PHY 635: Gravitation & Cosmology : Problem Sheet 4

Submit before 31st March 2025

1. In the Schwarzschild spacetime given by the line element

$$ds^2 = -c^2 \left(1 - \frac{2GM}{c^2 r}\right) dt^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2 d\Omega^2.$$

Express the Killing vector $R = (0, 0, 0, 1)$ in cartesian co-rdinates (t, x, y, z). Prove that $(0, z, 0, -x)$ and $0, 0, -z, y)$ are also Killing vectors.

2. In the Schwarzschild spacetime, a time-like particle feels to be put in a central potential $V_{eff}(r)$. Show that in absence of the angular momentum, there is no point when this potential has any extremum. What is the stable orbit's radius in the large angular momentum limit ? What is the unstable orbit's radius ? What are these values for *massless* particles ?

3. In Schwarzschild spacetime obtain the equation of motion for a massless particle (say photon) in the effective potential. Show that in the Newtonian limit (i.e. GR correction term $\to 0$) the trajectory is :

$$u(\phi) \equiv \frac{1}{r} \equiv \frac{M}{L^2}(1 + e \cos \phi) = \frac{1}{b} \sin \phi,$$

where b is the impact parameter of the photon. Show that if the GR induced correction is taken as perturbation, if we start with small angle $\phi \to 0$ in the $r \to \infty$ limit we get the angular deflection to be $-2M/b$.

4. In the newly defined time co-ordinate $\tilde{t}$, $\tilde{t} = t + 2M \log |r - 2M|$ show that for radially ingoing null rays $\frac{dr}{d\tilde{t}} < 0$ always, but for radially out-going rays $\frac{dr}{d\tilde{t}} > 0$ only when $r > R_S = 2M$.
[Therefore, in the interior of the black hole, both the radially ingoing and radially out-going null rays move towards $r = 0$. Nothing comes out really !]

5. In Schwarzschild spacetime, if a null ray starts from location (r, θ, ϕ) and moves radially, how much of frequency shift it would undergo when it lands up at (r_0, θ, ϕ) ? Find out the weak field limit of that. How does it scale with the *Newtonian* spatial distance of the points ?