Indian Institute of Science Education and Research Mohali

MTH101 : Linear Algebra (2023-24) **Exercise Sheet**

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1. Consider the linear transformation $T_A : \mathbb{R}^3 \to \mathbb{R}^3$ on the vector space $(\mathbb{R}^3, +, 0)$, which is given by $T_A(v) = Av$, where A is a 3×3 matrix. In each of the following cases, find eigenvalues and corresponding eigenvectors.
$$A := \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \text{ where } \cos\theta \neq 1; \qquad A := \begin{pmatrix} 0 & 0 & 6 \\ 1 & 0 & -11 \\ 0 & 1 & 6 \end{pmatrix}$$
$$A := \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix}$$

2. Determine if the following statement is true. *If A is $n \times n$ matrix and R is its row echelon matrix then the A and R have the same set of eigenvalues.* Justify.

3. Let $(V, +, \cdot)$ be a vector space and $S \subseteq V$ be a subspace. Recall that $(S, +, \cdot)$ is then a vector space itself. Argue that $\dim(S) \leq \dim(V)$. Give examples when V is infinite dimensional but S is finite dimensional.

4. On the vector space $(\mathbb{R}[x], +, \cdot)$ consider the linear transformation $\dfrac{d}{dx} : \mathbb{R}[x] \to \mathbb{R}[x]$. Determine all eigenvalues and the corresponding eigenvectors for this transformation. What are the kernel and image for this linear transformation.

5. On the vector space $(\mathbb{R}[x]_3, +, \cdot)$ consider the linear transformation $\dfrac{d}{dx} : \mathbb{R}[x]_3 \to \mathbb{R}[x]_3$. Show that $\mathcal{B} := \{1, (1+x), (1+x)^2, (1+x)^3\}$ is a basis for $(\mathbb{R}[x], +, \cdot)$. Determine the matrix of the linear transformation $\dfrac{d}{dx}$ with respect to the basis $\mathcal{B}$.

6. A linear transformation $T : V \to V$ on a finite vector space $(V, +, \cdot)$ is called *diagonalizable* if the there exists a basis $B := \{v_1, v_2, \cdots, v_n\}$ of V and scalars $\lambda_1, \lambda_2, \cdots, \lambda_n$, not necessarily distinct, such that $T(v_i) = \lambda_i v_i$ for each i. In other words, diagonalizable linear transformations are the ones where the vector space has a basis consisting of eigenvectors of $T : V \to V$.

   (a) Determine the matrix $A_{\mathcal{B}}$ of T with respect to such a basis.

   (b) Consider the vector space $(\mathbb{R}^2, +, \cdot)$ and the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T\left(\begin{pmatrix} a \\ b \end{pmatrix}\right) = \begin{pmatrix} b \\ a+b \end{pmatrix}$. Show that this linear trasformation is diagonalizable.
