used the orthonormal conditions

$$\langle z,+|z,+\rangle = 1, \quad \langle z,-|z,-\rangle = 1$$
$$\langle z,-|z,+\rangle = 0, \quad \langle z,+|z,-\rangle = 0$$

So $\langle\psi|\psi\rangle = 6$, and the state after normalization becomes –

$$|\tilde{\psi}\rangle = \frac{1}{\sqrt{6}}\left[(1+i)|z,+\rangle - (1+i\sqrt{3})|z,-\rangle\right].$$

$$\boxed{\text{Note} \quad (1+i) = \sqrt{2}\, e^{i\pi/4} \; ; \; (1+i\sqrt{3}) = 2\, e^{i\pi/3}}$$

So we have

$$|\tilde{\psi}\rangle = \frac{1}{\sqrt{3}} e^{i\pi/4}|z,+\rangle - \sqrt{\tfrac{2}{3}}\, e^{i\pi/3}|z,-\rangle$$
$$= e^{i\pi/4}\left[\frac{1}{\sqrt{3}}|z,+\rangle - \sqrt{\tfrac{2}{3}}\, e^{i\pi/12}|z,-\rangle\right]$$
$$= e^{i\pi/4}\left[\frac{1}{\sqrt{3}}|z,+\rangle + \sqrt{\tfrac{2}{3}}\, e^{i(\pi/12+\pi)}|z,-\rangle\right]$$

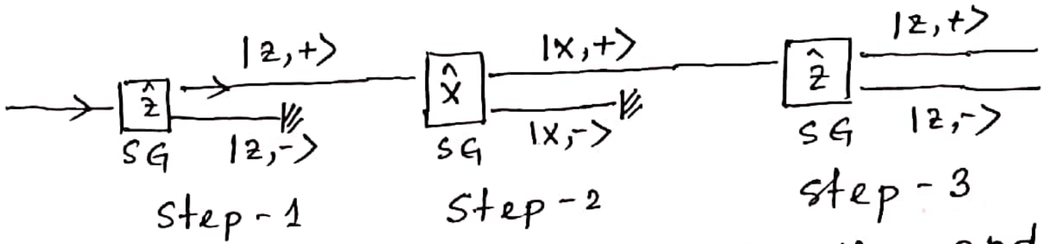
We ignore the global phase — then -

$$|\tilde{\psi}\rangle = \left[\frac{1}{\sqrt{3}}|z,+\rangle + \sqrt{\tfrac{2}{3}}\, e^{i13\pi/12}|z,+\rangle\right].$$

$$\boxed{\begin{array}{l}\text{A spin state along a vector } \vec{n} \text{ is given by} \\ |n,+\rangle = \sin\theta/2\,|z,+\rangle + \cancel{\cos\tfrac{\theta}{2}}\,\cos\tfrac{\theta}{2}\, e^{i\varphi}|z,+\rangle\end{array}}$$

Now you can find out the θ & φ.

(b)



* Of all atoms that entered the 2nd-magnet 50% will be found in the state ~~$|x,+\rangle$~~. $|x,+\rangle$.

~~Of that 50% of~~

Out of all these $|x,+\rangle$, 50% will be in $|z,+\rangle$ & 50% will be in $|z,-\rangle$ after passed through the third-magnet.

So 25% of all atoms entered in the second magnet will be found in $|z,+\rangle$ after third-magnet. This is also same for the state $|z,-\rangle$.

The fraction that never made it to the third-magnet is 50% and this is due to the fact that 50% in $|x,-\rangle$ did not enter in the third-magnet.

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