PHY201: Assignment 3

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Problems

Problem 1
Let the incident electromagnetic field be given by:

$$\vec{E}_I = \vec{E}_{0I}\cos(\vec{k}_I.\vec{r} - \omega t) \text{ and } \vec{B}_I = \frac{1}{v_1}(\vec{k}_I \times \vec{E}_I)$$

Similarly for reflected and transmitted waves:

$$\vec{E}_R = \vec{E}_{0R}\cos(\vec{k}_R.\vec{r} - \omega t) \text{ and } \vec{B}_R = \frac{1}{v_1}(\vec{k}_R \times \vec{E}_R)$$
$$\vec{E}_T = \vec{E}_{0T}\cos(\vec{k}_T.\vec{r} - \omega t) \text{ and } \vec{B}_T = \frac{1}{v_2}(\vec{k}_T \times \vec{E}_T)$$

Evidently,

$$k_I v_1 = k_T v_2 = \omega \implies k_I = \frac{v_2}{v_1}k_T = \frac{n_1}{n_2}k_T$$

Where n_i is the refractive indices of the medium. At the interface, we will apply the boundary conditions on the parallel and perpendicular components of the electromagnetic fields. As they all has the exponential factor in them, to guarantee that the boundary conditions are satisfied for all times, we must have:

$$\begin{aligned}&\vec{k}_I.\vec{r} = \vec{k}_R.\vec{r} = \vec{k}_T.\vec{r}\\ &\implies z(k_I)_z + y(k_I)_y = z(k_R)_z + y(k_R)_y = z(k_T)_z + y(k_T)_y\end{aligned}$$

at $x = 0$ (interface). For $z = 0$, we get:

$$(k_I)_y = (k_R)_y = (k_T)_y$$

For $y = 0$, we get:

$$(k_I)_z = (k_R)_z = (k_T)_z$$

For simplicity let us orient $\vec{k_I}$ in the XY plane. Therefore, $(k_I)_z = 0$ and we get the first law of refraction that incident, reflected and refracted ray - all of them has to reside on the same plane.
Again, $(k_I)_y = k_I \sin\theta_I$ and $(k_T)_y = k_T \sin\theta_T$ Therefore, we get

$$\frac{\sin\theta_I}{\sin\theta_T} = \frac{n_2}{n_1}$$

This is Snell's law.
The boundary conditions for the electromagnetic field at the interface yields:

$$i)\epsilon_1(E_{0I} + E_{0R})_x = \epsilon_2(E_{0T})_x$$
$$ii)(E_{0I} + E_{0R})_{y,z} = (E_{0T})_{y,z}$$
$$iii)\frac{1}{\mu_1}(B_{0I} + B_{0R})_{y,z} = \frac{1}{\mu_2}(B_{0T})_{y,z}$$
$$iv)(B_{0I} + B_{0R})_x = (B_{0T})_x$$

Problem 2

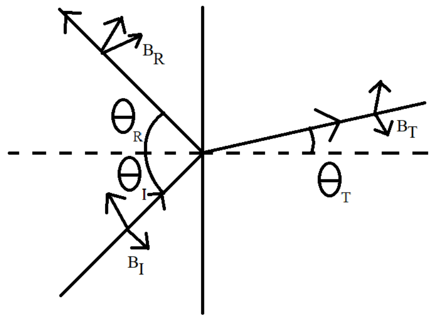

Figure 1: Incident, Reflected and Transmitted Ray at Interface

Let us take the magnetic field to be confined in the XY plane. Therefore, we can write from the third and fourth of the above conditions:

$$\frac{1}{\mu_1}(-B_{0I}\cos\theta_I + B_{0R}\cos\theta_R) = -\frac{1}{\mu_2}B_{0T}\cos\theta_T$$
$$B_{0I}\sin\theta_I + B_{0R}\sin\theta_R = B_{0T}\sin\theta_T$$

The second condition gives:

$$v_1(B_{0I} + B_{0R}) = v_2 B_{0T}$$
$$\implies B_{0I} + B_{0R} = \frac{n_1}{n_2}B_{0T}$$

which is just what we get from the fourth conditions with both sides being divided by $\sin\theta_I$ (We used $\theta_I = \theta_R$). Solving the above equations, we get:

$$B_{0R} = \frac{n-\beta}{n+\beta}B_{0I} \tag{1}$$
$$B_{0T} = \frac{2}{n+\beta}B_{0I} \tag{2}$$

Here, $n = \frac{n_1}{n_2}$ and $\beta = \frac{\mu_1\cos\theta_T}{\mu_2\cos\theta_I}$
Therefore, the reflectivity and transmittivity for the parallel component of the magnetic field is given by:

$$r_{||} = \frac{B_{0R}\cos\theta_R}{-B_{0I}\cos\theta_I} = -\frac{n-\beta}{n+\beta} \tag{3}$$
$$t_{||} = \frac{-B_{0T}\cos\theta_T}{-B_{0I}\cos\theta_I} = \frac{2}{n+\beta}\frac{\cos\theta_T}{\cos\theta_I} \tag{4}$$

The reflectivity and the transmittivity for the perpendicular components are similarly given by:

$$r_{pd} = \frac{B_{0R}\sin\theta_R}{B_{0I}\sin\theta_I} = \frac{n-\beta}{n+\beta} \tag{5}$$
$$t_{pd} = \frac{B_{0T}\sin\theta_T}{B_{0I}\sin\theta_I} = \frac{2n}{n+\beta} \tag{6}$$

Problem 3
Let us take the superposition of two electromagnetic waves given by:

$$\vec{E}_1 = \hat{\imath}E_0\sin(\omega t - kx)$$
$$\vec{E}_2 = \hat{\jmath}E_0\sin(\omega t - kx + \phi)$$

The resultant vector is therefor given by

$$\vec{E} = \hat{\imath}E_0 \sin(\omega t - kx) + \hat{\jmath}E_0 \sin(\omega t - kx + \phi)$$

1) **Clockwise Circular Polarization:**

$$\phi = \frac{\pi}{2}$$

Therefore, $\vec{E} = \hat{\imath}E_0 \sin(\omega t - kx) + \hat{\jmath}E_0 \cos(\omega t - kx)$
Note that at $x = 0$, for example, $\vec{E}$ points towards $\hat{\imath}$ for $\omega t = (4n+1)\frac{\pi}{2}$ and it points towards $\hat{\jmath}$ for $\omega t = 2n\pi$. Again $\frac{d\vec{E}}{dt}$ points towards $\hat{\imath}$ at $x = 0$ and $\omega t = 2n\pi$. Therefore, we get a clockwise circular polarisation.

2) **Anti-clockwise Circular Polarization:**

$$\phi = -\frac{\pi}{2}$$

Therefore, $\vec{E} = \hat{\imath}E_0 \sin(\omega t - kx) - \hat{\jmath}E_0 \cos(\omega t - kx)$
At $x = 0$, for example, $\vec{E}$ points towards $\hat{\imath}$ for $\omega t = (4n+1)\frac{pi}{2}$ and it points towards $-\hat{\jmath}$ for $\omega t = 2n\pi$. Again, $\frac{d\vec{E}}{dt}$ points towards $\hat{\imath}$ at $x = 0$ and $\omega t = 2n\pi$. Therefore, we get an anti- clockwise circular polarisation.

3) **Clockwise Elliptic Polarization:**
A general superposition as given earlier is:

$$\vec{E} = \hat{\imath}E_0 \sin(\omega t - kx) + \hat{\jmath}E_0 \sin(\omega t - kx + \phi)$$

To analyze the situation let's focus on $x = 0$ at $t = 0$.

$$\begin{aligned}
\vec{E}(0,0) &= \hat{\jmath}E_0 \sin\phi \\
\frac{d\vec{E}}{dt}(0,0) &= \omega E_0 \hat{\imath} + \omega E_0 \cos\phi \hat{\jmath}
\end{aligned}$$

Note that the expression controlling whether the polarization is clockwise or anticlockwise is $\vec{E}(0,0)$ as the second expression above (the derivative one) points towards the "right" side of the plane. It can easily be observed that for Clockwise Elliptic Polarization, $\vec{E}(0,0)$ has to point towards $\hat{\jmath}$. Therefore, we must have,

$$\begin{aligned}
&\sin\phi \geq 0 \\
\implies &0 \leq \phi \leq \pi
\end{aligned}$$

We get linear polarization for $\phi = 0, \pi$. For, $\phi = 0$, we get diagonal polarization and for $\phi = \pi$, we get anti-diagonal polarization.

Problem 4
Let the waves be represented by:

$$\begin{aligned}
&E_1 \cos(\omega t - kx) \text{ and} \\
&E_2 \cos(\omega t - kx + \Delta\phi(x))
\end{aligned}$$

As the frequencies are same, the wavelengths, hence k, are same.
Upon superposition, the resultant electric field is given by:

$$E = E_1 \cos(\omega t - kx) + E_2 \cos(\omega t - kx + \Delta\phi(x))$$

Therefore, the intensity at point x at time t is given by:

$$\begin{aligned}
I(x,t) &= E^2 \\
&= E_1^2 \cos^2(\omega t - kx) + E_2^2 \cos^2(\omega t - kx + \Delta\phi(x)) + 2E_1E_2 \cos(\omega t - kx)\cos(\omega t - kx + \Delta\phi(x)) \\
&= \frac{E_1^2}{2}[1 + \cos(2\omega t - 2kx)] + \frac{E_2^2}{2}[1 + \cos(2\omega t - 2kx + 2\Delta\phi(x))] \\
&+ E_1E_2[\cos(2\omega t - 2kx + \Delta\phi(x)) + \cos(\Delta\phi(x))]
\end{aligned}$$

The net intensity at point x can be calculated by taking the average of $I(x,t)$ over a complete cycle. Therefore,

$$I(x) = \frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} I(x,t)dt$$

Let's examine the first term of this integral:

$$\begin{aligned}
&\frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} \frac{E_1^2}{2}[1+\cos(2\omega t - 2kx)]dt \\
=&\frac{\omega}{2\pi}\frac{E_1^2}{2}\left[\frac{2\pi}{\omega} + \frac{1}{2\omega}\{\sin(4\pi - 2kx) - \sin(-2kx)\}\right] \\
=&\frac{E_1^2}{2}
\end{aligned}$$

(Here we used $\sin(4\pi - \theta) = -\sin\theta$)
We similarly evaluate the other two terms and the result is given by,

$$I(x) = \frac{E_1^2}{2} + \frac{E_2^2}{2} + E_1E_2\cos[\Delta\phi(x)]$$

Clearly, the maxima occurs when

$$\cos[\Delta\phi(x)] = 1 \implies \Delta\phi(x) = 2n\pi \text{ where } n \text{ is a natural number}$$

The minima occurs when

$$\cos[\Delta\phi(x)] = -1 \implies \Delta\phi(x) = (2n+1)\pi \text{ where } n \text{ is a natural number}$$

The distance between a maxima and minima is given by the derivatives of $\Delta\phi(x)$ with respect to x. To see this let x_0 be a point for maxima, therefore we can write the Taylor expansion for $\Delta\phi(x)$ around x_0 as

$$\Delta\phi(x) = \Delta\phi(x_0) + (x-x_0)\frac{d}{dx}\Delta\phi(x) + \frac{(x-x_0)^2}{2}\frac{d^2}{dx^2}\Delta\phi(x) + ...$$

How fast we achieve the minima intensity while going away from x_0 is described by the above expansion and that in turn gives the distance between a minimum and a maximum point.

Problem 5
For unequal frequencies, the intensity at point x at time t is given by

$$\begin{aligned}
I(x,t) =&\frac{E_1^2}{2}[1+\cos(2\omega_1 t - 2k_1x)] + \frac{E_2^2}{2}[1+\cos(2\omega_2 t - 2k_2 x + 2\Delta\phi(x))] \\
&+ E_1E_2[\cos\{(\omega_1+\omega_2)t - (k_1+k_2)x + \Delta\phi(x)\} + \cos\{(\omega_2-\omega_1)t - (k_2-k_1)x + \Delta\phi(x)\}]
\end{aligned}$$

Without loss of generality, let's consider: $\omega_1 > \omega_2$
We can take the average intensity in between two consecutive beats, which are separated by time interval: $\frac{2\pi}{\omega_1-\omega_2}$. Therefore, we can write:

$$\begin{aligned}
I(x) =& \frac{\omega_1-\omega_2}{2\pi}\int_0^{\frac{2\pi}{\omega_1-\omega_2}} I(x,t)dt \\
=& \frac{1}{2}(E_1^2+E_2^2) + \frac{E_1^2}{2}\frac{\omega_1-\omega_2}{4\pi\omega_1}[\sin\left(\frac{4\pi\omega_1}{\omega_1-\omega_2} - 2k_1x\right) + \sin(2k_1x)] \\
&+ \frac{E_2^2}{2}\frac{\omega_1-\omega_2}{4\pi\omega_2}[\sin\left(\frac{4\pi\omega_2}{\omega_1-\omega_2} - 2k_2x + 2\Delta\phi(x)\right) + \sin(2k_1x - 2\Delta\phi(x))] \\
&+ E_1E_2\frac{\omega_1-\omega_2}{2\pi(\omega_1+\omega_2)}[\sin\{2\pi\frac{\omega_1+\omega_2}{\omega_1-\omega_2} - (k_1+k_2)x + \Delta\phi(x)\} + \sin\{(k_1+k_2)x - \Delta\phi(x)\}]
\end{aligned}$$

From the above expression and the exact form of the function $\Delta\phi(x)$, we deduce the points of maximum and minimum intensity.

Problem 6

As both $\vec{E}_1$ and $\vec{E}_2$ are solutions to Maxwell's wave equation:

$$\nabla^2 \vec{E} = \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} \tag{7}$$

with propagation vector $\vec{k}_1$ and $\vec{k}_2$ respectively, we can write:

$$\nabla^2 \vec{E}_1 = -k_1^2 \vec{E}_1 \tag{8}$$

$$\nabla^2 \vec{E}_2 = -k_2^2 \vec{E}_2 \tag{9}$$

Multiplying Eq.() by a and Eq.() by b and adding, we get:

$$\nabla^2 (a\vec{E}_1 + b\vec{E}_2) = -ak_1^2 \vec{E}_1 - bk_2^2 \vec{E}_2$$

If the addition rule for the propagation rule was valid, then the wave equation we would've got is given by:

$$\nabla^2 (a\vec{E}_1 + b\vec{E}_2) = -(a\vec{k}_1 + b\vec{k}_2)^2 (a\vec{E}_1 + b\vec{E}_2)$$

which is quite different from what we got by superposing the individual waves using Maxwell's equations. Therefore, the proposed rule for the propagation vector of the resultant field is invalid.