P.165.22. 求 $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $(0<b\le a)$ 分别绕长轴、短轴所成旋转体的侧面积。 dem 17/4 —76.

解: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $y=\pm\frac{b}{a}\sqrt{a^2-x^2}$, 取 $y=\frac{b}{a}\sqrt{a^2-x^2}$

$$y'=-\frac{b}{a}\cdot\frac{x}{\sqrt{a^2-x^2}}$$

绕长轴旋转, 侧面积为 F_1, $\sqrt{1+y'^2}=\sqrt{1+\frac{b^2}{a^2}\cdot\frac{x^2}{a^2-x^2}}$

$$F_1=4\pi\int_0^a y\cdot\sqrt{1+y'^2}\,dx=4\pi\int_0^a\frac{b}{a}\cdot\sqrt{a^2-x^2}\cdot\sqrt{1+\frac{b^2}{a^2}\cdot\frac{x^2}{a^2-x^2}}\,dx$$

$$=4\pi\int_0^{\frac{\pi}{2}}\frac{b}{a}\cdot a\cdot\cos t\cdot\sqrt{1+\frac{b^2}{a^2}\cdot\frac{\sin^2 t}{\cos^2 t}}\,d(a\sin t)$$

$$=4\pi b\int_0^{\frac{\pi}{2}}\sqrt{a^2\cos^2 t+b^2\sin^2 t}\,d\sin t$$

$$=4\pi ab\int_0^{\frac{\pi}{2}}\sqrt{1-(1-\frac{b^2}{a^2})\sin^2 t}\,d\sin t$$

$$=\frac{4\pi ab}{\sqrt{1-\frac{b^2}{a^2}}}\int_0^{\frac{\pi}{2}}\sqrt{1-(1-\frac{b^2}{a^2})\sin^2 t}\,d(\sqrt{1-\frac{b^2}{a^2}}\sin t)$$

$$=\frac{4\pi ab}{\sqrt{1-\frac{b^2}{a^2}}}\left[\frac{\sqrt{1-\frac{b^2}{a^2}}\cdot\sin t}{2}\cdot\sqrt{1-(1-\frac{b^2}{a^2})\sin^2 t}\,\Big|_0^{\frac{\pi}{2}}+\frac{1}{2}\arcsin\sqrt{1-\frac{b^2}{a^2}}\sin t\,\Big|_0^{\frac{\pi}{2}}\right]$$

$$=\frac{2\pi ab}{\sqrt{1-\frac{b^2}{a^2}}}\left[\sqrt{1-\frac{b^2}{a^2}}\cdot\frac{b}{a}+\arcsin\sqrt{1-\frac{b^2}{a^2}}\right]$$

$$=2\pi ab\left(\frac{b}{a}+\frac{\arcsin\sqrt{1-\frac{b^2}{a^2}}}{\sqrt{1-\frac{b^2}{a^2}}}\right)$$

$$=2\pi ab\left(\sqrt{1-\varepsilon^2}+\frac{\arcsin\varepsilon}{\varepsilon}\right)$$

令 $\sqrt{1-\frac{b^2}{a^2}}=\varepsilon$, 则 $\frac{b}{a}=\sqrt{1-\varepsilon^2}$.

P.165.23 计算圆弧 $x^2+y^2=a^2$, $(a-h\le y\le a,\ 0<h<a)$ 绕y轴旋转所得球冠面积。

解: $x=\sqrt{a^2-y^2}$, $x'(y)=\frac{-y}{\sqrt{a^2-y^2}}$

$$F=2\pi\int_{a-h}^a x(y)\cdot\sqrt{1+x'^2(y)}\,dy=2\pi\int_{a-h}^a\sqrt{a^2-y^2}\cdot\frac{a\,dy}{\sqrt{a^2-y^2}}=2\pi a\int_{a-h}^a dy=2\pi ah.$$