Lecture 4.

§ Exact equations of the First order and of First degree.

An ODE of the first order and of the first degree may be expressed in form of a total differential equation:

$$P\,dx + Q\,dy = 0$$

where P, Q are fns of x, y and do not involve $p \equiv \frac{dy}{dx}$

If the eqn is exact and its primitive is $u = C$, the two expressions for du must be identical:

$$P\,dx + Q\,dy \quad \text{and} \quad \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy.$$

i.e. $P = \frac{\partial u}{\partial x}$, $Q = \frac{\partial u}{\partial y}$. Then $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$. (A)

provided that the equivalent expression $\frac{\partial^2 u}{\partial x \partial y}$ is Continuous.

(A) is "the condition of integrability" and is necessary.

• ~~Now~~ We show that it is sufficient too:

Let $$u(x,y) = \int_{x_0}^{x} P(x,y)\,dx + \phi(y)$$

where x_0 is an arbitrary Constant, and $\phi(y)$ is a fn. of y alone which for the moment, is also arbitrary.

Then $u = C$ will be primitive of $P\,dx + Q\,dy = 0$

if $\frac{\partial u}{\partial x} = P$, $\frac{\partial u}{\partial y} = Q$.

The first condition is satisfied. The second determines $\phi(y)$:

$$Q(x,y) = \frac{\partial u}{\partial y} = \int_{x_0}^{x} \frac{\partial P}{\partial y}\,dx + \phi'(y)$$

$$= \int_{x_0}^{x} \frac{\partial Q}{\partial x}\,dx + \phi'(y)$$

$$= Q(x,y) - Q(x_0,y) + \phi'(y).$$

So, $\phi(y) = \int_{y_0}^{y} Q(x_0, y)\, dy$. where y_0 is arbitrary.

The Condition is therefore sufficient and the eqn. is exact with primitive $\int_{x_0}^{x} P(x,y)\, dx + \int_{y_0}^{y} Q(x_0, y)\, dy = C$.

x_0, y_0 Can be chosen as Convenient. There is only one arbitr. Cont.

Example. $\frac{2x-y}{x^2+y^2}\, dx + \frac{2y+x}{x^2+y^2}\, dy = 0$.

Cond. of Integrability is satisfied. The primitive is

$$\int_{x_0}^{x} \frac{2x-y}{x^2+y^2}\, dx + \int_{y_0}^{y} \frac{2y+x_0}{x_0^2+y_0^2}\, dy = C$$

Clearly an advantage to take $x_0 = 0$. $y_0 = 1$ is also the gnidt

$$\int_{0}^{x} \frac{2x-y}{x^2+y^2}\, dx + 2\int_{1}^{y} \frac{dy}{y} = C.$$

i.e. $\left[\log(x^2+y^2) - \tan^{-1}\left(\frac{x}{y}\right)\right]_{x=0}^{x=x} + 2\log y = C$

Which reduces to $\log(x^2+y^2) - \tan^{-1}\left(\frac{x}{y}\right) = C$.

§ Separation of variables.

A particular instance is when P is a fn. of x alone & Q that of y alone.

The equation $P(x)\,dx + Q(y)\,dy = 0$ is then said to have Separated variables. Its primitive is $\int P(x)\,dx + \int Q(y)\,dy = C$

When eqn. is such that P can be factored into a fn. X of x alone & Y_1 a fn. of y alone and Q can similarly be factorised into X_1 & Y, the variables are said to be separable:

$$X Y_1 dx + X_1 Y dy = 0$$

may be written in separated form:

$$\frac{X}{X_1} dx + \frac{Y}{Y_1} dy = 0$$

It must be noticed however that a number of solutions are lost in the division of the eqn by $X_1 Y_1$.

If for example; $x = a$ is root of $X_1 = 0$; it would give us solution of $XY_1 dx + X_1 Y dy = 0$ but not of separable form.

Example $(x^2+1)(y^2-1) dx + xy\,dy = 0$.

The variables are separable:

$$\frac{x^2+1}{x} dx + \frac{y}{y^2-1} dy = 0$$

Integrating: $x^2 + \log x^2 + \log(y^2-1) = C$.

or if $C = \log C$,

$$y^2 = 1 + C\frac{e^{-x^2}}{x^2}$$

In addition $x = 0$, $y = 1$, $y = -1$ are real solutions of the given equation. $x = 0$ is not included in the general solution. The latter two are included.

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