PHY 101 : Worksheet-3

Bold faced objects are vectors

A particle of mass 1 unit starts its motion from $\mathbf{x} = 0$ at time $t = 0$ as

$$\mathbf{r}(t) = \alpha t^3(\hat{\mathbf{i}} + \hat{\mathbf{j}})$$

where α is a constant. Find out

1. its position and velocity in cylindrical polar co-ordinates.
2. the time taken by the particle to travel a *distance* of 1 unit.
3. How much further time Δt it takes for the particle to double its speed at any time t ?
4. The amount of work done by a force $\mathbf{F}$ in moving an object by a displacement vector $\mathbf{dr}$ is $\mathbf{F} \cdot \mathbf{dr}$. If the motion of the above mentioned particle is caused by some force according to Newton's law, how much work has to be done by the time particle's $x-$ cordinate value crosses 1 unit ?

A particle of mass m is moving on a circular trajectory on a spehre of radius R and $\phi = 0$ such that its co-ordinate θ changes as $\theta(t) = \beta t^4$ with a constant β,

1. Write down the position vector and velocity vector of the particle at any time t.
2. Find out the force acting on the particle in spherical polar basis using Newton's law.
3. What is the work done by the force in completing one full circle ?
4. What would have been the work done in making one complete circle, if the angle ϕ was changing as $\phi(t) = \beta t^4$ and the co-ordinate θ was fixed at $\pi/2$?

Q. Set 3

In cartesian co-ordinates $\qquad \vec{r} = \alpha t^3 (\hat{i} + \hat{j})$

Cylindrical polar co-ordinates

$$\hat{\rho} = \cos\theta\, \hat{i} + \sin\theta\, \hat{j} \qquad \hat{k} = \hat{k}$$

$$\hat{\phi} = -\sin\theta\, \hat{i} + \cos\theta\, \hat{j}$$

$$\cos\theta\, \hat{\rho} - \sin\theta\, \hat{\phi} = (\cos^2\theta + \sin^2\theta)\, \hat{i} + (\cos\theta \sin\theta - \cos\theta \sin\theta)$$

$$\Rightarrow \quad \hat{i} = \cos\theta\, \hat{\rho} - \sin\theta\, \hat{\phi}$$

$$\sin\theta\, \hat{\rho} + \cos\theta\, \hat{\phi} = (\cos\theta \sin\theta - \cos\theta \sin\theta)\, \hat{i} + (\sin^2\theta + \cos^2\theta)\, \hat{j}$$

$$\Rightarrow \quad \hat{j} = \sin\theta\, \hat{\rho} + \cos\theta\, \hat{\phi}$$

Thus,

$$\vec{r} = \alpha t^3 \left(\cos\theta\, \hat{\rho} - \sin\theta\, \hat{\phi} + \sin\theta\, \hat{\rho} + \cos\theta\, \hat{\phi} \right)$$

$$= \alpha t^3 (\cos\theta + \sin\theta)\, \hat{\rho} + (\cos\theta - \sin\theta)\, \hat{\phi}$$

(i) $\dot{\vec{r}} = 3\alpha t^2 (\hat{i} + \hat{j})$

$= 3\alpha t^2 \left[(\cos\theta + \sin\theta)\,\hat{\rho} + (\cos\theta - \sin\theta)\,\hat{\phi} \right]$

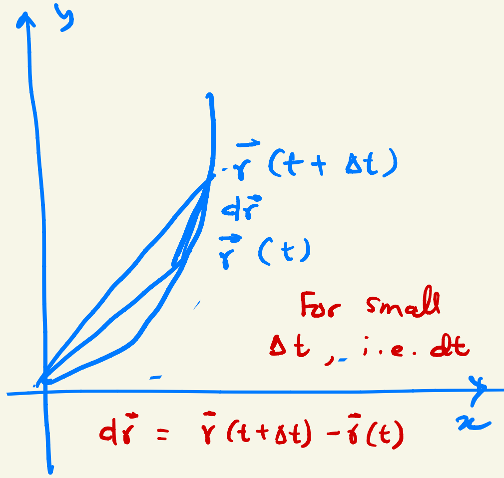
(ii) For small dt, dr is the distance travelled

where $dr = \sqrt{d\vec{r} \cdot d\vec{r}}$

$dr = \sqrt{\frac{d\vec{r}}{dt} \cdot \frac{d\vec{r}}{dt}}\; dt$

$= \sqrt{\vec{v} \cdot \vec{v}}\; dt$

$= 3\sqrt{2}\alpha t^2\, dt$



$\int_0^1 dr = \int_0^{t_0} 3\sqrt{2}\alpha t^2\, dt \Rightarrow 1 = \sqrt{2}\alpha t^3$

$\Rightarrow t = \left(\frac{1}{\sqrt{2}\alpha}\right)^{1/3}$

(iii) $\vec{v}(t) = 3\alpha t^2 (\hat{i} + \hat{j})$

$|\vec{v}(t)| = 3\alpha t^2 \sqrt{2}$

$|\vec{v}(t + \delta t)| = 2\,|\vec{v}(t)|$

$3\alpha (t + \delta t)^2 \sqrt{2} = 2 \times 3\alpha t^2 \sqrt{2}$

$\Rightarrow \frac{(t + \Delta t)}{t} = \sqrt{2}$

$\Rightarrow 1 + \frac{\Delta t}{t} = \sqrt{2} \Rightarrow \Delta t = (\sqrt{2} - 1)\, t$

(iv) $\ddot{\vec{r}} = 6\alpha t\ (\hat{i} + \hat{j})$

$\vec{F} = m\ddot{\vec{r}} = 6\alpha t\ (\hat{i} + \hat{j})$ $\boxed{\text{as } m = 1 \text{ unit}}$

$\dfrac{d\vec{r}}{dt} = 3\alpha t^2\ (\hat{i} + \hat{j})$

$\therefore\ d\vec{r} = 3\alpha t^2\ (\hat{i} + \hat{j})\ dt$

$dW = \vec{F} \cdot d\vec{r} = 18\alpha^2 t^3\ (2)\ dt$

$= 36\alpha^2 t^3\ dt$

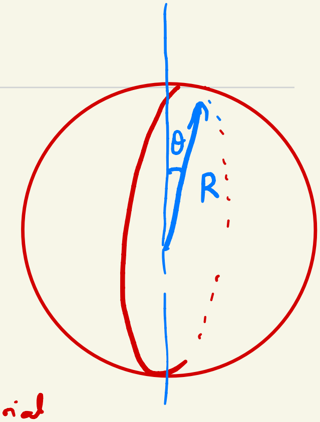
$$\int_0^W dW = \int_0^{t_0} 36\alpha^2 t^3\ dt = 36\alpha^2 \frac{t^4}{4}\Big|_0^{t_0}$$

t_0 is the time at which $x(t_0) = 1$

$\alpha t_0^3 = 1 \Rightarrow t_0 = \left(\frac{1}{\alpha}\right)^{1/3}$

$$W = \frac{36\alpha^2 t^4}{4}\Big|_0^{(1/\alpha)^{1/3}} = 9\alpha^2 \left(\frac{1}{\alpha}\right)^{4/3}$$

$= 9\alpha^{2/3}$



$$\theta(t) = \beta t^4$$

(i.) $\vec{r} = R\,\hat{r}$

$$\dot{\vec{r}} = \dot{R}\,\hat{r} + R\big(\underbrace{\dot{\theta}\,\hat{\theta} + \dot{\phi}\sin\theta\,\hat{\phi}}_{\text{Derived in last tutorial}}\big)$$

But since R is fixed and so is ϕ

$$\dot{R} = 0 = \dot{\phi}$$

Thus $\vec{v} = \dot{\vec{r}} = R\,\dot{\theta}\,\hat{\theta}$

(ii) $\vec{a} = \dot{\vec{v}} = R\ddot{\theta}\,\hat{\theta} + R\dot{\theta}\underbrace{\frac{d\hat{\theta}}{dt}}_{}$ derive this

$$= R\ddot{\theta}\,\hat{\theta} + R\dot{\theta}\left(-\dot{\theta}\,\hat{r} + \cos\theta\,\dot{\phi}\,\hat{\phi}\right)$$

$$= R\ddot{\theta}\,\hat{\theta} - R\dot{\theta}^2\,\hat{r} + R\dot{\theta}\dot{\phi}\cos\theta\,\hat{\phi}$$

For $\dot{\phi} = 0$

$$\vec{a} = R\ddot{\theta}\,\hat{\theta} - R\dot{\theta}^2\,\hat{r}$$

$$\vec{f} = m\vec{a} = mR\ddot{\theta}\,\hat{\theta} - mR\dot{\theta}^2\,\hat{r}$$

(iii) $d\vec{r} = \frac{d\vec{r}}{dt}\,dt = (R\dot{\theta})\,\hat{\theta}\,dt$

$$dW = \vec{f}\cdot d\vec{r} = mR^2\,\dot{\theta}\,\ddot{\theta}\,dt$$

$$\theta(t) = \beta t^4, \quad \dot{\theta} = 4\beta t^3, \quad \ddot{\theta} = 12\beta t^2$$

$$\theta(0) = 0, \quad \theta(t_0) = 2\pi \quad \Rightarrow \quad t_0^4 = \frac{2\pi}{\beta}$$

$$\Rightarrow t_0 = \left(\frac{2\pi}{\beta}\right)^{1/4}$$

$$W = \int_0^W dW = \int_0^{t_0} \vec{f} \cdot d\vec{r} = \int_0^{t_0} \vec{f} \cdot \frac{d\vec{r}}{dt}\, dt$$

$$= mR^2 \int_0^{t_0} \dot{\theta}\,\ddot{\theta}\, dt = mR^2 (48)\, \beta^2 \int_0^{t_0} t^5\, dt$$

$$= 48\, mR^2 \beta^2 \frac{t_0^6}{6} = 8\, mR^2 \beta^2 \left(\frac{2\pi}{\beta}\right)^{\frac{6}{4}}$$

$$= 8\,(2\pi)^{3/2}\, mR^2 \beta^{1/2}$$

(iv) If $\dot{\theta} = 0$, $\dot{R} = 0$, $\theta = \pi/2$

$$\dot{\vec{r}} = R \sin\pi/2\, \dot{\phi}\, \hat{\phi} = R\dot{\phi}\,\hat{\phi}$$

$$\ddot{\vec{r}} = R\ddot{\phi}\,\hat{\phi} + R\dot{\phi}\frac{d\hat{\phi}}{dt} = R\ddot{\phi}\,\hat{\phi} + R\dot{\phi}\left\{-\dot{\phi}\,(\sin\theta\,\hat{r} + \cos\theta\,\hat{\theta})\right\}$$

$$= R\ddot{\phi}\,\hat{\phi} - R\dot{\phi}^2\,\hat{r}$$ (as $\cos\theta = 0$, $\sin\theta = 1$ for $\pi/2$

$$\vec{f} = mR\ddot{\phi}\,\hat{\phi} - mR\dot{\phi}^2\,\hat{r}$$

$$\vec{f} \cdot d\vec{r} = \vec{f} \cdot \frac{d\vec{r}}{dt}\, dt = mR^2 \dot{\phi}\,\ddot{\phi}\, dt$$

↳ Same function → Same integral

→ Same work