### Discrete Mathematics: Lecture 2

- Last time:
  - Chap 6.1: The basics of counting
  - Chap 6.2: The pigeonhole principle
- Today:
  - Chap 6.3: Permutations and combinations
  - Chap 6.4: Binomial coefficients and identities
- Assignment 1 due in two weeks

#### Review of last time

- Basic counting principles: the product, sum, substraction and division rules
- Tree diagrams
- The pigeonhole principle and its generalized version

### Permutations (排列)

- ullet Definition: A permutation of a set of distinct objects is an ordered arrangement of these objects. An ordered arrangement of r elements of a set is called an r-permutation.
- Example: Let  $S = \{1, 2, 3\}$ . The sequence 3,1,2 is a permutation of S. The sequence 3,1 is a 2-permutation of S.
- Theorem: Let P(n,r) denote the number of r-permutations of a set with n elements. Then  $P(n,r) = n(n-1)\dots(n-r+1) = \frac{n!}{(n-r)!}$ .
- Note: P(n,0) = 1
- Example: How many ways are there to select a first-prize winner, a second-prize winner, and a third-prize winner from 100 different people who have entered a contest.
- Example: How many permutations of the letters ABCDEFGH contain the string ABC?

# Combinations (组合)

- Definition: An unordered selection of r elements of a set is called an r-combination.
- Theorem: Let C(n,r) denote the number of r-combinations of a set with n elements, where n and r are nonnegative integers with  $r \le n$ . Then

$$C(n,r) = \frac{n!}{r!(n-r)!}$$

- Corollary: Let n and r be nonnegative integers with  $r \le n$ . Then C(n,r) = C(n,n-r).
- Example: How many poker hands of 5 cards can be dealt from a standard deck of 52 cards? How many ways are there to select 47 cards from a standard deck of 52 cards?
- How many bit strings of length n contain exactly r 1s?

# Combinatorial proofs (组合证明)

- Definition: A combinatorial proof of an identity is a proof that uses counting arguments to prove that both sides of the identity count the same objects but in different ways.
- Example application: C(n,r) = C(n,n-r)

### The binomial theorem

- A binomial expression is the sum of two terms, such as x + y.
- Theorem: Let x and y be variables, and let n be a nonnegative integer. Then  $(x+y)^n = \sum_{j=0}^n C(n,j) x^{n-j} y^j$ . Proof:
  - The terms in the product  $(x+y)(x+y)\dots(x+y)$  are of the form  $x^{n-j}y^j$
  - ullet To obtain a term  $x^{n-j}y^j$ , we choose j ys from the n sums.
  - Thus the coefficient of  $x^{n-j}y^j$  is C(n,j).
- Example:  $(x+y)^4$
- Example: what is the coefficient of  $x^{12}y^{13}$  in the expansion of  $(2x-3y)^{25}$



### Corollaries

- Corollary 1:  $\sum_{j=0}^{n} C(n,j) = 2^n$ 
  - combinatorial proof
- Corollary 2:  $\sum_{j=0}^{n} (-1)^{j} C(n,j) = 0$
- Corollary 3:  $\sum_{j=0}^{n} 2^{j} C(n,j) = 3^{n}$

### Pascal's identity and triangle

Theorem: Let n and k be positive integers with  $n \ge k$ . Then C(n+1,k) = C(n,k-1) + C(n,k)The McGraw-Hill Companies. Inc. all rights reserved.

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$
By Pascal's identity:  $1 \quad 2 \quad 1$ 

$$\begin{pmatrix} 3 \\ 0 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} \begin{pmatrix} 3 \\ 3 \end{pmatrix} \begin{pmatrix} 3 \\ 3 \end{pmatrix} \begin{pmatrix} 6 \\ 4 \end{pmatrix} + \begin{pmatrix} 6 \\ 5 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$$

$$1 \quad 3 \quad 3 \quad 1$$

$$\begin{pmatrix} 4 \\ 0 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \end{pmatrix} \begin{pmatrix} 4 \\ 4 \end{pmatrix} \begin{pmatrix} 4 \\ 4 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$$

$$1 \quad 4 \quad 6 \quad 4 \quad 1$$

$$\begin{pmatrix} 5 \\ 0 \end{pmatrix} \begin{pmatrix} 5 \\ 1 \end{pmatrix} \begin{pmatrix} 5 \\ 2 \end{pmatrix} \begin{pmatrix} 5 \\ 3 \end{pmatrix} \begin{pmatrix} 5 \\ 4 \end{pmatrix} \begin{pmatrix} 5 \\ 5 \end{pmatrix} \begin{pmatrix} 5 \\ 6 \end{pmatrix}$$

$$1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1$$

$$\begin{pmatrix} 6 \\ 0 \end{pmatrix} \begin{pmatrix} 6 \\ 1 \end{pmatrix} \begin{pmatrix} 6 \\ 2 \end{pmatrix} \begin{pmatrix} 6 \\ 3 \end{pmatrix} \begin{pmatrix} 6 \\ 4 \end{pmatrix} \begin{pmatrix} 6 \\ 5 \end{pmatrix} \begin{pmatrix} 6 \\ 6 \end{pmatrix} \begin{pmatrix} 6 \\ 7 \end{pmatrix} \begin{pmatrix} 7 \\ 7 \end{pmatrix}$$

$$1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1$$

$$\begin{pmatrix} 8 \\ 0 \end{pmatrix} \begin{pmatrix} 8 \\ 1 \end{pmatrix} \begin{pmatrix} 8 \\ 2 \end{pmatrix} \begin{pmatrix} 8 \\ 3 \end{pmatrix} \begin{pmatrix} 8 \\ 4 \end{pmatrix} \begin{pmatrix} 8 \\ 5 \end{pmatrix} \begin{pmatrix} 8 \\ 6 \end{pmatrix} \begin{pmatrix} 8 \\ 7 \end{pmatrix} \begin{pmatrix} 8 \\ 8 \end{pmatrix}$$

$$1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1$$

7/18

### Some other identities

• Vandermonde's identity: Let m, n and r be nonnegative integers with  $r \le m, n$ . Then  $C(m+n,r) = \sum_{k=0}^{r} C(m,r-k)C(n,k)$ 

• Let n and r be nonnegative integers with  $r \le n$ . Then  $C(n+1,r+1) = \sum_{j=r}^n C(j,r)$