Assignment 9

PHY310: Mathematical Methods for Physicists I

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1. Solve the following boundary value problem

$$\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y}$$

given $u(0, y) = 8e^{-3y}$, by the method of separation of variables.

2. Solve by the method of separation of variables

$$\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u(x, t)$$

under the condition $u(x, 0) = 6e^{-3x}$.

3. Suppose the following differential equation refers to a problem of $2d$ steady flow of heat:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0.$$

Solve for $T(x, y)$ with the following boundary conditions: $T(0, y) = 0$; $T(x, \infty) = 0$; $T(a, y) = 0$ and $T(x, 0) = \sin(\pi x/a)$.

4. The edge $r = a$ of a circular plate is kept at a temperature $f(\theta)$. The plate is insulated so that there is no loss of heat from either surface. Find the temperature distribution in the steady state.

5. By letting the operator $\nabla^2 + k^2$ act on the general form $a_1\psi_1(x, y, z) + a_2\psi_2(x, y, z)$, show that it is linear, i.e., that $(\nabla^2+k^2)(a_1\psi_1+a_2\psi_2) = a_1(\nabla^2+k^2)\psi_1+a_2(\nabla^2+k^2)\psi_2$.

6. Show that the Helmholtz equation $\nabla^2\psi + k^2\psi = 0$ is still separable in circular cylindrical coordinates (ρ, ϕ, z) if k^2 is generalized to $k^2 + f(\rho) + (1/\rho^2)g(\phi) + h(z)$.

7. Verify that

$$\nabla^2\psi(r, \theta, \phi) + \left[k^2 + f(r) + \frac{1}{r^2}g(\theta) + \frac{1}{r^2\sin^2\theta}h(\phi)\right]\psi(r, \theta, \phi) = 0$$

is separable in spherical polar coordinates. The functions f, g, and h are functions only of the variables indicated; k^2 is a constant.

8. If Ψ is a solution of Laplace's equation, $\nabla^2\Psi = 0$, show that $\partial\Psi/\partial z$ is also a solution.

9. Solve the wave equation,

$$\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = \frac{\partial^2\psi}{\partial x^2},$$

subject to the indicated conditions.

(a) Determine $\psi(x,t)$ given that at $t = 0$, $\psi_0(x) = \delta(x)$ and the initial time derivative of ψ is zero.

(b) Determine $\psi(x,t)$ given that at $t = 0$, $\psi_0(x)$ is a single square-wave pulse as defined below, and the initial time derivative of ψ is zero.

$$\psi_0(x) = 0, \quad |x| > a/2 \quad \text{and} \quad \psi_0(x) = 1/a, \quad |x| < a/2.$$

10. For a homoheneous spherical solid with constant thermal diffusivity, K, and no heat sources, the equation of heat conduction becomes

$$\frac{\partial T(r,t)}{\partial t} = K\nabla^2 T(r,t).$$

Assume a solution of the form $T(r,t) = R(r)\Gamma(t)$ and separate variables. Show that the radial equation may take on the standard form

$$r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} + \alpha^2 r^2 R = 0,$$

and that $\sin(\alpha r)/r$ and $\cos(\alpha r)/r$ are its solutions.

11. A semi-infinite rectangular metal plate occupies the region $0 \le x \le \infty$ and $0 \le y \le b$ in the xy-plane. The temperature at the far end of the plate and along its two long sides is fixed at 0°C. If the temperature of the plate at $x = 0$ is also fixed and is given by $f(y)$, find the steady-state temperature distribution $u(x,y)$ of the plate. Hence find the temperature distribution if $f(y) = u_0$, where u_0 is a constant.

12. Solve the PDE

$$\frac{\partial \psi(x,t)}{\partial t} = a^2\frac{\partial^2 \psi(x,t)}{\partial x^2},$$

to obtain $\psi(x,t)$ for a rod of infinite extent (in both $+x$ and $-x$ directions), with a heat pulse at time $t = 0$ that corresponds to $\psi_0(x) = A\delta(x)$.

13. A bar of length L is initially at a temperature of 0°C. One end of the bar ($x = 0$) is held at 0°C and the other is supplied with heat at a constant rate per unit area of H. Find the temperature distribution within the bar after a time t.