Week-1

Natural Numbers & Their Operations

Natural number are used to count.

$\mathbb{N} = \{0, 1, 2 \ldots\ldots\}$

Sometimes $\mathbb{N}_0$ is used to represent 0

Operations

Addition
→ sum of two natural no is $\mathbb{N}$
eg: $2+3=5$

Subtraction
→ Difference of two natural no is not a $\mathbb{N}$ but a $\mathbb{Z}$
↓
Extend $\mathbb{N}$ with -ve no

Multiplication
→ $m \times n = \underbrace{m+m+\ldots\ldots m}_{n \text{ times}}$
$m \times n$ = n groups of m
→ notation $m \times n, m \cdot n, mn$
→ repeated addition
→ $-m \times n = -(m \cdot n)$
$-m \times -n = m \cdot n$
→ $m^k = \underbrace{m \times m \times m \cdots m}_{k \text{ times}}$
→ Exponentiation
↓
repeated multiplication

Division
→ repeated subtraction
Eg:- $21 \div 6$
Quotient = 3
remainder = 3
$21 \bmod 6 = 3$
mod → remainder

Factors

→ a divides b if b mod a is 0
→ a is a factor of b if a/b
→ factors occur in pairs e.g:- factors of 12 are $\{1,12\}, \{2,6\}, \{3,4\}$
→ unless the number is a perfect square (Here factor of one case is not in pair)
Eg:- $36 = \{1,36\}, \{2,18\}, \{3,12\}, \{4,9\}, \{6\}$

Prime numbers

→ A number p is prime if it has only 2 factors $\{1, p\}$
• 1 is not a prime
→ Sieve of Eratosthenes - remove multiples of p
→ The decompostion of ~~pr~~ numbers is prime factorisation
Eg:- $12 = 2 \times 2 \times 3 = 2^2 \times 3$
$126 = 2 \times 3 \times 3 \times 7 = 2 \times 3^2 \times 7$

Summary

* Natural numbers $\mathbb{N}$
* Integers
* Arithmetic operators $+, -, \times, \div, m^n$, m mod n
* Quotient, remainder
* Divisibility, a/b
* Factors
* Prime number and Prime factorisation

Rational Numbers ($\mathbb{Q}$)

Eg: $19 \div 5$ can't be represented as integer

Fraction $= 3\frac{4}{5}$

- $\frac{p}{q}$ is a rational number where (p and q are integers, $q \neq 0$)
- Representation is not unique

Eg:- $\frac{3}{5} = \frac{6}{10} = \frac{30}{50} \dots$

Reduced form $\frac{p}{q}$ (where p, q have no common factor)

Greatest common divisor

$gcd(a,b)$ = common factor of a,b

Eg:- $gcd(18, 60) = 6$

- First find out prime factors of two number
- Separate common prime factors of two numbers
- If there is no common prime factor then $gcd = 1$
- Multiply the common prime factors to get gcd

Density

→ For each integer, we have a next integer and a previous integer.

Eg For 3, next $= 3+1 = 4$

previous $= 3-1 = 2$

→ There is no integer between N, next and previous N

→ The above ~~contidil~~ condition does not apply for rational no

- Between two $\mathbb{Q}$ we can find another rational

Eg $\frac{m}{n} < \frac{p}{q}$

their avg $= \dfrac{\left(\frac{m}{n} + \frac{p}{q}\right)}{2}$

→ Rationals are dense but integers are ~~decre~~ discrete

→ Since rationals are dense we can't talk of next or previous in their case

Real and Complex Numbers

Limitations of Rationals (Beyond Rationals)

→ For an integer m, square is $m^2 = m \cdot m$

Square root of m may not be always rational

Eg:- $\sqrt{2}, \sqrt{3}$ etc.

Here numbers like $\sqrt{2}, \sqrt{3}$ are irrational.

* Set of all rational and irrational numbers is Real numbers

→ Like rationals, real numbers are dense

If $r_1 < r_2$ then $\left(\frac{r_1+r_2}{2}\right)$ lies between r_1 & r_2

Limitations of Reals (Beyond real)

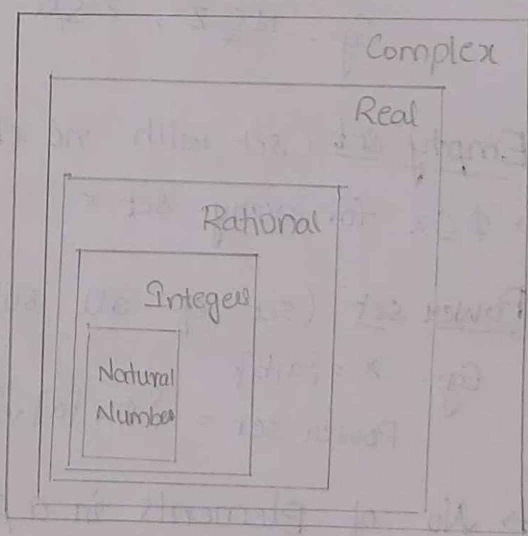
Square root of negative integers is not real because for any integer z, z^2 is positive.

* Set of ~~rat~~ real numbers and squareroot of negative integers is called Complex Numbers

Eg:- $\sqrt{-1}$ is a complex number.

Summary

→ Every natural no is an integer
→ Every integer is a rational
→ Every rational is a real
→ Every real is a complex
→ Typical irrational nos are sq. root of integers that are not perfect
→ Real are dense
→ Real nos extend rational nos
→ Complex no extend real nos



Set Theory.

Set a well defined collection of items

- Set may be infinite
- There is not requirement that members of a set have uniform type.

Eg:- ① Sets of object in a painting.
② Sets of objects in a room

Order, duplicate, Cardinality

→ Sets are unordered
→ Duplicates don't matter
→ Cardinality means number of items in a set

Subset

A part of a set

→ Eg:- Prime $\subseteq$ Natural,
$N \subseteq Z$, $Z \subseteq Q$, $Q \subseteq R$

→ Every set is a subset of itself.

Proper subset If $X \subset Y$ or $X \subsetneq Y$
Eg:- $N \subsetneq Z$, $Z \subsetneq Q$

Empty set (set with no elements)
→ $\phi \subseteq X$ for every set X

Power set (set of all subsets of a set)
Eg:- $X = \{a, b\}$
Power set = $\{\phi, \{a\}, \{b\}, \{a, b\}\}$

→ No of Elements in a power set $= 2^n$ ($n \rightarrow$ no of elements in set)

Binary Numbers

→ Digit i represent whether x_i is included in a subset
$X = \{a, b, c, d\}$ 0101 is $\{b, d\}$
0000 is ϕ and 1111 is X

→ 2^n n bit binary number

Construction of Subsets and Set Operations

Constructing Subsets

Set comprehension (set builder form)

→ Begin with an existing set, Z
→ Apply a condition to each element in that set
Eg: $x \in Z$ such that $x \bmod 2 = 0$
→ Collect all the elements that match the condition
Eg:- The set of perfect square
$\{m \mid m \in N, \sqrt{m} \in N\}$

Intervals

Eg:- Integers from -6 to +6
$\{z \mid z \in Z, -6 \le z \le 6$

→ Closed Interval $[a,b]$ (include end points)
$\{r \mid r \in R, a \le r \le b\}$
→ Open Interval (a,b) (exclude end points)
$\{r \mid r \in R, 0 < r < 1\}$
→ Left open
$\{r \mid r \in R, 0 < r \le 1\}$
→ Right open
$\{r \mid r \in R, 0 \le r < 1\}$

Union, Intersection, Complement

Union ($\cup$) combine X and Y → $X \cup Y$
Eg $X = \{a, b, c\}$ $Y = \{c, d, e\}$ $X \cup Y = \{a, b, c, d, e\}$

Intersection ($\cap$): common elements of X and Y → $X \cap Y$
Eg $X = \{a, b, d, c\}$ $Y = \{d, a, e, f\}$ $X \cap Y = \{a, b\}$

Complement ($\bar{X}$ / X^c)
element not in set X
Eg:- complement of even numbers is odd numbers

Set Difference
Elements in X but not in Y. denoted by $X - Y$ or X/Y
Eg:- $\{a, b, c, d\} - \{a, d, e, f\} = \{b, c\}$

Summary
- Sets may be finite or infinite
- Set operations are union, intersection, difference, compliment

Sets : Examples

Set Comprehension

→ Square of the even integers

$\{x^2 \mid x \in Z, x \bmod 2 = 0\}$

- Generate Elements drawn from existing set
- Filter select elements that satisfy a constraint
- Transform Modify selected elements

→ More filters

- Rationals in reduced form

  $\{\frac{p}{q} \mid p/q \in Q, \gcd(p,q)=1\}$

- Reals in interval $\{-1,2\}$

  $\{r \mid r \in R, -1 \leq r < 2\}$

→ Cubes of first 5 natural numbers

$Y = \{n^3 \mid n \in \{0,1,2,3,4\}\}$

→ Cube of first 500 natural numbers ?

$Y = \{n^3 \mid n \in \{0,1,2, \ldots\ldots\ldots 499\}\}$

- Use set comprehension to define first 500 natural numbers

  $X = \{n \mid n \in N, n < 500\}$

- Now a more readable version

  $X = \{n \mid n \in N, n < 500\}$

  $Y = \{n^3 \mid n \in X\}$

Perfect Square

→ Integers whose square root is also an integer

$\{z \mid z \in Z, \sqrt{z} \in Z$

→ All squares are positive, so this is the same as

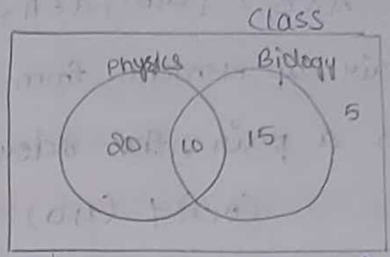
$\{n \mid n \in N, \sqrt{n} \in N\}$

→ Extend the definition to rationals

- $(9/16) = (3/4)^2$ is a squar $1/2 \neq \left(\frac{p}{q}\right)^2$ for any p,q
- $\{q \mid q \in Q, \sqrt{q} \in Q\}$ or $\{q^2 \mid q \in Q\}$

Counting Problems

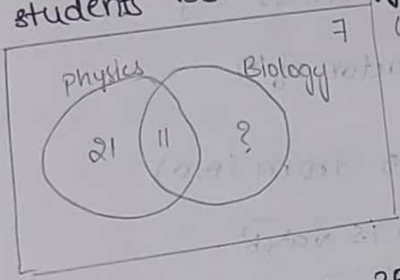
* In a class, 30 students took Physics, 25 took Biology and 10 took both, 5 took neither. How many students are there in class?



- Draw sets for Physics (P) & Biology (B)
- 10 students are in $P \cap B$
- 20 students in $P \setminus B$.
- This leaves 20 students [Took Physics but didn't take Biology]
- Likewise 15 students in $B \setminus P$ (Took Biology but not Physics)
- 5 students in $\overline{P \cup B}$ [neither physics nor Biology]

Class strength $20+10+15+5 = 50$

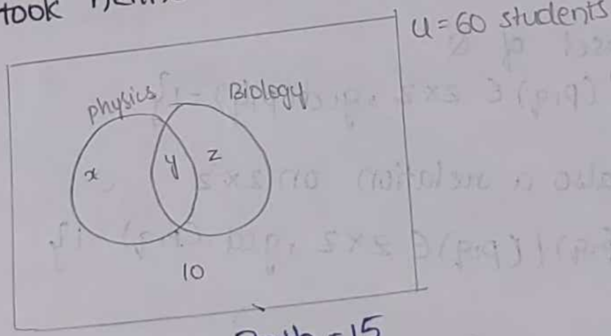
* In a class of 55 students, 32 students took Physics, 11 students took both Physics and Biology and 7 took neither. How many students took Biology but not Physics



$55-7 = A \cup B$
$48 = A \cup B$
$21+11+x = 48$
$x = 16$

16 students took only Biology

* In a class of 60 students, 35 students took Physics, 30 took Biology and 10 took neither. How many took both Physics and Biology.



$x+y+z = 60-10 = 50$
$x+y = 35$
$y+z = 30$
$x-z = 5$

$z = 50-35 = 15$
$x = 20$
$y = 15$

Both = 15

Summary

→ Set notation is useful way to concisely describe collection of objects

→ Set comprehension combines generators, filters & transformation to produce new sets from old

→ Venn diagrams can be useful to work out problems involving sets

Relations

Cartesian product

$A \times B = \{(a,b) \mid a \in A, b \in B\}$

Pair up elements from A and B

→ In a pair, the order is important

$(0,1) \neq (1,0)$

→ For sets of numbers, visualize product as 2-D space $N \times N$

Binary Relations

$R \subseteq A \times B$ Notation: $(a,b) \in R$, aRb

→ $\{(m,n) \mid (m,n) \in \mathbb{N} \times \mathbb{N}, n = m+1\}$

→ $\{(0,1), (1,2), (2,3) \dots\dots (17,18) \dots\}$

→ Pairs (d,n) where d is a factor of n

Eg ① Points at a distance 5 from $(0,0)$

→ Distance from $(0,0)$ to (a,b) is $\sqrt{a^2+b^2}$

→ $\{(a,b) \mid (a,b) \in R \times R, \sqrt{a^2+b^2} = 5$

② Rationals in reduced form

→ A subset of Q

$\{\frac{p}{q} \mid (p,q) \in Z \times Z, \gcd(p,q) = 1\}$

→ But also a relation on $Z \times Z$

$\{(p,q) \mid (p,q) \in Z \times Z, \gcd(p,q) = 1\}$

Binary Relations.

Identity Relation

$I \subseteq A \times A$

Eg $I = \{(a,b) \mid (a,b) \in A \times A, a = b\}$

$I = \{(a,a) \mid (a,a) \in A \times A\}$

$I = \{(a,a) \mid a \in A\}$

Reflexive Relation

$R \subseteq A \times A$ $I \subseteq R$

Eg: $\{(a,b) \mid (a,b) \in N \times N, a,b > 0, a \mid b\}$

$a \mid a$ for all $a > 0$

Symmetric

$(a,b) \in R$ if and only if $(b,a) \in R$

Transitive

If $(a,b) \in R$ and $(b,c) \in R$ then $(a,c) \in R$

Antisymmetric

If $(a,b) \in R$ and $a \neq b$, then $(b,a) \notin R$

Equivalence Relations

→ satisfy reflexive, symmetric and transitive

→ Same remainder module 5
  * If a mod 5 = b mod 5 then (b-a) is a multiple of 5
  * $z \bmod 5 = \{(a,b) \mid a,b \in Z, (b-a) \bmod 5 = 0\}$
  * Divide integers into 5 groups based on remainder when divided by 5

→ An equivalence relation partitions a set

→ Group of equivalent elements are called Equivalence classes

Beyond Binary Relations

→ Cartesian product of more than two sets

→ Pythagorean triplets
  * Square on the hypotenuse is the sum of the squares of ~~of~~ the opp. sides.
  * $\{(a,b,c) \mid (a,b,c) \in N \times N \times N \; a,b,c > 0, a^2+b^2=c^2\}$

→ Corners of squares
  * A corner is a point $(x,y) \in R \times R$
  * $((x_1,y_1)\,(x_2,y_2)\,(x_3,y_3)\,(x_4,y_4))$ are related if they are four corners of a square.
  * For instance
    * $((0,0), (0,4), (4,4), (4,0))$
  * $Sq < R^2 \times R^2 \times R^2 \times R^2$

Summary

→ Cartesian products generate n-tupples from n-sets
  * $(x_1, x_2, \ldots, x_n) \in X_1 \times X_2 \times X_3 \cdots \times X_n$

→ Relation → subset of cartesian product

→ Properties of relations: Reflexive, symmetric, transitive, Asymmetric

→ Equivalence relation partitions a set.

Functions

- A rule to map input to output
- Convert x to x^2
  - → This rule $x \to x^2$
  - → Give it a name = $sq(x) = x^2$

Input is a parameter

- Need to specify input and output sets

Domain: Input set   Codomain Output set of possible values
domain $(sq) = R$ codomain $(sq) = R$

Range: Actual values that output can take
range $(sq) = R \geq 0 = \{r \mid r \in R, r \geq 0\}$

- $f : X \to Y$, domain of f is X, codomain is Y

Functions and Relations

- Associate a relation R_f with each function f
- $R_{eq} = \{(x,y) \mid x, y \in R, y = x^2\}$
  - → Additional notation $Y = x^2$
- $R_f \subset$ domain$(f) \times$ range(f)
- Properties of R_f
  - → defined on the entire domain
    - * for each $x \in$ domain(f) there is ~~exactly one $Y \in$ codomain(f)~~ such that a pair $(x,y) \in R_f$
  - → Single valued
    - * for each $x \in$ domain(f), there is exactly one $Y \in$ codomain(f) such that $(x,y) \in R_f$
- Drawing f as a graph is plotting R_f

Lines

→ $f(x) = 3.5x + 5.7$

- 3.5 is slope
- 5.7 is intercept where the line crosses the y-axis where $x=0$

→ Changing the slope and intercept produce different lines

$f(x) = 3.5x - 1.2$
$f(x) = 2x + 5.7$
$f(x) = -4.5x + 2.5$

→ In all these cases
Domain $= R$ Codomain $=$ Range $= R$

More functions

- $x \to \sqrt{x}$
- Is this a function?
  1. $5^2 = (-5)^2 = 25$
  2. $\sqrt{25}$ takes 2 options
  3. By convention, take positive square root

What is the domain?
- Depends on codomain
- Negative numbers do not have real square roots
- If codomain is R domain is $R \geq 0$
- If codomain is the set C of complex numbers domain is R

Types of functions

Injective
- → Different inputs produce different outputs
- → one to one
- → If $x_1 \neq x_2$, $f(x_1) \neq f(x_2)$
- → $f(x) = 3x+5$ is injective
- → $f(x) = 7x^2$ is not, for any a $f(a) = f(-a)$

Surjective
- → Range is codomain
- → onto
- → For every $y \in$ codomain(f) there is an $x \in$ domain(f) such that $f(x) = y$
- → $f(x) = -7x+10$ is surjective
- → $f(x) = 5x^2+3$ is not surjective for codomain R

Bijective
- → 1-1 correspondence b/w domain and codomain
- → Every $x \in$ domain(f) maps to a distinct $y \in$ codomain(f) such that $y = f(x)$

Properties of functions

Theorem A function is bijective $\iff$ it is injective and surjective

→ From the definition, if a function is bijective it is injective and surjective

→ Suppose a function f is injective and surjective
- Injectivity guarantees that f ~~injective and surjective~~ satisfy first condition of a bijective
- surjectivity says every $y \in$ codomain(f) has a pre-image. Injectivity guarantees this pre-image is unique.

Bijections and Cardinality

- For finite sets we can count the items
- For infinite sets
  - → Number of lines is the same as $\mathbb{R}\times\mathbb{R}$
  - → Every line $y = mx + c$ is determined uniquely by (m, c) and vise versa
- For every pair of points (x_1, y_1) and (x_2, y_2) there is a unique line passing through both points
- Number of lines is same as cardinality of $R\times R$
- Does this show that $(R\times R)\times(R\times R) = R^2 \times R^2$ has the same cardinality as $R\times R$?
- The correspondence is not a bijection - many pairs of points describe the same line
- Be careful to establish that a function is a bijection

Summary

→ A function is given by a rule mapping inputs to out puts

→ Define the domain, codomain and range

→ Associate a relation R_f with each function f

→ Properties of functions: injective (one-to-one)
surjective (onto)

→ Bijections : injective and surjective (one-to-one and onto)

→ A bijection establishes that domain and codomain have same cardinality

Relations

- $A \times B \to$ Cartesian product ; all pairs (a,b) ; $a \in A$ and $b \in B$.
- $A = \{1,4,7\}$, $B = \{1,16,49\}$
  - $A \times B = \{(1,1)(1,16),(1,49),(4,1),(4,16),(4,49),(7,1),(7,16),(7,49)\}$
  - $B \times A = \{(1,1),(16,1)(49,1)(1,4),(16,4),(49,4),(1,7),(16,7),(49,7)\}$
  - $B \times B = \{(1,1),(1,16)(1,49),(16,1),(16,16),(16,49),(49,1)(49,16),(49,49)\}$
- Can take cartesian product of more than two sets.

  $A \times B \times A = \{(1,1,1),(1,1,4),\ldots\ldots\ldots(7,49,16)(7,49,49)\}$
- A relation picks out certain tuples in the Cartesian product
  - $S \subseteq A \times B = \{(1,1),(4,16),(7,49)\}$
  - $S = \{(a,b) \mid (a,b) \in A \times B, b = a^2\}$

Examples of relations

- Divisibility
  - → Pairs of natural numbers (d,n) such that $d|n$
  - → Pairs such as $(7,63)(17,85)(3,9)\ldots$
  - → $D = \{(d,n) \mid (d,n) \in \mathbb{Z} \times \mathbb{N}, d|n\}$
  - → Can also extend to integer divisors
  - → $E = \{(d,n) \mid (d,n) \in \mathbb{Z} \times \mathbb{N}, d|n\}$
  - → Now $(-7,63),(-17,85),(-3,9)\ldots$ are also in E
- Prime powers
  - → Pairs of natural numbers (p,n) such that p is prime and $n = p^m$ for some natural number m
  - → Examples $(3,1)(5,625)(7,343)$
  - → First define primes $P = \{p \mid p \in \mathbb{N}, factors(p) = \{1,p\}, p \neq 1\}$
  - → Prime powers $PP = \{(p,n) \mid (p,n) \in P \times \mathbb{N}, n = p^m \text{ for some } m \in \mathbb{N}\}$

Beyond numbers

Air line routes

- An airline flies to set of cities - eg:- Bangalore, Chennai ....
- Let C denote the set of cities served by airline
- Some cities are connected by direct flights

  $D \subseteq C \times C$
- Is D reflexive, irreflexive?

  Hopefully irreflexive.

• Is D symmetric
→ If there is a direct flight from Bangalore to Delhi, is there always a direct flight back from Delhi to Bangalore
→ For bigger cities, yes
→ For smaller cities may have triangular route
Chennai → Madurai → Salem → Chennai

Tables as relations

Example-1
Flying distances between cities

Source	Destination	Distance(km)
Bangalore	Chennai	290
Chennai	Delhi	1752
Delhi	Bangalore	1735
Delhi	Chennai	1752
⋮	⋮	

→ Table is a relation $Dist \subseteq C \times C \times N$
→ some entries are useless (Eg:- Delhi, Delhi, 0)
→ Restrict to cities served by direct flights
- $Dist = \{(a,b,c,d) \mid (a,b) \in D,$ d is distance from a to b$\}$
→ Distances are symmetric, even if D is not
→ Save space by representing only one direction in the table

Example-2

Roll no	Name	Date of birth
A71396	Abhay Shah	03-07-01
B82976	Payal Ghosh	18-06-99
F98989	Jeremy Pinto	22-02-03
C93986	Payal Ghosh	14-05-00
...	....	

→ Some columns are special (Eg:- each student has a unique roll number)
• Such a column is called a key
• Name is not a key, in general
→ Given the roll number, can retrieve the data for a student
• Function from Roll Numbers to (Name, Date of Birth)
• (key, value) pairs ~~{(r,n,s)}~~

Functions

- A rule to map inputs to outputs

  $x \to x^2$, $g(x) = x^2$

- Domain, codomain, range.
- Associated relation $\longrightarrow$ $R_{sq} = \{(x,y) \mid x,y \in \mathbb{R}, y = x^2\}$
- Can have functions on other sets.

Eg:- Mother: People $\to$ People.

Range of values

- What range of values does the output span
- $f(x) = x^2$ is always positive, range is 0 to $+\infty$
- $f(x) = x^3 - 3x^2 + 5$ ranges from $-\infty$ to $+\infty$
- $f(x) = 5\sin(x)$ has a bounded range from -5 to $+5$

Maxima and minima

- $f(x) = x^2$ attains a minimum value at $x = 0$, no maximum value
- $f(x) = x^3 - 3x^2 + 5$ has no global minimum or maximum, but a local maximum at $x = 0$ and local minimum at $x = 2$
- $f(x) = 5\sin(x)$ periodically attains minimum value -5 and maximum value $+5$ infinitely often

Comparing functions

- $f(x) = x^3 - 3x^2 + 5$ grows faster than $g(x) = x^2$
- Let $G(y)$ be the number of Data Science graduates in year y
- Let $J(y)$ be the number of new Data science jobs in year y
- Ideally $G(y)$ and $J(y)$ should grow at similar rates.
- If $J(y)$ grows faster than $G(y)$ more students will opt to study Data Science.

Summary

Many properties of functions are interesting

- Range of outputs
- Inputs for which function attains (local) maximum, minimum value
- Relative growth rates of functions

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