Topic: Classical Mechanics. — PHY 301 Date

Classical mechanics deals with the motion of a single or a collection of particles. Consider a point particle that is acted upon by force $\vec{F}$. The state of the particle is defined by the $(\vec{r}, \vec{p})$ and the corresponding evolution of these ~~coords~~ two variables are given by the Newton's equation of motion.

$$\vec{p} = m\vec{v} \qquad \frac{d\vec{p}}{dt} = \vec{F}$$

velocity vector: $\vec{v} = \frac{\vec{p}}{m} = \frac{d\vec{r}}{dt}$

Acceleration vector: $\vec{a} = \frac{d^2\vec{r}}{dt^2}$

$$\frac{d\vec{r}}{dt} = \frac{\vec{p}}{m}$$

clearly the set of equations is an initial value problem and has to be augmented by the initial conditions $(\vec{r}(0), \vec{p}(0))$. Once these two quantities are known then we get a unique trajectory $(\vec{r}(t), \vec{p}(t))$ defined for all times. ~~See~~ In the absence of ~~knowlled~~ knowledge of $(\vec{r}(0), \vec{p}(0))$ we get a family of curves $(\vec{r}(t), \vec{p}(t))$. Since the trajectory is uniquely defined, the motion is deterministic that is the knowledge about the particle position and momentum is known to us for all future times.

In writing the second law of motion and calculating the trajectory of motion, it is implicit that there is a frame of reference. ~~Since the definition~~ The equations are coordinate independent but there is a choice of a frame. A frame of reference where the $\vec{F} = \frac{d\vec{p}}{dt}$ is called an inertial frame.

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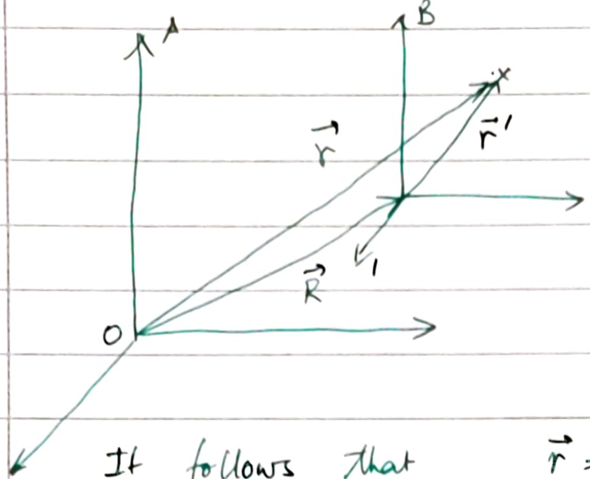
Consider two frames of reference A and B.



It follows that $\vec{r} = \vec{R} + \vec{r}\,'$

$$\vec{r}\,' = \vec{r} - \vec{R}$$

$$\frac{d\vec{r}\,'}{dt} = \frac{d\vec{r}}{dt} - \frac{d\vec{R}}{dt}$$

and $$\frac{d^2\vec{r}\,'}{dt^2} = \frac{d^2\vec{r}}{dt^2} - \frac{d^2\vec{R}}{dt^2}$$

$\because$ $\dfrac{d\vec{p}}{dt} = \vec{F}$ in frame A (inertial frame)

$$\frac{d\vec{p}\,'}{dt} = \frac{d\vec{p}}{dt} - M\frac{d^2R}{dt^2}$$

$$\Rightarrow \quad \vec{F}\,' = \vec{F} - M\ddot{\vec{R}}$$

Clearly the two forces are not equal. If they are to be equal it implies that

$$\ddot{\vec{R}} = 0 \quad \Rightarrow \quad \cancel{\vec{R}(t) = \vec{V}_0 t} \quad \dot{\vec{R}} = \vec{V}_0$$

$$\vec{R}(t) = \cancel{\vec{V}_0(t)}\ \vec{V}_0 t.$$

We choose the constant of integration to be zero if we assume that the origin of A & B coincides.

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Clearly, not all reference frames are equivalent. Frames of reference where $\dot{\vec{p}} = \vec{F}$ are valid are called inertial reference frames. The notion of inertial reference frame is somewhat an idealization. However, in practice, it is usually feasible to set up a coordinate system that is as close as possible to an inertial reference frame. For example, for a point particle on Earth, the reference frame fixed ~~on~~ on Earth is an inertial reference frame. However, for astronomical purposes, this may not be true.

Once we have set up a frame of reference, to describe the motion of a particle we need to assign a coordinate system. Note, in Newtonian Mechanics, time does not have the same footing as coordinate or momentum. Rather it is a parameter. Therefore, $\vec{r}(t)$ and $\vec{p}(t)$ are parametric equations of the trajectory. Clearly, the trajectory in the space of $(\vec{r}, \vec{p})$, the phase space of the system, is obtained by eleminating t between the ~~equat~~ two equations. We shall come back to this later. Setting up the coordinate system is important and is dictated by the ~~symmetry of in the problem~~. constraints in the system. For example, if a particle is moving ~~in~~ on a plane then the cartesian coordinates suffice to describe its motion. However, when a particle is rolling down the surface of a sphere, then (x, y, z) serves as a poor coordinate choice since whatever the value of (x, y, z) are $x^2+y^2+z^2 = R^2$ (radius of the sphere). Consequently it is more prudent to use the spherical polar coordinates.

Conservation theorems: Many of the important conclusions in mechanics can be expressed in terms of the conservation of theorems.

Linear Momentum: In the absence of an external force the linear of the particle $\vec{p}$ is conserved.

Angular Momentum: $\vec{L} = \vec{r} \times \vec{p}$ where $\vec{r}$ is the position vector and $\vec{p}$ is the momentum.

Similar to ~~translational~~ linear momentum.

Counterpart of $\vec{F} \longrightarrow \vec{\tau}$

$$\vec{\tau} = \vec{r} \times \vec{F}$$

$$= \vec{r} \times \frac{d\vec{p}}{dt} = \frac{d(\vec{r}\times\vec{p})}{dt} - \cancelto{0}{\vec{v} \times \vec{p}}$$

$$= \frac{d}{dt}(\vec{r}\times\vec{p})$$

$$\vec{\tau} = \frac{d\vec{L}}{dt}$$ $\rightarrow$ Equivalent to Newton's equation for translational motion.

In the absence of an external torque, the angular momentum is conserved.

$$\vec{\tau} = 0 \Rightarrow \frac{d\vec{L}}{dt} = 0 \Rightarrow \vec{L} = \text{Constant}.$$

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Conservation of Energy: Consider an external force $\vec{F}$ acting upon the particle ~~when~~ as it goes from point 1 to point 2. By definition the work done is

$$W_{12} = +\int_1^2 \vec{F}\cdot d\vec{r}$$

$$= \int_1^2 \frac{d\vec{p}}{dt}\cdot d\vec{r} = \int_1^2 \frac{d\vec{p}}{dt}\cdot\frac{\vec{p}}{m}\,dt.$$

$$d\vec{r} = \vec{v}\,dt \qquad = \int_1^2 \frac{d}{dt}\left(\frac{\vec{p}^2}{2m}\right)dt.$$

$$= \int_1^2 d\left(\vec{p}^2\right) = \frac{m}{2}\left(\vec{v}_2^{\,2} - \vec{v}_1^{\,2}\right)$$

$$W_{12} = T_2 - T_1$$

Now suppose the force field is such that it can be written as a derivative (directional one).

$$\vec{F} = -\vec{\nabla}U(r)$$

$$\int \vec{F}\cdot d\vec{r} = -\int \vec{\nabla}U\cdot d\vec{r} = -(U_2 - U_1).$$

$$-(U_2 - U_1) = (T_2 - T_1) \Rightarrow \underbrace{T_1 + U_1}_{\text{total energy}} = T_2 + U_2 = E$$

Hence for such an force the total energy of the system

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remains constant. There is yet another way of deriving this (rather more sophisticated) using the Hamiltonian of the system. For a single particle the Hamiltonian is a function of $(\vec{p}, \vec{q})$. $\mathcal{H}(\vec{p}, \vec{q}) \equiv T + V$

$$\mathcal{H}(\vec{p}, \vec{q}) = E$$

$$\Rightarrow \frac{dE}{dt} = \frac{\partial \mathcal{H}}{\partial p_i} \dot{p}_i + \frac{\partial \mathcal{H}}{\partial q_i} \dot{q}_i + \frac{\partial \mathcal{H}}{\partial t}$$

If $\mathcal{H}$ is explicitly not a function of t then the last term is zero. This means

$$\frac{dE}{dt} = \frac{\partial \mathcal{H}}{\partial p_i} \dot{p}_i + \frac{\partial H}{\partial q_i} \dot{q}_i$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} \qquad \dot{q}_i = \frac{\partial H}{\partial p_i}$$

$$\Rightarrow \frac{dE}{dt} = 0 \quad \Rightarrow \text{Energy remains conserved.}$$

Phase space and trajectory of a single particle.

It is very important that we understand the significance of phase space. ~~As~~ The phase space constitutes the space of ~~co~~ the coordinates and the momentum. For a many particle system three dimensions, the phase ~~ph~~ space is $6N$ dimensional with N being the number of particles. Unfortunately, it is not so easy to visualise such a space and therefore we restrict our discussion to ~~one~~ 1D. The qualitative behaviour of a particle motion can be analysed using the phase space. Here the conservation of energy comes to our rescue. The Hamiltonian which is the total energy of the system is a function of the variables (p, q) and is ~~conserved~~ a constant of motion.

$$\mathcal{H}(p, q) = E$$

This implies whatever the values of (p, q) are it must lie on the curve defined by the above equation. The more general case of an N particle system we have

$$\mathcal{H}(p_1, p_2, \dots p_{3N}, q_1, q_2, \dots q_{3N}) = E$$

and at any instant whatever the value of the vector $\vec{X}^N = \{\vec{p}_i, \vec{q}_i\}$ it must lie on the ~~surface~~ hyper surface that is defined by the above equation. Note that the value of E is determined by the initial condition $E = \mathcal{H}(\vec{X}^N(0))$, which uniquely determines the surface. Otherwise you have a family of closed surfaces. There are two things to note here

a) Unlike the conservation of momentum, the conservation of energy for a bound system defines a closed surface.

b) Once we have specified $\vec{X}^N(0)$, the evolution of the vector $\vec{X}^N(t)$ is merely a rotation operation.

Consider now the motion of a particle in a harmonic potential. The potential is $U(x) = \frac{1}{2} k x^2$. so that the Hamiltonian is $\mathcal{H} = \frac{p^2}{2m} + \frac{1}{2} k x^2$

$$\mathcal{H} = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2} = E$$

The curve defined by the equation $E = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2$ implies

$$\frac{p^2}{2mE} + \frac{m\omega^2 x^2}{2E} = 1. \rightarrow \text{Equation of an ellipse.}$$

$$p' = \frac{p}{\sqrt{2mE}} \qquad x' = \sqrt{\frac{m\omega^2}{2E}}\, x.$$

$$\Rightarrow \quad p'^2 + x'^2 = 1 \quad \text{Equation of a circle.}$$

$$x' = \cos(\omega t - \varphi) \quad p' = \sin(\omega t - \varphi)$$

$$\Rightarrow x = \sqrt{\frac{2E}{m\omega^2}} \cos(\omega t - \varphi)$$

$$p = \sqrt{2mE} \sin(\omega t - \varphi)$$

$$\ddot{x} + \omega^2 x = 0 \qquad x = A e^{\lambda t}.$$

$$(\lambda^2 + \omega^2) A = 0 \quad \Rightarrow \quad \lambda = \pm i\omega.$$

$$x = A e^{i\omega t} + B e^{-i\omega t}.$$

$$A + B = x(0).$$

$$\dot{x}(t) = \frac{\vec{p}}{m} = i\omega A\, e^{i\omega t} + i\omega B\, e^{-i\omega t}$$

$$\dot{x}(0) = \frac{\vec{p}(0)}{m} = i\omega A - i\omega B. = i\omega(A-B)$$

$$(A-B) = -i\frac{\vec{p}(0)}{m\omega}$$

$$A+B = x(0)$$

$$A = \frac{1}{2}\left[x(0) - i\frac{p(0)}{m\omega}\right]$$

$$B = \frac{1}{2}\left[x(0) + i\frac{p(0)}{m\omega}\right].$$

$$x(t) = \frac{1}{2}\left[x(0) - i\frac{p(0)}{m\omega}\right]e^{i\omega t} + \frac{1}{2}\left[x(0) + i\frac{p(0)}{m\omega}\right]e^{-i\omega t}.$$

$$= x(0)\cos\omega t + \frac{p(0)}{m\omega}\sin\omega t.$$

$$x(0) = a\cos\varphi \qquad \frac{p(0)}{m\omega} = a\sin\varphi$$

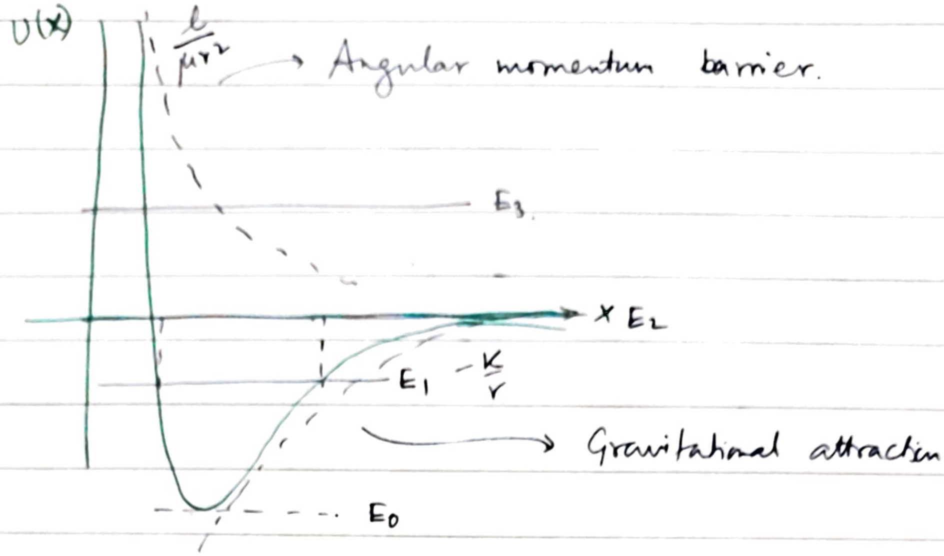
$$= a\cos(\omega t + \varphi)$$

$$a^2 = x^2(0) + \frac{p^2(0)}{m^2\omega^2} \qquad a = \sqrt{x^2(0) + \frac{p^2(0)}{m^2\omega^2}}$$

$$E = \frac{1}{2}m\omega^2 x^2(0) + \frac{p^2(0)}{2m} = \frac{m\omega^2}{2}\left[x^2(0) + \frac{p^2(0)}{m^2\omega^2}\right]$$

$$\frac{2E}{m\omega^2} = \left[x^2(0) + \frac{p^2(0)}{m^2\omega^2}\right].$$

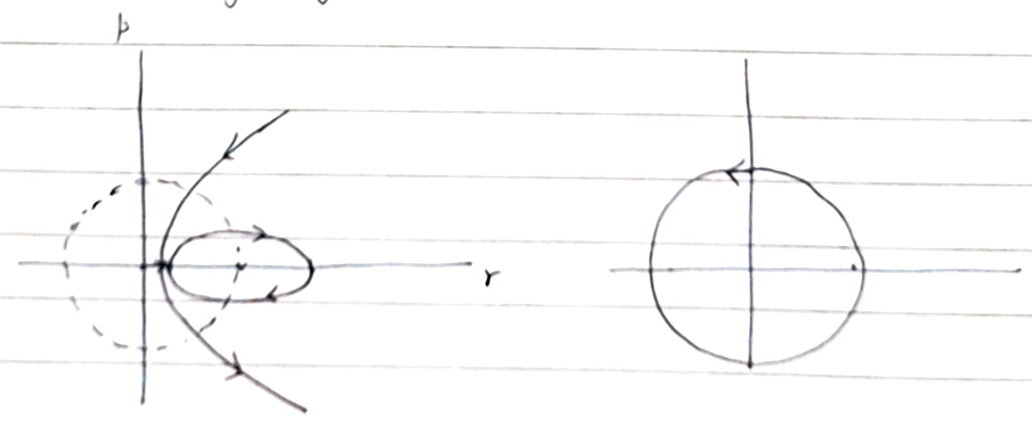
More complicated potentials



Consider the above potential which we will encounter when we study motion under gravitational potential. ~~Th~~ From the figure it is clear that we have to consider three different cases of initial ~~so~~ energy as indicated in the figure.

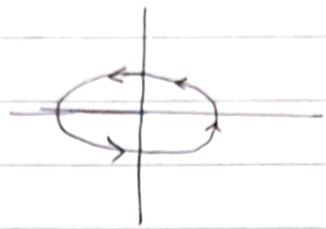
For a particle with energy E_1 There are two turning points. and the trajectory is bounded.

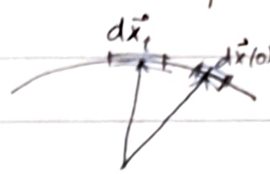
For a particle with energy E_0, There is only one turning point and the trajectory is a circle.



For ~~on~~ a particle with energy E_3, the particle sees a repulsive core.

Flow in the phase space: ~~The flow is~~ As the particle position and momentum evolves with time, we can ~~visualise~~ visualise the particle going around ~~around~~ and around in a circle. Such that we can think of a current, ~~or rather~~ and therefore a flow.





Let $d\vec{x}_t$ denote the volume element in the phase space at time t and $d\vec{x}(0)$ be the volume element ~~in time~~ at time $t=0$.

$$d\vec{x}_t = J(t, t_0)\, d\vec{x}_0$$

$J(t,t_0)$ is the Jacobian of the ~~transf~~ transformation between $\vec{x}(t)$ and $\vec{x}(0)$

$$\vec{x}_t = (\vec{p}, \vec{q})$$

$$\vec{x}(0) = (\vec{p}_{t_0}, \vec{q}_{t_0})$$

$$J = \frac{\partial(\vec{p}, \vec{q})}{\partial(\vec{p}_0, \vec{q}_0)}$$

$$J = \begin{vmatrix} \frac{\partial \vec{p}}{\partial \vec{p}_0} & \frac{\partial \vec{p}}{\partial \vec{q}_0} \\ \frac{\partial \vec{q}}{\partial \vec{p}_0} & \frac{\partial \vec{q}}{\partial \vec{q}_0} \end{vmatrix} \equiv J(t, t_0)$$

$$\vec{p}(t) = \vec{p}(t_0) + \dot{\vec{p}}_0 \Delta t + O(\Delta t^2)$$

$$\vec{q}(t) = \vec{q}(t_0) + \dot{\vec{q}}_0 \Delta t + O(\Delta t^2)$$

$$\vec{p}(t) = \vec{p}_0 + \dot{\vec{p}}_0 \Delta t + O(\Delta t^2)$$

$$\frac{\partial \vec{p}}{\partial \vec{p}_0} = 1 + \frac{\partial \dot{\vec{p}}}{\partial \vec{p}_0} \Delta t$$

$$\frac{\partial \vec{p}}{\partial \vec{q}_0} = 0.$$

$$\frac{\partial \vec{q}}{\partial \vec{p}_0} = 0.$$

$$\frac{\partial \vec{q}}{\partial \vec{q}_0} = 1 + \frac{\partial \dot{\vec{q}}}{\partial \vec{q}_0} \Delta t$$

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$$J(t,t_0) = 1 + \left[\frac{\partial \dot{p}_0}{\partial p_0} + \frac{\partial \dot{q}_0}{\partial \vec{q}_0} \right] \Delta t$$

$$\dot{p} = -\frac{\partial \mathcal{H}}{\partial \vec{q}}$$

$$\frac{\partial \dot{p}}{\partial \vec{p}} = -\frac{\partial \mathcal{H}}{\partial \vec{p}\,\partial \vec{q}}$$

Similarly $\frac{\partial \dot{q}}{\partial \vec{q}} = \frac{\partial \mathcal{H}}{\partial \vec{p}\,\partial \vec{q}}$

The quantity in bracket is zero. $\Leftarrow$
so that

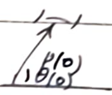
$$J(t,t_0) = 1.$$

Jacobian of the transformation is unit so that

No leak.

$d\vec{x}(t) = \vec{d}\,\vec{x}(0) \longrightarrow$ Volume is preserved. $\rightarrow$ No loss in probability.

$d\vec{x}(t) = p\,dp\,d\theta$ $\qquad$ $d\vec{x}(0) = p(t_0)\,dp\,d\theta$



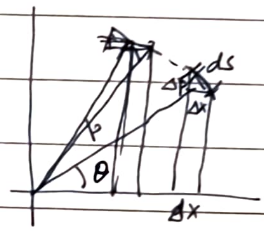
$\Delta x \overset{?}{\rightarrow} \Delta x(t)$

~~$d\theta = \omega dt$~~

$$dp = \dot{p}\,dt$$

$x(t_0),\ p(t_0)$
$\downarrow$
$x(t),\ p(t)$

$$x = \sqrt{\frac{2E}{m\omega^2}}\cos(\omega t - \varphi).$$



$x(t_0) + dx(t_0)$
$p(t_0) + dp(t_0)$

$$p = \sqrt{2mE}\,\sin(\omega t - \varphi).$$

$p(t) + \Delta p(t)$
$x(t) + \Delta x(t)$

$$ds^2 = (dx^2 + dy^2) = dx^2\left[1 + \left(\frac{dy}{dx}\right)^2\right]$$

$p\,\dot{p}\,\Delta t$

$p(t+\Delta t)\,\Delta p\,\Delta\theta.$

$$p(t_0+\Delta t) = p(t_0) + \dot{p}\,\Delta t + O(\Delta t^2) \qquad dp = \dot{p}\,\Delta t$$

$$p(t+\Delta t) = p(t_0) + \dot{p}\,\Delta t. \qquad \theta(t+\Delta t) = \theta(t_0) + \dot{\theta}(t_0)\,\Delta t.$$

$$\Delta\theta = \dot{\theta}(t_0)\,\Delta t$$

$$p(t+\Delta t)\,\Delta p\,\Delta\theta = \left(p(t_0) + \dot{p}_{t_0}\Delta t\right)\left(\Delta p\,\Delta\theta\right). \qquad \Delta p = \dot{p}_{t_0}\,\Delta t$$

$$= p(t_0)\,\underline{\dot{p}(t_0)}\,\Delta t\;\dot{\theta}(t_0)\,\Delta t + \dot{\theta}_{t_0}\,\dot{p}^{2}_{t_0}\,\Delta t^3$$

$$=$$

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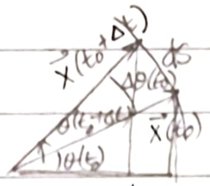
$$\Delta x = \dot{x}\,\Delta t$$

$$\vec{x} = (q, p)$$

$$\vec{x}(t_0+\Delta t) = \left(q(t_0+\Delta t),\ p(t_0+\Delta t)\right)$$



$$ds = \left[dq^2 + dp^2\right]^{1/2}$$

$$dq = q(t_0+\Delta t) - q(t_0) \qquad = dq(t_0)\left[1 + \frac{dp}{dq}\right]^{1/2}$$

$$dp = p(t_0+\Delta t) - p(t_0)$$

$$\left.\frac{dp}{dq}\right|_{t=t_0} = \frac{p(t_0+\Delta t) - p(t_0)}{q(t_0+\Delta t) - q(t_0)}$$

$$= \frac{\dot{p}(t_0)\,\Delta t}{\dot{q}(t_0)\,\Delta t} = \frac{\dot{p}(t_0)}{\dot{q}(t_0)}$$

$$\frac{p^2}{2mE} + q^2\frac{m\omega^2}{2E} = 1$$

$$\frac{1}{2mE}\,2p\frac{dp}{dq} + \frac{m\omega^2 q}{E} = 0$$

$$p\frac{dp}{dq} = -m^2\omega^2 q$$

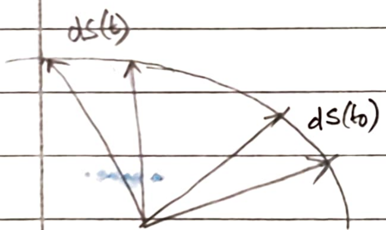
$$\frac{dp}{dq} = -m^2\omega^2\frac{q(t_0)}{p(t_0)}$$

$$= -m$$

$$ds(t_0) = dq(t_0)\left[1 + \frac{\dot{p}(t_0)}{\dot{q}(t_0)}\right]^{1/2}$$

$$\left.\dot{p}\right|_{t=t_0} = F(t_0) \qquad \left.\dot{q}\right|_{t=t_0} = \frac{p(t_0)}{m}$$

$$ds(t) = dq(t)\left[1 + \frac{dp}{dq}\right]^{1/2}$$



$$p(t) = p(t_0) + \dot{p}(t_0)\,(t - t_0)$$

$$p(t+\Delta t) = p(t_0) + \dot{p}(t_0)\left[t + \Delta t - t_0\right]$$

$$q(t) = q(t_0) + \dot{q}(t_0)\,(t - t_0)$$

$$q(t+\Delta t) = q(t_0) + \dot{q}(t_0)\,(t + \Delta t - t_0)$$

$$q(t+\Delta t) - q(t)$$

$$= \dot{q}(t_0)\left[(t-t_0) + \Delta t - (t-t_0)\right]$$

$$= \dot{q}(t_0)\,\Delta t$$

$$= dq(t_0)$$

$$dq(t) = q(t+\Delta t) - q(t)$$

$$= \dot{q}(t_0)\left[t + \Delta t - t_0\right]$$

$$= \dot{q}(t_0)\left[t - t_0 + \Delta t\right]$$

$$\frac{dp}{dq} = \frac{\dot{p}(t_0)\left[t - t_0 + \Delta t - (t-t_0)\right]}{\dot{q}(t_0)\left[t - t_0 + \Delta t - (t-t_0)\right]} = \frac{\dot{p}(t_0)}{\dot{q}(t_0)}$$

Therefore it follows that $dS(t_0) = dS(t)$. Which means that the volume element is preserved in such a flow on the phase space.

Mathematical Preliminaries

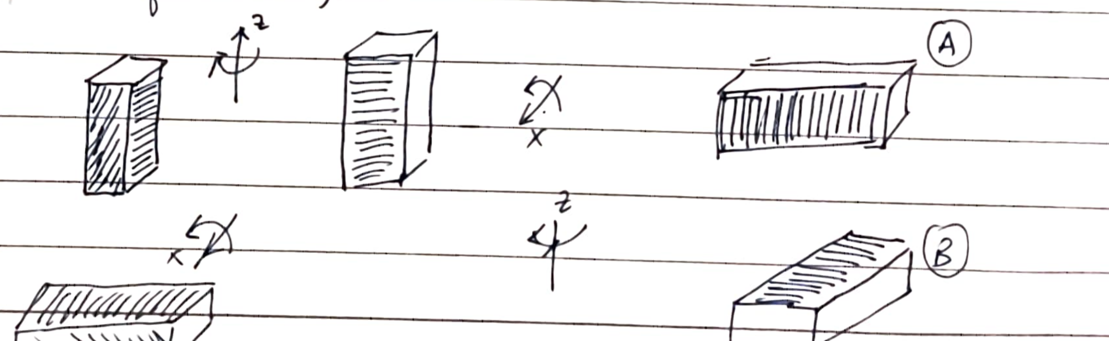
Scalars, Vectors and Tensors.

A scalar is a quantity which has only magnitude and no direction. For example mass, energy, work ~~[illegible]~~ etc.

A vector is a quantity which has a magnitude as well as a direction.

Both these definitions are inadequate. Consider a mountain and you wish to determine the slope of the mountain. The slope is $\lim_{\delta s \to 0} \frac{\delta h}{\delta s}$ and is a scalar quantity. However, its magnitude ~~at~~ is determined by the direction along ~~wh~~ which you are descending and therefore has a sense of direction to it.

Similarly for the definition of a vector, consider finite rotation of an object.

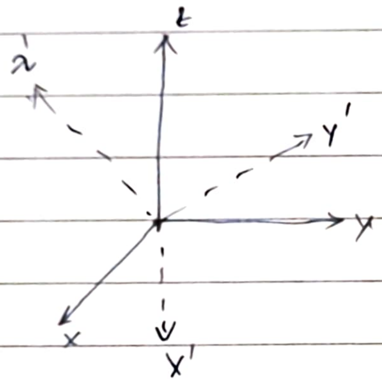


$A \neq B$

~~This~~ These operations have not only a magnitude but a sense of direction. Yet they are not a vector since the operations do not commute that is

$$\vec{A} + \vec{B} \neq \vec{B} + \vec{A}.$$

Consequently we need to revise our definition for scalars and vectors. The more elegant way to do this (and therefore consider the more general quantity tensor) is to look at a vector and how it transforms under rotation.



$$\vec{V} = V_x \hat{x} + V_y \hat{y} + V_z \hat{z}$$

$$\vec{V}' = V_x' \hat{x}' + V_y' \hat{y}' + V_z' \hat{z}'$$

$$V_x' = \vec{V} \cdot \hat{x}'$$

$$= V_x \, \hat{x} \cdot \hat{x}' + V_y \, \hat{x}' \cdot \hat{y} + V_z \, \hat{x}' \cdot \hat{z}$$

$$V_y' = \vec{V} \cdot \hat{y}'$$

$$V_z' = \vec{V} \cdot \hat{z}'$$

$$\vec{V}' = \underline{\mathbb{R}} \, \vec{V} \qquad \mathbb{R} = \underbrace{\begin{bmatrix} \hat{x} \cdot \hat{x}' & \hat{x}' \cdot \hat{y} & \hat{x}' \cdot \hat{z} \\ \hat{y}' \cdot \hat{x} & \hat{y} \cdot \hat{y}' & \hat{y}' \cdot \hat{z} \\ \hat{z}' \cdot \hat{x} & \hat{z}' \cdot \hat{y} & \hat{z}' \cdot \hat{z} \end{bmatrix}}_{\text{tensor}}$$

$$V_i' = R_{ij} V_j$$

Therefore, a scalar is a quantity which remains invariant under rotation whereas a vector is a quantity that obeys a transformation law given above.

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No matter how the coordinate axes are rotated or flipped, or both, the components of the vector are always related ~~to the~~ by an equation given above.

~~For~~ All possible rotations form a group $O(3)$. with the property that $|\mathbb{R}| = \pm 1$.

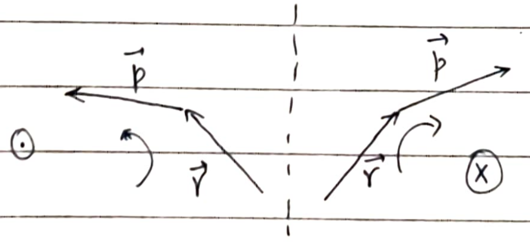
When ~~...~~ $|\mathbb{R}| = +1$ $\rightarrow$ this is a pure rotation and form a subgroup $SO(3)$.

For $|\mathbb{R}| = -1$, the operation corresponds to ~~refle~~ an improper rotation. For example reflection is an improper rotation. Under reflection

$$\begin{aligned} x &\rightarrow x' \\ y &\rightarrow y' \\ z &\rightarrow -z' \end{aligned} \qquad \mathbb{R} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Reflection in a mirror is a symmetry transformation effected by what is called 'parity.' The reflection symmetry, or symmetry under a parity has another important difference from rotation. While rotation is a continuous symmetry, reflection is a discrete symmetry.

Now take the example of angular momentum $\vec{L} = \vec{r} \times \vec{p}$



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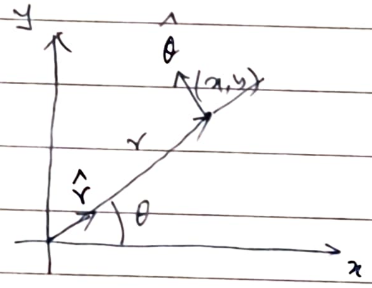
If everything in the universe undergoes a rotation ~~by~~ described by the rotation matrix $\mathbb{R}$, then the position vector also undergoes rotation $\vec{r}' = \mathbb{R}\vec{r}$. Therefore any vector must transform also transform $\vec{A}' = \mathbb{R}\vec{A}$. Such vectors are called polar vectors. But in the earlier example we saw that under reflection the vector $\vec{L}$ transforms in a completely different way. ~~Such~~ It changes direction when the coordinate system is rotated. Such vectors are called Axial vectors.

For polar vectors $\qquad \vec{A}' = \mathbb{R}\vec{A}$

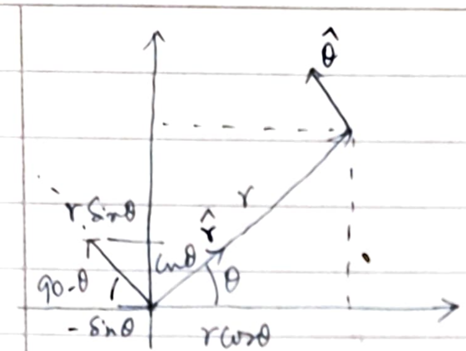
For Axial vector $\qquad \vec{A}' = \det(\mathbb{R})\,\mathbb{R}\vec{A}$

Non Cartesian Coordinate Systems

We first consider ~~the~~ a non Cartesian coordinate system in ~~2D~~ two-dimension - specifically the polar coordinates.



The position vector ~~$\vec{r}$~~ $\vec{r} = x\hat{x} + y\hat{y}$ in polar coordinates reads as $\vec{r} = r\hat{r}$. But there is a catch. Whereas the unit vector $(\hat{x}, \hat{y})$ are fixed in time, $(\hat{r}, \hat{\theta})$ are time dependendent quantities.



$$\vec r = r\cos\theta\,\hat x + r\sin\theta\,\hat y$$

$$\hat r = \frac{\vec r}{|\vec r|} = \cos\theta\,\hat x + \sin\theta\,\hat y$$

$$\vec r = r\hat r$$

$$\frac{d\hat r}{dt} = -\sin\theta\,\dot\theta\,\hat x + \cos\theta\,\dot\theta\,\hat y = \dot\theta\left[\underbrace{-\sin\theta\,\hat x + \cos\theta\,\hat y}_{\hat\theta}\right]$$

Velocity and acceleration in plane polar coordinates.

$$\vec v = \frac{d\vec r}{dt} = \frac{d(r\hat r)}{dt} = \dot r\hat r + r\frac{d\hat r}{dt} = \underbrace{\dot r\hat r}_{\text{radial Component}} + \underbrace{\overset{\omega r}{\dot\theta r}\,\hat\theta}_{\text{Angular Component}}$$

$$\vec a = \frac{d\vec v}{dt} = \frac{d}{dt}\left(\dot r\hat r + r\dot\theta\hat\theta\right)$$

$$= \ddot r\,\hat r + \dot r\dot\theta\,\hat\theta + \dot r\dot\theta\,\hat\theta + r\ddot\theta\,\hat\theta + r\dot\theta\frac{d\hat\theta}{dt}$$

$$\hat\theta = -\sin\theta\,\hat x + \cos\theta\,\hat y$$

$$\frac{d\hat\theta}{dt} = (-\cos\theta\,\hat x + \sin\theta\,\hat y)\dot\theta = -\dot\theta\hat r$$

$$\vec a = \ddot r\,\hat r + 2\dot r\dot\theta\,\hat\theta + r\ddot\theta\,\hat\theta - r\dot\theta^2\,\hat r$$

$$= (\ddot r - r\dot\theta^2)\,\hat r + 2\dot r\dot\theta\,\hat\theta + r\ddot\theta\,\hat\theta$$

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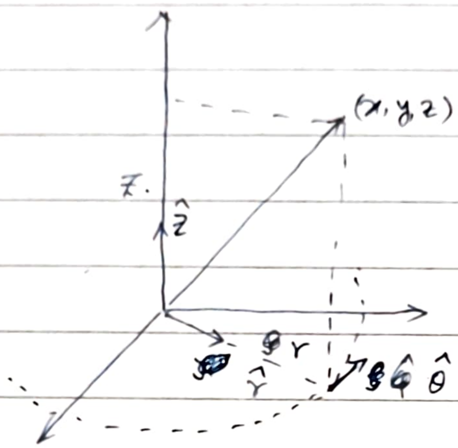
Cylindrical coordinate system

This is simply an extension of the plane polar coordinate to incorporate the 3rd dimension.



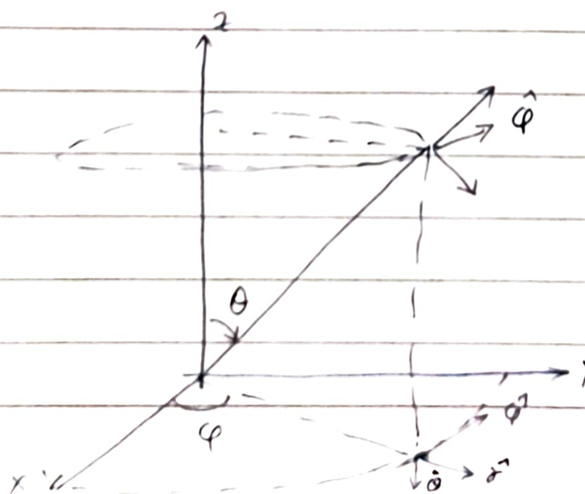
$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z} \equiv \cancel{r\hat{r} + \theta\hat{\theta} + z\hat{z}} \quad r\hat{r} + z\hat{z}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + \dot{z}\hat{z}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \ddot{r}\hat{r} + \dot{r}\dot{\theta}\hat{\theta} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} - r\dot{\theta}^2\hat{r} + \ddot{z}\hat{z}$$

$$= (\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta} + \ddot{z}\hat{z}$$

Spherical Polar Coordinates



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$(\hat{x}, \hat{y}, \hat{z}) \rightarrow (\hat{r}, \hat{\theta}, \hat{\varphi})$

$(x, y, z) \rightarrow (r, \theta, \varphi)$

$$\hat{r} = \sin\theta\cos\varphi\,\hat{x} + \sin\theta\sin\varphi\,\hat{y} + \cos\theta\,\hat{z}$$

$$\hat{\theta} = \cos\theta\cos\varphi\,\hat{x} + \cos\theta\sin\varphi\,\hat{y} - \sin\theta\,\hat{z} = \frac{\partial\hat{r}}{\partial\theta}$$

$$\varphi = -\sin\varphi\,\hat{x} + \cos\varphi\,\hat{y} = \frac{1}{\sin\theta}\frac{\partial\hat{r}}{\partial\varphi}$$

$$\vec{r} = r\hat{r}$$

$$\frac{d\vec{r}}{dt} = \dot{r}\hat{r} + r\frac{d\hat{r}}{dt}$$

$$\frac{d\hat{r}}{dt} = \dot{\theta}\cos\theta\cos\varphi - \dot{\varphi}\sin\theta\sin\varphi\,\hat{x} + \dot{\theta}\cos\theta\sin\varphi\,\hat{y} + \dot{\varphi}\sin\theta\cos\varphi\,\hat{y} - \sin\theta\,\hat{z}$$

$$= \dot{\theta}\hat{\theta} + \dot{\varphi}\hat{\varphi}\sin\theta$$

$$\frac{d\hat{\theta}}{dt} = \frac{d}{dt}\frac{\partial\hat{r}}{\partial\theta}$$

$$= \frac{\partial^2\hat{r}}{\partial\theta^2}\dot{\theta} + \frac{\partial^2\hat{r}}{\partial\varphi\partial\theta}\dot{\varphi}$$

$$= \frac{\partial\hat{\theta}}{\partial\theta}\dot{\theta} + \frac{\partial}{\partial\theta}\frac{\hat{\varphi}}{\sin\theta}$$

$$\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r\dot{\varphi}\hat{\varphi}\sin\theta$$

$$\vec{a} = \ddot{r}\hat{r} + \dot{r}\frac{d\hat{r}}{dt} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}\frac{d\hat{\theta}}{dt}$$

$$+ \dot{r}\dot{\varphi}\hat{\varphi} + r\ddot{\varphi}\hat{\varphi} + r\dot{\varphi}\frac{d\hat{\varphi}}{dt}$$

$$\frac{d\hat{\theta}}{dt} = -\dot{\theta}\hat{\theta} + \cos\theta\dot{\varphi}\hat{\varphi}$$

$$\frac{d\hat{\varphi}}{dt} = -\cos\varphi\dot{\varphi}\,\hat{x} - \dot{\varphi}\sin\varphi\,\hat{y} = -\dot{\varphi}\left[\cos\varphi\,\hat{x} + \sin\varphi\,\hat{y}\right]$$

$$= -\dot{\varphi}\left[\hat{r}\sin\theta + \hat{\theta}\cos\theta\right].$$

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Curvilinear Coordinates – General Approach.

(q_1, q_2, q_3) are the curvilinear coordinates.

Any vector can be written down as $\vec{A} = A_1 \hat{q}_1 + A_2 \hat{q}_2 + A_3 \hat{q}_3$.
except the position vector.

$$x(q_1, q_2, q_3) \qquad y(q_1, q_2, q_3) \qquad z(q_1, q_2, q_3)$$

$$dx = \sum_i \frac{\partial x}{\partial q_i} dq_i \qquad d\vec{r} = \sum_i \frac{\partial \vec{r}}{\partial q_i} dq_i$$

$$ds^2 = dx^2 + dy^2 + dz^2 = d\vec{r} \cdot d\vec{r}$$

$$= \frac{\partial \vec{r}}{\partial q_i} \cdot \frac{\partial \vec{r}}{\partial q_j} dq_i \, dq_j$$

$$= g_{ij} \, dq_i \, dq_j$$

$$g_{ij} = \frac{\partial \vec{r}}{\partial q_i} \cdot \frac{\partial \vec{r}}{\partial q_j}$$

This is a ~~scalar~~ products of the tangent vectors $\frac{\partial \vec{r}}{\partial q_i}$ to the curves $\vec{r}_j$ with q_j = constant $(j \neq i)$.

If we consider an orthogonal system then it is clear

$$g_{ij} = 0 \quad \text{for } j \neq i$$

$$g_{ii} = \frac{\partial \vec{r}}{\partial q_i} \cdot \frac{\partial \vec{r}}{\partial q_i} = h_i^2$$

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$$ds^2 = h_1^2\, dq_1^2 + h_2^2\, dq_2^2 + h_3\, dq_3^2$$

$$ds = h_i\, dq_i \qquad \text{and} \qquad \frac{\partial \vec{r}}{\partial q_i} = h_i\, \hat{q}_i$$

$$d\vec{r} = h_i\, dq_i\, \hat{q}_i$$

Gradient Operator

The starting point for ~~tondo~~ developing the gradient operator and related operators is ~~to~~ ~~consider~~ the geometric interpretation of the gradient as the vector having the magnitude and direction of ~~bearing~~ the maximum space rate of change. Therefore, $\nabla\psi(q_1, q_2, q_3)$ has components in the direction normal to the family of surfaces $q_1 =$ constant.

$$\hat{q}_1 \cdot \nabla\psi = \frac{\partial \psi}{\partial s_1} = \frac{1}{h_1}\frac{\partial \psi}{\partial q_1}$$

$$\Rightarrow \quad \nabla\psi = \frac{\partial \psi}{\partial s_1}\hat{q}_1 + \frac{\partial \psi}{\partial s_2}\hat{q}_2 + \frac{\partial \psi}{\partial s_3}\hat{q}_3$$

$$= \frac{1}{h_1}\frac{\partial \psi}{\partial q_1} + \frac{1}{h_2}\frac{\partial \psi}{\partial q_2}\hat{q}_2 + \frac{1}{h_3}\frac{\partial \psi}{\partial q_3}\hat{q}_3$$

$$\nabla = \frac{1}{h_i}\frac{\partial}{\partial q_i}\hat{q}_i .$$

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Mechanics of System of Particles

Consider a system of particles which ~~are not~~ have a pairwise interaction $\vec{F}_{ij}$. The total force on the system is given by

$$\vec{F}_i = \sum_j \vec{F}_{ij} + \vec{F}_i^{ext}$$

The total force on the system is given by

$$\vec{F} = \sum_i \vec{F}_i = \sum_i \sum_j \vec{F}_{ij} + \sum_i \vec{F}_i^{ext}$$

$$= \sum_{\substack{i,j \\ i<j \\ i \neq j}} \left(\vec{F}_{ij} + \vec{F}_{ji} \right) + \vec{F}^{ext}$$

Explicitly the pair wise interaction is written as

$$\vec{F}^{int} = \vec{F}_{12} + \vec{F}_{13} + \vec{F}_{14} + \dots \vec{F}_{23} + \vec{F}_{24} + \dots$$
$$+ \vec{F}_{31} + \dots$$

$$= \left(\vec{F}_{12} + \vec{F}_{21}\right) + \left(\vec{F}_{13} + \vec{F}_{31}\right) + \left(\vec{F}_{14} + \vec{F}_{41}\right) + \dots$$

By Newton's third law $\vec{F}_{ij} + \vec{F}_{ji} = 0$.

Hence the total force is $\vec{F} = \vec{F}^{ext}$.

The equation for the individual particle is given by

$$m_i \frac{d^2\vec{r}_i}{dt^2} = \vec{F}_i = \sum_j \vec{F}_{ij} + \vec{F}_i^{ext}$$

$$\sum_i m_i \frac{d^2\vec{r}_i}{dt^2} = \sum_i \sum_j \vec{F}_{ij} + \vec{F}_i^{ext} = \vec{F}^{ext}.$$

$$\sum_i m_i \frac{d^2}{dt^2} \frac{\sum m_i \vec{r}_i}{\sum m_i} = \vec{F}^{ext}$$

$$\Rightarrow M \frac{d^2 R}{dt^2} = \vec{F}^{ext}.$$

The equation means that we have reduced the problem to an effective single particle problem with position vector $\vec{R}$. If we solve for $\vec{R}(t)$, we get the trajectory of the center of mass ~~for~~ of the system. As we shall see later this is indeed very useful in reducing a two body problem to a one body problem.

The conservation of linear momentum now reads as: ~~the~~ If the total ~~external~~ external force is zero, then the total linear momentum of the system is conserved.

$$M \frac{d^2\vec{R}}{dt^2} = 0 \Rightarrow M\frac{d\vec{R}}{dt} = \text{Const.}$$

$$\Rightarrow \cancel{\vec{P}_{CM} = \text{Const}}$$

$$\cancel{\vec{P}_{cm}} \quad \sum m_i \vec{v}_i = \text{Constant.}$$

Angular momentum:

$$\vec{L}_i = \vec{r}_i \times \vec{p}_i$$

Total angular momentum: $\vec{L} = \sum \vec{r}_i \times \vec{p}_i$

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}\sum_i (\vec{r}_i \times \vec{p}_i) = \sum_i \frac{d\vec{r}_i}{dt}\times \vec{p}_i + \sum_i \vec{r}_i \times \frac{d\vec{p}_i}{dt}$$

$$= \sum_i \vec{r}_i \times \frac{d\vec{p}_i}{dt} = \sum_i \vec{r}_i \times \left[\vec{F}_i\right]$$

$$= \sum_i \vec{r}_i \times \left[\sum_j \vec{F}_{ij} + \vec{F}_i^{ext}\right]$$

$$= \sum_i \vec{r}_i \times \sum_j \vec{F}_{ij} + \sum_i \vec{r}_i \times \vec{F}_i^{ext}$$

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$$\frac{d\vec{L}}{dt} = \sum_{\substack{i,j \\ i<j \\ i\neq j}} \left[\vec{r}_i \times \vec{F}_{ij} + \vec{r}_j \times \vec{F}_{ji}\right] + \sum_i \vec{r}_i \times \vec{F}_i^{\,ext}$$

$$= \sum_{\substack{i,j,\, i<j \\ i\neq j}} \left[\underbrace{(\vec{r}_i - \vec{r}_j) \times \vec{F}_{ij}}_{\text{become } 0.}\right] + \sum_i \vec{r}_i \times \vec{F}_i^{\,ext}.$$

Depends on $\vec{r}_{ij} = \vec{r}_i - \vec{r}_j$ and therefore parallel to $\vec{r}_i - \vec{r}_j$.

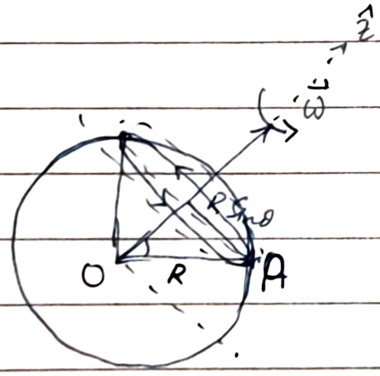
$$\vec{F}_{ij} = -\frac{\partial U}{\partial r_{ij}}\,\hat{r}_{ij}$$

$$\frac{d\vec{L}}{dt} = \sum_i \vec{r}_i \times \vec{F}_i^{\,ext} = \vec{\tau}^{\,ext}$$

Hence in the absence of a total external torque, the angular momentum is conserved.

$$\sum \vec{r}_i \times \vec{p}_i = \text{Constant}.$$

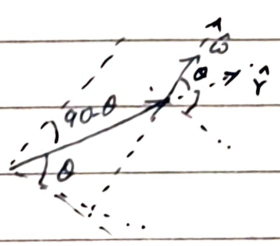
Consider now a rigid sphere that spinning about any axis with constant angular speed ω. Let us take the z-axis along that direction.



The point A moves along a circle of radius $R\sin\theta$. The velocity of point A

$$\vec{v} = v_r\,\hat{r} + v_\theta\,\hat{\theta}$$

$$v_r = \dot{r} = \frac{d}{dt}(R\sin\theta) = 0.$$



$$v_\theta = R\sin\theta\,\dot{\theta} = \omega R\sin\theta.$$

$$\vec{v} = \vec{\omega} \times \vec{R}.$$