General Relativity Fall 2019
Homework 5 solution

Exercise 1: Parallel transport on a sphere

The 2-sphere of radius r has line element $ds^2 = r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2$.

(i) Compute the Christoffel symbols of this metric.

The inverse metric is diagonal, with components $g^{\theta\theta} = 1/r^2, g^{\varphi\varphi} = 1/(r\sin\theta)^2$, $g^{\theta\varphi} = 0$.
There are only 6 independent Christoffel symbols to compute:

$$\Gamma^\theta_{\theta\theta} = \frac{1}{2r^2} g_{\theta\theta,\theta} = 0, \tag{1}$$

$$\Gamma^\theta_{\theta\varphi} = \frac{1}{2r^2} g_{\theta\theta,\varphi} = 0, \tag{2}$$

$$\Gamma^\theta_{\varphi\varphi} = \frac{1}{2r^2}\left(2g_{\theta\varphi,\varphi} - g_{\varphi\varphi,\theta}\right) = -\sin\theta\cos\theta, \tag{3}$$

$$\Gamma^\varphi_{\theta\theta} = \frac{1}{2r^2\sin^2\theta}(2g_{\theta\varphi,\theta} - g_{\theta\theta,\varphi}) = 0, \tag{4}$$

$$\Gamma^\varphi_{\theta\varphi} = \frac{1}{2r^2\sin^2\theta} g_{\varphi\varphi,\theta} = \frac{\cos\theta}{\sin\theta}, \tag{5}$$

$$\Gamma^\varphi_{\varphi\varphi} = \frac{1}{2r^2\sin^2\theta} g_{\varphi\varphi,\varphi} = 0. \tag{6}$$

(ii) Write the equations of parallel transport of a tangent vector $\overline{W} = W^\theta\partial_\theta + W^\varphi\partial_\varphi$ along a small circle with constant $\theta = \theta_0$ – the first thing you will need is to do is find a vector $\overline{V}$ tangent to this curve.

Suppose we parametrize the curve by some parameter λ. The tangent vector $d/d\lambda = \overline{V} = V^\theta\partial_\theta + V^\varphi\partial_\varphi$ has components $V^\theta = \frac{d\theta}{d\lambda}$ and $V^\varphi = d\varphi/d\lambda$. Since the curve has constant θ, we find that $V^\theta = 0$. Keeping the parametrization of the curve general for now, thus we have

$$\overline{V} = \frac{d\varphi}{d\lambda}\partial_\varphi. \tag{7}$$

The equations of parallel transport are

$$V^\mu\nabla_\mu W^\nu = 0 = \frac{dW^\nu}{d\lambda} + \Gamma^\nu_{\mu\sigma}V^\mu W^\sigma = \frac{dW^\nu}{d\lambda} + \Gamma^\nu_{\varphi\sigma}\frac{d\varphi}{d\lambda}W^\sigma. \tag{8}$$

We see that we can get rid of λ and write the parallel transport equation in terms of φ directly:

$$\frac{dW^\nu}{d\varphi} = -\Gamma^\nu_{\varphi\sigma}W^\sigma. \tag{9}$$

We thus have two coupled equations:

$$\frac{dW^\theta}{d\varphi} = -\Gamma^\theta_{\varphi\sigma}W^\sigma = \sin\theta_0\cos\theta_0\ W^\varphi, \tag{10}$$

$$\frac{dW^\varphi}{d\varphi} = -\Gamma^\varphi_{\varphi\sigma}\ W^\sigma = -\frac{\cos\theta_0}{\sin\theta_0}\ W^\theta, \tag{11}$$

where I used the expressions for the Christoffel symbols derived above, evaluated on the curve of constant $\theta = \theta_0$.

(iii) Solve these equations, starting with some initial conditions $W^\theta|_0$ and $W^\varphi|_0$ at $\varphi = 0$. How does the vector compare to itself after being parallel-transported once around a small circle?

We can combine these equations into the second-order linear ODE:

$$\frac{d^2W^\theta}{d\varphi^2} + \cos^2\theta_0\ W^\theta = 0, \tag{12}$$

and similarly for W^φ. These are just harmonic oscillators with frequency $\omega_0 = \cos\theta_0$, whose solution is

$$W^\theta(\varphi) = W^\theta|_0 \cos[(\cos\theta_0)\varphi] + \frac{1}{\cos\theta_0}\frac{dW^\theta}{d\varphi}\Big|_0 \sin[(\cos\theta_0)\varphi], \tag{13}$$

$$W^\varphi(\varphi) = W^\varphi|_0 \cos[(\cos\theta_0)\varphi] + \frac{1}{\cos\theta_0}\frac{dW^\varphi}{d\varphi}\Big|_0 \sin[(\cos\theta_0)\varphi]. \tag{14}$$

Now using the original equations, we can related the first derivative of W^θ to W^φ and vice-versa. Thus we obtain, finally,

$$W^\theta(\varphi) = W^\theta|_0 \cos[(\cos\theta_0)\varphi] + \sin\theta_0 \; W^\varphi|_0 \sin[(\cos\theta_0)\varphi], \tag{15}$$

$$W^\varphi(\varphi) = W^\varphi|_0 \cos[(\cos\theta_0)\varphi] - \frac{1}{\sin\theta_0} W^\theta|_0 \sin[(\cos\theta_0)\varphi]. \tag{16}$$

Let us moreover define $W^{\hat\varphi} = \sin\theta \; W^\varphi$. This is the component of $\overline{W}$ along $e_{\hat\varphi} = \frac{1}{\sin\theta}\partial_\varphi$, which has norm r, like ∂_θ. Thus, after being transported once, we have

$$\begin{pmatrix} W^\theta(2\pi) \\ W^{\hat\varphi}(2\pi) \end{pmatrix} = \begin{pmatrix} \cos(2\pi\cos\theta_0) & \sin(2\pi\cos\theta_0) \\ -\sin(2\pi\cos\theta_0) & \cos(2\pi\cos\theta_0) \end{pmatrix} \begin{pmatrix} W^\theta(0) \\ W^{\hat\varphi}(0) \end{pmatrix} \tag{17}$$

We see that the vector W is **rotated by an angle** $2\pi\cos\theta_0$ after being parallel-transported once around the small circle.

Let us consider a small circle near the pole, so $\theta_0 \ll 1$. We then have $\cos\theta_0 \approx 1 - \theta_0^2/2$, thus

$$\cos(2\pi\cos\theta_0) - 1 \approx \cos(\pi\theta_0^2) - 1 = \mathcal{O}(\theta_0^4), \tag{18}$$

$$\sin(2\pi\cos\theta_0) \approx -\sin(\pi\theta_0^2) \approx -\pi\theta_0^2. \tag{19}$$

We then get

$$W^\theta(2\pi) - W^\theta(0) \approx -\pi\theta_0^2 \; \times W^{\hat\varphi}|_0, \tag{20}$$

$$W^{\hat\varphi}(2\pi) - W^{\hat\varphi}(0) \approx \pi\theta_0^2 \times W^\theta|_0. \tag{21}$$

Therefore the difference if proportional to the area enclosed by the small circle.

Exercise 2: Killing vector fields and conservation laws

(*i*) *Suppose that, in some coordinate system* $\{x^\mu\}$*, the metric components do not depend on a specific coordinate* $x^{\sigma*}$*, i.e.* $\partial_{\sigma*} g_{\mu\nu} = 0$ *for any* μ, ν*. Show that this implies that* $P_{\sigma*}$ *(with the index downstairs!) is constant along geodesics with tangent vector* P^α*.*

Suppose $P = d/d/\lambda$, i.e. that λ is the parameter along the geodesic (proper time or affine parameter for null geodesics). Since $P^\mu \nabla_\mu P^\nu = 0$, and the covarariant derivative is metric-compatible, we also automatically have $P^\mu \nabla_\mu P_\nu = 0$. Applying this to $\nu = \sigma*$, we have

$$0 = P^\mu \nabla_\mu P_{\sigma*} = \frac{dP_{\sigma*}}{d\lambda} - \Gamma^\delta_{\mu\sigma*} P^\mu P_\delta. \tag{22}$$

Let's compute the relevant Christoffel symbol:

$$\Gamma^\delta_{\mu\sigma*} = \frac{1}{2} g^{\delta\rho}(g_{\rho\mu,\sigma*} + g_{\sigma*\rho,\mu} - g_{\sigma*\mu,\rho}) = g^{\delta\rho} g_{\sigma*[\rho,\mu]}, \tag{23}$$

since $g_{\rho\mu,\sigma*} = 0$. Thus we have

$$\frac{dP_{\sigma*}}{d\lambda} = g^{\delta\rho} g_{\sigma*[\rho,\mu]} P^\mu P_\delta = g_{\sigma*[\rho,\mu]} P^\mu P^\rho = 0, \tag{24}$$

as the last term is the contraction of a pair of symmetric indices with a pair of antisymmetric indices. We thus showed that $P_{\sigma*}$ is constant along the geodesic. **Important:** $P^\mu \nabla_\mu P_{\sigma*}$ really means $(\nabla P)^\mu{}_{\sigma*\mu}$, and is **not** equal to $dP_{\sigma*}/d\lambda$, which is the quantity that must be zero for $P_{\sigma*}$ to be constant along the geodesic.

(ii) A Killing vector field K^α is a vector field that satisfies Killings' equation, $\nabla_{(\alpha}K_{\beta)} = 0$. Show that, if K^α is a Killing vector field, then $K_\alpha P^\alpha$ is constant along geodesics with tangent vector P^α.

For an arbitrary vector field K^α, $P^\alpha K_\alpha$ is a scalar field, thus

$$\frac{d}{d\lambda}(K_\alpha P^\alpha) = P^\mu \nabla_\mu (K_\alpha P^\alpha) = P^\mu (\nabla_\mu K_\alpha) P^\alpha + P^\mu (\nabla_\mu P^\alpha) K_\alpha = (\nabla_\mu K_\alpha) P^\mu P^\alpha = (\nabla_{(\mu} K_{\alpha)}) P^\mu P^\alpha \tag{25}$$

since $P^\mu \nabla_\mu P^\alpha = 0$, and the tesnor product $P^\mu P^\alpha$ is symmetric. Thus, if K^α is a Killing vector field, this vanishes, thus $K^\alpha P_\alpha$ is constant along geodesics.

(iii) Show that, given a conserved *stress-energy tensor $T^{\alpha\beta}$, i.e. such that $\nabla_\alpha T^{\alpha\beta} = 0$, and a Killing vector field K^α, the current $J^\alpha \equiv K_\beta T^{\beta\alpha}$ is conserved (i.e. divergence-free).*

$$\nabla_\alpha J^\alpha = \nabla_\alpha (K_\beta T^{\beta\alpha}) = (\nabla_\alpha K_\beta) T^{\beta\alpha} + K_\beta \nabla_\alpha T^{\beta\alpha} = (\nabla_{(\alpha} K_{\beta)}) T^{\beta\alpha}, \tag{26}$$

since $T^{\alpha\beta}$ is divergence-free, and symmetric (so contractions with it only pick the symmetric part). Thus, if K^α is a Killing vector field, this vanishes.

(iv) In a coordinate system $\{x^\mu\}$, define $\overline{K} \equiv \partial_{\sigma*}$, for some specific coordinate $x^{\sigma*}$. Show that K^α satisfies Killing's equation if and only if $\partial_{\sigma*} g_{\mu\nu} = 0$ for all μ, ν.

Let us write explicitly the symmetric part of the gradient of K_α:

$$\nabla_{(\alpha} K_{\beta)} = \partial_{(\alpha} K_{\beta)} - \Gamma^\delta_{\alpha\beta} K_\delta, \tag{27}$$

where the second term is automatically symmetric. Now since $\overline{K} = \partial_{\sigma*}$, we have

$$K^\mu = \delta^\mu_{\sigma*}, \qquad K_\beta = g_{\beta\mu} K^\mu = g_{\beta\sigma*}. \tag{28}$$

So we found

$$\nabla_{(\alpha} K_{\beta)} = \partial_{(\alpha} g_{\beta)\sigma*} - g_{\delta\sigma*} \Gamma^\delta_{\alpha\beta} = \partial_{(\alpha} g_{\beta)\sigma*} - \frac{1}{2}(g_{\alpha\sigma*,\beta} + g_{\beta\sigma*,\alpha} - g_{\alpha\beta,\sigma*}), \tag{29}$$

where we used the expression for the Christoffel symbol and simplified the contraction with the inverse metric. The first two terms in the parenthesis are exactly $\partial_{(\alpha} g_{\beta)\sigma*}$. Thus we have

$$\nabla_{(\alpha} K_{\beta)} = \frac{1}{2} g_{\alpha\beta,\sigma*}. \tag{30}$$

Hence we see that $\overline{K} = \partial_{\sigma*}$ is a Killing vector field (i.e. the left-hand-side vanishes for any pair α, β, by definition) if and only if the metric components do not depend on $\sigma*$.

The concept of Killing vectors is important and you should remember it!

Exercise 3: Ideal fluid

Preliminaries: *A* projector *is a tensor $\mathcal{P}^\alpha{}_\beta$ which satisfies $\mathcal{P}^\alpha{}_\beta \mathcal{P}^\beta{}_\gamma = \mathcal{P}^\alpha{}_\gamma$.*
A projector parallel to *a vector X^α is a projector $(\mathcal{P}_{||})^\alpha{}_\beta$ satisfying $(\mathcal{P}_{||})^\alpha{}_\beta Y^\beta \propto X^\alpha$ for any vector Y^α.*
A projector perpendicular to *a vector X^α is a projector $(\mathcal{P}_\perp)^\alpha{}_\beta$ satisfying $X_\alpha \mathcal{P}^\alpha{}_\beta Y^\beta = 0$, for any vector Y^α.*

(i) Suppose an ideal fluid has 4-velocity u^α, normalized as usual such that $u^\alpha u_\alpha = -1$. Construct the projectors parallel to and perpendicular to the fluid's 4-velocity. Rewrite the stress-energy tensor of the ideal fluid $T^{\alpha\beta} = \rho_{\rm rf}\, u^\alpha u^\beta + P_{\rm rf}\, (g^{\alpha\beta} + u^\alpha u^\beta)$ in terms of these projectors.

The desired projectors are

$$(\mathcal{P}_{||})^\alpha{}_\beta = -u^\alpha u_\beta, \qquad (\mathcal{P}_\perp)^\alpha{}_\beta = \delta^\alpha_\beta + u^\alpha u_\beta. \tag{31}$$

Let us first prove that they are indeed projectors:

$$(\mathcal{P}_{||})^\alpha{}_\beta(\mathcal{P}_{||})^\beta{}_\gamma = u^\alpha u_\beta u^\beta u_\gamma = -u^\alpha u_\gamma = (\mathcal{P}_{||})^\alpha{}_\gamma, \tag{32}$$

since $u_\mu u^\mu = -1$.

$$(\mathcal{P}_\perp)^\alpha{}_\beta(\mathcal{P}_\perp)^\beta{}_\gamma = (\delta^\alpha_\beta + u^\alpha u_\beta)(\delta^\beta_\gamma + u_\beta u_\gamma) = \delta^\alpha_\gamma + 2u^\alpha u_\gamma + u^\alpha u_\beta u_\beta u_\gamma = \delta^\alpha_\gamma + u^\alpha u_\gamma = (\mathcal{P}_\perp)^\alpha{}_\gamma. \tag{33}$$

It should be clear that $(\mathcal{P}_{||})^\alpha{}_\beta Y^\beta = -(Y^\beta u_\beta)u^\alpha$ is along u^α, as it should. Moreover,

$$u_\alpha(\mathcal{P}_\perp)^\alpha{}_\beta Y^\beta = u_\alpha(\delta^\alpha_\beta + u^\alpha u_\beta)Y^\beta = u_\beta Y^\beta(1 + u_\alpha u^\alpha) = 0. \tag{34}$$

Recalling that $g^\alpha_\beta = \delta^\alpha_\beta$, we have

$$T^{\alpha\beta} = -\rho_{\rm rf}(\mathcal{P}_{||})^{\alpha\beta} + P_{\rm rf}\ (\mathcal{P}_\perp)^{\alpha\beta}. \tag{35}$$

(ii) What physical quantity does the scalar field $\dot{Q} \equiv -u_\alpha \nabla_\beta T^{\alpha\beta}$ represent? Show that $\dot{Q} = \nabla_\alpha(\rho_{\rm rf}\ u^\alpha) + P_{\rm rf}\ \nabla_\alpha u^\alpha$.

In the rest-frame of the fluid, the components of the 4-velocity are, by definition $u^\mu = (1, 0, 0, 0)$. Thus, taking inertial coordinates in that rest-frame, we have $u_\mu = (-1, 0, 0, 0)$ in the fluid's rest frame. Thus, $\dot{Q} = +\nabla_\beta T^{0\beta}$ in the fluid's rest-frame. This is the zero-th component of the 4-force density (see lecture 10). A 4-force is a rate of change of 4-momentum. The zero-th component of 4-momentum is energy. Thus, **$\dot{Q}$ is the rate of heating per unit volume, in the fluid's rest-frame** – heating is just a rate of change of energy! In general, whenever we multiply a quantity by $-u_\alpha$, it represents the value of the zero-th component of this quantity in the rest-frame of the "observer" whose 4-velocity is u^α.

Let us now compute this heating rate:

$$\dot{Q} = -u_\alpha \nabla_\beta T^{\beta\alpha} = -\nabla_\beta(u_\alpha T^{\beta\alpha}) + T^{\beta\alpha}\nabla_\beta u_\alpha. \tag{36}$$

The first term in parenthesis only picks the component of $T^{\beta\alpha}$ parallel to u^α. In addition,

$$u^\alpha \nabla_\beta u_\alpha = \frac{1}{2}\nabla_\beta(u^\alpha u_\alpha) = \frac{1}{2}\nabla_\beta(-1) = 0. \tag{37}$$

Thus the second term only picks the components of $T^{\alpha\beta}$ proportional to $g^{\alpha\beta}$. Hence we find the desired expression.

(iii) Suppose that the ideal fluid is made of particles with conserved number. *This means that there exists a number 4-current N^α, such that N^0 is the number density and $\vec{N}$ is the number flux, which satisfies $\nabla_\alpha N^\alpha = 0$. Denoting by $n_{\rm rf}$ the particle number density in the fluid's rest-frame, derive the expression for the particle number current N^α in an arbitrary frame.*

By definition, in the fluid's rest frame, $N^0 = n_{\rm rf}$ and $N^i = 0$, since this frame was defined as being *isotropic*, hence there is no preferred direction for the number flux. Now in the fluid's rest frame we have $u^0 = 1$ and $u^i = 0$. So in that frame, $N^\mu = n_{\rm rf} u^\mu$. This is a vector equality, which should hold in any frame – note that $n_{\rm rf}$ is a *scalar* field, defined as the fluid number density *in the fluid's rest frame.*

(iv) We define $\epsilon_{\rm rf} \equiv \rho_{\rm rf}/n_{\rm rf}$. What physical quantity does this represent? Combining particle number conservation with question (ii), derive an equation involving $\dot{Q}$ and $\epsilon_{\rm rf}$, and explain why it represents the first law of thermodynamics.

$\epsilon_{\rm rf}$ is the ratio of energy density in the fluid's rest frame to the number density in the fluid's rest frame. It is thus the **mean energy per particle in the fluid's rest frame**.

$$\dot{Q} = \nabla_\alpha(\epsilon_{\rm rf} n_{\rm rf} u^\alpha) + P_{\rm rf}\nabla_\alpha\left(\frac{1}{n_{\rm rf}} n_{\rm rf} u^\alpha\right) = \nabla_\alpha(\epsilon_{\rm rf} N^\alpha) + P_{\rm rf}\nabla_\alpha\left(\frac{N^\alpha}{n_{\rm rf}}\right) = n_{\rm rf} u^\alpha\left(\nabla_\alpha \epsilon_{\rm rf} + P_{\rm rf}\nabla_\alpha(1/n_{\rm rf})\right), \tag{38}$$

where we used $\nabla_\alpha N^\alpha = 0$ and substituted $N^\alpha = n_{\rm rf} u^\alpha$. Let us now consider a small volume V enclosing a fixed number of particles $N_{\rm part} \equiv n_{\rm rf} V$, where we work in the fluid's rest frame. We may thus replace $1/n_{\rm rf} = V/N_{\rm part}$. We also define the total energy of these particles by $U = N_{\rm part}\epsilon_{\rm rf}$. Multiplying the above equation by V, we get

$$V\dot{Q} = u^\alpha \nabla_\alpha U + P_{\rm rf} u^\alpha \nabla_\alpha V = \frac{dU}{d\tau} + P_{\rm rf}\frac{dV}{d\tau} \tag{39}$$

The left-hand side is the rate of heating of the volume V. This is exactly the first law of thermodynamics: $\delta Q = dU + PdV$, written in differential form.