MTH302, Integers, Polynomials & Matrices

Monsoon 2020

Deadline: 07/09/2020, 6:00PM.

Homework Assignment I

(1) For any positive integer n, verify that $\mathbb{Z}_n$ forms a group under addition and multiplication modulo[1] n.

(2) Prove that every Boolean ring is commutative.

(3) Suppose that R is a commutative ring such that $R[X]$ has a nontrivial zero-divisor $f(X)$. Show that there is a nonzero element $a \in R$ such that $a \cdot f(X) = 0$. Is commutativity of R essential?

(4) An element $a \in R$ is said to be *nilpotent* if $a^n = 0$ for some positive integer n. Let R be a commutative ring with 1. Prove that $f(X) = a_n X^n + \cdots + a_1 X + a_0 \in R[X]$ is a unit if and only if a_0 is a unit in R and a_i are nilpotent for $i = 1, 2, \cdots, n$.

(5) Let R be a commutative ring with 1. If $a \in R$ is nilpotent then show that $1 + a$ is a unit. Deduce that the sum of a unit and a nilpotent element is a unit.

(6) Let $a, b \in R$ be commuting elements (that is, $ab = ba$). For any positive integer n, prove the binomial expansion

$$(a+b)^n = \sum_{i=1}^{n} \binom{n}{i} a^i \cdot b^{n-i}.$$

(7) Let R be a ring with finitely many elements. If R is an integral domain then prove that R has 1 and that R is a division ring.

[1] In particular, check that these operations are well-defined