Supplemental Material of "Robust Relation Discovery from Focusing Seeds on Large Networks"

1 Theorems and Proofs

Theorem 1 *Algorithm 1 (OPT-α) finds an α-relation core containing $V(G_s) \cap S$ with the largest α value.*

Proof. Indeed, the relation strength of a vertex is node-monotonic non-increasing [Sozio and Gionis, 2010]. For a vertex v, its only changing factor of its relation strength $rs_{G_l,S}(v,t) = \sum_{u \in N_{G_l}(v)} filter_{G_l,S}(u,v) \cdot attr_{G_l,S}(v,t)$ is $N_{G_l}(v)$. Since $N_{G_l}(v)$ is a subset of $N_{G_s}(v)$ in any induced subgraph G_l of G_s, $rs_{G_l,S}(v,t) \leq rs_{G_s,S}(v,t)$.

Due to the node-monotonic non-increasing property of the relation strength, iteratively removing the vertex with the minimum relation strength can find an α-relation core with the largest α value. The proof is inline with that of [Sozio and Gionis, 2010]. □

Theorem 2 *Algorithm 1 (OPT-α) runs in $O(|S \cap V(G_s)|m_s + n_s m' + m_s log n_s)$ where $n_s = |V(G_s)|$, $m_s = |E(G_s)|$, and m' ($m' \leq m_s$) is the number of visited edges to check the connectivity of $S \cap V(G_s)$ in a subgraph.*

Proof. According to Equation 3 in the submitted paper, computing the relation strength for each vertex takes $O(|S \cap V(G_s)|m_s)$ using D and H. Checking the connectivity of $S \cap V(G_s)$ in G_l can be done in $O(m')$. As only one vertex is deleted in each iteration, OPT-α iterates $O(n_s)$ times. Therefore, checking the connectivity takes $O(n_s m')$ totally. If the binary heap is adopted, deleting u with the minimum relation strength and adjusting the relation strength values of u's neighbors can be done in $O(m_s log n_s)$. As a result, OPT-α runs in $O(|S \cap V(G_s)|m_s + n_s m' + m_s log n_s)$. □

Theorem 3 *Algorithm 2 (OPT-split) runs in $O(|S'|m_s + n_s m'_s + m_s log n_s)$ where $S' = V(G_s) \cap S$, $m_s = |E(G_s)|$, $n_s = |V(G_s)|$, and m'_s is the number of visited edges to check the connectivity of the seeds contained in each connected component of $\mathbb{G}_c$ ($\mathbb{G}_c$ in Line 17 of OPT-split).*

Proof. For a vertex, computing its preference difference takes $O(|S'|)$. Hence computing the preference difference for each vertex and obtaining all the preference gaps can be done in $O(n_s|S'|)$. Fetching the largest preference gap takes $O(|S'|^2)$. Making a partition of G_s runs in $O(n_s|S'|)$. Using the breadth-first search, $\mathbb{G}_c$ can be obtained in $O(m_s)$. As each subgraph in $\mathbb{G}_c$ contains at least one seed, $|\mathbb{G}_c| \leq |S'|$. For each subgraph $G_i \in \mathbb{G}_c$, OPT-α takes $O(|S \cap V(G_i)|m_i + n_i m'_i + m_i log n_i)$ where $m_i = |E(G_i)|$, $n_i = |V(G_i)|$, and m'_i is the number of traversed edges to check the connectivity of $S \cap V(G_i)$ in G_i. Hence $\sum_{i=1}^{|\mathbb{G}_c|} |S \cap V(G_i)|m_i + n_i m'_i + m_i log n_i \leq |S'|m_s + n_s m'_s + m_s log n_s$. Therefore, Algorithm 2 can be done in $O(|S'|m_s + n_s m'_s + m_s log n_s)$. □

Theorem 4 *Denote $G'(V,E)$ as the subgraph induced from $G(V,E)$ by $V_h = \{v | v \in V(G) \wedge \exists s \in S, heat_{G,s}(v,t) > 0\}$ where the heat values are computed by hk-relax. Then OPT-R-RDFS runs in $O(|S|T(t,\varepsilon) + |S|m_h + n_h m'_h + m_h log n_h)$ where $T(t,\varepsilon)$ is the time complexity of hk-relax, $m_h = |E(G')|$, $n_h = |V(G')|$ and m'_h is the number of visited edges to check the connectivity of the seeds contained in each connected component of $G'(V,E)$.*

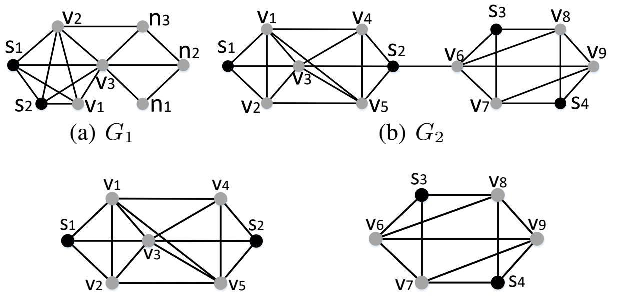
Proof. Computing the heat value of each vertex from each seed takes $O(|S|T(t,\varepsilon))$ (Line 3 of OPT-R-RDFS). Using the vertices with positive heat values, inducing a subgraph $G'(V,E)$ from $G(V,E)$ runs in $O(m_h)$ (Line 4 of OPT-R-RDFS). The shortest distance from each vertex to each seed can be computed by $|S|$ breadth-first search (Line 6 of OPT-R-RDFS), which takes $O(|S|m_h)$.

Next, the loop of OPT-R-RDFS (Lines 8-16 of OPT-R-RDFS) runs in $O(|S|m_h + n_h m'_h + m_h log n_h)$. In each iteration, both OPT-α and OPT-split run in $O(|Q_c|m_c + n_c m'_c + m_c log n_c)$ where $Q_c = V(G_c) \cap S$, $m_c = |E(G_c)|$, and $n_c = |V(G_c)|$ (Theorem 2 and Theorem 3). Only the subgraphs in $\mathbb{S}_c$ may join in the further computation. Hence the time complexity caused by a G_c during the loop is the following:

$$T(G_c) = O(|Q_c|m_c + n_c m'_c + m_c log n_c) + \sum_{i=1}^{|\mathbb{S}_c|} T(S_{ci}) \quad (1)$$

where S_{ci} is the ith element of $\mathbb{S}_c$ and $T(S_{ci})$ is the time complexity caused by S_{ci}. By expanding $T(S_{ci})$, Equation 1 can be rewritten as:

$$T(G_c) = O(|Q_c|m_c + n_c m'_c + m_c log n_c) + \sum_{i=1}^{|\mathbb{S}_c|} O(|Q_{ci}|m_{ci} + n_{ci} m'_{ci} + m_{ci} log n_{ci}) + \sum_{i=1}^{|\mathbb{S}_c|} \sum_{j=1}^{|\mathbb{S}_{ci}|} T(S_{cij}) \quad (2)$$



(a) G_1 (b) G_2

(c) G_3 ($G_3 \subseteq G_2$) (d) G_4 ($G_4 \subseteq G_2$)

Figure 1: Examples for illustrating the proposed methods. In each subgraph, black circles are the seeds.

As $|Q_c| \ge \sum_{i=1}^{\mathbb{S}_c} |Q_{ci}|$, $m_c \ge \sum_{i=1}^{\mathbb{S}_c} |m_{ci}|$, and $n_c \ge \sum_{i=1}^{\mathbb{S}_c} |n_{ci}|$, we have

$$T(G_c) \le O(|Q_c|m_c + n_c m'_c + m_c log n_c) + \sum_{i=1}^{|\mathbb{S}_c|}\sum_{j=1}^{|\mathbb{S}_{ci}|} T(S_{cij}) \tag{3}$$

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As $T(G_c)$ can be further expanded like Equation 2, the following equation is satisfied:

$$T(G_c) \le O(|Q_c|m_c + n_c m'_c + m_c log n_c) + \sum_{G_r \in \mathbb{R}} T(G_r) \tag{4}$$

As $\sum_{G_r \in \mathbb{R}} T(G_r) = 0$, $T(G_c) \le O(|Q_c|m_c + n_c m'_c + m_c log n_c)$. Similar to Equation 3, the loop (Lines 8-16) of Algorithm 3 (OPT-R-RDFS) runs in $O(|S|m_h + n_h m'_h + m_h log n_h)$.

Totally, OPT-R-RDFS runs in $O(|S|T(t,\varepsilon) + |S|m_h + n_h m'_h + m_h log n_h)$. □

2 Running Examples

A running example of OPT-α. For instance, given the graph G_1 in Fig.1(a) with $S = \{s_1, s_2\}$ and $t = 3$, the vertices n_1, n_2, n_3, v_1 and s_2 are removed sequentially. As S is no longer connected when s_2 is removed, the loop is terminated. Thereafter, the subgraph G_s consisting of s_1, s_2, v_1, v_2 and v_3 with $min\{rs_{G_s,S}(v,t)|v \in V(G_s)\} = rs_{G_s,S}(s_1,t) = 12$ is returned.

A running example of OPT-R-RDFS. Given the graph G_2 in Fig.1(b) with $S = \{s_1, s_2, s_3, s_4\}$, $t = 3$, and $\beta = 2$, $\mathbb{G}_c = \{G_2\}$ is obtained after Lines 1-7 in Algorithm 3 (OPT-R-RDFS). With G_2 fetched from $\mathbb{G}_c$, $R_\alpha = G_2$ is obtained applying OPT-α. Utilizing OPT-split, $\mathbb{S}_c = \mathbb{S}_\alpha = \{G_3, G_4\}$ is computed. As $\{G_3, G_4\}$ has larger effective relation than $\{G_2\}$, $\mathbb{G}_c = \{G_3, G_4\}$. In the following iterations, since dealing G_3 or G_4 with OPT-split will result in 0-relation cores (a connected subgraph containing only one seed is a 0-relation core), $\mathbb{R} = \{G_3, G_4\}$ is returned with $ER(\{G_3, G_4\}, S, \beta, t) = 2^2 \times 3.1 + 2^2 \times 6.2 = 37.2$.

3 Parameter Evaluation

The parameters of OPT-R-RDFS are evaluated in this section. t and ε are the parameters of heat kernel diffusion which affects the attraction of a vertex (For a vertex, its relation strength is affected by its attraction according to Equation 3 in the submitted paper). β is the weighting factor of the effective relation. 200 queries are generated for each network in Table 1 . In detail, each query is generated as follows: (1) Five communities are selected from 5,000 ground-truth communities. (2) In each selected community, three nodes are selected as the seeds. (3) Five extra seeds are selected from the network as outliers. All the communities and seeds are selected using the drawn-by-drawn method (a simple sampling method). As a result, each query consists of 20 seeds. The answer for the query is all the members from the selected communities.

Network	Abbr.	$\|V\|$	$\|E\|$	Diameter
DBLP	DP	317K	1M	21
Youtube	YT	1.1M	3M	20
LiveJournal	LJ	4M	35M	17
Orkut	OR	3.1M	117M	9

Table 1: Network statistics(K=10^3 and M=10^6)

The following criteria are used to measure the quality of the returned result:

$F_1 = \frac{2\cdot|C \cap C_T|}{|C|+|C_T|}$ measures the similarity between a set of discovered members C and the set of members from the selected ground-truth communities C_T.

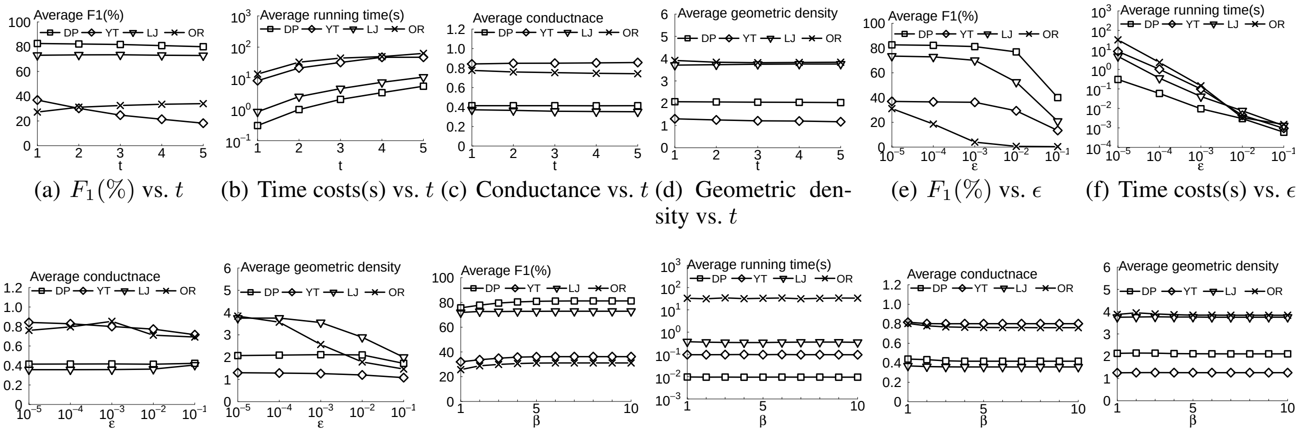
Given a graph $G(V,E)$ and a subgraph $G_s(V,E)$, the conductance of G_s is $\frac{c_s}{2\cdot m_s + c_s}$ where $m_s = |E(G_s)|$ and $c_s = |\{(u,v) \in E(G_s) : u \in V(G_s), v \notin V(G_s)\}|$. If the conductance of G_s is small, G_s is well separated from G.

The geometric density $\frac{2\cdot|E(G_s)|}{|V(G_s)|\cdot(|V(G_s)|-1)^{0.5}}$ of a subgraph $G_s(V,E)$ is the geometric mean of the average degree $\frac{|E(G_s)|}{|V(G_s)|}$ and the internal density $\frac{|E(G_s)|}{|V(G_s)|\cdot|V(G_s)-1|}$. A subgraph with large $\frac{|E(G_s)|}{|V(G_s)|\cdot|V(G_s)-1|}$ tends to be small in size, which cannot reveal rich relations among the seeds. In addition, study [Wu *et al.*, 2015] shows that a subgraph with the large average degree may contain vertices not related to the seeds. Hence the geometric density is used to measure the density of a subgraph while alleviating the drawbacks of the average degree and internal density.

In this experiment, the conductance and geometric density are used to measure the quality of the discovered subgraphs. As OPT-R-RDFS may result in several connected subgraphs for a query, the conductance for a query is the average conductance of each subgraph and the geometric density for a query is the average geometric density of each subgraph.

Evaluating t. Fig.2(a)-(d) show the performance of OPT-R-RDFS with t ranging from 1 to 5. ε is set to 10^{-5} and β is set to 8 in this experiment.

Fig.2(a) shows the average F_1. In different networks, OPT-R-RDFS achieves the best F_1 with different t values. The performance of OPT-R-RDFS is sensitive to t in Youtube and Orkut. As t gets larger, the relation strength values of the vertices not close to the seeds increase faster. Therefore, the performance of OPT-R-RDFS in Youtube shows that vertices closer to the seeds are better to reveal the relations among the seeds. In Orkut, the ground-truth communities are large in size. In order to retrieve the members in the ground-truth



(a) $F_1(\%)$ vs. t (b) Time costs(s) vs. t (c) Conductance vs. t (d) Geometric density vs. t (e) $F_1(\%)$ vs. ϵ (f) Time costs(s) vs. ϵ

(g) Conductance vs. ϵ (h) Geometric density vs. ϵ (i) $F_1(\%)$ vs. β (j) Time costs(s) vs. β (k) Conductance vs. β (l) Geometric density vs. β

Figure 2: Parameter evaluation of OPT-R-RDFS

communities, t should be larger to make the members have larger relation strength values. Therefore, F_1 in Orkut increases with the increasing t.

Fig.2(b) presents the average time costs. As t increases, OPT-R-RDFS becomes slower in each network. For one thing, a large t value requires OPT-R-RDFS visit more vertices to approximate heat values. For another, with a large t value, the large amount of vertices having heat values increase the time costs of OPT-α and OPT-split in OPT-R-RDFS.

Fig.2(c) and Fig.2(d) report the average conductance and the average geometric density respectively. Since the relation strength considers the dense connectivity of a vertex to the seeds, the subgraphs discovered by OPT-R-RDFS are also densely connected. As a result, the conductance and geometric density vary little with t.

Evaluating ε. Fig.2(e)-(h) present the performance of OPT-R-RDFS with ε ranging from 10^{-5} to 10^{-1}. In this experiment, β is set to 8. Besides, considering F_1 in Fig.2(a) and time costs in Fig.2(b), t is set to 1, 1, 3, and 2 for DBLP, Youtube, LiveJournal, and Orkut respectively.

Fig.2(e) reports the average F_1. F_1 decreases with the increasing ε. This is because some of the vertices related to the seeds are removed due to the large ε (which indicates a large error tolerance). Hence F_1 decreases.

Fig.2(f) shows the average running time. As ε increases, the efficiency of OPT-R-RDFS is improved. The larger the ε is, the fewer vertices should be visited to approximate the heat values. Besides, with fewer vertices having heat values, OPT-α and OPT-split speed up. Therefore, OPT-R-RDFS accelerates with increasing ε.

Fig.2(g) presents the average conductance. As ε increases, many vertices densely connected to the seeds are removed. As a result, the conductance of DBLP and LiveJournal increases. In Orkut, when $\varepsilon \geq 10^{-3}$, the conductance decreases. This is because ground-truth communities in Orkut are large in size. With the increasing ε, many members will not have heat values. As a result, only small and densely connected subgraphs are returned, which decreases the conductance. In Youtube, the seeds are far from each other and not densely connected in general, when the ε gets larger, only small and densely connected subgraphs can be returned. As a result, the conductance in Youtube decreases with the increasing t.

Fig.2(h) provides the average geometric density. As ε raises, many vertices which are densely connected to the seeds are pruned. Hence the geometric density declines with the increasing ε in each network.

Evaluating β**.** Fig.2(i)-(l) report the performance of OPT-R-RDFS with β ranging from 1 to 10. In this experiment, t is set to the same values as the ones in the experiment of evaluating ε. Considering the performance of OPT-R-RDFS varying with ε, ε is set to 10^{-3}, 10^{-3}, 10^{-4}, and 10^{-5} for DP, YT, LJ, and OR respectively.

Fig.2(i) show the average F_1. With the increasing β, OPT-R-RDFS tends to discover a set of α-relation cores with more seeds, which helps to discover more relations. Hence F_1 raises as β increases. When β is large enough, relations can be effectively discovered. Therefore, F_1 becomes stable with large β.

Fig.2(j)-(l) present the average time costs, the average conductance and the average geometric density respectively. As OPT-R-RDFS can effectively find the subgraphs in which the seeds are well related to each other, the metrics vary little with β.

4 More Comparison Results

In this section, a natural method called BASE is designed to compare with OPT-R-RDFS. Given a graph $G(V, E)$, a set of seeds S, and a threshold c, BASE proceeds in a sequence of steps as follows:

1. For each seed $s \in S$, a subgraph G_i is discovered using a method of community search. Denote $\mathbb{G}_s$ as the set of discovered subgraphs.
2. The subgraphs in $\mathbb{G}_s$ are merged iteratively by Jaccard similarity $J(G_i, G_j) = \frac{|V(G_i) \cap V(G_j)|}{|V(G_i) \cup V(G_j)|}$ where $G_i, G_j \in \mathbb{G}_s$:
   2.1. Find two subgraphs G_i and G_j from $\mathbb{G}_s$ achieving the largest Jaccard similarity.
   2.2. If $J(G_i, G_j) \geq c$, $G_{i \cup j} = G_i \cup G_j$, $\mathbb{G}_s = \mathbb{G}_s \cup \{G_{i \cup j}\} \backslash \{G_i, G_j\}$; Otherwise, $\{G' | G' \in \mathbb{G}_s \wedge |V(G') \cap S| > 1\}$ is returned.

Table 2: Average $F_1(\%)$ of the comparing methods

Network	BASE-0.3	BASE-0.6	BASE-0.9	OPT-R-RDFS
DP	66.7	64	62	**81.2**
YT	17.3	8.7	4.3	**36.2**
LJ	70	65.7	59.3	**72.9**
OR	7.1	0.9	0.2	**31.1**

LCTC (A state-of-the-art of community search) is applied in Step 1 due to its outstanding performance in discovering ground-truth communities [Huang *et al.*, 2015].

With the same queries used in Section 5.2 in the submitted paper, we report results using 3 different c settings, 0.3, 0.6, and 0.9, for BASE. As BASE may also return multiple subgraphs, we remove the subgraphs containing a single seed for accuracy.

Table 2 reports the average F_1. Using the similarity of each community discovered by LCTC, BASE is effective to discover relations among the seeds which are strongly related to each other. However, the members discovered for a single seed are very close to the seed in BASE. As a consequence, BASE cannot discover relations among the seeds which are related but not very close to each other. Therefore, it is not as effective as OPT-R-RDFS.

References

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