§§ Differential Equn:

Suppose f is a $\mathbb{C}$-valued fn. defined for all real $x \in I$ and and for complex $y \in \Omega$, $\in \mathbb{R}^2$.
To find a fn. φ on I which is diff. s.t. $\forall x \in I$

(i) $\phi(x) \in \Omega$.
(ii) $\varphi'(x) = f(x, \varphi(x))$.

This is called an ordinary diff. eqn of first order.
and denoted $y' = f(x, y)$.

ordinary : means only ord der. no partial der.
If φ exists then its called a soln.

An example is when f is independ. of y, i.e.

$$y' = f(x).$$

If f is cts. on I. then $\varphi_0(x) = \int_{x_0}^{x} f(t)\,dt$.

Second ex.

$$\varphi'(x) = k\varphi(x).$$
$$y' = ky.$$
$$\varphi(x) = e^{kx}, \quad \text{growth if } k > 0, \quad \text{decay if } k < 0.$$

$y' = f(x, y)$ has geometrical interpretation.

Most general: $F(x, \varphi(x), \varphi'(x)) = 0$.
ODE of first order.
$F(x, y, y') = y' - f(x, y)$ was earlier discussed

Lecture 2.

§§ Problems associated with diff. eqns.

One is tempted to find all solns when presented with a diff. eqn. ~~frequently~~ ~~we~~

For eg. $y' = f(x)$ (f continuous)

- all solns are given by $\varphi(x) = \int_{x_0}^{x} f(t)\,dt + c$ where x_0 is some pt. in the interval where f is defined and c is any constant.
- All solns of $y' = ky$ are $\varphi(x) = ce^{kx}$ and c can be any constant. We will prove this later.
- Every soln of $y'' + y = 0$ has the form $\varphi(x) = c_1 \cos x + c_2 \sin x$ where c_1, c_2 are arbitr. const. We will see a proof of this fact a bit later.

Frequently one is not interested in all solns of an eqn but only those satisfying certain other conditions. These conditions may take many forms but two of the most important types are:

Type 1: initial conditions.

Type 2: boundary conditions.

- An initial condition is a condition is a condition on the soln at one point. For eg. a soln $\varphi(x)$ of $y' = ky$ having the property that $\varphi(0) = 2$ (the initial cond.) is given by $\varphi(x) = 2 \cdot e^{kx}$.

Such an initial value problem would be denoted by $y' = ky$, $y(0) = 2$.

Mly ~~the~~ soln of $y'' + y = 0$, $y(0) = 1$, $y'(0) = 2$ is $\varphi(x) = \cos x + 2\sin x$.

- A boundary condition is a condition on the solution at two or more points.
For ex. the soln. φ of $y''+y=0$ satisfying $\varphi(0)=1,\ \varphi'(2\pi)=-1$
is $\varphi(x) = \cos x - \sin x$.

There are many equations for which it is not obvious that solutions exist at all and if they do, it might not be possible to write down "nice" formulas for them. For ex.

$$y''+y'+\sin y = 0$$

is found in the study of the motion of a pendulum. It can be shown that this has solns satisfying any given real initial conditions which exist for all real x, although we can't express them in terms of fns we meet in calculus. How do we solve such eqns i.e. find the solns?

One method is to develop math procedures which allow us to compute the value of a soln. at any given x to any desired degree of accuracy. The method should be sufficiently general to cover a large no. eqns. We will study such a gen. method for computing solns to initial value problems later.

Even in cases it is impossible to express solns. of some eqns. in nice formulas, it is often the case that we can say a good deal about the properties of solutions and this may suffice for our purposes.
For ex. without solving we can show that any soln. φ of $y''+y'+\sin y=0$ for which $-\pi < \varphi(0) < \pi,\ \varphi'(0)=0$, will tend to zero as $x\to\infty$. This corresponds to the fact that the oscillations of a pendulum are damped and eventually the pendulum will stay arbitrarily close to its equilibrium position, zero.

Def: <u>Order</u> of a diff. eq. is the order of the highest diff. coeff. involved.
When an eqn. is a polynomial in all the diff. coeff. involved, the power to which the highest diff. coeff. is raised is known as the degree of the eqn.

When in ODE or PDE, the dependent variable and it's derivates occur to the first degree only and not as higher power or products, the eqn. is said to be <u>linear</u>.

The coefficients of a linear eqn. are therefore either constants or fns. of the independent variable or variables.

For eg. $y'' + y = x^3$ is a linear ODE of second order.

$(x+y)^2 \frac{dy}{dx} = 1$ is a non-linear ODE of the first order and first degree.

§ Linear eqn. of the first order:

Consider $y' + a(x)y = b(x)$ where a, b are certain fns. defined on an interval I.

If $b(x) = 0$ the eqn. is called a <u>homogeneous</u> eqn. whereas if b is not identically zero on I, the eqn. is called <u>non-homogeneous</u>.

$y' = -a(x)y + b(x)$ then $f(x,y) = -a(x)y + b(x)$

for $b(x) = 0$ is s.t. $f(x, y_1+y_2) = f(x,y_1) + f(x,y_2)$
and $f(x, cy) = c f(x,y)$
for any constant c.

We first solve the above when $a(x)$ is a constant.
The eqn: $y' + ay = 0$, a is a constant.
If φ is a soln:
$\varphi' + a\varphi = 0 \Rightarrow e^{ax}(\varphi' + a\varphi) = 0$.
or $(e^{ax}\varphi)' = 0$.
$\Rightarrow \exists\, c$, a constant s.t. $e^{ax}\varphi(x) = c$ or $\varphi(x) = ce^{-ax}$.

Conversely, if c is any constant the fn. φ defined by $\varphi(x) = ce^{-ax}$ is a soln. since $\varphi' + a\varphi(x) = -ace^{-ax} + ace^{-ax} = 0$.
We have proved:

Theorem. Consider the eqn. $y' + ay = 0$ where a is a complex constant. If c is any complex no., the fn. φ defined by $\varphi(x) = ce^{-ax}$ is a soln. of this eqn and moreover, every soln. has this form.

- Note that all solns exist for all real x, i.e. for $-\infty < x < \infty$.
and also that c is the value of φ at 0, $c = \varphi(0)$.

The eqn. $y' + ay = b(x)$, a = constant, b is continuous on some interval I.

By same method, if φ is a soln. then;
$e^{ax}(\varphi' + a\varphi) = e^{ax} b$ or $(e^{ax}b)' = e^{ax} b$.

Let $B(x) = \int_{x_0}^{x} e^{at} b(t)\,dt$, i.e. a fn. s.t. $B'(x) = e^{ax} b(x)$
$x_0 \in I$ fixed.

Follows that $e^{ax} b = B(x) + c$ for some constant c.

Therefore $\varphi(x) = e^{-ax} B(x) + c e^{-ax}$.
and any φ defined like this is a soln.

Theorem Consider the eqn.
$y' + ay = b(x)$. where a is constant and b is cts fn. on an interval I. If x_0 is a pt. in I and c is any constant, the fn. φ defined by $\varphi(x) = e^{-ax} \int_{x_0}^{x} e^{at} b(t)\, dt + c e^{-ax}$

is a soln. of this eqn. Every soln has this form.

Exercises :

1. Find all solns of the following eqns:
(a) $y' - 2y = 1$ (b) $y' + y = e^{x}$
(c) $y' - 2y = x^2 + x$ (d) $3y' + y = 2e^{-x}$
(e) $y' + 3y = e^{ix}$.

2. Let φ be a soln. of $y' + iy = x$ s.t. $\varphi(0) = 2$.
Find $\varphi(\pi)$.

3. Consider $Ly' + Ry = E$ where L, R, E are positive constants
(a) Solve the eqn.
(b) Find the soln. φ satisfying $\varphi(0) = I_0$ where I_0 is a given positive constant.
(c) Sketch a graph of the soln. in (b) for the case $I_0 > E/R$.
(d) Show that every soln. tends to E/R as $x \to \infty$.

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