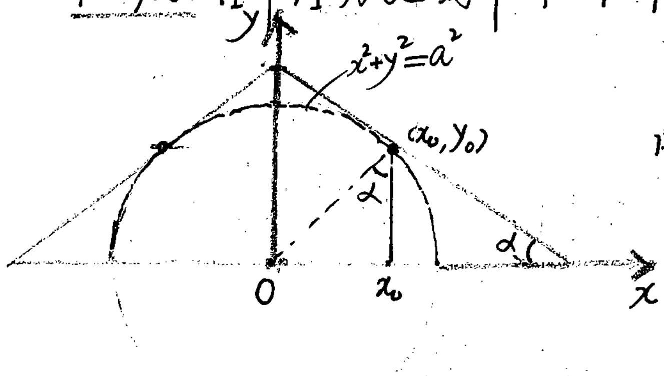
中山大學 本科生考试草稿纸 2011 20/7 —97

P.207.8. 在半径为a的半球外作一外切圆锥体，问其高及底半径取多少才能使圆锥体的体积最小？



解：过点(x_0,y_0)的切线斜率为：$\frac{dy}{dx}=-\frac{x_0}{y_0}$

圆过(x_0,y_0)点的切线：

$$y-y_0=-\frac{x_0}{y_0}(x-x_0)$$

$$y_0y-y_0^2=-x_0\cdot x+x_0^2$$

$$x_0x+y_0\cdot y=x_0^2+y_0^2=a^2$$

$$\frac{x}{\frac{a^2}{x_0}}+\frac{y}{\frac{a^2}{y_0}}=1$$

切线在x轴、y轴上的截距分别为：$\frac{a^2}{x_0}$，$\frac{a^2}{y_0}$.

$$V_{圆锥}=\frac{1}{3}\pi\cdot\left(\frac{a^2}{x_0}\right)^2\cdot\left(\frac{a^2}{y_0}\right)=\frac{\pi a^6}{3}\cdot\frac{1}{y_0\cdot x_0^2}=\frac{\pi a^6}{3}\cdot\frac{1}{y_0(a^2-y_0^2)}$$

$$V'(y_0)=\frac{\pi a^6}{3}\cdot(-1)\cdot\frac{(a^2-y_0^2)+y_0(-2y_0)}{y_0^2(a^2-y_0^2)^2}=\frac{\pi a^6}{3}\cdot\frac{-a^2+y_0^2+2y_0^2}{y_0^2(a^2-y_0^2)^2}=\frac{\pi a^6(3y_0^2-a^2)}{3y_0^2\cdot(a^2-y_0^2)^2}$$

令 $V'(y_0)=0$，得 $3y_0^2=a^2$，$y_0=\frac{a}{\sqrt{3}}$

代入 $x_0^2+y_0^2=a^2$，得 $x_0^2=a^2-\frac{a^2}{3}=\frac{2a^2}{3}$，$x_0=\sqrt{\frac{2}{3}}a$.

从而，圆锥的半径为：$\frac{a^2}{x_0}=\frac{a^2}{\sqrt{\frac{2}{3}}a}=\sqrt{\frac{3}{2}}a$

高为：$\frac{a^2}{y_0}=\frac{a^2}{\frac{a}{\sqrt{3}}}=\sqrt{3}a$. $\tan\alpha=\frac{\sqrt{3}a}{\sqrt{\frac{3}{2}}a}=\sqrt{2}$,

$\alpha=\arctan\sqrt{2}$.