1. Let $G = \{1, f_X, f_Y, f_{d_1}, f_{d_2}, r_{\pi/2}, r_\pi, r_{3\pi/2}\}$ be the group of symmetries of a square. Let

$$S = \{(x, y) \in \mathbb{R} \times \mathbb{R} : |x| \leq 1, |y| \leq 1\}$$

It is a square centred at origin, with each side of length 2 units. For $g \in G$ and $P \in S$ let $g.P$ denote the point after operating the symmetry g on P. (Observe that $g.P \in S$, else g would not be a symmetry!).

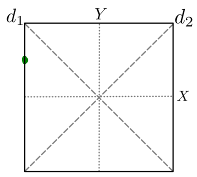
(a) Take some explicit $P \in S$ and $g, h \in G$, and show that $h.(g.P) = (hg).P$.

(b) Fix a point $P = \left(\frac{1}{2}, \frac{1}{3}\right) \in S$. As g varies, mark $g.P$ on the square S. What you have marked is called the *orbit* of P.

(c) Fix a point $P = \left(\frac{1}{2}, \frac{1}{2}\right) \in S$. Mark the orbit of P. Identify all $g \in G$ which do not move P, *i.e.*, find the set $\{g \in G : g.P = P\}$. This set will be called the *stabilizer* of P.

– worth recalling Exercise 04 of Tutorial 02.

(a). Lets take $P = (-1, \frac{1}{2}) \in S$, $g = f_{d_1}$, $h = r_{\pi/2}$



then $g \cdot P = f_{d_1} \cdot (-1, \frac{1}{2}) = (-\frac{1}{2}, 1)$

and $h \cdot (g \cdot P) = r_{\pi/2} \cdot (-\frac{1}{2}, 1) = (-1, -\frac{1}{2})$

Now $hg = r_{\pi/2} f_{d_1} = f_X$.

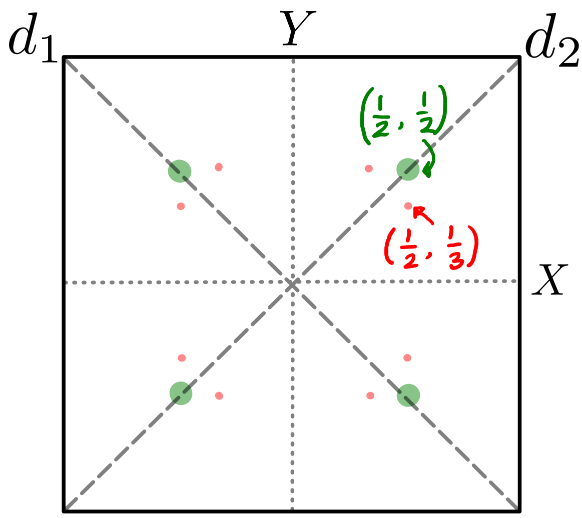
You may use rotation/reflection matrices here.

so $hg \cdot P = f_X \cdot (-1, \frac{1}{2}) = (-1, -\frac{1}{2})$

Thus $h \cdot (g \cdot P) = (-1, -\frac{1}{2}) = hg \cdot P$

(b) and (c).

- Orbit of $(\frac{1}{2}, \frac{1}{2})$
- Orbit of $(\frac{1}{2}, \frac{1}{3})$



For $P = (\frac{1}{2}, \frac{1}{2})$ if $g \cdot P = P$

then $g = 1$ or f_{d_2}

Therefore, stabilizer of P is

$\{1, f_{d_2}\}$.

– Just think: Is there a point on the square whose stabilizer has exactly 4 elements?

2. Take the Klein 4-group $V_4 := \{1, a, b, c\}$. The composition table of this group is given by $a^2 = b^2 = c^2 = 1$, $ab = ba = c$, $ac = ca = b$, $bc = cb = a$. Consider the set

$$S = \{(1,1,1), (1,1,-1), (1,-1,1), (1,-1,-1), (-1,1,1), (-1,1,-1), (-1,-1,1), (-1,-1,-1)\}.$$

For each $P = (x, y, z) \in S$, define

$$1.P := (x,y,z), \quad a.P := (x,-y,-z), \quad b.P := (-x,y,-z), \quad c.P := (-x,-y,z).$$

Show that the association of a $g \in V_4$ and $P \in S$ to $g.P \in S$ as defined above, is a group action.

2.

We are already given that $1.P = P$. We check $hg.P = h.(g.P)$ for other possibilities of $g, h \in V_4$.

h	g	hg	$g.P$	$h.(g.P)$	$hg.P$
a	b	c	$(-x, y, -z)$	$(-x, -y, z)$	$(-x, -y, z)$
b	a	c	$(x, -y, -z)$	$(-x, -y, z)$	$(-x, -y, z)$
a	c	b	$(-x, -y, z)$	$(-x, y, -z)$	$(-x, y, -z)$
c	a	b	$(x, -y, -z)$	$(-x, y, -z)$	$(-x, y, -z)$
b	c	a	$(-x, -y, z)$	$(x, -y, -z)$	$(x, -y, -z)$
c	b	a	$(-x, y, -z)$	$(x, -y, -z)$	$(x, -y, -z)$
a	a	1	$(x, -y, -z)$	(x, y, z)	(x, y, z)
b	b	1	$(-x, y, -z)$	(x, y, z)	(x, y, z)
c	c	1	$(-x, -y, z)$	(x, y, z)	(x, y, z)