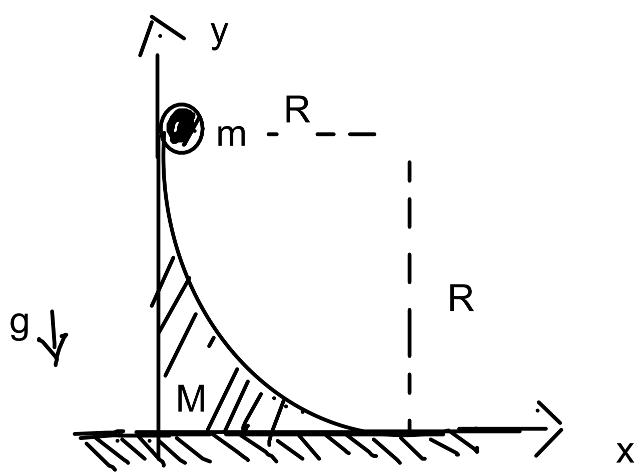
PHY 101 : Mechanics
Mid Semester Exam
21^{st} **October 2023, IISER Mohali**

There are 4 problems (with sub parts) and all problems carry equal weightage. Symbols have their usual meanings.

1. A particle moves on a trajectory with a profile $\rho = \alpha t^2,\ \phi = \beta t, z = \gamma t$ in cylindrical polar co-ordinates. Parameters α, β, γ are **positive** constants.
(i) Find out the position of the particle, its velocity and the acceleration at any time t.
(ii) Find out the instants of time when the position vector is orthogonal to the velocity.
(iii) Find out the instants of time when the position vector is orthogonal to the acceleration. $[\mathbf{3+1+1}]$

2. A ball of mass m rolls down under the action of gravity g from the top of a wedge shaped in circular arc profile (as shown in the figure) of radius R. The wedge has mass M and is put on a desk on which it can slide. (Assume all the surfaces to be frictionless)
(i) Draw the free body force diagram of the ball and the wedge both, when the ball has rolled down to



half the path length of the wedge.
(ii) Write down the equations of motion for the ball and the wedge.
(iii) Find out the constraint relation between the components of the acceleration of the ball with that of the wedge when it is leaving the wedge. $[\mathbf{1+2+2}]$

3. The Hamiltonian of a particle of some special (constant) charge e moving in 1-D under the influence of a constant field A is given as

$$\mathcal{H} = \frac{(p - eAx)^2}{2m}$$

(i) Find out the velocity of the particle $\dot{x}(t)$.
(ii) Find out the rate of change of the momentum of the particle $\dot{p}(t)$.
(iii) From the above two expressions find out the acceleration $\ddot{x}(t)$ of the particle.
(iv) Prove that $p - eAx$ is a conserved quantity. $[\mathbf{1+1+2+1}]$

4. In an inertial frame, the Hamiltonian of a free particle is given as

$$\mathcal{H} = \frac{\mathbf{p} \cdot \mathbf{p}}{2m}.$$

The free particle is moving along the x-axis with speed v_0 in the inertial frame according to an observer at the origin. The same particle is viewed by an observer sitting at the origin of a rotating frame whose origin and the z-axis coincides with those of the inertial frame but which is rotating about the z-axis with angular speed Ω_0. At an instant of time when the x- and y- axes of both the frames also coincide :
(i) Find out the momentum of the particle $\mathbf{p}'$ as seen in the non-inertial frame.
(ii) Find out the Hamiltonian expressed in the phase space of the non-inertial frame.
(iii) In this phase space find out the rate of change of the phase space variables of the non-inertial frame from the Hamitonian expressed above. $[\mathbf{1+2+2}]$

Mid Semester Exam

Q.1 Cylindrical polar co-ordinates

$$\rho = \alpha t^2 \,, \quad \phi = \beta t \,, \quad z = \gamma t$$

(i) $$\vec{r} = \rho \hat{\rho} + z \hat{k} = \alpha t^2 \hat{\rho} + \gamma t \hat{k}$$

$$\dot{\vec{r}} = 2\alpha t \hat{\rho} + \alpha t^2 \dot{\hat{\rho}} + \gamma \hat{k}$$

$$= 2\alpha t \hat{\rho} + (\alpha t^2) \dot{\phi} \hat{\phi} + \gamma \hat{k} \qquad \Big\{ \text{since } \dot{\hat{\rho}} = \dot{\phi} \hat{\phi}$$

$$= 2\alpha t \hat{\rho} + \alpha \beta t^2 \hat{\phi} + \gamma \hat{k}$$

$$\ddot{\vec{r}} = 2\alpha \hat{\rho} + 2\alpha t \dot{\hat{\rho}} + 2\alpha\beta t \hat{\phi} + \alpha \beta t^2 \dot{\hat{\phi}} \qquad 3$$

$$= 2\alpha \hat{\rho} + 2\alpha\beta t \hat{\phi} + 2\alpha\beta t \hat{\phi} - \alpha \beta^2 t^2 \hat{\rho} \qquad \Big\{ \text{since } \dot{\hat{\phi}} = -\dot{\phi} \hat{\rho}$$

$$= (2\alpha - \alpha\beta^2 t^2) \hat{\rho} + 4\alpha\beta t \hat{\phi}$$

(ii) When $\vec{r}$ is $\perp$ to $\dot{\vec{r}}$; $\vec{r} \cdot \dot{\vec{r}} = 0$ for non-zero $|\vec{r}|$ and $|\dot{\vec{r}}|$

$$2\alpha^2 t^3 + \gamma^2 t = 0$$

$$\Rightarrow t(2\alpha^2 t^2 + \gamma) = 0$$

Either $t = 0$ or $2\alpha^2 t^2 = -\gamma$ $\Big\{$ not possible as $\alpha^2, \gamma > 0$

At $t = 0$, $|\vec{r}| = 0$, $\vec{v} = \gamma \hat{k}$

$\therefore$ At no time $\vec{r} \perp \vec{v}$ 1

(iii) $\vec{r} \perp \ddot{\vec{r}} \Rightarrow \vec{r} \cdot \vec{r} = 0$ for non-zero $|\vec{r}|, |\ddot{\vec{r}}|$

$$\alpha t^2 (2\alpha - \alpha\beta^2 t^2) = 0$$

$$\underbrace{t = 0}_{|\vec{r}| = 0} \quad \text{or} \quad 2\alpha = \alpha\beta^2 t^2 \Rightarrow \underbrace{t = \pm \frac{\sqrt{2}}{\beta}}_{\text{both } |\vec{r}| \text{ and } |\ddot{\vec{r}}| \text{ are non-zero}}$$

hence $\perp$ not guaranteed

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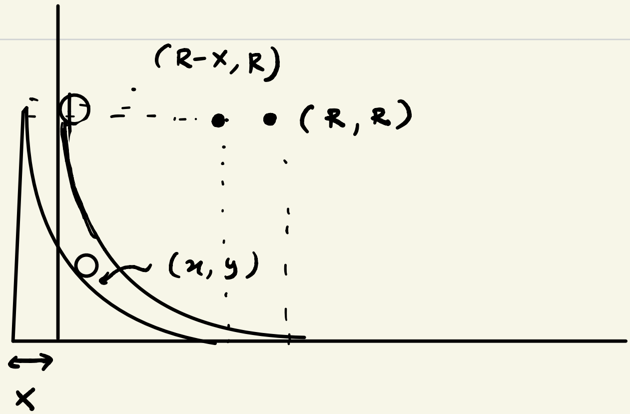
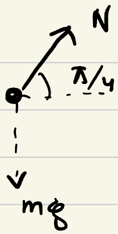
Q. 2

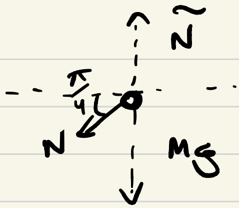


(i) Free-body diagram

For the ball



For the wedge



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(ii) E.O.M for the ball

$$m\ddot{x} = \frac{N}{\sqrt{2}}$$

$$m\ddot{y} = \frac{N}{\sqrt{2}} - mg$$

For the wedge

$$M\ddot{X} = -N/\sqrt{2}$$

$$M\ddot{Y} = \frac{N}{\sqrt{2}} + Mg - \tilde{N} = 0$$

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(iii) When the wedge moves back by X, the center of circular arc comes at (R - X, R). If the ball is at (x, y)

$$(R - X - x)^2 + (R - y)^2 = R^2$$

$$\Rightarrow \quad -2(R - X - x)(\dot{X} + \dot{x}) - 2(R - y)\dot{y} = 0$$

$$\Rightarrow \quad -2(R - X - x)(\ddot{X} + \ddot{x}) + 2(\dot{X} + \dot{x})^2 - 2(R - y)\ddot{y} + 2\dot{y}^2 = 0$$

As the ball is about to leave

$$y \to 0, \quad x \to R - X$$

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Q. 3

$$\mathcal{H} = \frac{(p - eAx)^2}{2m}$$

(i) $$\dot{x} = \frac{\partial \mathcal{H}}{\partial p} = \frac{p - eAx}{m}$$ 1

(ii) $$\dot{p} = -\frac{\partial \mathcal{H}}{\partial x} = -\frac{eA\,(eAx - p)}{m}$$ 1

$$= eA\dot{x}$$

(iii) $$\ddot{x} = \frac{d}{dt}\dot{x} = \frac{1}{m}(\dot{p} - eA\dot{x})$$ (from (i))

$$= 0$$ (from (ii)) 2

(iv) $$\dot{p} - eA\dot{x} = \ddot{x} = 0$$ (from (iii))

$$\frac{d}{dt}(p - eAx) = 0$$

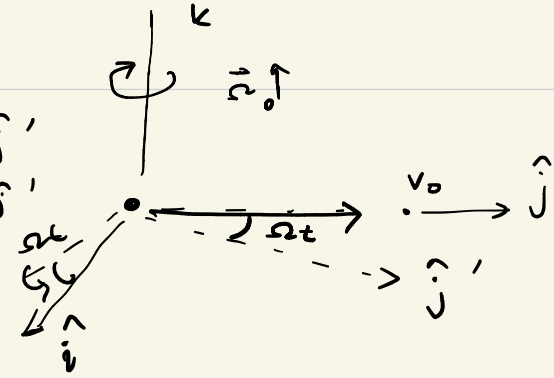
$p - eAx$ is conserved quantity 1

* Alternatively in $\mathcal{H}$ there is no explicit time dependence hence $\mathcal{H}$ is consered

$\frac{(p - eAx)^2}{m}$ is a conserved quantity

$\Rightarrow$ $p - eAx$ is a conserved quantity

Q.4



$$\hat{i} = \cos\Omega t\,\hat{i}' + \sin\Omega t\,\hat{j}'$$
$$\hat{j} = -\sin\Omega t\,\hat{i}' + \cos\Omega t\,\hat{j}'$$

For inertial observer

$$\vec{P} = m\vec{v} = m v_0 \hat{i}$$

(i) Since

$$\vec{v}_{rot} = \vec{v}_{in} - (\vec{\Omega} \times \vec{r})$$

$$\vec{r} = x\hat{i} + y\hat{j} = v_0 t\,\hat{i} = v_0 t(\cos\Omega t\,\hat{i}' + \sin\Omega t\,\hat{j}')$$

$$= \boxed{\underbrace{v_0 t\cos\Omega t}_{x'}\,\hat{i}' + \underbrace{v_0 t\sin\Omega t}_{y'}\,\hat{j}'} \quad (\text{upto a constant vector})$$

$$\therefore \vec{v}_{rot} = v_0\hat{i} - \Omega_0\hat{k} \times v_0 t\,\hat{i}$$
$$= v_0\hat{i} - \Omega_0 v_0 t\,\hat{j}$$

$$\therefore (i)\quad \vec{P}' = \vec{P} - m\Omega_0 v_0 t\,\hat{j} = \vec{P} - m\Omega_0 r\,\hat{j} \qquad 1$$

If $\vec{P}' = P_x'\hat{i}' + P_y'\hat{j}'$

(ii)
$$\vec{P} = \vec{P}' + m\Omega_0 r\,\hat{j} = (P_x'\hat{i}' + P_y'\hat{j}') + m\Omega_0 v_0 t(-\sin\Omega t\,\hat{i}' + \cos\Omega t\,\hat{j}')$$
$$= (P_x' - m\Omega_0 v_0 t\sin\Omega t)\hat{i}' + (P_y' + m\Omega_0 v_0 t\cos\Omega t)\hat{j}'$$
$$= (P_x' - m\Omega_0 y')\hat{i}' + (P_y' + m\Omega_0 x')\hat{j}'$$

$$\mathcal{H} = \frac{\vec{P}\cdot\vec{P}}{2m} = \frac{1}{2m}\left[(P_x' - m\Omega_0 y')^2 + (P_y' + m\Omega_0 x')^2\right] \qquad 2$$

(iii)
$$\dot{x}' = \frac{\partial\mathcal{H}}{\partial P_x'} = \frac{(P_x' - m\Omega_0 y')}{m}; \quad \dot{P}_x' = -\Omega_0(P_y' + m\Omega_0 x')$$
$$\dot{y}' = \frac{\partial\mathcal{H}}{\partial P_y'} = \frac{(P_y' + m\Omega_0 x')}{m}; \quad \dot{P}_y' = \Omega_0(P_x' - m\Omega_0 y')$$

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