QCQI PH631 Jan-April 2024: Assignment 1 August 24 2024
Given: August 23, 2024 Due: August 27, 2024

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1. Consider the simplest complex linear vector space with dimension equal to one. If this is the state space of a quantum system describe all the allowed quantum states. Can there be mixed states for this system. Describe the state space geometrically. What are the possible transformations i.e. unitary operators which will take states to states. How many distict physical situations exist for this system. Construct all possible observables and what are their possible outcomes.

2. Represent the following states on the Poincare Bloch sphere: $|0\rangle + |1\rangle$, $|0\rangle - |1\rangle$, $|0\rangle + i|1\rangle$, $|0\rangle + 2|1\rangle$

3. We have one qubit with an orthonormal basis given to us $\{|0\rangle, |1\rangle\}$ We are given two states

$$|0'\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle), \quad |1'\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$$

(a): Show that $\{|0'\rangle, |1'\rangle\}$ is also an orthonormal basis.

(b): Work out the matrix representations of the following operators in the original basis $\{|0\rangle, |1\rangle\}$:

$$\begin{aligned} A &= |0\rangle\langle 1| - |1\rangle\langle 0| \\ A &= |0'\rangle\langle 1'| - |1'\rangle\langle 0'| \\ A &= |0\rangle\langle 0| + |1\rangle\ \langle 1| \\ A &= |0'\rangle\langle 0'| + |1'\rangle\langle 1'| \end{aligned}$$

4. A system has a four-dim Hilbert space and we are given a set of orthonormal basis vectors in this space $\{|a_j\rangle, j = 1,2,3,4\}$ $\langle a_j|a_k\rangle = \delta_{jk}$ Given an operator

$$A = aI + \epsilon_1|a_1\rangle\langle a_2| + \epsilon_1|a_2\rangle\langle a_1| + \epsilon_2|a_3\rangle\langle a_4| + \epsilon_2|a_4\rangle\langle a_3|$$

(a): Find the matrix representation of A in the given orthonormal basis.

(b): Find the eigenvalues and eigenvectors of A.

(c): Given a state of the system $|\alpha\rangle = \cos\theta|a_1\rangle + \sin\theta|a_2\rangle$, imagine the measurement of A. What are the possible outcomes? What is the probability of obtaining each outcome? What is the state after measuring each outcome?

(d): Construct projectors whose 'Yes'/'No' measurement will be equivalent to measuring the operator A. Relate the 'Yes' 'No' sequences with the corresponding outcomes for A. What is the minimum number of projectors that you have to measure in this case? Measure the information gain in terms of bits from these measurements.

(e): Now, imagine that this system is actually a two qubit system and its Hilbert space has been obtained via tensor product of two one-qubit (2-dim) hilbert spaces and we have

$$\begin{aligned} |a_1\rangle = |0\rangle \otimes |0\rangle, \quad |a_2\rangle = |0\rangle \otimes |1\rangle, \\ |a_3\rangle = |1\rangle \otimes |0\rangle, \quad |a_4\rangle = |1\rangle \otimes |1\rangle. \end{aligned}$$

Is the operator A 'factorizable' or 'nonfactorizable' i.e. is it $I \otimes A_2, A_1 \otimes I, A_1 \otimes A_2$ type of something which cannot be written in this form. Also examine each of the projectors that you have constructed for their factorisability.

5. Consider a general state of two qubits

$$|\alpha\rangle = c_{00}|00\rangle + c_{01}|01\rangle + c_{10}|10\rangle + c_{11}|11\rangle$$

Find the general conditions on the coefficients c_{jk} so that the state is a product state of the form $|\alpha_1\rangle \otimes |\alpha_2\rangle$

6. Consider a general state on the Bloch sphere

$$|w^+\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|0\rangle$$

(a): Find the state orthogonal to this state $|w^-\rangle$.

(b): Show that

$$|w^+\rangle|w^-\rangle - |w^-\rangle|w^+\rangle = |0\rangle|1\rangle - |1\rangle|0\rangle$$

Any idea as to what does what does this invariance mean?