Indian Institute of Science Education and Research Mohali

MTH307: Topology
End Semester Examination: 01 May 2025

Instructions:

- Time Allowed: Three Hours.
- Maximum Marks: 50.
- Make sure that the particulars required are entered on your answer book.
- The numbers in the margin indicate how many marks are available for each question.
- $\mathbb{R}$ denotes the set of real numbers.
- $\mathbb{R}^\omega$ denotes the cartesian product of countably infinite copies of $\mathbb{R}$.
- $\mathbb{Z}$ denotes the set of integers.
- $\mathbb{N}$ denotes the set of natural numbers.
- $X \setminus A$ denotes the set $\{x \in X \mid x \notin A\}$.

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(1) State without justification whether the following statements are TRUE or FALSE:
  (i) Let τ_1 and τ_2 be two topologies on a set X such that $\tau_1 \subset \tau_2$. If (X, τ_1) is connected, then so is (X, τ_2).
  (ii) Let τ_1 and τ_2 be two topologies on a set X such that $\tau_1 \subset \tau_2$. If (X, τ_2) is Hausdorff, then so is (X, τ_1).
  (iii) Every topology on a countable set is Lindelöf.
  (iv) Every topology on a finite set is discrete.
  (v) Every homeomorphism is a quotient map.
  (vi) Every regular topological space is normal.
  (vii) Every open map between topological spaces is continuous.
  (viii) The sets $\mathbb{Z}$ and $\mathbb{Q}$ equipped with discrete topologies are homeomorphic.
  (ix) Every subspace of a path connected topological space is connected.
  (x) The discrete topology on a set is second countable if and only if the set is finite.
  [10 Marks]

(2) Let $\mathbb{R}_K$ denote the set of real numbers with K-topology, where $K = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$. Show that $[0, 1]$ is not compact as a subspace of $\mathbb{R}_K$.
[6 Marks]

(3) Consider $X = \mathbb{R} \times \mathbb{R} \setminus \{(0,0)\}$ with the usual topology. Let $\sim$ be an equivalence relation on X such that $(x, y) \sim (x', y')$ iff they lie on the ray emanating from the origin $(0,0)$. Determine the quotient space $X/\sim$ up to homeomorphism.
[6 Marks]

(4) Let $(x_n)_{n \geq 1}$ be a sequence in a topological space X converging to an element $x \in X$. Show that the set $\{x_n \mid n \in \mathbb{N}\} \cup \{x\}$ is a compact subspace of X. Further, show that the limit of a sequence in a topological space need not be unique.
[4+3 Marks]

(5) Show that $\mathbb{R} \times \mathbb{R}$ in the dictionary order topology is homeomorphic to the product space $\mathbb{R}_d \times \mathbb{R}_u$, where $\mathbb{R}_d$ is the set of reals equipped with the discrete topology and $\mathbb{R}_u$ is the set of reals with the usual topology. Deduce that $\mathbb{R} \times \mathbb{R}$ in the dictionary order topology is metrizable
[4+3 Marks]

(6) Show that the set $\mathbb{N}$ viewed as a subspace of $\mathbb{R}$ with the usual topology is locally compact. Further, determine the one-point compactification of $\mathbb{N}$ upto homeomorphism.
[3+4 Marks]

(7) Let $\mathbb{R}^\infty$ be the subset of $\mathbb{R}^\omega$ consisting of all sequences that are eventually zero. Define the product and the box topologies on $\mathbb{R}^\omega$. Determine the closures of $\mathbb{R}^\infty$ with respect to the product and the box topologies on $\mathbb{R}^\omega$.
[3+4 Marks]