MTH302: INTEGERS, POLYNOMIALS AND MATRICES
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1. Basic Properties of Ideals

Let R be a ring with 1 and I be a left ideal of R. Then the following are equivalent.

(1) $I = R$.
(2) $1 \in I$.
(3) I contains a unit.
(4) I contains an element which has a left inverse.

The proof is straightforward. A similar equivalent statements can be obtained for right ideals and two-sided ideals.

Let $(x)_l, (x)_r$ and (x) denote the left, right and the two-sided ideals generated by $x \in R$ respectively. Even for a ring R with 1 there may be elements $x \in R$ having no left or right inverses and that the ideal generated by x equals R. To see this, let $R = M_2(\mathbb{R})$ and let

$$x := E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = x + E_{21} \cdot x \cdot E_{12} \ \in (x). \tag{1.1}$$

and thus $(x) = R$. However, since x is a left as well as a right zero-divisor, x has neither a left inverse nor a right inverse.

Theorem 1.1. *For a ring R with 1, the following statements are equivalent.*

(1) R is a division ring.
(2) (0) and R are the only left ideals.
(3) (0) and R are the only right ideals.

Proof. Let R be a division ring and I be a left ideal of R. If $0 \neq x \in I$ then there is a $y \in R$ such that $y \cdot x = 1$ and thus $1 = y \cdot x \in (x)_l$. This implies $(x)_l = R$ and we have shown that (1) $\implies$ (2). Let (2) hold and $0 \neq x \in R$. Since $(x)_l = R$, we have $y \cdot x = 1$ for some $y \in R$ and since $(y)_l = R$, for some $z \in R$, we have $z \cdot y = 1$. It follows that $z = x$ and thus x is a unit. Thus R is a division ring and this proves (2) $\implies$ 1. Use similar arguments to prove $1 \iff 3$. $\square$

Corollary 1.2. *Let R be a commutative ring with 1. Then R is a field if and only if (0) and R are the only ideals of R.*

Remark 1.3. If R is a division ring then clearly, (0) and R are the only two-sided ideals of R. However, the converse is not true. For example, consider $R = M_2(\mathbb{R})$. Then R is not a division ring (why?). However, for any nonzero two-sided ideal I of R and

$$0 \neq x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in I, \quad \text{say } \ c \neq 0,$$

observe that $E_{11} = c^{-1}E_{12} \cdot x \cdot E_{11} \in I$. But we have already noted in Equation 1.1 that if $E_{11} \in I$ then $(x) \supseteq (E_{11}) = R$.

Theorem 1.4. *Let R be a ring with a nontrivial multiplication. That is, $x \cdot y \neq 0$ for some $x, y \in R$. Then R is a division ring if and only if (0) and R are the only left ideals.*

Proof. In view of Theorem 1.1, we only need to show that if (0) and R are the only left ideals then R has 1. Let (0) and R be the only left ideals of R and $x, y \in R$ be such that $x \cdot y \neq 0$. Then $Ry := \{r \cdot y \mid r \in R\}$ is a nonzero left ideal and therefore $Ry = R$. Therefore, there is an element $e \in R$ such that $e \cdot y = y$. Then, $y = e \cdot y = e^2 \cdot y$ and thus $(e^2 - e) \cdot y = 0$.

Let $L(Ann(y)) := \{z \in R \mid z \cdot y = 0\}$ be the set of all *left annihilators* of y. It is easy to see that $L(Ann(y))$ is a left ideal of R. Since $x \cdot y \neq 0$, $L(Ann(y)) \neq R$ and thus $L(Ann(y)) = (0)$. This shows that $e^2 = e$, that is, e is an idempotent.

Let $I = \{a \cdot e - a \mid a \in R\}$. Since $0 = 0 \cdot e - 0 \in I$, we have I is a nonempty subset of R. It is easy to see that I is a left ideal (check). Now since $(a \cdot e - a) \cdot y = a \cdot e \cdot y - a \cdot y = a \cdot y - a \cdot y = 0$, we see that $I \subseteq L(Ann(y)) = (0)$. Thus $a \cdot e = a$ for all $a \in R$ and this proves that e is a right identity of R.

Let $J = \{e \cdot a - a \mid a \in R\}$. It is easy to see that $0 \in J$ and that J is a left ideal of R. Then $J = (0)$ or $J = R$. If $J = R$ then $y = e \cdot a - a$ for some $a \in R$ and we have $x \cdot y = x \cdot (e \cdot a - a) = x \cdot a - x \cdot a = 0$, a contradiction. Thus $J = (0)$ and we obtain $e \cdot a = a$ for all $a \in R$. Hence e is an identity of R. $\square$