PHY302: Quantum mechanics
Tutorial-3

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Problem 1: The Hamiltonian operator for a two-state system is given by

$$H = a\left(|1\rangle\langle 1| - |2\rangle\langle 2| + |1\rangle\langle 2| + |2\rangle\langle 1|\right),$$

where a is a number with the dimension of energy. Find the energy eigenvalues and the corresponding energy eigenkets (as linear combinations of $|1\rangle$ and $|2\rangle$).

Problem 2: A two-state system is characterized by the Hamiltonian

$$H = H_{11}|1\rangle\langle 1| + H_{22}|2\rangle\langle 2| + H_{12}\left[|1\rangle\langle 2| + |2\rangle\langle 1|\right],$$

where H_{11}, H_{22}, and H_{12} are real numbers with the dimension of energy, and $|1\rangle$, and $|2\rangle$ are eigenkets of some observable ($\neq H$). Find the energy eigenkets and the corresponding energy eigenvalues. Make sure that your answer makes good sense for $H_{12} = 0$. (You need not solve this problem from scratch. The following fact may be used without proof:

$$\mathbf{S}\cdot\mathbf{n}\,|\mathbf{n};+\rangle = \frac{\hbar}{2}|\mathbf{n};+\rangle,$$

with given by

$$|\mathbf{n};+\rangle = \cos\frac{\beta}{2}|+\rangle + e^{i\alpha}\sin\frac{\beta}{2}|-\rangle,$$

where β and α are the polar and azimuthal angles, respectively, that characterize $\mathbf{n}$.

Problem 3: A spin $\frac{1}{2}$ system is known to be in an eigenstate of $\mathbf{S}\cdot\mathbf{n}$ with eigenvalue $\frac{\hbar}{2}$, where $\mathbf{n}$ is a unit vector lying in the xz-plane that makes an angle γ with the positive z-axis.
(a). Suppose S_x is measured. What is the probability of getting $+\frac{\hbar}{2}$?
(b). Evaluate the dispersion in S_x, that is,

$$\langle (S_x - \langle S_x\rangle)^2\rangle.$$

(For your own peace of mind, check your answers for the special cases $\gamma = 0, \frac{\pi}{2}$ and π.

Problem 4: A certain observable in quantum mechanics has a 3 x 3 matrix representation as follows:

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

a. Find the normalized eigenvectors of this observable and the corresponding eigenvalues. Is there any degeneracy?
b. Give a physical example where all this is relevant.

Problem 5: Let A and B be observables. Suppose the simultaneous eigenkets of A and B $\{|a', b'\rangle\}$ form a *complete* orthonormal set of base kets. Can we always conclude that

$$[A, B] = 0?$$

If your answer is yes, prove the assertion. If your answer is no, give a counterexample.

Problem 6: Two Hermitian operators anticommute:

$$\{A, B\} = AB + BA = 0.$$

Is it possible to have a simultaneous (that is, common) eigenket of A and B? Prove or illustrate your assertion.