PHY401:
Nuclear And Particle Physics

Quark Model

IISER

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Symmetries and Conservation Laws

★Suppose physics is invariant under the transformation

$$\psi \rightarrow \psi' = \hat{U}\psi \qquad \text{e.g. rotation of the coordinate axes}$$

•To conserve probability normalisation require

$$\langle\psi|\psi\rangle = \langle\psi'|\psi'\rangle = \langle\hat{U}\psi|\hat{U}\psi\rangle = \langle\psi|\hat{U}^\dagger\hat{U}|\psi\rangle$$

$$\Rightarrow \boxed{\hat{U}^\dagger\hat{U} = 1}$$ i.e. $\hat{U}$ has to be unitary

•For physical predictions to be unchanged by the symmetry transformation, also require all QM matrix elements unchanged

$$\boxed{\langle\psi|\hat{H}|\psi\rangle = \langle\psi'|\hat{H}|\psi'\rangle} = \langle\psi|\hat{U}^\dagger\hat{H}\hat{U}|\psi\rangle$$

i.e. require $\quad \hat{U}^\dagger\hat{H}\hat{U} = \hat{H}$

$\times\hat{U} \quad \hat{U}\hat{U}^\dagger\hat{H}\hat{U} = \hat{U}\hat{H} \quad \Rightarrow \quad \hat{H}\hat{U} = \hat{U}\hat{H}$

therefore $\quad \boxed{[\hat{H},\hat{U}] = 0}$ $\quad$ $\boxed{\hat{U} \text{ commutes with the Hamiltonian}}$

★Now consider the infinitesimal transformation (ε small)

$$\hat{U} = 1 + i\varepsilon\hat{G}$$

($\hat{G}$ is called the generator of the transformation)

Symmetries and Conservation Laws

- For $\hat{U}$ to be unitary

$$\hat{U}\hat{U}^{\dagger} = (1 + i\varepsilon\hat{G})(1 - i\varepsilon\hat{G}^{\dagger}) = 1 + i\varepsilon(\hat{G} - \hat{G}^{\dagger}) + O(\varepsilon^2)$$

neglecting terms in ε^2 $\quad UU^{\dagger} = 1 \quad \Rightarrow \quad \boxed{\hat{G} = \hat{G}^{\dagger}}$

i.e. $\hat{G}$ is Hermitian and therefore corresponds to an observable quantity ! G

- Furthermore, $[\hat{H}, \hat{U}] = 0 \quad \Rightarrow [\hat{H}, 1 + i\varepsilon\,\hat{G}] = 0 \quad \Rightarrow [\hat{H}, \hat{G}] = 0$

But from QM $\quad \dfrac{\mathrm{d}}{\mathrm{d}t}\langle\hat{G}\rangle = i\langle[\hat{H}, \hat{G}]\rangle = 0$

i.e. G is a conserved quantity.

Symmetry $\Longleftrightarrow$ Conservation Law

★ For each symmetry of nature have an observable <u>conserved</u> quantity

<u>Example:</u> Infinitesimal spatial translation $\quad x \rightarrow x + \varepsilon$

i.e. expect physics to be invariant under $\quad \psi(x) \rightarrow \psi' = \psi(x + \varepsilon)$

$$\psi'(x) = \psi(x + \varepsilon) = \psi(x) + \frac{\partial\psi}{\partial x}\varepsilon = \left(1 + \varepsilon\frac{\partial}{\partial x}\right)\psi(x)$$

but $\quad \hat{p}_x = -i\frac{\partial}{\partial x} \quad \Rightarrow \quad \psi'(x) = (1 + i\varepsilon\hat{p}_x)\psi(x)$

The generator of the symmetry transformation is , $\hat{p}_x \Rightarrow p_x$ is conserved

- Translational invariance of physics implies momentum conservation !

Symmetries and Conservation Laws

- In general the symmetry operation may depend on more than one parameter

$$\hat{U} = 1 + i\vec{\varepsilon}.\vec{G}$$

For example for an infinitesimal 3D linear translation : $\vec{r} \rightarrow \vec{r} + \vec{\varepsilon}$

$$\Rightarrow \quad \hat{U} = 1 + i\vec{\varepsilon}.\vec{p} \qquad \vec{p} = (\hat{p}_x, \hat{p}_y, \hat{p}_z)$$

- So far have only considered an infinitesimal transformation, however a finite transformation can be expressed as a series of infinitesimal transformations

$$\hat{U}(\vec{\alpha}) = \lim_{n\rightarrow\infty}\left(1 + i\frac{\vec{\alpha}}{n}.\vec{G}\right)^n = e^{i\vec{\alpha}.\vec{G}}$$

Example: Finite spatial translation in 1D: $x \rightarrow x + x_0$ with $\hat{U}(x_0) = e^{ix_0\hat{p}_x}$

$$\left(p_x = -i\frac{\partial}{\partial x}\right)$$

$$\begin{aligned}
\psi'(x) = \psi(x + x_0) &= \hat{U}\psi(x) = \exp\left(x_0\frac{\mathrm{d}}{\mathrm{d}x}\right)\psi(x) \\
&= \left(1 + x_0\frac{\mathrm{d}}{\mathrm{d}x} + \frac{x_0^2}{2!}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \ldots\right)\psi(x) \\
&= \psi(x) + x_0\frac{\mathrm{d}\psi}{\mathrm{d}x} + \frac{x_0^2}{2}\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + \ldots
\end{aligned}$$

i.e. obtain the expected Taylor expansion

Symmetries: Conserved quantities

Time dependence of observable U:

$$\begin{aligned}\frac{d}{dt}\langle U\rangle &= \frac{d}{dt}\langle\Psi|U|\Psi\rangle \\ &= \langle\frac{\partial\Psi}{\partial t}|U|\Psi\rangle + \langle\Psi|\frac{\partial U}{\partial t}|\Psi\rangle + \langle\Psi|U|\frac{\partial\Psi}{\partial t}\rangle \\ &= -\frac{1}{i\hbar}\langle H\Psi|U|\Psi\rangle + \langle\Psi|\frac{\partial U}{\partial t}|\Psi\rangle + \langle\Psi|U|H\Psi\rangle \\ &= \frac{1}{i\hbar}\langle[U,H]\rangle + \langle\Psi|\frac{\partial U}{\partial t}|\Psi\rangle\end{aligned}$$

Hamilton formalism: $$i\hbar\frac{\mathrm{d}}{\mathrm{dt}}\Psi = H\Psi$$

If *U* commutes with *H*, *[U,H]=0*
(and if *U* does not depend on time, dU/dt=0)

Then U is conserved: d/dt<U> = 0

U conserved → U generates a symmetry of the system

Transformation	Conserved quantity
Translation (space)	Momentum
Translation (time)	Energy
Rotation (space)	Orbital momentum
Rotation (iso-spin)	Iso-spin

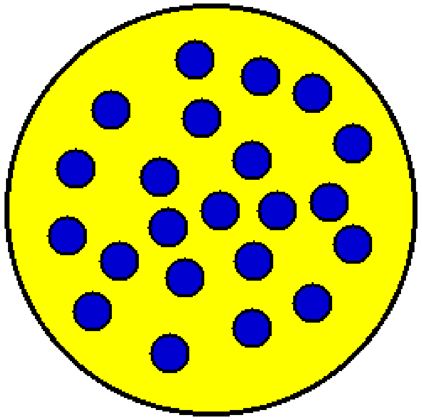
1894 - 1897: Discovery of the electron

Study of "cathode rays": electric current in tubes at very low gas pressure ("glow discharge")

Measurement of the electron mass: $m_e \approx M_H / 1836$

"Could anything at first sight seem more impractical than a body which is so small that its mass is an insignificant fraction of the mass of an atom of hydrogen?" (J.J. Thomson)

ATOMS ARE NOT ELEMENTARY



Thomson's atomic model:

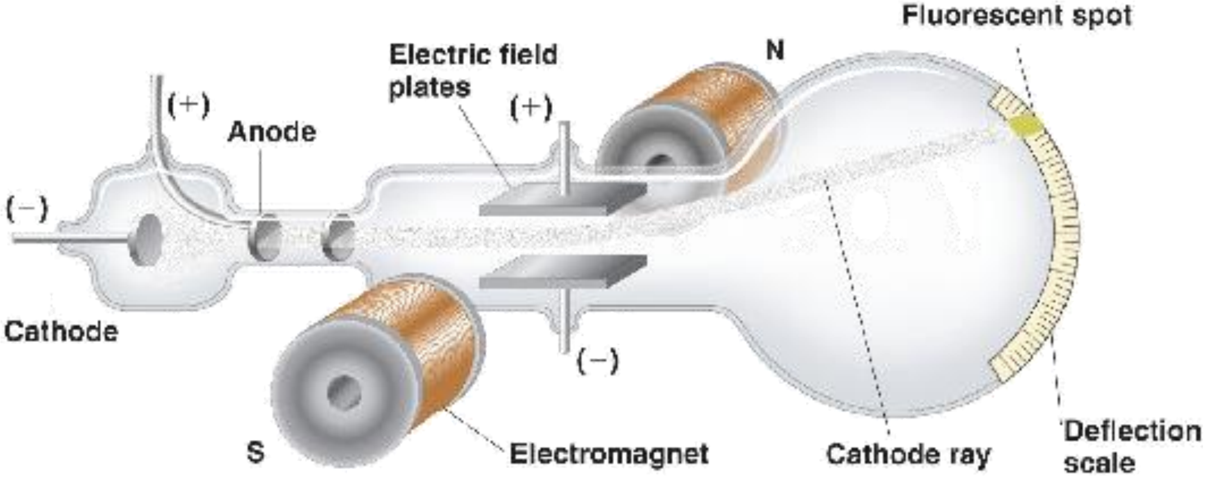
- Electrically charged sphere
- Radius ~ 10^{-8} cm
- Positive electric charge
- Electrons with negative electric charge embedded in the sphere

Electrons - 1897

Experiments by J.J. Thomson in 1897 led to the discovery of a fundamental building block of matter.

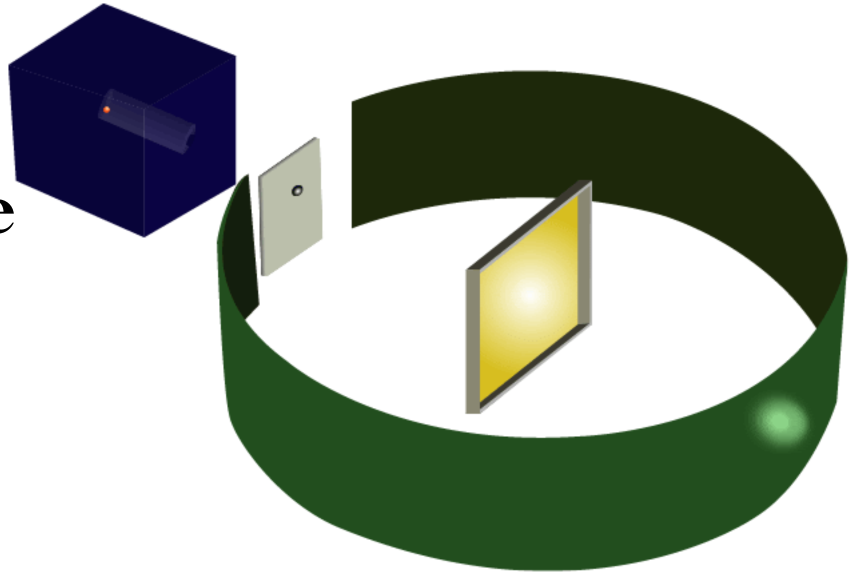
Excuse me... how can you discover a particle so small that nobody has ever seen one?

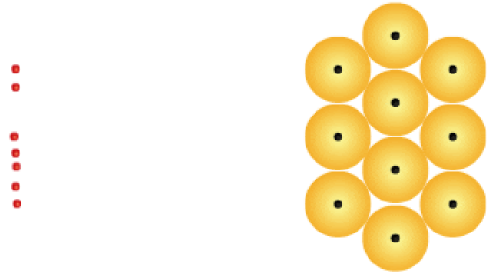
Accelerate in electric field, deflect in magnetic field and observe by having the e interact in a detector – the basic steps. Lesson: We can see electrons with our own eyes!



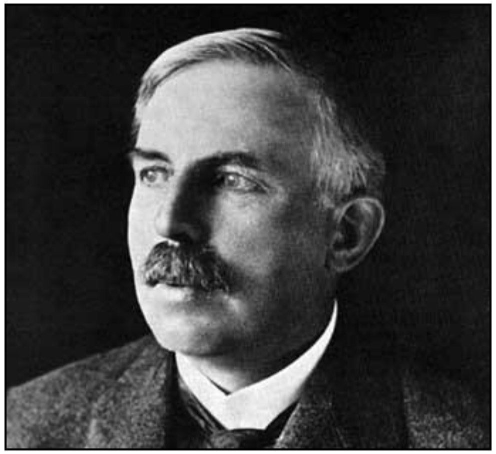
Nuclei - Rutherford, 1908

- Rutherford established the structure of the atom
  - by scattering particles off the atoms,
  - seeing wide angle deflections in a detector
  - Lesson: infer structure by scattering.
- Atoms have small, positively charged nuclei, 100,000 times smaller than the electron distribution!

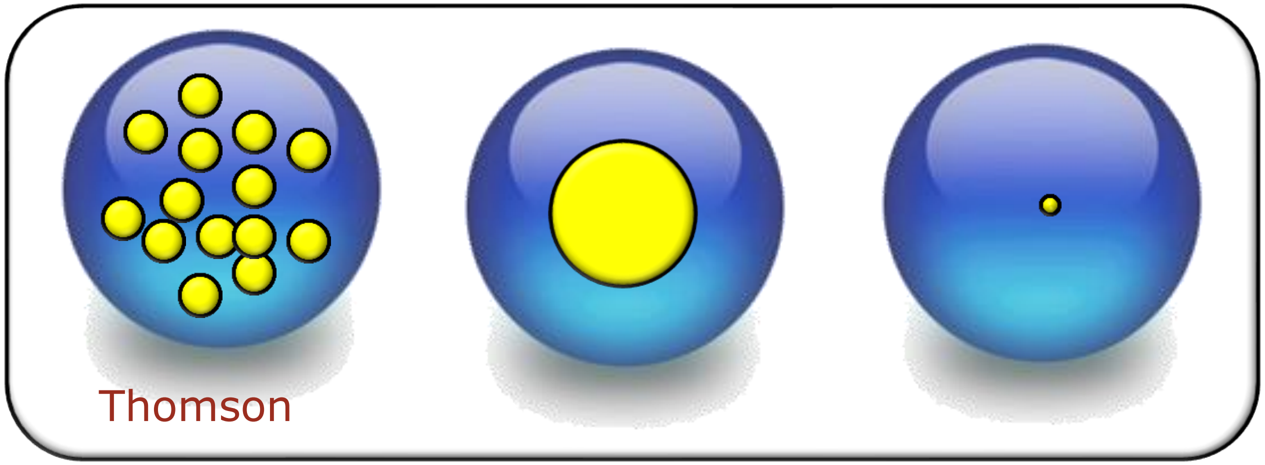




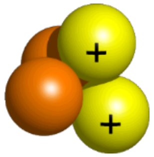
Rutherford



It was as if you fired a 15-inch shell at a sheet of tissue paper and it came back to hit you.

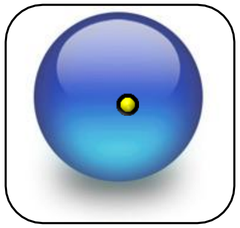


'Bullet':



6 MeV alpha particle

Hypothesis:



$$V \propto \frac{1}{r}$$

$$\frac{d\sigma}{d\Omega}(\theta) = \left(\frac{ZZ'}{4E}\right)^2 \frac{1}{\sin^4(\theta/2)}$$

Measurement:

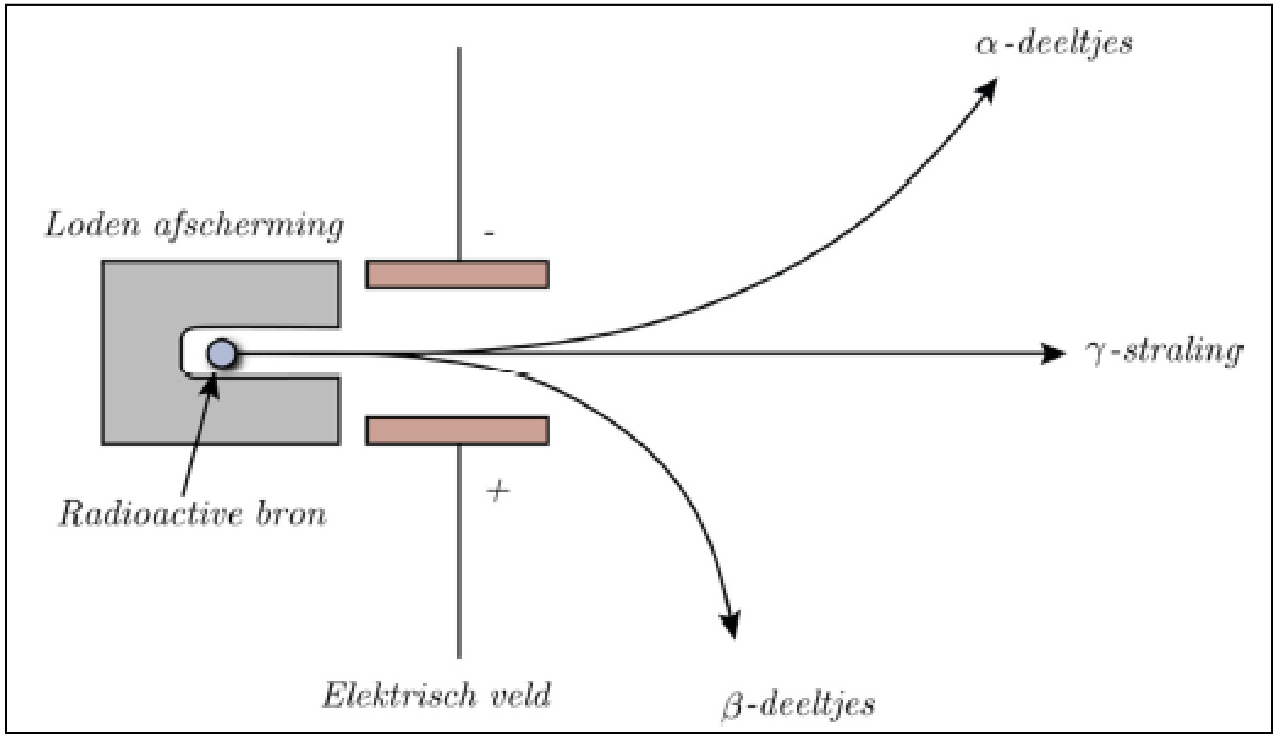
Scattering of α and β Particles by Matter. 681

Metal.	Atomic weight.	z.	$z/A^{3/2}$.
Lead	207	62	208
Gold	197	67	242
Platinum	195	63	232
Tin	119	34	226
Silver	108	27	241
Copper	64	14·5	225
Iron	56	10·2	250
Aluminium ...	27	3·4	243
		Average	233

Number of back-scattered particles ~ $A^{3/2}$

Several Particle discovery

- 1895: Röntgen discovered radiation from vaccum tubes (**γ**)
- 1895: Bequerel measured radiation from ^{238}U (**n**)
- 1898: Curie measured radiation from ^{232}Th (**α**)
- 1899: Rutherford concluded α ≠ **β**
- 1914: Rutherford determined wavelength of γ (scattering of crystals)



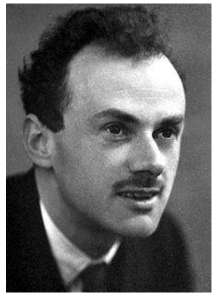
… looking at Rutherford's results, one notices that the number of electrons per atom
is precisely halve the atom mass.'

Dirac Equation particle-antiparticle

The famous Dirac equation:

$$\boxed{H\psi = (\vec{\alpha}\cdot\vec{p} + \beta m)\ \psi}$$

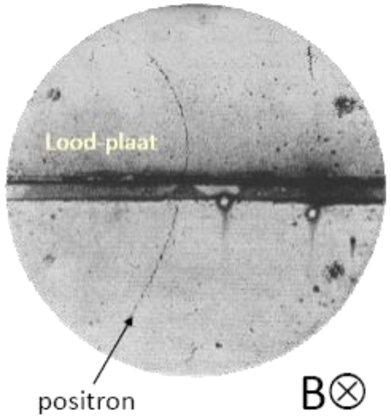
$$\boxed{\begin{aligned} (i\gamma^{\mu}\partial_{\mu} - m)\ \psi &= 0 \\ \text{with}:\ \gamma^{\mu} = (\beta, \beta\vec{\alpha}) &\equiv \text{Dirac } \gamma\text{-matrices} \end{aligned}}$$

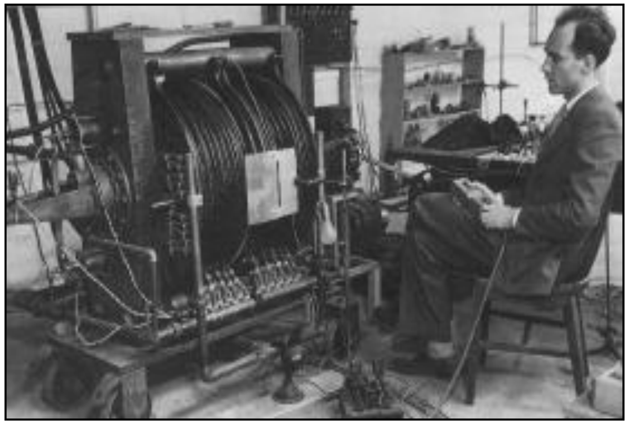


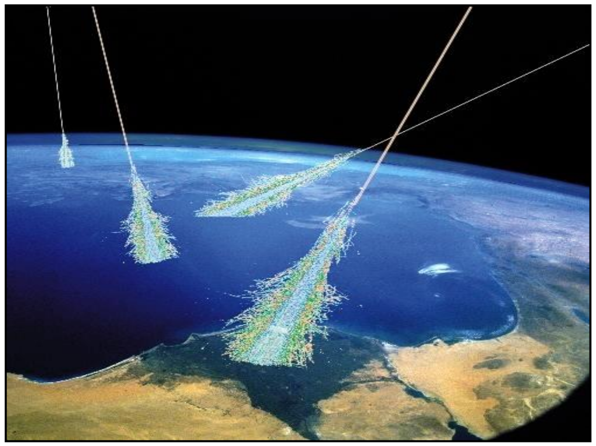
ψ is 4-component spinor

4 solutions correspond to fermions and anti-fermions with spin+1/2 and -1/2

$$\boxed{u^{(1)} = \begin{pmatrix} 1 \\ 0 \\ \vec{\sigma}\bullet\vec{p}/(E+m) \\ 0 \end{pmatrix}} \qquad \boxed{u^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vec{\sigma}\bullet\vec{p}/(E+m) \end{pmatrix}}$$







Dirac found his equation in 1928

The existence of anti-matter was not taken serious until 1932, when Anderson discovered the anti-electron: the positron

Dirac Fomalism

★The formulation of relativistic quantum mechanics starting from the linear Dirac equation
$$\hat{H}\psi = (\vec{\alpha}.\vec{p} + \beta m)\psi = i\frac{\partial\psi}{\partial t}$$

New degrees of freedom : found to describe Spin ½ particles

★ In terms of 4x4 gamma matrices the Dirac Equation can be written:

$$(i\gamma^\mu \partial_\mu - m)\psi = 0$$

★ Introduces the 4-vector current and adjoint spinor:

$$j^\mu = \psi^\dagger \gamma^0 \gamma^\mu \psi = \overline{\psi}\gamma^\mu \psi$$

★ With the Dirac equation: forced to have two positive energy and two negative energy solutions

★Feynman-Stückelberg interpretation: -ve energy particle solutions propagating backwards in time correspond to physical +ve energy anti-particles propagating forwards in time

$$u_1,\ u_2,\ v_1,\ v_2$$

Summary of Solutions to the Dirac Equation

★ Four solutions of form: $\psi_i = u_i(E,\vec{p})e^{i(\vec{p}.\vec{r}-Et)}$

$$\underbrace{u_1 = N\begin{pmatrix}1\\0\\\frac{p_z}{E+m}\\\frac{p_x+ip_y}{E+m}\end{pmatrix};\ u_2 = N\begin{pmatrix}0\\1\\\frac{p_x-ip_y}{E+m}\\\frac{-p_z}{E+m}\end{pmatrix}}_{E=+\left|\sqrt{|\vec{p}|^2+m^2}\right|};\ \underbrace{u_3 = N\begin{pmatrix}\frac{p_z}{E-m}\\\frac{p_x+ip_y}{E-m}\\1\\0\end{pmatrix};\ u_4 = N\begin{pmatrix}\frac{p_x-ip_y}{E-m}\\\frac{-p_z}{E-m}\\0\\1\end{pmatrix}}_{E=-\left|\sqrt{|\vec{p}|^2+m^2}\right|}$$

★ Four solutions of form $\psi_i = v_i(E,\vec{p})e^{-i(\vec{p}.\vec{r}-Et)}$

$$\underbrace{v_1 = N\begin{pmatrix}\frac{p_x-ip_y}{E+m}\\\frac{-p_z}{E+m}\\0\\1\end{pmatrix};\ v_2 = N\begin{pmatrix}\frac{p_z}{E+m}\\\frac{p_x+ip_y}{E+m}\\1\\0\end{pmatrix}}_{E=+\left|\sqrt{|\vec{p}|^2+m^2}\right|};\ \underbrace{v_3 = N\begin{pmatrix}1\\0\\\frac{p_z}{E-m}\\\frac{p_x+ip_y}{E-m}\end{pmatrix};\ v_4 = N\begin{pmatrix}0\\1\\\frac{p_x-ip_y}{E-m}\\\frac{-p_z}{E-m}\end{pmatrix}}_{E=-\left|\sqrt{|\vec{p}|^2+m^2}\right|}$$

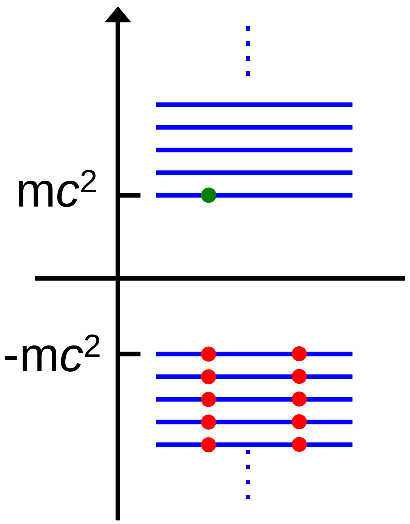
★ Since we have a four component spinor, only four are linearly independent
- Could choose to work with $\{u_1,u_2,u_3,u_4\}$ $\{v_1,v_2,v_3,v_4\}$ …
- Natural to use choose +ve energy solutions

$$\{u_1,u_2,v_1,v_2\}$$

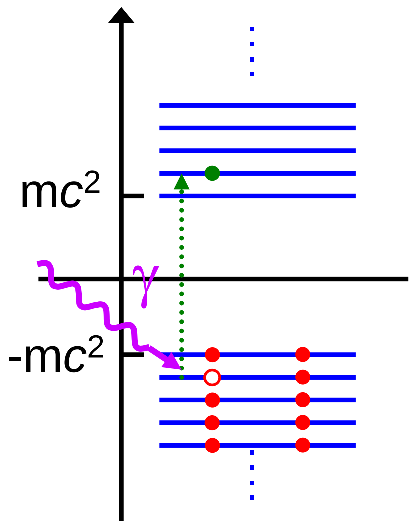
Particle and anti-partile

★The Dirac equation has negative energy solutions. Unlike the KG equation these have positive probability densities. But how should –ve energy solutions be interpreted? Why don′t all +ve energy electrons fall into to the lower energy –ve energy states?

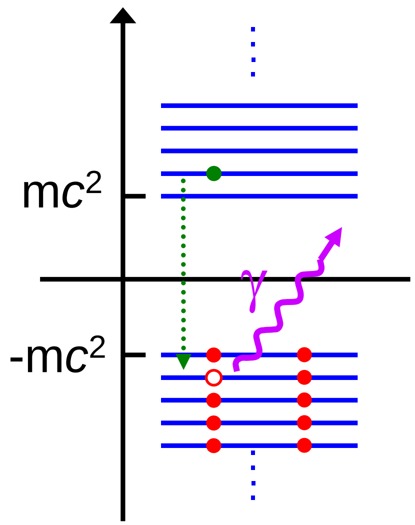
Dirac Interpretation: the vacuum corresponds to all –ve energy states being full with the Pauli exclusion principle preventing electrons falling into -ve energy states. Holes in the –ve energy states correspond to +ve energy anti-particles with opposite charge. Provides a picture for pair-production and annihilation.



$\gamma \rightarrow e^- e^+$



$e^- e^+ \rightarrow \gamma$



Experimental confirmation of antimatter

Detector: a Wilson cloud – chamber (visual detector based on a gas volume containing vapour close to saturation) in a magnetic field, exposed to cosmic rays

Carl D. Anderson

Measure particle momentum and sign of electric charge from magnetic curvature

Lorentz force $\vec{f} = e\vec{v} \times \vec{B}$ ➡ projection of the particle trajectory in a plan perpendicular to $\vec{B}$ is a circle

Circle radius for electric charge $|e|$:

$$R\,[\mathrm{m}] = \frac{10\, p_{\perp}\,[\mathrm{GeV/c}]}{3B\,[\mathrm{T}]}$$

$p_{\perp}$: momentum component perpendicular to magnetic field direction

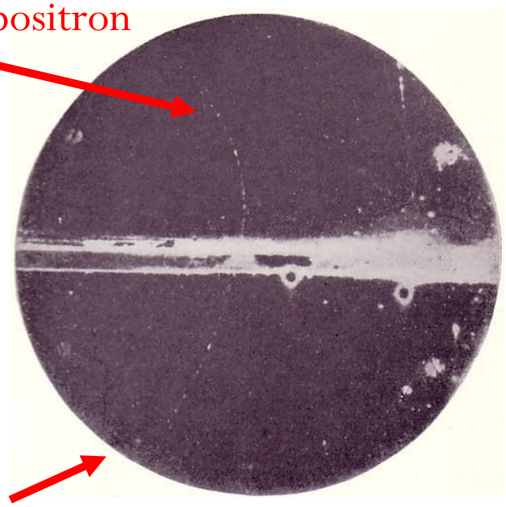
NOTE: impossible to distinguish between positively and negatively charged particles going in opposite directions

$-e$

$+e$

need an independent determination of the particle direction of motion

23 MeV positron



6 mm thick Pb plate

63 MeV positron

Summary of Solutions to the Dirac Equation

- The normalised free PARTICLE solutions to the Dirac equation:

$$\psi = u(E,\vec{p})e^{+i(\vec{p}.\vec{r}-Et)} \quad \text{satisfy} \quad \boxed{(\gamma^\mu p_\mu - m)u = 0}$$

with

$$u_1 = \sqrt{E+m}\begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix}; \qquad u_2 = \sqrt{E+m}\begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix}$$

- The ANTI-PARTICLE solutions in terms of the physical energy and momentum:

$$\psi = v(E,\vec{p})e^{-i(\vec{p}.\vec{r}-Et)} \quad \text{satisfy} \quad \boxed{(\gamma^\mu p_\mu + m)v = 0}$$

with

$$v_1 = \sqrt{E+m}\begin{pmatrix} \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix}; \qquad v_2 = \sqrt{E+m}\begin{pmatrix} \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \\ 1 \\ 0 \end{pmatrix}$$

For these states the spin is given by $\hat{S}^{(v)} = -\hat{S}$

- For both particle and anti-particle solutions: $E = \sqrt{|\vec{p}|^2 + m^2}$

Spin States

- In general the spinors u_1, u_2, v_1, v_2 are not Eigenstates of $\hat{S}_z$

$$\hat{S}_z = \frac{1}{2}\Sigma_z = \frac{1}{2}\begin{pmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

- However particles/anti-particles travelling in the z-direction: $p_z = \pm|\vec{p}|$

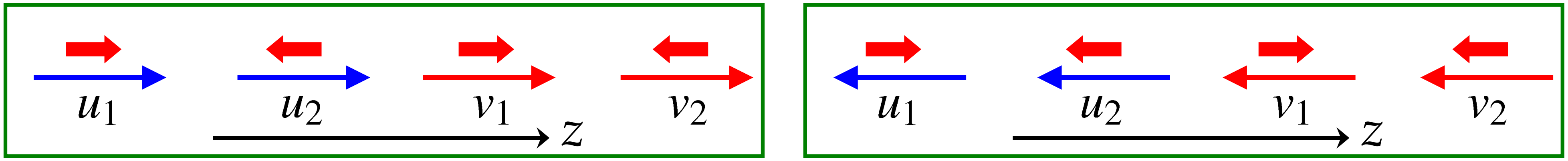
$$u_1 = N\begin{pmatrix} 1 \\ 0 \\ \frac{\pm|\vec{p}|}{E+m} \\ 0 \end{pmatrix}; \quad u_2 = N\begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{\mp|\vec{p}|}{E+m} \end{pmatrix}; \quad v_1 = N\begin{pmatrix} 0 \\ \frac{\mp|\vec{p}|}{E+m} \\ 0 \\ 1 \end{pmatrix}; \quad v_2 = N\begin{pmatrix} \frac{\pm|\vec{p}|}{E+m} \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

are Eigenstates of $\hat{S}_z$

$$\hat{S}_z u_1 = +\tfrac{1}{2}u_1 \qquad \hat{S}_z^{(v)} v_1 = -\hat{S}_z v_1 = +\tfrac{1}{2}v_1$$

$$\hat{S}_z u_2 = -\tfrac{1}{2}u_2 \qquad \hat{S}_z^{(v)} v_2 = -\hat{S}_z v_2 = -\tfrac{1}{2}v_2$$

Note the change of sign of $\hat{S}$ when dealing with antiparticle spinors



★ Spinors u_1, u_2, v_1, v_2 are only eigenstates of $\hat{S}_z$ for $p_z = \pm|\vec{p}|$

SPIN

- For a Dirac spinor is orbital angular momentum a good quantum number? i.e. does $L = \vec{r} \wedge \vec{p}$ commute with the Hamiltonian?

$$
\begin{aligned}
[H, \vec{L}] &= [\vec{\alpha}.\vec{p} + \beta m, \vec{r} \wedge \vec{p}] \\
&= [\vec{\alpha}.\vec{p}, \vec{r} \wedge \vec{p}]
\end{aligned}
$$

Consider the x component of L:

$$
\begin{aligned}
[H, L_x] &= [\vec{\alpha}.\vec{p}, (\vec{r} \wedge \vec{p})_x] \\
&= [\alpha_x p_x + \alpha_y p_y + \alpha_z p_z, y p_z - z p_y]
\end{aligned}
$$

The only non-zero contributions come from: $[x, p_x] = [y, p_y] = [z, p_z] = i$

$$
\begin{aligned}
[H, L_x] &= \alpha_y p_z [p_y, y] - \alpha_z p_y [p_z, z] \\
&= -i(\alpha_y p_z - \alpha_z p_y) \\
&= -i(\vec{\alpha} \wedge \vec{p})_x
\end{aligned}
$$

Therefore $\boxed{[H, \vec{L}] = -i\vec{\alpha} \wedge \vec{p}}$ (A.1)

★Hence the angular momentum does not commute with the Hamiltonian and is not a constant of motion

Introduce a new 4x4 operator:

$$\vec{S} = \frac{1}{2}\vec{\Sigma} = \frac{1}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$$

where $\vec{\sigma}$ are the Pauli spin matrices: i.e.

$$\Sigma_x = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}; \quad \Sigma_y = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}; \quad \Sigma_x = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Now consider the commutator

$$[H, \vec{\Sigma}] = [\vec{\alpha}.\vec{p} + \beta m, \vec{\Sigma}]$$

here $$[\beta, \vec{\Sigma}] = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} - \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} = 0$$

and hence $$[H, \vec{\Sigma}] = [\vec{\alpha}.\vec{p}, \vec{\Sigma}]$$

Consider the x comp: $$\begin{aligned} [H, \Sigma_x] &= [\alpha_x p_x + \alpha_y p_y + \alpha_z p_z, \Sigma_x] \\ &= p_x[\alpha_x, \Sigma_x] + p_y[\alpha_y, \Sigma_x] + p_z[\alpha_z, \Sigma_x] \end{aligned}$$

Discovery of Neutron

Rutherford was leading Chadwick. Rutherford guessed that protons were carrying the charge of the nucleus after a nitrogen atom expelled hydrogen on being hit by alphas. Chadwick discovered the actual neutron in 1932. The Curies had seen it but misinterpreted it as x-rays… Saw electrically neutral radiation from alphas on beryllium. Chadwick added paraffin, which ejected protons, too heavy to be removed by x-rays. He guessed they were neutrons: same weight as proton, and what Rutherford had hypothesized.

- Discovery of the neutron, by J. Chadwick

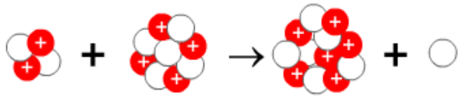
$$\boxed{\alpha + Be \rightarrow \text{non-ionizing radiation}}$$

Nobel prize 1935

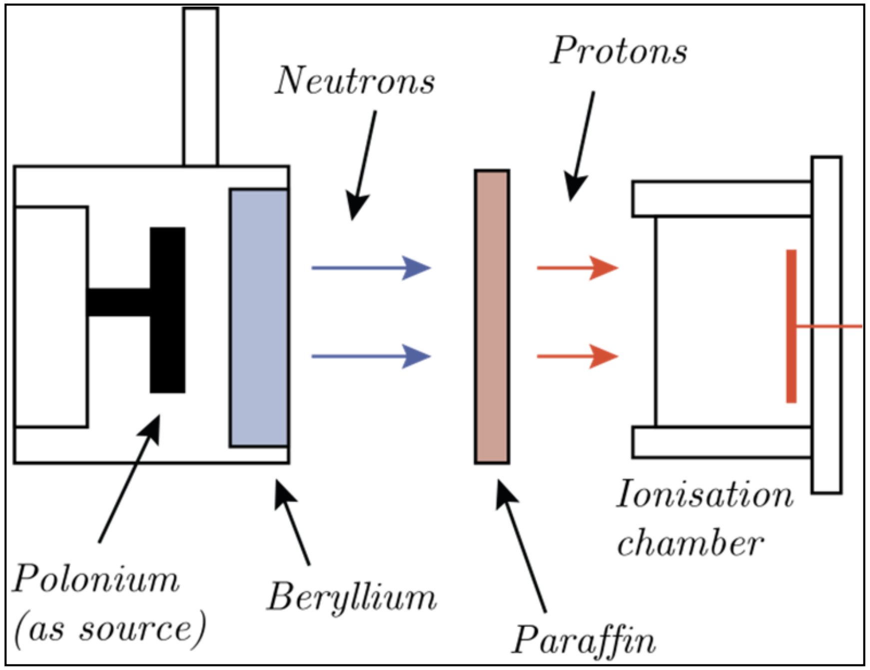
1) Neutral
   - Gamma? No! protons too energetic
2) $m_n \sim m_p$

➢ Interpretation:

$$\boxed{{}^{4}_{2}He + {}^{9}_{4}Be \rightarrow {}^{1}_{0}n + {}^{12}_{6}C}$$



${}^{4}He + {}^{9}Be \rightarrow {}^{12}C$ + neutron



Discovery of Neutron

Electron, proton spin = ½ħ (measured)

Nitrogen nucleus ($A = 14$, $Z = 7$): 14 protons + 7 electrons = 21 spin ½ particles

TOTAL SPIN MUST HAVE HALF-INTEGER VALUE

Measured spin = 1 (from hyperfine splitting of atomic spectral lines)

Neutron: a particle with mass ≈ proton mass but with zero electric charge

Solution to the nuclear structure problem:

Nucleus with atomic number Z and mass number A: a bound system of Z protons and ($A - Z$) neutrons

Nitrogen anomaly: no problem if neutron spin = ½ħ

Nitrogen nucleus (A = 14, Z = 7): 7 protons, 7 neutrons = 14 spin ½ particles

⇒ total spin has integer value

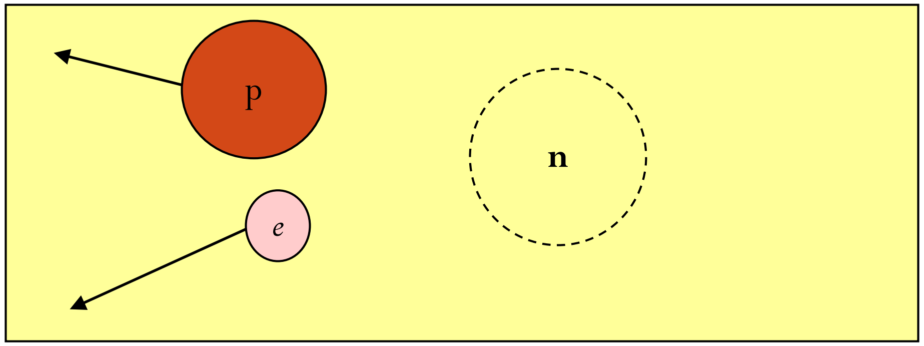
Neutron source in Chadwick's experiments: a ^{210}Po radioactive source (5 MeV α – particles) mixed with Beryllium powder ⇒ emission of electrically neutral radiation capable of traversing several centimetres of Pb:

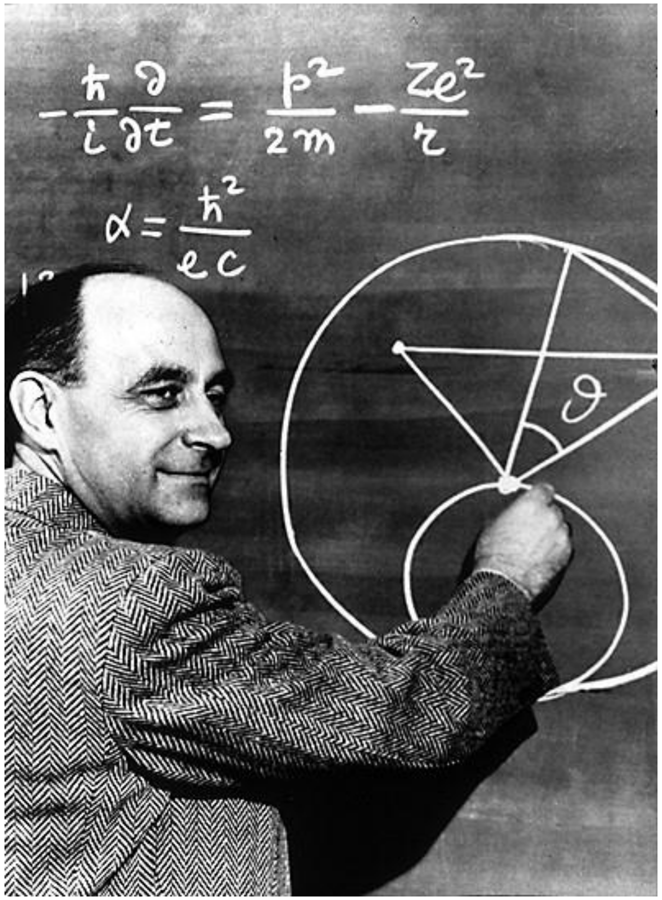
$$\underset{\substack{\uparrow \\ \alpha\text{ - particle}}}{{}^{4}He_{2}} + {}^{9}Be_{4} \rightarrow {}^{12}C_{6} + \text{neutron}$$

Neutrinos

1934: To account for the "unseen" momentum in the reaction (decay):

$$n \rightarrow p + e^- + X$$





Nobel Laureate: Enrico Fermi

Fermi proposed that the unseen momentum (X) was carried off by a particle dubbed the *neutrino* (ν).

(means "*little neutral one*")

Lepton Picture: Three happy family

- In **1975**, researchers at the Stanford Linear Accelerator discovered a **third charged lepton**, with a mass about **3500 times** that of the electron. It was named the **τ-lepton**.

- In **2000**, first evidence of the τ's partner, the **tau-neutrino (ν_τ)** was announced at Fermi National Accelerator Lab.

Family	Leptons		Antileptons	
	Q = -1	Q = 0	Q = +1	Q = 0
1	e^-	ν_e	e^+	ν_e
2	μ^-	ν_μ	μ^+	ν_μ
3	τ^-	ν_τ	τ^+	ν_τ

Lepton (particle)

e^- electron

μ^- muon-minus

τ^- tau-minus

ν_e electron neutrino

ν_μ muon neutrino

ν_τ tau neutrino

Anti-lepton (anti-particle)

e^+ positron

μ^+ muon-plus

τ^+ tau-plus

$\overline{\nu}_e$ electron anti-neutrino

$\overline{\nu}_\mu$ muon anti-neutrino

$\overline{\nu}_\tau$ tau anti-neutrino

Leptons

If neutrons & protons are not fundamental, what about electrons?

Are they made up of smaller constituents also ? **As far as we can tell, electrons appear to be indivisible.**

- **Electrons belong to a general class of particles, called "Leptons"**
- **As far as we can tell, the leptons are "fundamental".**
- **Each charged lepton has an uncharged partner called the "neutrino"**
- **The leptons behave quite differently than the quarks**
  -They don't form hadrons (no binding between leptons)

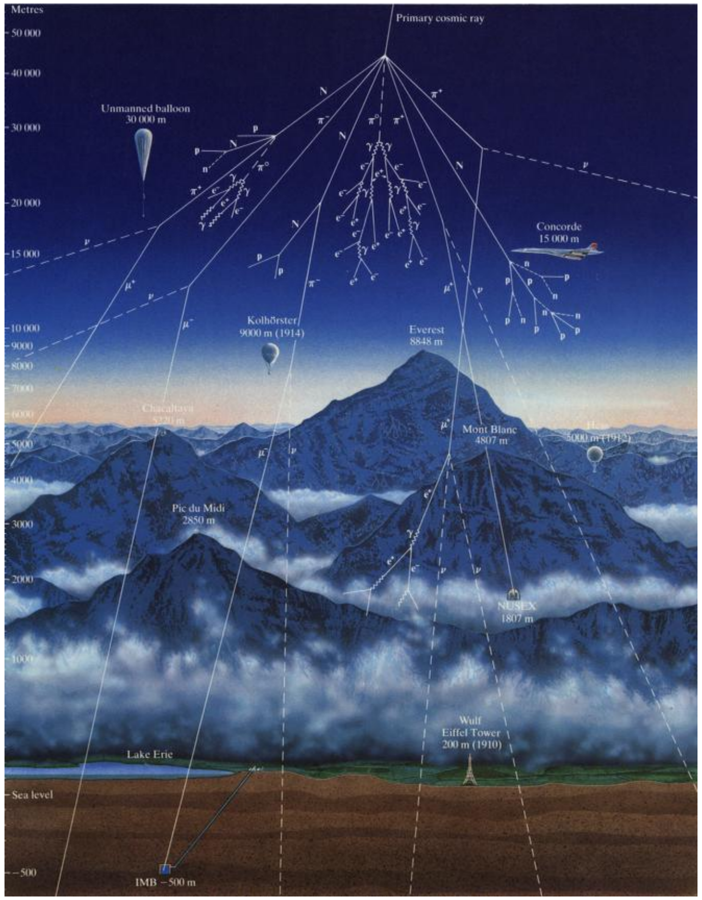
❑ **1932**: Discovery of the positron, the "anti-particle" of the electron.

Anti-particles really exist !!!!!

❑ **1937** – Muons (μ^+ and μ^-) discovered in cosmic rays.

$M(\mu) \sim 200{*}M(e)$

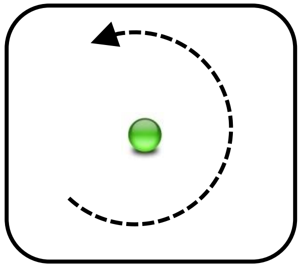
❑ The muon behaves very similarly to the electron (i.e., it's a lepton).



Quantum mechanics: orbital momentum

$$L = \vec{r} \times \vec{p} = -i\hbar(\vec{r} \times \vec{\nabla})$$

$$\begin{aligned} L_x &= -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) = yp_z - zp_y \\ L_y &= -i\hbar\left(z\frac{\partial}{\partial y} - x\frac{\partial}{\partial z}\right) = zp_x - xp_z \\ L_z &= -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) = xp_y - yp_x \end{aligned}$$



$$\boxed{[L_x, L_y] = i\hbar L_z}$$

L_x and L_y cannot be known simultaneously
Sequence matters!

$$\boxed{[x, p_x] = i\hbar}$$

$$\boxed{[L^2, L_i] = 0}$$

L^2 and L_i (i=x,y,z) can be known simultaneously
Can both be used to label states

$$\boxed{[L^2,H]=[L_z,H]=0}$$

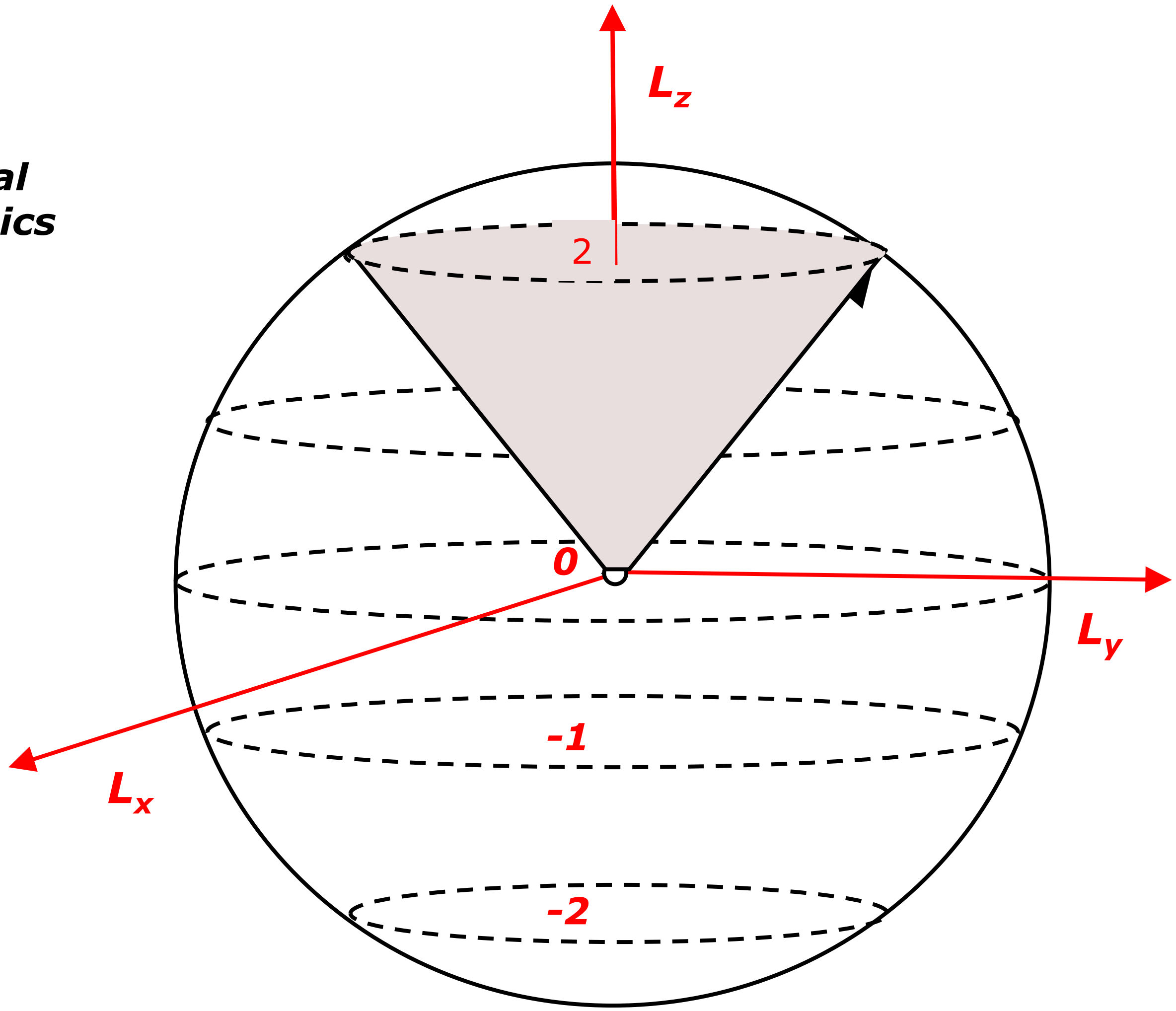
Provided V = V(r), ie not θ dependent
L^2 and L_z label eigenstates

Quantum mechanics: orbital momentum

$$\begin{aligned} L^2 f_l^m &= \hbar^2 l(l+1) f_l^m \\ L_z f_l^m &= \hbar m f_l^m \end{aligned}$$

$f_l^m = Y_l^m$ spherical harmonics

m = -l, -l+1, …, 0, … , l-1, l



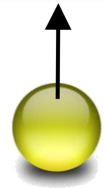
Different notation:

$$\begin{aligned} L^2 |l,m\rangle &= \hbar^2 l(l+1) |l,m\rangle \\ L_z |l,m\rangle &= \hbar m |l,m\rangle \end{aligned}$$

Quantum mechanics: (intrinsic) spin

Spin is characterized by: - total spin S

- spin projection S_z



Rotations: ***SO(3) group***

Internal symmetry: ***SU(2) group***

$\Big\}$ ***similar***

$[S_x, S_y] = i\hbar S_z$ $\qquad$ $[S^2, S_i] = 0$

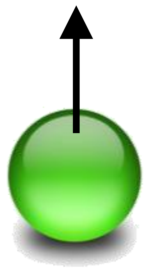
Spin is quantized, just as orbital momentum

$$\begin{aligned} S &= 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots \\ S_z &= -S, -S+1, \ldots, S-1, S \end{aligned}$$

Eigenfunctions $|s,m_s>$:

$$\begin{aligned} S^2|s,m_s\rangle &= \hbar^2 s(s+1)|s,m_s\rangle \\ S_z|s,m_s\rangle &= \hbar m_s|s,m_s\rangle \end{aligned}$$

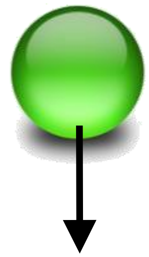
Spin-½ particles



$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Spin-up

$$|\uparrow\rangle = |\frac{1}{2}, +\frac{1}{2}\rangle$$



$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Spin-down

$$|\downarrow\rangle = |\frac{1}{2}, -\frac{1}{2}\rangle$$

spin-$\frac{1}{2}$ particles

Complex numbers

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \alpha\begin{pmatrix}1\\ 0\end{pmatrix} + \beta\begin{pmatrix}0\\ 1\end{pmatrix}$$

general

$|\alpha|^2$ prob for $S_z = +\frac{1}{2}\hbar$

$|\beta|^2$ prob for $S_z = -\frac{1}{2}\hbar$

$$S_x = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \qquad S_y = \frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} \qquad S_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

Pauli matrices: any complex 2x2 matrix can be written as: $A = a\sigma_1 + b\sigma_2 + c\sigma_3$

Protons and neutrons: Birth of Iso-spin

Proton and neutron <u>identical</u> under strong interaction

proton

m_p = 938.272 MeV

neutron

m_n = 939.565 MeV

?

Nucleon

+ internal degree of freedom
to distinguish the two

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$$

Nucleon

+ internal degree of freedom

Introduce new quantum number: isospin

Proton and neutron ('nucleons'): I en I_3

$$p = |\frac{1}{2},+\frac{1}{2}\rangle \text{ en } n = |\frac{1}{2},-\frac{1}{2}\rangle$$

Isospin 'up' ***Isospin 'down'***

Possible states for given value of the Isospin

$$I = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$$
$$I_z = -I, -I+1, \ldots, I-1, I$$

$\mathbf{I_z = \frac{1}{2}}$

$\mathbf{I_z = 1}$

$\mathbf{I_z = \frac{3}{2}}$

$\mathbf{I_z = \frac{1}{2}}$ $\begin{cases} I_z = +1/2 \\ I_z = -1/2 \end{cases}$

$\mathbf{I_z = 1}$ $\begin{cases} I_z = +1 \\ I_z = 0 \\ I_z = -1 \end{cases}$

$\mathbf{I_z = \frac{3}{2}}$ $\begin{cases} I_z = +3/2 \\ I_z = +1/2 \\ I_z = -1/2 \\ I_z = -3/2 \end{cases}$

proton $|\frac{1}{2}, +\frac{1}{2}\rangle$

neutron $|\frac{1}{2}, -\frac{1}{2}\rangle$

π^+ $|1, +1\rangle$

π^0 $|1, 0\rangle$

π^- $|1, -1\rangle$

Δ^{++} $|\frac{3}{2}, +\frac{3}{2}\rangle$

Δ^{+} $|\frac{3}{2}, +\frac{1}{2}\rangle$

Δ^{0} $|\frac{3}{2}, -\frac{1}{2}\rangle$

Δ^{-} $|\frac{3}{2}, -\frac{3}{2}\rangle$

m_p ~ 939 MeV

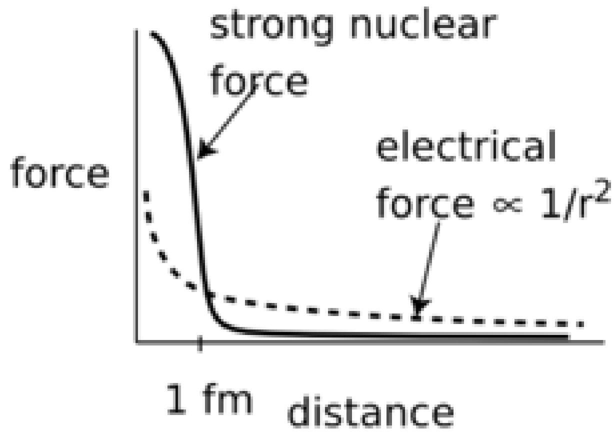
m_π ~ 140 MeV

m_Δ ~ 1232 MeV

Nuclear force model (1934): Yukawa

- Evidently, some force is holding the nucleus together: the "strong force."
- Inside the nucleus, the strong force has to overwhelm the EM force, but outside, on the atomic scale, it should have almost no effect.
- How to accomplish this? Assume the strong force has a very short range, falling off rapidly to zero for distances greater than 1 fm.
- H. Yukawa: force may vary as:

$$F_{\text{strong}} \propto -\frac{1}{r^2} e^{-r/a}$$



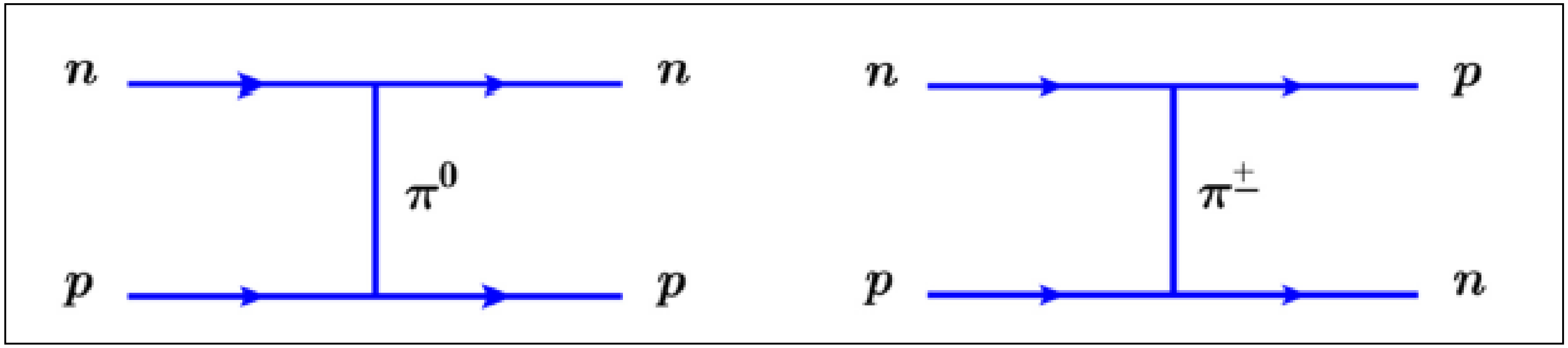
Nuclear force model (1934): Yukawa

- 1935: Introduced *strong* carriers on *small* distances Massive particle, that exists only shortly => 'virtual' particle
  - Electro-magnetism
    - Infinite range
    - Transmitted by massless photon
    - ➢ <u>Coulomb</u> potential $R \to \infty$

$$\boxed{V(r) = -e^2 \frac{1}{r}}$$

  - Strong force
    - Finite range
    - Transmitted by massive pion
    - ➢ <u>Yukawa</u> potential *R: range*

$$\boxed{U(r) = -g^2 \frac{e^{-r/R}}{r}}$$



Yukawa: Yukawa's pions – pictures

Powell used a new detection technique

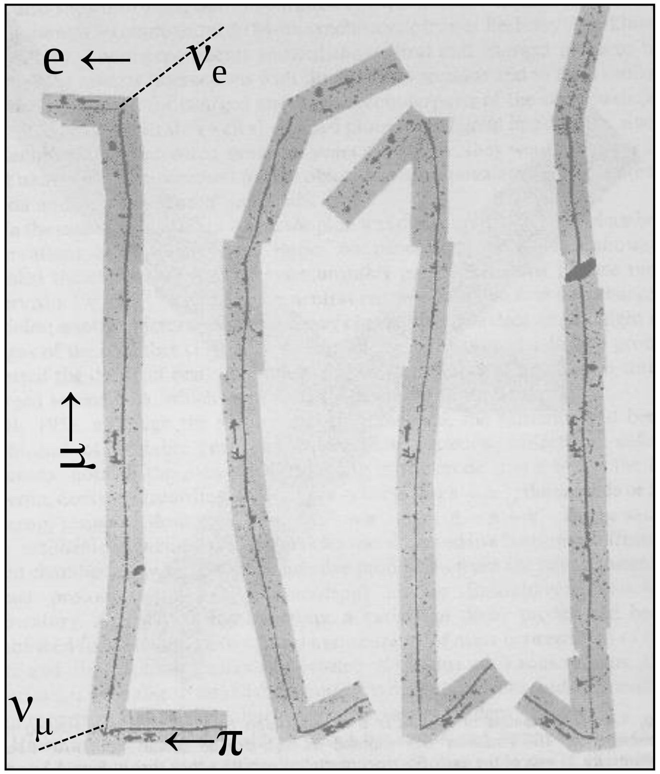
Photographic emulsion:
- Thick photosensitive film
- Charged particles leave tracks

Results: two particles (pion and muon)

$$\pi^+ \to \mu^+ \upsilon_\mu$$

$$\hookrightarrow \mu^+ \to e^+ + \upsilon_e + \overline{\upsilon}_\mu$$

- 1947 Discovery of pion (Powell): Nobel Prize 1950
- 1935 Prediction of pion (Yukawa): Nobel Prize 1949

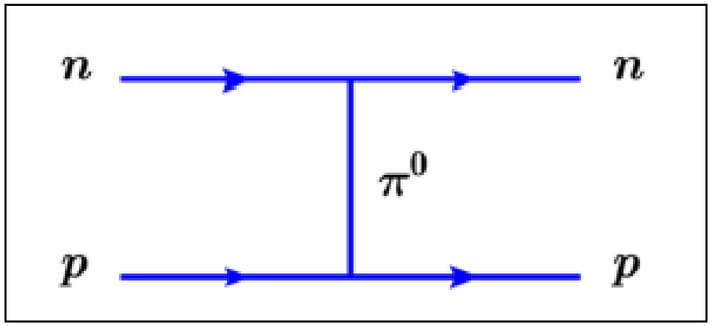


- π-meson, m=140 MeV, short lifetime

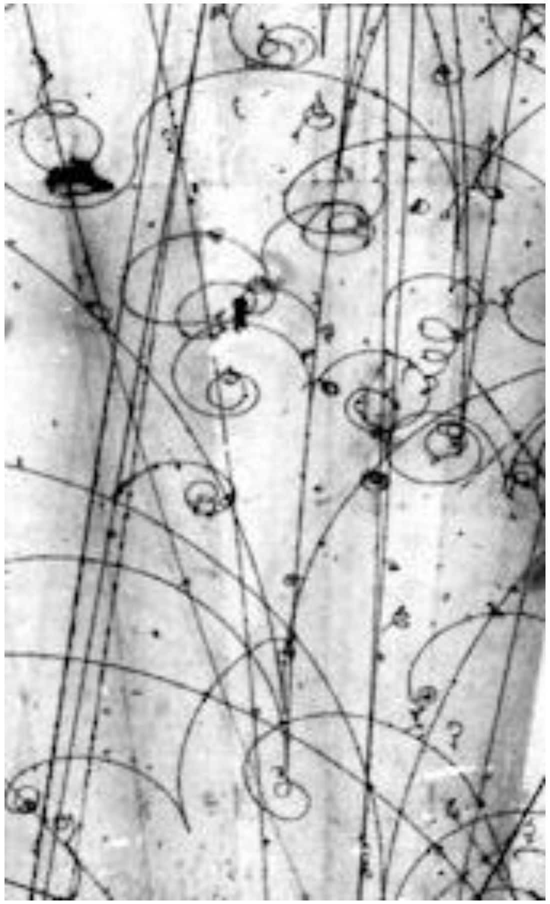
Produced high in atmosphere and decays before reaching sealevel.

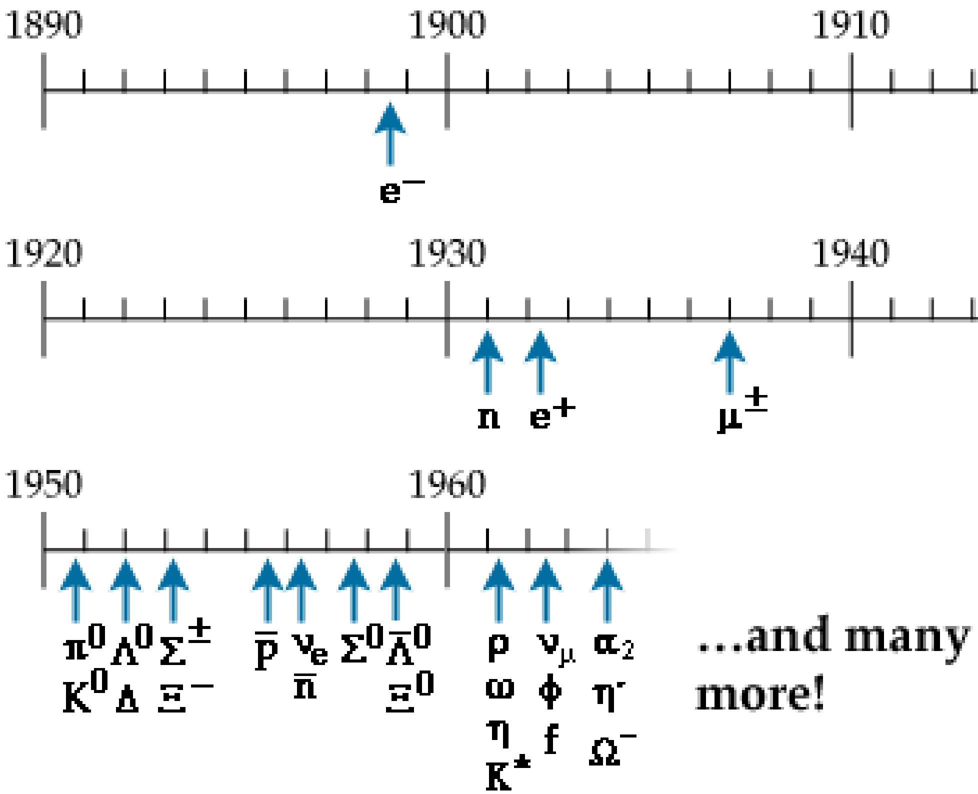
- muon (μ), m=105 MeV, long lifetime

Reaches sea-level and weakly interacts with matter



More and More Mystery particles





Fermilab: Bubble Chamber Photo

CONSERVED QUANTUM NUMBERS

Why is the free proton stable? Possible proton decay modes (allowed by all known conservation laws: energy – momentum, electric charge, angular momentum):

$$p \rightarrow \pi^{\circ} + e^{+}$$
$$p \rightarrow \pi^{\circ} + \mu^{+}$$
$$p \rightarrow \pi^{+} + \nu$$
$$\ldots\ldots$$

No proton decay ever observed – the proton is STABLE

Limit on the proton mean life: $\tau_p > 1.6 \times 10^{25}$ years

Invent a new quantum number : "Baryonic Number" B

B = 1 for proton, neutron

B = -1 for antiproton, antineutron

B = 0 for $e^{\pm}$, $\mu^{\pm}$, neutrinos, mesons, photons

$$\sum_i B_i = \sum_f B_f$$

(i : initial state particle ; f : final state particle)

Require conservation of baryonic number in all particle processes:

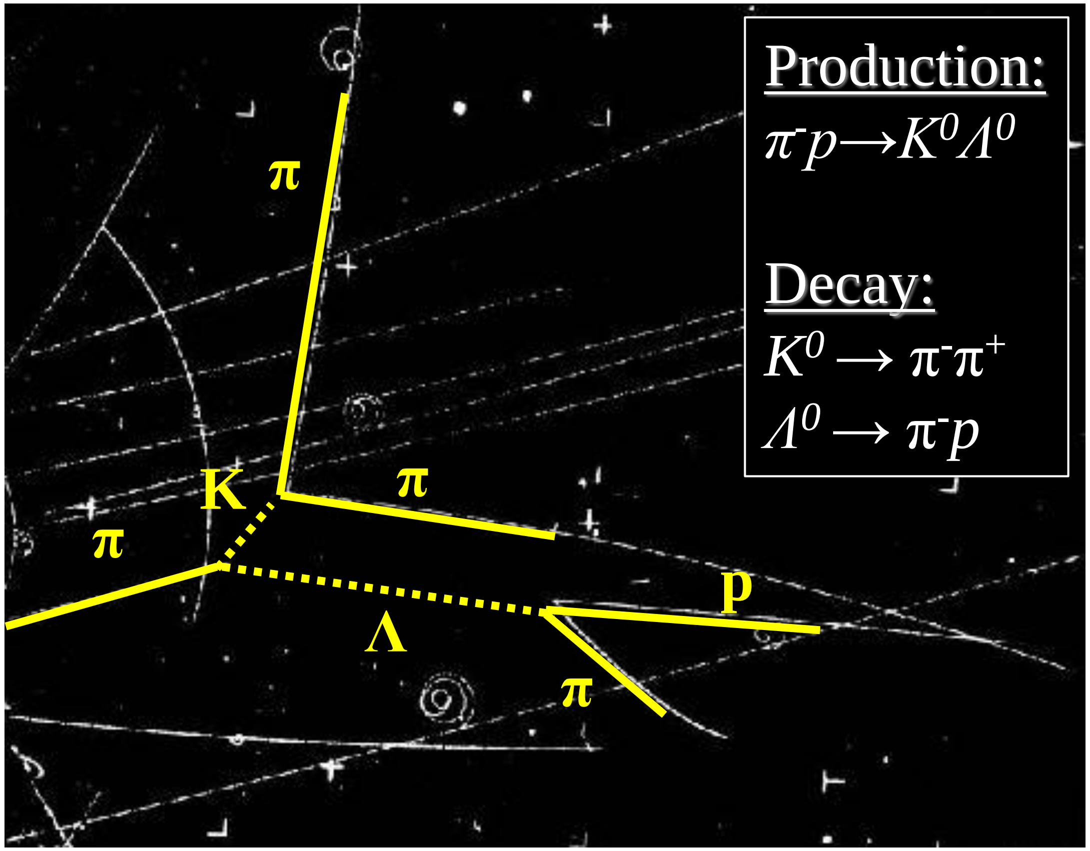
Discovery strange particles

- Why were these particles called strange?
- Large production cross section (10^{-27} cm^2)
- Long lifetime (corresponding to process with cross section 10^{-40} cm^2)

- *Associated* production!

New quantum number:

- Strangeness, S
- Conserved in the strong interaction, ΔS=0
  - Particles with S=+1 and S=-1 simultaneously produced
- Not conserved in individual decay, ΔS=1



Strangeness

Late 1940's: discovery of a variety of heavier mesons (K – mesons) and baryons ("hyperons") – studied in detail in the 1950's at the new high-energy proton synchrotrons (the 3 GeV "cosmotron" at the Brookhaven National Lab and the 6 GeV Bevatron at Berkeley)

Examples of mass values

Mesons (spin = 0): $m(K^{\pm}) = 493.68\ \mathrm{MeV}/c^2$; $m(K^{\circ}) = 497.67\ \mathrm{MeV}/c^2$

Hyperons (spin = ½): $m(\Lambda) = 1115.7\ \mathrm{MeV}/c^2$; $m(\Sigma^{\pm}) = 1189.4\ \mathrm{MeV}/c^2$

$m(\Xi^{\circ}) = 1314.8\ \mathrm{MeV}/c^2$; $m(\Xi^{-}) = 1321.3\ \mathrm{MeV}/c^2$

Properties

- Abundant production in proton – nucleus , π – nucleus collisions
- Production cross-section typical of strong interactions ($\sigma > 10^{-27}\ \mathrm{cm}^2$)
- Production in pairs (example: $\pi^- + p \rightarrow K^{\circ} + \Lambda$; $K^- + p \rightarrow \Xi^- + K^+$)
- Decaying to lighter particles with mean life values $10^{-8} – 10^{-10}$ s (as expected for a weak decay)

Examples of decay modes

$K^{\pm} \rightarrow \pi^{\pm}\pi^{\circ}$; $K^{\pm} \rightarrow \pi^{\pm}\pi^{+}\pi^{-}$; $K^{\pm} \rightarrow \pi^{\pm}\pi^{\circ}\pi^{\circ}$; $K^{\circ} \rightarrow \pi^{+}\pi^{-}$; $K^{\circ} \rightarrow \pi^{\circ}\pi^{\circ}$; . . .

$\Lambda \rightarrow p\,\pi^-$; $\Lambda \rightarrow n\,\pi^{\circ}$; $\Sigma^+ \rightarrow p\,\pi^{\circ}$; $\Sigma^+ \rightarrow n\,\pi^+$; $\Sigma^+ \rightarrow n\,\pi^-$; . . .

$\Xi^- \rightarrow \Lambda\,\pi^-$; $\Xi^{\circ} \rightarrow \Lambda\,\pi^{\circ}$

Antiproton discovery (1955)

Threshold energy for antiproton (p) production in proton – proton collisions

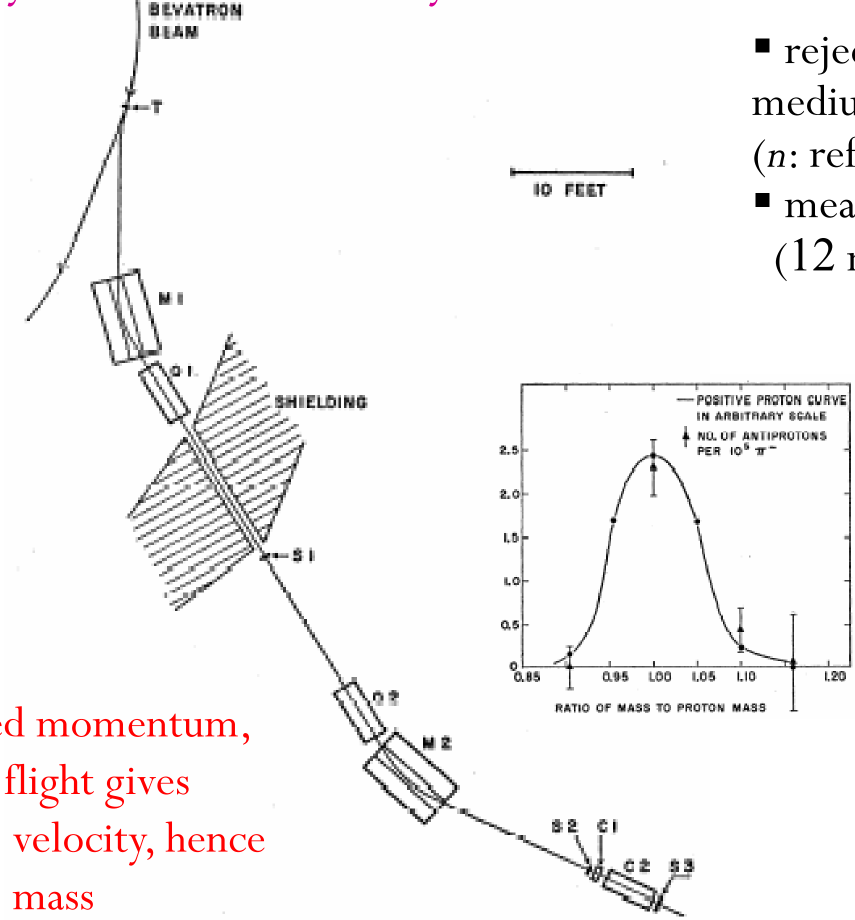
Baryon number conservation $\Rightarrow$ simultaneous production of p and p (or p and n)

Example: $p + p \rightarrow p + p + \overline{p} + p$ Threshold energy ~ 6 GeV

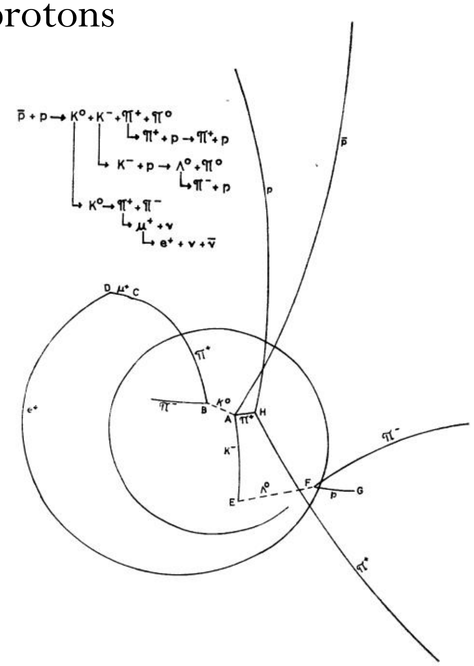
"Bevatron": 6 GeV

proton synchrotron in Berkeley

- build a beam line for 1.19 GeV/c momentum
- select negatively charged particles (mostly π^-)
- reject fast π^- by Čerenkov effect: light emission in transparent medium if particle velocity $v > c/n$ (n: refraction index) – antiprotons have $v < c/n \Rightarrow$ no Čerenkov light
- measure time of flight between counters S_1 and S_2 (12 m path): 40 ns for π^-, 51 ns for antiprotons



For fixed momentum, time of flight gives particle velocity, hence particle mass



Invention of a new, additive quantum number "Strangeness" (S)

(Gell-Mann, Nakano, Nishijima, 1953)

- conserved in strong interaction processes: $\sum_i S_i = \sum_f S_f$

- not conserved in weak decays: $\left| S_i - \sum_f S_f \right| = 1$

S = +1: K^+, K° ; S = –1: Λ, $\Sigma^\pm$, Σ° ; S = –2 : Ξ°, Ξ^- ; S = 0 : all other particles
(and opposite strangeness –S for the corresponding antiparticles)

Example of a K^- stopping in liquid hydrogen:

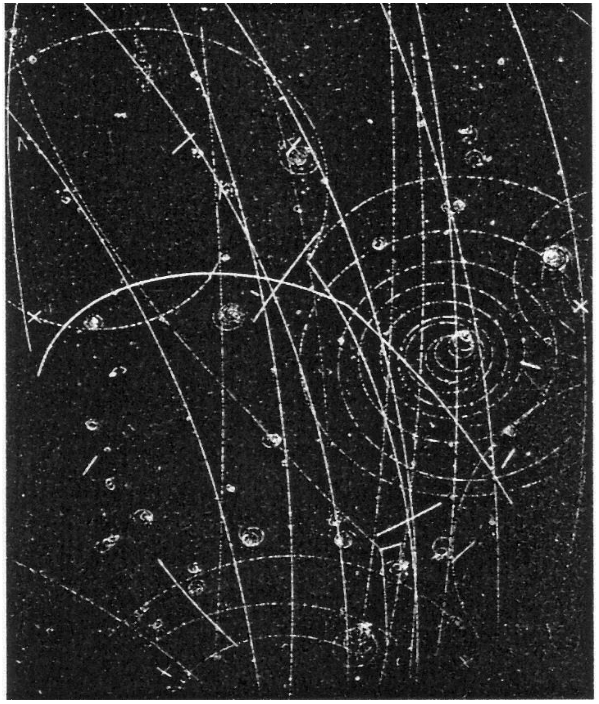
$$K^- + p \rightarrow \Lambda + \pi^\circ$$

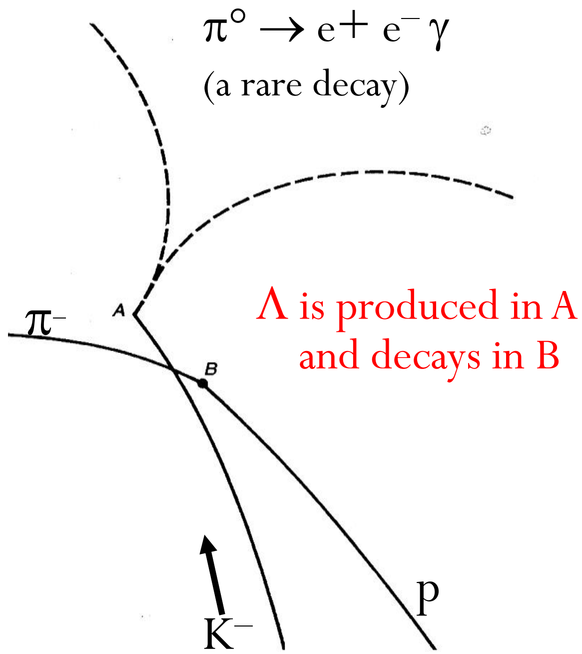
(strangeness conserving)

followed by the decay

$$\Lambda \rightarrow p + \pi^-$$

(strangeness violation)

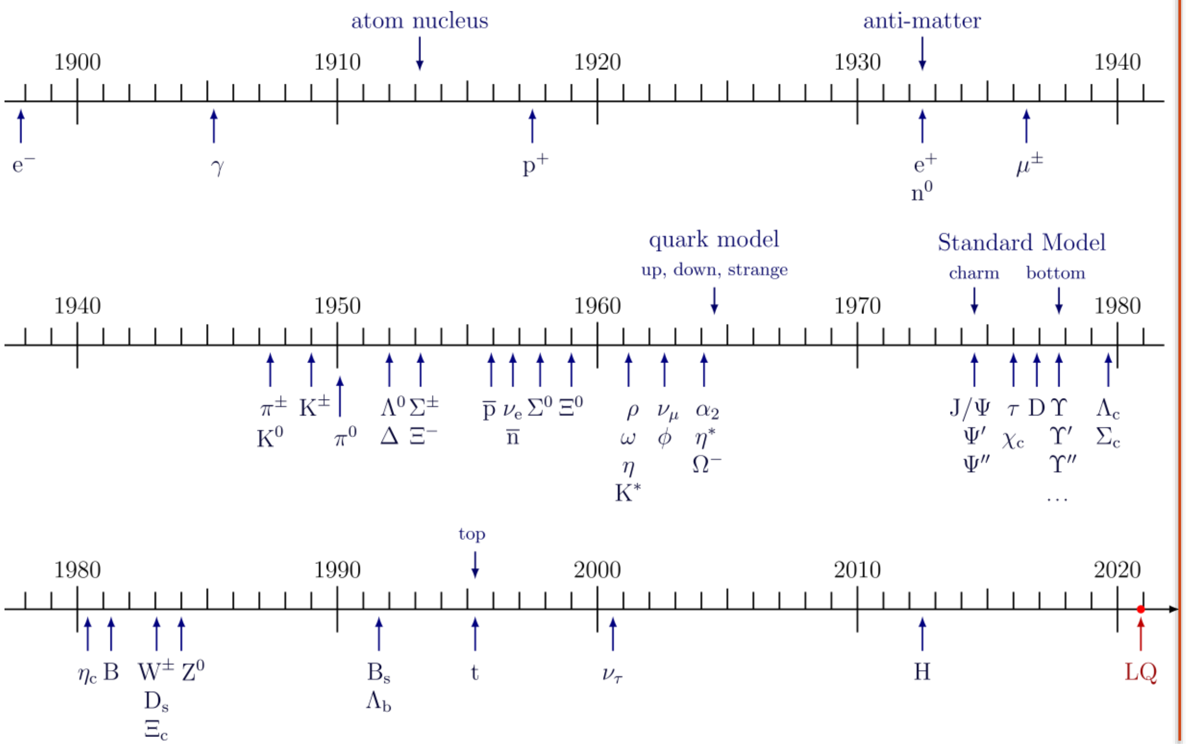




The Particle Zoo

Late 1950's – early 1960's: discovery of many strongly interacting particles at the high energy proton accelerators (Berkeley Bevatron, BNL AGS, CERN PS), all with very short mean life times (10^{-20} – 10^{-23} s, typical of strong decays)

⇒ catalog of > 100 strongly interacting particles (collectively named "hadrons")



Magnetic Moments

p	spin 1/2, charge $+e$	$\mu_s = \mu_N$
n	spin 1/2, charge 0	$\mu_s = 0$

$p \quad \mu_s = +2.793\mu_N \quad \rightarrow \quad g_s = +5.586$

$n \quad \mu_s = -1.913\mu_N \quad \rightarrow \quad g_s = -3.826$

Observation shows that p and n are not point-like ⇒ evidence for structure

There was a requirement for particle Table for 100s of particles

Force carrier: γ

Leptons: $e^-, \mu^-, \tau^-, \nu_e, \nu_\mu, \nu_\tau$

Mesons: $\pi^+, \pi^0, \pi^-, K^+, K^-, K^0, \rho^+, \rho^0, \rho^-$

Baryons: $p, n, \Lambda, \Sigma^+, \Sigma^-, \Sigma^0, \Delta^{++}, \Delta^+, \Delta^0, \Delta^-, \ldots$

ARE HADRONS ELEMENTARY PARTICLES?

1964 (Gell-Mann, Zweig): Hadron classification into "families"; observation that all hadrons could be built from three spin ½ "building blocks" (named "quarks" by Gell-Mann):

	u	d	s
Electric charge (units $\lvert e \rvert$)	$+2/3$	$-1/3$	$-1/3$
Baryonic number	$1/3$	$1/3$	$1/3$
Strangeness	0	0	-1

and three antiquarks ($\bar{u}$, $\bar{d}$, $\bar{s}$) with opposite electric charge and opposite baryonic number and strangeness

Mesons: quark – antiquark pairs

Examples of non-strange mesons:

$$\pi^+ \equiv u\bar{d} \quad ; \quad \pi^- \equiv \bar{u}d \quad ; \quad \pi^0 \equiv (d\bar{d} - u\bar{u})/\sqrt{2}$$

Examples of strange mesons:

$$K^- \equiv s\bar{u} \quad ; \quad \overline{K}^0 \equiv s\bar{d} \quad ; \quad K^+ \equiv \bar{s}u \quad ; \quad K^0 \equiv \bar{s}d$$

Baryons: three quarks bound together

Antibaryons: three antiquarks bound together

Examples of non-strange baryons:

$$\text{proton} \equiv uud \quad ; \quad \text{neutron} \equiv udd$$

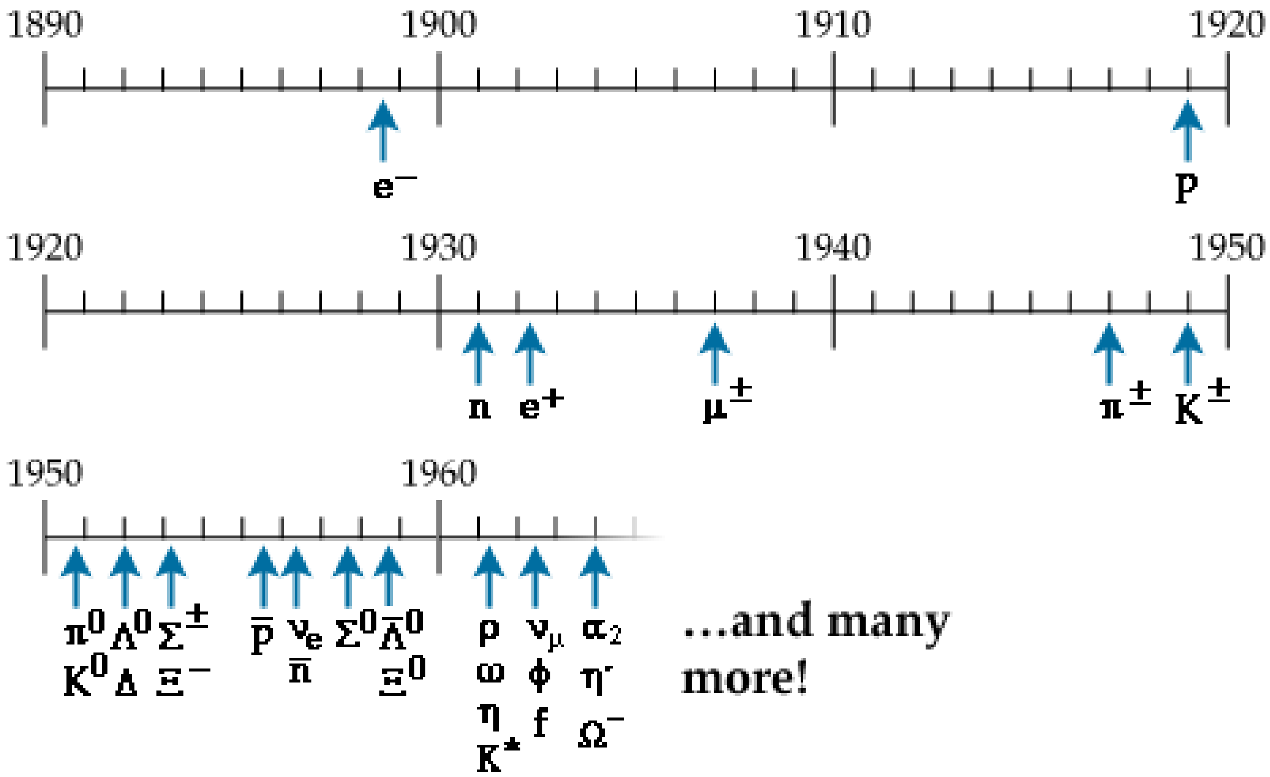
Examples of strangeness -1 baryons:

$$\Sigma^+ \equiv suu \quad ; \quad \Sigma^0 \equiv sud \quad ; \quad \Sigma^- \equiv sdd$$

Examples of strangeness -2 baryons:

$$\Xi^0 \equiv ssu \quad ; \quad \Xi^- \equiv ssd$$

Why quarks?



Murray Gell-Mann
1969 Nobel Prize
in Physics

<u>Why should nature be this complicated?</u>

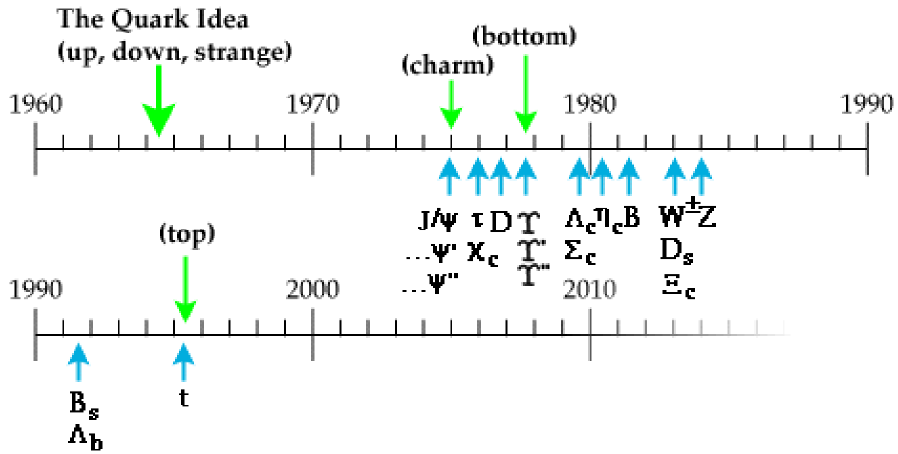
To simplify the picture, and still account for this plethora of particles which were observed, Murray Gell-Mann proposed all these particles were composed of just 3 smaller constituents, called quarks.

But even Gell-Mann doubted that they were real…

An excerpt from Gell-Mann's 1964 paper:

"A search for stable quarks of charge –1/3 or +2/3 and/or stable di-quarks of charge –2/3 or +1/3 or +4/3 at the highest energy accelerators would help to reassure us of the non-existence of real quarks".

In 1969, an experiment at SLAC uncovered the first evidence that protons in fact had substructure



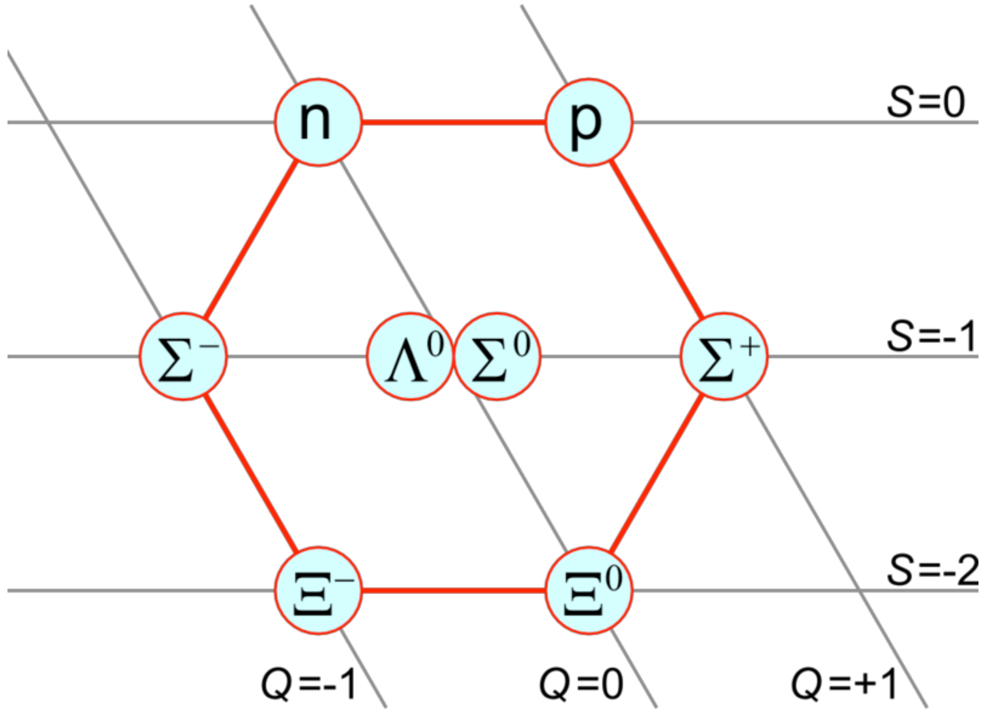
- Quarks can have 3 color values: red, green & blue
- Quarks have total spin S = ½ (S_Z = -½ or +½)
- Anti-quarks have the same mass as their quark does.

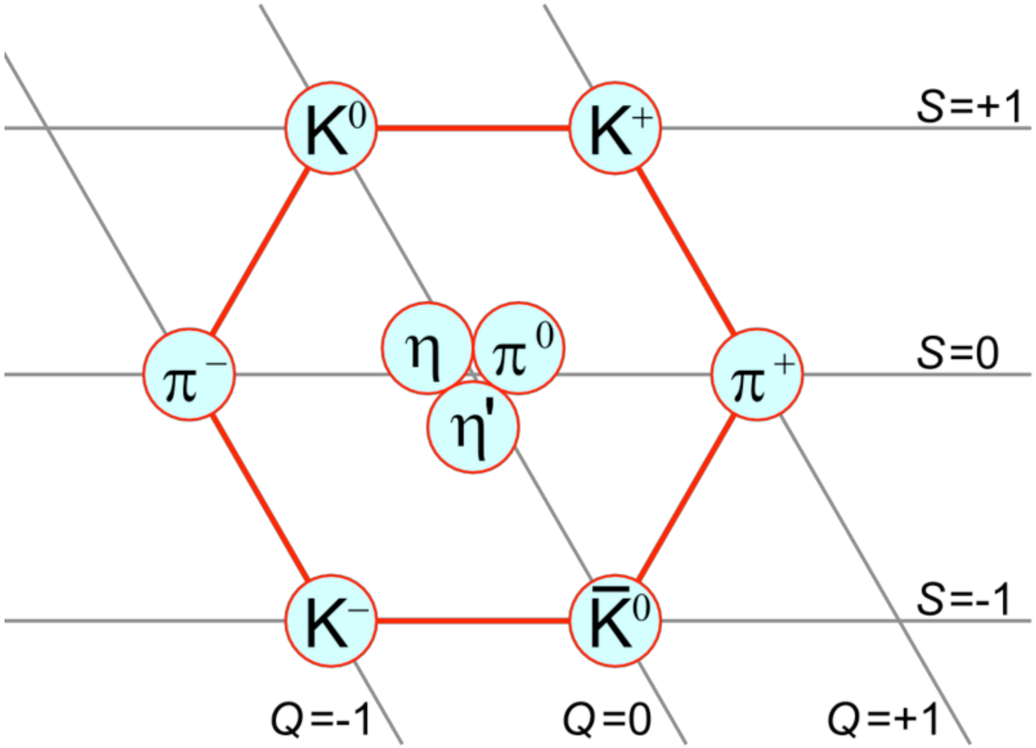
- Hadrons = Baryons + Mesons
- Baryons (antibaryons) contain 3 quarks (3 antiquarks)
- Mesons contain a quark and an antiquark

Quarks		Antiquarks	
Q = +2/3	Q = -1/3	Q = -2/3	Q = +1/3
u	d	$\bar{u}$	$\bar{d}$
c	s	$\bar{c}$	$\bar{s}$
t	b	$\bar{t}$	$\bar{b}$

The Eightfold Way

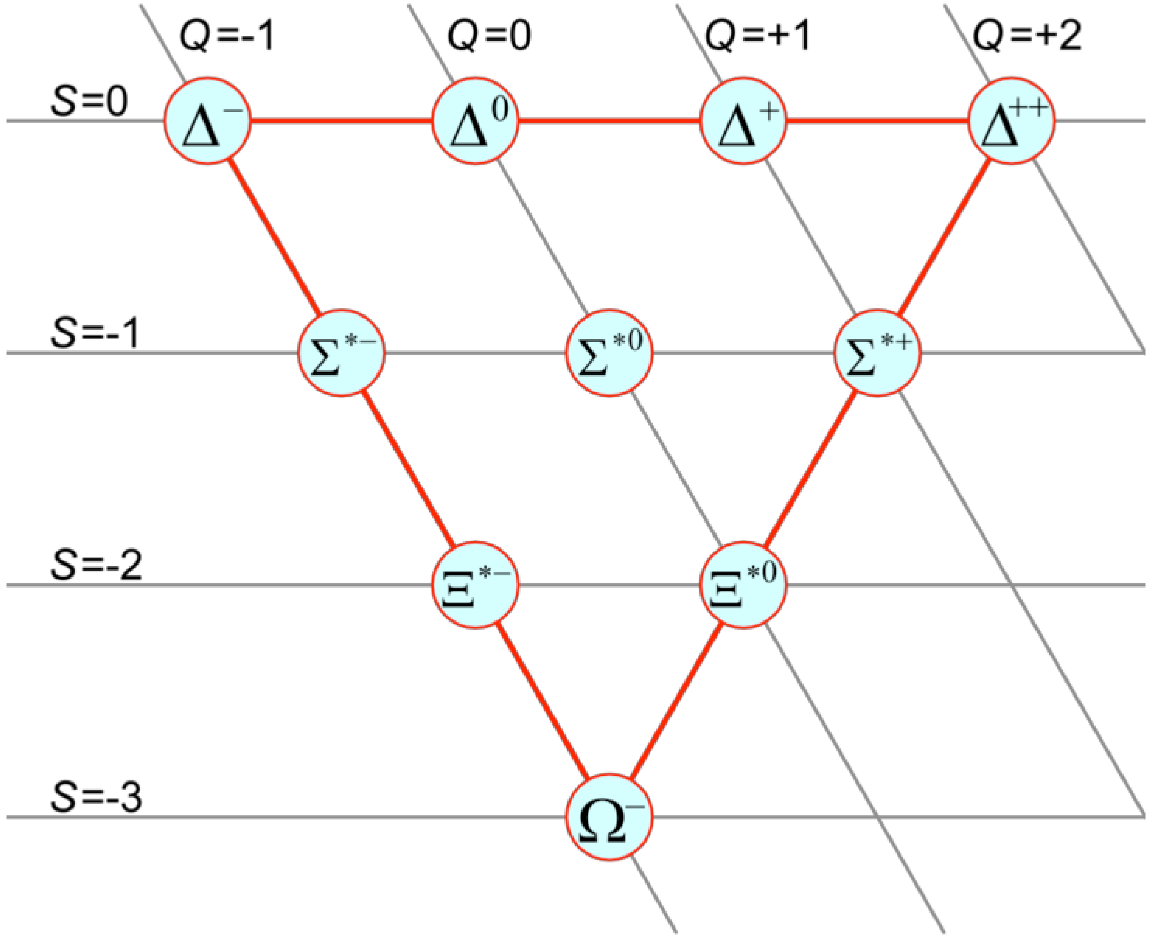
- Finally, in 1961, Gell-Mann brought some order to the chaos by developing a systematic ordering of the elementary particles.
- He noticed that if he plotted the mesons and baryons on a grid of strangeness S vs. charge Q, geometrical patterns emerged. The lightest mesons and baryons fit into hexagonal arrays:





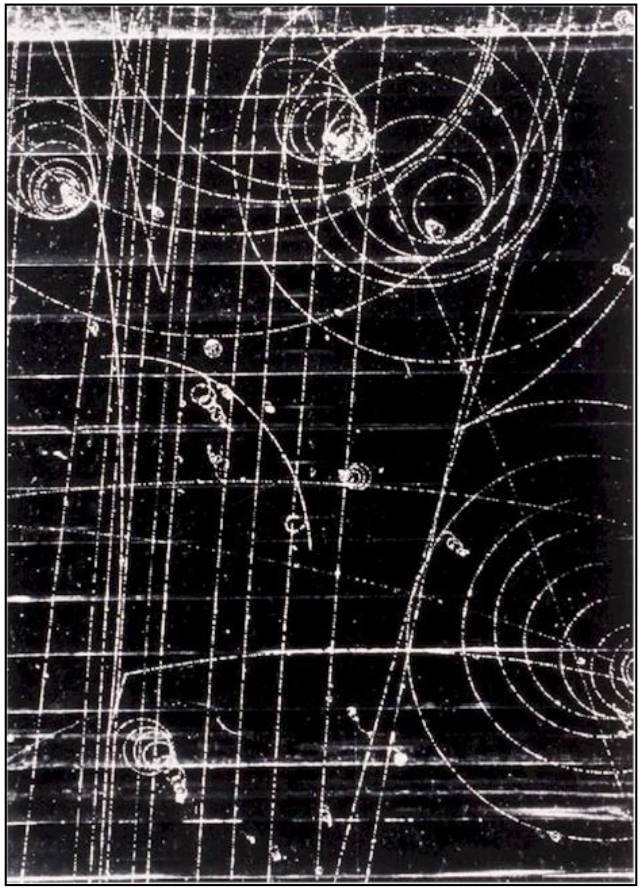
Baryon Decuplet

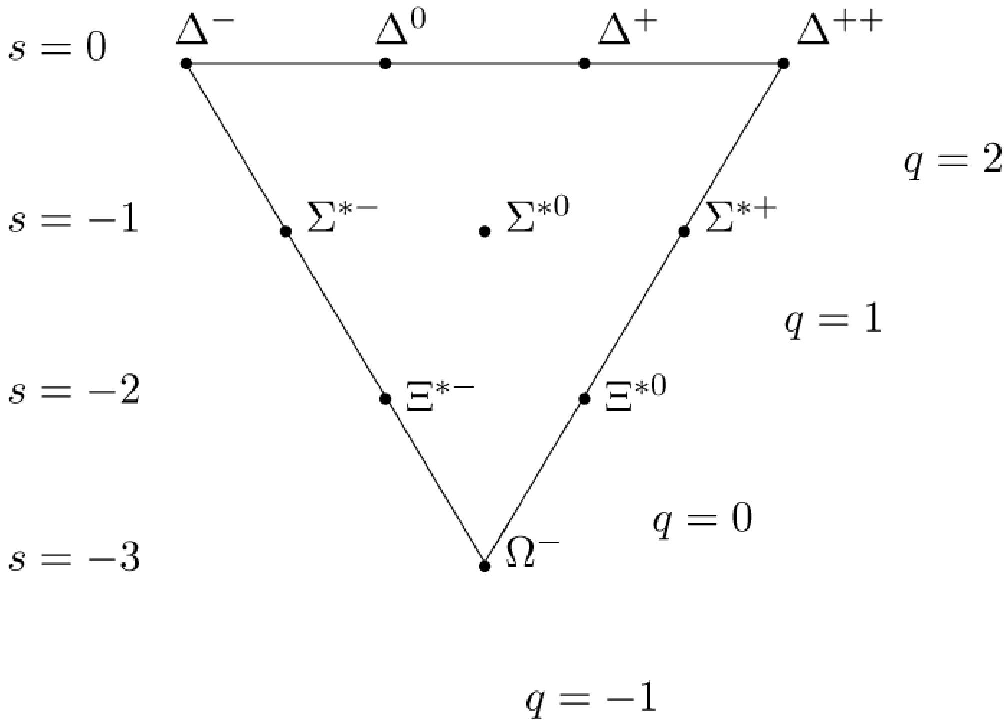
- Gell-Mann called his organizational scheme the "Eightfold Way".
- Note that other figures were allowed in this system, like a triangular array incorporating 10 of the heavier baryons.



Strange particles

The 8 lightest strange baryons: baryon octet



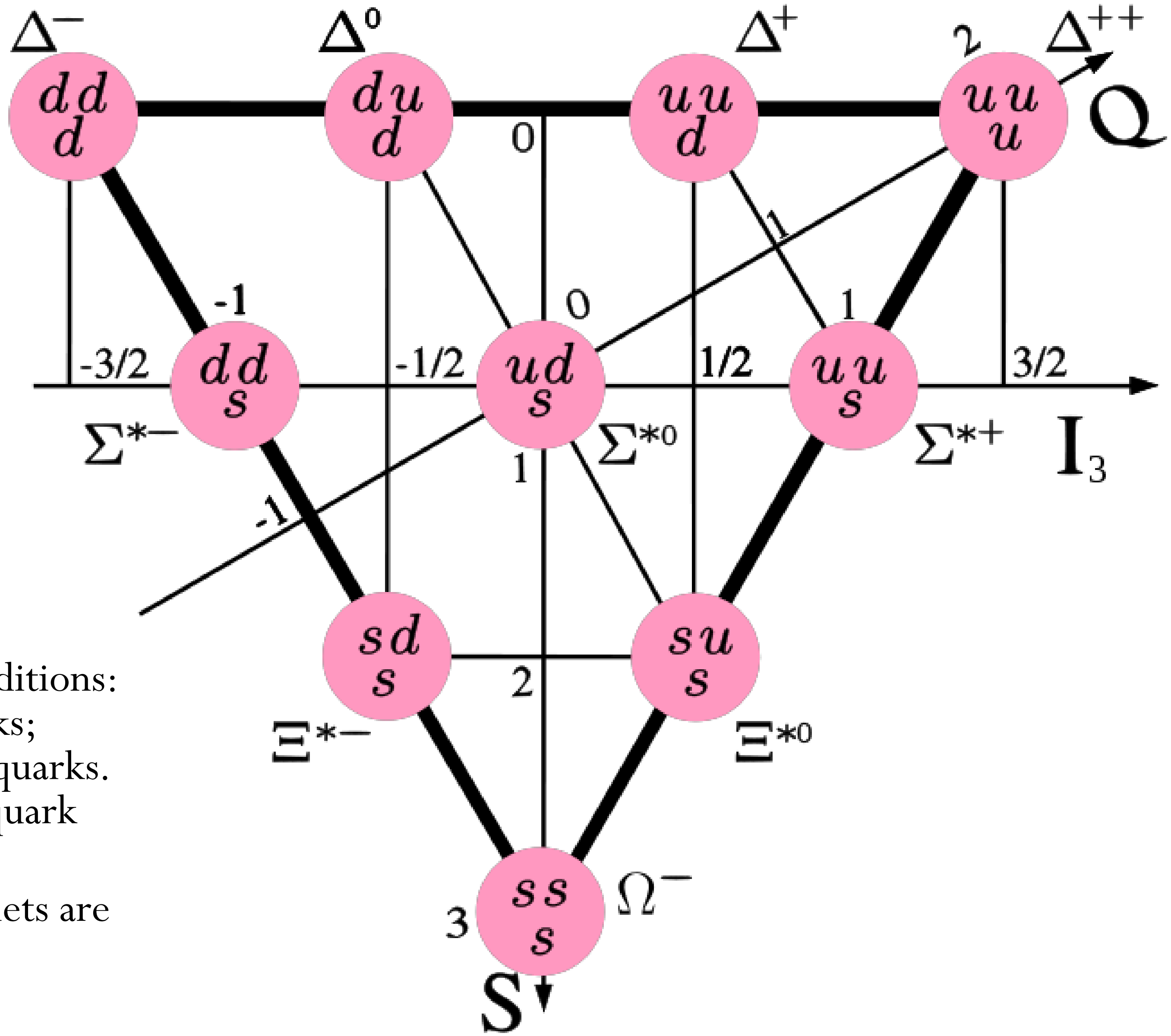


Breakthrough in 1961 (Murray Gell-Mann): **<u>"The eight-fold way"</u>** (Nobel prize 1969)

Also works for: Eight lightest mesons - meson octet
Other baryons - baryon decuplet

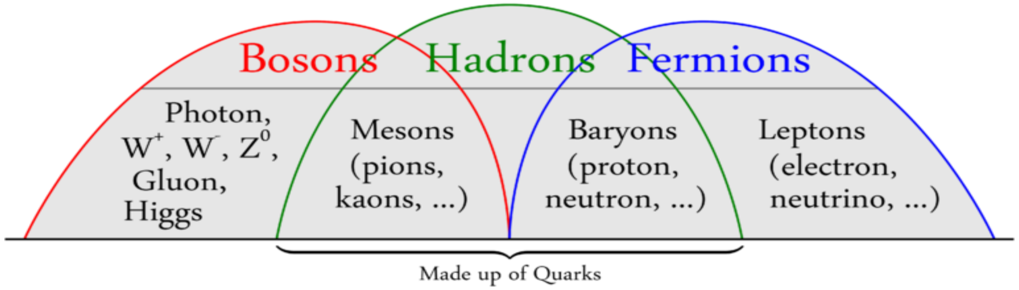
Baryon Decuplet

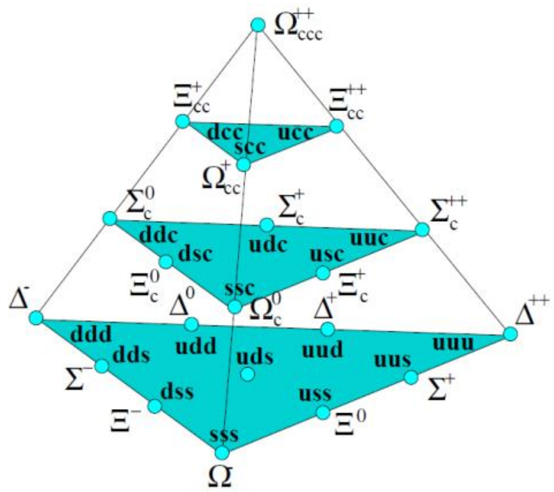
NOTE: quarks have never actually been observed! There is no such thing as a free quark (more on this later…). However, scattering experiments suggest hadrons do have a substructure (analogous to Rutherford scattering of atoms).



- The quark model has the following conditions:
  1) Baryons are composed of three quarks; antibaryons are composed of three antiquarks.
  2) Mesons are composed of quark-antiquark pairs.
- Using these rules, the hadronic multiplets are easily constructed…

Quark Model



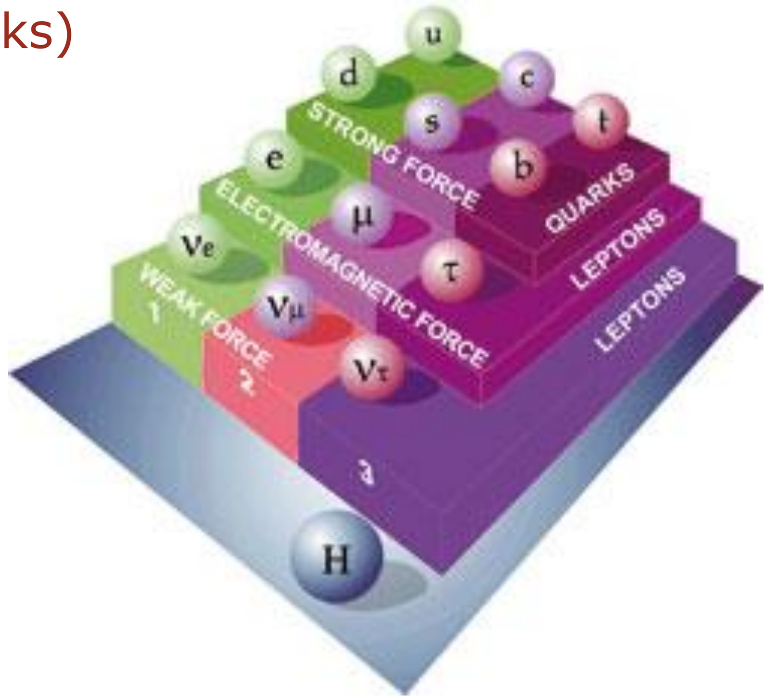


Gell-Mann en Zweig (1964):

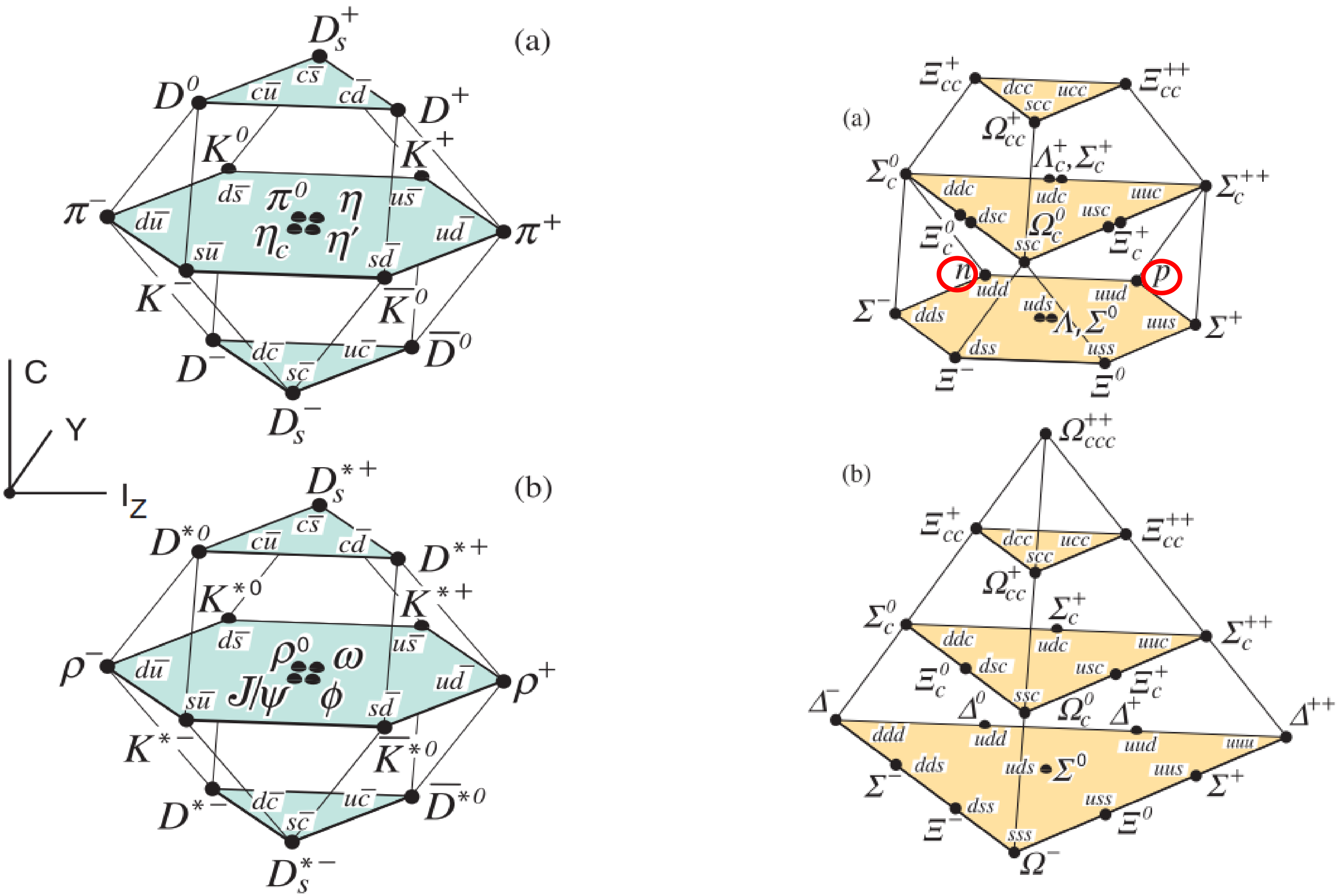
“All multiplet patterns can be explained if you assume hadrons are composite particles built from more elementary constituents: quarks”

- First quark model:
  - 3 types: up, down en strange (and anti-quarks)
  - Baryons: 3 quarks
  - Mesons: 2 quarks

	d	u	s
Q – electric charge	$-\frac{1}{3}$	$+\frac{2}{3}$	$-\frac{1}{3}$
I – isospin	$\frac{1}{2}$	$\frac{1}{2}$	0
I_z – isospin z-component	$-\frac{1}{2}$	$+\frac{1}{2}$	0
S – strangeness	0	0	-1



Some Mesons and Baryons quark content



Building Block: Leptons - Quarks (Anti particle)

Particles and Anti-particles

	I	II	III	Charge	Charge	I	II	III
quarks	u	c (1976)	t (1995)	+2/3 e	-2/3 e	$\bar{u}$	$\bar{c}$	$\bar{t}$
	d	s (1947)	b (1978)	-1/3 e	+1/3 e	$\bar{d}$	$\bar{s}$	$\bar{b}$
leptons	e (1895)	μ (1936)	τ (1973)	-1 e	+1 e	$\bar{e}$	$\bar{\mu}$	$\bar{\tau}$
	ν_e (1956)	ν_μ (1963)	ν_τ (2000)	0 e	0 e	$\bar{\nu}_e$	$\bar{\nu}_\mu$	$\bar{\nu}_\tau$

Symmetries in Particle Physics : Isospin

- The proton and neutron have very similar masses and the nuclear force is found to be approximately charge-independent, i.e.

$$V_{pp} \approx V_{np} \approx V_{nn}$$

- To reflect this symmetry, Heisenberg (1932) proposed that if you could "switch off" the electric charge of the proton

There would be no way to distinguish between a proton and neutron

- Proposed that the neutron and proton should be considered as two states of a single entity; the nucleon

$$p = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad n = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

★ Analogous to the spin-up/spin-down states of a spin-½ particle

ISOSPIN

★ Expect physics to be invariant under rotations in this space

- **The neutron and proton form an isospin doublet with total isospin $I = ½$ and third component $I_3 = \pm ½$**

Adding spin

$$|s_1,m_1> + |s_2,m_2> \;\rightarrow\; |s,m>$$

1) Conditions:

$m = m_1 + m_2$
$S = |s_1-s_2|, |s_1-s_2|+1, .., s_1+s_2 -1, s_1+s_2$

- S_z add up
- S can vary between difference and sum

2) Notation:

$$|s,m\rangle = \sum_{m_1+m_2=m} C^{s_1,s_2,s}_{m_1,m_2,m}|s_1,m_1\rangle|s_2,m_2\rangle$$

→ ***C: Clebsch-Gordan coefficient***

$\uparrow$ $\alpha = |\frac{1}{2},+\frac{1}{2}\rangle$

$\downarrow$ $\beta = |\frac{1}{2},-\frac{1}{2}\rangle$

		S_z	S
(1)	$\uparrow\uparrow$	+1	1
(2)	$\uparrow\downarrow$	0	?
(3)	$\downarrow\downarrow$	-1	1

Adding spin of two spin-$\frac{1}{2}$ particles

$\uparrow$ $\quad \alpha = |\frac{1}{2}, +\frac{1}{2}\rangle$

$\downarrow$ $\quad \beta = |\frac{1}{2}, -\frac{1}{2}\rangle$

		S_z	S
(1)	$\uparrow\uparrow$	+1	1
(2a)	$\sqrt{\frac{1}{2}}\left(\uparrow\downarrow + \downarrow\uparrow\right)$	0	1
(2b)	$\sqrt{\frac{1}{2}}\left(\uparrow\downarrow - \downarrow\uparrow\right)$	0	0
(3)	$\downarrow\downarrow$	-1	1

Adding spin of two spin-$\frac{1}{2}$ particles

$\uparrow \quad \alpha = |\frac{1}{2}, +\frac{1}{2}\rangle$

$\downarrow \quad \beta = |\frac{1}{2}, -\frac{1}{2}\rangle$

S=1 *Triplet (symmetric)*

$$\begin{cases} |1,+1\rangle = \uparrow\uparrow \\ |1,\ \ 0\rangle = \sqrt{\frac{1}{2}}\left(\uparrow\downarrow + \downarrow\uparrow\right) \\ |1,-1\rangle = \downarrow\downarrow \end{cases}$$

S=0 *Singlet (anti-symmetric)*

$$\left\{ |0,\ \ 0\rangle = \sqrt{\frac{1}{2}}\left(\uparrow\downarrow - \downarrow\uparrow\right) \right.$$

Triplet (symmetric)

$$\begin{cases} |1,+1\rangle = \alpha(1)\alpha(2) \\ |1,\ \ 0\rangle = \sqrt{\frac{1}{2}}[\alpha(1)\beta(2) + \beta(1)\alpha(2)] \\ |1,-1\rangle = \beta(1)\beta(2) \end{cases}$$

Singlet (anti-symmetric)

$$\left\{ |0,\ \ 0\rangle = \sqrt{\frac{1}{2}}[\alpha(1)\beta(2) - \beta(1)\alpha(2)] \right.$$

$$\boxed{2 \otimes 2 = 3 \oplus 1}$$

Quantum mechanics: adding spin

$$|s, m\rangle = \sum_{m_1+m_2=m} C^{s_1,s_2,s}_{m_1,m_2,m} |s_1, m_1\rangle |s_2, m_2\rangle$$

$C^{s_1,s_2,s}_{m_1,m_2,m}$ → ***Clebsch-Gordan coefficient***

***Specific:* adding spin of two spin-1/2 particles:**

$\uparrow \quad \alpha = |\frac{1}{2}, +\frac{1}{2}\rangle$

$\downarrow \quad \beta = |\frac{1}{2}, -\frac{1}{2}\rangle$

Triplet (symmetric)

$$\begin{cases} |1, +1\rangle = \uparrow\uparrow \\ |1, \ \ 0\rangle = \frac{1}{\sqrt{2}}\left(\uparrow\downarrow + \downarrow\uparrow\right) \\ |1, -1\rangle = \downarrow\downarrow \end{cases}$$

Singlet (anti-symmetric)

$$\left\{ |0, \ \ 0\rangle = \frac{1}{\sqrt{2}}\left(\uparrow\downarrow - \downarrow\uparrow\right) \right.$$

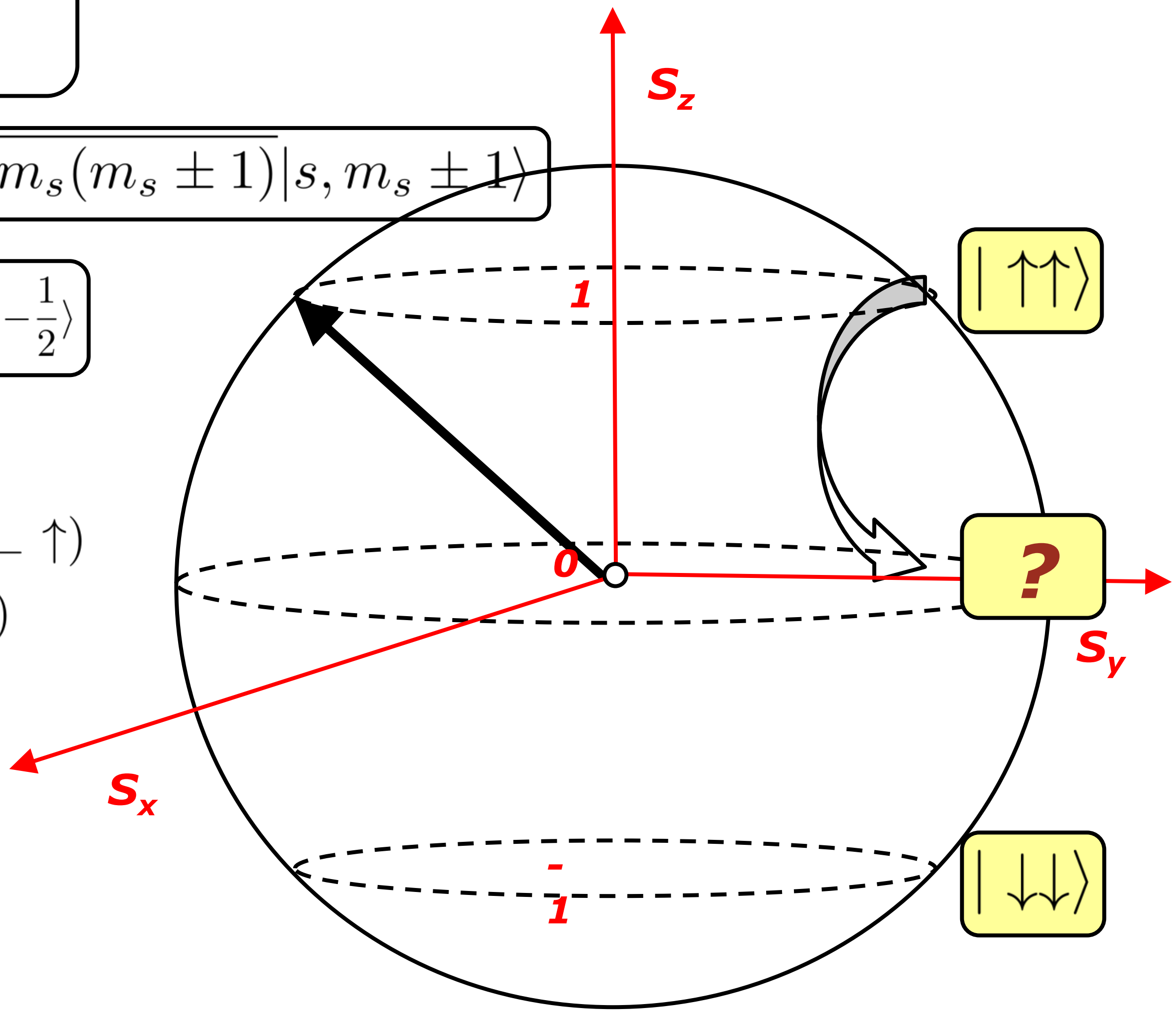
Why is $|1,0\rangle = \frac{1}{\sqrt{2}}|\uparrow\downarrow + \downarrow\uparrow\rangle$ and not $|1,0\rangle = \frac{1}{\sqrt{2}}|\uparrow\downarrow - \downarrow\uparrow\rangle$?

$$\begin{aligned} S^2|s,m_s\rangle &= \hbar^2 s(s+1)|s,m_s\rangle \\ S_z|s,m_s\rangle &= \hbar m_s|s,m_s\rangle \end{aligned}$$

$$S_\pm|s,m_s\rangle = \hbar\sqrt{s(s+1)-m_s(m_s\pm 1)}|s,m_s\pm 1\rangle$$

$$S_-|\tfrac{1}{2},+\tfrac{1}{2}\rangle = \hbar|\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

$$\begin{aligned} S_-|\uparrow\uparrow\rangle &= (S_-\uparrow)\uparrow + \uparrow(S_-\uparrow) \\ &= (\hbar\downarrow)\uparrow + \uparrow(\hbar\downarrow) \\ &= \hbar(\downarrow\uparrow + \uparrow\downarrow) \end{aligned}$$



Clebsch-Gordan coefficients

Coefficients can be used "both ways":

1) add

$$\left|\frac{3}{2},\frac{1}{2}\right\rangle|1,0\rangle = \sqrt{\frac{3}{5}}\left|\frac{5}{2},\frac{1}{2}\right\rangle + \sqrt{\frac{1}{15}}\left|\frac{3}{2},\frac{1}{2}\right\rangle - \sqrt{\frac{1}{3}}\left|\frac{1}{2},\frac{1}{2}\right\rangle$$

$$\boxed{|s_1,m_1\rangle + |s_2,m_2\rangle \rightarrow |s,m\rangle}$$

2) decay

$$|3,0\rangle = \sqrt{\frac{1}{5}}|2,1\rangle|1,-1\rangle + \sqrt{\frac{3}{5}}|2,0\rangle|1,0\rangle - \sqrt{\frac{1}{5}}|2,-1\rangle|1,1\rangle$$

$$\boxed{|s,m\rangle \rightarrow |s_1,m_1\rangle + |s_2,m_2\rangle}$$

Flavour Symmetry of the Strong Interaction

We can extend this idea to the quarks:

★ Assume the strong interaction treats all quark flavours equally (it does)

•Because $m_u \approx m_d$:

The strong interaction possesses an approximate flavour symmetry i.e. from the point of view of the strong interaction nothing changes if all up quarks are replaced by down quarks and *vice versa*.

- Choose the basis
$$u = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad d = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
- Express the invariance of the strong interaction under $u \leftrightarrow d$ invariance under "rotations" in an abstract isospin space

$$\begin{pmatrix} u' \\ d' \end{pmatrix} = \hat{U} \begin{pmatrix} u \\ d \end{pmatrix} = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} u \\ d \end{pmatrix}$$

The 2x2 unitary matrix depends on 4 complex numbers, i.e. 8 real parameters
But there are four constraints from $\hat{U}^\dagger \hat{U} = 1$

➡ 8 – 4 = 4 independent matrices

•In the language of group theory the four matrices form the U(2) group

- One of the matrices corresponds to multiplying by a phase factor

$$\hat{U}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} e^{i\phi}$$

not a flavour transformation and of no relevance here.

- The remaining three matrices form an SU(2) group (special unitary) with $\boxed{\det U = 1}$
- For an infinitesimal transformation, in terms of the Hermitian generators $\hat{G}$

$$\hat{U} = 1 + i\varepsilon\hat{G}$$

- $\det U = 1 \quad \Rightarrow \quad Tr(\hat{G}) = 0$
- A linearly independent choice for $\hat{G}$ are the Pauli spin matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- The proposed flavour symmetry of the strong interaction has the same transformation properties as SPIN !
- Define ISOSPIN: $\vec{T} = \frac{1}{2}\vec{\sigma} \qquad \hat{U} = e^{i\vec{\alpha}.\vec{T}}$
- Check this works, for an infinitesimal transformation

$$\hat{U} = 1 + \frac{1}{2}i\vec{\varepsilon}.\vec{\sigma} \quad = \quad 1 + \frac{i}{2}(\varepsilon_1\sigma_1 + \varepsilon_2\sigma_2 + \varepsilon_3\sigma_3) = \begin{pmatrix} 1 + \frac{1}{2}i\varepsilon_3 & \frac{1}{2}i(\varepsilon_1 - i\varepsilon_2) \\ \frac{1}{2}i(\varepsilon_1 + i\varepsilon_2) & 1 - \frac{1}{2}i\varepsilon_3 \end{pmatrix}$$

Which is, as required, unitary and has unit determinant

$$U^\dagger U = I + O(\varepsilon^2) \qquad \det U = 1 + O(\varepsilon^2)$$

Properties of Isopin: Through Quarks

- Isospin has the exactly the same properties as spin

$$[T_1, T_2] = iT_3 \quad [T_2, T_3] = iT_1 \quad [T_3, T_1] = iT_2$$

$$[T^2, T_3] = 0 \qquad T^2 = T_1^2 + T_2^2 + T_3^2$$

As in the case of spin, have three non-commuting operators, T_1, T_2, T_3
even though all three correspond to observables, can't know them simultaneously.
So label states in terms of total isospin I and the third component of isospin I_3

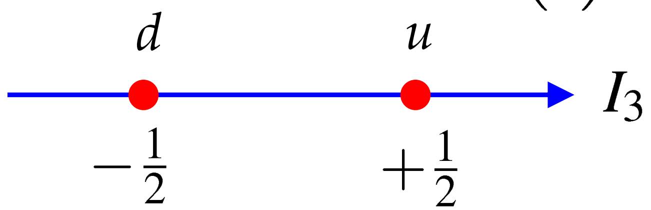
NOTE: isospin has nothing to do with spin – just the same mathematics

- The eigenstates are exact analogues of the eigenstates of ordinary angular momentum $|s, m\rangle \rightarrow |I, I_3\rangle$

  with $\quad T^2|I, I_3\rangle = I(I+1)|I, I_3\rangle \qquad T_3|I, I_3\rangle = I_3|I, I_3\rangle$

- In terms of isospin:

$$u = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |\tfrac{1}{2}, +\tfrac{1}{2}\rangle \qquad d = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |\tfrac{1}{2}, -\tfrac{1}{2}\rangle$$



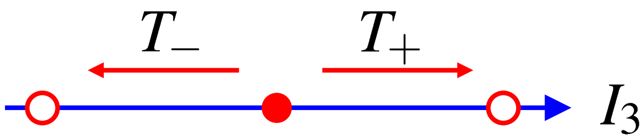
$$\boxed{I = \tfrac{1}{2}, \quad I_3 = \pm\tfrac{1}{2}}$$

- In general $\quad I_3 = \frac{1}{2}(N_u - N_d)$

- Can define isospin ladder operators – analogous to spin ladder operators

$$\boxed{T_- \equiv T_1 - iT_2}$$

u → d



$$\boxed{T_+ \equiv T_1 + iT_2}$$

d → u

$$T_+|I, I_3\rangle = \sqrt{I(I+1) - I_3(I_3+1)}\,|I, I_3+1\rangle$$

$$T_-|I, I_3\rangle = \sqrt{I(I+1) - I_3(I_3-1)}\,|I, I_3-1\rangle$$

Step up/down in I_3 until reach end of multiplet $\quad T_+|I, +I\rangle = 0 \quad T_-|I, -I\rangle = 0$

$$\boxed{T_+ u = 0 \qquad T_+ d = u \qquad T_- u = d \qquad T_- d = 0}$$

- Ladder operators turn $u \longrightarrow d$ and $d \longrightarrow u$

★ Combination of isospin: e.g. what is the isospin of a system of two d quarks, is exactly analogous to combination of spin (i.e. angular momentum)

$$|I^{(1)}, I_3^{(1)}\rangle |I^{(2)}, I_3^{(2)}\rangle \longrightarrow |I, I_3\rangle$$

- I_3 additive : $\quad I_3 = I_3^{(1)} + I_3^{(2)}$
- I in integer steps from $|I^{(1)} - I^{(2)}|$ to $|I^{(1)} + I^{(2)}|$

★ Assumed symmetry of Strong Interaction under isospin transformations implies the existence of conserved quantites

- In strong interactions I_3 and I are conserved, analogous to conservation of J_z and J for angular momentum

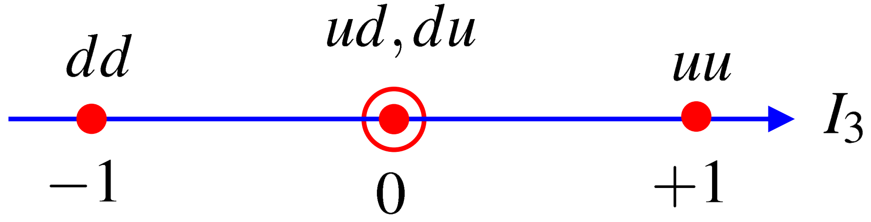
Combining Quarks

Goal: derive proton wave-function

- First combine two quarks, then combine the third
- Use requirement that fermion wave-functions are anti-symmetric

Isospin starts to become useful in defining states of more than one quark.

e.g. two quarks, here we have four possible combinations:



Note: ⦿ represents two states with the same value of I_3

- We can immediately identify the extremes (I_3 additive)

$$uu \equiv |\tfrac{1}{2},\tfrac{1}{2}\rangle|\tfrac{1}{2},\tfrac{1}{2}\rangle = |1,+1\rangle \qquad dd \equiv |\tfrac{1}{2},-\tfrac{1}{2}\rangle|\tfrac{1}{2},-\tfrac{1}{2}\rangle = |1,-1\rangle$$

To obtain the $|1,0\rangle$ state use ladder operators

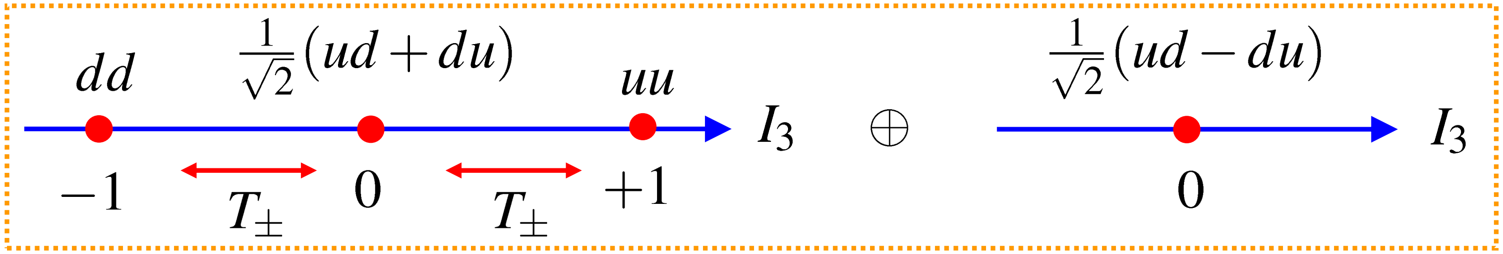
$$T_-|1,+1\rangle = \sqrt{2}|1,0\rangle = T_-(uu) = ud + du$$

$$\Rightarrow \quad |1,0\rangle = \frac{1}{\sqrt{2}}(ud+du)$$

The final state, $|0,0\rangle$ can be found from orthogonality with $|1,0\rangle$

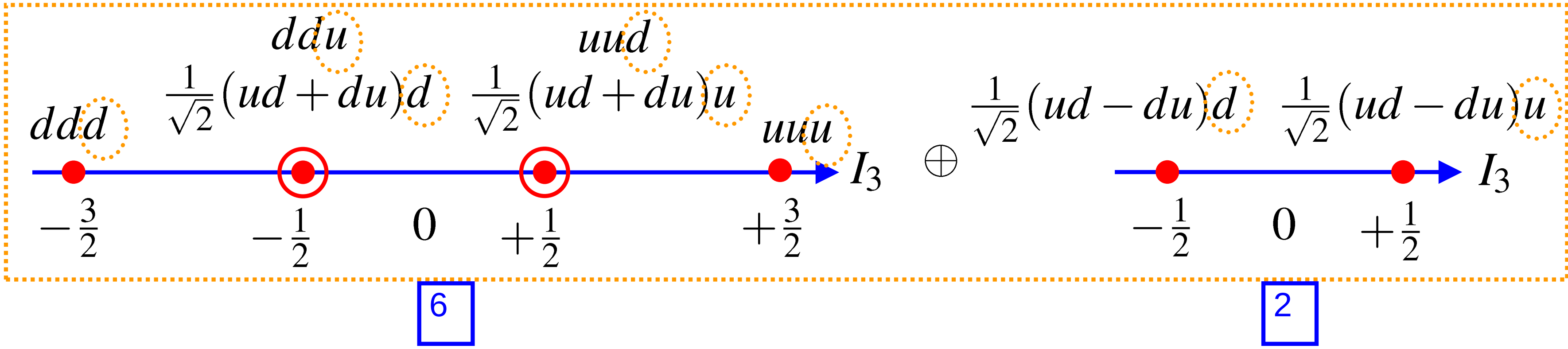
$$\Rightarrow \quad |0,0\rangle = \frac{1}{\sqrt{2}}(ud-du)$$

- From four possible combinations of isospin doublets obtain a triplet of isospin 1 states and a singlet isospin 0 state $\quad 2 \otimes 2 = 3 \oplus 1$



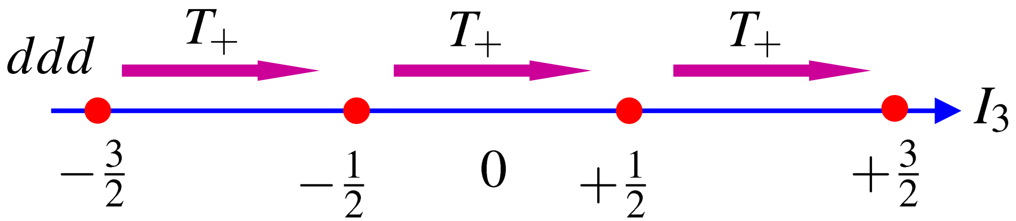
- Can move around within multiplets using ladder operators
- note, as anticipated $\quad I_3 = \frac{1}{2}(N_u - N_d)$
- States with different total isospin are physically different – the isospin 1 triplet is symmetric under interchange of quarks 1 and 2 whereas singlet is anti-symmetric

★ Now add an additional up or down quark. From each of the above 4 states get two new isospin states with $\quad I_3' = I_3 \pm \frac{1}{2}$



- Use ladder operators and orthogonality to group the 6 states into isospin multiplets, e.g. to obtain the $I = \frac{3}{2}$ states, step up from $\quad ddd$

★Derive the $I = \frac{3}{2}$ states from $ddd \equiv |\frac{3}{2}, -\frac{3}{2}\rangle$



$$
\begin{aligned}
T_+|\tfrac{3}{2}, -\tfrac{3}{2}\rangle &= T_+(ddd) = (T_+d)dd + d(T_+d)d + dd(T_+)d \\
\sqrt{3}|\tfrac{3}{2}, -\tfrac{1}{2}\rangle &= udd + dud + ddu \\
\boxed{|\tfrac{3}{2}, -\tfrac{1}{2}\rangle = \tfrac{1}{\sqrt{3}}(udd + dud + ddu)} & \\
T_+|\tfrac{3}{2}, -\tfrac{1}{2}\rangle &= \tfrac{1}{\sqrt{3}}T_+(udd + dud + ddu) \\
2|\tfrac{3}{2}, +\tfrac{1}{2}\rangle &= \tfrac{1}{\sqrt{3}}(uud + udu + uud + duu + udu + duu) \\
\boxed{|\tfrac{3}{2}, +\tfrac{1}{2}\rangle = \tfrac{1}{\sqrt{3}}(uud + udu + duu)} & \\
T_+|\tfrac{3}{2}, +\tfrac{1}{2}\rangle &= \tfrac{1}{\sqrt{3}}T_+(uud + udu + duu) \\
\sqrt{3}|\tfrac{3}{2}, +\tfrac{3}{2}\rangle &= \tfrac{1}{\sqrt{3}}(uuu + uuu + uuu) \\
\boxed{|\tfrac{3}{2}, +\tfrac{3}{2}\rangle = uuu} &
\end{aligned}
$$

★ From the $\boxed{6}$ states on previous page, use orthoganality to find $|\frac{1}{2}, \pm\frac{1}{2}\rangle$es

★ The $\boxed{2}$ states on the previous page give another $|\frac{1}{2}, \pm\frac{1}{2}\rangle$ublet

★The eight states $uuu,\ uud,\ udu,\ udd,\ duu,\ dud,\ ddu,\ ddd$
are grouped into an isospin quadruplet and two isospin doublets

$$2 \otimes 2 \otimes 2 = 2 \otimes (3 \oplus 1) = (2 \otimes 3) \oplus (2 \otimes 1) = 4 \oplus 2 \oplus 2$$

- Different multiplets have different symmetry properties

$$\left.\begin{aligned} |\tfrac{3}{2},+\tfrac{3}{2}\rangle &= uuu \\ |\tfrac{3}{2},+\tfrac{1}{2}\rangle &= \tfrac{1}{\sqrt{3}}(uud+udu+duu) \\ |\tfrac{3}{2},-\tfrac{1}{2}\rangle &= \tfrac{1}{\sqrt{3}}(ddu+dud+udd) \\ |\tfrac{3}{2},-\tfrac{3}{2}\rangle &= ddd \end{aligned}\right\} \boxed{S}$$

A quadruplet of states which are symmetric under the interchange of any two quarks

$$\left.\begin{aligned} |\tfrac{1}{2},-\tfrac{1}{2}\rangle &= -\tfrac{1}{\sqrt{6}}(2ddu-udd-dud) \\ |\tfrac{1}{2},+\tfrac{1}{2}\rangle &= \tfrac{1}{\sqrt{6}}(2uud-udu-duu) \end{aligned}\right\} \boxed{M_S}$$

Mixed symmetry.
Symmetric for 1 $\longleftrightarrow$ 2

$$\left.\begin{aligned} |\tfrac{1}{2},-\tfrac{1}{2}\rangle &= \tfrac{1}{\sqrt{2}}(udd-dud) \\ |\tfrac{1}{2},+\tfrac{1}{2}\rangle &= \tfrac{1}{\sqrt{2}}(udu-duu) \end{aligned}\right\} \boxed{M_A}$$

Mixed symmetry.
Anti-symmetric for 1 $\longleftrightarrow$ 2

- Mixed symmetry states have no definite symmetry under interchange of quarks $1 \leftrightarrow 3$ etc.

Combining Spin

- Can apply exactly the same mathematics to determine the possible spin wave-functions for a combination of 3 spin-half particles

$$\left.\begin{aligned}
|\tfrac{3}{2},+\tfrac{3}{2}\rangle &= \uparrow\uparrow\uparrow \\
|\tfrac{3}{2},+\tfrac{1}{2}\rangle &= \tfrac{1}{\sqrt{3}}(\uparrow\uparrow\downarrow + \uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow) \\
|\tfrac{3}{2},-\tfrac{1}{2}\rangle &= \tfrac{1}{\sqrt{3}}(\downarrow\downarrow\uparrow + \downarrow\uparrow\downarrow + \uparrow\downarrow\downarrow) \\
|\tfrac{3}{2},-\tfrac{3}{2}\rangle &= \downarrow\downarrow\downarrow
\end{aligned}\right\} \boxed{S}$$

A quadruplet of states which are symmetric under the interchange of any two quarks

$$\left.\begin{aligned}
|\tfrac{1}{2},-\tfrac{1}{2}\rangle &= -\tfrac{1}{\sqrt{6}}(2\downarrow\downarrow\uparrow - \uparrow\downarrow\downarrow - \downarrow\uparrow\downarrow) \\
|\tfrac{1}{2},+\tfrac{1}{2}\rangle &= \tfrac{1}{\sqrt{6}}(2\uparrow\uparrow\downarrow - \uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow)
\end{aligned}\right\} \boxed{M_S}$$

Mixed symmetry.
Symmetric for 1 $\longleftrightarrow$ 2

$$\left.\begin{aligned}
|\tfrac{1}{2},-\tfrac{1}{2}\rangle &= \tfrac{1}{\sqrt{2}}(\uparrow\downarrow\downarrow - \downarrow\uparrow\downarrow) \\
|\tfrac{1}{2},+\tfrac{1}{2}\rangle &= \tfrac{1}{\sqrt{2}}(\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow)
\end{aligned}\right\} \boxed{M_A}$$

Mixed symmetry.
Anti-symmetric for 1 $\longleftrightarrow$ 2

- Can now form total wave-functions for combination of three quarks

Baryon Wave-functions (ud)

★ Quarks are fermions so require that the total wave-function is anti-symmetric under the interchange of any two quarks

★ the total wave-function can be expressed in terms of:

$$\psi = \phi_{\text{flavour}}\chi_{\text{spin}}\xi_{\text{colour}}\eta_{\text{space}}$$

★ The colour wave-function for all bound qqq states is anti-symmetric (see handout 8)

- Here we will only consider the lowest mass, ground state, baryons where there is no internal orbital angular momentum.
- For L=0 the spatial wave-function is symmetric (-1)^L.

➡ $\xi_{\text{colour}}\eta_{\text{space}}$ anti-symmetric

➡ $\phi_{\text{flavour}}\chi_{\text{spin}}$ symmetric

Overall anti-symmetric

★ Two ways to form a totally symmetric wave-function from spin and isospin states:

❶ combine totally symmetric spin and isospin wave-functions $\phi(S)\chi(S)$

$$ddd \qquad \frac{1}{\sqrt{3}}(ddu+dud+udd) \qquad \frac{1}{\sqrt{3}}(uud+udu+duu) \qquad uuu$$

$$\Delta^- \qquad \Delta^0 \qquad \Delta^+ \qquad \Delta^{++}$$

I_3: $-\frac{3}{2}$, $-\frac{1}{2}$, 0, $+\frac{1}{2}$, $+\frac{3}{2}$

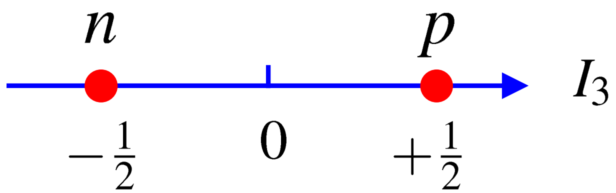
Spin 3/2
Isospin 3/2

❷ combine mixed symmetry spin and mixed symmetry isospin states

- Both $\phi(M_S)\chi(M_S)$ and $\phi(M_A)\chi(M_A)$ are sym. under inter-change of quarks $1 \leftrightarrow 2$
- Not sufficient, these combinations have no definite symmetry under $1 \leftrightarrow 3, \ldots$
- However, it is not difficult to show that the (normalised) linear combination:

$$\frac{1}{\sqrt{2}}\phi(M_S)\chi(M_S) + \frac{1}{\sqrt{2}}\phi(M_A)\chi(M_A)$$

is totally symmetric (i.e. symmetric under $1 \leftrightarrow 2;\; 1 \leftrightarrow 3;\; 2 \leftrightarrow 3$



Spin 1/2
Isospin 1/2

- The spin-up proton wave-function is therefore:

$$|p\uparrow\rangle = \frac{1}{6\sqrt{2}}(2uud - udu - duu)(2\uparrow\uparrow\downarrow - \uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow) + \frac{1}{2\sqrt{2}}(udu - duu)(\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow)$$

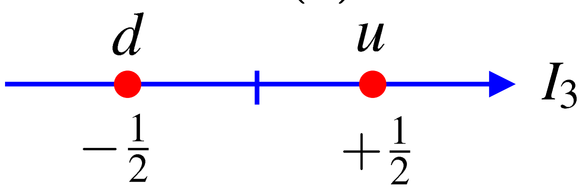
$$\Rightarrow \boxed{\begin{aligned} |p\uparrow\rangle = \frac{1}{\sqrt{18}}(\; & 2u\uparrow u\uparrow d\downarrow - u\uparrow u\downarrow d\uparrow - u\downarrow u\uparrow d\uparrow + \\ & 2u\uparrow d\downarrow u\uparrow - u\uparrow d\uparrow u\downarrow - u\downarrow d\uparrow u\uparrow + \\ & 2d\downarrow u\uparrow u\uparrow - d\uparrow u\downarrow u\uparrow - d\uparrow u\uparrow u\uparrow) \end{aligned}}$$

NOTE: not always necessary to use the fully symmetrised proton wave-function,
e.g. the first 3 terms are sufficient for calculating the proton magnetic moment

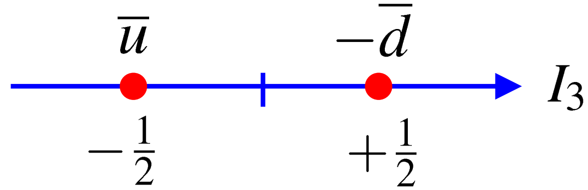
Anti-quarks and Mesons (u and d)

★The u, d quarks and $\bar{u}$, $\bar{d}$ anti-quarks are represented as isospin doublets

$$q = \begin{pmatrix} u \\ d \end{pmatrix}$$



$$\bar{q} = \begin{pmatrix} -\bar{d} \\ \bar{u} \end{pmatrix}$$



$$\bar{u} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \bar{d} = -\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

- Subtle point: The ordering and the minus sign in the anti-quark doublet ensures that anti-quarks and quarks transform in the same way (see Appendix I). This is necessary if we want physical predictions to be invariant under $u \leftrightarrow d;\ \bar{u} \leftrightarrow \bar{d}$

- Consider the effect of ladder operators on the anti-quark isospin states

e.g $$T_+\bar{u} = T_+\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -\bar{d}$$

- The effect of the ladder operators on anti-particle isospin states are:

$$\boxed{T_+\bar{u} = -\bar{d} \quad T_+\bar{d} = 0 \quad T_-\bar{u} = 0 \quad T_-\bar{d} = -\bar{u}}$$

Compare with $$\boxed{T_+u = 0 \quad T_+d = u \quad T_-u = d \quad T_-d = 0}$$

Appendix: the SU(2) anti-quark representation

Non-examinable

- Define anti-quark doublet $\overline{q} = \begin{pmatrix} -\overline{d} \\ \overline{u} \end{pmatrix} = \begin{pmatrix} -d^* \\ u^* \end{pmatrix}$

- The quark doublet $q = \begin{pmatrix} u \\ d \end{pmatrix}$ transforms as $q' = Uq$

$$\begin{pmatrix} u' \\ d' \end{pmatrix} = U \begin{pmatrix} u \\ d \end{pmatrix} \xrightarrow[\text{conjugate}]{\text{Complex}} \begin{pmatrix} u'^* \\ d'^* \end{pmatrix} = U^* \begin{pmatrix} u^* \\ d^* \end{pmatrix}$$

- Express in terms of anti-quark doublet

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \overline{q}' = U \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \overline{q}$$

- Hence $\overline{q}$ transforms as

$$\overline{q}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} U^* \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \overline{q}$$

- In general a 2x2 unitary matrix can be written as

$$U = \begin{pmatrix} c_{11} & c_{12} \\ -c_{12}^* & c_{11}^* \end{pmatrix}$$

- Giving

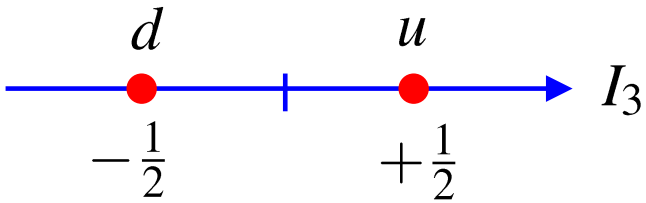
$$\begin{aligned} \bar{q}' &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} c_{11}^* & c_{12}^* \\ -c_{12} & c_{11} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \bar{q} \\ &= \begin{pmatrix} c_{11} & c_{12} \\ -c_{12}^* & c_{11}^* \end{pmatrix} \\ &= U\bar{q} \end{aligned}$$

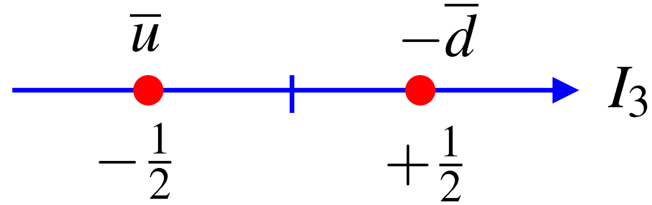
- Therefore the anti-quark doublet $\bar{q} = \begin{pmatrix} -\bar{d} \\ \bar{u} \end{pmatrix}$

transforms in the same way as the quark doublet $q = \begin{pmatrix} u \\ d \end{pmatrix}$

Light ud Mesons

★ Can now construct meson states from combinations of up/down quarks





- Consider the $q\bar{q}$ combinations in terms of isospin

$$|1,+1\rangle = |\tfrac{1}{2},+\tfrac{1}{2}\rangle\overline{|\tfrac{1}{2},+\tfrac{1}{2}\rangle} = -u\bar{d}$$

$$|1,-1\rangle = |\tfrac{1}{2},-\tfrac{1}{2}\rangle\overline{|\tfrac{1}{2},-\tfrac{1}{2}\rangle} = d\bar{u}$$

The bar indicates this is the isospin representation of an anti-quark

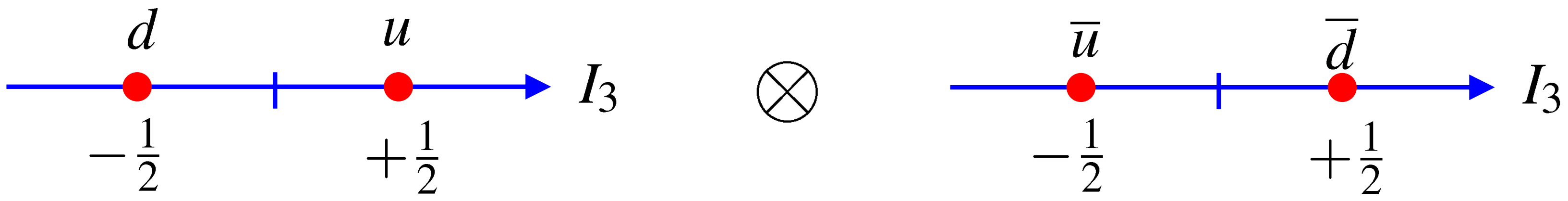
To obtain the $I_3 = 0$ states use ladder operators and orthogonality

$$\begin{aligned} T_-|1,+1\rangle &= T_-\left[-u\bar{d}\right] \\ \sqrt{2}|1,0\rangle &= -T_-[u]\bar{d} - uT_-\left[\bar{d}\right] \\ &= -d\bar{d} + u\bar{u} \end{aligned}$$

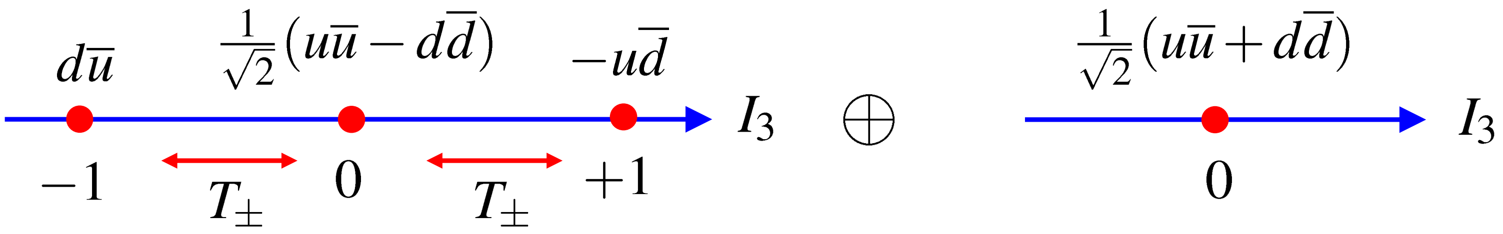
$$\Longrightarrow \quad |1,0\rangle = \frac{1}{\sqrt{2}}\left(u\bar{u} - d\bar{d}\right)$$

- **Orthogonality gives:** $|0,0\rangle = \frac{1}{\sqrt{2}}\left(u\bar{u} + d\bar{d}\right)$

★To summarise:



Triplet of $I=1$ states and a singlet $I=0$ state



•You will see this written as $2\otimes\bar{2}=3\oplus 1$

Quark doublet

Anti-quark doublet

•To show the state obtained from orthogonality with $|1,0\rangle$ singlet use ladder operators

$$T_+|0,0\rangle = T_+\frac{1}{\sqrt{2}}(u\bar{u}+d\bar{d}) = \frac{1}{\sqrt{2}}\left(-u\bar{d}+u\bar{d}\right)=0$$

similarly $\quad T_-|0,0\rangle = 0$

★ A singlet state is a "dead-end" from the point of view of ladder operators

SU(3) Flavour

★ Extend these ideas to include the strange quark. Since $m_s > m_u, m_d$ don't have an exact symmetry. But m_s not so very different from m_u, m_d and can treat the strong interaction (and resulting hadron states) as if it were symmetric under $u \leftrightarrow d \leftrightarrow s$

- NOTE: any results obtained from this assumption are only approximate as the symmetry is not exact.
- The assumed uds flavour symmetry can be expressed as

$$\begin{pmatrix} u' \\ d' \\ s' \end{pmatrix} = \hat{U} \begin{pmatrix} u \\ d \\ s \end{pmatrix} = \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & U_{33} \end{pmatrix} \begin{pmatrix} u \\ d \\ s \end{pmatrix}$$

- The 3x3 unitary matrix depends on 9 complex numbers, i.e. 18 real parameters There are 9 constraints from $\hat{U}^\dagger \hat{U} = 1$

➡ Can form 18 – 9 = 9 linearly independent matrices

These 9 matrices form a U(3) group.

- As before, one matrix is simply the identity multiplied by a complex phase and is of no interest in the context of flavour symmetry
- The remaining 8 matrices have $\det U = 1$ and form an SU(3) group
- The eight matrices (the Hermitian generators) are: $\vec{T} = \frac{1}{2}\vec{\lambda}$ $\hat{U} = e^{i\vec{\alpha}.\vec{T}}$

★In SU(3) flavour, the three quark states are represented by:

$$\boxed{u = \begin{pmatrix}1\\0\\0\end{pmatrix} \quad d = \begin{pmatrix}0\\1\\0\end{pmatrix} \quad s = \begin{pmatrix}0\\0\\1\end{pmatrix}}$$

★In SU(3) uds flavour symmetry contains SU(2) ud flavour symmetry which allows us to write the first three matrices:

$$\lambda_1 = \begin{pmatrix} \sigma_1 & \begin{matrix}0\\0\end{matrix} \\ 0\ 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} \sigma_2 & \begin{matrix}0\\0\end{matrix} \\ 0\ 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} \sigma_3 & \begin{matrix}0\\0\end{matrix} \\ 0\ 0 & 0 \end{pmatrix}$$

i.e. u ↔ d $\quad \lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

- The third component of isospin is now written $\boxed{I_3 = \frac{1}{2}\lambda_3}$

with $\quad I_3 u = +\frac{1}{2}u \quad I_3 d = -\frac{1}{2}d \quad I_3 s = 0$

- I_3 "counts the number of up quarks – number of down quarks in a state
- As before, ladder operators $\quad T_\pm = \frac{1}{2}(\lambda_1 \pm i\lambda_2) \quad d \bullet \xleftrightarrow{T_\pm} \bullet\ u$

- Now consider the matrices corresponding to the u ↔ s and d ↔ s

$\boxed{u \leftrightarrow s}$ $$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$\boxed{d \leftrightarrow s}$ $$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

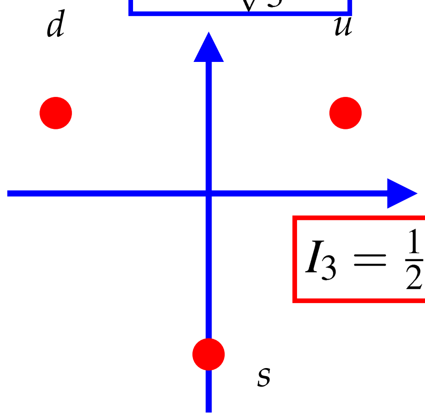
- Hence in addition to $\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ have two other traceless diagonal matrices
- However the three diagonal matrices are not be independent.
- Define the eighth matrix, λ_8 as the linear combination:

$$\lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} + \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

which specifies the "vertical position" in the 2D plane

"Only need two axes (quantum numbers) to specify a state in the 2D plane": (I_3, Y)

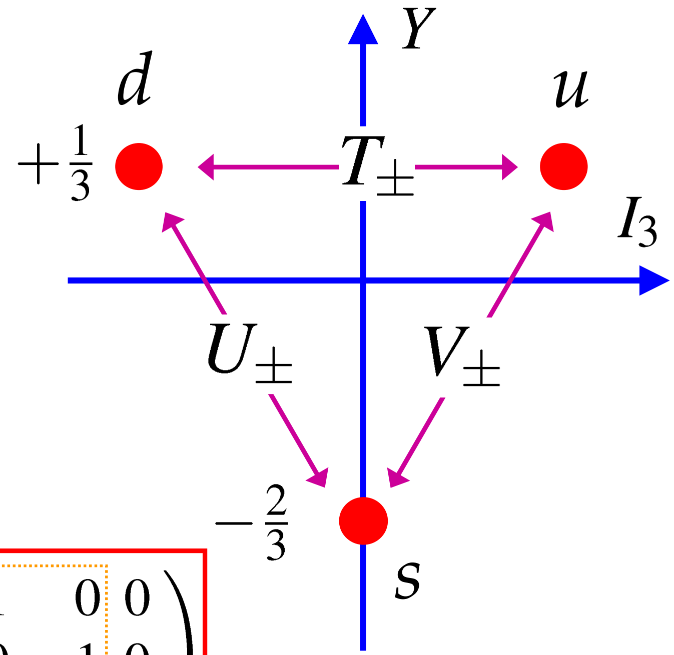
$\boxed{Y = \frac{1}{\sqrt{3}} \lambda_8}$



$\boxed{I_3 = \frac{1}{2} \lambda_3}$

★The other six matrices form six ladder operators which step between the states

$$T_\pm = \frac{1}{2}(\lambda_1 \pm i\lambda_2)$$
$$V_\pm = \frac{1}{2}(\lambda_4 \pm i\lambda_5)$$
$$U_\pm = \frac{1}{2}(\lambda_6 \pm i\lambda_7)$$



with $\quad I_3 = \frac{1}{2}\lambda_3 \quad Y = \frac{1}{\sqrt{3}}\lambda_8$

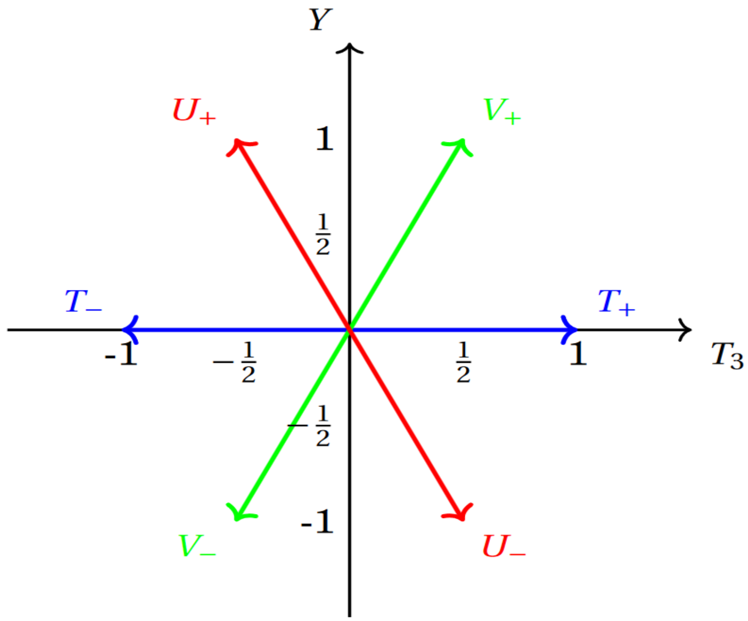
and the eight Gell-Mann matrices

u ↔ d
$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

u ↔ s
$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$$

d ↔ s
$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$\lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

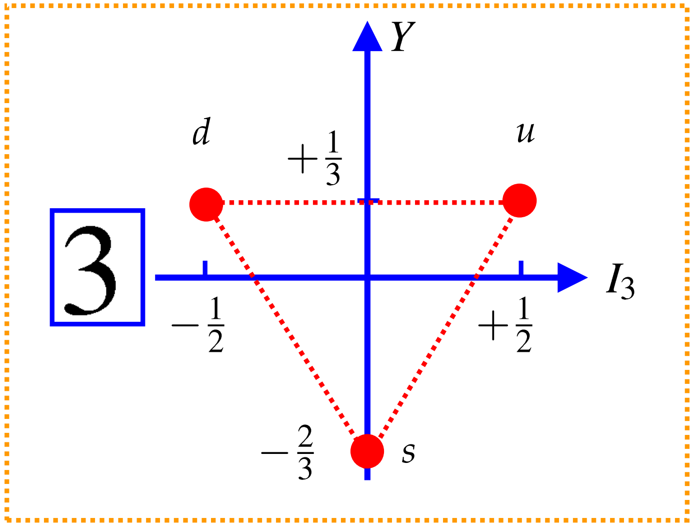


Quark	B	T	T_3	σ	S	Y	Q
u	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{3}$	$\frac{2}{3}$
d	$\frac{1}{3}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{3}$	$-\frac{1}{3}$
s	$\frac{1}{3}$	0	0	$\frac{1}{2}$	-1	$-\frac{2}{3}$	$-\frac{1}{3}$
$\overline{u}$	$-\frac{1}{3}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{3}$	$-\frac{2}{3}$
$\overline{d}$	$-\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{3}$	$\frac{1}{3}$
$\overline{s}$	$-\frac{1}{3}$	0	0	$\frac{1}{2}$	1	$\frac{2}{3}$	$\frac{1}{3}$

$$Q = I_z + \frac{Y}{2} = I_3 + \frac{B+S}{2}$$

$$Q = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{-1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & -\frac{2}{3} \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3}. \end{pmatrix}$$

Quarks and anti-quarks in SU(3) Flavour

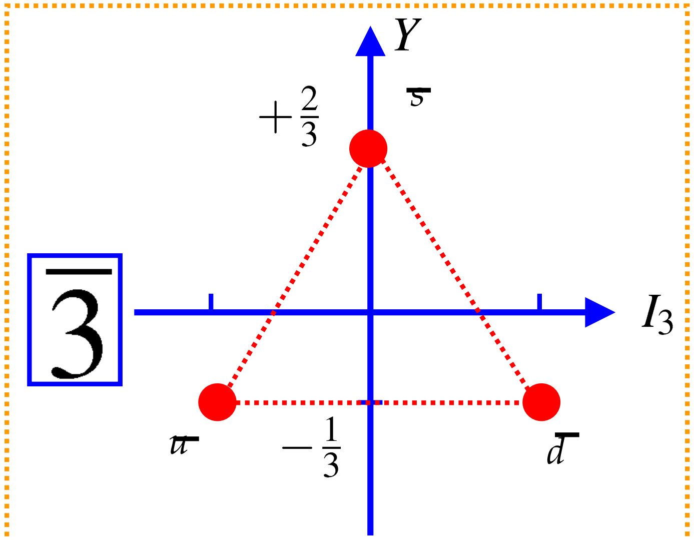


Quarks

$$\boxed{I_3 u = +\tfrac{1}{2}u; \quad I_3 d = -\tfrac{1}{2}d; \quad I_3 s = 0}$$

$$\boxed{Y u = +\tfrac{1}{3}u; \quad Y d = +\tfrac{1}{3}d; \quad Y s = -\tfrac{2}{3}s}$$

- The anti-quarks have opposite SU(3) flavour quantum numbers



Anti-Quarks

$$\boxed{I_3 \bar{u} = -\tfrac{1}{2}\bar{u}; \quad I_3 \bar{d} = +\tfrac{1}{2}\bar{d}; \quad I_3 \bar{s} = 0}$$

$$\boxed{Y \bar{u} = -\tfrac{1}{3}\bar{u}; \quad Y \bar{d} = -\tfrac{1}{3}\bar{d}; \quad Y \bar{s} = +\tfrac{2}{3}\bar{s}}$$

SU(3) Ladder Operators

- SU(3) *uds* flavour symmetry contains *ud*, *us* and *ds* SU(2) symmetries
- Consider the $u \leftrightarrow s$ symmetry "V-spin" which has the associated $s \longrightarrow u$ ladder operator

$$V_+ = \tfrac{1}{2}(\lambda_4 + i\lambda_5) = \tfrac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \tfrac{i}{2}\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with
$$V_+ s = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = +u$$

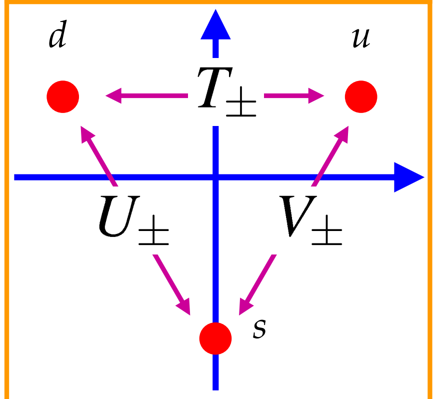
★The effects of the six ladder operators are:

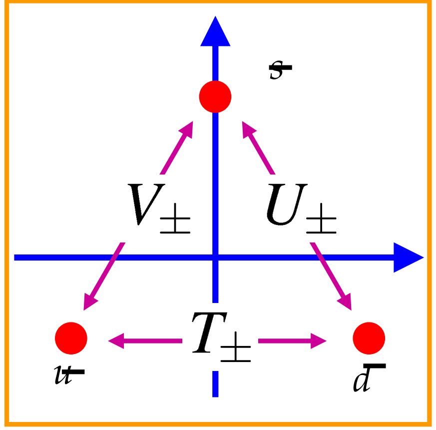
$T_+ d = u; \quad T_- u = d;$	$T_+ \bar{u} = -\bar{d}; \quad T_- \bar{d} = -\bar{u}$
$V_+ s = u; \quad V_- u = s;$	$V_+ \bar{u} = -\bar{s}; \quad V_- \bar{s} = -\bar{u}$
$U_+ s = d; \quad U_- d = s;$	$U_+ \bar{d} = -\bar{s}; \quad U_- \bar{s} = -\bar{d}$

all other combinations give zero

SU(3) LADDER OPERATORS

$$T_\pm = \tfrac{1}{2}(\lambda_1 \pm i\lambda_2)$$
$$V_\pm = \tfrac{1}{2}(\lambda_4 \pm i\lambda_5)$$
$$U_\pm = \tfrac{1}{2}(\lambda_6 \pm i\lambda_7)$$



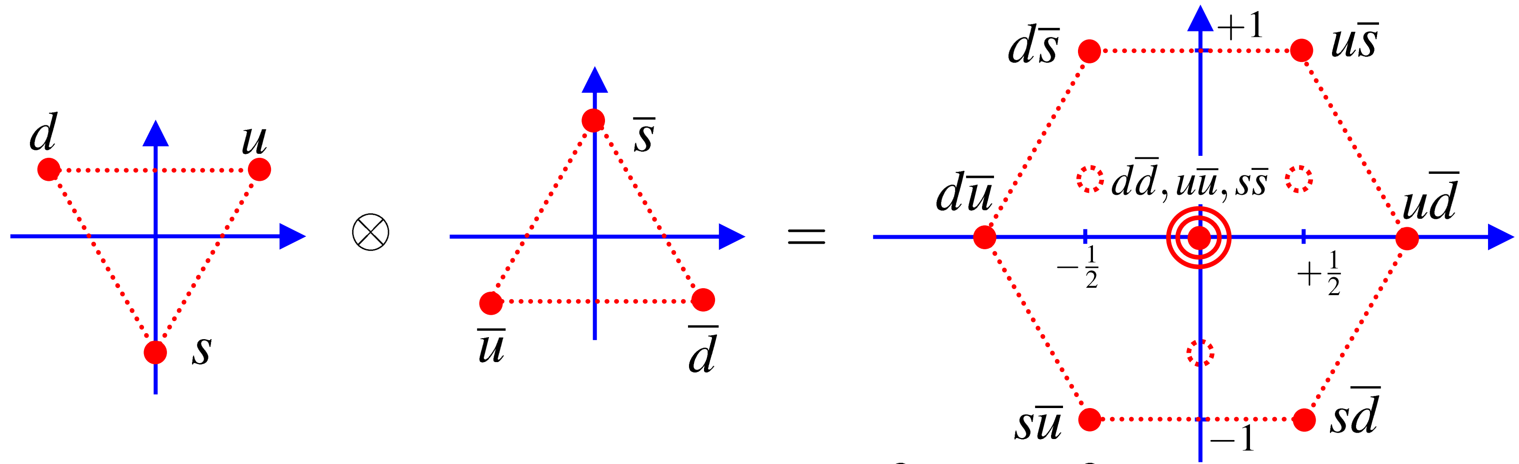


Hadrons

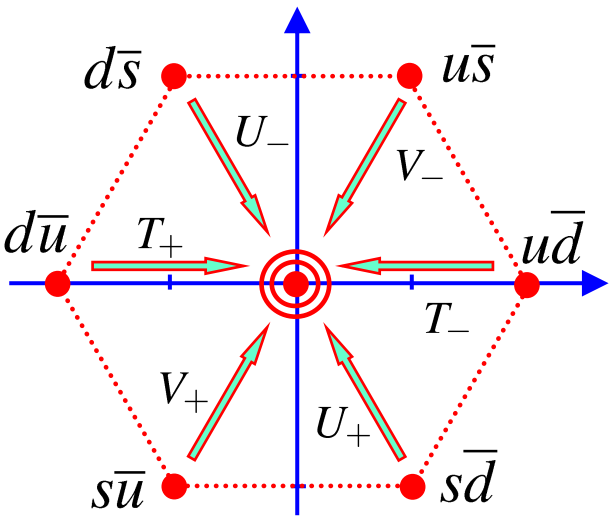
Particle	Mass(MeV)	J^P	Isospin	Strangeness
pseudoscalar Mesons: 8 + 1				
$\pi^{\pm,0}$	140	0^-	1	0
K^+, K^0	495	0^-	1/2	1
$\bar{K}^0, K^-$	495	0^-	1/2	-1
η^0	550	0^-	0	0
η'^0	960	0^-	0	0
vector Mesons: 8 + 1				
$\rho^{\pm,0}$	770	1^-	1	0
K^{*+}, K^{*0}	890	1^-	1/2	1
$\bar{K}^{*0}, K^{*-}$	890	1^-	1/2	-1
ω^0	780	1^-	0	0
ϕ^0	1020	1^-	0	0
spin 1/2 Baryons: 8				
p,n	940	$1/2^+$	1/2	0
Λ^0	1115	$1/2^+$	0	-1
$\Sigma^{\pm,0}$	1190	$1/2^+$	1	-1
$\Xi^{0,-}$	1315	$1/2^+$	1/2	-2
spin 3/2 Baryons: 10				
$\Delta^{++,+,0,-}$	1232	$3/2^+$	3/2	0
$\Sigma^{*\pm,0}$	1385	$3/2^+$	1	-1
$\Xi^{*0,-}$	1523	$3/2^+$	1/2	-2
Ω^-	1672	$3/2^+$	0	-3

Light (uds) Mesons

- Use ladder operators to construct uds mesons from the nine possible st$q\bar{q}$s



- The three central states, all of which have $Y = 0;\ I_3 = 0$n be obtained using the ladder operators and orthogonality. Starting from the outer states can reach the centre in six ways



$$T_+|d\bar{u}\rangle = |u\bar{u}\rangle - |d\bar{d}\rangle \qquad T_-|u\bar{d}\rangle = |d\bar{d}\rangle - |u\bar{u}\rangle$$
$$V_+|s\bar{u}\rangle = |u\bar{u}\rangle - |s\bar{s}\rangle \qquad V_-|u\bar{s}\rangle = |s\bar{s}\rangle - |u\bar{u}\rangle$$
$$U_+|s\bar{d}\rangle = |d\bar{d}\rangle - |s\bar{s}\rangle \qquad U_-|d\bar{s}\rangle = |s\bar{s}\rangle - |d\bar{d}\rangle$$

- Only two of these six states are linearly independent.
- But there are three states with $Y = 0;\ I_3 = 0$
- Therefore one state is not part of the same multiplet, i.e. cannot be reached with ladder ops.

- First form two linearly independent orthogonal states from:

$$\boxed{|u\bar{u}\rangle - |d\bar{d}\rangle} \qquad |u\bar{u}\rangle - |s\bar{s}\rangle \qquad |d\bar{d}\rangle - |s\bar{s}\rangle$$

★ If the SU(3) flavour symmetry were exact, the choice of states wouldn′t matter. However, $m_s > m_{u,d}$ and the symmetry is only approximate.

- Experimentally observe three light mesons with m~140 MeV: $\pi^+,\ \pi^0,\ \pi^-$
- Identify one state (the π^0) with the isospin triplet (derived previously)

$$\boxed{\psi_1 = \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})}$$

- The second state can be obtained by taking the linear combination of the other two states which is orthogonal to the π^0

$$\psi_2 = \alpha(|u\bar{u}\rangle - |s\bar{s}\rangle) + \beta(|d\bar{d}\rangle - |s\bar{s}\rangle)$$

with orthonormality: $\langle\psi_1|\psi_2\rangle = 0; \quad \langle\psi_2|\psi_2\rangle = 1$

$$\Rightarrow \boxed{\psi_2 = \frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s})}$$

- The final state (which is not part of the same multiplet) can be obtained by requiring it to be orthogonal to ψ_1 and ψ_2

$$\Rightarrow \boxed{\psi_3 = \frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s})}$$

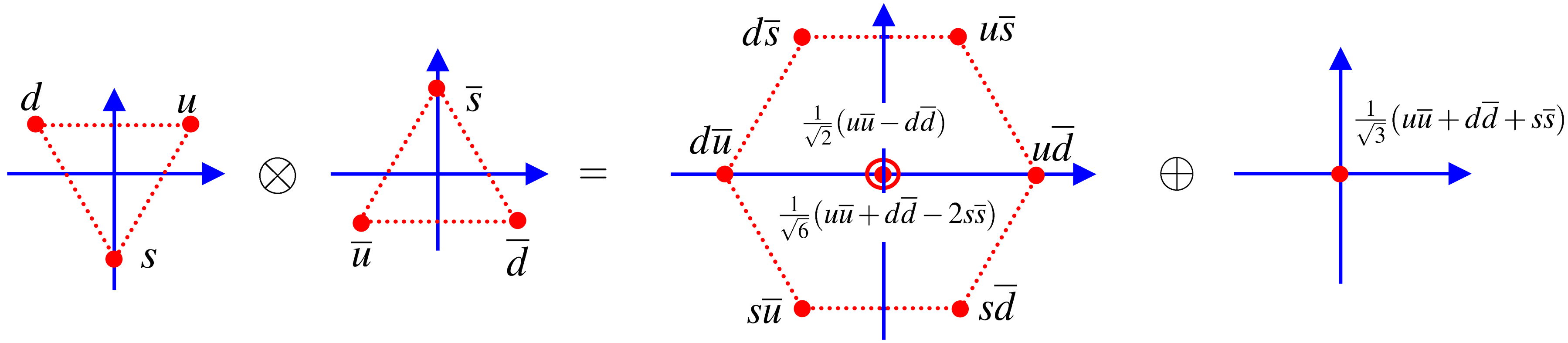
SINGLET

★It is easy to check that ψ_3 is a singlet state using ladder operators

$$T_+\psi_3 = T_-\psi_3 = U_+\psi_3 = U_-\psi_3 = V_+\psi_3 = V_-\psi_3 = 0$$

which confirms that $\psi_3 = \frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s})$ is a "flavourless" singlet

•Therefore the combination of a quark and anti-quark yields nine states which breakdown into an OCTET and a SINGLET



• In the language of group theory: $3 \otimes \bar{3} = 8 \oplus 1$

★ Compare with combination of two spin-half particles $2 \otimes 2 = 3 \oplus 1$

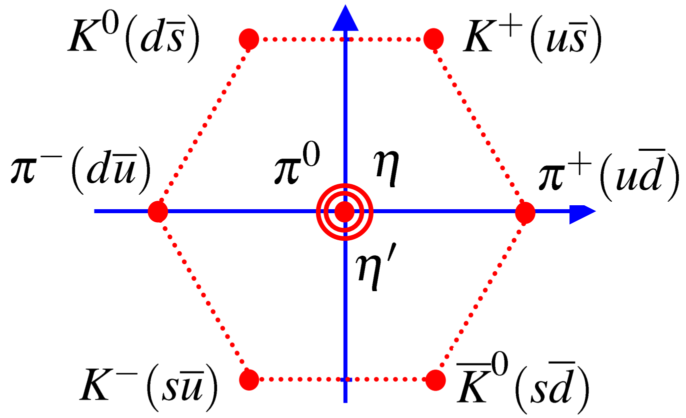
TRIPLET of spin-1 states: $|1,-1\rangle,\ |1,0\rangle,\ |1,+1\rangle$

spin-0 SINGLET: $|0,0\rangle$

•These spin triplet states are connected by ladder operators just as the meson *uds* octet states are connected by SU(3) flavour ladder operators

•The singlet state carries no angular momentum – in this sense the SU(3) flavour singlet is "flavourless"

PSEUDOSCALAR MESONS (L=0, S=0, J=0, P= –1)



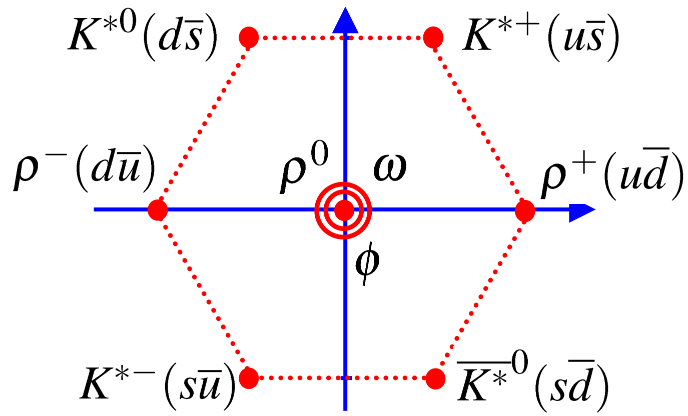
- Because SU(3) flavour is only approximate the physical states with $I_3 = 0,\ Y = 0$ can be mixtures of the octet and singlet states.

Empirically find:

$$\pi^0 = \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})$$
$$\eta \approx \frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s})$$
$$\eta' \approx \frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s})$$

singlet

VECTOR MESONS (L=0, S=1, J=1, P= –1)



- For the vector mesons the physical states are found to be approximately "ideally mixed":

$$\rho^0 = \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})$$
$$\omega \approx \frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d})$$
$$\phi \approx s\bar{s}$$

MASSES

$\pi^\pm$: 140 MeV	π^0 : 135 MeV
$K^\pm$: 494 MeV	$K^0/\overline{K}^0$: 498 MeV
η : 549 MeV	η' : 958 MeV

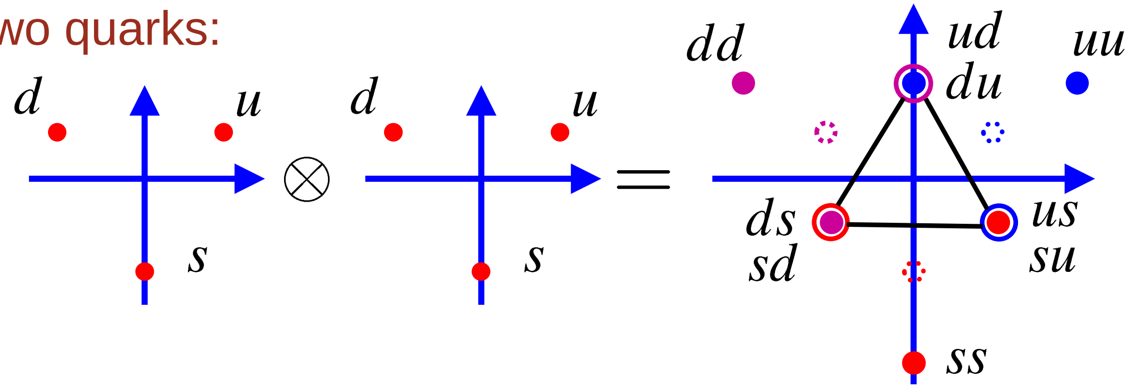
$\rho^\pm$: 770 MeV	ρ^0 : 770 MeV
$K^{*\pm}$: 892 MeV	$K^{*0}/\overline{K^*}^0$: 896 MeV
ω : 782 MeV	ϕ : 1020 MeV

Combining uds Quarks to form Baryons

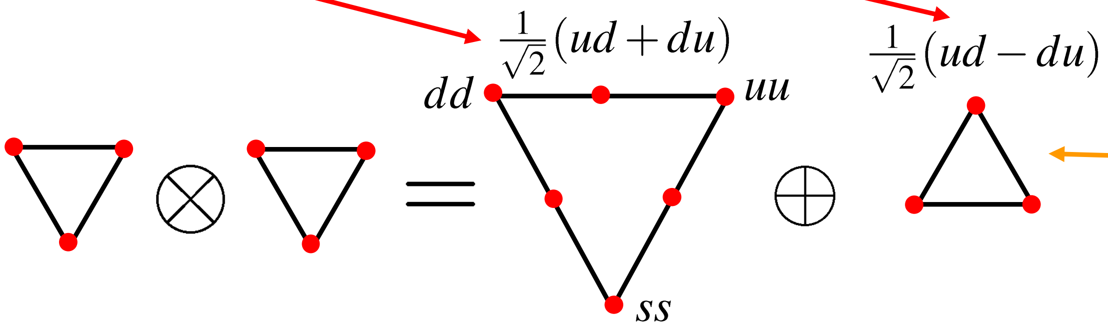
★ Have already seen that constructing Baryon states is a fairly tedious process when we derived the proton wave-function. Concentrate on multiplet structure rather than deriving all the wave-functions.

★ <u>Everything we do here is relevant to the treatment of colour</u>

- First combine two quarks:



★Yields a <u>symmetric</u> sextet and <u>anti-symmetric</u> triplet: $3 \otimes 3 = 6 \oplus \overline{3}$

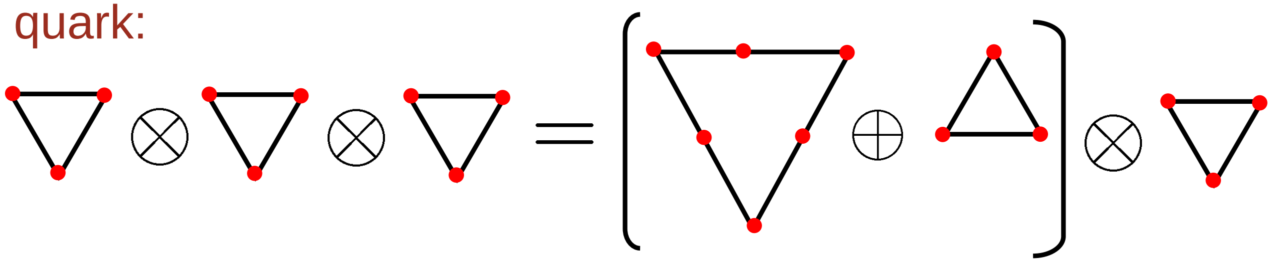


Same "pattern" as the anti-quark representation

SYMMETRIC

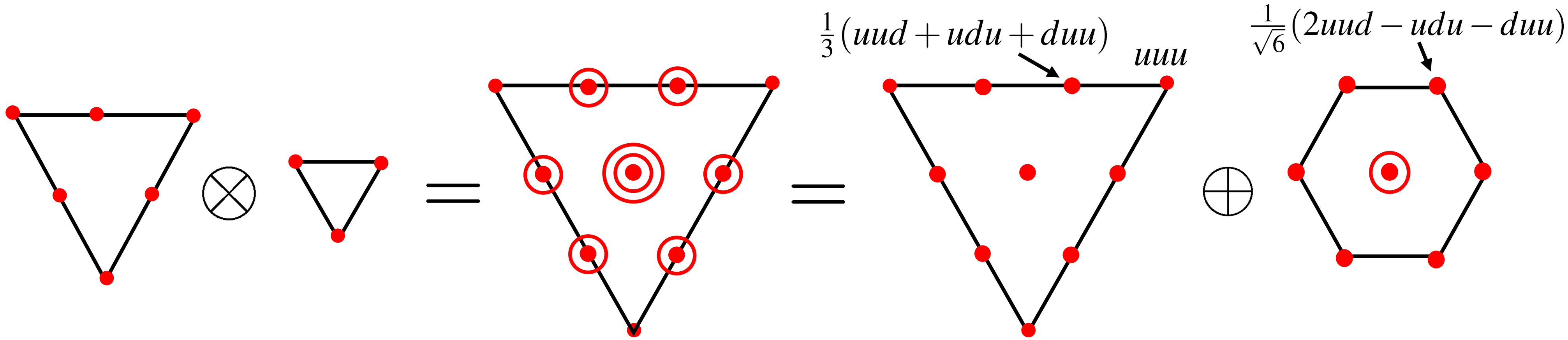
ANTI-SYMMETRIC

•Now add the third quark:



•Best considered in two parts, building on the sextet and triplet. Again concentrate on the multiplet structure (for the wave-functions refer to the discussion of proton wave-function).

❶ Building on the sextet: $3 \otimes 6 = 10 \oplus 8$



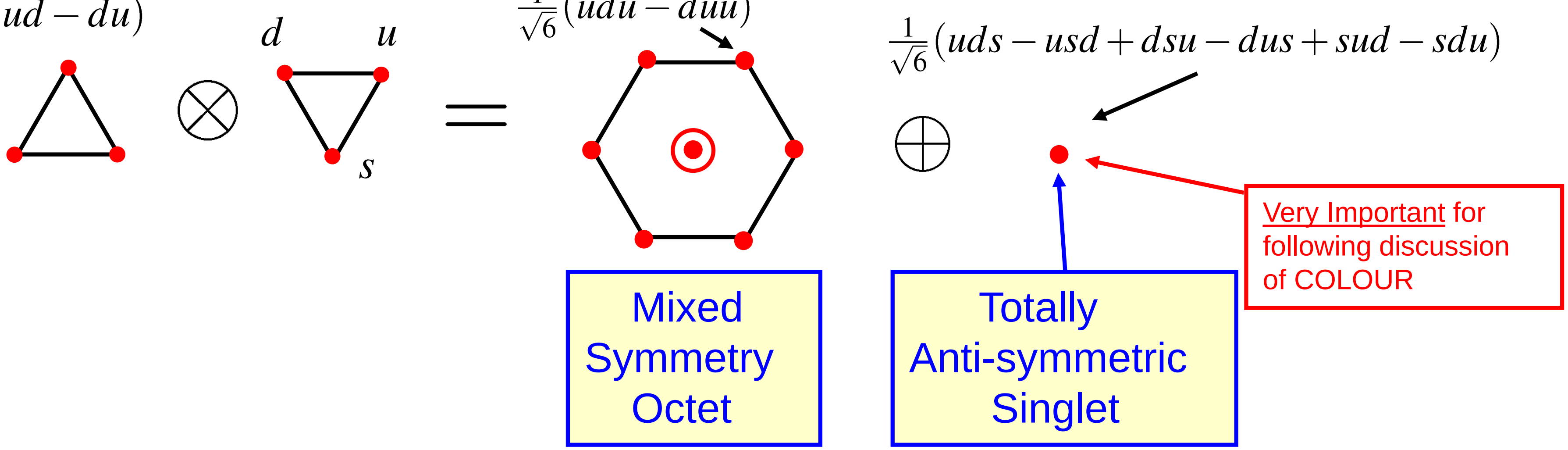
Symmetric Decuplet

Mixed Symmetry Octet

❷ Building on the triplet:

- Just as in the case of uds mesons we are combining $\bar{3} \times 3$ again
  obtain an octet and a singlet

$\frac{1}{\sqrt{2}}(ud - du)$ $\frac{1}{\sqrt{6}}(udu - duu)$ $\frac{1}{\sqrt{6}}(uds - usd + dsu - dus + sud - sdu)$



- Can verify the wave-function $\psi_{\text{singlet}} = \frac{1}{\sqrt{6}}(uds - usd + dsu - dus + sud - sdu)$
  is a singlet by using ladder operators, e.g.

$$T_+ \psi_{\text{singlet}} = \frac{1}{\sqrt{6}}(uus - usu + usu - uus + suu - suu) = 0$$

★ In summary, the combination of three *uds* quarks decomposes into

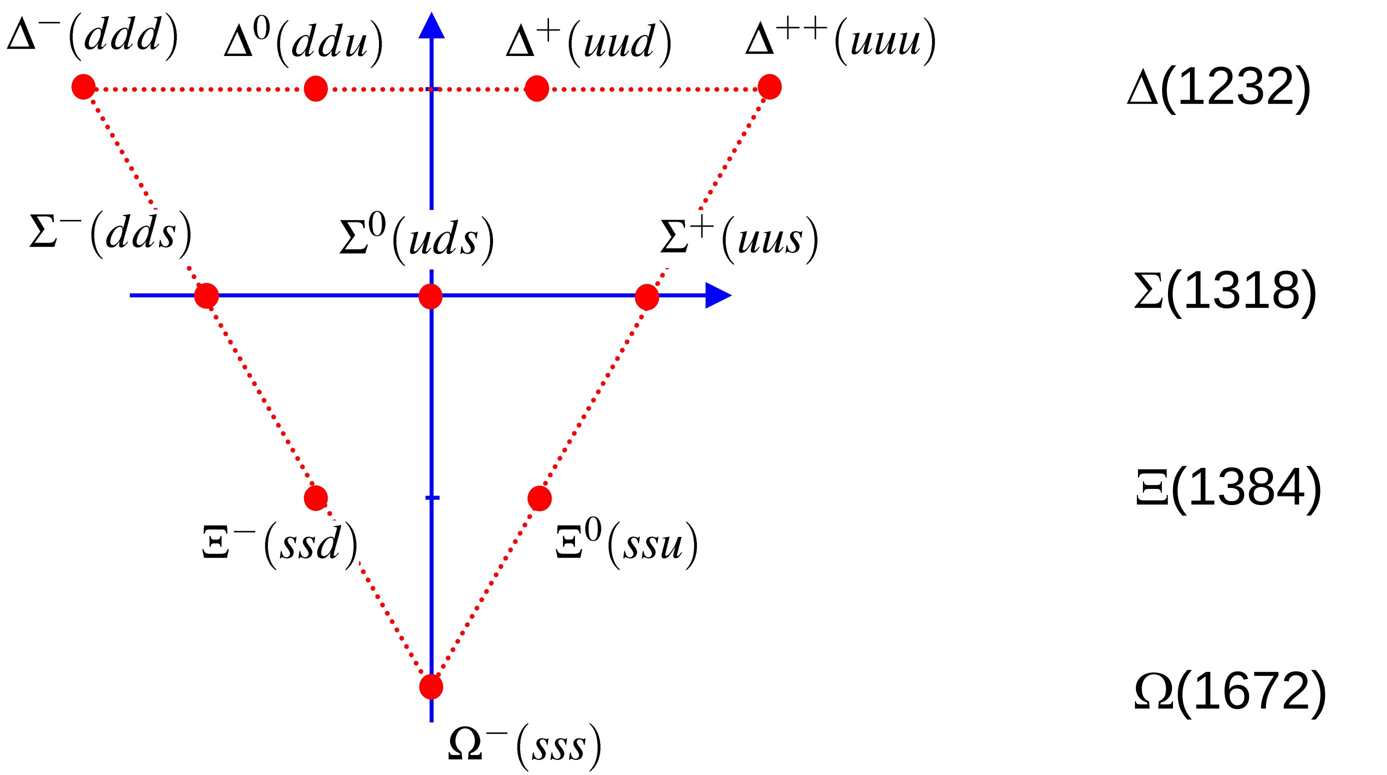
$$3 \otimes 3 \otimes 3 = 3 \otimes (6 \oplus \bar{3}) = 10 \oplus 8 \oplus 8 \oplus 1$$

Baryon Decuplet

★ The baryon states (L=0) are:

- the spin 3/2 decuplet of symmetric flavour and symmetric spin wave-functions $\phi(S)\chi(S)$

<u>BARYON DECUPLET</u> (L=0, S=3/2, J=3/2, P= +1)



★ If SU(3) flavour were an exact symmetry all masses would be the same (broken symmetry)

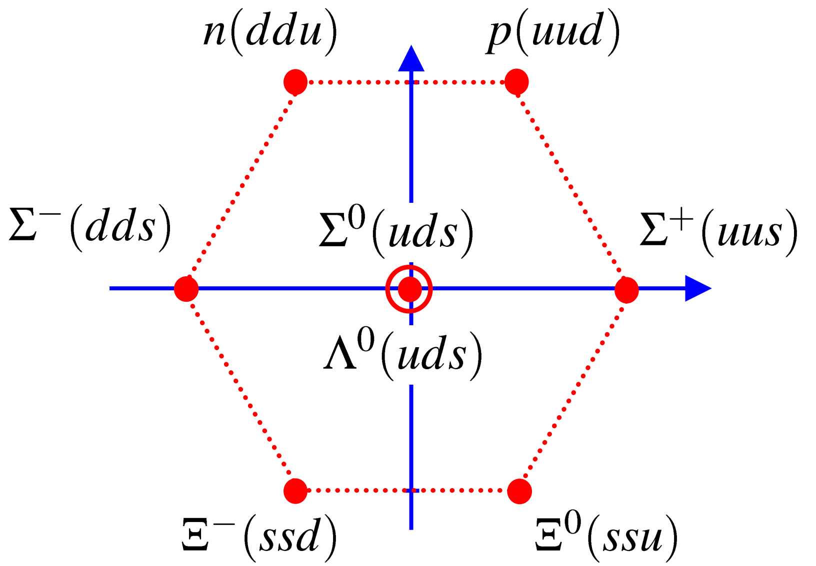
Baryon Octet

★ The spin 1/2 octet is formed from mixed symmetry flavour and mixed symmetry spin wave-functions

$$\alpha\phi(M_S)\chi(M_S)+\beta\phi(M_A)\chi(M_A)$$

See previous discussion proton for how to obtain wave-functions

BARYON OCTET (L=0, S=1/2, J=1/2, P= +1)



Mass in MeV

939

Σ(1193)
Λ(1116)

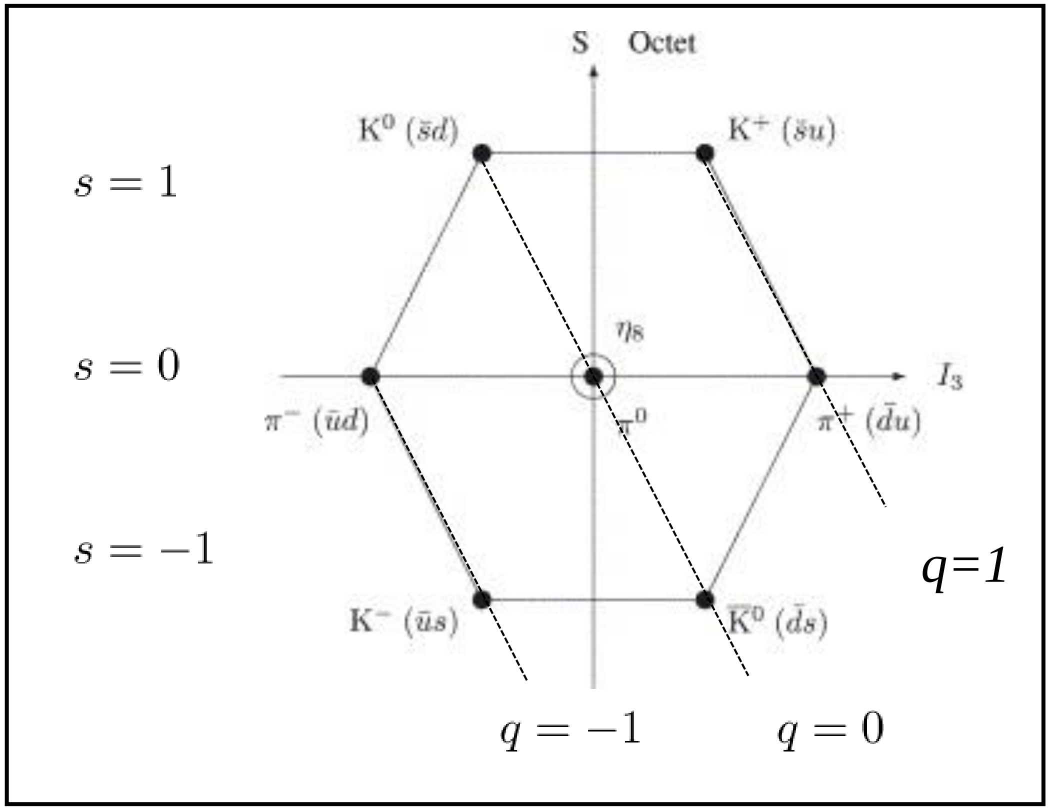
Ξ(1318)

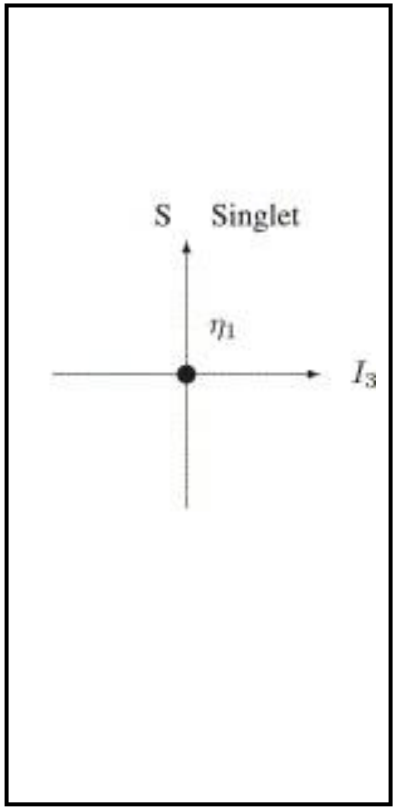
★ NOTE: Cannot form a totally symmetric wave-function based on the anti-symmetric flavour singlet as there no totally anti-symmetric spin wave-function for 3 quarks

Pictorial arrangements

- Mesons:
  - 2 quarks, with 3 possible flavours: u, d, s
  - 3^2 =9 possibilities = 8 + 1

$$\boxed{3 \otimes \overline{3} = 8 \oplus 1}$$





$$3 \otimes \bar{3} = 8 \oplus 1$$

- Mesons:
  - 2 quarks, with 3 possible flavours: u, d, s
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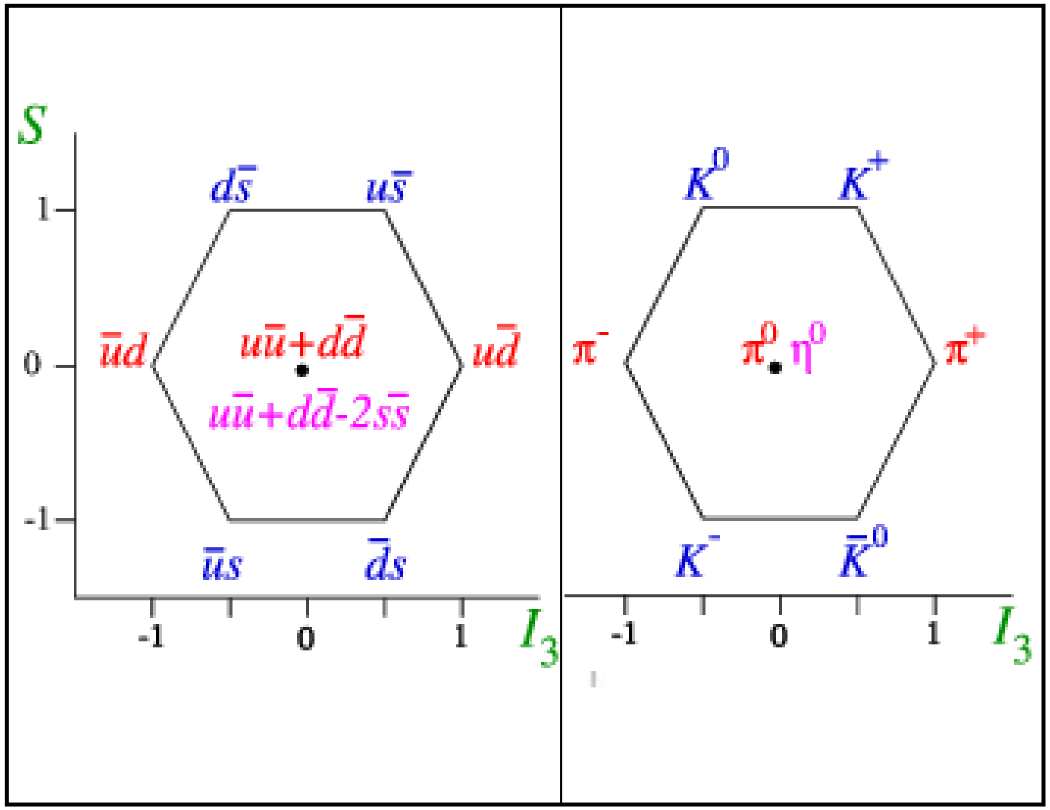
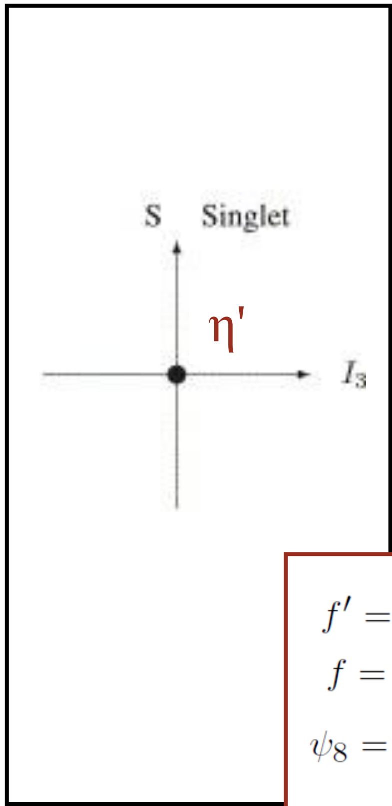


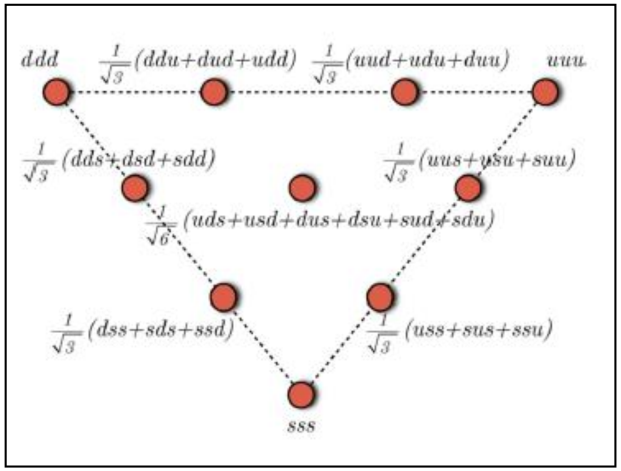


$$f' = \psi_8 \cos\theta - \psi_1 \sin\theta$$
$$f = \psi_8 \sin\theta + \psi_1 \cos\theta$$
$$\psi_8 = \frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s})$$
$$\psi_1 = \frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s})$$

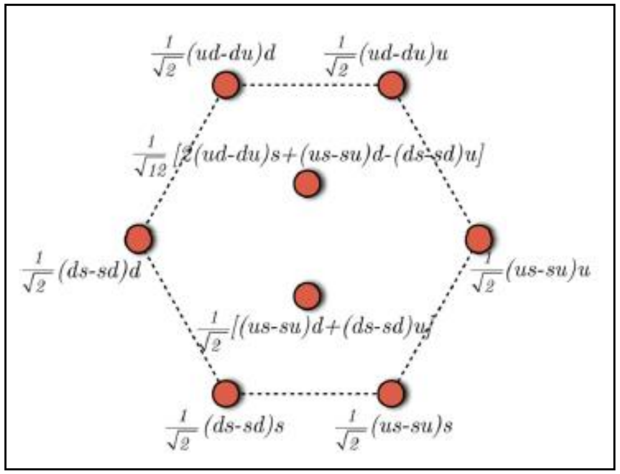
- Baryons:
  - 3 quarks, with 3 possible flavours: u, d, s
  - 3^3 =27 possibilities = 10 + 8 + 8 + 1

$$\boxed{3\otimes 3\otimes 3 = 10_S \oplus 8_M \oplus 8_M \oplus 1_A}$$

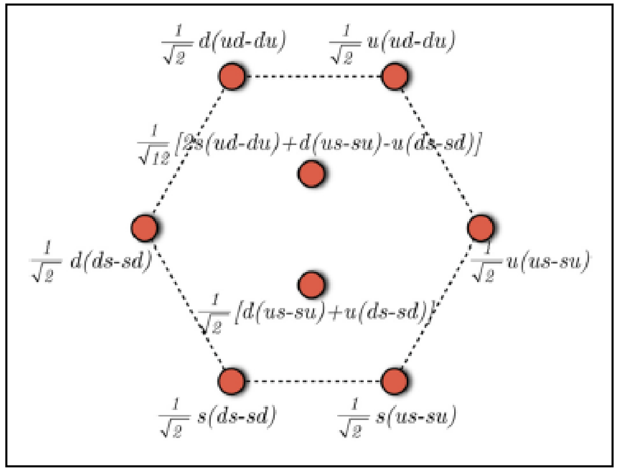
$$\boxed{\psi_{sym}}$$

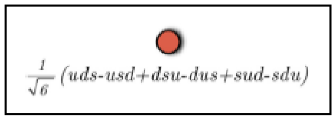


$$\boxed{\psi_{anti-sym}(1\leftrightarrow 2)}$$



$$\boxed{\psi_{anti-sym}(2\leftrightarrow 3)}$$

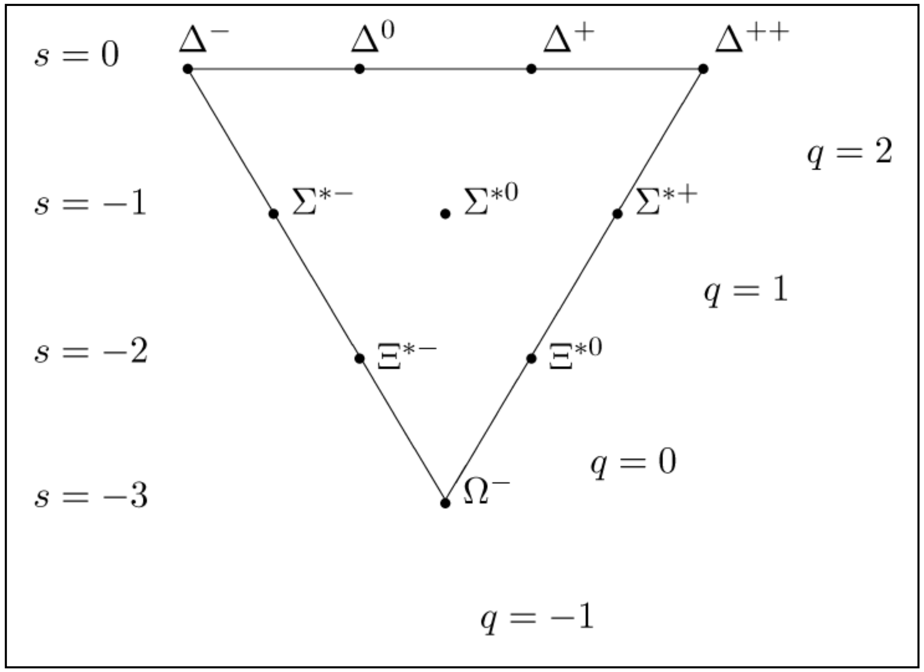


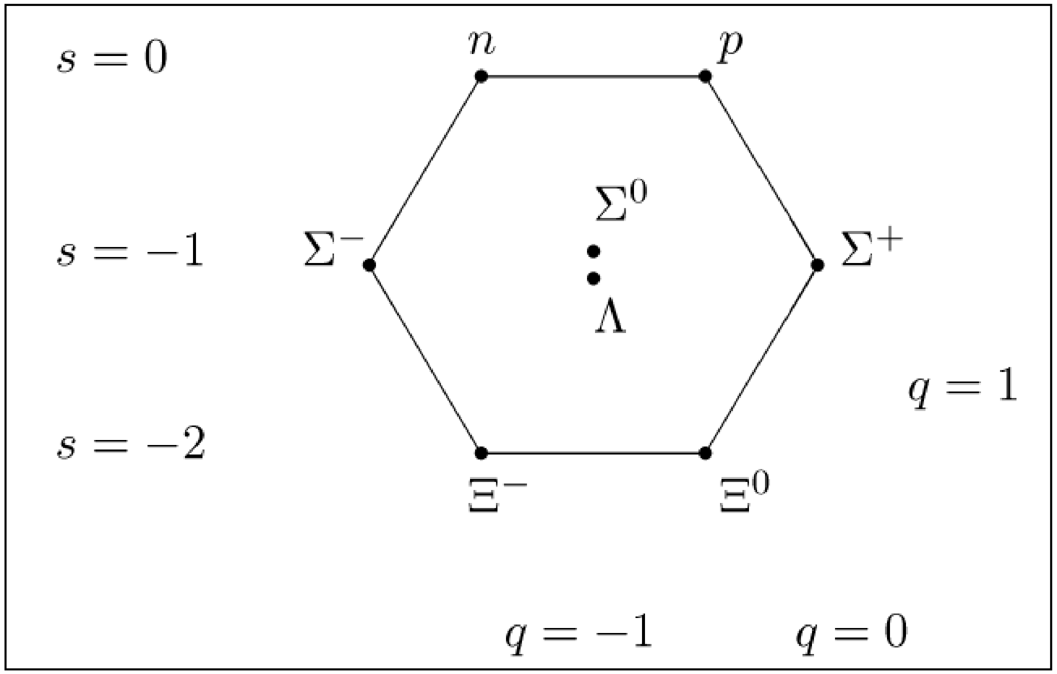


- Baryons:
  - 3 quarks, with 3 possible flavours: u, d, s
  - $3^3 = 27$ possibilities = 10 + 8 + 8 + 1

$$\boxed{3 \otimes 3 \otimes 3 = 10_S \oplus 8_M \oplus 8_M \oplus 1_A}$$

$$\boxed{\psi_{sym}} \qquad \boxed{\psi_{anti-sym}(1 \leftrightarrow 2)} \qquad \boxed{\psi_{anti-sym}(2 \leftrightarrow 3)}$$





What did we learn about quarks

Quarks:

- Associate production, but long lifetime: strangeness
- Many (degenerate) particles: isospin
- Pauli exclusion principle: color

	d	u	s
Q – electric charge	$-\frac{1}{3}$	$+\frac{2}{3}$	$-\frac{1}{3}$
I – isospin	$\frac{1}{2}$	$\frac{1}{2}$	0
I_z – isospin z-component	$-\frac{1}{2}$	$+\frac{1}{2}$	0
S – strangeness	0	0	-1

- How they combine into hadrons: multiplets
- How to add (iso)spin: Clebsch-Gordan