Quiz-3 and HW-5 solans

Quiz - 3 Solution:

Q1 (1) Find the $\limsup a_n$ & $\liminf a_n$, where

$$a_n = \sin\left(\frac{n\pi}{6}\right).$$

Sol: • Consider the subsequence $(a_{12n+3})_{n\geq 1}$:

Then, $\dfrac{(12n+3)\pi}{6} = \dfrac{3(4n+1)\pi}{6} = \dfrac{(4n+1)\pi}{2}$, and so $\sin\left(\dfrac{(12n+3)\pi}{6}\right) = 1$.

• Also consider the subsequence $(a_{12n-3})_{n\geq 1}$:

Then, $\dfrac{(12n-3)\pi}{6} = \dfrac{(4n-1)\pi}{2}$, and so $\sin\left(\dfrac{(12n-3)\pi}{6}\right) = -1$.

• Thus we have found subsequences of (a_n) converging to 1 & -1.

• Finally, note that $-1 \leq a_n \leq 1$ $\forall n$ (since $|\sin x| \leq 1$ $\forall x \in \mathbb{R}$).

Thus, both $\limsup a_n$ & $\liminf a_n \in [-1, 1]$.

• Hence, $\limsup a_n = 1$ & $\liminf a_n = -1$.

$\square$

Q1 (2). Determine if the series $\displaystyle\sum_{n=1}^{\infty} \left[\sin\left(\frac{n\pi}{6}\right)\right]^{2n^2}$ converges.

Sol: • From (1) above we observe that $\exists$ a subseq of

$b_n = \left(\sin\left(\frac{n\pi}{6}\right)\right)^{2n^2}$ that converges to 1.

- Thus, $b_n \not\to 0$ as $n \to \infty$ (since $b_n \to 0$ implies that all subsequences converge to 0).

- Thus, $\sum_{n \geq 1} b_n$ diverges (since $\sum_{n\geq 1} b_n$ conv $\Rightarrow b_n \to 0$ as $n \to \infty$).

$\boxtimes$

HOMEWORK-5 Solutions (Sequence & Series):

§14 :

Q1 (b) $\sum_{n=1}^{\infty} \frac{2^n}{n!}$: By Ratio test,

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2^{n+1}}{(n+1)!} \frac{n!}{2^n}$$

$= \lim_{n\to\infty} \frac{2}{n+1} = 0$. Limit of $\left|\frac{a_{n+1}}{a_n}\right|$ exists

$\Rightarrow \limsup \left|\frac{a_{n+1}}{a_n}\right| = 0 < 1 \Rightarrow$ series converges (to $e^2 - 1$).

(d) $\sum_{n=1}^{\infty} \frac{n!}{n^4+3}$

Sol: Ratio test: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)\cdot(n^4+3)}{(n+1)^4+3}$

$$= \lim_{n\to\infty} \frac{(n+1)\,(1+3/n^4)}{(1+1/n)^4+3/n^4} = \infty.$$

Thus, $\liminf \left|\frac{a_{n+1}}{a_n}\right| = \infty > 1$ giving that the series diverges.

(e) $\sum_{n\geq 1} \frac{\cos^2 n}{n^2}$:

Sol: $|a_n| = \left|\frac{\cos^2 n}{n^2}\right| \leq \frac{1}{n^2} \quad \forall n.$

& so by the Comparison test $\sum_{n\geq 1} |a_n|$ is conv (since $\sum 1/n^2 < \infty$). Thus, $\sum_{n\geq 1} a_n$ converges (absolute conv $\Rightarrow$ conv of series).

(f) $\sum_{n=2}^{\infty} \frac{1}{\log n}$

Sol: Note that $\log n < n \quad \forall n \geq 1$

$$\Rightarrow \frac{1}{\log n} > \frac{1}{n} \quad \forall n \geq 2$$

$$\underset{\substack{\text{Comparison}\\\text{Test}\\\text{with } \sum 1/n}}{\Rightarrow} \sum_{n\geq 2} 1/\log n \text{ diverges.}$$

Q3:

(b) $\sum_{n\geq 1} \frac{2+\cos n}{3^n}$

Sol: $|a_n| = \left|\frac{2+\cos n}{3^n}\right| \underset{\Delta\text{-ineq}}{\leq} \frac{1}{3^n}\cdot(2+|\cos n|)$

$\leq \frac{3}{3^n} = \frac{1}{3^{n-1}} \quad \forall\, n\geq 1$

Thus by Comparison test, $\sum_{n\geq 1} |a_n|$ converges $\left(\sum_{n\geq 0} \frac{1}{3^n} < \infty\right)$

$\Rightarrow \sum_{n\geq 1} a_n$ converges.

(d) $\sum_{n\geq 1} \frac{(50+2/n)}{2^n}$

Sol: $0 \leq a_n \leq \frac{51}{2^n} \underset{\substack{\text{Comparison} \\ \text{Test}}}{\Longrightarrow} \sum_{n\geq 1} a_n$ converges.

with $51 \sum \frac{1}{2^n}$

(e) $\sum_{n\geq 1} \sin\left(\frac{n\pi}{9}\right)$

Sol: Clearly, $\lim_{n\to\infty} a_n \neq 0$, where $a_n = \sin\left(\frac{n\pi}{9}\right)$

(consider the subseq $a_{18n+3} \longrightarrow \sqrt{3}/2$).

Thus the series $\sum \sin(n\pi/9)$ is divergent.

$\square$

Q4 (a) $\displaystyle\sum_{n\geq 2} \frac{1}{(n+(-1)^n)^2}$

Sol: Let $a_n = \dfrac{1}{(n+(-1)^n)^2}$, $n \geq 2$.

- Then $a_n = \begin{cases} \dfrac{1}{(n-1)^2} & \text{if } n \text{ is odd} \\ \dfrac{1}{(n+1)^2} & \text{if } n \text{ is even} \end{cases}$

- Thus, $0 \leq a_n \leq \dfrac{1}{(n-1)^2}$ $\forall n \geq 2$ since $\dfrac{1}{(n+1)^2} < \dfrac{1}{(n-1)^2}$ $\forall n \geq 2$.

- Now, $\displaystyle\sum_{n\geq 2} \frac{1}{(n-1)^2} = \sum_{n\geq 1} \frac{1}{n^2} < \infty$ by p-series theorem.

- Hence, by the comparison test, $\displaystyle\sum_{n\geq 2} a_n$ converges.

$\square$

(b) $\displaystyle\sum_{n\geq 1} \left(\sqrt{n+1} - \sqrt{n}\right)$

<u>Sol:</u> $(0 \leq)\ a_n = \sqrt{n+1} - \sqrt{n} = \dfrac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} \cdot \left(\sqrt{n+1} + \sqrt{n}\right)$

$$= \frac{1}{\sqrt{n+1} + \sqrt{n}} > \frac{1}{2\sqrt{n+1}}.$$

- Now, $\displaystyle\sum_{n\geq 1} \frac{1}{2\sqrt{n+1}} = \frac{1}{2} \sum_{n\geq 2} \frac{1}{\sqrt{n}}$, and

  thus diverges by the p-series theorem $(p = 1/2)$.

- So by the Comparison test, $\sum a_n$ diverges.

(c) $\displaystyle\sum_{n\geq 1} \frac{n!}{n^n}$

<u>Sol:</u> $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!}$

$$= \lim_{n\to\infty} \frac{n^n}{(n+1)^n} = \lim_{n\to\infty} \frac{1}{(1+1/n)^n} = \frac{1}{e} < 1.$$

Thus, by the Ratio test, the series converges.

$\square$

Q5 Suppose $\sum a_n = A$ & $\sum b_n = B$, $A, B \in \mathbb{R}$.

(a) P.T: $\sum (a_n + b_n) = A + B$

Pf: The SOPS $S_n = \sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$.

Now by hypothesis, $\lim_{n \to \infty} \left(\sum_{k=1}^{n} a_k \right) = A$ & $\lim_{n \to \infty} \left(\sum_{k=1}^{n} b_k \right) = B$,

and thus as both limits exist,

$$\sum_{k=1}^{\infty} (a_k + b_k) = \lim_{n \to \infty} S_n = \lim_{n \to \infty} \sum_{k=1}^{n} a_k + \lim_{n \to \infty} \sum_{k=1}^{n} b_k$$

$$= A + B. \quad \blacksquare$$

(b) SOPS $S_n = \sum_{r=1}^{n} k a_r = k \sum_{r=1}^{n} a_r$, and so

$$\lim_{n \to \infty} S_n = \sum_{r=1}^{\infty} k a_r = k \lim_{n \to \infty} \sum_{r=1}^{n} a_r = kA, \forall k \in \mathbb{R}. \quad \blacksquare$$

(c) Discussion:

- Firstly, $\sum a_n$, $\sum b_n$ convergent $\not\Rightarrow$ $\sum a_n b_n$ is convergent.

Eg: $a_n = b_n = \frac{(-1)^n}{\sqrt{n}}$. Then by the alternating series test, $\sum_{n\geq 1} a_n = \sum_{n\geq 1} b_n$ both converge.

However, $\sum_{n\geq 1} a_n b_n = \sum_{n\geq 1} \frac{1}{\sqrt{n}}$ diverges by p-series.

- Suppose now that both $a_n \geq 0$ & $b_n \geq 0 \ \forall n$.

Then, the SOPS for $\sum a_n b_n$ is $S_n = \sum_{k=1}^{n} a_k b_k$

& $0 \leq a_1 b_1 + a_2 b_2 + \dots + a_n b_n \leq (a_1 + a_2 + \dots + a_n) \cdot (b_1 + \dots + b_n)$.

Above inequality gives us :

1. $\lim_{n\to\infty} S_n$ exists since S_n is a monotonically $\uparrow$ seq with upper bound AB.

2. $\lim_{n\to\infty} S_n \leq AB$

3. The inequality above shows us that too many "cross multiplication terms" are missing in $\sum a_n b_n$ for it to

(always) converge to AB.

$\blacksquare$

Q6 (a) P.T.: If (b_n) is bounded & $\sum |a_n|$ converges then $\sum a_n b_n$ converges.

Pf: • (b_n) bdd sequence $\iff \exists\, M > 0$ s.t

$$|b_n| \leq M \quad \forall\, n \geq 1.$$

• Consider the SOPS of $\sum_{n\geq 1} |a_n b_n|$, say

$$S_n = \sum_{k=1}^{n} |a_k| |b_k|$$

Then, $S_n \leq M \sum_{k=1}^{n} |a_k| \leq M \sum_{k=1}^{\infty} |a_k| < \infty$

Thus, S_n is convergent (S_n is $\uparrow$ & bdd above).

Thus, $\sum a_n b_n$ is absolutely conv $\Rightarrow \sum a_n b_n$ is conv.

(Taking $b_n = 1 \;\forall\, n \geq 1$, we get $\underbrace{\sum |a_n| < \infty \Rightarrow \sum a_n}_{\text{Cor 14.7 in book}}$ conv)

$\square$

Q8: $a_n, b_n \geq 0$ & $\sum a_n < \infty$ $\sum b_n < \infty$

$\Rightarrow \sum \sqrt{a_n b_n} < \infty$.

Pf: AM – GM ineq gives:

$$0 \leqslant \sqrt{a_n b_n} \leqslant \frac{a_n + b_n}{2} \leqslant a_n + b_n$$

Thus, by Comparison test, as $\sum (a_n + b_n)$ is also conv,

$\sum \sqrt{a_n b_n}$ is conv. ☐

Q9 Suppose (a_n) & (b_n) are 2 sequences s.t

$S = \{ n \in \mathbb{N} : a_n \neq b_n \}$ is finite.

Show that $\sum a_n < \infty \iff \sum b_n < \infty$.

Pf: $S = \{ n \in \mathbb{N} : a_n \neq b_n \}$ is finite, choose $N \in \mathbb{N}$ s.t $n \in S \Rightarrow n < N$. Thus, $a_m = b_m \ \forall \ m \geqslant N$.

Then for any $n \geqslant N$, letting A_n & B_n be the SOPS for $\sum a_n$ & $\sum b_n$ resp., we see that:

$$A_n = \sum_{k=1}^{N-1} a_k + \sum_{k=N}^{n} a_k = A + \sum_{k=N}^{n} a_k$$

where $A = \sum_{k=1}^{N-1} a_k$ is a constant real number.

Similarly, $$B_n = \sum_{k=1}^{N-1} b_k + \sum_{k=N}^{n} b_k$$

$$= B + \sum_{k=N}^{n} b_k \text{ for constant } B.$$

Now, we observe that:

$$\sum_{k=N}^{n} b_k = \sum_{k=N}^{n} a_k \quad \forall \quad n \geq N$$

Thus, $\sum a_k < \infty \overset{\text{defn}}{\iff} A_n \text{ conv} \iff \sum_{k=N}^{\infty} a_k < \infty$

$\iff \sum_{k=N}^{\infty} b_k < \infty \iff B_n \text{ conv} \overset{\text{defn}}{\iff} \sum b_k < \infty.$

$\boxtimes$

Q13 (b) Prove $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$

Pf: Observe that $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$

Thus, $S_n = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right)$

$$= \left(1 - \cancel{\tfrac{1}{2}}\right) + \left(\cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{3}}\right) + \left(\cancel{\tfrac{1}{3}} - \tfrac{1}{4}\right) + \dots + \left(\cancel{\tfrac{1}{n}} - \tfrac{1}{n+1}\right)$$

$= 1 - \frac{1}{n+1}$, giving us that $\lim_{n \to \infty} S_n = 1$, ie,

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$$

$\square$

(c) Prove $\sum_{n=1}^{\infty} \frac{n-1}{2^{n+1}} = \frac{1}{2}$

Pf: Note that $\frac{k-1}{2^{k+1}} = \frac{k}{2^k} - \frac{k+1}{2^{k+1}}$ &

So $S_n = \sum_{k=1}^{n} \left(\frac{k}{2^k} - \frac{k+1}{2^{k+1}} \right)$

$\underset{\text{and cancel terms}}{\overset{\text{expand}}{=}} \frac{1}{2} - \frac{n+1}{2^{n+1}} \longrightarrow \frac{1}{2}$ ∎

§ 15:

Q1 (a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges by the Alternating series test (since $\left(\frac{1}{n}\right)$ is ↓ & conv to 0).

(b) $\sum_{n=1}^{\infty} \frac{(-1)^n n!}{2^n}$ is divergent since

$a_n = \frac{n!}{2^n} \not\to 0$ as $n \to \infty$

& so $(-1)^n a_n \not\to 0$ as $n \to \infty$

Rmk: You could also use the Ratio test.

Q6 (a) $a_n = 1/n$: $\sum \frac{1}{n}$ div but $\sum \frac{1}{n^2}$ conv.

(b) See the discussion of §14, Q5 (c).

(c) $a_n = \dfrac{(-1)^n}{\sqrt{n}}$: $\sum a_n$ conv by Alt series test

but $\sum a_n^2 = \sum \frac{1}{n}$ div by p-series.