THINKING MATHEMATICALLY : I

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WHAT IS MATHEMATICS?

What is mathematics? Is it the collection of symbols $1, 2, 3, \cdots$ that we call numbers? Or it is a study of shapes such as lines, circles, stars or cones? Or it comprises of manipulating symbols $x, y, z, \cdots$ which may assume some numerical values? Or it is about doing fast calculations? Or it is everything described above? Or none!

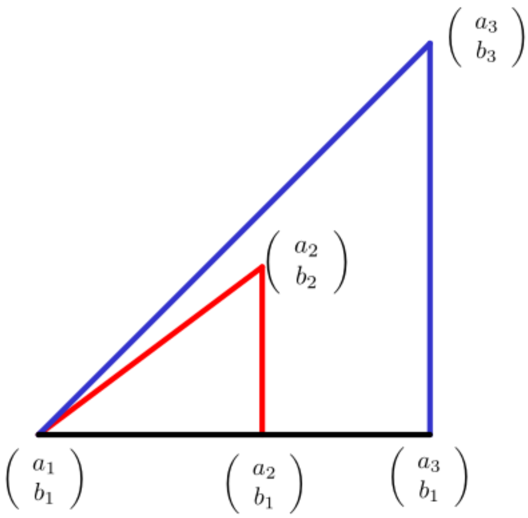
We all have our imagery of mathematics. We all have our expectations from mathematics. It is difficult to disengage with mathematics. Everyone remains either a user of it or a practitioner, or perhaps both.

Nonetheless, the following features of mathematics are hardly disputed.

(1) *Raw material*. Mathematics begins with assumptions. These assumptions are called *axioms* when the intention is to build a large theory, or *hypothesis* when the intention is to make a single statement. Axiom are foundational statements. These are not to be questioned.

(2) *Definitions*. These are the words which express a concept in indisputable manner. Following is an example of definition.

Let $S \subseteq \mathbb{R}^2$. Three distinct points $p_1 := \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$, $p_2 := \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$, $p_3 := \begin{pmatrix} a_3 \\ b_3 \end{pmatrix} \in S$ are called *collinear* if the triangles with blue and red hypotenuse in the following picture are similar.[1]



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[1]Recall that two triangles are called similar if their corresponding sides are in equal proportion.

Let us use the concept of collinearity to define a line. We start with the following definition.

D1. A subset $S \subseteq \mathbb{R}^2$ is called a *line* if any arbitrarily picked triplet $p, q, r \in S$ is collinear.

Are you happy with this definition? In the beginning it looks like that we are happy. Then we encounter the following questions.

Q1. Consider the set $S := \{(1,2),(3,4)\}$. This set consists of only two points. Would you like to call it a line?
Q2. Consider the set $S := \{(0,0),(1,1),(2,2),(3,3)\}$. Would you like to call it a line?
Q3. Consider the set $S := \{(1,1),(\frac{1}{2},\frac{1}{2}),(\frac{1}{3},\frac{1}{3}),(\frac{1}{4},\frac{1}{4}),\cdots\}$. This set consists of infinitely many points. Still, would you like to call it a line?

Which of the Q1, Q2, Q3 satisfy the definition D1? Do these examples inspire you to modify the definition D1 of line? You may have reasons to include or not include more conditions D2, D3, $\cdots$, in the definition of line while constructing *your* theory of lines. If a definition is too restrictive then there are very few examples satisfying the definition. On the other hand a definition that is less restrictive is likely to admit examples that sound odd.

(3) *Deduction*. This is the most important feature of modern day mathematics. One *deduces* from existing axioms, definitions and known facts other interesting statements called Theorems, Corollaries or Lemmas. Theorems are the statements of wider importance, while a corollary is a statement that is more or less an immediate consequence of a Theorem (or more theorems). Lemmas are technical statements which help in presenting ideas and contain some crucial steps in the proof of a theorem. Lemmas may not be interesting in themselves.

(4) *Organization*. Once there are enough theorems and corollaries, it is the time to organize the entire "database" of the theory much systematically. One therefore constructs mathematical objects to deal with. For example, once we understand enough properties of vectors and their transformations (such as rotation or reflection), it is desirable to organize ideas as vector spaces and linear transformations.

(5) *Drawing parallels*. A theory in its sufficient depth may resemble like another theory. This allows cross fertilization of ideas in the sense that the features of a theory may be utilized to enrich the other. Mathematics provides tools to efficiently draw parallels. *Functors* are examples of such tools and are beyond the scope of MTH101.

Application is *not* a fundamental feature of a theory. A theory may have been constructed to be applicable for a predefined purpose, or may find immediate applications. There are theories which are not meant to be applicable, or are not applicable yet. But that does not undermine the process of systematic construction of mathematics. The Perron-Frobenius theorem was not constructed to be utilized in Google PageRank algorithm. Eventually, after 90 years of its existence it turned out that the theorem is indispensable for Google search. This was the need of the Google search, and not the test of beauty of Perron-Frobenius theorem. This is the perspective most mathematicians will take!