1. $\underline{u}(\underline{x},t) = -\frac{1}{2}\alpha r\hat{e}_r + u_\theta(r)\hat{e}_\theta + \alpha z\hat{e}_z$

(a) $\underline{\nabla}\cdot\underline{u} = \frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{\partial u_z}{\partial z}$

$= \frac{1}{r}\frac{\partial}{\partial r}\left[r.\left\{-\frac{1}{2}\alpha r\right\}\right] + 0 + \frac{\partial}{\partial z}(\alpha z)$

$= \frac{1}{\cancel{r}}\cdot\left(-\frac{1}{\cancel{2}}\alpha\right).\cancel{2}\cancel{r} + \alpha.$

$= -\alpha + \alpha \quad = 0$

(b) Vorticity Eqn : $\frac{D\underline{\omega}}{Dt} = (\underline{\omega}\cdot\underline{\nabla})\underline{u} + \nu\nabla^2\underline{\omega}$

$\underline{\omega} = \omega\hat{e}_z \quad ; \quad \omega = \frac{1}{r}\frac{\partial}{\partial r}(ru_\theta). \equiv \omega(r)$

$\frac{D\underline{\omega}}{Dt} = \left(\frac{\partial}{\partial t} + \underline{u}\cdot\underline{\nabla}\right)\underline{\omega}$

$(\underline{u}\cdot\underline{\nabla})\omega = \left(u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial\theta} + u_z\frac{\partial}{\partial z}\right)\omega$

$= -\frac{1}{2}\alpha r\frac{\partial\omega}{\partial r}$

$\therefore \frac{D\underline{\omega}}{Dt} = \left(\frac{\partial\omega}{\partial t} - \frac{1}{2}\alpha r\frac{\partial\omega}{\partial r}\right)\hat{e}_z$

$(\underline{\omega}\cdot\underline{\nabla})\underline{u} = \omega\frac{\partial}{\partial z}\left[-\frac{1}{2}\alpha r\hat{e}_r + u_\theta(r)\hat{e}_\theta + \alpha z\hat{e}_z\right]$

$= \alpha\omega\hat{e}_z$

$\nabla^2\underline{\omega} = \left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2}\right)\omega\hat{e}_z$

$= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\omega}{\partial r}\right)\hat{e}_z \quad (\because \omega\equiv\omega(r))$

$\therefore \frac{\partial\omega}{\partial t} - \frac{1}{2}\alpha r\frac{\partial\omega}{\partial r} = \alpha\omega + \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\omega}{\partial r}\right).$

At steady state, $+\frac{1}{2}\alpha r\frac{\partial\omega}{\partial r} + \alpha\omega + \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\omega}{\partial r}\right) = 0$

Scanned with CamScanner

$$\Rightarrow \quad \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{1}{2}\alpha r^2 \omega\right) + \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \omega}{\partial r}\right) = 0$$

$$\Rightarrow \quad \frac{1}{r}\frac{\partial}{\partial r}\left[\frac{1}{2}\alpha r^2 \omega + \nu r \frac{\partial \omega}{\partial r}\right] = 0.$$

$$\Rightarrow \quad \frac{\partial \omega}{\partial r} = -\frac{\alpha r}{2\nu}\omega + k_1$$ (assuming constant of integration is zero).

i.e. ω & $\frac{\partial \omega}{\partial r}$ decays fast as $r \to \infty$.

Integrating,

$$\omega = \omega_0 e^{-\alpha r^2/4\nu} + k_1 r$$

To determine ω_0, we use,

$$\Gamma = 2\pi \int_0^\infty dr\, r\, \omega(r) = 2\pi \omega_0 \int_0^\infty r e^{-\alpha r^2/4\nu}\, dr$$

$$= 2\pi \omega_0 \frac{\Gamma(1)}{2\cdot(\alpha/4\nu)} = \frac{4\pi \nu \omega_0}{\alpha}$$

$$\Rightarrow \quad \omega_0 = \frac{\alpha \Gamma}{4\pi\nu}.$$

$$\therefore \quad \omega = \frac{\alpha \Gamma}{4\pi\nu} e^{-\alpha r^2/4\nu} + k_1 \frac{r^3}{3}$$

(c) $$\omega = \frac{1}{r}\frac{\partial}{\partial r}(r u_\theta).$$

$$\Rightarrow \quad \frac{\alpha\Gamma}{4\pi\nu} e^{-\alpha r^2/4\nu} = \frac{1}{r}\frac{d}{dr}(r u_\theta).$$

$$\Rightarrow \quad r u_\theta = \frac{\alpha \Gamma}{4\pi\nu}\int_0^r r' e^{-\alpha r'^2/4\nu}\, dr'$$

$$\Rightarrow \quad r u_\theta = \frac{\Gamma}{2\pi}\left(1 - e^{-\alpha r^2/4\nu}\right)$$

$$\Rightarrow \quad u_\theta = \frac{\Gamma}{2\pi r}\left(1 - e^{-\alpha r^2/4\nu}\right).$$

$$\psi(x,y) = Uy + \frac{\Gamma}{2\pi}\ln\sqrt{x^2+(y-b)^2} - \frac{\Gamma}{2\pi}\ln\sqrt{x^2+(y+b)^2}$$

(a) $$\sqrt{x^2+(y\pm b)^2} = r\sqrt{1+\left(\frac{\pm 2yb+b^2}{r^2}\right)} \approx r\left(1\pm\frac{by}{r^2}\right)$$

with $r^2 = x^2+y^2$

$$\therefore \ln\sqrt{x^2+(y\pm b)^2} \approx \ln(r) + \ln\left(1\pm\frac{by}{r^2}\right).$$

$$\approx \ln(r) \pm \frac{by}{r^2}.$$

$$\therefore \psi(x,y) = Uy + \frac{\Gamma}{2\pi}\left\{\ln r - \frac{by}{r^2}\right\} - \frac{\Gamma}{2\pi}\left\{\ln r + \frac{by}{r^2}\right\}$$

$$= Uy - \frac{\Gamma by}{\pi r^2} = Uy\left(1 - \frac{\Gamma b}{\pi r^2 U}\right).$$

$$= Uy\left(1 - \frac{c}{2U\pi r^2}\right).$$

(b) In polar coordinates, $y = r\sin\theta$.

$$\therefore \psi(r,\theta) = Ur\sin\theta\left(1-\frac{c}{2\pi Ur^2}\right).$$

$$\therefore u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = U\cos\theta\left(1-\frac{c}{2\pi Ur^2}\right)$$

$$u_\theta = -\frac{\partial\psi}{\partial r} = -U\sin\theta\left(1+\frac{c}{2\pi Ur^2}\right).$$

(c) $u_r = 0$ when $\theta = \pm\frac{\pi}{2}$ or $r = \sqrt{\frac{c}{2\pi U}}$

$u_\theta = 0$ when $\theta = 0, \pm\pi$ or $r = \sqrt{\frac{-c}{2\pi U}}$

Scanned with CamScanner

$$\phi = A\cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi y}{b}\right)\cosh[k(z+H)]e^{-i\omega t}$$

(a) $u = \frac{\partial\phi}{\partial x} = -A\left(\frac{m\pi}{L}\right)\sin\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi y}{b}\right)\cosh[k(z+H)]e^{-i\omega t}$

$u = 0$ at $x = 0$ and $x = L$.

$v = \frac{\partial\phi}{\partial y} = -A\left(\frac{n\pi}{b}\right)\cos\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi y}{b}\right)\cosh[k(z+H)]e^{-i\omega t}$

$v = 0$ at $y = 0$ and $y = b$.

(b) Laplacian: $\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$

$$= \left(-\frac{m^2\pi^2}{L^2} - \frac{n^2\pi^2}{b^2} + k^2\right)A\cos\left(\frac{m\pi x}{L}\right)\cdot\cos\left(\frac{n\pi y}{b}\right)\cosh[k(z+H)]e^{-i\omega t}$$

$$= 0$$

when $k^2 = \frac{m^2\pi^2}{L^2} + \frac{n^2\pi^2}{b^2}$

(c) Linearized free surface boundary conditions:

$\frac{\partial\eta}{\partial t} = \frac{\partial\phi}{\partial z}$ & $\frac{\partial\phi}{\partial t} = -g\eta$ at $z = 0$.

$\Rightarrow \frac{\partial^2\phi}{\partial t^2} = -g\frac{\partial\eta}{\partial t} = -g\frac{\partial\phi}{\partial z}$ at $z = 0$

Using the given potential,

$$\left[A(-i\omega)^2\cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi y}{b}\right)\cosh(k(z+H))e^{-i\omega t}\right]_{z=0}$$

$$= -g\left[Ak\cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi y}{b}\right)\sinh(k(z+H))e^{-i\omega t}\right]_{z=0}.$$

$\Rightarrow -\omega^2\cosh kH = -gk\sinh(kH)$

$\Rightarrow \omega^2 = gk\tanh(kH).$

Scanned with CamScanner