1. Calculate the number of accessible microstates of a system of two localized independent quantum oscillators with fundamental frequencies ω_0 and $3\omega_0$, and total energy of $10\hbar\omega_0$.

2. The spin Hamiltonian of a system of N localized magnetic ions is given by

$$\mathcal{H} = D \sum_{j=1}^{N} S_j^2,$$

where $D > 0$ and spin variables S_j may assume values ± 1 or 0, for $j = 1, 2, 3.....$ This spin Hamiltonian describes the effects of the electrostatic environment on spin-1 ions. An ion in state ± 1 has energy $D > 0$ and in state 0 has zero energy.

   a) Show that the number of accessible microstates of the system is given by

$$\Omega(U, N) = \frac{N!}{(N - U/D)!} \sum_{N_-} \frac{1}{(U/D - N_-)! N_-!}$$

   b) Calculate the binomial sum and obtain an exact result for $\Omega(U, N)$

   c) Now use Sterling's approximation and obtain the entropy of the system. Is it extensive ?

   d) Calculate the specific heat of the system as a function of temperature.

3. Consider N particles distributed in a volume V. Now divide the volume into cell of size b, with $N \leq V/b$. Suppose that each cell may be either empty or can be occupied by a single particle.

   a) calculate the number of microstates accessible to the system.

   b) from the above result, calculate the entropy of the system and hence the quantity P/T, where the symbols have their usual meaning.

   c) do you see any difference with an ideal gas? If yes what do think is the reason behind this difference?

4. In the class we have worked out the problem of N quantum harmonic oscillators in the microcanonical ensemble. Now assume that the fundamental frequency has volume dependence given by:

$$\omega = \omega(v) = \omega_0 - A \ln\left(\frac{v}{v_0}\right),$$

where $v = V/N$, and ω_0, A, and v_0 are positive constants. Calculate the expansion coefficient and the compressibility of the system.

5. Consider system of ideal gas of N particles which can be in discrete energy states ϵ_j; $j = 0, 1, 2, 3, ...$

   a) how would you specify the microscopic state of the system?

   b) calculate the number of accessible microstates for the system.

   c) assuming that the total particle number and total energy of the system is fixed, write down the constraint equations.

   d) can you now extremize the system and find out probability of finding N_i particle in an energy state ϵ_i? Do you recognize the distribution.

6. Consider a magnetic system with a total energy E and having N spins. The Hamiltonian for the system is $\mathcal{H} = -\mu H \sum_i \sigma_i$, with $\sigma_i = \pm 1$. In a microcanonical ensemble, we want to calculate the total number of accessible microstates.

   a) Assume that there are N_+ up spins and N_- down spins. Express the N_+ and N_- in terms of E and N.

   b) From this calculate the total number of microstates accessible to the system for $E, N \to \infty$ and $E/N = u$ fixed.

   c) Hence calculate the entropy per spin and derive an expression for the energy per spin of the system.

   d) Using all the above informations (not all are required though) derive an expression for the magnetization of the system.

7. We want to look at the above problem using a different approach.

   a) The total number of accessible microstates is given by $\Omega = \sum_{\{\sigma_i = \pm 1\}} \delta(E + h \sum_i \sigma_i)$. Using the integral representation of the delta function $\delta(x - a) = \int_{-\infty}^{\infty} e^{ik(x-a)} \frac{\mathrm{d}k}{2\pi}$, rewrite the expression for Ω.

   b) Now do the sum over the states of the individual spins σ_i to express Ω as $\Omega = 2^N \int_{-\infty}^{\infty} e^{f(k)} \frac{\mathrm{d}k}{2\pi}$. Determine $f(k)$.

   c) Verify that the argument of the exponential goes as $\mathcal{O}(N)$. We can therefore use a saddle point approximation. To do this find the minimum k_0 of the function $f(k)$ and verify that it is indeed a minimum (i.e. $f''(k_0) < 0$. You need the following identity : $\arctan(ix) = \frac{1}{2} \log\left(\frac{1+x}{1-x}\right)$

   d) Once you have the value of k_0, you want to re-express the integral in Ω. Recall how we did it while establishing the connection between the canonical partition function and the free energy F. You have to approximate the function $f(k)$ by a Taylor series around k_0 and retain the second order term. Once you have done that carry out the integral over and determine Ω as function of $e = E/Nh$.