PHY 310 - Mathematical Methods for Physicists I

Odd Term 2019, IISER Mohali

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Quiz 1 - Solutions

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1. Find the first three coefficients in the expansion of the function

$$f(x) = \begin{cases} 0 & -1 \le x \le 0 \\ 1 & 0 \le x \le 1 \end{cases}$$

in a series of Legendre polynomials $P_l(x)$ over the interval $(-1, 1)$.

Solution:

We have

$$f(x) \simeq c_0 P_0(x) + c_1 P_1(x) + c_2 P_2(x).$$

The coefficients are

$$\begin{aligned} c_l &= \frac{2l+1}{2} \int_{-1}^{1} dx \ f(x) P_l(x) \\ &= \frac{2l+1}{2} \int_{-1}^{0} dx \ f(x) P_l(x) + \frac{2l+1}{2} \int_{0}^{1} dx f(x) P_l(x) \\ &= \frac{2l+1}{2} \int_{0}^{1} dx \ x P_l(x). \end{aligned}$$

$$c_0 = \frac{1}{2} \int_0^1 dx \ x P_0(x) = \frac{1}{2} \int_0^1 dx = \frac{1}{2}.$$

$$c_1 = \frac{3}{2} \int_0^1 dx \ x P_1(x) = \frac{3}{2} \int_0^1 dx \ x = \frac{3}{2} \cdot \frac{1}{2} = \frac{3}{4}.$$

$$
\begin{aligned}
c_2 &= \frac{5}{2}\int_0^1 dx\ xP_2(x) \\
&= \frac{5}{2}\int_0^1 dx\ xP_2(x) \\
&= \frac{5}{2}\int_0^1 dx\ \frac{1}{2}(3x^3 - x) \\
&= 0.
\end{aligned}
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2. Express $\tan\theta$ using Legendre polynomials $P_0(\cos\theta)$ and $P_1(\cos\theta)$.

Solution:

We have

$$
\begin{aligned}
P_0(\cos\theta) &= 1, \\
P_1(\cos\theta) &= \cos\theta.
\end{aligned}
$$

Using these we can write $\tan\theta$ as

$$
\begin{aligned}
\tan\theta &= \frac{\sin\theta}{\cos\theta} \\
&= \frac{\sqrt{1-\cos^2\theta}}{\cos\theta} \\
&= \frac{\sqrt{1-P_1^2(\cos\theta)}}{P_1(\cos\theta)} \\
&= \sqrt{\frac{P_0(\cos\theta)}{P_1^2(\cos\theta)} - 1}.
\end{aligned}
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3. Express the function

$$f(x) = x^4$$

in terms of Legendre polynomials, $P_l(x)$.

Solution:

We have the first few Legendre polynomials

$$\begin{aligned} P_0 &= 1, \\ P_1 &= x, \\ P_2 &= \frac{1}{2}(3x^2 - 1), \\ P_3 &= \frac{1}{2}(5x^3 - 3x), \\ P_4 &= \frac{1}{8}(35x^4 - 30x^2 + 3). \end{aligned}$$

From P_4 we get

$$8P_4 = 35x^4 - 30x^2 + 3.$$

Bringing x^4 to the left hand side

$$x^4 = \frac{1}{35}(8P_4 + 30x^2 - 3).$$

From $P_2 = \frac{1}{2}(3x^2 - 1)$ we get

$$\begin{aligned} x^2 &= \frac{1}{3}(2P_2 + 1) \\ &= \frac{1}{3}(2P_2 + P_0). \end{aligned}$$

Thus we have

$$\begin{aligned} x^4 &= \frac{1}{35}\left(8P_4 + 30\left[\frac{1}{3}(2P_2 + P_0)\right] - 3P_0\right) \\ &= \frac{1}{35}\left(8P_4 + 20P_2 + 10P_0 - 3P_0\right) \\ &= \frac{1}{35}(8P_4 + 20P_2 + 7P_0). \end{aligned}$$

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4. Consider an electric charge q located at position $\mathbf{R}$ from the origin. We need to compute the electric potential due to this point charge at some other position $\mathbf{r}$. Let us take the polar angle θ to be the angle between $\mathbf{r}$ and $\mathbf{R}$.

We have, from Gauss' law in electromagnetism

$$\nabla^2\Phi(r,\theta,\phi) = -\frac{\rho(r,\theta,\phi)}{\epsilon_0},$$

with $\Phi(r,\theta,\phi)$ denoting the electric potential and $\rho(r,\theta,\phi)$ the charge density.

For all $r < R$ the charge density is zero. This gives us

$$\nabla^2\Phi = \left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\right]\Phi(r,\theta) = 0.$$

We can find a solution to this using the separation of variables ansatz. We take $\Phi(r,\theta) = R_l(r)P_l(\theta)$. Then the general solution is

$$\Phi(r,\theta) = \sum_{l=0}^{\infty} a_l R_l(r) P_l(\cos\theta),$$

with $R_l(r) = Ar^l + Br^{-l-1}$ and $P_l(\cos\theta)$ is the l-th Legendre polynomial. A finite solution at $r = 0$ requires $B = 0$. Thus

$$\Phi(r,\theta) = \sum_{l=0}^{\infty} a_l r^l P_l(\cos\theta).$$

Determine the constants a_l using the boundary condition that when $\theta = 0$ we must recover the potential of a point charge

$$\Phi(r) = \frac{1}{4\pi\epsilon_0}\frac{q}{R-r}.$$

Solution:

Expanding

$$\begin{aligned}
\Phi(r,0) &= \sum_{l=0}^{\infty} a_l r^l P_l(1) = \frac{1}{4\pi\epsilon_0}\frac{q}{R-r} \\
&= \frac{q}{4\pi\epsilon_0}\left(\frac{1}{R} + \frac{r}{R^2} + \frac{r^2}{R^3} + \cdots\right) \\
&= \sum_{l=0}^{\infty} \frac{q}{4\pi\epsilon_0}\frac{r^l}{R^{l+1}}.
\end{aligned}$$

This gives

$$a_l = \frac{q}{4\pi\epsilon_0}\frac{1}{R^{l+1}}.$$

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5. Express the Cartesian coordinates x, y and z in terms of spherical harmonics.

Hint: We have

$$\begin{aligned} Y_1^{\pm 1} &= \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{\pm i\phi}, \\ Y_1^0 &= \sqrt{\frac{3}{4\pi}}\cos\theta. \end{aligned}$$

and

$$\begin{aligned} x &= r\sin\theta\cos\phi, \\ y &= r\sin\theta\sin\phi, \\ z &= r\cos\theta. \end{aligned}$$

Solution:

We have

$$\begin{aligned} \cos\phi &= \frac{x}{r\sin\theta}, \\ \sin\phi &= \frac{y}{r\sin\theta}, \\ \cos\theta &= \frac{z}{r}. \end{aligned}$$

Using these we express the spherical harmonics as

$$\begin{aligned} Y_1^{\pm} &= \mp\sqrt{\frac{3}{8\pi}}\sin\theta\left(\cos\phi \pm i\sin\phi\right), \\ &= \mp\sqrt{\frac{3}{8\pi}}\frac{x \pm iy}{r}. \\ Y_1^0 &= \sqrt{\frac{3}{4\pi}}\frac{z}{r}, \\ &= \sqrt{2}\sqrt{\frac{3}{8\pi}}\frac{z}{r}. \end{aligned}$$

Inverting these expression we get

$$\begin{aligned} x &= \frac{1}{2}\sqrt{\frac{8\pi}{3}}r\left(Y_1^{-1} - Y_1^1\right) \\ y &= -\frac{1}{2i}\sqrt{\frac{8\pi}{3}}r\left(Y_1^{-1} + Y_1^1\right) \\ z &= \frac{1}{\sqrt{2}}\sqrt{\frac{8\pi}{3}}rY_1^0. \end{aligned}$$

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