Linear Algebra and Its Applications
Midterm Exam (Paper A)

November 12, 2010

Name: Student ID: Class:

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1. True or False? (Mark T if it is true or F if it is false)

   (1) The equation $(A+B)^2 = A^2 + 2AB + B^2$ holds for all $n \times n$ matrices A and B. (F)

   (2) If two invertible matrices A and B commute (i.e., $AB = BA$), then A^{-1}. (T)

   (3) Since $A + B = A + B$ holds for for all matrices A and B, $det(A+B) = det(A) + det(B)$ holds for all matrices A and B. (F)

   (4) For every invertible $n \times n$ matrix A, there must exist a nonzero $n \times n$ matrix B such that AB is the zero matrix. (F)

   (5) If a homogeneous system $Ax = 0$ (A is an $n \times n$ matrix) has non-trivial solutions, then the rank of $NulA$ is not 0. (T)

2. Fill in the single correct choice, and explain the reason.

   (1) If equation $det(2A) = 2detA$ holds for a non-zero $n \times n$ matrix A ($n > 1$), then A is ___c___.

   a. invertible b. any matrix c.singular d. diagonal

   Reason:

   Combine $det(2A) = 2^n det(A)$ (matrix multiplication rule) and $det(2A) = 2det(A)$ (given), then $2^{n-1}det(A) = det(A) \Rightarrow det(A) = 0 \Rightarrow$ A is not invertible, i.e., A is a singular matrix.

   (2)If a homogeneous system $Ax = 0$ (A is an $m \times n$ matrix) has only trivial solution, then the columns of A ___b___.

   a. are linearly dependent b.are linearly independent c.span $\mathbb{R}^n$ d. span $\mathbb{R}^m$

   Reason:

   Suppose $A = [a_1 a_2 ... a_n]$, then $Ax = 0$ has only trivial solution means that $x_1, x_2 ... x_n areall 0$ in $x_1 a_1 x_2 a_2 ... x_n a_n = 0$. Hence, the columns of A are linearly independent.

3. Consider the linear system:

$$\begin{cases} (5-k)x + y = 1 \\ 6x + (6-k)y = k \end{cases}$$

   For which value(s) of k, does this system have a unique solution? (use determinant or matrix) The coefficient matrix of the system is

$$A = \begin{vmatrix} 5-k & 1 \\ 6 & 6-k \end{vmatrix}$$

The system has a unique solution iff A is invertible, i.e., $detA \neq 0$, i.e., $(5-k)(6-k)-6 = k^2-11k+24 = (k-3)(k-8) \neq 0$, Thus $k \neq 3$ and $k \neq 8$.

Unique solution if $\underline{k \neq 3 \textit{ and } k \neq 8}$.

4.

$$A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 2 & 1 \end{bmatrix}$$

Prove that A is invertible and compute A^{-1}.

Proof:

$$detA = \begin{vmatrix} 5 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{vmatrix} = 5 \neq 0$$

Hence, A is invertible.

$$A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$$

where

$$A_1 = \begin{bmatrix} 5 \end{bmatrix}, A_2 = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & A_2^{-1} \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -2 & 3 \end{bmatrix}$$

5. Use the concepts of linear system and determinant to prove the following three vectors are linearly independent: $\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$.

Proof:

In the linear system $x_1 \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$

The determinant of the coefficient matrix is

$$A = \begin{vmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 2 \times (2 \times 4 - 3 \times 1) = 10 \neq 0$$

Therefore, the system has only trivial solution, the vectors of A are thus linearly independent.