21050

$\subseteq \mathcal{P}(N)$ $\{1\}\ \{2\} \ldots$

$a_n \in B(x, \frac{\varepsilon}{2^n})$

0 $\frac{1}{9}$ $\frac{2}{9}$ $\frac{1}{3}$ $\frac{2}{3}$ $\frac{7}{9}$

1. $l/3$ $2l/3$
2. $2l/9$ $4l/9$
3. $4l/27$

Dirichlet

$f(x) = 1$; x

0 ; 0

Thomae

$f(p/q)$

End-Semester Examination

Course : MTH301 (Real Analysis) Date : November 29, 2023

Duration of the Examination - Three Hours Maximum Marks - 50

Instructions : The question paper has TEN questions. Each question carries equal marks. Your answers should include justifications. You need to explain the relevant terms in the definitions and statement of theorems. Marks distribution is shown at the right side of the questions.

Notations: Let $a, b \in \mathbb{R}$ with $a < b$ and $[a, b]$ be a compact interval. Let $\mathcal{R}[a, b]$ be the set of all Riemann integrable functions on $[a, b]$. For a differentiable real-valued function f on an interval, f' (or $f^{(m)}$) denotes derivative (resp. mth derivative) of f. If S is a subset of a metric space, then $\overline{S}$ denotes the closure of S in the metric space. Unexplained notations and terminologies are as discussed in the lectures.

1. (a) Let (X, d) be a metric space and $\emptyset \neq S \subseteq X$. Define an accumulation point of S. If $x \in X$ is an accumulation point of S, then show that there exists a sequence $(a_n)_{n\in\mathbb{N}}$ of distinct points in S such that $\lim_{n\to\infty} a_n = x$. (1+2=3)

   (b) Let $f: (X, d) \to (Y, \rho)$ be a function between metric spaces. If $f(\overline{S}) \subseteq \overline{f(S)}$ for every $S \subseteq X$, then show that f is continuous. (2)

2. (a) Let $\mathbb{N}$ be the set of natural numbers and $\mathcal{P}(\mathbb{N})$ be the power set of $\mathbb{N}$. Show that $\mathcal{P}(\mathbb{N})$ is uncountable. (2)

   (b) Define Cantor's ternary set. Show that no point of the Cantor's ternary set is an interior point. (1+2=3)

3. State Riemann-Lebesgue theorem explaining all the relevant terms. If f and g are respectively the Dirichlet and the Thomae functions on $[0, 1]$, then deduce that $f \notin \mathcal{R}[0, 1]$ but $g \in \mathcal{R}[0, 1]$. (2+1+2=5)

   (**OR**) Let $A: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Define the operator norm $||A||$. If $||Ax|| \le \lambda ||x||$ for all $x \in \mathbb{R}^n$ and a real number λ, then show that $||A|| \le \lambda$. If $mat(A) = [a_{ij}]_{m\times n}$ is the matrix of A w.r.t. the standard bases, then show that $||A|| \le \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}^2}$. (1+2+2=5)

4. State Taylor's theorem. If $P(x)$ is a real polynomial in x of degree 6, such that $P(-1) = P(0) = 0$, $P(1) = 1$ and $P'(0) = 0$, then deduce that there exists $c \in (-1, 1)$ with $P^{(3)}(c) = P'''(c) \ge 3$. (2+3=5)

   (**OR**) Define Lebesgue covering number of an open covering. Verify that every open covering of a finite set has a Lebesgue number. In general, show that every open covering of a compact subset has a Lebesgue number. (1+1+3=5)

$\ldots + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$

(Please Turn Over)

$\ldots - a_5 + a_4 - a_3 + a_2 \ldots = 0$

$a_6 + a_5 + a_4 + a_3 + a_2 \ldots = 1$ $\Rightarrow 2(a_6 + a_4 + a_2) = \frac{1}{2}$ $\Rightarrow 2(a_5 + a_3) = \frac{1}{2}$

$a_1 = 0$

$6a_6x^5 + 5a_5x^4 + 4a_4x^3 + 3a_3x^2 + 2a_2x$

$30a_6x^4 + 20a_5x^3 + 12a_4x^2 + 6a_3x + 2a_2$

$120a_6x^3 + \ldots$

$400a_6x^3 + 200a_5x^2 + \ldots$

$200a_6x^3 + 10a_5x^2 + \ldots$

$l - l/3 - \frac{4l}{9}$

$l - \frac{l}{3} - \frac{2l}{9} - \frac{4l}{27}$

$|f_n - f| < \varepsilon$

$< f_n < f + \varepsilon$

$f < 2\varepsilon(b-a)$

$\frac{1}{n^2}$

5. Let $f_n \in \mathcal{R}[a,b]$ for $n \in \mathbb{N}$ and $f_n \to f$ uniformly on $[a,b]$. Show that $f \in \mathcal{R}[a,b]$ and $\int_a^b f\,dx = \lim_{n\to\infty} \int_a^b f_n dx$. (3+2=5)

6. For $n \in \mathbb{N}$, let $f_n : [a,b] \to \mathbb{R}$ be continuous and $f_n(x) \geq f_{n+1}(x) \geq 0$ for all x and n. If $\lim_{n\to\infty} f_n(x) = 0$ for all $x \in [a,b]$, then show that $f_n \to 0$ uniformly on $[a,b]$. Let $g_n(x) = \frac{x}{nx+1}$ and $h_n(x) = \frac{1}{nx+1}$ for $x \in (0,1)$. Show that $g_n \to 0$ uniformly but $h_n \to 0$ only pointwise on $(0,1)$. (3+1+1=5)

7. Consider the metric space $C[a,b]$ of real-valued continuous functions on $[a,b]$ under the sup-metric. Prove that $C[a,b]$ is complete. Let $f \in C[0,1]$ and $\int_0^1 f(x)x^n dx = 0$ for $n = 0, 1, 2, \ldots$. Show that $f(x) = 0\ \forall x \in [0,1]$. (3+2=5)

8. If an equicontinuous family $\mathcal{E} \subseteq C[a,b]$ is pointwise bounded, then show that it is uniformly bounded. Let $f_n(x) = \frac{x^2}{x^2+(nx-1)^2}$ and $g_n(x) = \frac{x^n}{n}$ for $x \in [0,1]$ and $n \in \mathbb{N}$. Show that $\{f_n : n \in \mathbb{N}\}$ is not equicontinuous, while $\{g_n : n \in \mathbb{N}\}$ is an equicontinuous family of $C[0,1]$. (2+1+2=5)

9. Let $\{e_1, \ldots, e_n\} \subset \mathbb{R}^n$ and $\{u_1, \ldots, u_m\} \subset \mathbb{R}^m$ be standard bases. Let $E \subseteq \mathbb{R}^n$ be open, $f : E \to \mathbb{R}^m$ be a mapping and $f(x) = \sum_{i=1}^m f_i(x)u_i$ for $x \in E$. Let $a \in E$. Define the total derivative $f'(a)$ and the partial derivatives $D_j f_i(a)$ for $1 \leq i \leq m$; $1 \leq j \leq n$. Show that the matrix $mat(f'(a))$ of $f'(a)$ w.r.t. the standard bases is the Jacobian matrix $J_f(a) = [D_j f_i(a)]_{m\times n}$ of f at a. (1+1+3=5)

(OR) Prove multivariate chain-rule of differentiation. Also, prove Banach contraction theorem. (3+2=5)

10. State inverse function theorem. Let $E = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 + x_2 \neq -1\}$ be open and for $x = (x_1, x_2) \in E$, let $f(x) = \left(\frac{x_1}{1+x_1+x_2}, \frac{x_2}{1+x_1+x_2}\right)$. Show that the inverse function theorem can be applied to f at $a = (1,1) \in E$ and thus deduce that f is (locally) invertible in a neighbourhood of the point a. Does the inverse f^{-1} of $f : E \to f(E)$ exist? Is f^{-1} a C^1-mapping? Explain. (1+2+1+1=5)

(OR) State implicit function theorem. Let $f : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ be the mapping given by $f(x,y) = (x_1y_1^2 + x_2, x_1x_2^2 + y_1^2y_2^2)$ for $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Let $a = (-1,1)$ and $b = (1,1)$. Determine a neighbourhood V of $a = (-1,1)$ in $\mathbb{R}^2$ and a C^1-mapping $g : V \to \mathbb{R}^2$ such that $g(a) = b$ and $f(x, g(x)) = 0$ for all $x \in V$. (2+1+2=5)

(End of the Question Paper)

$f(x_1) = \frac{x_1}{1+x_1+x_2}$

$\frac{\varepsilon x}{1-\varepsilon} < \frac{\varepsilon x}{x-\varepsilon}$

$\frac{1}{n^2}$

$\frac{1}{n^2} - \frac{1}{(n+1)^2}$

$\frac{1}{n}\left(\frac{nx+1}{nx+1} - \frac{1}{nx+1}\right)$

$\frac{1}{n}\left(1 - \frac{1}{nx+1}\right)$

$\left|\frac{tx}{x+t}\right| < \frac{\delta x}{x+\delta}$

$\frac{tx}{x+t} < \frac{dx}{x+d}$

$f(\bar{t}) - f(\bar{x}) = (f'(\bar{x}) + u(\bar{t}))(\bar{t} - \bar{x})$

$f(t) - f(x) = f'(x) + u(t)$

$n \to \infty$

$|t - 0| < \delta$

$\left|\frac{x}{nx+1} - 0\right| < \varepsilon \ \forall\ n \geq n_0$

$-\varepsilon < \frac{x}{nx+1} < \varepsilon$

$\lim_{t\to 0} \frac{tx}{x+t}$