PHY302: Quantum mechanics
Tutorial-2

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Problem 1:: Energy must exceed the minimum value of the potential.

Consider the time-independent *Schrödinger* equation(TISE) for a particle of Energy E in a Potential $V(x)$, with $x \in (-\infty, \infty)$:

$$\frac{d^2\psi(x)}{dx^2} = \frac{2m}{\hbar^2}\left[V(x) - E\right]\psi(x). \tag{1}$$

Without loss of generality one can assume that $\psi(x)$ is real. Assume the potential is bounded below,

$$V(x) \geq V_{min}, \qquad\qquad for \;\; all \;\; x,$$

where V_{min} is the minimum value of the potential.
Prove that $E > V_{min}$ for *normalizable* solutions to exist. To do this, assume $E \leq V_{min}$ and try using equation (1) and integration to reach a clear contradiction.

Problem 2:: Expectation value $\langle \hat{p} \rangle$ of the momentum. Consider the wavefunction

$$\psi(x) = e^{ikx}\phi(x),$$

with k real and constant and $\phi(x)$ real. Calculate $\langle \hat{p} \rangle$

Problem 3: Conserved probability current.: Suppose $\Psi(x,t)$ obeys the one-dimensional *Schrödinger* equation,

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x,t) + V(x)\Psi(x,t) = i\hbar\frac{\partial}{\partial t}\Psi(x,t).$$

- In the following we consider stationary states with spatial wavefunction $\psi(x)$. Compute the probability current J for $\psi(x) = e^{i\alpha(x)}\phi(x)$ where $\alpha(x)$ and $\phi(x)$ are real. Show that

$$\frac{J(x)}{\rho(x)} = \frac{\hbar}{m}\alpha'(x).$$

Explain why the ratio J/ρ can be viewed as the local velocity of the quantum particle describe by $\psi(x)$.

- Consider $\psi(x) = Ae^{ipx/\hbar} + Be^{-ipx/\hbar}$, with A and B complex constant. Calculate $J(x)$. Are there cross terms in J between the left and right-moving parts of ψ?

Problem 4: A property of matrices: We can define a function of a matrix $\mathbf{M}$ by a power series. If $f(z)$ is a function with a Taylor series expansion $f(z) = \sum_{n=0}^{\infty} f_n z^n$, then we define $f(M) \equiv \sum_{n=0}^{\infty} f_n M^n$.
Let $\mathbf{M}$ be the matrix

$$\mathbf{M} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

Show that $e^{i\mathbf{M}\theta}$ takes the form

$$e^{i\mathbf{M}\theta} = A(\theta)\mathbf{1} + B(\theta)\mathbf{M},$$

where $\mathbf{1}$ is the 2×2 identity matrix and A and B are functions you must determine. What is the algebraic property of a matrix $\mathbf{M}$ of arbitrary size that would lead to this result ?.

Problem 5: Spin one-half states:

(a) An (unnormalized) spin state is given by

$$(1+i)\,|Z;+\rangle - (1+i\sqrt{3})\,|Z;-\rangle\,.$$

What direction does this spin state point to?

(b) Consider the following sequence of experiments:

1. First, prepare a beam of spin-1/2 atoms which are all in the state $|Z;+\rangle$ by passing a beam through a Stern-Gerlach device oriented in the $\hat{z}$ direction, and keeping only those atoms measured to have eigenvalue $+\hbar/2$.

2. Then, pass these atoms through a second Stern-Gerlach device designed to measure the spin along the x-direction $\hat{S}_x$. That is, $\phi = 0$ and $\theta = \pi/2$. Keep only those atoms which have eigenvalue $+\hbar/2$

3. Finally, pass the atoms which remain through a third Stern-Gerlach device, oriented in the same (z) direction as the first.

Of all the atoms that entered the second magnet, what fraction are found by the third magnet to be in the state $|Z;+\rangle$? What fraction are found by the third magnet to be in the state $|Z;-\rangle$? What fraction never made it to the third magnet?

Problem 6: Overlap of two spin one-half states.:

Consider a spin state $|\mathbf{n};+\rangle$ where $\mathbf{n}$ is the unit vector defined by the polar and azimuthal angles θ and ϕ and the spin state $|\mathbf{n}';+\rangle$ where $\mathbf{n}'$ is the unit vector defined by the polar and azimuthal angles θ' and ϕ'. Let γ denote the angle between vectors $\mathbf{n}$ and $\mathbf{n}'$:

$$\mathbf{n} \cdot \mathbf{n}' = \cos\gamma.$$

Show by direct computation that the overlap of the associated spin states is controlled by $half$ the angle between the unit vectors:

$$|\langle\mathbf{n}';+|\mathbf{n};+\rangle\,|^2 = \cos^2\frac{\gamma}{2}.$$