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Roll No:

MS21050

50 marks, 3 hours

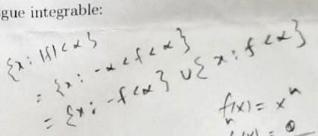
Show that all non-empty open sets in R^d have positive measure.

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(ii) Check if the following are Lebesgue integrable:

(a)
$$u(x) = \frac{1}{x}$$
, over $[1, \infty)$

(b)
$$v(x) = \frac{1}{\sqrt{x}}$$
, over $(0, 1]$.



$$f(x) = x \qquad \int f(x) = \frac{x}{x}$$

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$$f(x) = x \qquad \int f(x) = \frac{x}{x}$$

$$f(x) = x \qquad \int f(x) = 0 \qquad (2x)$$

2. Let $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ be a function.

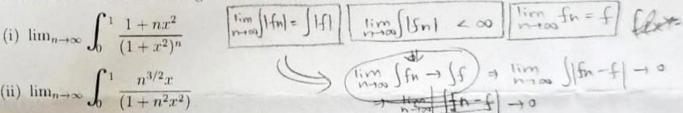
(i) If
$$f$$
 is measurable, then show that $|f|$ is measurable.

(ii) Show that |f| measurable does not imply that f is measurable, by giving an example.

Use the dominated convergence theorem to calculate the limit.

i)
$$\lim_{n\to\infty} \int_0^1 \frac{1+nx^2}{(1+x^2)^n}$$

$$\begin{cases}
5 & (1+x^2)^n \\
5 & (1+x^2)^n
\end{cases}$$
(ii) $\lim_{n\to\infty} \int_0^1 \frac{n^{3/2}x}{(1+n^2x^2)}$



4. Let $f_n \in L^1(\mathbb{R}^d)$ be a sequence of functions and $\lim_{n\to\infty} f_n = f$ a.e. If $\lim_{n\to\infty} \int |f_n| = \int |f|$, then show that $\lim_{n\to\infty} \int f_n = \int f$. Show that the converse is not true by constructing a couterexample.

5. Let f be a function which is absolutely continuous on the interval [a, b]. If $f \ge c$ for some c > 0. Then show that 1/f is also absolutely continuous on [a, b].

6. Let
$$f(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$
 Calculate $D^+(f)(0), D_-(f)(0).$

 \mathcal{T} . Let (X, \mathcal{M}, μ) be a measure space. Consider a sequence of sets $\{E_n\}_n$ in \mathcal{M} , such that $\sum_{n=1}^{\infty} \mu(E_n) < \infty$.

(i) Show that
$$\mu\left(\bigcap_{n=1}^{\infty}\bigcup_{k\geq n}E_{n}\right)=0.$$

$$\mu\left(\bigcap_{n=1}^{\infty}\bigcup_{k\geq n}E_{k}\right)=\mu\left(\bigcap_{n=1}^{\infty}\bigcup_{k\geq n}E_{k}\right)=\mu\left(\bigcap_{n=1}^{\infty}\bigcup$$

(ii) If f > 0 and is an integrable function on X, then show that μ is a σ -finite measure on X.

8. Let (X, \mathcal{M}, μ) be a measure space with $\mu(X) = 1$. Consider the collection of subsets

$$\mathcal{E} = \{ A \in \mathcal{M} : \mu(A) = 0 \text{ or } 1 \}.$$

Show that \mathcal{E} is a σ -algebra.

- 9. Let (X, \mathcal{M}, μ) be a measure space. Let $f: X \longrightarrow [0, \infty]$ be a measurable function such that $\int_{X} f d\mu < \infty$.
 - (i) Let $A = \{x \in X : f(x) = \infty\}$. Show that A is measurable, and $\mu(A) = 0$.
- . (ii) For any $\varepsilon > 0$, show that there exists $\alpha > 0$ such that

$$\int_{E} f d\mu < \varepsilon \text{ whenever } E \in \mathcal{M} \text{ with } \mu(E) \leq \alpha.$$

10. Let g be a measurable function on [0,1] such that the function

$$f(x,y) = 3g(y) - 2g(x)$$

is integrable on $[0,1] \times [0,1]$. Show that g is Lebesgue integrable on [0,1].

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** TAE