Indian Institute of Science Education and Research, Mohali
Integrated MSc, Semester: IV
Probability and Statistics: MTH 202

Tutorial 8 (March 15, 2023)

Summary

Failure rate or Hazard rate '$\lambda_X(t)$' of a positive random variable X with probability density function f_X and cumulative distribution function F_X is defined by

$$\lambda_X(t) = \frac{f_X(t)}{1 - F_X(t)} \chi_{(0,\infty)}(t).$$

Hazard rate λ of a random variable X, uniquely determine the cumulative distribution function F_X by

$$F_X(t) = [1 - e^{-\int_0^t \lambda(s)ds}]\chi_{[0,\infty)}(t).$$

For exponential random variable (X) with parameter λ the Failure rate is the constant λ and it is memoryless: condional probability $P(X \geq s+t)|x \geq s) = P(X \geq r+t)|x \geq r) = P(X \geq t)$.

Gamma distribution with parameters $(\alpha, \lambda), \alpha$ and λ are positive :

$$f(x) = \frac{\lambda e^{-\lambda x}(\lambda x)^{(\alpha-1)}}{\Gamma(\alpha)} \chi_{[0,\infty)}(x).$$

Here Γ is the **Gamma** function defined by

$$\Gamma(\alpha) = \int_0^\infty e^{-s} s^{\alpha-1} ds.$$

It satisfies $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$ and hence $\Gamma(n+1) = n!$.

The exponential distribution is the Gamma distribution with $\alpha = 1$.

Random Vector or Multivariate random variable: Let $(\Omega, \mathcal{A}, P)$ be a probability space. A function $X = (X_1, X_2, \cdots, X_n) : S \to \mathbb{R}^n$ is said to be a random vector if each co-ordinate $X_k : \Omega \to \mathbb{R}$ is a random variable. Here we restrict to bivariate case, i.e., $X = (X_1, X_2)$. For general n, the theory is very similar.

Joint Cumulative Distribution Function F_X of $X = (X_1, X_2)$: $F_X : \mathbb{R}^2 \to [0,1]$, is defined by

$$F_X(t_1, t_2) = P(X_1 \leq t_1, X_2 \leq t_2) = P(X_1^{-1}(-\infty, t_1], X_2^{-1}(-\infty, t_2]) = P(X^{-1}((-\infty, t_1] \times (-\infty, t_2])).$$

This function F_X completely describes the probability law for the random vector X, i.e., joint probabilistic behavior of the random variables X_1 and X_2. The cumulative distribution functions of X_1 and X_2 are given

$$\text{by } F_{X_1}(t_1) = \lim_{t_2 \to \infty} F_X(t_1, t_2) \text{ and } F_{X_2}(t_2) = \lim_{t_1 \to \infty} F_X(t_1, t_2)$$

and these are called **Marginal Distribution** of X.

Discrete Random Vector ($X(\Omega)$ is countable, equivalently X_1 and X_2 are discrete random variables):

$$P_X(k_1, k_2) = P(X_1 = k_1, X_2 = k_2) = P[(X_1 = k_1) \cap (X_2 = k_2)]$$

is the **Joint Probability Mass Function** of $X = (X_1, X_2)$. The **(Marginal) Probability Mass Function** $P_{X_1}(k_1) = \sum_{k_2 \in X_2(\Omega)} P_X(k_1, k_2)$ and other marginal can be defined similatrly.

Continuous Random Vector X**:** There exists a non-negative function $f_X(x_1, x_2)$ whose Riemann integration on $\mathbb{R}^2$ is 1 such that

$$F_X(t_1, t_2) = \int_{-\infty}^{t_1} \int_{-\infty}^{t_2} f_X(x_1, x_2) dx_2 dx_1.$$

This F_X is called the **Joint Probability Density Function** of X. By fundamental theorem of calculus

$$f_X(x_1, x_2) = \frac{\partial^2}{\partial x_2 \partial x_1} F_X(x_1, x_2) = \frac{\partial^2}{\partial x_1 \partial x_2} F_X(x_1, x_2).$$

The **(Marginal) Probability Density Functions** $f_{X_1}(x_1) = \int_{-\infty}^{\infty} f_X(x_1, x_2) dx_2$ and $f_{X_2}(x_2) = \int_{-\infty}^{\infty} f_X(x_1, x_2) dx_1$.

Independence: Two random variables X_1 and X_2 are said to be independent if the events $(X_1^{-1}[s_1, t_1])$ and $(X_2^{-1}[s_2, t_2])$ are independent. The following are equivalent:

- X_1 and X_2 are independent.
- $F_X(t_1, t_2) = F_{X_1}(t_1) F_{X_2}(t_2)$.
- (**Discrete**) $P_X(k_1, k_2) = P_{X_1}(k_1) P_{X_2}(k_2)$,
  (**Continuous**) $f_X(x_1, x_2) = f_{X_1}(x_1) f_{X_2}(x_2)$.

Question

1. Suppose life span (X : in years) of an instrument has hazard rate $\lambda(t) = t^3 \chi_{(0,\infty)}(t)$. Find the probability that the instrument life time lie in between $\sqrt{2}$ to 2.
2. Let X and Y are two non-negative continuous random variables with hazard rates $\lambda_X(t) = 3\lambda_Y(t)$. What is the relation between conditional probability $P(X \geq 10 | X \geq 5)$ and $P(Y \geq 10 | Y \geq 5)$.
3. Find the expectation of Gamma random variable with parameters λ, α).
4. Two fair dice rolled. Find the joint probability mass function of $X = (S, L)$. Where S is the sum and L is the largest of the two values obatained.
5. Suppose joint density of continuous bi-variate random variable $X = (X_1, X_2)$ is

   $$f_X(x_1, x_2) = c(x_1 + x_2) \chi_A(x_1, x_2),$$

   where $A = \{(x_1, x_2) : 0 \leq x_1, x_2 \leq 1\}$. Find the value of c and the joinf distribution function of X. Compute the marginal probability density functions f_{X_1} and f_{X_2}. Compute $\mathbb{E}(X_1)$. Are X_1 and X_2 independent?