**Sample qualifier set**
Mathematics for Data Science - 1

1. Suppose the cardinality of the power set of $A$ is 1 more than the cardinality of the power set of $B$. Which of the following is(are) correct?
(MSQ) [2]

   ◯ **Cardinality of $A$ is 1.**

   ◯ Cardinality of $A$ is 2.

   ◯ **$B$ is an empty set.**

   ◯ Cardinality of $A$ is 0.

   **Solution:** Suppose the set $A$ has cardinality $a$ and the set $B$ has the cardinality $b$, where both $a$ and $b$ are integers and $a > b$, then the difference between their cardinalities is $2^a - 2^b$, which must be greater than 2 if $a > b > 0$. So the only possibility for the condition "the cardinality of the power set of $A$ is 1 more than the cardinality of the power set of $B$", to be satisfied is $b$ should be 0 and $a$ should be 1, i.e. $B$ is an empty set and the cardinality of set $A$ is 1.

2. Find out the correct statements.
(MSQ) [2]

   ◯ **$\mathbb{N}$ is subset of $\mathbb{Z}$.**

   ◯ **$\mathbb{N}$ is proper subset of $\mathbb{Z}$.**

   ◯ $\mathbb{N} \cap \mathbb{Z} = \mathbb{Z}$.

   ◯ $\mathbb{N} \cup \mathbb{Z} = \mathbb{N}$.

   **Solution:** Clear from the definitions of subset and proper subset.

3. Which of the following statements is (are) correct?
(MSQ) [4]

   ◯ **$3x + 7y + 5 = 0$ and $\frac{x}{3} - \frac{y}{7} = 1$ are perpendicular to each other.**

   ◯ $y - 5 = 3(x - 4)^2$ is an equation of a parabola whose vertex is at the origin.

   ◯ **$y - 5 = 3(x - 4)^2$ is an equation of a parabola whose vertex is at the point $(4, 5)$.**

   ◯ $y - 5 = 3(x - 4)^2$ is an equation of a parabola whose vertex is at the point $(5, 4)$.

   **Solution:**

   - The slope of the straight line $3x + 7y + 5 = 0$ is $-\frac{3}{7}$. The slope of the second straight line $\frac{x}{3} - \frac{y}{7} = 1$ is $\frac{7}{3}$. Hence the product of their slope is $-1$. So they are perpendicular to each other.

- The given equation $y-5=3(x-4)^2$ can be written as $y=3x^2-24x+53$, which is in the form $y=ax^2+bx+c$. Now the $x$ coordinate of the vertex is $-\frac{b}{2a}=-\frac{-24}{6}=4$. Substituting $x=4$ in the given equation we get $y=5$. So the vertex of the parabola is $(4,5)$.

4. Let $S_1$ be the set of irrational numbers between 0 and 1, $S_2$ be the set of rational numbers between 0 and 1 (including both 0 and 1), and $S$ be the closed interval $[0,1]$. Which of the following options are correct?
(MSQ) [3]

   ◯ $\boldsymbol{S_1 \cap S_2}$ **is an empty set.**

   ◯ $\boldsymbol{S_1 \cup S_2 = S}$

   ◯ $S_1 \cup S_2$ is a proper subset of $S$.

   ◯ $S_2$ is a finite set.

   ◯ $\boldsymbol{S \setminus S_2 = S_1}$

   **Solution:** The given set $S=[0,1]$ is nothing but the set of real numbers between 0 and 1, including 0 and 1.
   The complement of the set of rational numbers in the universe of real numbers is the set of irrational numbers. Hence the following statements are true:

   - $S_1 \cap S_2$ is an empty set.
   - $S_1 \cup S_2 = S$
   - $S \setminus S_2 = S_1$

5. Which of the following functions is (are) onto (surjective)?
(MSQ) [4]

   ◯ Option 1

$$f : \mathbb{R} \longrightarrow \mathbb{N}$$
$$f(n) = \begin{cases} 0 & \text{if } n \in \mathbb{Q} \\ 1 & \text{otherwise} \end{cases}$$

   ◯ Option 2

$$f : \mathbb{N} \longrightarrow \mathbb{N}$$
$$f(n) = \begin{cases} n & \text{if } n \text{ is even} \\ n+1 & \text{otherwise} \end{cases}$$

   ◯ Option 3

$$f : \mathbb{N} \longrightarrow \mathbb{N}$$
$$f(n) = \begin{cases} n & \text{if } n \text{ is odd} \\ 2n & \text{otherwise} \end{cases}$$

◯ **Option 4**

$$f : \mathbb{N} \longrightarrow \mathbb{N}$$
$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ n+1 & \text{otherwise} \end{cases}$$

◯ **Option 5**

$$f : \mathbb{R} \longrightarrow \mathbb{R}$$
$$f(n) = n$$

**Solution:** A function is called onto if the range of the function is same as the codomain of the function.

- $$f : \mathbb{R} \longrightarrow \mathbb{N}$$
$$f(n) = \begin{cases} 0 & \text{if } n \in \mathbb{Q} \\ 1 & \text{otherwise} \end{cases}$$
$Range(f) = \{0, 1\}$, is not the same as $Codomain(f) = \mathbb{N}$.
- $$f : \mathbb{N} \longrightarrow \mathbb{N}$$
$$f(n) = \begin{cases} n & \text{if } n \text{ is even} \\ n+1 & \text{otherwise} \end{cases}$$
$Range(f) =$ Set of even numbers, is not same as $Codomain(f) = \mathbb{N}$.
- $$f : \mathbb{N} \longrightarrow \mathbb{N}$$
$$f(n) = \begin{cases} n & \text{if } n \text{ is odd} \\ 2n & \text{otherwise} \end{cases}$$
$Range(f) = \mathbb{N} \setminus \{2\}$, is not the same as $Codomain(f) = \mathbb{N}$.
- $$f : \mathbb{N} \longrightarrow \mathbb{N}$$
$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ n+1 & \text{otherwise} \end{cases}$$
$Range(f) = \mathbb{N}$, is same as $Codomain(f) = \mathbb{N}$. Hence $f$ is onto.
- $$f : \mathbb{R} \longrightarrow \mathbb{R}$$
$$f(n) = n$$
$Range(f) = \mathbb{R}$, is same as $Codomain(f) = \mathbb{R}$. Hence $f$ is onto.

6. A newspaper company currently sells 20000 newspapers per day. When it sells one newspaper at ₹5, it makes a profit of ₹2 per paper. It is observed that if the selling price of a newspaper is increased by ₹$x$, then the number of newspapers sold per day decreases by $2000x$. What should be the selling price per paper (in ₹) to obtain the maximum profit?
(NAT) [4]

**Answer: 9**

**Solution:** When the selling price of one newspaper is ₹5, the profit is ₹2. So the cost price of one newspaper is ₹3. So Cost price of 20000 newspaper is ₹60000.
When the selling price of a newspaper is increased by ₹$x$, then the number of newspapers sold per day decreases by $2000x$. So new selling price is ₹$5 + x$ and the number of sold newspapers become $20000 - 2000x$. The new profit per newspaper is ₹$(5 + x) - 3$= ₹$(2 + x)$.
So the total profit per day=
₹$(20000 - 2000x)(2 + x)$= ₹$2000(10 - x)(2 + x)$= ₹$2000(20 + 8x - x^2)$
The profit is maximized at $x = 4$.
So the selling price should be ₹$(5 + 4)$=₹9.

7. Suppose a laser beam, that travels in a straight line, passes through the points (-4,-6) and (2,6). Which of the following options is true?
(MCQ) [2]

- ◯ The laser beam passes through the point (0,-2).
- ◯ The laser beam is parallel to the straight line $y = -2x + 10$.
- ◯ **The laser beam is parallel to the straight line $y = 2x + 10$.**
- ◯ The laser beam passes through the point (10,20).

**Solution:** The slope of the straight line through which the laser beam passes

$$(y - 6) = \frac{6 - (-6)}{2 - (-4)}(x - 2)$$
$$y - 6 = 2(x - 2)$$
$$y = 2x + 2$$

The points $(0, -2)$ and $(10, 20)$ do not satisfy the equation of the straight line. The laser beam is parallel to the straight line $y = 2x + 10$, is the only correct option.

8. A thread of length 40m is placed on the coordinate plane such that it is parallel to the $X$-axis. One end of the thread is at (-8,5). Suppose the thread is cut into two smaller threads of lengths 10m and 30m. Which of the following options is(are) correct?
(MSQ) [3]

- ◯ **The thread is cut at the point $(2, 5)$.**
- ◯ The thread is cut at the point (10,5).

◯ **The thread is cut at the point (22, 5).**

◯ The thread is cut at the point (30,5).

**Solution:** As the thread is parallel to the $X$-axis, we only have to calculate the $x$-coordinate, according to the given information.

$$-8 + 10 = 2 \text{ and } -8 + 30 = 22.$$

Hence the thread is cut at the points $(2, 5)$ and $(22, 5)$.

9. Suppose two highways perpendicular to each other are considered as the coordinate axes. Arpita's house is at the point $(5, 5)$ and her friend Puja's house is at the point $(-5, k)$, where $k$ is a real number. The distance between their houses is 10 km. What is the value of $k$? (Consider the unit length to be 1 km on both the axes)
(NAT) [2]

**Answer: 5**

**Solution:** As the distance between their house if 10 km, we have,

$$\begin{aligned}
\sqrt{(5+5)^2 + (5-k)^2} &= 10 \\
\sqrt{10^2 + (5-k)^2} &= 10 \\
100 + (5-k)^2 &= 100 \\
(5-k)^2 &= 0 \\
5 - k &= 0 \\
k &= 5
\end{aligned}$$

10. Body Mass Index ($BMI$) is a measure of the body fat of adult humans based on their height and weight. The formula for $BMI$ of a person is given by the equation $BMI = \frac{W}{H^2}$ where $W$ is the weight of the person in kg and $H$ is the height of that person in metres(m). Table 1 lists commonly used $BMI$ ranges for an adult person.

| $BMI(\text{kg/m}^2)$ | Category |
|---|---|
| $16.0 - 18.4$ | Underweight |
| $18.5 - 24.9$ | Normal |
| $25.0 - 29.9$ | Overweight |

Table 1

Suppose the three curves in Figure 1 represent Height (in m) versus Weight (in kg) for three different fixed $BMI$ in which one curve represents underweight, another curve represents normal, and yet another curve represents overweight.

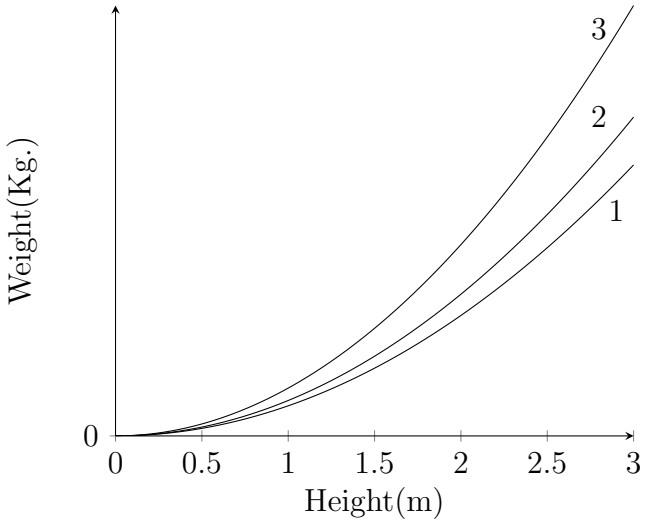



Figure 1

Which of the following options is most likely to be correct?
(MCQ) [3]

- ◯ Curve 1 represents overweight, Curve 2 represents normal, and Curve 3 represents underweight.
- ◯ **Curve 3 represents overweight, Curve 2 represents normal, and Curve 1 represents underweight.**
- ◯ Curve 2 represents overweight, Curve 1 represents normal, and Curve 3 represents underweight.
- ◯ Curve 3 represents overweight, Curve 1 represents normal, and Curve 2 represents underweight.

**Solution:** As $W = (BMI)H^2$, it is clear that the 2nd option is the correct. The higher the value of $BMI$, the corresponding parabola is nearer towards the positive direction of $Y$-axis.

Use the following information to answer the questions 11 to 13.
Let us define a function $f : \mathbb{N} \to \mathbb{N}$ such that $f(0) = 1$, $f(1) = 0$,
and $f(n) = (n-1)[f(n-1) + f(n-2)]$ for all $n \geq 2$. Define the following sets,

- $S_1 = \{n \mid f(n) \text{ is even}, n \leq 6\}$
- $S_2 = \{n \mid f(n) \text{ is odd}, n \leq 6\}$
- $T = \{n \mid f(n) \text{ is prime}\}$

11. Choose the correct option(s) from the following.
(MSQ) [3]

- ◯ $T$ is an infinite set.
- ◯ **$T$ is a finite set.**
- ◯ Cardinality of $T$ is 7.
- ◯ Cardinality of $T$ is 2.
- ◯ **Cardinality of $T$ is 1.**

12. Choose the correct option(s) from the following.
(MSQ) [4]

- ◯ Cardinality of $S_2$ is 3.
- ◯ **Cardinality of $S_2$ is 4.**
- ◯ **Cardinality of $S_1 \cup S_2 \cup T$ is 7.**
- ◯ $S_1 \cup S_2 \cup T$ is an infinite set.
- ◯ **$S_1 \cap T$ is a finite set.**
- ◯ $S_2 \cap T$ is an infinite set.

13. Choose the correct option(s) from the following.
(MSQ) [3]

- ◯ $f$ is an injective function.
- ◯ $f$ is a surjective function.
- ◯ $f$ is injective but not surjective function.
- ◯ $f$ is surjective but not injective function.
- ◯ **$f$ is neither injective nor surjective function.**
- ◯ $f$ is a bijective function.

**Solution:** $f(0) = 1, f(1) = 0, f(2) = 1, f(3) = 2, f(4) = 9, f(5) = 44, f(6) = 265$
$S_1 = \{1, 3, 5\}$
$S_2 = \{0, 2, 4, 6\}$
$f(n)$ is product of $(n-1)$ and $f(n-1) + f(n-2)$. So $f(n)$ will be prime if and only if one of $n-1$ and $f(n-1) + f(n-2)$ is prime and other one is 1.
$n-1$ is 1, if $n = 2$, and in this case $f(n-1) + f(n-2)$ is 1.
$n-1$ is 2, if $n = 3$, and in this case $f(n-1) + f(n-2)$ is 1
These are the only two cases where $f(n-1) + f(n-2)$ is 1. Hence, $f(n)$ is prime only for $n = 3$. Hence $T = \{3\}$

Solution of 11: $T$ is a finite set with cardinality 1.

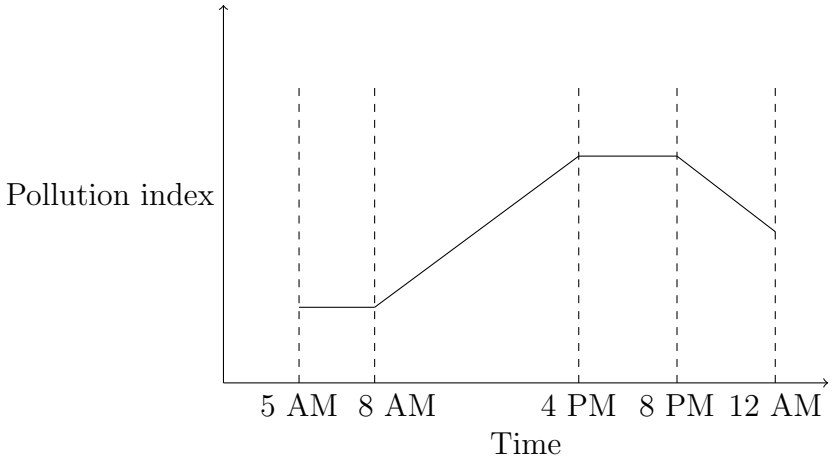
Solution of 12: cardinality of $S_2$ is 4. Moreover, $S_1 \cup S_2 \cup T = \{0, 1, 2, 3, 4, 5, 6\}$. Hence the cardinality of $S_1 \cup S_2 \cup T$ is 7. We also have $S_1 \cap T = \{3\}$, which is a finite set.

Solution of 13: $f(0) = 1 = f(2)$. So $f$ gives the same value for two different values in the domain. Hence $f$ is not injective. From the pattern it is clear that the output values of $f$ increase as the values of $n$ increase. So there is not any value in the domain for which it gives 3 as the output value. So Codomain and Range are different for $f$. Hence $f$ is not surjective. Hence $f$ is neither injective nor surjective.

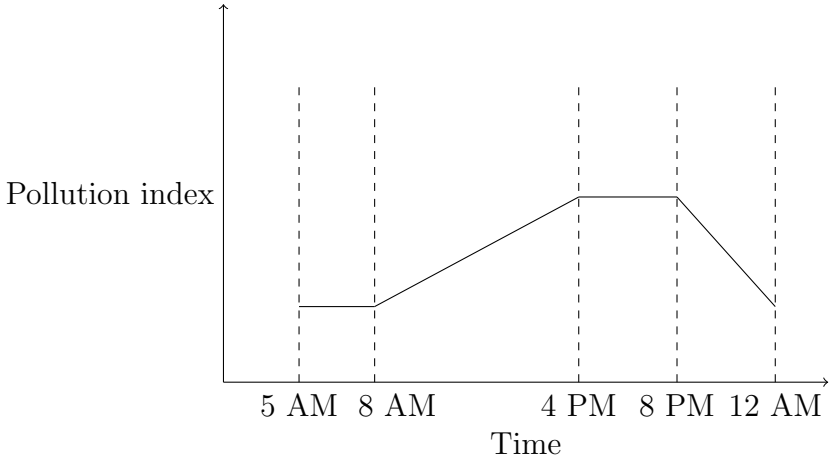
Use the paragraph given below to answer the questions 14 to 16.
During a summer day in a large city, the pollution index remained constant from 5 AM until 8 AM, then increased linearly until 4 PM, then again remained constant until 8 PM, and started decreasing linearly until 12 midnight when it reached the same value as at 5 AM. On that day, the reading at 10 AM was around 40 ppm (parts per million) and at 1 PM the reading was 80 ppm.

14. Which of the following graphs represents the pollution index with respect to time correctly?
(MCQ) [3]

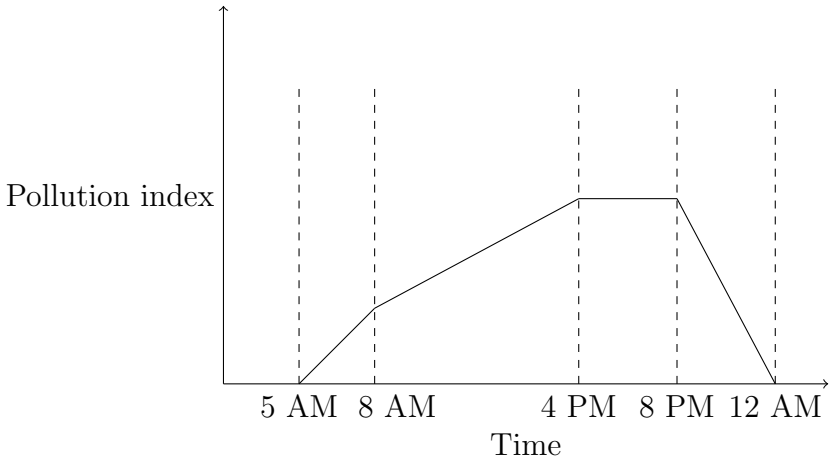
◯ Option 1



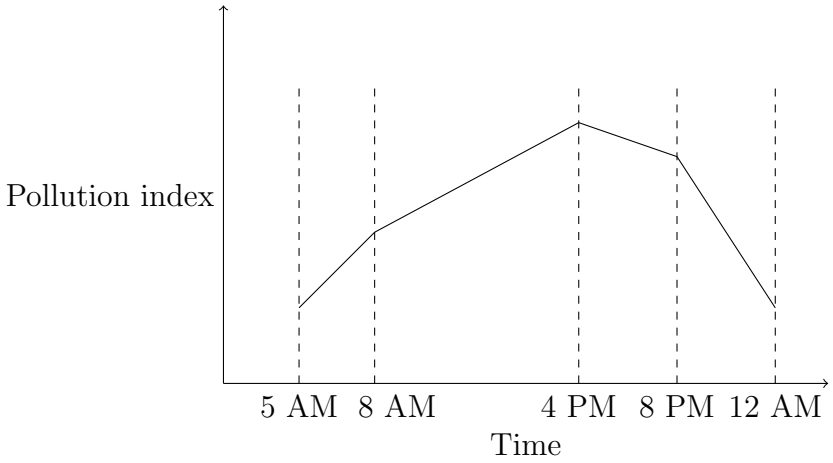

◯ **Option 2**



◯ Oprion 3




◯ Option 4




**Solution:** From the description it is clear that the second option is correct.

15. What is the slope of the line segment representing the pollution index between 4 PM and 8 PM?
(NAT) [3]
**Answer: 0**
**Solution:** The pollution index between 4 PM and 8 PM is constant. So the line segment representing the pollution index between 4 PM and 8 PM is parallel to $X$-axis which represents the time. So the slope of that line segment is 0.

16. What will be the pollution index at 10 PM?
(MCQ) [5]

◯ 70 parts per million

◯ $\frac{50}{3}$ parts per million

◯ $\frac{170}{3}$ parts per million

◯ **$\frac{200}{3}$ parts per million**

**Solution:** Let us consider time 5 AM as $x = 5$, 8 AM as $x = 8$, 4 PM as $x = 16$, 8 PM as $x = 20$, and 12 AM as $x = 24$. So the line segment representing the pollution index in between 8 AM and 4 PM passes thorugh the two points $(10, 40)$ and $(13, 80)$. Hence the equation of the line segment, $(y - 80) = \frac{40}{3}(x - 13)$. The Pollution index at 4 PM, i.e. at $x = 16$,

$$\begin{aligned} y - 80 &= \frac{40}{3}(16 - 13) \\ y - 80 &= \frac{40}{3}(3) \\ y &= 120 \end{aligned}$$

Hence the pollution index at 4 PM is 120 ppm. As the pollution index is constant from 4 PM to 8 PM, the pollution index at 8 PM is 120 ppm.
The pollution index at 8 AM , i.e. at $x = 8$,

$$\begin{aligned} y - 80 &= \frac{40}{3}(8 - 13) \\ y &= 80 - \frac{200}{3} \\ y &= \frac{40}{3} \end{aligned}$$

Hence the pollution index at 8 AM is $\frac{40}{3}$ ppm. As the pollution index at 12 AM is same as the pollution index at 8 AM, the pollution index at 12 AM is also $\frac{40}{3}$ ppm. The equation of line segment representing the population index in between 8 PM and 12 AM is,

$$y - \frac{40}{3} = \frac{\frac{40}{3} - 120}{24 - 20}(x - 24)$$

The pollution index at 10 PM, i.e. $x = 22$,

$$\begin{aligned} y - \frac{40}{3} &= \frac{\frac{40}{3} - 120}{24 - 20}(22 - 24) \\ y - \frac{40}{3} &= \frac{-320}{12}(-2) \\ y - \frac{40}{3} &= \frac{320}{6} \\ y - \frac{40}{3} &= \frac{160}{3} \\ y &= \frac{160}{3} + \frac{40}{3} \\ y &= \frac{200}{3} \end{aligned}$$

Hence the pollution index at 10 PM is $\frac{200}{3}$ ppm.