PHY304 - Statistical Mechanics

Spring 2021, IISER Mohali

Instructor: Dr. Anosh Joseph

PHY304: Homework 3 Solutions

Due: Friday, February 5, 2021 at 11:00pm.

(Upload your solutions to Moodle as a single `.pdf` file.)

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1. Consider a system consisting of two particles of mass m. They interact weakly and are allowed to move in one dimension. Let us denote the position coordinates of particle 1 by x_1 and particle 2 by x_2. We also denote their momenta by p_1 and p_2. The boundaries of the one-dimensional space are at $x = 0$ and $x = L$. The total energy of the system is between E and $E + \delta E$. Draw the phase space regions that are accessible to the system. Since the phase space is four dimensional, draw separately the part of the phase space involving x_1 and x_2 and that involving p_1 and p_2.

Solution:

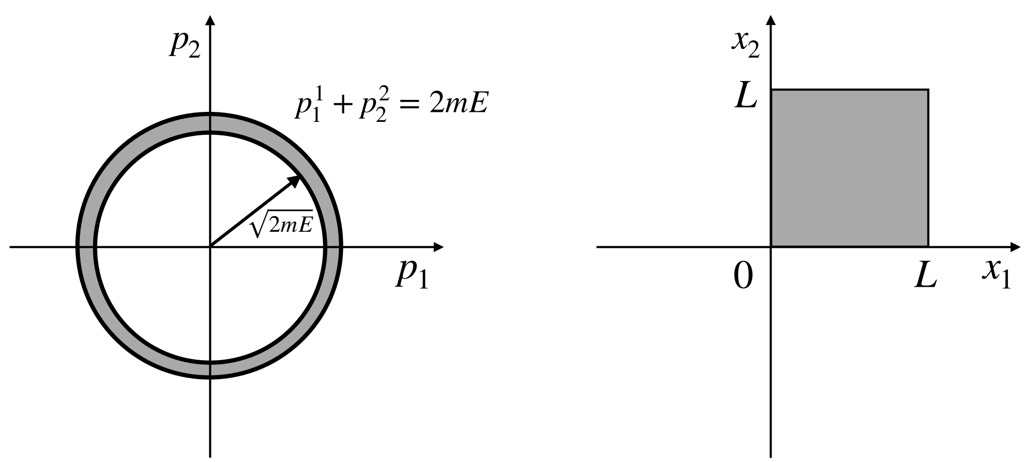

Figure 1: The four-dimensional phase space, shown as two two-dimensional spaces.

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2. Let us consider a system made out of a very large number N of distinguishable molecules. Let us assume that they are non-moving and non-interacting. Each molecule can have two energy levels: 0 and δ, with $\delta > 0$. In the large N limit, $N \to \infty$, E/N represents the mean energy per molecule. Compute the mean entropy per molecule, S/N, as a function of E/N.

Solution:

When the mean energy per molecule is E/N, we have E/δ molecules occupying the energy δ. The number of microstates is then

$$\Omega = \frac{N!}{\left(\frac{E}{\delta}\right)!\left(N - \frac{E}{\delta}\right)!}. \tag{1}$$

The entropy is

$$\begin{aligned} S &= k_B \ln \Omega \\ &= k_B \ln \frac{N!}{\left(\frac{E}{\delta}\right)!\left(N - \frac{E}{\delta}\right)!}. \end{aligned} \tag{2}$$

When $E/\delta \gg 1$, $N - (E/\delta) \gg 1$, we have

$$\begin{aligned} \frac{S}{N} &= k_B \left[\ln N - \frac{E/\delta}{N}\ln(E/\delta) - (1 - (E/\delta)/N)\ln(N - (E/\delta))\right] \\ &= k_B \left[\ln N - \frac{E}{\delta N}\ln\left(\frac{E}{\delta}\right) - \left(1 - \frac{E}{N\delta}\right)\ln\left\{N\left(1 - \frac{E}{N\delta}\right)\right\}\right] \\ &= k_B \left[\ln N - \frac{E}{N\delta}\ln\left(\frac{E}{\delta}\right) - \ln N - \ln\left(1 - \frac{E}{N\delta}\right)\right. \\ &\quad \left. + \frac{E}{N\delta}\ln N + \frac{E}{N\delta}\ln\left(1 - \frac{E}{N\delta}\right)\right] \\ &= k_B \left[-\frac{E}{N\delta}\ln\left(\frac{E}{N\delta}\right) - \ln\left(1 - \frac{E}{N\delta}\right) + \frac{E}{N\delta}\ln\left(1 - \frac{E}{N\delta}\right)\right] \\ &= k_B \left[-\frac{E}{N\delta}\ln\left(\frac{E}{N\delta}\right) + \ln\frac{1}{1 - E/(N\delta)} - \frac{E}{N\delta}\ln\ln\frac{1}{1 - E/(N\delta)}\right] \\ &= k_B \left[\frac{E}{N\delta}\ln\frac{N\delta}{E} + \left(1 - \frac{E}{N\delta}\right)\ln\frac{1}{1 - (E/N\delta)}\right]. \end{aligned} \tag{3}$$

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