QCQI PHY631 August-December 2024: Assignment 2
Given: September 26 2024 Due: October 11, 2024

Prof. Arvind, IISERM, arvind@quantumphys.org

1. Prove that the set of density operators ρ satisfying the conditions $\rho^\dagger = \rho$, $\rho \geq 0$, and $Tr(\rho) = 1$ is a convex set. (A set S is convex if $s_1 \in S$, $s_2 \in S$ implies that $ks_1 + (1-k)s_2 \in S$ for $0 \leq \lambda \leq 1$).

2. The Werner state ρ is defined as follows:
$$\rho = x|\phi_-\rangle\langle\phi_-| + \frac{1}{4}(1-x)I$$
where $|\phi_- = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$. Using the Peres partial transpose map analyze the entanglement properties of this state as a function of x.

3. Reduction map is defined as
$$\rho \xrightarrow{R} \rho' = (\operatorname{tr}\rho)I - \rho$$
Is this map positive? Is this map completely positive?

4. Simon's map written in a specific basis is defined as
$$\begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} \\ \rho_{21} & \rho_{22} & \rho_{23} \\ \rho_{31} & \rho_{32} & \rho_{33} \end{pmatrix} \rightarrow \begin{pmatrix} \rho_{11}+\rho_{22} & -\rho_{12} & -\rho_{13} \\ -\rho_{21} & \rho_{22}+\rho_{33} & -\rho_{23} \\ -\rho_{31} & -\rho_{32} & \rho_{33}+\rho_{11} \end{pmatrix}$$
Is this map positive? Is this map completely positive?

5. **(a):** *Consider a two-qubit controlled phase gate:* it applies $U = e^{i\alpha}I$ to the second qubit if the first qubit has value $|1\rangle$, and acts trivially otherwise. Show that it is actually a one-qubit gate.

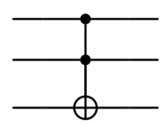
   (b): Draw a circuit using controlled-NOT gate and single-qubit gates that implements controlled-U, where U is an arbitrary 2×2 unitary transformation. The controlled-U operation means apply U on second qubit when first qubit is in a state $|1\rangle$.(This may not be a very easy problem. **Hint:** Euler angle decomposition of U might be useful).

6. Show that the action of the Hadamard gate on n qubits, ($H^n = H\otimes H \otimes H \cdots \otimes H$) can be written in the following form
$$H^n|x\rangle = \frac{1}{2^{\frac{n}{2}}}\sum_{y=0}^{2^n-1}(-1)^{x.y}|y\rangle$$
where x and y are binary strings labeling 2^n product eigen states of n-qubit in the standard basis. For example the state $|010010001\cdots\rangle$ will correspond to the string $010010001\cdots$.

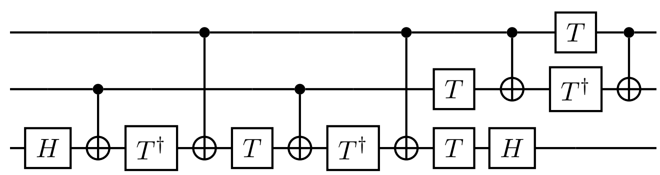
7. Consider the DJ problem in the classical setting. If we query the oracle k times and we obtain the same value every time, and thus classify the function as constant, what is the probability that we have arrived at a wrong conclusion. How does this probability scale with n and k. What do you conclude from your analysis.

8. For the quantum algorithm to solve Simon's problem we end up with a situation where we have the final state of the n qubit register is
$$|\psi_{\text{final}}\rangle = \frac{1}{2^{n-1}}\sum_{y.a=0}|y\rangle.$$
This expansion has only those states $|y\rangle$ in the superposition for which $y.a = 0$. The dot product is defined as $y_1a_1 \oplus y_2a_2 \oplus \cdots \oplus y_na_n$ where y_j and a_j are the jth bit values of y and a respectively. Show that there are 2^{n-1} terms in this superposition. Simulate a quantum measurement on this register by choosing random y such that $y.a = 0$. Keep doing this process till you find n independent vectors y such that $y.a = 0$. Repeat the simulation and obtain the average value of trials required to find such n independent ys. Plot this average and its log as a function of n. You can choose a random a vector and vary that too if you desire and repeat the simulation. Results should not depend upon the values of a.

9. Consider the Toffoli gate which is a Controlled-Controlled NOT gate represented by the following symbol. The third qubit undergoes a NOT operation when the first two qubits are in the state $|1\rangle$



Show that the following circuit involving one qubit gates and six C-NOT gates is equivalent to the Toffoli gate.



Here T represent a $\pi/4$ phase gate on one qubit i.e. $|0\rangle \xrightarrow{T} |0\rangle$ and $|1\rangle \xrightarrow{T} e^{i\pi/4}|1\rangle$.

10. Setup a simulation for the Search Algorithm to highlight the reflection about the mean idea. Consider a database comprising of N items. Choose a marked item w as one of of the items. Start the system in a quantum state $|s\rangle$ which is an equal superposition of all the basis states with amplitude $1/\sqrt{N}$ for each of the state. Apply $U_w = I - 2|w\rangle\langle w|$ followed by $U_s = 2|s\rangle\langle s| - I$. Calculate the amplitude for each state after this application. Calculate the amplitudes after k such iterations. For different values of N plot the overlap $|\langle w|s\rangle|^2$ as a function of k. For each N obtain the values of k for which the overlap is maximum. Plot this optimal value of k which we call k_{opt}, as a function of N. Also plot $\log(k_{\text{opt}})$ as function of N, what is its slope?