§ Homogeneous Equations: If P, Q are homogeneous fns of x, y of the same degree n, the eqn is reducible by substitution:

$y = vx$ to one whose vars are separable.

$$P(x,y) = x^n P(1,v), \quad Q(x,y) = x^n Q(1,v).$$

So $P\,dx + Q\,dy = 0$

$$\Rightarrow \{P(1,v) + v\,Q(1,v)\}\,dx + x\,Q(1,v)\,dv = 0$$

or $$\frac{dv}{\varphi(v)} + \frac{dx}{x} = 0 \quad \text{where } \varphi(v) = v + \frac{P(1,v)}{Q(1,v)}.$$

Soln. is $$\int \frac{dv}{\varphi(v)} = \log \frac{c}{x}.$$

Example $(y^4 - 2x^3y)\,dx + (x^4 - 2xy^3)\,dy = 0$.

$y = vx$ reduces it to $$\frac{dx}{x} = \frac{1-2v^3}{v+v^4}\,dv$$

$$= \left(\frac{1}{v} - \frac{3v^2}{1+v^3}\right)dv.$$

So $\log x = \log v - \log(1+v^3) + \log c$

or $x(1+v^3) = cv$

So primitive is $\boxed{x^3 + y^3 = cxy}$.

§ When the eqn $P\,dx + Q\,dy = 0$ is both homogeneous and exact, it is immediately integrable ~~without the integration~~ provided degree of homogeneity $n \neq -1$. Its primitive in fact is $Px + Qy = c$.

Let $u = Px + Qy$. Then $$\frac{\partial u}{\partial x} = P + x\frac{\partial P}{\partial x} + y\frac{\partial Q}{\partial x}$$

$$= P + x\frac{\partial P}{\partial x} + y\frac{\partial P}{\partial y} = (n+1)P.$$

by Euler's theorem:

Polynomial

Observe (Euler's theorem) For a homogeneous fn. $P(x,y)$.

$$x\frac{\partial P}{\partial x} + y\frac{\partial P}{\partial y} = nP \quad \text{where } n = \text{deg of homog.}$$

$|||^{ly}$. $\frac{\partial u}{\partial y} = (n+1)Q$.

So $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$.

$= (n+1)(Pdx + Qdy)$.

So $Pdx + Qdy = \frac{d(Px + Qy)}{n+1}$

If $n \neq -1$ the primitive is $Px + Qy = C$.

Example $x(x^2+3y^2)dx + y(y^2+3x^2)dy = 0$.

Soln is $x^4 + 6x^2y^2 + y^4 = C$.

- When $n = -1$. ~~Some effort needed (1?)~~ :

The homogeneous eqn $\frac{Pdx + Qdy}{Px + Qy} = 0$

is exact (check that cond. of integrab is satisfied)

Thus any homogeneous eqn. may be made exact by introducing the integrating factor $\frac{1}{Px+Qy}$.

The deg of homog. is -1, so integration is involved.

6. An eqn. of the type: $\frac{dy}{dx} = F\left(\frac{Ax+By+C}{ax+by+c}\right)$

in which A, B, C, a, b, c are const. s.t. $Ab - aB \neq 0$

may be brought into homogeneous form by a linear transfn. of the vars:

$$x = h + \xi, \quad y = k + \eta$$

where ξ, η are new vars & h, k const. s.t.

$$Ah + Bk + C = 0 \quad \& \quad ah + bk + c = 0$$

Eqn. becomes $\dfrac{d\eta}{d\xi} = F\left(\dfrac{A\xi + B\eta}{a\xi + b\eta}\right)$

So that F is homogen. fn. of ξ, η of degree zero.
h, k are determinate since $Ab - aB \neq 0$.

When $Ab - aB = 0$, let η be a new dependent var. defined by $\eta = x + By/A = x + by/a$

then $\dfrac{d\eta}{dx} = 1 + \dfrac{b}{a} F\left(\dfrac{A\eta + C}{a\eta + c}\right)$

The vars are now separable.

Example: $(3y - 7x + 7)\,dx + (7y - 3x + 3)\,dy = 0$.

Subst. $x = \xi + 1$, $y = \eta$ gives:

$(3\eta - 7\xi)\,d\xi + (7\eta - 3\xi)\,d\eta = 0$.

This homog. the transform. $\eta = v\xi$ change it into: $(7v - 3)\xi\,dv + (7v^2 - 7)\,d\xi = 0$

or $\left(\dfrac{2}{v-1} + \dfrac{5}{v+1}\right)dv + \dfrac{7}{\xi}\,d\xi = 0$

$\Rightarrow (v-1)^2 (v+1)^5 \xi^7 = c$. The primitive is:

$(y - x + 1)^2 (y + x - 1)^5 = c$. //

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Lecture 6

§ The Integrating factor.

Let $Pdx + Qdy = 0$ be a diff. eqn which is not exact. The theoretical method of integrating such an equation is to find a function $\mu(x,y)$ such that the expression

$$\mu(Pdx + Qdy) \text{ is a total diff } du.$$

When μ has been found, the problem reduces to integration.

Main question: whether or not integrating factor exist?

Theorem. On the assumption that the equation itself has a unique solution which depends upon one arbitrary constant, there exists an infinity of integrating factors.

Proof. Let the general soln be $\varphi(x,y) = C$. where C is an arbitrary constant

Then $\frac{\partial \varphi}{\partial x} dx + \frac{\partial \varphi}{\partial y} dy = 0$ or $\varphi_x dx + \varphi_y dy = 0$.

Since $\varphi(x,y) = C$ is the gen soln. of $Pdx + Qdy = 0$

The reln $\frac{\varphi_x}{P} = \frac{\varphi_y}{Q}$ must hold identically, it follows

that $\exists$ fn μ s.t. $\varphi_x = \mu P$, $\varphi_y = \mu Q$.

Therefore $\mu(Pdx + Qdy) = d\varphi$

i.e. an integrating factor exists.

Let $F(\varphi)$ be any fn of φ. Then $\mu F(\varphi)\{Pdx + Qdy\} = F(\varphi)d\varphi$ is exact.

If, therefore, μ is any integrating factor, giving rise to the solution $\varphi(x,y) = C$, then $\mu F(\varphi)$ is also an integrating factor. Since F is arbitrary, $\exists$ infinitely many integrating factors //

Since the equation $\mu(P\,dx + Q\,dy) = 0$ is exact the integrating factor satisfies the relⁿ

$$\frac{\partial(\mu P)}{\partial y} = \frac{\partial(\mu Q)}{\partial x}$$

or $$P\frac{\partial \mu}{\partial y} - Q\frac{\partial \mu}{\partial x} + \mu\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0.$$

So μ satisfies a PDE of the first order. In general therefore, the direct evaluation of μ depends upon an equation of a more advanced character than the ~~ODE~~ ordinary linear equation under consideration. It is, however, to be noted that any particular solution and not necessarily the general soln. of the PDE is sufficient to furnish an integrating factor.

In many particular cases, the PDE has an obvious solution which gives the reqd. integrating factor.

Example: Suppose μ is a fn. of x alone. then

$$\frac{1}{\mu}\frac{d\mu}{dx} = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$$

It is therefore necessary that the RHS should be independent of y. When this is the case then μ is easily obtained by an integration.

Now suppose also that $Q \equiv 1$, then P must be a linear fn. of y. The equation is therefore of the form: $dy + (py - q)\,dx = 0$. where p, q are fons. of x alone. The eqn is linear, and the integrating factor determined by $\frac{d\mu}{dx} = p\mu$ is $\mu = e^{\int p\,dx}$.

Example. $\alpha x\,dy + \beta y\,dx + x^m y^n (a x\,dy + b y\,dx) = 0$

Consider first of all the expression $\alpha x\,dy + \beta y\,dx$; an integrating factor is $x^{\beta-1}y^{\alpha-1}$ and

since $x^{\beta-1}y^{\alpha-1}(\alpha x\,dy + \beta y\,dx) = d(x^\beta y^\alpha)$,

the more general expression

$$x^{\beta-1}y^{\alpha-1}\Phi(x^\beta y^\alpha)$$ is also an integrating factor.

In the same way

$$x^{b-m-1}y^{a-n-1}F(x^b y^a)$$ is an I.F. for

$x^m y^n (a x\,dy + b y\,dx)$.

So if Φ & F are so determined that

$$x^{\beta-1}y^{\alpha-1}\Phi(x^\beta y^\alpha) = x^{b-m-1}y^{a-n-1}F(x^b y^a)$$

then an I.F. of the original eqn. will have been obtained.

Let $\Phi(z) = z^\rho$, $F(z) = z^\gamma$. Then

$x^\lambda y^\mu$ will be an I.F if

$$\lambda = (\rho+1)\beta - 1 = (\gamma+1)b - m - 1$$
$$\mu = (\rho+1)\alpha - 1 = (\gamma+1)a - n - 1$$

These eqns. determine ρ, γ and hence λ & μ iff $\alpha b - \beta a \neq 0$.

- If OTOH, $a = k\alpha$, $b = k\beta$, the original eqn. is $(1 + kx^m y^n)(\alpha x\,dy + \beta y\,dx) = 0$. I.F. is $\dfrac{x^{\beta-1}y^{\alpha-1}}{1 + kx^m y^n}$ //