PRE-INTERNSHIP TEST PROJECT

DERIVATION AND SOLUTION OF THE PDEs GOVERNING FLUID FLOW IN PETROLEUM RESERVOIRS

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TABLE OF CONTENTS

DERIVATION OF THE PARTIAL DIFFERENTIAL EQUATION GOVERNING FLUID FLOW IN PETROLEUM RESERVOIRS.

SUMMARY

In this project, the diffusivity equation for the flow of slightly compressible fluids in a porous medium was derived. Following that, an exhaustive explanation of the nature of the solutions of the diffusivity equation, namely: (I) constant rate solution: infinite cylindrical reservoir with line-source well (transient state flow); (II) constant rate solution: cylindrical reservoir with constant pressure at the outer boundary (steady state flow); (III) constant rate solution: bounded cylindrical reservoir (pseudo-steady state flow), was given. Finally, the equation was solved using Laplace transform approach for the infinite-acting homogeneous reservoir case.

Derivation of the Diffusivity Equation for the Flow of Slightly Compressible Fluids in a Porous Media.

Considering flow through the radial volume element shown below where r_w is at the centre of the wellbore, Δr is the thickness of the volume element and $(q\rho)$is the mass flow rate of the liquid in transit.

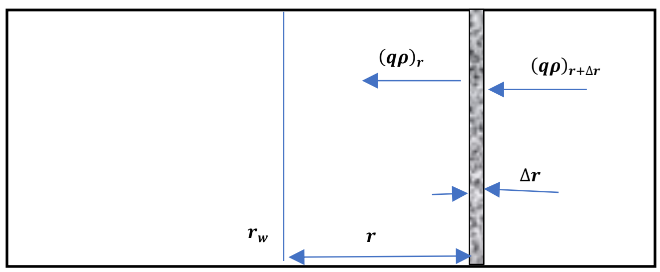

Figure 1: The radial flow of a single phase fluid in the vicinity of a producing well (Source: Dake 1998)

Assumptions

1. Mass is conserved
2. Flow is in single phase
3. Flow is laminar.
4. The reservoir is homogenous with respect to permeability, k.
5. Reservoir porosity changes with respect to pressure but constant with respect to position.
6. Fluid viscosity is constant
7. The fluid is slightly compressible.

Since mass is conserved,

$$[Mass\ entering\ volume\ element\ during\ interval\ \Delta t] - [Mass\ leaving\ volume\ element\ during\ interval\ \Delta t] = [Rate\ of\ mass\ accumulation\ within\ the\ interval\ \Delta t]$$

Mass entering the volume element during the interval Δt is given by:

$$(q\rho)_{r+\Delta r} = 2\pi(r + \Delta \mathrm{r})h(\rho v)_{r+\Delta r}$$

Mass leaving the volume element during the interval Δt is given by: $(q\rho)_r = 2\pi rh(\rho v)_r$

Rate of mass accumulation within the interval Δt: $\frac{(2\pi rh)\Delta r[(\emptyset\rho)_{t+\Delta t} - (\emptyset\rho)_t]}{\Delta t}$

Therefore,

$$2\pi h(r+\Delta r)\Delta t(v\rho)_{r+\Delta r} - 2\pi hr\Delta t(v\rho)_r = (2\pi rh)\Delta r[(\emptyset\rho)_{t+\Delta t} - (\emptyset\rho)_t] \quad (1)$$

Dividing equation (1) by $(2\pi rh)\Delta t$:

$$\left(\frac{1}{(2\pi rh)\Delta t}\right)[2\pi h(r+\Delta r)\Delta t(v\rho)_{r+\Delta r} - 2\pi hr\Delta t(v\rho)_r] = \left(\frac{1}{(2\pi rh)\Delta t}\right)(2\pi rh)\Delta r[(\emptyset\rho)_{t+\Delta t} - (\emptyset\rho)_t]$$

$$\frac{1}{r}[(r+\Delta r)(v\rho)_{r+\Delta r} - r(v\rho)_r] = \frac{\Delta r}{\Delta t}[(\emptyset\rho)_{t+\Delta t} - (\emptyset\rho)_t]$$

$$\frac{1}{r\Delta r}[(r+\Delta r)(v\rho)_{r+\Delta r} - r(v\rho)_r] = \frac{1}{\Delta t}[(\emptyset\rho)_{t+\Delta t} - (\emptyset\rho)_t]$$

Taking limit as Δr *and* Δt *approaches zero and* simplifying yields:

$$\frac{1}{r}\frac{\partial}{\partial r}[r(v\rho)] = \frac{\partial}{\partial t}(\emptyset\rho) \quad (2)$$

From Darcy's law, we have an expression for the velocity of fluid flow in a porous media under the influence of pressure differential stated as follows;

$$v = \frac{K}{\mu}\frac{\partial P}{\partial r} \quad (3)$$

Substituting for velocity in equation (3) into equation (2);

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\frac{K}{\mu}\rho\frac{\partial P}{\partial r}\right)\right] = \frac{\partial}{\partial t}(\emptyset\rho)$$

or

$$\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{K}{\mu}r\rho\frac{\partial P}{\partial r}\right) = \frac{\partial}{\partial t}(\emptyset\rho) \quad (4)$$

Equation (4) above is the ***continuity equation***.

Since we assumed constant permeability and fluid viscosity, $\left(\frac{k}{\mu}\right)$ is constant w.r.t. time and position. Applying product rule of differentiation to the RHS,

$$\frac{k}{\mu r}\left(r\frac{\partial}{\partial r}\left(\rho\frac{\partial P}{\partial r}\right) + \rho\frac{\partial P}{\partial r}\right) = \emptyset\frac{\partial \rho}{\partial t} + \rho\frac{\partial \emptyset}{\partial t}$$

Applying product rule of differentiation to the $\frac{\partial}{\partial r}\left(\rho\frac{\partial P}{\partial r}\right)$ term in the LHS of the above equation and simplifying;

$$\frac{k}{\mu r}\left[r\left(\rho\frac{\partial^2 P}{\partial r^2} + \frac{\partial P}{\partial r}\frac{\partial \rho}{\partial r}\right) + \rho\frac{\partial P}{\partial r}\right] = \emptyset\frac{\partial \rho}{\partial t} + \rho\frac{\partial \emptyset}{\partial t}$$

$$\frac{k}{\mu}\left[\left(\rho\frac{\partial^2 P}{\partial r^2} + \frac{\partial P}{\partial r}\frac{\partial \rho}{\partial r}\right) + (1/r)\rho\frac{\partial P}{\partial r}\right] = \emptyset\frac{\partial \rho}{\partial t} + \rho\frac{\partial \emptyset}{\partial t}$$

Applying chain rule on both sides of the equation;

$$\frac{k}{\mu}\left[\rho\frac{\partial^2 P}{\partial r^2}+\frac{\partial P}{\partial r}\frac{\partial \rho}{\partial P}\frac{\partial P}{\partial r}+\frac{1}{r}\rho\frac{\partial P}{\partial r}\right]=\emptyset\frac{\partial \rho}{\partial P}\frac{\partial P}{\partial t}+\rho\frac{\partial \emptyset}{\partial P}\frac{\partial P}{\partial t}$$

Dividing through by density, ρ;

$$\frac{k}{\mu}\left[\frac{\partial^2 P}{\partial r^2}+\frac{1}{\rho}\frac{\partial \rho}{\partial P}\left(\frac{\partial P}{\partial r}\right)^2+\frac{1}{r}\frac{\partial P}{\partial r}\right]=\emptyset\frac{1}{\rho}\frac{\partial \rho}{\partial P}\frac{\partial P}{\partial t}+\frac{\partial \emptyset}{\partial P}\frac{\partial P}{\partial t} \qquad (5)$$

Fluid compressibility, $c=\frac{1}{\rho}\frac{\partial \rho}{\partial P}$ and formation compressibility, $c_f=\frac{1}{\emptyset}\frac{\partial \emptyset}{\partial P}$ so that $\emptyset c_f=\frac{\partial \emptyset}{\partial P}$

Substituting for c and $\emptyset c_f$ in equation (5);

$$\frac{k}{\mu}\left[\frac{\partial^2 P}{\partial r^2}+c\left(\frac{\partial P}{\partial r}\right)^2+\frac{1}{r}\frac{\partial P}{\partial r}\right]=\emptyset c\frac{\partial P}{\partial t}+\emptyset c_f\frac{\partial P}{\partial t}$$

or

$$\frac{k}{\mu}\left[\frac{\partial^2 P}{\partial r^2}+c\left(\frac{\partial P}{\partial r}\right)^2+\frac{1}{r}\frac{\partial P}{\partial r}\right]=\emptyset\frac{\partial P}{\partial t}(c+c_f) \qquad (6)$$

The $c\left(\frac{\partial P}{\partial r}\right)^2$ term in equation (6) is negligible (Craft and Hawkins 1991), (B. F. Towler 2002). Also $\left(c+c_f\right)=c_t$ (total compressibility). Therefore, equation (6) becomes;

$$\frac{k}{\mu}\left[\frac{\partial^2 P}{\partial r^2}+\frac{1}{r}\frac{\partial P}{\partial r}\right]=\emptyset\frac{\partial P}{\partial t}c_t \quad \text{or}$$

$$\left[\frac{\partial^2 P}{\partial r^2}+\frac{1}{r}\frac{\partial P}{\partial r}\right]=\frac{\emptyset\mu C_t}{k}\frac{\partial P}{\partial t} \qquad (7)$$

Equation (7) is the ***diffusivity equation.*** It is the partial differential equation which describes the flow of slightly compressible fluid: oil and water, under isothermal conditions in hydrocarbon reservoirs. For the compressible (gas) phase, real gas pseudo pressure, $\boldsymbol{m}(\boldsymbol{p})$ replaces the $\boldsymbol{p}$ terms in equation (7) (Craft and Hawkins, 1990). This results in the equation below.

$$\left[\frac{\partial^2 m(P)}{\partial r^2}+\frac{1}{r}\frac{\partial m(P)}{\partial r}\right]=\frac{\emptyset\mu C_t}{k}\frac{\partial m(P)}{\partial t}$$

NATURE OF THE SOLUTIONS OF THE DIFFUSIVITY EQUATION

Equation (7) above is a second-order, parabolic, linear, partial differential equation. Its solution, therefore, requires two boundary conditions and an initial condition (B. F. Towler 2002). The table below summarizes the nature of the solutions and their initial and boundary conditions for slightly compressible fluids flow as depicted in Figure 1.

TABLE 1: NATURE OF THE SOLUTIONS OF THE DIFFUSIVITY EQUATION

S/N	NATURE OF SOLUTIONS	FLOW REGIME	BOUNDARY CONDITIONS	INITIAL CONDITION
1	Constant rate solutions: Infinite Cylindrical reservoirs with line-source well	Transient State flow	(i) $p = p_i \; at \; r = \infty$ (ii) $Q = C, at \; r = r_w$ C=constant, Q = flow rate,	$p(r, 0) = \; p_i; \; t = 0,$ p= reservoir pressure. p_i=initial reservoir pressure. t = time.
2	Constant rate solution: Cylindrical Reservoirs with constant pressure at the outer boundary	Steady state flow	(i) $p(r_e, t) = p_i$ at $r = r_e$ (ii) $Q = C, at \; r = r_w$ $r_e = reservoir \; external \; boundary$	$p(r, 0) = \; p_i; \; t = 0,$
3	Constant rate solutions: Bounded cylindrical reservoir	Pseudosteady steady state flow	(i)$\frac{\partial p}{\partial r} = 0 \; at \; r = r_e$ (ii) $Q = C, at \; r = r_w$	$p(r, 0) = \; p_i; \; t = 0,$
4	Constant pressure solution: Bounded cylindrical reservoir	Pseudosteady state flow	(i)$\frac{\partial p}{\partial r} = 0 \; at \; r = r_e$ (ii) $P = p_{wf} = C \; at \; r = r_w$	$P(r,0) = P_i$, t=0

When flow is initiated or altered at the wellbore, pressure disturbance travels away from the vicinity of the wellbore towards the volume element shown in Figure 1. Based on the extent of the radius of invasion, r, of the pressure disturbance and the nature of an outer boundary at $r = r_e$, the flow is classified into three regimes. Different solutions of Equation (7) exists for each of these regimes as given in Table 1 and discussed below.

1. Constant Rate Solutions: Infinite Cylindrical Reservoirs with Line-Source Well (Transient State Flow)

The transient flow equation model the flow early in the flow period while the reservoir behaves as though it were infinite because the pressure disturbance has not reached the outer boundary (B. F. Towler 2002), (Craft and Hawkins, 1990). This flow regime is called the infinite acting period or transient state regime. The boundary conditions and initial conditions required to solve for the particular solution of the diffusivity equation, equation (7), for the infinite acting period is as given in Table 1. That is, at time, $t = 0$, the reservoir pressure is equal to the initial reservoir pressure, p_i. The first boundary condition is that the rate, the rate as given by Darcy's equation, is constant at the wellbore. The second boundary condition is given by the fact that the desired solution is for the transient period so that at $r = \infty$, the reservoir pressure will remain equal to the initial reservoir pressure (Craft and Hawkins, 1990).

The solution for equation (7) for this flow regime is given by (Craft and Hawkins, 1990) as:

$$p(r,t) = p_i - \frac{162.6q\mu B}{kh}\left[log\left(\frac{kt}{\emptyset\mu c_t r_w{}^2}\right) - 3.23\right] \qquad \text{(i)}$$

Where: $k =$ permeability in millidarcy(md)
$\mu =$ fluid viscosity in centipoise(cp)
$\emptyset =$ porosity, %
$h =$ drainage area thickness
$B =$ Formation volume factor, resbbl/STB

2. Constant rate solution: Cylindrical Reservoirs with constant pressure at the outer boundary (Steady State Flow)

The time required for the reservoir to reach the external boundary is called the **infinite acting time**. When this period is reached and the outer boundary is reached, the flow behaviour in the reservoir is dependent on the nature of the outer boundary. If flow occurs across the boundary, it is assumed to be a constant pressure boundary and the flow is said to be in steady state. This occurs when there is pressure maintenance or there is pressure support from an active aquifer.

To obtain the flow equation describing this flow regime, the boundary conditions assumes constant production rate at the wellbore and that at all times, the pressure at the outer boundary is constant and equals the initial reservoir pressure. This solution is presented below:

From (7), $$\left[\frac{\partial^2 P}{\partial r^2} + \frac{1}{r}\frac{\partial P}{\partial r}\right] = \frac{\emptyset\mu C_t}{k}\frac{\partial P}{\partial t}.$$

Under steady state flow, pressure does not change in the reservoir with time. Therefore, $\frac{\partial P}{\partial t} = 0$

Substituting $0\ for\ \frac{\partial P}{\partial t}$ in (7) gives: $$\left[\frac{\partial^2 P}{\partial r^2} + \frac{1}{r}\frac{\partial P}{\partial r}\right] = 0. \qquad \text{(ii)}$$

Let $P' = \frac{\partial P}{\partial r}$. Then $\frac{\partial P'}{\partial r} = \frac{\partial^2 P}{\partial r^2}$.

Expressing equation (ii) in for of P' gives: $\frac{\partial P'}{\partial r} + \frac{1}{r}\ P' = 0$

Or $\int \frac{\partial P'}{P'} = -\int \frac{\partial r}{r} + C.$ C = Constant of integration

$\text{Ln}\, P' = -\ln r + C$ so that $P' = r^{-1}e^{C}$ or $P' = \frac{A}{r},$ A=e^{C}

But $P' = \frac{\partial P}{\partial r},$ Therefore, $\frac{\partial P}{\partial r} = \frac{A}{r}$ (iii)

Integrating equation (iii) across the wellbore center and the drainage radius;

$$\int_{p_{wf}}^{p_e} \partial p = A \int_{r_w}^{r_e} \frac{1}{r} \partial r$$

Thus, $p_{e-}\, p_{wf} = A \ln\left({}^{r_e}/_{r_w}\right)$ (iv)

From Darcy's law, $Q = 0.00708\left(\frac{kh}{\mu B}\, \frac{p_{e-}\, p_{wf}}{\ln\left({}^{r_e}/_{r_w}\right)}\right).$ (v)

Substituting (iv) in (v); we have $Q = 0.00708\left(\frac{kh}{\mu B}\, \frac{A\, \cancel{\ln\left({}^{r_e}/_{r_w}\right)}}{\cancel{\ln\left({}^{r_e}/_{r_w}\right)}}\right)$ (vi)

Therefore, $A = 141.2\, \frac{Q\mu B}{kh}$

Substituting for A in equation (iv) gives the steady state solution of the diffusivity equation, expressed in terms of the drainage radius external pressure, p_e, when skin, S=0 given below.

$$p_{e-}\, p_{wf} = 141.2\, \frac{Q\mu B}{kh} \ln\left({}^{r_e}/_{r_w}\right) \quad \text{(vii)}$$

3. Constant Rate Solutions: Bounded Cylindrical Reservoir (Pseudo-Steady State Flow)

As earlier stated, the pressure disturbance created in the wellbore, radiating radially away from it, will eventually reach the outer boundary of the reservoir. When this happens, fluid begins to flow from the boundary towards the wellbore. This can result in a constant reduction in reservoir pressure. This occurs when there is no flow across the boundary of the reservoir. The reservoir, with this flow characteristics, is said to possess no-flow boundary or in a pseudo-steady state condition. The solution to the diffusivity equation under this flow regime is presented as follows:

From (7), $\left[\frac{\partial^2 P}{\partial r^2} + \frac{1}{r}\frac{\partial P}{\partial r}\right] = \frac{\emptyset \mu C_t}{k}\frac{\partial P}{\partial t},$ or $\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial P}{\partial r}\right) = \frac{\emptyset \mu C_t}{k}\frac{\partial P}{\partial t}$

Under pseudo-steady state flow, pressure decline is constant. That is $\frac{\partial P}{\partial t} = C$ (viii)

'C' in equation (viii) above can be obtained from Material Balance using compressibility definition **(**Dake L. P. 1998) as shown below:

$$Cumulative\ production = change\ in\ pore\ volume$$

$q\partial t = -\partial V$ or $-q = \frac{\partial V}{\partial t}$ or $-q = C_t V \frac{\partial P}{\partial t}$. So that $\frac{\partial P}{\partial t} = \frac{-q}{C_t V}$ or $\frac{\partial P}{\partial t} = \frac{-q}{C_t(\pi r_e^2 h \emptyset)}$

Substituting $\frac{-q}{C_t(\pi r_e^2 h \emptyset)}$ for $\frac{\partial P}{\partial t}$ in equation (7) gives;

$$\frac{\partial}{\partial r}\left(r\frac{\partial P}{\partial r}\right) = \frac{\emptyset \mu C_t}{k}\frac{-qr}{C_t(\pi r_e^2 h \emptyset)}$$

or $$\partial\left(r\frac{\partial P}{\partial r}\right) = -\frac{q\mu r}{\pi k r_e^2 h}\partial r$$

Taking integrals of both sides, i.e. $\int \partial\left(r\frac{\partial P}{\partial r}\right) = -\int \frac{q\mu r}{\pi k r_e^2 h}\partial r$ we have;

$$r\frac{\partial P}{\partial r} = -\frac{q\mu r^2}{2\pi k r_e^2 h} + C \quad \text{(ix)}$$

Where C= integration constant.

From the stated boundary conditions (Table 1), $\frac{\partial P}{\partial r} = 0\ at\ r = r_e$. Therefore, $o = -\frac{q\mu \cancel{r_e^2}}{2\pi k \cancel{r_e^2} h} + C$ [From (ix)]

So that $C = \frac{q\mu}{2\pi k h}$.

Substituting for C in equation (ix) gives; $r\frac{\partial P}{\partial r} = -\frac{q\mu r^2}{2\pi k r_e^2 h} + \frac{q\mu}{2\pi k h} = \frac{q\mu}{2\pi k h}\left(1 - \frac{r^2}{r_e^2}\right)$

or $\frac{\partial P}{\partial r} = \frac{q\mu}{2\pi k h}\left(\frac{1}{r} - \frac{r}{r_e^2}\right)$. or $\partial p = \frac{q\mu}{2\pi k h}\left(\frac{1}{r} - \frac{r}{r_e^2}\right)\partial r$ (x)

Integrating equation (x) between $r = r_e\ and\ r = r_w$; i.e. $\int_{p_{wf}}^{p_e} \partial p = \int_{r_w}^{r_e} \frac{q\mu}{2\pi k h}\left(\frac{1}{r} - \frac{r}{r_e^2}\right)\partial r;$

$$p_{e-}\ p_{wf} = \frac{q\mu}{2\pi k h}\left[\ln\left(\frac{r_e}{r_w}\right) - \frac{1}{2r_e^2}(r_e^2 - r_w^2)\right] = \frac{q\mu}{2\pi k h}\left[\ln\left(\frac{r_e}{r_w}\right) - \frac{1}{2} + \frac{r_w^2}{r_e^2}\right]$$

Since $r_e \gg r_w$, The $\frac{r_w^2}{r_e^2}$ term in the above equation can be neglected (Dake 1998).

So that; $$p_{e-}\ p_{wf} = \frac{q\mu}{2\pi k h}\left[\ln\left(\frac{r_e}{r_w}\right) - \frac{1}{2}\right] \quad \text{(xi)}$$

In oil field unit, Equation (xi) is expressed as;

$$p_{e-} \, p_{wf} = \frac{141.2 q\mu B}{kh}\left[\ln\left(\frac{r_e}{r_w}\right) - \frac{1}{2}\right] \quad \text{(xii)}$$

Equation (xii) is the pseudo-steady state solution of the radial diffusivity equation, expressed in terms $p = p_e$ at $r = r_e$. Skin factor $S = 0$. If skin factor is considered, it is expressed as:

$$p_{e-} \, p_{wf} = \frac{141.2 q\mu B}{kh}\left[\ln\left(\frac{r_e}{r_w}\right) - \frac{1}{2} + S\right]$$

LAPLACE TRANSFORM SOLUTION OF THE PARTIAL DIFFERENTIAL EQUATION GOVERNING FLUID FLOW IN PETROLEUM RESERVOIRS.

Infinite-Acting Homogeneous Reservoir Case

Adapted from (Blasingame 1994)

From equation (7), the equation governing fluid flow in petroleum reservoirs have been derived as:

$\left[\frac{\partial^2 P}{\partial r^2} + \frac{1}{r}\frac{\partial P}{\partial r}\right] = \frac{\emptyset \mu C_t}{k}\frac{\partial P}{\partial t}$ which can be written as $\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial P}{\partial r}\right] = \frac{\emptyset \mu C_t}{k}\frac{\partial P}{\partial t}$

In dimensionless form, equation (7) is expressed as: $\frac{1}{r_D}\frac{\partial}{\partial r_D}\left[r_D\frac{\partial P_D}{\partial r_D}\right] = \frac{\partial P_D}{\partial t_D}$ (8)

where:

$r_D = \frac{r}{r_w}$ (9); $P_D = P_{Dc}\frac{kh}{q\mu B}(p_i - p_r)$ (10); $t_D = (t_{DC})\frac{kt}{\emptyset \mu C_t {r_w}^2}$ (11)

and t_{DC} and P_{Dc} have the values presented as given in Table 2.

TABLE 2: VALUES OF t_{DC} AND P_{Dc}

	Darcy Units	Field Units	SI Units
t_{DC}	1	2.637×10^{-4}	3.557×10^{-6}
P_{Dc}	2π	7.081×10^{-3}	5.336×10^{-4}

The initial condition is given as $p_D(r_D, t_D \le 0) = 0$ (i.e. uniform pressure distribution) (12)

The constant rate inner boundary condition is:

$\left[r_D\frac{\partial P_D}{\partial r_D}\right]_{r_D=1} = -1$ (i.e. constant flow rate at the well) (13)

The infinite acting outer boundary condition is given by: $p_D(r_D \to \infty, t_D) = 0$ (14)

Laplace transform formulation: $\overline{p_D} = L\{P_D(r_D, t_D)\}$.

Taking the Laplace transform of Equation (8) gives;

$\frac{1}{r_D}\frac{d}{dr_D}\left[r_D\frac{d\overline{p_D}}{dr_D}\right] = s(\overline{p_D}) - p_D(t_D = 0)$ Since $\left[L\left\{\frac{\partial f(t)}{\partial r_D}\right\} = \frac{d\bar{f}(t)}{dr_D}\right]$ (15)

From the initial condition, equation (12), $p_D(t_D = 0) = 0$. Substituting this in equation (15) yields:

$\frac{1}{r_D}\frac{\partial}{\partial r_D}\left[r_D\frac{d\overline{p_D}}{dr_D}\right] = s(\bar{p}_D)$ (16)

Taking the Laplace transform of equation (13), the inner boundary condition yields:

$\left[r_D\frac{d\overline{p_D}}{dr_D}\right]_{r_D=1} = \frac{-1}{s}$ (17)

Taking the Laplace transform of equation (14), the outer boundary condition yields:

$\bar{p}_D(r_D \to \infty, s) = 0$ (18)

Multiplying equation (16) through by ${r_D}^2$ yields:

$$r_D \frac{d}{dr_D}\left[r_D \frac{d\overline{p_D}}{dr_D}\right] = s{r_D}^2(\bar{p}_D) \tag{19}$$

Let $z = \sqrt{s}(r_D)$ (20); or $r_D = \frac{z}{\sqrt{s}}$ (20)

Applying chain rule on the $\frac{d}{dr_D}$ terms in equation (19) results in:

$$r_D \frac{dz}{dr_D}\frac{d}{dz}\left[r_D \frac{dz}{dr_D}\frac{d\bar{p}_D}{dz}\right] = s{r_D}^2(\bar{p}_D) \tag{21}$$

where $\frac{dz}{dr_D} = \frac{d}{dr_D}\left[\sqrt{s}(r_D)\right] = \sqrt{s}$ (22)

substituting equation (20) and (22) into equation (21), we have;

$\frac{z}{\sqrt{s}}(\sqrt{s})\frac{d}{dz}\left[\frac{z}{\sqrt{s}}\sqrt{s}\frac{d\bar{p}_D}{dz}\right] = z^2(\bar{p}_D)$ so that; $z\frac{d}{dz}\left[z\frac{d\bar{p}_D}{dz}\right] = z^2(\bar{p}_D)$ (23)

But $z\frac{d}{dz}\left[z\frac{d\bar{p}_D}{dz}\right] = z^2\frac{d\bar{p}_D}{dz^2} + z\frac{d\bar{p}_D}{dz}$. (24)

Therefore, equation (23) becomes $z^2\frac{d\bar{p}_D}{dz^2} + z\frac{d\bar{p}_D}{dz} = z^2(\bar{p}_D)$ (25)

Equation (25) is in the form of the Modified Bessel differential equation, $z^2\frac{dw}{dz^2} + z\frac{dw}{dz} = (z^2 + v^2)w$ (26)

with the general solution given by;

$$w = AI_V(z) + BK_v(z) \tag{27}$$

where $I_V(z)$ and $K_v(z)$ are the modified Bessel functions of the first and second kinds respectively. A and B are constants.

Comparing equations (25) and (26), we have $v = 0$, $w = \bar{p}_D(r_D, s)$ and from equation (20), $z = \sqrt{s}(r_D)$. Substituting these into equation (27) gives:

$$\bar{p}_D(r_D, s) = AI_0\left(\sqrt{s}(r_D)\right) + BK_0\left(\sqrt{s}(r_D)\right) \tag{28}$$

Equation (28) is the general solution of equation (8), the radial diffusivity equation in dimensionless form. The particular solution is derived as shown below:

From chain rule, $\frac{d\bar{p}_D}{dr_D} = \frac{dz}{dr_D}\frac{d\bar{p}_D}{dz}$ (29)

But $\frac{dz}{dr_D} = \sqrt{s}$ from equation (22). Therefore, equation (29) becomes:

$$\frac{d\bar{p}_D}{dr_D} = \sqrt{s}\frac{d\bar{p}_D}{dz} \tag{30}$$

Differentiating equation (28) results in $\frac{d\bar{p}_D}{dz} = A\frac{dI_0(z)}{dz} + B\frac{dK_0(z)}{dz}$ (31)

Abramowitz, Stegun, and Miller (1965) presents the following relationships:

$\frac{dI_0(z)}{dz} = I_{\mathrm{I}}(z)$ (32); and $\frac{dK_0(z)}{dz} = -K_{\mathrm{I}}(z)$ (33)

Substituting (32) and (33) in (31) reduces equation (31) to:

$$\frac{d\bar{p}_D}{dz} = AI_{\mathrm{I}}(z) - BK_{\mathrm{I}}(z) \tag{34}$$

Substituting the RHS of equation (34) for $\frac{\partial \bar{p}_D}{\partial z}$ in equation (30) results in:

$$\frac{\partial \bar{p}_D}{\partial r_D} = A\sqrt{s}I_{\mathrm{I}}(z) - B\sqrt{s}K_{\mathrm{I}}(z) \tag{35}$$

Substituting $z = \sqrt{s}(r_D)$ in equation (35) gives:

$$\frac{\partial \bar{p}_D}{\partial r_D} = A\sqrt{s}I_{\mathrm{I}}\left(\sqrt{s}(r_D)\right) - B\sqrt{s}K_{\mathrm{I}}(\sqrt{s}(r_D)) \tag{36}$$

Multiplying through by r_D, equation (36) can be expressed as:

$$r_D\frac{\partial \bar{p}_D}{\partial r_D} = Ar_D\sqrt{s}I_{\mathrm{I}}\left(\sqrt{s}(r_D)\right) - Br_D\sqrt{s}K_{\mathrm{I}}(\sqrt{s}(r_D)) \tag{37}$$

From equation (17), the Laplace transform of the inner boundary was given as:

$$\left[r_D\frac{d\bar{p}_D}{dr_D}\right]_{r_D=1} = \frac{-1}{s} \tag{17}$$

Substituting the RHS of (17) in (37) for the inner boundary condition at $r_D = 1$, results in:

$$A\sqrt{s}I_{\mathrm{I}}\left(\sqrt{s}\right) - B\sqrt{s}K_{\mathrm{I}}\left(\sqrt{s}\right) = \frac{-1}{s}. \tag{38}$$

Substituting equation (18) in (28) for the outer boundary condition results in:

$$\lim_{r_D\to\infty}\left[AI_0\left(\sqrt{s}(r_D)\right) + BK_0\left(\sqrt{s}(r_D)\right)\right] = 0 \tag{39}$$

Solving for *A* and *B* in equations (38) and (39).

Since $\lim_{z\to\infty} I_0(z) = \infty$ and $\lim_{z\to\infty} K_0(z) = 0$, then form equation (39);

$$A = 0.$$

Substituting 0 for A in equation (38) gives;

$$B\sqrt{s}K_{\mathrm{I}}\left(\sqrt{s}\right) = \frac{1}{s}. \qquad \text{or} \qquad B = \frac{1}{s}\frac{1}{\sqrt{s}K_{\mathrm{I}}(\sqrt{s})}$$

Substituting for A and B in the general solution or equation (28) results in the particular solution in Laplace domain given below:

$$\bar{p}_D(r_D, s) = \frac{1}{s}\frac{K_0\left(\sqrt{s}(r_D)\right)}{\sqrt{s}K_{\mathrm{I}}\left(\sqrt{s}\right)} \tag{40}$$

Equation (40) is the cylindrical source solution in Laplace domain. Its inversion is done by Residue Methods as illustrated by(Everdingen and Hurst (1949) and presented below:

$$P_D(r_D, t_D) = \frac{2}{\pi}\int_0^\infty\left[1-\exp(-\mu^2 t_D)\right]\frac{\left[J_{\mathrm{I}}(\mu)y_0(\mu r_D - y_{\mathrm{I}}(\mu)J_0(\mu r_D)\right]}{\mu^2\left[J_{\mathrm{I}}{}^2(\mu)+y_{\mathrm{I}}{}^2(\mu)\right]}\partial U \tag{41}$$

When $r_D = 1$, the **wellbore solution is:**

$$P_D(r_D = 1, t_D) = \frac{4}{\pi}\int_0^\infty\frac{\left[1-\exp(-\mu^2 t_D)\right]}{\mu^3\left[J_{\mathrm{I}}{}^2(\mu)+y_{\mathrm{I}}{}^2(\mu)\right]}\partial U \tag{42}$$

Line Source Solution

Equation (40) gives the cylindrical line source solution as:

$$\bar{p}_D(r_D, s) = \frac{1}{s}\frac{K_0(\sqrt{s}(r_D))}{\sqrt{s}K_{\mathrm{I}}(\sqrt{s})} \qquad (40)$$

From Abramowitz and Stegun, Handbook of Mathematical Functions,

$$K_{\mathrm{I}}(z) = \frac{1}{z} \quad \text{as} \quad z \to 0.$$

or $$zK_{\mathrm{I}}(z) = 1 \quad \text{as} \quad z \to 0.$$

If $z = \sqrt{s}$, then $\sqrt{s}K_{\mathrm{I}}(\sqrt{s}) = 1$ as s or $\sqrt{s} \to 0.$ (43)

Substituting equation (43) in equation (40) yields;

$$\bar{p}_D(r_D, s) = \left(\frac{1}{s}\right) K_0(\sqrt{s}(r_D)) \quad \text{as} \quad s \to 0. \qquad (44)$$

Equation 44 is the line source solution in Laplace domain, its inversion into Real domain, through Laplace Table lookup, is shown below (Abramowitz, Stegun, and Miller (1965), Roberts and Kaufman (1966) Carslaw and Jaegar (1958);

If $\bar{f}(s) = \frac{1}{s}K_0(\sqrt{s}a)$, then $f(t) = \frac{1}{2t}E_{\mathrm{I}}\left(\frac{a^2}{4t}\right)$.

Applying the above on equation (41) results in:

$$p_D(r_D, t_D) = \frac{1}{2}E_{\mathrm{I}}\left(\frac{{r_D}^2}{4t_D}\right) \qquad (45)$$

Equation (45) is the dimensionless form of the solution of the diffusivity equation for infinite acting homogenous reservoirs.

Substituting the values of r_D, t_D and p_D, as defined in eqns. {9, 10 and 11}, in equation {45} gives the pressure at the wellbore after $'t'$ hours of flow as:

$$\boldsymbol{p_{wf} = p_i - \frac{162.6q\mu B}{kh}\left[log\left(\frac{kt}{\emptyset\mu c_t {r_w}^2}\right) - 3.23\right]} \qquad \textbf{(46)}$$

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