

IJCAI-19 Paper #2331 - Supplemental Material

Exploiting Social Influence to Control Elections Based on Scoring Rules

Pseudo-code of the greedy algorithm for maximizing the score.

Algorithm 1 GREEDY

Require: Social graph $G = (V, E)$; Budget B ; Score function F

- 1: $A_0 = \emptyset$
 - 2: **while** $|A_0| \leq B$ **do**
 - 3: $v = \arg \max_{w \in V \setminus A_0} F(A_0 \cup \{w\}) - F(A_0)$
 - 4: $A_0 = A_0 \cup \{v\}$
 - 5: **return** A_0
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Deferred proofs

Some of the proofs contained in this section are also present in the paper but we give here a more detailed version.

Proof of Lemma 1

Lemma 1. *For any set of initially active nodes A_0 and any node v ,*

$$\begin{aligned} \mathbf{P}(v \in R(A_0)) &= \sum_{G' \in \mathcal{G}} \mathbf{P}(G') \cdot \mathbf{1}_{(G', v)} \\ &= \sum_{U \subseteq N_v} \sum_{u \in U} b_{uv} \cdot \mathbf{P}((R \cap N_v) = U). \end{aligned}$$

Proof. By the law of total probability

$$\mathbf{P}(v \in R(A_0)) = \sum_{G' \in \mathcal{G}} \mathbf{P}(v \in R(A_0) \mid G') \cdot \mathbf{P}(G').$$

Given a live-edge graph G' sampled from \mathcal{G} , the value of $\mathbf{P}(v \in R(A_0) \mid G')$ is equal to 1 if v is reachable from A_0 in G' , and it is 0 otherwise. Then

$$\sum_{G' \in \mathcal{G}} \mathbf{P}(v \in R(A_0) \mid G') \cdot \mathbf{P}(G') = \sum_{G' \in \mathcal{G}} \mathbf{P}(G') \cdot \mathbf{1}_{(G', v)}.$$

which shows the first part of the lemma. We now show the following equality:

$$\sum_{G' \in \mathcal{G}} \mathbf{P}(G') \cdot \mathbf{1}_{(G', v)} = \sum_{U \subseteq N_v} \sum_{u \in U} b_{uv} \cdot \mathbf{P}((R(A_0) \cap N_v) = U). \quad (1)$$

We can re-write the left hand side as

$$\sum_{G' \in \mathcal{G}} \mathbf{P}(G') \cdot \mathbf{1}_{(G', v)} = \sum_{U \subseteq N_v} \sum_{\substack{G' \in \mathcal{G} \text{ s.t.} \\ R_{G'}(A_0) \cap N_v = U}} \mathbf{P}(G') \cdot \mathbf{1}_{(G', v)}.$$

In each live-edge graph G' for which $\mathbf{P}(G') \cdot \mathbf{1}_{(G', v)} \neq 0$ node v selected one of its incoming edges and then $\mathbf{P}(v \text{ selected } u \text{ in } LCF) = b_{uv}$, for each $u \in N_v$. Therefore, the above value is equal to

$$\begin{aligned} & \sum_{U \subseteq N_v} \sum_{\substack{G' \in \mathcal{G} \text{ s.t.} \\ R_{G'}(A_0) \cap N_v = U}} \sum_{u \in U} \mathbf{P}(G' \mid v \text{ selected } u \text{ in } LCF) b_{uv} \\ &= \sum_{U \subseteq N_v} \sum_{u \in U} b_{uv} \sum_{\substack{G' \in \mathcal{G} \text{ s.t.} \\ R_{G'}(A_0) \cap N_v = U}} \mathbf{P}(G' \mid v \text{ selected } u \text{ in } LCF), \end{aligned}$$

where the first equality is due to the law of total probability and the last one is just reordering of the terms of the sums.

In each live-edge G' that does not contain the edge (u, v) , the probability $\mathbf{P}(G' \mid v \text{ selected } u \text{ in } LCF)$ is equal to zero. Then,

$$\begin{aligned} & \sum_{\substack{G' \in \mathcal{G} \text{ s.t.} \\ R_{G'}(A_0) \cap N_v = U}} \mathbf{P}(G' \mid v \text{ selected } u \text{ in } LCF) \\ &= \sum_{\substack{G' \in \mathcal{G} \text{ s.t.} \\ R_{G'}(A_0) \cap N_v = U \\ (u, v) \in E'}} \mathbf{P}(G' \mid v \text{ selected } u \text{ in } LCF). \end{aligned}$$

By definition of conditional probability we have that the above sum is equal to:

$$\sum_{\substack{G' \in \mathcal{G} \text{ s.t.} \\ R_{G'}(A_0) \cap N_v = U \\ (u, v) \in E'}} \frac{\mathbf{P}(G' \cap (v \text{ selected } u \text{ in } LCF))}{b_{uv}}.$$

Since, in each G' considered in the sum, edge (u, v) belongs to G' , this is equal to:

$$\sum_{\substack{G' \in \mathcal{G} \text{ s.t.} \\ R_{G'}(A_0) \cap N_v = U \\ (u, v) \in E'}} \frac{\mathbf{P}(G')}{b_{uv}}. \quad (2)$$

Let us now consider the right hand side of Equality (1). For each $U \subseteq N_v$, the probability that $(R(A_0) \cap N_v) = U$ is given by the sum of the probabilities of all the live-edge graphs that satisfy this property, since these graphs represent disjoint events, we have:

$$\mathbf{P}((R(A_0) \cap N_v) = U) = \sum_{\substack{G' \in \mathcal{G} \text{ s.t.} \\ R_{G'}(A_0) \cap N_v = U}} \mathbf{P}(G').$$

Let us fix a node $u \in U$. For each $G' \in \mathcal{G}$ such that $(R_{G'}(A_0) \cap N_v) = U$, there exists a live-edge graph G'' that has the same edges as G' but has edge (u, v) as incoming edge of v . Since all the other edges of G'' are equal to those of G' , then $(R_{G''}(A_0) \cap N_v) = U$.

We have that

$$\mathbf{P}(G') = \begin{cases} \frac{\mathbf{P}(G'')}{b_{uv}} \cdot b_{u_i v} & \text{if } \exists(u_i, v) \text{ in } G', \\ \frac{\mathbf{P}(G'')}{b_{uv}} \cdot \left(1 - \sum_{u_i \in N_v} b_{u_i v}\right) & \text{otherwise.} \end{cases}$$

Therefore,

$$\begin{aligned} \sum_{\substack{G' \in \mathcal{G} \text{ s.t.} \\ R_{G'}(A_0) \cap N_v = U}} \mathbf{P}(G') &= \sum_{\substack{G'' \in \mathcal{G} \text{ s.t.} \\ R_{G''}(A_0) \cap N_v = U \\ (u, v) \in E'}} \left(\sum_{u_i \in N_v} b_{u_i v} \frac{\mathbf{P}(G'')}{b_{uv}} + \left(1 - \sum_{u_i \in N_v} b_{u_i v}\right) \frac{\mathbf{P}(G'')}{b_{uv}} \right) \\ &= \sum_{\substack{G'' \in \mathcal{G} \text{ s.t.} \\ R_{G''}(A_0) \cap N_v = U \\ (u, v) \in E'}} \frac{\mathbf{P}(G'')}{b_{uv}}. \end{aligned}$$

Equality (1) follows since the above expression is equal to (2). \square

Proof of Theorem 1

Theorem 1. *Given a set of initially active nodes A_0 , let A'_{LTR} and A'_{LCF} be the set of nodes such that $\tilde{\pi}_v(c_\star) = 1$ at the end of LTR and LCF, respectively, both starting from A_0 . Then, for each $v \in V$, $\mathbf{P}(v \in A'_{LTR}) = \mathbf{P}(v \in A'_{LCF})$.*

Proof. We exclude from the analysis the nodes v with $\pi_v(c_\star) = 1$ since they keep their original ranking in both models. Let us start by analyzing the LTR process. Let A be the set of active nodes that starts from A_0 . If U is the maximal subset of active neighbors of v (i.e., $U = A \cap N_v$), then we can write the probability that $v \in A'_{LTR}$ given U , as

$$\begin{aligned} \mathbf{P}(v \in A'_{LTR} \mid (A \cap N_v) = U) \\ = \mathbf{P}\left(t_v \leq \frac{\alpha(\pi_v(c_\star))}{\pi_v(c_\star) - 1} \sum_{u \in U} b_{uv}\right) = \frac{\alpha(\pi_v(c_\star))}{\pi_v(c_\star) - 1} \sum_{u \in U} b_{uv}. \end{aligned}$$

The overall probability that $v \in A'_{LTR}$ is

$$\begin{aligned} \mathbf{P}(v \in A'_{LTR}) \\ = \sum_{U \subseteq N_v} \mathbf{P}(v \in A'_{LTR} \mid (A \cap N_v) = U) \mathbf{P}(U = (A \cap N_v)) \\ = \frac{\alpha(\pi_v(c_\star))}{\pi_v(c_\star) - 1} \sum_{U \subseteq N_v} \sum_{u \in U} b_{uv} \cdot \mathbf{P}((A \cap N_v) = U). \end{aligned}$$

Let us now analyze the LCF process. In order for v to be in A'_{LCF} it must hold that the coin toss has a positive outcome and that $v \in R$. Thus,

$$\mathbf{P}(v \in A'_{LCF}) = \frac{\alpha(\pi_v(c_\star))}{\pi_v(c_\star) - 1} \mathbf{P}(v \in R(A_0)). \quad (3)$$

By Lemma 1, we have

$$\begin{aligned} \mathbf{P}(v \in A'_{LCF}) \\ = \frac{\alpha(\pi_v(c_\star))}{\pi_v(c_\star) - 1} \sum_{U \subseteq N_v} \sum_{u \in U} b_{uv} \cdot \mathbf{P}((R(A_0) \cap N_v) = U). \end{aligned}$$

Using [1, Proposition 4.1] we get

$$\mathbf{P}((R(A_0) \cap N_v) = U) = \mathbf{P}((A \cap N_v) = U),$$

and hence the theorem follows. \square

Proof of Lemma 2

Lemma 2. *Given a graph $G' \in \mathcal{G}$ and a positive integer $r \leq m$, the size of $R_{G'}(A_0, V_{c_\star}^r)$ in G' is a monotone submodular function of A_0 .*

Proof. Given $A_0 \subseteq V$, for any $v \in V \setminus A_0$, the nodes in $V_{c_\star}^r$ that are reachable from A_0 in G' are reachable also from $A_0 \cup \{v\}$. Therefore, $|R_{G'}(A_0 \cup \{v\}, V_{c_\star}^r)| \geq |R_{G'}(A_0, V_{c_\star}^r)|$.

Let us consider two sets of nodes S, T such that $S \subseteq T \subseteq V$ and a node $v \in V \setminus T$. We show that $|R_{G'}(S \cup \{v\}, V_{c_\star}^r)| - |R_{G'}(S, V_{c_\star}^r)| \geq |R_{G'}(T \cup \{v\}, V_{c_\star}^r)| - |R_{G'}(T, V_{c_\star}^r)|$. Since $v \in S \cup \{v\}$, we have that

$$\begin{aligned} & |R_{G'}(S \cup \{v\}, V_{c_\star}^r)| - |R_{G'}(S, V_{c_\star}^r)| \\ &= |R_{G'}(S \cup \{v\}, V_{c_\star}^r) \setminus R_{G'}(S, V_{c_\star}^r)|. \end{aligned}$$

Moreover, for any two sets of nodes B, C we have that $R_{G'}(B \cup C, V_{c_\star}^r) = R_{G'}(B, V_{c_\star}^r) \cup R_{G'}(C, V_{c_\star}^r)$. Hence

$$\begin{aligned} & R_{G'}(S \cup \{v\}, V_{c_\star}^r) \setminus R_{G'}(S, V_{c_\star}^r) \\ &= [R_{G'}(S, V_{c_\star}^r) \cup R_{G'}(\{v\}, V_{c_\star}^r)] \setminus R_{G'}(S, V_{c_\star}^r) \\ &= R_{G'}(\{v\}, V_{c_\star}^r) \setminus R_{G'}(S, V_{c_\star}^r). \end{aligned}$$

Similarly,

$$\begin{aligned} & |R_{G'}(T \cup \{v\}, V_{c_\star}^r)| - |R_{G'}(T, V_{c_\star}^r)| \\ &= |R_{G'}(\{v\}, V_{c_\star}^r) \setminus R_{G'}(T, V_{c_\star}^r)|. \end{aligned}$$

Since $S \subseteq T$, then $R_{G'}(S, V_{c_\star}^r) \subseteq R_{G'}(T, V_{c_\star}^r)$ and then $R_{G'}(\{v\}, V_{c_\star}^r) \setminus R_{G'}(S, V_{c_\star}^r) \supseteq R_{G'}(\{v\}, V_{c_\star}^r) \setminus R_{G'}(T, V_{c_\star}^r)$, which implies the statement. \square

Proof of Theorem 2

Theorem 2. *Given a set of initially active nodes A_0 and a node $v \in V$, let $\tilde{\pi}_v^{LTR}(c_\star)$ and $\tilde{\pi}_v^{LDR}(c_\star)$ be the position of node v at the end of LTR and LDR, respectively, both starting from A_0 . Then, $\mathbf{P}(\tilde{\pi}_v^{LTR}(c_\star) = \ell) = \mathbf{P}(\tilde{\pi}_v^{LDR}(c_\star) = \ell)$, for each $\ell = 1, \dots, \pi_v(c_\star)$.*

Proof. Let A be the set of active nodes at the end of the LTR process that starts from A_0 . The probability that an active node moves candidate c_\star from position r to position ℓ is:

$$\mathbf{P}(r, \ell) := \begin{cases} \frac{\alpha(r)}{r-1} & \text{if } \ell = 1, \\ \frac{\alpha(r)}{r-\ell} - \frac{\alpha(r)}{r-\ell+1} & \text{if } \ell = 2, \dots, r-1, \\ 1 - \alpha(r) & \text{if } \ell = r, \end{cases}$$

for each $r, \ell \in \{1, \dots, m\}$, $\ell \leq r$. In particular, for a node v , the probability that the second step of LDR yields $\tilde{\pi}_v(c_\star) = \ell$, for $\ell = 1, \dots, \pi_v(c_\star)$, is $\mathbf{P}(\pi_v(c_\star), \ell)$.

We have that

$$\mathbf{P}(\tilde{\pi}_v^{LTR}(c_\star) = \ell) = \sum_{U \subseteq N_v} \mathbf{P}(\tilde{\pi}_v^{LTR}(c_\star) = \ell \mid (A \cap N_v) = U) \mathbf{P}((A \cap N_v) = U).$$

If U is the maximal subset of active neighbors of v (i.e., $U = A \cap N_v$), then we can write the probability that $\tilde{\pi}_v^{LTR}(c_\star) = \ell$ given U as follows:

$$\mathbf{P}(\tilde{\pi}_v^{LTR}(c_\star) = \ell \mid (A \cap N_v) = U) = \mathbf{P}\left(t_v \leq \frac{\alpha(\pi_v(c_\star))}{\pi_v(c_\star) - 1} \sum_{u \in U} b_{uv}\right)$$

if $\ell = 1$;

$$\mathbf{P}(\tilde{\pi}_v^{LTR}(c_\star) = \ell \mid (A \cap N_v) = U) = \mathbf{P}\left(\frac{\alpha(\pi_v(c_\star))}{\pi_v(c_\star) - \ell + 1} \sum_{u \in U} b_{uv} < t_v \leq \frac{\alpha(\pi_v(c_\star))}{\pi_v(c_\star) - \ell} \sum_{u \in U} b_{uv}\right)$$

if $\ell = 2, \dots, \pi_v(c_\star) - 1$;

$$\mathbf{P}(\tilde{\pi}_v^{LTR}(c_\star) = \ell \mid (A \cap N_v) = U) = \mathbf{P}\left(t_v > \alpha(\pi_v(c_\star)) \sum_{u \in U} b_{uv}\right)$$

if $\ell = \pi_v(c_\star)$. In other words,

$$\mathbf{P}(\tilde{\pi}_v^{LTR}(c_\star) = \ell \mid (A \cap N_v) = U) = \mathbf{P}(\pi_v(c_\star), \ell) \sum_{u \in U} b_{uv}.$$

Therefore,

$$\mathbf{P}(\tilde{\pi}_v^{LTR}(c_\star) = \ell) = \mathbf{P}(\pi_v(c_\star), \ell) \sum_{U \subseteq N_v} \sum_{u \in U} b_{uv} \mathbf{P}((A \cap N_v) = U).$$

Recall that, in LDL , $\mathbf{P}(\tilde{\pi}_v^{LDL}(c_\star) = \ell)$ is equal to $\mathbf{P}(v \in R(A_0)) \cdot \mathbf{P}(\pi_v(c_\star), \ell)$. By Lemma 1, it follows that

$$\mathbf{P}(v \in R) = \sum_{U \subseteq N_v} \sum_{u \in U} b_{uv} \mathbf{P}((R \cap N_v) = U)$$

and hence $\mathbf{P}(\tilde{\pi}_v^{LDL}(c_\star) = \ell)$ is equal to

$$\mathbf{P}(\pi_v(c_\star), \ell) \sum_{U \subseteq N_v} \sum_{u \in U} b_{uv} \mathbf{P}((R(A_0) \cap N_v) = U).$$

Finally, using [1, Proposition 4.1], we get that $\mathbf{P}((R(A_0) \cap N_v) = U) = \mathbf{P}((A \cap N_v) = U)$. \square

Proof of Theorem 3

Theorem 3. GREEDY (Algorithm 1) is a $\frac{1}{3}(1 - 1/e)$ -approximation algorithm for the problem of election control in arbitrary scoring rule voting systems.

Proof. Let A_0 be the solution found by GREEDY (Algorithm 1) in the election control problem and let A_0^\star be the optimal solution. Let \bar{c} and \hat{c} respectively be the candidates that minimize the second term of $\mathbf{E}[\text{MoV}_{G'}(A_0)]$ and $\mathbf{E}[\text{MoV}_{G'}(A_0^\star)]$. Note that

$$\begin{aligned} \mathbf{E}[\text{MoV}_{G'}(A_0)] &= F(A_0) - F(\emptyset) + |V_c^1| - |V_{\bar{c}}^1| \\ &\quad + \sum_{r=2}^m \sum_{\ell=1}^{r-1} \sum_{h=\ell}^{r-1} \mathbf{P}(r, \ell) |R_{G'}(A_0, V_{c_\star}^r \cap V_{\bar{c}}^h)| (f(h) - f(h+1)), \end{aligned}$$

where c is the most voted candidate before the process. Since $F(A_0) - F(\emptyset) = \sum_{r=2}^m \sum_{\ell=1}^{r-1} \mathbf{P}(r, \ell) |R_{G'}(A_0, V_{c_\star}^r)| (f(\ell) - f(r))$ and $F(A_0) - F(\emptyset) \geq (1 - 1/e)(F(A_0^\star) - F(\emptyset))$, we get

$$\begin{aligned} \mathbf{E}[\text{MoV}_{G'}(A_0)] &\geq F(A_0) - F(\emptyset) + |V_c^1| - |V_{\bar{c}}^1| \\ &\geq (1 - 1/e) \left[\sum_{r=2}^m \sum_{\ell=1}^{r-1} \mathbf{P}(r, \ell) |R_{G'}(A_0^\star, V_{c_\star}^r)| (f(\ell) - f(r)) \right] + |V_c^1| - |V_{\bar{c}}^1| \\ &\geq \frac{1}{3}(1 - 1/e) \left[\sum_{r=2}^m \sum_{\ell=1}^{r-1} \mathbf{P}(r, \ell) |R_{G'}(A_0^\star, V_{c_\star}^r)| (f(\ell) - f(r)) \right] \end{aligned}$$

$$\begin{aligned}
& + \sum_{r=2}^m \sum_{\ell=1}^{r-1} \sum_{h=\ell}^{r-1} \mathbf{P}(r, \ell) |R_{G'}(A_0^*, V_{c_\star}^r \cap V_{\hat{c}}^h)| (f(h) - f(h+1)) \\
& + \sum_{r=2}^m \sum_{\ell=1}^{r-1} \sum_{h=\ell}^{r-1} \mathbf{P}(r, \ell) |R_{G'}(A_0^*, V_{c_\star}^r \cap V_{\hat{c}}^h)| (f(h) - f(h+1)) + |V_{\hat{c}}^1| - |V_{\hat{c}}^1| \Big].
\end{aligned}$$

Note that this is possible thanks to Theorem 2 and because the last two terms in the last inequality are smaller than the first term for any solution A_0 and candidate c_i since the solution A_0 can only increase the score of c_\star . Therefore, for any other candidate c_i the score can only decrease. With some additional algebra we get that

$$\begin{aligned}
\mathbf{E} [\text{MoV}_{G'}(A_0)] & \geq \frac{1}{3}(1 - 1/e)\text{MoV}_{G'}(A_0^*) + |V_{\hat{c}}^1| - |V_{\hat{c}}^1| \\
& + \sum_{r=2}^m \sum_{\ell=1}^{r-1} \sum_{h=\ell}^{r-1} \mathbf{P}(r, \ell) |R_{G'}(A_0^*, V_{c_\star}^r \cap V_{\hat{c}}^h)| (f(h) - f(h+1)).
\end{aligned}$$

By definition of \hat{c} we have that

$$\begin{aligned}
|V_{\hat{c}}^1| - |V_{\hat{c}}^1| & \geq \sum_{r=2}^m \sum_{\ell=1}^{r-1} \sum_{h=\ell}^{r-1} \mathbf{P}(r, \ell) |R_{G'}(A_0^*, V_{c_\star}^r \cap V_{\hat{c}}^h)| (f(h) - f(h+1)) \\
& - \sum_{r=2}^m \sum_{\ell=1}^{r-1} \sum_{h=\ell}^{r-1} \mathbf{P}(r, \ell) |R_{G'}(A_0^*, V_{c_\star}^r \cap V_{\hat{c}}^h)| (f(h) - f(h+1))
\end{aligned}$$

and therefore

$$\mathbf{E} [\text{MoV}_{G'}(A_0)] \geq \frac{1}{3}(1 - 1/e)\text{MoV}_{G'}(A_0^*).$$

□

Proof of Theorem 4

Theorem 4. GREEDY (Algorithm 1) is a $\frac{1}{2}(1 - 1/e)$ -approximation algorithm for the problem of destructive election control in arbitrary scoring rule voting systems.

Lemma 3. $F_D(\emptyset) - F_D(A_0) = F'(A_0) - F'(\emptyset)$, for every A_0 .

Proof. Observe that $\pi_v^{\uparrow}(c_\star) = \pi_v^{\downarrow}(c_\star)$ and that $\pi_v(c_\star) = m - \pi_v'(c_\star) + 1$. It follows that

$$\begin{aligned}
F'(A_0) - F'(\emptyset) & = \mathbf{E} \left[\sum_{v \in V} [f_{\max} - f(m - (\pi_v'(c_\star) - \pi_v^{\uparrow}(c_\star)) + 1)] \right] \\
& - \mathbf{E} \left[\sum_{v \in V} [f_{\max} - f(m - \pi_v'(c_\star) + 1)] \right] \\
& = \mathbf{E} \left[\sum_{v \in V} [f(m - \pi_v'(c_\star) + 1) - f(m - (\pi_v'(c_\star) - \pi_v^{\uparrow}(c_\star)) + 1)] \right] \\
& = \mathbf{E} \left[\sum_{v \in V} [f(m - \pi_v'(c_\star) + 1) - f(m - \pi_v'(c_\star) + 1 + \pi_v^{\uparrow}(c_\star))] \right] \\
& = \mathbf{E} \left[\sum_{v \in V} [f(\pi_v(c_\star)) - f(\pi_v(c_\star) + \pi_v^{\downarrow}(c_\star))] \right] = F(\emptyset) - F_D(A_0).
\end{aligned}$$

□

Proof of Theorem 4. Let A_0 be the solution found by GREEDY (Algorithm 1) in the election control problem and let A_0^* be the optimal solution. Let \bar{c} and \hat{c} respectively be the candidates that minimize the first term of $\mathbf{E} [\text{MOV}_D(A_0)]$ and $\mathbf{E} [\text{MOV}_D(A_0^*)]$. By Lemma 3 we have that

$$\begin{aligned}
\mathbf{E} [\text{MOV}_D(A_0)] &= \mathbf{E} [\mu(A_0) - \mu(\emptyset)] \\
&= F(\emptyset) - F_D(A_0) - |V_c^1| + |V_{\bar{c}}^1| \\
&\quad + \sum_{r=1}^{m-1} \sum_{h=r+1}^m \sum_{\ell=r+1}^m \mathbf{P}(r, \ell) |R_{G'}(A_0^*, V_{c_\star}^r \cap V_{\bar{c}}^h)| (f(h+1) - f(h+1)) \\
&= F'(A_0) - F'(\emptyset) - |V_c^1| + |V_{\bar{c}}^1| \\
&\quad + \sum_{r=1}^{m-1} \sum_{h=r+1}^m \sum_{\ell=r+1}^m \mathbf{P}(r, \ell) |R_{G'}(A_0^*, V_{c_\star}^r \cap V_{\bar{c}}^h)| (f(h+1) - f(h+1))
\end{aligned}$$

where c is the most voted candidate before the process. Since $F'(A_0) - F'(\emptyset)$ is an instance of the score in the constructive case we are able to approximate this value, thus we get

$$\begin{aligned}
\mathbf{E} [\text{MOV}_D(A_0)] &\geq \left(1 - \frac{1}{e}\right) [F'(A_0^*) - F'(\emptyset) - |V_c^1| + |V_{\bar{c}}^1| \\
&\quad + \sum_{r=1}^{m-1} \sum_{h=r+1}^m \sum_{\ell=r+1}^m \mathbf{P}(r, \ell) |R_{G'}(A_0^*, V_{c_\star}^r \cap V_{\bar{c}}^h)| (f(h+1) - f(h+1))] \\
&\geq \frac{1}{2} \left(1 - \frac{1}{e}\right) [F(\emptyset) - F_D(A_0^*) - |V_c^1| + |V_{\bar{c}}^1| \\
&\quad + \sum_{r=1}^{m-1} \sum_{h=r+1}^m \sum_{\ell=r+1}^m \mathbf{P}(r, \ell) |R_{G'}(A_0^*, V_{c_\star}^r \cap V_{\bar{c}}^h)| (f(h+1) - f(h+1)) + |V_{\hat{c}}^1| - |V_{\bar{c}}^1|] \\
&\geq \frac{1}{2} \left(1 - \frac{1}{e}\right) [\text{MOV}_D(A_0^*) + |V_{\bar{c}}^1| \\
&\quad + \sum_{r=1}^{m-1} \sum_{h=r+1}^m \sum_{\ell=r+1}^m \mathbf{P}(r, \ell) |R_{G'}(A_0^*, V_{c_\star}^r \cap V_{\bar{c}}^h)| (f(h+1) - f(h+1)) - |V_{\hat{c}}^1|]
\end{aligned}$$

By definition of \bar{c} we have that

$$\begin{aligned}
|V_{\bar{c}}^1| - |V_{\hat{c}}^1| &\geq \sum_{r=1}^{m-1} \sum_{h=r+1}^m \sum_{\ell=r+1}^m \mathbf{P}(r, \ell) |R_{G'}(A_0^*, V_{c_\star}^r \cap V_{\bar{c}}^h)| (f(h+1) - f(h+1)) \\
&\quad - \sum_{r=1}^{m-1} \sum_{h=r+1}^m \sum_{\ell=r+1}^m \mathbf{P}(r, \ell) |R_{G'}(A_0^*, V_{c_\star}^r \cap V_{\hat{c}}^h)| (f(h+1) - f(h+1))
\end{aligned}$$

and therefore

$$\text{MOV}_D(A_0) \geq \frac{1}{2} \left(1 - \frac{1}{e}\right) \text{MOV}_D(A_0^*).$$

□

Simulations Results

We performed the experiments on four heterogeneous social and communication networks, namely:

1. *facebook*,¹ an undirected network of 10 Facebook users, with 2,888 nodes and 2,981 edges;
2. *irvine*,¹ a directed network of instant messages exchanged between students at U.C. Irvine, with 1,899 nodes and 20,296 edges;
3. *netscience*,² an undirected network of research collaborations in network science, with 1,461 nodes and 2,742 edges;
4. *polblogs*,² a directed network of hyperlinks between web blogs on US politics, with 1,224 nodes and 19,025 edges.

We made the two undirected networks directed, by doubling the edges and orienting them; moreover, to adhere to the Linear Threshold Model, we assigned random weights to edges of the graphs since they are unweighted.

We considered three different scenarios, each with a different number of candidates, i.e., $k = 2, 5, 10$. For each scenario, we assigned a random preference list to each node of the networks; this assignment was performed 10 distinct times, by randomly permuting its preference list. We separately analyzed three different initial budgets, i.e., $B = 5, 10, 15$, and three different values of α (the rate at which the position of candidate c^* changes in the preference list of each node), i.e., $\alpha = 0.1, 0.5, 1$. For each combination of parameters (dataset, number of candidates, preference list assignment, budget, and α) we performed 20 experiments for each of the two considered voting systems, namely *plurality rule* and *borda count* (as example for the *scoring rule*), in the *constructive election control* scenario. We measured the *probability of victory* (POV), i.e., the fraction of times c^* won out of the 20 experiments, and the *margin of victory* (MOV), average value of the difference between the score of candidate c^* and the score of the most voted opponent.

All the experiments were run in parallel on a machine with four 16-core AMD Opteron™ 6376 with 2.3 GHz CPU, 16 MB L2 cache, 64 GB RAM, running Ubuntu 16.04.5 LTS. Overall, we performed 43,200 distinct runs for an average running time of approximately 4 hours per run.

Tables 1, 2, and 3 report detailed results of the experiments. Table 1 reports unified results for *plurality rule* and *borda count* voting systems, given their equivalence in the scenario with only $k = 2$ candidates running for the elections. Figure 1 gives a visual interpretation of the results, considering the scenario with $k = 10$, which is the “hardest” among the considered ones, since it has the maximum number of candidates and the minimum budget. The boxplots consider 200 observations each, i.e., the results obtained by permuting the preference list of each voter 10 times and repeating 20 experiments on each of them, and show the results for all considered values of B .

¹<http://konect.uni-koblenz.de>

²<http://www.umich.edu/~mejn/netdata/>

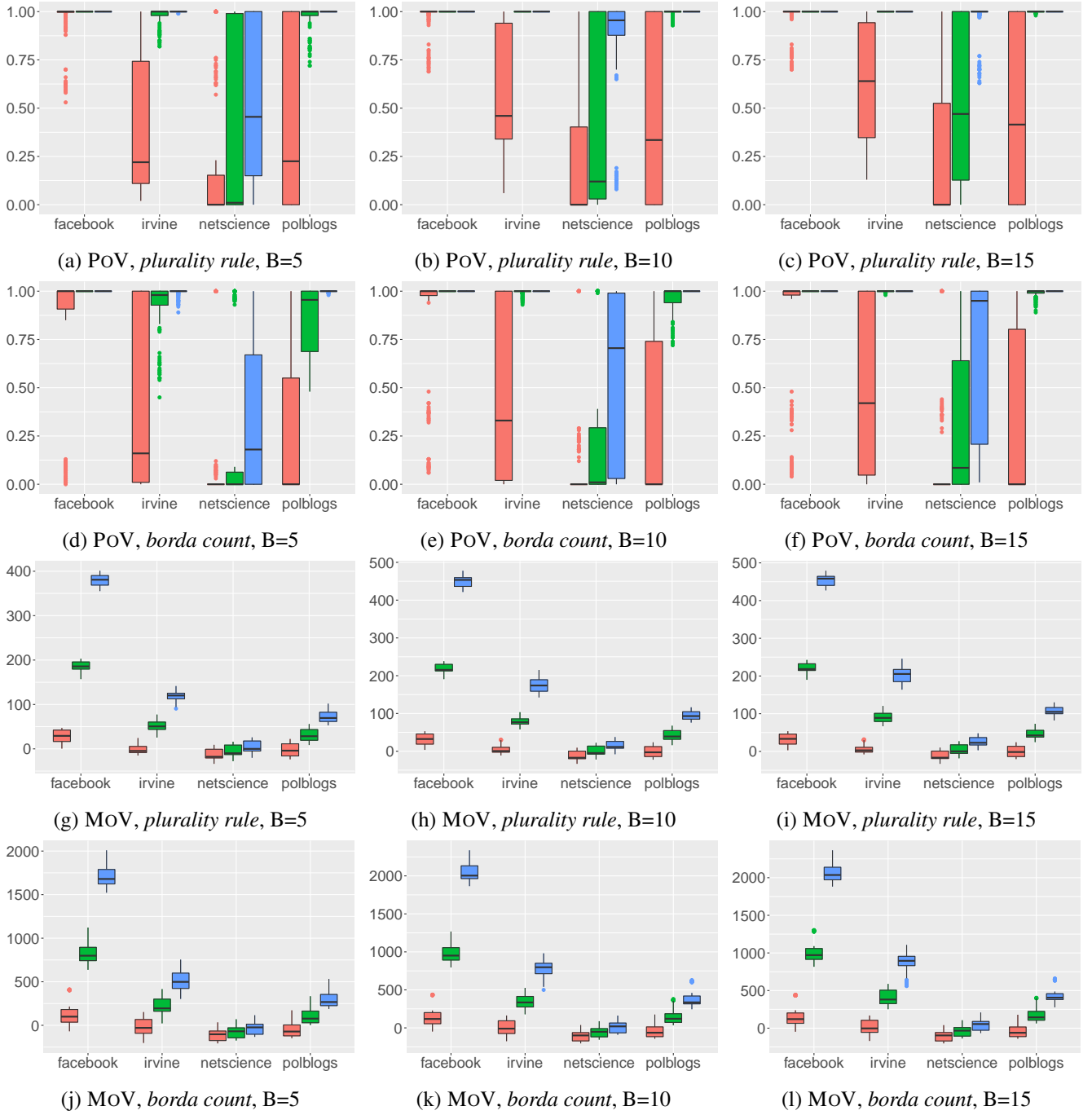


Figure 1: POV and MoV with $m = 10$. Each plot compares the results on different datasets (*facebook*, *irvine*, *netscience*, *polblogs*) and for different values of α . For each dataset, from left to right: $\alpha = 0.1$ (red), $\alpha = 0.5$ (green), $\alpha = 1.0$ (blue).

Table 1: PoV and MoV values relative to the experiments with $m = 2$.

	α	PoV						MoV					
		$B = 5$		$B = 10$		$B = 15$		$B = 5$		$B = 10$		$B = 15$	
		μ	σ	μ	σ	μ	σ	μ	σ	μ	σ	μ	σ
<i>facebook</i>	0.1	1.00	0.00	1.00	0.00	1.00	0.00	149.78	46.10	165.63	47.32	168.22	46.14
	0.5	1.00	0.00	1.00	0.00	1.00	0.00	631.90	39.11	743.61	35.70	751.42	38.55
	1.0	1.00	0.00	1.00	0.00	1.00	0.00	1234.91	31.05	1472.39	28.42	1476.18	28.37
<i>irvine</i>	0.1	0.95	0.15	0.96	0.13	0.99	0.03	66.93	39.44	82.04	40.21	89.84	38.75
	0.5	1.00	0.00	1.00	0.00	1.00	0.00	230.48	36.68	313.38	23.76	350.47	19.72
	1.0	1.00	0.00	1.00	0.00	1.00	0.00	441.67	40.74	624.90	39.16	693.16	59.91
<i>netscience</i>	0.1	0.74	0.42	0.78	0.39	0.81	0.36	29.95	40.73	32.55	40.80	34.71	40.86
	0.5	0.99	0.04	1.00	0.00	1.00	0.00	55.38	39.76	71.31	40.03	86.40	40.77
	1.0	1.00	0.00	1.00	0.00	1.00	0.00	85.98	39.57	122.36	40.12	151.88	41.31
<i>polblogs</i>	0.1	0.91	0.29	0.92	0.25	0.93	0.22	47.16	30.39	52.69	30.57	55.24	30.40
	0.5	1.00	0.00	1.00	0.00	1.00	0.00	150.11	29.34	178.11	28.84	196.28	27.95
	1.0	1.00	0.00	1.00	0.00	1.00	0.00	280.13	27.29	344.81	23.40	387.12	17.19

μ and σ are, respectively, the mean and the standard deviation of the observations averaged over the 10 preference list permutations.

Table 2: PoV and MoV values relative to the experiments with $m = 5$.

		PoV							MoV					
		α	$B = 5$		$B = 10$		$B = 15$		$B = 5$		$B = 10$		$B = 15$	
			μ	σ	μ	σ	μ	σ	μ	σ	μ	σ	μ	σ
<i>facebook</i>	Plurality	0.1	0.86	0.29	0.95	0.12	0.95	0.10	47.52	30.88	55.56	29.91	57.08	29.96
		0.5	1.00	0.00	1.00	0.00	1.00	0.00	302.92	30.50	362.70	28.65	366.13	28.50
		1.0	1.00	0.00	1.00	0.00	1.00	0.00	621.27	29.22	745.84	30.36	749.99	28.11
	Borda	0.1	0.80	0.42	0.80	0.42	0.80	0.42	80.93	114.03	96.43	114.99	99.04	114.68
		0.5	1.00	0.00	1.00	0.00	1.00	0.00	629.83	109.08	751.63	106.74	757.55	106.06
		1.0	1.00	0.00	1.00	0.00	1.00	0.00	1314.98	103.33	1572.05	98.62	1588.18	99.25
<i>irvine</i>	Plurality	0.1	0.68	0.40	0.77	0.36	0.81	0.34	10.96	22.64	18.46	22.42	22.53	22.34
		0.5	0.99	0.03	1.00	0.00	1.00	0.00	98.33	22.87	143.74	16.19	171.63	15.92
		1.0	1.00	0.00	1.00	0.00	1.00	0.00	211.20	25.12	296.33	25.73	341.69	31.26
	Borda	0.1	0.61	0.41	0.71	0.41	0.74	0.41	15.32	76.67	29.68	74.70	37.03	75.88
		0.5	0.96	0.12	1.00	0.01	1.00	0.00	203.44	74.23	307.97	70.30	359.88	64.62
		1.0	1.00	0.01	1.00	0.00	1.00	0.00	443.95	77.76	624.60	76.87	716.32	81.49
<i>netscience</i>	Plurality	0.1	0.32	0.47	0.35	0.47	0.38	0.46	-7.98	22.00	-6.65	21.73	-5.54	21.76
		0.5	0.64	0.42	0.82	0.32	0.88	0.31	4.14	21.87	13.25	21.39	20.42	21.12
		1.0	0.87	0.31	0.92	0.24	0.96	0.13	19.21	21.01	38.58	20.78	53.11	22.03
	Borda	0.1	0.40	0.46	0.44	0.46	0.47	0.46	-21.14	74.37	-18.69	74.14	-16.79	74.47
		0.5	0.63	0.48	0.69	0.46	0.74	0.43	4.71	73.90	20.49	73.70	34.58	73.56
		1.0	0.74	0.42	0.85	0.33	0.90	0.29	36.54	74.51	73.44	73.24	101.91	73.86
<i>polblogs</i>	Plurality	0.1	0.50	0.44	0.56	0.44	0.60	0.42	3.46	21.11	6.45	20.71	7.58	20.87
		0.5	1.00	0.00	1.00	0.00	1.00	0.00	56.12	20.49	74.57	18.99	82.99	18.32
		1.0	1.00	0.00	1.00	0.00	1.00	0.00	125.56	19.15	159.31	16.60	180.31	19.28
	Borda	0.1	0.54	0.46	0.62	0.44	0.65	0.45	10.63	63.13	15.99	62.35	19.55	61.99
		0.5	0.98	0.04	1.00	0.01	1.00	0.00	126.71	58.98	163.79	62.94	180.19	59.44
		1.0	1.00	0.00	1.00	0.00	1.00	0.00	271.19	62.85	339.74	59.25	375.73	55.27

μ and σ are, respectively, the mean and the standard deviation of the observations averaged over the 10 preference list permutations.

Table 3: PoV and MoV values relative to the experiments with $m = 10$.

		PoV							MoV					
		α	$B = 5$		$B = 10$		$B = 15$		$B = 5$		$B = 10$		$B = 15$	
			μ	σ	μ	σ	μ	σ	μ	σ	μ	σ	μ	σ
<i>facebook</i>	Plurality	0.1	0.96	0.12	0.97	0.08	0.97	0.08	27.99	14.88	31.81	15.55	32.16	15.50
		0.5	1.00	0.00	1.00	0.00	1.00	0.00	183.94	12.90	217.54	13.77	220.02	15.09
		1.0	1.00	0.00	1.00	0.00	1.00	0.00	378.91	13.12	450.20	15.08	454.48	14.53
	Borda	0.1	0.80	0.39	0.84	0.33	0.84	0.33	111.75	133.60	130.78	134.36	134.20	135.25
		0.5	1.00	0.00	1.00	0.00	1.00	0.00	816.22	134.28	973.03	128.19	987.42	134.47
		1.0	1.00	0.00	1.00	0.00	1.00	0.00	1701.58	133.16	2036.01	131.07	2055.55	131.70
<i>irvine</i>	Plurality	0.1	0.40	0.38	0.55	0.34	0.62	0.31	-1.04	11.82	4.09	12.00	6.14	12.05
		0.5	0.98	0.04	1.00	0.00	1.00	0.00	51.49	12.23	79.38	11.78	91.22	15.15
		1.0	1.00	0.00	1.00	0.00	1.00	0.00	118.46	9.50	175.53	18.33	203.05	19.76
	Borda	0.1	0.36	0.45	0.42	0.43	0.46	0.42	-25.12	101.84	-4.95	99.47	3.34	99.69
		0.5	0.93	0.13	1.00	0.01	1.00	0.00	216.60	100.45	347.52	98.22	408.77	107.29
		1.0	0.99	0.02	1.00	0.00	1.00	0.00	508.13	114.34	780.43	102.47	885.19	125.87
<i>netscience</i>	Plurality	0.1	0.18	0.36	0.23	0.39	0.24	0.41	-13.66	12.85	-12.97	12.84	-12.41	12.90
		0.5	0.31	0.48	0.39	0.45	0.53	0.41	-6.05	13.20	-0.57	13.21	3.75	13.24
		1.0	0.51	0.41	0.86	0.27	0.97	0.09	3.17	13.69	15.00	13.33	25.42	13.36
	Borda	0.1	0.11	0.31	0.12	0.32	0.14	0.32	-108.47	81.50	-105.35	81.60	-102.35	81.83
		0.5	0.20	0.41	0.25	0.41	0.32	0.42	-75.12	81.05	-53.48	80.35	-36.07	81.25
		1.0	0.33	0.41	0.54	0.46	0.66	0.42	-32.80	79.95	14.51	83.71	48.93	84.95
<i>polblogs</i>	Plurality	0.1	0.41	0.46	0.45	0.47	0.48	0.48	-2.22	15.10	-0.83	15.12	0.14	14.97
		0.5	0.97	0.06	1.00	0.01	1.00	0.00	31.21	14.73	41.51	14.67	46.43	13.25
		1.0	1.00	0.00	1.00	0.00	1.00	0.00	73.13	14.60	94.24	11.49	106.88	12.39
	Borda	0.1	0.25	0.41	0.27	0.44	0.28	0.45	-48.35	92.53	-42.21	92.46	-38.94	92.17
		0.5	0.87	0.18	0.96	0.07	0.99	0.02	103.80	94.58	141.53	93.96	170.65	92.91
		1.0	1.00	0.00	1.00	0.00	1.00	0.00	293.93	95.50	372.21	100.52	423.54	93.06

μ and σ are, respectively, the mean and the standard deviation of the observations averaged over the 10 preference list permutations.

References

- [1] David Kempe, Jon Kleinberg, and Éva Tardos. Maximizing the spread of influence through a social network. *Theory of Computing*, 11(4):105–147, 2015.