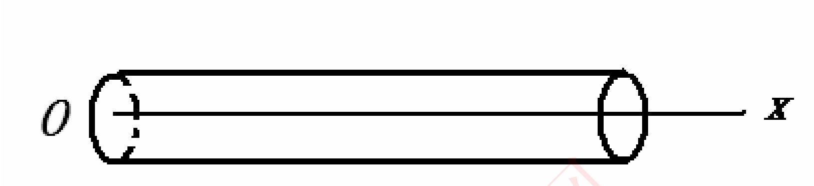
习题 2.1

1.设一物质细杆的长为l,其质量在横截面的分布上可以看作均匀的.现取杆的左端点为坐标原点O,杆所在直线为x轴.设从左端点到细杆上任一点x之间那一段的质量为$m(x)=2x^2(0\le x\le l)$

(1)给自变量x一个增量Δx,求的相应增量Δm;

(2)求比值$\dfrac{\Delta m}{\Delta x}$,问它的物理意义是什么?

(3)求极限$\lim\limits_{\Delta x\to 0}\dfrac{\Delta m}{\Delta x}$,问它的物理意义是什么?



解(1)$\Delta m=2(x+\Delta x)^2-2x^2=2(x^2+2x\Delta x+\Delta x^2)-2x^2=2(2x\Delta x+\Delta x^2)$.

(2)$\dfrac{\Delta m}{\Delta x}=\dfrac{2(2x\Delta x+\Delta x^2)}{\Delta x}=2(2x+\Delta x)$.$\dfrac{\Delta m}{\Delta x}$是$x$到$x+\Delta x$那段细杆的平均线密度.

(3)$\lim\limits_{\Delta x\to 0}\dfrac{\Delta m}{\Delta x}=\lim\limits_{\Delta x\to 0}2(2x+\Delta x)=4x$.$\lim\limits_{\Delta x\to 0}\dfrac{\Delta m}{\Delta x}$是细杆在点$x$的线密度.

2.根据定义,求下列函数的导函数:

(1)$y=ax^3$;(2)$y=\sqrt{2px},p>0$;(3)$y=\sin 5x$.

解(1)$y'=\lim\limits_{\Delta x\to 0}\dfrac{a(x+\Delta x)^3-ax^3}{\Delta x}$

$$=a\lim_{\Delta x\to 0}\frac{(x^3+3x^2\Delta x+3x\Delta x^2+\Delta x^3)-x^3}{\Delta x}=a\lim_{\Delta x\to 0}(3x^2+3x\Delta x+\Delta x^2)=3ax^2.$$

(2)$y'=\lim\limits_{\Delta x\to 0}\dfrac{\sqrt{2p(x+\Delta x)}-\sqrt{2px}}{\Delta x}=\sqrt{2p}\lim\limits_{\Delta x\to 0}\dfrac{\sqrt{x+\Delta x}-\sqrt{x}}{\Delta x}$

$$=\sqrt{2p}\lim_{\Delta x\to 0}\frac{(\sqrt{x+\Delta x}-\sqrt{x})(\sqrt{x+\Delta x}+\sqrt{x})}{\Delta x(\sqrt{x+\Delta x}+\sqrt{x})}=\sqrt{2p}\lim_{\Delta x\to 0}\frac{\Delta x}{\Delta x(\sqrt{x+\Delta x}+\sqrt{x})}$$

$$=\sqrt{2p}\lim_{\Delta x\to 0}\frac{1}{\sqrt{x+\Delta x}+\sqrt{x}}=\frac{\sqrt{2p}}{2\sqrt{x}}.$$

(3)$y'=\lim\limits_{\Delta x\to 0}\dfrac{\sin 5(x+\Delta x)-\sin 5x}{\Delta x}=\lim\limits_{\Delta x\to 0}\dfrac{2\cos\dfrac{5(2x+\Delta x)}{2}\sin\dfrac{5\Delta x}{2}}{\Delta x}$

$$=\lim_{\Delta x\to 0}\frac{2\cdot\frac{5}{2}\cos\frac{5(2x+\Delta x)}{2}\sin\frac{5\Delta x}{2}}{\frac{5\Delta x}{2}}=5\lim_{\Delta x\to 0}\cos\frac{5(2x+\Delta x)}{2}\lim_{\Delta x\to 0}\frac{\sin\frac{5\Delta x}{2}}{\frac{5\Delta x}{2}}=5\cos 5x.$$

3.求下列曲线$y=f(x)$在指定点$M(x_0,f(x_0))$处的切线方程:

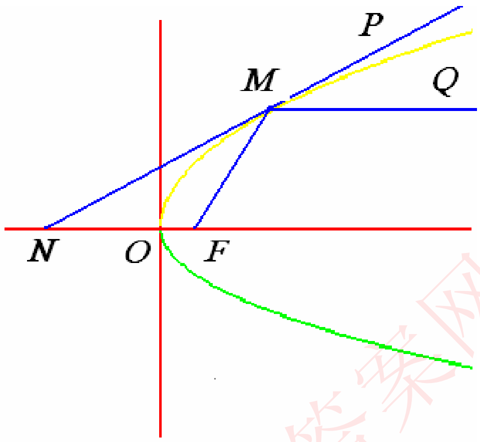
(1)$y=2^x, M(0,1)$;　(2)$y=x^2+2, B(3,11)$.

解(1)$y'=2^x\ln 2, y'(0)=\ln 2$,切线方程$y-1=\ln 2(x-0), y=(\ln 2)x+1$.

(2)$y'=2x, y'(3)=6$,切线方程:$y-11=6(x-3)$.

4.试求抛物线$y^2=2px(p>0)$上任一点$M(x,y)(x>0,y>0)$处的切线斜率,

并证明:从抛物线的焦点$F\left(\frac{p}{2},0\right)$发射光线时,其反射线一定平行于$x$轴.



证$y=\sqrt{2px}, y'=\frac{2p}{2\sqrt{2px}}=\frac{p}{y}$,过点$M$的切线$PMN$方程:$Y-y=\frac{p}{y}(X-x)$.

切线与x轴交点$N(X_0,0)$,$-y=\frac{p}{y}(X_0-x), X_0=x-\frac{y^2}{p}=-x$.

$$FN=\frac{p}{2}+x, FM=\sqrt{\left(x-\frac{p}{2}\right)^2+y^2}=\sqrt{\left(x-\frac{p}{2}\right)^2+2px}$$

$$=\sqrt{x^2+px+\left(\frac{p}{2}\right)^2}=\sqrt{\left(x+\frac{p}{2}\right)^2}=x+\frac{p}{2}=FN,\text{故}\angle FNM=\angle FMN.$$

过M作PQ平行于x轴,则$\angle PMQ=\angle FNM=\angle FMN$.

5.曲线$y=x^2+2x+3$上哪一点的切线与直线$y=4x-1$平行,并求曲线在该点的切线和法线方程.

解　$y'=2x+2=4, x_0=1, y_0=6, k=4$

切线方程:$y-6=4(x-1), y=4x+2$.法线方程:$y-6=\left(-\frac{1}{4}\right)(x-1), y=-\frac{1}{4}x+\frac{25}{4}$.

6.离地球中心r处的重力加速度g是r的函数,其表达式为

$$g(r)=\begin{cases}\dfrac{GMr}{R^3}, r<R;\\ \dfrac{GM}{r^2},\ r\geq R\end{cases}$$ 其中R是地球的半径,M是地球的质量,G是引力常数.

(1)问$g(r)$是否为r的连续函数:

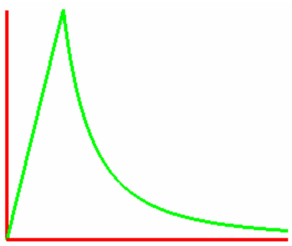
(2)作$g(r)$的草图;

(3)$g(r)$是否是r的可导函数.

解 明显地,$r\neq R$时$g(r)$连续. $\lim\limits_{r\to R-}g(r)=\lim\limits_{r\to R-}\dfrac{GMr}{R^3}=\dfrac{GM}{R^2}$,

$\lim\limits_{r\to R+}g(r)=\lim\limits_{r\to R+}\dfrac{GM}{r^2}=\dfrac{GM}{R^2}=\lim\limits_{r\to R-}g(r)$, $g(r)$在$r=R$连续.

(2)



(3)$r\neq R$时$g(r)$可导.

$g'_-(R)=\dfrac{GM}{R^3}, g'_+(R)=-\dfrac{2GM}{R^3}\neq g'_-(R)$, $g(r)$在$r=R$不可导.

7.求二次函数$P(x)$,已知:点$(1,3)$在曲线$y=P(x)$上,且$P'(0)=3, P'(2)=1$.

解 $P(x)=ax^2+bx+c, P'(x)=2ax+b.\begin{cases}a+b+c=3\\ b=3\\ 4a+b=1\end{cases}$

$b=3, a=-\dfrac{1}{2}, c=3-(a+b)=\dfrac{1}{2}, P(x)=-\dfrac{1}{2}x^2+3x+\dfrac{1}{2}.$

8.求下列函数的导函数:

(1)$y=8x^3+x+7, y'=24x^2+1.$

(2)$y=(5x+3)(6x^2-2), y'=5(6x^2-2)+12x(5x+3)=90x^2+36x-10.$

(3)$y=(x+1)(x-1)\tan x=(x^2-1)\tan x, y'=(2x)\tan x+(x^2-1)\sec^2 x.$

(4)$y=\dfrac{9x+x^2}{5x+6}, y'=\dfrac{(9+2x)(5x+6)-5(9x+x^2)}{(5x+6)^2}=\dfrac{5x^2+12x+54}{(5x+6)^2}.$

(5)$y=\dfrac{1+x}{1-x}=-1+\dfrac{2}{1-x}(x\neq 1), y'=\dfrac{2}{(1-x)^2}.$

$(6)y=\dfrac{2}{x^3-1}(x\neq 1), y'=\dfrac{-6x^2}{(x^3-1)^2}.$

$(7)y=\dfrac{x^2+x+1}{e^x}, y'=\dfrac{(2x+1)e^x-e^x(x^2+x+1)}{e^{2x}}=\dfrac{-x^2+x-1}{e^x}.$

$(8)y=x\cdot 10^x, y'=10^x+x\cdot 10^x\ln 10=10^x(1+x\ln 10).$

$(9)y=x\cos x+\dfrac{\sin x}{x}, y'=\cos x-x\sin x+\dfrac{x\cos x-\sin x}{x^2}.$

$(10)y=e^x\sin x, y'=e^x\sin x+e^x\cos x=e^x(\sin x+\cos x).$

9.定义：若多项式$P(x)$可表为$P(x)=(x-x_0)^m g(x), g(x_0)\neq 0$
则称x_0是$P(x)$的m重根.今若已知x_0是$P(x)$的k重根，证明x_0是$P'(x)$的$(k-1)$重根
$(k>2)$.

证$P(x)=(x-x_0)^k g(x), g(x_0)\neq 0$

$P'(x)=k(x-x_0)^{k-1}g(x)+(x-x_0)^k g'(x)$

$=(x-x_0)^{k-1}(kg(x)+(x-x_0)g'(x))=(x-x_0)^{k-1}h(x),$

$h(x_0)=kg({}_0x)\neq 0$,由定义x_0是$P'(x)$的$(k-1)$重根.

10.若$f(x)$在$(-a,a)$中有定义,且满足$f(-x)=f(x)$,则称$f(x)$为偶函数.设$f(x)$
是偶函数，且$f'(0)$存在,试证明$f'(0)=0$.

证$f'(0)=\lim\limits_{x\to 0}\dfrac{f(x)-f(0)}{x}=\lim\limits_{x\to 0}\dfrac{f(-x)-f(0)}{x}=-\lim\limits_{x\to 0}\dfrac{f(-x)-f(0)}{-x}=-f'(0), f'(0)=0.$

11.设$f(x)$在x_0处可导,证明$\lim\limits_{\Delta x\to 0}\dfrac{f(x_0+\Delta x)-f(x_0-\Delta x)}{2\Delta x}=2f'(x_0).$

证 $\lim\limits_{\Delta x\to 0}\dfrac{f(x_0+\Delta x)-f(x_0-\Delta x)}{2\Delta x}=\dfrac{1}{2}\lim\limits_{\Delta x\to 0}\left[\dfrac{f(x_0+\Delta x)-f(x_0)}{\Delta x}-\dfrac{f(x_0-\Delta x)-f(x_0)}{\Delta x}\right]$

$=\dfrac{1}{2}\lim\limits_{\Delta x\to 0}\left[\dfrac{f(x_0+\Delta x)-f(x_0)}{\Delta x}+\dfrac{f(x_0-\Delta x)-f(x_0)}{-\Delta x}\right]$

$=\dfrac{1}{2}\left[\lim\limits_{\Delta x\to 0}\dfrac{f(x_0+\Delta x)-f(x_0)}{\Delta x}+\lim\limits_{\Delta x\to 0}\dfrac{f(x_0-\Delta x)-f(x_0)}{-\Delta x}\right]=\dfrac{1}{2}[f'(x_0)+f'(x_0)]=f'(x_0).$

12.一质点沿曲线$y=x^2$运动,且已知时刻$t(0<t<\pi/2)$时质点所在位置
$P(t)=(x(t),y(t))$满足：直线$\overline{OP}$与x轴的夹角恰为t.求时刻t时质点的位置速度及
加速度.

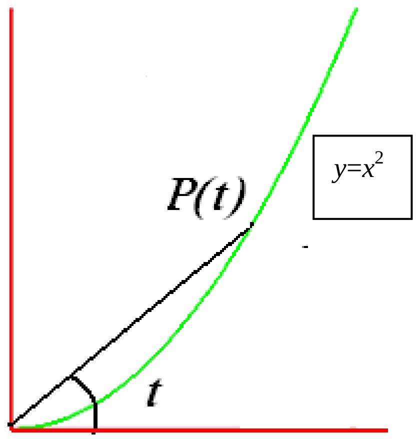
解 $\dfrac{y(t)}{x(t)}=\dfrac{x^2(t)}{x(t)}=x(t)=\tan t, y(t)=\tan^2 t,$

位置$(\tan t,\tan^2 t),$

$v'(t)=(\sec^2 t, 2\tan t\sec^2 t),$

$v''(t)=(2\sec^2 t\tan t, 2\sec^4 t+4\tan^2 t\sec^2 t)$

$=2\sec^2 t(\sec^2 t, 2\tan^2 t).$



13.求函数

$$f(x)=\begin{cases}\dfrac{x}{1+e^{1/x}}, & x\neq 0\\ 0, & x=0\end{cases}$$

在$x=0$的左右导数.

解$f'_-(0)=\lim\limits_{x\to 0-}\dfrac{\dfrac{x}{1+e^{1/x}}}{x}=\lim\limits_{x\to 0-}\dfrac{1}{1+e^{1/x}}=1,\ f'_+(0)=\lim\limits_{x\to 0+}\dfrac{\dfrac{x}{1+e^{1/x}}}{x}=\lim\limits_{x\to 0+}\dfrac{1}{1+e^{1/x}}=0.$

14.设$f(x)=|x-a|\varphi(x)$,其中$\varphi(x)$在$x=a$处连续且$\varphi(a)\neq 0$.证明$f(x)$在$x=a$不可导.

证$f'_-(a)=\lim\limits_{x\to a-}\dfrac{(a-x)\varphi(x)}{x-a}=-\varphi(a),\mathrm{f}'_+(a)=\lim\limits_{x\to a-}\dfrac{(x-a)\varphi(x)}{x-a}=\varphi(a)\neq f'_-(a).$

第二章总练习题

1.讨论函数$f(x)=\begin{cases}|x-3| & x\ge 1\text{时}\\ \dfrac{x^2}{4}-\dfrac{3}{2}x+\dfrac{13}{4}, & x<1\text{时}\end{cases}$的连续性和可导性.

解$x\ne 1$时$f(x)$可导. $f(1-0)=\lim\limits_{x\to 1}\left(\dfrac{x^2}{4}-\dfrac{3}{2}x+\dfrac{13}{4}\right)=2;$

$f(1+0)=\lim\limits_{x\to 1}|x-3|=2=f(1-0)=f(1)$, f在$x=1$连续.

$f'_+(1)=(3-x)'|_{x=1}=-1$, $f'_-(1)=\left(\dfrac{x^2}{4}-\dfrac{3}{2}x+\dfrac{13}{4}\right)'\Bigg|_{x=1}=\left(\dfrac{x}{2}-\dfrac{3}{2}\right)\Bigg|_{x=1}=-1=f'_+(1)$, $f'(1)=-1.$

f在$x=1$可导.

2.设函数$f(x)=\begin{cases}2x-2 & x<-1\text{时}\\ Ax^3+Bx^2+Cx+D, & -1\le x\le 1\text{时}\\ 5x+7 & x>1\text{时}\end{cases}$

试确定常数A,B,C,D的值,使$f(x)$在$(-\infty,+\infty)$可导.

解$f(-1-0)=\lim\limits_{x\to -1}(2x-2)=-4=f(-1)=-A+B-C+D.$

$f'_-(-1)=(2x-2)'|_{x=-1}=2=f'_+(-1)=(Ax^3+Bx^2+Cx+D)'|_{x=-1}$

$=(3Ax^2+2Bx+C)|_{x=-1}=3A-2B+C.$

$f(1-0)=A+B+C+D=f(1+0)=12,$

$f'_-(1)=3A+2B+C=f'_+(1)=5.$

$$\begin{cases}-A+B-C+D=-4\\ 3A-2B+C=2\\ A+B+C+D=12\\ 3A+2B+C=5.\end{cases}$$

$\{A=-9/4,\ B=3/4,\ C=41/4, D=13/4\}.$

3.设函数$g(x)=(\sin 2x)f(x)$,其中$f(x)$在$x=0$连续.问$g(x)$在$x=0$是否可导,若可导,求出$g'(0)$.

解$\dfrac{g(\Delta x)-g(0)}{\Delta x}=2\dfrac{f(\Delta x)\sin 2\Delta x}{2\Delta x}\to 2f(0)(\Delta x\to 0)$, $g'(0)=2f(0).$

4.问函数$\mathrm{f}(\mathrm{x})=\dfrac{\mathrm{x}^2+\sin^2 x}{1+\mathrm{x}^2}$与$\mathrm{g}(\mathrm{x})=\dfrac{-\cos^2 x}{1+\mathrm{x}^2}$为什么有相同得导数?

解因为$f(x)-g(x)=1.$

5,.设函数$f(x)$在$[-1,1]$上有定义,且满足$x\le f(x)\le x^2+x, x\in[-1,1]$.证明存在且等于1.

证$0\le f(0)\le 0$, $f(0)=0.\Delta x>0,$

$\dfrac{f(\Delta x)-f(0)}{\Delta x}=\dfrac{f(\Delta x)}{\Delta x}\le\dfrac{\Delta x^2+\Delta x}{\Delta x}=\Delta x+1\to 1(\Delta x\to 0+0)$, $f'_+(0)=1$,类似$f'_-(0)=1,$

故$f'(0)=1.$

6.设$f(x)=|x^2-4|$,求$f'(x)$.

解$|x|>2$ 时, $f(x)=x^2-4,\ f'(x)=2x. f'_+(2)=(x^2-4)'|_{x=2}=4,$

$f'_-(2)=(4-x^2)'|_{x=2}=-4,\ f'(2)$不存在,同理$f'(-2)$不存在.

7. 设$y=\dfrac{1+x}{1-x}$,求$\dfrac{d^2y}{dx^2}$.

解$y=-1+\dfrac{2}{1-x},\dfrac{dy}{dx}=\dfrac{2}{(1-x)^2},\dfrac{d^2y}{dx^2}=-\dfrac{4}{(1-x)^3}.$

8.设函数$f(x)$在$(-\infty,+\infty)$上有定义,且满足下列性质:

(1) $f(a+b)=f(a)f(b)(a,b$为任意实数$)$;(2) $f(0)=1$;(3)在$x=0$处可导.证明:对于任意$x\in(-\infty,+\infty)$都有$f'(x)=f'(0)\cdot f(x)$.

证$\dfrac{f(x+\Delta x)-f(x)}{\Delta x}=\dfrac{f(x)f(\Delta x)-f(x)f(0)}{\Delta x}$

$=f(x)\dfrac{f(\Delta x)-f(0)}{\Delta x}\to f'(0)\cdot f(x)(\Delta x\to 0),\ f'(x)=f'(0)\cdot f(x).$

9.设$f(x)=\begin{cases}1/2^{2n}, & x=1/2^n,\\ 0, & x\neq 1/2^n\end{cases}(n=1,2,\cdots);\ g(x)=\begin{cases}1/2^{n+1}, & x=1/2^n,\\ 0, & x\neq 1/2^n\end{cases}(n=1,2,\cdots);$

问$f(x)$在$x=0$处是否可导?$g(x)$在$x=0$处是否可导?

解$\dfrac{f(1/2^n)-f(0)}{1/2^n}=\dfrac{1/2^{2n}}{1/2^n}=\dfrac{1}{2^n}\to 0(n\to\infty),$

$\dfrac{f(x)-f(0)}{x}=0\to 0(x\neq 1/2^n,x\to 0).\lim\limits_{x\to 0}\dfrac{f(x)-f(0)}{x}=0,\ f'(0)=0.$

$\dfrac{g(1/2^n)-g(0)}{1/2^n}=\dfrac{1/2^{n+1}}{1/2^n}=\dfrac{1}{2}\to\dfrac{1}{2}(n\to\infty),$

$\dfrac{g(x)-g(0)}{x}=0\to 0(x\neq 1/2^n,x\to 0),\lim\limits_{x\to 0}\dfrac{g(x)-g(0)}{x}.g'(0)$不存在.

10.设$y=f(x)$及$y=g(x)$在$[a,b]$上连续,证明:

$$\left(\int_a^b f(x)g(x)dx\right)^2\le\int_a^b f^2(x)dx\int_a^b g^2()xdx.$$

证$\int_a^b[f(x)+tg(x)]^2dx=\left(\int_a^b g^2(x)dx\right)t^2+\left(2\int_a^b f(x)g(x)dx\right)t+\int_a^b f^2(x)dx\ge 0(*),$

如果$\int_a^b g^2(x)dx=0$,则由g的连续性$g(x)=0,x\in[a,b]$,不等式两端都是0.

如果$\int_a^b g^2(x)dx>0$,(*)左端的二次函数恒非负,故其判别式非正,

$$\left(2\int_a^b f(x)g(x)dx\right)^2-4\left(\int_a^b g^2(x)dx\right)\int_a^b f^2(x)dx\le 0,$$

$$\left(\int_a^b f(x)g(x)dx\right)^2\le\int_a^b f^2(x)dx\int_a^b g^2()xdx.$$

11.求出函数

$$f(x)=\frac{1}{2}x+\frac{1}{2^2}x^2+\cdots+\frac{1}{2^n}x^n$$

在点$x=1$的导数，再将函数$f(x)$写成$f(x)=\dfrac{x/2-(x/2)^{n+1}}{1-x/2}$的形式，再求$f'(1)$，由此证明下列等式：

$$\frac{1}{2}+\frac{2}{2^2}+\cdots+\frac{n}{2^n}=2-\frac{n+2}{2^n}.$$

证$f'(x)=\dfrac{1}{2}+\dfrac{2}{2^2}x+\cdots+\dfrac{n}{2^n}x^{n-1}$, $f'(1)=\dfrac{1}{2}+\dfrac{2}{2^2}+\cdots+\dfrac{n}{2^n}$.

$$f(x)=\frac{x/2-(x/2)^{n+1}}{1-x/2},$$

$$f'(x)=\frac{(1/2-(n+1)(x/2)^n(1/2))(1-x/2)+(1/2)(x/2-(x/2)^{n+1})}{(1-x/2)^2},$$

$$f'(1)=\frac{(1/2-(n+1)(1/2^{n+1}))(1/2)+(1/2)(1/2-1/2^{n+1})}{1/2^2}$$

$$=(1-(n+1)/2^n)+1-1/2^n=2-\frac{n+2}{2^n}.$$

12.由类似上题的办法证明$1+2x+3x^2+\cdots+nx^{n-1}=\dfrac{1-(n+1)x^n+nx^{n+1}}{(1-x)^2}(x\neq 1)$.

证 由等比级数求和公式$x+x^2+\cdots+x^n=\dfrac{x-x^{n+1}}{1-x}$,

两端求导得$1+2x+3x^2+\cdots+nx^{n-1}$

$$=\frac{(1-(n+1)x^n)(1-x)+(x-x^{n+1})}{(1-x)^2}=\frac{1-(n+1)x^n+nx^{n+1}}{(1-x)^2}(x\neq 1).$$

13.设$y=f(x)$在$[0,1]$连续且$f(x)>0$证明$\displaystyle\int_0^1\frac{1}{f(x)}dx\geq\frac{1}{\int_0^1 f(x)dx}$.

证$\displaystyle 1=\int_0^1 1dx=\int_0^1\sqrt{f(x)}\frac{1}{\sqrt{f(x)}}dx\leq\int_0^1 f(x)dx\int_0^1\frac{1}{f(x)}dx.$

14.$\ln x=\displaystyle\int_1^n\frac{dt}{t}$

$(a)\dfrac{1}{n+1}<\ln\left(1+\dfrac{1}{n}\right)<\dfrac{1}{n}(n>0)$

$(b)\dfrac{1}{2}+\dfrac{1}{3}+\cdots+\dfrac{1}{n}<\ln n<1+\dfrac{1}{2}+\dfrac{1}{3}+\cdots+\dfrac{1}{n-1}$;$(c)e^{1-\frac{1}{n+1}}<\left(1+\dfrac{1}{n}\right)^n<e.$

证$(1)\displaystyle\frac{1}{n+1}=\int_1^{1+1/n}\frac{dt}{1+1/n}\ln\left(1+\frac{1}{n}\right)=\int_1^{1+1/n}\frac{dt}{t}<\int_1^{1+1/n}\frac{dt}{1}=\frac{1}{n}.$

$(2)\displaystyle\ln n=\ln\frac{2}{1}\cdot\frac{3}{2}\cdots\cdot\frac{n}{n-1}=\ln\left(1+\frac{1}{1}\right)+\cdots+\left(1+\frac{1}{n}\right)<1+\frac{1}{2}+\cdots+\frac{1}{n},$

$$\ln n=\ln\left(1+\frac{1}{1}\right)+\cdots+\left(1+\frac{1}{n}\right)>\frac{1}{2}+\cdots+\frac{1}{n}.$$

(3) $\left(1+\frac{1}{n}\right)^{n}=e^{n\ln\left(1+\frac{1}{n}\right)}>e^{n\cdot\frac{1}{n+1}}=e^{1-\frac{1}{n+1}}.$

第二章总练习题

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解$x\ne 1$时$f(x)$可导. $f(1-0)=\lim\limits_{x\to 1}\left(\dfrac{x^2}{4}-\dfrac{3}{2}x+\dfrac{13}{4}\right)=2;$

$f(1+0)=\lim\limits_{x\to 1}|x-3|=2=f(1-0)=f(1)$, f在$x=1$连续.

$$f_+'(1)=(3-x)'|_{x=1}=-1,\ f_-'(1)=\left(\frac{x^2}{4}-\frac{3}{2}x+\frac{13}{4}\right)'\bigg|_{x=1}=\left(\frac{x}{2}-\frac{3}{2}\right)\bigg|_{x=1}=-1=f_+'(1),\ f'(1)=-1.$$

f在$x=1$可导.

2.设函数$f(x)=\begin{cases}2x-2 & x<-1\text{时}\\ Ax^3+Bx^2+Cx+D, & -1\le x\le 1\text{时}\\ 5x+7 & x>1\text{时}\end{cases}$

试确定常数A,B,C,D的值,使$f(x)$在$(-\infty,+\infty)$可导.

解$f(-1-0)=\lim\limits_{x\to -1}(2x-2)=-4=f(-1)=-A+B-C+D.$

$f_-'(-1)=(2x-2)'|_{x=-1}=2=f_+'(-1)=(Ax^3+Bx^2+Cx+D)'|_{x=-1}$

$=(3Ax^2+2Bx+C)|_{x=-1}=3A-2B+C.$

$f(1-0)=A+B+C+D=f(1+0)=12,$

$f_-'(1)=3A+2B+C=f_+'(1)=5.$

$$\begin{cases}-A+B-C+D=-4\\ 3A-2B+C=2\\ A+B+C+D=12\\ 3A+2B+C=5.\end{cases}$$

$\{A=-9/4,\ B=3/4,\ C=41/4, D=13/4\}.$

3.设函数$g(x)=(\sin 2x)f(x)$,其中$f(x)$在$x=0$连续.问$g(x)$在$x=0$是否可导,若可导,求出$g'(0)$.

解$\dfrac{g(\Delta x)-g(0)}{\Delta x}=2\dfrac{f(\Delta x)\sin 2\Delta x}{2\Delta x}\to 2f(0)(\Delta x\to 0), g'(0)=2f(0).$

4.问函数$\mathrm{f}(\mathrm{x})=\dfrac{\mathrm{x}^2+\sin^2 x}{1+\mathrm{x}^2}$与$\mathrm{g}(\mathrm{x})=\dfrac{-\cos^2 x}{1+\mathrm{x}^2}$为什么有相同得导数?

解因为$f(x)-g(x)=1.$

5,.设函数$f(x)$在$[-1,1]$上有定义,且满足$x\le f(x)\le x^2+x, x\in[-1,1]$.证明存在且等于1.

证$0\le f(0)\le 0, f(0)=0.\Delta x>0,$

$\dfrac{f(\Delta x)-f(0)}{\Delta x}=\dfrac{f(\Delta x)}{\Delta x}\le\dfrac{\Delta x^2+\Delta x}{\Delta x}=\Delta x+1\to 1(\Delta x\to 0+0),\ f_+'(0)=1,$类似$f_-'(0)=1,$

故$f'(0)=1.$

6.设$f(x)=|x^2-4|$,求$f'(x)$.

解$|x|>2$ 时, $f(x)=x^2-4,\ f'(x)=2x.\ f'_+(2)=(x^2-4)'|_{x=2}=4,$

$f'_-(2)=(4-x^2)'|_{x=2}=-4,\ f'(2)$不存在,同理$f'(-2)$不存在.

7. 设$y=\dfrac{1+x}{1-x}$,求$\dfrac{d^2y}{dx^2}$.

解$y=-1+\dfrac{2}{1-x},\ \dfrac{dy}{dx}=\dfrac{2}{(1-x)^2},\ \dfrac{d^2y}{dx^2}=-\dfrac{4}{(1-x)^3}.$

8.设函数$f(x)$在$(-\infty,+\infty)$上有定义,且满足下列性质:

(1) $f(a+b)=f(a)f(b)(a,b$为任意实数$)$;(2) $f(0)=1$;(3)在$x=0$处可导.证明:对于任意$x\in(-\infty,+\infty)$都有$f'(x)=f'(0)\cdot f(x)$.

证$\dfrac{f(x+\Delta x)-f(x)}{\Delta x}=\dfrac{f(x)f(\Delta x)-f(x)f(0)}{\Delta x}$

$=f(x)\dfrac{f(\Delta x)-f(0)}{\Delta x}\to f'(0)\cdot f(x)(\Delta x\to 0),\ f'(x)=f'(0)\cdot f(x).$

9.设$f(x)=\begin{cases}1/2^{2n}, & x=1/2^n,\\ 0, & x\neq 1/2^n\end{cases}(n=1,2,\cdots);\ g(x)=\begin{cases}1/2^{n+1}, & x=1/2^n,\\ 0, & x\neq 1/2^n\end{cases}(n=1,2,\cdots);$

问$f(x)$在$x=0$处是否可导?$g(x)$在$x=0$处是否可导?

解$\dfrac{f(1/2^n)-f(0)}{1/2^n}=\dfrac{1/2^{2n}}{1/2^n}=\dfrac{1}{2^n}\to 0(n\to\infty),$

$\dfrac{f(x)-f(0)}{x}=0\to 0(x\neq 1/2^n,x\to 0).\lim\limits_{x\to 0}\dfrac{f(x)-f(0)}{x}=0,\ f'(0)=0.$

$\dfrac{g(1/2^n)-g(0)}{1/2^n}=\dfrac{1/2^{n+1}}{1/2^n}=\dfrac{1}{2}\to\dfrac{1}{2}(n\to\infty),$

$\dfrac{g(x)-g(0)}{x}=0\to 0(x\neq 1/2^n,x\to 0),\lim\limits_{x\to 0}\dfrac{g(x)-g(0)}{x}.g'(0)$不存在.

10.设$y=f(x)$及$y=g(x)$在$[a,b]$上连续,证明:

$\left(\int_a^b f(x)g(x)dx\right)^2\le\int_a^b f^2(x)dx\int_a^b g^2()xdx.$

证$\int_a^b[f(x)+tg(x)]^2dx=\left(\int_a^b g^2(x)dx\right)t^2+\left(2\int_a^b f(x)g(x)dx\right)t+\int_a^b f^2(x)dx\ge 0(*),$

如果$\int_a^b g^2(x)dx=0$,则由g的连续性$g(x)=0,x\in[a,b]$,不等式两端都是0.

如果$\int_a^b g^2(x)dx>0$,(*) 左端的二次函数恒非负, 故其判别式非正,

$\left(2\int_a^b f(x)g(x)dx\right)^2-4\left(\int_a^b g^2(x)dx\right)\int_a^b f^2(x)dx\le 0,$

$\left(\int_a^b f(x)g(x)dx\right)^2\le\int_a^b f^2(x)dx\int_a^b g^2()xdx.$

11.求出函数

$$f(x)=\frac{1}{2}x+\frac{1}{2^2}x^2+\cdots+\frac{1}{2^n}x^n$$

在点$x=1$的导数,再将函数$f(x)$写成$f(x)=\dfrac{x/2-(x/2)^{n+1}}{1-x/2}$的形式,再求$f'(1)$,

由此证明下列等式:

$$\frac{1}{2}+\frac{2}{2^2}+\cdots+\frac{n}{2^n}=2-\frac{n+2}{2^n}.$$

证$f'(x)=\dfrac{1}{2}+\dfrac{2}{2^2}x+\cdots+\dfrac{n}{2^n}x^{n-1}$, $f'(1)=\dfrac{1}{2}+\dfrac{2}{2^2}+\cdots+\dfrac{n}{2^n}$.

12.由类似上题的办法证明

$$f(x)=\frac{x/2-(x/2)^{n+1}}{1-x/2},$$

$$f'(x)=\frac{(1/2-(n+1)(x/2)^n(1/2))(1-x/2)+(1/2)(x/2-(x/2)^{n+1})}{(1-x/2)^2},$$

$$f'(1)=\frac{(1/2-(n+1)(1/2^{n+1}))(1/2)+(1/2)(1/2-1/2^{n+1})}{1/2^2}$$

$$=(1-(n+1)/2^n)+1-1/2^n=2-\frac{n+2}{2^n}.$$

$$\frac{1-(n+1)x^n+nx^{n+1}\,{}^2}{(1-x)^2}\quad(x\neq 1)$$

13.设$y=f(x)$在$[0,1]$连续且$f(x)>0$证明$\displaystyle\int_0^1\frac{1}{f(x)}dx\geq\frac{1}{\int_0^1 f(x)dx}$.

证$\displaystyle 1=\int_0^1 1dx=\int_0^1\sqrt{f(x)}\frac{1}{\sqrt{f(x)}}dx\leq\int_0^1 f(x)dx\int_0^1\frac{1}{f(x)}dx.$

14.$\displaystyle\ln x=\int_1^n\frac{dt}{t}$

$(a)\dfrac{1}{n+1}<\ln\left(1+\dfrac{1}{n}\right)<\dfrac{1}{n}(n>0)$

$(b)\dfrac{1}{2}+\dfrac{1}{3}+\cdots+\dfrac{1}{n}<\ln n<1+\dfrac{1}{2}+\dfrac{1}{3}+\cdots+\dfrac{1}{n-1};$

$(c)e^{1-\frac{1}{n+1}}<\left(1+\dfrac{1}{n}\right)^n<e.$

证(1)$\displaystyle\frac{1}{n+1}=\int_1^{1+1/n}\frac{dt}{1+1/n}\ln\left(1+\frac{1}{n}\right)=\int_1^{1+1/n}\frac{dt}{t}<\int_1^{1+1/n}\frac{dt}{1}=\frac{1}{n}.$

(2)$\displaystyle\ln n=\ln\frac{2}{1}\cdot\frac{3}{2}\cdots\frac{n}{n-1}=\ln\left(1+\frac{1}{1}\right)+\cdots+\left(1+\frac{1}{n}\right)<1+\frac{1}{2}+\cdots+\frac{1}{n},$

$\displaystyle\ln n=\ln\left(1+\frac{1}{1}\right)+\cdots+\left(1+\frac{1}{n}\right)>\frac{1}{2}+\cdots+\frac{1}{n}.$

$(c)e^{1-\frac{1}{n+1}} < \left(1+\frac{1}{n}\right)^n < e.$

证(1) $\frac{1}{n+1} = \int_1^{1+1/n} \frac{dt}{1+1/n} \ln\left(1+\frac{1}{n}\right) = \int_1^{1+1/n} \frac{dt}{t} < \int_1^{1+1/n} \frac{dt}{1} = \frac{1}{n}.$

(2) $\ln n = \ln \frac{2}{1} \cdot \frac{3}{2} \cdots \frac{n}{n-1} = \ln\left(1+\frac{1}{1}\right) + \cdots + \left(1+\frac{1}{n}\right) < 1 + \frac{1}{2} + \cdots + \frac{1}{n},$

$\ln n = \ln\left(1+\frac{1}{1}\right) + \cdots + \left(1+\frac{1}{n}\right) > \frac{1}{2} + \cdots + \frac{1}{n}.$

(3) $\left(1+\frac{1}{n}\right)^n = e^{n\ln\left(1+\frac{1}{n}\right)} > e^{n \cdot \frac{1}{n+1}} = e^{1-\frac{1}{n+1}}.$

习题 2.4

n

(1) $y=x^n, y^{(n)}=n!$.

(2) $y=e^x, y^{(n)}=e^n$.

(3) $y=\dfrac{1}{1+x}=(1+x)^{-1}(x\neq-1).y^{(n)}=(-1)(-1-1)\cdots(-1-n+1)(1+x)^{-1-n}=\dfrac{(-1)^n n!}{(1+x)^{n+1}}$.

(4) $y=\dfrac{1}{x(x+1)}=\dfrac{1}{x}-\dfrac{1}{x+1}, y^{(n)}=(-1)^n n!\left(\dfrac{1}{x^{n+1}}-\dfrac{1}{(1+x)^{n+1}}\right)$.

2.设$y(x)=e^x\cos x$,证明$y''-2y'+2y=0$.

证$y'=e^x\cos x-e^x\sin x=e^x(\cos x-\sin x)$,

$y''=e^x(\cos x-\sin x)+e^x(-\sin x-\cos x)=e^x(-2\sin x)$,

$y''-2y'+2y=e^x(-2\sin x)-2e^x(\cos x-\sin x)+2e^x\cos x=0$,

3.设$y=\dfrac{x-3}{x+4}(x\neq-4)$,证明$2y'^2=(y-1)y''$.

证$y=\dfrac{x-3}{x+4}=1-\dfrac{7}{x+4}, y'=\dfrac{7}{(x+4)^2}, y''=-\dfrac{14}{(x+4)^3}$.

$2y'^2=\dfrac{98}{(x+4)^4}, (y-1)y''=\left(-\dfrac{7}{x+4}\right)\left(-\dfrac{14}{(x+4)^3}\right)=\dfrac{98}{(x+4)^4}=2y'^2$.

4.设$y=(1-x)(2x+1)^2(3x-1)^3$,求$y^{(6)}, y^{(7)}$.

解$y^{(6)}=6!\,(-108), y^{(7)}=0$.

5.要使$y=e^{\lambda x}$满足方程$y''+py'+qy=0$(其中p,q为常数),λ该取哪些值?

解$y'=\lambda e^{\lambda x}, y''=\lambda^2e^{\lambda x}, y''+py'+qy=(\lambda^2+p\lambda+q)e^{\lambda x}=0, e^{\lambda x}\neq0$,

λ该取方程$\lambda^2+p\lambda+q=0$的根.

6.飞轮绕一定轴转动,转过的角度θ与时间t的关系为$\theta=t^3-2t^2+3t-1$,求飞轮转动的角速度与角加速度.

解角速度$\theta'=3t^2-4t+3$,角加速度$\theta''=6t-4$.

7.设$f(x)=\dfrac{1}{(1-x)^n}$,其中n为一个正整数,求$f^{(k)}(x)$,k为一个正整数.

解$f(x)=\dfrac{1}{(1-x)^n}=(1-x)^{-n}, f^{(k)}(x)=(-n)(-n-1)\cdots(-n-k+1)(1-x)^{-n-k}(-1)^k$

$=\dfrac{n(n+1)\cdots(n+k-1)}{(1-x)^{n+k}}, f^{(k)}(0)=n(n+1)\cdots(n+k-1)$.

8.设$y=x^2\ln(1+x)$,求$y^{(50)}$.

解由 Leibniz公式,

$$y^{(50)}=x^2\left(\ln(1+x)\right)^{(50)}+50\cdot(2x)\cdot\left(\ln(1+x)\right)^{(49)}+\frac{50\cdot 49}{2}\cdot 2\cdot\left(\ln(1+x)\right)^{(48)}$$

$$=x^2\left((1+x)^{-1}\right)^{(49)}+50\cdot(2x)\cdot\left((1+x)^{-1}\right)^{(48)}+\frac{50\cdot 49}{2}\cdot 2\cdot\left((1+x)^{-1}\right)^{(47)}$$

$$=x^2(-1)(-2)\cdots(-1-49+1)(1+x)^{-50}+100x(-1)(-2)\cdots(-1-48+1)(1+x)^{-49}+$$

$$2450(-1)(-2)\cdots(-1-47+1)(1+x)^{-48}$$

$$=-x^2\,49!(1+x)^{-50}+100\cdot 48!(1+x)^{-49}-2450\cdot 47!(1+x)^{-48}=\frac{-2\cdot 47!}{(1+x)^{50}}(x^2+50x+1225).$$

9.验证函数$y=C_1e^{ax}+C_2e^{bx}$(其中C_1与C_2为任意常数)是微分方程$y''-(a+b)y'+aby=0$的解.

证$y'=(C_1e^{ax}+C_2e^{bx})'=C_1ae^{ax}+C_2be^{bx}$, $y''=(C_1ae^{ax}+C_2be^{bx})'=C_1a^2e^{ax}+C_2b^2e^{bx}$,

$$y''-(a+b)y'+aby=C_1a^2e^{ax}+C_2b^2e^{bx}-(a+b)(C_1ae^{ax}+C_2be^{bx})+ab(C_1e^{ax}+C_2e^{bx})=0.$$

10.验证函数$y=(C_1x+C_2)e^{ax}$(其中C_1与C_2为任意常数)是微分方程$y''-2ay'+a^2y=0$的解.

证$y'=\left((C_1x+C_2)e^{ax}\right)'=C_1e^{ax}+a(C_1x+C_2)e^{ax}=e^{ax}(aC_1x+C_1+aC_2)$,

$$y''=e^{ax}a(aC_1x+C_1+aC_2)+e^{ax}(aC_1)=e^{ax}(a^2C_1x+a^2C_2+2aC_1),$$

$$y''-2ay'+a^2y$$

$$=e^{ax}(a^2C_1x_1+a^2C_2+2aC_1)-2ae^{ax}(aC_1x+C_1+aC_2)+a^2(C_1x+C_2)e^{ax}=0.$$

验证函数$y=C_1\cos\omega t+C_2\sin\omega t$(其中$C_1$与$C_2$为任意常数)是微分方程$y''+\omega^2y=0$的解.

证$y'=-C_1\omega\sin\omega t+C_2\omega\cos\omega t$, y''

$$=-C_1\omega^2\cos\omega t-C_2\omega^2\sin\omega t=-\omega^2(C_1\cos\omega t+C_2\sin\omega t)=-\omega^2y.$$

习题 2.2

1.下列各题的计算是否正确,指出错误并加以改正:

(1)$(\cos\sqrt{x})'=-\sin\sqrt{x}$,错.$(\cos\sqrt{x})'=-\sin\sqrt{x}\sqrt{x}'=-\dfrac{\sin\sqrt{x}}{2\sqrt{x}}$.

(2)$[\ln(1-x)]'=\dfrac{1}{1-x}$,错.$[\ln(1-x)]'=\dfrac{1}{1-x}(1-x)'=\dfrac{1}{x-1}$.

(3)$\left[x^2\sqrt{1+x^2}\right]'=\left(x^2\right)'\left(\sqrt{1+x^2}\right)'=2x\dfrac{x}{\sqrt{1+x^2}}$,错.

$\left[x^2\sqrt{1+x^2}\right]'=\left(x^2\right)'\left(\sqrt{1+x^2}\right)+\left(x^2\right)\left(\sqrt{1+x^2}\right)'=2x\sqrt{1+x^2}+x^2\dfrac{x}{\sqrt{1+x^2}}$

$=2x\sqrt{1+x^2}+\dfrac{x^3}{\sqrt{1+x^2}}=\dfrac{2x+3x^3}{\sqrt{1+x^2}}$.

(4)$\left[\ln|x+2\sin^2 x|\right]'=\dfrac{1}{x+2\sin^2 x}(1+4\sin x)\cos x$,错.

$\left[\ln|x+2\sin^2 x|\right]'=\dfrac{1}{x+2\sin^2 x}(1+4\sin x\cos x)$.

2.记$f'(g(x))=f'(u)|_{u=g(x)}$.现设$f(x)=x^2+1$.

(1)求$f'(x),f'(0),f'(x^2),f'(\sin x)$;

(2)求$\dfrac{d}{dx}f(x^2),\dfrac{d}{dx}f(\sin x)$;

(3)$f'(g(x))$与$[f(g(x))]'$是否相同?指出两者的关系.

解(1)$f'(x)=2x,f'(0)=0,f'(x^2)=2x^2,f'(\sin x)=2\sin x$.

(2)$\dfrac{d}{dx}f(x^2)=f'(x^2)\left(x^2\right)'=2x^2\cdot 2x=4x^3$.

$\dfrac{d}{dx}f(\sin x)=f'(\sin x)(\sin x)'=2\sin x\cos x=\sin 2x$.

(3)$f'(g(x))$与$[f(g(x))]'$不同,$[f(g(x))]'=f'(g(x))g'(x)$.

3.求下列函数的导函数:

(1)$y=\dfrac{2}{x^3-1},y'=-\dfrac{2\cdot 3x^2}{\left(x^3-1\right)^2}=-\dfrac{6x^2}{\left(x^3-1\right)^2}$.

(2)$y=\sec x,y'=\left((\cos x)^{-1}\right)'=-(\cos x)^{-2}(\cos x)'=-(\cos x)^{-2}(-\sin x)=\tan x\sec x$.

(3)$y=\sin 3x+\cos 5x,y'=3\cos 3x-5\sin 5x$.

(4)$y=\sin^3 x\cos 3x,y'=3\sin^2 x\cos x\cos 3x-3\sin^3 x\sin 3x$

$=3\sin^2 x(\cos x\cos 3x-\sin x\sin 3x)=3\sin^2 x\cos 4x$.

$$(5)\ y=\frac{1+\sin^2 x}{\cos x^2},\ y'=\frac{2\sin x\cos x\cos x^2-(1+\sin^2 x)(-\sin x^2)2x}{\cos^2 x^2}$$
$$=\frac{\sin 2x\cos x^2+2x(1+\sin^2 x)(\sin x^2)}{\cos^2 x^2}.$$

$$(6)\ y=\frac{1}{3}\tan^3 x-\tan x+x,\ y'=\tan^2 x\sec^2 x-\sec^2 x+1$$
$$=\tan^2 x\sec^2 x-\tan^2 x=\tan^2 x(\sec^2 x-1)=\tan^4 x.$$

$$(7)\ y=e^{ax}\sin bx,\ y'=ae^{ax}\sin bx+be^{ax}\cos bx=e^{ax}(a\sin bx+b\cos bx).$$

$$(8)\ y=\cos^5\sqrt{1+x^2},\ y'=5\cos^4\sqrt{1+x^2}(-\sin\sqrt{1+x^2})\frac{x}{\sqrt{1+x^2}}$$
$$=-\frac{5x\cos^4\sqrt{1+x^2}\sin\sqrt{1+x^2}}{\sqrt{1+x^2}}.$$

$$(9)\ y=\ln\left|\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right|,\ y'=\frac{1}{2}\frac{1}{\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)}\sec^2\left(\frac{x}{2}+\frac{\pi}{4}\right)$$
$$=\frac{1}{2}\frac{1}{\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)}\frac{1}{\cos^2\left(\frac{x}{2}+\frac{\pi}{4}\right)}=\frac{1}{2\sin\left(\frac{x}{2}+\frac{\pi}{4}\right)\cos\left(\frac{x}{2}+\frac{\pi}{4}\right)}$$
$$=\frac{1}{\sin(x+\frac{\pi}{2})}=\frac{1}{\cos x}=\sec x.$$

$$(10)\ y=\frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|(a>0,x\neq\pm a),\ y'=\frac{1}{2a}\frac{x+a}{x-a}\frac{(x+a)-(x-a)}{(x+a)^2}=\frac{1}{x^2-a^2}.$$

4.求下列函数的导函数:

$$(1)\ y=\arcsin\frac{x}{a}(a>0),\ y'=\frac{1}{\sqrt{1-\left(\frac{x}{a}\right)^2}}\cdot\frac{1}{a}=\frac{1}{\sqrt{a^2-x^2}}.$$

$$(2)\ y=\frac{1}{a}\arctan\frac{x}{a}(a>0),\ y'=\frac{1}{a}\frac{1}{1+\left(\frac{x}{a}\right)^2}\frac{1}{a}=\frac{1}{a^2+x^2}.$$

$$(3)\ y=x^2\arccos x(|x|<1),\ y'=2x\arccos x-\frac{x^2}{\sqrt{1-x^2}}.$$

$$(4)\ y=\arctan\frac{1}{x},\ y'=\frac{1}{1+\frac{1}{x^2}}\frac{-1}{x^2}=-\frac{1}{1+x^2}.$$

$$(5)\ y=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\arcsin\frac{x}{a}(a>0),$$

$$y'=\frac{1}{2}\sqrt{a^2-x^2}+\frac{x}{2}\frac{-2x}{\sqrt{a^2-x^2}}+\frac{a^2}{2}\frac{1}{\sqrt{1-\left(\frac{x}{a}\right)^2}}\frac{1}{a}$$

$$=\frac{1}{2}\sqrt{a^2-x^2}-\frac{x^2}{\sqrt{a^2-x^2}}+\frac{a^2}{\sqrt{a^2-x^2}}=\sqrt{a^2-x^2}.$$

$$(6)\,y=\frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\ln\frac{x+\sqrt{x^2+a^2}}{a}\,(a>0)$$

$$y'=\frac{1}{2}\sqrt{x^2+a^2}+\frac{x}{2}\frac{x}{\sqrt{x^2+a^2}}+\frac{a^2}{2}\frac{1}{x+\sqrt{x^2+a^2}}\left(1+\frac{x}{\sqrt{x^2+a^2}}\right)$$

$$=\frac{1}{2}\sqrt{x^2+a^2}+\frac{x^2}{2\sqrt{x^2+a^2}}+\frac{a^2}{2\sqrt{x^2+a^2}}=\sqrt{x^2+a^2}.$$

$$(7)\,y=\arcsin\frac{2x}{x^2+1},x\neq\pm1.$$

$$y'=\frac{1}{\sqrt{1-\frac{4x^2}{(x^2+1)^2}}}\frac{2(x^2+1)-2x\cdot 2x}{(x^2+1)^2}=2\frac{1}{\left|x^2-1\right|}\frac{1-x^2}{x^2+1}=\frac{2\operatorname{sgn}(1-x^2)}{x^2+1}.$$

$$(8)\,y=\frac{2}{\sqrt{a^2-b^2}}\arctan\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right)(a>b\geq 0).$$

$$y'=\frac{2}{\sqrt{a^2-b^2}}\frac{1}{1+\frac{a-b}{a+b}\tan^2\frac{x}{2}}\sqrt{\frac{a-b}{a+b}}\sec^2\frac{x}{2}\left(\frac{1}{2}\right)$$

$$=\frac{1}{a+b+(a-b)\tan^2\frac{x}{2}}\sec^2\frac{x}{2}=\frac{1}{(a+b)\cos^2\frac{x}{2}+(a-b)\sin^2\frac{x}{2}}$$

$$=\frac{1}{a+b\cos x}.$$

$$(9)\,y=(1+\sqrt{x})(1+\sqrt{2x})(1+\sqrt{3x}),\ln y=\ln(1+\sqrt{x})+\ln(1+\sqrt{2x})+\ln(1+\sqrt{3x})$$

$$y'/y=\frac{1}{2(1+\sqrt{x})\sqrt{x}}+\frac{2}{2(1+\sqrt{2x})\sqrt{2x}}+\frac{3}{2(1+\sqrt{3x})\sqrt{3x}},$$

$$y'=y\left[\frac{1}{2(1+\sqrt{x})\sqrt{x}}+\frac{2}{2(1+\sqrt{2x})\sqrt{2x}}+\frac{3}{2(1+\sqrt{3x})\sqrt{3x}}\right].$$

$$(10)\,y=\sqrt{1+x+2x^2},y'=\frac{1+4x}{2\sqrt{1+x+2x^2}}.$$

$$(11)\,y=\sqrt{x^2+a^2},y'=\frac{x}{\sqrt{x^2+a^2}}.$$

$$(12)\,y=\sqrt{a^2-x^2},y'=\frac{-x}{\sqrt{a^2-x^2}}.$$

$(13)\ y=\ln(x+\sqrt{x^2+a^2}),\ y'=\dfrac{1}{x+\sqrt{x^2+a^2}}\left(1+\dfrac{x}{\sqrt{x^2+a^2}}\right)=\dfrac{1}{\sqrt{x^2+a^2}}.$

$(14)\ y=(x-1)\sqrt[3]{(3x+1)^2(2-x)}.\ln y=\ln(x-1)+\dfrac{2}{3}\ln(3x+1)+\dfrac{1}{3}\ln(2-x),$

$\dfrac{y'}{y}=\dfrac{1}{x-1}+\dfrac{2}{3x+1}+\dfrac{1}{3}\dfrac{-1}{2-x}$

$y'=y\left[\dfrac{1}{x-1}+\dfrac{2}{3x+1}+\dfrac{1}{3}\dfrac{-1}{2-x}\right].$

$(15)\ y=e^x+e^{e^x},\ y'=e^x+e^{e^x}\cdot e^x=e^x(1+e^{e^x}).$

$(16)\ y=x^{a^a}+a^{x^a}+a^{a^x}\ (a>0).$

$y'=a^a x^{a^a-1}+a^{x^a}\ln a(ax^{a-1})+a^{a^x}\ln a a^x\ln a$

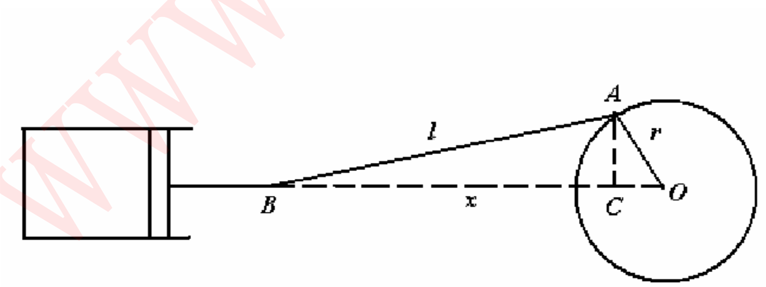
$=a^a x^{a^a-1}+a\ln a a^{x^a}x^{a-1}+a^{a^x}a^x\ln^2 a.$

5.一雷达的探测器瞄准着一枚安装在发射台上的火箭，它与发射台之间的距离是400m. 设t=0时向上垂直地发射火箭，初速度为0，火箭以的匀加速度8m/s^{2}垂直地向上运动；若雷达探测器始终瞄准着火箭. 问：自火箭发射后10秒钟时，探测器的仰角$\theta(t)$的变化速率是多少?

解$x(t)=\dfrac{1}{2}\cdot 8\cdot t^2=4t^2,\tan\theta(t)=\dfrac{x(t)}{400}=\dfrac{t^2}{100},$

$\theta(t)=\arctan\dfrac{t^2}{100},\theta'(t)=\dfrac{1}{1+\left(\dfrac{t^2}{100}\right)^2}\cdot\dfrac{t}{50},\theta'(10)=\dfrac{1}{1+\left(\dfrac{10^2}{100}\right)^2}\cdot\dfrac{10}{50}=0.1(\text{弧度}/s).$

6.在图示的装置中，飞轮的半径为2m且以每秒旋转4圈的匀角速度按顺时针方向旋转. 问：当飞轮的旋转角为$\theta=\dfrac{\pi}{2}$时，活塞向右移动的速率是多少?



解$x(t)=2\cos 8\pi t+\sqrt{36-4\sin^2 8\pi t}$,

$$x'(t)=-16\pi\sin 8\pi t+\frac{-8\sin 8\pi t\cos 8\pi t\cdot(8\pi)}{2\sqrt{36-4\sin^2 8\pi t}},$$

$\alpha(t)=8\pi t=\frac{\pi}{2},,t_0=\frac{1}{16},x'(\frac{1}{16})=-16\pi.$

活塞向右移动的速率是16πm/s.

习题 2.3

1.当$x \to 0$时,下列各函数是x的几阶无穷小量?

(1)$y = x + 10x^2 + 100x^3$.1阶.

(2)$y = (\sqrt{x+2} - \sqrt{2})\sin x = \dfrac{x\sin x}{\sqrt{x+2}+\sqrt{2}}$,2阶.

(3)$y = x(1-\cos x) = x\cdot 2\sin^2\dfrac{x}{2}$,2阶.

2.已知:当$x \to 0$时,$\alpha(x) = o(x^2)$.试证明$\alpha(x) = o(x)$.

证$\dfrac{\alpha(x)}{x} = \dfrac{\alpha(x)}{x^2}x = o(1)x = o(1)$.

3.设$\alpha(x) = o(x)(x \to 0)$, $\beta(x) = o(x)(x \to 0)$.试证明:$\alpha(x)+\beta(x) = o(x)(x \to 0)$.

证$\dfrac{\alpha(x)+\beta(x)}{x} = \dfrac{\alpha(x)}{x} + \dfrac{\beta(x)}{x} = o(1)+o(1) = o(1)$.

上述结果有时可以写成$o(x)+o(x) = o(x)$.

4.计算下列函数在指定点x_0处的微分:

(1)$y = x\sin x, x_0 = \pi/4$.$y' = \sin x + x\cos x$, $y'\left(\dfrac{\pi}{4}\right) = \dfrac{1}{\sqrt{2}}\left(1+\dfrac{\pi}{4}\right)$, $dy = \dfrac{1}{\sqrt{2}}\left(1+\dfrac{\pi}{4}\right)dx$.

(2)$y = (1+x)^\alpha$ ($\alpha > 0$是常数).

$y' = \alpha(1+x)^{\alpha-1}$, $y'(0) = \alpha$, $dy = \alpha dx$.

5.求下列各函数的微分:

(1)$y = \dfrac{1-x}{1+x} = -1 + \dfrac{2}{1+x}$, $y' = -\dfrac{2}{(1+x)^2}$, $dy = -\dfrac{2dx}{(1+x)^2}$.

(2)$y = xe^x$, $y' = e^x + xe^x = e^x(1+x)$.$dy = e^x(1+x)dx$.

6.设$y = \dfrac{2}{x-1}(x \ne 1)$,计算当$x$由3变到3.001时,函数的增量和向相应的微分.

解 $y' = -\dfrac{2}{(x-1)^2}$, $y'(3) = -\dfrac{1}{2}$.

$\Delta y = \dfrac{2}{2.001} - 1 = -\dfrac{0.001}{2.001}$, $dy = -\dfrac{0.001}{2}$.

7.试计算$\sqrt[5]{32.16}$的近似值.

解$\sqrt[5]{32.16} = 2\sqrt[5]{1+.16/32} = 2\cdot(1+\dfrac{1}{5}\cdot\dfrac{.16}{32}) = 2.002$.

8.求下列方程所确定的隐函数的导函数:

(1)$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}(a > 0)$.$\dfrac{1}{3}x^{-\frac{1}{3}} + \dfrac{1}{3}y^{-\frac{1}{3}}y' = 0$, $y' = -\left(\dfrac{y}{x}\right)^{\frac{1}{3}}$.

(2)$(x-a)^2 + (y-b)^2 = c^2$ (a,b,c为常数).

$2(x-a) + 2(y-b)y' = 0$, $y' = -\dfrac{x-a}{y-b}$.

$$(3)\arctan\frac{y}{x}=\ln\sqrt{x^2+y^2}.$$

$$\frac{-\frac{y}{x^2}+\frac{y'}{x}}{1+\left(\frac{y}{x}\right)^2}=\frac{x+yy'}{x^2+y^2},\frac{xy'-y}{x^2+y^2}=\frac{x+yy'}{x^2+y^2},xy'-y=x+yy',y'=\frac{x+y}{x-y}.$$

$(4)\,y\sin x-\cos(x-y)=0$

$y'\sin x+y\cos x+\sin(x-y)(1-y')=0,$

$$y'=\frac{y\cos x+\sin(x-y)}{\sin(x-y)-\sin x}.$$

9.求下列隐函数在指定的点M的导数:

$(1)\,y^2-2xy-x^2+2x-4=0,M(3,7)$

$$2yy'-2y-2xy'-2x+2=0,y'=\frac{y+x-1}{y-x},y'(3)=\frac{7+3-1}{7-3}=\frac{9}{4}.$$

$$(2)\,e^{xy}-5x^2y=0,M\left(\frac{e^2}{10},\frac{20}{e^2}\right).$$

$$e^{xy}(y+xy')-10xy-5x^2y'=0,y'=\frac{10xy-ye^{xy}}{xe^{xy}-5x^2},y'\left(\frac{e^2}{10}\right)=\frac{20-\frac{20}{e^2}e^2}{\frac{e^2}{10}e^2-5\cdot\frac{e^4}{100}}=0.$$

10.设$y=f(x)$由下列参数方程给出,求$y'=\dfrac{dy}{dx}$:

$$(1)\begin{cases}x=2t-t^2\\y=3t-t^3\end{cases}$$

$$\frac{dy}{dx}=\frac{3-3t^2}{2-2t}=\frac{3}{2}(1+t).(t\neq1).$$

$$(2)\begin{cases}x=t\ln t\\y=e^t\end{cases}\frac{dy}{dx}=\frac{e^t}{\ln t+1},t\neq1/e.$$

$$(3)\begin{cases}x=\arccos\dfrac{1}{\sqrt{1+t^2}}\\y=\arcsin\dfrac{t}{\sqrt{1+t^2}}\end{cases}$$

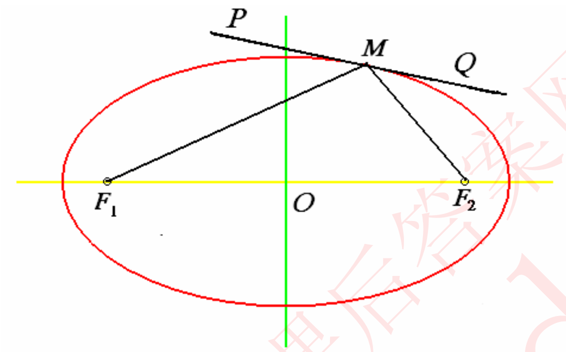
$$\frac{dy}{dx}=\frac{-\frac{1}{\sqrt{1-\frac{1}{1+t^2}}}\left(-\frac{1}{2}\right)\frac{2t}{(1+t^2)^{3/2}}}{\frac{1}{\sqrt{1-\left(\frac{t^2}{1+t^2}\right)}}\cdot\frac{\sqrt{1+t^2}-t\cdot\frac{t}{\sqrt{1+t^2}}}{1+t^2}}=\operatorname{sgn}(t),t\neq0.$$

11.试求椭圆周$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$上一点$M_0(x_0,y_0)$处的切线方程与法线方程. 并证明:从椭圆的一个焦点向椭圆周上任一点M发射的光线,其反射线必通过椭圆的另一个焦点.

$\dfrac{2x}{a^2}+\dfrac{2yy'}{b^2},y'=-\dfrac{b^2x}{a^2y}.$

切线方程：$y-y_0=\left(-\dfrac{b^2x_0}{a^2y_0}\right)(x-x_0),\dfrac{x_0x}{a^2}+\dfrac{y_0y}{b^2}=1.$

法线方程：$y-y_0=\left(\dfrac{a^2y_0}{b^2x_0}\right)(x-x_0),a^2y_0x-b^2x_0y=(a^2-b^2)x_0y_0$



焦点$F_1(-c,0),F_1(c,0),c^2=a^2-b^2(a>b)$.设$y_0\neq 0$.切线斜率$k=-\dfrac{b^2x_0}{a^2y_0}$.

MF_1的斜率$k_1=\dfrac{y_0}{x_0+c}$,MF_2的斜率$k_2=\dfrac{y_0}{x_0-c}$.

$$\tan\angle F_2MQ=\frac{k-k_2}{1+kk_2}=\frac{-\dfrac{b^2x_0}{a^2y_0}-\dfrac{y_0}{x_0-c}}{1-\dfrac{b^2x_0}{a^2y_0}\cdot\dfrac{y_0}{x_0-c}}=-\frac{b^2x_0(x_0-c)+a^2y_0^2}{a^2y_0(x_0-c)-b^2x_0y_0}$$

$$=-\frac{a^2b^2-b^2cx_0}{(a^2-b^2)x_0y_0-a^2cy_0}=-\frac{b^2(a^2-cx_0)}{c^2x_0y_0-a^2cy_0}=\frac{b^2(a^2-cx_0)}{cy_0(a^2-cx_0)}=\frac{b^2}{cy_0};$$

$$\tan\angle PMF_1=\frac{k_1-k}{1+kk_1}=\frac{\dfrac{b^2x_0}{a^2y_0}+\dfrac{y_0}{x_0+c}}{1-\dfrac{b^2x_0}{a^2y_0}\cdot\dfrac{y_0}{x_0+c}}=-\frac{b^2x_0(x_0+c)+a^2y_0^2}{a^2y_0(x_0+c)-b^2x_0y_0}$$

$$=\frac{a^2b^2+b^2cx_0}{(a^2-b^2)x_0y_0+a^2cy_0}=\frac{b^2(a^2+cx_0)}{c^2x_0y_0+a^2cy_0}=\frac{b^2(a^2+cx_0)}{cy_0(a^2+cx_0)}=\frac{b^2}{cy_0}=\tan\angle F_2MQ.$$

$\angle PMF_1$和$\angle F_2MQ$都在区间$\left(-\dfrac{\pi}{2},\dfrac{\pi}{2}\right)$,故$\angle PMF_1=\angle F_2MQ$.

习题 2.6

1.根据定积分的定义直接求下列积分:

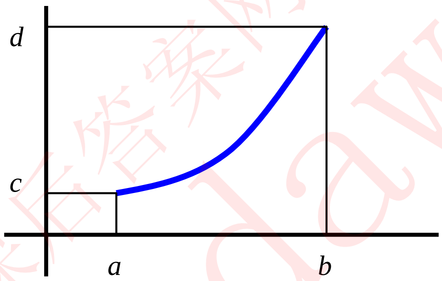
(1)$\int_a^b kdx = \lim\limits_{\lambda\to 0}\sum\limits_{i=1}^{n} k\Delta x_i = \lim\limits_{\lambda\to 0} k(b-a) = k(b-a).$

(2)$\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}(a+\frac{i(b-a)}{n})\frac{b-a}{n} = \lim\limits_{n\to\infty}(a(b-a)+\frac{(b-a)^2}{n^2}\sum\limits_{i=1}^{n} i)$

$= a(b-a)+(b-a)^2\lim\limits_{n\to\infty}(\frac{1}{n^2}\sum\limits_{i=1}^{n} i) = a(b-a)+(b-a)^2\lim\limits_{n\to\infty}\frac{1}{n^2}\frac{n(n+1)}{2}$

$= a(b-a)+(b-a)^2\lim\limits_{n\to\infty}\frac{(1+1/n)}{2} = a(b-a)+\frac{(b-a)^2}{2} = \frac{b^2-a^2}{2}.$

2.设函数$x=\varphi(y)$在$[c,d]$上连续且$\varphi(y)>0$.试用定积分表示曲线$x=\varphi(y), y=c,$ $y=d$及y轴所围的图形的面积;又设$c\geq 0$,函数$x=\varphi(y)$在$[c,d]$上严格递增,试求积分和$\int_c^d \varphi(y)dy+\int_a^b \psi(x)dx$,其中$y=\psi(x)$是$x=\varphi(y)$的反函数,$a=\varphi(c), b=\varphi(b)$.



解$\int_c^d \varphi(y)dy+\int_a^b \psi(x)dx = bd-ac.$

3.写出函数$y=x^2$在区间$[0,1]$上的Riemann和,其中分割为n等分,中间点ξ_i为分割小区间的左端点.求出当$n\to\infty$时Riemann和的极限.]

解$s_n=\sum\limits_{i=0}^{n-1}\frac{i^2}{n^2}\cdot\frac{1}{n}=\frac{1}{6n^3}(n-1)n(2n-1)=\frac{1}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)\to\frac{1}{3}(n\to\infty).$

4.求定积分$\int_0^1\sqrt{x}dx.$

解$y=\sqrt{x}$的反函数$x=y^2$,当$x=0$时,$y=0$,当$x=1$时,$y=1$.由2,3题

$\int_0^1\sqrt{x}dx = 1-\int_0^1 y^2dy = 1-\frac{1}{3}=\frac{2}{3}.$

5.证明下列不等式

(1)$\frac{\pi}{2}<\int_0^{\pi/2}(1+\sin x)dx<\pi.$

证$\int_0^{\pi/2}(1+\sin x)dx>\int_0^{\pi/2}(1)dx=\frac{\pi}{2}.\int_0^{\pi/2}(1+\sin x)dx<\int_0^{\pi/2}(2)dx=\pi.$

(2)$\sqrt{2}<\int_0^1\sqrt{2+x-x^2}dx<\frac{3}{2}.$

证$2+x-x^2=(1+x)(2-x)=0, x_1=-1, x_2=2$.当$x\in(-\infty,1/2)$时,$\sqrt{2+x-x^2}$递增,

.当$x\in(1/2,+\infty)$时,$\sqrt{2+x-x^2}$递减, 故

$$\sqrt{2}=\int_0^1\sqrt{2}dx<\int_0^1\sqrt{2+x-x^2}dx<\int_0^1\sqrt{2+1/2-1/4}dx=\frac{3}{2}.$$

6.判断下列各题中两个积分值之大小:

(1)$\int_0^1 e^x dx>\int_0^1 e^{x^2}dx.$

(2)$\int_0^{\pi/2}x^2dx>\int_0^{\pi/2}(\sin x)^2dx.$

(3)$\int_0^1 xdx<\int_0^1\sqrt{1+x^2}dx.$

7.设函数$y=f(x)$在$[a,b]$上有定义,并且假定$y=f(x)$在任何闭子区间上有最大值和最小值.对于任意一个分割:$T:x_0=a<x_1<x_2<\cdots<x_{n-1}<x_n=b$记$m_i$为$f(x)$在$[x_{i-1},x_i]$中的最小值,$M_i$为$f(x)$在$[x_{i-1},x_i]$中的最大值. 证明$y=f(x)$在$[a,b]$上可积的充要条件是极限$\lim\limits_{\lambda(T)\to}\sum\limits_{i=1}^n m_i\Delta x_i$与$\lim\limits_{\lambda(T)\to}\sum\limits_{i=1}^n M_i\Delta x_i$存在并且相等.

证设$y=f(x)$在$[a,b]$上可积, 则$\lim\limits_{\lambda(T)\to 0}\sum\limits_{i=1}^n m_i\Delta x_i=\lim\limits_{\lambda(T)\to 0}\sum\limits_{i=1}^n f(\xi_i)\Delta x_i=\int_a^b f(x)dx,$

$$\lim_{\lambda(T)\to 0}\sum_{i=1}^n M_i\Delta x_i=\lim_{\lambda(T)\to 0}\sum_{i=1}^n f(\eta_i)\Delta x_i=\int_a^b f(x)dx.$$

设$\lim\limits_{\lambda(T)\to 0}\sum\limits_{i=1}^n m_i\Delta x_i=\lim\limits_{\lambda(T)\to 0}\sum\limits_{i=1}^n M_i\Delta x=I$,则

$$\sum_{i=1}^n m_i\Delta x_i\le\sum_{i=1}^n f(\xi_i)\Delta x_i\le\sum_{i=1}^n M_i\Delta x,$$

由夹挤定理, $\lim\limits_{\lambda(T)\to 0}\sum\limits_{i=1}^n f(\xi_i)\Delta x_i=I.$

习题 2.7

1.求下列变上(下)限积分所定义的函数的导函数:

$(1)F(x)=\int_1^{x^2}\frac{dt}{1+t^2}, F'(x)=\frac{1}{1+x^2}.$

$(2)G(x)=\int_0^{1+x^2}\sin t^2 dt, G'(x)=2x\sin(1+x^2)^2.$

$(3)H(x)=\int_x^1 t^2\cos t dt, H'(x)=-x^2\cos x.$

$(4)L(x)=\int_x^{x^2} e^{-t^2}dt, L'(x)=2xe^{-x^2}-e^{-x^2}.$

2.设$y=f(x)$在$[a,b]$上连续.证明$F_0(x)=\int_a^x f(t)dt$在a处有右导数,且$F_+'(a)=f(a)$.

证 $\frac{F_0(a+\Delta x)-F_0(a)}{\Delta x}=\frac{1}{\Delta x}\int_a^{a+\Delta x} f(t)dt=\frac{1}{\Delta x}f(\xi)\Delta x(a\le\xi\le a+\Delta x)$

$=f(\xi)\to f(a)(\Delta x\to 0+)$故$F_+'(a)=f(a)$.

3.设f(x)在$[a,b]$上连续.假定$f(x)$有一个原函数$F(x)$且$F((a)=0$.证明

当$a\le x\le b$时$F(x)=\int_a^x f(t)dt$.

证G$(\mathrm{x})=\int_a^x f(t)dt.G(a)=0$,由变上限积分求导定理,$G'(x)=f(x).F'(x)=f(x),$

$F(a)=0.$

$(G(x)-F(x))'=G'(x)-F'(x)=f(x)-f(x)=0, G(x)-F(x)=C, x\in[a,b].$

$C=F(a)-G(a)=0, F(x)=G(x)=\int_a^x f(t)dt, x\in[a,b].$

4.证明:当$x\in(0,+\infty)$时,$\ln x=\int_1^x\frac{dt}{t}dt.$

证由于$(\ln x)'=\frac{1}{x},\left(\int_1^x\frac{dt}{t}dt\right)'=\frac{1}{x}, x\in(0,+\infty), \ln 1=\int_1^1\frac{dt}{t}dt=0$,故$\ln x=\int_1^x\frac{dt}{t}dt.$

5.设$y=f(x)$在$[a,b]$上可积, 且$|f(x)|\le L,(\forall x\in[a,b]), uqz$其中$L$为常数.证明

变上限积分$F(x)=\int_a^x f(t)dt$在$[a,b]$上满足Lipschiz 条件:

$|F(x_1)-F(x_2)|\le L|x_1-x_2|,(x_1,x_2\in[a,b]).$

证不妨设$x_1<x_2$,

$|F(x_1)-F(x_2)|=\left|\int_a^{x_2} f(t)dt-\int_a^{x_1} f(t)dt\right|=\left|\int_{x_1}^{x_2} f(t)dt\right|\le\int_{x_1}^{x_2}|f(t)|dt\le\int_{x_1}^{x_2}Ldt=x_2=x_1.$

6.求函数$G(x)=\int_0^x e^t\int_0^t\sin zdzdt$的二阶导数.

解$G'(x)=e^x\int_0^x\sin zdz, G''(x)=\mathrm{e}^x\int_0^x\sin zdz+e^x\sin x=\mathrm{e}^x(1-\cos x)+e^x\sin x.$

习题 2.8

4. 将下列积分改成若干个区间上定积分之和, 然后分别使用Newton-Leibniz公式求处其值:

(1)

1.用Newton-Leibniz公式计算下列定积分:

$$(1)\int_0^1 x^3dx=\left.\frac{x^4}{4}\right|_0^1=\frac{1}{4}.$$

$$(2)\int_a^b e^x dx=e^x\Big|_a^b=e^b-e^a.$$

$$(3)\int_0^{3\pi}\sin xdx=-\cos x\,|_0^{3\pi}=2.$$

$$(4)\int_1^2\frac{dx}{x}=\ln x\,|_1^2=\ln 2.$$

$$(5)\int_0^{\pi}(2\sin x+x^3)dx=\left[-2\cos x+\frac{x^4}{4}\right]_0^{\pi}=4+\frac{\pi^4}{4}.$$

$$(6)\int_0^1(x^5+\frac{1}{3}x^3+\frac{1}{2}x+1)dx=\left[\frac{x^6}{6}+\frac{x^4}{12}+\frac{x^2}{4}+x\right]_0^1=\frac{3}{2}.$$

2.验证$\frac{1}{2}x^2-\frac{1}{x}$是$x+\frac{1}{x^2}$的一个原函数并计算定积分$\int_2^4\left(x+\frac{1}{x^2}\right)dx$.试问下式

$\int_{-1}^1\left(x+\frac{1}{x^2}\right)dx=\left.\left(\frac{1}{2}x^2-\frac{1}{x}\right)\right|_{-1}^1$ 是否成立:为什么?

解 $\left(\frac{1}{2}x^2-\frac{1}{x}\right)'=\left(\frac{1}{2}x^2\right)'-\left(x^{-1}\right)'=x+x^{-2}=x+\frac{1}{x^2}$,故$\frac{1}{2}x^2-\frac{1}{x}$是$x+\frac{1}{x^2}$的一个原函数.

$$\int_2^4\left(x+\frac{1}{x^2}\right)dx=\left.\left(\frac{1}{2}x^2-\frac{1}{x}\right)\right|_2^4=\frac{25}{4}.$$

$\int_{-1}^1\left(x+\frac{1}{x^2}\right)dx=\left.\left(\frac{1}{2}x^2-\frac{1}{x}\right)\right|_{-1}^1$ 不成立.因为$x+\frac{1}{x^2}$在$[-1,1]$不可积.

3.将下列极限中的和式视作适当函数的Riemann和, 然后使用Newton-Leibniz公式求出其值:

$$(1)\lim_{n\to\infty}\sum_{k=1}^n\frac{1}{n}\sin\frac{k}{n}=\int_0^1\sin xdx=-\cos x\,|_0^1=1-\cos 1.$$

$$(2)\lim_{n\to\infty}\sum_{k=1}^n\frac{k^3}{n^4}=\lim_{n\to\infty}\sum_{k=1}^n\frac{1}{n}\left(\frac{k}{n}\right)^3=\int_0^1x^3dx=\left.\frac{x^4}{4}\right|_0^1=\frac{1}{4}.$$

$$(3)\lim_{n\to\infty}\sum_{k=1}^n\frac{1}{n+k}=\lim_{n\to\infty}\sum_{k=1}^n\frac{1}{n}\frac{1}{1+k/n}=\int_0^1\frac{dx}{1+x}=\ln(1+x)\,|_0^1=\ln 2.$$

4.将下列积分改成若干个区间上定积分之和,然后分别使用Newton-Leibniz公式求处其值:

$$(1)\int_{-1}^{1}|x|dx=\int_{0}^{1}xdx-\int_{-1}^{0}xdx=\left.\frac{x^2}{2}\right|_0^1-\left.\frac{x^2}{2}\right|_{-1}^0=1.$$

$$(2)\int_{-1}^{1}\operatorname{sgn}xdx=\int_{0}^{1}1dx+\int_{-1}^{0}(-1)dx=1-1=0.$$

$$(3)\int_{0}^{1}x\left|\frac{1}{2}-x\right|dx=\int_{0}^{1/2}x\left(\frac{1}{2}-x\right)dx+\int_{1/2}^{1}x\left(x-\frac{1}{2}\right)dx$$

$$=\left.\left(\frac{x^2}{4}-\frac{x^3}{3}\right)\right|_0^{1/2}+\left.\left(\frac{x^3}{3}-\frac{x^2}{4}\right)\right|_{1/2}^1=\frac{1}{16}-\frac{1}{24}+\frac{1}{3}-\frac{1}{4}-\frac{1}{24}+\frac{1}{16}=\frac{1}{8}.$$

$$(4)\int_{0}^{2\pi}|\sin x|dx=\int_{0}^{\pi}\sin xdx-\int_{\pi}^{2\pi}\sin xdx=-\cos x|_0^{\pi}+\cos x|_{\pi}^{2\pi}=2+2=4.$$

$$(5)\int_{0}^{2}(x-[x])dx=\int_{0}^{1}xdx+\int_{1}^{2}(x-1)dx=\left.\frac{x^2}{2}\right|_0^1+\left.\left(\frac{x^2}{2}-x\right)\right|_1^2$$

$$=\frac{1}{2}-(-\frac{1}{2})=1.$$

5.设$F(x)$在$[a,b]$上有连续的导函数$F'(x)$.试证明:存在一点$c\in[a,b]$,使得$F(b)-F(a)=F'(c)(b-a)$.

证$F(b)-F(a)=\int_a^b F'(x)dx$(Newton-Leibniz公式)

$=F'(c)(b-a)$(定积分中指中值公式).