Hints to HW 2 problems

1. (ii) $$\left|\frac{n}{2n+1}-\frac{1}{2}\right| = \frac{1}{2(2n+1)}$$

$\varepsilon = \frac{1}{10}$: $\frac{1}{2(2n+1)} < \frac{1}{10}$

iff $2n+1 > 5$

iff $n > 2$

iff $n \geq 3$

Hence $N \geq 3$

Someone may say $N = 20$. That's OK

(iii) $$\left|\frac{n-1}{n+1}-1\right| = \frac{2}{n+1} \leq 2/3$$

$\Leftrightarrow$ $n+1 \geq 3$ $\Leftrightarrow$ $n \geq 2$

So N is any integer bigger than 1.

2. (i) $$\left|(-1)^n\frac{1}{n}-0\right| = \frac{1}{n}$$

Given $\varepsilon > 0$ by Archimedean property of $\mathbb{R}$ $\exists N$ with $N\varepsilon > 1$ etc.

(ii) $$\left|\frac{2n}{n+1}-2\right| = \frac{2}{n+1}$$

Given $\varepsilon > 0$ $\frac{2}{n+1} < \varepsilon$

$\Leftrightarrow$ $n+1 > 2/\varepsilon$ $\Leftrightarrow$ $n+1 \geq \left[\frac{2}{\varepsilon}\right]$

since $n \in \mathbb{N}$.

$\Leftrightarrow$ $n \geq \left[\frac{2}{\varepsilon}\right] - 1$

Thus we may choose $N = \max\{1, [\frac{2}{\varepsilon}]-1\}$.
The max is need for $\varepsilon > 1$.

3 (ii) Divide numerator, denominator by n^2 etc.

4. (i), (ii), (iii) are clearly unbounded.
Convergent sequences are bounded.

(iv) $$1+(-1)^n = \begin{cases} 0 & \text{if } n \text{ odd} \\ 2 & \text{if } n \text{ even} \end{cases}$$

The case $(-1)^n$ was given as exercise in class. One shows that there is no limit by contradiction.

(i) $l = 0$, or 2 one can take $\varepsilon = 1$

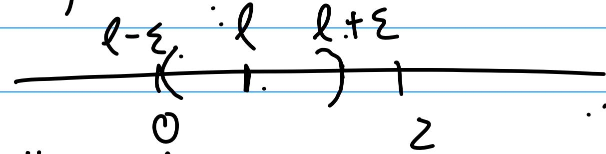
(ii) If $l < 0$ then take $\varepsilon = l$

(iii) If $l > 2$ take $\varepsilon = l-2$

(iv) If $0 < l < 2$ take $\varepsilon = \min\{l, 2-l\}$

In all these cases no N will exist such that $|x_n - l| < \varepsilon \;\; \forall n \geq N$.

Drawing a figure for each case helps.

e.g. (iv) 

$\boxed{l < 2-l}$

All x_{2n}'s fall outside $(l-\varepsilon, l+\varepsilon)$.

(v) $(1.01)^n \geq 1 + 0.01 \times n$ by binomial theorem etc.

5) (i) $1 + \frac{1}{2} + \cdots + \frac{1}{2^n} = 2 - \frac{1}{2^n}$ (G.P.)

By induction $\frac{1}{2^n} \leq \frac{1}{n} \;\; \forall n$

$\lim \frac{1}{n} = 0$. Hence, by sandwich thm

$\lim \frac{1}{2n} = 0 \Rightarrow \lim 2 - \frac{1}{2n} = 2$

(ii) The sequence is increasing and bounded by 2. Compare with (i).
I will prove later in class that such sequences converge.

6) $\exists M > 0$ s.t. $|y_n| \le M \quad \forall n$

$$\Rightarrow |x_n y_n| = |x_n| \cdot |y_n| \le \underbrace{|x_n| \cdot M}_{\to 0} \quad \forall n$$

By sandwich thm $\lim |x_n y_n| = 0$

$\Rightarrow \lim x_n y_n = 0$ since $-|x_n y_n| \le x_n y_n \le |x_n y_n|$

7) Given $\varepsilon > 0$ $\exists N$ such that

$$|x_n - 0| < \varepsilon^p \quad \forall n \geq N$$

$$\Rightarrow \quad x_n < \varepsilon^p$$

<u>claim</u>: $x_n^{1/p} < \varepsilon \quad \forall n$

If $x_n^{1/p} \geq \varepsilon$ then $(x_n^{1/p})^p \geq \varepsilon^p$ (check)

i.e. $x_n \geq \varepsilon^p \rightarrow\leftarrow$