PHY302: Quantum mechanics
Tutorial-5

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Que.1 Prove <u, T v> = <T† u, v> for any linear operator T acting on a complex vector space V and u,v $\in$ V.

Que.2 Translation operators
Consider the coordinate-space and momentum-space translation operators

$$T_x = exp\left(-\frac{i\hat{p}x}{\hbar}\right), \;\; \tilde{T}_p = exp\left(-\frac{ip\hat{x}}{\hbar}\right).$$

(a) Verify that the above are translation operators by calculation of

$$T_x^\dagger \hat{x} T_x \;\; and \;\; \tilde{T}_p^\dagger \hat{p} \tilde{T}_p$$

(b) Since $\hat{x}$ and $\hat{p}$ do not commute, the translation operators T_x and $\tilde{T}_p$ do not generally commute. But they sometimes do! Compute the commutator

$$\left[T_x, \tilde{T}_p\right] = ... \tag{1}$$

You should find the CBH formula useful. What is the condition satisfied by x and p that guarantees that T_x and $\tilde{T}_p$ commute?

Que.3 Projectors and the $P^2 = P$ condition
Consider a vector space V and a linear operator P that satisfies equation $P^2 = P$.

(a) Show that V= null P $\oplus$ range P.

The condition $P^2 = P$, however, is not enough to show that P is an orthogonal projectors. One must additionally prove that any vector in the first summand is orthogonal to any vector in the second summand.

(b) Show that any of the two conditions below guarantees that orthogonality:
(1) P is Hermitian.

(2) $|Pv| \leq |v|$ for any $v \in V$.

Case (2) is harder than case (1). You may find if useful to prove first the following result: Let $u, v \in V$. Then $\langle u, v\rangle = 0$ if and only if $|u| \leq |u + av|$ for any $a \in \mathbb{F}$.

(c) Invent a two-by-two matrix P that satisfies $P^2 = P$ but fails to be a projector because (as you will demonstrate) violates both conditions (1) and (2) of part (b).

Que. 4 Exercise with matrices.
Consider two hermitian matrices A_1 and A_2 that commute:

$$A_1 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{bmatrix}.$$

The matrix A_1 has eigenvalue and orthonormal eigenvectors

$$\lambda_1 = 2, |u_1\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}; \qquad \lambda_2 = 0, |u_2\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \qquad \lambda_3 = 0, |u_3\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

In the basis $\{|u_1\rangle, |u_2\rangle, |u_3\rangle\}$ the matrix A_2 takes the form

$$\begin{bmatrix} 3 & * & * \\ 0 & * & -\sqrt{2} \\ 0 & * & * \end{bmatrix}.$$

Determine the missing entries (denoted by *) in the above matrix. Use your result to find the eigenvalues of A_2.

Que. 5 Minimum uncertainty
We showed in class that for two hermitian operators A and B the uncertainty in equality

$$(\Delta A)^2 (\Delta B)^2 \geq \left(\langle\Psi| \frac{1}{2i}[A, B] |\Psi\rangle \right)^2$$

is saturated on a state $|\Psi\rangle$ that satisfies

$$(B - \langle B\rangle) |\Psi\rangle = i\gamma(A - \langle A\rangle) |\Psi\rangle, \qquad \textit{with} \qquad \gamma = \pm\frac{\Delta B}{\Delta A}.$$

Verify explicitly this claim for the Gaussian states

$$\psi(x) = N e^{i\langle p\rangle x/\hbar} e^{-x^2/(2\Delta^2)}$$

that saturate the uncertainty inequality for the product of $\hat{x}$ and $\hat{p}$ uncertainties.