Tutorial 03
25/01/22

1. Consider the following 3×3 matrices

$$S_{2,3} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad M_2(\lambda) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad L_{1,2}(\lambda) := \begin{pmatrix} 1 & \lambda & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A := \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

(a) Compute $S_{2,3}A$, $M_2(\lambda)A$ and $L_{1,2}(\lambda)A$.

(b) Compute $AS_{2,3}$, $AM_2(\lambda)$ and $AL_{1,2}(\lambda)$.

(c) Matrix $S_{2,3}$ is a **swapper**, $M_2(\lambda)$ is a **multiplier** and $L_{1,2}(\lambda)$ is a **product adder**. Why do you think we should call them by these names?

(a), (b)

$$S_{2,3}A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$$

$$AS_{2,3} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{13} & a_{12} \\ a_{21} & a_{23} & a_{22} \\ a_{31} & a_{33} & a_{32} \end{pmatrix}$$

$\Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ swaps rows when multiplied on left
swaps columns when multiplied on right

$$M_2(\lambda)A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ \lambda a_{21} & \lambda a_{22} & \lambda a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$AM_2(\lambda) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a_{11} & \lambda a_{12} & a_{13} \\ a_{21} & \lambda a_{22} & a_{23} \\ a_{31} & \lambda a_{32} & a_{33} \end{pmatrix}$$

$\Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{pmatrix}$ multiplies 2^{nd} row by λ when multiplied on left
multiplies 2^{nd} column by λ when multiplied on right

$$L_{1,2}(\lambda)\,A = \begin{pmatrix} 1 & \lambda & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{11}+\lambda a_{21} & a_{12}+\lambda a_{22} & a_{13}+\lambda a_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$A\,L_{1,2}(\lambda) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} 1 & \lambda & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a_{11} & \lambda a_{11}+a_{12} & a_{13} \\ a_{21} & \lambda a_{21}+a_{22} & a_{23} \\ a_{31} & \lambda a_{31}+a_{32} & a_{33} \end{pmatrix}$$

$\Rightarrow \begin{pmatrix} 1 & \lambda & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ substitutes 1^{st} row by 1^{st} row $+\lambda$ (2^{nd} row) when multiplied on left

substitute 2^{nd} column by 2^{nd} column $+\lambda$ (1^{st} column) when multiplied on right

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2. Find all angles θ for which $R_{x,\theta}R_{y,\theta} = R_{y,\theta}R_{x,\theta}$, where $R_{x,\theta}$ and $R_{y,\theta}$ are rotation matrices by θ about x and y axes, respectively.

$$R_{x,\theta}\,R_{y,\theta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ \sin^2\theta & \cos\theta & -\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \sin\theta & \cos^2\theta \end{pmatrix}$$

$$R_{y,\theta}\,R_{x,\theta} = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & \sin^2\theta & \sin\theta\cos\theta \\ 0 & \cos\theta & -\sin\theta \\ -\sin\theta & \sin\theta\cos\theta & \cos^2\theta \end{pmatrix}$$

So, $R_{x,\theta} R_{y,\theta} = R_{y,\theta} R_{x,\theta}$

$$\Leftrightarrow \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ \sin^2\theta & \cos\theta & -\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \sin\theta & \cos^2\theta \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\theta & \sin^2\theta & \sin\theta\cos\theta \\ 0 & \cos\theta & -\sin\theta \\ -\sin\theta & \sin\theta\cos\theta & \cos^2\theta \end{pmatrix}}_{\text{substitute } \theta = 0,\ \pi}$$

$$\Leftrightarrow \sin^2\theta = 0 \quad \Leftrightarrow \quad \sin\theta = 0$$

So $\theta = 0, \pi$ are the only such angles, in $0 \le \theta < 2\pi$ range

So $R_{x,0} R_{y,0} = R_{y,0} R_{x,0} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

= Identity matrix

$\Rightarrow$ No change in the configuration

and $R_{x,\pi} R_{y,\pi} = R_{y,\pi} R_{x,\pi} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$= R_{z,\pi}$

All this validates (in fact, strengthens) your experiments with tomatoes, Rubik's cube etc.)

3. Take three 2×2 matrices A, B, C of your choice and show that $A(BC) = (AB)C$. Do you think that for every choice of 2×2 matrices this equality will hold? What about 3×3 matrices?

$A = (a_{ij})$, $\quad B = (b_{jk})$ $\quad C = (c_{kl})$

- $(j,l)^{th}$ entry of BC is

$$\sum_k b_{jk} c_{kl}$$

$(i,l)^{th}$ entry of $A(BC)$ is

$$\sum_j a_{ij} \left(\sum_k b_{jk} c_{kl} \right) = \sum_j \sum_k a_{ij} b_{jk} c_{kl}$$

- $(i,k)^{th}$ entry of AB is

$$\sum_j a_{ij} b_{jk}$$

$(i,l)^{th}$ entry of $(AB)C$ is

$$\sum_k \left(\sum_j a_{ij} b_{jk} \right) c_{kl} = \sum_k \sum_j a_{ij} b_{jk} c_{kl}$$

4. Consider the following system of linear equations:

$$\begin{aligned} x + y - z &= 4 \\ 3x - 2z &= 6 \\ x + 2y - z &= 7 \end{aligned}$$

and express it in the matrix form. Now, compute

$$\begin{pmatrix} -4 & 1 & 2 \\ -1 & 0 & 1 \\ -6 & 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 & -1 \\ 3 & 0 & -2 \\ 1 & 2 & -1 \end{pmatrix}.$$

Can you use this computation to obtain values of x, y, z that satisfy the above system of equations?

The given system of equations in the matrix form is

$$\begin{pmatrix} 1 & 1 & -1 \\ 3 & 0 & -2 \\ 1 & 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \\ 7 \end{pmatrix}$$

Multiplying on left by $\begin{pmatrix} -4 & 1 & 2 \\ -1 & 0 & 1 \\ -6 & 1 & 3 \end{pmatrix}$

$$\underbrace{\begin{pmatrix} -4 & 1 & 2 \\ -1 & 0 & 1 \\ -6 & 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 & -1 \\ 3 & 0 & -2 \\ 1 & 2 & -1 \end{pmatrix}}_{= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -4 & 1 & 2 \\ -1 & 0 & 1 \\ -6 & 1 & 3 \end{pmatrix}\begin{pmatrix} 4 \\ 6 \\ 7 \end{pmatrix}$$

Therefore

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -4 & 1 & 2 \\ -1 & 0 & 1 \\ -6 & 1 & 3 \end{pmatrix}\begin{pmatrix} 4 \\ 6 \\ 7 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -16 + 6 + 14 \\ -4 + 0 + 7 \\ -24 + 6 + 21 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ 3 \end{pmatrix}; \text{ so } \begin{aligned} x &= 4 \\ y &= 3 \\ z &= 3 \end{aligned}$$

is a solution.

5. Resultant of multiplying a matrix A with itself is called the **square** of A. It is written as A^2. So $A^2 := AA$.

   (a) Can you find a 2×2 matrix A such that none of the entries of A is zero, but $A^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$?

   (b) Can you find a 2×2 matrix A such that $A^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$?

(a) Try $A = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$.

or $A = \begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix}$

Any such matrix will be of the form

$\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ with $a^2 = -bc$.

why?

(b) Try $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

How to come up with it?

Hint: Rotation by $\frac{\pi}{2}$, followed by rotation by $\frac{\pi}{2}$, results into net rotation by π.

There could be more ways.