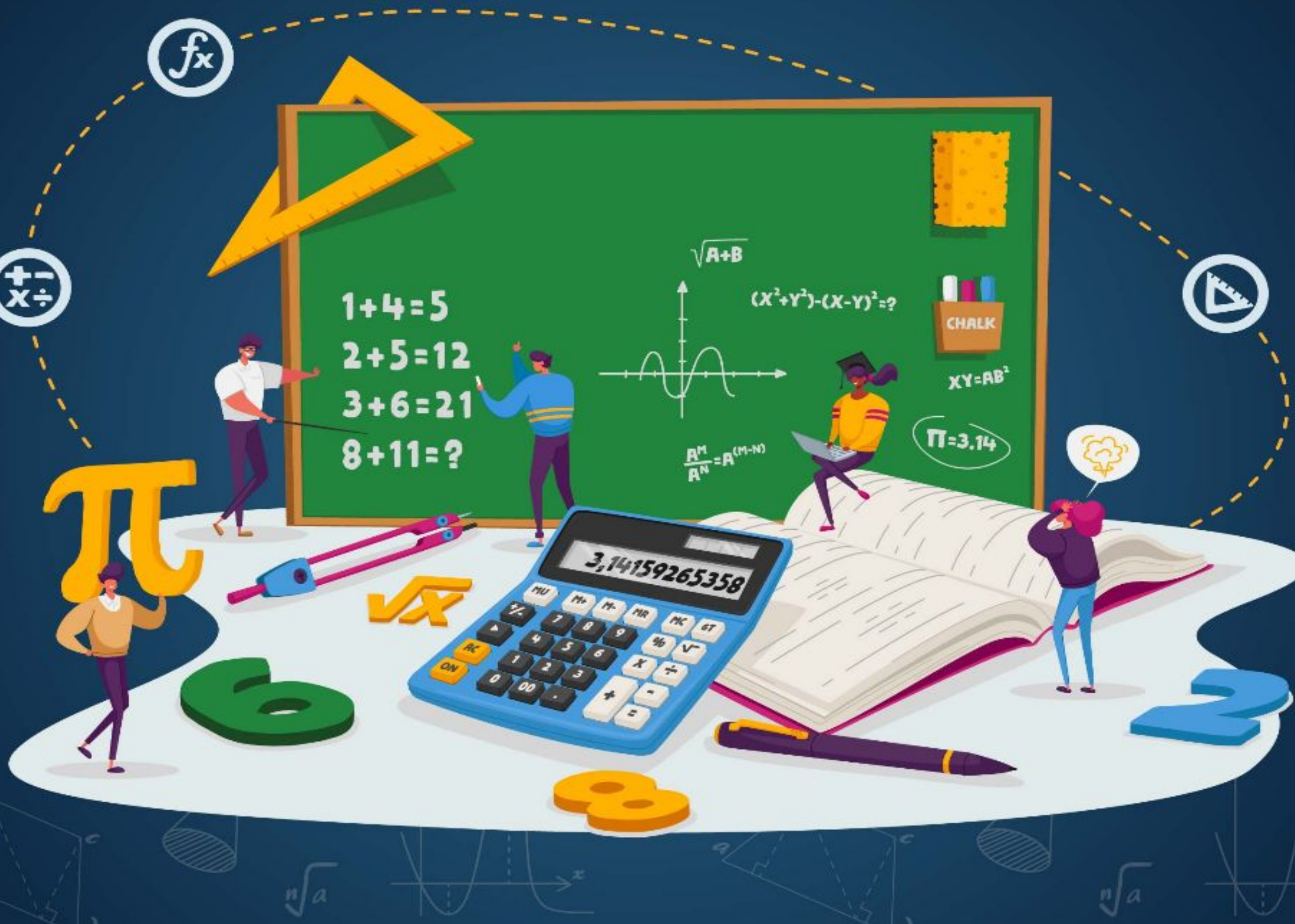
IIT MADRAS BS DEGREE



MATHEMATICS

Mathematics for Data Science

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Calculus of one variable

Contents

1. Limit and Continuity

"Nihil est sine ratione. [There is nothing without a reason.]"

— Gottfried Wilhelm Leibniz

1.1 Introduction

In school time, we usually deal with Real numbers, Integers and Natural numbers, etc. and deal with operations like; addition, subtraction, multiplication and division. We have familiar with sets, different types of relations, polynomial and functions which are nothing but a specific type of relations. In this section, we will see a brief introduce of functions and some type of functions. And then we will know about curves and tangent of curves, and slowly move to an abstract notion which is sequences and limit of sequences and broader notion, that is, limit of functions and continuity of functions.

1.2 What is a function?

As we know, a **relation** R from set A to set a B is a subset of $A \times B$, where $A \times B$ is the Cartesian product of A and B. The sets A and B are called domain and codomain of R, respectively. If R is a relation on $A \times B$, we use the notation aRb to denote that $(a, b) \in A \times B$.
(**Note:** aRb is pronounced "a is related to b".)
A function is a special kind of relation. Abstractly it can be thought of as a machine which produces an output for a given input.

Definition 1.2.1. Let f be a relation on $A \times B$. We say f is a **function** from A to B (denoted by $f : A \to B$) if every for $a \in A$, there is exactly one $b \in B$ such that $(a, b) \in f$. If $(a, b) \in f$, then we write $f(a) = b$, that is, b is the image of a under f.

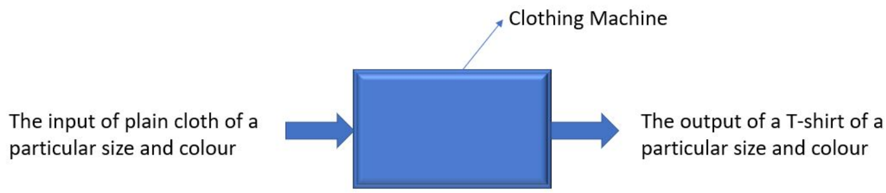

Figure 1.1:

Let $f : A \to B$ be a function. Then

(1) The set A is called **domain** of f, which is denoted by domain(f).

(2) The set B is called the **codomain** of f.

(3) The **range** or **image** of f is the set

$$\{f(a) \,|\, a \in A\} = \{b \in B : b = f(a) \text{ for some } a \in A\} \subset B,$$

which is denoted by range(f) or im(f).

(**Note:** All functions are relations but all relations are not functions.)

Example 1.2.1. The relation $R = \{(1, a), (2, b), (3, a), (1, b)\}$ from $A = \{1, 2, 3\}$ to $B = \{(a, b)\}$ is not a function as the element $1 \in A$ has two different images a and b in B. This violates the one input-one output property of function.

Example 1.2.2. (1) $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = x^2$, is a function. Both domain and codomain of f are $\mathbb{R}$. The range of f is

$$\{x^2 \,|\, x \in \mathbb{R}\} = \{x \geq 0 \,|\, x \in \mathbb{R}\}.$$

(2) $h : \mathbb{R} \to \mathbb{R}$ such that $h(x) = x^3 + 5$, is also a function. The range of h is $\mathbb{R}$ (Hint: Use intermediate value theorem).

1.2.1 Graph of functions

In school time, as you used to draw (graphs) the line, circle, ellipse, or parabola etc. geometrically on your note book. It is the very similar case that if we want to draw the graph of a function ($f(x)$) on your note book then we will collect the inputs (i.e. values of x) and corresponding outputs (i.e. value of $f(x)$) and drawing the

$X-$axis and $Y-$ axis and indicate the points $(x, f(x))$ in the $XY-$ plane and draw the rough diagram of the function which is approximately graph of the function. If we want to see it algebraically, then the graph of a function is noting but the set "$\{(x, f(x)) \mid x \in D\}$, **where D is the domain of function f.**"

Definition 1.2.2. Let $f : X \to Y$ be function. Then the graph of the function is the subset

$$\lceil(f) = \{(x, f(x)) \mid x \in X\} \subset X \times Y$$

Example 1.2.3. Let $f(x) = 7x + 2$ then graph of the function is the set

$$\lceil(f) = \{(x, 7x + 2) \mid x \in \mathbb{R}\}$$

1.2.2 Types of functions

There are several types of functions which we will see in a course of calculus. The most common types of functions include:

- Linear function: $f(x) = ax + b$, where $a, b \in \mathbb{R}$ and $a \neq 0$.
- Quadratic function: $f(x) = ax^2 + bx + c$, where $a, b, c \in \mathbb{R}$ and $a \neq 0$.
- Polynomial function of degree n: $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$, where $a_n \neq 0$, and $a_i \in \mathbb{R}$ for all $i \in \{0, 1, 2, \ldots, n\}$.
- Exponential function: $f(x) = a^x$, where $a \in \mathbb{R}$ and $a > 0$.
- Logarithmic function: $f(x) = log_a(x)$, where a is a positive real and $a \neq 1$. The domain is the set of all positive real numbers.
- Trigonometric functions: $\sin x, \cos x, \tan x$ etc.
- Step functions:

  i) Floor function: floor function is denoted by $f(x) = \lfloor x \rfloor$= greatest integer value less than or equal to x.

  Example 1.2.4. Consider the floor function in the domain [-1, 2]

  $$\lfloor x \rfloor = \begin{cases} -1 & \text{if } -1 \leq h < 0 \\ 0 & \text{if } 0 \leq h < 1 \\ 1 & \text{if } 1 \leq h < 2 \end{cases}$$

  ii) Ceiling function: ceiling function is denoted by $f(x) = \lceil x \rceil$= smallest integer value greater than or equal to x.

Example 1.2.5. Consider the ceiling function in the domain (-1, 2)

$$\lceil x \rceil = \begin{cases} 0 & \text{if } -1 < h \leq 0 \\ 1 & \text{if } 0 < h \leq 1 \\ 2 & \text{if } 1 < h < 2 \end{cases}$$

- Absolute value function: It denoted by $f(x) = |x|$;

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

1.2.3 Bounded function

Definition 1.2.3. Let $f : D \to \mathbb{R}$, where $D \subset \mathbb{R}$. We say f is **bounded** if there are numbers m and M such that $m \leq f(x) \leq M$ for all $x \in D$. A function that is not bounded is said to be **unbounded**.

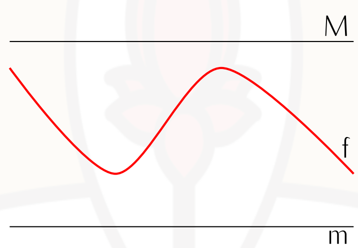

Figure 1.2: **Bounded**

Example 1.2.6. (1)

$$f : \mathbb{R} \to \mathbb{R}$$
$$f(x) = \frac{1}{x^2 + 1}$$

We want to check whether the above function is bounded or not. Let x be a real number. Then

$$x^2 + 1 \geq 1$$
$$\implies m = 0 \leq f(x) = \frac{1}{x^2 + 1} \leq M = 1.$$

We have two numbers $m = 0$ and $M = 1$ such that $m \leq f(x) \leq M$ for all $x \in$ domain$(f) = \mathbb{R}$. Therefore f is a bounded function.

(2)
$$f : (0, \infty) \to \mathbb{R}$$
$$f(x) = \frac{1}{x}$$

The function f is not bounded in $(0, \infty)$. We will show this by method of contradiction. Suppose there exists a $M > 0$ such that $f(x) \leq M$ for all $x \in (0, \infty)$. Take $x = \frac{1}{M+1} \in (0, \infty)$, then $f(x) = M + 1 > M$, which is a contradiction. Therefore f is an unbounded function.

1.2.4 Monotonicity of functions

Definition 1.2.4. (1) A function f is called **increasing** if $x \leq y$ then $f(x) \leq f(y)$, for all $x, y \in \text{domain}(f)$.

(2) A function f is called **decreasing function** if $x \leq y$ implies $f(x) \geq f(y)$, for all such $x, y \in \text{domain}(f)$.

Example 1.2.7.
$$f : [0, \infty) \to \mathbb{R}$$
$$f(x) = x^2$$

Pick two elements of $[0, \infty)$, say x and y. If $x \leq y$, then $f(x) = x^2 \leq y^2 = f(y)$. Hence f is increasing on $[0, \infty)$.

Example 1.2.8.
$$f : \mathbb{R} \to \mathbb{R}$$
$$f(x) = 7 - 4x$$

Let $x, y \in \mathbb{R}$. If $x \leq y$, then
$$-4y \leq -4x$$
$$\implies f(y) = 7 - 4y \leq f(x) = 7 - 4x.$$

Hence f is decreasing on $\mathbb{R}$.

Example 1.2.9.
$$f : \mathbb{R} \to \mathbb{R}$$
$$f(x) = |x|$$

- $-3 \leq -2$, but $f(-3) = 3 \geq 2 = f(-2)$. Hence f is not increasing on $\mathbb{R}$.
- $2 \leq 3$, but $f(2) = 2 \leq 3 = f(3)$. Hence f is not decreasing on $\mathbb{R}$.

Therefore $f(x) = |x|$ is neither increasing nor decreasing on $\mathbb{R}$.

(**Note:** It is not easy to check the monotonicity of a function without the concept of differentiability.)

1.2.5 Arithmetic operations on function

As you have seen that we can add, subtract multiply or divide (denominator should not be 0) any two real numbers. Similarly we can define the operations on two functions.
Consider two functions $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$, where $A \subseteq \mathbb{R}$ then follow the following operations on the functions

i) The sum function $f+g$ is defined on the set A as $(f+g)(x) = f(x)+g(x)$

ii) The subtraction of two functions $f-g$ defined on the set A as $(f-g)(x) = f(x)-g(x)$

iii) The product function fg is defined on the set A as $(fg)(x) = f(x)g(x)$

iv) The quotient function $\frac{f}{g}$ is defined on the set A as $\frac{f}{g}(x) = \frac{f(x)}{g(x)}$, where $g(x) \neq 0, \forall x \in A$.

Example 1.2.10. let $f(x) = x^3+5x+1$ and $g(x) = 3x^2+2x+5$ then

$$(f-g)(x) := f(x)-g(x) = x^3-3x^2+3x-4$$

1.2.6 Composition of two functions

Definition 1.2.5. Let $f : D_1 \to \mathbb{R}$, and $g : D_2 \to \mathbb{R}$ be two functions, such that, $range(f) \subseteq D_2$, where $D_1, D_2 \subseteq \mathbb{R}$. The composition of two functions is defined as:

$$g \circ f : D_1 \to \mathbb{R}$$

$$(g \circ f)(x) = g(f(x))$$

Example 1.2.11. Let $f(x) = x^2$ and $g(x) = 3x^3+2x$. Then find $(g \circ f)(x)$.

$$(g \circ f)(x) = g(f(x)) = 3(f(x))^3 + 2f(x) = 3(x^2)^3 + 2x^2 = 3x^6 + 2x^2$$

Question 1. Let $f(x) = \frac{x}{x+a}$, where $x > 0$ and $a > 0$. If $f(f(x)) = \frac{x}{3x+4}$, then find the value of a. [Ans: 2]

1.3 Curve

Suppose there is a temple in your village and you are at home and want to reach to the temple. So the path whatever you followed to reach the temple can think as a curve between home and temple.

" A curve is a figure that is obtained as a path of a moving point."

Example 1.3.1. The graph of function $f(x) = x^2$ represents a curve in $XY-$ plane.

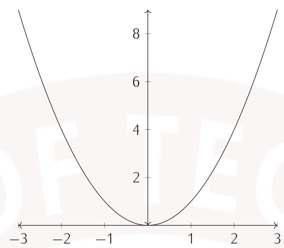

Figure 1.3:

Example 1.3.2. The graph of the floor function $f(x) = \lfloor x \rfloor$ is not a curve in $XY-$ plane because as there are jumps in between the graph itself so this cannot be a path of a moving point.

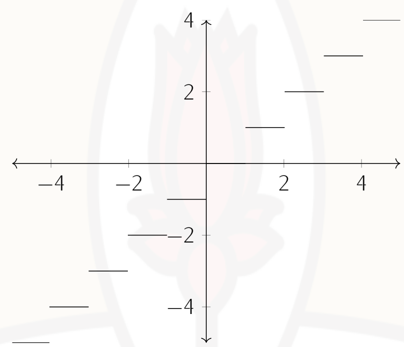

Figure 1.4:

1.3.1 Tangent to a curve

A tangent line to a curve C at a point P (on C) is a line represented the instantaneous direction in which the curve C moves at a point P.
Let's understand it via an example:
Suppose you and your friend are riding bikes on a hilly road and you both are at a distant, and want to meet. There is a very sharp turn on the road. If you both didn't take a break then you and your friend may fall down in the trench. Now think, hilly road as a curve and you are moving you pen along the curve very fast (one by one either side as the case of riding) and sharp turn is denoted by the point P on the curve (Figure 1.5).

Then at point P, your pen draw two straight lines (because of instantaneous direction and which touche the curve only at point P) one by one in different directions (direction means slope). What it means? It means when your pen is at the point P you have two directions to move (two different lines of different slopes) instantly. But see the case at the point Q, the possible lines at the point Q (because of instantaneous direction) which touches the curve is only one and which is unique also.

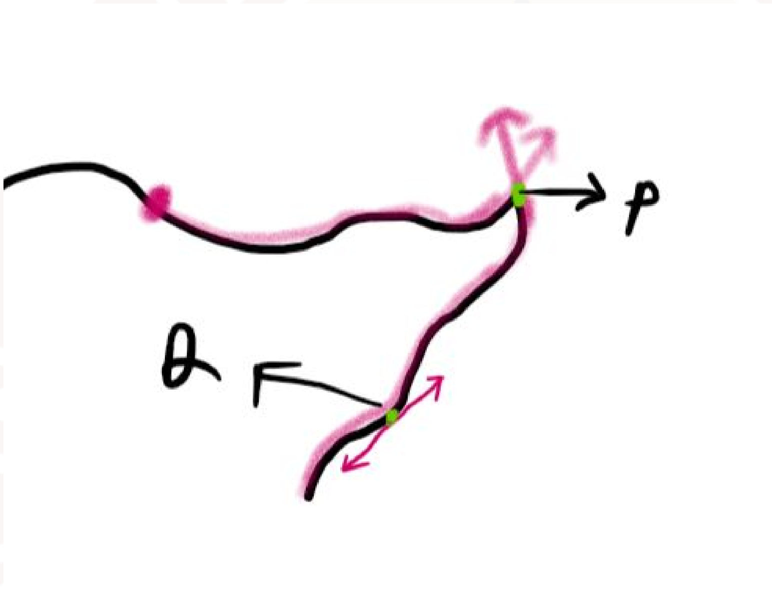

Figure 1.5:

Often, **A line which touches a curve at a point then the line is called tangent line to the curve at that point.**

Now, back to our example, that line which touches the curve at point Q is called tangent line to the curve at point Q. But the case at point P, there is no unique line i.e. there are two different lines of different slopes touche the curve at the point P. So in that case, there is no tangent line to the curve at point P.

Lets see some other identification of the tangent line to a curve at a point:

Tangent lines - Some means of identification: A tangent line to a curve C at a point p (on C) often has the property that it passes through the point p but does not intersect the curve C, in any other point close to the point p, and the lines parallel and close to it either do not intersect C close to p, intersect the curve C in two or more points close to p.

Note: Tangent (line) to a function: Let $f : D \to R$ be a function, where D is a subset of $\mathbb{R}$. Assume that $\lceil(f)$, the graph of the function is a curve. Then the tangent (line) to the function f at $a \in D$ is the tangent (line) to the graph of the function at the point $(a, f(a))$

Example 1.3.3. Let C be a curve in the following Figure 1.6

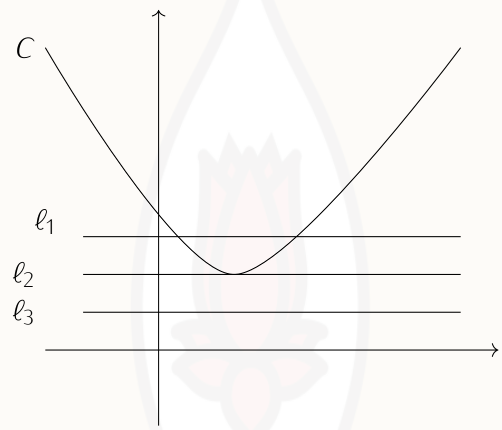

Figure 1.6:

Then from the lines ℓ_1, ℓ_2 and ℓ_2 which one is the tangent at some point to the curve C?
As you can see that in the diagram, instead of touching the curve C, line ℓ_1 is intersecting the curve at two points and line ℓ_3 passes for away from the curve and not even intersecting i.e., there does not exist any point on the curve C for which ℓ_1 and ℓ_3 satisfy the properties of tangent to the curve C at that point. And only the line ℓ_2 touches the curve C once at a particular point. Hence, ℓ_2 is the tangent to the curve C.

Example 1.3.4. Let $f(x) = \lfloor x \rfloor$ then is tangent possible to the function at point $x = 2$ and the point $x = 3.5$? (See the figure 1.7)

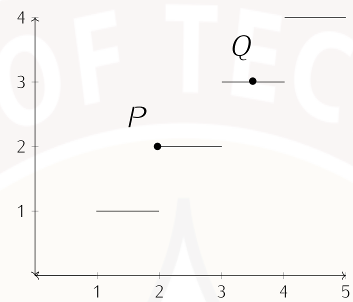

Figure 1.7:

As you can see the in Figure 1.8, at point $x = 2$ (i.e. (2, 2), denoted by P),there are two ways, we can say there is no tangent. Fist, you can see in neighbourhood points for $x = 2$, there is a jump in the graph of the function itself, in such type of case we can not talk about tangent at $x = 2$. Second, you can see in Figure 1.8, there are many lines which are touches the graph of the function at point (2,2) i.e., there is no unique line which only touches the graph of the function at (2,2) (i.e., we can not fix the instantaneous direction of the particle which is moving along the graph of the function in very small neighbourhood points for $x = 2$.)

At point $x = 3.5$ (i.e., (3.5, 3), denoted by Q), if a particle is moving along the graph of the function, the instantaneous direction of the particle at point (3.5, 3) is the line itself $y = 3$ i.e., unique line $y = 3$ (you can observe that the tangent line at a point of a straight line is the line itself which is the similar case here.). So at $x = 3.5$ tangent line is possible.

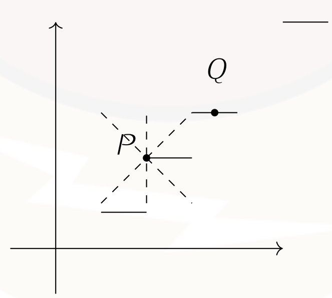

Figure 1.8:

1.4 What is a sequence?

As we have seen the the end behavior of the polynomials or number of turning points of a polynomial which are almost depend on coefficient of the highest degree term and degree of the polynomial. Similarly, we can not bound our self to only on polynomials, we have to find some tools so that we can find the behavior of any function, not only end behavior of the function, also at a particular point; which can be known using some different tools, like; existence of limit of the function at the particular point, continuity, and differentiability which we will learn later.
Now, to know the limit of the function, continuity or differentiability at a particular point, we can use mathematical definition but before to know these concepts there is a concept " limit of a sequence" which is very much helpful to learn the above concepts.
As we have gone through the definition of a function $f : A \to B$, where domain of the function is the set A and codomain of the function is the set B.

Similarly, **real sequence (f_n) is a function which have domain as a Natural number $(\mathbb{N})$ (starting from 1) and codomain is a real number $(\mathbb{R})$ $(f : \mathbb{N} \to \mathbb{R})$. Symbolically, we denote a sequence as $\{f_n\}, n \in \mathbb{N}$.**

Example 1.4.1. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined as

$$f(x) = \frac{x+5}{x^2+2}$$

Then we can think a corresponding sequence $\{f_n\}$ as $f : \mathbb{N} \to \mathbb{R}$ and defined as

$$f_n = \frac{n+5}{n^2+2}$$

Here, $f_1 = \frac{1+5}{1^2+2} = 2, f_2 = \frac{2+5}{2^2+2}, \ldots$

Example 1.4.2. $\{a_n\}$, where $a_n = 1 - \frac{1}{n^2}, n \in \mathbb{N}$ is a sequence.

1.4.1 Limit of a sequence

Let's start this section with an interesting example to understand the idea of limit of sequence.

Example 1.4.3. A scientist creates a human robot to handover the human robot to the Indian Army. In physical test, the capacity to perform long jump of the robot in n^{th} try is given by the formula $(3 - \frac{1}{n})$ (in meter, $n \in \mathbb{N}$) i.e., in first try, robot can jump (3- 1) meter, in second try, can jump $(3 - \frac{1}{2})$ meter and in third try, can jump $(3 - \frac{1}{3})$ meter.

Question 2. Will the Indian Army pass the human robot?

As you can see that if robot tries 1000 times still it perform $3 - \frac{1}{1000}$ meter long jump which is less than 3.
Since we know that the real number $\frac{1}{n}, n \in \mathbb{N}$ is greater 0 but less than 1 for any natural number and as you put a very big natural number in $\frac{1}{n}$, the corresponding real number will be very small (saying to very close to 0). In other word we can say as n tends to infinity (means very big natural number which we can not imagine) the real number $\frac{1}{n}$ tends to 0 (saying to very small real number which is very close to 0). (Symbolically, we denote it as $\lim\limits_{n\to\infty}(\frac{1}{n}) = \mathbf{0}$)
So we can say that the number $3 - \frac{1}{n}$ will never cross to 3 (means, will never be grater than 3) for any natural number.
So can we say, limit of the robot to perform long jump is at most 3 meter (if it tries infinitely many times)?
So Indian Army will never pass the human robot as it does not pass the minimum criteria for long jump (as if robot tries infinitely many times still long jump distance will not be grater than 3 meter.)

Definition 1.4.1. Let $\{a_n\}$ be a sequence of real numbers. We say that $\{a_n\}$ has limit $a \in \mathbb{R}$ if as n increase, the number a_n come closer and closer to a (unique).

Other equivalent terminology

1. $\{a_n\}$ tends to a
2. $\{a_n\}$ converges to a
3. $\lim\limits_{n\to\infty} a_n = a$
4. $a_n \to a$
5. $\lim a_n = a$

Definition 1.4.2. A sequence is $\{a_n\}$ is called convergent if it converges to a limit (i.e a real number).
A sequence is $\{a_n\}$ is called divergent if it is not convergent.

Example 1.4.4. Is sequence $\{n\}$ not convergent?
In this sequence, if we substitute a big natural number, we get the same big natural number. If we substitute different big natural number we get, in out put, the same big different natural number that means as n in increasing sequence does not come closer to any real number and we can conclude sequence $\{n\}$ is not convergent or sequence is divergent.

Example 1.4.5. Is sequence $\{a_n\}$ convergent, where $a_n = \frac{1}{n^2}, n \in \mathbb{N}$?

We know that $n^2 > n$ is true for any natural number $n \in \mathbb{N}$ and we have seen in the scientist example as n tends to big natural number $\frac{1}{n}$ comes closer and closer to 0. This is the same case for the sequence $\frac{1}{n^2}$. And so sequence $\{a_n\}$ is convergent and limit of the sequence of sequence is 0.

Example 1.4.6. Is sequence $\{a_n\}$ convergent, where $a_n = (-1)^n, n \in \mathbb{N}$?

$$a_n = \begin{cases} 1 & \text{if } n \text{ is an even number} \\ -1 & \text{if } n \text{ is an add number} \end{cases}$$

In this sequence, as n is increasing sequence $\{a_n\}$ does not come closer to a unique real number so this sequence is not convergent or this sequence is divergent.

Example 1.4.7. Is sequence $\{a_n\}$ convergent, where $a_n = \frac{n+1}{n}$?

We can write

$$a_n = \frac{n+1}{n} = 1 + \frac{1}{n}$$

and $\frac{1}{n} \to 0$ as $n \to \infty$.
And so we can conclude that $a_n \to 1$ as $n \to \infty$

1.4.2 Subsequence

Consider a sequence $\{a_n\}$. We can write sequence as $\{a_n\} = \{a_1, a_2, a_3, \ldots\}$. A sequence $\{b_n\} = \{b_1, b_2, b_3, \ldots\}$ is said to be subsequence of the sequence $\{a_n\}$ if the set $\{b_1, b_2, b_3, \ldots\}$ picked out in any way from the set $\{a_1, a_2, a_3, \ldots\}$ (but order of the term (elements) should be preserved for example if $b_1 = a_3$ then next b_2 will be after the term a_3 , like $b_2 = a_5$, but it can never be a_1 or a_2).

Theorem 1.4.1. *A sequence $\{a_n\}$ is converges to a limit a if and only if its every subsequence converges to a.*

Example 1.4.8. Sequence $\{a_n\}$ such that $a_n = 5n^2 + 1$. Then subsequences of the sequence $\{a_n\}$ can be

- $\{b_n\} = \{5 \times 2^2 + 1, 5 \times 4^2 + 1, 5 \times 6^2 + 1, \ldots\}$ or $b_n = 5(2n)^2 + 1$ or $\{b_n\} = \{a_2, a_4, a_6, \ldots\}$
- $\{c_n\} = \{5 \times 1^2 + 1, 5 \times 3^2 + 1, 5 \times 5^2 + 1, \ldots\}$ or $c_n = 5(2n-1)^2 + 1$ or $\{c_n\} = \{a_1, a_3, a_5, \ldots\}$

- $\{d_n\} = \{5\times1^2+1, 5\times7^2+1, 5\times8^2+1, 5\times10^2+1, 5\times25^2+1, 5\times29^2+1\dots\}$ or $\{d_n\} = \{a_1, a_7, a_8, a_{10}, a_{25}, a_{29}, \dots\}$

There can be many more subsequences of the sequence $\{a_n\}$ but many times we can not find a proper formula (as the case for the subsequence $\{d_n\}$ of the sequence $\{a_n\}$) for a subsequence. But in the case for the subsequence $\{b_n\}$ or $\{c_n\}$ of the sequence $\{a_n\}$, we got a proper formula $b_n = 5(2n-1)^2+1$ and $c_n = 5(2n-1)^2+1$

Example 1.4.9. Consider a sequence $\{n^2\}$. Which of the following options are subsequences of the given sequence?

- $\{b_n\}, b_n = (n+5)^2$, this is subsequence of sequence $\{n^2\}$ as terms of subsequence are picked from original sequence and order of the term is also preserved.
- $\{c_n\}, c_n = (2n+1)^2$, this is subsequence of sequence $\{n^2\}$ as terms of subsequence are picked from original sequence and order of the term is also preserved.
- $\{w_n\}, w_n = (2n+1)^3$, this is not a subsequence of $\{n^2\}$ as $w_2 = 125$ not a term of the original sequence.
- $\{e_n\}, e_n = n^2$, every sequence is a subsequence itself.
- $\{f_n\}, f_n = (2n)^2$, this is subsequence of sequence $\{n^2\}$ as terms of subsequence are picked from original sequence and order of the term is also preserved.
- $\{g_n\} = \{1, 9, 25, 16, 49, \dots\}$, this is not a subsequence as term are picked from original sequence but order of therms are not preserved.

Question 3. Let $\{a_n\}$ be a sequence converging to the limit 1. Then which of the following option(s) about the mentioned subsequences is(are) true?

[**Hint:** If a sequence $\{a_n\}$ is convergent and limit of sequence is a, then every subsequence of the sequence is also convergent and limit of each subsequence is also a.]

- **Option 1:** $\lim_{n\to\infty} a_{3n} = 1$
- Option 2: $\lim_{n\to\infty} a_{2n} = 2$
- Option 3: $\lim_{n\to\infty} a_{3n+1} = 2$
- **Option 4:** $\lim_{n\to\infty} a_{6n} = 1$

1.4.3 Some useful tools regrading limit of sequence

Consider two convergent sequences $\{a_n\}$ and $\{b_n\}$ such that $a_n \to a$ and $b_n \to b$, then

i) $a_n + b_n \to a + b$

ii) $ca_n \to ca$

iii) $a_n - b_n \to a - b$

iv) $a_n b_n \to ab$

v) $f(a_n) \to f(a)$, where f is any polynomial function.

vi) $\frac{a_n}{b_n} \to \frac{a}{b}$, if $b \neq 0$

vii) $c^{a_n} \to c^a$ i.e., $\lim c^{a_n} = c^{\lim a_n}$, for any real number c, where c^{a_n} is real number for each $n \in \mathbb{N}$.

viii) $log_c(a_n) \to log_c(a)$, if $a_n > 0$ for all $n \in \mathbb{N}$, and $a, c > 0$.

ix) **The sandwich principle:** If $a_n \leq c_n \leq b_n$, for all $n \in \mathbb{N}$, then $c_n \to a$.

Definition 1.4.3. A sequence $\{a_n\}$ is increasing if $a_{n+1} \geq a_n, \forall n$ and a sequence $\{a_n\}$ is decreasing if $a_{n+1} \leq a_n, \forall n$

Example 1.4.10. Sequence $\{n\}$ is an increasing sequence.

Example 1.4.11. Sequence $\left\{\frac{1}{n}\right\}$ is a decreasing sequence.

Definition 1.4.4. A sequence $\{a_n\}$ is bounded above if $a_n \leq M, \forall n$, where $M \in \mathbb{R}$ and M is called upper bound of the sequence $\{a_n\}$. A sequence $\{a_n\}$ is bounded below if $a_n \geq m, \forall n$, where $m \in \mathbb{R}$ and m is called lower bound of the sequence $\{a_n\}$.

Example 1.4.12. Sequence $\left\{1 - \frac{1}{n}\right\}$ is bounded above and upper bound is 1. (check!)

Example 1.4.13. Sequence $\left\{\frac{1}{n}\right\}$ is bounded below and lower bound is 0. (Check!)

Question 4. Which of the following option(s) is(are) true?

- Option 1: $\left\{\dfrac{3n-5}{4n+2}\right\}$ is a decreasing sequence.
- **Option 2:** $\left\{-\dfrac{1}{6n-5}\right\}$ is an increasing sequence.

- Option 3: $\left\{\sqrt{n+1}-\sqrt{n}\right\}$ is an increasing sequence.
- **Option 4:** $\left\{\sqrt{2n}\right\}$ is an increasing sequence.

Theorem 1.4.2. *If a increasing sequence is bounded above then is convergent and if a decreasing sequence is bounded below then it is also a convergent sequence.*

Example 1.4.14. Find the limit of sequence $\{a_n\}$, where $a_n = \frac{2}{n} + \frac{n^3}{3n^4+8}$.

Let

$$b_n = \frac{2}{n} \rightarrow \quad \text{as } n \rightarrow \infty$$

and

$$c_n = \frac{n^3}{3n^4+8} = \frac{n^3}{n^4(3+\frac{8}{n^4})} = \frac{1}{n(3+\frac{8}{n^4})}$$

$$\text{in denominator } \left(3+\frac{8}{n^4}\right) \rightarrow 3 \text{ as } n \rightarrow \infty \text{ and finally } \frac{1}{3n} \rightarrow 0 \text{ as } n \rightarrow \infty$$

Hence $c_n \rightarrow 0$ as $n \rightarrow \infty$.
And so $\lim a_n = \lim b_n + \lim c_n = 0 + 0 = 0$

Example 1.4.15. Find the limit of sequence a_n, where $a_n = 5^{\left(2+\frac{n+1}{n^2-1}\right)}, n > 1$.

Let

$$b_n = 2 + \frac{n+1}{n^2-1} = 2 + \frac{n+1}{(n+1)(n-1)} = 2 + \frac{1}{n-1}$$

and we know $\frac{1}{n} \rightarrow 0$ as $n \rightarrow \infty$
similarly,

$$\frac{1}{n-1} \rightarrow 0 \text{ as } n \rightarrow \infty$$

and so

$$\lim b_n = \lim(2) + \lim \frac{1}{n-1} = 2 + 0 = 2$$

Hence,

$$\lim 5^{\left(2+\frac{n+1}{n^2-1}\right)} = 5^{\lim b_n} = 5^2 = 25$$

Example 1.4.16. If limit of the sequence $\{a_n\}$ is e, where $a_n = n\left(\frac{\sqrt{2\pi n}}{n!}\right)^{\frac{1}{n}}$ Find the limit of sequence b_n, where $b_n = \log n + \frac{1}{n}\log\left(\frac{\sqrt{2\pi n}}{n!}\right)$
Note: Given log is of base 10.

Using the property of $\log(ab) = \log a + \log b$ and $\log a^b = b \log a$

$$b_n = \log n + \frac{1}{n} \log \left(\frac{\sqrt{2\pi n}}{n!} \right) = \log n \left(\frac{\sqrt{2\pi n}}{n!} \right)^{\frac{1}{n}}$$

i.e.,

$$b_n = \log a_n$$

Now, using property $log_c(a_n) \to log_c(a)$, if $a_n > 0$ for all $n \in \mathbb{N}$, and $a, c > 0$.

$$\lim b_n = \lim \log a_n = \log e$$

Note: Here, $b_n = \log n + \frac{1}{n} \log \left(\frac{\sqrt{2\pi n}}{n!} \right)$ and we can not use the property $\lim(a_n + b_n) \to \lim a_n + \lim b_n$ (when $\lim a_n$ and $\lim b_n$ exist) because $\lim \log n$ does not exist.

Example 1.4.17. Find the limit of sequence c_n, where $c_n = \frac{\sin n}{n}$.

We know that

$$-1 \leq \sin n \leq 1, \forall n$$
$$\implies -\frac{1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n}, \forall n$$

let $a_n = -\frac{1}{n} \to 0$ as $n \to \infty$
and $b_n = \frac{1}{n} \to 0$ as $n \to \infty$

We have $a_n \leq c_n \leq b_n$ and $\lim a_n = 0$ and $b_n = 0$ which are the criteria of the sandwich principle. And so we can conclude that $\lim c_n = 0$

Example 1.4.18. Find the limit of sequence c_n, where $c_n = \frac{2}{(n+1)^2} + \frac{2}{(n+2)^2} + \frac{2}{(n+3)^2} + \ldots + \frac{2}{(2n)^2}$

We can write,

$$c_n = 2 \left[\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \frac{1}{(n+3)^2} + \ldots + \frac{1}{(2n)^2} \right]$$

Since

$$\frac{1}{(2n)^2} \leq \frac{1}{(n+1)^2} \leq \frac{1}{n^2}, \forall n \in \mathbb{N},$$
$$\frac{1}{(2n)^2} \leq \frac{1}{(n+2)^2} \leq \frac{1}{n^2}, \forall n \in \mathbb{N},$$

$$\frac{1}{(2n)^2} \leq \frac{1}{(n+3)^2} \leq \frac{1}{n^2}, \forall n \in \mathbb{N},$$

$$\vdots$$

$$\frac{1}{(2n)^2} \leq \frac{1}{(n+n)^2} \leq \frac{1}{n^2}, \forall n \in \mathbb{N}$$

After adding all inequalities according to their sides, we get

$$\frac{2n}{(2n)^2} \leq c_n \leq \frac{2n}{n^2}, \forall n \in \mathbb{N}$$

let $a_n = \frac{2n}{(2n)^2}$ and $b_n = \frac{2n}{n^2}$ and we can write,

$$a_n = \frac{2n}{(2n)^2} = \frac{1}{2n} \to 0 \text{ as } n \to \infty$$

and

$$b_n = \frac{2n}{n^2} = \frac{2}{n} = 0 \text{ or } b_n \to 0 \text{ as } n \to \infty$$

We have $a_n \leq c_n \leq b_n$ and $\lim a_n = 0$ and $b_n = 0$ which are the criteria of the sandwich principle. And so we can conclude that $\lim c_n = 0$

1.4.4 Some important theorem

Theorem 1.4.3. *If* $\lim a_n = \ell$, *then* $\lim \frac{a_1+a_2+\ldots+a_n}{n} = \ell$

Example 1.4.19. Find the limit of sequence c_n, where $c_n = \frac{2}{\sqrt{n^2}} + \frac{2}{\sqrt{n^2+1}} + \frac{2}{\sqrt{n^2+2}} + \ldots + \frac{2}{\sqrt{n^2+n}}$

We can write

$$c_n = \frac{1}{n}\left(\frac{2n}{\sqrt{n^2}} + \frac{2n}{\sqrt{n^2+1}} + \frac{2n}{\sqrt{n^2+2}} + \ldots + \frac{2n}{\sqrt{n^2+n}}\right)$$

Now, let

$$a_k = \frac{2n}{\sqrt{n^2+k}} = \frac{2}{\sqrt{1+\frac{k}{n}}}$$

$$\implies a_n = \frac{2}{\sqrt{1+\frac{1}{n}}} \to 2 \text{ as } n \to \infty$$

So using above theorem,

$$\begin{aligned} c_n &= \frac{2}{\sqrt{n^2}} + \frac{2}{\sqrt{n^2+1}} + \frac{2}{\sqrt{n^2+2}} + \ldots + \frac{2}{\sqrt{n^2+n}} \\ &= \frac{1}{n}\left(\frac{2n}{\sqrt{n^2}} + \frac{2n}{\sqrt{n^2+1}} + \frac{2n}{\sqrt{n^2+2}} + \ldots + \frac{2n}{\sqrt{n^2+n}}\right) \\ &= \frac{\left(\frac{2n}{\sqrt{n^2}} + \frac{2n}{\sqrt{n^2+1}} + \frac{2n}{\sqrt{n^2+2}} + \ldots + \frac{2n}{\sqrt{n^2+n}}\right)}{n} \to 2 \text{ as } n \to \infty \end{aligned}$$

Theorem 1.4.4. *If a sequence $\{a_n\}$ such that $\lim \frac{a_{n+1}}{a_n} = \ell$, where $|\ell| < 1$ then $\lim a_n = 0$ and if $\lim \frac{a_{n+1}}{a_n} = \ell$, where $\ell > 1$ then $\lim a_n = \infty$.*

1.4.5 Exercise

Question 5. Find the limit of sequence $\{a_n\}$, where $a_n = \frac{5+3\sqrt{n}}{\sqrt{n}}$.

Question 6. Find the limit of sequence $\{a_n\}$, where $a_n = 5^{\frac{1}{n}}$.

Question 7. Find the limit of sequence $\{a_n\}$, where $a_n = (\frac{1}{2})^n$.

Question 8. Find the limit of sequence $\{a_n\}$, where $a_n = \frac{(-1)^n}{2n}$.

Question 9. If sequences $\{b_n\}$ and $\{c_n\}$ are such that $b_n \to 1$ and $c_n \to \infty$ as $n \to \infty$, then find the limit of the sequence $\{a_n\}$, where $a_n = \frac{b_n}{c_n}$

1.5 Limit of function

In the previous section, we studied the limit of a sequence. Limits are used to make all the basic definitions of calculus. In this section we will introduce the limit of a function. Roughly, a function f is said to have a limit L at a if the values of $f(x)$ are close to L as x gets close to (but different from) a.

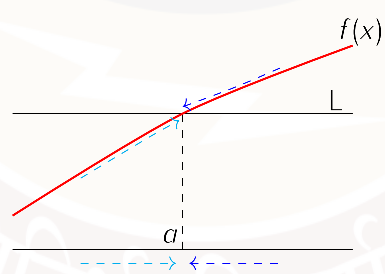

Figure 1.9:

Notice that x can approaches to a from both the directions, the left or the right. No matter from which side x approaches to a, the value of the function $f(x)$ must approach to L. If x comes closer and closer to a only on one side of a, then we get one sided limit.

Definition 1.5.1. (Left (hand) limit) Let $f : D \to \mathbb{R}$ be a function. If there is a real number L_1 such that $f(a_n) \to L_1$ for every sequence $\{a_n\}$ in D with $a_n \to a$ and $a_n < a$, then we say that the **left limit** of f at a exists and equals L_1. In this case we write

$$\lim_{x \to a^-} f(x) = L_1.$$

If there is no such L_1 then we say that the limit of f at a from the left from the left does not exist.

Definition 1.5.2. (Right (hand) limit) Let $f : D \to \mathbb{R}$ be a function. If there is a real number L_2 such that $f(a_n) \to L_2$ for every sequence $\{a_n\}$ in D with $a_n \to a$ and $a_n > a$, then we say that the **right limit** of f at a exists and equals L_2. In this case we write

$$\lim_{x \to a^+} f(x) = L_2.$$

If there is no such L_2 then we say that the limit of f at a from the left from the left does not exist.

Definition 1.5.3 (Limit of a function at a point). Let $f : D \to \mathbb{R}$ be a function. Suppose the limit of f from both the sides exist and are equal to L, that is

$$\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L$$

Then we say that the limit of f at a exist and equals to L, and in that case we write

$$\lim_{x \to a} f(x) = L$$

or

$$f(x) \to L \quad \text{as} \quad x \to a.$$

Otherwise, $\lim_{x \to a} f(x)$ does not exist.

Limit of a function in terms of sequences:

Definition 1.5.4. Let $f : D \to \mathbb{R}$ be a function. If there is a real number L such that $f(a_n) \to L$ for every sequence $\{a_n\}$ in D with $a_n \to a$, then we say that the **limit** of f at a exists and equals L. In this case we write

$$\lim_{x \to a} f(x) = L.$$

(**Note:** Both the definitions(1.5.3 and 1.5.4) are equivalent.)
Now we will work few examples illustrating how to use the definition to compute the limit of a function at a point.

Example 1.5.1 (**A Limit That Exists**).

$$f : [-2, 2] \to \mathbb{R}$$

$$f(x) = x^2$$

We want to compute the limit of f at the point 1, if it exists. The graph of this function is as follows.

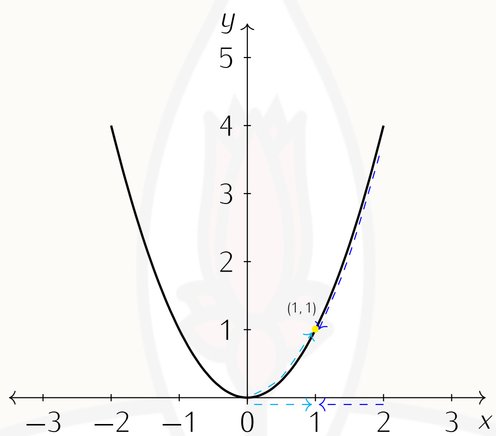

Figure 1.10: $f(x) = x^2$

One can easily see that as x approaches to 1 from the left or the right, the value of the $f(x)$ gets closer and closer to 1. Therefore we can say that $\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^-} f(x) = 1$. Let us do it more rigorously.

For every sequence a_n in domain(f) $= [-2, 2]$ such that $a_n \to 1$, the sequence $f(a_n) = {a_n}^2$ will converge 1. Hence $\lim\limits_{x\to 1} x^2 = 1$.

Example 1.5.2 (**A Limit That Does Not Exist**).

$$f : [-2, 2] \to \mathbb{R}$$

$$f(x) = \lfloor x \rfloor$$

We want to compute the limit of f at the point -1, if it exists. The graph of this function is given below.

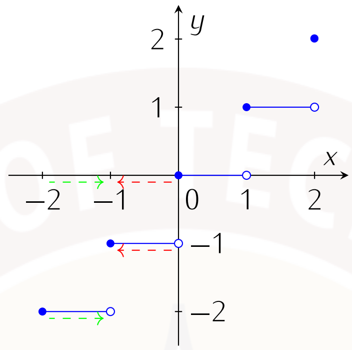

Figure 1.11: $f(x) = \lfloor x \rfloor$

We can see that as x approaches to -1 from the left the value of the $f(x)$ gets closer and closer to -2, that is, $\lim\limits_{x \to 1^-} \lfloor x \rfloor = -2$. On the other hand, as x approaches to -1 from the left the value of the $f(x)$ gets closer and closer to -1, that is, $\lim\limits_{x \to 1^+} \lfloor x \rfloor = -1$. As $\lim\limits_{x \to 1^-} \lfloor x \rfloor \neq \lim\limits_{x \to 1^+} \lfloor x \rfloor$, the limit of f at -1 does not exist.

Sequential argument: Consider two sequences $a_n = -1 + \frac{1}{2n}$ and $b_n = -1 - \frac{1}{2n}$, both a_n and b_n converges to -1. Then we have $f(a_n) = -1$ and $f(b_n) = -2$ for all n. So both the sequences $f(a_n)$ and $f(b_n)$ converges to two different points. Hence, $\lim\limits_{x \to -1} \lfloor x \rfloor$ does not exist.

Example 1.5.3 (**A Limit That Does Not Exist**).

$$f(x) = \begin{cases} 1 & \text{if x is a rational number} \\ 0 & \text{if x is not a rational number.} \end{cases}$$

We want to compute the limit of f at the point $\sqrt{2}$, if it exists. It isn't easy to draw the graph of the above function. We will use a sequential argument to know about the existence of limit.

Consider the sequence $a_0 = 1, a_1 = 1.4, a_2 = 1.41, a_3 = 1.414, \ldots$. The n-th term of the sequence is $\sqrt{2}$ rounded down to n decimal places. Now each term of this sequence is rational and $\lim\limits_{n \to \infty} a_n = \sqrt{2}$. Then we have $f(a_n) = 1$ for all n, by the definition of f. Hence, $\lim\limits_{n \to \infty} f(a_n) = 1$.

Now consider an another sequence $b_n = \sqrt{2} + \frac{1}{n}$. Each term of this sequence is irrational and $\lim\limits_{n \to \infty} b_n = \sqrt{2}$. Then we have $f(a_n) = 0$ for all n, by the definition of f. Hence, $\lim\limits_{n \to \infty} f(b_n) = 1$. So both the sequences $f(a_n)$ and $f(b_n)$ converges to two different points. As a result, $\lim\limits_{x \to \sqrt{2}} f(x)$ does not exist.

Example 1.5.4 (**A Limit That Does Not Exist**).

$$f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$$

$$f(x) = \sin\left(\frac{1}{x}\right).$$

We want to compute the limit of f at the point 0, if it exists. The graph of this function is given below.

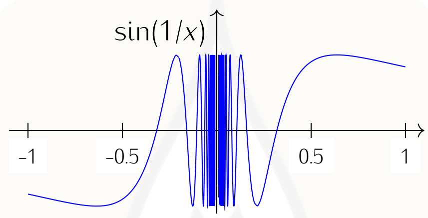

Figure 1.12: $f(x) = \sin\left(\frac{1}{x}\right)$

The above function is not periodic. As x gets closer and closer 0, the function oscillates faster between -1 and 1. Therefore, $\lim_{x\to 0} f(x)$ does not exist.

Sequential argument: Consider two sequences $a_n = \dfrac{1}{\frac{\pi}{2} + n\pi}$ and $b_n = \dfrac{1}{\frac{\pi}{2} - n\pi}$, both a_n and b_n converges to 0. Then we have $f(a_n) = \sin(\frac{\pi}{2} + n\pi) = 1$ and $f(b_n) = \sin(\frac{\pi}{2} - n\pi) = -1$ for all n. So both the sequences $f(a_n)$ and $f(b_n)$ converges to two different points. Hence, $\lim_{x\to 0} f(x)$ does not exist.

(**Note:** The value of a function at a has nothing to do with $\lim_{x\to a} f(x)$. Sometimes, the value (or existence) of $\lim_{x\to a} f(x)$ need not depend on the value (or existence) $f(a)$.

1.5.1 Limit at Infinity:

Definition 1.5.5. Let $f : (a, \infty) \to \mathbb{R}$ be a function for some $a \in \mathbb{R}$. We say f has the limit L as $x \to \infty$, if for every sequence x_n diverging to ∞, the sequence $f(x_n)$ converges to L. In this case we write

$$\lim_{x\to\infty} f(x) = L.$$

Informally it says the values of f(x) become closer and closer to L, as x becomes larger and larger.

Definition 1.5.6. Let $f : (-\infty, a) \to \mathbb{R}$ be a function for some $a \in \mathbb{R}$. We say f has the limit L as $x \to -\infty$, if for every sequence x_n diverging to $-\infty$, the sequence $f(x_n)$ converges to L. In this case we write

$$\lim_{x \to -\infty} f(x) = L.$$

Informally it says the values of f(x) become closer and closer to L, as x becomes smaller and smaller.

Example 1.5.5.

$$f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$$

$$f(x) = \frac{1}{x}$$

Some part of the graph of the above function is given below.

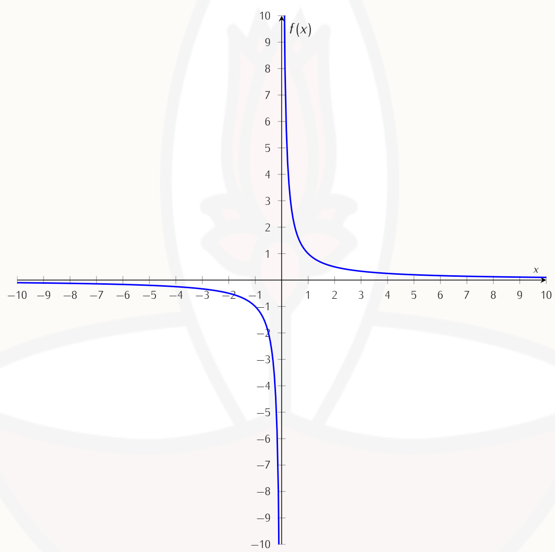

Figure 1.13: $f(x) = \frac{1}{x}$

From the above graph, it is clear that as x becomes larger and larger, the value of the function $f(x) = \frac{1}{x}$ gets closer and closer to 0. Therefore

$$\lim_{x \to \infty} \frac{1}{x} = 0.$$

Similarly one can see that $f(x) = \frac{1}{x} \to 0$ as x diverges to $-\infty$.

Sequential argument:

Recall: If a sequence $a_n (a_n \neq 0$ for all $n)$ diverges to ∞ or $-\infty$ then the sequence $\frac{1}{a_n}$ will always converge to 0.

Consider a sequence a_n such that a_n diverges ∞. Then the sequence $f(a_n) = \frac{1}{a_n}$ converges 0. Hence, $\lim\limits_{x\to\infty} \dfrac{1}{x} = 0$. Similarly if we consider a consider a sequence b such that b_n diverges to $-\infty$, then the sequence $f(b_n) = \frac{1}{b_n}$ will converge to 0. consequently, $\lim\limits_{x\to-\infty} \dfrac{1}{x} = 0$.

Theorem 1.5.1 (Algebra of Limits). *let f and g be functions defined on $D \subset \mathbb{R}$ to $\mathbb{R}$. Suppose a is a real number and $\lim\limits_{x\to a} f(x)$ and $\lim\limits_{x\to a} g(x)$ exist. If $\lim\limits_{x\to a} f(x) = F$ and $\lim\limits_{x\to a} g(x) = G$, then*

(1) $\lim\limits_{x\to a}(f \pm g)(x) = \lim\limits_{x\to a} f(x) \pm \lim\limits_{x\to a} g(x) = F \pm G.$

(2) If $c \in \mathbb{R}$, then $\lim\limits_{x\to a}(cf)(x) = c \times \lim\limits_{x\to a} f(x) = cF.$

(3) $\lim\limits_{x\to a} f(x)g(x) = \big(\lim\limits_{x\to a} f(x)\big)\big(\lim\limits_{x\to a} g(x)\big) = FG.$

(4) If $G \neq 0$, then $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)} = \dfrac{F}{G}.$

Example 1.5.6. Find

$$\lim_{x\to 2} 5x + 9.$$

From Theorem 1.5.1(1) and (2), we have

$$\begin{aligned}
\lim_{x\to 2}(5x+9) &= \lim_{x\to 2} 5x + \lim_{x\to 2} 9 \\
&= 5\lim_{x\to 2} x + \lim_{x\to 2} 7 \\
&= 5 \cdot 2 + 9 \\
&= 19.
\end{aligned}$$

Example 1.5.7. Find

$$\lim_{x\to -3} x^4.$$

From Theorem 1.5.1(3), we have

$$\begin{aligned}
\lim_{x\to -3} x^4 &= \big(\lim_{x\to -3} x\big)^4 \\
&= (-3)^4 \\
&= 81.
\end{aligned}$$

Example 1.5.8. Find

$$\lim_{x\to 5} \frac{25}{x^2}.$$

From Theorem 1.5.1(4), we have

$$\lim_{x\to 5} \frac{25}{x^2} = \frac{\lim\limits_{x\to 5} 25}{\lim\limits_{x\to 5} x^2} = \frac{25}{25} = 1.$$

Example 1.5.9. Find

$$\lim_{x\to 5} \frac{5x^3 + 0.45x^2 - 2x + 100}{x^2 - 5x + 6}.$$

$$\lim_{x\to 0} \frac{5x^3 + 0.45x^2 - 2x + 100}{x^2 - 5x + 6} = \frac{\lim\limits_{x\to 0} 5x^3 + \lim\limits_{x\to 0} 0.45x^2 - \lim\limits_{x\to 0} 2x + \lim\limits_{x\to 0} 100}{\lim\limits_{x\to 0} x^2 - \lim\limits_{x\to 0} 5x + \lim\limits_{x\to 0} 6} = \frac{0+0-0+100}{0-0+6} = \frac{100}{6}.$$

Theorem 1.5.2 (Sandwich Theorem). *If $\lim\limits_{x\to a} f(x) = L$, $\lim\limits_{x\to a} g(x) = L$, and if h is a function such that $f(x) \le h(x) \le g(x)$, then $\lim\limits_{x\to a} h(x) = L$.*

Example 1.5.10. Find

$$\lim_{x\to 0} \frac{\log_e(x)}{x}.$$

To find the limit of the above function, consider the inequality

$$\frac{x}{1+x} \le \log_e(x) \le x, \quad \text{for all } x > -1.$$

If we divide by x, then

$$\frac{1}{1+x} \le \frac{\log_e(x)}{x} \le 1, \quad \text{for all } x > -1.$$

Since $\lim\limits_{x\to 0} \frac{1}{1+x} = 0$ and $\lim\limits_{x\to 0} 1 = 1$, By Sandwich Theorem

$$\lim_{x\to 0} \frac{\log_e(x)}{x} = 1.$$

Note: We will see a different proof of the above fact in the next chapter.

Example 1.5.11. Find

$$\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right).$$

Note that

$$\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) \neq \lim_{x\to 0} x^2 \times \lim_{x\to 0} \sin\left(\frac{1}{x}\right),$$

because $\lim_{x\to 0} \sin\left(\frac{1}{x}\right)$ does not exist (see Example 1.5.4). Consider the inequality

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1, \quad \text{for } x \neq 0.$$

If we multiply by x^2, then

$$-x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2, \quad \text{for } x \neq 0.$$

By Sandwich Theorem, we get

$$\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0.$$

Example 1.5.12. Find

$$\lim_{x\to 0} \frac{\sin(x)}{x}.$$

Note: In the expressions above, x is in radians. Here we will discuss a geometric idea to find out the limit. Consider the diagram given below.

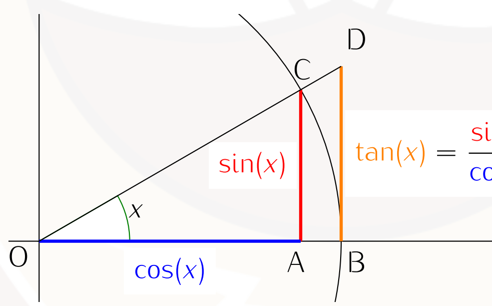

Figure 1.14:

In the above diagram, Both ΔOAC and ΔOBD are right angled triangle. The lengths of OC and OB are 1,and |Arc(CA)| = x. In ΔOAC,

$$\sin(x) = \frac{|AC|}{|OD|} = \frac{|AC|}{1} \implies |AC| = \sin(x).$$

Clearly, the length of |AC| is less than or equal to the length of Arc(CA), that is,

$$\sin(x) \leq x \tag{1.1}$$

In ΔOBD,

$$\tan(x) = \frac{|BD|}{|OB|} = \frac{|BD|}{1} \implies |BD| = \sin(x).$$

The area of the sector ACB is $\frac{x}{2}$ and the area of ΔOBD is $\frac{1}{2} \times |OB| \times |BD| = \frac{\tan(x)}{2}$. Observe that the area of the sector ACB is less than or equal to the area of ΔOBD, that is,

$$\frac{x}{2} \leq \frac{\tan(x)}{2} \iff x \leq \tan(x) \tag{1.2}$$

From equations 1.1 and 1.2, we have

$$\sin(x) \leq x \leq \tan(x).$$

If we divide by $\sin(x)$, then

$$1 \leq \frac{x}{\sin(x)} \leq \frac{\tan(x)}{\sin(x)} = \frac{1}{cos(x)}.$$

By taking the reciprocals of all terms in the above inequality, we get

$$\cos(x) \leq \frac{\sin(x)}{x} \leq 1.$$

Finally, by Sandwich Theorem, we get

$$\lim_{x \to 0} \cos(x) = 1 \leq \lim_{x \to 0} \frac{\sin(x)}{x} \leq \lim_{x \to 0} 1 = 1.$$

Therefore $\lim_{x \to 0} \frac{\sin(x)}{x} = 1$.

Note: We will see a easy proof of the above fact in the next chapter.

Example 1.5.13. Find

$$\lim_{x \to 0} \frac{\tan(x)}{x}.$$

Note that here we can not use Theorem 1.5.1(4) because $\lim\limits_{x\to 5} x = 0$. However, by slightly modifying the function, we can apply Theorem 1.5.1(4).

$$\begin{aligned}
\lim_{x\to 0}\frac{\tan(x)}{x} &= \lim_{x\to 0}\frac{\sin(x)}{x}\times\frac{1}{\cos(x)} \qquad \left(\textbf{Note: } \tan(x) = \frac{\sin(x)}{\cos(x)}\right)\\
&= \lim_{x\to 0}\frac{\sin(x)}{x}\times\lim_{x\to 0}\frac{1}{\cos(x)}\\
&= 1\times\frac{\lim\limits_{x\to 0} 1}{\lim\limits_{x\to 0}\cos(x)}\\
&= 1\times\frac{1}{1} = 1
\end{aligned}$$

Example 1.5.14. Find

$$\lim_{x\to 0}\frac{1-\cos(x)}{x^2}.$$

$$\begin{aligned}
\lim_{x\to 0}\frac{1-\cos(x)}{x^2} &= \lim_{x\to 0}\frac{2\sin^2\left(\frac{x}{2}\right)}{x^2} \qquad \left(\textbf{Note:} 1-\cos(x) = 2\sin^2\left(\frac{x}{2}\right)\right)\\
&= \frac{2}{4}\times\lim_{x\to 0}\frac{\sin^2\left(\frac{x}{2}\right)}{(\frac{x}{2})^2}\\
&= \frac{2}{4}\times\lim_{x\to 0}\left(\frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}}\right)^2\\
&= \frac{1}{2}\times 1 = \frac{1}{2}.
\end{aligned}$$

1.6 Continuity

Definition 1.6.1. Let $f : D \to \mathbb{R}$ be a function, where $D \subset \mathbb{R}$. We say f is continuous at $a \in D$ if

(1) $\lim\limits_{x\to a^+} f(x)$ and $\lim\limits_{x\to a^-} f(x)$ exist.

(2) $\lim\limits_{x\to a^+} f(x) = \lim\limits_{x\to a^-} f(x) = f(a)$.

When f is continuous for each point inside D=domain(f) then we say that the function is continuous on D.

Equivalent definition of continuity:

Definition 1.6.2. Let $f : D \to \mathbb{R}$ be a function, where $D \subset \mathbb{R}$. Then f is continuous at $a \in D$ if

$$\lim_{n\to\infty} f(x_n) = f(a)$$

for every sequence $\{x_n\}$ in D such that $x_n \to a$ as $n \to \infty$.

In particular, to show that f is not continuous at a point $a \in D$, we need to find a sequence x_n in the domain D of f such that $x_n \to a$ but $f(x_n) \not\to f(a)$.

<u>Note:</u>

(1) For a function to be continuous at $x = a$, it must be defined at that point. To have a limit as $x \to a$, it need not be.

(2) Continuity means "the limit" of the function at a can be obtained by evaluating the function at a.

Example 1.6.1. Consider the function $f : \mathbb{R} \to \mathbb{R}$, such that

$$f(x) = |x|.$$

We want to check whether the function is continuous at $x = 0$ or not. The graph of the function is as follows.

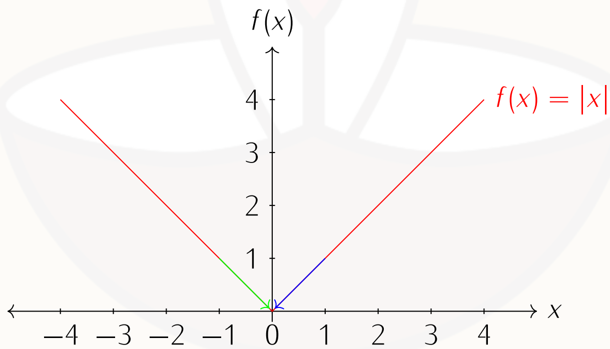

Figure 1.15: $f(x) = |x|$

From the graph, we see that when we make x close to 0 from the left or the right, the value $f(x)$ gets closer to 0. So $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = 0 = f(0)$. Hence, f is continuous at $x = 0$.

<u>**Sequential argument:**</u> Consider a sequence a_n such that $a_n \to 0$. Then the sequence $f(a_n) = |a_n|$ will also converge to $0 = f(0)$. Hence, f is continuous at $x = 0$.

Example 1.6.2.

$$f : \mathbb{R} \to \mathbb{R}$$

$$f(x) = \lfloor x \rfloor$$

Let us check whether the function is continuous at $x = -1$ or not. We have seen in Example 1.5.2 that $\lim_{x\to 1^-} \lfloor x \rfloor \neq \lim_{x\to 1^+} \lfloor x \rfloor$. So the second condition of continuity in Definition 1.6.1 does not hold. Hence f is not continuous.

Question 10. The function $f(x) = \lfloor x \rfloor$ is continuous at $x = a$ if and only if a is an integer.

Example 1.6.3.

$$f : [-4, 3] \to \mathbb{R}$$

$$f(x) = \begin{cases} x + 1 & \text{if} - 4 \leq x < 2 \\ x^2 - 4 & \text{if}.2 \leq x \leq 3 \end{cases}$$

We want to check whether the function is continuous at $x = 2$ or not. The graph of the function is given below.

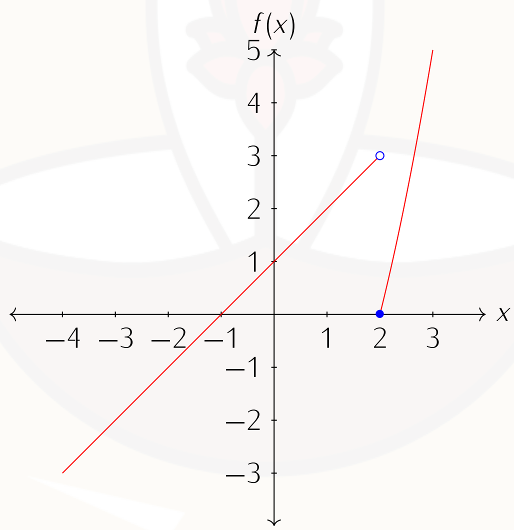

Figure 1.16:

As x approaches to 2 from the right, the value of the function $f(x)$ to 0, that is, $\lim_{x\to 2^+} f(x) = 0$. Similarly, if x approaches to 2 from the left, the value of the function $f(x)$ approaches to 3, that is, $\lim_{x\to 2^-} f(x) = 3$. Hence the function is not continuous $x = 0$.

Theorem 1.6.1. *let f and g be functions defined on $D \subset \mathbb{R}$ to $\mathbb{R}$. Suppose f and g are continuous at $x = a \in D$, then*

(1) $f \pm g$ are also continuous at $x = a$.

(2) $f \cdot g$ is continuous at $x = a$.

(3) if $g(a) \neq 0$, then $\frac{f}{g}$ is continuous at $x = a$.

Theorem 1.6.2. *If g is continuous at $x = a$ and f is continuous at $g(a)$, then the composite function $(f \circ g)(x) := f(g(x))$ is continuous at $x = a$.*

Example 1.6.4.

$$f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$$

$$f(x) = \frac{\sin(x^2)}{x^2 - 1}$$

We want to know whether the function is continuous at $x = 0$ or not.

- The function $g(x) = x^2$ is continuous at $x = 0$ and $f(x) = \sin(x)$ is continuous at $0 = g(0)$. By Theorem 1.6.2, $f \circ g(x) = \sin(x^2)$ is also continuous at 0.
- Both x^2 and 1 are continuous at $x = 0$. Therefore, $x^2 - 1$ is also continuous at $x = 0$, by Theorem 1.6.1(1).
- Both $\sin(x^2)$ and $x^2 - 1$ is continuous at $x = 0$, and the value of the function $x^2 - 1$ is non-zero at $x = 0$. Therefore $f(x) = \frac{\sin(x^2)}{x^2-1}$ is continuous at $x = 0$, by Theorem 1.6.1(3).

1.6.1 Exercise

Question 11. Define a function

$$f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$$

Which of the following option(s) is(are) true?

- Option 1: $\lim_{x \to 2} f(x) = -1$
- Option 2: $\lim_{x \to 0} f(x) = 1$
- **Option 3:** $\lim_{x \to -0.5} f(x) = -1$

- **Option 4:** Left limit at 0 i.e., $\lim_{x\to 0^-} f(x) = -1$
- Option 5: Right limit at 0 i.e., $\lim_{x\to 0^+} f(x) = -1$
- Option 6: Right limit at 0 i.e., $\lim_{x\to 0^+} f(x)$ does not exist.
- **Option 7:** Limit of the function at 0 does not exist.
- Option 8: Limit of the function does not exist at any real number.

Feedback:

- **For Option 1:** If x approaches to 2 from left or from the right the value of the function remains the same and that is 1.
- **For Option 2:** $\lim_{x\to 0-} f(x) = -1$, as $f(x) = -1$, if $x < 0$, and $\lim_{x\to a+} f(x) = 1$, as $f(x) = 1$, if $x \geq 0$. Hence, $\lim_{x\to a-} f(x) \neq \lim_{x\to a+} f(x)$.

Question 12. Which of the following option(s) is(are) true?

- **Option 1:** $\lim_{x\to\infty} \frac{1}{x} = 0$
- Option 2: $\lim_{x\to\infty} \frac{x^2}{1+x} = 1$
- **Option 3:** $\lim_{x\to-\infty} \frac{1+x}{x^2} = 0$
- **Option 4:** $\lim_{x\to\infty} \frac{1+x+x^2}{5x^2+1} = \frac{1}{5}$
- Option 5: $\lim_{x\to\infty} \frac{x^{2021}+x^{2020}+\ldots+x+1}{x^{2021}+2021x^{2020}+\ldots+2021} = 2021$

Question 13. Consider the following graph of a function in the Figure 1.17, where bullet point represents the point included in the curve and circle represents the point does not included in the line segment. Answer the following 2 questions

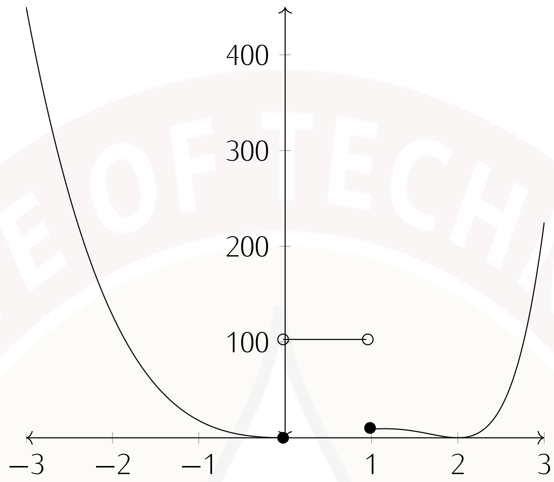

Figure 1.17:

Which of the following is(are) true?

- **Option 1:** Left limit at $x = 0$ is 0.
- Option 2: Limit exists at $x = 1$.
- **Option 3:** Limit exists at $x = 2$.
- Option 4: Limit exists at $x = 0$.
- **Option 5:** Limit exists at $x = \frac{1}{2}$ which is 100.
- Option 6: Limit exists at $x = \frac{1}{2}$ which is 0.

Question 14. Which of the following option(s) is(are) true?

[Hint: If the given function is of the form $\frac{f(x)}{g(x)}$ and $g(x) \to 0$, as $x \to a$. In this case, if possible, try to cancel the factor $(x - a)$ from both $f(x)$ and $g(x)$, and then try to evaluate the function at $x = a$.]

- **Option 1:** $\lim\limits_{x \to -1} \dfrac{x^2 - 6x - 7}{x^2 + 3x + 2} = -8$
- Option 2: $\lim\limits_{x \to 0} \dfrac{x^2 - 6x - 7}{x^2 + 3x + 2} = -8$
- **Option 3:** $\lim\limits_{x \to 2} (x^3 + 4x^2 - 6x - 7) = 5$
- Option 4: $\lim\limits_{x \to 3} \dfrac{x^2 - 6x + 9}{x - 3} = 1$

Feedback:

- **For Option 1:** $\lim_{x\to -1} \dfrac{x^2-6x-7}{x^2+3x+2} = \lim_{x\to -1} \dfrac{(x+1)(x-7)}{(x+1)(x+2)} = \lim_{x\to -1} \dfrac{(x-7)}{(x+2)}$.

- **For Option 2:** To find the value of: $\lim_{x\to 0} \dfrac{x^2-6x-7}{x^2+3x+2}$, evaluate the given function at $x=0$, as the function is defined at $x=0$.

- **For Option 4:** $\lim_{x\to 3} \dfrac{x^2-6x+9}{x-3} = \lim_{x\to 3} \dfrac{(x-3)^2}{x-3} = \lim_{x\to 3}(x-3)$

Question 15. Let f be a function and the Figure 1.18 represent the graph of function f. The solid points denote the value of the function at the points, and the values denoted by the hollow points are not taken by the functions.

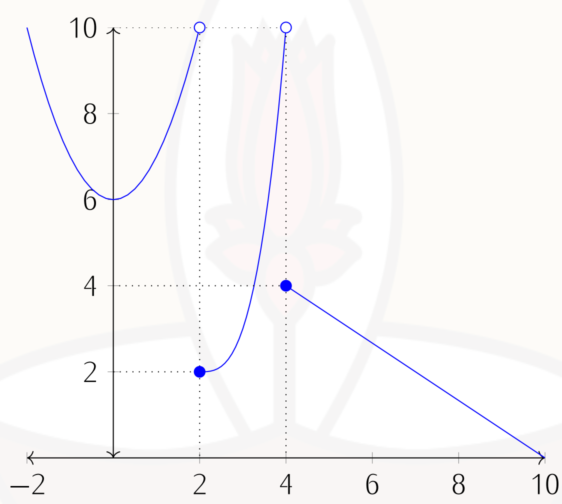

Figure 1.18:

Choose the set of correct options.

[**Hint:** Try to find out the value of the function, as x approaches a point from the left or the right.]

- **Option 1:** $\lim_{t\to 2-} f(t) = 10$
- **Option 2:** $\lim_{t\to 2+} f(t) = 2$
- Option 3: $\lim_{t\to 4-} f(t) = 4$
- **Option 4:** $\lim_{t\to 4+} f(t) = 4$

- Option 5: f is continuous at $x = 2$.
- **Option 6:** f is continuous at $x = 6$

Feedback:

- **For option 3:** Observe that as t approaches 4 from the left side, the value of the function in the graph approaches 10.
- **For option 5:** Observe that as t approaches 2 from the left side, the value of the function in the graph approaches 10 and as t approaches 2 from the right side, the value of the function in graph approaches 2. It means the left limit and the right limit exist but are not equal.
- **For option 6:** Observe that as t approaches 6 from the left side or the right side, the value of the function in the graph approaches a value which is equal to the value of the function at $x = 6$.

Question 16. Define a function f as follows:

$$f(x) = \begin{cases} \frac{x}{tan^{-1}2x} & \text{if } x > 0 \\ b & \text{if } x = 0 \\ \frac{sin(ax)}{x} & \text{if } x < 0, \end{cases}$$

Which of the following options is true if f is continuous?

[**Hint:** It is enough to check continuity at 0. Use $\lim\limits_{x\to 0} \frac{x}{\tan^{-1} x} = 1$ and $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$ to compute left and right limit.]

- Option 1: $a = b = 2$
- Option 2: $a = 2, b = \frac{1}{2}$
- Option 3: $a = \frac{1}{2}, b = 2$
- **Option 4:** $a = b = \frac{1}{2}$

Feedback:

- **For option 1, 2, 3:** $\lim\limits_{x\to 0^+} \frac{x}{\tan^{-1} 2x} = \lim\limits_{x\to 0^+} \frac{2x}{2\tan^{-1} 2x}$. Denote $2x$ as y, then as $x \to 0^+ \implies 2x = y \to 0^+$ and so $\lim\limits_{x\to 0^+} \frac{x}{\tan^{-1} 2x} = \lim\limits_{y\to 0^+} \frac{y}{2\tan^{-1} y}$.

  Similarly $\lim\limits_{x\to 0^-} \frac{\sin ax}{x} = \lim\limits_{y\to 0^-} \frac{a \sin y}{y} = a$ by putting $ax = y$.

  Hence for continuity $b = \lim\limits_{x\to 0^+} \frac{x}{\tan^{-1} 2x} = \lim\limits_{x\to 0^-} \frac{\sin ax}{x}$

Question 17. Consider a function $f : \mathbb{R} \to \mathbb{R}$ such that $f(cx) = cf(x)$ for all $c, x \in \mathbb{R}$. Which of the following option(s) is(are) correct?

- **Option 1:** $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$.
- Option 2: f is not continuous in $\mathbb{R}$.
- **Option 3:** f is continuous in $\mathbb{R}$.
- Option 4: $\lim\limits_{x \to a} f(x)$ exists for all $a \in R$, but f is not continuous in $\mathbb{R}$.

Question 18. Define a function f as follows:

$$f(x) = \begin{cases} \dfrac{1}{e^{\frac{1}{x}} + 1} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$$

Which of the following option(s) is(are) true?

- **Option 1**: $\lim\limits_{x \to 0^-} f(x) \neq \lim\limits_{x \to 0^+} f(x)$
- Option 2: $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^+} f(x)$
- **Option 3:** f is a bounded function on $\mathbb{R}$.
- Option 4: f is continuous at $x = 0$.

Question 19. Let C represent a curve in the Figure 1.19. Answer the questions (i) and (ii) using the curve C.

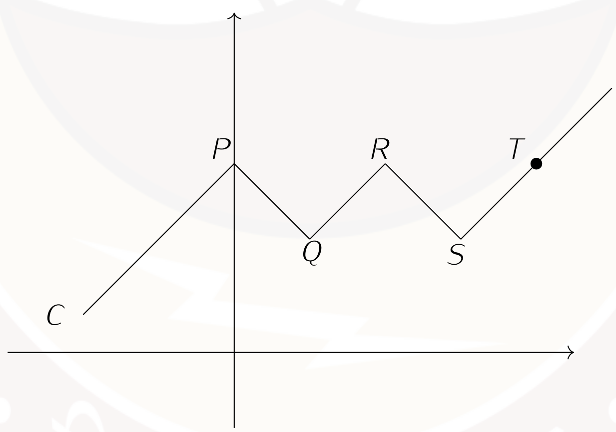

Figure 1.19:

i) Which of the following option(s) is(are) true?

- Option 1: At point P, there is a tangent.
- **Option 2:** At point Q, there is no tangent.
- Option 3: At point P, there are an infinite number of tangents.
- Option 4: None of the above.

ii) Let A be the set points where the curve C has no tangents. Find a lower bound for the cardinality of the set A. [Ans: 4]

2. Differentiation

"What we know is a drop, what we don't know is an ocean."

— Isaac Newton

The problem of finding tangent lines and the seemingly unrelated problem of finding maximum or minimum values were first seen to have a connection by Fermat in the 1630s. Furthermore, the relation between tangent lines to curves and the velocity of a moving particle was discovered in the late 1660s by Isaac Newton. Newton's theory of "fluxions," which was based on an intuitive idea of limit (discussed in the previous chapter), would be familiar to students of different branches once some changes in terminology and notation were made. But the vital observation made by Newton and, independently, by Gottfried Leibniz in the 1680s was that areas under curves could be calculated by reversing the differentiation process.

This chapter will discuss the concept of differentiability, derivatives, and application of derivatives for finding the maxima and minima of a function.

2.1 Differentiability and the derivative

We start this section with an example to understand the idea of rate of change.

Example 2.1.1. A truck travels the 2900 km distance from Jalandhar, Punjab, to Tiruchirappalli (Trichy), Tamil Nadu, in about 72 hrs.

Question 20. What is its speed?

$$\text{speed} = \frac{\text{Distance covered}}{\text{time}} = \frac{2900 \text{ km}}{72 \text{ hrs}} = 40.28 \text{ km/hr. (approx.)}$$

After reaching Tiruchirappalli, the driver tells his friend that a traffic constable fined him near Nagpur, Maharashtra, for speeding (assume that if speed is more than 80 km/hr, one can be fined). The friend calculates his speed (40.28 km/hr., as we have calculated above) from the above data and says the constable is wrong and that the fine was unjustified.

Question 21. Is the friend correct?

The driver then mentions that he covered the 260 km. stretch in Maharashtra in 4 hours with about an hour's break for breakfast. The friend now recalculates and takes back his opinion that the fine is unjustified.

Question 22. Is the friend correct now? Give reason for your answer.

From the data given above, we know that the driver covered the 260 km. stretch in Maharashtra in 3 hours only. So the average speed in that stretch is $\frac{260 \text{ km}}{3 \text{ hrs}} =$ 86.67 km/hr. (approx.). Hence, there is a possibility that the truck's speed crossed the permissible speed limit at some point in Maharashtra.
So it is clear that average speed and instantaneous speed are different concepts, and one cannot deduce instantaneous speed from average speed. To calculate the instantaneous speed at some time, we have to compute the distance travelled in a very short period of time around that time and divide by that period of time. e.g., one could take the distance travelled in 1 minute after that time; if it is y km., then the instantaneous speed would be approximately $60y$ km/hr. Ideally, one should take as small a time interval as possible, i.e., an infinitesimal time. Thus, we obtain a limit!

$$\text{Infinitesimal speed} = \lim_{\Delta t \to 0} \frac{\text{Distance travelled in time } \Delta t}{\Delta t}$$

(where Δt is measured in seconds and distance in km).

2.1.1 Differentiability of one variable function

Let f be a function defined on an interval around a (i.e., $(a-h, a+h)$ for some $h \in \mathbb{R}$). Motivated from the example above, let us try to define the differentiability of f at the point a. When we move towards the right of a to $a+h$, the increment of f is $f(a+h) - f(a)$. The rate of change of the function (See Figure 2.1) in the interval $(a, a+h)$ is

$$\frac{f(a+h) - f(a)}{(a+h) - a} = \frac{f(a+h) - f(a)}{h}$$

Similarly, when we move towards the left of a to $a-h$, the increment of f is $-(f(a-h) - f(a)) = f(a) - f(a-h)$. The rate of change of the function (See Figure 2.2) in the interval $(a-h, a)$ is

$$\frac{f(a) - f(a-h)}{h}$$

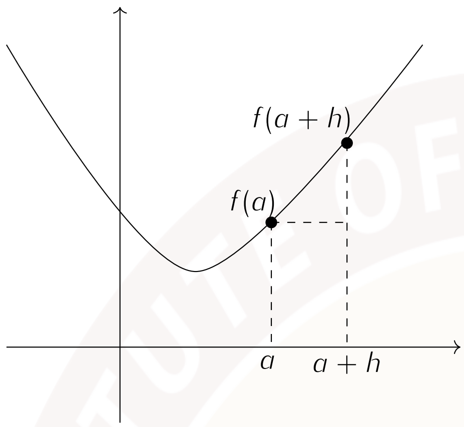

Figure 2.1:

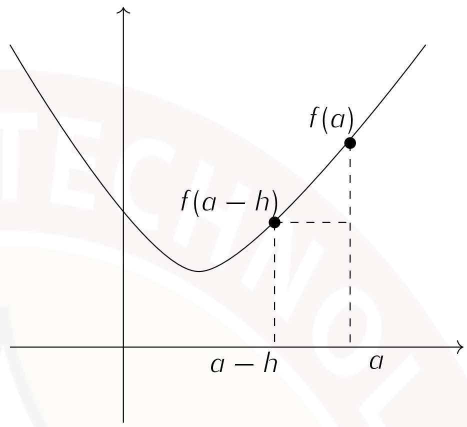

Figure 2.2:

Definition 2.1.1. Let f be a function defined on an interval around a. Then f is differentiable at a if $\lim_{h\to 0} \dfrac{f(a+h)-f(a)}{h}$ exists.

Using this definition let us try to understand whether the following functions are differentiable at some given points or not.

Example 2.1.2.

$$f : \mathbb{R} \to \mathbb{R}$$

$$f(x) = x$$

Let us try to check whether f is differentiable at any arbitrary point a or not.

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{a+h-a}{h} = \lim_{h\to 0} \frac{h}{h} = 1$$

Hence, f is differentiable at any real number.

Example 2.1.3.

$$f : \mathbb{R} \to \mathbb{R}$$

$$f(x) = sin\ x$$

Let us try to check whether f is differentiable at the point 0 or not.

$$\lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{sin\ h - sin\ 0}{h} = \lim_{h\to 0} \frac{sin\ h - 0}{h} = \lim_{h\to 0} \frac{sin\ h}{h} = 1$$

Hence, f is differentiable at $x = 0$.

Example 2.1.4.

$$f : \mathbb{R} \to \mathbb{R}$$

$$f(x) = |x|$$

Let us try to check whether f is differentiable at the point 0 or not.

$$\lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{|h| - 0}{h} = \lim_{h\to 0} \frac{|h|}{h}$$

Question 23. Can you evaluate this limit?

Recall,

$$|h| = \begin{cases} h & \text{if } h \geq 0 \\ -h & \text{if } h < 0 \end{cases}$$

So we have,

$$\lim_{h\to 0^+} \frac{|h|}{h} = \lim_{h\to 0^+} \frac{h}{h} = \lim_{h\to 0^+} 1 = 1$$

$$\lim_{h\to 0^-} \frac{|h|}{h} = \lim_{h\to 0^+} \frac{-h}{h} = \lim_{h\to 0^+} -1 = -1$$

Hence, $\lim\limits_{h\to 0} \frac{|h|}{h}$ does not exist. So $f(x) = |x|$ is not differentiable at $x = 0$.

Question 24. Is $f(x) = |x|$ differentiable at every point other than 0?

Example 2.1.5.

$$f : \mathbb{R} \to \mathbb{R}$$

$$f(x) = x^{\frac{1}{3}}$$

Let us try to check whether f is differentiable at the point 0 or not.

$$\lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{h^{\frac{1}{3}} - 0}{h} = \lim_{h\to 0} \frac{h^{\frac{1}{3}}}{h} = \lim_{h\to 0} h^{-\frac{2}{3}} = \lim_{h\to 0} \frac{1}{h^{\frac{2}{3}}},$$

$h^{-\frac{2}{3}}$ diverges to infinity as h tends to 0. So we can say that $\lim\limits_{h\to 0} h^{-\frac{2}{3}}$ does not exist. Hence $f(x) = x^{\frac{1}{3}}$ is not differentiable at $x = 0$.

Example 2.1.6.

$$f : \mathbb{R} \to \mathbb{R}$$

$$f(x) = \lfloor x \rfloor$$

Let us try to check whether f is differentiable at the point 0 or not.

$$\lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{\lfloor h \rfloor - 0}{h} = \lim_{h\to 0} \frac{\lfloor h \rfloor}{h}$$

We recall the behavior of the function $\lfloor h \rfloor$ in the interval $[-1, 1)$.

$$\lfloor h \rfloor = \begin{cases} 0 & \text{if } 0 \leq h < 1 \\ -1 & \text{if } -1 \leq h < 0 \end{cases}$$

So we have

$$\lim_{h \to 0^+} \frac{\lfloor h \rfloor}{h} = \lim_{h \to 0^+} \frac{0}{h} = 0$$

$$\lim_{h \to 0^-} \frac{\lfloor h \rfloor}{h} = \lim_{h \to 0^-} \frac{-1}{h} \text{ does not exist}$$

Hence, $\lim\limits_{h \to 0} \frac{\lfloor h \rfloor}{h}$ does not exist. So $f(x) = \lfloor x \rfloor$ is not differentiable at $x = 0$.

Question 25. Is $f(x) = \lfloor x \rfloor$ differentiable at $x = 1$?

Question 26. Is $f(x) = \lfloor x \rfloor$ differentiable at $x = a$, where a is an integer?

In the previous chapter, we have studied the continuity of a function. The following theorem states a relation between continuity and differentiability, which gives us a class of functions that are not differentiable.

Theorem 2.1.1. *If f is differentiable at a, then it is continuous at a.*

Proof. If f is differentiable at a then $\lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$ exists, say L.

$$\lim_{h \to 0} f(a+h) - f(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} h = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \lim_{h \to 0} h = L \cdot 0 = 0.$$

Hence $\lim\limits_{h \to 0} f(a+h) = f(a)$. That is, both left limit and right limits exist and are equal to $f(a)$, which implies that f is continuous at a. □

Recall that the function $f(x) = \lfloor x \rfloor$ is not continuous at $x = a$, if a is an integer. Hence it is evident from Theorem 2.1.1 that f is not differentiable at $x = a$, where a is an integer, which gives us the answer of both the questions mentioned above.

2.1.2 Exercise

Question 27. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined as $f(x) = 5x$. Which of the following options shows the step wise solution to check whether or not the function f is differentiable at $x = 2$?

- **Option 1:** $\lim\limits_{h \to 0} \dfrac{f(2+h) - f(2)}{h} = \lim\limits_{h \to 0} \dfrac{5(2+h) - 10}{h} = \lim\limits_{h \to 0} \dfrac{5h}{h} = 5.$

- Option 2: $\lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{5(0+h)-0}{h} = \lim_{h\to 0} \frac{5h}{h} = 5.$
- Option 3: $\lim_{h\to 0} \frac{f(2+h)+f(2)}{h} = \lim_{h\to 0} \frac{5(2+h)+10}{h} = \lim_{h\to 0} \frac{20+5h}{h} = \infty$, so function is not differentiable
- Option 4: $\lim_{h\to 0} \frac{f(2+h)}{h} = \lim_{h\to 0} \frac{5(2+h)}{h} = \lim_{h\to 0} \frac{10+5h}{h} = 10.$
- Option 5: $\lim_{h\to 0} \frac{f(2+h)-f(2)}{h^2} = \lim_{h\to 0} \frac{5(2+h)-10}{h^2} = \lim_{h\to 0} \frac{5h}{h^2} = 0.$

Feedback:

- **For option 2** We have to check the differentiability at the point $x = 2$.
- **For option 3, 4, 5** Check whether we are using the correct definition of differentiability or not.

Question 28. Which of the following option shows the step wise solution to check whether or not the functions mentioned in the option are differentiable at the point mentioned in the option?

[**Hint:** Use the definition of differentiability of a function and apply the concept of limit.]

- Option 1: $f(x) = a$, at any real c: $\lim_{h\to 0} \frac{f(c+h)-f(c)}{h} = \lim_{h\to 0} \frac{a+h-a}{h} = \lim_{h\to 0} \frac{h}{h} = 1.$
- **Option 2:** $f(x) = x - c$, at c for some real number c: $\lim_{h\to 0} \frac{f(c+h)-f(c)}{h} = \lim_{h\to 0} \frac{(c+h-c)-(c-c)}{h} = \lim_{h\to 0} \frac{h}{h} = 1.$
- Option 3: $f(x) = x^2$, at $c \in \mathbb{R}$: $\lim_{h\to 0} \frac{f(c+h)-f(c)}{h} = \lim_{h\to 0} \frac{(c+h)^2-c^2}{h} = \lim_{h\to 0} \frac{2ch+h^2}{h} = 0$
- **Option 4:** $f(x) = e^x$, $c \in \mathbb{R}$: $\lim_{h\to 0} \frac{f(c+h)-f(c)}{h} = \lim_{h\to 0} \frac{e^{c+h}-e^c}{h} = \lim_{h\to 0} \frac{e^c(e^h-1)}{h} = e^c \lim_{h\to 0} \frac{e^h-1}{h} = e^c \cdot 1 = e^c.$

Feedback:

- **For option 1:** Observe that the given function is a constant function.
- **For option 3:** Check the limit of $\frac{2ch+h^2}{h}$ as $h \to 0$.

Question 29. Consider the graphs given below:

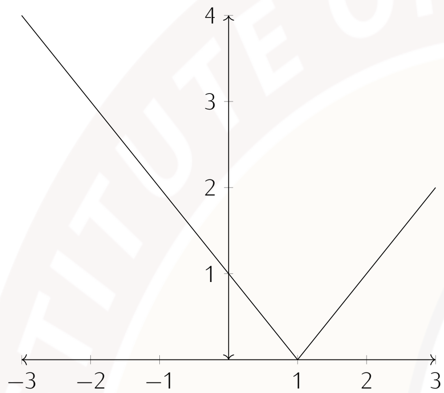

Figure 2.3: Figure: f_1

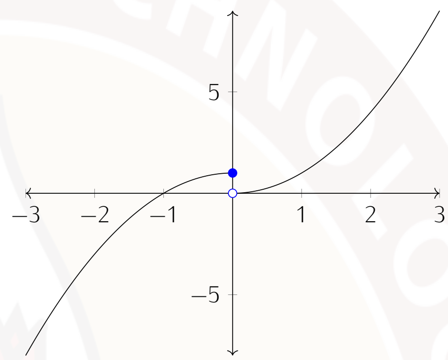

Figure 2.4: Figure: f_2

Choose the set of correct options.

[**Hint:** Check whether they are continuous at each point in their domain, and also check whether there is any sharp corner in the graphs.]

- Option 1: f_1 is continuous and differentiable at each real number.
- **Option 2:** f_1 is not differentiable at 1.
- **Option 3:** f_2 is not continuous at 0.
- **Option 4:** f_2 is differentiable in the interval $[1, 2]$.

Question 30. Choose the set of correct options.

- **Option 1:** If $\lim\limits_{h\to 0} \dfrac{f(a+h) - f(a)}{h}$ exists, then f is differentiable at a.
- Option 2: A function f may be differentiable at a point a, even if it is not continuous at a.
- **Option 3:** If a function is differentiable at a point a, then it must be continuous at a.
- **Option 4:** There exist some continuous functions which are not differentiable at some points in the domain.

Question 31. Consider the function $f : \mathbb{R} \to \mathbb{R}$, defined as follows:

$$f(x) = \begin{cases} \lfloor x \rfloor & \text{if } x \geq 0 \\ \lceil x \rceil & \text{if } x < 0, \end{cases}$$

Choose the set of correct options:

- **Option 1:** f is differentiable at $x = 0$.
- Option 2: f is differentiable at $x = 1$.
- Option 3: f is differentiable at $x = -1$.
- **Option 4:** f is differentiable at $x = 1.5$.

Question 32. The following curve shown in Figure 2.5 represents the function $f : \mathbb{R} \to \mathbb{R}$, such that $f(x) = x^{\frac{1}{3}}$.

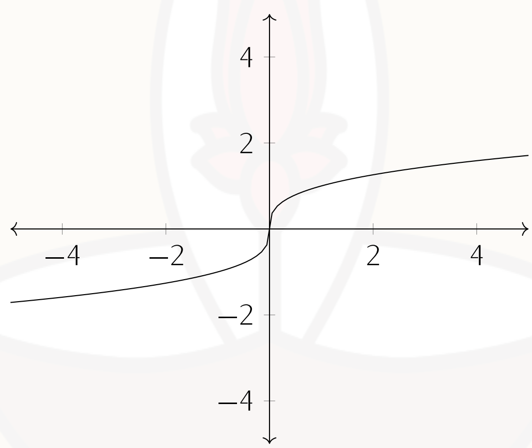

Figure 2.5:

Which of the following options is(are) correct?

- **Option 1:** f is continuous on $\mathbb{R}$.
- Option 2: f is differentiable everywhere on $\mathbb{R}$.
- Option 3: f is nowhere differentiable on $\mathbb{R}$.
- **Option 4:** f is not differentiable at 0.

Question 33. Define a function f as follows:

$$f(x) = \begin{cases} kx^2 + l & \text{if } x \leq 1 \\ lx^2 + kx + m & \text{if } x > 1 \end{cases}$$

If f is continuous and differentible at $x = 1$ then the value of $k - 2l + m$ is [Answer: 0]

2.1.3 Derivatives

Calculating derivatives is a fundamental tool of calculus. As we have discussed at the beginning of Section 2.1, the derivative of the position of a moving object with respect to time is the object's velocity: this measures how quickly the object's position changes when time advances. We begin this subsection with the definition of the derivative of a function.

Definition 2.1.2. The Derivative of a function f at the point $x \in$domain (f) is denoted by $f'(x)$ and defined as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Using the definition let us calculate the derivatives of some functions.

Example 2.1.7. Consider $f(x) = c$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c - c}{h} = \lim_{h \to 0} \frac{0}{h} = 0$$

Example 2.1.8. Consider $f(x) = x$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{x + h - x}{h} = \lim_{h \to 0} \frac{h}{h} = 1$$

Example 2.1.9. Consider $f(x) = x^2$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} =$$

$$= \lim_{h \to 0} \frac{2xh + h^2}{h} = \lim_{h \to 0} (2x + h) = 2x$$

Example 2.1.10. Consider $f(x) = \sin\ x$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sin(x+h) - \sin\ x}{h}$$

$$= \lim_{h\to 0} \frac{\sin\ x\ \cos\ h + \cos\ x\ \sin\ h - \sin\ x}{h}$$
$$= \lim_{h\to 0} \frac{\sin\ x(\cos\ h-1)}{h} + \lim_{h\to 0} \frac{\cos\ x\ \sin\ h}{h}$$
$$= \sin\ x \lim_{h\to 0} \frac{(\cos\ h-1)}{h} + \cos\ x \lim_{h\to 0} \frac{\sin\ h}{h}$$
$$= (\sin\ x)\cdot 0 + (\cos\ x)\cdot 1 = \cos\ x$$

Here are some helpful rules about derivatives mentioned in the following proposition.

Proposition 2.1.2. *Let $f(x)$ and $g(x)$ be differentiable at the point a, then we have the following linearity property:*

i) *$(f+g)(x)$ is differentiable at a, and $(f+g)'(a) = f'(a) + g'(a)$.*

ii) *$(f-g)(x)$ is differentiable at a, and $(f-g)'(a) = f'(a) - g'(a)$.*

iii) *$(cf)(x)$ is differentiable at the point a, for all $c \in \mathbb{R}$, and $(cf)'(a) = cf'(a)$.*

Proposition 2.1.3. *Let $f(x)$ and $g(x)$ be differentiable at the point a, then we have*

i) ***The product rule:** $(fg)(x)$ is differentiable at a, and $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$.*

ii) ***The quotient rule:** $\frac{f}{g}(x)$ is differentiable at a (assuming $g(a) \neq 0$), and $\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{g(a)^2}$.*

Example 2.1.11. Consider $h(x) = x^3$. We can write $h(x) = f(x)g(x)$, where $f(x) = x$ and $g(x) = x^2$. Now using the product rule we have, $h'(x) = f'(x)g(x) + f(x)g'(x)$. In Example 2.1.8 and Example 2.1.9, we have calculated $f'(x) = 1$ and $g'(x) = 2x$. So $h'(x) = f'(x)g(x) + f(x)g'(x) = 1\cdot x^2 + x\cdot 2x = 3x^2$

Question 34. Can you find out the derivative of $f(x) = x^n$?

Example 2.1.12. Consider $h(x) = 5x^2$. We can write $h(x) = f(x)g(x)$, where $f(x) = 5$ and $g(x) = x^2$. Now using the product rule we have, $h'(x) = f'(x)g(x) + f(x)g'(x)$. In Example 2.1.7 and Example 2.1.9, we have calculated $f'(x) = 0$ and $g'(x) = 2x$. So $h'(x) = f'(x)g(x) + f(x)g'(x) = 0\cdot x^2 + 5\cdot 2x = 10x$

Question 35. Can you find out the derivative of $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$, where $a_n, a_{n-1}, \ldots, a_2, a_1, a_0$ are real numbers?

Remark 2.1.1. Write $f(x)$ as $f_n(x) + f_{n-1}(x) + \ldots + f_2(x) + f_1(x) + f_0(x)$, where

$$\begin{aligned} f_n(x) &= a_n x^n & f_n'(x) &= a_n\ nx^{n-1} \\ f_{n-1}(x) &= a_{n-1}x^{n-1} & f_{n-1}'(x) &= a_{n-1}\ (n-1)x^{n-2} \\ &\vdots & &\vdots \\ f_2(x) &= a_2 x^2 & f_2'(x) &= a_2\ 2x \\ f_1(x) &= a_1 x & f_1'(x) &= a_1 \\ f_0(x) &= a_0 & f_0'(x) &= 0. \end{aligned}$$

Using the linearity property mentioned in Proposition 2.1.2, we have

$$f'(x) = f_n'(x) + f_{n-1}'(x) + \ldots + f_2'(x) + f_1'(x) + f_0'(x).$$

Hence,

$$f'(x) = a_n\ nx^{n-1} + a_{n-1}\ (n-1)x^{n-2} + \ldots + a_2\ 2x + a_1.$$

Example 2.1.13. Consider $h(x) = x^7\ \sin(x)$. We can write $h(x) = f(x)g(x)$, where $f(x) = x^7$ and $g(x) = \sin(x)$. Now using the product rule we have, $h'(x) = f'(x)g(x) + f(x)g'(x)$. Using Question 34 and Example 2.1.10, we calculate $f'(x) = 7x^6$ and $g'(x) = \cos(x)$. So $h'(x) = f'(x)g(x) + f(x)g'(x) = 7x^6\ \sin(x) + x^7\ \cos(x)$

Example 2.1.14. Consider $h(x) = \tan\ x$. We can write $h(x) = \frac{f(x)}{g(x)}$, where $f(x) = \sin\ x$ and $g(x) = \cos\ x$. Now using the quotient rule (in Proposition 2.1.3) we have,

$$h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

$$\begin{aligned} f(x) &= \sin\ x & f'(x) &= \cos\ x \\ g(x) &= \cos\ x & g'(x) &= -\sin\ x \end{aligned}$$

$$\begin{aligned} h'(x) &= \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2} \\ &= \frac{\cos\ x\ \cos\ x - \sin\ x\ (-\sin\ x)}{\cos^2\ x} \\ &= \frac{\cos^2\ x + \sin^2\ x}{\cos^2\ x} \\ &= \frac{1}{\cos^2\ x} \\ &= \sec^2\ x \end{aligned}$$

Question 36. Can you find the derivative of $f(x) = \dfrac{1}{x^r}$, for $r > 0$?

Calculating the derivative of a composite function requires some special attention, which the following proposition can summarize

Proposition 2.1.4. ***Composition: the chain rule:*** *If $f(x)$ and $g(x)$ are differentiable functions, then so is $f(g(x))$, and its derivative is : $f(g(x))' = f'(g(x))g'(x)$.*

Example 2.1.15. Consider $h(x) = \tan\ 2x$. We can write $h(x)$ as the composite function $f(g(x))$, where $f(x) = \tan\ x$ and $g(x) = 2x$.

$$\begin{aligned} f(x) &= \tan\ x & f'(x) &= \sec^2\ x \\ g(x) &= 2x & g'(x) &= 2 \end{aligned}$$

$$h'(x) = f'(g(x))g'(x) = (\sec^2\ 2x)\ 2 = 2\sec^2\ 2x$$

The following table summarizes derivatives of some special functions. We recommend you verify each one yourself.

Functions	Their derivatives
$f(x) = x^n$	$f'(x) = nx^{n-1}$
$f(x) = \sin ax$	$f'(x) = a\cos ax$
$f(x) = \cos ax$	$f'(x) = -a\sin ax$
$f(x) = e^{ax}$	$f'(x) = ae^{ax}$
$f(x) = \ln ax$	$f'(x) = \frac{a}{ax} = \frac{1}{x}$
$f(x) = \log_b ax$	$f'(x) = \frac{a}{ax\ln b} = \frac{1}{x\ln b}$

2.1.4 Exercise

Question 37. Which of the following statements are correct ?

- Option 1: Derivative of $e^{x^2 \cos x}$ is $e^x(2x\cos x - x^2 \sin x)$.
- **Option 2:** Derivative of $e^{x \sin x}$ is $e^{x\sin x}(\sin x + x\cos x)$.
- Option 3: Derivative of $e^{x \sin x}$ is $e^{x\sin x}(\sin x - x\cos x)$.
- **Option 4:** Derivative of $e^{x^2 \cos x}$ is $e^{x^2\cos x}(2x\cos x - x^2 \sin x)$.

Question 38. Consider a function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = x|x|$. Which of the following option(s) is(are) true?

- Option 1: f is not differentiable at any point of $\mathbb{R}$.
- **Option 2:** f is differentiable on $\mathbb{R}$.
- **Option 3:** $f'(x) = \begin{cases} 2x & \text{if } x \geq 0 \\ -2x & \text{if } x < 0, \end{cases}$.
- Option 4: $f'(x) = \begin{cases} -2x & \text{if } x \geq 0 \\ 2x & \text{if } x < 0, \end{cases}$
- Option 5: $f'(x) = 2x$
- Option 6: $f'(x) = \begin{cases} 2x & \text{if } x > 0 \\ -2x & \text{if } x < 0, \end{cases}$, and f is not differentiable at 0.

Question 39. If $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$ and $f(9) = 6$, $f'(0) = 4$, then find out the value of $f'(9)$. [Answer: 24]

2.2 Indeterminate limits and L'Hopital's rule

In mathematics, more specifically calculus, L'Hôpital's rule or L'Hospital's rule is a theorem that provides a technique to evaluate the limits of indeterminate forms. The rule's application (or repeated application) often converts an indeterminate form to an expression that can be quickly evaluated by substitution. The rule is named after the 17th-century French mathematician Guillaume de l'Hôpital.

Definition 2.2.1. Suppose $\lim\limits_{x \to a} f(x) = 0$ and $\lim\limits_{x \to a} g(x) = 0$ or both diverge to ∞ or both diverge to $-\infty$, then the $\lim\limits_{x \to a} \frac{f(x)}{g(x)}$ is said to be an indeterminate limit.

Example 2.2.1. As $\lim_{x\to 0} x = 0$ and $\lim_{x\to 0}(x + sin\ x) = 0$, the $\lim_{x\to 0} \frac{x}{x+sin\ x}$ is indeterminate limit.

Example 2.2.2. As $\lim_{x\to 0} sin^2\ x = 0$ and $\lim_{x\to 0}(1 - cos\ 2x) = 0$, the $\lim_{x\to 0} \frac{sin^2\ x}{1-cos\ 2x}$ is indeterminate limit.

L'Hopital's rule: In the situation of an indeterminate limit, suppose the following conditions hold:

i) $f'(x)$ and $g'(x)$ exist on an interval I containing a (except possibly at a).

ii) $g'(x) \neq 0$ in the interval I.

iii) $\lim_{x\to a} \dfrac{f'(x)}{g'(x)} = L$.

Then $\lim_{x\to a} \dfrac{f(x)}{g(x)} = L$.

Example 2.2.3. Find out the limit: $\lim_{x\to 0} \dfrac{\log_e(1+x)}{x}$.

$log_e(1+x) \to 0$, as $x \to 0$, so $\lim_{x\to 0} \dfrac{log_e(1+x)}{x}$ is an indeterminate limit. The derivative of both the functions $log_e(1+x)$ and x exist on an interval I containing 0. Derivative of $log_e(1+x)$ is $\dfrac{1}{1+x}$ and the derivative of x is 1, which is non-zero. Hence by L'Hospital's rule we have,

$$\lim_{x\to 0} \frac{log_e(1+x)}{x} = \lim_{x\to 0} \frac{\frac{1}{1+x}}{1} = 1$$

Example 2.2.4. Find out the limit: $\lim_{x\to 0} \dfrac{\sin\ x}{x}$.

$\sin\ x \to 0$, as $x \to 0$, so $\lim_{x\to 0} \dfrac{\sin\ x}{x}$ is an indeterminate limit. The derivative of both the functions $\sin\ x$ and x exist on an interval I containing 0. Derivative of $\sin\ x$ is $\cos\ x$ and the derivative of x is 1, which is non-zero.
Hence by L'Hospital's rule we have,

$$\lim_{x\to 0} \frac{\sin\ x}{x} = \lim_{x\to 0} \frac{\cos\ x}{1} = 1$$

Example 2.2.5. Find out the limit: $\lim_{x\to\infty} \frac{a+be^x}{c+de^x}$, where b and d are non-zero positive.

Both $(a + be^x)$ and $(c + de^x)$ diverge to infinity, so $\lim_{x\to\infty} \dfrac{a+be^x}{c+de^x}$ is an indeterminate limit. The derivative of both the functions $(a + be^x)$ and $(c + de^x)$ exist. Derivative

of $a + be^x$ is be^x and the derivative of $c + de^x$ is de^x, which is non-zero.
Hence by L'Hospital's rule we have,

$$\lim_{x\to\infty} \frac{a + be^x}{c + de^x} = \lim_{x\to\infty} \frac{be^x}{de^x} = \frac{b}{d}$$

Example 2.2.6. Find out the limit: $\lim\limits_{x\to 0} \dfrac{1 - \cos\ x}{x^2}$.

$1 - \cos\ x \to 0$, and $x^2 \to 0$, as $x \to 0$. So $\lim\limits_{x\to 0} \dfrac{1 - \cos\ x}{x^2}$ is an indeterminate limit. The derivative of both the functions $1 - \cos\ x$ and x^2 exist on an interval I containing 0. Derivative of $1 - \cos\ x$ is $\sin\ x$ and the derivative of x^2 is $2x$.
Hence by L'Hospital's rule we have,

$$\lim_{x\to 0} \frac{1 - \cos\ x}{x^2} = \lim_{x\to 0} \frac{\sin\ x}{2x} = \frac{1}{2} \lim_{x\to 0} \frac{\sin\ x}{x} = \frac{1}{2}$$

2.2.1 Exercise

Question 40. If $f(x) = \sqrt{9 - x^2}$, then find out the value of $\sqrt{8} \times \lim\limits_{x\to 1} \dfrac{f(x) - f(1)}{x - 1}$. [Answer: -1]

Question 41. In which of the following, one can apply L'Hospital's rule to evaluate the limits?

- Option 1: $\lim\limits_{x\to\infty} \dfrac{x}{x + \sin x}$
- **Option 2**: $\lim\limits_{x\to 0} \dfrac{\sin^2 x}{1 - \cos(2x)}$
- **Option 3:** $\lim\limits_{x\to\infty} \dfrac{7 + \ln x}{x^3 + 6}$
- Option 4: $\lim\limits_{x\to 0^+} \dfrac{1}{x}$

Question 42. What is the value of $\lim\limits_{x\to\infty} xe^{-x}$? [Answer: 0]

2.3 Tangents and linear approximation

We begin this section by recalling the definition of the tangent to a function $f(x)$. A tangent to $f(x)$ at point a is a line that represents the instantaneous direction in which the graph $\Gamma(f)$ moves at $(a, f(a))$. Traditionally, the tangent to $f(x)$ at a is thought of as a line which just touches $\Gamma(f)$ at $(a, f(a))$. If it exists, we can think of the tangent as a "limit" of secants joining $(a, f(a))$ and nearby points $(a+h, f(a+h))$.

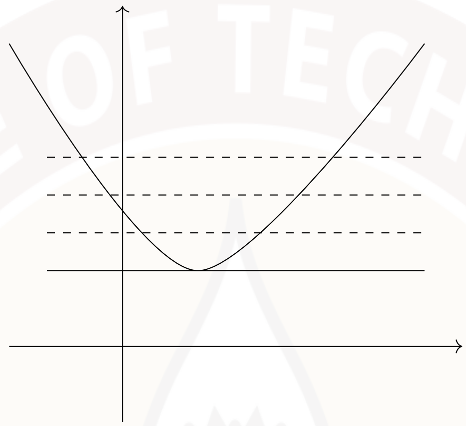

Figure 2.6:

The equation of secant joining $(a, f(a))$ and $(a+h, f(a+h))$ is,

$$y - f(a) = \frac{f(a+h) - f(a)}{(a+h) - a}(x - a) = \frac{f(a+h) - f(a)}{h}(x - a)$$

Question 43. What happens as the point $(a+h, f(a+h))$ comes closer to the point $(a, f(a))$?

Remark 2.3.1. As the point $(a+h, f(a+h))$ comes closer to the point $(a, f(a))$ (same as $h \to 0$), the secant comes closer to the tangent. Hence by evaluating the equation as $h \to 0$, we get the equation of the tangent. Let us evaluate the equation when $h \to 0$.

$$y - f(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}(x - a)$$

If the $\lim\limits_{h\to 0} \frac{f(a+h)-f(a)}{h}$ exists, then f is differentiable at a, and $\lim\limits_{h\to 0} \frac{f(a+h)-f(a)}{h} = f'(a)$. Hence the equation of tangent is

$$y - f(a) = f'(a)(x - a).$$

2.3.1 Equation of the tangent at a point on a curve

Let $f(x)$ be differentiable at the point a. Then the tangent to f at a exists and unique. It is given by the equation

$$y = f'(a)(x - a) + f(a).$$

Conversely, if the tangent to f at a exists and it is not vertical (i.e, not the line $x = a$). Then f is differentiable at a and hence the equation of the tangent is

$$y = f'(a)(x - a) + f(a).$$

Example 2.3.1. Let us calculate the tangent of $f(x) = \cos\ x$ at $x = \frac{\pi}{3}$.

$$f(x) = \cos\ x \qquad f\left(\frac{\pi}{3}\right) = \cos\ \left(\frac{\pi}{3}\right) = \frac{1}{2}$$

$$f'(x) = -\sin\ x \qquad f'\left(\frac{\pi}{3}\right) = -\sin\ \left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

The equation of tangent of $f(x) = \cos\ x$ at $x = \frac{\pi}{3}$ is,

$$y = f'\left(\frac{\pi}{3}\right)\left(x - \frac{\pi}{3}\right) + f\left(\frac{\pi}{3}\right)$$

$$y = -\frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) + \frac{1}{2}$$

Example 2.3.2. Let us calculate the tangent of $f(x) = x\ \tan\ x$ at $x = \frac{\pi}{4}$.

$$f(x) = x\ \tan\ x \qquad f\left(\frac{\pi}{4}\right) = \frac{\pi}{4}\tan\ \left(\frac{\pi}{4}\right) = \frac{\pi}{4}$$

$$f'(x) = \tan\ x + x\ \sec^2\ x \qquad f'\left(\frac{\pi}{4}\right) = \tan\ \left(\frac{\pi}{4}\right) + \frac{\pi}{4}\ \sec^2\left(\frac{\pi}{4}\right) = 1 + \frac{\pi}{2}$$

The equation of tangent of $f(x) = x\ \tan\ x$ at $x = \frac{\pi}{4}$ is,

$$y = f'\left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right) + f\left(\frac{\pi}{4}\right)$$

$$y = \left(1 + \frac{\pi}{2}\right)\left(x - \frac{\pi}{4}\right) + \frac{\pi}{4}.$$

2.3.2 Linear approximation

Recall that a linear function is a function of the form $L(x) = c + dx$. Let $f(x)$ be a function and a be a point in the domain of f. A linear function which takes the value $f(a)$ at a will have the form $L(x) = f(a) + m(x - a)$. We want to choose a linear function which best approximates the function $f(x)$ around the point a. That is, $f(x) \approx L(x)$ for all x close to a.

Definition 2.3.1. If $f(x)$ is differentiable at a, then the best linear approximation is given by

$$L_f(x) = f(a) + f'(a)(x - a)$$

Conversely, if there is a best linear approximation for f at a, then f is differentiable at a and hence the equation of best linear approximation of f at a is given by

$$L_f(x) = f(a) + f'(a)(x - a)$$

Example 2.3.3. Find out the linear approximation of $f(x) = x^3$ at 1.

$$f(x) = x^3 \qquad f(1) = 1$$
$$f'(x) = 3x^2 \qquad f'(1) = 3$$

The linear approximation $f(x) = x^3$ at 1 is

$$L_f(x) = 1 + 3(x - 1) = 3x - 2$$

Example 2.3.4. Find out the linear approximation of $f(x) = \sec\ x$ at 0.

$$f(x) = \sec\ x \qquad f(0) = 1$$
$$f'(x) = \sec\ x\ \tan\ x \qquad f'(0) = 0$$

The linear approximation $f(x) = \sec\ x$ at 0 is

$$L_f(x) = 1 + 0(x - 0) = 1$$

2.3.3 Exercise

Question 44. What will be the equation of the tangent of a parabola given by $f(x) = 4x^2$ at the point $(2, 16)$($i.e.,\ x = 2$)?

[**Hint:** Differentiate the function $f(x)$ at $x = 2$ and use the tangent formula $y = f'(2)(x - 2) + f(2)$.]

- **Option 1:** $y = 16(x - 1)$
- Option 2: $y = 8x + 8$
- Option 3: $y = 16x$
- Option 4: $y = 4x$

Question 45. Consider the function $f : \mathbb{R} \to \mathbb{R}$, such that $f(x) = 2x + 5$. Which of the following expression represents the best linear approximation $L_f(x)$ at the origin?

[**Hint:** Differentiate the function $f(x)$ at $x = 0$ and use the best linear approximation formula formula $L_f(x) = f'(0)(x - 0) + f(0)$.]

- Option 1: $L_f(x) = 5$
- Option 2: $L_f(x) = 2x$
- **Option 3:** $L_f(x) = 2x + 5$

- Option 4: $L_f(x) = 0$

Question 46. Let $f(x)$ be differentiable at $x = 1$ and f be represented by the curve C. If a line which passes through the point $(5, 8)$, is the tangent to the curve C at the point $(1, 0)$, then find the value of $f'(1)$. [Answer:2]

[**Hint:** Find the equation of the line using formula $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$ and compare with the equation of tangent at $x = 1$ which is $y = f'(1)(x - 1) + f(1)$.]

Question 47. Suppose m_1, m_2, and m_3 denote the slopes of the tangents of the curve represented by the function $f(x) = x^3 + 3x$, at the points $(-1, f(1))$, $(0, f(0))$, and $(1, f(1))$, respectively. The value of $m_1 + m_2 + m_3$ is [Answer:15]

Question 48. Let f be a differentiable function at $x = 1$. The tangent line to the curve represented by the function f at the point $(1, 0)$ passes through the point $(5, 8)$. What will be the value of $f'(1)$? [Answer:2]

Question 49. Suppose the tangent of the curve represented by a function f at the point $(1, f(1))$ is given by the equation $y = 3x + 2$. What is the value of $f(1)$? [Answer:5]

2.4 Finding critical points : an application

An interval of increase (respectively decrease) is an interval on which the function $f(x)$ is increasing (respectively decreasing) i.e. if $x_1 < x_2$ are points belonging to that interval, then $f(x_1) < f(x_2)$ (respectively $f(x_1) > f(x_2)$). A point x is a turning point of $f(x)$ if one of the following phenomena take place :

1) There is an interval of increase ending at x and an interval of decrease beginning at x.

2) There is an interval of decrease ending at x and an interval of increase beginning at x.

A turning point as in 1 above is called a local maximum, and a turning point as in 2 above is called a local minimum. Let us observe these points graphically in the following two examples.

Example 2.4.1. In the Figure 2.7 observe that,

- In the interval (a, c_1), the function increases and in the interval (c_1, c_2), the function decreases. So c_1 is a turning point and it is a local maximum.
- In the interval (c_1, c_2), the function decreases and in the interval (c_2, c_3), the function increases. So c_2 is a turning point and it is a local minimum.

- In the interval (c_2, c_3), the function increases and in the interval (c_3, b), the function decreases. So c_3 is a turning point and it is a local maximum.

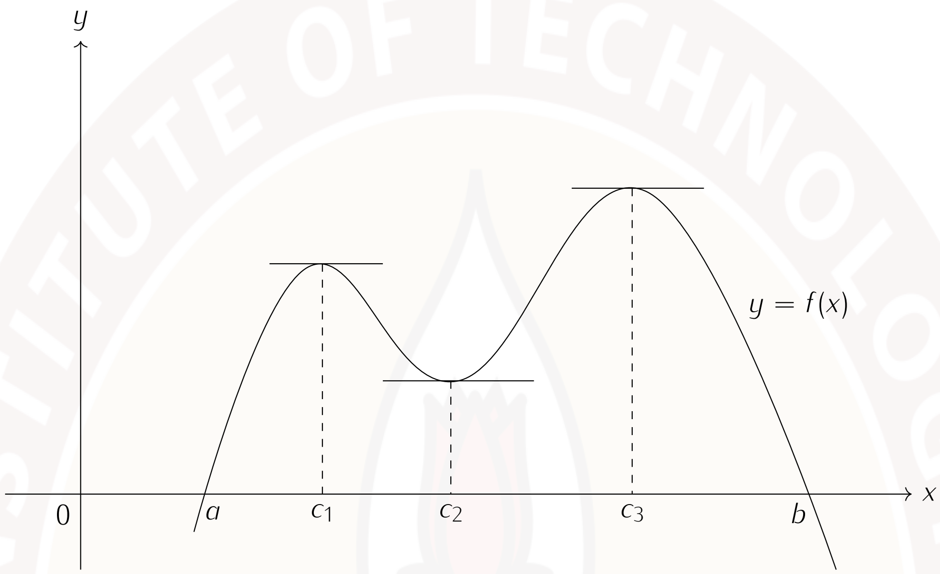

Figure 2.7:

Example 2.4.2. In Figure 2.8, point out the local maxima and local minima in the interval (a, b), as we have done in the previous example.

- c_2, c_4, c_6, c_8 and c_{10} are local maxima in the interval (a, b).
- c_1, c_3, c_5, c_7 and c_9 are local minima in the interval (a, b).

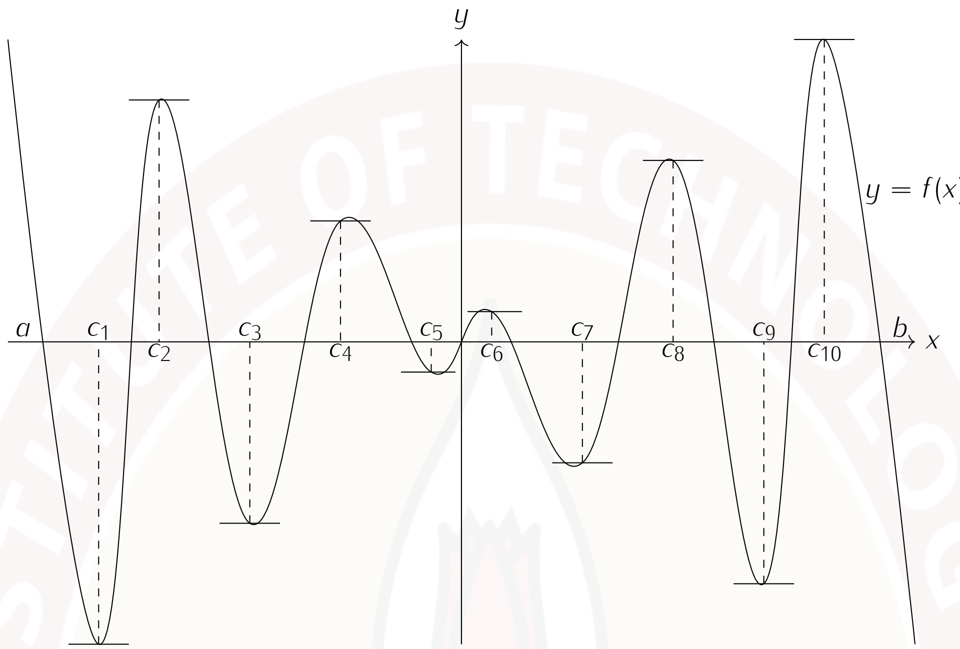

Figure 2.8:

Remark 2.4.1. Observe that in both examples, tangents at the turning points are horizontal (i.e., flat or parallel to the X-axis).

Theorem 2.4.1. *The tangent (if it exists) is horizontal at a turning point.*

Proof. Suppose a is a turning point of the function f and $f'(a)$ exists. Recall that the tangent to f at a exists (and is not vertical) is equivalent to f being differentiable at a and its equation is given by $y = f'(a)(x - a) + f(a)$. Proving that the tangent (if it exists) is horizontal at a turning point a is equivalent to showing that $f'(a) = 0$. Suppose at a, the function f has a local maximum. Then in the interval $(a - \epsilon, a)$ the function is increasing and in the interval $(a, a + \epsilon)$ the function is decreasing, where ϵ is a positive real number.
If $0 < h < \epsilon$, then

- $f(a - h) < f(a)$, i.e., $f(a - h) - f(a) < 0$
- $f(a) > f(a + h)$, i.e., $f(a + h) - f(a) < 0$

(Observe, Figure 2.9)

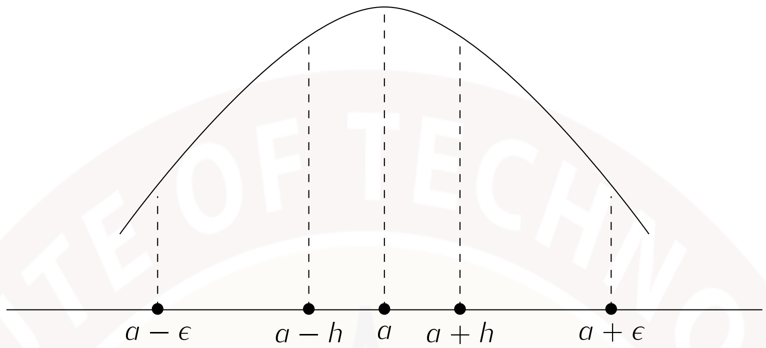

Figure 2.9:

As $f'(a)$ exists, we have

$$f'(a) = \lim_{h\to 0^-} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0^+} \frac{f(a)-f(a-h)}{h} \geq 0$$

Again,

$$f'(a) = \lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h} \leq 0$$

Hence, $f'(a) = 0$. □

2.4.1 Critical points and Saddle points

We begin this subsection with the definition of critical points.

Definition 2.4.1. A point a is called a critical point of a function $f(x)$ if either f is not differentiable at a or $f'(a) = 0$.

Thus using the Theorem 2.4.1, we can conclude that every turning point is a critical point. Therefore, we will first find the critical points to find turning points (i.e., local maxima/minima). But is the converse also true? We summarise the converse statement as the following question.

Question 50. Suppose f is differentiable. Is every critical point a turning point?

Example 2.4.3. Consider the function:

$$f(x) = (x^2 - 4x + 3.8)(x+2)x^3$$

Observe in Figure 2.10 at $x = 0$, there is no turning point of the function. But $f'(0) = 0$. So at $x = 0$, the function has a critical point, which is not a turning point. The point $x = 0$ is neither a local maximum nor a local minimum. We classify these points as saddle points, the definition of which is given below.

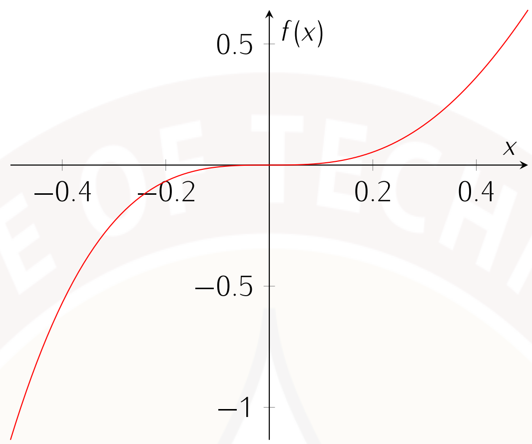

Figure 2.10:

Definition 2.4.2. A saddle point is a critical point which is neither a local maximum nor a local minimum.

Suppose f is differentiable. The question that arises is, "How do we classify critical points?" Just like the first derivative f' checks for the monotonicity of f, the second derivative f'' checks for the monotonicity of f'. So if f is twice differentiable, we check f'' at all the critical points.

Second derivative test: Let f be a function defined on an interval I and $c \in I$. Let f be twice differentiable at c. Then,

- $x = c$ is a local maximum if $f'(c) = 0$ and $f''(c) < 0$.
- $x = c$ is a local minimum if $f'(c) = 0$ and $f''(c) > 0$.
- The test fails if $f'(c) = 0$ and $f''(c) = 0$.
- A saddle point or inflection point is a critical point which is not a local maximum or a local minimum.

Example 2.4.4. Find the critical points of the function $f(x) = x^3 - 12x$ in its domain and then classify those using second derivative test.

$$f(x) = x^3 - 12x$$

$$f'(x) = 3x^2 - 12$$

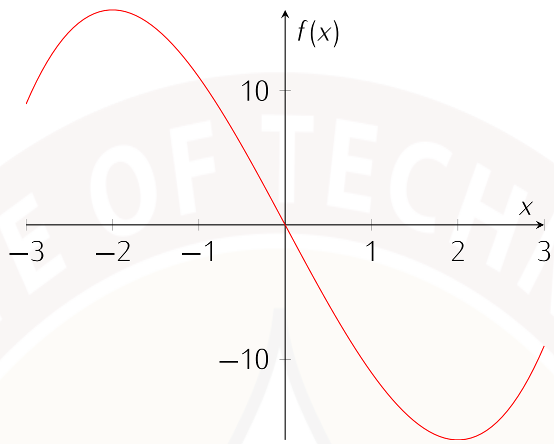

Figure 2.11:

To find the critical points of the function f, we have to solve the equation $f'(x) = 0$, i.e., $3x^2 - 12 = 0$. The roots of the equation are 2 and -2. These two are the critical points.

$$f''(x) = 6x$$

$$\begin{aligned} f''(2) &= 12 && \text{i.e., } f''(2) > 0 \\ f''(-2) &= -12 && \text{i.e., } f''(-2) < 0 \end{aligned}$$

Hence, $x = 2$ is local minimum and $x = -2$ is local maxima.

Example 2.4.5. Find the critical points of the function $f(x) = \cos\ x$ in its domain and then classify those using second derivative test.

$$f(x) = \cos\ x$$

$$f'(x) = -\sin\ x$$

To find the critical points of the function f, we have to solve the equation $f'(x) = 0$, i.e., $\sin\ x = 0$. The set of critical points is $\{k\pi \mid k \in \mathbb{Z}\}$.

$$f''(x) = -\cos\ x$$

$$f''(k\pi) = -\cos\ (k\pi) = \begin{cases} -1, & k \text{ is even integer} \\ 1, & k \text{ is odd integer} \end{cases}$$

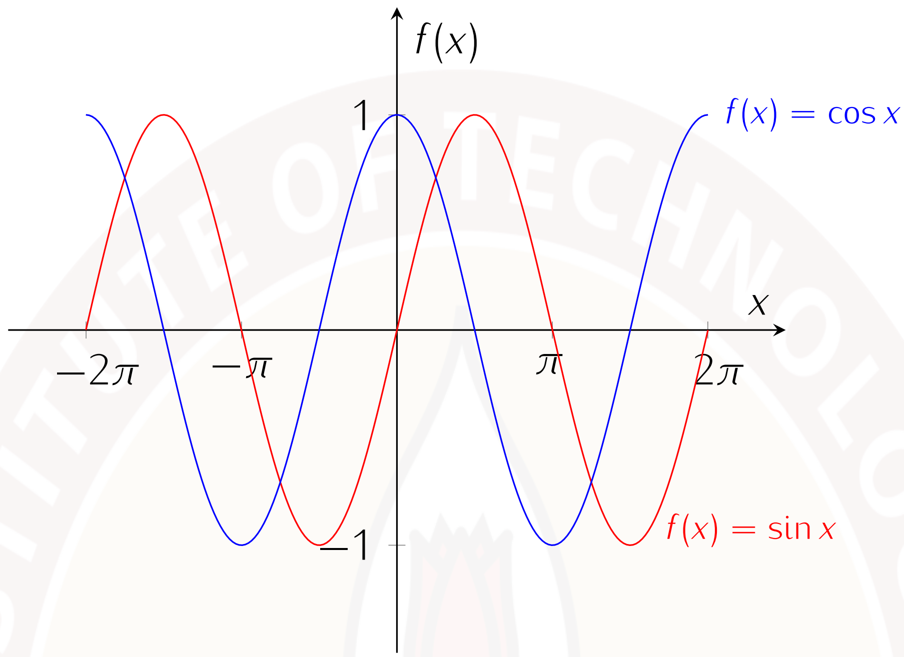

Figure 2.12:

Set of local maxima= $\{k\pi \mid k \text{ is even integer }\}$ and set of local minima= $\{k\pi \mid k \text{ is odd integer }\}$

Example 2.4.6. Find the critical points of the function $f(x) = x^3 + x^2 - x + 5$ in its domain and then classify those using second derivative test.

$$f(x) = x^3 + x^2 - x + 5$$

$$f'(x) = 3x^2 + 2x - 1$$

To find the critical points of the function f, we have to solve the equation $f'(x) = 0$, i.e., $3x^2 + 2x - 1 = 0$. The roots of the equation are -1 and $\frac{1}{3}$. These two are the critical points.

$$f''(x) = 6x + 2$$

$$f''(-1) = -4 \qquad \text{i.e., } f''(-1) < 0$$

$$f''\left(\frac{1}{3}\right) = 4 \qquad \text{i.e., } f''\left(\frac{1}{3}\right) > 0$$

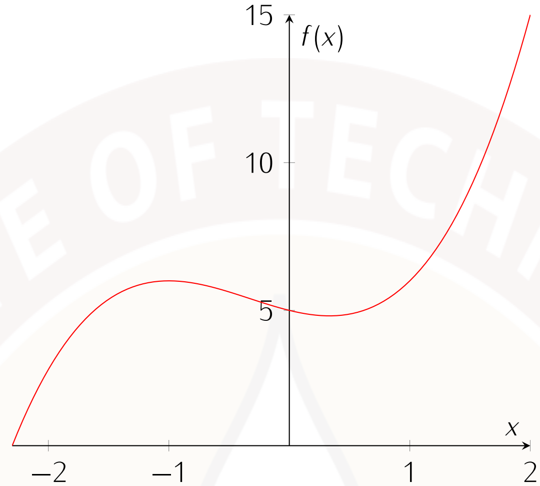

Figure 2.13:

Hence, $x = -1$ is local maximum and $x = \frac{1}{3}$ is local minima.

2.4.2 Local maxima/minima on closed interval

Sometimes we want to find the local extrema of a function f on a closed interval $I = [a, b]$. In addition to finding the extrema via the previously discussed method, it is possible that the end points could also be local extrema and so we have to consider them separately. If c is a boundary point of the domain (in this case, c is either a or b), then

- c is called a local maximum if there is an $h > 0$ such that $f(c) \geq f(x)$ for all $x \in (c - h, c]$ or $x \in [c, c + h)$, whichever is in the domain.

- c is called a local minimum if there is an $h > 0$ such that $f(c) \leq f(x)$ for all $x \in (c - h, c]$ or $x \in [c, c + h)$, whichever is in the domain.

Example 2.4.7. Find the local maxima and minima of the function $f(x) = x^2$ in the interval $[-1, 1]$.

$$f(x) = x^2$$
$$f'(x) = 2x$$

Setting $f'(x) = 0$, we get $x = 0$. So 0 is a critical point of the function.

$$f''(x) = 2$$

Hence, $f''(0) = 2 > 0$, which implies that $x = 0$ is a local minimum.
Observe that,

- $f(1) \geq f(x)$ for all $x \in \left(\frac{1}{2}, 1\right]$. Hence, $x = 1$ is local maximum.
- $f(-1) \geq f(x)$ for all $x \in \left[-1, -\frac{1}{2}\right)$. Hence, $x = -1$ is local maximum.

2.4.3 Global maximum and minimum

Sometimes we want to find the maximum or minimum value of a function f on a particular interval I.

Definition 2.4.3. The maximum or minimum over the entire function is called an "absolute" or "global" maximum or minimum.

In general this may not exist e.g. $f(x) = \frac{1}{x}$ on $I = (0, \infty)$. Here we mention one theorem without any proof which gives a sufficient condition for the existence of the maximum and minimum.

Theorem 2.4.2. *If the interval I is closed and bounded, and f is continuous, the maximum and minimum must exist.*

Note that the maximum and minimum are, in particular, local maxima or local minima unless they are on boundary points. Thus to find the maximum and minimum, we need to find all the critical points and the boundary points and check the value of f on all of them. We can do this on any function which is defined piecewise continuously with finitely many pieces on a closed and bounded interval.

Example 2.4.8. Consider the function

$$f(x) = \begin{cases} x^3 + x^2 - x + 5, & \text{if } 0 \leq x \leq 100 \\ x^3 + 2x^2 + x - 5, & \text{if } -100 \leq x < 0 \end{cases}$$

Differentiating the function we get,

$$f'(x) = \begin{cases} 3x^2 + 2x - 1, & \text{if } 0 \leq x \leq 100 \\ 3x^2 + 4x + 1, & \text{if } -100 \leq x < 0 \end{cases}$$

To find out the critical points, first we set $f'(x) = 0$.

$$3x^2 + 2x - 1 = 0$$

$$x = -1, \frac{1}{3}$$

But $x = -1$ is not inside the given interval $[0, 100]$.

$$3x^2 + 4x + 1 = 0$$

$$x = 0, -\frac{2}{3}$$

But $x = 0$ is not inside the given interval $[-100, 0)$.
So the critical points are $x = \frac{1}{3}, -\frac{2}{3}$. We need check the values of f on all critical points along with the boundary points $x = -100, 0, 100$.

$$f''(x) = \begin{cases} 6x + 2, & \text{if } 0 \le x \le 100 \\ 6x + 4, & \text{if } -100 \le x < 0 \end{cases}$$

$f''\left(\frac{1}{3}\right) = 4 > 0$, hence $x = \frac{1}{3}$ is a local minima. $f''\left(-\frac{2}{3}\right) = 0$, hence the classification is inconclusive at $x = -\frac{2}{3}$.

$$f\left(\frac{1}{3}\right) = 4.814 \text{ (approx.)}$$
$$f\left(-\frac{2}{3}\right) = -5.074 \text{ (approx.)}$$
$$f(-100) = -980105$$
$$f(0) = 5$$
$$f(100) = 1009905$$

Hence global maximum is at $x = 100$ and the maximum value is 1009905, and global minimum is at $x = -100$ and the minimum value is -980105.

2.4.4 Exercise

Question 51. Let $f(x) = \frac{1}{3}x^3 - x^2 + x$. Choose the set of correct option(s).

[Hint: Find out c, such that $f'(c) = 0$. Check for the sign of $f''(c)$.]

- Option 1: The number of critical points is 2.
- **Option 2:** The number of critical points is 1.
- Option 3: The second derivative test yields that $x = 1$ is a local maximum for $f(x)$.
- Option 4: The second derivative test yields that $x = 1$ is a local minimum for $f(x)$.

Feedback:

- **For Option 3 and 4:** Observe that, $f''(1) = 0$. So the second derivative test is inconclusive here. Also observe that, $f'(x)$ does not change sign in a small neighbourhood of 1 (in this case, it is always non-negative).

Question 52. Consider the function defined as follows:

$$f(x) = \begin{cases} -x^2 + 2x + 3 & \text{if } 0 \leq x \leq 50 \\ x^3 + 3 & \text{if } -50 \leq x < 0, \end{cases}$$

Which of the following options are correct?

[Hint: Consider the boundary points also.]

- **Option 1:** $x = 1$ is a local maximum.
- **Option 2:** $x = -50$ is a global minimum.
- Option 3: $x = 0$ is a global maximum.
- Option 4: $x = 50$ is a global minimum.
- **Option 5:** $x = 1$ is a global maximum.

Feedback:

- **For Option 3:** $f(0) = 3$, and $f(1) = 4$, so 0 cannot be a global maximum.
- **For Option 4:** $f(-50) < f(50)$, so 50 cannot be a global minimum.

Question 53. Let $f(x)$ be a function. If the function $f(x)$ has a local minimum at the point $x = 2$ and a local maximum at the point $x = 5$, then which of the following option(s) is(are) true?

- Option 1: Slope of the tangent at the point $x = 2$ is 2.
- **Option 2:** Slope of the tangent at the point $x = 5$ is 0.
- **Option 3:** Slope of the tangent at the point $x = 2$ is 0.
- Option 4: Slope of the tangent at the point $x = 5$ is 5.

Question 54. Let $f(x)$ be a differentiable function defined as $f(x) = \frac{x^4}{2} - \frac{13x^3}{3} + 11x^2 - 8x$. Which of the following option(s) is (are) true?

- **Option 1:** The number of critical points are 3.
- Option 2: The number of critical points are 4.
- **Option 3:** The point $x = 4$ is a critical point.
- Option 4: The point $x = 1$ is a critical point.

- **Option 5:** The point $x = 2$ is a critical point.

Question 55. Let A be the set of local minima and B be the set of local maxima for the function $f(x) = \frac{1}{4}x^4 - 2x^3 + \frac{11}{2}x^2 - 6x$. Choose the correct option.

- Option 1: $A = \{1, 2\}, B = \{3\}$
- Option 2: $A = \{3\}, B = \{1, 2\}$
- Option 3: $A = \{2, 3\}, B = \{1\}$
- **Option 4**: $A = \{1, 3\}, B = \{2\}$

Question 56. The minimum value of the polynomial $P(x) = (x - \alpha)(x - \beta)$ occurs at

- Option 1: $x = \dfrac{\alpha - \beta}{2}$
- Option 2: $x = \alpha - \beta$
- **Option 3:** $x = \dfrac{\alpha + \beta}{2}$
- Option 4: $x = \alpha + \beta$

Question 57. If $x + y = 50$, then find the maximum value of $2xy$. [Answer: 1250]

Question 58. Consider the function defined as follows:

$$f(x) = \begin{cases} -x^2 + 2x + 3 & \text{if } 0 \leq x \leq 50 \\ x^3 + 3 & \text{if } -50 \leq x < 0. \end{cases}$$

Which of the following options are correct?

- **Option 1:** 1 is a local maximum.
- **Option 2:** -50 is the global minimum.
- Option 3: 0 is the global maximum.
- Option 4: 50 is the global minimum.

3. Integration

"If only I had the Theorems! Then I should find the proofs easily enough"

— Bernhard Riemann

3.1 Introduction

How do we calculate the amount of water required to fill a large swimming pool? The answer is simple or complex based on the shape and size of the swimming pool. If its base is a rectangle, square or a fancy triangle, then finding the amount of water required is not difficult in any way because these geometric shapes are regular. But what if the shape of the swimming pool is not a regular geometric shape! Most of the pools start with a slight gradient and then the slope descends steeply. The sides of the pool become curved, or semi-elliptical. Then it tends to rise slightly. Is it now easy to find the size of the water to fill this pool? Yes!! In this chapter, we shall see how to compute areas using the powerful tool of integration. We shall also see how integration is nothing but the "reverse process" of differentiation.

3.2 Computing areas

In this section, we shall compute the areas of certain objects using some different techniques.

We shall start by computing the area of a rectangle, given its length l and breadth b. The formula for the area of a rectangle, as taught is school, is lb square units. We shall now use the concept of sequences and obtain the same formula.

Figure 3.1:

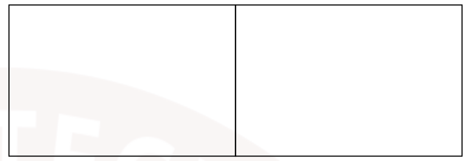

Figure 3.2:

The area of any rectangle is a function of its length l and breadth b. So let us call the area as $A(l, b)$.

Question 59. Now suppose we double the length, what effect does it have on the area?

This is equivalent to finding the area of 2 rectangles both of length l and breadth b. Thus we have $A(2l, b) = 2A(l, b)$. Similarly, if the breadth is doubled, in a similar fashion, $A(l, 2b) = 2A(l, b)$.

Question 60. What happens to the area on doubling the length and breadth?

In this case, note that we can divide the large rectangle into 4 smaller rectangles of length l and breadth b. Thus $A(2l, 2b) = 4A(l, b) = 2A(l, 2b) = 2A(2l, b)$. Thus, in general, we have $A(nl, mb) = nmA(l, b)$, for any $n, m \in \mathbb{N}$.

Question 61. Does this stop with natural numbers? Can we multiply the length with a rational number and get some "nice" relationship between their areas?

Yes, extending the same concept, we can easily show that $A(rl, b) = rA(l, b)$, where r is any rational number.

Now that we have arrived at this relationship for rational numbers, we can conclude the same for any real number as well, because every real number can be written as a limit of a sequence of rational numbers (How?!?).

Thus for any positive real numbers p, q, we have $A(pl, qb) = pqA(l, b)$. Now take $l = p$ and $b = q$. Thus we have $A(l, b) = A(l \times 1, b \times 1) = lbA(1, 1)$. $A(1, 1)$ is nothing but the area of the rectangle whose length and breadth equals 1 unit, i.e., it is nothing but the area of the unit square. Thus $A(1, 1)$ is equal to 1 square unit. This way, we have shown that the area of any rectangle of length l and breadth b is nothing but lb square units, the one that we have seen in school!

Using the same idea, we can compute areas of other geometrical objects. For example, consider a parallelogram.

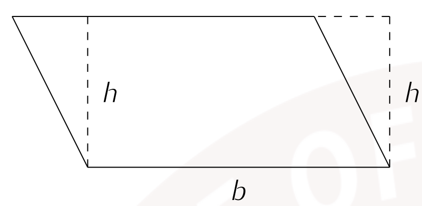

Figure 3.3:

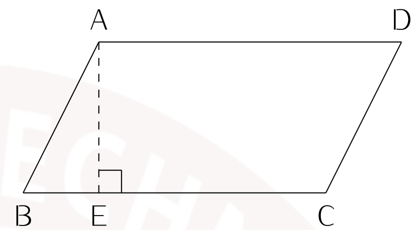

Figure 3.4:

As it is clear from the figure, the area of a parallelogram with base length b and height h is same as the area of a rectangle with length b and breadth h. Since we know to calculate the area of a rectangle, we can calculate the area of a parallelogram easily as bh square units.

If the two sides and a base angle of the parallelogram are given, i.e., base length b and side length s and $\angle AOB$, then the height can be calculated using the fact that $\triangle AOB$ is a right angled triangle.

Question 62. How do you calculate the area of a triangle, using the fact that the area of a parallelogram is bh square units?

Question 63. How do you calculate the area of a trapezium, using the areas of shapes already known to you? [Hint: Divide the trapezium into rectangles and/or triangles]

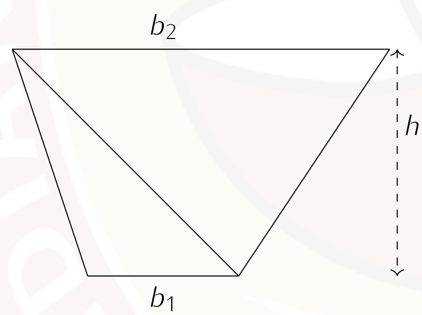

Figure 3.5:

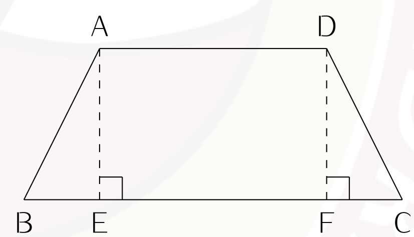

Figure 3.6:

Note that in the above discussion on computing areas, we tried to find the area of a figure using the area of a shape as basic as a rectangle. Extending this technique of "divide and conquer", we can compute areas of any quadrilateral or polygons.

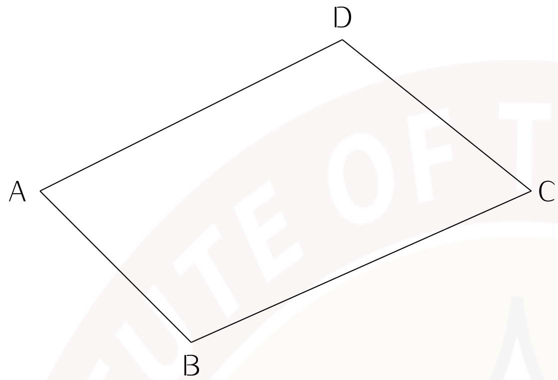

Figure 3.7:

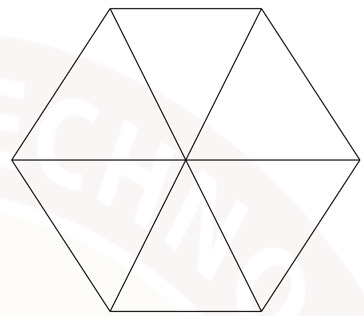

Figure 3.8:

Question 64. What about shapes of other objects which cannot be cut into rectangles or triangles, i.e., those objects which are not based on lines? What about the area of a circle?

To calculate the area of non-linear shapes (shapes that are not based on lines), we use the "divide and gradually conquer" technique. We cannot divide a circle into shapes whose area we already know. So we use the concept of limit and try to find the area of the circle. Archimedes came up with this technique. He first covered the circle by a triangle. The area of the triangle is definitely larger than that of the circle. A square gave a better estimate to the area. But a regular polygon of n sides will give a much better approximation. In this case, all the areas were larger than that of the circle. In the limiting case, i.e., as $n \to \infty$, we will get the area of the circle. The same procedure was done by inscribing these shapes in the circle. In both the cases, the limiting value was found to be equal and equal to πr^2, where r is the radius of the circle.

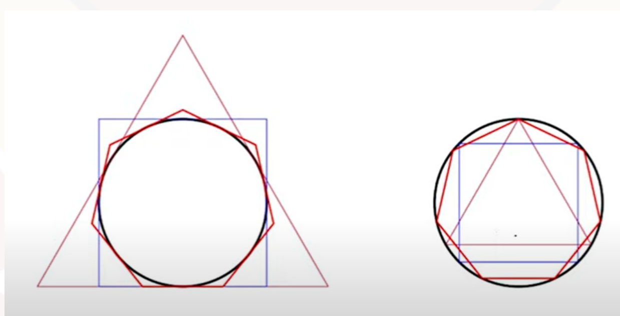

Figure 3.9:

In summary, these were the steps followed in calculating the area of an object.

1. Divide the shape in to rectangles. There may be portions left out or extra portions included.
2. Calculate the area covered by the rectangles. This may be more or lesser than the area of the actual object based on whether extra portions were included or portions were left out.
3. Keep decreasing the size of the rectangles so that the error in the area keeps decreasing.
4. The object gets approximated better as the size of the rectangles are decreased.
5. In the "limit", we get the required area.

Using these techniques, we shall study what is called as the Riemann sum in the next section.

3.2.1 Exercise

Question 65. Consider the two trapeziums $ACDB$ and $PQRS$ in the following figures. In trapezium $ACDB$, $AB = 4$ units, $CD = 2$ units and height $h_1 = 2$ units. In trapezium $PQRS$, $\theta_1 = 60°$, $\theta_2 = 45°$, $PT = TU = 3$ units.

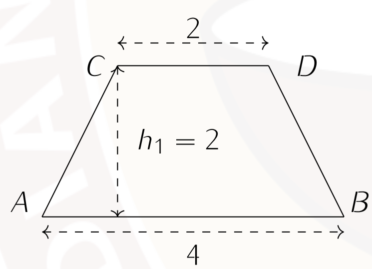

Figure 3.10:

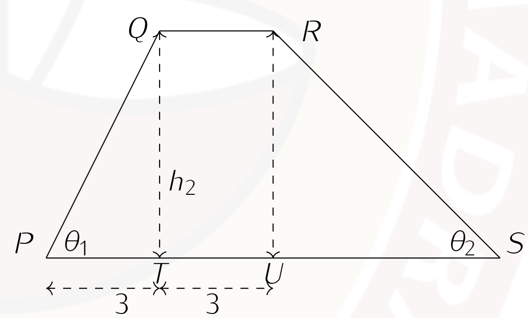

Figure 3.11:

a) Find the area of the trapezium $ACDB$. [Answer: 6 square units]

[Hint: Area of a trapezium= $\frac{1}{2} \times h \times (a + b)$, where a and b are the lengths of the parallel of sides, and h is the height, i.e., the distance between the parallel sides.]

b) Choose the correct option(s).

- Option 1: $h_2 = 3$ units
- Option 2: $h_2 = \sqrt{3}$ units
- **Option 3:** $US = 3\sqrt{3}$ units
- **Option 4:** Area of $PQRS = \frac{27(\sqrt{3}+1)}{2}$ square units
- Option 5: Area of $PQRS = \frac{27}{2}$ square units

Question 66. Suppose there are infinitely many circles $C_1, C_2, \ldots, C_n, \ldots$. The sequence $\{r_n\}$ of the radii of these circles is defined as $r_n = \frac{2n-1}{2n+2}$ units. Now as $n \to \infty$, the radius of the circles is increasing so areas of the circles are also increasing. (for example if $n = 1$, then the radius of the circle C_1 is $\frac{1}{4}$ units so are area of circle(C_1) $= \frac{\pi}{4^2}$ square units, and if $n = 2$, then the radius of the circle C_2 is $\frac{3}{6} = \frac{1}{2} > \frac{1}{4}$ units and area of circle(C_2) $= \frac{\pi}{2^2} > \frac{\pi}{4^2}$ square units and so on).

Which of the following option(s) is(are) true about the circles $C_1, C_2, \ldots$ [**Hint:** Find the limit of the sequence $\frac{2n-1}{2n+1}$ and since the sequence is increasing so the radius of any circle is less than the limit of the sequence.]

- **Option 1:** Area of the biggest circle $\leq \pi$.
- Option 2: Area of the biggest circle $> \pi$.
- **Option 3:** Area of the smallest circle is $\frac{\pi}{16}$ square units.
- Option 4: Area of the smallest circle is π square units.

3.3 Riemann Sums and the integral

In this section, we try to estimate the area under any given curve $y = f(x)$ using the techniques discussed earlier. Suppose we are given a function $f(x) = e^{-x}$ and are asked to estimate the area under this curve between -1 and 2. (The shaded region in the figure.)

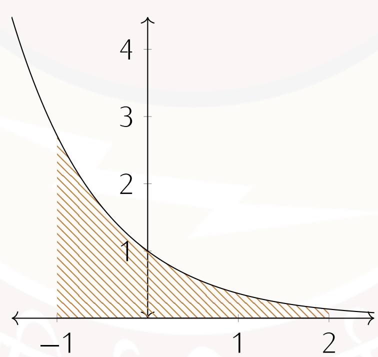

Figure 3.12:

One way of doing it is to divide the shaded region into rectangles and compute the total area of all the rectangles (since we know how to compute the area of figures that can be divided into rectangles). How do we divide this region into rectangles? We may divide the interval $[-1, 2]$ into sub-intervals and take the value of the function at the left end point as the height of the rectangle and calculate the area (as shown in the figure).

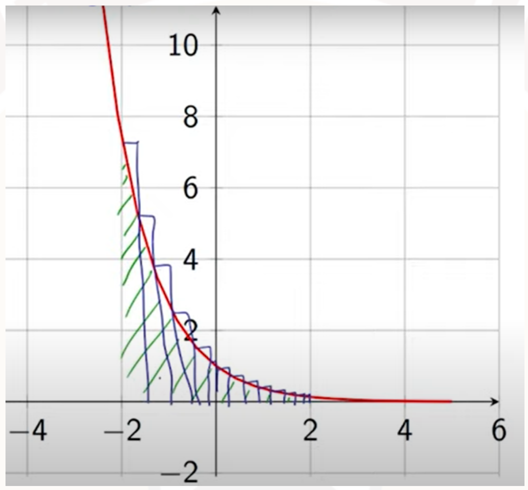

Figure 3.13:

It is clear from the figure that there is a lot of unwanted area that is included while adding up the areas of the rectangles, but if we reduce the base length of each rectangle, the approximation gets better. In the limit, we get the exact area under the curve.

Question 67. What is special about the left end points of the intervals? Why not the right end points or any other point in the interval?

To answer this question, we now formally define what are called as Riemann sums. We begin with the definition of a partition of an interval $[a, b]$.

Definition 3.3.1. Choose a finite ordered set of points including the end points from the interval $[a, b]$ i.e., let $x_0, x_1, x_2, \ldots, x_n \in [a, b]$ such that $x_{i-1} < x_i$, $x_0 = a$ and $x_n = b$. $P = \{x_0, x_1, \ldots, x_n\}$ is called a *partition of the interval.*

Definition 3.3.2. Let $f : D \to \mathbb{R}$ be a function defined on some domain $D \subseteq \mathbb{R}$ and suppose the interval $[a, b]$ is contained in the domain D. Let P be a partition of the interval $[a, b]$ defined as $a = x_0 < x_1 < \ldots < x_n = b$, $x_i^* \in [x_{i-1}, x_i]$. Define $\triangle x_i = x_i - x_{i-1}$, and $||P|| = \max_i\{\triangle x_i\}$ (called the norm of P). The *Riemann sum* of f w.r.t. the above data is defined as $S(P) = \sum_{i=1}^{n} f(x_i^*)\triangle x_i$.

Note that we have chosen an arbitrary point x_i^* in each of the sub-intervals.

In our Figure 3.13, the point that is chosen in each sub-interval is the left end point of the interval. So the area of each rectangle is $f(x_{i-1})(x_i - x_{i-1})$. Summing up the areas, we get the Riemann sum with respect to that partition.

As we reduce the size of the sub-intervals in the partition, we get a better approximation to the area under the curve. This way, the Riemann sum approximates the area under the graph of $f(x)$.

If x_i^* is chosen to be that point in $[x_{i-1}, x_i]$ such that $f(x_i^*) \geq f(x)$ for all $x \in [x_{i-1}, x_i]$, that is, $f(x_i^*)$ is the maximum value in that interval, then the Riemann sum always over estimates the region and is called the *Riemann upper sum*. Similarly, we can define the *Riemann lower sum*, where the point x_i^* is chosen such that $f(x_i^*)$ is minimum.

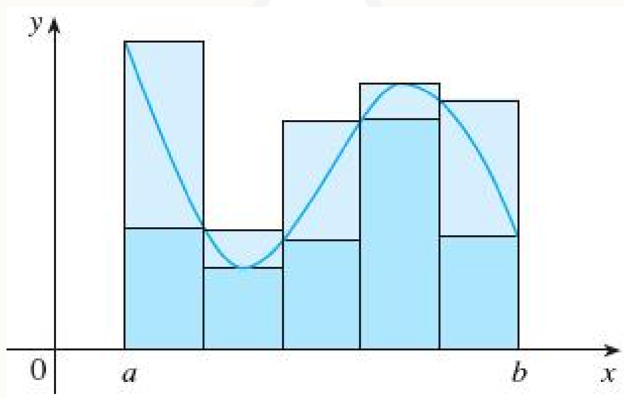

Figure 3.14:

There are also other ways of approximating the area under the curve. One such way is to divide the region into trapeziums, as shown in the figure.

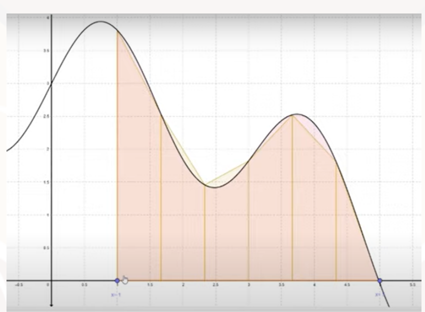

Figure 3.15:

3.3.1 Exercise

Question 68. Consider an interval [0, 10] and the following partitions
$P_1 : 0 = x_0 < x_1 < \ldots < x_{10} = 10$, where $x_i = x_{i-1} + 1$, $i = 1, 2, \ldots, 10$.
$P_2 : 0 = x_0 < x_1 < \ldots < x_5 = 10$, where $x_i = x_{i-1} + 2$, $i = 1, 2, \ldots, 5$.
$P_3 : 0 = x_0 < x_1 < x_2 < x_3 = 10$, where $x_1 = 3, x_2 = 5, x_3 = 10$.
If $\triangle x_i = x_i - x_{i-1}$ and for a partition P_k, define $||P_k|| = \max_i\{\triangle x_i\}$, then which of the following option(s) is (are) true?

- **Option 1:** For P_1, $\triangle x_i = 1$ for all i.
- **Option 2:** For P_2, $\triangle x_i = 2$ for all i.
- Option 3: For P_3, $\triangle x_1 = 5$.
- **Option 4:** For P_3, $\triangle x_3 = 5$.
- Option 5: $||P_1|| = 2$.
- Option 6: $||P_2|| = 1$.
- **Option 7:** $||P_3|| = 5$.

Question 69. Consider a closed interval $[3, 9]$. Let $P = \{x_0, x_1, \ldots, x_n\}$ be an ordered set, where $n \in \mathbb{N}$. Which of the following option(s) is (are) true regarding the partition of the interval $[3, 9]$?

- **Option 1:** If $x_0 = 3$ and $n = 1$($i.e.$, $x_1 = 9$), then P is a partition of the interval $[3, 9]$.
- **Option 2:** If $x_0 = 3$ and $x_n = 9$ and $3 < x_i < 9$ with $x_i < x_{i+1}$ for $i = 1, 2, \ldots, n-1$, then P is a partition of the interval $[3, 9]$.
- Option 3: If $x_0 = 3$ and $x_n = 9$ and $3 < x_i < 9$ for $i = 2, 3 \ldots, n-1$, and $x_1 = 2$, then P is a partition of the interval $[3, 9]$.
- Option 4: If S is the collection of all partitions of the interval $[3, 9]$, then cardinality of the set S is finite.

Feedback:

- **For option 3:** Observe that $2 = x_1 \in P$, but $2 \notin [3, 9]$
- **For option 4:** There are many ways to partition a set. For example, let $P_1^m = \{3 = x_0, x_1 = 4 + 1/m, x_2 = 5, x_3 = 9\}$, where m is any natural number.

Question 70. Let $f(x) = x$ be a function defined on the domain D which contains the interval $[0, 5]$. Let P be a partition of the interval $[0, 5]$ defined as $0 = x_0 < x_1 < \ldots < x_n = 5$ where $x_i = \frac{i\times 5}{n}$ i.e., $x_0 = 0 < \frac{5}{n} < \frac{2\times 5}{n} < \frac{3\times 5}{n} < \ldots < \frac{5(n-1)}{n} < \frac{5n}{n} = 5 = x_n$, $i = 1, 2, \ldots, n$ and $x_i^* \in [x_{i-1}, x_i]$, $\triangle x_i = x_i - x_{i-1}$. Answer the questions based on the above information.

a) Which of the following option(s) is(are) true?

- **Option 1:** If $x_i^* = x_i$, then the Riemann sum $\displaystyle\sum_{i=1}^{n} f(x_i^*)\triangle x_i$ equals $\dfrac{25(n+1)}{2n}$.
- Option 2: If $x_i^* = x_i$, then the Riemann sum $\displaystyle\sum_{i=1}^{n} f(x_i^*)\triangle x_i$ equals $\dfrac{25(n+1)}{n}$.
- **Option 3:** If $x_i^* = x_{i-1}$, then the Riemann sum $\displaystyle\sum_{i=1}^{n} f(x_i^*)\triangle x_i$ equals $\dfrac{25(n-1)}{2n}$.
- Option 4: If $x_i^* = x_{i-1}$, then the Riemann sum $\displaystyle\sum_{i=1}^{n} f(x_i^*)\triangle x_i$ equals $\dfrac{25(n-1)}{n}$.
- Option 5: If $x_i^* = x_{i-1}$, then the Riemann sum $\displaystyle\sum_{i=1}^{n} f(x_i^*)\triangle x_i$ equals $\dfrac{25}{2}$.

b) If $x_i^* = \dfrac{x_i + x_{i-1}}{2}$, then find the Riemann sum $\sum_{i=1}^{n} f(x_i^*)\triangle x_i$. [**Hint:** Observe that n will get canceled from $\sum_{i=1}^{n} f(x_i^*)\triangle x_i$.] [Answer: 12.5]

3.4 Definite Integrals

Let f be a function defined on a domain D and $[a, b] \subset D$. The definite integral of the function f from a to b is defined as,

$$\int_a^b f(x)\, dx = \lim_{||P||\to 0} S(P) = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\triangle x_i,$$

where $S(P)$ is the same as how it was defined earlier.

Let $f \geq 0$ (respectively $f \leq 0$) be a piecewise continuous function on the interval $[a, b]$. Then the area under the graph of the function, above the interval $[a, b]$ is measured by $\displaystyle\int_a^b f(x)\, dx$ (respectively -$\displaystyle\int_a^b f(x)\, dx$).

If $f \leq 0$, then the integral will represent the negative of the area bounded by the graph of $f(x)$ and the x-axis. In general, $\int_a^b f(x)\, dx$ represents the signed area. In other words, we use the integral to compute areas.

Example 3.4.1. The following figures represent the graphs of two functions. It is clear that for Curve 1, the area is positive (the shaded region lies above the x-axis) and for Curve 2, the area is negative (the shaded region lies below the x-axis).

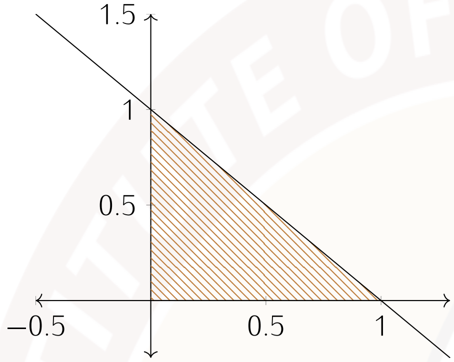

Figure 3.16: Curve 1

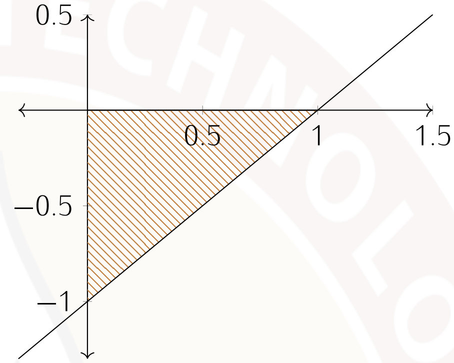

Figure 3.17: Curve 2

Example 3.4.2. Let $f(x) = 2x - 1$ be defined on the interval $[1, 2]$. The area under the graph of $f(x)$ and above the x-axis is shown in the figure.

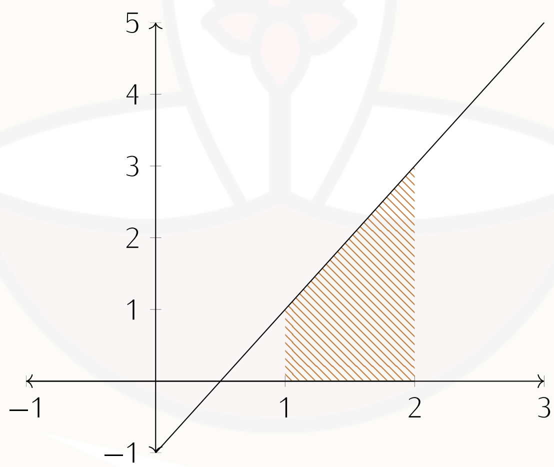

Figure 3.18:

Now, we find the area by using Riemann sums. Let us partition the interval into n pieces by taking the partition $P_n : x_0 = 1 < x_1 < \ldots < x_i = 1 + \frac{i}{n} < \ldots < x_n = 2$. Each sub-interval is of length $\frac{1}{n}$, i.e., $\triangle x_i = \frac{1}{n}$. Thus $P_n = \frac{1}{n}$. As $n \to \infty$, $P_n \to 0$. Take $x_i^* = \dfrac{x_{i-1} + x_i}{2}$, the mid points of each of the sub-intervals. Then the Riemann sum corresponding to P_n is

$$\begin{aligned}
S(P_n) &= \sum_{i=1}^{n} f(\frac{x_{i-1}+x_i}{2})\triangle x_i \\
&= \sum_{i=1}^{n} (x_{i-1}+x_i-1)\frac{1}{n} \\
&= \frac{1}{n}(x_0+2(x_1+\ldots+x_{n-1})+x_n)-n) \\
&= \frac{1}{n}(1+2(1+\frac{1}{n}+1+\frac{2}{n}+\ldots+1+\frac{n-1}{n})+2-n) \\
&= 2. \\
\lim_{P_n\to 0} S(P_n) &= \lim_{n\to\infty} 2 = 2.
\end{aligned}$$

Thus we have $\displaystyle\int_1^2 (2x-1)dx = 2.$

3.4.1 Exercise

Question 71. Let $f(x) = 3x+1$ then find the value of the integral $\int_0^2 f(x)\,dx$ using limit of Riemann sums as $n \to \infty$, for the given partition $P = \{0 = x_0, x_1 = \frac{2}{n}, \ldots, x_i = \frac{2\times i}{n}, \ldots, x_n = 2\}, i = 1, 2, \ldots, n$ and $x_i^* \in [x_{i-1}, x_i]$, where $x_i^* = \frac{2\times i}{n}$. [Answer: 8]

3.5 Anti-derivatives aka (indefinite) integrals

Let f be a function defined from a domain D in $\mathbb{R}$ to $\mathbb{R}$. An anti-derivative of the function f is a function F defined on the domain D such that $F'(x) = f(x)$ for all $x \in D$. Note that here, we assume F is differentiable.

Example 3.5.1. An anti-derivative of $f(x) = x^7 + 2x^6 - \pi x^5 + 0.5x^4 - 9$ is $F(x) = \frac{x^8}{8} + 2\frac{x^7}{7} - \pi\frac{x^6}{6} + 0.5\frac{x^5}{5} - 9x + C$, for some constant C.

Remark 3.5.1. If F is an anti-derivative of f , then so is $F_1(x) = F(x) + c$ where c is any constant (i.e. a real number). In fact, every anti-derivative has this form. Because of this, We will use the instead of an for the anti-derivative, since any two anti-derivatives only differ by a constant.

The anti-derivative for a function f is more often called the (indefinite) integral of f and denoted by $\int f(x)dx$. This can be summarised by the fundamental theorem of calculus.

Theorem 3.5.1. *(Fundamental Theorem of Calculus) Suppose f is continuous on the domain D which includes the interval $[a, b]$. Then an anti-derivative for f on (a, b) is given by*

$$F(x) = \int_a^x f(t)dt$$

Conversely, if f is continuous on the domain D which includes the interval $[a, b]$ and F is the (indefinite) integral of f, then the (definite) integral from a to b of f can be computed by

$$\int_a^b f(x)dx = F(b) - F(a)$$

Using the above theorem we can derive some important formulas for integration:

- $\displaystyle\int 1\ dx = x + c$
- $\displaystyle\int a\ dx = ax + c$
- $\displaystyle\int x^n\ dx = \frac{x^{n+1}}{n+1} + c$, for $n \neq -1$.
- $\displaystyle\int sin\ x\ dx = -cos\ x + c$
- $\displaystyle\int cos\ x\ dx = sin\ x + c$
- $\displaystyle\int sec^2\ x\ dx = tan\ x + c$
- $\displaystyle\int cosec^2\ x\ dx = -cot\ x + c$
- $\displaystyle\int sec\ x\ tan\ x\ dx = sec\ x + c$
- $\displaystyle\int cosec\ x\ cot\ x\ dx = -cosec\ x + c$
- $\displaystyle\int \frac{1}{x}\ dx = ln|x| + c$
- $\displaystyle\int e^x\ dx = e^x + c$

- $\displaystyle\int a^x \; dx = \frac{a^x}{lna} + c$, for $a > 0$, and $a \neq 1$
- $\displaystyle\int \frac{1}{\sqrt{1-x^2}} \; dx = sin^{-1} \; x + c$
- $\displaystyle\int \frac{1}{1+x^2} \; dx = tan^{-1} \; x + c$
- $\displaystyle\int \frac{1}{|x|\sqrt{x^2-1}} \; dx = sec^{-1} \; x + c$

As an immediate corollary of the fundamental theorem of calculus we get,

Corollary 3.5.2. *If $f(x)$ and $g(x)$ are two continuous function, and $F(x)$ and $G(x)$ are their anti-derivatives, i.e., $F'(x) = f(x)$ and $G'(x) = g(x)$, then*

- $\int (f(x) + g(x)) \; dx = \int f(x) \; dx + \int g(x) \; dx = F(x) + G(x) + c$
- $\int Af(x) \; dx = A \int f(x) \; dx = AF(x) + c,$ *where A is a constant.*

In each of the above cases c is the constant of integration.

3.5.1 Exercise

Question 72. Suppose f is a continuous function on the domain D which includes interval $[4, 9]$ and F is the anti-derivative of f such that $F(4) = 3$ and $F(9) = 6$. Then which of the following option(s) is (are) true?

- **Option 1:** $F'(x) = f(x)$
- **Option 2:** $F(x) - F(4) = \int_4^x f(x) \, dx$, where $x \in (4, 9)$.
- Option 3: $\int_4^9 f(x) \, dx = 9$
- **Option 4:** $\int_4^9 f(x) \, dx = 3$

Question 73. Let $f(x) = x^3 - x^2 + x$. Let $F(x)$ be the anti-derivative of $f(x)$ such that $F(2) = 6$. Then $F(x)$ equals

- Option 1: $\frac{x^4}{4} - \frac{x^3}{3} + \frac{x^2}{2} + 8$
- Option 2: $\frac{x^5}{5} - \frac{x^4}{4} + \frac{x^3}{3} + \frac{8}{3}$
- Option 3: $\frac{x^3}{3} - \frac{x^2}{2} + x + \frac{8}{3}$
- **Option 4:** $\frac{x^4}{4} - \frac{x^3}{3} + \frac{x^2}{2} + \frac{8}{3}$

3.5.2 Two important techniques

If we have to integrate the product of two differentiable functions then the method we use is known as integration by parts.

Theorem 3.5.3. *(**Integration by Parts**) Let $f(x)$ and $g(x)$ be two differentiable functions. Then*

$$\int f(x)g(x)\ dx = f(x)\int g(x)\ dx - \int \left\{ f'(x)\left(\int g(x)\ dx\right)\right\}\ dx$$

or we can write the same above thing as

$$\int f(x)g'(x)\ dx = f(x)g(x) - \int f'(x)g(x)\ dx$$

Proof. In the previous chapter we have mentioned the derivative of product of two functions. Recall that,

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$$

Integrating both the sides we get,

$$\begin{aligned} f(x)g(x) &= \int \big(f'(x)g(x) + f(x)g'(x)\big)\ dx \\ &= \int f'(x)g(x)dx + \int f(x)g'(x)dx \\ \int f(x)g'(x)dx &= f(x)g(x) - \int f'(x)g(x)dx \end{aligned}$$

$\square$

Example 3.5.2. Evaluate $\displaystyle\int x^2.2^x\, dx$

Assume $f(x) = x^2$ and $g(x) = 2^x$, and apply the integration by parts rule

$$\int f(x)g(x)\ dx = f(x)\int g(x)\ dx - \int \left\{ f'(x)(\int g(x)\ dx)\right\}\ dx$$

$\displaystyle\int g(x)\ dx = \int 2^x\ dx = \frac{2^x}{ln\ 2}$ and $f'(x) = 2x$

Hence we have,

$$\int x^2.2^x\ dx = \frac{x^2.2^x}{ln\ 2} - \int 2x\frac{2^x}{ln\ 2}\ dx + A$$

$$\int x^2.2^x \ dx = \frac{x^2.2^x}{ln\ 2} - \frac{2}{ln\ 2}\int x.2^x \ dx + A$$

where A is integration constant.

Now we have to evaluate $\int x.2^x \ dx$. Assume $f(x) = x$ and $g(x) = 2^x$.

$\int g(x) \ dx = \int 2^x \ dx = \frac{2^x}{ln\ 2}$ and $f'(x) = 1$

So,

$$\int x.2^x \ dx = \frac{x.2^x}{ln\ 2} - \int 1.\frac{2^x}{ln\ 2} \ dx + B$$

$$\int x.2^x \ dx = \frac{x.2^x}{ln\ 2} - \frac{2^x}{(ln\ 2)^2} + B$$

where B is integration constant.

Hence,

$$\begin{aligned}\int x^2.2^x \ dx &= \frac{x^2.2^x}{ln\ 2} - \frac{2}{ln\ 2}\int x.2^x \ dx \\ &= \frac{x^2.2^x}{ln\ 2} - \frac{2}{ln\ 2}\left[\frac{x.2^x}{ln\ 2} - \frac{2^x}{(ln\ 2)^2}\right] \\ &= \frac{x^2.2^x}{\ln 2} - \frac{x.2^{x+1}}{(\ln 2)^2} + \frac{2^{x+1}}{(\ln 2)^3} + C\end{aligned}$$

where C is integration constant.

Theorem 3.5.4. *(**Integration by Substitution**) Let $f(x)$ and $g(x)$ be two differential functions. Assume $g(x) = u$. Then*

$$\int f(g(x))g'(x) \ dx = \int f(u) \ du$$

Algorithm using integration by substitution:
Suppose we have to evaluate

$$\int f(g(x))g'(x) \ dx.$$

- Step 1: Substitute $g(x) = u$.
- Step 2: Replace $g'(x)dx$ by du. (Heuristically the explanation is: $g'(x) = \frac{du}{dx}$ and we bring the dx to the other side).
- Step 3: Then the original integral becomes

$$\int f(g(x))g'(x) \ dx = \int f(u) \ du.$$

- Step 4: After evaluating $\displaystyle\int f(u)\ du$, substitute back $u = g(x)$ in the final expression.

Example 3.5.3. Evaluate $\displaystyle\int sin(5x)\ dx$.
Let $u = 5x$. Then $du = 5dx$.

$$\int sin(5x)\ dx = \int sin(u)\ 5du = 5\int sin(u)\ du = 5(-cos(u)) + c = -5cos(5x) + c.$$

3.5.3 Exercise

Question 74. Which of the following option(s) is (are) true for some constant c?

- Option 1: $\int x^3 \ln x\, dx = x^4[\ln x - 1] + c$
- Option 2: $\int x^3 \ln x\, dx = \frac{x^4}{4}[\ln x + \frac{1}{4}] + c$
- Option 3: $\int x^2 2^{3x}\, dx = \frac{6x\ (2^{3x})}{\ln 2} - \frac{18\ (2^{3x})}{(\ln 2)^2} + c$
- **Option 4:** $\int 3cosec\ 2x + 2\cot 3x - x^4\, dx = \frac{3\ln|cosec\ 2x - \cot 2x|}{2} + \frac{2\ln|\sin 3x|}{3} - \frac{x^5}{5} + c$
- **Option 5:** $\int 5\tan 3x - \sec 4x\, dx = \frac{5\ln|\sec 3x|}{3} - \frac{\ln|\sec 4x + \tan 4x|}{4} + c$

3.5.4 Basic properties of definite integral

- $\displaystyle\int_a^b cf(x)dx = c\int_a^b f(x)dx$, for any $c \in \mathbb{R}$.
- $\displaystyle\int_a^b (f+g)(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$
- $\displaystyle\int_a^b (f-g)(x)dx = \int_a^b f(x)dx - \int_a^b g(x)dx$
- $\displaystyle\int_a^b f(x)dx = -\int_b^a f(x)dx$
- $\displaystyle\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ for any $c \in \mathbb{R}$.
- If $f(x) \geq g(x)$ for all but finitely many points on the interval $[a, b]$, then $\displaystyle\int_a^b f(x)dx \geq \int_a^b g(x)dx$

Example 3.5.4. When one of the limits of the definite integral is ∞ or $-\infty$, the integral is evaluated as follows:

$$\int_a^\infty f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx.$$

Such integrals are called improper integrals. Let us now evaluate $\displaystyle\int_1^\infty e^{-x}dx$.

$$\int_1^\infty e^{-x}dx = \lim_{b\to\infty} \int_1^\infty e^{-x}dx = \lim_{b\to\infty} (e^{-b} - e^{-1}) = -e^{-1}.$$

3.5.5 Exercise

Question 75. What is the value of $\displaystyle\int_2^3 x^2\ dx$?

[Hint: Use the formula: $\displaystyle\int x^n\ dx = \frac{x^{n+1}}{n+1} + c$, for $n \neq -1$.]

- Option 1: $\frac{1}{3}$
- **Option 2:** $\frac{19}{3}$
- Option 3: $\frac{5}{2}$
- Option 4: 1

Question 76. What is the value of $\displaystyle\int_1^2 (3x^2 + \frac{1}{x})\ dx$?

[Hint: Use $\int(f(x)+g(x))\ dx = \int f(x)\ dx + \int g(x)\ dx$ and the formulas, $\displaystyle\int x^n\ dx = \frac{x^{n+1}}{n+1} + c$, for $n \neq -1$, and $\displaystyle\int \frac{1}{x}\ dx = ln|x| + c$.]

- Option 1: 7
- Option 2: $\frac{7}{3} + ln\ 2$
- **Option 3:** $7 + ln\ 2$
- Option 4: $\frac{8}{3} + ln\ 2$

Question 77. Which of the following option(s) is(are) true?

- Option 1: $\int_2^3 x^2\, dx = \frac{3}{2}$

- **Option 2:** $\int_1^2 \frac{1}{x}\, dx = \ln 2$
- **Option 3:** $\int_0^{\frac{\pi}{3}} \tan x \sec x\, dx = 1$
- **Option 4:** $\int_0^2 \frac{1}{\sqrt{4-x^2}}\, dx = \frac{\pi}{2}$
- Option 5: $\int_{2020}^{2021} 1\, dx = 2021$
- **Option 6:** $\int_1^2 x^{-5}\, dx = \frac{15}{64}$
- **Option 7:** $\int_0^1 \sqrt{x}\, dx = \frac{2}{3}$
- **Option 8:** $\int_0^1 x^{\frac{3}{2}}\, dx = \frac{2}{5}$

Question 78. Which of the following option(s) is(are) true?

- **Option 1:** $\int_1^{\infty} e^{-x}\, dx = \frac{1}{e}$
- Option 2: $\int_0^{\infty} \frac{1}{4+x^2}\, dx = \pi$
- **Option 3:** $\int_1^{\infty} \frac{1}{x}\, dx$ does not exist.
- Option 4: $\int_1^{\infty} x^{-3}\, dx = -\frac{1}{2}$

Question 79. If $f(x) = x^2$ then find the area under the curve represented by $f(x)$, which lie above to X- axis in the interval [-1,2]. [Ans: 3]

Question 80. Which of the following option(s) is(are) true?
[**Hint:** Use the function $\frac{1}{1-\frac{x}{2}}$ and $\frac{1}{1+x^2}$]

- Option 1: $\int_0^1 \frac{1}{1+x^3}\, dx \leq \frac{\pi}{4}$
- **Option 2:** $\int_0^1 \frac{1}{1+x^3}\, dx \geq \frac{\pi}{4}$
- Option 3: $\int_0^1 \frac{1}{1+x^3}\, dx \geq 2\ln 2$
- **Option 4:** $\int_0^1 \frac{1}{1+x^3}\, dx \leq 2\ln 2$

3.5.6 Integration of piecewise defined functions

If a function f is defined piecewise on $[a, b]$, then the integral of f from a to b can be computed by calculating the definite integral of f on each of the sub-intervals and adding them up.

Example 3.5.5. Let $f(x) = \begin{cases} x & \text{if } 0 \leq x \leq 1 \\ 3 - x & \text{if } 1 < x \leq 2. \end{cases}$

$$\begin{aligned}
\int_0^2 f(x)dx &= \int_0^1 xdx + \int_1^2 (3-x)dx \\
&= \left[\frac{x^2}{2}\right]_0^1 + \left[3x - \frac{x^2}{2}\right]_1^2 \\
&= \frac{1}{2} + (6-3) - (2 - \frac{1}{2}) = 2
\end{aligned}$$

With these basics of integration, we conclude this chapter with the following example. We had seen that the area of the circle with radius a is πa^2 square units as obtained using limits. Now we can show the same using integration.

Example 3.5.6. Consider a circle of radius a units. The area of this circle $x^2+y^2 = a^2$ is four times the area in the first quadrant.

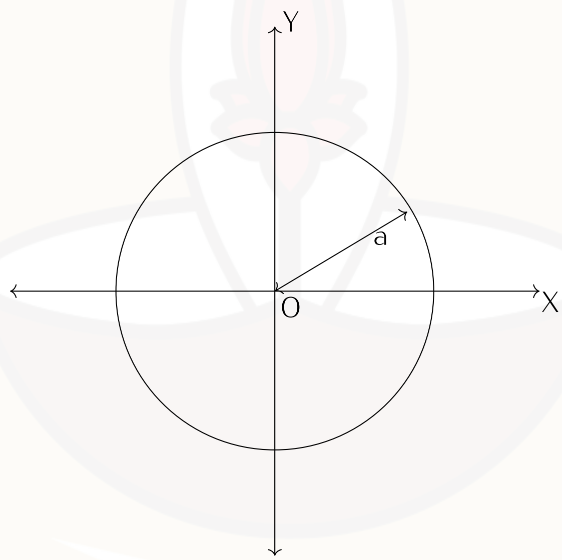

Figure 3.19:

The area in the first quadrant is nothing but the region bounded by the curve

$y = \sqrt{a^2 - x^2}$ and the x-axis. This can be obtained by evaluating $\int_0^a \sqrt{a^2 - x^2}dx$.

$$\begin{aligned}
\text{Area} &= 4\int_0^a \sqrt{a^2 - x^2}dx \\
&= 4\int_0^{\frac{\pi}{2}} acos\ \theta acos\ \theta\ d\theta, (\text{putting } x = asin\ \theta) \\
&= 4a^2 \int_0^{\frac{\pi}{2}} cos^2\ \theta\ d\theta \\
&= 4a^2 \int_0^{\frac{\pi}{2}} \frac{1 - sin2\ \theta}{2}\ d\theta \\
&= \pi a^2
\end{aligned}$$

This is same as what we had obtained earlier.

3.5.7 Exercise

Question 81. Let $f_1(x) = 3x^2$ and $f_2(x) = 4 - x^2$ represent two curves (see Figure 3.20 for reference). If A is the area which is enclosed by the curves $f_1(x)$ and $f_2(x)$, then find the value of $3A$. [Answer: 16]

[**Hint:** Find the area under both curve$f(x) = 3x^2$ and $f(x) = 4 - x^2$, and between the intersection point of the curves and above to X- axis.]

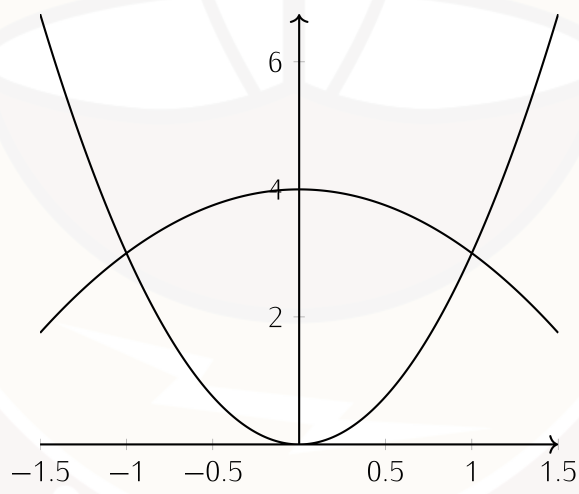

Figure 3.20: