MTH308: Rings and Modules

Final Examination, spring 2024

Venue: AB1-2A. Time: 9:00AM-12:00noon.

$p|ab \Rightarrow p|a$ or $p|b$

~~$p = ar$ or $p =$~~

$a = pr$ or $b = ps$

$p|ab \Rightarrow p|aps$ or $p|prb$

let $a = pr$

$p \mid prb$

$p(1-rb) = 0.$

Instructions.

Prime ideal $\to$ Irr $\Rightarrow p|ab \Rightarrow p|a$ or $p|b \Rightarrow p = ab$

$p|a$ or $p|b$

$p = a$ or $p = b$

$ab \in (p) \Rightarrow a \in (p)$ or $b \in (p)$

- This question paper has 9 questions.
- The symbols $\mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$ will denote the field of rational, real and complex numbers.
- Justify your answers clearly to obtain maximum credits. Only elegant and correct solutions will receive full credits.

$\frac{R}{(r)} \cong \frac{R}{(s)}$

Questions

(1) (4 pts) Show that every nonzero ring endomorphism of $M_2(\mathbb{R})$ is an automorphism.

(2) (4 pts) Prove that $\mathbb{Z}-$module $\dfrac{\mathbb{Q}}{\mathbb{Z}}$ is a torsion module and that it is not a finitely generated $\mathbb{Z}-$module.

(3) (4 pts) Let R be an integral domain. Prove or disprove the following statements.
   (a) Every prime element of R is irreducible.
   (b) Every irreducible element of R is prime.

Irr: $p = ab \Rightarrow p = a$ or $p = b$

Prime: $p|ab \Rightarrow p|a$ or $p|b$.

(4) (4 pts) Let R be a commutative ring with 1. Prove or disprove the following statements
   (a) Every maximal ideal of R is a prime ideal of R. T
   (b) Every nonzero prime ideal of R is a maximal ideal of R. F

(5) (4 pts) Let F be a field, $\phi : V \to V$ be a map of finite dimensional $F-$vector spaces and $\lambda \in F$ be a root of the characteristic polynomial of ϕ. Prove that λ is also a root of the minimal monic polynomial of ϕ.

(6) (4 pts) Let R be a PID. For $r, s \in R$, if $R/(r)$ is isomorphic to $R/(s)$ as $R-$modules then show that r and s are associates. $r \neq us$

(7) (4 pts) Prove or disprove the following statements.
   (a) A homomorphic image of a PID is a PID. T
   (b) A homomorphic image of a UFD is a UFD. T

$\frac{1}{3} \Big| \frac{2}{5} \times \frac{5}{6}$

$abr = a$

$abs = b$

$pr = a$

$ps = b$

$p = ab$

(8) (8 pts) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by $T(e_1) = e_1 + 2e_2 + 3e_3$, $T(e_2) = 2e_2 + 3e_3$ and $T(e_3) = 3e_3$, where e_i are the standard unit vectors.

(a) Find the matrix of T with respect to the basis $e_1 + e_2, e_2 + e_3, e_3$.
(b) Find all the eigenvalues and corresponding eigenvectors of T.
(c) Find the Jordan canonical form of T.
(d) Find the elementary divisors of T.

(9) (4 pts) Let ϕ, ψ be endomorphisms of a nonzero finite dimensional vector space over $\mathbb{C}$. Prove that there is a $\lambda \in \mathbb{C}$ which is an eigenvalue for both ϕ and ψ. $\phi \circ \psi = \psi \circ \phi$