Assignment 4 Solutions

1. (a) $\vec{\nabla}\cdot(\phi\vec{E}) = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\cdot\left(\hat{x}\,\phi E_x + \hat{y}\,\phi E_y + \hat{z}\,\phi E_z\right)$

$$= \frac{\partial}{\partial x}(\phi E_x) + \frac{\partial}{\partial y}(\phi E_y) + \frac{\partial}{\partial z}(\phi E_z)$$

$$= \left(\frac{\partial \phi}{\partial x}E_x + \frac{\partial \phi}{\partial y}E_y + \frac{\partial \phi}{\partial z}E_z\right) + \phi\left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right)$$

$$= \vec{\nabla}\phi\cdot\vec{E} + \phi\,\vec{\nabla}\cdot\vec{E}$$

(b) $\vec{E} = -\vec{\nabla}\phi$ & $\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0$.

$$\therefore \vec{\nabla}\cdot(\phi\vec{E}) = \vec{\nabla}\phi\cdot\vec{E} + \phi\,\vec{\nabla}\cdot\vec{E} = -\vec{E}\cdot\vec{E} + \phi\frac{\rho}{\epsilon_0}.$$

$$= -E^2 + \frac{\phi\rho}{\epsilon_0}.$$

Integrating over a large volume V,

$$\int_V \vec{\nabla}\cdot(\phi\vec{E})\,dv = -\int_V E^2\,dv + \frac{1}{\epsilon_0}\int_V \rho\phi\,dv$$

Using divergence theorem,

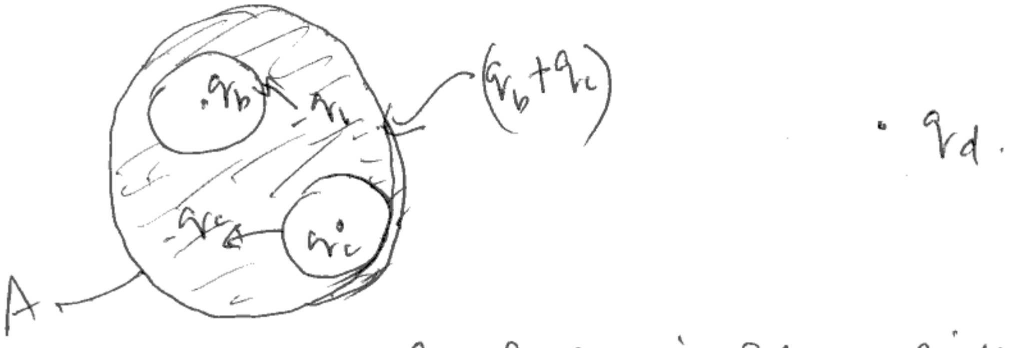
$$\int_V \vec{\nabla}\cdot(\phi\vec{E})\,dv = \int_S \phi\vec{E}\cdot d\vec{a}.$$

In the limit of S going to infinity, the surface integral vanishes : $\phi \sim \frac{1}{r}$, $E \sim \frac{1}{r^2}$, $a \sim r^2$

$\therefore$ The integrand goes as $\frac{1}{r}$. & in the limit of large volume, $\int_S \phi\vec{E}\cdot d\vec{a} \longrightarrow 0$.

$$\therefore \frac{\epsilon_0}{2}\int_V E^2\,dv = \frac{1}{2}\int_V \rho\,\phi\,dv$$

2.



Force on q_b & q_c is zero, since $\vec{E} = 0$ inside conductor.

The induced charges follow from the argument in the previous problem.

The charge induced on the outer surface of A is $(q_b + q_c)$ distributed in a spherically symmetric manner. The field due to A : $\dfrac{(q_b + q_c)}{4\pi\epsilon_0 r^2}$.

q_d will disturb the distribution of charge but not the amount ~~of~~ of charge on surface of A.

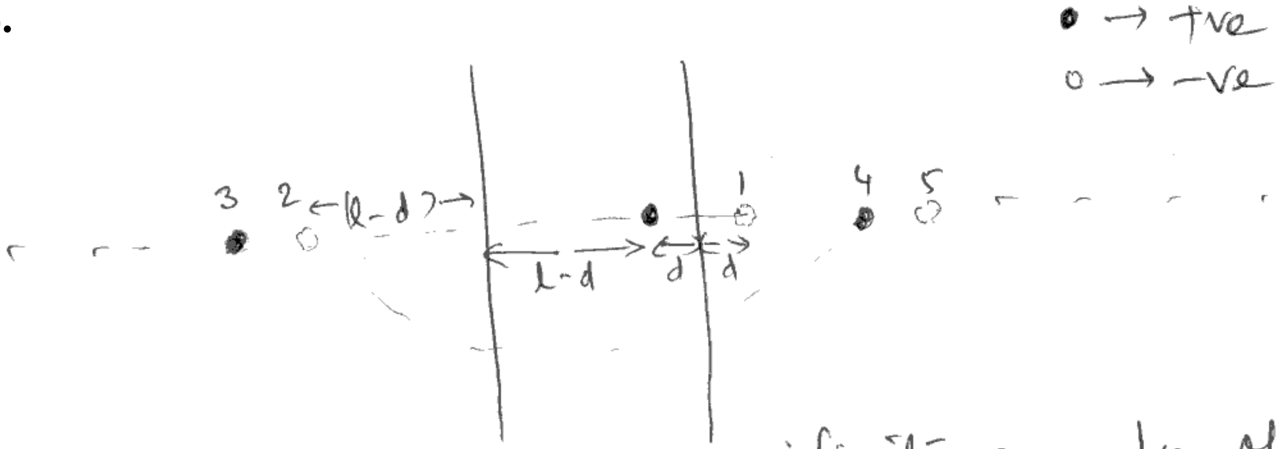
If q_d is placed far enough, then,

force on q_d = $q_d \cdot \dfrac{(q_b+q_c)}{4\pi\epsilon_0 r^2}$

$$\therefore \vec{F}_d = \frac{q_d(q_b+q_c)}{4\pi\epsilon_0 r^2}\hat{r}.$$

Force on A : $\vec{F}_A = -\vec{F}_d$.

4.



We will need an infinite number of image charges as shown.

3. Field outside the outer shell = 0.

$\therefore$ Potential at outer shell = potential at infinity.

$\therefore$ When outer shell is grounded, charge will not move.

If inner shell is grounded, then,

potential diff between inner and outer shells

= potential diff between outer shell & infinity

If Q_f is final charge on inner shell, then,

electric field between shells $= \frac{Q_f}{4\pi\epsilon_0 r^2}\hat{r}$

$\therefore$ potential diff between inner and outer shells

$$= -\frac{Q_f}{4\pi\epsilon_0}\int_{R_1}^{R_2}\frac{dr}{r^2} = -\frac{Q_f}{4\pi\epsilon_0}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)$$

Potential diff between outer shell & infinity

$$= \frac{(-Q+Q_f)}{4\pi\epsilon_0}\int_{R_2}^{\infty}\frac{dr}{r^2} = \frac{(-Q+Q_f)}{4\pi\epsilon_0}\cdot\frac{1}{R_2}$$

$$\therefore \frac{-Q_1+Q_F}{4\pi\varepsilon_0} \cdot \frac{1}{R_2} = -Q_F\left(\frac{1}{R_1}-\frac{1}{R_2}\right) \Rightarrow Q_F = \frac{R_1}{R_2} Q_1.$$

5. Energy stored in the field of the disk is,

$$U_1 = \frac{Q^2}{2C} = \frac{Q^2}{2(8\epsilon_0 a)} = \frac{Q^2}{16\epsilon_0 a}.$$

For a uniformly charged non-conducting disk,

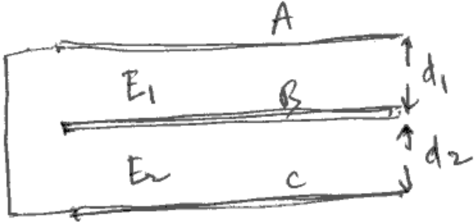
show that, $$U_2 = \frac{2}{3}\cdot\frac{Q^2}{\pi^2 \epsilon_0 a}.$$

$$\therefore \frac{U_2}{U_1} = \frac{2/3\pi^2}{1/16} = \frac{32}{3\pi^2} \simeq 1.081.$$

$\therefore U_2 > U_1.$

On the conducting disk, the charge distributes itself to minimize energy.

7.



Let's call the three conducting plates A, B and C.

Since A & C are connected by a wire, they are at the same potential. Therefore, if B is at some potential, then the potential difference between A & B and between B & C are same.

Now, the electric fields between the plates are given as E_1 & E_2. If σ_1 is the surface charge on the upper surface of B & σ_2 that in the lower surface then,

$$E_1 = \frac{\sigma_1}{\epsilon_0} \quad \& \quad E_2 = \frac{\sigma_2}{\epsilon_0}.$$

Now, potential difference between plates A & B and between B & C :

$$\phi = E_1 d_1 = E_2 d_2 . \qquad (\because E = \phi/d)$$

uniform fields!

$$\therefore \frac{\sigma_1}{\epsilon_0} d_1 = \frac{\sigma_2}{\epsilon_0} d_2 \Rightarrow \sigma_1 d_1 = \sigma_2 d_2 .$$

Also, $\sigma = \sigma_1 + \sigma_2$.

Solving, $\sigma_1 = \dfrac{\sigma d_2}{d_1 + d_2}$; $\sigma_2 = \dfrac{\sigma d_1}{d_1 + d_2}$.

8. $V_1 = 100$ volts. $C_1 = 100\ pF = 100 \times 10^{-12} F = 10^{-10} F$

$\therefore Q = V_1 C_1 = 100 \times 10^{-10}\ C = 10^{-8}\ C$.

After charging battery is disconnected & the capacitor is connected in $\parallel^{l}$ to another capacitor of capacitance C_2 then, the total charge remain same. If V_2' is the final voltage, then,

$Q = V_2'(C_1 + C_2)$ ($\parallel^{l}$: capacitance add up).

$V_2' = 30$ volts.

$$\therefore 10^{-8} = 30.(10^{-10} + C_2).$$

$$C_2 = \frac{10^{-8}}{30} - 10^{-10} = 10^{-10}\left(\frac{100}{30} - 1\right).$$

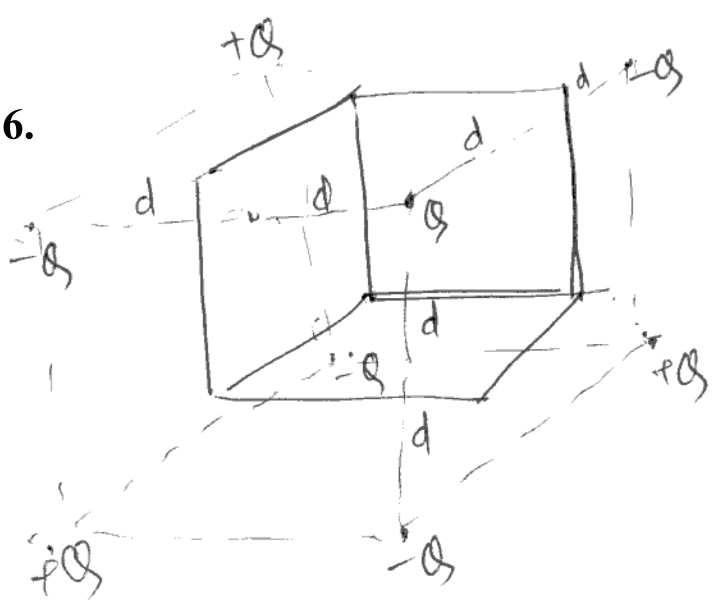
$$= \frac{7}{3} \times 10^{-10}\ F = \frac{7}{3} C_1$$

$$E_i = \frac{Q^2}{2C_1} = \frac{1}{2} Q V_1 \quad ; \quad E_f = \frac{1}{2} Q V_2.$$

$$\therefore \text{Energy lost} = E_i - E_f = \frac{1}{2} Q (V_1 - V_2).$$

$$= \frac{1}{2} \cdot 10^{-8} \cdot (100 - 30) = 35 \times 10^{-8}\ J.$$

6.



7 image charges forming 7 corners of the cube of side $2d$.

Calculate force.