1. Consider the group $GL_3(\mathbb{R})$ (of 3×3 matrices whose entries are real numbers, under multiplication of matrices) and the set $\mathbb{R}^3$ (of triplets of real numbers). For $A \in GL_3(\mathbb{R})$ and $(a, b, c) \in \mathbb{R}^3$, define

$$A.(a,b,c) := A \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

• Here we are writing an element of $\mathbb{R}^3$ as a row as well as a column.

   (a) Show that above is an action of $GL_3(\mathbb{R})$ on $\mathbb{R}^3$.

   (b) What is the orbit of $(1, 2, 3)$? Is it true that the orbit of $(1, 2, 3)$ is same as that of $(-1, -2, -3)$?

   (c) Apart from the identity element of $GL_3(\mathbb{R})$, find an element in the stabilizer of $(1, 2, 3)$. How many elements are there in this stabilizer?

   (d) Take the 3×3 rotation matrix $R_{x,\theta}$. Find all $P := (a, b, c) \in \mathbb{R}$ such that the stabilizer of P contains $R_{x,\theta}$.

(a) • For identity matrix $I_3 \in GL_3(\mathbb{R})$ and $(a,b,c) \in \mathbb{R}^3$.

$$I_3 \cdot (a,b,c) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

• For $A = (a_{ij})$, $B = (b_{kl}) \in GL_3(\mathbb{R})$, just check that matrix multiplication gives

$$A\left(B\begin{pmatrix} a \\ b \\ c \end{pmatrix}\right) = (AB)\begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

(b) Orbit of $(1,2,3)$ consists of all elements of the form $A\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, as A varies over $GL_3(\mathbb{R})$.

<u>Observe</u>

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & -1/3 \\ 1 & -1/2 & 0 \end{pmatrix}}_{\in GL_3(\mathbb{R})} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Hence $(1,0,0)$ is in the orbit of $(1,2,3)$.

Now, take $(a, b, c) \neq (0,0,0)$.

Then we can find an invertible matrix of the form

$$\begin{pmatrix} a & * & * \\ b & * & * \\ c & * & * \end{pmatrix}$$

and

$$\begin{pmatrix} a & * & * \\ b & * & * \\ c & * & * \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

Thus (a,b,c) is in the orbit of $(1,0,0)$.

$\Rightarrow$ (a,b,c), $(1,0,0)$, $(1,2,3)$ all are in the same orbit.

Further, $(0,0,0) \notin$ orbit of $(1,0,0)$. - why?

$\Rightarrow$ orbit of $(1,2,3)$ consists of all $(a,b,c) \neq (0,0,0)$ $\in \mathbb{R}^3$.

(c) $\underset{\underset{(a_{ij})}{\shortparallel}}{A} \in GL_3(\mathbb{R})$ is in the stabilizer of $(1,2,3)$ if

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

One such case is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \alpha & \beta & \gamma \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

where $\alpha + 2\beta + 3\gamma = 3$

Now choose $\gamma \neq 0$; and α, β such that

$\alpha + 2\beta = 3(1-\gamma)$ - Infinitely many such choices

A particular choice is

$\alpha = -2, \beta = 1, \gamma = 1$, so that

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 1 & 1 \end{pmatrix}}_{\in GL_3(\mathbb{R})} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

(d) We look at

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

This is equivalent to

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} b \\ c \end{pmatrix} = \begin{pmatrix} b \\ c \end{pmatrix}$$

i.e., rotation by θ does not move the point (b, c).

Thus,

either $\theta = 0$ (i.e. $R_{x,\theta}$ = identity matrix)

In which case $R_{x,\theta}$ is in the stabilizer of every $(a, b, c) \in \mathbb{R}^3$.

or $(b,c) = (0,0)$ In which case $R_{x,\theta}$ is in the stabilizer of $(a, 0, 0)$, for every $a \in \mathbb{R}$.

(Indeed, rotation about x-axis will stabilize all points on x-axis).

2. Consider the group S_4, consisting of permutations of four elements $1, 2, 3, 4$. Show that the permutation action of S_4 on $S := \{1, 2, 3, 4\}$ has only one orbit. Determine the stabilizer of $3 \in S$. Show that $\#\text{orbit}(3) \times \#\text{stab}(3) = \#S_4$. Here the symbol $\#$ signifies the number of elements.

$$\left.\begin{array}{l} (1\ 2)\cdot 1 = 2 \\ (1\ 3)\cdot 1 = 3 \\ (1\ 4)\cdot 1 = 4 \end{array}\right\}$$

$\Rightarrow$ 1, 2, 3, 4 all are in the same orbit $\{1, 2, 3, 4\}$.
So, there is only one orbit.

Let $\sigma \in S_4$ be in the stabilizer of 3. Then σ is allowed to move points 1, 2, 4, but not 3. Possibilities of such σ are

$$\left.\begin{array}{l} \text{Identity permutation} \\ (1\ 2) \\ (1\ 4) \\ (2\ 4) \\ (1\ 2\ 4) \\ (1\ 4\ 2) \end{array}\right\} \text{six such elements}$$

so $\#\text{orbit}(3) = 4$, $\#\text{stab}(3) = 6$

and $\#\text{orbit}(3) \times \#\text{stab}(3) = \#S_4 = 4! = 24$

indeed holds.