Q1. The ground state of a SHO

$$\psi_0 = \left(\frac{\alpha}{\pi}\right)^{1/4} e^{-\alpha x^2/2} \qquad \alpha = \frac{m\omega_0}{\hbar}, \quad \omega_0^2 = \frac{k}{m}.$$

For ground state, $E = \frac{\hbar\omega_0}{2}$ and the non-classical region is

$$\frac{1}{2}\hbar\omega_0 < \frac{1}{2} m\omega_0^2 x^2$$

$$\Rightarrow \quad x^2 > \frac{\hbar}{m\omega_0} = \frac{1}{\alpha}, \quad \text{or} \quad \left\{ \begin{array}{l} x > \sqrt{\frac{1}{\alpha}} \\ x < -\sqrt{\frac{1}{\alpha}} \end{array} \right.$$

The probability of finding the particle in this region —

$$P = \int_{|x| > \sqrt{\frac{1}{\alpha}}} |\psi_0|^2 \, dx$$

$$= \int_{-\infty}^{-\sqrt{1/\alpha}} |\psi_0|^2 dx + \int_{\sqrt{1/\alpha}}^{\infty} |\psi_0|^2 dx$$

$$= 2\int_{\sqrt{1/\alpha}}^{\infty} \sqrt{\frac{\alpha}{\pi}}\, e^{-\alpha x^2} dx$$

$$= 2\left(\int_0^{\infty} \sqrt{\frac{\alpha}{\pi}}\, e^{-\alpha x^2} dx - \int_0^{\sqrt{1/\alpha}} \sqrt{\frac{\alpha}{\pi}}\, e^{-\alpha x^2} dx\right)$$

$$= 1 - 2\int_0^{\sqrt{1/\alpha}} \sqrt{\frac{\alpha}{\pi}}\, e^{-\alpha x^2} dx = 1 - \mathrm{Erf}(1) \approx 0.16$$

We have -

84. $H = -\lambda B S_z = \cancel{d} - \lambda B \frac{\hbar}{2} \sigma_z = \frac{1}{2}\hbar\omega \sigma_z$,

where $\omega = -\lambda B$.

We want to ~~can~~ calculate the Heisenberg operators $S_x^{(H)}(t)$, $S_y^{(H)}(t)$, and $S_z^{(H)}(t)$.

$\Rightarrow$ The $S_z^{(H)}(t) = S_z$, because $[S_z, H] = 0$.

We start with time-evolution of the ~~Schrödinger~~ Heisenberg operators — $S_x^{(H)}(t) = \frac{\hbar}{2}\sigma_x(t)$

$$i\hbar \frac{1}{2}\frac{d\sigma_x(t)}{dt} = \frac{1}{2}[\sigma_x(t), H] = -i\hbar\frac{\omega}{2}\sigma_y(t).$$

Similarly, for $S_y^{(H)}(t) = \frac{\hbar}{2}\sigma_y(t)$,

$$i\hbar\frac{1}{2}\frac{d\sigma_y(t)}{dt} = \frac{1}{2}[\sigma_y(t), H] = i\hbar\frac{\omega}{2}\sigma_x(t).$$

Now we have two coupled linear eqns.

$$\cancel{\dot{\sigma}_x = -\omega\sigma_y} \quad \dot{\sigma}_x(t) = -\omega\sigma_y(t), \quad \dot{\sigma}_y(t) = \omega\sigma_x(t).$$

Let us define two new operators

$\sigma_\pm(t) = \sigma_x(t) \mp i\sigma_y(t)$, and obtain

$$\dot{\sigma}_\pm(t) = -\omega\sigma_y(t) \pm i\omega\sigma_x(t) = \pm i\omega\sigma_\pm(t).$$

From these eqns. we have

$$\sigma_\pm(t) = \sigma_\pm(0)e^{\pm i\omega t},$$

and $\sigma_x(t) = (\sigma_+(t) + \sigma_-(t))/2$

$= \sigma_x(0)\cos\omega t - \sigma_y(0)\sin\omega t.$

Note, at $t=0$, $\sigma_x(0) = \sigma_x$ & $\sigma_y(0) = \sigma_y$.

Then, $\sigma_x(t) = \sigma_x \cos \omega t - \sigma_y \sin \omega t$,

and $S_x^{(H)}(t) = \frac{\hbar}{2} \sigma_x(t)$.

following the same steps –

$\sigma_y(t) = \sigma_y \cos \omega t + \sigma_x \sin \omega t$, and

$S_y^{(H)}(t) = \frac{\hbar}{2} \sigma_y(t)$.

(ii) $[a,(a^{+})^{n}] = n(a^{+})^{n-1}$

Solution.

$$[a, a^{+}(a^{+})^{n-1}] = [a, a^{+}](a^{+})^{n-1} + a^{+}[a, (a^{+})^{n-1}]$$

$$= (a^{+})^{n-1} + (a^{+})[a,(a^{+})^{n-1}].$$

$$= (a^{+})^{n-1} + (a^{+})\{[a, a^{+}](a^{+})^{n-2} + (a^{+})[a,(a^{+})^{n-2}]\}$$

$$= (a^{+})^{n-1} + (a^{+})(a^{+})^{n-2} + (a^{+})^{2}[a, (a^{+})^{n-2}]$$

$$= (a^{+})^{n-1} + (a^{+})^{n-1} + (a^{+})^{2}[a, (a^{+})^{n-2}].$$

So, we can say that,

$$\boxed{[a,(a^{+})^{n}] = n.(a^{+})^{n-1}}$$

⇒ (iii) $[a^{+},(a)^{n}] = -n\, a^{n-1}$

Solution

$[a^{+} a] = -1.$

$$[a^{+}, a^{n}] = [a^{+}, a a^{n-1}] = [a^{+} a] a^{n-1} + a[a^{+}, a^{n-1}]$$

$$= -a^{n-1} + a[a^{+}, a^{n-1}]$$

$$= -a^{n-1} + a\{[a^{+} a] a^{n-2} + a[a^{+} a^{n-2}]\}$$

$$= -a^{n-1} - a^{n-1} + a^{2}[a^{+}, a^{n-2}]$$

$$= -n a^{n-1}$$

(iii) $[N, a^n] = [a^+ a, a^n] = [a^+, a^n] a + a^+ \underbrace{[a, a^n]}_{0}$.

$= \cancel{[a^+ a^n] a + [a^+}$

$[a^+, a^n] a = -n (a^{n-1}) \cdot a = -n a^n$.

(iv) $[N, (a^+)^n] = [a^+ a, (a^+)^n]$

$= \cancel{[a^+, (a^+)^n]}^{\to 0} a + a^+ [a, (a^+)^n]$

$= a^+ n (a^+)^{n-1} = n (a^+)^n$

(b) matrix Representation :-

$\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle$ action of $\hat{a}$

$\hat{a}^{+}|n\rangle = \sqrt{n+1}\,|n+1\rangle$ " " $\hat{a}^{+}$

So, matrix element O_{mn}.

So,

1) O_{mn} for $\hat{a} = \langle m|\hat{a}|n\rangle = \langle m|\sqrt{n}|n-1\rangle = \sqrt{n}\,\delta_{m,n-1}$ —— ①

2) " " $\hat{a}^{+} = \langle m|\hat{a}^{+}|n\rangle = \langle m|\sqrt{n+1}|n+1\rangle = \sqrt{n+1}\,\delta_{m,n+1}$ —— ②

" " $\hat{x}$.

we know $\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\,(\hat{a}+\hat{a}^{+})$

3) O_{mn} for $\hat{x} = \langle m|\hat{x}|n\rangle$

$= \sqrt{\frac{\hbar}{2m\omega}}\,[\langle m|\hat{a}+\hat{a}^{+}|n\rangle] = \sqrt{\frac{\hbar}{2m\omega}}\,[\langle m|\hat{a}|n\rangle + \langle m|\hat{a}^{+}|n\rangle]$

$= \sqrt{\frac{\hbar}{2m\omega}}\left[\delta_{m,n-1}\sqrt{n} + \delta_{m,n+1}\sqrt{n+1}\right]$. —— ③

similarly for $\hat{p} = i\sqrt{\frac{m\hbar\omega}{2}}\,(\hat{a}^{+}-\hat{a})$

4) O_{mn} for $\hat{p} = \langle m|\hat{p}|n\rangle = i\sqrt{\frac{m\hbar\omega}{2}}\,[\langle m|\hat{a}^{+}-\hat{a}|n\rangle]$

$= i\sqrt{\frac{m\hbar\omega}{2}}\left[\delta_{m,n+1}\sqrt{n+1} - \sqrt{n}\,\delta_{m,n-1}\right]$. —— ④

5) $\hat{X}^2 = \left(\sqrt{\frac{\hbar}{2m\omega}}\right)^2 (\hat{a}+\hat{a}^+)\ (\hat{a}+\hat{a}^+)$

$= \frac{\hbar}{2m\omega}\left(\hat{a}^2 + (\hat{a}^+)^2 + aa^+ + a^+a\right)$

we know $[a\,a^+] = \mathbb{1}$, $aa^+ - a^+a = \mathbb{1}$

$aa^+ + \mathbb{1} = a^+a$

$= \frac{\hbar}{2m\omega}\left(\hat{a}^2 + (\hat{a}^+)^2 + 2\hat{a}^+a + \mathbb{1}\right).$

$O_{mn} = \frac{\hbar}{2m\omega} \langle m | \hat{a}^2 + (\hat{a}^+)^2 + 2\hat{a}^+a + \mathbb{1} | n\rangle$

$= \frac{\hbar}{2m\omega}.\left[\sqrt{n}\sqrt{n-1}\,\delta_{m,n-2} + \sqrt{n+1}\sqrt{n+2}\,\delta_{m,n+2} + 2n\delta_{m,n} + \delta_{m,n}\right].$ —— 5

similarly for $\hat{p}^2$

$\hat{p}^2 = i\sqrt{\frac{m\hbar\omega}{2}}\,(\hat{a}^+ - \hat{a})\ i\sqrt{\frac{m\hbar\omega}{2}}\,(\hat{a}^+ - \hat{a})$

$= -\frac{m\hbar\omega}{2}\left[\hat{a}^2 + (\hat{a}^+)^2 - \hat{a}\hat{a}^+ - \hat{a}^+a\right]$

$= -\frac{m\hbar\omega}{2}\left[(\hat{a})^2 + (\hat{a}^+)^2 - 2\hat{a}^+a - \mathbb{1}\right].$

$O_{mn} = -\frac{m\hbar\omega}{2} \langle m | (\hat{a})^2 + (\hat{a}^+)^2 - 2\hat{a}^+a - \mathbb{1} | n\rangle$

$= -\frac{m\hbar\omega}{2}\left[\delta_{m,n-2}\sqrt{n}\sqrt{n-1} + \delta_{m,n+2}\sqrt{n+1}\sqrt{n+2} - 2n\,\delta_{m,n} + \delta_{mn}\right].$ —— 6

5) $\hat{x}^2 = \left(\sqrt{\frac{\hbar}{2m\omega}}\right)^2 (\hat{a}+\hat{a}^+)\ (\hat{a}+\hat{a}^+)$

$$= \frac{\hbar}{2m\omega}\left(\hat{a}^2 + (\hat{a}^+)^2 + aa^+ + a^+a\right)$$

we know $[a\,a^+] = \mathbb{1}$, $aa^+ - a^+a = \mathbb{1}$

$aa^+ + \mathbb{1} = a^+a$.

$$= \frac{\hbar}{2m\omega}\left(\hat{a}^2 + (\hat{a}^+)^2 + 2\hat{a}^+a + \mathbb{1}\right).$$

$$O_{mn} = \frac{\hbar}{2m\omega} \langle m|\, \hat{a}^2 + (\hat{a}^+)^2 + 2\hat{a}^+a + \mathbb{1}\,|n\rangle$$

$$= \frac{\hbar}{2m\omega}.\Big[\sqrt{n}\sqrt{n-1}\,\delta_{m,n-2} + \sqrt{n+1}\sqrt{n+2}\,\delta_{m,n+2} + 2n\delta_{m,n} + \delta_{m,n}\Big] \quad \text{—— } (5)$$

similarly for $\hat{p}^2$

$$\hat{p}^2 = i\sqrt{\frac{m\hbar\omega}{2}}\,(\hat{a}^+ - \hat{a})\ i\sqrt{\frac{m\hbar\omega}{2}}\,(\hat{a}^+ - \hat{a})$$

$$= -\frac{m\hbar\omega}{2}\left[\hat{a}^2 + (\hat{a}^+)^2 - \hat{a}\hat{a}^+ - \hat{a}^+a\right]$$

$$= -\frac{m\hbar\omega}{2}\left[(\hat{a})^2 + (\hat{a}^+)^2 - 2\hat{a}^+a - \mathbb{1}\right].$$

$$O_{mn} = -\frac{m\hbar\omega}{2} \langle m|\, (\hat{a})^2 + (\hat{a}^+)^2 - 2\hat{a}^+a - \mathbb{1}\,|n\rangle$$

$$= -\frac{m\hbar\omega}{2}\Big[\delta_{m,n-2}\sqrt{n}\sqrt{n-1} + \delta_{m,n+2}\sqrt{n+1}\sqrt{n+2} - 2n\,\delta_{m,n} - \delta_{mn}\Big] \quad \text{—— } (6)$$

matrix for $|n\rangle$

$$\hat{a} = \begin{array}{c|c} & \begin{array}{cccc} |0\rangle & |1\rangle & |2\rangle & |3\rangle \end{array} \\ \begin{array}{c} \langle m| \\ \langle 0| \\ \langle 1| \\ \langle 2| \\ \langle 3| \end{array} & \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

using $\hat{a} = \delta_{m,n-1}\sqrt{n}$

$$\hat{a}^{+} = \begin{array}{c|c} & \begin{array}{cccc} |0\rangle & |1\rangle & |2\rangle & |3\rangle \end{array} \\ \begin{array}{c} \langle 0| \\ \langle 1| \\ \langle 2| \\ \langle 3| \end{array} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ \sqrt{1} & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix} \end{array}$$

using $\hat{a}^{+} = \delta_{m,n+1}\sqrt{n+1}$

$$\hat{x} = \begin{array}{c|c} & \begin{array}{cccc} |0\rangle & |1\rangle & |2\rangle & |3\rangle \end{array} \\ \begin{array}{c} \langle 0| \\ \langle 1| \\ \langle 2| \\ \langle 3| \end{array} & \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 \\ \sqrt{1} & 0 & \sqrt{2} & 0 \\ 0 & \sqrt{2} & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix} \end{array}$$

Let $\sqrt{\frac{\hbar}{2m\omega}} = 1$

using

$\hat{x} = \delta_{m,n-1}\sqrt{n} + \delta_{m,n+1}\sqrt{n+1}$

$$\hat{p} = \begin{array}{c|c} & \begin{array}{cccc} |0\rangle & |1\rangle & |2\rangle & |3\rangle \end{array} \\ \begin{array}{c} \langle 0| \\ \langle 1| \\ \langle 2| \\ \langle 3| \end{array} & \begin{pmatrix} 0 & -\sqrt{1} & 0 & 0 \\ \sqrt{1} & 0 & -\sqrt{2} & 0 \\ 0 & \sqrt{2} & 0 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix} \end{array}$$

Let $i\sqrt{\frac{m\hbar\omega}{2}} = 1$

$\hat{p} = \delta_{m,n+1}\sqrt{n+1} - \delta_{m,n-1}\sqrt{n}$

$$\hat{x}^2 = \begin{array}{c} \\ \langle 0| \\ \langle 1| \\ \langle 2| \\ \langle 3| \end{array} \begin{array}{c} \begin{array}{cccc} |0\rangle & |1\rangle & |2\rangle & |3\rangle \end{array} \\ \begin{pmatrix} 1 & 0 & \sqrt{2} & 0 \\ 0 & 3 & 0 & \sqrt{6} \\ \sqrt{2} & 0 & 5 & 0 \\ 0 & \sqrt{6} & 0 & 7 \end{pmatrix} \end{array}$$

taking $\frac{\hbar}{2m\omega} = 1$

Using eq^n (5)

$$\hat{p}^2 = \begin{array}{c} \\ \langle 0| \\ \langle 1| \\ \langle 2| \\ \langle 3| \end{array} \begin{array}{c} \begin{array}{cccc} |0\rangle & |1\rangle & |2\rangle & |3\rangle \end{array} \\ \begin{pmatrix} -1 & 0 & \sqrt{2} & 0 \\ 0 & -3 & 0 & \sqrt{6} \\ \sqrt{2} & 0 & -5 & 0 \\ 0 & \sqrt{6} & 0 & -7 \end{pmatrix} \end{array}$$

let. $-\frac{m\hbar\omega}{2} = 1$

Using eq^n (6)

for commutator :-

$$[\hat{x}, \hat{p}] = \hat{x}\hat{p} - \hat{p}\hat{x}$$

$$\sum_{k=0}^{\infty} \langle m|\hat{x}|k\rangle \langle k|\hat{p}|n\rangle - \langle m|\hat{p}|k\rangle\langle k|\hat{x}|n\rangle$$

So,

$$\sum_{k} \langle m|\hat{x}|k\rangle \langle k|\hat{p}|n\rangle =$$

$$\sum_{k} \sqrt{\frac{\hbar}{2m\omega}}\, i\sqrt{\frac{m\hbar\omega}{2}} \left[\delta_{m,k-1}\sqrt{k} + \delta_{m,k+1}\sqrt{k+1}\right]\left[\delta_{k,n+1}\sqrt{n+1} - \sqrt{n}\,\delta_{m,n-1}\right]$$

$$= \sum_{k} \frac{i\hbar}{2}\left[\delta_{m,k-1}\,\delta_{k,n+1}\sqrt{k}\sqrt{n+1} - \delta_{m,k-1}\sqrt{k}\,\delta_{k,n-1}\sqrt{n} + \delta_{m,k+1}\sqrt{k+1}\,\delta_{k,n+1}\sqrt{n+1} - \delta_{m,k+1}\sqrt{k+1}\,\delta_{k,n-1}\sqrt{n}\right]$$

$$= \frac{i\hbar}{2}\left[\delta_{m,n}(n+1) - \delta_{m,n-2}\sqrt{n-2}\sqrt{n} + \delta_{m,n+2}\sqrt{n+2}\sqrt{n+1} - \delta_{mn}\, n\right]$$

$$= \frac{i\hbar}{2}\left[\delta_{m,n} - \delta_{m,n-2}\sqrt{n-2}\sqrt{n} + \delta_{m,n+2}\sqrt{n+2}\sqrt{n+1}\right]$$

similarly we can calculate second term, will get

$$= \frac{i\hbar}{2}\left[-\delta_{m,n} - \delta_{m,n-2}\sqrt{n-2}\sqrt{n} + \delta_{m,n+2}\sqrt{n+2}\sqrt{n+1}\right]$$

$$[\hat{x}, \hat{p}] = \frac{i\hbar}{2}\{\delta_{mn} + \delta_{mn}\} = i\hbar\,\delta_{m,n}$$

$$\boxed{\langle m|[\hat{x}, \hat{p}]|n\rangle = i\hbar\,\delta_{m,n}}$$

$$\hat{x} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & \sqrt{2} & 0 \\ 0 & \sqrt{2} & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix} \qquad \hat{p} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & -\sqrt{2} & 0 \\ 0 & \sqrt{2} & 0 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}.$$

So,

$$[\hat{x}, \hat{p}] = \hat{x}\hat{p} - \hat{p}\hat{x}$$

$$= \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \boxed{-6} \end{pmatrix}$$ due to this shifting.

(d) $\langle n|x^2|n\rangle = \frac{\hbar}{m\omega}(n+1/2)$.

we know that,

$$\langle m|x^2|n\rangle = \frac{\hbar}{2m\omega}\Big[\sqrt{n}\sqrt{n-1}\,\delta_{m,n-2} + \sqrt{n+1}\sqrt{n+2}\,\delta_{m,n+2} + 2n\,\delta_{m,n} + \delta_{mn}\Big]$$

put $m=n$.

$$\langle n|x^2|n\rangle = \frac{\hbar}{2m\omega}\Big[\sqrt{n}\sqrt{n-1}\,\delta_{n,n-2} + \sqrt{n+1}\sqrt{n+2}\,\delta_{n,n+2} + 2n\,\delta_{n,n} + \delta_{n,n}\Big]$$

$$= \frac{\hbar}{2m\omega}\Big[(2n+1)\Big].$$

similarly for

$$\langle m|p^2|n\rangle = -\frac{m\hbar\omega}{2}\Big[\delta_{m,n-2}\sqrt{n}\sqrt{n-1} + \delta_{m,n+2}\sqrt{n+1}\sqrt{n+2} + -2n\,\delta_{m,n} - \delta_{m,n}\Big]$$

put $m=n$.

$$\langle n|p^2|n\rangle = -\frac{m\hbar\omega}{2}\Big[\delta_{n,n-2}\sqrt{n}\sqrt{n-1} + \delta_{n,n+2}\sqrt{n+1}\sqrt{n+2} + -2n\,\delta_{nn} - \delta_{nn}\Big]$$

$$= -\frac{m\hbar\omega}{2}\Big[(-2n-1)\Big] = \frac{m\hbar\omega}{2}(2n+1)$$

$$\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{\langle x^2\rangle} \cdot \sqrt{\frac{\hbar}{2m\omega}(2n+1)}$$

$$\Delta p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \sqrt{\langle p^2\rangle} = \sqrt{\frac{m\hbar\omega}{2}(2n+1)}$$

$$\boxed{\Delta x\,\Delta p = \left(n+\frac{1}{2}\right)\hbar}$$

1\. (a) First we note $\vec{\mu} = \gamma \vec{S}$ where $\gamma = \frac{qg}{2m}$

$$\hat{H}(t) = -\gamma B_0 \cos(\omega t) \hat{S}_z$$

Since $[H(t), H(t')] = 0 \quad \forall t, t'$, we have

$$U(t) = \exp\left[-\frac{i}{\hbar}\int_0^t dt' \, \hat{H}(t')\right]$$

$$= \exp\left[-\frac{i}{\hbar}\int_0^t -\gamma B_0 \cos(\omega t') \hat{S}_z \, dt'\right]$$

$$= \exp\left[\frac{i\gamma B_0}{\hbar} \left.\frac{\sin(\omega t')}{\omega}\right|_{t'=0}^{t} \hat{S}_z\right]$$

$$\boxed{= \exp\left[\frac{i\gamma B_0 \sin(\omega t) \hat{S}_z}{\hbar\omega}\right]}$$

(b) $|\Psi, t\rangle = U(t)\,|\Psi, 0\rangle$

$$= \exp\left[\frac{i\gamma B_0 \sin(\omega t)}{\hbar\omega}\hat{S}_z\right]\left(\frac{1}{\sqrt{2}}|z;+\rangle + \frac{1}{\sqrt{2}}|z;-\rangle\right)$$

$$= \exp\left[\frac{i\gamma B_0 \sin(\omega t)}{2\omega}\right]\frac{1}{\sqrt{2}}|z;+\rangle + \exp\left[\frac{-i\gamma B_0 \sin(\omega t)}{2\omega}\right]\frac{1}{\sqrt{2}}|z;-\rangle$$

$$= \frac{1}{\sqrt{2}}\exp\left[\frac{i\gamma B_0 \sin(\omega t)}{2\omega}\right]\left(|z;+\rangle + \exp\left[\frac{-i\gamma B_0 \sin(\omega t)}{\omega}\right]|z;-\rangle\right)$$

$$\boxed{\theta(t) = \frac{\pi}{2} \qquad \phi(t) = -\frac{\gamma B_0 \sin(\omega t)}{\omega}}$$

(c) $\langle x;-|\Psi, t\rangle = \left(\frac{1}{\sqrt{2}}\langle z;+| - \frac{1}{\sqrt{2}}\langle z;-|\right)|\Psi, t\rangle$

$$= \frac{1}{2}\exp\left[\frac{i\gamma B_0 \sin(\omega t)}{2\omega}\right]\left(1 - \exp\left[\frac{-i\gamma B_0 \sin(\omega t)}{\omega}\right]\right)$$

$$|\langle x;-|\Psi, t\rangle|^2 = \frac{1}{4}\left|1 - \cos\frac{\gamma B_0 \sin(\omega t)}{\omega} + i \sin\frac{\gamma B_0 \sin(\omega t)}{\omega}\right|^2$$

$$= \frac{1}{4}\left[\left(1 - \cos\frac{\gamma B_0 \sin(\omega t)}{\omega}\right)^2 + \sin^2\frac{\gamma B_0 \sin(\omega t)}{\omega}\right]$$

$$= \frac{1}{4}\left(2 - 2\cos\frac{\gamma B_0 \sin(\omega t)}{\omega}\right)$$

$$= \frac{1 - \cos\frac{\gamma B_0 \sin(\omega t)}{\omega}}{2}$$

$$\boxed{= \sin^2\frac{\gamma B_0 \sin(\omega t)}{2\omega}}$$

(d) We need $\frac{\gamma B_0}{2\omega} \geq \frac{\pi}{2}$ i.e. $\omega \leq \frac{\gamma B_0}{\pi}$

Largest value of ω that allows full flip in S_x is $\boxed{\frac{\gamma B_0}{\pi}}$.

2. Heisenberg equation of motion for time-independent $\hat{A}_S$:

$$i\hbar \frac{d\hat{A}_H(t)}{dt} = [\hat{A}_H(t), \hat{H}_H(t)]$$

First, we have $\boxed{\hat{S}_z(t) = \hat{S}_z}$ because $\hat{S}_z$ commutes with $\hat{H}$.

Next, for $\hat{S}_x(t)$, it satisfies

$$\begin{aligned} i\hbar \frac{d\hat{S}_x(t)}{dt} &= [\hat{S}_x(t), -\lambda B \hat{S}_z(t)] \\ &= -\lambda B [\hat{S}_x(t), \hat{S}_z(t)] \\ &= i\hbar\lambda B \hat{S}_y(t) \end{aligned}$$

("Heisenberg-ing" preserves commutation relations)

and for $\hat{S}_y(t)$, it satisfies

$$\begin{aligned} i\hbar \frac{d\hat{S}_y(t)}{dt} &= [\hat{S}_y(t), -\lambda B \hat{S}_z(t)] \\ &= -\lambda B i\hbar \hat{S}_x(t) \end{aligned}$$

Thus, $$\frac{d^2\hat{S}_x(t)}{dt^2} = \lambda B \frac{d\hat{S}_y(t)}{dt} = -\lambda^2 B^2 \hat{S}_x(t)$$

$$\hat{S}_x(t) = \hat{c}_1 \cos \lambda B t + \hat{c}_2 \sin \lambda B t$$

$$\hat{S}_y(t) = \frac{1}{\lambda B} \frac{d\hat{S}_x(t)}{dt} = -\hat{c}_1 \sin \lambda B t + \hat{c}_2 \cos \lambda B t$$

At $t=0$, $\hat{S}_x(t) = \hat{S}_x$ and $\hat{S}_y(t) = \hat{S}_y$. $\Rightarrow \hat{c}_1 = \hat{S}_x$, $\hat{c}_2 = \hat{S}_y$

$$\therefore \boxed{\begin{aligned} \hat{S}_x(t) &= \hat{S}_x \cos \lambda B t + \hat{S}_y \sin \lambda B t \\ \hat{S}_y(t) &= -\hat{S}_x \sin \lambda B t + \hat{S}_y \cos \lambda B t \end{aligned}}$$