PHY 310 - Mathematical Methods for Physicists I

Odd Term 2019, IISER Mohali

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Homework 1 - Solutions

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1. Find the expression for Legendre polynomial $P_2(x)$ from Rodrigues' formula

$$P_l(x) = \frac{1}{2^l l!}\frac{d^l}{dx^l}\left[(x^2-1)^l\right].$$

Solution:

We have

$$P_2(x) = \frac{1}{2^2 2!}\frac{d^2}{dx^2}\left[(x^2-1)^2\right].$$

Expanding

$$\begin{aligned} P_2(x) &= \frac{1}{2^2 2!}\frac{d^2}{dx^2}\left[x^4+1-2x^2\right] \\ &= \frac{1}{2^2 2!}\frac{d}{dx}\left[4x^3-4x\right] \\ &= \frac{1}{8}\left[12x^2-4\right] \\ &= \frac{3}{2}x^2-\frac{1}{2} \\ &= \frac{1}{2}(3x^2-1). \end{aligned}$$

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2. Obtain the Legendre polynomial $P_2(x)$ directly from Legendre's equation of order 2 by assuming a polynomial of degree 2

$$y(x) = ax^2+bx+c.$$

Solution:

We have Legendre's equation

$$(1-x^2)y'' - 2xy' + l(l+1)y = 0.$$

We have

$$\begin{aligned} y(x) &= ax^2 + bx + c \\ y'(x) &= 2ax + b \\ y''(x) &= 2a \end{aligned}$$

Inserting these in Legendre's equation

$$\begin{aligned} &(1-x^2)(2a) - 2x(2ax+b) + 2(2+1)(ax^2+bx+c) = 0, \\ &2a - 2ax^2 - 4ax^2 + 2bx + 6ax^2 + 6bx + 6c = 0 \end{aligned}$$

Collecting terms with same powers of x

$$(-2a - 4a + 6a)x^2 + (2b + 6b)x + 2a + 6c = 0.$$

This gives the constraints

$$\begin{aligned} (6-6)a &= 0, \\ 8b &= 0, \\ a + 3c &= 0. \end{aligned}$$

Solving for unknowns

$$\begin{aligned} a &= -3c, \\ b &= 0. \end{aligned}$$

Setting $a = 1$ we get the unnormalized solution

$$y(x) = x^2 - \frac{1}{3}.$$

That is

$$P_2(x) = N\left(x^2 - \frac{1}{3}\right),$$

with N a normalization constant.

Upon using

$$\int_{-1}^{1} dx\, [P_l(x)]^2 = \frac{2}{2l+1},$$

we have

$$N^2 \int_{-1}^{1} \left(x^2 - \frac{1}{3}\right)^2 = \frac{2}{5}.$$

This gives

$$\begin{aligned}
&N^2 \frac{5}{2} \int_{-1}^{1} dx \left(x^4 + \frac{1}{9} - \frac{2}{3}x^2\right) = 1 \\
&N^2 \frac{5}{2} \left(\frac{2}{5} + \frac{2}{9} - \frac{4}{9}\right) = 1, \\
&\frac{4}{9}N^2 = 1 \to N = \frac{3}{2}.
\end{aligned}$$

This gives

$$\begin{aligned}
P_2(x) &= \frac{3}{2}\left(x^2 - \frac{1}{3}\right) \\
&= \frac{1}{2}(3x^2 - 1)
\end{aligned}$$

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3. Obtain the Legendre polynomial $P_4(x)$ by application of the recurrence formula

$$lP_l(x) = (2l-1)xP_{l-1}(x) - (l-1)P_{l-2}(x).$$

Solution:

We have

$$P_0(x) = 1, \quad P_1(x) = x.$$

The recurrence formula for $l = 2$ gives

$$2P_2(x) = 3xP_1(x) - P_0(x).$$

That is

$$P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}.$$

The recurrence relation for $l = 3$ gives

$$3P_3(x) = 5xP_2(x) - 2P_1(x).$$

That is

$$\begin{aligned} P_3(x) &= \frac{5}{3}x\left(\frac{3}{2}x^2 - \frac{1}{2}\right) - \frac{2}{3}x \\ &= \frac{5}{2}x^3 - \frac{3}{2}x \\ &= \frac{1}{2}\left(5x^3 - 3x\right) \end{aligned}$$

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4. Find the first three coefficients in the expansion of the function

$$f(x) = \begin{cases} 0 & -1 \le x \le 0 \\ 1 & 0 \le x \le 1 \end{cases}$$

in a series of Legendre polynomials $P_l(x)$ over the interval $(-1, 1)$.

Solution:

We have

$$f(x) \simeq c_0P_0(x) + c_1P_1(x) + c_2P_2(x).$$

The coefficients are

$$\begin{aligned} c_l &= \frac{2l+1}{2}\int_{-1}^{1} dx\ f(x)P_l(x) \\ &= \frac{2l+1}{2}\int_{-1}^{0} dx\ f(x)P_l(x) + \frac{2l+1}{2}\int_{0}^{1} dx f(x)P_l(x) \\ &= \frac{2l+1}{2}\int_{0}^{1} dx\ xP_l(x). \end{aligned}$$

$$c_0 = \frac{1}{2}\int_0^1 dx\ xP_0(x) = \frac{1}{2}\int_0^1 dx = \frac{1}{2}.$$

$$c_1 = \frac{3}{2}\int_0^1 dx\ xP_1(x) = \frac{3}{2}\int_0^1 dx\ x = \frac{3}{2}\cdot\frac{1}{2} = \frac{3}{4}.$$

$$\begin{aligned} c_2 &= \frac{5}{2}\int_0^1 dx\ xP_2(x) \\ &= \frac{5}{2}\int_0^1 dx\ xP_2(x) \\ &= \frac{5}{2}\int_0^1 dx\ \frac{1}{2}(3x^3 - x) \\ &= 0. \end{aligned}$$

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5. Find the first three coefficients in the expansion of the function

$$f(\theta) = \begin{cases} \cos\theta & 0 \le \theta \le \pi/2 \\ 0 & \pi/2 \le \theta \le \pi \end{cases}$$

in a series of the form

$$f(\theta) = \sum_{n=0}^{\infty} a_l P_l(\cos\theta), \quad 0 \le \theta \le \pi.$$

Solution:

We have

$$a_l = \frac{2l+1}{2}\int_0^{\pi} d\theta \sin\theta f(\theta) P_l(\cos\theta), \quad l = 0, 1, 2, \cdots$$

In our case

$$a_l = \frac{2l+1}{2}\int_0^{\frac{\pi}{2}} d\theta \sin\theta f(\theta) P_l(\cos\theta), \quad l = 0, 1, 2.$$

$$\begin{aligned}
a_0 &= \frac{1}{2}\int_0^{\frac{\pi}{2}} d\theta \sin\theta \cos\theta P_0(\cos\theta) \\
&= \frac{1}{2}\int_0^{\frac{\pi}{2}} d\theta \sin\theta \cos\theta \\
&= -\frac{1}{2}\int_1^{0} d(\cos\theta)\cos\theta = \frac{1}{2}\int_0^1 dx\ x = \frac{1}{4}
\end{aligned}$$

$$\begin{aligned}
a_1 &= \frac{3}{2}\int_0^{\frac{\pi}{2}} d\theta \sin\theta \cos\theta P_1(\cos\theta) \\
&= \frac{3}{2}\int_0^{\frac{\pi}{2}} d\theta \sin\theta \cos^2\theta \\
&= -\frac{3}{2}\int_1^{0} d(\cos\theta)\cos^2\theta \\
&= \frac{3}{2}\int_0^1 dx\ x^2 = \frac{1}{2}.
\end{aligned}$$

$$\begin{aligned}
a_2 &= \frac{5}{2}\int_0^{\frac{\pi}{2}} d\theta \sin\theta \cos\theta P_2(\cos\theta) \\
&= \frac{5}{2}\int_0^{\frac{\pi}{2}} d\theta \sin\theta\ \frac{1}{4}(3\cos 2\theta + 1) \\
&= -\frac{5}{2}\int_1^{0} d(\cos\theta)\ \frac{1}{4}(6\cos^2\theta) \\
&= \frac{5}{2}\int_0^1 dx\ \frac{3}{2}x^2 = \frac{5}{4}.
\end{aligned}$$

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6. Obtain the associated Legendre functions $P_1^1(x)$, $P_1^2(x)$ and $P_1^{-1}(x)$.

Solution:

We have

$$P_l^m(x) = (1-x^2)^{m/2}\frac{d^m}{dx^m}P_l(x). \tag{1}$$

$$\begin{aligned} P_1^1(x) &= (1-x^2)^{1/2}\frac{d}{dx}P_1(x), \\ &= (1-x^2)^{1/2}\frac{d}{dx}x, \\ &= (1-x^2)^{1/2}. \end{aligned}$$

$$\begin{aligned} P_1^2(x) &= (1-x^2)\frac{d^2}{dx^2}P_1(x), \\ &= (1-x^2)\frac{d^2}{dx^2}x, \\ &= 0. \end{aligned}$$

We have

$$P_l^{-m}(x) = (-1)^m\frac{(l-m)!}{(l+m)!}P_l^m(x).$$

Thus,

$$\begin{aligned} P_1^{-1}(x) &= (-1)^1\frac{(1-1)!}{(1+1)!}P_1^1(x) \\ &= -\frac{1}{2}(1-x^2)^{1/2} \end{aligned}$$

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7. Verify that the Legendre polynomials $P_1(x)$ and $P_2(x)$ are solutions of Legendre's equation for $l = 1, 2$.

Solution:

We have

$$\begin{aligned} P_1 &= x, \\ P_2 &= \frac{1}{2}(3x^2 - 1) \end{aligned}$$

Legendre's equation is

$$(1-x^2)\frac{d^2}{dx^2}P_l - 2x\frac{d}{dx}P_l + l(l+1)P_l = 0.$$

When $l = 1$ we have

$$\begin{aligned} &(1-x^2)\frac{d^2}{dx^2}P_1 - 2x\frac{d}{dx}P_1 + 2P_1 \\ &= (1-x^2)\frac{d^2}{dx^2}x - 2x\frac{d}{dx}x + 2x \\ &= 0 - 2x + 2x = 0. \end{aligned}$$

Thus $P_1(x)$ satisfies Legendre's equation.

When $l = 2$ we have

$$y \equiv P_2 = \frac{1}{2}(3x^2 - 1).$$

$$\begin{aligned} y' &= 3x \\ y'' &= 3. \end{aligned}$$

Plugging in y for Legendre's equation for $l = 2$

$$\begin{aligned} &(1-x^2)y'' - 2xy' + 6y \\ &= (1-x^2)3 - 2x3x + 6\frac{1}{2}(3x^2 - 1) \\ &= 3 - 3x^2 - 6x^2 + 9x^2 - 3 = 0. \end{aligned}$$

Thus $P_2(x)$ satisfies Legendre's equation.

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8. Verify that the associated Legendre polynomial P_1^1 is a solution of associated Legendre's equation for $l = 1$ and $m = 1$.

Solution:

We have

$$P_l^m = (1-x^2)^{m/2}\frac{d^m}{dx^m}P_l(x).$$

When $l = 1$ and $m = 1$, we have

$$\begin{aligned} P_1^1 &= (1-x^2)^{1/2}\frac{d}{dx}P_1(x) \\ &= (1-x^2)^{1/2}\frac{d}{dx}x \\ &= \sqrt{(1-x^2)}. \end{aligned}$$

The associated Legendre's equation is

$$(1-x^2)\frac{d^2}{dx^2}P_l^m - 2x\frac{d}{dx}P_l^m + \left[l(l+1) - \frac{m^2}{(1-x^2)}\right]P_l^m = 0.$$

When $l = 1$ and $m = 1$ it becomes

$$(1-x^2)\frac{d^2}{dx^2}P_1^1 - 2x\frac{d}{dx}P_1^1 + \left[2 - \frac{1}{(1-x^2)}\right]P_1^1 = 0.$$

Let us denote $\alpha = (1-x^2)$. Then

$$y = P_1^1 = \alpha^{\frac{1}{2}}$$

$$y' = \frac{1}{2}\alpha^{-\frac{1}{2}} \cdot -2x = -x\alpha^{-\frac{1}{2}}$$

$$y'' = -\alpha^{-\frac{1}{2}} + x\frac{1}{2}\alpha^{-\frac{3}{2}} \cdot -2x = -\alpha^{-\frac{1}{2}} - x^2\frac{1}{2}\alpha^{-\frac{3}{2}}$$

Plugging in the solutions in the associated Legendre's equation is

$$
\begin{aligned}
&(1-x^2)\left(-\alpha^{-\frac{1}{2}} - x^2\alpha^{-\frac{3}{2}}\right) - 2x\left(-x\alpha^{-\frac{1}{2}}\right) + \left[2 - \frac{1}{\alpha}\right]\alpha^{\frac{1}{2}} \\
&= -\alpha^{\frac{1}{2}} - x^2\alpha^{-\frac{1}{2}} + 2x^2\alpha^{-\frac{1}{2}} + 2\alpha^{\frac{1}{2}} - \alpha^{-\frac{1}{2}} \\
&= \alpha^{\frac{1}{2}}(-1+2) + \alpha^{-\frac{1}{2}}(-x^2+2x^2-1) = \alpha^{\frac{1}{2}} + \alpha^{-\frac{1}{2}}(x^2-1) = \alpha^{\frac{1}{2}} - \alpha^{\frac{1}{2}} = 0.
\end{aligned}
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Thus P_1^1 is a solution of associated Legendre's equation for $l=1$ and $m=1$.

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9. Verify that the associated Legendre polynomial P_2^2 is a solution of associated Legendre's equation for $l=2$ and $m=2$.

Solution:

We have

$$P_l^m = (1-x^2)^{m/2}\frac{d^m}{dx^m}P_l(x).$$

When $l=2$ and $m=2$, we have

$$
\begin{aligned}
P_2^2 &= (1-x^2)\frac{d}{dx}P_2(x) \\
&= (1-x^2)\frac{d}{dx}3x \\
&= 3(1-x^2).
\end{aligned}
$$

The associated Legendre's equation is

$$(1-x^2)\frac{d^2}{dx^2}P_l^m - 2x\frac{d}{dx}P_l^m + \left[l(l+1) - \frac{m^2}{(1-x^2)}\right]P_l^m = 0.$$

When $l=2$ and $m=2$ it becomes

$$(1-x^2)\frac{d^2}{dx^2}P_2^2 - 2x\frac{d}{dx}P_2^2 + \left[6 - \frac{4}{(1-x^2)}\right]P_1^1 = 0.$$

Let us denote $\alpha = (1-x^2)$. Then

$$y = P_2^2 = 3\alpha$$

$$y' = -6x$$

$$y'' = 6$$

Plugging in the solutions in the associated Legendre's equation is

$$\begin{aligned}&(1-x^2)(-6) - 2x(-6x) + \left[6 - \frac{4}{\alpha}\right]3\alpha \\ &= -6\alpha + 12x^2 + 18\alpha - 12 \\ &= 12\alpha + 12x^2 - 12 = 12(1-x^2) + 12(x^2-1) = 0.\end{aligned}$$

Thus P_2^2 is a solution of associated Legendre's equation for $l = 2$ and $m = 2$.

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10. Express x, x^2, x^3, x^4 using the set of Legendre polynomials

$$\{P_0(x), P_1(x), P_2(x), P_3(x), P_4(x)\}.$$

Solution:

We have the general form of the Legendre polynomials

$$P_l(x) = \sum_{n=0}^{N}(-1)^n \frac{(2l-2n)!}{2^l n!(l-n)!(l-2n)!}x^{l-2n},$$

where $N = l/2$ for l even and $(l-1)/2$ for l odd.

This gives

$$\begin{aligned} P_0(x) &= 1, \\ P_1(x) &= x, \\ P_2(x) &= \frac{1}{2}(3x^2 - 1), \\ P_3(x) &= \frac{1}{2}(5x^3 - 3x), \\ P_4(x) &= \frac{1}{8}(35x^4 - 3 - x^2 + 3). \end{aligned}$$

From the above we can write the following linear combinations

$$\begin{aligned} x &= P_1, \\ x^2 &= \frac{1}{3}(P_0 + 2P_2), \\ x^3 &= \frac{1}{5}(3P_1 + 2P_3), \\ x^4 &= \frac{1}{35}(7P_0 + 20P_2 + 8P_4). \end{aligned}$$

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11 Express the following function using Legendre polynomials

$$f(x) = \sigma + \omega x^2 - \lambda x^4,$$

where σ, ω and λ are constants.

Solution:

We have

$$\begin{aligned} 1 &= P_0(x), \\ x^2 &= \frac{1}{3}\left(P_0(x) + 2P_2(x)\right), \\ x^4 &= \frac{1}{35}\left(7P_0(x) + 20P_2(x) + 8P_4(x)\right). \end{aligned}$$

Thus the function is

$$\begin{aligned}
f(x) &= \sigma + \omega x^2 - \lambda x^4, \\
&= \sigma P_0(x) + \omega \frac{1}{3}\left(P_0(x) + 2P_2(x)\right) - \lambda \frac{1}{35}\left(7P_0(x) + 20P_2(x) + 8P_4(x)\right), \\
&= \left(\sigma + \frac{1}{3}\omega - \frac{1}{5}\lambda\right) P_0(x) + \left(\frac{2}{3}\omega - \frac{4}{7}\lambda\right) P_2(x) - \frac{8}{35}\lambda P_4(x).
\end{aligned}$$

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12. Express the function

$$f(x) = 30x^2 - 6$$

using Legendre polynomials. Use the method of solving algebraic equations to get the solution.

Solution:

We have

$$\begin{aligned}
f(x) &= 30x^2 - 6 \\
&= 30\ \frac{1}{3}(P_0(x) + 2P_2(x)) - 6P_0(x) \\
&= 10P_0(x) + 20P_2(x)) - 6P_0(x) \\
&= 4P_0(x) + 20P_2(x)).
\end{aligned}$$

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13. Express the function

$$f(x) = 30x^2 - 6$$

using Legendre polynomials. Use the orthogonality integral for Legendre polynomials to get the solution.

Solution:

We have

$$f(x) = \sum_{l=0}^{\infty} a_l P_l(x).$$

From the orthogonality integral for Legendre polynomials we have the coefficients

$$a_l = \frac{2l+1}{2} \int_{-1}^{+1} dx f(x) P_l(x). \tag{2}$$

$$\begin{aligned} a_0 &= \frac{1}{2} \int_{-1}^{+1} dx f(x) P_0(x) \\ &= \frac{1}{2} \int_{-1}^{+1} dx (30x^2 - 6) P_0(x) \\ &= \frac{1}{2} \int_{-1}^{+1} dx (30x^2 - 6) \\ &= \frac{1}{2}(30 \times \frac{2}{3} - 6 \times 2) \\ &= \frac{1}{2}(20 - 12) = 4. \end{aligned}$$

$$\begin{aligned} a_1 &= \frac{3}{2} \int_{-1}^{+1} dx f(x) P_1(x) \\ &= \frac{3}{2} \int_{-1}^{+1} dx (30x^2 - 6) x \\ &= 0 \end{aligned}$$

since the integrand is odd.

$$
\begin{aligned}
a_2 &= \frac{5}{2}\int_{-1}^{+1} dx f(x) P_2(x) \\
&= \frac{5}{2}\int_{-1}^{+1} dx (30x^2-6) P_2(x) \\
&= 5\int_{-1}^{+1} dx (15x^2-3)\frac{1}{2}(3x^2-1) \\
&= 5\times\frac{1}{2}\int_{-1}^{+1} dx (45x^4-24x^2+3) \\
&= 5\times\frac{1}{2}(18-16+6) \\
&= 20
\end{aligned}
$$

Thus

$$
\begin{aligned}
f(x) &= \sum_{l=0}^{\infty} a_l P_l(x) \\
&= a_0 P_0(x) + a_2 P_2(x) \\
&= 4P_0(x) + 20P_2(x).
\end{aligned}
$$

14. Obtain the first two Legendre coefficients of

$$f(x) = Ae^{-mx}.$$

Solution:

We have

$$a_0 = \frac{1}{2}\int_{-1}^{+1} dx\ Ae^{-mx} = -\frac{A}{2m}(e^{-m}-e^{m}) = A\frac{\sinh m}{m}.$$

$$a_1 = \frac{3}{2}\int_{-1}^{+1} dx\ xAe^{-mx} = -3A\left(\frac{\cosh m}{m} - \frac{\sinh m}{m^2}\right).$$

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15. Compute

$$\int_{-1}^{+1} dx\ P_1^1(x)P_2^1(x), \quad \text{and}$$
$$\int_{-1}^{+1} dx\ \left[P_2^1(x)\right]^2.$$

Solution:

$$\int_{-1}^{+1} dx\ P_1^1(x)P_2^1(x) = \int_{-1}^{+1} dx\ 3x(1-x^2) = 0,$$
$$\int_{-1}^{+1} dx\ \left[P_2^1(x)\right]^2 = \int_{-1}^{+1} dx\ 9x^2(1-x^2) = 2\frac{9}{3} - 2\times\frac{9}{5} = \frac{12}{5}.$$

16. Using $P_0(x) = 1$, $P_1(x) = x$ and the recurrence relation

$$(2l+1)xP_l(x) = (l+1)P_{l+1}(x) + lP_{l-1}(x),$$

find $P_3(x)$.

Solution:

$$(2l+1)xP_l(x) = (l+1)P_{l+1}(x) + lP_{l-1}(x).$$

Using $P_0(x)$ and $P_1(x)$ and setting $l = 1$ the recurrence relation becomes

$$3xP_1(x) = 2P_2(x) + P_0(x).$$

$$P_2(x) \;=\; \frac{1}{2}(3xP_1(x) - P_0(x)) = \frac{1}{2}(3x^2 - 1).$$

Setting $l = 2$ the recurrence relation becomes

$$5xP_2(x) = 3P_3(x) + 2P_1(x).$$

$$\begin{aligned} P_3(x) &= \frac{1}{3}(5xP_2(x) - 2P_1(x)) \\ &= \frac{1}{3}(5x\frac{1}{2}(3x^2 - 1) - 2x) \\ &= \frac{1}{6}(15x^3 - 9x) = \frac{1}{2}(5x^3 - 3x). \end{aligned}$$

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17. Find P_0^0, P_1^1, P_1^0 and P_1^{-1} in terms of the angle variable $\cos\theta$.

Solution:

We have

$$\begin{aligned} P_l^m(x) &= (1 - x^2)^{m/2}\frac{d^m}{dx^m}P_l(x) \\ &= \frac{1}{2^l l!}(1 - x^2)^{m/2}\frac{d^{l+m}}{dx^{l+m}}(x^2 - 1)^l. \end{aligned}$$

$$P_0^0(x) = 1.$$

$$\begin{aligned} P_1^1(x) &= (1 - x^2)^{1/2}\frac{d}{dx}P_1(x) \\ &= \frac{1}{2}(1 - x^2)^{1/2}\frac{d^2}{dx^2}(x^2 - 1) \\ &= \frac{1}{2}(1 - x^2)^{1/2}2 \\ &= (1 - x^2)^{1/2} = \sin\theta. \end{aligned}$$

$$\begin{aligned}
P_1^0(x) &= \frac{d}{dx}P_1(x) = \frac{1}{2}\frac{d}{dx}(x^2-1) \\
&= \frac{1}{2}2x \\
&= \cos\theta.
\end{aligned}$$

$$\begin{aligned}
P_1^{-1}(x) &= (-1)^1\frac{(1-1)!}{(1+1)!}P_1^1(x) \\
&= -\frac{1}{2}\sin\theta.
\end{aligned}$$

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18. Show that
$$\int_{-1}^{+1} dx\ P_l(x) = 0,\ \text{for } l > 0.$$

Solution:

We have for
$$\int_{-1}^{+1} dx\ P_l(x)P_{l'}(x) = \frac{2}{(2l+1}\delta_{ll'}.$$

For $l > 0$
$$\int_{-1}^{+1} dx\ P_l(x)P_0(x) = \frac{2}{(2l+1}\delta_{l0} = 0.$$

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19. Let us consider the following potential
$$V = \frac{K}{r},$$
generated by two masses separated by a distance r
$$r = |\mathbf{A} - \mathbf{B}|.$$

The masses are located at the heads of the two vectors $\mathbf{A}$ and $\mathbf{B}$ originating from the origin of the coordinate system $\mathbf{O}$ and $\mathbf{r}$ is the distance vector starting at the head of vector $\mathbf{B}$ and ending at the head of vector $\mathbf{A}$. The angle between the two vectors $\mathbf{A}$ and $\mathbf{B}$ is θ. Then from the low of cosines we have

$$
\begin{aligned}
r &= |\mathbf{A} - \mathbf{B}| \\
&= \sqrt{A^2 - 2AB\cos\theta + B^2}.
\end{aligned}
$$

Let us consider the case $|\mathbf{B}| \ll |\mathbf{A}|$. Then we can make the following change of variables

$$t = \frac{B}{A}, \quad x = \cos\theta.$$

In this situation

a.) Show that the gravitational potential is

$$V(r) = \frac{K}{A}\phi(x,t),$$

where $\phi(x,t)$ is the generating function for the Legendre polynomials.

b.) Expand the the potential using $P_l(\cos\theta)$.

Solution:

a.) We have

$$r = A\sqrt{1 - 2tx + t^2},$$

$$V(r) = \frac{K}{A}\frac{1}{\sqrt{1-2tx+t^2}} = \frac{K}{A}\phi(x,t).$$

b.) Expanding the generating function

$$\begin{aligned}
V(r) &= \frac{K}{A}\frac{1}{\sqrt{1-2tx+t^2}} \\
&= \frac{K}{A}\sum_{l=0}^{\infty} t^l P_l(x) \\
&= \frac{K}{A}\sum_{l=0}^{\infty} \frac{B^l}{A^l} P_l(\cos\theta) \\
&= K\sum_{l=0}^{\infty} \frac{B^l}{A^{l+1}} P_l(\cos\theta).
\end{aligned}$$

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20. Show from the generating function for Legendre polynomials

$$\phi(x,t) = \frac{1}{\sqrt{1-2xt+t^2}}, \quad |t| < 1,$$

that

$$(x-t)\frac{\partial\phi}{\partial x} = t\frac{\partial\phi}{\partial t}.$$

Solution:

Denoting $\alpha = (1-2xt+t^2)$, we have

$$\begin{aligned}
\frac{\partial\phi}{\partial x} &= t\alpha^{-3/2}, \\
\frac{\partial\phi}{\partial t} &= (x-t)\alpha^{-3/2}.
\end{aligned}$$

This gives

$$\begin{aligned}
(x-t)\frac{\partial\phi}{\partial x} &= (xt-t^2)\alpha^{-3/2} \\
&= t(x-t)\alpha^{-3/2} \\
&= t\frac{\partial\phi}{\partial t}.
\end{aligned}$$

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