PHY638 MidSem II Date : March 7, 2025 Inst: Abhishek Chaudhuri

- **Time : 60 minutes, Max Marks : 20**
- **Attempt all questions.**

1. Consider the flow $\boldsymbol{u}(\boldsymbol{x},t) = -\frac{1}{2}\alpha r\boldsymbol{e}_r + u_\theta(r)\boldsymbol{e}_\theta + \alpha z\boldsymbol{e}_z$, where α is a positive constant. Given that the vorticity is $\boldsymbol{\omega} = \omega\boldsymbol{e}_z$ with $\omega = \frac{1}{r}\frac{d}{dr}(ru_\theta)$, answer the following:
   (a) What is $\nabla\cdot\boldsymbol{u}$? [2]
   (b) Write down the vorticity equation for this flow in a steady state in terms of ω? (Hint: The equation can be reduced to a first-order differential equation). Your integration constant will need to be set by the condition that the circulation of the flow is given as: $\Gamma = \int_S \boldsymbol{\omega}\cdot d\boldsymbol{S} = 2\pi\int_0^\infty dr r\omega(r)$. [4]
   (c) Hence determine u_θ. [2]

2. Consider the following two-dimensional stream function composed of a uniform horizontal stream of speed U and two vortices of equal and opposite strength in (x,y)-Cartesian coordinates.

$$\psi(x,y) = Uy + (\Gamma/2\pi)\ln\sqrt{x^2+(y-b)^2} - (\Gamma/2\pi)\ln\sqrt{x^2+(y+b)^2}$$

   (a) Simplify this stream function for the combined limit of $b\to 0$ and $\Gamma\to\infty$ when $2b\Gamma = C$ (a constant) to find $\psi(x,y)$. (Hint: It may be useful to consider $r^2 = x^2+y^2$ while simplifying. [2]
   (b) Switch to (r,θ) polar coordinates and find both components of the velocity using the simplified stream function. [2]
   (c) Determine where $u_r = 0$ and $u_\theta = 0$ and hence sketch the streamlines for the flow. [2]

3. Consider stationary surface gravity waves in a rectangular container of length L and breadth b, containing water of undisturbed depth H. The velocity potential is given by

$$\phi = A\cos(m\pi x/L)\cos(n\pi y/b)\cosh[k(z+H)]e^{-i\omega t},$$

   where m and n are integers.
   (a) Do the velocities obey the boundary conditions at the wall? [2]
   (b) Under what conditions does the velocity potential satisfy the Laplace equation? [2]
   (c) What would be the dispersion relation to satisfy the linearised free surface boundary condition? [2]

$$\int_0^\infty x^n e^{-bx^2}\, dx = \frac{\Gamma(\frac{n+1}{2})}{2b^{\frac{n+1}{2}}}$$

The Navier–Stokes equations in cylindrical polar coordinates
ure:

$$\frac{\partial u_r}{\partial t} + (\boldsymbol{u}\cdot\nabla)u_r - \frac{u_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right),$$

$$\frac{\partial u_\theta}{\partial t} + (\boldsymbol{u}\cdot\nabla)u_\theta + \frac{u_r u_\theta}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} + \nu\left(\nabla^2 u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2}\right), \qquad \text{(A.35)}$$

$$\frac{\partial u_z}{\partial t} + (\boldsymbol{u}\cdot\nabla)u_z = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\,\nabla^2 u_z,$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0.$$

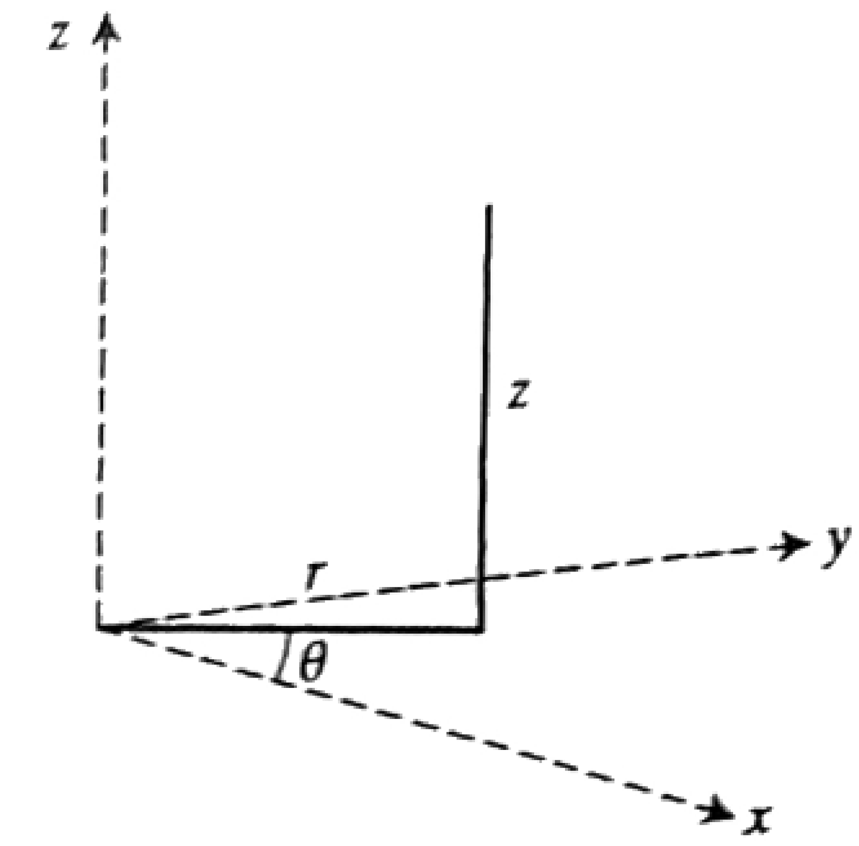

Fig. A.2 Cylindrical polar coordinates.

Also,

$$\nabla\phi = \frac{\partial \phi}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial \phi}{\partial \theta}\boldsymbol{e}_\theta + \frac{\partial \phi}{\partial z}\boldsymbol{e}_z,$$

$$\nabla\cdot\boldsymbol{F} = \frac{1}{r}\frac{\partial}{\partial r}(rF_r) + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta} + \frac{\partial F_z}{\partial z},$$

$$\nabla\wedge\boldsymbol{F} = \frac{1}{r}\begin{vmatrix} \boldsymbol{e}_r & r\boldsymbol{e}_\theta & \boldsymbol{e}_z \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial z} \\ F_r & rF_\theta & F_z \end{vmatrix},$$

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2},$$

$$\boldsymbol{u}\cdot\nabla = u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial \theta} + u_z\frac{\partial}{\partial z}.$$

The components of the rate-of-strain tensor are given by:

$$e_{rr} = \frac{\partial u_r}{\partial r}, \qquad e_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}, \qquad e_{zz} = \frac{\partial u_z}{\partial z},$$

$$2e_{\theta z} = \frac{1}{r}\frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z}, \qquad 2e_{zr} = \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}, \qquad \text{(A.36)}$$

$$2e_{r\theta} = r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial \theta}.$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\boldsymbol{u} + \boldsymbol{g},$$

$$\nabla\cdot\boldsymbol{u} = 0.$$

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{\omega} = (\boldsymbol{\omega}\cdot\nabla)\boldsymbol{u} + \nu\nabla^2\boldsymbol{\omega}.$$

For surface gravity waves of uniform depth H, we had the following conditions:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0.$$

subject to the conditions

$$\frac{\partial \phi}{\partial z} = 0 \quad \text{at} \quad z = -H,$$

$$\frac{\partial \phi}{\partial z} = \frac{\partial \eta}{\partial t} \quad \text{at} \quad z = 0,$$

$$\frac{\partial \phi}{\partial t} = -g\eta \quad \text{at} \quad z = 0.$$

$$\frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} = 0 \qquad \text{(continuity)},$$

$$\frac{1}{r}\frac{\partial}{\partial r}(ru_\theta) - \frac{1}{r}\frac{\partial u_r}{\partial \theta} = 0 \qquad \text{(irrotationality)},$$

$$u_r = \frac{\partial \phi}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta},$$

$$u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r},$$

$$\nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0,$$

$$\nabla^2\psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} = 0,$$