QCQI PH631 August-December 2024: Assignment 2
Given: September 1 2024 Due: September 6, 2024

Prof. Arvind, IISERM, arvind@quantumphys.org

1. For the state given in part c problem 4 of Assignment 1, compute the reduced density matrix for the state $|\alpha\rangle$. Can the mesurement outcomes of A be determined using these reduced density matrices?

2. We learnt in the class that quantum states are represented by density operators ρ:
$$\rho^\dagger = \rho;\ \rho \geq 0;\ \mathrm{Tr}(\rho) = 1.$$
For a pure state where the density operator can be written as $\rho = |\alpha\rangle\langle\alpha|$ we have the condition that $\rho^2 = \rho$. Show if $\rho^2 = \rho$ it implies that the density operator represents a pure state ie. it can be written as $\rho = |\alpha\rangle\langle\alpha|$.

3. We have seen in the class that all the pure states of a qubit can be represented on the Poincare'-Bloch sphere surface while the mixed states are represented in the interior. Consider a general pure state of a qubit
$$|\alpha(\theta,\phi)\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{i\phi}|1\rangle$$
Pure states find a very compact expression on the Bloch sphere given by
$$\rho = |\alpha\rangle\langle\alpha| = \frac{1}{2}(I + \hat{n}.\sigma)$$
where $\hat{n}$ is the unit vector representing the state $|\alpha\rangle$ on the Bloch sphere and σ represents the three Pauli matrices.

   Join the point representing this state with the origin of the Bloch sphere and consider the family of states from the orgin to the point (θ,ϕ) represented by the distance from the center r.

   (a): Represent this family of states geometrically.

   (b): Can the density matrix corresponding to this family of states be written in a compact form like the one for the pure states?

   (c): Calculate the von Neumann entropy $S\left(\rho(\theta,\phi,r)\right)$.

4. Consider the following two quibt states
$$|B1\rangle = \frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1||1\rangle)$$
$$|B2\rangle = \frac{1}{\sqrt{2}}(|0\rangle|0\rangle - |1||1\rangle)$$
$$|B3\rangle = \frac{1}{\sqrt{2}}(|0\rangle|1\rangle + |1||0\rangle)$$
$$|B4\rangle = \frac{1}{\sqrt{2}}(|0\rangle|1\rangle - |1||0\rangle)$$
For each of the states above

   (a): Calculate the density matrix in the computational basis $\{|0\rangle|0\rangle, |0\rangle|1\rangle, |1\rangle|0\rangle, |1\rangle|1\rangle\}$.

   (b): Calculate the reduced desnsity operator for each qubit by taking the partial trace over the other qubit.

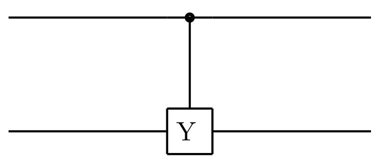
   (c): Calculate $S(\rho_1^{\mathrm{Red}})$ and $S(\rho_2^{\mathrm{Red}})$ in each case.

   (d): Calcualte the partial transpose of density operators corresponding to each case with respect to each of the qubits ie $\rho^{\mathrm{PT}(1)}$ and $\rho^{\mathrm{PT}(2)}$.

   (e): Compute the eigen values of the 'Partially Transposed' density matrices.

5. Consider a general state of two qubits
$$|\alpha\rangle = c_{00}|00\rangle + c_{01}|01\rangle + c_{10}|10\rangle + c_{11}|11\rangle$$
Find the general conditions on the coefficients c_{jk} so that the state is a product state of the form $|\alpha_1\rangle \otimes |\alpha_2\rangle$

6. Find the 4×4 unitary matrix corresponding to the following circuit



7. Compute the circuit corresponding to the following two qubit unitary matrix.
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & 1 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}$$

8. Construct a circuit which can be used to teleport two-qubit states $|\alpha\rangle \otimes |\beta\rangle$, where $|\alpha\rangle$ is the state for the first qubit and $|\beta\rangle$ is the state for second qubit. (Hint: In this case Bob may have to use four qubits) Can this circuit be used to teleport an entangled state of the two qubits? For example, the state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$