2 $S_T = S_1 + S_2 = (N_1 V_1 U_1)^{1/3} + (N_2 V_2 U_2)^{1/3}$

$$\frac{\partial S_T}{\partial U_1} = \frac{1}{3}(N_1 V_1)^{1/3} U_1^{-2/3} - \frac{1}{3}(N_2 V_2)^{1/3}(U_0 - U_1)^{-2/3} = 0$$

$$\left(\frac{N_1 V_1}{N_2 V_2}\right)^{1/3} = \frac{U_1^{2/3}}{(U_0 - U_1)^{2/3}}$$

$$\left(\frac{U_1}{U_0 - U_1}\right) = \left(\frac{N_1 V_1}{N_2 V_2}\right)^{1/2} = x$$

$$\frac{U_0}{U_1} - 1 = \frac{1}{x}$$

$$\frac{U_0}{U_1} = 1 + \frac{1}{x} \Rightarrow \frac{U_1}{U_0} = \frac{x}{1+x}$$

$$U_2 = U_0 - U_1 = U_0\left(1 - \frac{U_1}{U_0}\right)$$

$$\frac{U_2}{U_0} = 1 - \frac{U_1}{U_0} = 1 - \frac{x}{1+x} = \frac{1}{1+x}$$

Total entropy $= (N_1 V_1)^{1/3} U_0^{1/3} \dfrac{x^{1/3}}{(1+x)^{1/3}} + (N_2 V_2)^{1/3} \dfrac{U_0^{1/3}}{(1+x)^{1/3}}$

$$x^2 = \frac{N_1 V_1}{N_2 V_2} \qquad N_2 V_2 = \frac{N_1 V_1}{x^2}$$

$$= (N_1 V_1)^{1/3} \left(\frac{U_0}{1+x}\right)^{1/3} \left[x^{1/3} + x^{-2/3} \right]$$

$$\left.\frac{\partial S_1}{\partial U_1}\right) = \frac{1}{T_1} \qquad\qquad U_1 = \frac{U_0 x}{1+x}.$$

$$\frac{1}{3}(N_1V_1)^{1/3}\, U_1^{-2/3} = \frac{1}{T_1}$$

$$\frac{1}{3}(N_1V_1)^{1/3}\, \frac{U_0^{-2/3}\, x^{-2/3}}{(1+x)^{-2/3}} = \frac{1}{T_1} \quad \Rightarrow \quad \frac{1}{3}(N_1V_1)^{1/3}\, \frac{(1+x)^{2/3}}{U_0^{2/3}\, x^{2/3}} = \frac{1}{T_1}$$

$$T_1 = \frac{3\, U_0^{2/3}\, x^{2/3}}{(N_1V_1)^{1/3}\,(1+x)^{2/3}}$$

3. $\left.\frac{\partial U}{\partial S}\right) = T$ $\Rightarrow$ $T = 3k\frac{S^2}{NV} = \frac{U}{S}$

$\left.\frac{\partial U}{\partial V}\right) = -p$ $\left.\frac{\partial U}{\partial V}\right) = -k\frac{S^3}{NV^2} = -p$

$$\boxed{p = +k\frac{S^3}{NV^2} = \frac{U}{V}}$$

$S = \left(\frac{UNV}{k}\right)^{1/3}$

$S^2 = \left(\frac{UNV}{k}\right)^{2/3}$ $\Rightarrow$ $T = 3k\left(\frac{UNV}{k}\right)^{2/3}\frac{1}{NV}$

$= 3k^{1/3}\frac{U^{2/3}}{(NV)^{1/3}}$

$T^3 = 27k\frac{U^2}{NV}$

$$\boxed{U^2 = \frac{T^3 NV}{27k}}$$

4. $u = \frac{A}{v^2}e^{s/R}$. $u(s,v)$. $du = \left.\frac{\partial u}{\partial v}\right)dv + \left.\frac{\partial u}{\partial s}\right)\cancel{ds}$ 0.

$Tds = du + pdv$ $\left.\frac{\partial u}{\partial v}\right) = -p$

$= -\frac{2A}{v^2}e^{s/R} + pdv$ $-\frac{2A}{v^3}e^{s/R} = -p$

$p = \frac{2A}{v^3}e^{s/R}$.

$\left.\frac{\partial u}{\partial s}\right) = \frac{1}{T}$ $\Rightarrow \frac{1}{T} = \frac{A}{v^2}e^{s/R}\frac{1}{R}$.

$\frac{A}{v^2}e^{s/R} = \frac{R}{T}$ $\Rightarrow$ $p = \frac{2}{v}\frac{R}{T}$ $pv = \frac{2R}{T}$

$u = pv$ $\Rightarrow 0 = \cancel{p}\,dv + vdp + pdv$ $\Rightarrow$ $vdp + 3pdv = 0$.

$pv^3 = \text{const.}$

$p^{-2} T^{-3} = \text{con.}$

$p^2 T^3 = \text{const.}$

$p_0^2 T_0^3 = \left(\frac{p_0}{2}\right)^2 T_f^3$

$T_f^3 = 4 T_0^3$

$$\boxed{T_f = 4^{1/3} T_0}$$

$\left.\frac{\partial u}{\partial s}\right) = \frac{1}{T}$

$\Rightarrow \frac{A}{v^2} e^{s/R} \frac{1}{R} = \frac{1}{T}$

$\frac{A}{v^2} e^{s/R} = \frac{R}{T}$

$u = \frac{R}{T}$

$pv = \frac{2R}{T}$

$v = \frac{2R}{Tp}$

5. Initial $\boxed{m}\ T_f \qquad \boxed{m}\ T_f$

$$\downarrow$$

Final

$\boxed{m}\ T_1 \qquad \boxed{m}\ T_2 \qquad \& \ T_1 < T_2$

total $\Delta S = \Delta S_1 + \Delta S_2$

$$= \int_{T_f}^{T_1} \frac{dQ_1}{T} + \int_{T_f}^{T_2} \frac{dQ_2}{T}$$

$\left.\begin{matrix} dQ_1 \\ dQ_2 \end{matrix}\right\}$ are heat exchange.

$$= mc \int_{T_f}^{T_1} \frac{dT}{T} + mc \int_{T_f}^{T_2} \frac{dT}{T}$$

$$= mc \left[\ln \frac{T_1}{T_f} + \ln \frac{T_2}{T_f} \right] = mc \ln \frac{T_1 T_2}{T_f^2}$$

But by first law $T_f = \frac{T_1+T_2}{2}$.

$$\therefore \frac{T_1 T_2}{T_f^2} < 1.$$

$$\Rightarrow \because \quad T_1 T_2 < \frac{(T_1+T_2)^2}{4}$$

$$4T_1T_2 < (T_1+T_2)^2 \quad \because (T_1-T_2)^2 > 0.$$

$$\Rightarrow \underline{\Delta S < 0.}$$

Hence entropy change is (-)ve.

$\left.\frac{\partial P}{\partial V}\right)_S$

$\left.\frac{\partial P}{\partial V}\right)_S \left.\frac{\partial V}{\partial S}\right)_P \left.\frac{\partial S}{\partial P}\right)_V = -1$

$\left.\frac{\partial P}{\partial V}\right)_S = -\left.\frac{\partial P}{\partial S}\right)_V \left.\frac{\partial S}{\partial V}\right)_P$

$\left.\frac{\partial V}{\partial S}\right)_P$

~~S.dP~~ ~~VdP~~
TdS VdP

$\frac{\partial S}{\partial V}$

SdT pdV.

$T \; \cancel{\left.\frac{\partial S}{\partial V}\right)_P} = \cancel{\left.\frac{\partial U}{\partial V}\right)_P}$

$-\left.\frac{\partial P}{\partial S}\right)_V$

(V is constant).
& Derivative
w.r.t. S).
So a potential
i.e. function of
U (S,V)

$du = TdS - pdV$

$-\left.\frac{\partial P}{\partial S}\right)_V = \left.\frac{\partial T}{\partial V}\right)_S$

$\left.\frac{\partial S}{\partial V}\right)_P = \frac{1}{\left.\frac{\partial V}{\partial S}\right)_P}$

P held constant
Derivative w.r.t. S.
H(S,P)

$\frac{1}{\left.\frac{\partial T}{\partial P}\right)_S} = \left.\frac{\partial P}{\partial T}\right)_S$.

$\left.\frac{\partial T}{\partial V}\right)_S$

$\left.\frac{\partial T}{\partial V}\right)_S \left.\frac{\partial V}{\partial S}\right)_T \left.\frac{\partial S}{\partial T}\right)_V = -1.$

$\left.\frac{\partial T}{\partial V}\right)_S = -\left.\frac{\partial S}{\partial V}\right)_T \left.\frac{\partial T}{\partial S}\right)_V$

$\left.\frac{\partial P}{\partial T}\right)_S \left.\frac{\partial T}{\partial S}\right)_P \left.\frac{\partial S}{\partial P}\right)_T = -1$

$\left.\frac{\partial P}{\partial T}\right)_S = -\left.\frac{\partial S}{\partial T}\right)_P \left.\frac{\partial P}{\partial S}\right)_T$

$\left.\frac{\partial T}{\partial S}\right)_V$

$-\left.\frac{\partial S}{\partial V}\right)_T$

T held Const.
Derivative w.r.t. V.
F(T, V)
$dF = -pdV - SdT$

$\left.\frac{\partial S}{\partial V}\right)_T = \left.\frac{\partial P}{\partial T}\right)_V$

$-\left.\frac{\partial P}{\partial S}\right)_T = -\frac{1}{\left.\frac{\partial S}{\partial P}\right)_T}$.

$\left.\frac{\partial S}{\partial T}\right)_P \dashrightarrow C_P/T$.

[T held Const
& Derivative w.r.t. P]
G(T,P)

$-\left.\frac{\partial S}{\partial P}\right)_T = \left.\frac{\partial V}{\partial T}\right)_P$

$\dashrightarrow -\frac{T}{C_V}$

$\left.\frac{\partial P}{\partial T}\right)_V \dashrightarrow \frac{\alpha}{\kappa}$

$\left.\frac{\partial P}{\partial T}\right)_V \left.\frac{\partial T}{\partial V}\right)_P \left.\frac{\partial V}{\partial P}\right)_T = -1.$

$\left.\frac{\partial P}{\partial T}\right)_V = -\left.\frac{\partial V}{\partial T}\right)_P \left.\frac{\partial P}{\partial V}\right)_T$

$\frac{1}{\left.\frac{\partial V}{\partial T}\right)_P} = \frac{1}{V\alpha}$

$\left.\frac{\partial T}{\partial V}\right)_P \left.\frac{\partial V}{\partial P}\right)_T \left.\frac{\partial P}{\partial T}\right)_V = -1.$

$\left.\frac{\partial T}{\partial V}\right)_P = -\left.\frac{\partial P}{\partial V}\right)_T \left.\frac{\partial T}{\partial P}\right)_V$

$\left.\frac{\partial V}{\partial T}\right)_P \dashrightarrow \alpha V$

$-\left.\frac{\partial P}{\partial V}\right)_T \dashrightarrow \kappa V.$

$\frac{1}{\left.\frac{\partial P}{\partial T}\right)_V} \dashrightarrow \frac{\kappa}{\alpha}$

1

$$H = D \sum S_i^2$$

N_+, N_0, N_-

N_0 ions with spin $S=0$

N_+ ions with spin $S=1$

N_- ions with spin $S=-1$

$$N_+ + N_0 + N_- = N$$

$$D(N_+ + N_-) = U$$

$$N_+ + N_- = U/D$$

$$\Omega(N_0, N_+, N_-) = \frac{N!}{N_+!\,N_-!\,N_0!}$$

[for a given value of N_0, N_+, N_-]

Number of microscopic configurations with energy U, ions N is given by

$$\Omega(U,N) = \sum_{N_0, N_+, N_-} \frac{N!}{N_+!\,N_-!\,N_0!} \quad \delta_{N,\, N_+ + N_0 + N_-}\ \delta_{N_+ + N_-,}$$

$$N_0 = N - (N_+ + N_-) = (N - U/D)$$

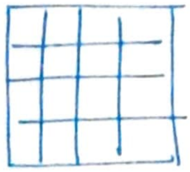
$$N_+ = U/D - N_-$$

$$\Omega(U,N) = \sum_{N_-=0}^{U/D} \frac{N!}{\left(N - U/D\right)!\,\left(\frac{U}{D} - N_-\right)!\,N_-!}$$

$$= \frac{N!}{\left(N - \frac{U}{D}\right)!\,\left(\frac{U}{D}\right)!} \sum_{N_-=0}^{U/D} \frac{(U/D)!}{\left(\frac{U}{D} - N_-\right)!\,N_-!}$$

$$= \frac{N!}{\left(N - \frac{U}{D}\right)!\,\left(\frac{U}{D}\right)!} \left[1 + \frac{U}{D} + \left(\frac{U}{D} - 1\right)\frac{U}{D}\frac{1}{2} + \cdots \right] = ?$$

2. $M = \frac{V}{b}$



M cells
and N balls.

There are M cells and you have to fill N-balls.

$$\Omega(M,N) = \frac{M!}{(M-N)!}$$

$$= \frac{(V/b)!}{(V/b - N)!}.$$

Since the particles are indistinguishable

$$\Omega(V,N) = \frac{(V/b)!}{\left(\frac{V}{b}-N\right)!\,N!}.$$

$$\therefore \quad S = k_B \ln \Omega = k_B \ln \frac{(V/b)!}{\left(\frac{V}{b}-N\right)!\,N!}.$$

$$= k_B\Big[\left(\frac{V}{b}\ln\frac{V}{b} - \frac{V}{b}\right) - \left(\frac{V}{b}-N\right)\ln\left(\frac{V}{b}-N\right) + \left(\frac{V}{b}-N\right) - N\ln N + N\Big].$$

$$= k_B\left[\frac{V}{b}\ln\frac{V}{b} - \left(\frac{V}{b}-N\right)\ln\left(\frac{V}{b}-N\right)\right].$$

$$= k_B\left[N\frac{v}{b}\ln N\frac{v}{b} - N\left(\frac{v}{b}-1\right)\ln N\left(\frac{v}{b}-1\right)\right].$$

$$= Nk_B\left[\frac{v}{b}\ln\frac{v}{b} + \frac{v}{b}\ln N - \frac{v}{b}\ln N + \ln N - \left(\frac{v}{b}-1\right)\ln\left(\frac{v}{b}-1\right)\right].$$

$$= Nk_B\left[\frac{v}{b}\ln\frac{v}{b} - \left(\frac{v}{b}-1\right)\ln\left(\frac{v}{b}-1\right) + \ln N\right].$$

$$S = N k_B \left[\frac{v}{b} \ln \frac{v}{b} - \left(\frac{v}{b} - 1\right) \ln\left(\frac{v}{b} - 1\right) + \ln N \right].$$

~~In the thermodynamic~~

$$\left.\frac{\partial S}{\partial V}\right) = \frac{p}{T} \quad \Rightarrow \quad \left.\frac{\partial s}{\partial v}\right) = \frac{p}{T}.$$

$$\Rightarrow k_B \ln \frac{v/b}{(v/b - 1)} = \frac{p}{T}.$$

4. Total number of microstates

$$\Omega(E,N) = \sum_{N_1, N_2, ..} \frac{N!}{N_1! \, N_2! \dots}$$

where N_i is the number of particles in energy level ϵ_i.

$$S = k_B \ln \Omega(E,N) \equiv k_B \ln \sum_{N_j} \frac{N!}{\prod_j N_j!}$$

→ In the thermodynamic limit we replace the sum by its largest term $\{\tilde{N}_1, \tilde{N}_2, \dots\}$.

The constraint equations are

$$\sum N_j = N$$

and $\sum \epsilon_j N_j = E$.

To extremize the entropy subjected to ~~using~~ the contraints we use Lagrange multipliers.

this gives me $\tilde{N}_j$.

$$f(\{N_j\}, \lambda_1, \lambda_2) = \ln \frac{N!}{\prod N_j!} + \lambda_1 (N - \textstyle\sum_j N_j) + \lambda_2 (E - \textstyle\sum_j \epsilon_j N_j)$$

$$= \ln N! - \sum_j \ln N_j! + \lambda_1 (N - \textstyle\sum_j N_j) + \lambda_2 (E - \textstyle\sum_j \epsilon_j N_j).$$

$$f = \ln N! - \sum_j N_j \ln N_j + \sum_j N_j + \lambda_1 (N - \textstyle\sum_j N_j) + \lambda_2 (E - \textstyle\sum_j \epsilon_j N_j).$$

$$\frac{\partial f}{\partial N_j} = -\ln N_j - \cancel{1} + \cancel{1} - \lambda_1 - \lambda_2 \epsilon_j = 0.$$

$$\ln \tilde{N}_j = -(\lambda_1 + \lambda_2 \epsilon_j).$$

$$\tilde{N}_j = e^{-\lambda_1 - \lambda_2 \epsilon_j} = e^{-\lambda_1} e^{-\lambda_2 \epsilon_j}$$

$$\sum_j \tilde{N}_j = N.$$

$$\sum_j e^{-\lambda_2 \epsilon_j} e^{-\lambda_1} = N.$$

$$e^{-\lambda_1} = \frac{N}{\sum_j e^{-\lambda_2 \epsilon_j}}$$

$$\frac{\tilde{N}_j}{N} = \frac{e^{-\lambda_2 \epsilon_j}}{\sum_j e^{-\lambda_2 \epsilon_j}} \qquad Z = \sum_j e^{-\lambda_2 \epsilon_j}$$

$$\sum \epsilon_j \frac{\tilde{N}_j}{N} = \frac{E}{N} = u. \qquad \Rightarrow u = -\frac{\partial Z}{\partial \lambda_2}$$

For ideal gas (continuum limit).

$$Z = \sum_j e^{-\lambda_2 \epsilon_j} = \int d^3p \, e^{-\lambda_2 \vec{p}^2/2m} = \left(\frac{2\pi m}{\lambda_2}\right)^{3/2}$$

$$\Rightarrow \lambda_2 = \beta.$$

$$Z = \sum_j e^{-\beta \epsilon_j}$$

Canonical distribution.

$\downarrow$

Results from constraints

$\sum N_j = N \qquad \sum \epsilon_j N_j = E.$