PHY 310 - Mathematical Methods for Physicists I

Odd Term 2019, IISER Mohali

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Homework 3 - Solutions

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1. Evaluate the following integral involving Bessel function

$$I = \int_0^\infty dt\ e^{-at} J_0(bt), \quad a, b > 0$$

using the integral representation

$$J_0(x) = \frac{1}{\pi}\int_0^\pi d\theta\ \cos(x \sin\theta).$$

Hint: Use the expression

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta \frac{a}{a^2 + b^2 \sin^2\theta} = \frac{1}{\sqrt{a^2+b^2}}.$$

Solution:

Let us plug in the expression for $J_0(x)$ in the integral I. We get

$$I = \int_0^\infty dt\ e^{-at} \frac{1}{\pi}\int_0^\pi d\theta\ \cos(bt \sin\theta).$$

Upon using

$$\int_0^\pi d\theta\ \cos(bt\sin\theta) = \frac{1}{2}\int_{-\pi}^{\pi} d\theta\ \cos(bt\sin\theta)$$

and

$$\int_{-\pi}^{\pi} d\theta\ \sin(bt\sin\theta) = 0$$

the integral I can be written as

$$\begin{aligned} I &= \frac{1}{2\pi}\int_0^\infty dt\ e^{-at}\int_{-\pi}^{\pi} d\theta\ \exp(ibt\sin\theta) \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta\ \int_0^\infty dt\ \exp(t[-a+ib\sin\theta]) \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta\ \frac{-1}{-a+ib\sin\theta}. \end{aligned}$$

Note that the integral I we started with,

$$I = \int_0^\infty dt\ e^{-at}\frac{1}{\pi}\int_0^\pi d\theta\ \cos(bt\sin\theta).$$

is real. and therefore we must have

$$\begin{aligned} I &= \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta\ \mathrm{Re}\left[\frac{-1}{-a+ib\sin\theta}\right] \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta \frac{a}{a^2+b^2\sin^2\theta} \\ &= \frac{1}{\sqrt{a^2+b^2}}. \end{aligned}$$

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2. The l-th spherical Bessel function is given by

$$j_l(x) = (-1)^l x^l \left(\frac{1}{x}\frac{d}{dx}\right)^l j_0(x). \tag{1}$$

Compute $j_1(x)$ and $j_2(x)$. Note that $j_0(x) = x^{-1}\sin x$.

Solution:

We have for $l = 1$

$$j_1(x) = -x\left(\frac{1}{x}\frac{d}{dx}\right) j_0(x). \tag{2}$$

That is,

$$
\begin{aligned}
j_1(x) &= -\frac{d}{dx}\left(\frac{\sin x}{x}\right) \\
&= \frac{\sin x}{x^2} - \frac{\cos x}{x}.
\end{aligned} \tag{3}
$$

For $l = 2$ we have

$$
j_2(x) = (-1)^2 x^2 \left(\frac{1}{x}\frac{d}{dx}\right)^2 j_0(x). \tag{4}
$$

That is

$$
\begin{aligned}
j_2(x) &= x^2 \left(\frac{1}{x}\frac{d}{dx}\right)\left(\frac{1}{x}\frac{d}{dx}\right) j_0(x) \\
&= x^2 \left(\frac{1}{x}\frac{d}{dx}\right)\left(\frac{1}{x}\left[-\frac{\sin x}{x^2} + \frac{\cos x}{x}\right]\right) \\
&= x^2 \left(\frac{1}{x}\frac{d}{dx}\right)\left[-\frac{\sin x}{x^3} + \frac{\cos x}{x^2}\right] \\
&= x\left[-\frac{\cos x}{x^3} + 3\frac{\sin x}{x^4} - \frac{\sin x}{x^2} - 2\frac{\cos x}{x^3}\right] \\
&= -\frac{\cos x}{x^2} + 3\frac{\sin x}{x^3} - \frac{\sin x}{x} - 2\frac{\cos x}{x^2} \\
&= \left(\frac{3}{x^3} - \frac{1}{x}\right)\sin x - \frac{3}{x^2}\cos x.
\end{aligned}
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3. Obtain the series expansion formula for the n-th Laguerre polynomial

$$
L_n(x) = \sum_{m=0}^{n} (-1)^m \frac{n!}{(m!)^2 (n-m)!} x^m, \tag{5}
$$

from the Rodrigues' formula

$$
L_n(x) = \frac{e^x}{n!}\frac{d^n}{dx^n}\left(x^n e^{-x}\right).
$$

Solution:

Upon using Leibnitz' theorem we can evaluate the n-th derivative

$$\begin{aligned} L_n(x) &= \frac{e^x}{n!}\sum_{r=0}^{n} {}^nC_r \frac{d^r x^n}{dx^r}\frac{d^{n-r}e^{-x}}{dx^{n-r}} \\ &= \frac{e^x}{n!}\sum_{r=0}^{n} \frac{n!}{r!(n-r)!}\frac{n!}{(n-r)!}x^{n-r}(-1)^{n-r}e^{-x} \\ &= \sum_{r=0}^{n}(-1)^{n-r}\frac{n!}{r!(n-r)!(n-r)!}x^{n-r}. \end{aligned}$$

Using the index $m = n - r$ we get

$$L_n(x) = \sum_{m=0}^{n}(-1)^m \frac{n!}{(m!)^2(n-m)!}x^m.$$

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4. Derive the following recurrence relation for Laguerre polynomials

$$(n+1)L_{n+1}(x) = (2n+1-x)L_n(x) - nL_{n-1}(x)$$

using the generating function

$$\phi(x,t) = \frac{e^{xt/(1-t)}}{(1-t)} = \sum_{n=0}^{\infty} L_n(x)t^n.$$

Solution:

Differentiating the generating function with respect to t we get

$$\frac{\partial \phi}{\partial t} = \frac{(1-x-t)e^{xt/(1-t)}}{(1-t)^3} = \sum_{n=0}^{\infty} nL_n(x)t^{n-1}.$$

We can write this as

$$(1-x-t)\sum_{n=0}^{\infty} L_n(x)t^n = (1-t)^2\sum_{n=0}^{\infty} nL_n(x)t^{n-1}.$$

Equating the coefficients of t^n on each side we get

$$(1-x)L_n(x) - L_{n-1}(x) = (n+1)L_{n+1}(x) - 2nL_n(x) + (n-1)L_{n-1}(x).$$

Rearranging the terms we get the recurrence relation

$$(n+1)L_{n+1}(x) = (2n+1-x)L_n(x) - nL_{n-1}(x).$$

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5. The generating function for associate Laguerre polynomials $L_n^m(x)$ is given by

$$\phi(x,t) = \frac{e^{-xt/(1-t)}}{(1-t)^{m+1}} = \sum_{n=0}^{\infty} L_n^m(x)t^n.$$

Use this generating function to find $L_n^m(0)$.

Solution:

Setting $x = 0$ in the generating function and expansing the series using Binomial theorem

$$\begin{aligned}\sum_{n=0}^{\infty} L_n^m(0)t^n &= \frac{1}{(1-t)^{m+1}} \\ &= 1 + (m+1)t + \frac{(m+1)(m+2)}{2!}t^2 + \cdots \\ &\quad + \frac{(m+1)(m+2)\cdots(m+n)}{n!}t^n + \cdots .\end{aligned}$$

On equating coefficients of t^n we get

$$L_n^m(0) = \frac{(n+m)!}{n!m!}.$$

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6. Evaluate

$$\int_{-\infty}^{\infty} dx \; e^{-x^2} \left[H_2(x)\right]^2 ,$$

where $H_2(x)$ is the Hermite polynomial of degree 2.

Solution:

We have

$$\int_{-\infty}^{\infty} dx \; e^{-x^2} \left[H_n(x)\right] = 2^n n! \sqrt{\pi}. \tag{6}$$

Setting $n = 2$ we get

$$\int_{-\infty}^{\infty} dx \; e^{-x^2} \left[H_2(x)\right] = 2^2 2! \sqrt{\pi} = 8\sqrt{\pi}. \tag{7}$$

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