PHY 310 - Mathematical Methods for Physicists I

Odd Term 2019, IISER Mohali

Instructor: Dr. Anosh Joseph

First Mid-Semester Examination

13th September, 2019 from 8:00 AM - 9:55 AM in LH3/LH4

Maximum Marks 100

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1. [**4 + 4 + 2 = 10 Marks**] Consider the following function of complex variable $z = x+iy$

$$f(z) = \begin{cases} x^3 y(y - ix)/(x^6 + y^2) & \text{when } z \neq 0, \\ 0 & \text{when } z = 0. \end{cases}$$

(1a.) Find the derivative of the function $f'(z) = df/dz$, at $z = 0$, using the approximation

$$f'(z)|_{z=0} = \left.\frac{df}{dz}\right|_{z=0} = \lim_{z\to 0}\left[\frac{f(z) - f(0)}{z}\right],$$

along the path $y = Ax$.

(1b.) Find the derivative of the function using the same approximation given above along the path $y = x^3$.

(1c.) Is the function $f(z)$ differentiable at $z = 0$?

2. [**2 + 2 + 2 + 2 + 2 = 10 Marks**] Are the following statements true or false?

(2a.) A function, which is analytic everywhere (for all z in the complex plane) is known as an entire function. (True/False)

(2b.) Bessel function $J_p(x)$ is an odd function when p is even and is an even function p is odd. (True/False)

(2c.) A branch point of a multi-valued function is a point such that the function becomes discontinuous once we go around an arbitrarily small closed path around this point. (True/False)

(2d.) $P_n^0(x) = P_n(x)$. (True/False)

(2e.) Helmholtz' equation in spherical polar coordinates leads to Hermite polynomials. (True/False)

3. [**5 + 5 = 10 Marks**] The function $f(z) = 1/z$ is analytic everywhere in the complex plane.

   (3a.) Is the above statement **true** or **false**?

   (3b.) Provide the justification for your answer.

4. [**10 Marks**] For a function $f(z) = u + iv$, the Cauchy-Riemann conditions take the following form

$$\begin{aligned} \frac{\partial u}{\partial r} &= A\frac{\partial v}{\partial \theta}, \\ \frac{\partial u}{\partial \theta} &= B\frac{\partial v}{\partial r}, \end{aligned}$$

   in polar coordinates, $(x, y) = (r\cos\theta, r\sin\theta)$. Find A and B.

5. [**10 Marks**] Evaluate the integral

$$I = \oint_C dz(z-a)^n,$$

   where the contour C is a circle with center a and radius r and the direction of the contour is counterclockwise. Find the value of integral when $n \neq -1$ and when $n = -1$.

6. [**10 Marks**] Show that

$$\sqrt{\frac{1}{2}\pi x}\; J_{\frac{3}{2}}(x) = \frac{\sin x}{x} - \cos x.$$

7. [**10 Marks**] The Ber and Bei functions are defined as

$$J_0(i^{\frac{3}{2}}x) = \text{Ber } x + i\text{Bei } x,$$

   where

$$\begin{aligned} \text{Ber } x &= 1 + \sum_{p=1}^{\infty}(-1)^p \frac{x^{4p}}{2^2 \times 4^2 \times 6^2 \times \cdots \times (4p)^2}, \\ \text{Bei } x &= -\sum_{p=1}^{\infty}(-1)^p \frac{x^{4p-2}}{2^2 \times 4^2 \times 6^2 \times \cdots \times (4p-2)^2}. \end{aligned}$$

   Show that

$$\frac{d}{dx}\left(x \text{ Ber}' \, x\right) = -x\text{Bei } x.$$

8. [**10 Marks**] Consider the integral

$$I = \int_0^{\pi} d\theta \sin 2\theta P_n(\cos\theta),$$

with $n > 1$ and $P_n(x)$ is the Legendre polynomial of degree n. Evaluate I.

9. [**10 Marks**] Consider a particle of mass m that moves vertically under the influence of the Earth's gravitational field. It bounces elastically off some hard surface, which occupies the plane $x = 0$. The potential is given by

$$V(x) = \begin{cases} mgx & \text{when } x > 0, \\ \infty & \text{when } x < 0. \end{cases}$$

The Schroedinger equation for the particle under this potential is

$$-\frac{\hbar^2}{2m}\psi''(x) + V(x)\psi(x) = E\psi(x).$$

Use the substitution

$$z = \alpha\left(x - \frac{E}{mg}\right),$$

with

$$\alpha = \left(\frac{2m^2 g}{\hbar^2}\right)^{\frac{1}{3}},$$

to show that the Schroedinger's equation reduces to Airy differential equation.

10. [**10 Marks**] Let Π be the parity operator. A function $f(\theta, \phi)$ transforms the following way under parity

$$\Pi f(\theta, \phi) \to f(\pi - \theta, \pi + \phi).$$

Show that $Y_l^l(\theta, \phi)$ is an eigenfunction of Π with eigenvalue $(-1)^l$.