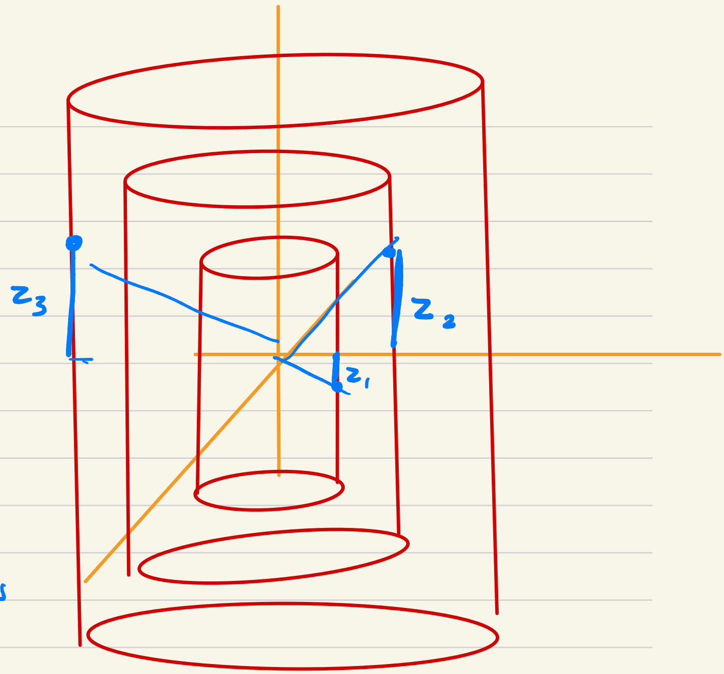
Polar Co-ordinates (3-D)

★ Cylindrical Polar co-ordinates

Imagine the 3-D space be filled with cylinders, such that each point sits on the surface of some cylinder

Each point (x, y, z) can also be mapped by (ρ, θ, z) where

z is the height along the z-axis of the cylinder.



Unit vectors $(\hat{r}, \hat{\theta}, \hat{k})$

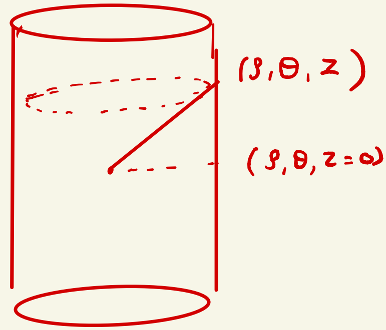
Relations w.r.t. Cartesian co-ordinates

$$r = \sqrt{x^2+y^2+z^2} = \sqrt{\rho^2+z^2}$$

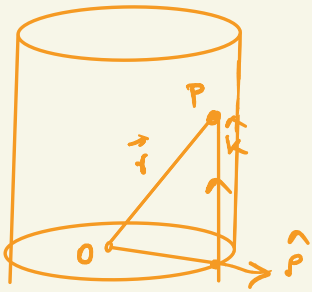
$$\rho = \sqrt{x^2+y^2} \quad , \quad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Position in cylindrical polar co-ordinates

$$\vec{r} = z\hat{k} + \rho\hat{\rho}$$



* The projection of $\vec{r}$ has no component along the $\hat{\theta}$ direction

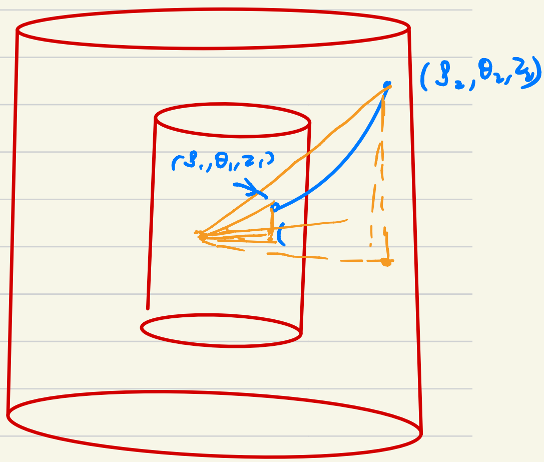


Velocity in cylindrical Polar co-ordinates

$$\vec{V} = \frac{d\vec{r}}{dt}$$

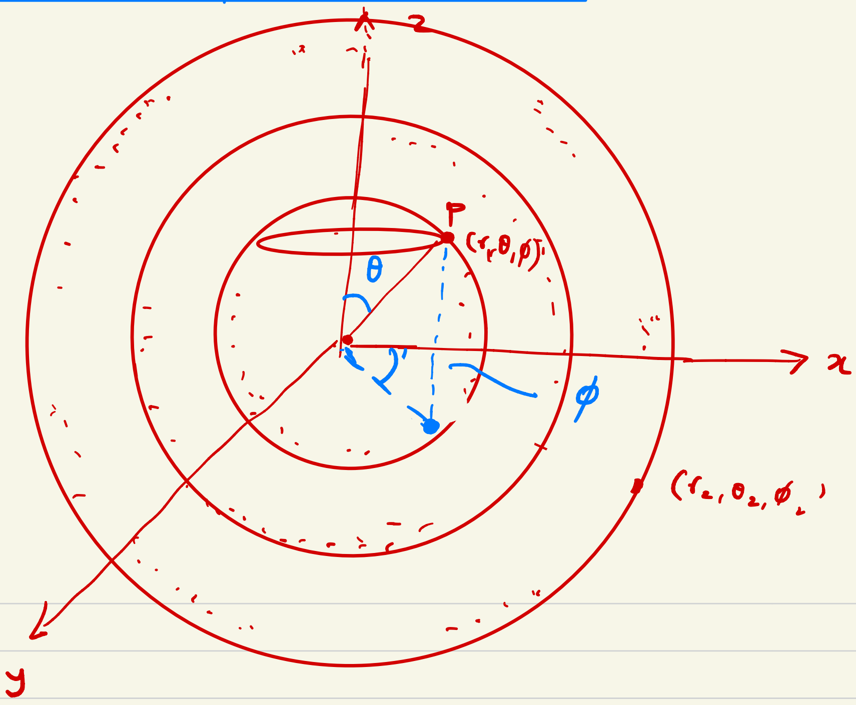
$$= \frac{dz}{dt}\hat{k} + \frac{d\rho}{dt}\hat{\rho} + \rho\frac{d\hat{\rho}}{dt}$$

$$= \underbrace{\dot{z}\,\hat{k}}_{\text{ability to move in } \hat{z} \text{ direction}} + \underbrace{\dot{\rho}}_{\text{ability to move in } \hat{\rho} \text{ direction}}\hat{\rho} + \rho\underbrace{\dot{\theta}}_{\text{ability to change } \theta \text{ (move in } \hat{\theta} \text{ direction)}}\hat{\theta}$$



Exercise : Find the expression of acceleration in terms of $\hat{k}$, $\hat{\rho}$ and $\hat{\theta}$.

Cylindrical polar co-ordinates



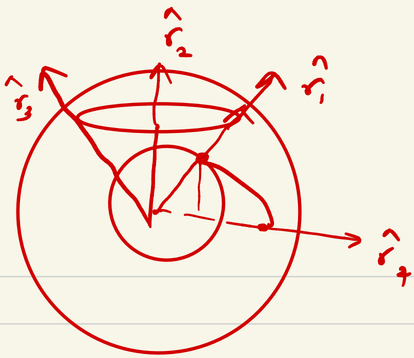
Imagine the 3-D space be filled up by spheres with center on yourself. Then each point on the 3-D space will lie on the surface of a sphere.

It will be identified by (r = radius of sphere, θ = angle w.r.t. some z-axis, ϕ = angle of the projection of point onto the x-y plane w.r.t. the x-axis)

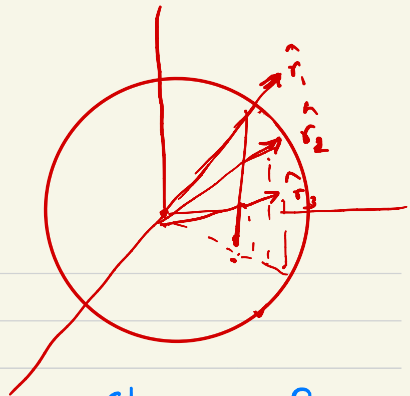
Position (r, θ, ϕ)

Position vector $\vec{r} = r\,\hat{r}$

This time $\hat{r}$ is different at each θ and each ϕ.

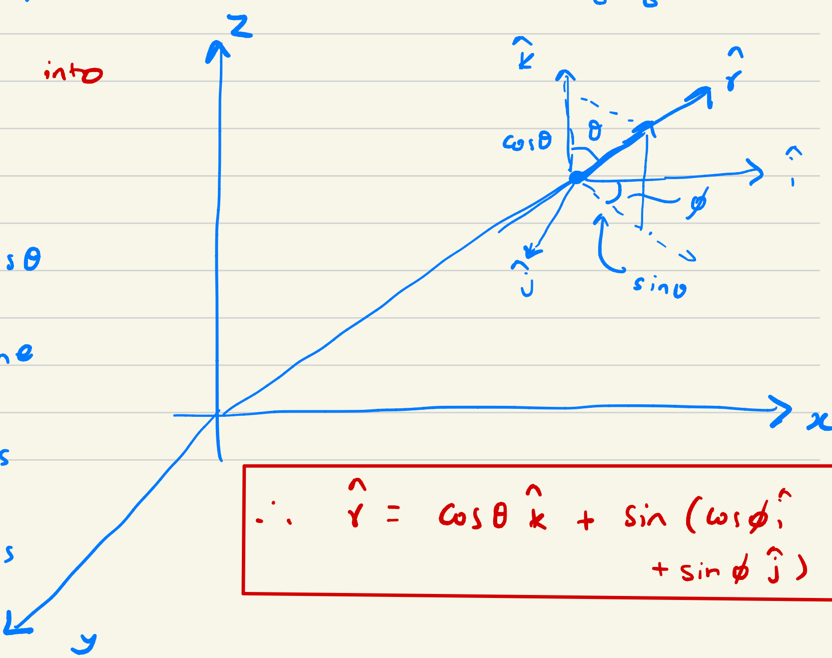


Changing ϕ



Changing θ

★ Projecting $\hat{r}$ into $\hat{i}$, $\hat{j}$, $\hat{k}$



Along z-axis
$\cos\theta$

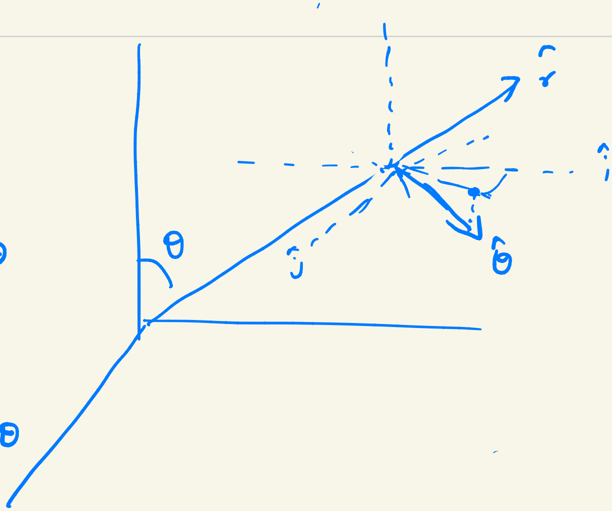
In x-y plane
$\sin\theta$

Along x-axis
$\sin\theta \cos\phi$

Along y-axis
$\sin\theta \sin\phi$

$$\boxed{\therefore\ \hat{r} = \cos\theta\,\hat{k} + \sin(\cos\phi\,\hat{i} + \sin\phi\,\hat{j})}$$

★ Projecting $\hat{\theta}$ into $\hat{i}, \hat{j}, \hat{k}$



Along z-axis

$\cos\left(\frac{\pi}{2}+\theta\right) = -\sin\theta$

In the x-y plane

$\sin\left(\frac{\pi}{2}+\theta\right) = \cos\theta$

Along x-axis

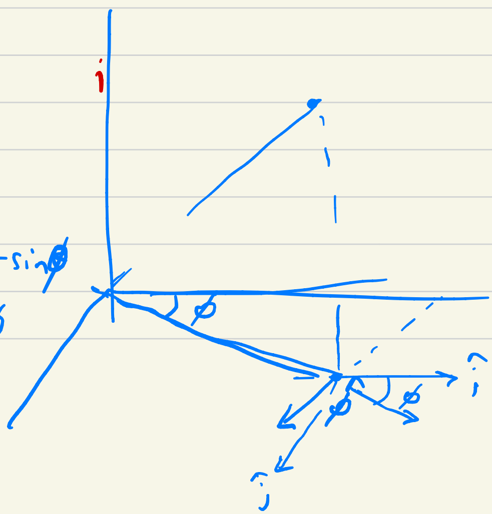
$\cos\theta \cos\phi$

Along y-axis

$\cos\theta \sin\phi$

$$\boxed{\hat{\theta} = -\sin\theta\,\hat{k} + \cos\theta\left(\cos\phi\,\hat{i} + \sin\phi\,\hat{j}\right)}$$

★ Projecting $\hat{\phi}$ into $\hat{i}, \hat{j}, \hat{k}$



Along z-axis : 0

Along x-axis : $\cos\left(\frac{\pi}{2}+\phi\right) = -\sin\phi$

Along y-axis : $\sin\left(\frac{\pi}{2}+\phi\right) = \cos\phi$

$$\boxed{\hat{\phi} = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}}$$

Velocity vector $\vec{v} = \frac{d\vec{r}}{dt}$

$$= \frac{d}{dt}(r\hat{r}) = \frac{d}{dt}\left[r\left(\cos\theta\,\hat{k} + \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j}\right)\right]$$

$$= \dot{r}\,\hat{r} + r\left[-\sin\theta\,\dot{\theta}\,\hat{k} + \dot{\theta}\cos\theta\cos\phi\,\hat{i} - \dot{\phi}\sin\theta\sin\phi\,\hat{i} + \dot{\theta}\cos\theta\sin\phi\,\hat{j} + \dot{\phi}\sin\theta\cos\phi\,\hat{j}\right]$$

$$= \dot{r}\hat{r} + r\dot{\theta}\left(\cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,\hat{k}\right)$$

$$+ r\sin\theta\,\dot{\phi}\left(-\sin\phi\,\hat{i} + \cos\phi\,\hat{j}\right)$$

$$\boxed{\vec{v} = \dot{r}\,\hat{r} + r\dot{\theta}\,\hat{\theta} + r\sin\theta\,\dot{\phi}\,\hat{\phi}}$$

Exercise: Find out acceleration in terms of $\hat{r}$, $\hat{\theta}$, $\hat{\phi}$.