End Sem Exam: MTH306

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1. (i) Show that all non-empty open sets in $\mathbb{R}^d$ have positive measure. $|f| < -2$ [2]

(ii) Check if the following are Lebesgue integrable: [2x1...]

(a) $u(x) = \frac{1}{x}$, over $[1, \infty)$

(b) $v(x) = \frac{1}{\sqrt{x}}$, over $(0, 1]$.

$|A| > 0$, $A \to$ open in $\mathbb{R}^d$

$\{x : |f| < \alpha\} = \{x : -\alpha < f < \alpha\}$
$= \{x : -f < \alpha\} \cup \{x : f < \alpha\}$

$x \in [-1, \cdots$

$f_n(x) = x^n$ $\quad \int f_n(x) = \frac{x^{n+1}}{n+1}$

$f(x) = 0$ $\quad \int f = 0$

$|f| = |x^n| = |x|^n$

2. Let $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ be a function. [2x2...]

(i) If f is measurable, then show that $|f|$ is measurable.

(ii) Show that $|f|$ measurable does not imply that f is measurable, by giving an example.

3. Use the dominated convergence theorem to calculate the limit. [2x2...]

(i) $\lim_{n\to\infty} \int_0^1 \frac{1 + nx^2}{(1 + x^2)^n}$

$\boxed{\lim_{n\to\infty} \int |f_n| = \int |f|}$ $\boxed{\lim_{n\to\infty} \int |f_n| < \infty}$ $\boxed{\lim_{n\to\infty} f_n = f}$

(ii) $\lim_{n\to\infty} \int_0^1 \frac{n^{3/2} x}{(1 + n^2 x^2)}$

$\lim_{n\to\infty} \int f_n \to \int f \Rightarrow \lim_{n\to\infty} \int |f_n - f| \to 0$

$\{x : |f| < \alpha\}$ $\neq$ $\{x : f < \alpha\}$

4. Let $f_n \in L^1(\mathbb{R}^d)$ be a sequence of functions and $\lim_{n\to\infty} f_n = f$ a.e. If $\lim_{n\to\infty} \int |f_n| = \int |f|$, then show that $\lim_{n\to\infty} \int f_n = \int f$. Show that the converse is not true by constructing a couterexample. [5]

5. Let f be a function which is absolutely continuous on the interval $[a, b]$. If $f \geq c$ for some $c > 0$. Then show that $1/f$ is also absolutely continuous on $[a, b]$. [5]

6. Let $f(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$ [5]

Calculate $D^+(f)(0), D_-(f)(0)$.

$f_n(x) = \frac{x}{n}$ $\quad f_n(x) = \frac{x(x-1)\cdots}{n}$ $\quad f = 0$

7. Let $(X, \mathcal{M}, \mu)$ be a measure space. Consider a sequence of sets $\{E_n\}_n$ in $\mathcal{M}$, such that $\sum_{n=1}^{\infty} \mu(E_n) < \infty$. [2x2]

(i) Show that $\mu\left(\bigcap_{n=1}^{\infty} \bigcup_{k\geq n} E_k\right) = 0$. $\quad \mu\left(\bigcap_{n=1}^{\infty} \bigcup_{k\geq n} E_k\right) = \mu\left[\bigcap_{n=1}^{\infty} (E_k \cup E_{k+1} \cup \cdots)\right]$

(ii) If $f > 0$ and is an integrable function on X, then show that μ is a σ-finite measure on X.

$X = \sum X_i \to$ finite measure

8. Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) = 1$. Consider the collection of subsets

$$\mathcal{E} = \{A \in \mathcal{M} : \mu(A) = 0 \text{ or } 1\}.$$

Show that $\mathcal{E}$ is a σ-algebra.

$\lim |f_n - f| \to 0$ $\quad \mu\left[\bigcap_{n=1}^{\infty} (E_n \cup E_{n+1} \cup \cdots\right.$

9. Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $f : X \longrightarrow [0, \infty]$ be a measurable function such that $\int_X f d\mu < \infty$. [2x2

(i) Let $A = \{x \in X : f(x) = \infty\}$. Show that A is measurable, and $\mu(A) = 0$.

(ii) For any $\varepsilon > 0$, show that there exists $\alpha > 0$ such that

$$\int_E f d\mu < \varepsilon \text{ whenever } E \in \mathcal{M} \text{ with } \mu(E) \leq \alpha.$$

10. Let g be a measurable function on $[0, 1]$ such that the function [5]

$$f(x, y) = 3g(y) - 2g(x)$$

is integrable on $[0, 1] \times [0, 1]$. Show that g is Lebesgue integrable on $[0, 1]$.