PHY 310 - Mathematical Methods for Physicists I

Odd Term 2019, IISER Mohali

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Homework 7 - Solutions

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1. Consider a string stretched under unit tension between fixed endpoints $x = 0$ and $x = L$, and subject to a force distribution $\phi(x)$ Newtons per unit length. The static deflection of of the string, $y(x)$, solves the following boundary value problem

$$y'' = \phi(x), \quad y(0) = y(L) = 0. \tag{1}$$

(i.) Find the Green's function of this system by applying the boundary conditions and the constraints in the continuity and the jump in the derivative.

(ii.) Find the solution to Eq. (1) using the method of direct integration.

Solution:

(i.) The Green's function $G(x, s)$ will solve

$$\frac{d^2}{dx^2}G(x, s) = \delta(x - s), \quad G(0, s) = 0 = G(L, s). \tag{2}$$

The homogeneous equation has the solution

$$y_1(x) = x, \quad y_2(x) = 1. \tag{3}$$

We can write the Green's function using four unknown constants $\alpha_1, \alpha_2, \beta_1, \beta_2$

$$G(x, s) = \begin{cases} \alpha_1 \cdot x + \alpha_2 \cdot 1 & : x < s \\ \beta_1 \cdot x + \beta_2 \cdot 1 & : x > s \end{cases} \tag{4}$$

Imposing $G(0, s) = 0$ for the first and $G(L, s) = 0$ for the second gives

$$G(x, s) = \begin{cases} \alpha_1 x & : x < s \\ \beta_1 (x - L) & : x > s \end{cases} \tag{5}$$

Applying the *continuity constraint* at $x = s$ we get

$$\alpha_1 s = \beta_1(s - L). \tag{6}$$

The constraint on the *jump in the derivative* gives

$$\frac{d}{ds}\alpha_1 s - \frac{d}{ds}[\beta_1(s - L)] = 1 \implies \alpha_1 - \beta_1 = 1. \tag{7}$$

The above two constraints give

$$\alpha_1 = (s - L), \quad \beta_1 = s. \tag{8}$$

Thus the Green's function is

$$G(x, s) = \begin{cases} (s - L)x & : x < s \\ s(x - L) & : x > s \end{cases} \tag{9}$$

(ii.) The solution to Eq. 1 can be written as

$$y(x) = \frac{1}{L}\left(\int_0^x s(x - L)\phi(s)ds + \int_x^L x(s - L)\phi(s)ds\right). \tag{10}$$

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2. Find the Green's function for the one-dimensional Helmholtz problem

$$\frac{d}{dx^2}\phi(x) - k^2\phi(x) = h(x), \quad \lim_{|x|\to\infty} \phi(x) = 0, \tag{11}$$

upon using the boundary conditions, and constraints on continuity and jump in the derivative.

Solution:

The Green's function for the problem solves the equation

$$\frac{d}{dx^2}G(x, s) - k^2 G(x) = \delta(x - s), \tag{12}$$

together with the boundary conditions $\lim_{|x|\to\infty G(x,s)=0}$ and the continuity and jump in the derivative conditions.

The homogeneous equation has the solutions

$$y_1(x) = e^{-kx}, \quad y_2(x) = e^{+kx}. \tag{13}$$

Thus the Green's function is

$$G(x,s) = \begin{cases} \alpha_1 e^{-kx} + \alpha_2 e^{+kx} & : x < s \\ \beta_1 e^{-kx} + \beta_2 e^{+kx} & : x > s \end{cases} \tag{14}$$

Imposing the two boundary conditions gives us

$$G(x,s) = \begin{cases} \alpha_2 e^{+kx} & : x < s \\ \beta_1 e^{-kx} & : x > s \end{cases} \tag{15}$$

The continuity condition gives us

$$\alpha_2 e^{+ks} = \beta_1 e^{-ks}, \tag{16}$$

and the jump in the derivative condition gives us

$$k\alpha_2 e^{+ks} + k\beta_1 e^{-ks} = 1. \tag{17}$$

This gives

$$\alpha_2 = \frac{1}{2k} e^{-ks}, \quad \beta_1 = \frac{1}{2k} e^{ks}. \tag{18}$$

Thus the Green's function is

$$G(x,s) = \frac{1}{2k} e^{-k|x-s|}. \tag{19}$$

The solution to inhomogeneous equation is

$$\phi(x) = \frac{1}{2k} \int_{-\infty}^{\infty} h(s) e^{-k|x-s|} ds. \tag{20}$$

(Note: The Green's function here depends on the separation between x and s. This is a very common feature in problems which have translational symmetry that live on infinite domains.)

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3. Find the Green's function for the Laplacian in two dimensions,

$$\nabla^2 \phi(r,\theta) = h(r,\theta), \tag{21}$$

with the boundary conditions $\lim_{r\to\infty} (\phi(r,\theta) - \phi_r(r,\theta) r \ln r) = 0$.

(*Hint: Use Gauss' theorem to get the normalization condition*

$$1 = \int_{\partial D} \frac{\partial_x}{\partial n} G(\mathbf{x}, \mathbf{x}_0)\ dx, \tag{22}$$

where D is the unit disk, $\vec{n}$ is the outward normal, and the derivative is with respect to x in the notation $r = |\mathbf{x} - \mathbf{x}_0|$.)

Solution:

The system has rotational symmetry. Thus we look for a Green's function of the form

$$G = g(|\mathbf{x} - \mathbf{x}_0|) = g(r). \tag{23}$$

In polar coordinates the Laplacian takes the form

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}. \tag{24}$$

$$\frac{1}{r}\left(r g'(r)\right)' = \delta(r - r'), \quad \lim_{r\to\infty} \left(g(r) - r\ln r g'(r)\right) = 0. \tag{25}$$

We get the solution as

$$G = a\ln r + b. \tag{26}$$

The boundary condition sets $b = 0$.

The normalization condition gives

$$1 = \int_{\partial D} \frac{\partial_x}{\partial n} G(\mathbf{x}, \mathbf{x}_0) dx = \int_{\partial D} a dx = 2\pi a, \tag{27}$$

where D is the unit disk, so that $a = 1/(2\pi)$.

Thus the Green's function is

$$G(\mathbf{x}, \mathbf{x}_0) = \frac{1}{2\pi} \ln |\mathbf{x} - \mathbf{x}_0|, \tag{28}$$

and the solution to equation is

$$\phi(x) = \int \frac{1}{2\pi} \ln |\mathbf{x} - \mathbf{x}_0| \; h(\mathbf{x}_0) \; dx_0^2. \tag{29}$$

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4. Show that the Green's function for the system

$$y''(x) + 3y'(x) + 2y(x) = f(x), \tag{30}$$

with the boundary conditions $y(0) = 0 = y'(0)$ is

$$G(x, s) = \begin{cases} 0 & : x < s, \\ e^{2(s-x)} + e^{s-x} & : x > s. \end{cases} \tag{31}$$

Solution:

Let us first find the general solution of the homogeneous equation

$$y''(x) + 3y'(x) + 2y(x) = 0. \tag{32}$$

The two linearly independent solutions of this equation are e^{-2x} and e^{-x}.

Let us write the Green's function as

$$G(x, s) = \begin{cases} \alpha_1 e^{-2x} + \alpha_2 e^{-x} & : x < s \\ \beta_1 e^{-2x} + \beta_2 e^{-x} & : x > s \end{cases} \tag{33}$$

Applying the two boundary conditions

$$\alpha_1 e^{-2x} + \alpha_2 e^{-x} \Big|_{x=0} = 0 \implies \alpha_1 + \alpha_2 = 0, \tag{34}$$

$$-2\alpha_1 e^{-2x} - \alpha_2 e^{-x} \Big|_{x=0} = 0 \implies -2\alpha_1 - \alpha_2 = 0. \tag{35}$$

This implies that $\alpha_1 = 0$ and $\alpha_2 = 0$.

Applying the continuity condition

$$\begin{aligned} \alpha_1 e^{-2x} + \alpha_2 e^{-x} &= \beta_1 e^{-2x} + \beta_2 e^{-x} \\ \implies 0 &= \beta_1 e^{-2s} + \beta_2 e^{-s} \end{aligned} \tag{36}$$

Applying the jump condition

$$\begin{aligned} \left(-2\beta_1 e^{-2s} - \beta_2 e^{-s}\right) - \left(-2\alpha_1 e^{-2s} - \alpha_2 e^{-s}\right) &= 1, \\ \implies -2\beta_1 e^{-2s} - \beta_2 e^{-s} &= 1. \end{aligned} \tag{37}$$

Thus we have two equations in two unknowns

$$\beta_1 e^{-2s} + \beta_2 e^{-s} = 0, \tag{38}$$

$$-2\beta_1 e^{-2s} - \beta_2 e^{-s} = 1. \tag{39}$$

The solution is

$$\beta_1 = -e^{2s}, \quad \beta_2 = e^{s}. \tag{40}$$

Thus the Green's function is

$$G(x,s) = \begin{cases} 0 & : x < s \\ e^{2(s-x)} + e^{s-x} & : x > s \end{cases} \tag{41}$$

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5. Construct the Green's function for the equation

$$\frac{d}{dx^2} y(x) - a^2 y(x) = 0, \quad \lim_{|x|\to\infty} y(x) = 0, \quad a > 0, \tag{42}$$

using the Fourier transform method.

Solution:

Let $G(x)$ be the Green's function. Then for the homogeneous equation we have

$$\frac{d}{dx^2} y(x) - a^2 G(x) = \delta(x). \tag{43}$$

The Fourier transform of $G(x)$ is

$$G(x) = \frac{1}{\sqrt{2\pi}} \int e^{isx} g(s) ds. \tag{44}$$

We also have the integral representation of the Dirac delta function

$$\delta(x) = \frac{1}{2\pi} \int e^{isx} ds. \tag{45}$$

Plugging into the equation for the Green's function, we find

$$\frac{1}{\sqrt{2\pi}} \int (-s^2 - a^2) e^{isx} g(s) ds = \frac{1}{2\pi} \int e^{isx} ds. \tag{46}$$

Thus we have

$$g(s) = -\frac{1}{\sqrt{2\pi}} \frac{1}{s^2 + a^2} \tag{47}$$

The Green's function is

$$G(x) = -\frac{1}{2\pi} \int e^{isx} \frac{1}{(s+ia)(s-ia)} ds. \tag{48}$$

We can perform this integral using contour integration in the complex s plane. The integrand has two simple poles at $s = \pm ia$. The contour integration must respect the boundary conditions $\lim_{|x|\to\infty} G(x) = 0$.

When $x > 0$, we can close the contour by adding an additional arc at the infinity on the upper half plane since the arc gives an exponentially damped contribution. This choice encircles the pole at $s = +ia$. The integral is

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{e^{isx}}{s^2 + a^2} ds &= \oint_{s=ia} \frac{e^{isx}}{(s+ia)(s-ia)} ds \\ &= 2\pi i \frac{e^{-ax}}{2ia} = \pi \frac{e^{-ax}}{a}. \end{aligned} \tag{49}$$

When $x < 0$, the arc will give an exponentially enhanced contribution. Thus, instead we close the contour using an arc at infinity in the lower half plane. The contour encircles the pole at $s = -ia$, in the clockwise direction. Thus

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{e^{isx}}{s^2+a^2} ds &= -\oint_{s=-ia} \frac{e^{isx}}{(s+ia)(s-ia)} ds \\ &= -2\pi i \frac{e^{ax}}{-2ia} = \pi \frac{e^{ax}}{a}.\end{aligned} \tag{50}$$

We can combine both the contributions and we have the Green's function

$$G(x) = \frac{\pi}{a} e^{a|x|}. \tag{51}$$

(Note: This is the Yukawa potential or screened Coulomb potential in one dimension.)

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6. Express the following equations in their respective Sturm-Liouville form

(i.) Bessel's equation

$$x^2y'' + xy' + (a^2x^2 - n^2)y(x) = 0. \tag{52}$$

(ii.) Chebyshev's equation

$$(1-x^2)y'' - xy' + n^2y(x) = 0. \tag{53}$$

Solution:

(i.) Bessel's equation

$$x^2y'' + xy' + (a^2x^2 - n^2)y(x) = 0. \tag{54}$$

(This is the parametric Bessel's equation, with $0 \leq x \leq a$. It solves the vibrations on a circular drum. Note that the parameter n is coming from separation of variables, while arriving at this equation from another partial differential equation. Thus we should treat n as a constant.)

Dividing by x we get

$$xy'' + y' - \frac{n^2}{x}y(x) + a^2xy(x) = 0. \tag{55}$$

This is in Sturm-Liouville form

$$p_0(x)u'' + p_1(x)u' + p_2(x)u + \lambda w(x)u(x) = 0, \tag{56}$$

The coefficient $p(x)$ is the coefficient of the second derivative of the eigenfunction. The eigenvalue λ is the parameter that is available in a term of the form $\lambda w(x)u(x)$; any x dependance apart from the eigenfunction becomes the weight function $w(x)$. If there is another term containing the eigenfunction (not the derivatives), the coefficient of the eigenfunction in this additional term is identified as $q(x)$. If no such term is present then $q(x)$ is zero.

Thus we have

$$\begin{aligned}
&p_0(x) = p(x) = x,\\
&p_1(x) = 1 = p',\\
&p_2(x) = q(x) = -\frac{n^2}{x},\\
&\lambda = a^2,\\
&w(x) = x.
\end{aligned}$$

(ii.) Chebyshev's equation

$$(1-x^2)y'' - xy' + n^2 y(x) = 0. \tag{57}$$

Dividing Chebyshev's equation by $\sqrt{1-x^2}$

$$\sqrt{(1-x^2)}y'' - \frac{x}{\sqrt{1-x^2}}y' + \frac{n^2}{\sqrt{1-x^2}}y(x) = 0, \tag{58}$$

and comparing with the Sturm-Liouville form

$$p_0(x)u'' + p_1(x)u' + p_2(x)u + \lambda w(x)u(x) = 0, \tag{59}$$

we get

$$\begin{aligned}
&p_0(x) = p(x) = \sqrt{(1-x^2)}, \quad p_1(x) = -\frac{x}{\sqrt{(1-x^2)}} = p',\\
&p_2(x) = q(x) = 0, \quad w(x) = \frac{1}{\sqrt{1-x^2}}, \quad \lambda = n^2.
\end{aligned}$$

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