Solution to HW4

1. (i) $1+(-1)^n = \begin{cases} 0 & \text{if } n \text{ odd} \\ 2 & \text{if } n \text{ even} \end{cases}$

If $S_K = \{x_i : i > K\}$ clearly $\sup S_K = 2$, $\inf S_K = 0$

Hence $\limsup x_n = \lim(\sup S_K) = 2$
and $\liminf x_n = \lim(\inf S_K) = 0$

(ii) $\sin \frac{n\pi}{2} = \begin{cases} 0 & \text{if } n \text{ even} \\ 1 & \text{if } n = 4K+1 \\ -1 & \text{else} \end{cases}$

Hence, $\sup S_K = 1$, $\inf S_K = -1$ $\forall K$
$\Rightarrow$ $\limsup \sin \frac{n\pi}{2} = 1$, $\liminf \sin \frac{n\pi}{2} = -1$

(iii) $n(1+(-1)^n) = \begin{cases} 0 & \text{if } n \text{ odd} \\ 2n & \text{if } n \text{ even} \end{cases}$

Hence, $\sup S_K = \infty$, $\inf S_K = 0$ $\forall K$
$\Rightarrow$ $\limsup = \infty$, $\liminf = 0$ in this case.

(iv) $\frac{n}{n+1} \cdot (-1)^n = \begin{cases} \frac{n}{n+1} & \text{for } n \text{ odd} \\ -\frac{n}{n+1} & \text{for } n \text{ even} \end{cases}$

Note: $\frac{n}{n+1} = 1 - \frac{1}{n+1}$ is an increasing sequence.
check that $\sup \left\{\frac{n}{n+1} : \underset{n \text{ even}}{n > K}\right\} = 1$ $\forall K \in \mathbb{N}$.

Then clearly $\sup S_K = \sup\left\{\frac{n}{n+1} : \begin{matrix} n \geq K \\ n \text{ even}\end{matrix}\right\}$
$= 1$

Similarly $-\frac{n}{n+1}$ is an increasing sequence

Thus $\inf S_K = \inf\left\{-\frac{n}{n+1} : n > K, n \text{ odd}\right\}$

$$= \begin{cases} -\frac{K+1}{K+2} & \text{if } K \text{ is even} \\ -\frac{K+2}{K+3} & \text{if } K \text{ is odd.}\end{cases}$$

check that $\lim\left(\inf S_K\right) = -1$ in this case.

Hence, $\liminf (-1)^n \cdot \frac{n}{n+1} = -1$

$\limsup (-1)^n \cdot \frac{n}{n+1} = 1$

2. This question is wrong.
Take $x_n = \frac{1}{n}$, $y = n$ $\forall n$

Note. $0 \cdot \infty$ is not defined.

However, assume that x_n, y_n are bounded.

Let $z_n = x_n y_n$ $\forall n \in \mathbb{N}$,

Let $S_K^x = \{x_n : n > K\}$, $S_K^y = \{y_n : n > K\}$

and $S_K^z = \{z_n = x_n y_n : n > K\}$

Note: $x_n \leq \sup S_K^x$ $\forall n > K$

$y_n \leq \sup S_K^y$ $\forall n > K$

$\Rightarrow$ $z_n = x_n y_n \leq (\sup S_K^x)(\sup S_K^y)$ $\forall n > K$
$\left(\because \begin{matrix} x_n \geq 0 \\ y_n \geq 0\end{matrix}\right.$

$\Rightarrow$ $\sup S_K^z \leq (\sup S_K^x)(\sup S_K^y)$ and all the terms are real no.s$)$

Taking limits (maybe use sandwich theorem)

$\limsup S_K^z \leq \lim\left((\sup S_K^x)(\sup S_K^y)\right)$

$= \lim(\sup S_K^x) \cdot \lim(\sup S_K^y)$

$\Rightarrow \quad \limsup (x_n y_n) \leq (\limsup x_n)(\limsup y_n)$

<u>Example</u>: Take $x_n = (-1)^n$, $y_n = (-1)^{n+1}$

Then $\limsup x_n = \limsup y_n = 1$

$x_n y_n = (-1)^{2n+1} = -1 \quad \forall n$

$\Rightarrow \limsup x_n y_n = -1$

3. Let $K \in \mathbb{N}$. Then for $n > K$

$$s_n = \frac{a_1 + \cdots + a_K}{n} + \frac{a_{K+1} + \cdots + a_n}{n}$$

Clearly:

$$\frac{a_1 + \cdots + a_K}{n} + \frac{n-K}{n} \cdot \min\{a_{K+1}, \cdots, a_n\} \leq s_n \leq \frac{a_1 + \cdots + a_K}{n} + \frac{n-K}{n} \max\{a_{K+1}, \cdots, a_n\}$$

Let $y_n = \dfrac{a_1 + \cdots + a_K}{n}$; $z_n = \min\{a_{K+1}, \cdots, a_n\}$; $w_n = \max\{a_{K+1}, \cdots, a_n\}$

and $t_n = \dfrac{n-K}{n}$ for $n > K$.

Observe that y_n is a decreasing sequence, $y_n \to 0$.

z_n is a decreasing sequence, w_n is an increasing sequence

and t_n is an increasing sequence.

Moreover, $z_n \to \inf\{a_{K+1}, \cdots\} = \inf S_K$ (S_K defined for $\{a_n\}$)

$w_n \to \sup\{a_{K+1}, \cdots\} = \sup S_K$

$t_n \to 1$

Check that we have

$$y_n + \inf S_K \leq y_n + z_n \leq s_n \leq y_n + t_n w_n$$
$$\leq y_n + w_n$$
$$\leq y_n + \sup S_K \rightarrow (*)$$

$\Rightarrow \inf\{y_n + \inf S_K : n > K\} \leq \inf\{s_n : n > K\}$ (Using the first two inequalities in $(*)$)

$\Rightarrow \inf\limits_{n > K} y_n + \inf S_K \leq \inf\{s_n : n > K\}$

Since $\{y_n\}$ is decreasing, $y_n \to 0$,

$$\inf_{n>K} y_n = 0$$

Thus $\inf S_K \leq \inf_{n>K} s_n$

Taking limits $\liminf a_n \leq \liminf s_n$

Similarly, show that $\limsup s_n \leq \limsup a_n$ (using the last three inequalities from (*))

We always have $\liminf s_n \leq \limsup s_n$.

Putting all these together we have the required result!

When $\lim a_n$ exists then $\limsup a_n = \liminf a_n = \lim a_n$.

In that case it follows from the inequalities that

$$\liminf s_n = \limsup s_n = \lim a_n$$
$$\Rightarrow \lim s_n = \lim a_n.$$

Example: Let $a_n = (-1)^n$. Then

$$s_1 = -1, \; s_2 = 0, \; s_3 = -\frac{1}{3} \text{ etc.}$$

In general $|s_n| = \left|\frac{-1+1-1+\cdots}{n}\right| \leq \frac{1}{n}$

Check that thus $\lim s_n = 0$, but $\lim a_n$ does not exist.

4) $$x_n = \begin{cases} \frac{1}{2^n} & n \text{ odd} \\ \frac{1}{2^{n-2}} = \frac{4}{2^n} & n \text{ even} \end{cases}$$

$$\Rightarrow |x_n|^{1/n} = \begin{cases} \frac{1}{2} & n \text{ odd} \\ \frac{1}{2} \cdot 4^{1/n} & n \text{ even} \end{cases}$$

We know that $\lim 4^{1/n} = 1$

Using that check that $\lim |x_n|^{1/n} = 1/2$

For n even $\quad \dfrac{x_n}{x_{n+1}} = \dfrac{1/2^{n-2}}{1/2^{n+1}} = \dfrac{2^{n+1}}{2^{n-2}} = 2^3 = 8$

For n odd $\quad \dfrac{x_n}{x_{n+1}} = \dfrac{1/2^n}{1/2^{n+1-2}} = \dfrac{2^{n-1}}{2^n} = \dfrac{1}{2}$

Hence the sequence $\left\{ \left| \dfrac{x_n}{x_{n+1}} \right| \right\}$ is $\left\{ \frac{1}{2}, 8, \frac{1}{2}, 8, \ldots \right\}$

i.e. an alternating sequence.

So $\lim \left| \dfrac{x_n}{x_{n+1}} \right|$ does not exist although $\lim |x_n|^{1/n}$ does exist.