Which of the following are groups? In case not, find which condition(s) is/are not satisfied.

✗ (A). $\mathbb{Z}$, under the operation $*$, where $*$ denotes the multiplication of integers.

✗ (B). $\mathbb{R}$, under the operation $*$, where $*$ denotes the multiplication of real numbers.

} inverses?

✗ (C). The collection of irrational numbers under addition.

- two irrational numbers may not add up to an irrational number
- $0 \notin$ irrational numbers

✓ (D). The set of clock hours $\{12, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$ under the addition of clock hours. (Therefore $10 + 3 = 1$ under this operation).

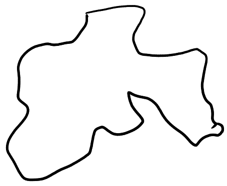
identity element of this group is 12

✓ (E). The set $\{1, 3, 7, 9\}$ under the operation "*rightmost digit in the multiplication of numbers*."

identity $= 1$, $3^{-1} = 7$, $7^{-1} = 3$, $9^{-1} = 9$

✓ (F). Symmetries of an amoeba.

Only identity



✗ (G). The collection $\{a, b, c\}$ of three alphabets with the operation given by the following composition table:

	a	b	c
a	b	a	c
b	c	b	a
c	a	c	b

where is the identity?

✓ (H). The collection $\{\square, \hexagon, \bullet, \circ\}$ of four symbols with the operation given by the following composition table:

	$\square$	$\hexagon$	$\bullet$	$\circ$
$\square$	$\bullet$	$\square$	$\circ$	$\hexagon$
$\hexagon$	$\square$	$\hexagon$	$\bullet$	$\circ$
$\bullet$	$\circ$	$\bullet$	$\hexagon$	$\square$
$\circ$	$\hexagon$	$\circ$	$\square$	$\bullet$

✗ (J). The subset $\{1, (1\ 2), (1\ 3), (1\ 4), (2\ 3), (2\ 4), (3\ 4)\}$ of S_4 under the composition of permutations.

$(1\ 2)(1\ 3) = (1\ 3\ 2)$ is not there in the subset.

✓ (K). The subset

$\{1, (1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3), (1\,2\,3), (1\,3\,2), (1\,2\,4), (1\,4\,2), (1\,3\,4), (1\,4\,3), (2\,3\,4), (2\,4\,3)\}$

of S_4 under the composition of permutations. even permutations

- $(1\;2\;3)(2\;3\;4) = (1\;2)(3\;4)$
- $(1\;2\;3)^{-1} = (1\;3\;2)$ etc.

✗ (L). The collection of 2×2 matrices having nonzero determinant and entries in $\mathbb{Z}$, under the operation of matrix multiplication; i.e. $\{A \in M_2(\mathbb{Z}) : \det(A) \neq 0\}$, under multiplication of matrices.

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}^{-1} \notin M_2(\mathbb{Z})$$

✓ (M). $\mathrm{GL}_n(\mathbb{R}) := \{A \in M_n(\mathbb{R}) : \det(A) \neq 0\}$, under multiplication of matrices. invertible matrices

✓ (N). $\mathrm{SL}_n(\mathbb{R}) := \{A \in M_n(\mathbb{R}) : \det(A) = 1\}$, under multiplication of matrices.

- Thanks to $\det(AB) = \det(A)\det(B)$

✗ (O). $\mathrm{Sym}_3(\mathbb{R}) := \{A \in M_3(\mathbb{R}) : A^t = A\}$, under multiplication of matrices. symmetric matrices

✓ (P). $\mathrm{Sym}_3(\mathbb{R}) := \{A \in M_3(\mathbb{R}) : A^t = A\}$, under addition of matrices.

✓ (Q). $\mathrm{Skew}_3(\mathbb{R}) := \{A \in M_3(\mathbb{R}) : A^t = -A\}$, under addition of matrices. skew-symmetric matrices

- $A^t = A,\ B^t = A \not\Rightarrow (AB)^t = AB$

$$\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 3 & 4 \\ 4 & 5 \end{pmatrix} = \begin{pmatrix} 11 & 14 \\ 18 & 23 \end{pmatrix} \notin \mathrm{Sym}_2(\mathbb{R})$$

– find such A, B for $\mathrm{Sym}_3(\mathbb{R})$

✗ (R). $\mathrm{Sym}_3(\mathbb{R}) \cap \mathrm{GL}_3(\mathbb{R}) := \{A \in M_3(\mathbb{R}) : A \text{ is invertible and } A^t = A\}$, under multiplication of matrices.

- Same argument as (O).

✓ (S). $O_3(\mathbb{R}) := \{A \in M_3(\mathbb{R}) : A \text{ is invertible and } A^t = A^{-1}\}$, under multiplication of matrices.

✓ (T). $SO_3(\mathbb{R}) := \{A \in M_3(\mathbb{R}) : A \text{ is invertible, } A^t = A^{-1} \text{ and } \det(A) = 1\}$, under multiplication of matrices.

- $(AB)^t = B^t A^t = B^{-1}A^{-1} = (AB)^{-1}$
- $\det(A) = 1 \Rightarrow \det(A^{-1}) = 1$

✓ (U). The collection of rotations R_θ of a circular disc, under composition of symmetries.

✗ (V). The collection of reflections f_θ of a circular disc, under composition of symmetries.

- $R_\theta R_\phi = R_{\theta+\phi}$, $R_0 = 1$, $R_\theta^{-1} = R_{-\theta}$

Associative

- No reflection is "doing nothing".

so identity $\notin$ set of reflections

✓ (W). The collection of 2×2 matrices of the form $\begin{pmatrix} a & a \\ a & a \end{pmatrix}$, where a is a nonzero element in $\mathbb{Q}$, under the operation of matrix multiplication.

✗ (X). The collection of 2×2 matrices of the form $\begin{pmatrix} a & a \\ a & a \end{pmatrix}$, where a is a nonzero element in $\mathbb{Q}$, under the operation of matrix addition.

(W) $\begin{pmatrix} a & a \\ a & a \end{pmatrix}\begin{pmatrix} b & b \\ b & b \end{pmatrix} = \begin{pmatrix} a & a \\ a & a \end{pmatrix} \quad \forall a$

$\Rightarrow \begin{pmatrix} 2ab & 2ab \\ 2ab & 2ab \end{pmatrix} = \begin{pmatrix} a & a \\ a & a \end{pmatrix}$

$\Rightarrow b = \frac{1}{2}$

so $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ is the identity in this case!

And $\begin{pmatrix} a & a \\ a & a \end{pmatrix}^{-1} = \begin{pmatrix} 1/4a & 1/4a \\ 1/4a & 1/4a \end{pmatrix}$ – strange!

(X). $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \notin$ the given set.

(Y). Collection of all polynomials in one variable with coefficients in $\mathbb{R}$, under the addition of polynomials.

(Z). Collection of all polynomials in one variable with coefficients in $\mathbb{R}$, under the multiplication of polynomials.

- constants are also considered polynomials.
  0 = identity of polynomials under addition
- What is the multiplicative inverse of the polynomial x?