$$\psi(x) = N\, x\, \exp\left(-\tfrac{1}{2}\alpha x^2\right)$$

(a) Normalize $\longrightarrow$

$$\int_{-\infty}^{+\infty} \psi^* \psi \, dx = 1$$

$$\Rightarrow \int_{-\infty}^{+\infty} N^2 \underbrace{x^2 \exp\left(-\alpha x^2\right)}_{\text{Even function}} dx = 1.$$

$$2\int_{0}^{\infty} N^2 x^2 \exp\left(-\alpha x^2\right) dx = 1. \quad \text{—(1)}$$

We know
$$\boxed{\begin{aligned} \int_0^{\infty} e^{-\alpha x^2} x^n \, dx &= \frac{(n-1)!}{2^{n/2+1}\,\alpha^{n/2}} \sqrt{\frac{\pi}{\alpha}} \quad \text{For } n \text{ even} \\ &= \frac{\left[\frac{1}{2}(n-1)\right]!}{2\, a^{(n+1)/2}} \quad \text{For } n \text{ odd.} \end{aligned}}$$

From (1) $\Rightarrow$
$$2N^2 \frac{1}{2^2 \alpha} \sqrt{\pi/\alpha} = 1.$$

$$N^2 \frac{1}{2\alpha} \sqrt{\pi/\alpha} = 1.$$

$$N = \left[\sqrt{\alpha/\pi} \cdot 2\alpha\right]^{1/2}$$

$$= \left(\frac{2\,\alpha^{3/2}}{\pi^{1/2}}\right)^{1/2} = \frac{\sqrt{2}\,\alpha^{3/4}}{\pi^{1/4}}$$

$$\boxed{N = \frac{\sqrt{2}\,\alpha^{3/4}}{\pi^{1/4}}}$$

What is $\langle \hat{x} \rangle$ and $\langle \hat{x}^2 \rangle$

$$\langle \hat{x} \rangle = \int_{-\infty}^{+\infty} \psi^* \hat{x} \psi \, dx = N^2 \int_{-\infty}^{+\infty} x \exp - \tfrac{1}{2}\alpha x^2 \cdot \hat{x} \left[x \exp - \tfrac{1}{2}\alpha x^2 \right] dx$$

$$= N^2 \int_{-\infty}^{+\infty} \underbrace{x^3 \exp - \alpha x^2}_{\text{odd}} \, dx$$

$$\boxed{\langle \hat{x} \rangle = 0}$$

$$\langle x^2 \rangle = \int_{-\infty}^{+\infty} \psi^* x^2 \psi \, dx = N^2 \int_{-\infty}^{+\infty} \underbrace{x^4 \exp(-\alpha x^2)}_{\text{Even}} \, dx$$

$$= 2N^2 \int_{0}^{+\infty} x^4 \exp - \alpha x^2 \, dx \, .$$

Apply the formulation mentioned in box for $\boxed{n=4}$

(b) $\langle \hat{p} \rangle$ and $\langle \hat{p}^2 \rangle$

$$\langle \hat{p} \rangle = \int_{-\alpha}^{+\alpha} \psi^* \hat{p} \psi \, dx = N^2 \int_{-\alpha}^{+\alpha} x \exp -\frac{1}{2}\alpha x^2 \left(-i\hbar \frac{\partial}{\partial x}\right) \times \exp\left(-\frac{1}{2}\alpha x^2\right) dx$$

$$= N^2 \int_{-\alpha}^{+\alpha} (-i\hbar) \times \exp -\frac{\alpha x^2}{2} \left[\left[1 + x\left(-\frac{1}{2}\alpha . 2x\right)\right] \exp - \frac{\alpha x^2}{2}\right] dx$$

$$= N^2 \int -i\hbar \left[\underbrace{x \exp - \alpha x^2}_{\text{odd}} - \underbrace{\alpha x^3 \exp - \alpha x^2}_{\text{odd}}\right] dx$$

$$\boxed{\langle \hat{p} \rangle = 0}$$

$$\langle p^2 \rangle = \int_{-\alpha}^{+\alpha} \psi^* \hat{p}^2 \psi \, dx = \int_{-\alpha}^{+\alpha} \psi^* (-i\hbar)^2 \frac{\partial^2}{\partial x^2} \psi \, dx$$

$$= -\hbar^2 \int_{-\alpha}^{+\alpha} \psi^* \frac{\partial^2 \psi}{\partial x^2} \, dx.$$

On evaluation of $\psi^* \frac{\partial^2 \psi}{\partial x^2}$ you will get even integrand. It can be integrated easily via the formula for Gaussian integration.

(d) For $V(x) = 0 \longrightarrow \langle H \rangle = ?$

$$\langle H \rangle = \langle p^2/2m + \underset{\substack{\uparrow \\ 0}}{V(x)} \rangle = \langle p^2/2m \rangle = \frac{-\hbar^2}{2m} \langle \partial^2/\partial x^2 \rangle$$

$$= -\frac{\hbar^2}{2m} \int_{-a}^{+a} \psi^* \frac{\partial^2}{\partial x^2} \psi \, dx$$

Putting $\psi(x) = N x \, Exp\left(-\frac{1}{2} \alpha x^2\right)$

You can work out the value of $\langle H \rangle$

(e) Find $v(x) = ?$

$$H\,\psi(x) = E\,\psi(x)$$

$$\left[+\frac{p^2}{2m} + V(x)\right]\psi(x) = E\,\psi(x)$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x) + V(x)\,\psi(x) = E\,\psi(x)$$

Now, $\psi(x) = N\,x\,\exp -\frac{\alpha x^2}{2}$

$$\frac{\partial\psi}{\partial x} = N\exp -\frac{\alpha x^2}{2} + N\,x\,(-\alpha x)\exp -\frac{\alpha x^2}{2}$$

$$= N\exp -\frac{\alpha x^2}{2} - N\alpha x^2\exp -\frac{\alpha x^2}{2}$$

$$\frac{\partial^2\psi}{\partial x^2} = N(-\alpha x)\exp -\frac{\alpha x^2}{2} - 2N\alpha x\exp -\frac{\alpha x^2}{2} + N\alpha^2 x^3\exp -\frac{\alpha x^2}{2}$$

$$= \left(-3N\alpha x + N\alpha^2 x^3\right)\exp -\frac{\alpha x^2}{2}$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V(x)\,\psi(x) = E\,\psi(x)$$

$$-\frac{\hbar^2}{2m}\left(-3N\alpha x + N\alpha^2 x^3\right)\exp -\frac{\alpha x^2}{2} + V(x)\,N.x\exp -\frac{\alpha x^2}{2}$$

(1)

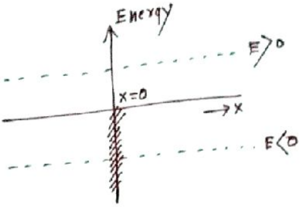
$$= NE\,x\exp -\frac{\alpha x^2}{2}$$

Equate. Taking $V(x) = \frac{\hbar^2\alpha^2 x^2}{2m} \rightarrow$ satisfies $V(0) = 0$

we get $\Rightarrow \boxed{E = \frac{3\hbar^2\alpha}{2m}}$

⇒ A delta-function is an infinitely high, infinitesimal narrow spike.

⇒ Form = $V(x) = -\alpha\delta(x)$



⇒ Schrödinger Eq^n

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} - \alpha\delta(x)\psi = E\psi(x) \quad -(1)$$

looking only for $E<0$ cases. Forget about $E>0$. → (2)

⇒ For $x>0$ and $x<0$, $V(x)=0$

⇒ Eq^1 becomes ⟶ $\boxed{\frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi}$ ⟶ $\frac{d^2\psi}{dx^2} = k^2\psi$

where $\boxed{k^2 = -\frac{2mE}{\hbar^2}}$

⇒ Since $E<0$; k^2 is always positive.

⇒ By Solving, We get $\boxed{\psi_{\text{at } x<0} = Be^{kx} \text{ and } \psi_{\text{at } x>0} = Fe^{-kx}}$

⇒ Continuity at $x=0$ claims ⇒ $\psi_{x<0}\Big|_{\text{at } x=0} = \psi_{x>0}\Big|_{\text{at } x=0}$ ⇒ $\boxed{B=F}$

⇒ Taking (1) and integrating

$$-\frac{\hbar^2}{2m}\int_{-\epsilon}^{+\epsilon}\frac{d^2\psi}{dx^2}dx - \alpha\int_{-\epsilon}^{+\epsilon}\delta(x)\psi(x)\,dx = E\int_{-\epsilon}^{+\epsilon}\psi(x)\,dx$$

Make it narrow

$$-\frac{\hbar^2}{2m}\left[\frac{d\psi}{dx}\Big|_{+\epsilon} - \frac{d\psi}{dx}\Big|_{-\epsilon}\right] - \alpha\psi(0) = 0$$

$$\boxed{\Delta\left(\frac{d\psi}{dx}\right) = -\frac{2m\alpha}{\hbar^2}\psi(0)}$$

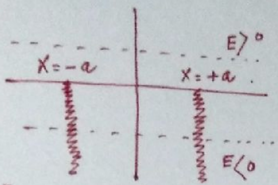
⇒ $x<0$: $\frac{\partial\psi}{\partial x}\Big|_{-\epsilon} = Bk$; $x>0$: $\frac{\partial\psi}{\partial x}\Big|_{\epsilon} = -Bk$, $\psi(0) = B$

⇒ $\Delta\left(\frac{d\psi}{dx}\right) = 2Bk = -\frac{2m\alpha}{\hbar^2}B$ ⇒ $\boxed{k = \frac{m\alpha}{\hbar^2}}$

$$\boxed{E = -\frac{m\alpha^2}{2\hbar^2}}$$

Double Delta

Problem NO. 2



$$V(x) = -\alpha\left[\delta(x+a) + \delta(x-a)\right] \quad \text{where } \alpha = \frac{\hbar^2}{ma}$$

(2)

$x < -a$: $\psi(x) = Ae^{kx}$

$-a \le x \le a$: $\psi(x) = Ce^{kx} + De^{-kx}$

$x > a$: $\psi(x) = Fe^{-kx}$

$$\boxed{k^2 = -\frac{2mE}{\hbar^2}}$$

Continuity at $x = -a \Rightarrow \boxed{Ae^{-ka} = Ce^{-ka} + De^{ka}}$ — (A)

$x = +a \Rightarrow \boxed{Fe^{-ka} = Ce^{ka} + De^{-ka}}$ — (B)

Discontinuity of $\frac{d\psi}{dx}$ at $x = -a$

$$\boxed{k\left(Ce^{-ka} - De^{ka}\right) - kAe^{-ka} = -\frac{2m\alpha}{\hbar^2}Ae^{-ka} = -\frac{2}{a}Ae^{-ka}}$$

at $x = +a \Rightarrow \boxed{-kFe^{-ka} - k\left(Ce^{-ka} - De^{ka}\right) = -\frac{2m\alpha}{\hbar^2}Fe^{-ka} = -\frac{2}{a}Fe^{-ka}}$

Discontinuous equations can be simplified as

For $x = -a$: $Ae^{-ka}\left(k - \frac{2}{a}\right) = k\left(Ce^{-ka} - De^{ka}\right)$ — (C)

$x = +a$: $Fe^{-ka}\left(\frac{2}{a} - k\right) = k\left(Ce^{-ka} - De^{ka}\right)$ — (D)

Eliminating C and D for A and F

$$\left.\begin{aligned} Ce^{-ka} &= De^{ka}(ka-1) \\ De^{-ka} &= Ce^{ka}(ka-1) \end{aligned}\right\} \longrightarrow C^2 = D^2 \Rightarrow \boxed{C = \pm D}$$

For $C = D$: $\psi = C\left(e^{kx} + e^{-kx}\right) = C' \cosh kx$

$C = -D$: $\psi = C\left(e^{kx} - e^{-kx}\right) = C' \sinh kx$

Solving for Bound states

Even $C = D$: $\boxed{e^{-2ka} = ka - 1}$

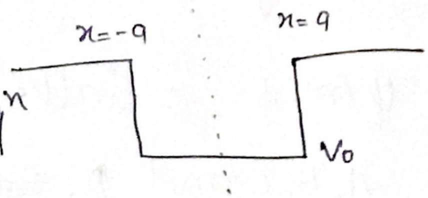
Odd $C = -D$: $\boxed{e^{-2ka} = 1 - ka}$

From intersection work out the value of ka and get E_{Bound}.

3 (a). For finite square well potential

$$V(x) = \begin{cases} -V_0 & \text{for} \quad -a \le x \le a \\ 0 & \text{otherwise} \quad (|x| < a) \end{cases}$$

Solⁿ :-



(i) In Region $|x| < a$, schrödinger eqⁿ

$$\left(\frac{-\hbar^2}{2m}\nabla^2 + V\right)\psi = E\psi \qquad V=0 \text{ in this Region}$$

$$-\frac{\hbar^2}{2m}\nabla^2\psi = E\psi \qquad \text{Let } K = \sqrt{\frac{-2mE}{\hbar^2}}$$

$$\frac{\partial^2\psi}{\partial x^2} = \frac{-2mE}{\hbar^2}\psi$$

$$\frac{\partial^2\psi}{\partial x^2} = K^2\psi$$

solution of this eqⁿ.

$$\psi(x) = Ae^{Kx} + Be^{-Kx}$$

But $x \to \infty$, ~~first~~ first term blows up and similarly $x \to -\infty$, second term diverge.

So,

$$\psi(x) = \begin{cases} Ae^{-Kx} & \text{for} \quad x < -a \\ Be^{Kx} & \text{for} \quad x > a \end{cases}$$

(ii) Now In Region $-a \le x \le a$ $\quad V(x) = -V_0$

Schrödinger eqⁿ

$$\left(\frac{-\hbar^2}{2m}\nabla^2 - V_0\right)\psi(x) = E\psi(x).$$

$$\frac{\partial^2 \psi'}{\partial x^2} = \frac{-2m(E+V_0)}{\hbar^2}\psi \qquad \text{Let } l = \sqrt{\frac{2m\,E+V_0}{\hbar^2}}$$

$$\frac{\partial^2 \psi'}{\partial x^2} = -l^2 \psi$$

Solution of this equation is.

$$\psi(x) = C\sin(lx) + D\cos(lx)$$

So, A, B, C and D. ~~cant~~ Can be determined by using boundary conditions.

(i) contious at $x = a$ and $x = -a$

(ii) $\frac{\partial \psi}{\partial x}$ contious at $x = a$ and $x = -a$

As we know that potential is symmetric about $x=0$

So,

(i) $Be^{-Ka} = C\cos(la)$.

(ii) $-KBe^{-Ka} = C\sin(la).(-l)$

$$\frac{K}{l} = \tan(la)$$

Now for sufficiently deep well, we given that

$$z_0 = \frac{a}{\hbar}\sqrt{2mV_0} \text{ is fairly large}$$

$$z = la \quad \text{and} \quad z_0 = \frac{a}{\hbar}\sqrt{2mV_0}$$

as we know that value of K and l

$$K^2 + l^2 = 2mV_0/\hbar^2$$

$$\tan z = \sqrt{\left(\frac{z_0}{z}\right)^2 - 1} \quad \text{if} \quad z_0 \to \infty$$

So,

$$\tan(z) = \tan\left(\frac{(2n+1)\pi}{2}\right)$$

So,

$$la \simeq \frac{(2n+1)\pi}{2}$$

$$\sqrt{\frac{2m(E+V_0)}{\hbar^2}}\, a = \frac{(2n+1)\pi}{2}$$

$$\frac{2m(E+V_0)}{\hbar^2}\, a^2 = \frac{(2n+1)^2\pi^2}{4}$$

$$E = \frac{(2n+1)^2\pi^2\hbar^2}{8ma^2} - V_0$$

$$E = \frac{(2n+1)^2\pi^2\hbar^2}{2m(2a)^2} - V_0$$

for finite V_0, only finite no. of bound states will be possible as long as $E < 0$

or

$$\boxed{\frac{(2n+1)^2\pi^2\hbar^2}{2m(2a)^2} < V_0}$$

4a. $\psi(x) = e^{i\alpha}\phi(x)$

where $\phi(x)$ is Real and $\alpha =$ Constant.

Sol^n

$$\langle p\rangle = \int_{-\infty}^{\infty} e^{-i\alpha}\phi(x) \cdot -i\hbar\frac{\partial}{\partial x} e^{i\alpha}\phi(x)\, dx$$

$$= -i\hbar\int_{-\infty}^{\infty} \phi(x) \frac{\partial}{\partial x}\phi(x)\, dx$$

$$\langle p\rangle = -i\hbar\int_{-\infty}^{\infty} \phi(x)\cdot\phi'(x)\, dx.$$

So, as p is hermitian operator so, ~~eigen~~ expectation value of $(\langle p\rangle)$ p must be Real.

and $\phi(x)$ is Real ~~value~~ function. So,

$$\int_{-\infty}^{\infty}\phi(x)\,\phi'(x)\,dx = \text{Real value.}$$

$$\langle p\rangle = \underbrace{-i\hbar\,(\text{Real value})}_{\text{imaginary value}} \qquad \{\langle p\rangle = a+ib\}\text{ for Ex.}$$

So,

$$\langle p\rangle = 0$$

4b. $\psi(x) = \phi_1(x) + e^{i\alpha}\phi_2(x)$

similarly

$$\langle p\rangle = \int_{-\infty}^{\infty}\left(\phi_1(x) + e^{-i\alpha}\phi_2(x)\right) -i\hbar\frac{\partial}{\partial x}\left(\phi_1(x) + e^{i\alpha}\phi_2(x)\right) dx.$$

$$\langle p \rangle = -i\hbar \Big[\cancel{\int_{-\infty}^{\infty} \phi_1(x)\, \phi_1'(x)} \xrightarrow{} 0 + \int_{-\infty}^{\infty} \phi_1(x)\, \phi_2'(x)\, e^{i\alpha}$$

$$+ \int_{-\infty}^{\infty} \phi_2(x)\, \phi_1'(x)\, e^{-i\alpha} + \cancel{\int_{-\infty}^{\infty} \phi_2(x)\, \phi_2'(x)\, dx} \xrightarrow{} 0 \Big]$$

Because of these both term will give us Real value.

$$\langle p \rangle = -i\hbar e^{i\alpha} \int_{-\infty}^{\infty} \phi_1(x)\, \phi_2'(x)\, dx$$

$$+ -i\hbar \int_{-\infty}^{\infty} e^{-i\alpha}\, \phi_2(x)\, \phi_1'(x)\, dx$$

$\langle p \rangle =$ Real value.

$\int_{-\infty}^{\infty} \phi_i(x)\, \phi_j'(x) =$ Real value

So, if $\alpha = \pi/2\,(2n+1)$ $\quad e^{\pm i\alpha} = \pm i$

So,

$\langle p \rangle =$ Non zero when $\alpha = \left(\frac{2n+1}{2}\right)\pi$

Problem 5: The 1-D Schrödinger eqn.

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t) + V(x)\psi(x,t) = i\hbar\frac{\partial}{\partial t}\psi(x,t).$$

It can also be written as

$$\hat{H}\psi(x,t) = i\hbar\frac{\partial}{\partial t}\psi(x,t) = i\hbar\dot{\psi}(x,t)$$

where the Hamiltonian is given by

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial}{\partial x^2} + V(x)$$

(a) The probability density is

$$\rho(x,t) = \psi^*\psi.$$

Then

$$\frac{\partial\rho}{\partial t} = \dot{\psi}^*\psi + \psi^*\dot{\psi}.$$

$$= \left[(\hat{H}\psi)^*\psi - \psi^*(\hat{H}\psi)\right]\frac{i}{\hbar}$$

$$= \left[-\frac{\hbar^2}{2m}\left(\frac{\partial^2\psi^*}{\partial x^2}\psi - \psi^*\frac{\partial^2\psi}{\partial x^2}\right)\right]\frac{i}{\hbar}$$

$$+ \Big[\underbrace{V(x,t)\psi^*\psi - \psi^* V(x,t)\psi}_{=0}\Big]\frac{i}{\hbar}$$

Now

$$\frac{\partial\rho}{\partial t} = \frac{\hbar}{2im}\left(\frac{\partial^2\psi^*}{\partial x^2}\psi - \psi^*\frac{\partial^2\psi}{\partial x^2}\right)$$

$$= -\frac{\partial}{\partial x}\left[\frac{\hbar}{2im}\left(\psi^*\frac{\partial\psi}{\partial x} - \frac{\partial\psi^*}{\partial x}\psi\right)\right]$$

$$= -\frac{\partial}{\partial x}\left[\frac{\hbar}{2im}\,2i\,\mathrm{Im}\left(\psi^*\frac{\partial\psi}{\partial x}\right)\right] = -\frac{\partial}{\partial x}J$$

where the probability current is defined as (2)

$$J(x,t) = \frac{\hbar}{m} \, Im \left(\Psi^* \frac{\partial \Psi}{\partial x} \right).$$

Therefore we have shown.

$$\frac{\partial \rho}{\partial t} + \frac{\partial J}{\partial x} = 0.$$

—

Note $\int dx \, \Psi^* \Psi = 1$ is unit free.

Then, the $\rho = \Psi^* \Psi$ has the unit L^{-1}.

Similarly $\left[\Psi^* \frac{\partial \Psi}{\partial x} \right] = \frac{1}{L^2}$; $[\hbar] = \frac{ML^2}{T}$; $\left[\frac{\hbar}{m} \right] = \frac{L^2}{T}$.

Thus $[J] = \frac{1}{T} \Rightarrow$ probability per unit time.

(b) Now we demonstrate the conservation of probability — in time.

Say, $N(t)$ is the integral of the $\rho(x,t)$ throughout the space —

$$N(t) \equiv \int_{-\alpha}^{\alpha} \rho(x,t) \, dx.$$

The change in probability ~~ra~~ is then

$$\frac{dN(t)}{dt} = \int_{-\alpha}^{\alpha} \frac{\partial \rho(x,t)}{\partial t} dx = - \int_{-\alpha}^{\alpha} \frac{\partial J}{\partial x} dx$$

$$= - \left(J(\alpha,t) - J(-\alpha,t) \right)$$

The $\frac{dN(t)}{dt}$ vanishes if the probability current vanishes at infinity, $|x| \to \infty$.

Recall

$$J = \frac{\hbar}{2im}\left(\psi^* \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi^*}{\partial x}\right).$$

The current indeed vanishes as we restrict the wave fn. to satisfy

$$\lim_{x \to \pm\infty} \psi = 0 \quad \& \quad \lim_{x \to \pm\infty} \frac{\partial \psi}{\partial x} \text{ is bouded.}$$

Therefore, we prove $\frac{dN}{dt} = 0$.

Now we go for probability in a region $[a,b]$, and is given by

$$P_{ab}(t) = +\int_a^b \rho(x,t)\, dx.$$

The change in the probability –

$$\frac{dP_{ab}(t)}{dt} = -\int_a^b \frac{\partial J(x,t)}{\partial x}\, dt = -J(b,t) + J(a,t)$$

$-J(b,t) \to$ probability flowing out at $x = b$.

$J(a,t) \to$ probability flowing in at $x = a$.