PHY 101 : Worksheet-1

Bold faced objects are vectors

The position vector $\mathbf{x}(t)$ of a particle in a co-ordinate system is given as a function of time t as

$$\mathbf{x}(t) = \alpha t^2 \hat{\mathbf{i}} + \beta t^3 \hat{\mathbf{j}},$$

where α, β are constants. Find out

1. The velocity of the particle $\mathbf{v}(t) = d\mathbf{x}(t)/dt$ and the acceleration $\mathbf{a}(t) = d\mathbf{v}(t)/dt$.
2. Angle between the position vector and the velocity vector at $t = 2$.
3. Angle between the velocity vector and the acceleration vector at $t = 5$.
4. The angular momentum vector $\mathbf{L}(t)$ if the mass of the particle is m units.

A unit mass particle with a charge q enters a region of magnetic field $\mathbf{B}$ and starts experiencing a force $\mathbf{F}(t) = -2q\omega \sin(\omega t)\hat{\mathbf{i}} - 2q\omega \cos(\omega t)\hat{\mathbf{j}}$, where ω is a constant. Its location at any time t is given as the position vector $\mathbf{r}(t) = 2\sin(\omega t)\hat{\mathbf{i}} + 2\cos(\omega t)\hat{\mathbf{j}}$

1. Find out the magnetic field in this set-up.
2. Find out the angular momentum $\mathbf{L}$ of the particle w.r.t. the origin.
3. What is the relation between the angular momentum and the magnetic field?
4. Find out $\mathbf{r} \times \mathbf{L}$ and $\mathbf{v} \times \mathbf{L}$ in this set-up.

[**Hint :** $d\sin(y)/dy = \cos(y)$ and $d\cos(y)/dy = -\sin(y)$]

Worksheet - 1

Q.1

$$\vec{x}(t) = \alpha t^2 \hat{i} + \beta t^3 \hat{j}$$

(i) $\vec{v}(t) = \frac{d\vec{x}}{dt} = \frac{d}{dt}\left(\alpha t^2 \hat{i} + \beta t^3 \hat{j}\right)$

$$= \alpha \hat{i} \frac{d}{dt} t^2 + \beta \hat{j} \frac{d}{dt} t^3$$

Since $\alpha, \beta, \hat{i}, \hat{j}$ do not change with t

$$= 2\alpha t \hat{i} + 3\beta t^2 \hat{j}$$

Acceleration $\vec{a}(t) = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(2\alpha t \hat{i} + 3\beta t^2 \hat{j}\right)$

$$= 2\alpha \hat{i} + 6\beta t \hat{j}$$

(ii) Let the angle between $\vec{x}$ and $\vec{v}$ be $\theta(t)$

$$\Rightarrow \vec{x} \cdot \vec{v} = |\vec{x}||\vec{v}| \cos\theta(t)$$

$$|\vec{x}| = \sqrt{\alpha^2 t^4 + \beta^2 t^6}$$

$$|\vec{v}| = \sqrt{4\alpha^2 t^2 + 9\beta^2 t^4}$$

$$(\alpha t^2 \hat{i} + \beta t^3 \hat{j}) \cdot (2\alpha t \hat{i} + 3\beta t^2 \hat{j})$$

$$2\alpha^2 t^3 + 3\beta^2 t^5 = \sqrt{\alpha^2 t^4 + \beta^2 t^6}\sqrt{4\alpha^2 t^2 + 9\beta^2 t^4} \cos\theta(t)$$

$$\therefore \quad \cos\theta(t) = \frac{2\alpha^2 t^3 + 3\beta^2 t^5}{\sqrt{\alpha^2 t^4 + \beta^2 t^6}\,\sqrt{4\alpha^2 t^2 + 9\beta^2 t^4}}$$

At $t = 2$

$$\cos\theta(2) = \frac{8\alpha^2 + 106\beta^2}{\sqrt{16\alpha^2 + 64\beta^2}\,\sqrt{16\alpha^2 + 144\beta^2}}$$

(iii) Let the angle between $\vec{a}$ and $\vec{v}$ be $\tilde{\theta}(t)$

$$|\vec{a}| = \sqrt{4\alpha^2 + 36\beta^2 t^2}$$

$$\vec{a}\cdot\vec{v} = |\vec{a}|\,|\vec{v}|\cos\tilde{\theta}(t)$$

$$(2\alpha\hat{i} + 6\beta t\hat{j})\cdot(2\alpha t\,\hat{i} + 3\beta t^2\hat{j})$$

$$= 4\alpha^2 t + 18\beta^2 t^3 = \sqrt{4\alpha^2 + 36\beta^2 t^2}\,\sqrt{4\alpha^2 t^2 + 9\beta^2 t^4}\,\cos\tilde{\theta}(t)$$

$$\Rightarrow \cos\tilde{\theta}(t) = \frac{4\alpha^2 t + 18\beta^2 t^3}{\sqrt{4\alpha^2 + 36\beta^2 t^2}\,\sqrt{4\alpha^2 t^2 + 9\beta^2 t^4}}$$

(iv) Angular momentum

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v}$$

$$= m\left(\alpha t^2\hat{i} + \beta t^3\hat{j}\right) \times \left(2\alpha t\hat{i} + 3\beta t^2\hat{j}\right)$$

$$= m\left[3\alpha\beta t^4\hat{k} - 2\alpha\beta t^4\hat{k}\right]$$

$$= \alpha\beta t^4\hat{k}$$

Q.2 In the magnetic field a charged particle experiences a force

$$\vec{F} = q\left(\vec{v} \times \vec{B}\right)$$

$$-2q\omega\left(\sin\omega t\,\hat{i} + \cos\omega t\,\hat{j}\right) = q\left(\frac{d\vec{r}}{dt} \times \vec{B}\right)$$

$$\vec{r} = 2\sin\omega t\,\hat{i} + 2\cos\omega t\,\hat{j}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = 2\hat{i}\frac{d}{dt}\sin\omega t + 2\hat{j}\frac{d}{dt}\cos\omega t$$

because $2, \hat{i}, \hat{j}$ are constants

$$\frac{d}{dt}\sin\omega t = \frac{d(\omega t)}{dt}\frac{d}{d(\omega t)}\sin\omega t = \omega\cos\omega t$$

since ω is constant

$$\frac{d}{dt}\cos\omega t = \frac{d(\omega t)}{dt}\frac{d}{d(\omega t)}\cos\omega t = -\omega\sin\omega t$$

$$\vec{v} = 2w\cos wt\,\hat{i} - 2w\sin wt\,\hat{j}$$

$$-2qw(\sin wt\,\hat{i} + \cos wt\,\hat{j})$$

$$= 2qw(\cos wt\,\hat{i} - \sin wt\,\hat{j}) \times \vec{B}$$

If $\vec{B} = B_0\hat{k}$, the RHS is

$$2qwB_0(\cos wt(-\hat{j}) - \sin wt\,\hat{i})$$

$$= -2qwB_0(\sin wt\,\hat{i} + \cos wt\,\hat{j})$$

Comparing it to LHS, $B_0 = 1$ unit

$$\therefore B = \hat{k}$$

$$\vec{L} = \vec{r} \times m\vec{v} = (2\sin wt\,\hat{i} + 2\cos wt\,\hat{j}) \times (2w\cos wt\,\hat{i} - 2w\sin wt\,\hat{j})$$

$$= -4w\sin^2 wt\,\hat{k} - 4w\cos^2 wt\,\hat{k}$$

$$= -4w\,\hat{k}$$

(iii) $\vec{L} = -\dfrac{\vec{B}}{4w}$

(iv) $\vec{r} \times \vec{L}$

$$\Rightarrow (2\sin\omega t\,\hat{i} + 2\cos\omega t\,\hat{j}) \times (-4\omega\,\hat{k})$$

$$= 8\omega\sin\omega t\,\hat{j} - 8\omega\cos\omega t\,\hat{i}$$

$\vec{v} \times \vec{L}$

$$(2\omega\sin\omega t\,\hat{i} - 2\omega\cos\omega t\,\hat{j}) \times (-4\omega\,\hat{k})$$

$$= 8\omega^2\sin\omega t\,\hat{j} + 8\omega^2\sin\omega t\,\hat{i}$$