PHY304: Statistical Mechanics

Assignment 3

January 21, 2025

1. Let X be a continuous random variable with probability density function (PDF) for all real x
$$p_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2},$$
and let $Y = X^2$. Find the probability density function for Y i.e., $p_Y(y)$.

2. Let $X \sim Uniform(-\pi/2, \pi)$ (i.e., uniform PDF) and $Y = \sin(X)$. Find the PDF for Y i.e., $p_Y(y)$.

3. Calculate the characteristic function, the mean, and the variance of the following PDFs:

   (a) Uniform $\quad p(x) = \begin{cases} \frac{1}{2a} & \text{for} -a < x < a, \\ 0 & \text{otherwise.} \end{cases}$

   (b) Laplace $\quad p(x) = \frac{1}{2a} \exp\left(-\frac{|x|}{a}\right).$

   (c) Cauchy or Lorentz $\quad p(x) = \frac{a}{\pi(x^2 + a^2)}.$

4. Calculate the mean and variance of the following PDFs defined for $x \geq 0$.

   (a) Rayleigh $\quad p(x) = \frac{x}{a^2} \exp\left(-\frac{x^2}{2a^2}\right).$

   (b) Maxwell $\quad p(x) = \sqrt{\frac{2}{\pi}} \frac{x^2}{a^3} \exp\left(-\frac{x^2}{2a^2}\right).$

5. The amount of time, in hours, that a computer functions before breaking down is a continuous random variable with probability density function given by
$$p(x) = \begin{cases} \lambda e^{-x/100} & x \geq 0 \\ 0 & x < 0 \end{cases}$$
What is the probability that

   (a) a computer will function between 50 and 150 hours before breaking down;

   (b) it will function less than 100 hours?

6. **Buffon's needle problem.** A table is ruled with equidistant parallel lines a distance D apart. A needle of length L, where $L \leq D$, is randomly thrown on the table. What is the probability that the needle will intersect one of the lines (the other possibility being that the needle will be completely contained in the strip between two lines)?

7. The joint PDF of X and Y is given by

$$p(x, y) = \begin{cases} 15x(2 - x - y)/2 & 0 < x < 1, 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

   Compute the conditional PDF of X, given that $Y = y$, where $0 < y < 1$.