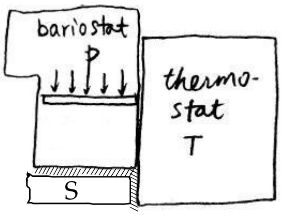
1. Consider a hydrostatic system $\mathcal{S}$ which is connected to a bath, with which it can exchange energy and volume.



   a) Recall, how we calculated the canonical and the grand canonical distribution functions. Using the same line of argument, calculate the probability $\rho(E,V)$ of finding the system with energy E and volume V.
   b) Using the relation $S = \langle S\rangle$ and the Boltzmann relation for the entropy S, relate the partition function Q, that defines the normalization in $\rho(E,V)$, to one of the thermodynamic potentials.
   c) Using the expression for $\rho(E,V)$ calculate the relative fluctuation $\sqrt{\langle \Delta V^2\rangle}/\langle V\rangle$.
   d) Now, the partition function Q can be written down as

$$Q = \int_0^\infty \mathrm{d}V \int \prod_{i=1}^{N} \mathrm{d}^3p_i \mathrm{d}^3q_i \;\; \rho(E,V).$$

   Using this expression, relate Q to the canonical partition function $Z(N,V,T)$. Hence, derive an expression for Q of an ideal gas, and the chemical potential μ for the same system.

2. The energy of a system of N localized magnetic spins, at temperature T and in the presence of magnetic field H is given by

$$\mathcal{H} = D\sum_{i=1}^{N} S_i^2 - \mu_0 H \sum_{i=1}^{N} S_i,$$

   where the parameters D, μ_0, H are positive and spin variables S_j may assume values ± 1 or 0, for $i = 1,2,3.....$

   a) Obtain an expression for the internal energy, the entropy and the magnetization per site.
   b) In zero field limit ($H = 0$), sketch the graphs of the internal energy, entropy and the specific heat as a function of temperature.
   c) Indicate the behavior of the these quantities in the limits of $T \to 0$ and $T \to \infty$.
   d) Calculate the average of the quadrupole moment defined as $\mathcal{Q} = \frac{1}{N}\sum_{i=1}^{N} S_i^2$.

3. Consider a one-dimensional magnetic system of N localized spins, at temperature T, with the Hamiltonian given by

$$\mathcal{H} = -J \sum_{i=1,3,5,\ldots N-1} \sigma_i\sigma_{i+1} - \mu_0 H \sum_{i=1}^{N} \sigma_i$$

   where the parameters J, μ_0, H are positive numbers, and $\sigma_i = \pm 1$ for all sites i. Assume that N is even and note that the first sum is over the odd spins.

 Instructor: Dipanjan Chakraborty

a) Obtain an expression for the canonical partition function and calculate the internal energy per spin $u = u(T, H)$. Sketch a graph of $u(T, H = 0)$ versus temperature.

b) Obtain an expression for the entropy per spin $s = s(T, H)$. Sketch a graph of $s(T, H = 0)$ versus temperature.

c) Calculate the magnetization per spin defined as

$$m = m(T, H) = \frac{1}{N}\left\langle \mu_0 \sum_{i=1}^{N} \sigma_i \right\rangle$$

and the magnetic susceptibility $\chi(T, H)$.

4. Consider a system of N classical and non-interacting particles in contact with a thermal reservoir at temperature T. Each particle may have energies $0, \epsilon >)$, or 3ϵ. Obtain the expression for the canonical partition function, and calculate the internal energy and the entropy per particle for the system. Sketch the results, indicating the values for $T \to 0$ and $T \to \infty$.

5. Consider N non-interacting spins in a magnetic field $\mathbf{B} = B\hat{z}$, and at a temperature T. The work done by the field is given by BM_z, with a magnetization $M_z = \sum_{i=1}^{N} m_i$. For each spin, m_i can takes only the $(2s + 1)$ values $-s, -s + 1, ...s - 1, s$.

   a) Calculate the canonical partition function and the free energy of the system.

   b) Expand the free energy for small magnitude of the magnetic field and show that the magnetic susceptibility obeys Curie's law $\chi = c/T$.

   c) Show that $C_B - C_M = cB^2/T^2$, where C_B and C_M are the heat capacities at constant field and magnetization, respectively.

6. A system of N one dimensional localized classical harmonic oscillators, at a given temperature T, is associated with the Hamiltonian

$$\mathcal{H} = \sum_{i=1}^{N}\left[\frac{p_i^2}{2m} + V(q_i)\right],$$

   where $V(q)$ has the form

$$V(q) = \begin{cases} \frac{1}{2}m\omega^2 q^2 & \text{for } q > 0 \\ \frac{1}{2}m\omega^2 q^2 + \epsilon & \text{for } q < 0 \end{cases}$$

   a) Plot the potential as a function of q. What happens when $\epsilon \to \infty$?

   b) Calculate the canonical partition function and the internal energy per oscillator u.

   c) What are the limiting values of the u for $\epsilon \to 0$ and $\epsilon \to \infty$.

   d) Now setting $\epsilon \to \infty$ in the potential, re-calculate the canonical partition function and the internal energy per particle and see whether they agree with the previous result.

   e) Consider now the quantum version of the problem, where the classical oscillators are replaced by quantum oscillators. Calculate the canonical partition function in the limit of $\epsilon \to \infty$.