

# Master Theorem: Practice Problems and Solutions

## Master Theorem

The Master Theorem applies to recurrences of the following form:

$$T(n) = aT(n/b) + f(n)$$

where  $a \geq 1$  and  $b > 1$  are constants and  $f(n)$  is an asymptotically positive function.

There are 3 cases:

1. If  $f(n) = O(n^{\log_b a - \epsilon})$  for some constant  $\epsilon > 0$ , then  $T(n) = \Theta(n^{\log_b a})$ .
2. If  $f(n) = \Theta(n^{\log_b a} \log^k n)$  with<sup>1</sup>  $k \geq 0$ , then  $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$ .
3. If  $f(n) = \Omega(n^{\log_b a + \epsilon})$  with  $\epsilon > 0$ , and  $f(n)$  satisfies the regularity condition, then  $T(n) = \Theta(f(n))$ .  
Regularity condition:  $af(n/b) \leq cf(n)$  for some constant  $c < 1$  and all sufficiently large  $n$ .

## Practice Problems

For each of the following recurrences, give an expression for the runtime  $T(n)$  if the recurrence can be solved with the Master Theorem. Otherwise, indicate that the Master Theorem does not apply.

1.  $T(n) = 3T(n/2) + n^2$

2.  $T(n) = 4T(n/2) + n^2$

3.  $T(n) = T(n/2) + 2^n$

4.  $T(n) = 2^n T(n/2) + n^n$

5.  $T(n) = 16T(n/4) + n$

6.  $T(n) = 2T(n/2) + n \log n$

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<sup>1</sup>most of the time,  $k = 0$

$$7. \ T(n) = 2T(n/2) + n/\log n$$

$$8. \ T(n) = 2T(n/4) + n^{0.51}$$

$$9. \ T(n) = 0.5T(n/2) + 1/n$$

$$10. \ T(n) = 16T(n/4) + n!$$

$$11. \ T(n) = \sqrt{2}T(n/2) + \log n$$

$$12. \ T(n) = 3T(n/2) + n$$

$$13. \ T(n) = 3T(n/3) + \sqrt{n}$$

$$14. \ T(n) = 4T(n/2) + cn$$

$$15. \ T(n) = 3T(n/4) + n\log n$$

$$16. \ T(n) = 3T(n/3) + n/2$$

$$17. \ T(n) = 6T(n/3) + n^2\log n$$

$$18. \ T(n) = 4T(n/2) + n/\log n$$

$$19. \ T(n) = 64T(n/8) - n^2\log n$$

$$20. \ T(n) = 7T(n/3) + n^2$$

$$21. \ T(n) = 4T(n/2) + \log n$$

$$22. \ T(n) = T(n/2) + n(2 - \cos n)$$

## Solutions

1.  $T(n) = 3T(n/2) + n^2 \implies T(n) = \Theta(n^2)$  (Case 3)
2.  $T(n) = 4T(n/2) + n^2 \implies T(n) = \Theta(n^2 \log n)$  (Case 2)
3.  $T(n) = T(n/2) + 2^n \implies \Theta(2^n)$  (Case 3)
4.  $T(n) = 2^n T(n/2) + n^n \implies$  Does not apply ( $a$  is not constant)
5.  $T(n) = 16T(n/4) + n \implies T(n) = \Theta(n^2)$  (Case 1)
6.  $T(n) = 2T(n/2) + n \log n \implies T(n) = n \log^2 n$  (Case 2)
7.  $T(n) = 2T(n/2) + n/\log n \implies$  Does not apply (non-polynomial difference between  $f(n)$  and  $n^{\log_b a}$ )
8.  $T(n) = 2T(n/4) + n^{0.51} \implies T(n) = \Theta(n^{0.51})$  (Case 3)
9.  $T(n) = 0.5T(n/2) + 1/n \implies$  Does not apply ( $a < 1$ )
10.  $T(n) = 16T(n/4) + n! \implies T(n) = \Theta(n!)$  (Case 3)
11.  $T(n) = \sqrt{2}T(n/2) + \log n \implies T(n) = \Theta(\sqrt{n})$  (Case 1)
12.  $T(n) = 3T(n/2) + n \implies T(n) = \Theta(n^{\lg 3})$  (Case 1)
13.  $T(n) = 3T(n/3) + \sqrt{n} \implies T(n) = \Theta(n)$  (Case 1)
14.  $T(n) = 4T(n/2) + cn \implies T(n) = \Theta(n^2)$  (Case 1)
15.  $T(n) = 3T(n/4) + n \log n \implies T(n) = \Theta(n \log n)$  (Case 3)
16.  $T(n) = 3T(n/3) + n/2 \implies T(n) = \Theta(n \log n)$  (Case 2)
17.  $T(n) = 6T(n/3) + n^2 \log n \implies T(n) = \Theta(n^2 \log n)$  (Case 3)
18.  $T(n) = 4T(n/2) + n/\log n \implies T(n) = \Theta(n^2)$  (Case 1)
19.  $T(n) = 64T(n/8) - n^2 \log n \implies$  Does not apply ( $f(n)$  is not positive)
20.  $T(n) = 7T(n/3) + n^2 \implies T(n) = \Theta(n^2)$  (Case 3)
21.  $T(n) = 4T(n/2) + \log n \implies T(n) = \Theta(n^2)$  (Case 1)
22.  $T(n) = T(n/2) + n(2 - \cos n) \implies$  Does not apply. We are in Case 3, but the regularity condition is violated. (Consider  $n = 2\pi k$ , where  $k$  is odd and arbitrarily large. For any such choice of  $n$ , you can show that  $c \geq 3/2$ , thereby violating the regularity condition.)