Physics Assignment-2.

MS21134
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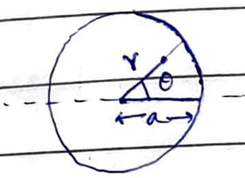

In polar co-ordinates, the wave equation becomes,

$$\partial_t^2 F = v^2 \nabla_r^2 F$$

Where, $\nabla_r^2 = \partial_r^2 + \frac{\partial_r}{r} + \frac{\partial_\theta^2}{r^2}$; $F(r,\theta,t)$.

The boundary conditions are :- $F(a,\theta,t) = 0$
$F(r,0,t) = F(r,2\pi,t)$.
↳ In general, $F(r,\theta+2\pi,t) = F(r,\theta,t)$.

For standing waves, the wave eqⁿ. $F(r,\theta,t)$ is separable.
$F(r,\theta,t) = \cancel{R(r)\Theta(\theta)f(t)}\ g(r,\theta)f(t)$.

$$\frac{\partial_t^2 (g(r,\theta)f(t))}{v^2} = \partial_r^2(g(r,\theta)f(t)) + \frac{\partial_r(g(r,\theta)f(t))}{r} + \frac{\partial_\theta^2 g(r,\theta)f(t)}{r^2}.$$

$$\Rightarrow g\frac{\partial_t^2 f}{v^2} = f\,\partial_r^2 g + f\frac{\partial_r g}{r} + f\frac{\partial_\theta^2 g}{r^2}$$

$$\Rightarrow \frac{\partial_t^2 f}{v^2 f} = \frac{\partial_r^2 g}{g} + \frac{\partial_r g}{rg} + \frac{\partial_\theta^2 g}{r^2 g}$$

Since the L.H.S & R.H.S expressions have no dependance on each other's variables, & both must be equal to a constⁿ.

$$\frac{\partial_t^2 f}{v^2 f} = -\lambda^2 \quad \& \quad \frac{\partial_r^2 g}{g} + \frac{\partial_r g}{rg} + \frac{\partial_\theta^2 g}{r^2 g} = -\lambda^2 \quad (\lambda \in \mathbb{R}) \qquad (1)$$

$$\Rightarrow \partial_t^2 f = -\omega^2 f. \quad (\lambda^2 v^2 = \omega^2)$$

$$\Rightarrow \boxed{f(t) = A\cos\omega t + B\sin\omega t}$$

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Eq^n ①

$$\partial_r^2 g + r\partial_r^2 g + \partial_\theta^2 g = -\lambda^2 r^2 g$$

$$\Rightarrow \left(\partial_r^2 g + r\partial_r^2 g + \lambda^2 r^2 g\right) = -\partial_\theta^2 g$$

We see that $g(r,\theta)$ is further separable, Let $g(r,\theta) = R(r)\Psi(\theta)$.

$$\Psi\,\partial_r^2 R + \Psi r\,\partial_r R + \lambda^2 r^2 R\Psi = -R\,\partial_\theta^2\Psi$$

$$\Rightarrow \frac{\partial_r^2 R}{R} + \frac{r\,\partial_r R}{R} + \frac{\lambda^2 r^2 R}{R} = -\frac{\partial_\theta^2 \Psi}{\Psi} = m^2 . \quad (m \in \mathbb{R})$$

We get,

$$\partial_\theta^2 \Psi = -m^2 \Psi \quad \& \quad \partial_r^2 R + r\,\partial_r R + (\lambda^2 r^2 - m^2) R = 0 \quad ②$$

$$\Rightarrow \boxed{\Psi(\theta) = A_2 \cos(m\theta) + B_2 \sin(m\theta)}$$

$*$ $\Psi(\theta + 2\pi) = \Psi(\theta) \Rightarrow \cos(m\theta + 2m\pi) = \cos(m\theta)$.

$\Rightarrow \boxed{m \in \mathbb{Z}}$

Eq^n ②,

$$\partial_r^2 R + r\partial_r^2 R + (\lambda^2 r^2 - m^2) R = 0 .$$

This is Bessel's equation of order 'm'.

The solutions to the above equation is given by the Bessel functions,

$$R_m(r) = c_1 J_m(\lambda r) + c_2 Y_m(\lambda r) .$$

The second Bessel function $Y_m(r)$ satisfies the limit,

$$\lim_{r\to 0^+} Y_m(r) = -\infty .$$

But we must have $\lim_{r\to 0^+} R(r) < \infty$. $\therefore \boxed{c_2 = 0}$.

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$\therefore R_m(r) = J_m(\lambda r)$

Such that $R_m(a) = 0 \Rightarrow \boxed{J_m(\lambda a) = 0}$.

Let b_{nm} be the n^{th} solution to $J_m(x) = 0$.

$\therefore \lambda_{nm} a = b_{nm}$

$\Rightarrow \boxed{\lambda_{nm} = \frac{b_{nm}}{a}}$

$\therefore$ solution to the wave eq^n, ~~Ignoring~~

$F_{nm}(r,\theta,t) = A J_m(\lambda_{nm} r) \cos(m\theta) \sin(\omega_{nm} t.)$ $\quad (\omega_{nm} = v\lambda_{nm})$.

$\Rightarrow F_{nm}(r,\theta,t) = A\, J_m\left(\frac{b_{nm} r}{a}\right) \cos(m\theta) \sin(\omega_{nm} t)$.

$F(r,\theta,t)$ is the displacement along z-direct^n, $\therefore \boxed{F_{nm}(r,\theta,t) = Z_{nm}(r,\theta,t)}$

Kinetic energy of an infinitesimal patch is,

$dK = \rho \dot{z}^2\, dA$, $\quad dA = r\,dr\,d\theta$

$\therefore K = \int_{r_1}^{r_2} \int_0^{2\pi} \frac{1}{2} \rho \dot{z}^2\, r\,dr\,d\theta$ is the K.E in annular region r_1 to r_2.

$\Rightarrow K = \frac{\rho}{2} \int_{r_1}^{r_2} \int_0^{2\pi} A^2 \omega_{nm}^2\, J_m^2(b_{nm} r/a) \cos^2(m\theta) \cos^2(\omega_{nm} t)\, r\,dr\,d\theta$

$= \frac{A^2 \rho\, \omega_{nm}^2 \cos^2(\omega_{nm} t)}{2} \int_{r_1}^{r_2} r^2 J_m^2(b_{nm} r/a)\,dr \cdot \int_0^{2\pi} \cos^2(m\theta)\,d\theta$.

$\Rightarrow \boxed{K = \frac{\rho A^2 \omega_{nm}^2 \cos^2(\omega_{nm} t)}{4} \int_{r_1}^{r_2} r^2 J_m^2(b_{nm} r/a)\,dr}$

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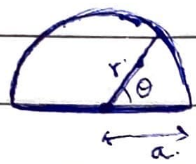

The equat^n. of motion that a wave satisfies here is,

$$\rho \ddot{z}(r,\theta,t) = S \nabla_r^2 z(r,\theta,t).$$

$\rho \to$ mass density $\quad S \to$ surface tension.

Where $z(r,\theta,t)$ can be written as a product of separable solutions,

$$\boxed{z(r,\theta,t) = R(r)\,\psi(\theta)\,f(t)}$$

$$f(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t).$$
$$\psi(\theta) = c_1 \cos(m\theta) + c_2 \sin(m\theta).$$

$$\psi(\theta) = \psi(\theta + 2\pi)$$

$\Rightarrow$ $\cos(m\theta) = \cos(m\theta + 2m\pi) \Rightarrow \boxed{m \in \mathbb{Z}}$

Boundary conditions for D-shaped membrane are,

$$z(a,\theta,t) = 0 \;;\; z(r, \pi \leq \theta \leq 2\pi, t) = 0.$$

$R(r) = J_m(\lambda r)$. Given $R(a) = 0$, we have.

$$J_m(\lambda a) = 0.$$

Let b_{nm} be solutions to $J_m(x) = 0$.

Then, $\boxed{\lambda_{nm} = b_{nm}/a}$

Also, $\psi(\pi) = 0$ & $\psi(0) = 0$.

$\Rightarrow \boxed{\psi(\theta) = \sin(m\theta)}$ (Normal mode conditions).

Soln. $\therefore$ $\boxed{z_{nm}(r,\theta,t) = A J_m(b_{nm} r/a) \sin(m\theta) \cos(\omega_{nm} t)}$

are the normal modes for this setup.

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(3) $y(x,0) = x\sin(\pi x/L)$

$$y(x,0) = \sum_n c_n \sin\left(\frac{n\pi x}{L}\right) \qquad (n \in \mathbb{Z}^+)$$

$$\Rightarrow \quad c_n = \int_0^L \sin\left(\frac{n\pi x}{L}\right) x \sin\left(\frac{\pi x}{L}\right) dx.$$

$$= \int_0^L x \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{\pi x}{L}\right) dx.$$

We know, $\displaystyle\int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right) dx = \frac{L}{2}\delta_{nm}.$

$$\int \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{\pi x}{L}\right) dx = \int \cos\left(\frac{(n+1)\pi x}{L}\right) - \cos\left(\frac{(n-1)\pi x}{L}\right) dx$$

$$= \frac{\sin\left(\frac{(n+1)\pi x}{L}\right)}{(n+1)\pi/L} - \frac{\sin\left(\frac{(n-1)\pi x}{L}\right)}{(n-1)\pi/L}$$

$$c_n = \left[x\int \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{\pi x}{L}\right)dx\right]_0^L - \int_0^L \frac{\sin\left(\frac{(n+1)\pi x}{L}\right)}{(n+1)\pi/L} - \frac{\sin\left(\frac{(n-1)\pi x}{L}\right)}{(n-1)\pi/L}\, dx.$$

$$= -\frac{L^2}{\pi^2}\left[\frac{-\cos((n+1)\pi x/L)\Big|_0^L}{(n+1)^2} + \frac{\cos((n-1)\pi x/L)\Big|_0^L}{(n-1)^2}\right]$$

$$= -\frac{L^2}{\pi^2}\left[\frac{1-\cos((n+1)\pi)}{(n+1)^2} + \frac{\cos((n-1)\pi) - 1}{(n-1)^2}\right]$$

For ~~even~~ odd n,

$\boxed{c_n = 0} \quad (n \neq 1)$

For even n,

$$c_n = -\frac{L^2}{\pi^2}\left[\frac{2}{(n+1)^2} - \frac{2}{(n-1)^2}\right] = \frac{2L^2}{\pi^2} \times \frac{1}{(n^2-1)^2} \times \left[\cancel{n^2} + \cancel{1} + 2n - \cancel{n^2} - \cancel{1} + 2n\right]$$

$$\Rightarrow \boxed{c_n = \frac{8L^2 n}{\pi^2(n^2-1)^2}}$$

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For $n = 1$,

$$c_1 = \int x \sin^2(\pi x/L)\, dx.$$

$$= \left[\frac{x}{2}\left(1 - \frac{\sin\left(\frac{2\pi x}{L}\right)}{(2\pi/L)}\right) \right]_0^L - \frac{1}{2}\int_0^L x - \frac{\sin(2\pi x/L)}{2\pi/L}\, dx.$$

$$= \frac{L^2}{2} - \frac{1}{2}\left[\frac{x^2}{2} + \frac{\cos(2\pi x/L)}{(2\pi/L)^2} \right]_0^L$$

$$= \frac{L^2}{2} - \frac{1}{2}\left[\frac{L^2}{2} + 0\right] = L^2/4$$

$$\Rightarrow \boxed{c_1 = L^2/4}$$

$$\therefore \boxed{c_1 = L^2/4, \quad c_{n=3,5,7,\ldots} = 0, \quad c_{n=2,4,6,\ldots} = \frac{8L^2 n}{\pi^2 (n^2-1)^2}}$$

The wave at any time t can be given as,

~~$\phi(x,t) = a \sin(2\pi x/L) \cos(\omega t + \phi)$~~

$$\boxed{y(x,t) = \sum_n c_n \sin(n\pi x/L) \cos(\omega_n t + \phi)}$$

where c_n's are given as above & $\boxed{\omega_n = \sqrt{\frac{T}{\rho}}\, \frac{n\pi}{L}}$

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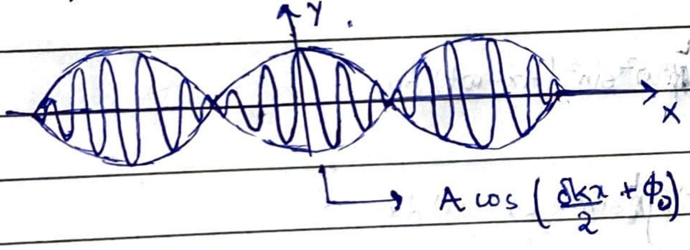
(4) Let the two waves be,

$$y_1 = A\cos(kx-\omega t) \;\&\; y_2 = A\cos((k+\delta k)x - (\omega+\delta\omega)t)$$

$$y = y_1 + y_2 = A\left[\cos(kx-\omega t) + \cos((k+\delta k)x - (\omega+\delta k)t)\right]$$

$$= \left[A\cos\left(\left(k+\frac{\delta k}{2}\right)x - \left(\omega+\frac{\delta\omega}{2}\right)t\right)\right]\cos\left(\frac{\delta k x}{2} - \frac{\delta\omega t}{2}\right)$$

At any time t, $y(x,t)$ has the following profile,



We have $\delta k/2 < (k+\delta k)/2$. The factor with lower wavenumber envelopes the whole wave profile.

$\therefore$ the group velocity $v_{group} = \dfrac{\delta\omega/2}{\delta k/2} = \delta\omega/\delta k$

$$\Rightarrow \boxed{v_{group} = \delta\omega/\delta k}$$

Let y_1 & y_2 have a relative phase ϕ,

$$y = A\cos(kx-\omega t+\phi) + A\cos((k+\delta k)x - (\omega+\delta\omega)t)$$

$$= \left[A\cos\left(\left(k+\frac{\delta k}{2}\right)x - \left(\omega+\frac{\delta\omega}{2}\right)t + \phi/2\right)\right]\cos\left(\frac{\delta k x}{2} - \frac{\delta\omega t}{2} - \phi/2\right)$$

y now has an envelope of the form $A\cos\left(\frac{\delta k x}{2} - \frac{\delta\omega t}{2} - \phi/2\right)$

which moves with speed, $\boxed{v_{group} = \delta\omega/\delta k}$

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Hence the answer remains same.

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(5) Eq^n. for a longitudinal wave in a cylinder is,

$$\partial_t^2 \psi(x,t) = \frac{1}{\rho K} \partial_x^2 \psi(x,t).$$

Sol^n. for a right-travelling wave is,

$$\psi(x,t) = \psi_0 \cos(kx-\omega t).$$

where, $\boxed{\omega^2 = \frac{k^2}{\rho K}}$

Kinetic energy, $T = \int_0^L \frac{\rho A \dot{\psi}^2}{2} dx.$

$$= \frac{\rho A \psi_0^2}{2} \int_0^L \omega^2 \sin^2(kx-\omega t)\, dx.$$

$$= \frac{\rho A \omega^2 \psi_0^2}{2} \int_0^L \sin^2(kx-\omega t)\, dx$$

$$= \frac{\rho A \omega^2 \psi_0^2}{2} \times \left(\frac{L}{2} - \left[\frac{\cos(2kL - 2\omega t)}{2k} - \frac{\cos(2\omega t)}{2k} \right] \right).$$

$$\Rightarrow T = \frac{\rho A \omega^2 \psi_0^2}{4} \left(L - \frac{1}{k} \left[\cos(2\omega t - 2kL) - \cos(2\omega t) \right] \right)$$

$$\Rightarrow \boxed{T = \frac{\rho A \omega^2 \psi_0^2}{4} \left(L - \frac{2}{k} \left[\cos(2\omega t - kL) \cos(kL) \right] \right).}$$

(6) Acceleration of an infinitesimal segment is,

$$a(x,t) = \ddot{y}(x,t).$$

For a travelling wave, $y(x,t) = A\cos(kx - \omega t + \phi).$

$$\ddot{y}(x,t) = -\omega^2 A \cos(kx - \omega t + \phi).$$

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$$\langle a(x,t)\rangle = \frac{\int_0^{2\pi/\omega} \ddot{y}(x,t)\,dt}{2\pi/\omega} = \frac{-\omega^2 A}{2\pi/\omega}\int_0^{2\pi/\omega} \cos(kx-\omega t+\phi)\,dt$$

$$= \frac{+\omega^3 A}{2\pi}\,\frac{\sin(kx-\omega t+\phi)}{\cancel{\alpha x+\phi}}\Big|_0^{2\pi/\omega}$$

$$= \frac{A\omega^2}{2\pi}\left[\sin(kx+\phi-2\pi) - \sin(kx+\phi)\right]$$

$$= 0$$

$$\langle a^2(x,t)\rangle = \frac{\int_0^{2\pi/\omega} \ddot{y}^2(x,t)\,dt}{2\pi/\omega} = \frac{A^2\omega^4}{2\pi/\omega}\int_0^{2\pi/\omega} \cos^2(kx-\omega t+\phi)\,dt$$

$$= \frac{A^2\omega^4}{2\pi/\omega}\times\frac{1}{2}\left[\cancel{t}+\cos(2kx-2\omega t+2\phi)\right]_0^{2\pi/\omega}$$

$$= \frac{A^2\omega^4}{2}\times\frac{2\pi/\omega}{2\pi/\omega} = \frac{A^2\omega^4}{2}$$

Variance (σ_a^2) $= \langle a^2(x,t)\rangle - \langle a(x,t)\rangle^2$

$= \frac{A^2\omega^4}{2}$

$\Rightarrow$ $\boxed{\sigma_a^2 = A^2\omega^4/2}$

(7) $y(x,t) = \sum_n c_n \sin(k_n x)\cos(\omega_n t+\phi)$.

where $k_n = \frac{n\pi}{l}$.

$$\int_0^{2l/v} \cos(\omega_m t+\phi)\cos(\omega_n t+\phi)\,dt = \frac{1}{2}\int_0^{2l/v} \cos((\omega_n+\omega_m)t+2\phi) + \cos((\omega_n-\omega_m)t)\,dt$$

$\left(v = \frac{\omega_1 l}{\pi}\right)$

$$= \frac{1}{2}\left[\frac{\sin((\omega_n+\omega_m)t+2\phi)}{(\omega_n+\omega_m)}\Big|_0^{2l/v} + \frac{\sin((\omega_n-\omega_m)t)}{(\omega_n-\omega_m)}\Big|_0^{2l/v}\right]$$

$= 0 \quad (n\neq m)$

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For $n=m$,

$$\int_0^{2l/v} \cos^2(\omega_n t+\phi)\,dt = \frac{1}{2}\times\frac{2l}{v} = \frac{l}{v}$$

$$\therefore \boxed{\int_0^{2l/v} \cos(\omega_m t+\phi)\cos(\omega_n t+\phi)\,dt = \frac{l}{v}\delta_{nm}}$$

$$y(x,t) = \sum_n c_n \sin(k_n x)\cos(\omega_n t+\phi)$$

$$y\left(\frac{l}{2m},t\right) = \sum_n c_n \sin\left(\frac{n\pi}{\cancel{l}}\times\frac{\cancel{l}}{2m}\right)\cos(\omega_n t+\phi)$$

$$= \sum_n c_n \sin\left(\frac{n\pi}{2m}\right)\cos(\omega_n t+\phi)$$

$$\int_0^{2l/v} \cos(\omega_m t+\phi)\,y(l/2m,t)\,dt = \int_0^{2l/v} \cos(\omega_m t+\phi)\left(\sum_n c_n \sin\left(\frac{n\pi}{2m}\right)\cos(\omega_n t+\phi)\right)dt$$

We proved, $\int_0^{2l/v} \cos(\omega_m t+\phi)\cos(\omega_n t+\phi)\,dt = \frac{l}{v}\delta_{nm}$.

$$\therefore \int_0^{2l/v} \cos(\omega_m t+\phi)\,y(l/2m,t)\,dt = \frac{l}{v}c_m \sin\left(\frac{\cancel{m}\pi}{2\cancel{m}}\right)$$

$$\Rightarrow \boxed{c_m = \frac{v}{l}\int_0^{2l/v} \cos(\omega_m t+\phi)\,y(l/2m,t)\,dt}$$

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(8) $y_1 = A_1 \sin(kx - \omega t)$ $\quad y_2 = A_2 \sin(kx + \omega t)$

$$y = y_1 + y_2 = A_1 \sin(kx - \omega t) + A_2 \sin(kx + \omega t)$$

For extremum $\partial y/\partial x = 0$,

$\Rightarrow \cancel{k}\left(A_1 \cos(kx - \omega t) + A_2 \cos(kx + \omega t) \right) = 0$

$\Rightarrow A_1 \cos kx \cos \omega t + A_1 \sin kx \sin \omega t + A_2 (\cos kx \cos \omega t - \sin kx \sin \omega t) = 0$

$\Rightarrow (A_1 + A_2) \cos kx \cos \omega t = (A_2 - A_1) \sin kx \sin \omega t$

$\Rightarrow$ ~~take~~ $\boxed{\tan kx = \left(\frac{A_2 + A_1}{A_2 - A_1} \right) \cot \omega t}$

The extremum satisfy the above eqn.

Max. amplitude $= A_1 + A_2$

Min. amplitude $= |A_1 - A_2|$

$$\boxed{\text{Ratio} = \left| \frac{A_1 + A_2}{A_2 - A_1} \right|}$$