# Discrete Mathematics: Lecture 9

- Last time:
  - Chap 2.3: Functions
  - Chap 2.4: Sequences and Summations
- Today:
  - Chap 2.5: Cardinality of Sets
  - Chap 2.6: Matrices
- Next time:
  - Chap 3.1: Algorithms
  - Chap 3.2: The Growth of functions

# Review of last time

- Injection, surjection, bijection
- Inverse function, composition of functions
- The graph of functions, partial functions
- Sequences, recurrence relation

## Summations

- To represent $a_m + a_{m+1} + \ldots + a_n$, we use

$$\sum_{j=m}^{n} a_j, \qquad \textstyle\sum_{j=m}^{n} a_j, \qquad \text{or } \textstyle\sum_{m \le j \le n} a_j$$

  where $j$ – the index of the summation, $m$ – lower limit, $n$ – upper limit,

- Examples: $\sum_{j=1}^{100} \frac{1}{j}, \qquad \sum_{k=4}^{8} (-1)^k$
- Shift the index of a summation, *e.g.*, $\sum_{j=1}^{5} j^2$
- Double summations, *e.g.*, $\sum_{i=1}^{4} \sum_{j=1}^{3} ij$
- Summations of function values where the index runs over all values in a set: $\sum_{s \in S} f(s), \qquad e.g., \sum_{s \in \{1,3,5\}} s^2$

**TABLE 2 Some Useful Summation Formulae.**

| *Sum* | *Closed Form* |
|---|---|
| $\sum_{k=0}^{n} ar^k \ (r \neq 0)$ | $\frac{ar^{n+1} - a}{r - 1}, r \neq 1$ |
| $\sum_{k=1}^{n} k$ | $\frac{n(n+1)}{2}$ |
| $\sum_{k=1}^{n} k^2$ | $\frac{n(n+1)(2n+1)}{6}$ |
| $\sum_{k=1}^{n} k^3$ | $\frac{n^2(n+1)^2}{4}$ |
| $\sum_{k=0}^{\infty} x^k, \vert x \vert < 1$ | $\frac{1}{1-x}$ |
| $\sum_{k=1}^{\infty}, kx^{k-1}, \vert x \vert < 1$ | $\frac{1}{(1-x)^2}$ |



Proofs of Equations 1,5,6

Example: Find $\sum_{k=50}^{100} k^2$

# A motivating example: Hilbert's Grand Hotel

- The Grand Hotel has an infinitely many rooms: Room 1, Room 2, Room 3, ...
- Suppose all rooms are occupied
- How can we accommodate a new guest without removing any of the current guests?

# Cardinality

- Definition: Let $S$ be a set. If there are exactly $n$ distinct elements in $S$ where $n \in \mathbf{N}$, we say $n$ is the cardinality of $S$.
- Chap 2.3 Ex 79: There is a bijection between any two sets with the same number of objects.
- Definition: We say that two sets $A$ and $B$ have the same cardinality, written $|A| = |B|$, if there is a bijection between them.
- Definition: We say that the cardinality of $A$ is less than or equal to the cardinality of $B$, written $|A| \leq |B|$, if there is an injection from $A$ to $B$.
- Definition: We say that the cardinality of $A$ is less than the cardinality of $B$, written $|A| < |B|$, if $|A| \leq |B|$ but $|A| \neq |B|$.

# Countable sets

- Definition: A set that is either finite or has the same cardinality as $\mathbf{Z}^+$ is called countable. A set that is not countable is called uncountable. When an infinite set $S$ is countable, we denote its cardinality by $\aleph_0$, we write $|S| = \aleph_0$
- A set $S$ is countable iff there exists an injection from $S$ to $\mathbf{Z}^+$
- A subset of a countable set is countable
- A set is countable iff it is possible to list the elements in a sequence where elements may be repeated

# Examples

1. the set of odd positive integers is countable.
2. the set of all integers is countable
3. the set of positive rational numbers is countable
4. the set of real numbers is uncountable
    - Use the fact that every real number has a unique decimal expansion with no tail of 9's

# Results about cardinality

1. Theorem: If $A$ and $B$ are countable sets, then $A \cup B$ is also counable.
2. Schr$\ddot{o}$der-Bernstein Theorem: If $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$. That is, if there are injections from $A$ to $B$ and from $B$ to $A$, then there is a bijection between $A$ and $B$.
   - No known proof is easy to explain, we omit a proof here.
3. Example: Show that $|(0,1)| = |(0,1]|$.

# Questions

Is $A - B$ a countable set?

1. when $A$ is countable
2. when $A$ is uncountable and $B$ is countable
3. when $A$ is uncountable and $B$ is uncountable

# More about countable sets

- Show that the union of a countable number of countable sets is countable
- Suppose that a countably infinite number of buses, each containing a countably infinite number of guests, arrive at Hilbert's fully occupied Grand Hotel. How will you accommodate these guests?
- Give a bijection from $\mathbf{Z}^+ \times \mathbf{Z}^+$ to $\mathbf{Z}^+$
- For any $n \geq 1$, $(\mathbf{Z}^+)^n$ is countable

- Definition: We say that a function is computable (可计算的) if there is a computer program in some programming language that finds the value of this function. If a function is not computable we say it is uncomputable (不可计算的).
- Theorem: There are uncomputable functions.
- Lemma 1: The set of all computer programs in any particular programming language is countable.
- Lemma 2: The set of functions from $\mathbf{Z}^+$ to $\mathbf{Z}^+$ is uncountable.

# Matrices

- A matrix (plural: matrices) is a rectangular array of numbers.
- A matrix with $m$ rows and $n$ columns is called an $m \times n$ matrix.
- A matrix with the same number of rows as columns is called square.
- Two matrices are equal if they have the same number of rows and the same number of columns and the corresponding entries in every position are equal.
- Example: $\begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 1 & 3 \end{bmatrix}$

# Matrices

- Let $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$
- The $i$th row of $\mathbf{A}$, the $j$th column of $\mathbf{A}$,
- The $(i, j)$th element or entry of $\mathbf{A}$
- We write $\mathbf{A} = [a_{ij}]$

# Matrix addition

- Let $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$ be $m \times n$ matrices. The sum of $\mathbf{A}$ and $\mathbf{B}$, denoted by $\mathbf{A} + \mathbf{B}$, is the $m \times n$ matrix that has $a_{ij} + b_{ij}$ as its $(i, j)$th element. That is, $\mathbf{A} + \mathbf{B} = [a_{ij} + b_{ij}]$.

# Matrix product

- Let $\mathbf{A} = [a_{ij}]$ be an $m \times k$ matrix and $\mathbf{B} = [b_{ij}]$ be a $k \times n$ matrix. The product of $\mathbf{A}$ and $\mathbf{B}$, denoted by $\mathbf{AB}$, is the $m \times n$ matrix that has $c_{ij}$ as its $(i, j)$th element, where

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{ik}b_{kj}.$$

- Matrix multiplication is not commutative.
  - it may be that only one of the products is defined
  - even if both are defined, they may not be the same size
  - even if both are the same size, they may not be equal

    Example: $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

# Powers of matrices

- The identity matrix of order $n$ is the $n \times n$ matrix $\mathbf{I}_n = [\delta_{ij}]$, where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$.
- $\mathbf{A}\mathbf{I}_n = \mathbf{I}_m\mathbf{A} = \mathbf{A}$, where $\mathbf{A}$ is an $m \times n$ matrix
- $\mathbf{A}^0 = \mathbf{I}_n$, $\mathbf{A}^{r+1} = \mathbf{A}^r\mathbf{A}$, where $\mathbf{A}$ is an $n \times n$ matrix

# Transposes (转置) of matrices

- Let $\mathbf{A} = [a_{ij}]$ be an $m \times n$ matrix. The transpose of $\mathbf{A}$, denoted by $\mathbf{A}^t$, is the $n \times m$ matrix obtained by interchanging the rows and columns of $\mathbf{A}$
- Example: $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$
- A square matrix $\mathbf{A}$ is called symmetric if $\mathbf{A} = \mathbf{A}^t$. Thus $\mathbf{A} = [a_{ij}]$ is symmetric if $a_{ij} = a_{ji}$ for all $i$ and $j$.

# Zero-one matrices

- A matrix all of whose entries are either 0 or 1 is called a zero-one matrix.
- Zero-one matrices are often used to represent discrete structures, such as relations.
- Operations on such structures are based on Boolean arithmetic with zero-one matrices, which is based on Boolean operations $\wedge$ and $\vee$ on bits
- Let $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$ be $m \times n$ zero-one matrices.
- The join of $\mathbf{A}$ and $\mathbf{B}$ is $\mathbf{A} \vee \mathbf{B} = [a_{ij} \vee b_{ij}]$.
- The meet of $\mathbf{A}$ and $\mathbf{B}$ is $\mathbf{A} \wedge \mathbf{B} = [a_{ij} \wedge b_{ij}]$.

# Boolean products

- Let $\mathbf{A} = [a_{ij}]$ be an $m \times k$ zero-one matrix and $\mathbf{B} = [b_{ij}]$ be a $k \times n$ zero-one matrix. The Boolean product of $\mathbf{A}$ and $\mathbf{B}$, denoted by $\mathbf{A} \odot \mathbf{B}$, is the $m \times n$ matrix that has $c_{ij}$ as its $(i, j)$th element, where

$$c_{ij} = a_{i1} \wedge b_{1j} \vee a_{i2} \wedge b_{2j} \vee \ldots \vee a_{ik} \wedge b_{kj}.$$

- Let $\mathbf{A}$ be a square zero-one matrix. The Boolean power of $\mathbf{A}$ is defined as follows:
  $\mathbf{A}^{[0]} = \mathbf{I}_n$, $\mathbf{A}^{[r+1]} = \mathbf{A}^{[r]} \odot \mathbf{A}$