General Relativity Fall 2019
Homework 4 solutions

Exercise 1: Index manipulation

(i) If the tensor $T_{\alpha\beta}$ is symmetric, show that $T^{\alpha}{}_{\beta} = T_{\beta}{}^{\alpha}$.

$$\begin{aligned} T^{\alpha}{}_{\beta} &= g^{\alpha\lambda}T_{\lambda\beta} \quad \text{[by definition]} \\ &= g^{\alpha\lambda}T_{\beta\lambda} \quad [T_{\alpha\beta} \text{ is symmetric}] \\ &= T_{\beta}{}^{\alpha} \quad \text{[by definition]}. \end{aligned} \tag{1}$$

(ii) Given a rank (0,2) tensor $T_{\alpha\beta}$, what is the rank of the tensor $T_{\alpha\beta}T_{\gamma}{}^{\sigma}T^{\beta\gamma}$?

Rank (1,1): there is one free index up and one free index down (all others are contracted).

How about $T_{\alpha\beta}T_{\gamma}{}^{\alpha}T^{\beta\gamma}$?

Rank (0,0), i.e. this is a scalar, becuase all indices are contracted.

Suppose that in some basis, the components of a tensor $T_{\alpha\beta}$ are given by

$$\begin{pmatrix} T_{00} & T_{01} & \cdots \\ T_{10} & T_{11} & \\ \vdots & & \ddots \\ & & \end{pmatrix} = \begin{pmatrix} -1 & 0 & 2 & 3 \\ 1 & 0 & 2 & 1 \\ -1 & 3 & 0 & -1 \\ -1 & 0 & -2 & 3 \end{pmatrix}.$$

(iii) Explicitly write the components of the tensors $T_{(\alpha\beta)}$ and $T_{[\alpha\beta]}$ in this basis (using the same matrix convention).

$$\begin{pmatrix} T_{(00)} & T_{(01)} & \cdots \\ & T_{(11)} & \\ \vdots & & \ddots \\ & & \end{pmatrix} = \begin{pmatrix} -1 & 1/2 & 1/2 & 1 \\ & 0 & 5/2 & 1/2 \\ & & 0 & -3/2 \\ & \text{(symmetric)} & & 3 \end{pmatrix}.$$

$$\begin{pmatrix} T_{[00]} & T_{[01]} & \cdots \\ & T_{[11]} & \\ \vdots & & \ddots \\ & & \end{pmatrix} = \begin{pmatrix} 0 & -1/2 & 3/2 & 2 \\ & 0 & -1/2 & 1/2 \\ & & 0 & 1/2 \\ & \text{(antisymmetric)} & & 0 \end{pmatrix}.$$

(iv) Compute $T^{\alpha}{}_{\alpha}$ and $T_{\alpha(\beta}T_{\gamma)}{}^{\alpha}T^{[\beta\gamma]}$.

Since we are not given the metric components in the basis, **we cannot compute $T^{\alpha}{}_{\alpha} = g^{\alpha\beta}T_{\alpha\beta}$**.

There is no need to explicitly write any calculation for $T_{\alpha(\beta}T_{\gamma)}{}^{\alpha}T^{[\beta\gamma]}$: the contraction of a symmetric pair of indices with an antisymmetric vanishes:

$$X_{(\beta\gamma)}Y^{[\beta\gamma]} = X_{(\gamma\beta)}Y^{[\beta\gamma]} \quad \text{[by symmetry of the lower two indices]} \tag{2}$$
$$= -X_{(\gamma\beta)}Y^{[\gamma\beta]} \quad \text{[by antisymmetry of the upper two indices]} \tag{3}$$
$$= -X_{(\beta\gamma)}Y^{[\beta\gamma]} \quad \text{[renaming dummy indices]}. \tag{4}$$

Hence $T_{\alpha(\beta}T_{\gamma)}{}^{\alpha}T^{[\beta\gamma]} = 0$.

Exercise 2: Covariant derivatives

(i) *Consider two covariant derivatives ${}_A\nabla$ and ${}_B\nabla$, with associated connection coefficients ${}_A\Gamma^{\mu}_{\nu\sigma}$ and ${}_B\Gamma^{\mu}_{\nu\sigma}$. Prove that the difference ${}_A\Gamma^{\mu}_{\nu\sigma} - {}_B\Gamma^{\mu}_{\nu\sigma}$ transforms as a tensor, even though neither connection separately is a tensor.*

This simply follows from the cancellation of the non-tensor part (the second derivatives of coordinates) when taking the difference.

(ii) *In class we explicitly derived the expression for the covariant derivative of a dual vector field, starting from the covariant derivative of a vector field. With the same procedure, derive the explicit expression for the components of the covariant derivative of a rank-(0, 2) tensor field, $\nabla_\mu T_{\nu\sigma}$, in a coordinate basis. Do the same thing for a rank-$(1,1)$ tensor field, i.e. derive the explicit expression for $\nabla_\mu T^{\nu}{}_{\sigma}$ in a coordinate basis.*

Let us pick two arbitrary vector fields X^α and Y^β. By definiton of a rank (0, 2) tensor field, the quantity $T(X,Y) = T_{\mu\nu}X^\mu Y^\nu$ is a scalar field. Therefore, we have

$$\nabla_\lambda(T_{\mu\nu}X^\mu Y^\nu) = \frac{\partial}{\partial x^\lambda}(T_{\mu\nu}X^\mu Y^\nu) = \frac{\partial T_{\mu\nu}}{\partial x^\lambda}X^\mu Y^\nu + T_{\mu\nu}\left(\frac{\partial X^\mu}{\partial x^\lambda}Y^\nu + X^\mu \frac{\partial Y^\nu}{\partial x^\lambda}\right), \tag{5}$$

where in the second equality we just used Leibniz' rule for partial derivatives. On the other hand, using Leibniz rule for the covariant derivative, we have

$$\begin{aligned}\nabla_\lambda(T_{\mu\nu}X^\mu Y^\nu) &= (\nabla_\lambda T_{\mu\nu})X^\mu Y^\nu + T_{\mu\nu}\left((\nabla_\lambda X^\mu)Y^\nu + X^\mu(\nabla_\lambda Y^\nu)\right)\\ &= (\nabla_\lambda T_{\mu\nu})X^\mu Y^\nu + T_{\mu\nu}\left(\frac{\partial X^\mu}{\partial x^\lambda}Y^\nu + \Gamma^{\mu}{}_{\lambda\sigma}X^\sigma Y^\nu + X^\mu\frac{\partial Y^\nu}{\partial x^\lambda} + X^\mu \Gamma^{\nu}{}_{\lambda\sigma}Y^\sigma\right).\end{aligned} \tag{6}$$

Equating the two, we see that partial derivatives of X and Y drop out, and we get

$$(\nabla_\lambda T_{\mu\nu})X^\mu Y^\nu + T_{\mu\nu}\left(\Gamma^{\mu}{}_{\lambda\sigma}X^\sigma Y^\nu + X^\mu\Gamma^{\nu}{}_{\lambda\sigma}Y^\sigma\right) - \frac{\partial T_{\mu\nu}}{\partial x^\lambda}X^\mu Y^\nu = 0. \tag{7}$$

Renaming dummy indices, we may factorize $X^\mu Y^\nu$. Since the equality holds for any vectors X, Y, we conclude that the quantity multiplying it vanishes, i.e.

$$\nabla_\lambda T_{\mu\nu} = \frac{\partial T_{\mu\nu}}{\partial x^\lambda} - \Gamma^{\sigma}{}_{\lambda\mu}T_{\sigma\nu} - \Gamma^{\sigma}{}_{\lambda\nu}T_{\mu\sigma}. \tag{8}$$

The proof for a rank-(1, 1) tensor is essentially identical.

(iii) *A metric-compatible covariant derivative satisfies $\nabla_\alpha g_{\beta\gamma} = 0$. Assuming the covariant derivative is moreover torsion-free, prove that the connection coefficients must be the Christoffel symbols.*

Applying the above expression to the metric tensor gives

$$0 = \nabla_\lambda g_{\mu\nu} = \frac{\partial g_{\mu\nu}}{\partial x^\lambda} - \Gamma^{\sigma}{}_{\lambda\mu}g_{\sigma\nu} - \Gamma^{\sigma}{}_{\lambda\nu}g_{\mu\sigma} \quad \Rightarrow \frac{\partial g_{\mu\nu}}{\partial x^\lambda} = \Gamma^{\sigma}{}_{\lambda\mu}g_{\sigma\nu} + \Gamma^{\sigma}{}_{\lambda\nu}g_{\sigma\mu}, \tag{9}$$

where we used the symmetry of the metric tensor in the last expression. Guided by the expression for the Christoffel connection, let us compute the sum

$$\begin{aligned}\frac{\partial g_{\mu\lambda}}{\partial x^\nu} + \frac{\partial g_{\lambda\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\lambda} &= \Gamma^{\sigma}{}_{\nu\mu}g_{\sigma\lambda} + \Gamma^{\sigma}{}_{\nu\lambda}g_{\sigma\mu} + \Gamma^{\sigma}{}_{\mu\nu}g_{\sigma\lambda} + \Gamma^{\sigma}{}_{\mu\lambda}g_{\sigma\nu} - \Gamma^{\sigma}{}_{\lambda\mu}g_{\sigma\nu} - \Gamma^{\sigma}{}_{\lambda\nu}g_{\sigma\mu}\\ &= \left(\Gamma^{\sigma}{}_{\nu\mu} + \Gamma^{\sigma}{}_{\mu\nu}\right)g_{\sigma\lambda} + \left(\Gamma^{\sigma}{}_{\nu\lambda} - \Gamma^{\sigma}{}_{\lambda\nu}\right)g_{\sigma\mu} + \left(\Gamma^{\sigma}{}_{\mu\lambda} - \Gamma^{\sigma}{}_{\lambda\mu}\right)g_{\sigma\nu}\\ &= 2\Gamma^{\sigma}{}_{(\mu\nu)}g_{\sigma\lambda} + 2\Gamma^{\sigma}{}_{[\nu\lambda]}g_{\sigma\mu} + 2\Gamma^{\sigma}{}_{[\mu\lambda]}g_{\sigma\nu} = 2\Gamma^{\sigma}{}_{(\mu\nu)}g_{\sigma\lambda} = 2\Gamma^{\sigma}{}_{\mu\nu}g_{\sigma\lambda},\end{aligned} \tag{10}$$

since we assumed the torsion vanishes. Multiplying by $\frac{1}{2}\, g^{\delta\lambda}$, we then find

$$\Gamma^{\delta}{}_{\mu\nu} = \frac{1}{2}g^{\delta\lambda}\left(\frac{\partial g_{\lambda\mu}}{\partial x^\nu} + \frac{\partial g_{\lambda\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\lambda}\right), \tag{11}$$

which is indeed the Christoffel symbol.

Exercise 3: Non-relativistic, perturbed geodesics

(i) Suppose a particle moves on a geodesic at low velocity, i.e. such that, in the coordinates $\{x^\mu\}$, $v^i = dx^i/dt \ll 1$, *where as usual* $t \equiv x^0$. *Compute the* coordinate *acceleration, i.e.* d^2x^i/dt^2, at second order in the velocity, *i.e. neglecting terms of order* v^3 *and higher. Specifically, find the coefficients* A^i, B^i_j *and* C^i_{jk} *such that*

$$\frac{d^2x^i}{dt^2} = A^i + B^i_j\ v^j + C^i_{jk}\ v^j v^k + \mathcal{O}(v^3). \tag{12}$$

For question (i), keep the Christoffel symbols when they appear, to keep compact expressions (i.e. do not write their explicit expression in terms of the metric). Note that it is d^2x^i/dt^2 *that we want to compute here,* not $d^2x^i/d\tau^2$.

Chain rule!

$$\frac{d^2x^i}{dt^2} = (dt/d\tau)^{-1}\frac{d}{d\tau}\left((dt/d\tau)^{-1}\frac{dx^i}{d\tau}\right) = (dt/d\tau)^{-2}\frac{d^2x^i}{d\tau^2} - (dt/d\tau)^{-3}\frac{dx^i}{d\tau}\frac{d^2t}{d\tau^2}. \tag{13}$$

Now let's recall that $dx^i/d\tau = (dt/d\tau)dx^i/dt = (dt/d\tau)v^i$, so we find

$$\frac{d^2x^i}{dt^2} = (dt/d\tau)^{-2}\left[\frac{d^2x^i}{d\tau^2} - v^i\frac{d^2t}{d\tau^2}\right]. \tag{14}$$

We can now plug in the geodesic equation:

$$\begin{aligned}\frac{d^2x^i}{dt^2} &= -(dt/d\tau)^{-2}\left[(\Gamma^i_{00} - v^i\Gamma^0_{00})(dt/d\tau)^2 + 2(\Gamma^i_{0j} - v^i\Gamma^0_{0j})\frac{dt}{d\tau}\frac{dx^j}{d\tau} + (\Gamma^i_{jk} - v^i\Gamma^0_{jk})\frac{dx^j}{d\tau}\frac{dx^k}{d\tau}\right]\\ &= -\left[(\Gamma^i_{00} - v^i\Gamma^0_{00}) + 2(\Gamma^i_{0j} - v^i\Gamma^0_{0j})v^j + (\Gamma^i_{jk} - v^i\Gamma^0_{jk})v^jv^k\right].\end{aligned} \tag{15}$$

So far this is an *exact expression.* We can put it in the desired form by defining

$$\boxed{A^i \equiv -\Gamma^i_{00}, \quad B^i_j \equiv \delta^i_j\Gamma^0_{00} - 2\Gamma^i_{0j}, \quad C^i_{jk} = 2\delta^i_{(j}\Gamma^0_{k)0} - \Gamma^i_{jk}}. \tag{16}$$

Note that I purposefully wrote the last term in a clearly symmetric way.

Now suppose that moreover, the metric components are nearly Minkowski in this coordinate system, i.e. $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, *where* $|h_{\mu\nu}| \ll 1$. *You can expect that the inverse metric components will also be close to Minkowski:* $g^{\mu\nu} = \eta^{\mu\nu} + k^{\mu\nu}$, *with* $|k^{\mu\nu}| \ll 1$.
(ii) Compute $k^{\mu\nu}$ *explicitly in terms of* $h_{\mu\nu}$, *at first order (i.e. neglecting terms quadratic in* $h_{\mu\nu}$*).*

We enforce that $g^{\mu\nu}g_{\nu\sigma} = \delta^\mu_\sigma$ and expand to linear order in perturbations:

$$\delta^\mu_\sigma = (\eta^{\mu\nu} + k^{\mu\nu})(\eta_{\nu\sigma} + h_{\nu\sigma}) = \eta^{\mu\nu}\eta_{\nu\sigma} + \eta^{\mu\nu}h_{\nu\sigma} + k^{\mu\nu}\eta_{\nu\sigma} + \mathcal{O}(h^2). \tag{17}$$

The first term on the right-hand-side is δ^μ_σ, thus we find

$$k^{\mu\nu}\eta_{\nu\sigma} = -\eta^{\mu\nu}h_{\nu\sigma}. \tag{18}$$

Multiplying on the right by the inverse-Minkowski metric, we find

$$\boxed{k^{\mu\lambda} = -\eta^{\mu\nu}h_{\nu\sigma}\eta^{\sigma\lambda}}. \tag{19}$$

Important: $h_{\mu\nu}$ **is not a tensor.** Only the full metric $g_{\mu\nu}$ is a general tensor, and $\eta_{\mu\nu}$ is just a symbol, not a tensor (it only behaves as a tensor for Lorentz transformations). Thus, $h^{\mu\nu}$ **is meaningless**: we only defined $h_{\mu\nu} \equiv g_{\mu\nu} - \eta_{\mu\nu}$, and it is **only meaningful to raise indices of tensors**.

(iii) Compute the Christoffel symbols at second order *in* $h_{\mu\nu}$, *i.e. neglecting terms cubic in* $h_{\mu\nu}$.

The partial derivatives in the Christoffel symbols only apply to the perturbation, so we just need to include terms linear in $h_{\mu\nu}$ in the inverse metric:

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2}(\eta^{\lambda\sigma} - \eta^{\lambda\rho}h_{\rho\delta}\eta^{\delta\sigma})\left(h_{\mu\sigma,\nu} + h_{\nu\sigma,\mu} - h_{\mu\nu,\sigma}\right). \tag{20}$$

(iv) Write explicit expressions of A^i, B^i_j and C^i_{jk} in terms of $h_{\mu\nu}$, at first order *in $h_{\mu\nu}$ only (i.e. neglecting terms quadratic in $h_{\mu\nu}$ and higher). You will thus* not *need the full expression for the Christoffel symbols derived in (iii).*

At first order in $h_{\mu\nu}$, we just need the first term in the first parenthesis. We can then susbtitute $\eta^{i\sigma} = \delta^{i\sigma}$ and $\eta^{0\lambda} = -\delta^{0\lambda}$, and find

$$A^i = \frac{1}{2} h_{00,i} - h_{0i,0}, \tag{21}$$

$$B^i_j = -\frac{1}{2} h_{00,0}\ \delta^i_j - h_{ij,0} + 2 h_{0[j,i]}, \tag{22}$$

$$C^i_{jk} = -h_{00,(j}\ \delta^i_{k)} + \frac{1}{2} h_{jk,i} - h_{i(j,k)}. \tag{23}$$

Note the use of symmetrization/ antisymmetrization brackets to keep compact expressions.