PHY 101 : Problem Sheet 2

For the discussion on **cylindrical polar co-ordinate** please refer to the Fig. 1 on the next page. Any point in the 3-Dimensional space can either be written as (x, y, z) or a point on a cylinder as (r, θ, z) where r is the magnitude of the projection ($\mathbf{r}$) of the position vector $\mathbf{R}$ into the $x-y$ plane, and θ is the angle the projected vector $\mathbf{r}$ makes with the $x-$ axis. Note that the unit vectors associated with $\hat{\mathbf{r}}$ and $\hat{\theta}$ is the same as we learnt for the polar co-ordinates in 2-dimensional case in the class. Thus,

$$\begin{aligned}\hat{\mathbf{r}} &= \cos\theta\ \hat{\mathbf{i}} + \sin\theta\ \hat{\mathbf{j}};\\ \hat{\theta} &= -\sin\theta\ \hat{\mathbf{i}} + \cos\theta\ \hat{\mathbf{j}},\end{aligned}$$

while $\hat{\mathbf{k}}$ keeps pointing along the $z-$ axis

1. Find out $\mathbf{v} = d\mathbf{R}/dt$ and $\mathbf{a} = d\mathbf{v}/dt$ in $\hat{\mathbf{r}}, \hat{\theta}, \hat{\mathbf{k}}$ basis.
2. Find out $\hat{\mathbf{r}} \times \hat{\theta}$, $\hat{\theta} \times \hat{\mathbf{k}}$ and $\hat{\mathbf{k}} \times \hat{\mathbf{r}}$.
3. Find out the expression of angular momentum $\mathbf{L}$ of a particle at a location $\mathbf{R}$ and moving with velocity $\mathbf{v}$ in cylindrical polar co-ordinates.
4. In cylindrical polar co-ordinates write down the expression for torque $\overrightarrow{\tau} = d\mathbf{L}/dt$.

For the discussion on **spherical polar co-ordinate** please refer to the Fig. 2 on the next page. Any point P in the 3-Dimensional space can either be written as (x, y, z) or a point on a sphere as (r, θ, ϕ) where r is the magnitude of position vector $\mathbf{r}$, θ is the angle the vector $\mathbf{r}$ makes with the $z-$ axis and ϕ is the angle the vector $\mathbf{p}$, which is projection of $\mathbf{r}$ into the $x-y$ plane, makes w.r.t. the $x-$ axis. Note that the projection of $\mathbf{r}$ along the $x-$ axis (OA) can be calculated as $OA = OB\cos\phi = r\sin\theta\cos\phi$. Find out the projection of $\mathbf{r}$ along the $y-$ axis and show for a unit radius sphere we have

$$\hat{\mathbf{r}} = \sin\theta\cos\phi\ \hat{\mathbf{i}} + \sin\theta\sin\phi\ \hat{\mathbf{j}} + \cos\theta\ \hat{\mathbf{k}};$$

Also note that the unit vector $\hat{\phi}$ plays the same role as the angular co-ordinate we learnt for the polar co-ordinates in 2-dimensional case in the class. Thus,

$$\hat{\phi} = -\sin\phi\ \hat{\mathbf{i}} + \cos\phi\ \hat{\mathbf{j}}.$$

Since $\hat{\theta}$ is perpendicular to $\hat{\mathbf{r}}$ [because $\hat{\mathbf{r}}$ along the radial direction while $\hat{\theta}$ is along the tanget to the sphere], show that

1. the projection of $\hat{\theta}$ along the $z-$axis is " - $\sin\theta$".
2. If the projection of $\hat{\theta}$ along the $x-$ and $y-$ axis are x and y respectively, i.e.

   $$\hat{\theta} = x\ \hat{\mathbf{i}} + y\ \hat{\mathbf{j}} - \sin\theta\ \hat{\mathbf{k}},$$

   then find out x and y from the conditions $\hat{\mathbf{r}} \cdot \hat{\theta} = 0$ and $\hat{\phi} \cdot \hat{\theta} = 0$.
3. Find out the expression for $d\hat{\mathbf{r}}/dt$ and $d\hat{\phi}/dt$ in terms of $\hat{\mathbf{r}}, \hat{\theta}, \hat{\phi}$.
4. For a general position vector $\mathbf{r} = r\hat{\mathbf{r}}$, find the expression of velocity $\hat{\mathbf{v}} = d\mathbf{r}/dt$.

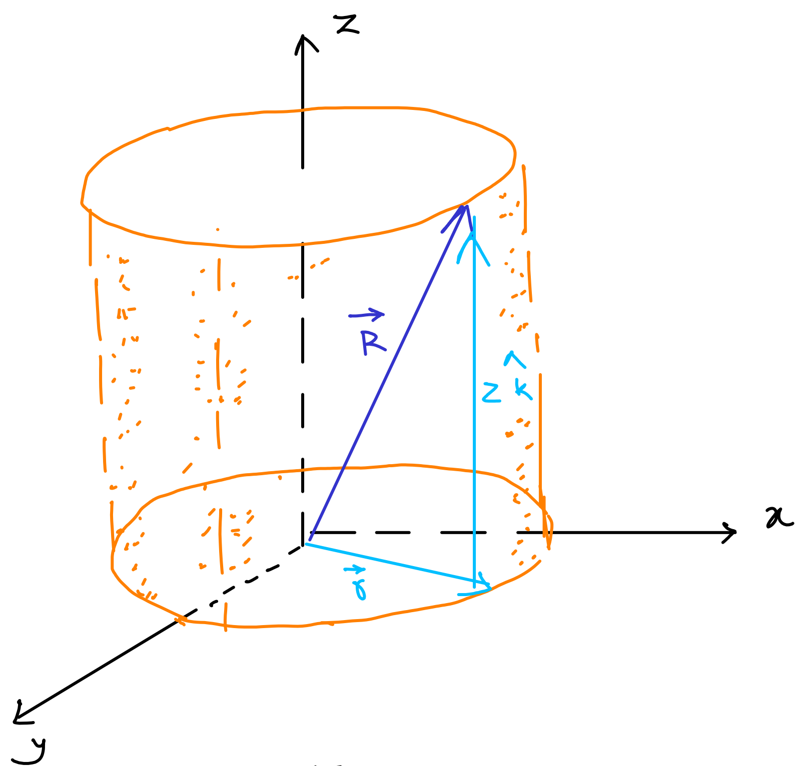


Fig 1

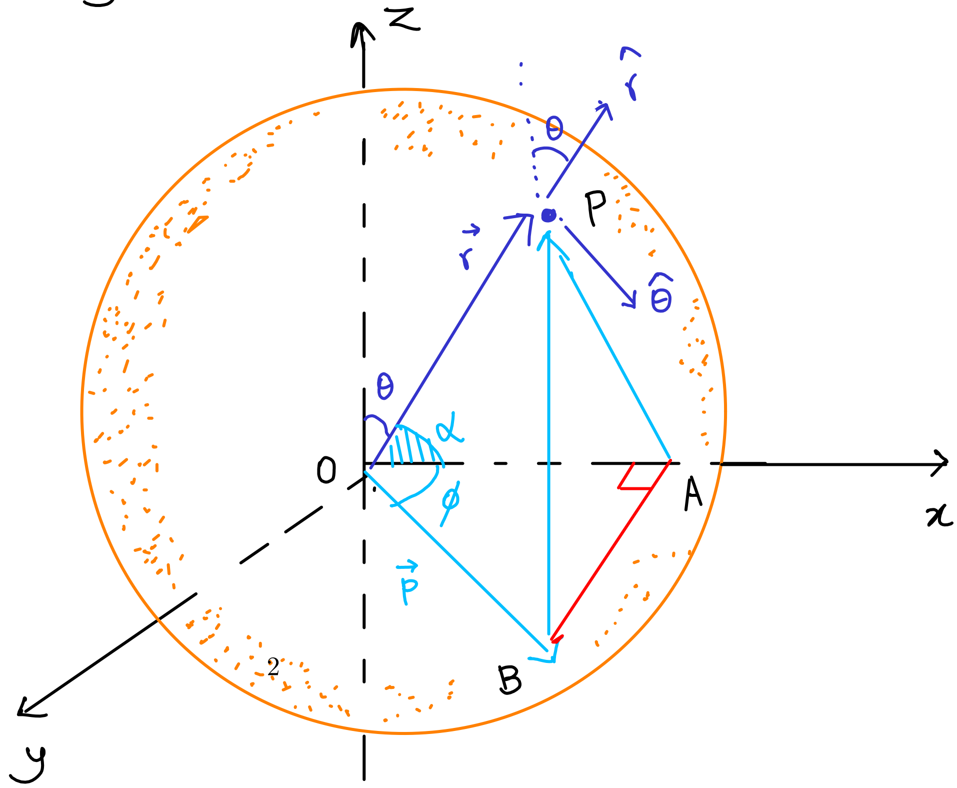
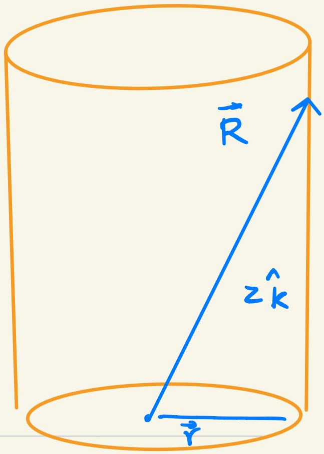


Fig 2

Worksheet - 2

1. The position vector

$$\vec{R} = \vec{r} + z\hat{k}$$



$$= r\hat{r} + z\hat{k}$$

$$\vec{V} = \frac{d\vec{R}}{dt} = \frac{d}{dt}(r\hat{r}) + \frac{d}{dt}(z\hat{k})$$

$$= \dot{r}\hat{r} + \underbrace{r\dot{\theta}\hat{\theta}}_{\text{because } \dot{\hat{r}} = \dot{\theta}\hat{\theta}} + \underbrace{\dot{z}\hat{k}}_{\text{because } \dot{\hat{k}} = 0}$$

Therefore,

$$\vec{a} = \frac{d\vec{V}}{dt} = \frac{d}{dt}(\dot{r}\hat{r}) + \frac{d}{dt}(r\dot{\theta}\hat{\theta}) + \frac{d}{dt}(\dot{z}\hat{k})$$

$$= \ddot{r}\hat{r} + \dot{r}\dot{\theta}\hat{\theta} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} - r\dot{\theta}^2\hat{r} + \ddot{z}\hat{k}$$

$$= (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta} + \ddot{z}\hat{k}$$

2. $\hat{r} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$

$\hat{\theta} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$

$\hat{k} = \hat{k}$

$$\vec{r} \times \hat{\theta} = (\cos\theta\,\hat{i} + \sin\theta\,\hat{j}) \times (-\sin\theta\,\hat{i} + \cos\theta\,\hat{j})$$
$$= (\cos^2\theta + \sin^2\theta)\,\hat{k}$$
$$= \hat{k}$$

$$\hat{\theta} \times \hat{k} = (-\sin\theta\,\hat{i} + \cos\theta\,\hat{j}) \times \hat{k}$$
$$= (\sin\theta\,\hat{j} + \cos\theta\,\hat{i}) = \hat{r}$$

$$\hat{k} \times \hat{r} = \hat{k} \times (\cos\theta\,\hat{i} + \sin\theta\,\hat{j})$$
$$= (\cos\theta\,\hat{j} - \sin\theta\,\hat{i}) = \hat{\theta}$$

3. $\vec{L} = \vec{R} \times m\vec{V}$

$$= (r\hat{r} + z\hat{k}) \times m\,(\dot{r}\hat{r} + r\dot{\theta}\,\hat{\theta} + \dot{z}\hat{k})$$

$$= mr^2\dot{\theta}\,\hat{k} + mr\dot{z}(-\hat{\theta})$$

4. $$\vec{\tau} = \frac{d\vec{L}}{dt} = \frac{d}{dt}\left(\vec{R} \times m\vec{v}\right)$$

$$= \underbrace{\frac{d\vec{R}}{dt} \times m\vec{v}}_{0 \text{ because } \vec{v} \times \vec{v}} + \vec{R} \times m\frac{d\vec{v}}{dt}$$

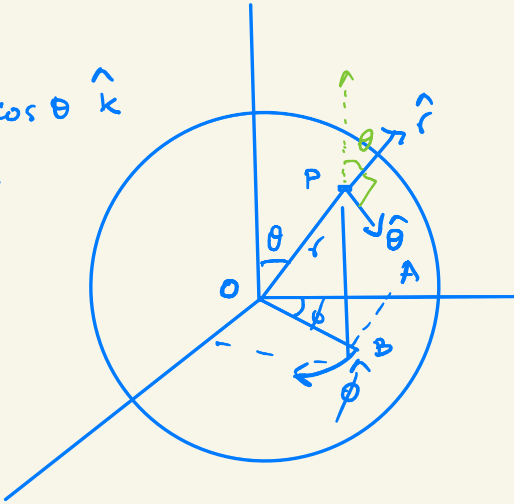
$$= \vec{R} \times m\vec{a} = (r\hat{r} + z\hat{k}) \times \left[(\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta} + \ddot{z}\hat{k}\right]$$

$$= (2r\dot{r}\dot{\theta} + r^2\ddot{\theta})\hat{k} - r\ddot{z}\hat{\theta} + (z\ddot{r} - zr\dot{\theta}^2)\hat{\theta}$$
$$- (2\dot{r}z\dot{\theta} + zr\ddot{\theta})\hat{r}$$

$$= -(2\dot{r}z\dot{\theta} + zr\ddot{\theta})\hat{r} + (z\ddot{r} - r\ddot{z} - zr\dot{\theta}^2)\hat{\theta}$$
$$+ (2r\dot{r}\dot{\theta} + r^2\ddot{\theta})\hat{k}$$

Q. Set 2

1.) $\hat{r} = \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}$



$\hat{\phi} = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$

The vector $\hat{\theta}$ makes an angle $90^\circ + \theta$ with z-axis

$\therefore \quad \hat{\theta} = x\,\hat{i} + y\,\hat{j} + \cos(90^\circ+\theta)\,\hat{k}$

$= x\,\hat{i} + y\,\hat{j} - \sin\theta\,\hat{k}$

2.) $\hat{r}\cdot\hat{\theta} = x\sin\theta\cos\phi + y\sin\theta\sin\phi - \sin\theta\cos\theta$

$= 0$ — (1)

$\hat{\theta}\cdot\hat{\phi} = -x\sin\phi + y\cos\phi = 0$

$\Rightarrow \quad x\sin\phi = y\cos\phi$ — (2)

$\sin\phi \times$ (1) $\Rightarrow \quad x\sin\theta\sin\phi\cos\phi + y\sin\theta\sin^2\phi$

$= \sin\theta\cos\theta\sin\phi$

Using (2) $\Rightarrow \quad y\sin\theta\cos^2\phi + y\sin\theta\sin^2\phi$

$= \sin\theta\cos\theta\sin\phi$

$\Rightarrow \quad y\sin\theta = \sin\theta\cos\theta\sin\phi$

$y = \cos\theta\sin\phi$

Using ② $\quad x \sin\phi = y \cos\phi$

$$= \cos\theta \sin\phi \cos\phi$$

$$\therefore \quad x = \cos\theta \cos\phi$$

$$\therefore \quad \hat{\theta} = \cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi - \sin\theta\,\hat{k}$$

3.) $\quad \hat{r} = \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}$

$$\frac{d\hat{r}}{dt} = \frac{d}{dt}(\sin\theta\cos\phi)\,\hat{i} + \frac{d}{dt}(\sin\theta\sin\phi)\,\hat{j} + \frac{d}{dt}(\cos\theta)\,\hat{k}$$

$$= (\dot{\theta}\cos\theta\cos\phi - \dot{\phi}\sin\theta\sin\phi)\,\hat{i} + (\dot{\theta}\cos\theta\sin\phi + \dot{\phi}\sin\theta\cos\phi)\,\hat{j}$$

$$- \dot{\theta}\sin\theta\,\hat{k}$$

$$= \dot{\theta}\,(\cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,\hat{k})$$

$$+ \dot{\phi}\sin\theta\,(-\sin\phi\,\hat{i} + \cos\phi\,\hat{j})$$

$$= \dot{\theta}\,\hat{\theta} + \dot{\phi}\sin\theta\,\hat{\phi}$$

$$\frac{d\hat{\phi}}{dt} = \frac{d}{dt}(-\sin\phi\,\hat{i} + \cos\phi\,\hat{j}) = -\dot{\phi}\,(\cos\phi\,\hat{i} + \sin\phi\,\hat{j})$$

$$= -\dot{\phi}\,(\sin\theta\,\hat{r} + \cos\theta\,\hat{\theta})$$

4. $$\frac{d\vec{r}}{dt} = \frac{d}{dt}(r\hat{r}) = \dot{r}\hat{r} + r\frac{d}{dt}\hat{r}$$

$$= \dot{r}\hat{r} + r(\dot{\theta}\hat{\theta} + \sin\theta\,\dot{\phi}\hat{\phi})$$

$$= \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r\sin\theta\,\dot{\phi}\hat{\phi}$$