Indian Institute of Science Education and Research Mohali

**MTH101 : Linear Algebra (2023-24)** **Tutorial 02 (September 08, 2023)**

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Let $A = (a_{ij})_{n\times n}$ and $B = (b_{ij})_{n\times n}$. The matrix multiplication of $A$ and $B$ is the matrix whose $i^{\text{th}}$ row and $j^{\text{th}}$ column has entry $v_i^t.w_j$ (dot product). Here $v_i$ is the $i^{\text{th}}$ row vector of $A$ and $w_j$ is the $j^{\text{th}}$ column vector of $B$.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \quad B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nj} & \cdots & b_{nn} \end{pmatrix}$$

Thus, $v_i = (a_{i1}, a_{i2}, \cdots, a_{in})$ and $w_j = \begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{pmatrix}$. We denote the matrix thus constructed by $AB$.

Observe that one may define matrix multiplication even when matrices are not square matrices. The only compatibility between $A$ and $B$ that is required to define matrix multiplication $AB$ is that the number of columns of $A$ should be equal to the number of rows of $B$. In this course we shall mostly focus on $2\times 2$ and $3\times 3$ matrices.

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1. Consider the matrix $A := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Compute $A^2, A^3, A^4, A^5$. Do the same calculation for the matrix $A := \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

2. A matrix $I$ is called a $2\times 2$ identity matrix if $AI = A = IA$ for every $A \in M_{2\times 2}(\mathbb{R})$. Argue that the *only* $2\times 2$ identity matrix is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

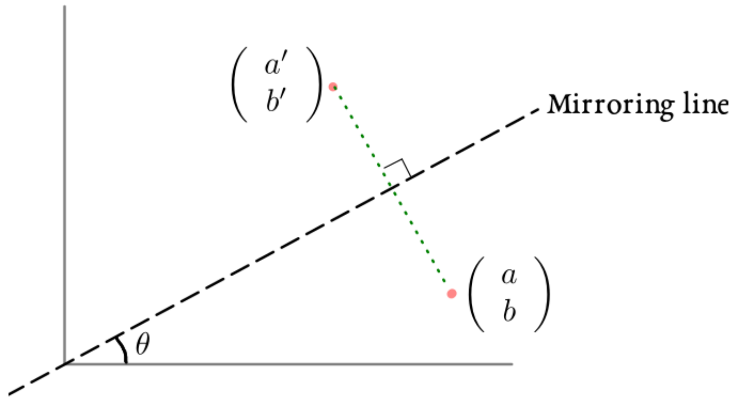
3. Can you find two $2\times 2$ matrices $A$ and $B$ such that $AB \neq BA$?

4. A matrix is called a *zero matrix* if all its entries are 0. As an instance of overuse of notation, a zero matrix is denoted by 0. Find a matrix $A$ which is not a zero matrix, but $A^2 := AA = 0$. Can you find a $2\times 2$ matrix $A$ such that $A^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

5. Take three $2\times 2$ matrices $A, B, C$ of your choice and show that $A(BC) = (AB)C$ and $A(B + C) = AB + AC$. Do you think that for every choice of $2\times 2$ matrices these equalities will hold? What about $3\times 3$ matrices?

6. Find a $2 \times 2$ matrix $A$, none of whose entries are zero, and $A^{100} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

7. Consider the reflection in a plane through a mirroring line placed at an angle $\theta$ from $x$-axis as shown in the figure below.




Express the process of reflection through matrix multiplication. That is, find a $2 \times 2$ matrix $F_\theta$ (in terms of $\theta$ and trigonometric functions) so that $F_\theta \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a' \\ b' \end{pmatrix}$. Notice that here we are treating vectors as $2 \times 1$ matrices, and multiplying a square matrix with non-square but compatible matrix.