PHY201 Assignment 1

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Contents

Question 1

Given $\omega = 10\ s^{-1}$ with $x(0) = 10\ cm,\ v(0) = 10\ cm\ s^{-1}, \beta = 12 s^{-1}$ for the damped harmonic oscillator:

$$\ddot{x}(t) + \beta\dot{x} + \omega^2 x = 0.$$

The general solution is

$$x(t) = e^{\frac{-\beta}{2}t}\tilde{A}\ \sin(\tilde{\omega}t + \phi)$$

where $\tilde{A}$ is the amplitude,

$$\tilde{\omega} := \frac{1}{2}\sqrt{4\omega^2 - \beta^2} = \frac{1}{2}\sqrt{4(10)^2 - 12^2}\ s^{-1} = 8\ s^{-1}$$

is the **angular frequency** of oscillation and ϕ is the phase. Now putting the initial values we get

$$\begin{aligned} x(0) = 10\ cm &= \tilde{A}\sin(\phi) \\ v(0) = 10\ cm\ s^{-1} &= \tilde{A}\ \tilde{\omega}\ \cos(\phi) - \frac{\beta\tilde{A}}{2}\sin(\phi) \\ &= \tilde{A}\ (8\ s^{-1})\ \cos(\phi) - (6\ s^{-1})(10\ cm) \end{aligned}$$

Thus $\tilde{A}\cos(\phi) = \frac{70}{8}\ cm$, thus

$$\begin{aligned} \tan(\phi) &= \frac{8}{7} \\ \implies \phi &= 0.852\ rad \\ \tilde{A}^2 &= 10^2 + \left(\frac{70}{8}\right)^2 \\ \implies \tilde{A} &= 13.288\ cm. \end{aligned}$$

Hence, these are the frequency, phase and amplitude of the oscillator,

$$\begin{aligned} f &= \frac{8}{2\pi}s^{-1} = 1.273\ s^{-1} \\ \phi &= 0.852\ rad \\ \tilde{A} &= 13.288\ cm. \end{aligned}$$

Question 2

For the forced oscillation equation

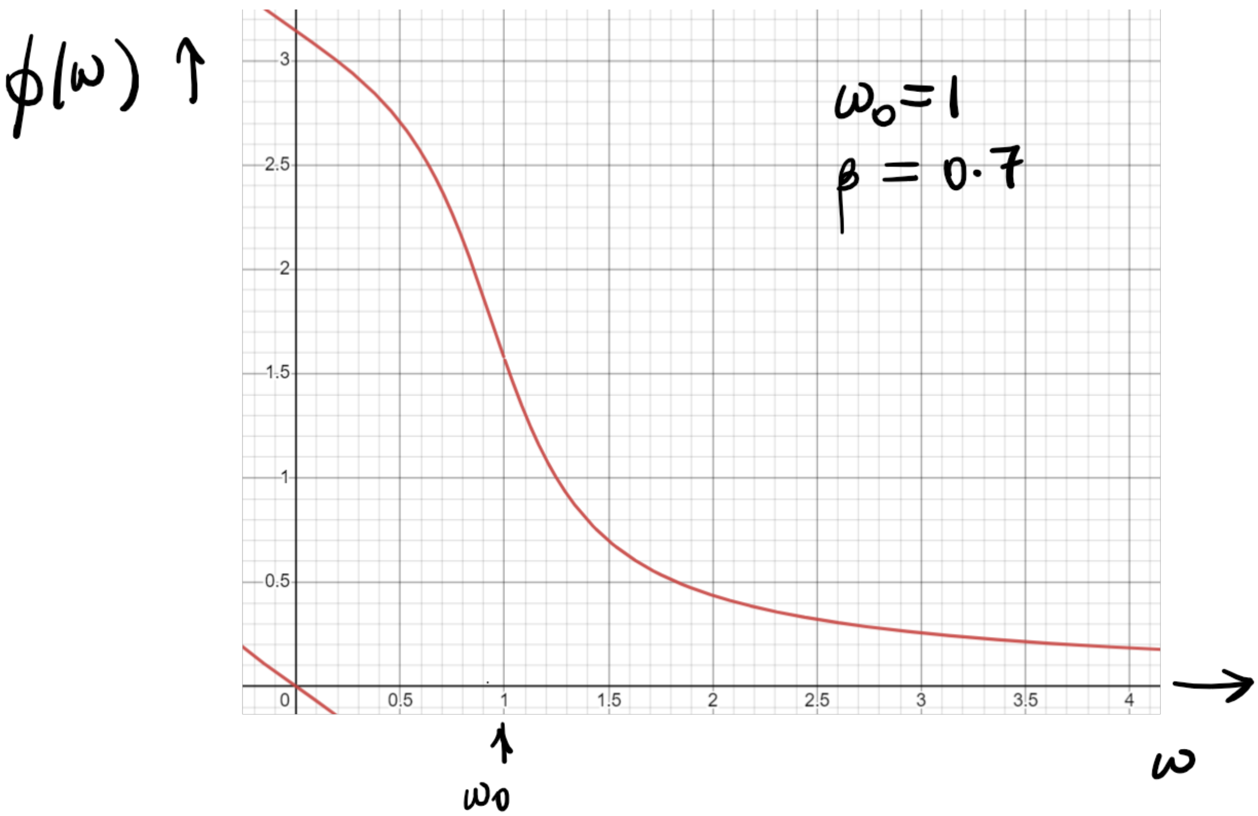
$$m\ddot{x}(t) + \beta\dot{x} + (\omega_0)^2 x = f_0\sin(\omega t)$$

the general solution is

$$x(t) = \underbrace{\tilde{A}e^{-\frac{\beta}{2m}t}\sin(\tilde{\omega}t + \phi)}_{\text{damped term} \to 0} + \underbrace{\frac{f_0}{m\sqrt{(\omega_0^2 - \omega^2)^2 + \left(\frac{\beta}{m}\omega\right)^2}}}_{\text{amplitude}}\sin(\omega t + \underbrace{\tilde{\phi}}_{\text{phase}})$$

where

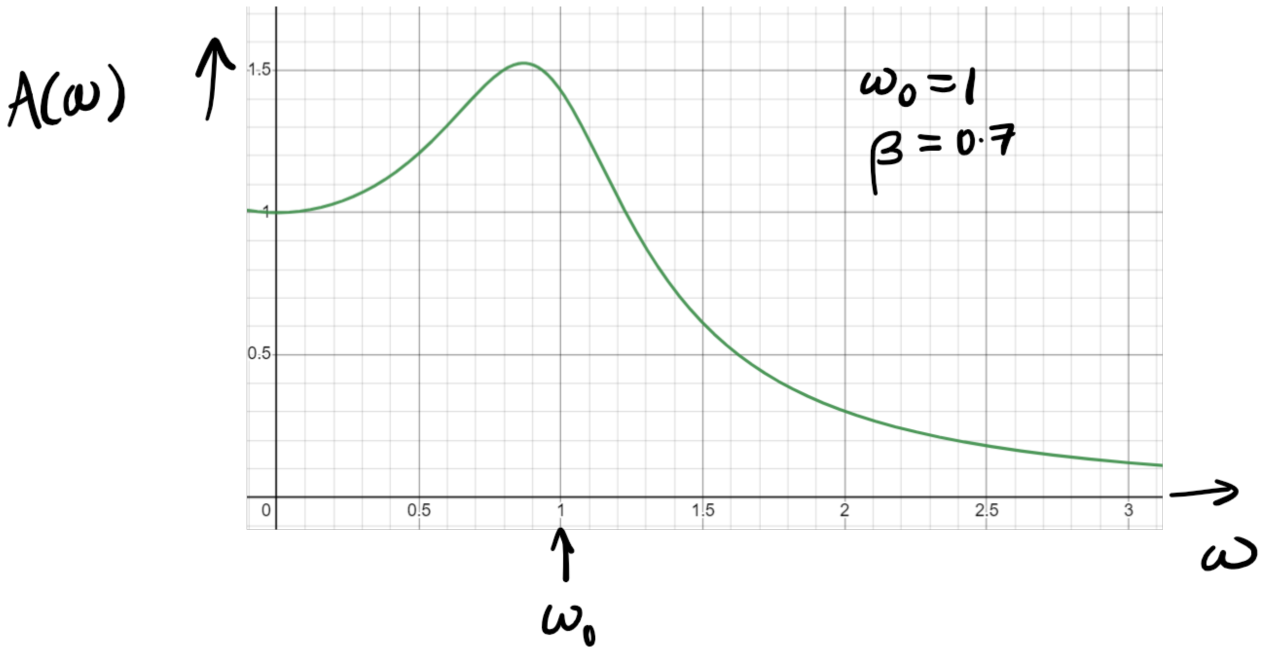
$$\tan\tilde{\phi}(\omega) = \frac{\beta}{m}\frac{\omega}{\omega^2 - \omega_0^2}$$



Question 3

$$A(\omega) = \frac{f_0}{m\sqrt{(\omega_0^2 - \omega^2)^2 + \left(\frac{\beta}{m}\omega\right)^2}}$$

This is the plot:



with a peak at

$$\omega = \sqrt{\omega_0^2 - \frac{\beta^2}{2}}$$

Question 4

[[PHY201.A01 solutions 2022-09-26-180842.excalidraw]]

The equations of motion are

$$\begin{aligned}
ml\ \ddot{\theta}_1 &= -mg\ \theta_1 - \kappa_1 \left(l\ \theta_1 - l\ \theta_2\right) \\
ml\ \ddot{\theta}_2 &= -mg\ \theta_2 - \kappa_1 \left(l\ \theta_2 - l\ \theta_1\right) - k_2 \left(l\ \theta_2 - l\ \theta_3\right) \\
ml\ \ddot{\theta}_3 &= -mg\ \theta_3 - \kappa_2 \left(l\ \theta_3 - l\ \theta_2\right)
\end{aligned}$$

where we define

$$\omega_0 := \sqrt{\frac{g}{l}},\ \ k_i := \frac{\kappa_i}{ml}$$

hence the equations become

$$\begin{aligned}
\ddot{\theta}_1 &= -\omega_0^2 \theta_1 - k_1 \left(\theta_1 - \theta_2\right) \\
\ddot{\theta}_2 &= -\omega_0^2 \theta_2 - k_1 \left(\theta_2 - \theta_1\right) - k_2 \left(\theta_2 - \theta_3\right) \\
\ddot{\theta}_3 &= -\omega_0^2 \theta_3 - k_2 \left(\theta_3 - \theta_2\right)
\end{aligned}$$

The equations can be written as a second order matrix differential equation:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \underbrace{\begin{pmatrix} \omega_0^2 + k_1 & -k_1 & 0 \\ -k_1 & \omega_0^2 + k_1 + k_2 & -k_2 \\ 0 & -k_2 & \omega_0^2 + k_2 \end{pmatrix}}_{:=\ M}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \mathbf{0}$$

where we can put $z_i(t) = (\mathbf{A})_i e^{i\omega t}$ as a test solution and get ω^2 as the eigenvalue, and $\mathbf{A}$ as the eigenvector of the matrix M. Here ω would be the frequencies. Thus, we find the eigenvalues of the matrix M:

$$\det(M - \lambda\ I) = 0$$

$$\implies \begin{vmatrix} \omega_0^2 + k_1 - \lambda & -k_1 & 0 \\ -k_1 & \omega_0^2 + k_1 + k_2 - \lambda & -k_2 \\ 0 & -k_2 & \omega_0^2 + k_2 - \lambda \end{vmatrix} = 0$$

$$\implies \begin{vmatrix} \omega_0^2 + k_1 - \lambda & -k_1 & 0 \\ -\lambda + \omega_0^2 & -\lambda + \omega_0^2 & -\lambda + \omega_0^2 \\ 0 & -k_2 & \omega_0^2 + k_2 - \lambda \end{vmatrix} = 0$$

$$\implies (-\lambda + \omega_0^2)\begin{vmatrix} \omega_0^2 + k_1 - \lambda & -k_1 & 0 \\ 1 & 1 & 1 \\ 0 & -k_2 & \omega_0^2 + k_2 - \lambda \end{vmatrix} = 0$$

$$\implies (-\lambda + \omega_0^2)[(-\lambda + \omega_0^2 + k_1)(-\lambda + \omega_0^2 + k_2 + k_2) + k_1(-\lambda + \omega_0^2 + k_2)] = 0$$

$$\implies (-\lambda + \omega_0^2)[(-\lambda + \omega_0^2)^2 + (2(k_1 + k_2))(-\lambda + \omega_0^2) + (3k_1k_2)] = 0$$

Hence, the solutions for $(-\lambda + \omega_0^2)$ are in the set

$$\{0, -(k_1 + k_2) + \sqrt{(k_1 + k_2)^2 - 3k_1k_2},\ -(k_1 + k_2) - \sqrt{(k_1 + k_2)^2 - 3k_1k_2}\ \}$$

Hence the solutions for $\lambda = \omega^2$ are among

$$\{\omega_0^2, \omega_0^2 + (k_1 + k_2) - \sqrt{(k_1 + k_2)^2 - 3k_1k_2},\ \omega_0^2 + (k_1 + k_2) + \sqrt{(k_1 + k_2)^2 - 3k_1k_2}\ \}$$

Thus the normal mode frequencies are

$$\begin{gathered}\omega_0 \\ \left(\omega_0^2 + (k_1 + k_2) - \sqrt{(k_1+k_2)^2 - 3k_1k_2}\right)^{\frac{1}{2}} \\ \left(\omega_0^2 + (k_1 + k_2) + \sqrt{(k_1+k_2)^2 - 3k_1k_2}\right)^{\frac{1}{2}}\end{gathered}$$

where

$$\omega_0 := \sqrt{\frac{g}{l}},\ \ k_i := \frac{\kappa_i}{ml}$$

Question 5

[[PHY201.A01 solutions 2022-09-27-144226.excalidraw]]

$$\begin{aligned} ml\ \ddot{\theta}_1 &= -mg\ \theta_1 - \kappa\,(l\ \theta_1 - l\ \theta_2) \\ ml\ \ddot{\theta}_2 &= -mg\ \theta_2 - \kappa\,(l\ \theta_2 - l\ \theta_1) - k\,(l\ \theta_2 - l\ \theta_3) \\ ml\ \ddot{\theta}_4 &= -mg\ \theta_4 - \kappa\,(l\ \theta_4 - l\ \theta_3) \end{aligned}$$

with the constraint

$$\theta_3 = \theta_2$$

thus

$$\begin{aligned} \ddot{\theta}_1 &= -\frac{g}{l}\ \theta_1 - \frac{\kappa}{ml}\,(\ \theta_1 -\ \theta_2) \\ \ddot{\theta}_2 &= -\frac{g}{l}\ \theta_2 - \frac{\kappa}{ml}\,(\ \theta_2 -\ \theta_1) \\ \ddot{\theta}_3 = \ddot{\theta}_2 &= -\frac{g}{l}\ \theta_2 - \frac{\kappa}{ml}\,(\ \theta_2 -\ \theta_1) \\ \ddot{\theta}_4 &= -\frac{g}{l}\ \theta_3 - \frac{\kappa}{ml}\,(\ \theta_4 -\ \theta_2) \end{aligned}$$

hence the matrix equation is

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\begin{bmatrix}\theta_1\\\theta_2\\\theta_3\\\theta_4\end{bmatrix} = \begin{pmatrix} -\frac{g}{l} - \frac{\kappa}{ml} & \frac{\kappa}{ml} & 0 & 0 \\ -\frac{\kappa}{ml} & -\frac{g}{l} - \frac{\kappa}{ml} & 0 & 0 \\ -\frac{\kappa}{ml} & -\frac{g}{l} - \frac{\kappa}{ml} & 0 & 0 \\ 0 & \frac{\kappa}{ml} & -\frac{g}{l} & -\frac{\kappa}{ml} \end{pmatrix}\begin{bmatrix}\theta_1\\\theta_2\\\theta_3\\\theta_4\end{bmatrix}$$

Question 6

For the forced oscillation equation

$$m\ddot{x}(t) + \beta\dot{x} + m(\omega_0)^2 x = f_0 \sin(\omega t)$$

the general solution is

$$x(t) = \underbrace{\tilde{A}e^{-\frac{\beta}{2m}t}\sin(\tilde{\omega}t + \phi)}_{\text{damped term} \to 0} + \underbrace{\frac{f_0}{m\sqrt{(\omega_0^2 - \omega^2)^2 + \left(\frac{\beta}{m}\omega\right)^2}}}_{\text{amplitude } A} \sin(\omega t + \underbrace{\tilde{\phi}}_{\text{phase}})$$

Here, $x(t)$ is charge on one capacitor plate, $m = L$, $\beta = R$, $m(\omega_0)^2 = \frac{1}{C}$ and $f_0 = V_0$.

Energy lost through resistor **after the damping dies down**:

$$\begin{aligned}
&= \int V(t)\ \dot{q}(t)\,\mathrm{d}\ t = \int V_0 \sin(\omega t)\ A\omega\cos(\omega t + \tilde{\phi})\,\mathrm{d}\ t \\
&= V_0 A \int_0^{2\pi/\omega} (\sin(\omega t)\cos(\omega t)\cos(\phi) - \sin^2(\omega t)\sin(\phi))\,\mathrm{d}\ t \\
&= -V_0 A \sin(\phi)\left(\frac{1}{2}\frac{2\pi}{\omega}\right) \\
&= \pi\frac{V_0 A}{\omega}\sin(\phi)
\end{aligned}$$

where

$$\frac{A}{\omega} = \frac{V_0}{\sqrt{\left(\left(\frac{1}{\omega C}\right) - \omega L\right)^2 + (R)^2}}$$

and

$$\sin(\phi) = \frac{R}{\omega L - \frac{1}{\omega C}}$$

Hence the energy lost in one cycle is

$$= \pi V_0 \times \frac{V_0}{\sqrt{\left(\left(\frac{1}{\omega C}\right) - \omega L\right)^2 + (R)^2}} \times \frac{R}{\omega L - \frac{1}{\omega C}}$$

Question 7

Energy dissipated through capacitor

$$
\begin{aligned}
&= \int \dot{q} V_C(t) \,\mathrm{d}\, t \\
&= \int (A\omega \cos(\omega t + \tilde{\phi})) \left(\frac{A}{C} \sin(\omega t + \tilde{\phi}) \right) \mathrm{d}\ t \\
&= \int_0^{\frac{2\pi}{\omega}} \frac{A^2 \omega}{C} \sin(\omega t + \tilde{\phi}) \cos(\omega t + \tilde{\phi}) \,\mathrm{d}\ t \\
&= 0
\end{aligned}
$$

Energy dissipated through inductor

$$
\begin{aligned}
&= \int \dot{q} V_L(t) \,\mathrm{d}\ t \\
&= \int (A\omega \cos(\omega t + \tilde{\phi})) \left(-\frac{LA}{\omega^2} \sin(\omega t + \tilde{\phi}) \right) \mathrm{d}\ x \\
&= \int_0^{\frac{2\pi}{\omega}} \frac{LA^2}{\omega} \sin(\omega t + \tilde{\phi}) \cos(\omega t + \tilde{\phi}) \,\mathrm{d}\ t \\
&= 0
\end{aligned}
$$

Hence, both inductor and capacitor have no net energy dissipation through themselves.

Question 8

For the forced oscillation equation

$$m\ddot{x}(t) + \beta\dot{x} + (\omega_0)^2 x = f_0 \sin(\omega t + \phi_0)$$

the general solution will just be a time translated equation of the previous one:

$$t \to t + t_0,\ t_0 := \frac{\phi_0}{\omega}$$

hence

$$x(t) = \underbrace{\tilde{A} e^{-\frac{\beta}{2m}(t+\phi_0)} \sin(\tilde{\omega} t + \phi + \phi_0)}_{\text{damped term} \to\ 0} + \underbrace{\frac{f_0}{m\sqrt{(\omega_0^2 - \omega^2)^2 + \left(\frac{\beta}{m}\omega\right)^2}}}_{\text{amplitude } A} \sin(\omega t + \underbrace{\tilde{\phi}}_{\text{phase difference}} + \phi_0)$$

Now given LCR circuit has a source

$$= V_0[\sin(\omega t) + \cos(\omega t)] = \sqrt{2}\ V_0 \left[\sin\left(\omega t + \frac{\pi}{2} \right) \right]$$

Hence, in the amplitude of charge oscillations is

$$= \frac{\sqrt{2}V_0}{L\sqrt{\left(\left(\frac{1}{LC}\right) - \omega^2\right)^2 + \left(\frac{R}{L}\omega\right)^2}}$$

and amplitude of current oscilations

$$= \omega \times \frac{\sqrt{2}V_0}{L\sqrt{\left(\left(\frac{1}{LC}\right) - \omega^2\right)^2 + \left(\frac{R}{L}\omega\right)^2}} = \frac{\sqrt{2}V_0}{\sqrt{\left(\left(\frac{1}{\omega C}\right) - \omega L\right)^2 + (R)^2}}$$