General Relativity Fall 2019
Homework 6 solution

Exercise 1: Commutator of vector fields

Given a vector field X, for any smooth function $f : \mathcal{M} \to \mathbb{R}$, we can define the smooth function $X(f)$ which, at each point $p \in \mathcal{M}$, associates $X|_p(f)$. Given two vector fields X and Y, we define their commutator *$[X, Y]$ such that*

$$[X, Y](f) \equiv X(Y(f)) - Y(X(f)). \tag{1}$$

(i) Prove that $[X, Y]$ is a vector field, i.e. satisfies all the properties of a vector at each event $p \in \mathcal{M}$, and is smooth.

Let us first show that $[X, Y]|_p$ is a tangent vector at p. It is clearly a *linear* operator on functions, since both X and Y are. We need to show that it satisfies Leibniz's rule. Since X is a vector, is satisfies Leibniz's rule at any point $q \in \mathcal{M}$: $X|_q(fg) = f(q)X|_q(g) + g(q)X|_q(f)$. We can rewrite this as a functional equality: $X(fg) = fX(g) + gX(f)$. Thus,

$$\begin{aligned}
[X, Y]|_p(fg) &= X|_p(Y(fg)) - Y|_p(X(fg)) = X|_p(fY(g) + gY(f)) - Y|_p(fX(g) + gX(f)) \\
&= f(p)X|_p(Y(g)) + Y|_p(g)X|_p(f) + g(p)X|_p(Y(f)) + Y|_p(f)X|_p(g) \\
&\quad - f(p)Y|_p(X(g)) - X|_p(g)Y|_p(f) - g(p)Y|_p(X(f)) - X|_p(f)Y|_p(g) \\
&= f(p)\left(X|_p(Y(g)) - Y|_p(X(g))\right) + g(p)\left(X|_p(Y(f)) - Y|_p(X(f))\right) \\
&= f(p)[X, Y]|_p(g) + g(p)[X, Y]|_p(f),
\end{aligned} \tag{2}$$

i.e. $[X, Y]|_p$ satisfies Leibniz's rule.

Next, to show that $[X, Y]|_p$ is smooth, we must show, by definition, that for any smooth function f, the function $p \mapsto [X, Y]|_p(f)$ is smooth. Since both X and Y are smooth, this is indeed satisfied.

(ii) Compute the components of $[X, Y]$ in a coordinate basis as a function of the components of X and Y. Show that the commutator of coordinate basis vectors vanishes, $[\partial_{(\mu)}, \partial_{(\nu)}] = 0$, for any μ, ν.

Let f be a smooth function. Then $X(f) = X^\mu \partial_\mu f$, hence

$$X(Y(f)) = X^\nu \partial_\nu (Y^\mu \partial_\mu f) = X^\nu Y^\mu \partial_\mu \partial_\nu f + X^\nu (\partial_\nu Y^\mu) \partial_\mu f. \tag{3}$$

As a consequence,

$$[X, Y](f) = \left[X^\nu \partial_\nu Y^\mu - Y^\nu \partial_\nu X^\mu\right] \partial_\mu f. \tag{4}$$

On the other hand, by definition, $[X, Y](f) = [X, Y]^\mu \partial_\mu f$. Thus we found

$$\boxed{[X, Y]^\mu = X^\nu \partial_\nu Y^\mu - Y^\nu \partial_\nu X^\mu}. \tag{5}$$

In particular, if $X = \partial_{(\mu)}$ and $Y = \partial_{(\nu)}$, then the components $X^\sigma = \delta^\sigma_\mu$ are constant, and so are the components Y^σ. Hence $[\partial_{(\mu)}, \partial_{(\nu)}] = 0$.

(iii) Recall the notation convention $\nabla_X Y^\alpha \equiv X^\beta \nabla_\beta Y^\alpha$. Prove that $[X, Y]^\alpha = \nabla_X Y^\alpha - \nabla_Y X^\alpha$.

Recall that

$$\nabla_X Y^\alpha = X^\beta \nabla_\beta Y^\alpha = X^\beta \partial_\beta Y^\alpha + \Gamma^\alpha_{\beta\sigma} X^\beta Y^\sigma. \tag{6}$$

We see that the Christoffel symbol pieces in $\nabla_X Y^\alpha$ and $\nabla_Y X^\alpha$ are identical, and cancel out, and we get

$$\nabla_X Y^\alpha - \nabla_Y X^\alpha = X^\beta \partial_\beta Y^\alpha - Y^\beta \partial_\beta X^\alpha = [X, Y]^\alpha. \tag{7}$$

(iv) Show that, for any three vector fields X, Y, Z, we have

$$\nabla_X \nabla_Y Z^\alpha - \nabla_Y \nabla_X Z^\alpha - \nabla_{[X,Y]} Z^\alpha = R^\alpha{}_{\beta\gamma\delta} Z^\beta X^\gamma Y^\delta. \tag{8}$$

This is just a matter of being careful with the placement of indices:

$$\nabla_X \nabla_Y Z^\alpha = X^\delta \nabla_\delta (Y^\gamma \nabla_\gamma Z^\alpha) = X^\delta Y^\gamma \nabla_\delta \nabla_\gamma Z^\alpha + (X^\delta \nabla_\delta Y^\gamma) \nabla_\gamma Z^\alpha, \tag{9}$$

$$\nabla_Y \nabla_X Z^\alpha = Y^\gamma \nabla_\gamma (X^\delta \nabla_\delta Z^\alpha) = X^\delta Y^\gamma \nabla_\gamma \nabla_\delta Z^\alpha + (Y^\gamma \nabla_\gamma X^\delta) \nabla_\delta Z^\alpha. \tag{10}$$

Therefore, renaming dummy indices in the last term,

$$\nabla_X \nabla_Y Z^\alpha - \nabla_Y \nabla_X Z^\alpha = X^\delta Y^\gamma \left(\nabla_\delta \nabla_\gamma Z^\alpha - \nabla_\gamma \nabla_\delta Z^\alpha \right) + (X^\delta \nabla_\delta Y^\gamma - Y^\delta \nabla_\delta X^\gamma) \nabla_\gamma Z^\alpha. \tag{11}$$

The first term in parenthesis is the difference between second derivatives of the vector field Z^α, and is $R^\alpha{}_{\beta\delta\gamma} Z^\beta$, by definition of the Riemann tensor. The second term in parenthesis is $[X, Y]^\gamma$. Thus we arrived at the desired expression.

Consider flat 3-D space, with cartesian coordinates x^1, x^2, x^3 and associated coordinate basis $\{\partial_{(i)}\}$, $i = 1, 2, 3$, and line element $d\ell^2 = \delta_{ij} dx^i dx^j$. Define the three vectors fields $V_{(i)} \equiv \epsilon_{ijk} x^j \partial_{(k)}$, where ϵ_{ijk} is the Levi-Civita symbol (fully antisymmetric, such that $\epsilon_{123} = 1$), and we sum over repeated indices, even if they are not up and down.

(*v*) *Show that these three vector fields are Killing vector fields.*

Recall that, by definition, a Killing vector field K^α must satisfy $\nabla_{(\alpha} K_{\beta)} = 0$. Let's define $V^{(i)}_\alpha \equiv g_{\alpha\beta} V^\beta_{(i)}$ – just to get the parenthesized label at the right position, i.e. upstairs for a dual vector. Recall that we are assuming flat 3D space with cartesian coordinates, thus the metric components are $g_{kl} = \delta_{kl}$, and Christoffel symbols all vanish. Hence, $V^{(i)}_k = V^k_{(i)}$ – this means that the *numerical values* of these components are equal, in a cartesian coordinate system. Now, by definition, $V^k_{(i)} = \epsilon_{ijk} x^j$, so we have $V^{(i)}_k = \epsilon_{ijk} x^j$. Let us now compute

$$\nabla_l V^{(i)}_k = \partial_l V^{(i)}_k \qquad \text{[Christoffel symbols vanish]} \tag{12}$$

$$= \epsilon_{ijk} \partial_l x^j = \epsilon_{ijk} \delta^j_l = \epsilon_{ilk}. \tag{13}$$

This is antisymmetric in k, l, thus $\nabla_{(l} V^{(i)}_{k)} = 0$. Hence $V_{(i)}$ is a Killing vector field.

(*vi*) *Show that they satisfy the commutation relations $[V_{(i)}, V_{(j)}] = -\epsilon_{ijk} V_{(k)}$.*
Using our previous results,

$$\begin{aligned}[V_{(i)}, V_{(j)}]^k &= V^l_{(i)} \partial_l V^k_{(j)} - V^l_{(j)} \partial_l V^k_{(i)} = \epsilon_{iml} x^m \partial_l (\epsilon_{jnk} x^n) - \epsilon_{jnl} x^n \partial_l (\epsilon_{imk} x^m) \\ &= \epsilon_{iml} x^m \epsilon_{jlk} - \epsilon_{jnl} x^n \epsilon_{ilk} = \left(\epsilon_{jml} \epsilon_{ikl} - \epsilon_{iml} \epsilon_{jkl} \right) x^m, \end{aligned} \tag{14}$$

where we used the antisymmetry of ϵ_{ijk} and renamed dummy indices. Now, a very useful relation to remember is the contraction of two Levi-Civita symbols on one component:

$$\boxed{\epsilon_{iml} \epsilon_{jkl} = \delta_{ij} \delta_{mk} - \delta_{ik} \delta_{mj}}. \tag{15}$$

Thus we find

$$[V_{(i)}, V_{(j)}]^k = (\delta_{ik} \delta_{jm} - \delta_{im} \delta_{jk}) x^m = \delta_{ik} x^j - \delta_{jk} x^i. \tag{16}$$

On the other hand,

$$-\epsilon_{ijl} V^k_{(l)} = -\epsilon_{ijl} \epsilon_{lmk} x^m = \epsilon_{ijl} \epsilon_{kml} x^m = (\delta_{ik} \delta_{jm} - \delta_{im} \delta_{jk}) x^m = \delta_{ik} x^j - \delta_{jk} x^i. \tag{17}$$

Thus, we found $[V_{(i)}, V_{(j)}]^k = -\epsilon_{ijl} V^k_{(l)}$, i.e. the k-th component of these two vectors are equal, which proves the desired equality.

Exercise 2: Rindler spacetime

Consider the two-dimensional spacetime with metric $ds^2 = -x^2 dt^2 + dx^2$, with $-\infty < t < \infty$ and $0 < x < \infty$.

(*i*) *Compute the connection coefficients in these coordinates with your favorite method.*

We'll use the fact that geodesics are extrema of

$$I = \frac{1}{2}\int d\tau\ g_{\mu\nu}\dot{x}_\mu \dot{x}_\nu = \frac{1}{2}\int d\tau \left[-x^2\dot{t}^2 + \dot{x}^2\right], \qquad [\dot{x}^\mu \equiv \frac{dx^\mu}{d\tau}]. \tag{18}$$

Euler-Lagrange equations give us

$$\frac{d}{d\tau}\left[x^2\dot{t}\ \right] = 0 = x^2\ddot{t} + 2x\dot{x}\dot{t}, \qquad \frac{d}{d\tau}\dot{x} = -x\ \dot{t}^2. \tag{19}$$

From this we directly read off the non-vanishing Christoffel symbols by matching with the geodesic equation:

$$\Gamma^t_{tx} = \frac{1}{x}, \quad \Gamma^x_{tt} = x. \tag{20}$$

(*ii*) *Compute the Ricci scalar of this spacetime (do so in the most efficient way, using results about the form of Riemann in dimension 2). What does your result imply? Make sure to invoque the Riemann tensor in your argumentation.*

$$\begin{aligned} R = g^{\sigma\nu}R^\mu{}_{\sigma\mu\nu} &= g^{\sigma\nu}\left[\Gamma^\mu_{\sigma\nu,\mu} - \Gamma^\mu_{\sigma\mu,\nu} + \Gamma^\mu_{\mu\lambda}\Gamma^\lambda_{\sigma\nu} - \Gamma^\mu_{\nu\lambda}\Gamma^\lambda_{\sigma\mu}\right] \\ &= -\frac{1}{x^2}\left[\Gamma^\mu_{tt,\mu} - \Gamma^\mu_{t\mu,t} + \Gamma^\mu_{\mu\lambda}\Gamma^\lambda_{tt} - \Gamma^\mu_{t\lambda}\Gamma^\lambda_{t\mu}\right] + \left[\Gamma^\mu_{xx,\mu} - \Gamma^\mu_{x\mu,x} + \Gamma^\mu_{\mu\lambda}\Gamma^\lambda_{xx} - \Gamma^\mu_{x\lambda}\Gamma^\lambda_{x\mu}\right] \\ &= -\frac{1}{x^2}\left[\Gamma^x_{tt,x} + \Gamma^t_{tx}\Gamma^x_{tt} - 2\Gamma^t_{tx}\Gamma^x_{tt}\right] + \left[-\Gamma^t_{xt,x} - (\Gamma^t_{xt})^2\right] = -\frac{1}{x^2}\left[1-1\right] + \left[\frac{1}{x^2} - \frac{1}{x^2}\right] = 0. \end{aligned} \tag{21}$$

So we find that the Ricci scalar of this metric vanishes everywhere. Since in two dimensions the Riemann tensor has only one independent component, and may be written as $R_{\mu\nu\rho\sigma} = \frac{R}{2}\left(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu}\right)$, we conclude that the Riemann tensor vanishes everywhere.

As a consequence, the spacetime represented by this metric is flat: the fact that the metric coefficients are not the 2-D Minkowski metric is simply a consequence of the coordinate choice. **Note**: the condition for the spacetime to be flat is that the Riemann tensor vanishes, not the Ricci scalar. In 2-D these are equivalent conditions but not in higher dimensions!

(*iii*) *What is the acceleration 2-vector of observers of constant x?*

Observers of constant x have 2-velocity $u^\mu = (u^t, u^x) = (\dot{t}, \dot{x}) = (\dot{t}, 0)$. The 2-acceleration vector is

$$a^\mu = u^\alpha \nabla_\alpha u^\mu = \frac{du^\mu}{d\tau} + \Gamma^\mu_{\lambda\sigma}u^\lambda u^\sigma = \frac{du^\mu}{d\tau} + \Gamma^\mu_{tt}\dot{t}^2 = (\ddot{t}, x\dot{t}^2). \tag{22}$$

Now, since the 2-velocity is normalized, $-1 = g_{\mu\nu}u^\mu u^\nu = -x^2\dot{t}^2$, we conclude that $\dot{t} = \pm 1/|x|$ is constant, and as a consequence $\ddot{t} = 0$. So the components of the 2-acceleration in these coordinates is

$$(a^t, a^x) = \left(0, \frac{1}{x}\right). \tag{23}$$

To conclude, stationary observers (i.e. which have constant spatial coordinate x) are accelerated, with a constant acceleration $a^x = 1/x$ – **they are therefore not on geodesics, as their 2-acceleration is non-zero!**

(*iv*) *Show that null geodesics are such that $t = \pm\ln(x) +$ constant. Define the coordinates $u \equiv t - \ln(x)$ and $v \equiv t + \ln(x)$. What are the null geodesics in these coordinates? What are the metric coefficients in these coordinates?*

Let us parametrize a null geodesic with the affine parameter λ, such that $p^\alpha = d/d\lambda$. Null geodesics are such that $g_{\mu\nu}p^\mu p^\nu = 0$, implying

$$-x^2\dot{t}^2 + \dot{x}^2 = 0, \tag{24}$$

where overdots denote $d/d\lambda$. We then get $\dot{t} = \pm\dot{x}/x$, i.e. $t = \pm\ln(x) +$ constant.

The geodesic $t = +\ln(x) +$ constant have $u =$ constant, $v = 2t-$ constant. Similarly, the geodesics with $t = -\ln(x)+$ constant have $u = 2t-$ constant and $v =$ constant. $\Rightarrow$ **lines of constant u or constant v are null geodesics**.

Differentiating, we have $du = dt - \frac{dx}{x}$ and $dv = dt + \frac{dx}{x}$. It is easy to then see that $dudv = dt^2 - \frac{dx^2}{x^2} = -\frac{1}{x^2}ds^2$. So we get $ds^2 = -x^2 dudv$. Now, we also have $2\ln(x) = v - u$, implying $x^2 = \mathrm{e}^{v-u}$. So we arrive at

$$ds^2 = -\mathrm{e}^{v-u}dudv, \qquad \Rightarrow g_{uu} = g_{vv} = 0, \quad g_{uv} = g_{vu} = -\frac{1}{2}\mathrm{e}^{v-u}. \tag{25}$$

(*v*) *Can you find new coordinates T, X in which $ds^2 = -dT^2 + dX^2$? [*Hint*: start by finding coordinates U, V in which $ds^2 = -dUdV$]. Give an explicit expression of the old coordinates in terms of the new coordinates.*

Let's rewrite the above result as

$$ds^2 = -\mathrm{e}^{-u}du \times \mathrm{e}^{v}dv = d(\mathrm{e}^{-u})d(\mathrm{e}^{v}) \tag{26}$$

Define $U \equiv -\mathrm{e}^{-u}$ and $V \equiv \mathrm{e}^{v}$. In these new coordinates the metric is $ds^2 = -dUdV$. Finally, let us define $X \equiv (V - U)/2$ and $T \equiv (V + U)/2$, amounting to $V = T + X$ and $U = T - X$. In these coordinates, we have $ds^2 = -dT^2 + dX^2$.

To recap, we have found an explicit coordinate transformation in which the metric is Minkowski:

$$T = \frac{1}{2}(\mathrm{e}^{v} - \mathrm{e}^{-u}) = \frac{1}{2}x(\mathrm{e}^{t} - \mathrm{e}^{-t}) = x\sinh(t), \qquad X \equiv \frac{1}{2}(\mathrm{e}^{v} + \mathrm{e}^{-u}) = \frac{1}{2}x(\mathrm{e}^{t} + \mathrm{e}^{-t}) = x\cosh(t). \tag{27}$$

Note that the range of these variables is $X \in (0, +\infty)$ and $-X < T < X$.

The inverse of this transformation is

$$\boxed{x = \sqrt{X^2 - T^2}, \qquad t = \tanh^{-1}(T/X)} \tag{28}$$

(*vi*) *In a spacetime diagram in the (T, X) coordinates, draw the lines corresponding to constant x and those corresponding to constant t.*

The observers of constant x (which, as we saw, are accelerated) ave coordinates such that $X^2 - T^2 = x^2 =$ constant, which are hyperbolas in the (T, X) plane. Lines of constant t are such that $T/X = \tanh(t) =$ constant. The line $x \to 0$ corresponds to the two lines $X = \pm T$. These lines also correspond to $t = \pm\infty$. We show these lines in Fig. 1. The initial coordinates $t \in (-\infty, \infty)$ and $x \in (0, \infty)$ only cover region I, with $X > 0$ and $-X < T < X$. The other regions are not covered by the initial coordinates, however, they are perfectly fine regions of flat spacetime.

When we discuss Schwarzschild black holes, we will go through a very similar process to "extend" the region of spacetime covered by the original coordinates. The apparent singularity at $x \to 0$ is very similar to the apparent singularity at $r \to 2M$, the Schwarzschild radius.

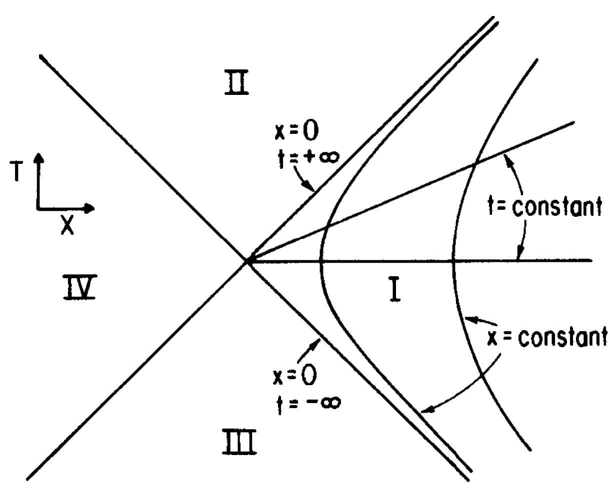

FIG. 1. Rindler spacetime (taken from Wald's textbook *General Relativity*).

Exercise 3: Ricci scalar of special metric

At linear order in Φ, compute the 10 components of the Ricci tensor, as well as the Ricci scalar of the metric

$$ds^2 = -\left[1 + 2\Phi(\vec{x})\right] dt^2 + \left[1 - 2\Phi(\vec{x})\right] \delta_{ij} dx^i dx^j. \tag{29}$$

Note that $\Phi(\vec{x})$ is assumed to depend only on the spatial coordinates (i.e. not on t).

Let us start by rewriting the metric components as $g_{\mu\nu} = \eta_{\mu\nu} - 2\Phi\delta_{\mu\nu}$. Since the Christoffel symbols vanish for Minkowski, they are at least linear in Φ. Thus, to linear order in Φ, we have

$$R^{\lambda}{}_{\gamma\mu\nu} = \partial_\mu \Gamma^{\lambda}_{\nu\gamma} - \partial_\nu \Gamma^{\lambda}_{\mu\gamma} + \mathcal{O}(\Phi^2). \tag{30}$$

Contracting, we find the components of the Ricci tensor:

$$R_{\gamma\nu} = \partial_\mu \Gamma^{\mu}_{\nu\gamma} - \partial_\nu \Gamma^{\mu}_{\mu\gamma} + \mathcal{O}(\Phi^2). \tag{31}$$

Now let us compute the Christoffel symbols at linear order in Φ:

$$\Gamma^{\mu}_{\nu\gamma} = \frac{1}{2}\eta^{\mu\sigma}\left(\partial_\nu g_{\gamma\sigma} + \partial_\gamma g_{\nu\sigma} - \partial_\sigma g_{\nu\gamma}\right) = -\eta^{\mu\sigma}\left(\partial_\nu \Phi \delta_{\gamma\sigma} + \partial_\gamma \Phi \delta_{\nu\sigma} - \partial_\sigma \Phi \delta_{\nu\gamma}\right). \tag{32}$$

From this we get

$$\partial_\mu \Gamma^{\mu}_{\nu\gamma} = -\eta^{\mu\sigma}\left(\partial_\mu\partial_\nu\Phi \ \delta_{\gamma\sigma} + \partial_\mu\partial_\gamma\Phi \ \delta_{\nu\sigma} - \partial_\mu\partial_\sigma\Phi \ \delta_{\nu\gamma}\right), \tag{33}$$

$$\partial_\nu \Gamma^{\mu}_{\mu\gamma} = -\eta^{\mu\sigma}\left(\partial_\nu\partial_\mu\Phi \ \delta_{\gamma\sigma} + \partial_\nu\partial_\gamma\Phi \ \delta_{\mu\sigma} - \partial_\nu\partial_\sigma\Phi \ \delta_{\mu\gamma}\right), \tag{34}$$

thus

$$\begin{aligned} R_{\gamma\nu} &= -\eta^{\mu\sigma}\left(\partial_\gamma\partial_\mu\Phi \ \delta_{\nu\sigma} - \partial_\gamma\partial_\nu\Phi \ \delta_{\mu\sigma} + \partial_\sigma\partial_\nu\Phi \ \delta_{\mu\gamma} - \partial_\sigma\partial_\mu\Phi \ \delta_{\nu\gamma}\right) \\ &= -\left(\partial_\gamma\partial^\sigma\Phi \ \delta_{\nu\sigma} - 2\partial_\gamma\partial_\nu\Phi + \partial_\nu\partial^\mu\Phi \ \delta_{\mu\gamma} - \partial^\mu\partial_\mu\Phi \ \delta_{\nu\gamma}\right), \end{aligned} \tag{35}$$

where I used $\eta^{\mu\sigma}\delta_{\mu\sigma} = \eta^{00} + \eta^{11} + \eta^{22} + \eta^{33} = 2$. Let's now compute individual components. Since $\partial_0\Phi = 0$, $\partial^\mu\partial_\mu\Phi = \nabla^2\Phi$. Only the last piece does not vanish in R_{00}:

$$R_{00} = \nabla^2\Phi. \tag{36}$$

We know that R_{0i} has to be proportional to $\partial_0\partial_i\Phi$ – there had to be only one free index i and a total of two derivatives. Thus, it must be that $R_{0i} = 0$. Lastly, the three first pieces cancel out for R_{ij}, and we get

$$R_{ij} = \nabla^2\Phi \ \delta_{ij} \tag{37}$$

The Ricci scalar is then, to linear order,

$$R = \delta^{ij}R_{ij} - R_{00} = 2\nabla^2\Phi. \tag{38}$$

For reference, the Ricci scalar of a sphere of radius r is $R = 2/r^2$. Thus, the characteristic radius of curvature of this spacetime is

$$r \sim 1/\sqrt{\nabla^2\Phi}. \tag{39}$$