Discrete Mathematics: Lecture 4

Today

- Chap 8.1: Applications of recurrence relations
- Chap 8.2: Solving linear recurrence relations

Recurrence relation (递推关系)

Some counting problems that cannot be solved using techniques from Chap 5 can be solved by using recurrence relations

- A recursive definition of a sequence specifies one or more initial terms and a rule for determining subsequent terms form those preceding them.
- Definition: A recurrence relation for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms, for all $n \geq n_0 \geq 0$. A sequence is a solution of a recurrence relation if its terms satisfies the relation.
- The initial conditions and recurrence relation uniquely determines a sequence.
- Example: The number of bacteria in a colony doubles every hour. If a colony begins with 5 bacteria, how many will be present in n hours? Solve using the iterative approach

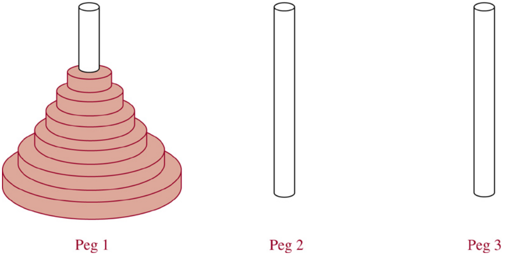
Modeling with recurrence relation

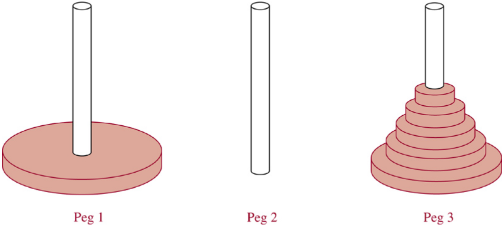
Rabbits and the Fibonacci numbers

- A young pair of rabbits (one of each sex) is placed on an island.
- A pair of rabbits does not breed until they are 2 months old.
- After they are 2 months old, each pair of rabbits produces another pair each month.
- Find a recurrence relation for the number of pairs of rabbits on the island after n months.

Reproducing pairs (at least two months old)	Young pairs (less than two months old)	Month	Reproducing pairs	Young pairs	Total pairs
		1	0	1	1
		2	0	1	1
		3	1	1	2
		4	1	2	3
		5	2	3	5
		6	3	5	8

The tower of Hanoi





Suppose there are 64 disks, and it takes a second to make a move. Then it takes $> 5 \cdot 10^{11}$ years to complete the transfer.

More examples

- The number of bit strings of length n that do not have two consecutive 0s
- The number of strings of decimal digits with an even number of 0 digits
- The number of ways to parenthesize the product of $n+1$ numbers $x_0, x_1, \ldots, x_n$, denoted by C_n, called Catalan numbers

Solving linear recurrence relation

- Definition: A linear homogeneous recurrence relation of degree k with constant coefficients is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k},$$

  where $c_1, c_2, \ldots, c_k$ are real numbers, and $c_k \neq 0$.
  - linear: each term of the sum is a_j multiplied by a function of n
  - homogeneous: every term of the sum is a multiple of a_j
  - degree k: a_n expressed in terms of the previous k terms
- A sequence is uniquely determined by this recurrence relation and the k initial conditions $a_0 = C_0, a_1 = C_1, \ldots, a_{k-1} = C_{k-1}$
- Examples: $P_n = 1.11 P_{n-1}$, $f_n = f_{n-1} + f_{n-2}$, $a_n = a_{n-5}$
- Counter-examples: $a_n = a_{n-1} + a_{n-2}^2$, $H_n = 2H_{n-1} + 1$, $B_n = nB_{n-1}$

Solving linear homogeneous recurrence relations with constant coefficients

- The basic approach: look for solutions of the form $a_n = r^n$
- $a_n = r^n$ is a solution iff r is a solution of the equation $r^k - c_1 r^{k-1} - c_2 r^{k-2} - \ldots - c_{k-1} r - c_k = 0$
- This equation is called the characteristic equation of the recurrence relation.
- The solutions of this equation is called the characteristic roots of the recurrence relation.

The case of degree 2 (1)

- Theorem 1: Let c_1 and c_2 be real numbers. Suppose that $r^2 - c_1 r - c_2 = 0$ has two distinct roots r_1 and r_2. Then the sequence $\{a_n\}$ is a solution of the recurrence relation $a_2 = c_1 a_{n-1} + c_2 a_{n-2}$ iff $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for $n \geq 0$, where α_1 and α_2 are constants.
- Proof
- Example: $a_n = a_{n-1} + 2a_{n-2}$, $a_0 = 2$, $a_1 = 7$
- Example: Fibonacci numbers

The case of degree 2 (2)

- Theorem 2: Let c_1 and c_2 be real numbers with $c_2 \neq 0$. Suppose that $r^2 - c_1 r - c_2 = 0$ has only one root r_0. Then the sequence $\{a_n\}$ is a solution of the recurrence relation $a_2 = c_1 a_{n-1} + c_2 a_{n-2}$ iff $a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$ for $n \geq 0$, where α_1 and α_2 are constants.
- Example: $a_n = 6a_{n-1} - 9a_{n-2}$, $a_0 = 1$, $a_1 = 6$

The general case

- Theorem 3: Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose that the characteristic equation

$$r^k - c_1 r^{k-1} - \ldots - c_k = 0$$

has k distinct roots $r_1, r_2, \ldots, r_k$. Then the sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} \text{ iff}$$

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \ldots + \alpha_k r_k^n$$

for $n \geq 0$, where $\alpha_1, \alpha_2, \ldots, \alpha_k$ are constants.

- Example: $a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$, $a_0 = 2$, $a_1 = 5$, $a_2 = 15$

The most general case

Theorem 4: Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose that the characteristic equation

$$r^k - c_1 r^{k-1} - \ldots - c_k = 0$$

has t distinct roots $r_1, r_2, \ldots, r_t$ with multiplicities $m_1, m_2, \ldots, m_t$, respectively, so that $m_i \geq 1$ for $i = 1, 2, \ldots, t$ and $m_1 + m_2 + \ldots + m_t = k$. Then the sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} \text{ iff}$$

$$\begin{aligned} a_n = &(\alpha_{1,0} + \alpha_{1,1} n + \ldots + \alpha_{1,m_1-1} n^{m_1-1}) r_1^n \\ &+ (\alpha_{2,0} + \alpha_{2,1} n + \ldots + \alpha_{2,m_2-1} n^{m_2-1}) r_2^n \\ &+ \ldots + (\alpha_{t,0} + \alpha_{t,1} n + \ldots + \alpha_{t,m_t-1} n^{m_t-1}) r_t^n \end{aligned}$$

for $n \geq 0$, where $\alpha_{i,j}$ are constants for $1 \leq i \leq t$ and $0 \leq j \leq m_i - 1$

Examples

- The roots are 2, 2, 2, 5, 5, 9. What is the form of the general solution?
- $a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$, $a_0 = 1, a_1 = -2, a_2 = -1$

Linear nonhomogeneous recurrence relations with constant coefficients

- A linear nonhomogeneous recurrence relation with constant coefficients is one of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} + F(n),$$

  where $c_1, c_2, \ldots, c_k$ are real numbers, and $F(n) \neq 0$ is a function depending only on n.

- The recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$$

  is called the associated homogeneous recurrence relation.

- Examples: $a_n = a_{n-1} + 2^n$, $a_n = a_{n-1} + a_{n-2} + n^2 + n + 1$, $a_n = 3a_{n-1} + n3^n$, $a_n = a_{n-1} + a_{n-2} + a_{n-3} + n!$

The general form of solutions

- Theorem 5: If $\{a_n^{(p)}\}$ is a particular solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} + F(n),$$

then every solution is of the form $\{a_n^{(p)} + \{a_n^{(h)}\}$, where $\{a_n^{(h)}\}$ is a solution of the associated homogenous recurrence relation.

- Proof

- Example: Find all solutions of $a_n = 3a_{n-1} + 2n$. What is the solution with $a_2 = 3$? Try a particular solution of the form $a_n^{(p)} = cn + d$

- Example: Find all solutions of $a_n = 5a_{n-1} - 6a_{n-2} + 7^n$. Try a particular solution of the form $a_n^{(p)} = C \cdot 7^n$

The form of a particular solution

Theorem 6: Suppose that $\{a_n\}$ satisfies

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} + F(n), \text{ where}$$

$$F(n) = (b_t n^t + b_{t-1} n^{t-1} + \ldots + b_n + b_0) s^n,$$

where $b_0, b_1, \ldots, b_t$ and s are real numbers. When s is not a root of the characteristic equation of the associated homogeneous recurrence relation, there is a particular solution of the form

$$(p_t n^t + p_{t-1} n^{t-1} + \ldots + p_n + p_0) s^n.$$

When s is a root of this characteristic equation and its multiplicity is m, there is a particular solution of the form

$$n^m (p_t n^t + p_{t-1} n^{t-1} + \ldots + p_n + p_0) s^n.$$

Examples

- $a_n = 6a_{n-1} - 9a_{n-2} + F(n)$, where $F(n) = 3^n$, $F(n) = n3^n$, $F(n) = n^2 2^n$, $F(n) = (n^2+1)3^n$
- $a_n = \sum_{k=1}^{n} k$