IIT Madras

ONLINE DEGREE

Mathematics for Data Science 1
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Week - 01
Lecture – 01
Natural Numbers and their operations

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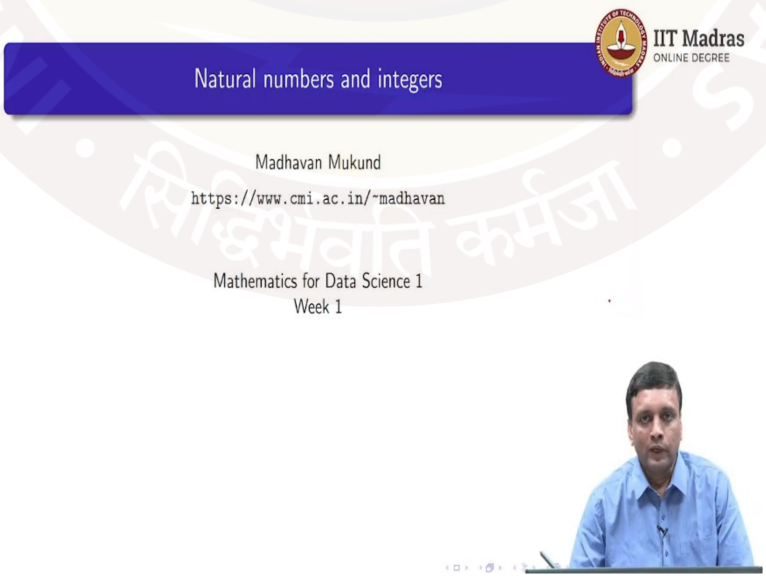

So, welcome to the 1st week of Mathematics 1 for Data Science. So, we are going to start with some very basic things which you probably know; right from the beginning we are going to start talking about numbers. So, in this 1st module what we are going to talk about is natural numbers and integers.

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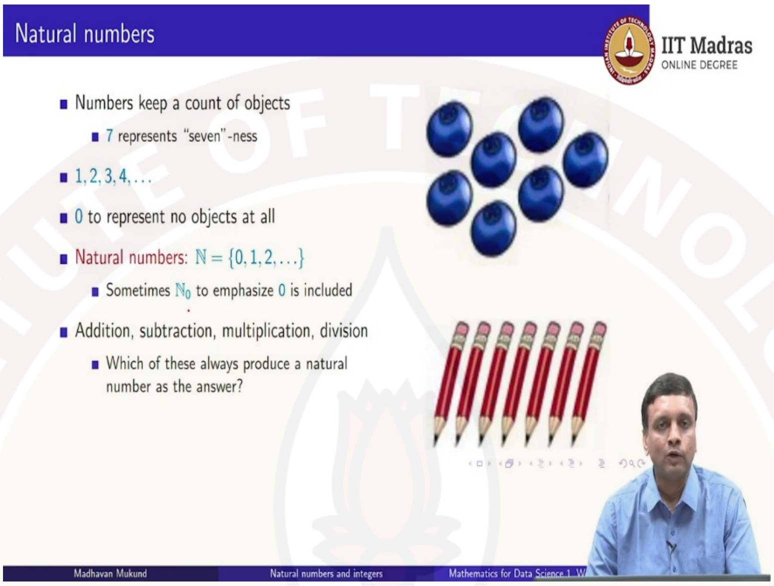

So, as you probably remember from as young as you were in school when you first came across numbers, we use numbers mainly for counting. So, for instance if we see 7 balls like this and then we see 7 pencils like this, then we need to know that these are the same number of things and for this we use this number 7. So, 7 represents what is common to these two objects that there are 7 balls and 7 pencils. So, 7 is an abstract concept in that sense and it refers to a quantity.

So, we all of course, know the numbers 1, 2, 3, 4 and all that. So, when we see a number of things, we can count them. But perhaps the most important number of all which is of Indian origin is 0. So, it is quite important to have a way to represent something when there is nothing to count because without a 0, we cannot use our place numbering system that we use to manipulate numbers.

So, these numbers starting with 0 are what are often called the natural numbers. Now there is some confusion in some books and many books will actually use only 1, 2, 3, 4 to represent the natural numbers. So, we use N to represent the set of natural numbers and in case there is any confusion whether 0 is included in this set or not, now sometimes people will not include 0 in the set of natural numbers.

So, sometimes to emphasize that we are using 0, we will actually put the subscript 0 below the N right. So, we will write either N or N_0, but whenever we are talking about natural numbers, it always includes a 0. Now what can we do with natural numbers? Well we can add them, we can subtract them, we can multiply them, we can divide them. So, these are the normal arithmetic operations which you have studied in school.

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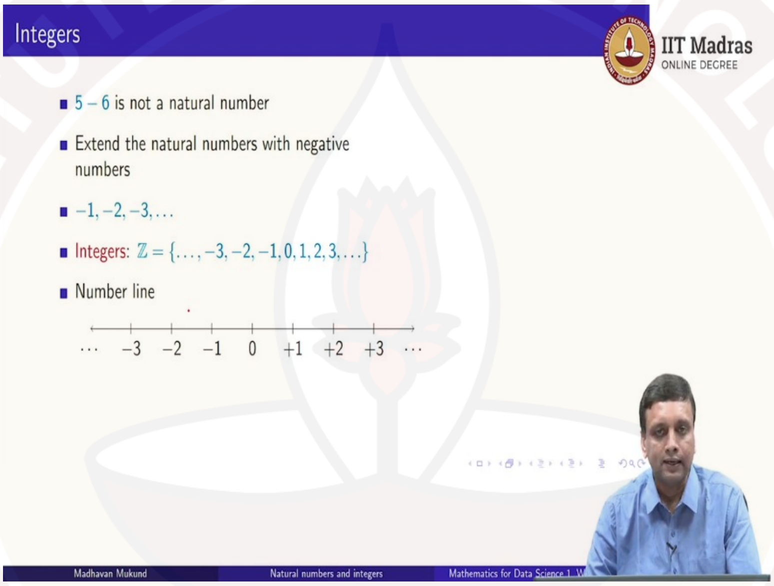

But what is really interesting from a mathematics perspective is, when we take natural numbers and we perform an operation on them, do we always get a natural number? So, if we add two natural numbers, do we get a natural number? If we subtract a number from another, we get a natural number? If we multiply them, do we get a natural number? If we divide one by another, do we get a natural number?

So, the first operation which fails this test is subtraction because if we subtract a larger number from a smaller number, so supposing we take 6 and subtract it from 5; then we go below 0 right. If you have 5 things and we take away 6 things, we will be cannot take away 6 things that is what subtraction means. So, we need to expand the scope of our numbers to allow these operations to work sensibly and this is how we get the negative numbers.

So, we had the positive numbers 0, 1, 2 the non-negative technically because 0 is neither positive nor negative. So, we had the positive numbers 1, 2, 3, 4. We added a 0 to account for the fact that we are counting nothing and now we add symmetrically on the other side negative numbers -1, -2, -3. So, this is just to illustrate why we get them of course, this is something that you should know from school.

So, this set which is the natural numbers extended with a negative numbers is what we call the integers and we use Z to indicate the set of integers. So, we have N the set of natural numbers which starts at 0 and goes forward 0, 1, 2, 3, 4 and we have the integers which start at no at minus infinity and go to plus infinity. So, these are both infinite sets, but the natural numbers have a starting point 0 and the integers extend to infinity in both directions.

So, it is very convenient mentally to think of the integers as forming this kind of a sequence where on the left you have the very small ones and on the right you have the very long ones and this is normally called the number line. So, as you go from left to right, the numbers are increasing and this is how the integers are arranged.

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So, we said that subtraction takes us away from natural numbers and we brought the integers. So, now, let us look at the other two operations that we talked about multiplication and division. So, let us start with multiplication. So, when we say 7 × 4 what we are really saying is take 7 objects and make 4 copies of them. So, for instance on the right, we have those 7 balls that we started with and then we have made 4 copies of them. So, if we want to know how many balls are here, then we have 7 from the first group, 7 from the second group and so on. So, we have 4 groups of 7 and this is if we add it up going to be 28.

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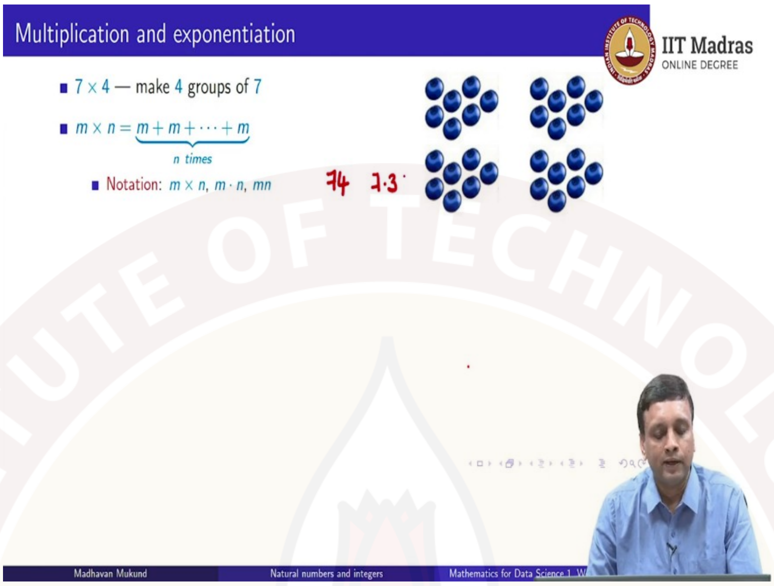

So, in general this is how we multiply when we take a number m and multiply it by n, what we are doing is we are making n copies of m. So, we are taking m + m + m… n times. So, in this sense multiplication is repeated addition.

So, we often use this time sign the × sign for multiplication, but this is often cumbersome when we write out equations. So, sometimes we replace this time sign by a . and sometimes we write nothing at all. So, if we just write two symbols together, we do not write this normally for numbers because imagine that if I write 7 4 like this, then you do not know whether it is a number 74 or its 7 × 4. So, if we have numbers, we will normally write a dot explicitly between them like 7 × 3. But when we have a names like m or n standing for numbers, then if we write mn; we assume that it is one number m multiplied by another number n.

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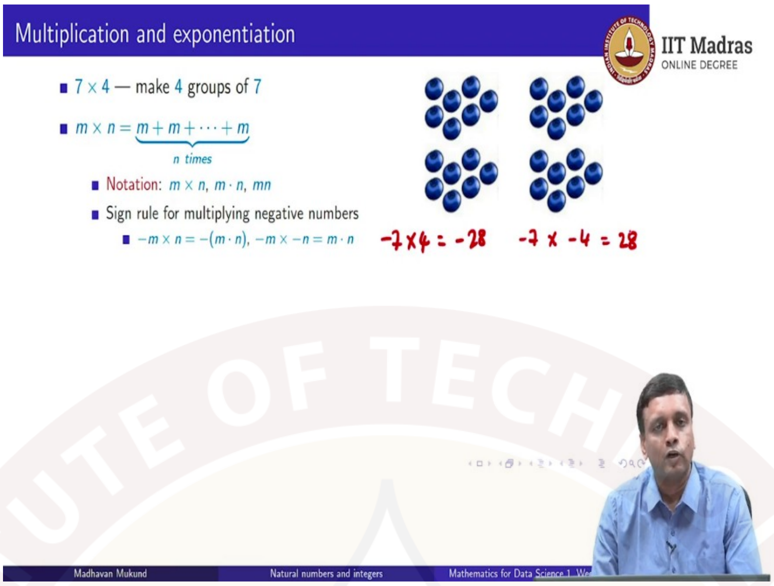

Now, we have integers, an integers have signs they are positive and negative numbers. So, we have to remember that when we multiply numbers with signs, the resulting number also has a sign and there is a sign rule which basically says that if we have one negative number multiplied by one positive number, then the result is a negative number. So, let us assume that m is a positive number so, - m is a negative number. So, say - 7 × 4 would be - 28. On the other hand if I take (- 7) × (- 4), then the two negations will cancel, and I will get 28.

So, if you have an even number of minus signs, you get a positive number; if you have an odd number of minus signs, you get a negative number.

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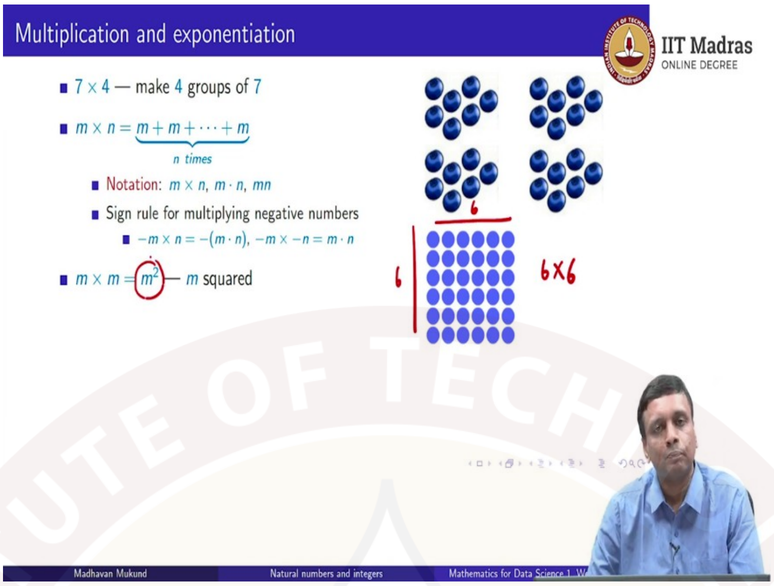

Now just like we have repeated addition, we can also do repeated multiplication. So, instead of doing m plus m, we can takes m times m and this is called m squared and the reason that it is called m squared is visible in the picture here. So, we have now here 6 balls and 6 balls. So, we have 6 × 6 right. So, this means that we can arrange these 6 times 6 balls in a square and this is why we call this square.

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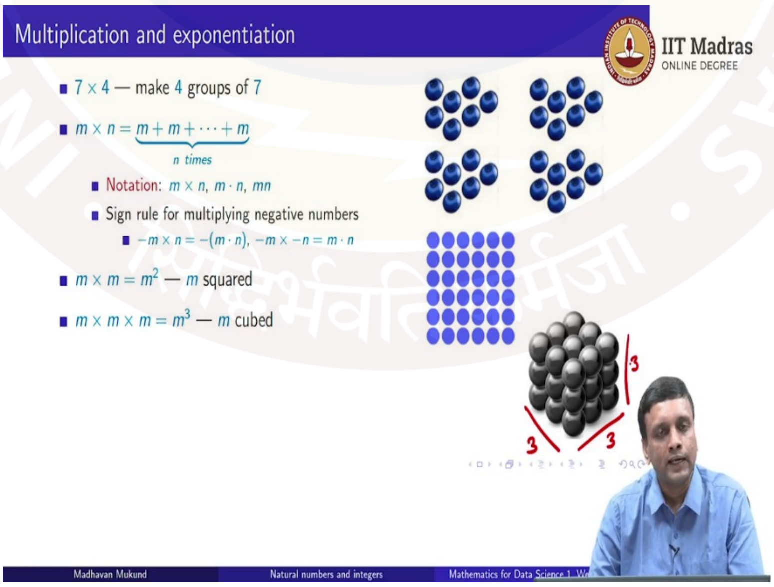

So, this notation m^2 stands to the fact that, m is multiplied by itself twice. Now if you multiply it by self 3 times, then we get a cube. So, here for instance we have 3 balls by 3 balls and then we have a height a stack of 3 such balls. So, we have a square of 3 by 3, 9 balls and we have 3 stacks of these one on top of the other. So, this naturally forms a cube so, m × m × m is written m^3.

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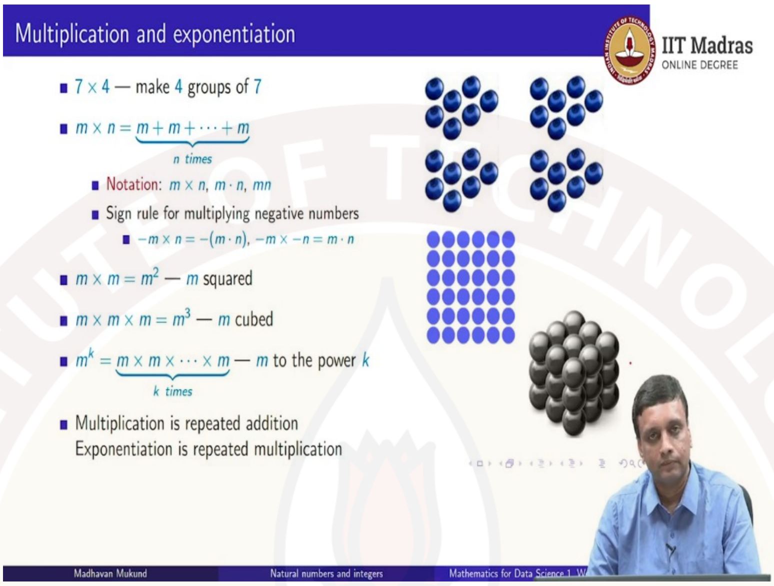

Now, unfortunately we live in a 3-dimensional world and we cannot imagine objects which have more than 3 dimensions. So, our vocabulary stops with cube. So, in general if we have m^k, then we write m × m × m…, k times and we just say it is m^k, we do not have a fancy name for it. We just say it is the kth power of m ok. So, to emphasize multiplication is repeated addition and exponentiation as we have seen is repeated multiplication.

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So, now let us come to division. So, you would have seen this familiar problem in school. You have a certain number of objects and you want to divide them among certain number of people. So, for example, supposing you have 20 mangoes and you want to give them to 5 friends. So, how many mangoes does each friend get?

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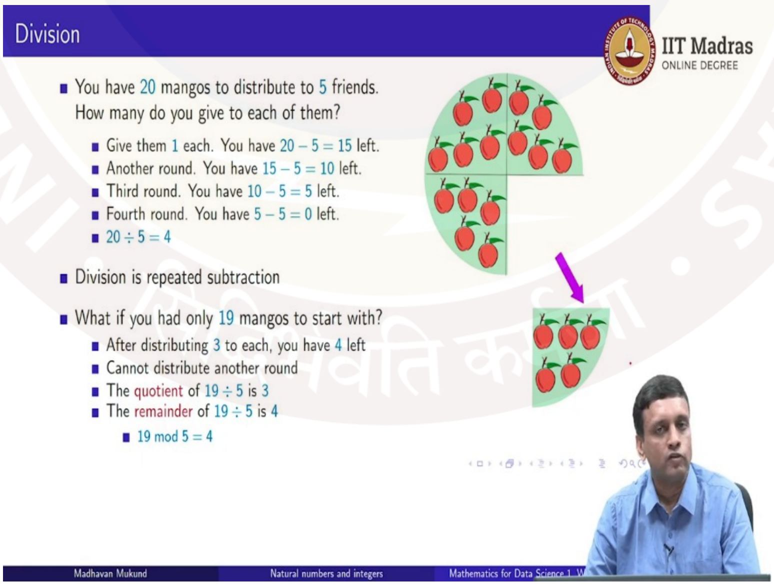

So, here on the right we have this picture and then, what you do is well you start by distributing one mango to each friend right. So, you take out 5 mangoes and you give them to each of your friends. So, now, you have given away 5 mangoes and you have only 15 mangoes left so, you repeat the process. Among the 15 mangoes, you give away 5 to your friends one each and now your 15 mangoes have become 10 and do it one more time and your 10 mangoes have become 5, do it a third time or fourth time rather and the 5 mangoes are now gone.

So, after 4 rounds of distributing mangoes, each time giving one mango each so, 5 mangoes per round, you have got rid of your 20 mangoes so, $20 \div 5$ is 4. So, here as we have illustrated, division is actually repeated subtraction. You keep subtracting by the number you are trying to divide and finally, if you hit 0, then you have divided it exactly.

Well, what if you had only 19 mangoes? Now you know very well that 19 mangoes cannot be evenly divided into 4 into 5 groups. So, if you would start distributing like we had above the first three rounds would go fine; you would come from 19 to 14 from 14 to 9 and then you will come from 9 to 4 and now you have only 4 mangoes left and you have 5 friends so, you cannot give 1 each.

So, we have managed to distribute 3 times and we have 4 left over. So, formally this is written as saying that the quotient the number of times you can actually divide without getting into a fractional part is 3 and the remainder that is after you have a little bit left over which you cannot subtract one more time is the remainder is 4. So, for $19 \div 5$, the quotient is 3 and the remainder is 4.

Now, very often we will need to use this remainder and there is a notation for remainder. So, this is this notation called modulus. So, modulus is another word for remainder and it is written as mod. So, 19 mod 5 is the same as the remainder when 19 is divided by 5. So, instead of saying the remainder of 19 divided by 5 is 4, we will often say 19 mod 5 is 4.

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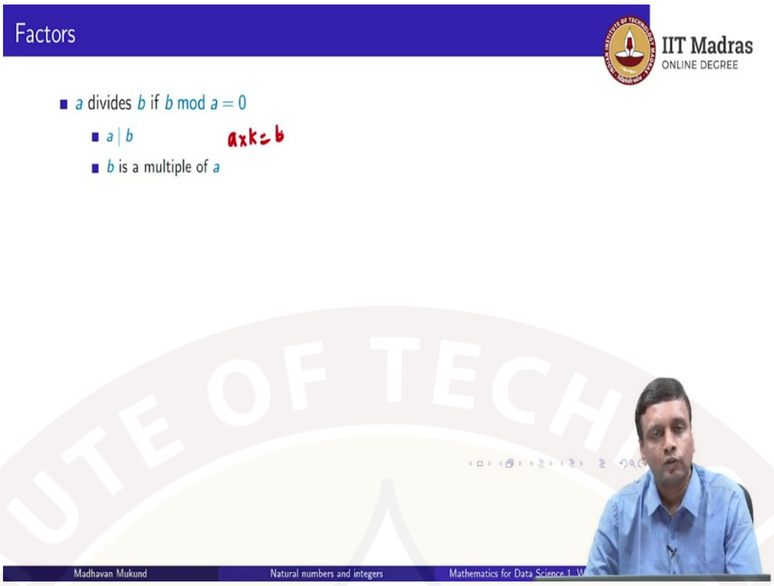

So, with this notation, we can now define what is a factor. So, a factor is a number which divides a bigger number evenly without any remainder. So, a | b, if b mod a is 0. Remember what this mean is means is that if b is divided by a, there is no remainder and we write this with this vertical bar |. So, on the left is the smaller number, on the right is the bigger number. So, a divides b this is what this is supposed to say and the other way of thinking about it is that b is some multiple of a. So, b if a | b then a.k=b ok. So, we have some multiple the some number of times that a goes into b. So, therefore, b is a multiple of a.

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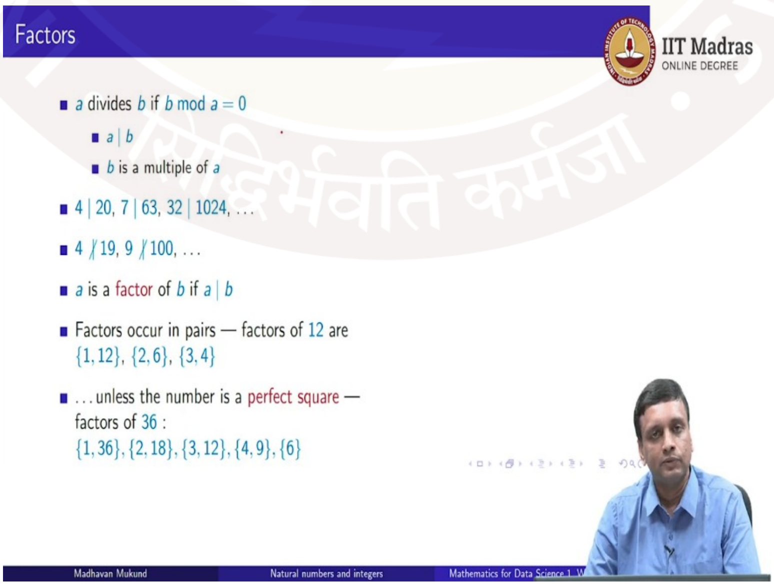

So, here are some examples we have already seen that $4 \mid 20$ because 4×5 is 20, $7 \mid 63$ because 7×9 is 63, $32 \mid 1024$ because 32×32 is 1024 and so on.

Now, the symbol that we use for not being a divisor is just to put a stroke across that vertical line. So, 4 does not divide 19 because there is no way to multiply anything by 4 and get 19. Similarly, 9 does not divide 100 evenly because we get $9 \times 11 = 99$ and then we go 108.

So, we say formally that a is a factor of b if $a \mid b$ right. So, $a \mid b$ is the same as saying that a is a factor of b and it is easy to see that factors must come in pairs because if $a \mid b$ then, a goes into b some k times. So, $k \mid b$ right so, $k \times a = b$ so, both k is a factor and a is a factor. So, for instance, if you take a number 12 then 1 is a factor because 1 divides everything and in fact, for every number n, $1 \times n$ is n so, the pair for 1 is always the number itself.

Now in this case, 12 is divisible by 2 and 2 goes in 6 times. So, the pair 2, 6 form 2 factors 6 times 2 is 12, 2×6 is 12 and similarly 3×4. Now, of course, there is an important side condition which is that sometimes the pair is the same as the number itself and this happens when the number actually happens to be a perfect square that is, it is some number multiplied by itself. So, for instance consider 36 so, 36 is 6×6. So, if you look at the factors of 36 and group them in pairs, then we have 1 and 36, we have 2 and 18, we have 3 and 12, we have 4 and 9 and finally, we have the factor 6, but 6 is multiplied by 6. So, 6 does not produce a new factor as its pair, it is just itself.

So, another way of thinking about it is that, if you have something which is not a square you will have an even number of factors, you will have $2 + 2 + 2 + 2$. If something is a square, you will have an odd number of factors, you will have $2 + 2 + 2$ and finally, when you come to the number of which it is a square that number will come only once in the list of factors.

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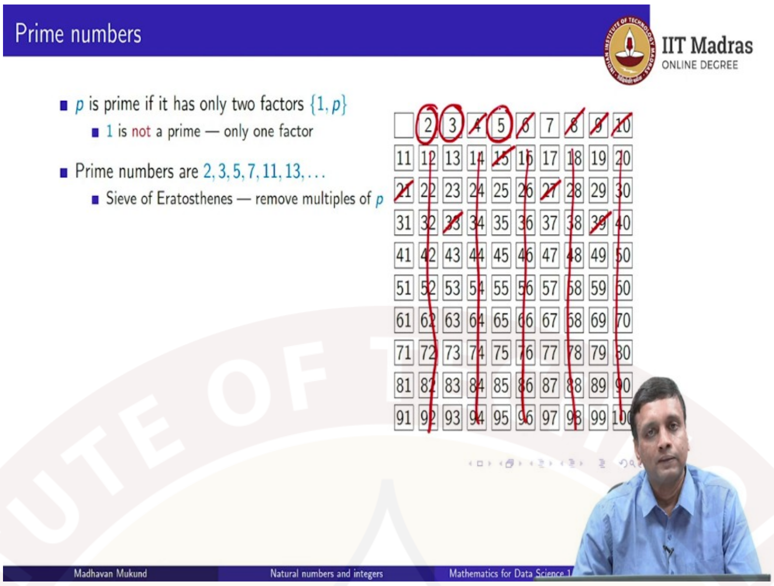

So, once we talk about factors, we come to a very interesting class of numbers which are the prime numbers. So, a prime number is one which has no factors other than 1 and itself. So, 1 is a factor always and $1 \times n$ is n. So, for we try to usually write p for a prime number. So, a prime number has only two factors 1 and p.

Now, it is important that it must have two factors, two separate factors. So, one technically is not a prime because it has only one factor one itself because 1×1 is 1 and so, the only factor that 1 has is 1. So, the smallest prime actually is 2 because it has two factors 1 and itself 2 and no other factors. 3 is also a prime because it has only 2 factors 1 and 3, 2 does not go into 3 and so on.

So, we are all familiar with the smaller prime numbers. So, 2 is the first prime number, 3 is the next prime number, then 5, then 7. Notice that, after 2 no even numbers can be primes because they are all multiples of 2 and so, 2 divides them. Now we come to 9 and 9 is not a prime number because it is a multiple of 3, but 11 is a prime number and so on.

So, there is actually one clever way which is call the sieve of Eratosthenes to generate prime numbers which is whenever you discover a prime, you knock off all the numbers which are multiples of it. So, we can do this for instance to get all the prime numbers from 1 to 100. So, what we do is we first lay out a grid like this right, we know that 1 is not a prime so, the first prime that we have as a candidate is 2 right. So, this is how the sieve of Eratosthenes works, you lay out the numbers in a grid and now we can try and mark off all the prime numbers which are up to 100.

So, we know that 1 is not a prime so, we leave 1 off the grid and we start with 2. So, 2 is our first prime number and what the sieve of Eratosthenes says is you knock off all multiples of 2. So, you knock off all the even numbers and of course, now you can do it in one shot so, you can knock off this whole column, this whole column so, all these numbers are not prime ok.

So, now once you have you have a target so, we are looking only up to 100. So, up to 100 we have knocked off all the powers of 2 or all the multiples of 2. So, now, we look at the first number which is not been marked off and we notice that 3 is a prime because 3 is not yet marked off. So, now, we start mark off multiples of 3, some of them are already marked off because they are multiples of 2. So, 6 is already gone, but 9 is also gone, 12 is already gone, but 15 is also gone and so on.

So, we can mark off all the other multiples of 2 which are not multiples of 3 and so on right. So, we get this kind of a picture and now having done this assuming we have done it all the way, then we will come and find that 5 is a prime right. So, this is the process by which if you want to know count all the primes up to a certain number n, you can write out all the numbers up to n and starting at the left you can take the first unmarked number, call it a prime and mark all its multiples to the right as non primes and the next unmarked number will be the next prime and so on.

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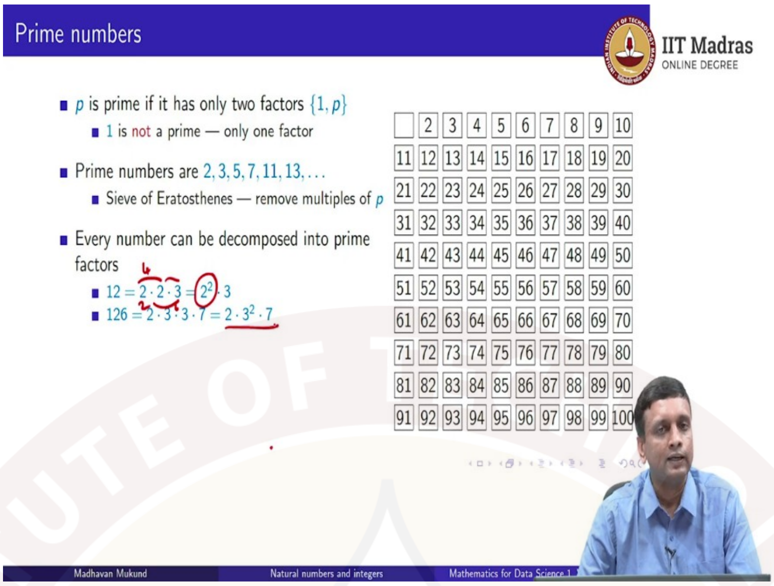

Now, this is not necessarily an efficient way to do the prime numbers, but this is a good way to generate them without missing out any. One of the important facts that we use all the time is that every number can not only be factorized as we have seen into a number of different pairs of factors it can actually factorize uniquely into the prime numbers that form it.

So, for instance if we look at 12, we said that 12 was 2 times 6, it was also 4 times 3, it was 1 times 12 and so on, but fundamentally it has 3 prime factors 2 2 again and 3. So, depending on how we combine them for instance, we can get 4 × 3 or we can get 2 × 6 and so on, but 2 × 2 × 3 is the absolute unique way of writing 12 as a product of prime numbers and using our exponentiation notation, we can condense this and put the 2 2's together and say it is $2^2 \times 3$.

Similarly, if we take a number like 126, then it is 2 × 3, 6 × 3, 18 ×7 ok. So, the prime factors are precisely 2 3 twice and 7 and we can write this as $2 \times 3^2 \times 7$. So, this is very important because we use it implicitly along a lot and we will see later how we use this.

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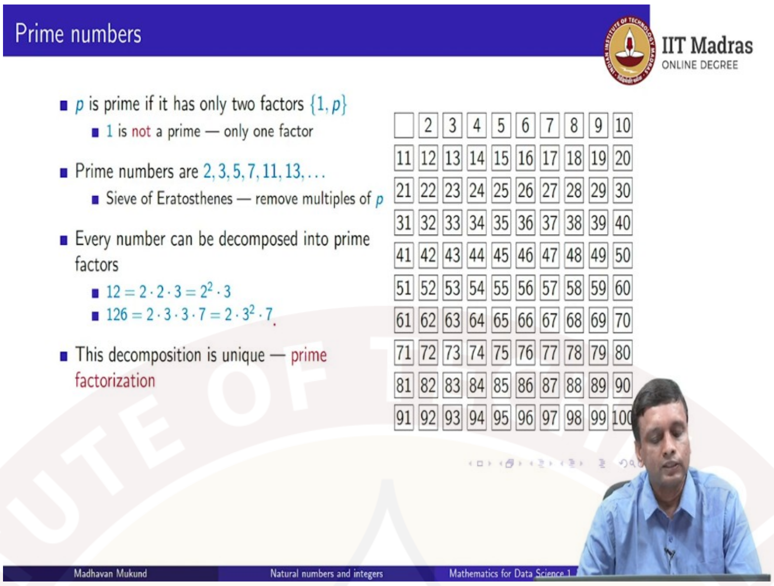

So, this is called the prime factorization right. So, every integer can be decomposed into a product of primes in a unique way.

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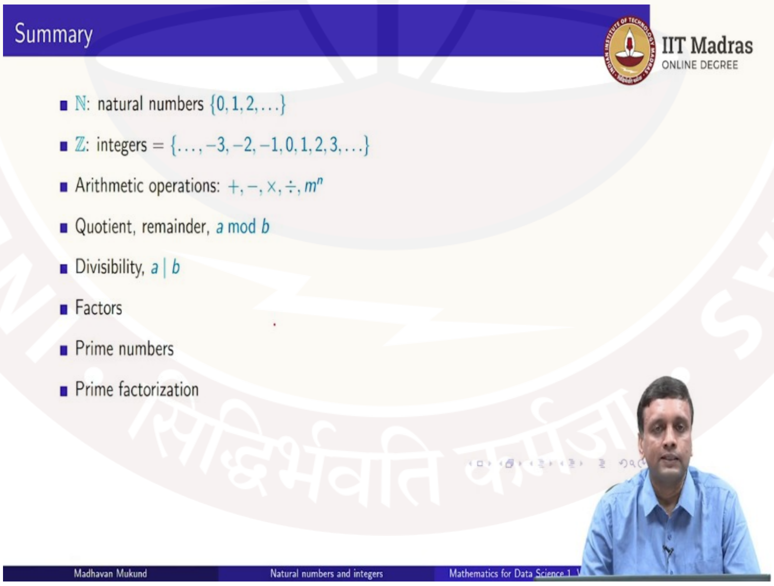

So, to summarize we started with a natural numbers which we use for counting which are the numbers 0, 1, 2, 3, 4 and so on. Then, we extended these numbers with a negative numbers and this gave us the set of integers. So, the integers include all the natural numbers as well as the negative numbers 0, 1, 2, 3 and so on -1, -2, -3 and so on. We saw some basic arithmetic operations on these the usual addition, subtraction, multiplication, division and exponentiation.

We also looked at what happens when we divide integers and we do not want to look at fractions, then we talk about the quotient which is the integer number of times that the dividend goes into the number and the remainder is also written as a mod b. So, using this notation of a mod b, we can talk about divisibility which we write with a vertical bar. So, a | b if a mod b is 0. So, the factors of a number are those numbers which divide it and a prime number has exactly two factors 1 and itself and we can always decompose any integer uniquely into the list of factors, prime factors that multiply out to form that number.