PHY306 Advanced Quantum Mechanics Jan-Apr 2025: Assignment 10

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1. Prove that the Dirac γ^i matrices ($i = 0, 1, 2, 3$) are hermitian, traceless matrices of even dimensionality, with eigenvalues ± 1. Also show that $(\sigma.\mathbf{p})^2 = |\mathbf{p}|^2$

2. Operate on the Dirac equation with $\gamma^\nu \partial_\nu$ and show that each of the four components ψ_i satisfy the Klein-Gordon equation

$$(\Box - \frac{m^2c^2}{\hbar^2})\psi_i = 0$$

where $\Box$ is the d'Alembertian operator.

3. Check that of the four plane wave solutions to the Dirac equation, u^1 and u^2 are orthogonal and similarly, u^3 and u^4 are orthogonal. Are u^1 and u^3 orthogonal?

4. Calculate the $\lambda = +\frac{1}{2}$ helicity eigenspinor of an electron of momentum $\mathbf{p} = (p\sin\theta, 0, p\cos\theta)$

5. First construct the Hamiltonian H for the Dirac equation. Find the commutator of H with the total orbital angular momentum ($L = r \times P$) and the spin angular momentum ($S = (\hbar/2)\Sigma$) where

$$\Sigma = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}$$

and show that L, S are not conserved. Show that the total angular momentum $J = L + \frac{\hbar}{2}\Sigma$ is conserved. Show that every bispinor is an eigenstate of S^2 and write down the corresponding eigenvalue.

6. For a nonrelativistic electron of velocity v, use the two-component spinor solutions of the Dirac equation u_A, u_B to show that u_A is larger than u_B by a factor of the order of v/c.

7. Show that the spinor representing an electron at rest is an eigenstate of the parity operator P. What is its intrinsic parity? What about the positron?

8. The charge conjugation operator C takes a Dirac spinor ψ into its charge conjugate spinor ψ_c given by

$$\psi_c = i\gamma^2\psi^*$$

Find the charge-conjugates of u^1 and u^2 and compare them with v^1 and v^2

9. Choose the z axis points along the direction of motion and write out the four independent plane wave equations of the Dirac equation. Confirm that these are eigenspinors of S_z and find the eigenvalues.