1. Calculate the number of accessible microstates of a system of two localized independent quantum oscillators with fundamental frequencies $\omega_0$ and $3\omega_0$, and total energy of $10\hbar\omega_0$.

2. The spin Hamiltonian of a system of $N$ localized magnetic ions is given by

$$\mathcal{H} = D\sum_{j=1}^{N} S_j^2,$$

where $D > 0$ and spin variables $S_j$ may assume values $\pm 1$ or 0, for $j = 1, 2, 3.....$ This spin Hamiltonian describes the effects of the electrostatic environment on spin-1 ions. An ion in state $\pm 1$ has energy $D > 0$ and in state 0 has zero energy.

   *a)* Show that the number of accessible microstates of the system is given by

$$\Omega(U, N) = \frac{N!}{(N - U/D)!}\sum_{N_-}\frac{1}{(U/D - N_-)!N_-!}$$

   *b)* Calculate the binomial sum and obtain an exact result for $\Omega(U, N)$

   *c)* Now use Sterling's approximation and obtain the entropy of the system. Is it extensive ?

   *d)* Calculate the specific heat of the system as a function of temperature.

3. Consider $N$ particles distributed in a volume $V$. Now divide the volume into cell of size $b$, with $N \leq V/b$. Suppose that each cell may be either empty or can be occupied by a single particle.

   *a)* calculate the number of microstates accessible to the system.

   *b)* from the above result, calculate the entropy of the system and hence the quantity $P/T$, where the symbols have their usual meaning.

   *c)* do you see any difference with an ideal gas? If yes what do think is the reason behind this difference?

4. In the class we have worked out the problem of $N$ quantum harmonic oscillators in the microcanonical ensemble. Now assume that the fundamental frequency has volume dependence given by:

$$\omega = \omega(v) = \omega_0 - A\ln\left(\frac{v}{v_0}\right),$$

where $v = V/N$, and $\omega_0, A$, and $v_0$ are positive constants. Calculate the expansion coefficient and the compressibility of the system.

5. Consider system of ideal gas of $N$ particles which can be in discrete energy states $\epsilon_j$; $j = 0, 1, 2, 3, ...$

   *a)* how would you specify the microscopic state of the system?

   *b)* calculate the number of accessible microstates for the system.

   *c)* assuming that the total particle number and total energy of the system is fixed, write down the constraint equations.

   *d)* can you now extremize the system and find out probability of finding $N_i$ particle in an energy state $\epsilon_i$? Do you recognize the distribution.

6. Consider a magnetic system with a total energy $E$ and having $N$ spins. The Hamiltonian for the system is $\mathcal{H} = -\mu H \sum_i \sigma_i$, with $\sigma_i = \pm 1$. In a microcanonical ensemble, we want to calculate the total number of accessible microstates.

   *a)* Assume that there are $N_+$ up spins and $N_-$ down spins. Express the $N_+$ and $N_-$ in terms of $E$ and $N$.

   *b)* From this calculate the total number of microstates accessible to the system for $E, N \to \infty$ and $E/N = u$ fixed.

   *c)* Hence calculate the entropy per spin and derive an expression for the energy per spin of the system.

   *d)* Using all the above informations (not all are required though) derive an expression for the magnetization of the system.

7. We want to look at the above problem using a different approach.

   *a)* The total number of accessible microstates is given by $\Omega = \sum_{\{\sigma_i = \pm 1\}} \delta(E + h \sum_i \sigma_i)$. Using the integral representation of the delta function $\delta(x - a) = \int_{-\infty}^{\infty} e^{ik(x-a)} \frac{\mathrm{d}k}{2\pi}$, rewrite the expression for $\Omega$.

   *b)* Now do the sum over the states of the individual spins $\sigma_i$ to express $\Omega$ as $\Omega = 2^N \int_{-\infty}^{\infty} e^{f(k)} \frac{\mathrm{d}k}{2\pi}$. Determine $f(k)$.

   *c)* Verify that the argument of the exponential goes as $\mathcal{O}(N)$. We can therefore use a saddle point approximation. To do this find the minimum $k_0$ of the function $f(k)$ and verify that it is indeed a minimum (i.e. $f''(k_0) < 0$. You need the following identity : $\arctan(ix) = \frac{1}{2} \log\left(\frac{1+x}{1-x}\right)$

   *d)* Once you have the value of $k_0$, you want to re-express the integral in $\Omega$. Recall how we did it while establishing the connection between the canonical partition function and the free energy $F$. You have to approximate the function $f(k)$ by a Taylor series around $k_0$ and retain the second order term. Once you have done that carry out the integral over and determine $\Omega$ as function of $e = E/Nh$.