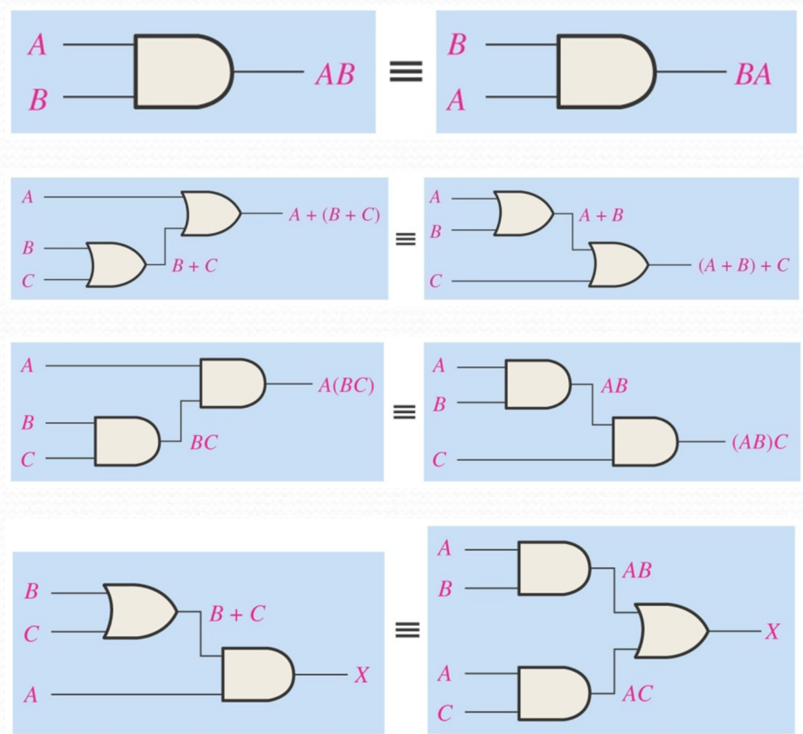
Chapter 4 Boolean Algebra and Logic Simplification

4.1 Boolean Operations and Expressions

- Variable: a symbol used to represent a logical quantity;
- Complement: the inverse of a variable
  - Be indicated by a bar over the variable: NOT operation
- Boolean Addition: OR operation
- Boolean Multiplication: equivalent to the AND operation.

4.2 Laws and Rules of Boolean Algebra

- Laws of Boolean Algebra
  - **Commutative Laws**
    - A+B=B+A
    - AB=BA
  - **Associative Laws**
    - A+(B+C)=(A+B)+C
    - A(BC)=(AB)C
  - **Distributive Law**
    - A(B+C)=AB+AC



$X = A(B + C)$ $X = AB + AC$

Rules of Boolean Algebra

$$0 \bullet A = 0$$

$$1 \bullet A = A$$

$$A \bullet A = A$$

$$A \bullet \overline{A} = 0$$

$$A \bullet B = B \bullet A$$

$$A \bullet (B \bullet C) = (A \bullet B) \bullet C$$

$$A \bullet (B + C) = A \bullet B + A \bullet C$$

$$\overline{A \bullet B} = \overline{A} + \overline{B}$$

$$\overline{\overline{A}} = A$$

$$\overline{1} = 0; \overline{0} = 1$$

$$1 + A = 1$$

$$0 + A = A$$

$$A + A = A$$

$$A + \overline{A} = 1$$

$$A + B = B + A$$

$$A + (B + C) = (A + B) + C$$

$$A + B \bullet C = (A + B) \bullet (A + C)$$

$$\overline{A + B} = \overline{A} \bullet \overline{B}$$

$$A + A \bullet B = A$$

$$A + \overline{A} \bullet B = A + B$$

$$A \bullet B + A \bullet \overline{B} = A$$

$$A \bullet (A + B) = A$$

$$A \bullet B + \overline{A} \bullet C + B \bullet C = A \bullet B + \overline{A} \bullet C$$

$$A \bullet B + \overline{A} \bullet C + B \bullet C \bullet D = A \bullet B + \overline{A} \bullet C$$

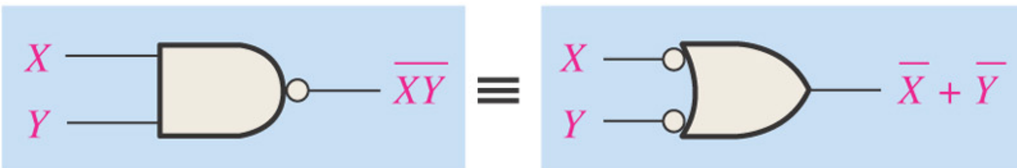
$$A \bullet \overline{A \bullet B} = A \bullet \overline{B}$$

4.3 Demorgan's Theorems

- The complement of a product of variables is equal to the sum of the complements of the variables
- The complement of a sum of variables is equal to the product of the complements of the variables

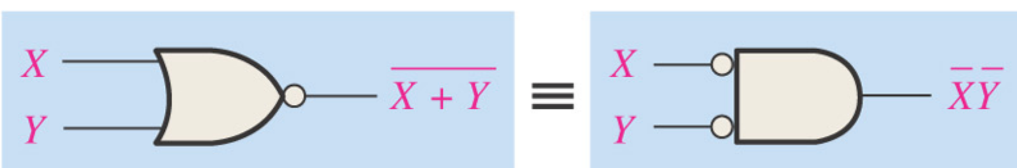
$$\overline{A \bullet B} = \overline{A} + \overline{B}$$

$$\overline{A + B} = \overline{A} \bullet \overline{B}$$



NAND Negative-OR

Inputs		Output	
X	Y	$\overline{XY}$	$\overline{X} + \overline{Y}$
0	0	1	1
0	1	1	1
1	0	1	1
1	1	0	0



NOR Negative-AND

Inputs		Output	
X	Y	$\overline{X + Y}$	$\overline{X}\,\overline{Y}$
0	0	1	1
0	1	0	0
1	0	0	0
1	1	0	0

Example: Apply DeMorgan’s theorems to the following expressions

$$\overline{XYZ} = ?$$

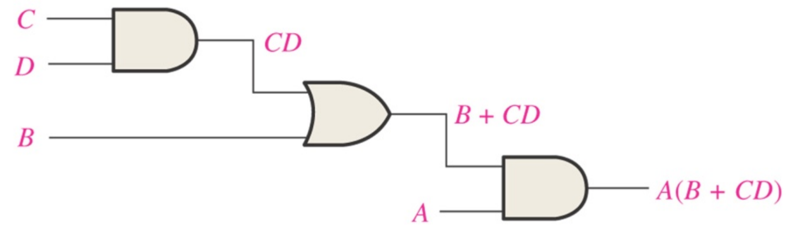
$$\overline{X + Y + Z} = ?$$

$$\overline{XYZ} = \overline{X} + \overline{Y} + \overline{Z}$$

$$\overline{X + Y + Z} = \overline{X}\,\overline{Y}\,\overline{Z}$$

4.4 Boolean Analysis of Logic Circuits

- Boolean Expression for a Logic Circuit
- Constructing a Truth Table for a Logic Circuit



A	B	C	D	A(B+CD)
0	0	0	0	0
0	0	0	1	0
0	0	1	0	0
0	0	1	1	0
0	1	0	0	0
0	1	0	1	0
0	1	1	0	0
0	1	1	1	0
1	0	0	0	0
1	0	0	1	0
1	0	1	0	0
1	0	1	1	1
1	1	0	0	1
1	1	0	1	1
1	1	1	0	1
1	1	1	1	1

4.5 Simplification Using Boolean Algebra

$$Examples:$$

$$Y_1 = A\overline{\overline{B}CD} + A\overline{B}CD$$

$$Y_2 = A\overline{B} + ACD + \overline{A}\,\overline{B} + \overline{A}CD$$

$$Y_3 = \overline{A}B\overline{C} + A\overline{C} + \overline{B}\,\overline{C}$$

$$Y_4 = B\overline{C}D + BC\overline{D} + \overline{B}CD + BCD$$

$$A + \overline{A} = 1$$

Solutions:

$$Y_1 = A\overline{\overline{B}CD} + A\overline{B}CD = A(\overline{\overline{B}CD} + \overline{B}CD) = A$$

$$Y_2 = A\overline{B} + ACD + \overline{A}\,\overline{B} + \overline{A}CD = A(\overline{B} + CD) + \overline{A}(\overline{B} + CD) = \overline{B} + CD$$

$$Y_3 = \overline{A}B\overline{C} + A\overline{C} + \overline{B}\,\overline{C} = \overline{A}B\overline{C} + (A + \overline{B})\overline{C} = (\overline{A}B)\overline{C} + (\overline{\overline{A}B})\overline{C} = \overline{C}$$

$$Y_4 = B\overline{C}D + BC\overline{D} + \overline{B}CD + BCD = B(\overline{C}D + C\overline{D}) + B(\overline{C}\,\overline{D} + CD)$$

$$= B(C \oplus D) + B(\overline{C \oplus D}) = B$$

Examples:

$$A + 1 = 1$$

$$Y_1 = (\overline{\overline{A}B} + C)\ ABD + AD$$

$$Y_2 = AB + AB\overline{C} + ABD + AB(\overline{C} + \overline{D})$$

$$Y_3 = A + \overline{\overline{A} \bullet \overline{BC}}(\overline{A} + \overline{\overline{B}\,\overline{C} + D}) + BC$$

Solutions:

$$Y_1 = (\overline{\overline{A}B} + C)\ ABD + AD = \left[(\overline{\overline{A}B} + C)\ B\right]AD + AD = AD$$

$$Y_2 = AB + AB\overline{C} + ABD + AB(\overline{C} + \overline{D}) = AB + AB\left[\overline{C} + D + (\overline{C} + \overline{D})\right] = AB$$

$$Y_3 = A + \overline{\overline{A} \bullet \overline{BC}}(\overline{A} + \overline{\overline{BC} + D}) + BC$$

$$= (A + BC) + (A + BC)\left(\overline{A} + \overline{\overline{BC} + D}\right) = A + BC$$

$$Examples:$$

$$A + \overline{A}B = A + B$$

$$Y_1 = \overline{B} + ABC$$

$$Y_2 = A\overline{B} + B + \overline{A}B$$

$$Y_3 = AC + \overline{A}D + \overline{C}D$$

$$Solution:$$

$$Y_1 = \overline{B} + ABC = \overline{B} + AC$$

$$Y_2 = A\overline{B} + B + \overline{A}B = A + B + \overline{A}B = A + B$$

$$Y_3 = AC + \overline{A}D + \overline{C}D = AC + (\overline{A} + \overline{C})\ D = AC + \overline{AC}D$$

$$= AC + D$$

$$Example:$$

$$A + A = A$$

$$Y = A\overline{B} + \overline{A}B + B\overline{C} + \overline{B}C$$

$$Solution:$$

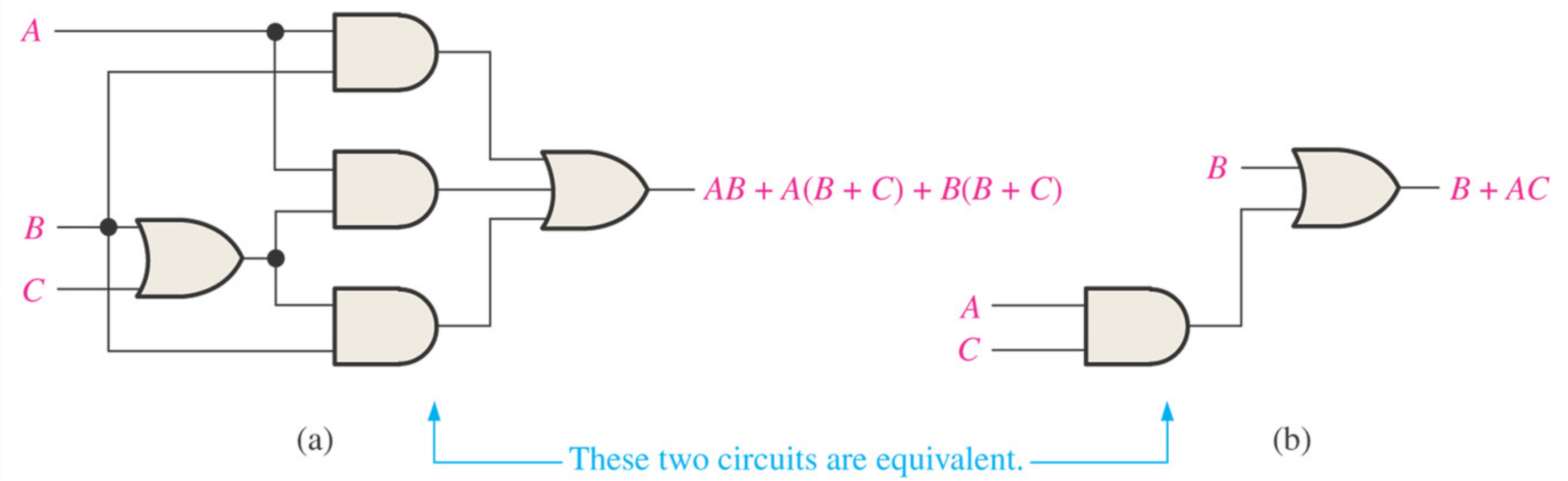
$$Y = A\overline{B} + \overline{A}B(C + \overline{C}) + B\overline{C} + (A + \overline{A})\overline{B}C$$

$$= A\overline{B} + \overline{A}BC + \overline{A}B\overline{C} + B\overline{C} + A\overline{B}C + \overline{A}\overline{B}C$$

$$= (A\overline{B} + A\overline{B}C) + (B\overline{C} + \overline{A}B\overline{C}) + (A\overline{B}C + \overline{A}\overline{B}C)$$

$$= A\overline{B} + B\overline{C} + \overline{A}C$$

Figure 4–17 Gate circuits for Example 4–8



$$AB + A(B+C) + B(B+C) = A(B+B+C) + B(B+C)$$
$$= A(B+C) + B(B+C) = AB + AC + B + BC$$
$$= B(A+1) + AC + B(1+C)$$
$$= B + AC$$

4.6 Standard Forms of Boolean Expressions

- The sum-of-products (SOP) form
  - A single overbar cannot extend over more than one variable;
  - More than one variable in a term can have an overbar
- The product-of-sums (POS) form
  - A single overbar cannot extend over more than one variable
  - More than one variable in a term can have an overbar

Example:

$$A\overline{B}\,\overline{C}D + \overline{A}CD + AC$$

$$(A + B)(B + C + D)(A + C).$$

- Conversion of a General Expression to SOP Form
  - Applying the distributive law
- The standard SOP Form
  - **All the variables in the domain appear in each product term in the expression**
- Converting Product Terms to Standard SOP
  - Using $A + \overline{A} = 1$

$Example$: Convert the following expression to the standard SOP form

$$Y = A\overline{B}\,\overline{C}D + \overline{A}CD + AC$$

$Solution$:

$$Y = A\overline{B}\,\overline{C}D + \overline{A}(B + \overline{B})\ CD + A(B + \overline{B})\ C$$

$$= A\overline{B}\,\overline{C}D + \overline{A}BCD + \overline{A}\,\overline{B}CD + ABC(D + \overline{D}) + ABC(D + \overline{D})$$

$$= A\overline{B}\,\overline{C}D + \overline{A}BCD + \overline{A}\,\overline{B}CD + ABCD + ABC\overline{D} + A\overline{B}CD + A\overline{B}C\overline{D}$$

$$= \sum_i m_i (i = 3, 7, 9, 10, 11, 14, 15)$$

$$Example \quad A\bar{B}C + \bar{A}\bar{B} + AB\bar{C}D$$

$$A\bar{B}C = A\bar{B}C\left(D + \bar{D}\right) = A\bar{B}CD + A\bar{B}C\bar{D}$$

$$\bar{A}\bar{B} = \bar{A}\bar{B}\left(C + \bar{C}\right)\left(D + \bar{D}\right) = \bar{A}\bar{B}CD + \bar{A}\bar{B}C\bar{D} + \bar{A}\bar{B}\bar{C}D + \bar{A}\bar{B}\bar{C}\bar{D}$$

$$A\bar{B}C + \bar{A}\bar{B} + AB\bar{C}D$$

$$= A\bar{B}CD + A\bar{B}C\bar{D} + \bar{A}\bar{B}CD + \bar{A}\bar{B}C\bar{D} + \bar{A}\bar{B}\bar{C}D + \bar{A}\bar{B}\bar{C}\bar{D} + AB\bar{C}D$$

- The Standard POS Form
  - **All the variables in the domain appear in each sum term in the expression.**
  - Converting a Sum Term to Standard POS
  - Use

$$A\overline{A} = 0 \qquad A + BC = (A+B)(A+C)$$

Example

$$\left(A+\overline{B}+C\right)\left(\overline{B}+C+\overline{D}\right)$$

$$=\left(A+\overline{B}+C+D\overline{D}\right)\left(A\overline{A}+\overline{B}+C+\overline{D}\right)$$

$$=\left(A+\overline{B}+C+D\right)\left(A+\overline{B}+C+\overline{D}\right)\left(A+\overline{B}+C+\overline{D}\right)\left(\overline{A}+\overline{B}+C+\overline{D}\right)$$

Converting standard SOP to standard POS

$$Y=\overline{A}\,\overline{B}\,\overline{C}+\overline{A}B\overline{C}+\overline{A}BC+A\overline{B}C+ABC$$ (standard SOP form)

$$=?$$ (standard POS form)

$$\because \overline{Y}+Y=1$$

$$\overline{A}\,\overline{B}\,\overline{C}+\overline{A}B\overline{C}+\overline{A}BC+A\overline{B}C+ABC+\overline{A}\,\overline{B}C+A\overline{B}\,\overline{C}+AB\overline{C}=1$$

$$Y=\overline{A}\,\overline{B}\,\overline{C}+\overline{A}B\overline{C}+\overline{A}BC+A\overline{B}C+ABC$$

$$\therefore \overline{Y}=\overline{A}\,\overline{B}C+A\overline{B}\,\overline{C}+AB\overline{C}$$

$$Y=\overline{\overline{A}\,\overline{B}C+A\overline{B}\,\overline{C}+AB\overline{C}}=\left(A+B+\overline{C}\right)\left(\overline{A}+B+C\right)\left(\overline{A}+\overline{B}+C\right)$$

4.7 Boolean Expressions and Truth Tables

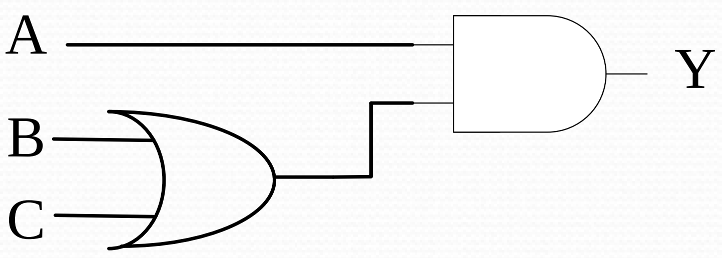
Truth Table

Input			Output
A	B	C	Y
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

Boolean Expression

$$Y=A(B+C)$$

Implementation



From Truth Table to Boolean Expression

Input			Output
A	B	C	Y
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	1 → $A\overline{B}C$
1	1	0	1 → $AB\overline{C}$
1	1	1	1 → ABC

$$Y = A\overline{B}C + AB\overline{C} + ABC$$
$$= A\overline{B}C + ABC + AB\overline{C} + ABC$$
$$= AC(\overline{B} + B) + AB(\overline{C} + C)$$
$$= AC + AB$$
$$= A(B + C)$$

SOP, POS expressions and truth table

Input			Output	P-Term	S-Term
A	B	C	X		
0	0	0	0		$(A+B+C)$
0	0	1	1	$\overline{A}\overline{B}C$	
0	1	0	0		$(A+\overline{B}+C)$
0	1	1	0		$(A+\overline{B}+\overline{C})$
1	0	0	1	$A\overline{B}\overline{C}$	
1	0	1	0		$(\overline{A}+B+\overline{C})$
1	1	0	0		$(\overline{A}+\overline{B}+C)$
1	1	1	1	ABC	

$$X = ?$$

4.8 The Karnaugh Map

Example

Simplify the following Boolean expression

$$Y = AC + \overline{B}C + B\overline{D} + C\overline{D} + A\left(B + \overline{C}\right) + \overline{A}BC\overline{D} + A\overline{B}DE$$

$$Y = AC + \overline{B}C + B\overline{D} + C\overline{D} + A\left(B + \overline{C}\right) + \overline{A}BC\overline{D} + A\overline{B}DE$$

$$= AC + \overline{B}C + B\overline{D} + C\overline{D} + A\left(B + \overline{C}\right) + A\overline{B}DE$$

$$= AC + \overline{B}C + B\overline{D} + C\overline{D} + A\overline{\overline{B}C} + A\overline{B}DE$$

$$= AC + \overline{B}C + B\overline{D} + C\overline{D} + A + A\overline{B}DE$$

$$= A + \overline{B}C + B\overline{D} + C\overline{D}$$

$$= A + \overline{B}C + B\overline{D}$$

$$C\overline{D} + \overline{A}BC\overline{D} = C\overline{D}\left(1 + \overline{A}B\right) = C\overline{D}$$

$$A\left(B + \overline{C}\right) = A\overline{\overline{B}C}$$

$$\overline{B}C + A\overline{\overline{B}C} = \overline{B}C + A$$

$$AC + A + A\overline{B}DE = A$$

$$\overline{B}C + B\overline{D} + C\overline{D} = \overline{B}C + B\overline{D}$$

How do you think about this method?

- Many Boolean rules are used
- A little hard
- …
- Are there any other methods?

Karnaugh Map

- A systematic method for simplifying Boolean expressions
- Produce the simplest SOP or POS expression
- Presents all of the possible values of input variables
  - An array of cells
  - Each cell represents a binary value of the input variables
  - Adjacency in position equivalents to adjacency in Boolean algebra

The Construction of Karnaugh Map

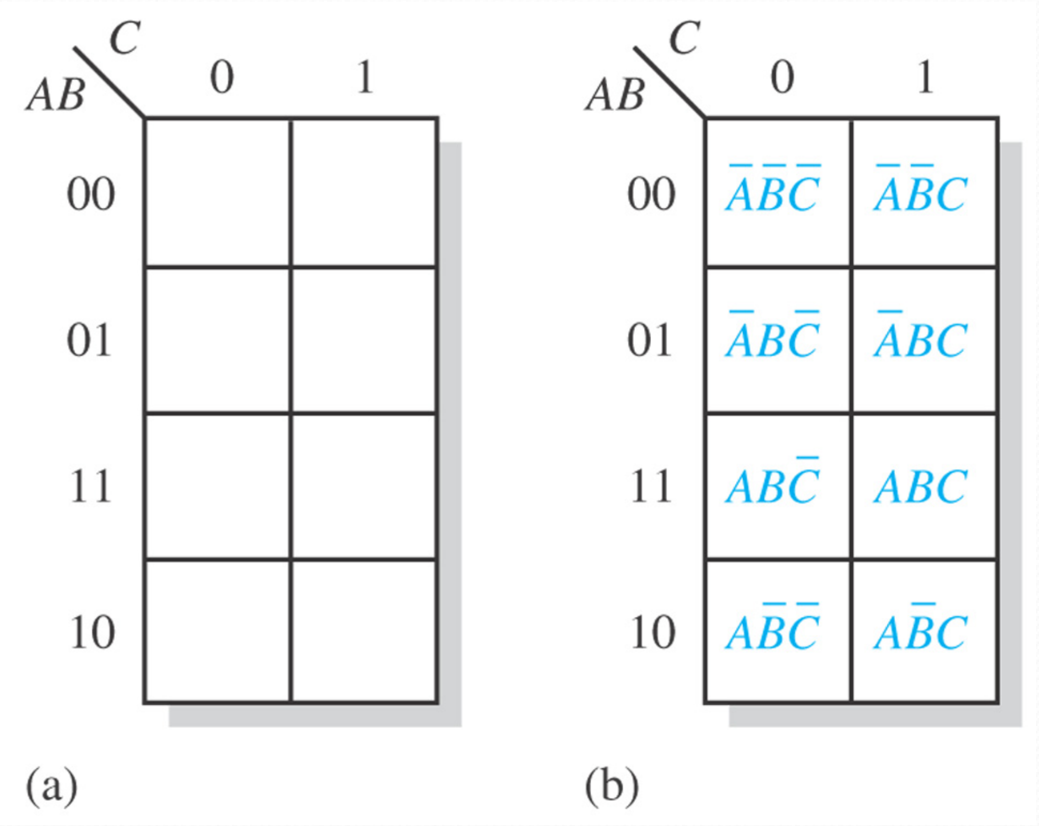


Figure 4–21 A ***3-variable Karnaugh map*** showing product terms.

The Construction of Karnaugh Map

AB \ CD	00	01	11	10
00				
01				
11				
10				

(a)

AB \ CD	00	01	11	10
00	$\bar{A}\bar{B}\bar{C}\bar{D}$	$\bar{A}\bar{B}\bar{C}D$	$\bar{A}\bar{B}CD$	$\bar{A}\bar{B}C\bar{D}$
01	$\bar{A}B\bar{C}\bar{D}$	$\bar{A}B\bar{C}D$	$\bar{A}BCD$	$\bar{A}BC\bar{D}$
11	$AB\bar{C}\bar{D}$	$AB\bar{C}D$	$ABCD$	$ABC\bar{D}$
10	$A\bar{B}\bar{C}\bar{D}$	$A\bar{B}\bar{C}D$	$A\bar{B}CD$	$A\bar{B}C\bar{D}$

(b)

Figure 4–22 A ***4-variable Karnaugh*** map.

Cell Adjacency

- Adjacency (in logic): a single-variable change
  - Cells that differ by only one variable are adjacency
  - Cells that differ by more than one variable are not adjacency
- Adjacency (in location)
  - Cells locate next to others

Cell Adjacency

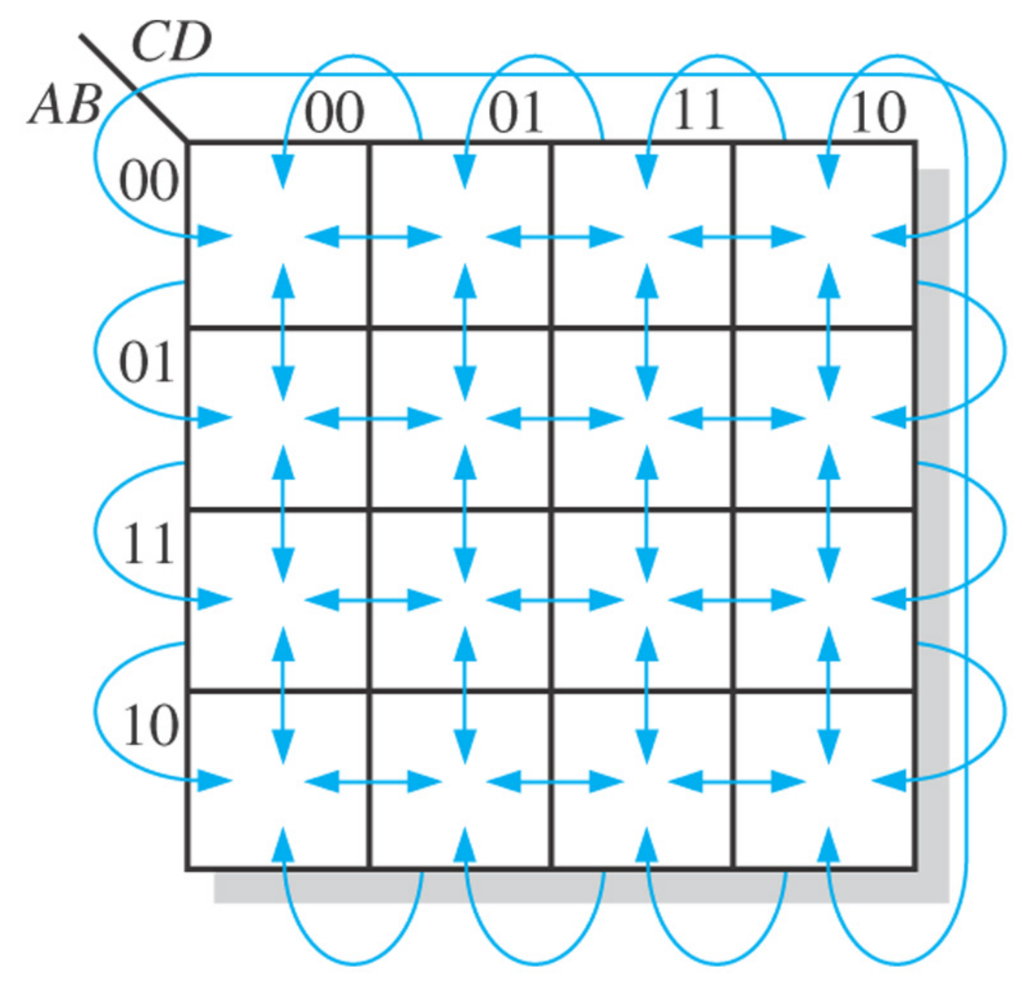


Figure 4–23 Adjacent cells on a Karnaugh map are those that differ by only one variable. Arrows point between adjacent cells.

$A \backslash B$	0	1
0	$A'B'$ m_0	$A'B$ m_1
1	AB' m_2	AB m_3

(a)

$A \backslash BC$	00	01	11	10
0	m_0	m_1	m_3	m_2
1	m_4	m_5	m_7	m_6

(b)

$AB \backslash CD$	00	01	11	10
00	m_0	m_1	m_3	m_2
01	m_4	m_5	m_7	m_6
11	m_{12}	m_{13}	m_{15}	m_{14}
10	m_8	m_9	m_{11}	m_{10}

(c)

$AB \backslash CDE$	000	001	011	010	110	111	101	100
00	m_0	m_1	m_3	m_2	m_6	m_7	m_5	m_4
01	m_8	m_9	m_{11}	m_{10}	m_{14}	m_{15}	m_{13}	m_{12}
11	m_{24}	m_{25}	m_{27}	m_{26}	m_{30}	m_{31}	m_{29}	m_{28}
10	m_{16}	m_{17}	m_{19}	m_{18}	m_{22}	m_{23}	m_{21}	m_{20}

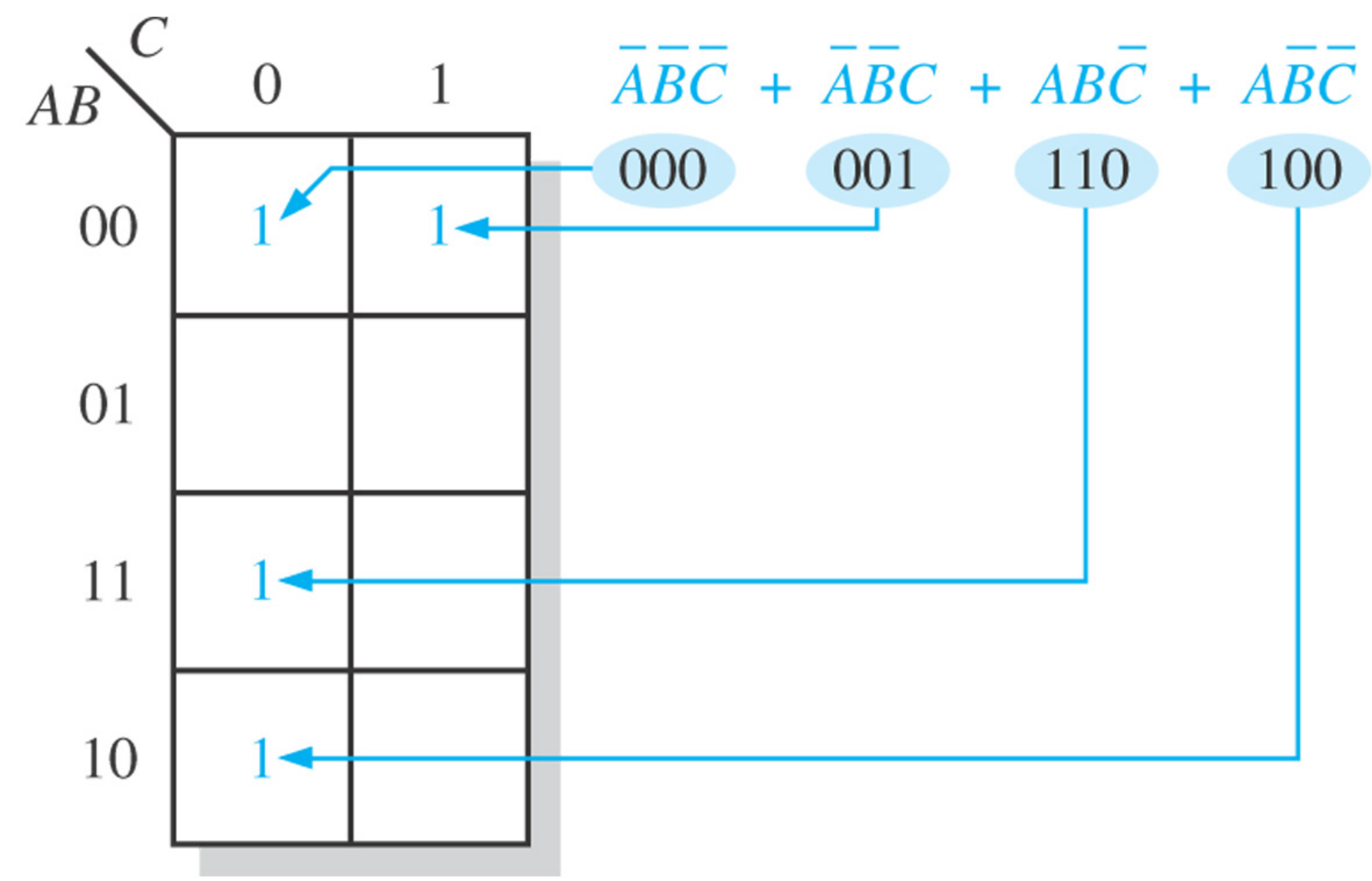
(d)

2-6-1

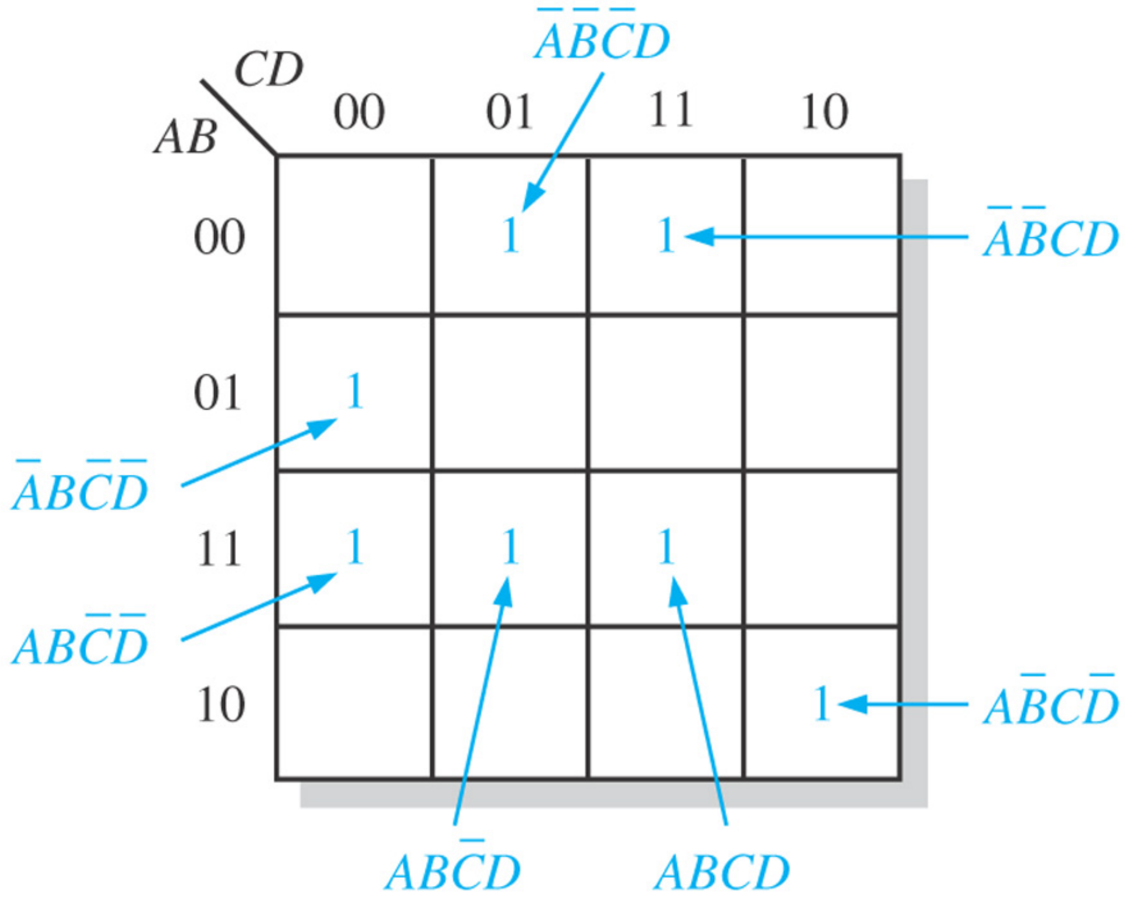
4.9 Karnaugh Map SOP Minization

- Mapping a SOP Expression
- Karnaugh Map Simplification of SOP Expressions

Example: Mapping a standard SOP expression



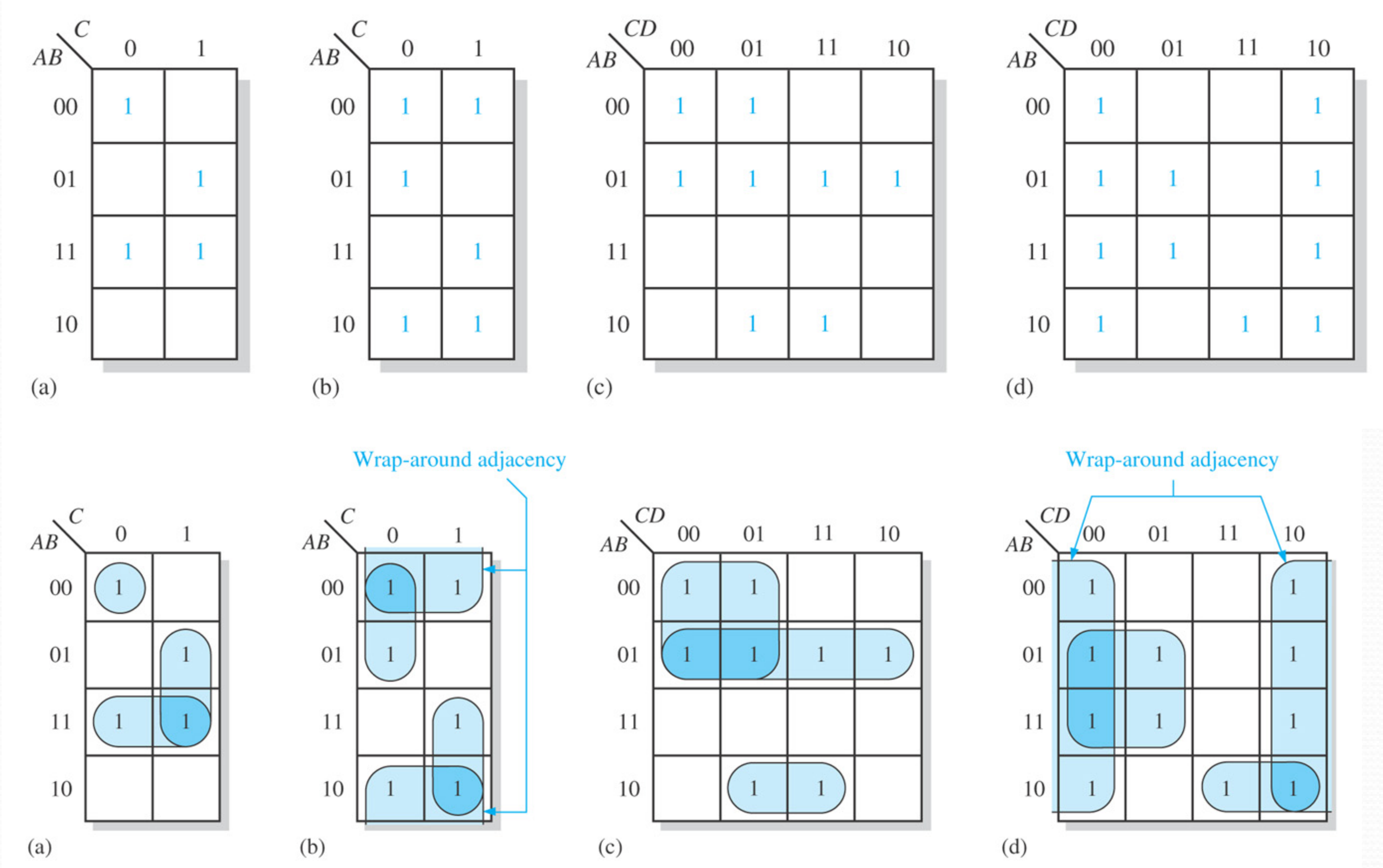
Another mapping example



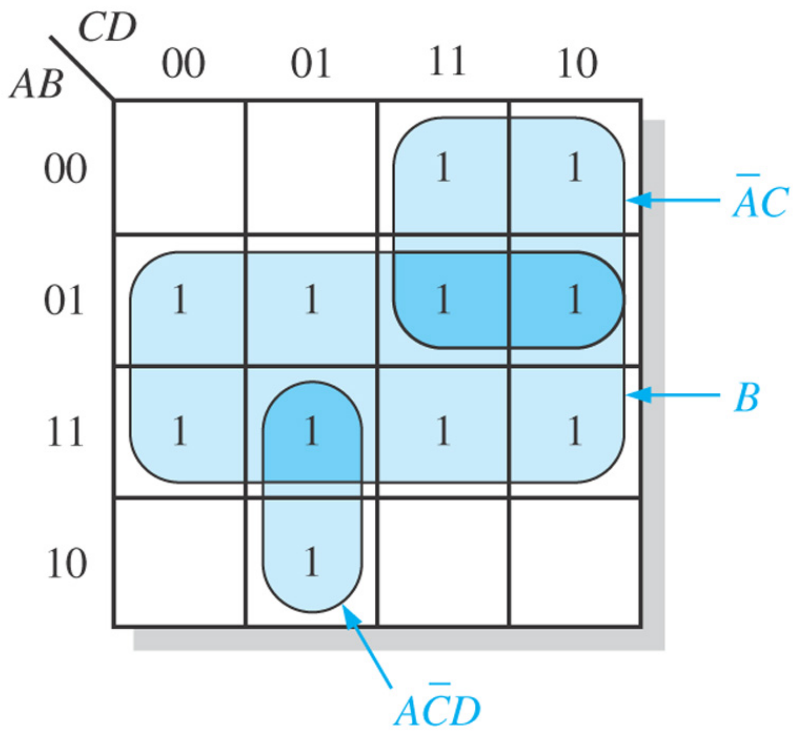
Karnaugh Map Simplification of SOP Expressions

- Group the 1s
  - Maximize the size of the groups
  - Minimize the number of groups
- Rules
  - A group must contain 2^n cells
  - Each cell must be adjacent to one or more cells in that group
  - Include 1s as much as possible
  - Each 1 on the map must be included at least one group
  - Cell with 1 can be included into more than one group

Example: Group the 1s in each of the Karnaugh maps in the following fi

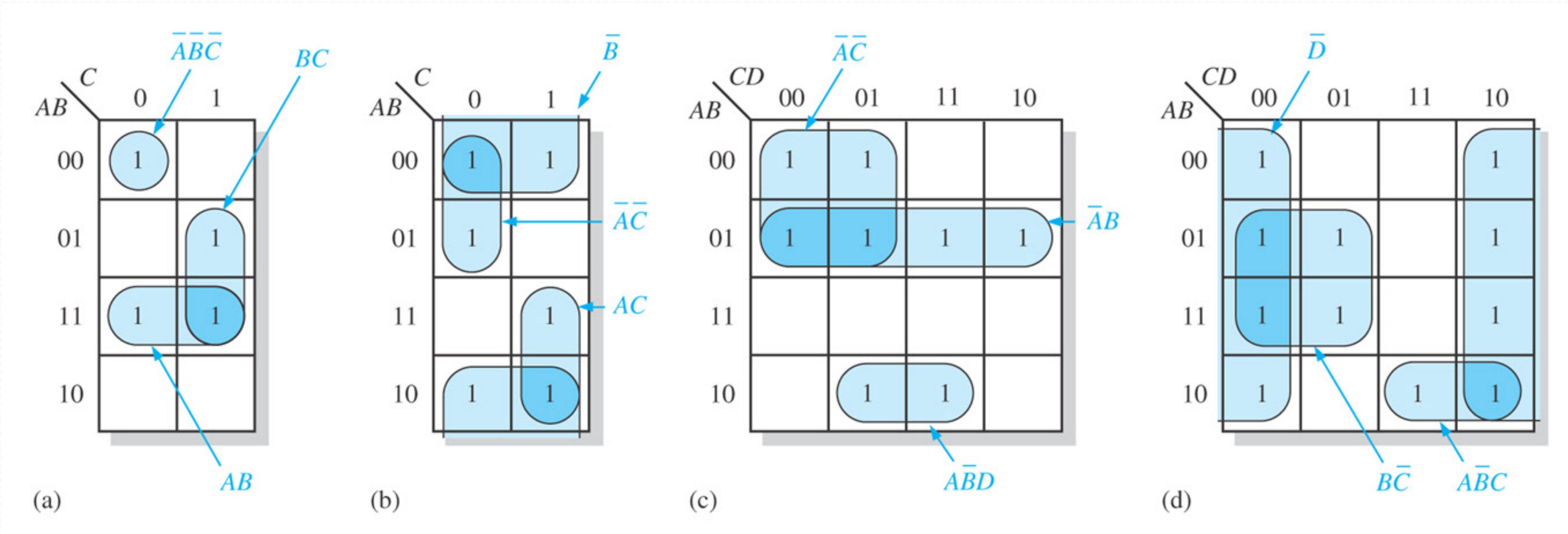


Example: Determine the product terms for the Karnaugh map in the following figure and write the resulting minimum SOP expression.



$$B + \overline{A}C + A\overline{C}D$$

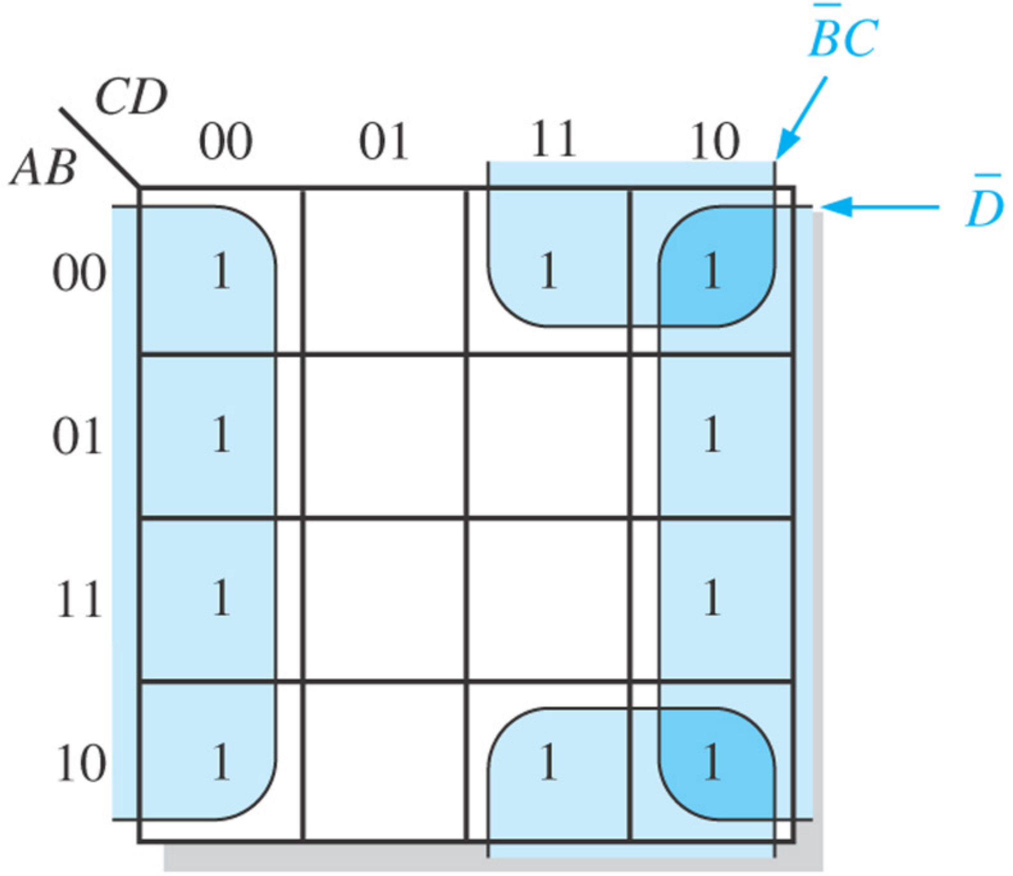
Example: Determine the product terms for the Karnaugh map in the following figure and write the resulting minimum SOP expression.



$$(a)\ AB + BC + \overline{A}\overline{B}\overline{C} \qquad (b)\ \overline{B} + \overline{A}\overline{C} + AC$$

$$(c)\ \overline{A}B + \overline{A}\overline{C} + A\overline{B}D \qquad (d)\ \overline{D} + A\overline{B}C + B\overline{C}$$

Example: Determine the product terms for the Karnaugh map in the following figure and write the resulting minimum SOP expression.

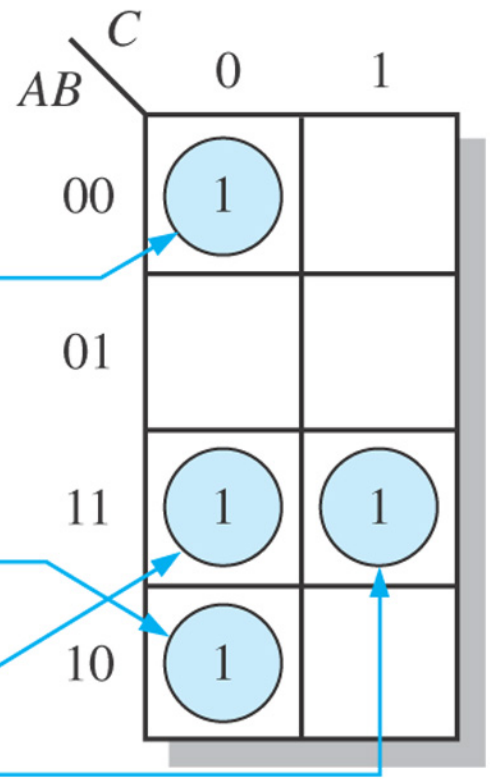


$$\overline{D} + \overline{B}C$$

Figure 4–35 Example of mapping directly from a truth table to a Karnaugh map.

$$X = \overline{A}\,\overline{B}\,\overline{C} + A\overline{B}\,\overline{C} + AB\overline{C} + ABC$$

Inputs A	B	C	Output X
0	0	0	1
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	1



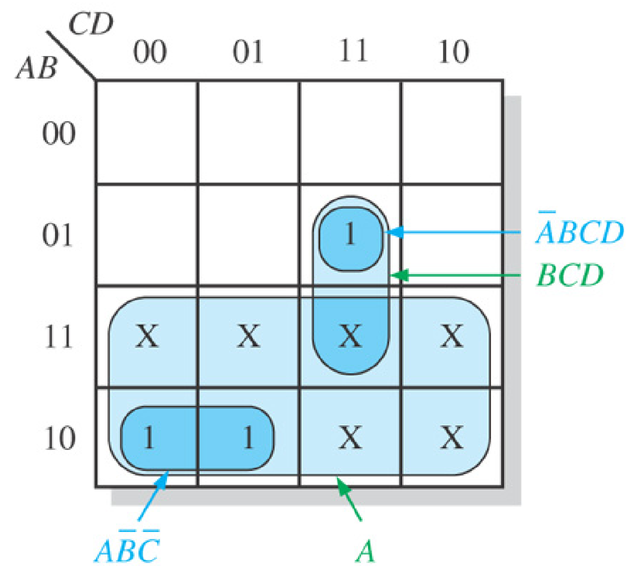
- Don't care conditions
  - Some input variable combinations are not allowed
    - These unallowed states will never occur in application
    - They can be treated as "don't care" terms
  - "don't care" terms either a 1 or 0 may be assigned to the output

Figure 4–36 Example of the use of ***"don't care" conditions*** to simplify an expression.

Inputs $ABCD$	Output Y
0 0 0 0	0
0 0 0 1	0
0 0 1 0	0
0 0 1 1	0
0 1 0 0	0
0 1 0 1	0
0 1 1 0	0
0 1 1 1	1
1 0 0 0	1
1 0 0 1	1
1 0 1 0	X
1 0 1 1	X
1 1 0 0	X
1 1 0 1	X
1 1 1 0	X
1 1 1 1	X

Don't cares

(a) Truth table



(b) Without "don't cares" $Y = A\bar{B}\bar{C} + \bar{A}BCD$
With "don't cares" $Y = A + BCD$

Summary

- Boolean Algebra
  - Variable
  - Operation
  - Laws and rules
- Simplification of Boolean expression
- SOP form and POS form
- Karnaugh maps

HW

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