PHY 310 - Mathematical Methods for Physicists I

Odd Term 2019, IISER Mohali

Instructor: Dr. Anosh Joseph

Second Mid-Semester Examination

16th October, 2019 from 8:30 AM to 10:15 AM in LH3/LH4

Maximum Marks 100

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1. [**2 + 2 + 2 + 2 + 2 = 10 Marks**]

   (a) The function $f(z) = \ln z$ has an essential singularity at $z = 0$. Is this statement true or false?

   (b) Find the Laurent series for the function

   $$f(z) = \frac{z+1}{z},$$

   and provide the region where the the Laurent series of this function is valid.

   (c) What term do we use to call a function that is analytic in a region R except for a set of poles of finite order?

   (d) Consider the function $f(z) = \cot(z)$. Find the residue of $f(z)$ at 42π.

   (e) Consider the function

   $$f(z) = \frac{1}{z\sqrt{z^2-1}}.$$

   Draw a diagram showing (i) all possible singularities of this function in the complex plane, and (ii) the closed contour Γ that would help us evaluate the integral

   $$\int_1^\infty \frac{dx}{x\sqrt{x^2-1}}.$$

2. [**5 Marks**] Obtain the series expansion for

   $$f(z) = \frac{1}{z^2+4}$$

   valid in the region $|z - 2i| < 4$.

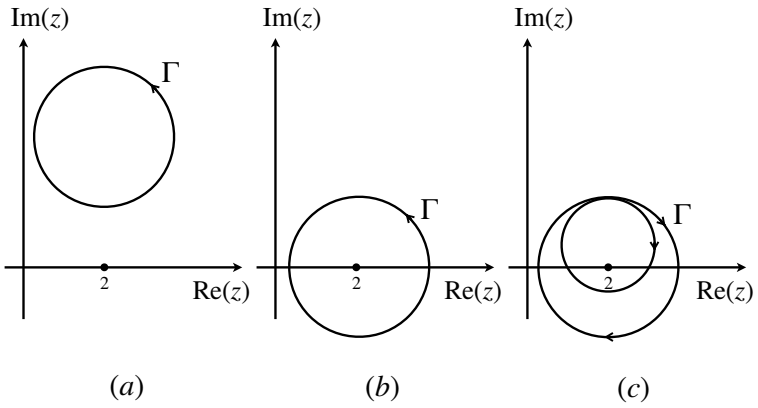

Figure 1: The three contours for computing the integral $I = \int_\Gamma \frac{e^{z^2}}{(z-2)} dz$.

3. [**10 Marks**] Evaluate the integral

$$I = \int_{-\infty}^{\infty} \frac{\sin x}{(x^2 + 2x + 2)} dx.$$

4. [**2 + 3 + 5 = 10 Marks**] Compute the following integral

$$I = \int_\Gamma dz \; \frac{e^{z^2}}{(z-2)}$$

for the three cases given in Fig. 1.

5. [**3 + 3 + 4 = 10 Marks**] Let us consider the following transformation

$$w = f(z) = \frac{z-i}{z+i}.$$

(a) What happens to the x-axis under this mapping?

(b) How is the upper half-plane mapped under this transformation?

(c) Complete the figure on the right-hand side of Fig. 2.

6. [**15 Marks**] Consider the one-dimensional heat equation in the context of a heat conducting wire of length L

$$u_t - k u_{xx} = 0,$$

where $k > 0$ is a constant, $0 < x < L$, and with the boundary conditions

$$u(0,t) = 0, \quad u(L,t) = 0, \quad u(x,0) = f(x).$$

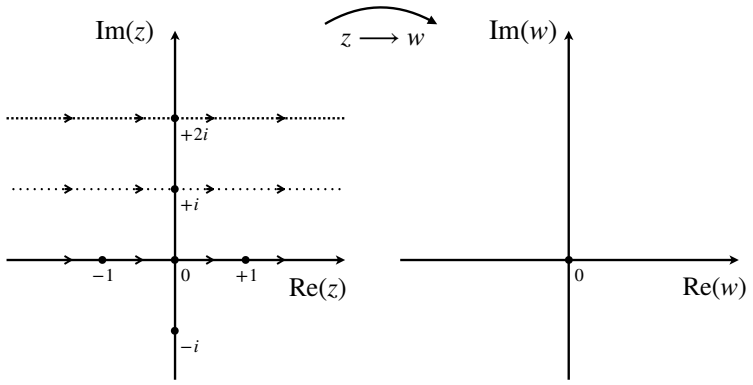

Figure 2: The map of the function $w = f(z) = (z - i)/(z + i)$.

Find the non-trivial solution $u(x, t)$ using the method of separation of variables.

7. [**10 Marks**] For an LR circuit with L the inductance and R the resistance (both taken as constants) the Kirchhoff's law leads to

$$L\frac{dI(t)}{dt} + RI(t) = V(t),$$

for the current $I(t)$. Find the current as a function of time by solving the differential equation using an appropriate integrating factor. Also take $I(0) = 0$ and $V(t) = V_0 =$ constant to arrive at the solution.

8. [**10 Marks**] Consider the differential equation

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0.$$

Find the roots of its indicial equation.

9. [**20 Marks**] Compute the integral

$$I(a) = \int_{-\infty}^{\infty} e^{-iax^2}$$

using saddle point method.