Indian Institute of Science Education and Research, Mohali
Integrated MSc, Semester: IV *Probability and Statistics: MTH 202*
Tutorial 11 (April 5, 2023)

Summary:
Let X_1, X_2 be two random variables with mean μ_1, μ_2 and variance σ_1^2, σ_2^2 respectively.
Covariance: $Cov(X_1, X_2) = \mathbb{E}([X_1 - E(X_1)][X_2 - E(X_2)]) = \mathbb{E}(X_1X_2) - \mu_1\mu_2$.
Correalation: $\rho_{X_1,X_2} = \frac{Cov(X_1,X_2)}{\sqrt{Var(X_1)Var(X_2)}} = \frac{Cov(X_1,X_2)}{\sigma_1\sigma_2}$.
Covariance matrix:

$$\Sigma = \begin{pmatrix} Cov(X_1, X_1) & Cov(X_1, X_2) \\ Cov(X_2, X_1) & Cov(X_2, X_2) \end{pmatrix} = \begin{pmatrix} \sigma_1^2 & \rho_{X_1,X_2}\sigma_1\sigma_2 \\ \rho_{X_1,X_2}\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}.$$

The determinant $Det(\Sigma) = \sigma_1^2\sigma_2^2(1 - \rho_{X_1,X_2}^2)$. Thus covariance matrix of X_1, X_2 is invertible when the correlation is not ±1. A bi-variate normal density can be describe in term of inverse of covariance matrix.

Markov's Inequality : Let X be a non negative random variable with expectation $E(X) = \mu$. Then for any $a > 0, P(X \geq a) \leq \frac{\mu}{a}$.

Chebyshev's Inequality : Let X be a random variable with mean μ and variance σ^2. Then for any $a > 0, P(|X - \mu| \geq a) \leq \frac{\sigma^2}{a^2}$.

Limit Theorems

Let $\{X_k\}$ be a sequence of independent random variables and identical probability distributions with mean μ and variance σ^2. Let $S_n = X_1 + X_2 + \cdots + X_n$. Then $\mathbb{E}(S_n) = n\mu$ and $Var(S_n) = n\sigma^2$.

Weak Law of Large Number: Let $\{X_k\}$ be as above. Then for any $\epsilon > 0, \lim_{n\to\infty} P(|\frac{S_n}{n} - \mu| \geq \epsilon) = 0$.

Strong Law of Large Number: Let $\{X_k\}$ be as above. Then $P(\lim_{n\to\infty} \frac{S_n}{n} = \mu) = 1$.

Central Limit Theorem: Let $\{X_k\}$ be as above. Then for any $t \in \mathbb{R}$,

$$\lim_{n\to\infty} P(\frac{S_n - n\mu}{\sigma\sqrt{n}} \leq t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{t} e^{-\frac{x^2}{2}}dx = \Phi(t).$$

Under the assumption that the moment generating function ϕ_{X_k} exists, you saw a proof of Central Limit Theorem using of the Continuity Theorem. Recall that Moment generating function

$$\phi_X(t) = \mathbb{E}(e^{tX}) = 1 + \sum_{k=1}^{\infty} \frac{t^k}{k!}E(X^k).$$

For standard normal random variable $X, \phi_X(t) = e^{\frac{t^2}{2}}$. Note that the n-th moment $E(X^n) = \frac{d^n\phi_X(t)}{dt}|_{t=0}$.
Continuity Theorem: Let $Z, Z_1, Z_2, \cdots$ be a family of random variable such that $\lim_{n\to\infty} \phi_{Z_n}(t) = \phi_Z(t)$ on an interval $(-\alpha, \alpha)$. Then $\lim_{n\to\infty} F_{Z_n}(t) = F_Z(t)$ for all t where F_Z is continuous.
Question

1. Let (X, Y) be a bi-variate random variable with joint density $f(x, y) = \frac{2}{x}e^{-2x}$ for $0 < y \leq x < \infty$. Find the Covariance matrix and Correlation.

2. Find the moment generating function of exponetial random variable X with parameter λ and compute the third moment.

3. Let $\{X_k\}$ be a sequence of independent Poisson random variables with parameter 1. Let $S_n = X_1 + X_2 + \cdots + X_n$. Estimate the probablity $P(S_{20} > 15)$ using Markov's inequality and Chebyshev's inequality. See if you can obtain a better estimate using Central limit theorem.

4. Let f be a continuous function on the interval $I = [0, 1]$. Consider the associaited Bernstein polynomials $B_n(x) = \sum_{k=0}^{n} f(\frac{k}{n}) \frac{n!}{k! \; n-k!} x^k (1-x)^{n-k}$. Using Strong Law of large number show that $\{B_n(x)\}$ converges to $f(x)$ for $0 < x < 1$.