PHY302 - Tutorial 6

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1. The wave function of the ground state of a harmonic oscillator of force constant k and mass m is

$$\psi_0 = (\alpha/\pi)^{1/4} e^{-\alpha x^2/2},\ \alpha = m\omega_0/2,\ \omega_0^2 = k/m. \tag{1}$$

   Obtain an expression for the probability of finding the particle outside the classical region ($E < V(x)$).

2. (a) Using $[a, a^\dagger] = \mathbb{I}$, find the commutations $[a, (a^\dagger)^n]$, $[a^\dagger, (a)^n]$, $[N, (a)^n]$, and $[N, (a^\dagger)^n]$, where $N = a^\dagger a$.

   (b) Using the number basis $\{|n\rangle)\}$ find the matrix representation for the operators $\hat{a}$, $\hat{a}^\dagger$, $\hat{x}$, $\hat{p}$, $\hat{x}^2$, $\hat{p}^2$, and the number operator $\hat{N} = \hat{a}^\dagger \hat{a}$. Do this by giving general formulae for the matrix elements O_{mn} of each operator $\hat{O}$. Write explicitly the corresponding four by four matrix truncations using $m, n = 0, 1, 2, 3$.

   (c) Use the four by four matrices for $\hat{x}$ and $\hat{p}$ to compute $[\hat{x}, \hat{p}]$. Do you get the matrix $i\mathbb{I}$? Explain.

   (d) (e) From your earlier result above you must have found that

$$\langle n|\hat{x}^2|n\rangle = \frac{\hbar}{m\omega}\left(n + \frac{1}{2}\right),\ \ \langle n|\hat{p}^2|n\rangle = \hbar m\omega\left(n + \frac{1}{2}\right). \tag{2}$$

   Find the uncertainties Δx and Δp in the state $|n\rangle$. Is the product of uncertainties saturated?

3. A spin is placed on an uniform but oscillating magnetic field

$$\vec{B} = B_0 \cos(\omega i)\vec{e}_z. \tag{3}$$

   The spin is initially in an eigenstate of S_x with eigenvalue $\hbar/2$.

   (a) Find the unitary operator $U(t)$ that generates time evolution. Note that the Hamiltonian is time-dependent but $[H(t), H(t')] = 0$.

   (b) Calculate the time evolution of the state and describe it by giving the time-dependent angles $\theta(t)$ and $\phi(t)$ that define the direction of the spin.

   (c) Find the time dependent probability to find the spin with $S_x = -\hbar/2$.

   (d) Find the largest value of ω that allows the full flip in S_x.

4. Consider the time-independent Schrodinger Hamiltonian for a spin in a uniform and constant magnetic field of magnitude B along the z-direction:

$$H = -\lambda B S_z, \tag{4}$$

   where λ is the (real) constant that relates the dipole moment of the spin. Find the explicit time evolution for the Heisenberg operators $S_x(t)$, $S_y(t)$, and $S_z(t)$ associated with the Schrodinger operators S_x, S_y, and S_z.