MA 403 MINOR REAL ANALYSIS

Practice Problem-1

Notations:
$\mathbb{R}$ denotes the set of reals.
$\mathbb{Q}$ stands for the set of rational numbers.
$\mathbb{Z}$ stands for the set of integers.

(1) If $x \in \mathbb{R}$, prove that
$$x = sup\{r \in \mathbb{Q} : r < x\} = \inf\{r \in \mathbb{Q} : r > x\}.$$

(2) Show that the least upper boud property also holds in $\mathbb{Z}$ (i.e. each non empty subset of $\mathbb{Z}$ with an upper bound in $\mathbb{Z}$ has a least upper bound in $\mathbb{Z}$).

(3) Let A be a non-empty subset of $\mathbb{R}$ that is bounded above. Show that there is a sequence (x_n) of elements of A which converges to $\sup A$.

(4) Prove that the field $\mathbb{Q}$ satisfies the Archimedean property but it doesn't satisfy the l.u.b. property.

(5) Prove that given $\alpha \in \mathbb{R}$, there exists a unique $m \in \mathbb{Z}$ such that $\alpha \in [m-1, m)$.

(6) If $a < b$, then there is also an $x \in \mathbb{R} \setminus \mathbb{Q}$ (i.e. irrational number) with $a < x < b$.

(7) Give an example of a set which has l.u.b. property but it has at least one non-empty subset which doesn't have l.u.b. property. Does the set of irrational numbers have l.u.b. property?

(8) Prove that every monotonically increasing sequence in $\mathbb{R}$ which is bounded above converges to some point in $\mathbb{R}$. (A sequence(a_n) is monotonically increasing (decreasing) if $a_k \leq (\geq) a_l$ whenever $k \leq l$)

(9) In an ordered field let x be any element s.t. $x \geq -1$. Prove that
$$(1 + x)^n \geq 1 + nx$$
for every positive integer n, where $nx := x + \cdot\cdot ntimes \cdot +x$.

(10) Define the sequence $a_n = \sqrt{n^2 + 1} - n$. Does this sequence converge? If yes, then find the limit point.

(11) If a_n is a monotone sequence in $\mathbb{R}$ that has a bounded subsequence then show that (a_n) is convergent. (Monotone sequence means either monotonically increasing or decreasing)

(12) Fix $b > 1$, prove that
$$(b^m)^{1/n} = (b^{1/n})^m,$$
where n is a positive integer and $m \in \mathbb{Z}$. $x^{1/n}$ is the unique positive solution of the equation $y^n = x$.

   (a) If m, n, p, q are integers such that $n > 0, q > 0$. Let $r = \frac{m}{n} = \frac{p}{q}$, prove that $(b^m)^{1/n} = (b^p)^{1/q}$, therefore it makes sense to define $b^r = (b^m)^{1/n}$.

   (b) If x is real, define $B(x)$ to be the set of b^t where $t \in \mathbb{Q}$ and $t \leq x$. Prove that $b^r = \sup B(r)$ where $r \in \mathbb{Q}$. Now we define $b^x := \sup B(x)$ for every $x \in \mathbb{R}$.

   (c) Prove that $b^{x+y} = b^x b^y$ for all $x, y \in \mathbb{R}$.

(13) Prove that the field of Complex numbers $\mathbb{C}$ cannot be made into an ordered field.