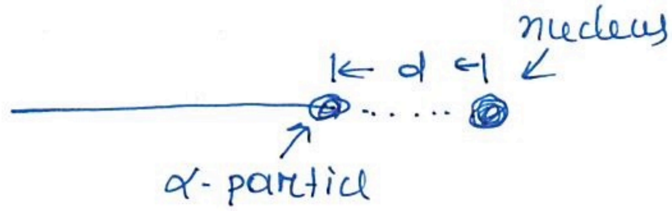
PHY401-Nuclear and Particle Physics			
Academic Year:	2024-25	Instructor:	Dr. Satyajit Jena,
Class	MS21	Lecture	Nuclear Properties

Nuclear propertise

Based on previous discussion, we will start discussing the properties of nucleus. In this we must first establi the size of nucleus. As a historical event I discussed about the importance of Rutherford scattering experim & the result out of its findings. let's find out some common facts about Rutherford scattering.

One of the mean to determine the size of object is to look into a scattering of small object into the object under observation. In that sence if Rutherford was using α-particle as probe (due to the availabily He must choose a nucleus bigger than α-particle precisely for same reason, he choose the Gold or higher element metal foil for his experiment.

starting with head-on collision of α-particle with the nucleus.



say at a distance d from the nucleus α-particle start reversing its direction due to the colomb repulsion. why?

Because α-particle is $2+$ charge $\Rightarrow 2e$
nucleus is +ve charged $\Rightarrow Ze$.

Therefore α-particle being 2+ charges will suffer a columb repulsion as it ~~come & closure to the~~ approaches to nucleus with +ve charges. ~~Even~~ eventually it will stop at particlure distance say d from nucleus & start coming back. In other word. we can say at d the kinetic Energy of α-particle, T, becomes equal to columb energy.

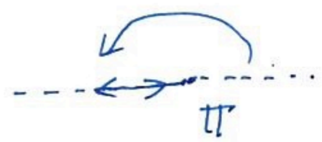
Thus

$$T = \frac{ze\,Ze}{4\pi\epsilon_0 d} = \frac{zZe^2}{4\pi\epsilon_0 d}$$

or

$$\boxed{d = \frac{zZe^2}{4\pi\epsilon_0}\frac{1}{T}}$$

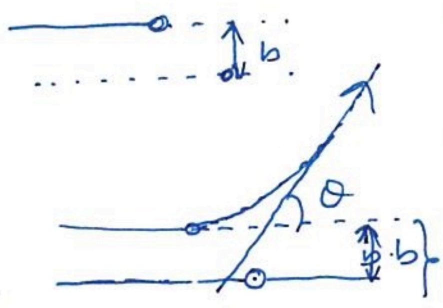
what is the scattering angle θ ?



$\boxed{\theta = \pi}$ for such situation.

what would be the situation if the α-particle is not in the line of sight?

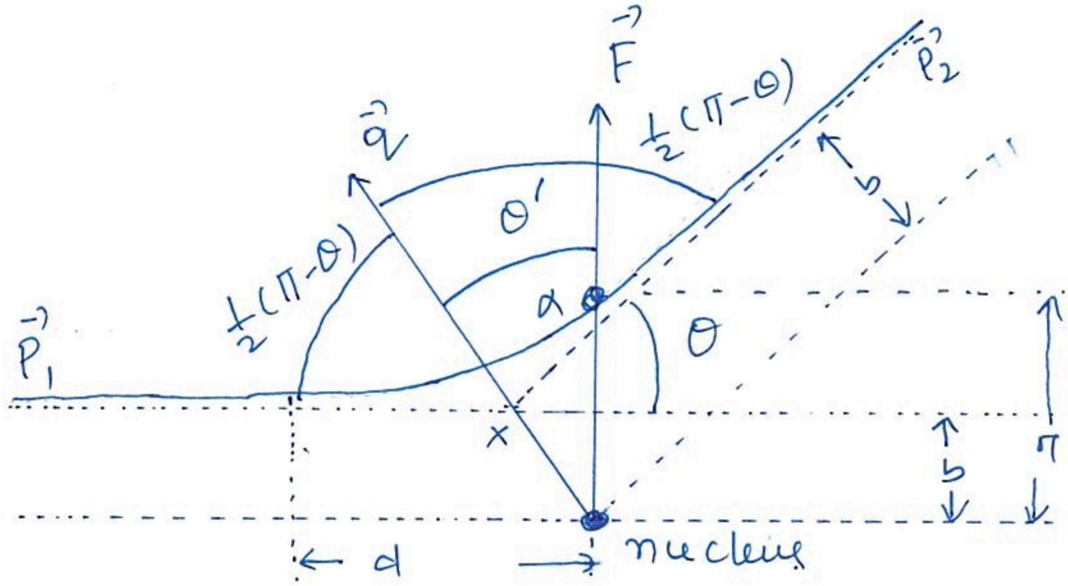
if the line of incidence of the α-particle is a distance b from the nucleus, then the scattering angle will be smaller.



b → This is called impact parameter.

By using Newton's Second Law of motion, columb's law for the force between α-particle and nucleus, and conservation of angular momentum, we can calculate the relation between θ, d & b. i.e. the relation between impact parameter (b), scattering angle (θ) & distance of columbic deflection.

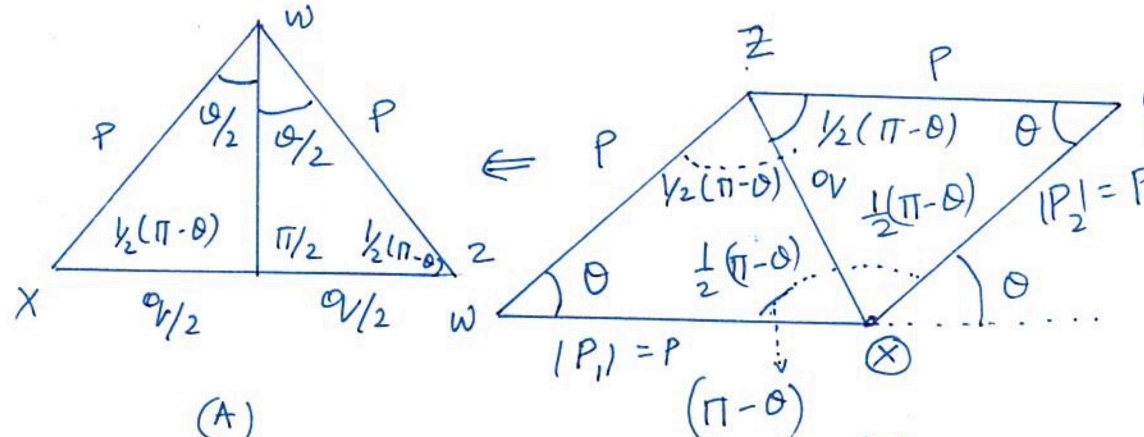
lets draw the picture bit carefully.



Here, note that if $\theta = \pi$, b becomes zero $\Rightarrow b = 0$
That is if b increases the α-particle "glances" the nucleus so that the scattering angle decreases.

Assuming elastic scattering, we can approximate that the magnitude of initial momentum $\vec{P_1}$ & final momentum $\vec{P_2}$ of α-particle will be of same value, p
$\vec{q}$, therefore is the resultant vector of $\vec{P_1}$ & $\vec{P_2}$

Tothere with $\vec{P_1}$; $\vec{P_2}$, the momentum change vector q will form an isoscele triangle with angle θ between P_1 & P_2 (initial & final) vectors momenta.



(A) (B)

Making more geometrical analysis we can mak fig (B) from the scattin picture, ~~this~~ & then to ~~not~~ have more clear picture we can make figure (A) which is a isoscele triangle. Now using pure geomtry we can write.

$$\sin\left(\frac{\theta}{2}\right) = \frac{\left(\frac{q}{2}\right)}{P}.$$

$$\Rightarrow \boxed{\frac{q}{P} = 2\sin\left(\frac{\theta}{2}\right)}$$

Just for clarificatin, the q is the resultant momentum vector, & the direction of this vector is along the line joining the nucleus to closest approach of the α- particle.

Since the target nucleus is heavier than α-particle we can neglect its recoil. we also neglect all relativist effects.

By Newton's second law, the rate of change of momentum in the direction of q is the component of the force acting on the α-particle due to the electric charge of the nucleus. Thus by columb's law the magnitude of the force is

$$F = \frac{zZe^2}{4\pi\epsilon_0 r^2}$$

The position of α-particle is in terms of 2-D polar coordinates (r, θ') with the nucleus at origin & $\theta' = 0$ chosen to be the point of closest approach

Putting kinetic energy T derived earlier.

$$T = \frac{zZe^2}{4\pi\epsilon_0}\frac{1}{d} \Rightarrow \boxed{\frac{zZe^2}{4\pi\epsilon_0} = Td}$$

$$\Rightarrow \boxed{F = \frac{Td}{r^2}}$$

The component of this force in the direction of q vector will be

$$F_q(t) = \frac{Td}{r^2}\cos[\theta'(t)]$$

Thus the change of momentum can be given by

$$q = \int \frac{dq}{dt} = \int F_q(t)\, dt.$$

$$\Rightarrow q = \int \frac{zZe^2}{4\pi\epsilon_0}\frac{1}{r^2}\cos\theta'(t)\, dt$$

we can replace integration of time over the θ' using

$$dt = \frac{d\theta'}{\dot{\theta}'}$$

$\dot{\theta}'$ can be obtained from the conservation of angular momentum.

$$L = m_\alpha r^2 \dot{\theta}' \qquad m_\alpha = \text{mass of } \alpha\text{- particle.}$$

& the initial angular momentum is given by

$$L = bp.$$

$$\Rightarrow \boxed{\dot{\theta}' = \frac{bp}{m_\alpha r^2}}$$

Now we can rewrite the q as

$$q = \int \frac{zZe^2}{4\pi\epsilon_0} \frac{1}{r^2} \cos\theta' dt$$

$$= \int \frac{zZe^2}{4\pi\epsilon_0} \frac{1}{r^2} \cos\theta' \frac{d\theta'}{\dot{\theta}'}$$

$$= \int \frac{zZe^2}{4\pi\epsilon_0} \frac{1}{r^2} \frac{\cos\theta' d\theta'}{\left(\frac{bp}{m_\alpha r^2}\right)}$$

$$= \int \frac{Td}{r^2} \frac{r^2 m_\alpha}{bp} \cos\theta' d\theta'$$

$$= \int \frac{p^2 d}{2m_\alpha r^2} \frac{r^2 m_\alpha}{bp} \cos\theta' d\theta' \qquad T = \frac{p^2}{2m_\alpha}$$

$\downarrow$ Kinetic energy in terms of momentum of α-particle.

$$\boxed{q = \int \frac{pd}{2b} \cos\theta' d\theta'}$$

Let's perform the integration of R.H.S

Before integration Let's decide what is the limit of θ', This can move from $-\frac{1}{2}(\pi-\theta)$ to $+\frac{1}{2}(\pi-\theta)$

$$q = \int_{-\frac{1}{2}(\pi-\theta)}^{+\frac{1}{2}(\pi-\theta)} \frac{pd}{2b} \cos\theta' \, d\theta.$$

$$q = \frac{pd}{2b}\left[2\sin\left(\tfrac{1}{2}(\pi-\theta)\right)\right]$$

Earlier we calculated $\frac{q}{p} = 2\sin\left(\frac{\theta}{2}\right)$

$$\Rightarrow \frac{q}{p} = \frac{d}{2b}\left[2\sin\left\{\tfrac{1}{2}(\pi-\theta)\right\}\right]$$

$$\Rightarrow 2\sin\left(\frac{\theta}{2}\right) = \frac{d}{b}\sin\left(\tfrac{1}{2}(\pi-\theta)\right)$$

$$\Rightarrow \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} = \frac{d}{2b}$$

$\sin\left(\pi/2 - \theta/2\right) = \cos\frac{\theta}{2}$

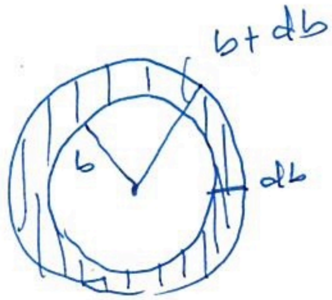
$$\Rightarrow \boxed{\tan\left(\frac{\theta}{2}\right) = \frac{d}{2b}}$$

Above calculation was for the single α-particle but in reality the ~~num~~ experiment was conducted with α-source. This α-source emits large number of α-particles. Thus we must have a way to calculate the scattering angle of all these. But since all these α-particle will have different lines of incidence, their scattering angle will also be very diffent.

Thus getting the θ_s calculated for each of them individually is not possible. Thus we can calculate the total number of α-particles passing in a particular area per unit time. This is precisely we call FLUX of α-particle.

Let's say $dN(b)$ is the number of α-particle within the impact parameter between b & $b+db$. Thus the $dN(b)$ will be the flux multiplied by the area between two concentric circle.

Flux ϕ is number of particle per unit area per second.



Area $= 2\pi b \times db$

$$dN(b) = \phi \, 2\pi b \, db.$$

Earlier we calculated

$$\tan\frac{\theta}{2} = \frac{d}{2b} \Rightarrow$$

$$\Rightarrow b = \frac{d}{2}\frac{1}{\tan\left(\frac{\theta}{2}\right)} = \frac{d}{2}\cot(\theta/2)$$

$$\Rightarrow \frac{db}{d\theta} = \frac{d}{2}\frac{d}{d\theta}\cot\left(\frac{\theta}{2}\right) = \frac{d}{2}\frac{d}{dx}\cot x \frac{dx}{d\theta} \quad \left(x = \frac{\theta}{2}\right)$$

$$= \frac{d}{2} \cdot -\operatorname{cosec}^2 x \frac{dx}{d\theta}$$

$$\Rightarrow \quad \frac{db}{d\theta} = -\frac{d}{2}\operatorname{cosec}^2\frac{\theta}{2}\,\frac{d\left(\frac{\theta}{2}\right)}{d\theta}$$

$$= -\frac{d}{2}\operatorname{cosec}^2\frac{\theta}{2}\,\frac{1}{2}\,\frac{d\theta}{d\theta}$$

$$\Rightarrow \quad \frac{db}{d\theta} = -\frac{d}{4}\,\frac{1}{\sin^2\left(\frac{\theta}{2}\right)}$$

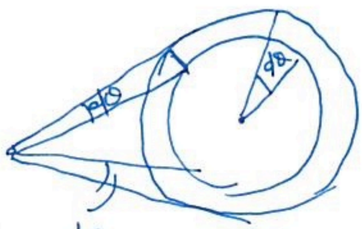
$$\Rightarrow \quad \boxed{db = -\frac{d}{4\sin^2\left(\frac{\theta}{2}\right)}\,d\theta.}$$

No $dN(b) = \phi\, 2\pi b\, db$ can be written in terms of θ & $d\theta$ as.

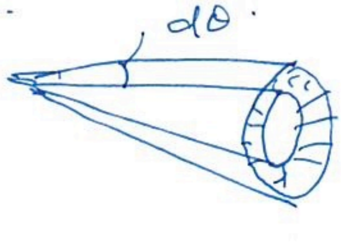
$$dN(\theta) = \phi\, 2\pi b\left(-\frac{d}{4\sin^2\frac{\theta}{2}}\right)d\theta.$$

$dN(\theta)$ This the number of α-particle between the angle area θ & $\theta + d\theta$. what will call it $\Rightarrow$ "solid angle".

Let's use $\tan\frac{\theta}{2} = \frac{d}{2b}$



$$\Rightarrow \quad dN(\theta) = -\phi\, 2\pi\,\frac{d}{4\sin^2\frac{\theta}{2}}\,\frac{d}{2\tan\frac{\theta}{2}}\,d\theta.$$



$$= -\phi\,\frac{\pi d^2}{4}\,\frac{\cos\left(\frac{\theta}{2}\right)}{\sin^3\left(\frac{\theta}{2}\right)}\,d\theta$$

$dN(\theta)$ is a number, so. this $-ve$ sign indicates that number should decreas as you increas θ & similarly increase the b θ will decrease.

The differential cross-seetin is defined by $\frac{d\sigma}{d\theta}$ with respeet to the scattering angle, which is the number of scatterings between θ and $\theta+d\theta$ per unit flux per unit range of angle.

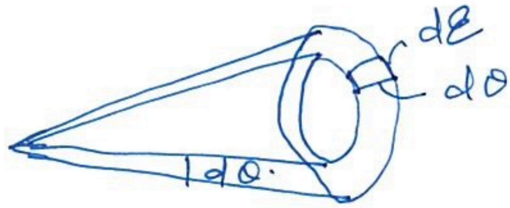
$$\frac{d\sigma}{d\theta} = \frac{dN(\theta)}{\Phi\, d\theta}.$$

$$\Rightarrow \quad \frac{d\sigma}{d\theta} = \pi \frac{d^2}{4} \frac{\cos\left(\frac{\theta}{2}\right)}{\sin^3\left(\frac{\theta}{2}\right)}$$

The diffential crossectin is given by the number within θ to $\theta+d\theta$. what exactly it is? this angle area is nothing but the diffential solid angle $d\Omega$

we know that

$$\boxed{\Omega = \frac{a}{R^2}}$$ area of sphere.

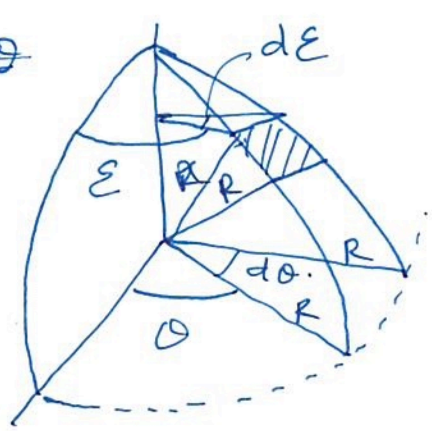


using polar co-ordinate system we can calculate that

ε is azimuthal angle.

$$\cancel{d\Omega = (R\sin\theta\, d\varepsilon)(R\,d\theta)}$$

$$d\Omega = \frac{(R\sin\theta\, d\varepsilon)(R\,d\theta)}{R^2}$$



$$\Rightarrow d\Omega = \sin\theta\, d\theta\, d\varepsilon$$

$$\Rightarrow d\Omega = \sin\left(\frac{\theta}{2}+\frac{\theta}{2}\right) d\theta\, d\varepsilon = 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) d\theta\, d\varepsilon$$

~~There for~~ The diffenential cross section with respect to diffential solid angle can be written as

$$N = \phi \sigma$$

$$\boxed{\frac{dN}{d\Omega} = \phi \frac{d\sigma}{d\Omega}}$$

Thus we need to write $\frac{d\sigma}{d\theta}$ in terms of $\frac{d\sigma}{d\Omega}$ which is more meaningful.

Thus
$$\frac{d\sigma}{d\theta} = \pi \frac{d^2}{4} \frac{\cos(\theta/2)}{\sin^3\left(\frac{\theta}{2}\right)}$$

$$\Rightarrow \quad \frac{d\sigma}{d\theta} = \frac{d\sigma}{d\Omega}\frac{d\Omega}{d\theta}$$

$$\frac{d\Omega}{d\theta} = 2\sin\left(\frac{\theta}{2}\right)\cos\frac{\theta}{2}\int d\varepsilon$$

$\int d\varepsilon$ is for total ring of $\theta + d\theta$. i.e. all azimu angle.

$$= 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\; 2\pi$$

$$= 4\pi \sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)$$

$$\Rightarrow \quad \frac{d\sigma}{d\Omega}\frac{d\Omega}{d\theta} = \frac{d\sigma}{d\theta}$$

$$\Rightarrow \quad \frac{d\sigma}{d\Omega}\, 4\pi \sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) = \pi \frac{d^2}{4}\frac{\cos(\theta/2)}{\sin^3\left(\frac{\theta}{2}\right)}$$

$$\Rightarrow \quad \boxed{\frac{d\sigma}{d\Omega} = \frac{d^2}{16}\frac{1}{\sin^4\left(\frac{\theta}{2}\right)}}$$

The diftential creosssectin have dimensr of an area. and thie is uuually quoted as baren

$$1 \text{ baren} = 10^{-28} m^2$$

$$\Rightarrow 1 \text{ millibaren} = 1 mb = 10^{-31} m^2$$

$$\Rightarrow 10^{-15} m = 1 fm$$

$$\boxed{1 fm^2 = 10 mb}$$

The difticulties of Rutherford scattin was multiple small angle scatterin. Thus to reeduce the multiple scattin one needs to make very thin foil. But nomatter how thin you make it will ceretainly ~~have~~ go though multiple scattin.

Say A is mass numbeer then the total number of nuclei per unit area canbe given as

$$\rho \delta \frac{1}{A m_p}$$

ρ is densty
δ is thickness.
A is atomic mas.

Fraction of α particle scattered fr. small intreval of solid angle will be

$$\frac{\delta n}{n} = \rho \delta \frac{1}{A m_p} \frac{d\sigma}{d\Omega} d\Omega$$

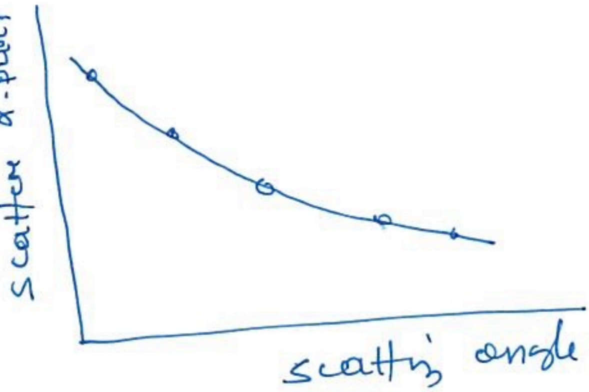
solid angle is defined such that an area element dA at a distance r from the scattin center subtends a solid angle.

$$d\Omega = \frac{dA}{r^2}$$

Putting a detector with θ & $\frac{dA}{r^2}$, the α-paretrcle canbe detected

Experimental result

This theoretical result compares very well with the data taken by Geiger & ~~Marsde~~ Marsden →



The colmb force can be written in terms of fine structure constent. i:

$$\boxed{\frac{e^2}{4\pi\epsilon_0} = \alpha \hbar c}$$

$\hbar c = 197.3$ Mev fm.

$$\Rightarrow \frac{e^2}{4\pi\epsilon_0} = \frac{197.3}{137} \text{ Mev fm.}$$

$$d = \frac{197.3}{137} \frac{zZ}{T} \text{ fm}$$

T is in MeV. which is experim value.

$d \approx 10^{-14}$ m i.e. 10 fm.

To be able to reach the distance of closest approach in order of fm, one needs to increase the enrgy of α-particle to very large kinetic Energy. Thus there are deviatins from the Rutherford scattig formulae for very high enrg α-particle. A quantum mechanical treatment is required to encounter it.

Finding the nuclear size:

Standard way of determining the shape (size) of an object is to look into scattered radiation, the similar way we see any object via scatted light. Doing so, we know the Rutherford's scattering formula where the scatting of α-particles from nuclei is modeled from the columb's force & treated as an orbit. By treating the scatting process statistically for the cross-section of interaction with nucleus (a point like charge Ze), all the quantities of scattering can be found out:

$$\frac{d\sigma}{d\Omega} = \frac{d^2}{16\sin^4\left(\frac{\theta}{2}\right)} = \left(\frac{Ze^2}{4\pi\epsilon_0 K_E}\right)^2 \frac{1}{\sin^4\frac{\theta}{2}}$$

$$N(\theta) = \frac{N_i n L Z^2 k^2 e^4}{4\pi^2 K_E^2 \sin^4(\theta/2)}$$

where

$\theta \rightarrow$ scattering angle

$N_i \rightarrow$ Number of α-particles

$n \rightarrow$ number of target atoms per unit volume

$L \rightarrow$ Thickness of the target

$Z \rightarrow$ atomic number of target atom.

$k \rightarrow$ colomb constant

$e \rightarrow$ charge of electron

$K_E \rightarrow$ Kinetic energy of α-particle

~~It is really difficult to produce α-particle with suffi~~

To see the object and its details the wave length of the readiation must be smallere than the dimention of the object. otherewise the effects of diffreaction will partially ore completely obscure the image.

From well known foremula

$$\lambda = \frac{hc}{E}$$

λ = wave length
h = plank's constant
E = is energy.

one can see that de-Broglie wave-length is inveresely preoperetional to ~~the~~ it's eneregy.

one can use this facet to resolve the structure of the paretcile under test if the de-Broglie wave length of ~~pa~~ testereparticle is camparable smallere than the size of paretcile under test. On contrarey if the energy of the tester particle is too low, due to its large de-Broglie wave-length it will just bend around the object under study and no its internal structure will be resolved.

Previous class we derived that, as per Rutherford's α-scattin experiment, the size of nucleus or the distance of closest approach $d \simeq 10$ fm.

So for the nuclei with 10 fm, we require $\lambda \leq 10$ fm.

corresponding to $p \geq 100$ MeV/c , Thus,

it is difficult to produce α-particle with sufficient energy to probe particle ~~B~~ size of ≤ 10 fm,-

Thus one can use high energy electrons. It is comparatively easier to produce electron's beam with 100 MeV to 1 GeV using some of the accelerators, and it can be analyzed with a precise spectrometer to select only those electrons that are elastically scattered from the selected nuclear target.

Considering electron as projectile, it would certainly move faster, thus relativistic treatment is required for the Rutherford's finding, which was first calculated by Mott

$$\left(\frac{d\sigma}{d\Omega}\right)_{Mott} = \left(\frac{d\sigma}{d\Omega}\right)_{Rutherford}\left[1-\frac{v^2}{c^2}\sin^2\left(\frac{\theta}{2}\right)\right]$$

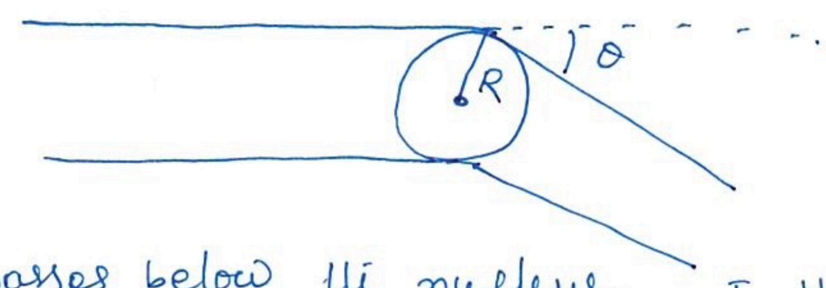
$$\Rightarrow \left(\frac{d\sigma}{d\Omega}\right)_{mott} = \frac{d^2}{\left(16\sin^4\frac{\theta}{2}\right)}\left(1-\beta^2\sin^2\left(\frac{\theta}{2}\right)\right)$$

The cross-section for scattering from a point-like target is given by the Rutherford scattering formula. If the target has a finite spatial extent, the cross-section can be divided into two factors (1) cross section times (2) squared of a term called "Electric Form Factor" → a correction factor; This takes care of the spatial extent & shape of the target. Thus the probability amplitude for a point like scattered gets modified by the form factor.

$$\left.\frac{d\sigma}{d\Omega}\right|_{expt} = \left.\frac{d\sigma}{d\Omega}\right|_{Mott} \cancel{|F(q^2)|}\,|F(q^2)|^2$$

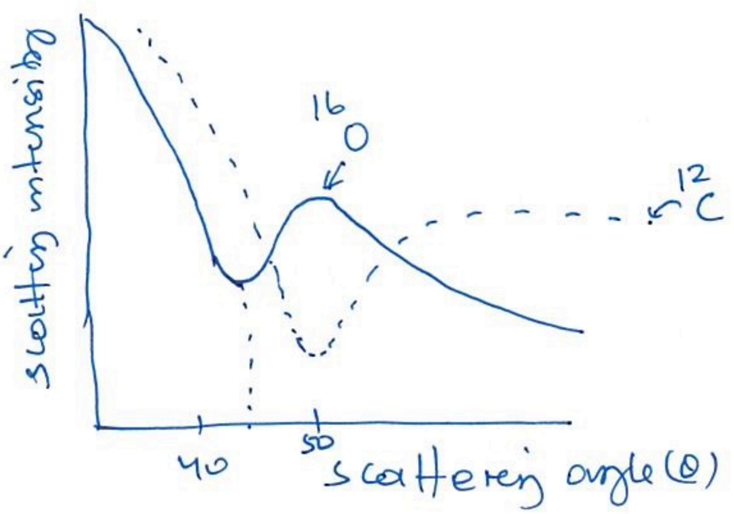
q^2 is the momentum ~~transforemered~~ transfered by the electrons in the scattering. Its magnitude is related to the scattering angle. Thus mathematically the Form Factor is defined as the ratio by which the scattering x-section is reduced when the charge Ze is spread out over a finite volume.

To understand the structure of the electric form factor we need to recall that the electron has a de-Broglie wavelength $\lambda = h/p$ and when this wavelength is of the order of the nuclear radius we get the diffraction pattern. Simple example that the nucleus were a solid sphere of Radius R with an infinite potential inside the sphere & zero outside, so that the electron can't penetrate the sphere. The wave that pass over nucleus travels a distance $2R\sin\theta$ further than the -



wave that passes below the nucleus. I this difference is near equal to $\lambda/2$, $3\lambda/2$ .... then we get destructive ~~interference~~ interference. At these angles the diffential cross-section vanishes.

Taking these into consideration, early experiments at SLAC Stanford linear Accelerator Center published results in Phy. Rev. 113, 666, (1959).



420 MeV electrons scattering from ^{16}O & ^{12}C was reported. (The paper reported other results like 240, 360 MeV electron data was also presented.)

If you consider 400 MeV electrons, the $\lambda = \frac{1240}{420}$ fm $\simeq 3$ fm putting this into the first minima one can find the raddii as

$$r_{O16} = 2.6 \text{ fm}$$
$$R_{C12} = 2.3 \text{ fm}$$

But experiment couldnot produce the results of destructive minima. That is these minima do not fall to zero like diffraction minima seen with light incident on an opaque disk. That refers.

nucleus does not have sharp boundary.

The above measurements are only rough measurement considering the fact that potential scattering is a three dimential problem only approximately related to the diffraction by a two dimential sphere.

The real case is a little more complicated than above. A proper quantum mechanical treatment (analogous to diffraction in optics) shows that the electric form factor is actually the Fourier transform of the charge distribution.

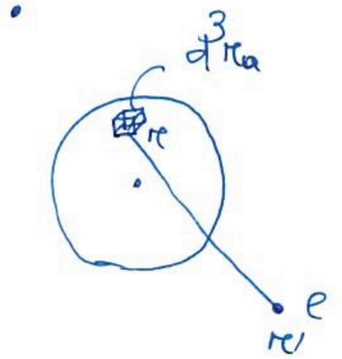
Quantum Mechanical Treatment of elastic scattering of electrons with a extent shape of charge distribution

From previous discussions, it is clear that nucleus is a not a point like object, rather a extended object of finite size. Within this size the charges are distributed charges means the protons.

Consider the charges are distributed uniformly with charge density $\rho(r)$.

What is the charges of a small volume element d^3r can be calculated as

$$\left[\rho(r)\, d^3r\right]$$



Say an electron is is at r'. & what is the potential energy is

$$V = \frac{q_1 q_2}{4\pi\epsilon_0 r}.$$

$$\Rightarrow V = \frac{(\rho(r)\, d^3r)(-e)}{4\pi\epsilon_0\, |\vec{r} - \vec{r}'|}$$

for all volume you integrate it.

Now let us look into the electreon scattering with the nucleus.

$$\psi_i = e^{i\vec{k}\cdot\vec{r}} \qquad \psi_f = e^{i\vec{k}'\cdot\vec{r}}$$

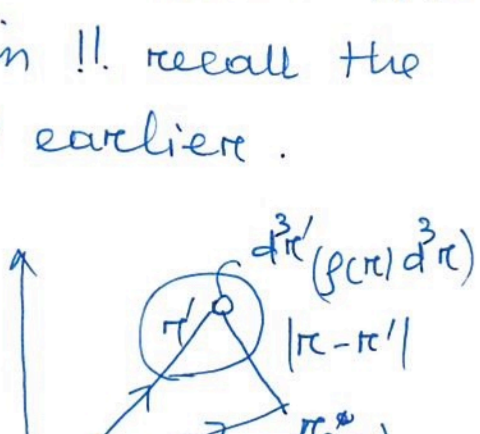
Say the ψ_i be the wave fenctin of electron before scattering & it becomes ψ_f, the final wave function of electreon after scattereing.

Since the scattering is elastic scattering the magnitude of $\vec{k}$ should be equal to the magnitude of $\vec{k}'$. The magnitude of kinetic energy remain same. only the direction changes.

What made the change of electreon wave fenctin ψ_i to ψ_f? it is the columb interaction !!. recall the potential energy that we discussed earlier.

The interaction enerergy can be writtten as

$$H' = \int_{\text{all volume}} \frac{(\rho(r')\,d^3r')(-e)}{4\pi\epsilon_0\,|r-r'|}$$



This interaction will change the electron wave fenctin ψ_i to ψ_f. What is the probability to find it.

$$P \propto |\langle \psi_f | H' | \psi_i \rangle|^2$$

This is nothing but the integreatin overe entiree space

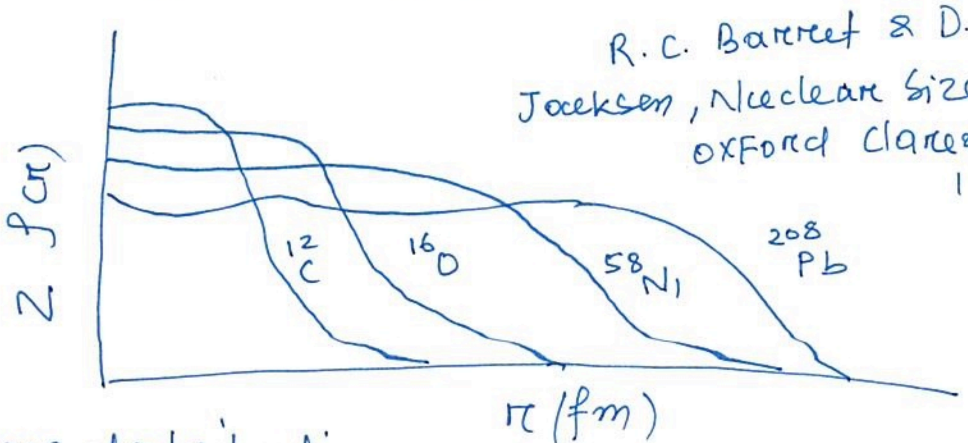
$$P = \int \psi_f^*(\vec{r})\, H'\, \psi_i(r)\, d^3r.$$

$$= \int e^{i(K-K')r}\, H'\, d^3r$$

$$= \int e^{i(K-K')r} d^3r \left[\int \frac{\rho(r')\, d^3r' (-e)}{4\pi\epsilon_0 (r-r')} \right]$$

This is the preobability how an eleetreen scatters freem nucleus. For this we should know the charege distreibution, But this is what donot know. Thus we look into the experimental data, that it ~~sto~~ experemtally measure the preobability. Before we discuss about the experimental reesults, Let's little bit visualise it. The electreen scattin will only be the colomb intereatin, why because it doesnot intereact strongly. Thus the experentaal out cones will mostly be the columbic intereatin.

At this stage the preoblem is reeveresed, the preobabilities are avalable freem the experiments. The questin is although we know the total charges (Ze) but don't know the charge density $\rho(r)$.



R.C. Barreet & D. F Jackson, Nuclear Size & structure, OXFORD Clarendon 1977.

Radial charege distreibutin of ^{12}C, ^{16}O, ^{58}Ni, ^{208}Pb

The above figure shows a more quantitative determinatin of the relationship between nuclear radius & total charge, based on electron scattering. what we observe that for different nuclei, for carbon, light nucleus the graph is starting higher & for heavier nucleus it is starting at lower charge density.

what does this means? For lighter nucleus more charge density at center, heavier nucleus less charge density (This is purely comming from the experimental data). And the radius upto which density is almost constant, is smaller for lighter & larger for heavier nuclei.

By the time these experiments were done, existance of neutrons were known. since electrons interact only with proton these curves are refection of protons distributin or we call them charge density. But in same volume there are presence of neutrons.

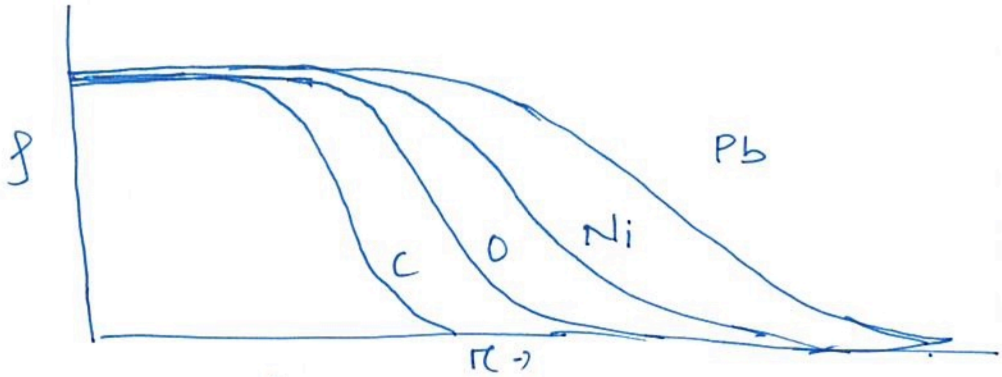
Thus, this charge distributin also depends how neutron are distributed. Let make an analogical explaination.

what is the mass of nucleus. it is the sum of total proton & neutron. Or we can say the total number of neucleon. & we call is mass number A

Now we can derive a mass density ρ, which is

$$\rho = \rho_p \left(\frac{A}{2}\right)$$

And if we redo the curve with same data but the mass density. The curve looks.

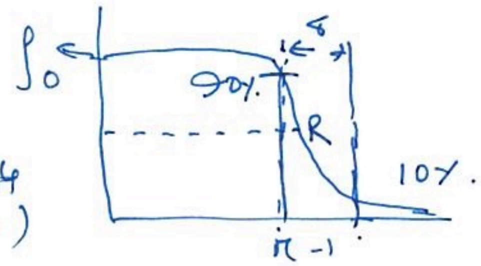


This describe how mass is distribution across the nucleus. Can we find a function to fit this data?

The charge, mass or matter distribution should take a form.

$$\rho(r) = \frac{\rho_0}{1 + e^{(r-R)/\delta}}$$

where ρ_0 is the matter density at center



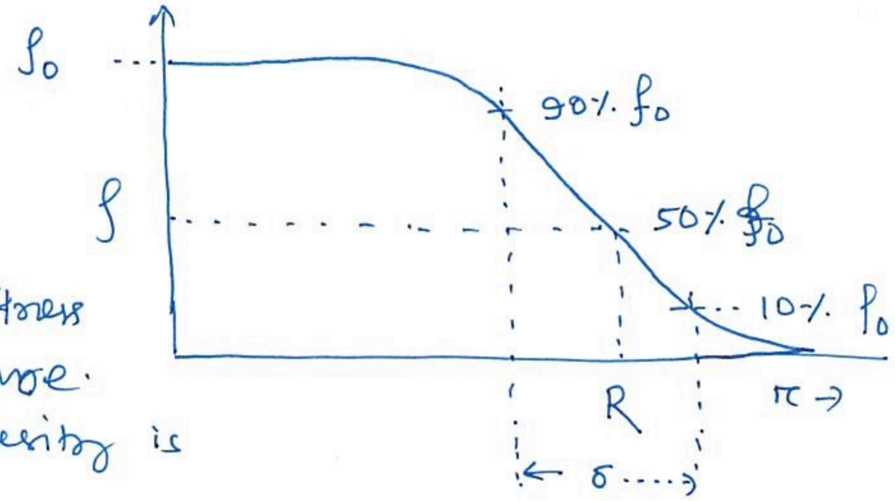
$R \rightarrow$ nuclear radius (~~density falls of 1/2 of ρ_0~~)

$\delta \rightarrow$ surface depth which measures the range in r over which the charge density / or matter density changes from the order of its value at the center to the much smaller than this value. This distribution is known as Wood-Saxon formula or Model.

Let us analyze little bit to find various quantities.

R = average radius

R is 50% of ρ_0



$\delta \rightarrow$ measure of stiffness of density change.
i.e. how fast the density is ~~Experi~~ changin from 90% to 10% of its central value ρ_0

Experimentally, $\rho_0 = 0.17 \, fm^{3}$

at 90% of ρ_0:

$$0.9\rho_0 = \frac{\rho_0}{1+e^{(r-R)/\delta}}$$

$$\Rightarrow \frac{1}{0.9} = 1 + e^{(r-R)/\delta}$$

$$\Rightarrow 0.1 = e^{(r-R)/\delta}$$

$$\Rightarrow \ln(0.1) = \ln\left[e^{(r-R)/\delta}\right]$$

$$\Rightarrow \frac{r-R}{\delta} = -2.303 \qquad \ln(0.1) = -2.303$$

$$\Rightarrow \boxed{r = R - 2.303\delta}$$

at 10% ρ_0 :

$$0.1\rho_0 = \frac{\rho_0}{1+e^{(r-R)/\delta}}$$

$$\Rightarrow \frac{1}{0.1} = 1 + e^{(r-R)/\delta}$$

ln

$\ln 9 = 2.197$

$$\Rightarrow 9 = e^{(r-R)/\delta} \Rightarrow \ln 9 = \ln\left[e^{(r-R)/\delta}\right] \Rightarrow r = R + 2.197\delta$$

After fitting various data, one can find out the value of δ ;

i.e. for almost all nuclear range.

$$\boxed{\delta = 0.4 \Leftrightarrow 0.5 \text{ fm}}.$$

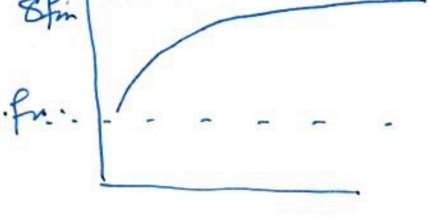
solving wood saxon $\rho(r)$, ~~one can find out~~ & incorporating experimental data.

$$r = r_0 A^{1/3}$$

where $\boxed{r_0 = 1.18 \text{ fm}}$

$$\boxed{\rho_0 = 0.17 \text{ nucleon}/\text{fm}^3}$$

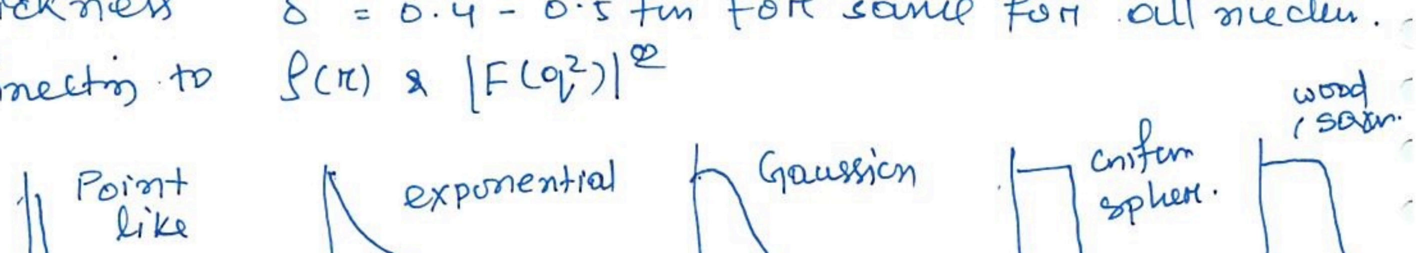
If you try to draw the radius of nucleus vs the mass number you will get a result similar to

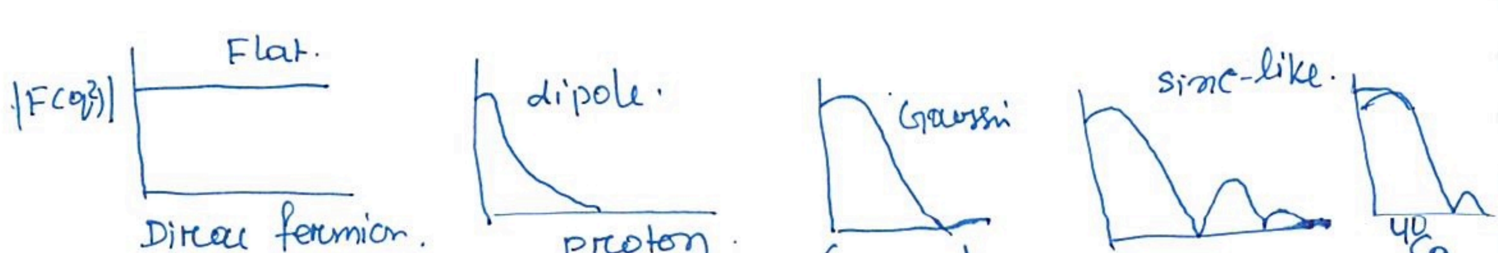


This gives interesting result that there is not much change in radius for higher Mass number. &

Experimentally what you will find that the skin thickness $\delta = 0.4 - 0.5$ fm for same for all nuclei.

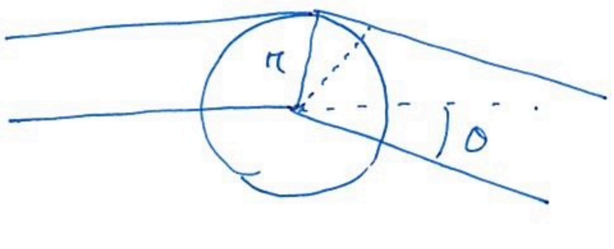
connecting to $\rho(r)$ & $|F(q^2)|^2$





what we learn from the above excersize ??
Qualitatively, the charge distribution is almost similar all accross. And experiment precisely varified these calculations. And the difraction pattern is shown in electron scattering of nucleus. The reason for this is that ~~the~~ (reson for interfence pattern), the part of wave front that passes through the nucleus at a distance r from the center and is scattered through an angle θ travels a further distance than the part of the wave that passes through the center, by an amount propertional to r and therefore suffers a phase change (relative to the part of the wave ~~to~~ passing through the center)

This phase change also depends on the scattering angle θ and is equal to $qr/\hbar$



This means that diffent part of wavefront suffer a different phase change (just as in optical diffraction). these different ~~ap~~ amplitudes are summed to get the total amplitude at some scattering angle θ and this gives rise to the diffraction pattern. The contribution to amplitude ~~part~~ from the part of wave front passes at a distance r from the center of the nucleus is propertional to charge density $\rho_p(r)$ at r or matter density $\rho(r)$ at r. The total scattering amplitude is therefore the sum of the amplitude from all these different parts, which is what the integral in some equation above

Other ways of calculating the Radius of Nucleus.

One can use purely Quantum mechanical approach to calculate the radius of nucleus using the similar ~~a~~ analogy that of Quantum mechanical treatment of Bohr's atomic structure. i.e. ~~using~~ the

this you can find in details in some of the books.

Before going into such calculation let me briefly discuss about the Quantum theory of Hydrogen atom (we will not discuss this in class - and this is a suplimentary material)

Hydrogen system is a system of one proton, one electron & the electrostatic (Columb) potential that holds them together.

Proton $q = +e$ electron $q = -e$

The potential energy in this case is

$$V = -\frac{e^2}{4\pi\epsilon_0 r}$$ [attractive potential between $+e$ & $-e$.

Notice that it is a function r, not (x, y, z)

we can simply write $r^2 = x^2 + y^2 + z^2$

$$r = \sqrt{x^2 + y^2 + z^2}$$

By going (x, y, z), one can go away with it, but things becomes more complecated. The appropriate approach is to let the ~~s~~ symmetry of the potential guide in solution & the spherically symmetric potential ~~to~~ make use of spherical polar coordinates.

Time independent schrodinger equation

$$H|\vec{E_i}\rangle = E_i|\vec{E_i}\rangle$$

E_i are eigen values & $|E_i\rangle$ are energy Eigenstates

By changing the notation $E_i \to E$ & $|E_i\rangle \to \psi$, the 3-dimentn schrodinger equation can be writen as

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\psi = E\psi$$

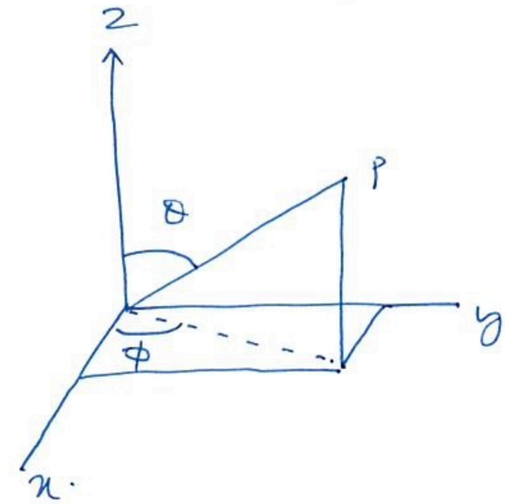
∇^2 is the laplacian operator.

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} + \frac{2m}{\hbar^2}(E-V)\psi = 0$$

we can do a spherical polar co-ordinate transformation

$$\theta = \cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$$

$$\phi = \tan^{-1}\left(\frac{y}{x}\right)$$



Here the co-ordinate is (r, θ, ϕ) and are related to (x, y, z) & can be expressed in ~~to~~

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta.$$

so the schrodinger's equation in spherical polar coordinates as

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\cos\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin\theta}\frac{\partial^2\psi}{\partial\phi^2} + \frac{2m}{\hbar^2}(E-V)=0$$

pluging V & multiply both side $r^2\sin^2\theta$

$$\sin^2\theta\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{\partial^2\psi}{\partial\phi^2} + \frac{2mr^2\sin^2\theta}{\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0 r} + E\right)=0$$

This equation does not look nice.

This equation gives the wave function ψ for the electron in the hydrogen atom. That means if we solve for ψ in principle we know everything about the hydrogen atom. when we solve one dimential. ψ, we get one quantum number & for above one is three dimentional equation; ~~we~~ we can get detail information about hydrogen atom.

The solution to ~~these~~ this equation for electron in the hydrogen atom by separation of variables. This is usually done to ease the complexity of linear algebric equation, differential equations, coupled differential equations etc.

when we have an equation like the one above, we like to see if we can "separate" the variable; i.e. "split" the equation into different parts, with only one variable in each part. Equation becomes simplified if we write.

$$\psi(r,\theta,\phi) = R(r)\,\Theta(\theta)\,\Phi(\phi) = R\Theta\Phi$$

The schrodinger equation will become:

$$\frac{\partial\psi}{\partial r} = \Theta\Phi\frac{\partial R}{\partial r} = \Theta\Phi\frac{dR}{dr}$$

$$\frac{\partial\psi}{\partial\theta} = R\Phi\frac{\partial\Theta}{\partial\theta} = R\Phi\frac{d\Theta}{d\theta}$$

$$\frac{\partial^2\psi}{\partial\phi^2} = R\Theta\frac{\partial^2\Phi}{\partial\phi^2} = R\Theta\frac{d^2\Phi}{d\phi^2}$$

where the partial derivatives become full derivatives because R, Θ, and Φ depend on r, θ and ϕ only.

To separate the variables, plug $\psi = R\Theta\Phi$ into the Schrodinger's equation and devide by ~~R~~ $R\Theta\Phi$

The schrodinger's equatin will become

$$\frac{\sin^2\theta}{2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}+\frac{2mr^2\sin^2\theta}{\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0 r}+E\right)=0$$

Note that we have separated out the ϕ variable, in term

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}$$

is a function of ϕ only. Let's put it over on the right hand side of the equatin. This gives us.

$$\frac{\sin^2\theta}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\frac{2mr^2\sin^2\theta}{\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0 r}+E\right)=-\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}$$

This equatin looks like.

$$f(r,\theta) = g(\phi)$$

f is a function of r and θ only, and g is a functin of ϕ only. If an r or θ shows up on the LHS, it can't be satisfied, because r and θ never show up on the RHS. Similarly if ϕ shows up on the RHS, the equation can't be satisfied because ϕ never shows up on the LHS. The only way for the equatin to be satisfied is for

$$f(r,\theta) = \text{a constant, independent of } r, \theta, \text{ and } \phi = g(\phi)$$

we have taken a complex equatin & using r, θ, and ϕ separation method separated it into two equations, one in r and θ, and the other in ϕ only.

It is a general practice that some differential equations can be solved only if certain conditions are satisfied. These conditions have led us to quantum numbers. In this context, we can use a similar concept. for $-\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}$

$$\Rightarrow \boxed{-\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = m_l^2}$$

~~New~~ The previous equation: we set the LHS equal to m_l^2, divide by $\sin^2\theta$ and rearrange, we get,

$$r \frac{1}{R}\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) + \frac{2mr^2}{\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0 r} + E\right) = \frac{m_l^2}{\sin^2\theta} - \frac{1}{\Theta \sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right)$$

Here we again separated variables. The LHS is a function of r only, and the RHS is a function of θ only. Again, the only way to satisfy this equation is for LHS = a constant = RHS

Conditions on the constant will arise from the solution of the differential equations. this constant $= l(l+1)$

Now the equations can be written:

$$\frac{d^2\Phi}{d\phi^2} + m_l^2 \Phi = 0 \qquad - (A)$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) + \left[l(l+1) - \frac{m_l^2}{\sin^2\theta}\right]\Theta = 0 \qquad - (B)$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) + \left[\frac{2m}{\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0 r} + E\right) - \frac{l(l+1)}{r^2}\right]R = 0 \qquad - (C)$$

At this stage we manage to make it ~~more~~ lot more simplier because the "big" partial differential equation into three simpler ones, each of which is a function of a single variable and each of which is much "easier" to handle. At this stage the schrodinger's equation for the hydrogen atom into three separate equations, one for each variable, R, Θ, Φ.

But we need to now solve three equation instead of one equation. But this is lot more easier than our starting equation. Let's quickly see the solutions-

~~we write down~~

Quantum numbers are constants which identify each solution to schrodinger's equation.

First quantum number: comes by solving the diffential equation for ϕ

$$\frac{d^2\Phi}{d^2\phi} + m_l^2 \Phi = 0$$

The general solution to above equation will be

$$\Phi(\phi) = A e^{j m_l \phi}$$

Now we know that ϕ and $\phi + 2\pi$ represent a single point in space, we must have

$$A e^{j m_l \phi} = A e^{j m_l (\phi + 2\pi)}$$

This happens only for $m_l = 0, \pm 1, \pm 2, \pm 3, \ldots$

m_l is called the magnetic quantum number.

Solving the diffential equation Θ will give ~~two~~ 2nd quantum number

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) + \left[l(l+1) - \frac{m_l^2}{\sin^2\theta}\right]\Theta = 0$$

it solves the term $l(l+1) - \frac{m_l^2}{\sin^2\theta}$

it turns out that from diffential equations that the equation for Θ can be solved only if l is an integer greater than equal to the absolute value of m_l.

l is called orbital quantum number & the requirement on l can be ~~restarted~~ restated as $m_l = 0, \pm 1, \pm 2 \ldots \pm l$.

The radial diffential equation is

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) + \left\{\left[\frac{2m}{\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0 r} + E\right)\right] - \frac{l(l+1)}{r^2}\right\} R = 0$$

It can be solved only for energies E which satisfy the same condition as we found on the energies for

for the Bohr atom:

$$E_n = -\frac{me^4}{32\pi^2 \epsilon_0 \hbar^2} \frac{1}{n^2} = \frac{E_1}{n^2} \qquad n = 1, 2, 3 \ldots$$

n is known as the principal quantum number.

By summarizing all above we can write the solutin of schrodinger equatin for the hydrogen atom must be of the form.

$$\psi = R_{nl} \Theta_{lm_l} \Phi_{m_l}$$

where n, l, m_l being the quantum number as discussed above. If we write it concisely.

First coordinate. $\Phi(\phi) = e^{jm\phi}$ $m \Rightarrow$ integer

second coorendl $\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial\Theta}{\partial\theta} + \left(\lambda - \frac{m^2}{\sin^2\theta}\right)\Theta = 0$

$\lambda = l(l+1)$ with $l = 0, 1, 2 \ldots$

$m = -l, -l+1 \ldots\ldots l-1, l.$

Angular part.

Third coordinate.

$$\frac{1}{R}\left[\frac{d}{dr} r^2 \frac{dR}{dr} + \frac{2mr^2}{\hbar^2}(E - V(r))R\right] = l(l+1)$$

$$E_n = -\frac{z^2}{n^2} R_\infty = -\frac{z^2}{n^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2 \frac{m_e}{2\hbar^2}$$

with $n > 0$

radial part.

Angular wave functin

$$L^2 = \frac{\hbar}{i}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right].$$

$$L_z = \frac{\hbar}{i}\frac{\partial}{\partial\phi} \quad \& \quad \vec{L} = (L_x, L_y, L_z)$$

Angular momentum.

There is a class of functions that are simultaneous eigenfunctions.

$$L^2 Y_{lm}(\theta,\phi) = l(l+1)\hbar^2 Y_{lm}(\theta,\phi) \quad l = 0, 1, 2 \ldots$$

$$L_z Y_{lm}(\theta,\phi) = m\hbar Y_{lm}(\theta,\phi) \quad m = -l, -l+1 \ldots l-1, l$$

spherical harmonics $Y_{lm}(\theta,\phi)$

$$Y_{00} = \sqrt{\frac{1}{4\pi}} \qquad Y_{11} = \sqrt{\frac{3}{8\pi}} \sin\theta \, e^{i\phi}$$

$$Y_{10} = \sqrt{\frac{3}{4\pi}} \cos\theta \qquad Y_{1,-1} = \sqrt{\frac{3}{8\pi}} \sin\theta \, e^{-i\phi}$$

The radial part.

$$\frac{1}{R}\left[\frac{d}{dr} r^2 \frac{dR}{dr} + \frac{2mr^2}{\hbar^2}(E - V(r))R\right] = \lambda \qquad \lambda = l(l+1)$$

For solution to ground state i.e. $l = 0, m = 0$.

$$-\frac{\hbar^2}{2m}\left(R'' + \frac{2}{r}R'\right) - \frac{Ze^2}{4\pi\epsilon_0 r} R = ER$$

$r \to \infty \Rightarrow$ no density.

$$R(r) = A e^{-r/a}.$$

$$R' = -\frac{A}{a} e^{-r/a} = -\frac{R}{A}$$

$$R'' = \frac{A}{a^2} e^{-r/a} = \frac{R}{a^2}$$

Prefactor for $1/r$

$$\frac{\hbar^2}{ma} - \frac{Ze^2}{4\pi\epsilon_0} = 0$$

solution for the length scale parameter

$$a_0 = \frac{4\pi\epsilon_0 \hbar^2}{Ze^2 m} \quad \text{Bohr radius.}$$

$$\Rightarrow -\frac{\hbar^2}{2m}\left(\frac{1}{a^2} - \frac{2}{ar}\right) - \frac{Ze^2}{4\pi\epsilon_0 r} = E$$

solution for the energy.

$$\boxed{E = -\frac{\hbar^2}{2ma} = -Z^2\left(\frac{e^2}{4\pi\epsilon_0}\right)^2 \frac{m_e}{2\hbar^2}}$$

Ground state in the Bohr's model $n = 1$.

Thus the ground state is $n=1$, $l=0$, $m=0$, i.e. there is only one quanten state with the wave functin for $n=1$ and its ψ_{100}; However for $n=2$ we have

$$l=0,\ m=0$$

$$l=1\quad m=-1,0,1.$$

Thus the allowed state for $n=2$, the states are ψ_{200}, ψ_{21-1}, ψ_{210}, ψ_{211}: Thus the wave functn are:

$$\psi_{100} = \frac{1}{\sqrt{\pi}}\frac{1}{a_0^{3/2}} e^{-r/a_0} \qquad n=1,\ l=0,\ m_l=0$$

$$\psi_{200} = \frac{1}{4\sqrt{2\pi}}\frac{1}{a_0^{3/2}}\left(2-\frac{r}{a_0}\right)e^{-r/2a_0} \qquad n=2,\ l=0,\ m_l=0$$

$$\psi_{21-1} = \frac{1}{8\sqrt{\pi}}\frac{1}{a_0^{3/2}}\frac{r}{a_0}e^{-r/2a_0}\sin\theta\, e^{-i\phi} \qquad n=2,\ l=1,\ m_l=-1$$

$$\psi_{210} = \frac{1}{4\sqrt{2\pi}}\frac{1}{a_0^{3/2}}\frac{r}{a_0}e^{-r/2a_0}\cos\theta \qquad n=2,\ l=1,\ m_l=0$$

$$\psi_{211} = \frac{1}{8\sqrt{\pi}}\frac{1}{a_0^{3/2}}\frac{r}{a_0}e^{-r/2a_0}\sin\theta\, e^{i\phi} \qquad n=2,\ l=1,\ m_l=1.$$

usually ~~ψ_0~~ the corresponding Orbital Name are

$$\psi_{100} \to \psi_{1s}$$

$$\psi_{200} \to \psi_{2s}$$

$$\psi_{210} \to \psi_{2p_z}$$

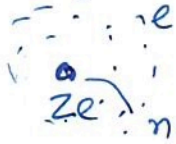
$$\psi_{21-1} \to \psi_{2p_y}$$

$$\psi_{211} \to \psi_{2p_x}.$$

Underestanding the nucleare volume using hydreogen or hydreogen like nucleus (one electreon in orbit)

The potential enereegy for such a system will be

$$V(r) = -\frac{Ze^2}{4\pi\epsilon_0 r}$$ } Note here this is a hydreogen like so $Z \neq 1$



we need to solve the schreodinger ~~eqn~~ wave function

$$H\psi = E\psi$$

$$H = -\frac{\hbar^2}{2m}\nabla^2 - \frac{Ze^2}{4\pi\epsilon_0 r}$$

The ground state of this hydreogen like nucleus.

$$\psi_{100} = \sqrt{\frac{1}{\pi}\left(\frac{Z}{a_0}\right)^3}\, e^{-Zr/a_0}$$

$a_0 \leftarrow$ Bohr's radius

This wave function doesn't go to zero ~~if you bring~~ at $r=0$
That means there is a probability of finding the electreon at the side of the nucleus. Thus if the ~~strucu~~ electren is penetreating the nucleus, the streuctures of nucleus will afect it. Thus this electreon is going aftere that energy states also.

For point charge the potential

$$V = -\frac{Ze^2}{4\pi\epsilon_0 r}$$

$\circ$ ze $\quad \circ$ e

For the extended charge distreibution in sphere the potential:

$r > R$

$$V(r) = \frac{Ze}{4\pi\epsilon_0 r}$$

$$r = R \Rightarrow V(r) = \frac{Ze}{4\pi\epsilon_0 R}$$

What happens to the potential inside surface?

i.e. $r < R$.

$$\Rightarrow \quad V(r) - V(R) = \int_R^r \vec{E} \cdot dr.$$

Potential at r & potential at R, The potential differnce is $\int \vec{E} \cdot dr$.

$$\vec{E}(r) = \frac{Ze}{4\pi\epsilon_0} \left(\frac{r \hat{r}}{R^3} \right)$$

$\hat{r} \cdot d\vec{r} = \hat{r} \cdot \hat{r}\, dr = dr$

$$\Rightarrow \quad V(r) - V(R) = \int_R^r \frac{Ze}{4\pi\epsilon_0} \frac{r}{R^3} dr$$

$$= -\frac{Ze}{4\pi\epsilon_0 R^3} \left[\frac{r^2 - R^2}{2} \right]$$

$$\Rightarrow \quad V(r) = V(R) - \frac{Ze}{4\pi\epsilon_0 R^3} \left[\frac{r^2 - R^2}{2} \right]$$

$$= \frac{Ze}{4\pi\epsilon_0 R} - \frac{Ze}{4\pi\epsilon_0 R^3} \left(\frac{r^2 - R^2}{2} \right)$$

$$= \frac{Ze}{4\pi\epsilon_0 R} \left[1 - \frac{r^2 - R^2}{2R^2} \right] = \frac{Ze}{4\pi\epsilon_0 R} \left[1 - \frac{1}{2} - \frac{r^2}{2R^2} \right]$$

$$\Rightarrow \quad \boxed{V(r) = \frac{Ze}{4\pi\epsilon_0 R} \left[\frac{3}{2} - \frac{r^2}{2R^2} \right]}$$

Extended charge sphere.

If we find the potential diffence from uniform ~~potential~~ charge distributed sphere to a point charge will be.

$$\Delta V = \left[\frac{Ze}{8\pi\epsilon_0 R} \left(3 - \frac{r^2}{R^2} \right) \right] - \left(\frac{Ze}{4\pi\epsilon_0 r} \right)$$

the change in energy

$$\Delta U = Ze^2 \quad \Delta V(-e)$$

$$= -\frac{Ze^2}{8\pi\epsilon_0 R} \left(3 - \frac{r^2}{R^2} \right) - \left(\frac{-Ze^2}{4\pi\epsilon_0 r} \right)$$

If the potential energy change by an amount of ΔU to bring point nucleus to a extended nucleus, the hamiltonian will also change $H' = H_0 + H_1$ $\quad$ H_0 = point like, H_1 = change ΔU

Thus the ~~wa~~ Schrodinger equation is

$$(H_0 + H_1)\psi = (E + \Delta E)\psi$$

For schrodinger technique provides a methods to find the solution for such a system through the perturbation treatment ~~proof~~ provided ΔE is very small. ~~By~~ o This done by knowing the solution for $H_0\psi = E\psi$ & appling that to whole equation. Thus.

$$\langle \Delta E \rangle = \langle \psi_0 | H' | \psi_0 \rangle$$ $\quad$ ψ_0 = solution for H_0

$$= \int \psi_0^* H' \psi_0 \, d\tau$$

$$\langle \Delta E \rangle = \int \psi_0^* \left[\frac{Ze^2}{4\pi\epsilon_0 r} - \frac{Ze^2}{8\pi\epsilon_0 R}\left(3 - \frac{r^2}{R^2}\right) \right] \psi_0 \, d\tau$$

Let's calculate the wave function for the ground state.

$$\psi_{100} = \psi_{1s} = \sqrt{\frac{Z^3}{\pi a_0^3}} \, e^{-Zr/a_0}$$

$$\Delta E = \int \sqrt{\frac{Z^3}{\pi a_0^3}} \, e^{-Zr/a_0} \left[\frac{Ze^2}{4\pi\epsilon_0 r} - \frac{Ze^2}{8\pi\epsilon_0 R}\left(3 - \frac{r^2}{R^2}\right) \right] \sqrt{\frac{Z^3}{4\pi a_0^3}} \, e^{-\frac{Zr}{a_0}} \, d\tau$$

$$d\tau = r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$\Rightarrow \Delta E = \int \frac{Z^3}{\pi a_0^3} e^{-2r/a_0} \left[\frac{Ze^2}{4\pi\epsilon_0 r} - \frac{Ze^2}{8\pi\epsilon_0 R}\left(3 - \frac{r^2}{R^2}\right) \right] r^2 \sin\theta \, dr \, d\theta \, d\phi$$

since the potential difference is calculated from r 0-to R out side of the sphere H_1 will become zero. Thus the integral of r will be of 0 to R.

$$\Delta E = \frac{z^3}{\pi a_0^3} \cdot \frac{Ze^2}{4\pi\epsilon_0} \int_0^R \left[\frac{1}{r} - \frac{1}{2R}\left(3 - \frac{r^2}{R^2}\right)\right] r^2 dr \underbrace{\int \sin\theta\, d\theta \int d\phi}_{2 \cdot 2\pi} \quad e^{-2r/a_0}$$

$$\boxed{\Delta E = \frac{Z^4 e^2}{\pi a_0^3 \epsilon_0} \int_0^R \left(r - \frac{3r^2}{2R} + \frac{r^4}{2R^3}\right) e^{-2r/a_0} dr.}$$

we could use the approximation for $e^{-2r/a}$ & its limit to solve or simplify this integration or one can do proper integration without putting $e^{-2r/a}$ as the limiting case to make it disappearing

$$e^{-2r/a} \rightarrow \quad r \sim 10^{-15} m$$
$$a_0 \sim 10^{-10} m.$$
$$\Rightarrow e^{-2\times 10^{-5}} \rightarrow 1.$$

$$\Rightarrow \quad \Delta E = \frac{Z^4 e^2}{\pi \epsilon_0 a_0^3} \int_0^R \left(r - \frac{3r^2}{2R} + \frac{r^4}{2R^3}\right) dr$$

$$= \frac{Z^4 e^2}{\pi \epsilon_0 a_0^3} \left[\int_0^R r\, dr - \frac{1}{R}\int \frac{3r^2 dr}{2} + \frac{1}{2R^3}\int r^4 dr \right]$$

$$= \frac{Z^4 e^2}{\pi \epsilon_0 a_0^3} \left[\frac{R^2}{2} - \frac{3}{2R}\frac{R^3}{3} + \frac{R^5}{2R^3 \cdot 5} \right]$$

$$= \frac{Z^4 e^2 R^2}{10 \pi \epsilon_0 a_0^3}$$

Mathematica solution to

$$\int_0^R \left[\left(\frac{1}{r} - \frac{1}{2R}\left(3 - \frac{r^2}{R^2}\right)\right] r^2 dr = \frac{R^2}{10}$$

If you donot want use the vanishing of e^{-2r/a_0} then I give the simple equation & its mathematical solution

$$\int_0^R \left[\frac{r^4}{2R^3} - \frac{3r^2}{2R} + r\right] e^{-2r/a_0}\, dr$$

$$= \int_0^R \left[\frac{r^4 e^{-2r/a_0}}{2R^3} - \frac{3r^2 e^{-2r/a_0}}{2R} + r e^{-2r/a_0}\right] dr$$

$$= \Big[-\frac{3a_0^5}{8R^3} e^{-2r/a_0} - \frac{3a^4 r}{4R^3} e^{-2r/a_0} - \frac{3a_0^3 r^2}{4R^3} e^{-2r/a_0} + \frac{3a^3}{8R} e^{-2r/a}$$
$$- \frac{a^2 r^3}{2R^3} e^{-2r/a_0} + \frac{3a^2 r}{4R} e^{-2r/a_0} - \frac{1}{4} a^2 e^{-2r/a_0} - \frac{ar^4}{4R^3} e^{-2r/a_0}$$
$$+ \frac{3ar^2}{4R} e^{-2r/a_0} - \frac{1}{2} ar e^{-2r/a_0}\Big]_0^R$$

$$= \Big[-\frac{1}{8R^3} a e^{-2r/a} \big[3a^4 + 6a^3 r + a^2(6r^2 - 3R^2) + 2a(2r^3 - 3rR^2 + R^3)$$
$$+ 2r(r-R)^2(r+2R)\big]\Big]_0^R$$

$$= \frac{1}{8R^3}\left[a_0^2(3a_0^3 - 3a_0R^2 - 3a_0 e^{-2R/a_0}(a_0+R)^2 + 2R^3)\right]$$

In any cases at the end of the day with some approximate

$$\boxed{\Delta E = \frac{Z^4 e^2 R^2}{10\pi \epsilon_0 a_0^3}}$$

At this stage we will take analogy from the atomic structure. like taking the diffrence of energy of various energy states like E_{1s} & E_{2sp}

2P electron goes to zero at 

$r = 0$

So what is the picture for the nuclear system (extended charge sphere)? this we can do with an analogues to the energy gap & when electron comes from 2s2p to 1s it emits some rays (x-ray) with fix energy.

2S2P

1S

This called K_α-Xray

What we found by now is

$$E_{1s} = E_{1s}^{Point} + \Delta E$$

$$E_{1s} = E_{1s}^{Point} + \frac{Z^4 e^2 R^2}{10\pi \epsilon_0 a_0^3}$$

If nucleus were a point particle, the expression for K_α-x-ray

$$E_{K\alpha} = E_{2P} - E_{1s}$$

$$= E_{2P} - E_{1s}^{Point} - \frac{Z^4 e^2 R^2}{10\pi \epsilon_0 a_0^3}$$

2P state exist at $r = 0$ ie. we can cn $E_{2P} = E_{2P}^{Point}$.

let's find an elements with same protons but severeral neutrons (isotopes). Then find Energy of K_α x-ray.

$$E_{K\alpha}(A') = E_{2P} - E_{1s}^{Point} - \frac{Z^4 e^2 R'^2}{10\pi \epsilon a_0^3} \quad A' \text{ iso}$$

A' is isotope for A

$$E_{K\alpha}(A) = E_{2P} - E_{1s}^{Point} - \frac{Z^4 e^2 R^2}{10\pi \epsilon_0 a_0^3}$$

E_{2P} does not change for A' to A as energy is not changed for E_{2P}.

E_{1s}^{Point} also doesnot change for A' to A as it is a point charge like state.

only change is in last term $\frac{Z^4 e^2 R'^2}{10\pi \epsilon_0 a_0^3}$ to $\frac{Z^4 e^2 R^2}{10\pi \epsilon_0 a_0^3}$

Now substract these two energies. $E_{K\alpha}(A')$ & $E_{K\alpha}(A)$

$$(\Delta E_{K\alpha}) = [E_{K\alpha}(A) - E_{K\alpha}(A')]$$

This is a measureable quantity as these are x-rays & you put proper detectore to ~~cont~~ count it & find the energy.

$$(\Delta E_{K\alpha}) = \frac{Z^4 e^2}{10\pi\epsilon_0 a_0^3}\left(R'^2 - R^2\right)$$

we donot need the point structure of nucleus. At this stage we donot know R' & R, but we know Z, & A

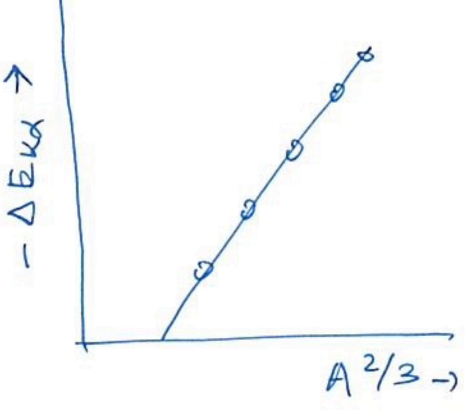
Now using $R = R_0 A^{1/3}$ from the smple volmatric calculati

$$\Delta E_{K\alpha} = \frac{Z^4 e^2}{10\pi\epsilon_0 a_0^3}\left[R_0^2 (A'^{1/3})^2 - R_0^2 (A^{1/3})^2\right]$$

$$\Delta E_{K\alpha} = \frac{Z^4 e^2}{10\pi\epsilon_0 a_0^3} R_0^{2}\left[A'^{2/3} - A^{2/3}\right]$$

If we measure several x-rays $\Delta E_{K\alpha}$ & try to draw a line, it will be a straight line with

$\left(\frac{Z^4 e^2}{10\pi\epsilon_0 a_0^3} R_0^2\right)$ as the slope of equati



Now fitting the curve one can find out what should be R_0.

The straight line goes as functin of $A^{2/3}$ of varioues isotope of some Z

This can also be done nicely with ~~a~~ muonic x-ray which is beyond the scope of our course.