Discrete Mathematics: Lecture 7

Today:

- Chap 9.1: Relations and their properties
- Chap 9.2: n-ary relations
- Chap 9.3: Representing relations

Relations (关系)

Relationships between elements of sets occur in many contexts, such as daily lives, mathematics, computer science, etc.

- Definition: Let A and B be sets. A binary relation (二元关系) from A to B is a subset of $A \times B$.
- Notation: We use aRb to denote that $(a, b) \in R$ and $a \not R\, b$ to denote that $(a, b) \notin R$.
- Example: Let A be the set of students in your school, and let B be the set of courses. Let R be the relation that consists of those pairs (a, b), where a is a student enrolled in course b.
- Example: Let $A = \{0, 1, 2\}$ and $B = \{a, b\}$. Then $\{(0, a), (0, b), (1, a), (2, b)\}$ is a relation from A to B.

Functions as relations

- Recall: A function from a set A to a set B is an assignment of exactly one element of B to each element of A.
- The graph of $f : A \to B$ is the set of ordered pairs $\{(a, b) \mid a \in A \text{ and } f(a) = b\}$.
- Thus the graph of $f : A \to B$ is a relation, which has the property that each element of A is the first element of exactly one element of the relation.
- Conversely, if R is a relation from A to B such that each element of A is the first element of exactly one element of R, then a function can be defined with R as its graph.

Relations on a set

- Definition: A relation on the set A is a relation from A to A.
- Example: the "divides" relation on $\{1, 2, 3, 4\}$
- Example: Relations on $\mathbf{Z}$:
  - $R_1 = \{(a, b) \mid a \leq b\}$
  - $R_2 = \{(a, b) \mid a > b\}$
  - $R_3 = \{(a, b) \mid a = b \text{ or } a = -b\}$
  - $R_5 = \{(a, b) \mid a = b + 1\}$
  - $R_6 = \{(a, b) \mid a + b \leq 3\}$
- How many relations are there on a set with n elements?

Reflexive relations

- Definition: A relation R on a set A is called reflexive (自反的) if $(a, a) \in R$ for every element $a \in A$.
- Remark: We use predicate $R(x, y)$ to represent that $(x, y) \in R$. Then R is reflexive iff $\forall x R(x, x)$ is true.
- Examples: $\leq$, the "divides" relation on $\mathbf{Z}^+$
- How many reflexive relations are there on a set with n elements?

Symmetric and antisymmetric relations

- Definition: A relation R on a set A is called symmetric (对称的) if $(b,a) \in R$ whenever $(a,b) \in R$, for all $a,b \in A$. A relation R on a set A is called antisymmetric (反对称的) if $(a,b) \in R$ and $(b,a) \in R$ implies $a = b$, for all $a,b \in A$.
- Remark: R is symmetric iff $\forall x \forall y (R(x,y) \to R(y,x))$ is true. R is antisymmetric iff $\forall x \forall y (R(x,y) \land R(y,x) \to x = y)$ holds.
- Examples:
- Is "divides" on $\mathbf{Z}^+$ symmetric? Is it antisymmetric?
- a relation which is both symmetric and antisymmetric;
  a relation which is neither symmetric nor antisymmetric
- How many symmetric (antisymmetric) relations are there on a set with n elements?

Transitive relations

- Definition: A relation R on a set A is called transitive (传递的) if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$, for all $a, b, c \in A$.
- Remark: R is transitive iff $\forall x \forall y \forall z (R(x, y) \wedge R(y, z) \rightarrow R(x, z))$ holds.
- Examples:
- Is "divides" on $\mathbf{Z}^+$ transitive?

Combining relations (关系的组合)

- Because relations from A to B are subsets of $A \times B$, two relations from A to B can be combined in any way two sets can be combined, such as $\cap, \cup, -, \oplus$, where $A \oplus B = A \cup B - A \cap B$
- Example: Let $R_1 = \{(a, b) \mid a \geq b\}$, and $R_2 = \{(a, b) \mid a \leq b\}$. What is $R_1 \oplus R_2$?

Composing relations (关系的复合)

- Definition: Let R be a relation from A to B, and S a relation from B to C. The composite of R and S, denoted by $S \circ R$, is the relation $\{(a, c) \mid \text{there exists } b \in B \text{ such that } (a, b) \in R \text{ and } (b, c) \in S\}$.
- Example: composing the parent relation with itself
- Definition: Let R be a relation on the set A. The power R^n, $n = 1, 2, 3, \ldots$ are defined recursively by: $R^1 = R$, and $R^{n+1} = R^n \circ R$.
- Example: Let $R = \{(a, b) \mid a, b \in \mathbf{Z}, \text{ and } b - a = 1\}$. Find R^n
- Theorem: The relation R on a set A is transitive iff $R^n \subseteq R$ for $n = 1, 2, 3, \ldots$

n-ary relations （n元关系）

- Relationships among elements of > two sets often arise.
- Let $A_1, A_2, \ldots, A_n$ be sets. An n-ary relation on these sets is a subset of $A_1 \times A_2 \times \ldots \times A_n$. The sets $A_1, A_2, \ldots, A_n$ are called the domains of the relation, and n is called its degree.
- Example: $R = \{(a, b, c) \mid a, b, c \in \mathbf{Z}, a + b = c\}$
- Example: Let R be the relation consisting of 5-tuples (A, N, S, D, T) representing airline flights, where A is the airline, N is the flight number, S is the starting point, D is the destination, and T is the departure time.
- Example: Let R be the relation consisting of 4-tuples (N, I, M, G) representing student records, where N is the name, I is the ID number, M is the major, G is the GPA.

Representing relations

- From now on, all relations we study are binary relations.
- Two ways to represent a relation between finite sets
- Using zero-one matrices, suitable for computer programs
- Using direct graphs

Representing relations using matrices

- A relation R from $A = \{a_1, a_2, \ldots, a_m\}$ to $B = \{b_1, b_2, \ldots, b_n\}$ can be represented by the matrix $M_R = [m_{ij}]$, where

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{if } (a_i, b_j) \notin R \end{cases}$$

- Example: Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3, b_4\}$. Let $R = \{(a_1, b_2), (a_2, b_1), (a_2, b_3), (a_2, b_4), (a_3, b_1), (a_3, b_3)\}$. What is M_R?

- The matrix of a relation on a set, which is a square matrix, can be used to determine if the relation has certain properties
  - R is reflexive iff all elements on the main diagonal of M_R are 1
  - R is symmetric iff M_R is a symmetric matrix
  - R is antisymmetric iff $m_{ij} = 0$ or $m_{ji} = 0$ when $i \neq j$

Operations on zero-one matrices

- Definition: Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be $m \times n$ zero-one matrices. Then the join of A and B, denoted by $A \vee B$, is a matrix with (i, j)th entry $a_{ij} \vee b_{ij}$. The meet of A and B, denoted by $A \wedge B$, is a matrix with (i, j)th entry $a_{ij} \wedge b_{ij}$.

- The Boolean product of two matrices is defined in an analogous way to the ordinary product of two matrices.

- Definition: Let $A = [a_{ij}]$ be an $m \times k$ zero-one matrix and $B = [b_{ij}]$ be a $k \times n$ zero-one matrix. Then the Boolean product of A and B, denoted by $A \odot B$, is the $m \times n$ matrix with (i, j)th entry c_{ij}, where

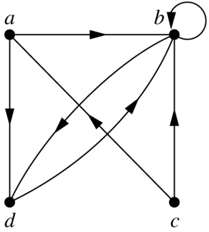
$$c_{ij} = a_{i1} \wedge b_{1j} \vee a_{i2} \wedge b_{2j} \vee \ldots \vee a_{ik} \wedge b_{kj}.$$

Relation operations and matrix operations

- $M_{R \cup S} = M_R \vee M_S$, $M_{R \cap S} = M_R \wedge M_S$,
- $M_{S \circ R} = M_R \odot M_S$, proof
- Example:

$$M_R = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$ and $$M_S = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$ What is $M_{S \circ R}$?

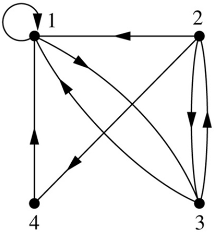
- Definition: A directed graph (有向图), or digraph, consists of a set V of vertices (or nodes) together with a set E of ordered pairs of elements of V called edges (or arcs).
- An edge of the form (a, a) is represented using an arc from a back to itself. Such an edge is called a loop (环).
- Example



Representing relations using digraphs

- The relation R on a set A is represented by the digraph that has the elements of A as its vertices and the ordered pairs $(a,b) \in R$ as edges.

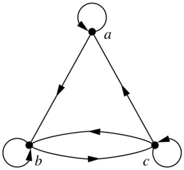
- Example: Let $A = \{1,2,3,4\}$ and $R = \{(1,1),(1,3),(2,1),(2,3),(2,4),(3,1),(3,2),(4,1)\}$.



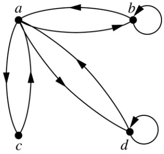
Representing relations using digraphs

The digraph representing a relation can be used to determine whether the relation has various properties:

- A relation is reflexive iff there is a loop at every vertex
- A relation is symmetric iff for every edge between distinct vertices, there is an edge in the opposite direction
- A relation is antisymmetric iff there are never two edges in opposite directions between distinct vertices
- A relation is transitive iff whenever there is an edge from x to y and an edge from y to z, there is an edge from x to z



(a) Directed graph of R



(b) Directed graph of S