PHY304: Statistical Mechanics
End Semester Examination 2025 (Part B)

April 29, 2025

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Max. Marks 30

- All questions are compulsory.
- Some important results are given at the end.

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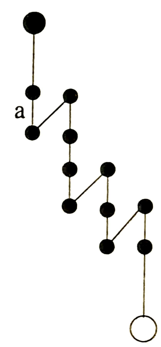
1. A chain made of N massless segments of equal length a hangs from a fixed point. a mass m is attached to the other end under gravity. Each segment can be in either of two states, up or down, as illustrated in the sketch. The segments have no mass, and the chain can go as far up as it can; there is no ceiling.



   (a) Obtain the partition function at temperature T. [2]

   (b) Find the entropy of the chain. [1]

   (c) Find the internal energy, and determine the length of the chain. [3]

2. Consider a molecule, such as carbon monoxide, made up of two different atoms, one carbon and one oxygen separated by a distance d. Such a molecule can exist in quantum states of different orbital angular momentum with each state having energy

$$\epsilon_\ell = \frac{\hbar^2 \ell(\ell+1)}{2I}.$$

Here $I = \mu d^2$ (μ is the reduced mass) is the moment of inertia of the molecule about an axis through its centre of mass, and $\ell = 0, 1, 2, 3, \ldots$ is the quantum number associated with the orbital angular momentum. Each energy level of the rotating molecule has a degeneracy $g_\ell = (2\ell + 1)$. Calculate the following quantities in both the low temperature ($kT \ll \hbar^2/I$) and the high temperature ($kT \gg \hbar^2/I$) limits

   (a) the partition function. [3]

   (b) the free energy. [1]

   (c) the entropy. [1]

   (d) the heat capacity at high temperatures. [1]

3. (a) Find the density matrix ρ of a partially polarized beam of spin-$\frac{1}{2}$ atoms containing a mixture of 50% of $|\uparrow_z\rangle$, 25% of $|\uparrow_x\rangle$ and 25% of $|\uparrow_y\rangle$. Show that the obtained matrix is a valid density matrix. [4]

   (b) Check whether ρ is a pure or a mixed state? [2]

4. The asymptotic expansion for the chemical potential of a spinless ideal Fermi gas at temperature T is given by

$$\mu = kT \log z = \epsilon_F \left[1 - \frac{\pi^2}{12} \left(\frac{kT}{\epsilon_F} \right)^2 + \cdots \right],$$

where ϵ_F is the Fermi energy. Show that the internal energy of the gas has the following asymptotic expansion: [6]

$$U = \frac{3}{5} N \epsilon_F \left[1 + \frac{5}{12} \pi^2 \left(\frac{kT}{\epsilon_F} \right)^2 + \cdots \right].$$

5. The Einstein model is a cruder version of the Debye model. It assumes that all phonons have the same frequency ω_0.

(a) Obtain the internal energy of this model. [2]

(b) Find the heat capacity. [1]

(c) How does heat capacity behaves near absolute zero and at high temperatures? [3]

Some useful results:

$$\int_{-\infty}^{\infty} e^{-ax^2}\, dx = \sqrt{\frac{\pi}{a}}$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}; \qquad \sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

Equations of state for Fermi system

$$\frac{P}{kT} = \frac{1}{\lambda^3} f_{5/2}(z); \qquad \frac{1}{v} = \frac{1}{\lambda^3} f_{3/2}(z)$$

$$f_{5/2}(z) \equiv \frac{4}{\sqrt{\pi}} \int_0^{\infty} x^2 \log\left(1 + z e^{-x^2}\right) dx; \qquad f_{3/2}(z) \equiv z \frac{\partial}{\partial z} f_{5/2}(z) = \frac{4}{\sqrt{\pi}} \int_0^{\infty} \frac{x^2}{z^{-1} e^{x^2} + 1} dx$$

Asymptotic expressions for $f_{5/2}(z)$ and $f_{3/2}(z)$:

$$f_{5/2}(z) = \frac{8}{15\sqrt{\pi}} (\ln z)^{5/2} \left[1 + \frac{5\pi^2}{8} (\ln z)^{-2} + \cdots \right]; \quad f_{3/2}(z) = \frac{4}{3\sqrt{\pi}} (\ln z)^{3/2} \left[1 + \frac{\pi^2}{8} (\ln z)^{-2} + \cdots \right]$$