Quantum Measurements:

→ Von-Neuman Measurements:

Projective measurement

$$|\phi\rangle \in \mathcal{H}_s$$

$$A \in \mathcal{B}(\mathcal{H}_s)$$

$$A = A^{+} = \sum_j \alpha_j |\psi_j\rangle\langle\psi_j|$$

$$|\phi\rangle = \sum_n C_n |\psi_n\rangle$$

$$p_n = |\langle\psi_n|\phi\rangle|^2 = Tr[|\psi_n\rangle\langle\psi_n|\phi\rangle\langle\phi|] = |C_n|^2$$

≡ Probability of getting n^{th} outcome.

→ Define projectors $P_n = |\psi_n\rangle\langle\psi_n|$

→ $P_n^2 = P_n$

→ $eig(P_n) = 1, 0$

→ Probability p_n for a Density matrix ρ

$$p_n = Tr[P_n \rho]$$

→ After the measurement the state of the system is $\tilde{\rho} = \dfrac{P_n \rho P_n}{Tr[P_n \rho]} = \dfrac{1}{p_n} |\psi_n\rangle\langle\psi_n|\rho|\psi_n\rangle\langle\psi_n|$

$$\boxed{\rho_n = |\psi_n\rangle\langle\psi_n|}$$

→ After repeating the measurements we get

$$\boxed{\rho_{total} = \sum_n p_n \rho_n = \sum_n p_n |\psi_n\rangle\langle\psi_n|}$$

→ Projector $P^{(i)}$ can be $P^{(i)} = \sum_{n_i} |\psi_{n_i}\rangle\langle\psi_{n_i}|$

Still $P^{(i)2} = P^{(i)}$, $P^{(i)} = P^{(i)\dagger}$

$\text{eigs}(P^{(i)}) = 0, 1$

→ If we perform measurements in the projector $P^{(i)}$ such that $P^{(i)} P^{(j)} = \delta_{ij}$ and $\sum_i P^{(i)} = \mathbb{1}$

Then the measurement is complete and the outcome will be $\rho^{(i)} = \dfrac{P^{(i)} \rho P^{(i)}}{\mathrm{Tr}[P^{(i)} \rho]}$

→ $\rho_{final} = \sum_i p_i \rho^{(i)} = \sum_i P^{(i)} \rho P^{(i)}$

→ For composite systems:

$$|\Psi\rangle = \sum_{ij} \alpha_{ij} |i\rangle \otimes |j\rangle$$

→ Performing measurement on subsystem B in basis $\{|\psi_n\rangle\}$ will result in the B system collapsing to state $|\psi_n\rangle$ and corresponding projected state of A.

$$\Rightarrow \quad |\Psi\rangle \to (\mathbb{1} \otimes |\psi_n\rangle\langle\psi_n|) |\Psi\rangle$$

or $\rho_{AB} = |\Psi\rangle\langle\Psi| \to \rho_n = \dfrac{(\mathbb{1} \otimes |\psi_n\rangle\langle\psi_n|)\, \rho_{AB}\, (\mathbb{1} \otimes |\psi_n\rangle\langle\psi_n|)}{\mathrm{Tr}[\mathbb{1} \otimes |\psi_n\rangle\langle\psi_n|\, \rho_{AB}]}$

$$\boxed{\rho_{final} = \sum_n (\mathbb{1} \otimes |\psi_n\rangle\langle\psi_n|)\, \rho_{AB}\, (\mathbb{1} \otimes |\psi_n\rangle\langle\psi_n|)}$$

Using a Probe to make a measurement.

→ System + (ancilla = probe)

$$\rho_{AS} = \rho_A \otimes \rho_S$$ → initial state of the system + ancilla

→ Interaction b/w $S + A \rightarrow U$

$$\rho_{AS} \rightarrow U \rho_{AS} U^{\dagger} = U(\rho_A \otimes \rho_S) U^{\dagger}$$

→ Projective measurements on the probe.

$$\tilde{\rho}_n = (P_n \otimes \mathbb{1})\, U \rho_{AS} U^{\dagger}\, (P_n \otimes \mathbb{1})$$ → Un Normalised

$$\equiv \tilde{\rho}_n = |n\rangle\langle n| \otimes A_n \rho_S A_n^{\dagger} \; ; \quad p_n = \mathrm{Tr}[A_n^{\dagger} A_n \rho_S]$$

$$A_n = (\langle n| \otimes \mathbb{1} \; U \; |\psi\rangle \otimes \mathbb{1})$$

where $$\boxed{\begin{aligned} \rho_A &= |\psi\rangle\langle\psi| \\ |n\rangle\langle n| &= P_n \end{aligned}}$$

→ $\{A_n\}$ are the measurement operators.

Prove it. $$\boxed{\sum_n A_n^{\dagger} A_n = \mathbb{1}}$$

→ State after measurement

$$\rho_{fin} = \sum_n |n\rangle\langle n| \otimes A_n \rho_S A_n^{\dagger}$$

$$\boxed{\rho_{S,fin} = \sum_n A_n \rho_S A_n^{\dagger}}$$

→ In the measurement, the $|n\rangle$ clicks with probability $p_n = \mathrm{Tr}[A_n^{\dagger} A_n \rho_S]$

→ Interestingly, $p_n = Tr[A_n^\dagger A_n \rho_s]$ remains same for any set of Measurement operators $\{B_n = W_n A_n\}$ where $W_n \in$ Unitary.

→ There fore, for the same experimental out come p_n, we have a freedom in choosing the measurement operators.

⇒ We need to have a formalism which is free of this problem.

→ <u>Positive operator valued measurements (POVM):</u>

→ A POVM is a set of +ve semidefinite operator $\{E_i\}$ called effect, acting on the Hilbert space $\mathcal{H}_d$ that sum to the identity operator

$$\Rightarrow \boxed{\sum_i E_i = \mathbb{1} \quad \text{and} \quad E_i \geq 0}$$

→ The measurement out come is the probability $p_n = \langle E_n \rangle = Tr[E_n \rho]$

→ For every effect E_i we can choose a measurement operator A_i such that

$$E_i = A_i^\dagger A_i$$

→ The state after the measurements is $\boxed{\dfrac{A_i \rho A_i^\dagger}{\langle E_i \rangle}}$

→ Neumark's Dilation theorem:

Neumark's Dilation theorem states that a POVM can be lifted to a projective-measurements on an extended Hilbert space.

(The proof is similar to the Kraus-operators - Unitary link).

→ Expectation values of observables using POVM:

(For qubits only)

→ let the observable is σ_z or and $\hat{n}\cdot\vec{\sigma}$ with $|\hat{n}|=1$

we can define the effects.

$$E_{\pm} = \frac{1}{2}\left[\mathbb{1} \pm \lambda\, \hat{n}\cdot\vec{\sigma}\right] \qquad 0 \leq \lambda \leq 1$$

→ $\lambda = 1$ implies projective measurements.

→ $\lambda < 1$ " unsharp (weak) measurement

$$\boxed{\langle \hat{n}\cdot\vec{\sigma}\rangle = \frac{\langle E_+\rangle - \langle E_-\rangle}{\lambda}}$$

→ Quantum state tomography using POVM:

→ For state tomography we need a POVM with d^2 effects $\{E_i\}_{i=1}^{d^2}$

$$\langle E_i\rangle = \mathrm{Tr}[\rho E_i] = \langle E_i | \rho\rangle = p_i$$

→ Define a matrix $F = \begin{bmatrix} \langle E_1 | \\ \langle E_2 | \\ \vdots \\ \langle E_{d^2} | \end{bmatrix}$

$$\Rightarrow \quad F|\rho\rangle = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_{d^2} \end{bmatrix}$$

→ If E_i are linearly independent

$\Rightarrow$ F^{-1} exists

$\Rightarrow$ $\boxed{|\rho\rangle = F^{-1} p}$ From here we can find the density matrix

→ One example of POVM for qubits

$$\frac{|0\rangle\langle 0|}{2}, \quad \frac{|+\rangle\langle +|}{2}, \quad \frac{|+_y\rangle\langle +_y|}{2}, \quad \mathbb{I} - \frac{|0\rangle\langle 0|}{2} - \frac{|+\rangle\langle +|}{2} - \frac{|+_y\rangle\langle +_y|}{2}$$

→ Symmetric informationally complete POVM

SIC - POVM

A set of d^2 Rank one POVM $\{E_i = |\psi_i\rangle\langle\psi_i|\}$ which satisfy the relation.

$$\mathrm{Tr}[E_i E_j] = |\langle\psi_i|\psi_j\rangle|^2 = \frac{d\delta_{ij} + 1}{d+1}$$

→ SIC - POVM gives you the best estimate of the density operators.

→ Optical examples of generalized measurement

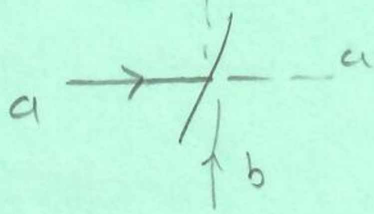
→ Projective measurement ($d \to 1$ limit)

→ Consider a photon in the polarization state

$|\psi\rangle = \alpha|H\rangle + \beta|V\rangle$

→ A Beam-Splitter as two input spatial modes and two output modes.

→ $|a\rangle, |b\rangle$ are the two states of a beam-splitter.



→ photon incoming in mode $a \Rightarrow$ state $|a\rangle$

// $b \Rightarrow |b\rangle$

→ For a balanced (50:50) beam splitter

$$|a\rangle \to \frac{|a\rangle + |b\rangle}{\sqrt{2}} \qquad\qquad |b\rangle \to \frac{|a\rangle - |b\rangle}{\sqrt{2}} \Rightarrow B = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Beam-Splitter matrix.

→ This Beam splitter is independent of Polarization.

→ Polarizing beam splitter:

$$|Ha\rangle \to |Ha\rangle$$
$$|Va\rangle \to |Vb\rangle$$
$$|Hb\rangle \to |Hb\rangle$$
$$|Vb\rangle \to |Va\rangle$$

$$\Rightarrow \begin{array}{c|cccc} & H_a & V_a & H_b & V_b \\ \hline H_a & 1 & 0 & 0 & 0 \\ V_a & 0 & 0 & 0 & 1 \\ H_b & 0 & 0 & 1 & 0 \\ V_b & 0 & 1 & 0 & 0 \end{array}$$

$\Rightarrow$ A polarizing beam splitter will take the input state $(\alpha|H\rangle + \beta|V\rangle)\otimes|a\rangle \to \alpha|H\,a\rangle + \beta|V\,b\rangle$

$\Rightarrow$ Placing the detectors in $|a\rangle$ and $|b\rangle$ outputs will cause the state to collapse to $|H\rangle$ and $|V\rangle$.

$\Rightarrow$ Projective measurement.

$\to$ Weak (Unsharp) measurement.

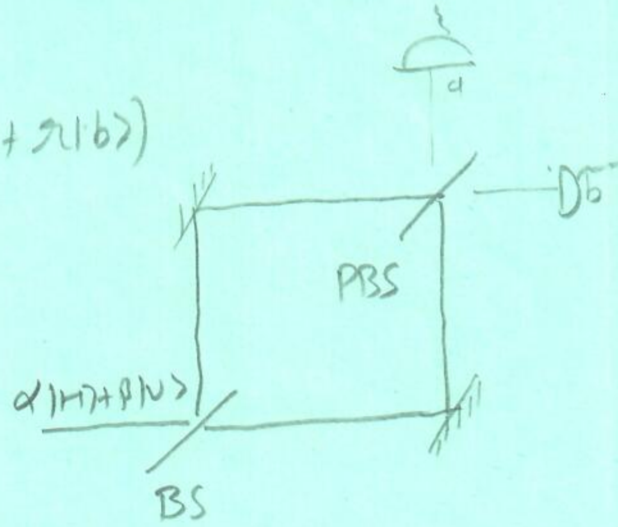
Consider a beam splitter with transmission t^2 and reflection r^2 $\Rightarrow t^2 + r^2 = 1$; let $t, r \in \mathbb{R}$

$\to$ $B = \begin{bmatrix} t & r \\ r & -t \end{bmatrix}$

$\Rightarrow$ $|a\rangle \to t|a\rangle + r|b\rangle$
$|b\rangle \to r|a\rangle - t|b\rangle$

$\to$ Consider $|\psi\rangle = (\alpha|H\rangle + \beta|V\rangle)\otimes|a\rangle$.

$\to$ After BS
$|\psi\rangle \to (\alpha|H\rangle + \beta|V\rangle)\otimes(t|a\rangle + r|b\rangle)$



$\to$ After PBS

$(t\alpha|H\rangle + \beta r|V\rangle)\otimes|a\rangle$
$+ (\beta t|V\rangle + \alpha r|H\rangle)\otimes|b\rangle$

$\Rightarrow$ Detecting photon in $|a\rangle \Rightarrow (t\alpha|H\rangle + \beta r|V\rangle)$
$|b\rangle \Rightarrow \beta t|V\rangle + \alpha r|H\rangle$

$\Rightarrow$ Measurement operator

$$K_1 = \begin{bmatrix} t & \\ & r \end{bmatrix} \; ; \; K_2 = \begin{bmatrix} r & \\ & t \end{bmatrix}$$

$$\boxed{K_1^{+} K_1 + K_2^{+} K_2 = \mathbb{1}}$$

$\rightarrow$ The probability of click in $|a\rangle \equiv p_a = |t\alpha|^2 + |\beta r|^2$

$$= \langle\psi| K_1^{+} K_1 |\psi\rangle$$

// $\quad |b\rangle \equiv p_b = |\beta t|^2 + |\alpha r|^2$

$$= \langle\psi| K_2^{+} K_2 |\psi\rangle$$

$\rightarrow$ if $t = r = \frac{1}{\sqrt{2}}$ $\Rightarrow$ weakest measurement

if $t = 1,\ r = 0$
or $r = 1 \ \ t = 0$ $\Rightarrow$ Projective measurements

$\rightarrow$ <u>Distinguishing non-orthogonal states</u>:

$\rightarrow$ Given two non-orthogonal states $|\psi\rangle, |\phi\rangle$
$\langle\psi|\phi\rangle \neq 0$, with equal probability, can we design an experimental scheme which distinguishes b/w $|\psi\rangle$ and $|\phi\rangle$.

$\rightarrow$ The discrimination should by unambiguous.

$\rightarrow$ We can chose the basis $\{|\psi\rangle, |\psi^{\perp}\rangle\}$ for the measurements. So that if we get a click in $|\psi^{\perp}\rangle$ then the state was $|\phi\rangle$. Whereas if we get a click in $|\psi\rangle$ then we donot know if it was $|\psi\rangle$ or $|\phi\rangle$.

→ Probability of Success is $p_s = \langle \psi_\perp | \rho_s | \psi_\perp \rangle$

where $\boxed{\rho_s = \frac{|\psi\rangle\langle\psi| + |\phi\rangle\langle\phi|}{2}}$

$\Rightarrow \quad p_s = \frac{1}{2} |\langle \psi_\perp | \phi \rangle|^2$

→ Similarly, we can choose $\{|\phi\rangle, |\phi_\perp\rangle\}$ basis.

→ It seems like the best we can do.

→ For simplicity, lets consider

$$|\psi\rangle = \begin{bmatrix} \cos \theta/2 \\ \sin \theta/2 \end{bmatrix} ; \quad |\phi\rangle = \begin{bmatrix} \cos \theta/2 \\ -\sin \theta/2 \end{bmatrix} \quad 0 \leq \theta \leq \pi/2$$

$\Rightarrow \quad \boxed{p_s = \frac{1}{2} \sin^2 \theta}$

→ We now try generalized measurement to distinguish them.

→ Let the state of ancilla be $|0\rangle$.

→ Total State of system + ancilla

$$|\chi\rangle = \cos \frac{\theta}{2} |00\rangle \pm \sin \frac{\theta}{2} |10\rangle$$

→ Consider a unitary U such that

$$U(\cos\tfrac{\theta}{2}|00\rangle) \rightarrow \sin \frac{\theta}{2} |00\rangle + \sqrt{\cos\theta}\, |11\rangle$$

and $U|01\rangle = |01\rangle$ and $U|10\rangle = |10\rangle$

$$\Rightarrow U|\chi\rangle = \left(\sin \frac{\theta}{2} |00\rangle \pm \sin \frac{\theta}{2} |10\rangle \right) + \sqrt{\cos\theta}\, |11\rangle$$

$$= \sqrt{2}\sin\frac{\theta}{2}\left[\frac{|0\rangle \pm |1\rangle}{\sqrt{2}}\right]|0\rangle + \sqrt{\cos\theta}\ |1\rangle$$

$\Rightarrow$ if upon measurement on the ancilla we get $|1\rangle$ then the measurement result is inconclusive

→ if the out come is $|0\rangle$ then the system will collapse to $\frac{|0\rangle \pm |1\rangle}{\sqrt{2}}$. Therefore, performing measurement on the system will result in the discrimination b/w $|\psi\rangle$ and $|\varphi\rangle$

→ Success probability $p_s = 2\sin^2\frac{\theta}{2}$

$$\boxed{p_s = 1 - |\langle\psi|\varphi\rangle|^2}$$

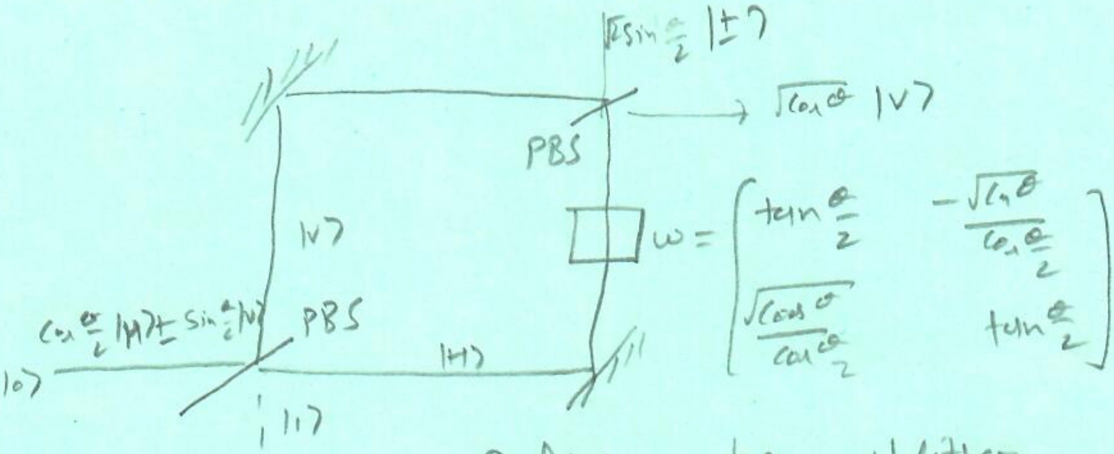
→ Surely this probability is more than what we got earlier

$$U = \left[\begin{array}{cc|cc} \tan\frac{\theta}{2} & 0 & 0 & -\frac{\sqrt{\cos\theta}}{\cos\theta/2} \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ \frac{\sqrt{\cos\theta}}{\cos\theta/2} & 0 & 0 & \tan\frac{\theta}{2} \end{array}\right]$$

→ <u>Optical implementation.</u>

→ $|\psi\rangle, |\varphi\rangle$ are the polarization states of a single photon. Two spatial modes of a beam-splitter forms the ancilla.

→ Initially the photon is in the mode $|0\rangle$; therefore, the total state is $\left(\cos\frac{\theta}{2}|H\rangle \pm \sin\frac{\theta}{2}|V\rangle\right)\otimes|0\rangle$



→ Pass it through the Polarizing beam-splitter.

$$|X\rangle \to \cos\frac{\theta}{2}\,|H0\rangle \pm \sin\frac{\theta}{2}\,|V1\rangle$$

⇒ Apply Rotation on $|H\rangle$ in the $|0\rangle$ such that

$$|H\rangle \longrightarrow \tan\frac{\theta}{2}|H\rangle + \frac{\sqrt{\cos\theta}}{\cos\frac{\theta}{2}}|V\rangle$$

$$|V\rangle \longrightarrow -\frac{\sqrt{\cos\theta}}{\cos\frac{\theta}{2}}|H\rangle + \tan\frac{\theta}{2}|V\rangle$$

→ let it interfere at another PBS.

→ In the language of POVM:

$E_1 = a|\psi_\perp\rangle\langle\psi_\perp|$

$E_2 = a|\phi_\perp\rangle\langle\phi_\perp|$; a is chosen optimally so that $E_3 \geq 0$ and Rank 1.

$E_3 = \mathbb{1} - E_1 - E_2$

⇒ $\langle E_1\rangle = \langle E_2\rangle = p_s$

$\langle E_3\rangle \to$ inconclusive.

Entanglement Witness.

→ Pure state (Schmidt Rank)

→ Mixed states

→ +ve but not CP maps.

→ Partial transpose on 2×2 or 2×3 syst.

→ Entanglement witness.

Entanglement measure.

→ For pure states:

Von-Neumann entropy

→ Concurrence

$$|\psi\rangle = |x\rangle \otimes |y\rangle.$$

$$\Rightarrow (\sigma_2 \otimes \sigma_2) C |\psi\rangle = \sigma_2 |x^*\rangle \otimes \sigma_2 |y^*\rangle = |\tilde{\psi}\rangle$$

$|\tilde{\psi}\rangle$ is orthogonal to $|\psi\rangle$.

if $|\psi\rangle = \sum_{ij} \alpha_{ij} |i\rangle \otimes |j\rangle$.

$$\cdot |\tilde{\psi}\rangle = \sum_{ij} \alpha^*_{ij} (\sigma_2 \otimes \sigma_2) |i\rangle \otimes |j\rangle.$$

→ $|\langle\tilde{\psi}|\psi\rangle|$ will be zero for product states

→ It can have values b/w 0 and 1.

$$|\langle\tilde{\psi}|\psi\rangle| = |\mathrm{tr}(-\alpha\, \sigma_2 \alpha^* \sigma_2)|$$

Magic basis.

$$\left.\begin{aligned}
|e_1\rangle &= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \\
|e_2\rangle &= \frac{i}{\sqrt{2}}(|00\rangle - |11\rangle) \\
|e_3\rangle &= \frac{i}{\sqrt{2}}(|01\rangle + |10\rangle) \\
|e_4\rangle &= \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)
\end{aligned}\right\}$$

$(\sigma_2 \otimes \sigma_2) C |e_i\rangle = -|e_i\rangle$

if $|\psi\rangle = \sum_i \gamma_i |e_i\rangle$

$|\langle\tilde{\psi}|\psi\rangle| = |\sum_i \gamma_i|^2$

→ If γ_i are real $\Rightarrow$ $|\sum_i \gamma_i^2| = 1$ → Maximally entangled state.

→ Properties of an ideal entanglement measure.

→ An entanglement Measure $E \equiv E(\rho)$; $E(\rho) \to$ real number

and $E(\rho) = 0$ implies separable states

→ E should not increase under any LOCC.

→ $E(\rho) = E(u \otimes v \, \rho \, u^{\dagger} \otimes v^{\dagger})$

→ Convexity. $E(\alpha \rho_1 + (1-\alpha)\rho_2) \leq \alpha E(\rho_1) + (1-\alpha) E(\rho_2)$

→ Continuity: if $\|\rho_1 - \rho_2\| \to 0 \Rightarrow |E(\rho_1) - E(\rho_2)| \to 0$.

→ Additivity: $E(\rho_1 \otimes \rho_2) = E(\rho_1) + E(\rho_2)$.

→ Entanglement of formation (for mixed states)

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \Rightarrow E(\rho) = \sum_i p_i E(|\psi_i\rangle)$$

$$E_{of}(\rho) = \min_{\{p_i, |\psi_i\rangle\}} \sum p_i E(|\psi_i\rangle)$$

→ Concurrence for mixed states:

$$\tilde{\rho} = (\sigma_2 \otimes \sigma_2) \rho^{*} (\sigma_2 \otimes \sigma_2)$$

$$C(\rho) = \max(0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4)$$

where $\lambda_1^2, \lambda_2^2, \lambda_3^2, \lambda_4^2$ are the eigenvalues of $\boxed{\tilde{\rho} . \rho = R}$

$$E_{of}(\rho) = h\left(\frac{1 + \sqrt{1 - C^2}}{2}\right) \cdot \quad \left(\text{Entanglement of formation.}\right)$$

$$h(x) = -x \log_2 x - (1-x) \log_2 (1-x)$$