IISER Mohali

[Session 2018-19, Even Semester]

PHY 304 (Statistical Mechanics)

Problem Set 04: Quantum Statistics

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1. Write down the symmetric and anti-symmetric combinations of wave functions $\phi_1(x), \phi_2(x)$ and $\phi_3(x)$.

2. Write down the partition function for a system of three identical particles each of which can be in a state with energy 0 or ϵ.

3. Show that in a quantum ideal gas of N particles the classical limit corresponds to $z = e^{\beta\mu} << 1$.

4. Relate canonical and grand canonical partition functions using the relation between the thermodynamic potentials of the two ensembles. Show that in the classical limit, the canonical partition function for a quantum gas of N particles reduces into,
$$Z \approx \frac{Z_{\text{MB}}}{N!}$$

5. Show that the Bose function can be written as
$$\begin{aligned} g_m^-(z) &= \frac{1}{\Gamma(m)} \int_0^\infty dx \, \frac{x^{m-1}}{z^{-1}e^x - 1} \\ &= \sum_{\ell=1}^{\infty} \frac{z^\ell}{\ell^m}. \end{aligned}$$
Further, show that
$$\frac{d}{dz} g_m^-(z) = \frac{1}{z} g_{m-1}^-(z).$$
[**Hint:** For Bose gas $0 < z < 1$.]

6. Plot $g_{1/2}^-(z)$, $g_{3/2}^-(z)$ and $g_{5/2}^-(z)$ numerically.

7. Show that the specific heat at constant volume for an ideal Bose gas follows $C_V \sim T^{3/2}$ below critical temperature, T_c. [**Hint:** Below T_c, $z \approx 1$.]

8. Starting from
$$U = g_s \frac{3}{2} k_B T \frac{V}{\lambda_T^3} g_{5/2}^-(z)$$

for an ideal Bose gas, show that above critical temperature ($T > T_c$)

$$C_V = \frac{15}{4} N k_B \frac{g^-_{5/2}(z)}{g^-_{3/2}(z)} - \frac{9}{4} N k_B \frac{g^-_{3/2}(z)}{g^-_{1/2}(z)}.$$

Plot $C_V/(Nk_B)$ as a function of T/T_c numerically. [**Hint**: Above T_c, $N = g_s \frac{V}{\lambda_T^3} g^-_{3/2}(z)$]

9. Starting from

$$N = g_s \frac{V}{\lambda_T^3} g^+_{3/2}(z)$$

for an ideal Fermi gas, show that at very low temperature ($T \neq 0$) the chemical potential is given by

$$\mu(T) \approx \epsilon_F \Big[1 - \frac{\pi^2}{12}\Big(\frac{T}{T_F}\Big)^2\Big]; \quad T_F = \epsilon_F / k_B.$$

Assuming that conduction electrons in metals can be modelled as ideal Fermi gas, estimate the temperature dependence of C_V at very low temperature ? How is it different from the temperature dependence of C_V for metals due to lattice vibrations ? [**Hint**: Use the low temperature expansion of Fermi function given below.]

Useful expressions:

1. Thermal de Broglie wavelength

$$\lambda_T = \sqrt{\frac{2\pi\hbar^2}{m k_B T}}$$

2. Mean occupation number in FD and BE statistics

$$\langle n_i \rangle = \frac{1}{e^{\beta(\epsilon_i - \mu)} \pm 1}$$

3. Spin degeneracy factor for a massive particle of spin s

$$g_s = (2s+1)$$

4. Low temperature expansion of Fermi function

$$g^+_n(z) = \frac{(\ln z)^n}{\Gamma(n+1)} \Big[1 + \frac{\pi^2}{6} \frac{n(n-1)}{(\ln z)^2} + \ldots\Big]$$
