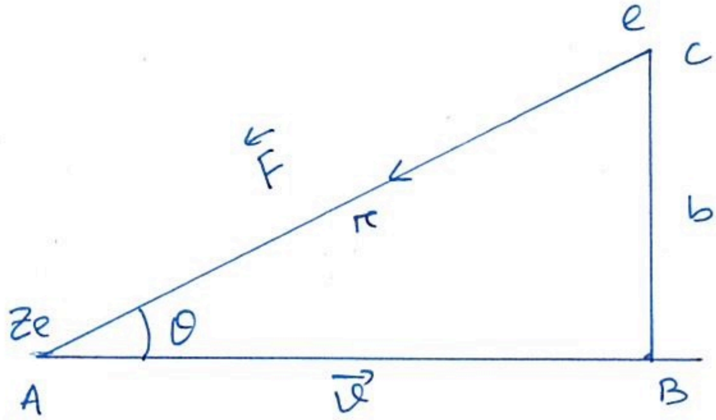
Let's zoom the picture & do a detail study: the geometry will look as follows.

$AB = vt$

$BC = b$

$AC = r$



$F_\perp = F\sin\theta$

$\sin\theta = \frac{BC}{AC}$

$\Rightarrow \sin\theta = \frac{b}{r}$

$\Rightarrow \boxed{r = \frac{b}{\sin\theta}}$

$\tan\theta = \frac{BC}{AB} \Rightarrow \tan\theta = \frac{b}{vt} \Rightarrow t = \left(\frac{b}{v}\right)\cot\theta \Rightarrow dt = \left(\frac{b}{v}\right)\operatorname{cosec}^2\theta\, d\theta$

Now we can write the equation as

$$I = \Delta P = \int F_\perp dt$$

$$\Rightarrow I = \Delta P = \int F \sin\theta\, dt$$

$$= \int F \sin\theta \left(\frac{b}{v}\right) \operatorname{cosec}^2\theta\, d\theta$$

$$= \int_0^\pi \left(\frac{Ze.e}{4\pi\epsilon_0 r}\right) \sin\theta \left(\frac{b}{v}\right) \operatorname{cosec}^2\theta\, d\theta$$

$$= \int_0^\pi \frac{Ze^2}{4\pi\epsilon_0 \left(\frac{b}{\sin\theta}\right)^2} \sin\theta \left(\frac{b}{v}\right) \operatorname{cosec}^2\theta\, d\theta$$

$$= \int_0^\pi \frac{Ze^2}{4\pi\epsilon_0\, bv} \sin\theta\, d\theta = \frac{Ze^2}{4\pi\epsilon_0 vb} \int_0^\pi \sin\theta\, d\theta$$

$$= \frac{Ze^2}{4\pi\epsilon_0} \frac{2}{bv}.$$

$$\boxed{I = \Delta P = \frac{Ze^2}{2\pi\epsilon_0 bv}}.$$

we need to calculate the kinetic energy transfer to electron mass me.

The v is taken outside the integral, as the change in r during the collision with a single electron is very small.

what is the total energy transfer? This can be aproximated ~~to the~~ using Bohr's classical approach:

The energy transfer is $\Delta E = \frac{(\Delta P)^2}{2 m_e}$ ~~me~~ in ~~denomi~~

me in denominatore, which is realy small than the incoming mass of the incuin particle (if it is not the electreen), thus it gives a negnigible contributin

$$\Rightarrow \quad \Delta E = \frac{(\Delta P)^2}{2 m_e}$$

ΔE is b dependent.

$$\Rightarrow \quad \Delta E(b) = \left(\frac{1}{4\pi\epsilon_0}\right)^2 \frac{2 z^2 e^4}{m_e v^2 b^2}$$

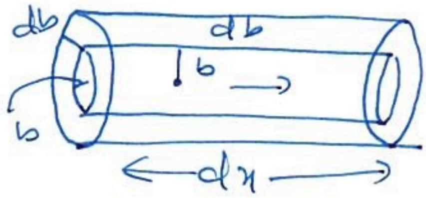
me is electreen mass which is takin the KE frem this processs.

None that this quantity depends on z^2/m_e

$$\Rightarrow \boxed{\Delta E(b) \propto \frac{z^2}{m_e}}$$

To findout the effect of all electrons with the given impact parameter b, we construct a cylinderical shell of electrons of radii b & b + db & the lngth dx along the path of the incoming particle.

considering the volme element $dV = 2\pi b \, db \, dx$



and the electreon number density n_e, we can write the energy transfer as

$$dE(b) = \Delta E(b) \, n_e \, dV$$

This is the enrgy transfer to the electreens within the cylinder. ~~This quantit~~

$$\Rightarrow dE(b) = \frac{1}{(4\pi\epsilon_0)^2} \frac{4\pi z^2 e^4}{m_e v^2} n_e \frac{db}{b} dx \quad - (A)$$

Thus the energy loss per unit path length of the particle due to the electrons at impact parameter between b & $b+db$ is given by above equation ~~(A)~~

$$\frac{dE(b)}{dx} = \frac{1}{(4\pi\epsilon_0)^2} \frac{4\pi z^2 e^4}{m_e v^2} n_e \frac{db}{b}.$$

$\frac{dE}{dx}$ is often negative to symbolize the energy loss in nature due to the passage through the material. Now the total rate of energy loss per unit path length can be calculated by integrating right hand side from b_{min} to b_{max}.

$$\Rightarrow -\frac{dE}{dx} = \frac{1}{(4\pi\epsilon_0)^2} \frac{4\pi z^2 e^4}{m_e v^2} n_e \int_{b_{min}}^{b_{max}} \frac{db}{b}$$

b_{min} & b_{max} are the limits for which the equation for ~~ene~~ ~~valid~~. energy loss is valid.

$$\Rightarrow \boxed{-\frac{dE}{dx} = \frac{1}{(4\pi\epsilon_0)^2} \frac{4\pi z^2 e^4}{m_e v^2} n_e \ln \frac{b_{max}}{b_{min}}} \quad - (B)$$

$-\frac{dE}{dx}$ also called as stopping power

$$\Rightarrow \boxed{-\frac{dE}{dx} = \frac{z^2 e^4 n_e}{4\pi\epsilon_0^2 m_e v^2} \ln\left(\frac{b_{max}}{b_{min}}\right)} \quad - (C)$$

can we put $b_{min} = 0$ & $b_{max} = \infty$; Think practically this will lead to a situation where the energy loss per unit path length is infinity, which is a unnatural case as incoming particle has finite energy. Thus we must avoid this situation.

Thus b_{min} is b for which $\Delta E(b)$ has its maximum possible value, i.e. maximum possible energy transfer which occurs for a head on collision. i.e.

$$\frac{1}{2} m_e (2v)^2 = 2 m_e v^2 \quad \text{classically}$$

the relativistic approximation is $2\gamma^2 m_e v^2$.

where $\quad \gamma = (1-\beta^2)^{-1/2} \quad \beta = v/c$.

$$\Delta E_{max} = \frac{2 z^2 e^4}{m_e v^2 b^2} \left(\frac{1}{4\pi\epsilon_0}\right)^2 = 2\gamma^2 m_e v^2$$

$$\Rightarrow b_{min} = \frac{z e^2}{\gamma m_e v^2} \left(\frac{1}{4\pi\epsilon_0}\right)$$

b_{max}: The interaction time $\frac{b}{\gamma v}$ should be less than the free mean orbital period of atomic electron $\frac{1}{\bar{\nu}}$ in order for electron to absorb energy "adiabatic invariance": $\bar{\nu}$ is the characteristic orbital frequency.

$$\Rightarrow \frac{b_{max}}{\gamma v} = \frac{1}{\bar{\nu}} \Rightarrow b_{max} = \frac{\gamma v}{\bar{\nu}}$$

This is so as electrons are not free. $\left\{ \begin{array}{l} \bar{\nu} \text{ is the} \\ \text{characteristic} \\ \text{orbital frequency} \end{array} \right.$

Further we can calculate the number of electrons n_e in terms of density of material as

$$n_e = \frac{\rho N_A Z}{A}$$

ρ = Density of the absorber matterial

N_A = Avagadro number to know how many atoms are there

$Z, A \equiv$ Atomic and mass number of atoms of matterial

Putting all these quantiies b_{min}, b_{max}, n_e etc in equatin (B) of $-\frac{dx}{dt}$:

$$\Rightarrow -\frac{dE}{dx} = \left(\frac{1}{4\pi\epsilon_0}\right)^2 \frac{4\pi z^2 e^4}{m_e v^2} n_e \ln\left(\frac{b_{max}}{b_{min}}\right)$$

$$= 4\pi \left(\frac{e^2}{4\pi\epsilon_0 m_e c^2}\right)^2 m_e z^2 c^2 \left(\frac{c^2}{v^2}\right) \rho \frac{N_A Z}{A} \ln\left(\frac{b_{max}}{b_{min}}\right)$$

$$= 4\pi N_A r_e^2 m_e c^2 \frac{Z}{A} \frac{\rho}{\beta^2} z^2 \ln\left(\frac{\gamma v}{\bar{v}} \Big/ \left(\frac{z e^2}{\gamma m_e v^2} \frac{1}{4\pi\epsilon_0}\right)\right)$$

$$\Rightarrow \boxed{-\frac{dE}{dx} = 4\pi N_A r_e^2 m_e c^2 z^2 \frac{Z}{A} \frac{\rho}{\beta^2} \ln\left(\frac{4\pi\epsilon_0 \gamma^2 m_e v^2}{z e^2 \bar{v}}\right)}$$

This is the Bohr's classical formula for the enerrgy loss for heavy charge particle.

In quantum mechanical ~~correction~~ treatment, the inelastic collision with atomic electrons is divided into two classes hard & soft. In former, a large energy is transfer and the quantum mechanical spin and exchange effects enters into picture. In later type of collisions, the energy transfer extends from some arbitary energy to the minimum energy. In 1930-33 Bethe & Block deduced an approximatin using quantum-mechanical treatment & Born approximatin. More complete formula is:

$$-\frac{dE}{dx} = 2\pi N_A r_e^2 m_e c^2 \rho \frac{Z}{A} \frac{z^2}{\beta^2} \left[\ln\left(\frac{2 m_e \gamma^2 v^2 W_{max}}{I^2}\right) - 2\beta^2 - \delta - 2\frac{C}{Z}\right]$$

where $W_{max} = \dfrac{2 m_e (c\beta\gamma)^2}{1 + \left(\frac{m_e}{M}\right)\sqrt{1 + (\beta\gamma)^2 + (m_e/M)^2}} \approx 2 m_e (c\beta\gamma)^2$

I is ionizatin potential.

Let's discuss about each quantities in the above equation

r_e : classical electron radius

m_e : mass of electron

N_a : Avogadro's number $6.022 \times 10^{23}/mol$

c : speed of light.

$$2\pi N_a r_e^2 m_e c^2 = 0.1535 \ MeV.cm^2/g.$$

Z : atomic number

A : Mass number

ρ : Density of absorbing material

z : charge of the incident particle

γ : $\frac{1}{\sqrt{1-\beta^2}}$: $\beta = \left(\frac{v}{c}\right)$

I : The mean excitation potential : it is determined empirically for different material from the measurement of dE/dx. This is a main parameter of the Bethe-Bloch formula and is essentially the average orbital frequency $\bar{\nu}$ from the Bohr's formula times the planck constant., $h\bar{\nu}$. It is theoretically a logarithmic average of ν weighted by the so called oscillator strength of the atomic levels.

From the empirical estimation $\quad I = h\bar{\nu} \Rightarrow b_{max} = \frac{v\gamma}{\bar{\nu}}$

$\Rightarrow b_{max} = \frac{v\gamma h}{I}$

$$\frac{I}{Z} = 12 + \frac{7}{Z} \ eV \qquad Z < 13$$

$$= 9.76 + 58.8 Z^{-1.19} \ eV \qquad Z \geq 13$$

δ : Density correction: the parameter which describes how transverse electric field of incident particle is screened by the charge density of the electrons in the medium. This is kind of polarization effects of atom in the medium caused by E field of the particle, more distance atom are shielded from the full E field intensity - contribute less to energy loss.

C: it is the shell correction for the case where the velocity of the incident particle is comparable (or less) to the orbital velocity of the bound electron ($\beta \approx Z\alpha$). It has very small contribution as compared to the other effect.

W_{max}: Maximum energy ~~loss~~ transfer in one collision

$$W_{max} = \frac{2 m_e c (\beta\gamma)^2}{1 + (m_e/M)\sqrt{1+(\beta\gamma)^2 + (m_e/M)^2}}$$

also written as

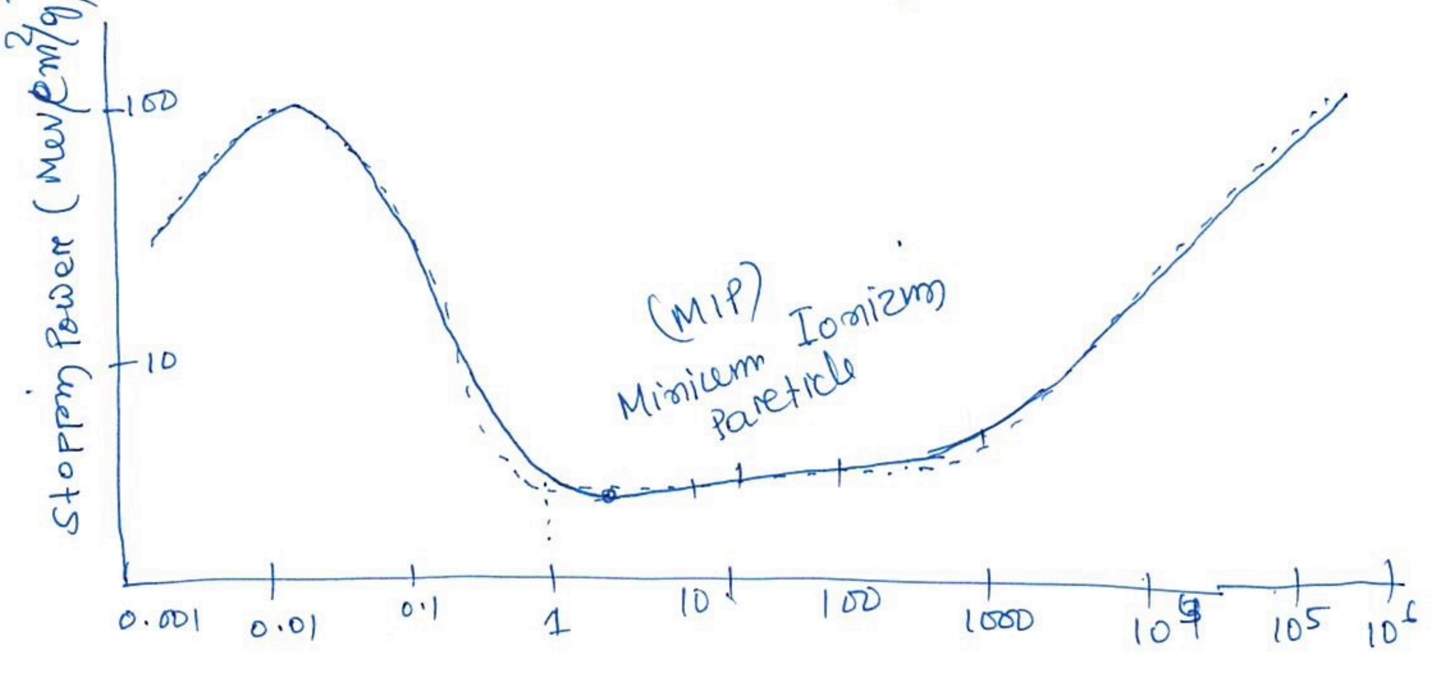
$$= \frac{2 m_e c^2 \eta^2}{1 + 2s\sqrt{1+\eta^2+s^2}} \qquad s = m_e/M \qquad \eta = \beta\gamma$$

$$\text{if } M \gg m_e \Rightarrow W_{max} \approx 2 m_e c^2 \eta^2$$

- The ionization energy loss does not depend on the mass of ionizing particle
- There is only a weak dependence on the medium because $\frac{Z}{A} \approx 0.5$ for most materials.

→ The energy loss depends only on β and one needs the mass of incoming particle to convert it to momentum

let's look at the famous muon energy loss mechanism.



- The Bethe-Block equation dictates that the enerergy loss increases as $\sim\beta^{-2}$ at the particle speed decreases. This is shown in the muon energy loss figure as above. This means that the slow particles will be more ionizing than fast particles.

- As particle speed increases the energy loss reaches a minimum of about 1.5 – 2.0 MeV cm^2/g remains at this level for almost the entire range of muon momenta one would observe in a high energy physics experiment. Hence the concept of minimum ionizing particle, which is used EHEP to refer to particles whose speed is in this regime and suffer similar energy losses (1.5-2.0 MeV cm^2/g)

- After that it start rising following the so called relativistic rise. However radiative effects are also important in this region.

→ The relativist rise comes from the fact that the electric field of the ionizing particle in the lab frame is important & propertional to the relativist gamma. Hence the faster the particle is, the stronger the field becomes and therefore the particle can ionize atoms at larger distances and losses more energy. This rise of the energy loss is logarithmic as can be seen by approximating $\frac{dE}{dx}$ equation.

$$-\frac{dE}{dx} \propto \ln\left(\frac{2m_ec^2\beta^2\gamma^2 W}{I}\right) \sim \ln\frac{2m_ec^2 W_m}{I} + \ln(\beta^2\gamma^2)$$

$$\sim \ln\left(\frac{p}{M}\right)$$

Eventually the medium polarizes and cancels this effect. This prevents the energy loss from rising perpetually and eventually flattens at very high particle speeds. This the origin of density terms, δ - correction, in the Bethe-Block equation.

It is a fact that the energy loss dE/dx is given in terms of MeV cm^2/g in unit. It really depends very weakly on the type of medium. However to convert it to MeV/cm one has to take in to ~~acc~~ account how dense the material is as the Bethe-Block equation is given normalised to unit density.

There is a short notation of Bethe-Block equation you may encounter in some reference.

$$-\frac{dE}{dx} = K z^2 \frac{Z}{A} \frac{1}{\beta^2} \rho \left[\frac{1}{2} \ln\left(\frac{2 m_e \beta^2 \gamma^2 W_{max}}{I^2}\right) - \beta^2 - \frac{\delta}{2} - \frac{C}{Z} \right]$$

$$\frac{K}{A} = 4\pi N_A r_e^2 m_e c^2 / A \approx 0.307075 \frac{\text{MeV cm}^2}{g}$$

$$\Rightarrow -\frac{1}{\rho}\frac{dE}{dx} = K z^2 \frac{Z}{A} \frac{1}{\beta^2} \left[\frac{1}{2} \ln\left(\frac{2 m_e \beta^2 \gamma^2 W_{max}}{I^2}\right) - \beta^2 - \frac{\delta}{2} - \frac{C}{Z} \right]$$

$\rho dx = X \Rightarrow$ Radiation Length, [This can be uniformly found out while working for elements i.e. X_0]

~~Energy loss of Electrons~~:

The Ionization ~~ste~~ energy loss is always given in terms of radiation length which is basically the $\left(\frac{dE}{dx}\right)$ per unit density thus the radiation length is ~~given~~ expressed in g/cm^2 unit.