① Say $|\varphi\rangle$ is an element of a set of linearly independent vectors spanning the entire $\mathcal{H}$-space.
Say this set is $\{|\varphi_i\rangle\}$, where $|\varphi_1\rangle = |\varphi\rangle$.
Now using Gram-schmidt orthonormalization, we convert the set with orthonormal elements, $\{|\tilde{\varphi}_i\rangle\}$, where $|\tilde{\varphi}_1\rangle = \dfrac{|\varphi\rangle}{\sqrt{\langle\varphi|\varphi\rangle}}$ & $\sum_{i=1}^{d} |\tilde{\varphi}_i\rangle\langle\tilde{\varphi}_i| = \mathbb{I}$

Consider the innerporduct of $|\psi\rangle$,

$$\langle\psi|\psi\rangle = \langle\psi|\sum_i |\varphi_i\rangle\langle\varphi_i|\psi\rangle = \sum_{i=1}^{d} \langle\psi|\tilde{\varphi}_i\rangle\langle\tilde{\varphi}_i|\psi\rangle$$

$$= \langle\psi|\tilde{\varphi}_1\rangle\langle\tilde{\varphi}_1|\psi\rangle + \sum_{i=2}^{d} |\langle\psi|\tilde{\varphi}_i\rangle|^2$$

$$\geq \langle\psi|\tilde{\varphi}_1\rangle\langle\tilde{\varphi}|\psi\rangle = \frac{\langle\psi|\varphi\rangle\langle\psi|\varphi\rangle}{\langle\varphi|\varphi\rangle}$$

$$\Rightarrow \langle\psi|\psi\rangle\langle\varphi|\varphi\rangle \geq |\langle\psi|\varphi\rangle|^2$$

② ⓐ M_i is hermitian, then $M_i = \sum \lambda_m |e_m\rangle\langle e_m|$.

The property $M_i^2 = \mathbb{I}$ implies $\lambda_m^2 = 1$.

Thus, $\lambda_m = \pm 1$.

ⓑ $Tr(M_i) = Tr(M_j M_j M_i)$, with $i \neq j$

$$= -Tr(M_j M_i M_j)$$

$$= -Tr(M_i M_j M_j) = -Tr(M_i)$$

This is satisfied if $Tr(M_i) = 0$.

ⓒ Since the eigen values are ± 1 & trace equals to zero, the 'd' has to be even to make equal number of $+1$ & -1 eigenvalues.

③ The unnormalized vector is

$$|\psi\rangle = (1+i)|z,+\rangle + (1+i\sqrt{3})|z,-\rangle \text{ and } \langle\psi|\psi\rangle = 6$$

The normalized vector, ie the spin state, is

$$|\tilde{\psi}\rangle = \frac{1}{\sqrt{\langle\psi|\psi\rangle}}|\psi\rangle = \frac{1}{\sqrt{6}}\left[(1+i)|z,+\rangle + (1+i\sqrt{3})|z,-\rangle\right]$$

$$\boxed{\begin{aligned} 1+i &= \sqrt{2}\, e^{i\pi/4} \\ 1+i\sqrt{3} &= 2\, e^{i\pi/3} \end{aligned}}$$

$$= \frac{1}{\sqrt{3}} e^{i\pi/4}|z,+\rangle + \sqrt{\frac{2}{3}}\, e^{i\pi/3}|z,-\rangle$$

$$= e^{i\pi/4}\left[\frac{1}{\sqrt{3}}|z,+\rangle + \sqrt{\frac{2}{3}}\, e^{i\pi/12}|z,-\rangle\right]$$

$$\approx \frac{1}{\sqrt{3}}|z,+\rangle + \sqrt{\frac{2}{3}}\, e^{i\pi/12}|z,-\rangle$$

$$= \sin\frac{\theta}{2}|z,+\rangle + \cos\frac{\theta}{2}\, e^{i\pi/12}|z,-\rangle.$$

Thus, $\theta = 2\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$, $\varphi = \pi/12$.

④ Initial state and Hamiltonian are

$$|\psi(0)\rangle = \alpha|r\rangle + \beta|l\rangle$$

$$H = \delta\left(|r\rangle\langle l| + |l\rangle\langle r|\right)$$

ⓐ The normalized eigenkets are

$\frac{1}{\sqrt{2}}\left(|r\rangle + |l\rangle\right)$ & $\frac{1}{\sqrt{2}}\left(|r\rangle - |l\rangle\right)$ with the corresponding eigenvalues $+\delta$ and $-\delta$ respectively.

Note, $H = \delta\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, for $|r\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ & $|l\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

$$= \delta\,\sigma_x .$$

(b) The time evolution operator is

$$U(t) = e^{-iHt/\hbar} = e^{-i\delta\sigma_x t/\hbar}.$$

$$= \cos\left(\frac{\delta t}{\hbar}\right)\mathbb{I} - i\sin\left(\frac{\delta t}{\hbar}\right)\sigma_x.$$

So the state at time t is

$$|\psi(t)\rangle = U(t)|\psi(0)\rangle$$

$$= \cos\left(\frac{\delta t}{\hbar}\right)|\psi(0)\rangle - i\sin\left(\frac{\delta t}{\hbar}\right)\sigma_x|\psi(0)\rangle$$

$$= \cos\left(\frac{\delta t}{\hbar}\right)\alpha|r\rangle + \cos\left(\frac{\delta t}{\hbar}\right)\beta|l\rangle - i\sin\left(\frac{\delta t}{\hbar}\right)\alpha|l\rangle$$
$$- i\sin\left(\frac{\delta t}{\hbar}\right)\beta|r\rangle$$

$$= \left(\alpha\cos\left(\frac{\delta t}{\hbar}\right) - i\beta\sin\left(\frac{\delta t}{\hbar}\right)\right)|r\rangle + \left(\cos\left(\frac{\delta t}{\hbar}\right)\beta - i\sin\left(\frac{\delta t}{\hbar}\right)\alpha\right)|l\rangle$$

(c) At time $t=0$, the particle is in $|r\rangle$. It means $\alpha=1$ & $\beta=0$. Thus the time-evolved state is

$$|\Psi(t)\rangle = \alpha\cos\left(\frac{\delta t}{\hbar}\right)|r\rangle - i\alpha\sin\left(\frac{\delta t}{\hbar}\right)|l\rangle.$$

The time-dependent probability of finding the particle in $|l\rangle$ is

$$p_l(t) = \alpha^2\sin^2\left(\frac{\delta t}{\hbar}\right).$$

(d) Say $|\psi(t)\rangle = \alpha(t)|r\rangle + \beta(t)|l\rangle$.

Then according to Schrödings equation -

$$i\hbar \frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle$$

$$\Rightarrow i\hbar\left(\dot{\alpha}(t)|r\rangle + \dot{\beta}(t)|l\rangle\right) = \delta\sigma_x\left(\alpha(t)|r\rangle + \beta(t)|l\rangle\right)$$

$$= \delta\left(\alpha(t)|l\rangle + \beta(t)|r\rangle\right)$$

Thus the coupled equations are

$$\dot{\alpha}(t) = \frac{\delta}{i\hbar}\beta(t) \quad \& \quad \dot{\beta}(t) = \frac{\delta}{i\hbar}\alpha(t).$$

(e) $A_S = |r\rangle\langle r| = \frac{1}{2}(\mathbb{I} + \sigma_z)$.

$$\boxed{\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$$

The corresponding Heisenberg operator is

$$A_H(t) = U^\dagger(t) A_s U(t) = \frac{1}{2} U^\dagger(t) \sigma_z U(t) + \frac{1}{2}\mathbb{I}$$

$$= \frac{1}{2}\left[\cos\left(\frac{\delta t}{\hbar}\right)\mathbb{I} + i \sin\left(\frac{\delta t}{\hbar}\right)\sigma_x\right]\sigma_z\left[\cos\left(\frac{\delta t}{\hbar}\right)\mathbb{I} - i \sin\frac{\delta t}{\hbar}\sigma_x\right] + \frac{1}{2}\mathbb{I}$$

$$= \frac{1}{2}\left[\cos^2\left(\frac{\delta t}{\hbar}\right)\sigma_z - i\cos\frac{\delta t}{\hbar}\sin\frac{\delta t}{\hbar}\sigma_z\sigma_x + i\sin\frac{\delta t}{\hbar}\cos\frac{\delta t}{\hbar}\sigma_x\sigma_z + \sin^2\frac{\delta t}{\hbar}\sigma_x\sigma_z\sigma_x\right] + \frac{1}{2}\mathbb{I}$$

$$= \frac{1}{2}\left[\cos^2\frac{\delta t}{\hbar}\sigma_z + i\cos\frac{\delta t}{\hbar}\sin\frac{\delta t}{\hbar}[\sigma_x, \sigma_z] + \sin^2\frac{\delta t}{\hbar}(-\sigma_z)\right] + \frac{1}{2}\mathbb{I}$$

$$= \frac{1}{2}\left[\cos\left(\frac{2\delta t}{\hbar}\right)\sigma_z + \sin\left(\frac{2\delta t}{\hbar}\right)\sigma_y + \mathbb{I}\right].$$

The Heisenberg equation of motion is

$$\frac{d A_H(t)}{dt} = \frac{i}{\hbar}[H_H, A_H(t)] = \frac{i}{\hbar}[H, A_H(t)]$$

$$= \frac{i}{\hbar}\left[\delta \sigma_x , \cos\left(\frac{2\delta t}{\hbar}\right)\sigma_z + \sin\left(\frac{2\delta t}{\hbar}\right)\sigma_y + \mathbb{I}\right]\frac{1}{2}$$

$$= \frac{i}{2\hbar}\left(\delta \cos\left(\frac{2\delta t}{\hbar}\right)[\sigma_x, \sigma_z] + \delta \sin\left(\frac{2\delta t}{\hbar}\right)[\sigma_x, \sigma_y]\right)$$

$$= \frac{i}{2\hbar}\left(\delta \cos\left(\frac{2\delta t}{\hbar}\right)(-2i\,\sigma_y) + \delta \sin\left(\frac{2\delta t}{\hbar}\right)(2i\,\sigma_z)\right)$$

$$= \frac{1}{\hbar}\left[\delta \cos\left(\frac{2\delta t}{\hbar}\right)\sigma_y - \delta \sin\left(\frac{2\delta t}{\hbar}\right)\sigma_z\right].$$

(f) The 'unitary' operator, corresponding to the Hamiltonian $H = \delta |e\rangle\langle r|$, is

$$V(t) = e^{-iHt/\hbar} = \sum_{n=0}^{\infty} \frac{(-iHt/\hbar)^n}{n!} = \mathbb{I} - \frac{i}{\hbar} H t,$$

because $H^2 = H \cdot H = \delta^2 |e\rangle\langle r|e\rangle\langle r| = 0$ and all other higher powers vanish.

Thus $V(t) = \mathbb{I} - \frac{i\delta t}{\hbar} |e\rangle\langle r|$. Now apply the operator on a normalized state $|r\rangle$. Then

$$|r'\rangle = V(t)|r\rangle = |r\rangle - \frac{i\delta t}{\hbar}|e\rangle.$$

Now for any $\delta > 0$, $t > 0$, the $\langle r'|r'\rangle > 1$.

So, the probability is not conserved.

(You shall have similar observation with $|\psi(0)\rangle$)