INDIAN INSTITUTE OF SCIENCE EDUCATION AND RESEARCH MOHALI

MTH101 (Symmetry)

End Semester Examination / April 10, 2022 50 marks / 180 minutes

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Instructions

1. Write your name and roll number on the top of **every page**.
2. Write all arguments precisely and do not leave anything to the evaluator's imagination.
3. **Mysterious or unsupported answers will not receive credit**. A correct answer, unsupported by calculations or explanation will receive no credit; an incorrect answer supported by substantially correct calculations and explanations *might* still receive partial credit.
4. Stop writing at 12:00 noon, and submit your answers by 12:15 PM. Submission must be in the form of a **single PDF file**.

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1. A point $P = \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2$ undergoes the following operations.

   (a) First, P is rotated about origin $(0, 0)$ by an angle θ. The point thus obtained is P_1.

   (b) Then P_1 is reflected about x-axis, to obtain the point P_2.

   (c) Finally, the point P_2 is rotated by an angle φ, to obtain the point P_3.

   Express the entire process through matrices. What are the coordinates of P_3? [3+2]

2. Consider the group $G := \{1, 2, 3, 4, 5, 6, 7, 8, 9, 0\}$, where the group law $+$ is defined by *"unit digit after addition"*. Thus, $6 + 7 = 3$, $4 + 5 = 9$, $6 + 4 = 0$ and $9 + 8 = 7$, etc. (This is very much like group of clock hour addition, except that in this case 12 has been replaced by 0).

   (a) What is the identity element of G? [1]

   (b) Find all $a \in G$ such that $a + a + a + a$ is equal to the identity of G. [2]

   (c) Draw a shape whose group of rotational symmetries is given by G. [2]

3. Consider the matrix $A := S_{1,2}S_{2,3}S_{3,1}$, where $S_{p,q}$ are 3×3 swapper matrices. Now, let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by $T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = A\begin{pmatrix} a \\ b \\ c \end{pmatrix}$. Is T a rigid linear transformation? [5]?

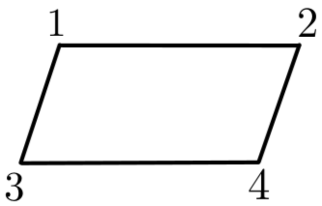
4. Let $(V, +) = (\mathbb{Z}, +)$ be the abelian group of integers under addition. Define

$$\cdot : \mathbb{R} \times \mathbb{Z} \to \mathbb{Z}$$
$$(\alpha, n) \mapsto \alpha.n,$$

   where $\alpha.n :=$ largest integer that is smaller than the product αn. Is $(\mathbb{Z}, +, \cdot)$ a vector space over $\mathbb{R}$? [5]

5. Let $u = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \in \mathbb{R}^3$. Find two vectors $v, w \in \mathbb{R}^3$ such that the set $S = \{u, v, w\}$ is a basis of the vector space $(\mathbb{R}^3, +, \cdot)$. [5]

6. Consider a paralleleogram P whose nonparallel sides are not equal.



Determine the group of symmetries of P. [5]

7. Find all angles θ for which the rotation matrix $R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ has an eigenvector in $\mathbb{R}^2$. [4+1]

8. On the collection $G := \{□, ⬡, ●, ○\}$ of four symbols, an operation is defined by the following composition table:

	□	⬡	●	○
□	●	□	○	⬡
●	○	●	⬡	□
⬡	□	⬡	●	○
○	⬡	○	□	●

Is G a group under this operation? [5]

9. Determine, if the following statement is true about group actions. "*Consider a group action $G \times S \to S$. If there is $s \in S$ such that $g.s = h.s$ for all $g, h \in G$, then $g.t = h.t$ for all $t \in S$ and all $g, h \in G$.*" [5]

10. For the vector space $(M_2(\mathbb{R}), +, \cdot)$, find $S \subseteq M_2(\mathbb{R})$ such that $\text{span}(S) = M_2(\mathbb{R})$ and none of the entries of any element of S is equal to 0. [5]