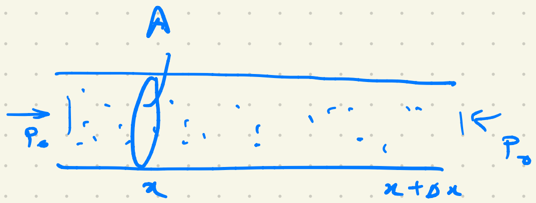
Longitudinal Waves

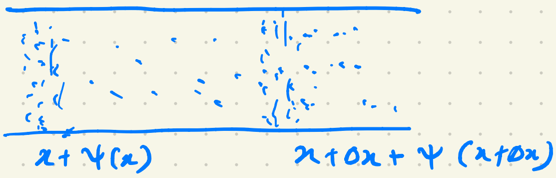
Horizontal displacement of a column at x is $\psi(x)$



For small forces

$$\Delta V = -k V \phi$$

$$\therefore \quad \frac{\Delta V}{V} = -k\phi$$



$$p(x) = P_0 + \phi(x)$$

$$P(x+\Delta x) = P_0 + \phi(x+\Delta x)$$

$$\frac{A\,\Delta\psi}{A\,\Delta x} = -k\phi \Rightarrow \frac{\partial\psi}{\partial x} = -k\phi$$

$$\boxed{\Rightarrow \frac{\partial^2\psi}{\partial x^2} = -k\frac{\partial\phi}{\partial x}} \quad - \quad 1$$

$$\boxed{\begin{aligned} &\text{Since} \quad \phi(x+\Delta x) = \phi(x) + \frac{\partial\phi}{\partial x}\Delta x \\ &\Rightarrow \quad \Delta\phi = \frac{\partial\phi}{\partial x}\Delta x = -\frac{\Delta x}{k}\frac{\partial^2\psi}{\partial x^2} \end{aligned}}$$

Force on gas column between x and $x+\Delta x$

$$\Rightarrow \underbrace{\rho V}_{m} \underbrace{\frac{\partial^2\psi}{\partial t^2}}_{a} = -A\Delta\phi$$

$$= +\frac{A}{k}\frac{\partial^2\psi}{\partial x^2}\Delta x$$

$$\therefore \quad \frac{\partial^2\psi}{\partial t^2} = \frac{A}{kV\rho}\frac{\partial^2\psi}{\partial x^2}\Delta x = \frac{\cancel{A\Delta x}}{k\rho\cancel{V}}\frac{\partial^2\psi}{\partial x^2}$$

Thus the horizontal disturbance ψ satisfies a wave equation

$$\boxed{\frac{\partial^2 \psi}{\partial t^2} = \underbrace{\left(\frac{1}{K\rho}\right)}_{\to\, v^2} \frac{\partial^2 \psi}{\partial x^2}}$$

Travelling waves

We have the expression for a standing wave

$$y_n(x,t) = A_n \sin(k_n x) \cos(\omega_n t + \phi)$$

$$= \frac{A_n}{2}\left[\sin(k_n x + \omega_n t + \phi) + \sin(k_n x - \omega_n t - \phi)\right]$$

$$= \underbrace{\frac{A_n}{2} \sin(k_n x + \omega_n t + \phi)}_{y_n^{(1)}} + \frac{A_n}{2} \underbrace{\sin(k_n x - \omega_n t - \phi)}_{y_n^{(2)}}$$

$$\frac{\partial^2 y_n^{(1)}}{\partial t^2} = -\omega_n^2 y_n^{(1)} \quad ; \quad \frac{\partial^2 y_n^{(1)}}{\partial x^2} = -k_n^2 y_n$$

Since $\omega_n^2 = k_n^2 v^2$

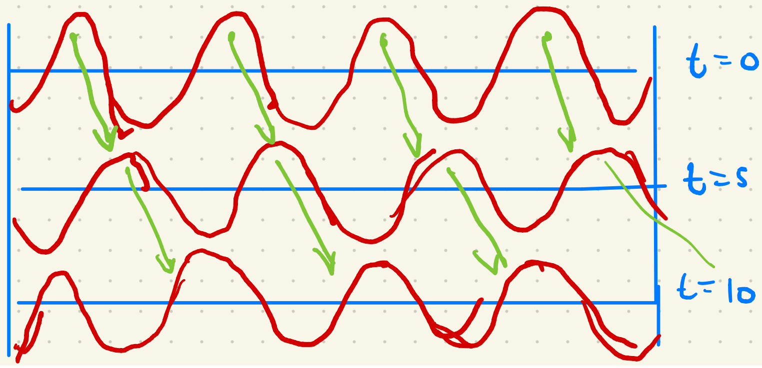
$\Rightarrow \quad \dfrac{\partial^2 y_n^{(1)}}{\partial t^2} = v^2 \dfrac{\partial^2 y_n^{(1)}}{\partial x^2}$ (Satisfies the wave eqn)

Similarly $\quad \dfrac{\partial^2 y_n^{(2)}}{\partial t^2} = v^2 \dfrac{\partial^2 y_n^{(2)}}{\partial x^2}$ (satisfies the wave eqn)

However for $y_n^{(1)}$, the maxima/antimaxima is obtained from $\dfrac{\partial y_n^{(1)}}{\partial x} = 0 \Rightarrow \dfrac{A_n k_n}{2} \cos(k_n x + \omega_n t + \phi) = 0$

$\therefore \quad k_n x + \omega_n t + \phi = (2n+1)\pi/2$

$\rightarrow \quad k_n x = (2n+1)\pi/2 - \phi - \omega_n t$ (locations shift !!)



Travelling Waves

For $y_n^{(1)} = \frac{A_n}{2} \sin(k_n x + \omega_n t + \phi)$

the extrema are at $k_n x = (2n+1)\frac{\pi}{2} - \phi - \omega_n t$

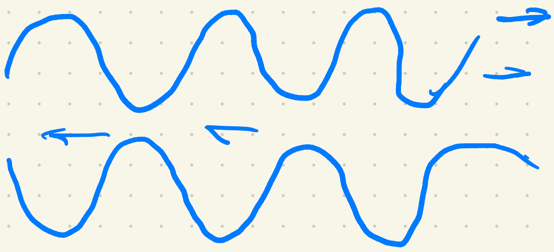
Thus, if x_* is an extrema at time t_*
then at $t_* + \Delta t$, $x_* + \Delta x$ will be the new location of extrema s.t.

$$k_n \Delta x = -\omega_n \Delta t \quad (-\text{ve})$$

Hence, the extrema shift 'leftwards' in time!

$y_n^{(1)}$ is a left-moving wave!

Ex: $y_n^{(2)}$ is a right-moving wave



The left moving and the right moving waves in equal amplitude constitute a standing wave.

Rate of shift of extrema for $y_n^{(1)}$

$$\frac{\Delta x}{\Delta t} = -\frac{\omega_n}{k_n} = -v$$

; Thus $y_n^{(1)}$ travels left with speed v.

Similarly $y_n^{(2)}$ travels right with speed v.

$$v^2 = \frac{T}{\rho} \quad (\text{string})$$

$$= \frac{S}{\sigma} \quad (\text{2-D membrane})$$

$$= \frac{1}{k_B \rho} \quad (\text{fluid})$$

In fact any function

$$f(x \pm vt) \Rightarrow \frac{\partial^2 f}{\partial x^2} = f''$$

$$\frac{\partial^2 f}{\partial t^2} = (\pm v)^2 f''$$

$$\Rightarrow \frac{\partial^2 f}{\partial t^2} = v^2 \frac{\partial^2 f}{\partial x^2}$$; satisfies the wave eqn.

$f(x+vt)$ is left moving wave.

$f(x-vt)$ is right moving wave.

Thus functions of $x \pm vt$ are waves (still we needs to take care of boundary conditions)

Any $$\boxed{f(x+vt) = \sum_n A_n \sin(k_n(x+vt)+\phi)}$$

$$= \sum_n A_n \left[\sin(k_n x)\cos(\omega_n t+\phi) + \cos(k_n x)\sin(\omega_n t+\phi)\right]$$

Exercise: Find out A_2, A_3 !!

Waves in free space

Can be obtained by taking $L \to \infty$

$$k_{n+1} - k_n = \frac{\pi}{L} \to 0$$

Hence k will become continuous

$$y_k(x,t) = A \sin(kx) \cos(\omega t + \phi)$$

with $\omega = kv$

A general wave

$$\Rightarrow \quad y(x,t) = \int_{-\infty}^{\infty} dk \; A(k) \sin(kx) \cos(\omega t + \phi)$$

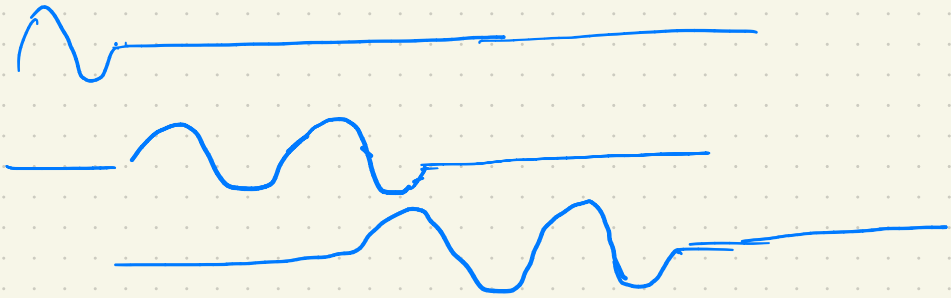
In order to find $A(k)$ we use

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} dx \; e^{ik(x-x')} = \delta(x - x')$$

Dirac Delta function

Where $\displaystyle\int_{-\infty}^{\infty} dx \; f(x) \, \delta(x-a) = f(a)$

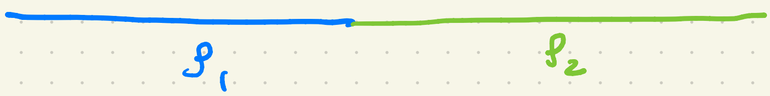
$$\int_{-\infty}^{\infty} dx \; \delta(x-a) = 1$$



Travelling waves : $f\left(k(x-vt)\right)$

$k = \dfrac{\omega}{v} = \omega\sqrt{\dfrac{\rho}{T}}$ depends on media through ρ.

If we have two media, joined at a junction

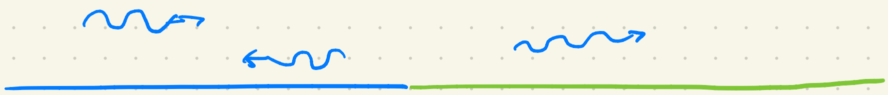


If same ω wave is sent, it will have two different k in the two media

At the junction the wave may pass into the new region with change in k, but to conserve energy

$$\therefore \quad E = \frac{1}{2}\int_0^L dx \left[\rho \dot{y}^2 + T y'^2 \right]$$

Some part should reflect back as well.



$$y_I = A_1 \sin(kx - \omega t) + A_2 \sin(kx + \omega t) \qquad y_{II} = A_3 \sin(k'x - \omega t)$$

$x < 0$ $\qquad$ $x = 0$ $\qquad$ $x > 0$

At the junction they should match

$$y_I(0, t) = y_{II}(0, t)$$

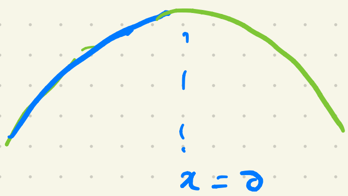
$$\Rightarrow -A_1 \sin \omega t + A_2 \sin \omega t = -A_3 \sin \omega t$$

$$\Rightarrow -A_1 + A_2 = -A_3$$

$$\boxed{\Rightarrow \quad A_1 = A_2 + A_3} \quad - \text{(a)}$$

It should be a smooth matching

$$\left.\frac{\partial y_I}{\partial x}\right|_{x=0} = \left.\frac{\partial y_{II}}{\partial x}\right|_{x=0}$$



$$\Rightarrow \left. A_1 k \cos(kx - \omega t) + A_2 k \cos(kx + \omega t)\right|_{x=0} = \left. A_3 k' \cos(k'x - \omega t)\right|_{x=0}$$

$$\Rightarrow A_1 k \cos \omega t + A_3 k \cos \omega t = A_3 k' \cos \omega t$$

$$\boxed{\Rightarrow \quad (A_1 + A_2) k = A_3 k'} \quad - \text{(b)}$$

Exercise: From (a) and (b); show

$$\frac{A_2}{A_1} = \frac{k' - k}{k' + k}$$

$$\frac{A_3}{A_1} = \frac{2k}{k + k'}$$

Since $k = \frac{\omega}{v} = \frac{\omega}{\sqrt{T}}\sqrt{\rho}$

$$\frac{A_2}{A_1} = \frac{\omega\left(\frac{\sqrt{\rho'}}{\sqrt{T}} - \frac{\sqrt{\rho}}{\sqrt{T}}\right)}{\omega\left(\frac{\sqrt{\rho'}}{\sqrt{T}} + \frac{\sqrt{\rho}}{\sqrt{T}}\right)} = \frac{\sqrt{T\rho'} - \sqrt{T\rho}}{\sqrt{T\rho'} + \sqrt{T\rho}}$$

The quantity $\sqrt{T\rho}$ is called impedance (Z) of the medium

$\therefore \frac{A_2}{A_1} = \frac{Z' - Z}{Z + Z}$; Reflectivity : $r = \left(\frac{A_2}{A_1}\right)^2 = \frac{(Z' - Z)^2}{(Z + Z')^2}$

Similarly $\frac{A_3}{A_1} = \frac{2Z'}{(Z + Z')}$ $\Rightarrow$ Transmitivity $t = \frac{Z'}{Z}\left(\frac{A_3}{A_1}\right)^2 = \frac{4ZZ'}{(Z + Z')^2}$

$$\boxed{\text{Hence, } \quad r + t = 1}$$

Energy carried in a wave

$$E = \frac{1}{2}\int dx \left[\rho \dot{y}^2 + T y'^2 \right]$$

Further, the wave eqn.

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}$$

$\Rightarrow$ Multiplying through $\rho \frac{\partial y}{\partial t}$

$$\rho \frac{\partial y}{\partial t} \frac{\partial^2 y}{\partial t^2} = \rho \frac{\partial y}{\partial t} v^2 \frac{\partial^2 y}{\partial x^2} = T \frac{\partial y}{\partial t} \frac{\partial^2 y}{\partial x^2}$$

Adding $T \frac{\partial y}{\partial x} \frac{\partial^2 y}{\partial t \partial x}$

$$\Rightarrow \rho \frac{\partial y}{\partial t} \frac{\partial^2 y}{\partial t^2} + T \frac{\partial y}{\partial x} \frac{\partial^2 y}{\partial t \partial x} = T \left(\frac{\partial y}{\partial t} \frac{\partial^2 y}{\partial x^2} + \frac{\partial y}{\partial x} \frac{\partial^2 y}{\partial t \partial x} \right)$$

$$\Rightarrow \frac{1}{2} \frac{\partial}{\partial t} \left[\rho \left(\frac{\partial y}{\partial t} \right)^2 + T \left(\frac{\partial y}{\partial x} \right)^2 \right]$$

$$= \frac{\partial}{\partial x} \left(T \frac{\partial y}{\partial t} \frac{\partial y}{\partial x} \right)$$

$$\int_{x_1}^{x_2} dx \frac{\partial}{\partial t} \left[\frac{1}{2} \left(\rho \dot{y}^2 + T y'^2 \right) \right] = \int_{x_1}^{x_2} dx \frac{\partial}{\partial x} \left(T \frac{\partial y}{\partial t} \frac{\partial y}{\partial x} \right)$$

$$\Rightarrow \frac{\partial}{\partial t} E = T \frac{\partial y}{\partial t} \frac{\partial y}{\partial x} \Big|_{x_2} - T \frac{\partial y}{\partial t} \frac{\partial y}{\partial x} \Big|_{x_1}$$

$$\Rightarrow \frac{\partial E}{\partial t} + \mathcal{L}(x_2) - \mathcal{L}(x_1) = 0$$

where $\mathcal{L}(x) = -T \frac{\partial y}{\partial t} \frac{\partial y}{\partial x}$

$\mathcal{Z}(x)$ is the flux of energy carried out !

<u>Exercise:</u> Calculate the flux of energy for

(i) A stationary wave $A \sin kx \cos(\omega t + \phi)$

(ii) A right-moving wave $A \sin(\omega t - kx + \phi)$

Mean flux carried out for one cycle

For $y = A \cos(\omega t + kx + \phi)$

$$\Rightarrow \mathcal{Z}(x) = -T A^2 \omega k \sin^2(\omega t + kx + \phi)$$

$$\langle \mathcal{Z}(x) \rangle = \frac{\int_0^{2\pi/\omega} dt \, \mathcal{Z}(x)}{\frac{2\pi}{\omega}}$$

$$= \frac{\omega}{2\pi} \int_0^{2\pi/\omega} dt \, (-T A^2 \omega k) \sin^2(\omega t + kx + \phi)$$

$$= -\frac{T A^2 \omega k}{2} = -\frac{T A^2 \omega^2}{2v} = -\frac{\sqrt{T\rho}\, A^2 \omega^2}{2}$$

$$= -\frac{Z A^2 \omega^2}{2}$$

$Z = \sqrt{T\rho}$ is the impedance