Problem no : 2

We know $\left[\hat{A}, \hat{B}\right] = c\hat{\mathbb{1}}$, $e^{\hat{A}}\hat{B}e^{-\hat{A}} = \hat{B} + c\hat{\mathbb{1}}$

$\uparrow$ Number

(a)

$$T_a^{+}\hat{x}\,T_a = \exp\frac{i\hat{p}a}{\hbar}\,\hat{x}\,\exp\frac{-i\hat{p}a}{\hbar} \qquad \left[\frac{i\hat{p}a}{\hbar}, \hat{x}\right] = \frac{ia}{\hbar}\left(-i\hbar\right) = a\hat{\mathbb{1}}$$

$$= \hat{x} + a\hat{\mathbb{1}}$$

Similarly, $\hat{T}_q^{+}\hat{p}\,\hat{T}_q = \exp\left(\frac{-iq\hat{x}}{\hbar}\right)\hat{p}\,\exp\left(\frac{iq\hat{x}}{\hbar}\right)$ $\qquad \left[\frac{-iq\hat{x}}{\hbar}, \hat{p}\right] = -\frac{iq}{\hbar}\,i\hbar = q\hat{\mathbb{1}}$

$$= \hat{p} + q\hat{\mathbb{1}}$$

(b)

$$\left[\hat{T}_a, \hat{T}_q\right] = \exp\frac{-i\hat{p}a}{\hbar}\exp\frac{iq\hat{x}}{\hbar} - \exp\frac{iq\hat{x}}{\hbar}\exp\frac{-i\hat{p}a}{\hbar}$$

Employing BCH $\Rightarrow$ $e^{\hat{x}}e^{\hat{y}} = e^{\hat{x}+\hat{y}+\frac{1}{2}[\hat{x},\hat{y}]+\cdots}$

$$= \exp\left[\frac{i}{\hbar}\left(q\hat{x} - a\hat{p}\right) - \frac{iaq}{2\hbar}\right] - \exp\left[\frac{i}{\hbar}\left(q\hat{x} - a\hat{p}\right) + \frac{iaq}{2\hbar}\right]$$

$$= \left[\exp\left(\frac{-iaq}{2\hbar}\right) - \exp\left(\frac{iaq}{2\hbar}\right)\right]\exp\frac{i}{\hbar}\left(q\hat{x} - a\hat{p}\right)$$

Commutation is guranteed if

$$\exp\left(\frac{-iaq}{2\hbar}\right) = \exp\left(\frac{iaq}{2\hbar}\right)$$

$$\boxed{aq = 2\pi n\hbar}$$

Q.3(a). In a vector space V, a linear operator P satisfies $P^2 = P$. Now we can define another operator $\mathbb{I} - P$ & it also satisfies $(\mathbb{I}-P)^2 = \mathbb{I} - P$. Now we have to show that

$$\begin{cases} \text{range}(\mathbb{I}-P) = \text{null}(P), & \text{--- (1)} \\ \text{null}(\mathbb{I}-P) = \text{range}(P). & \text{--- (2)} \end{cases}$$

Note, $A = B$ if $A \subseteq B$ & $B \subseteq A$. Now take a vector v such that $Pv = 0$, then $(\mathbb{I}-P)v = v - Pv = v$. Thus any vector in the null space of P also in the range of $\mathbb{I}-P$;

$$\boxed{\text{null}(P) \subset \text{range}(\mathbb{I}-P)}.$$

Again, consider any $x \in \text{range}(\mathbb{I}-P)$, then

$$x = (\mathbb{I}-P)v \quad \text{for some } v,$$

$\Rightarrow$ $X = v - Pv = -(Pv - v) \in \text{null}(P)$, because $P(Pv - v) = 0$. Therefore, if $x \in \text{range}(\mathbb{I}-P)$ then $x \in \text{null}(P)$;

$$\boxed{\text{range}(\mathbb{I}-P) \subseteq \text{null}(P)}.$$

Now with (1) & (2), we may write

$$\begin{aligned} \text{null}(P) + \text{range}(P) &= \text{null}(P) + \text{null}(\mathbb{I}-P) \\ &= \text{range}(\mathbb{I}-P) + \text{range}(P) \\ &= V. \end{aligned}$$

Q3.(b) A projector P is called to be orthogonal if P & $\mathbb{I}-P$ acts on the disjoint ~~spaces~~ sub-spaces of the vector space V.

① P is Hermitian, ie, $P = P^{\dagger}$. Then $P = \sum_{i=1}^{n} \lambda_i |i\rangle\langle i|$. Again $P^2 = P$ implies $P^2 = \sum_i \lambda_i^2 |i\rangle\langle i| = \sum_i \lambda_i |i\rangle\langle i|$. So we have $\lambda_i = 1$. Now,

$\boxed{n<d}$

$$P = \sum_{i=1}^{n} |i\rangle\langle i| \quad \& \quad \mathbb{I} - P = \sum_{i=n+1}^{d} |i\rangle\langle i| .$$

Consider a vector $|v\rangle = \sum_{i=1}^{n} c_i |i\rangle$ & apply the projector P $\Rightarrow P|v\rangle = |v\rangle$. ~~how ever~~ However $(\mathbb{I}-P)|v\rangle = 0$. Similarly, for any vector $|u\rangle = \sum_{i=n+1}^{d} e_i |i\rangle$, we see that

$(\mathbb{I}-P)|u\rangle = |u\rangle$ & $P|u\rangle = 0$. Therefore P is orthogonal.

② Consider an arbitrary vector $|w\rangle = \sum_{i=1}^{d} h_i |i\rangle$. ~~How~~ Then $P|w\rangle = \sum_{i=1}^{n} h_i |i\rangle$ & $(\mathbb{I}-P)|w\rangle = \sum_{i=n+1}^{d} h_i |i\rangle$.

Now we have

$$|Pw| \leq |w| \quad \text{for all } w \in V$$

or, $$|Pw| \leq |(P + (\mathbb{I}-P))w|$$

$$= |\underline{Pw} + \underline{(\mathbb{I}-P)w}|.$$

Therefore, $\langle Pw, (\mathbb{I}-P)w\rangle = 0$ and implies P is an orthogonal projector.

I leave the proof of the following –
$|u| \leq |u + av|$ if, and only if, $\langle u, v\rangle = 0$.

Q.3(c) A 2×2 matrix that satisfies $P^2 = P$ ~~buts do not~~ but is not a projector is

$$P = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

Q 4 Ans

$$A_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

$$[A_1 A_2] = 0$$

A_1 have eigenvalues and eigenvectors,

$$\lambda_1 = 2, \ |u_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \qquad \lambda_2 = 0, \ |u_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\lambda_3 = 0, \ |u_3\rangle = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$$

if $\lambda_i \neq \lambda_j$ for $i \neq j$ suppose $[T, S] = 0$

Two T and S linear operators.

$$T u_i = \lambda_i u_i$$

$$T(S u_i) = S(T u_i)$$

$$\Rightarrow S(\lambda_i u_i)$$

so, $\boxed{S u_i = w_i u_i}$

with new eigenvalue w_i

So,

$$A_2' = \begin{pmatrix} 3 & * & * \\ 0 & * & -\sqrt{2} \\ 0 & * & * \end{pmatrix} = U^{\dagger} A_2 V$$

$$A_2' = \begin{pmatrix} 3 & * & * \\ 0 & * & -\sqrt{2} \\ 0 & * & * \end{pmatrix} = U^+ A_2 U = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 1/\sqrt{2} \\ -\frac{1}{\sqrt{2}} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{pmatrix} A_2 \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{pmatrix}$$

$$A_2' = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & -\sqrt{2} \\ 0 & -\sqrt{2} & 0 \end{pmatrix}$$

So, Now, if we find eigen value of A_2'

$$\boxed{A_2' \quad (-1,\ 2,\ 3)}$$

we get A_2' is Non-diagonal Because A_1 have some degenrate eigenvalue.

(i) for degenrate eigenvalue,

if we have two linear operator $([T, S] = 0)$ T and S,

So, $T u_i = \lambda_i u_i$

$S w_j = \lambda_j w_j$

So, $\boxed{W_j = \sum_k C_k u_k}$ for degerate eigenvalue.

Q 5. $\psi(x) = N e^{i\langle p\rangle x/\hbar} e^{-x^2/2\Delta^2}$

Normalization,

$$\int_{-\infty}^{\infty} \psi^*(x)\psi(x)\,dx = 1$$

using $\int_0^{\infty} x^n e^{-\alpha x^2} = \dfrac{\Gamma\frac{n+1}{2}}{2(\alpha)^{n+1/2}}$ — Ⓐ

$$|N|^2 \int_{-\infty}^{\infty} e^{-x^2/\Delta^2}\,dx = 1$$

$$|N|^2 \cancel{2} \cdot \frac{\Gamma 1/2}{\cancel{2}\cdot\left(\frac{1}{\Delta^2}\right)^{1/2}} = 1, \quad |N|^2 = (\sqrt{\pi}\,\Delta)^{-1}$$

$$|N|^2 = \frac{1}{\sqrt{\pi}\,\Delta}$$

So,

$$(\Delta x)^2(\Delta p)^2 \geq \left(\langle\psi|\frac{1}{2i}[x,p]|\psi\rangle\right)^2$$

$$(\Delta x)^2 = (x - \langle x\rangle \mathbb{1})^2 = \langle x^2\rangle - \langle x\rangle^2.$$

$$\langle x\rangle = \int_{-\infty}^{\infty} \psi^* x \psi\,dx = \int_{-\infty}^{\infty} |N|^2 x e^{-x^2/\Delta^2}\,dx.$$

$$\Rightarrow \frac{1}{\sqrt{\pi}\,\Delta}\int_{-\infty}^{\infty} x e^{-x^2/\Delta^2}\,dx = 0 \quad \text{(odd function)}$$

$$\langle x^2\rangle = \int_{-\infty}^{\infty} |N|^2 x^2 e^{-x^2/\Delta^2}\,dx = \frac{1}{\sqrt{\pi}\,\Delta}\,2\int_0^{\infty} x^2 e^{-x^2/\Delta^2}\,dx$$

$$= \frac{1}{\sqrt{\pi}\,\Delta}\cancel{2}\,\frac{\Gamma 3/2}{\cancel{2}\cdot\left(\frac{1}{\Delta^2}\right)^{3/2}} \quad \text{using Ⓐ}$$

$\Gamma 3/2 = \frac{1}{2}\sqrt{\pi}$

$$\Rightarrow \frac{1}{\sqrt{\pi}\cdot\Delta}\,\frac{1}{2}\,\Delta^3\sqrt{\pi}$$

$$= \frac{\Delta^2}{2}.$$

$$\boxed{(\Delta x)^2 = \frac{\Delta^2}{2} - 0 = \frac{\Delta^2}{2}}$$

Similarly we will find $(\Delta p)^2$

$$\hat{p} = -i\hbar \frac{\partial}{\partial x}.$$

$$(\Delta p)^2 = \langle p^2 \rangle - \langle p \rangle^2$$

$$\langle p \rangle = \int_{-\infty}^{\infty} \psi^* \, \hat{p} \, \psi \, dx$$

$$\hat{p}\psi = -i\hbar \frac{\partial}{\partial x} \left[N e^{+i\langle p \rangle x/\hbar} e^{-x^2/2\Delta^2} \right]$$

$$= -i\hbar N \left[e^{i\langle p \rangle x/\hbar} e^{-x^2/2\Delta^2} \cdot \left(-\frac{\cancel{2}x}{\cancel{2}\Delta^2} \right) + e^{-x^2/2\Delta^2} \frac{i\langle p \rangle}{\hbar} e^{i\langle p \rangle x/\hbar} \right]$$

$$= N e^{i\langle p \rangle x/\hbar} e^{-x^2/2\Delta^2} \left[\frac{ix\hbar}{\Delta^2} + \frac{\hbar \langle p \rangle}{\hbar} \right]$$

$$\langle p \rangle = \int_{-\infty}^{\infty} |N|^2 e^{-x^2/2\Delta^2} \cdot e^{-x^2/2\Delta^2} \left(\frac{ix\hbar}{\Delta^2} + \langle p \rangle \right) dx$$

$$\Rightarrow |N|^2 \int_{-\infty}^{\infty} e^{-x^2/\Delta^2} \langle p \rangle \, dx = \frac{1}{\Delta\sqrt{\pi}} \cdot \frac{\sqrt{\pi}\,\Delta}{} \langle p \rangle$$

$$= \langle p \rangle$$

Similarly we will find $\langle p^2 \rangle$

$$\langle p^2 \rangle = \int_{-\infty}^{\infty} \psi^* p^2 \psi \, dx$$

$$\hat{p}^2 = (-i\hbar)(-i\hbar) \frac{\partial^2}{\partial x^2} = -\hbar^2 \frac{\partial}{\partial x^2}$$

$$\hat{p}^2\psi = \frac{-\hbar^2\partial^2}{\partial x^2}\left[N e^{i\langle p\rangle x/\hbar} e^{-x^2/2\Delta^2}\right].$$

$$= -\hbar^2 \frac{\partial}{\partial x}\left[N e^{-x^2/2\Delta^2} e^{+i\langle p\rangle x/\hbar}\left(-\frac{2x}{2\Delta^2}\right) + e^{-x^2/2\Delta^2} e^{i\langle p\rangle x/\hbar}\frac{i\langle p\rangle}{\hbar}\right]$$

$$= -N\hbar^2 \frac{\partial}{\partial x}\left[\left(\left(\frac{-x}{\Delta^2}\right) + \frac{i\langle p\rangle}{\hbar}\right) e^{-x^2/2\Delta^2} e^{i\langle p\rangle x/\hbar}\right]$$

$$= -N\hbar^2\left[\left\{\frac{-1}{\Delta^2}+0\right\} e^{-x^2/2\Delta^2} e^{i\langle p\rangle x/\hbar} + \left(\frac{-x}{\Delta^2} + i\frac{\langle p\rangle}{\hbar}\right)\right.$$

$$\left.\left\{e^{-x^2/2\Delta^2} e^{i\langle p\rangle x/\hbar}\left(\frac{-2x}{2\Delta^2}\right) + e^{i\langle p\rangle x/\hbar} e^{-x^2/2\Delta^2}\frac{i\langle p\rangle}{\hbar}\right\}\right]$$

$$= -N\hbar^2\left[e^{-x^2/2\Delta^2} e^{i\langle p\rangle x/\hbar}\left\{-\frac{1}{\Delta^2} + \left(\frac{-x}{\Delta^2} + i\frac{\langle p\rangle}{\hbar}\right)\left(\frac{-x}{\Delta^2} + i\frac{\langle p\rangle}{\hbar}\right)\right\}\right]$$

$$\Rightarrow -N\hbar^2\left[\frac{-1}{\Delta^2} + \left(\frac{i\langle p\rangle}{\hbar} - \frac{x}{\Delta^2}\right)^2\right] e^{-x^2/2\Delta^2} e^{i\langle p\rangle x/\hbar}].$$

$$\hat{p}^2\psi \Rightarrow N e^{-x^2/2\Delta^2} e^{i\langle p\rangle x/\hbar}\left[\frac{\hbar^2}{\Delta^2} - \hbar^2\left(-\frac{\langle p\rangle^2}{\hbar^2} + \frac{x^2}{\Delta^4} - \frac{2i\langle p\rangle x}{\hbar\Delta^2}\right)\right]$$

$$= N e^{-x^2/2\Delta^2} e^{i\langle p\rangle x/\hbar}\left[\frac{\hbar^2}{\Delta^2} + \langle p\rangle^2 - \frac{\hbar^2 x^2}{\Delta^2} + \frac{2i\langle p\rangle\hbar x}{\Delta^2}\right].$$

$$\langle \hat{p}^2\rangle = \langle\psi|\hat{p}^2|\psi\rangle = \int \psi^*(\hat{p}^2\psi)\,dx$$

$$= |N|^2 \int_{-\infty}^{\infty} e^{-x^2/2\Delta^2} e^{-i\langle p\rangle x/\hbar} \cdot e^{-x^2/2\Delta^2} e^{i\langle p\rangle x/\hbar}$$

$$\left(\frac{\hbar^2}{\Delta^2} + \langle p\rangle^2 - \frac{\hbar^2 x^2}{\Delta^2} + \frac{2i\langle p\rangle\hbar x}{\Delta^2}\right)dx$$

$$\langle p^2\rangle = |N|^2 \int_{-\infty}^{\infty} e^{-x^2/\Delta^2} \left\{ \frac{\hbar^2}{\Delta^2} + \langle p\rangle^2 - \frac{\hbar^2 x^2}{\Delta^4} + \underbrace{\frac{2i\langle p\rangle x\hbar}{\Delta^2}}_{0} \right\} dx$$

0 odd function.

$$|N|^2 = \frac{1}{\sqrt{\pi}\,\Delta} \quad \text{or} \quad \frac{1}{|N|^2} = \int_0^{\infty} e^{-x^2/\Delta^2}\, dx.$$

$$\Rightarrow \frac{\hbar^2}{\Delta^2} + \langle p\rangle^2 - \frac{\hbar^2}{\Delta^4}\cdot\frac{\Delta^2}{2}$$

$$= \frac{\hbar^2}{\Delta^2} + \langle p\rangle^2 - \frac{\hbar^2}{\Delta^2 2} = \frac{\hbar^2}{\Delta^2}\left(1-\frac{1}{2}\right) + \langle p\rangle^2$$

$$= \frac{1}{2}\frac{\hbar^2}{\Delta^2} + \langle p\rangle^2.$$

So,

$$(\Delta p)^2 = \langle p^2\rangle - \langle p\rangle^2$$

$$= \frac{1}{2}\frac{\hbar^2}{\Delta^2} + \cancel{\langle p\rangle^2} - \cancel{\langle p\rangle^2}$$

$$\Rightarrow \frac{\hbar^2}{2\Delta^2}$$

$$(\Delta x)^2 (\Delta p)^2 = \frac{\Delta^2}{2}\cdot\frac{\hbar^2}{2\Delta^2} = \frac{\hbar^2}{4}.$$

$$\boxed{\Delta x \,\Delta p = \frac{\hbar}{2}}$$

$$\Rightarrow \left(\langle \psi | \frac{1}{2i} [x, p] | \psi \rangle \right)^2 = (\Delta x)^2 (\Delta p)^2$$

$$[x, p] = i\hbar$$

$$\left(\langle \psi | \frac{1}{2i} i\hbar | \psi \rangle \right)^2 = (\Delta x)^2 (\Delta p)^2$$

$$= \left(\frac{\hbar}{2} \right)^2 \langle \psi | \psi \rangle = (\Delta x)^2 (\Delta p)^2$$

$$\Rightarrow \boxed{\Delta x \, \Delta p = \frac{\hbar}{2}}$$

<u>verified</u>.

3 Projectors

If $P \in \mathbb{C}^{m \times m}$ is a square matrix such that $P^2 = P$ then P is called a *projector*. A matrix satisfying this property is also known as an *idempotent* matrix.

Remark It should be emphasized that P need not be an orthogonal projection matrix. Moreover, P is usually not an orthogonal matrix.

Example Consider the matrix

$$P = \begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix},$$

where $c = \cos\theta$ and $s = \sin\theta$. This matrix projects perpendicularly onto the line with inclination angle θ in $\mathbb{R}^2$.

We can check that P is indeed a projector:

$$\begin{aligned} P^2 &= \begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix}\begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix} \\ &= \begin{bmatrix} c^4 + c^2s^2 & c^3s + cs^3 \\ c^3s + cs^3 & c^2s^2 + s^4 \end{bmatrix} \\ &= \begin{bmatrix} c^2(c^2+s^2) & cs(c^2+s^2) \\ cs(c^2+s^2) & s^2(c^2+s^2) \end{bmatrix} = P. \end{aligned}$$

Note that P is not an orthogonal matrix, i.e., $P^*P = P^2 = P \neq I$. In fact, $\operatorname{rank}(P) = 1$ since points on the line are projected onto themselves.

Example The matrix

$$P = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

is clearly a projector. Since the range of P is given by all points on the x-axis, and any point (x, y) is projected to $(x + y, 0)$, this is clearly not an orthogonal projection.

In general, for any projector P, any $v \in \operatorname{range}(P)$ is projected onto itself, i.e., $\boldsymbol{v} = P\boldsymbol{x}$ for some $\boldsymbol{x}$ then

$$P\boldsymbol{v} = P(P\boldsymbol{x}) = P^2\boldsymbol{x} = P\boldsymbol{x} = \boldsymbol{v}.$$

We also have

$$P(P\boldsymbol{v} - \boldsymbol{v}) = P^2\boldsymbol{v} - P\boldsymbol{v} = P\boldsymbol{v} - P\boldsymbol{v} = \boldsymbol{0},$$

so that $P\boldsymbol{v} - \boldsymbol{v} \in \operatorname{null}(P)$.

3.1 Complementary Projectors

In fact, $I - P$ is known as the *complementary projector* to P. It is indeed a projector since

$$(I - P)^2 = (I - P)(I - P) = I - \underbrace{IP}_{=P} - \underbrace{PI}_{=P} + \underbrace{P^2}_{=P} = I - P.$$

Lemma 3.1 *If P is a projector then*

$$\text{range}(I-P) = \text{null}(P), \tag{10}$$

$$\text{null}(I-P) = \text{range}(P). \tag{11}$$

Proof We show (10), then (11) will follow by applying the same arguments for $P = I-(I-P)$. Equality of two sets is shown by mutual inclusions, i.e., $A = B$ if $A \subseteq B$ and $B \subseteq A$.

First, we show $\text{null}(P) \subseteq \text{range}(I-P)$. Take a vector $\boldsymbol{v}$ such that $P\boldsymbol{v} = \mathbf{0}$. Then $(I-P)\boldsymbol{v} = \boldsymbol{v} - P\boldsymbol{v} = \boldsymbol{v}$. In words, any $\boldsymbol{v}$ in the nullspace of P is also in the range of $I-P$.

Now, we show $\text{range}(I-P) \subseteq \text{null}(P)$. We know that any $\boldsymbol{x} \in \text{range}(I-P)$ is characterized by

$$\boldsymbol{x} = (I-P)\boldsymbol{v} \quad \text{for some } \boldsymbol{v}.$$

Thus

$$\boldsymbol{x} = \boldsymbol{v} - P\boldsymbol{v} = -(P\boldsymbol{v} - \boldsymbol{v}) \in \text{null}(P)$$

since we showed earlier that $P(P\boldsymbol{v}-\boldsymbol{v}) = \mathbf{0}$. Thus if $\boldsymbol{x} \in \text{range}(I-P)$, then $\boldsymbol{x} \in \text{null}(P)$. ■

3.2 Decomposition of a Given Vector

Using a projector and its complementary projector we can decompose any vector $\boldsymbol{v}$ into

$$\boldsymbol{v} = P\boldsymbol{v} + (I-P)\boldsymbol{v},$$

where $P\boldsymbol{v} \in \text{range}(P)$ and $(I-P)\boldsymbol{v} \in \text{null}(P)$. This decomposition is *unique* since $\text{range}(P) \cap \text{null}(P) = \{\mathbf{0}\}$, i.e., the projectors are complementary.

3.3 Orthogonal Projectors

If $P \in \mathbb{C}^{m\times m}$ is a square matrix such that $P^2 = P$ and $P = P^*$ then P is called an *orthogonal projector*.

Remark In some books the definition of a projector already includes orthogonality. However, as before, P is in general *not* an orthogonal matrix, i.e., $P^*P = P^2 \neq I$.

3.4 Connection to Earlier Orthogonal Decomposition

Earlier we considered the orthonormal set $\{\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n\}$, and established the decomposition

$$\begin{aligned}\boldsymbol{v} &= \boldsymbol{r} + \sum_{i=1}^{n} (\boldsymbol{q}_i^*\boldsymbol{v})\boldsymbol{q}_i \\ &= \boldsymbol{r} + \sum_{i=1}^{n} (\boldsymbol{q}_i\boldsymbol{q}_i^*)\boldsymbol{v} \qquad (12)\end{aligned}$$

with $\boldsymbol{r}$ orthogonal to $\{\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n\}$. This corresponds to the decomposition

$$\boldsymbol{v} = (I - P)\boldsymbol{v} + P\boldsymbol{v}$$

with $P = \sum_{i=1}^{n}(\boldsymbol{q}_i\boldsymbol{q}_i^*)$.

Note that $\sum_{i=1}^{n}(\boldsymbol{q}_i\boldsymbol{q}_i^*) = QQ^*$ with $Q = [\boldsymbol{q}_1\boldsymbol{q}_2 \ddots \boldsymbol{q}_n]$. Thus the orthogonal decomposition (12) can be rewritten as

$$\boldsymbol{v} = (I - QQ^*)\boldsymbol{v} + QQ^*\boldsymbol{v}. \tag{13}$$

It is easy to verify that QQ^* is indeed an orthogonal projection:

1. $(QQ^*)^2 = Q\underbrace{Q^*Q}_{=I}Q^* = QQ^*$ since Q has orthonormal columns (but not rows).
2. $(QQ^*)^* = QQ^*$.

Remark The orthogonal decomposition (13) will be important for the implementation of the QR decomposition later on. In particular we will use the rank-1 projector

$$P_{\boldsymbol{q}} = \boldsymbol{q}\boldsymbol{q}^*$$

which projects onto the direction $\boldsymbol{q}$ and its complement

$$P_{\perp\boldsymbol{q}} = I - \boldsymbol{q}\boldsymbol{q}^*.$$

Thus,

$$\boldsymbol{v} = (I - \boldsymbol{q}\boldsymbol{q}^*)\boldsymbol{v} + \boldsymbol{q}\boldsymbol{q}^*\boldsymbol{v},$$

or, more generally, orthogonal projections onto an arbitrary direction $\boldsymbol{a}$ is given by

$$\boldsymbol{v} = \left(I - \frac{\boldsymbol{a}\boldsymbol{a}^*}{\boldsymbol{a}^*\boldsymbol{a}}\right)\boldsymbol{v} + \frac{\boldsymbol{a}\boldsymbol{a}^*}{\boldsymbol{a}^*\boldsymbol{a}}\boldsymbol{v},$$

where we abbreviate $P_{\boldsymbol{a}} = \frac{\boldsymbol{a}\boldsymbol{a}^*}{\boldsymbol{a}^*\boldsymbol{a}}$ and $P_{\perp\boldsymbol{a}} = (I - \frac{\boldsymbol{a}\boldsymbol{a}^*}{\boldsymbol{a}^*\boldsymbol{a}})$.

As a further generalization we can consider orthogonal projection onto the range of a (full-rank) matrix A. Earlier, for the orthonormal basis $\{\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n\}$ (the columns of Q) we had $P = QQ^*$. Now we require only that $\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n\}$ be linearly independent. In order to compute the projection P for this case we start with an arbitrary vector $\boldsymbol{v}$. We need to ensure that $P\boldsymbol{v} - \boldsymbol{v} \perp \text{range}(A)$, i.e., if $P\boldsymbol{v} \in \text{range}(A)$ then

$$\boldsymbol{a}_j^*(P\boldsymbol{v} - \boldsymbol{v}) = 0, \qquad j = 1, \ldots, n.$$

Now, since $P\boldsymbol{v} \in \text{range}(A)$ we know $P\boldsymbol{v} = A\boldsymbol{x}$ for some $\boldsymbol{x}$. Thus

$$\begin{aligned} \boldsymbol{a}_j^*(A\boldsymbol{x} - \boldsymbol{v}) &= 0, \qquad j = 1, \ldots, n \\ A^*(A\boldsymbol{x} - \boldsymbol{v}) &= 0 \end{aligned}$$

or

$$A^*A\boldsymbol{x} = A^*\boldsymbol{v}.$$

One can show that $(A^*A)^{-1}$ exists provided the columns of A are linearly independent (our assumption). Then

$$\boldsymbol{x} = (A^*A)^{-1}A^*\boldsymbol{v}.$$

Finally,

$$P\boldsymbol{v} = A\boldsymbol{x} = \underbrace{A(A^*A)^{-1}A^*}_{=P}\boldsymbol{v}.$$

Remark Note that this includes the earlier discussion when $\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n\}$ is orthonormal since then $A^*A = I$ and $P = AA^*$ as before.