习题 6.1

1.确定下列函数的定义域并且画出定义域的的图形:

$(1) z=(x^2+y^2-2x)^{1/2}+\ln(4-x^2-y^2); x^2+y^2-2x\geq 0, x^2-y^2<4.$

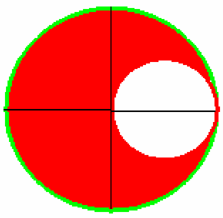
$(2) z=(x^2-y^2)^{-1}; x^2\neq y^2.$

$(3) z=\ln(y-x^2)+\ln(1-y); y-x^2>0, y<1.$

$(4) z=\arcsin\dfrac{x}{a}+\arccos\dfrac{y}{b}(a>0,b>0); |x|\leq a, |y|\leq b.$

$(5) z=\sqrt{1-x^2-y^2}+\ln(x+y); x^2+y^2\leq 1, x+y>0.$

$(6) z=\arcsin(x^2+y^2)+\sqrt{xy}.x^2+y^2\leq 1, xy\geq 0.$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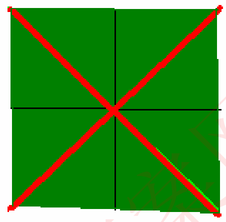
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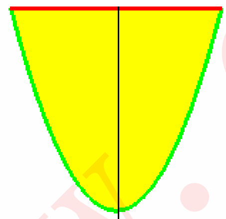
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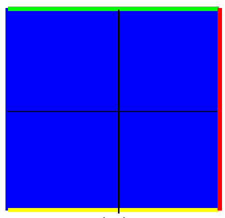
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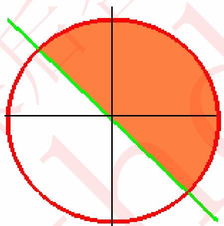
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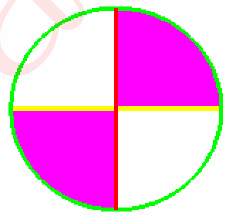

1(6)

2.指出下列集合中哪些集合在中是开集, 哪些是区域?哪些是有界区域?哪些是有界闭区域?

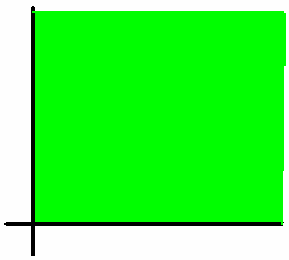
$(1) E_1=\{(x,y)\mid x>0, y>0\}$;开集, 区域.

$(2) E_2=\{(x,y)\mid |x|<1, |y-1|<2\}$;开集, 区域, 有界区域.

$(3) E_3=\{(x,y)\mid y\geq x^2, x\geq y^2\}$;有界闭区域.

$(4) E_4=\{(x,y)\mid y\neq\sin\dfrac{1}{x}$ 且 $x\neq 0\}$.区域, 边界点集合

$\partial E_4=\{(x,\sin\dfrac{1}{x})\mid x\neq 0\}\cup\{(0,y)\mid -1\leq y\leq 1\}.$


2(1)

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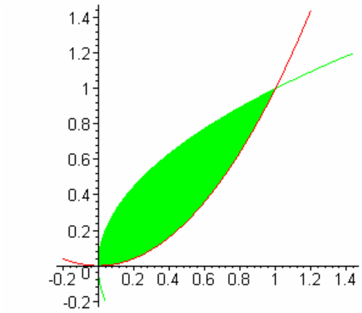

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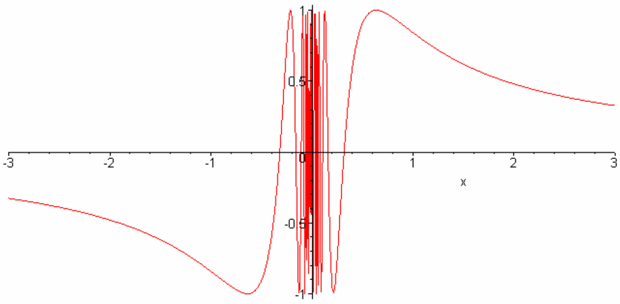

2(4)

3.设$E\subset\mathbf{R}^n$,∂E为E的边界点集合.试证明$\overline{E}=E\cup\partial E$是一个闭集.

证设 $P_0\notin\overline{E}$,则 $P_0\notin E$且 $P_0\notin\partial E$.于是存在$r>0$,使得$U_r(P_0)$不含E的点,从而不含∂E的点.否则,存在$Q\in U_r(P_0)\cap\partial E$,$Q$作为$E$的边界点,存在 $U_\rho(Q)\subseteq U_r(P_0)$,$U_\rho(Q)$含$E$的点,于是$U_r(P_0)$含$E$的点,矛盾.因此,$U_r(P_0)$不含$E\cup\partial E=\overline{E}$的点,$P_0$不是$\overline{E}$的的边界点.这表明$\overline{E}$的边界点全属于$\overline{E}$.故$\overline{E}$是闭集合.

4.像在 $\mathbf{R}^2$中一样,我们把$\mathbf{R}^n$中的点$(x_1,\cdots,x_n)$同时也视作一个向量,并定义两个向量$\alpha=(x_1,\cdots,x_n)$及$\beta=(y_1,\cdots,y_n)$的加法运算

$\alpha+\beta=(x_1+y_1,\cdots,x_n+y_n)$

及数乘运算

$\lambda\alpha=(\lambda x_1,\cdots,\lambda x_n),\forall\lambda\in\mathbf{R}.$

此外=,我们也可以定义两个向量之内积

$\alpha\cdot\beta=x_1y_1+\cdots+x_ny_n$, 并规定

$\sqrt{\alpha\cdot\alpha}=|\alpha|$ 作为向量的模. 试证明

(1)$|\alpha\cdot\beta|\le|\alpha||\beta|,\forall\alpha,\beta\in\mathbf{R}^n$;

(2)$|\alpha-\beta|\le|\alpha-\gamma|+|\gamma-\beta|,\forall\alpha,\beta,\gamma\in\mathbf{R}^n$;

(3)将点$P(x_1,\cdots,x_n)$及$Q(y_1,\cdots,y_n)$分别看成向量α及β,则有P到Q的距离$d(P,Q)=|\alpha-\beta|$.由此,可由(2)中之不等式导出三角不等式.

证 (1)$\beta=0$ 时结论显然成立.设$\beta\neq 0$.考虑二次函数

$|\alpha+\lambda\beta|^2=|\beta|^2\lambda^2+2\alpha\cdot\beta\lambda+|\alpha|^2\ge 0,\forall\lambda\in\mathbf{R}.$

其判别式$|\alpha\cdot\beta|^2-|\alpha|^2|\beta|^2\le 0,|\alpha\cdot\beta|\le|\alpha||\beta|.$

(2)$|\alpha+\beta|^2=|\alpha|^2+|\beta|^2+2\alpha\cdot\beta\le|\alpha|^2+|\beta|^2+2|\alpha||\beta|=(|\alpha|+|\beta|)^2,$

$|\alpha+\beta|\le|\alpha|+|\beta|.$

$|\alpha-\beta|=|(\alpha-\gamma)-(\beta-\gamma)|\le|(\alpha-\gamma)|+|\beta-\gamma|=|\alpha-\gamma|+|\gamma-\beta|.$

(3)$P=\alpha,Q=\beta,R=\gamma,d(P,R)=|\alpha-\gamma|\le|\alpha-\beta|+|\beta-\gamma|=d(P,Q)+d(Q,R).$

习题 6.2

1.求下列极限:

(1) $\lim\limits_{(x,y)\to(0,0)}\dfrac{3x^2-y^2+5}{x^2+y^2+2}=\dfrac{5}{2}.$

(2) $\lim\limits_{(x,y)\to(0,0)}\dfrac{\sin(x^2+y^2)}{x^2+y^2}=\lim\limits_{u\to 0}\dfrac{\sin u}{u}=1.$

(3) $\lim\limits_{(x,y)\to(0,0)}(x^2+y^2)\sin\dfrac{1}{x^2+y^2}=0$(有界变量乘无穷小量得无穷小量).

(4) $\lim\limits_{(x,y)\to(0,1)}\dfrac{x^3+(y-1)^3}{x^2+(y-1)^2}=\lim\limits_{(x,y)\to(0,1)}\dfrac{x^3}{x^2+(y-1)^2}+\lim\limits_{(x,y)\to(0,1)}\dfrac{(y-1)^3}{x^2+(y-1)^2}$

$=\lim\limits_{(x,y)\to(0,1)}x\dfrac{x^2}{x^2+(y-1)^2}+\lim\limits_{(x,y)\to(0,1)}(y-1)\dfrac{(y-1)^2}{x^2+(y-1)^2}=0+0=0$

$\left(0\le\dfrac{x^2}{x^2+(y-1)^2}\le 1, 0\le\dfrac{(y-1)^2}{x^2+(y-1)^2}\le 1\right).$

(5) $\lim\limits_{(x,y)\to(1,1)}\dfrac{xy-y-2x+2}{x-1}=\lim\limits_{(x,y)\to(1,1)}\dfrac{y(x-1)-2(x-1)}{x-1}=\lim\limits_{(x,y)\to(1,1)}(y-2)=-1.$

(6) $\lim\limits_{(x,y,z)\to(1-2,0)}\ln\sqrt{x^2+y^2+z^2}=\ln\sqrt{5}.$

2.证明:当$(x,y)\to(0,0)$时下列函数无极限:

(1) $f(x,y)=\dfrac{x^4-y^2}{x^4+y^2}$.由于

$\lim\limits_{\substack{(x,y)\to(0,0)\\ y=x^2}}\dfrac{x^4-y^2}{x^4+y^2}=0,\ \lim\limits_{\substack{(x,y)\to(0,0)\\ y=x}}\dfrac{x^4-y^2}{x^4+y^2}=\lim\limits_{x\to 0}\dfrac{x^4-x^2}{x^4+x^2}=\lim\limits_{x\to 0}\dfrac{x^2-1}{x^2+1}=-1\ne 0,$

故当$(x,y)\to(0,0)$时上述函数无极限.

(2) $f(x,y)=\begin{cases}\dfrac{x+y}{x-y}, & y\ne x,\\ 0, & y=x.\end{cases}$

$\lim\limits_{\substack{(x,y)\to(0,0)\\ y=x}}f(x,y)=0,\ \lim\limits_{\substack{(x,y)\to(0,0)\\ y=2x}}f(x,y)=\lim\limits_{x\to 0}\dfrac{3x}{-x}=-3\ne 0.$

3.讨论当$(x,y)\to(0,0)$时下列函数是否有极限,若有极限,求出其值:

(1) $f(x,y)=(x+2y)\ln(x^2+y^2)=x\ln(x^2+y^2)+2y\ln(x^2+y^2),$

$|x\ln(x^2+y^2)|\le 2\sqrt{|x|^2+|y|^2}\,|\ln\sqrt{|x|^2+|y|^2}|\to 0((x,y)\to(0,0)),$

$\lim\limits_{(x,y)\to(0,0)}x\ln(x^2+y^2)=0$.类似有$\lim\limits_{(x,y)\to(0,0)}2y\ln(x^2+y^2)=0.$

故$\lim\limits_{(x,y)\to(0,0)}f(x,y)=0.$

(2)

(2) $f(x,y)=\dfrac{1-\cos(x^2+y^2)}{(x^2+y^2)x^2y^2}$.$1-\cos(x^2+y^2)\sim\dfrac{1}{2}(x^2+y^2)^2$.

只需讨论$\dfrac{(x^2+y^2)^2}{(x^2+y^2)x^2y^2}=\dfrac{x^2+y^2}{x^2y^2}=\dfrac{1}{y^2}+\dfrac{1}{x^2}$极限存在与否.

$\lim\limits_{(x,y)\to(0,0)}\left(\dfrac{1}{y^2}+\dfrac{1}{x^2}\right)=+\infty,\ \lim\limits_{(x,y)\to(0,0)}f(x,y)=+\infty.$

不存在有限极限.

(3) $f(x,y)=(x^2+y^2)^{x^2y^2}=e^{x^2y^2\ln(x^2+y^2)}$,

$|x^2y^2\ln(x^2+y^2)|=x^2y^2|\ln(x^2+y^2)|\le\dfrac{1}{2}(x^2+y^2)|\ln(x^2+y^2)|\to 0$

$((x,y)\to(0,0))$. $\lim\limits_{(x,y)\to(0,0)}x^2y^2\ln(x^2+y^2)=0$,

$\lim\limits_{(x,y)\to(0,0)}f(x,y)=\lim\limits_{(x,y)\to(0,0)}e^{x^2y^2\ln(x^2+y^2)}=e^{\lim\limits_{(x,y)\to(0,0)}x^2y^2\ln(x^2+y^2)}=e^0=1.$

(4) $f(x,y)=\dfrac{P_n(x,y)}{\rho^{n-1}}$, $n\ge1$,其中$\rho=\sqrt{x^2+y^2}$, $P_n(x,y)$为n次齐次多项式.

$0\le\alpha\le n,\left|\dfrac{x^\alpha y^{n-\alpha}}{\rho^{n-1}}\right|=\dfrac{|x|^\alpha|y|^{n-\alpha}}{\rho^{n-1}}\le\dfrac{\rho^\alpha\rho^{n-\alpha}}{\rho^{n-1}}=\rho\to0(\rho\to0).$

故极限存在,并且等于零.

4.求下列函数的累次极限$\lim\limits_{x\to0}\lim\limits_{y\to0}f(x,y)$及$\lim\limits_{y\to0}\lim\limits_{x\to0}f(x,y)$:

(1) $f(x,y)=\dfrac{|x|-|y|}{|x|+|y|}$.

$\lim\limits_{x\to0}\lim\limits_{y\to0}\dfrac{|x|-|y|}{|x|+|y|}=\lim\limits_{x\to0}\dfrac{|x|}{|x|}=1,\lim\limits_{y\to0}\lim\limits_{x\to0}\dfrac{|x|-|y|}{|x|+|y|}=\lim\limits_{y\to0}\dfrac{-|y|}{|y|}=-1.$

(2) $f(x,y)=\dfrac{y^3+\sin x^2}{x^2+y^2}$.

$\lim\limits_{x\to0}\lim\limits_{y\to0}\dfrac{y^3+\sin x^2}{x^2+y^2}=\lim\limits_{x\to0}\dfrac{\sin x^2}{x^2}=1,$

$\lim\limits_{y\to0}\lim\limits_{x\to0}\dfrac{y^3+\sin x^2}{x^2+y^2}=\lim\limits_{y\to0}\dfrac{y^3}{y^2}=0.$

(3) $f(x,y)=(1+x)^{\frac{y}{x}}(x\ne0),\ f(0,y)=1.$

$\lim\limits_{x\to0}\lim\limits_{y\to0}(1+x)^{\frac{y}{x}}=\lim\limits_{x\to0}1=1,$

$\lim\limits_{y\to0}\lim\limits_{x\to0}(1+x)^{\frac{y}{x}}=\lim\limits_{y\to0}e^y=1.$

习题 6.10

在指定的各点求曲面的切平面：

(1)$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1(a>0,b>0,c>0)$，在$\left(0,\dfrac{b}{\sqrt{2}},\dfrac{c}{\sqrt{2}}\right)$点.

$\boldsymbol{n}=\left(\dfrac{2x}{a^2},\dfrac{2y}{b^2},\dfrac{2z}{c^2}\right)=\left(0,\dfrac{\sqrt{2}}{b},\dfrac{\sqrt{2}}{c}\right)$,

$\dfrac{\sqrt{2}}{b}(x-\dfrac{b}{\sqrt{2}})+\dfrac{\sqrt{2}}{c}(x-\dfrac{c}{\sqrt{2}})=0$,

$\dfrac{\sqrt{2}}{b}x+\dfrac{\sqrt{2}}{c}y-2=0.$

(2)$z=x^2-y^2,(2,1,3).x^2-y^2-z=0$

$\boldsymbol{n}=(2x,-2y,-1)=(4,-2,-1)$,

$4(x-2)-2(y-1)-(z-3)=0,4x-2y-3=0.$

(3)$x=\cosh\rho\cos\theta,y=\cosh\rho\sin\theta,z=\rho(\rho>0,0\le\theta\le2\pi),\rho=1,\theta=\dfrac{\pi}{2}.$

$$\boldsymbol{n}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ \sinh\rho\cos\theta & \sinh\rho\sin\theta & 1\\ -\cosh\rho\sin\theta & \cosh\rho\cos\theta & 0\end{vmatrix}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ 0 & \sinh 1 & 1\\ -\cosh 1 & 0 & 0\end{vmatrix}=(0,-\cosh 1,\sinh 1\cosh 1),$$

$(0,\cosh 1,1)$,

$-\cosh 1(y-\cosh 1)+\sinh 1\cosh 1(z-1)=0.$

(4)$e^z-2z+xy=3,(2,1,0)$

$\boldsymbol{n}=(y,x,e^z-2)=(1,2,-1)$,

$(x-2)+2(y-1)-z=0,x+2y-z-4=0.$

2.试证明曲面$\sqrt{x}+\sqrt{y}+\sqrt{z}=\sqrt{a}(a>0)$上任一点的切平面在各坐标轴上截距之和等于$a$.

证$\boldsymbol{n}=\left(\dfrac{1}{2\sqrt{x}},\dfrac{1}{2\sqrt{y}},\dfrac{1}{2\sqrt{z}}\right)$,

$\dfrac{1}{\sqrt{x}}(X-x)+\dfrac{1}{\sqrt{y}}(Y-y)+\dfrac{1}{\sqrt{z}}(Z-z)=0$,

$\dfrac{1}{\sqrt{x}}X+\dfrac{1}{\sqrt{y}}Y+\dfrac{1}{\sqrt{z}}Z-\sqrt{a}=0$,

x轴上截距$X_0=\sqrt{x}\sqrt{a},Y_0=\sqrt{y}\sqrt{a},Z_0=\sqrt{z}\sqrt{a}$,

$X_0+Y_0+Z_0=\sqrt{x}\sqrt{a}+\sqrt{y}\sqrt{a},+\sqrt{z}\sqrt{a}=(\sqrt{x}+\sqrt{y},+\sqrt{z})\sqrt{a}=\sqrt{a}\sqrt{a}=a.$

习题 6.4

1.求下列函数的一阶偏导数：

(1)$z=\ln(x+\sqrt{x^2+y^2})$.

$$\frac{\partial z}{\partial x}=\frac{1+\frac{x}{\sqrt{x^2+y^2}}}{x+\sqrt{x^2+y^2}}=\frac{1}{\sqrt{x^2+y^2}},$$

$$\frac{\partial z}{\partial y}=\frac{\frac{y}{\sqrt{x^2+y^2}}}{x+\sqrt{x^2+y^2}}=\frac{y}{\sqrt{x^2+y^2}(x+\sqrt{x^2+y^2})}.$$

(2)$z=\frac{x}{\sqrt{x^2+y^2}}$.

$$\frac{\partial z}{\partial x}=\frac{1}{\sqrt{x^2+y^2}}-\frac{x^2}{\sqrt{x^2+y^2}^3}=\frac{y^2}{\sqrt{x^2+y^2}^3},$$

$$\frac{\partial z}{\partial y}=-\frac{xy}{\sqrt{x^2+y^2}^3}.$$

(3)$z=x^{x^y},\ln z=x^y\ln x$.

$$\frac{1}{z}=yx^{y-1}\ln x+x^{y-1},\frac{\partial z}{\partial x}=z(yx^{y-1}\ln x+x^{y-1}),$$

$$\frac{1}{z}\frac{\partial z}{\partial y}=x^y\ln x\ln x,\frac{\partial z}{\partial y}=z(x^y\ln^2 x).$$

(4)$z=\frac{xy}{x-y}$.

$$\frac{\partial z}{\partial x}=y\left(\frac{x-y-x}{(x-y)^2}\right)=\frac{-y^2}{(x-y)^2},$$

$$\frac{\partial z}{\partial y}=x\left(\frac{x-y+y}{(x-y)^2}\right)=\frac{x^2}{(x-y)^2}.$$

(5)$z=\arcsin(x\sqrt{y})$.

$$\frac{\partial z}{\partial x}=\frac{\sqrt{y}}{\sqrt{1-x^2y}},\frac{\partial z}{\partial y}=\frac{x}{2\sqrt{y}\sqrt{1-x^2y}}.$$

(6)$z=xe^{-xy}$.

$$\frac{\partial z}{\partial x}=e^{-xy}+xe^{-xy}(-y)=e^{-xy}(1-xy),\frac{\partial z}{\partial y}=-x^2e^{-xy}.$$

(7)$u=\frac{y}{x}+\frac{z}{y}-\frac{x}{z}$.

$$\frac{\partial u}{\partial x}=-\frac{y}{x^2}-\frac{1}{z},\frac{\partial u}{\partial y}=\frac{1}{x}-\frac{z}{y^2},\frac{\partial u}{\partial z}=\frac{1}{y}+\frac{x}{z^2}.$$

$(8) u=(xy)^z.$

$$\frac{\partial u}{\partial x}=yz(xy)^{z-1},\frac{\partial u}{\partial y}=xz(xy)^{z-1},\frac{\partial u}{\partial z}=(xy)^z.\ln(xy)$$

2求下列函数在指定点的偏导数:

$(1) z=\dfrac{x\arccos(y-1)-(y-1)\cos x}{1+\sin x+\sin(y-1)}$,求$\left.\dfrac{\partial z}{\partial x}\right|_{(0,1)}$及$\left.\dfrac{\partial z}{\partial y}\right|_{(0,1)}$.

$$\left.\frac{\partial z}{\partial x}\right|_{(0,1)}=\left.\frac{d}{dx}\frac{x}{1+\sin x}\right|_{x=0}=\left.\frac{d}{dx}\frac{1+\sin x-x\cos x}{(1+\sin x)^2}\right|_{x=0}=1,$$

$$\left.\frac{\partial z}{\partial y}\right|_{(0,1)}=\left.\frac{d}{dy}\frac{-(y-1)}{1+\sin(y-1)}\right|_{y=1}=\left.\frac{d}{dy}\frac{-(1+\sin(y-1))+(y-1)\cos(y-1)}{(1+\sin(y-1))^2}\right|_{y=1}=-1.$$

$(2) z=\dfrac{2y}{y+\cos x}$,求$\left.\dfrac{\partial z}{\partial x}\right|_{(\frac{\pi}{2},1)}$及$\left.\dfrac{\partial z}{\partial y}\right|_{(\frac{\pi}{2},1)}$.

$$\frac{\partial z}{\partial x}=\frac{2y\sin x}{(y+\cos x)^2},\left.\frac{\partial z}{\partial y}\right|=2\times\frac{y+\cos x-y}{(y+\cos x)^2}=\frac{2\cos x}{(y+\cos x)^2}$$

$$\left.\frac{\partial z}{\partial x}\right|_{(\frac{\pi}{2},1)}=2,\left.\frac{\partial z}{\partial y}\right|_{(\frac{\pi}{2},1)}=0.$$

$(3) f(x,y,z)=\ln(xy+z)$,求$f_x(2,1,0),\ f_y(2,1,0),\ f_z(2,1,0)$.

$$f_x(x,y,z)=\frac{y}{xy+z},\ f_y(x,y,z)=\frac{x}{xy+z},\ f_z(x,y,z)=\frac{1}{xy+z}.$$

$$f_x(2,1,0)=\frac{1}{2},\ f_y(2,1,0)=1,\ f_z(x,y,z)=\frac{1}{2}.$$

3.证明函数$f(x,y)=\begin{cases}\dfrac{x^2+y^2}{|x|+|y|},(x,y)\neq(0,0),\\ 0,\qquad (x,y)=(0,0)\end{cases}$

在$(0,0)$连续,但是$f_x(0,0)$不存在.

证$|f(x,y)|=\dfrac{x^2+y^2}{|x|+|y|}\leq|x|+|y|\to 0((x,y)\to(0,0)),$

$f(x,y)\to f(0,0)=0((x,y)\to(0,0)),$

$f(x,y)$在$(0,0)$连续.

$$f_x(0,0)=\lim_{\Delta x\to 0}\frac{\frac{\Delta x^2}{|\Delta x|}}{\Delta x}=\lim_{\Delta x\to 0}\frac{\Delta x}{|\Delta x|}\text{不存在.}$$

4.设$z=\sqrt{x}\sin\frac{y}{x}$, 证明$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=\frac{z}{2}$.

证为齐1/2次函数,根据关于齐次函数微分的一个定理,立得结论.

直接计算如下.

$$\frac{\partial z}{\partial x}=\frac{1}{2\sqrt{x}}\sin\frac{y}{x}+\sqrt{x}\cos\frac{y}{x}\left(-\frac{y}{x^2}\right),\frac{\partial z}{\partial y}=\sqrt{x}\cos\frac{y}{x}\left(\frac{1}{x}\right),$$

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=\frac{\sqrt{x}}{2}\sin\frac{y}{x}-\frac{y}{\sqrt{x}}\cos\frac{y}{x}+\frac{y}{\sqrt{x}}\cos\frac{y}{x}=\frac{\sqrt{x}}{2}\sin\frac{y}{x}=\frac{z}{2}.$$

5.求下列函数的二阶混合偏导数f_{xy}：

(1) $f(x,y)=\ln(2x+3y)$.

$f_x=\dfrac{2}{2x+3y},\ f_{xy}=\dfrac{-6}{(2x+3y)^2}$.

(2) $f(x,y)=y\sin x+e^x$.

$f_x=y\cos x+e^x,\ f_{xy}=\cos x$.

(3) $f(x,y)=x+xy^2+4x^3-\ln(x^2+1)$.

$f_x=1+y^2+12x^2-\dfrac{2x}{x^2+1},\ f_{xy}=2y$.

(4) $f(x,y)=x\ln(xy)=x\ln x+x\ln y$.

$f_x=\ln y+\ln x+1,\ f_{xy}=\dfrac{1}{y}$.

6.设$u=e^{-3y}\cos 3x$,证明u满足平面Laplace方程$\Delta u=\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0$.

证$\because \dfrac{\partial u}{\partial x}=-3e^{-3y}\sin 3x,\dfrac{\partial^2 u}{\partial x^2}=-9e^{-3y}\cos 3x,$

$\dfrac{\partial u}{\partial y}=-3e^{-3y}\cos 3x,\dfrac{\partial^2 u}{\partial y^2}=9e^{-3y}\cos 3x,$

$\therefore \Delta u=\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0.$

7.证明函数$u(x,t)=e^{x+ct}+4\cos(3x+3ct)$满足波动方程$\dfrac{\partial^2 u}{\partial t^2}=c^2\dfrac{\partial^2 u}{\partial x^2}$.

证$\dfrac{\partial u}{\partial t}=ce^{x+ct}-12c\sin(3x+3ct),\dfrac{\partial^2 u}{\partial t^2}=c^2e^{x+ct}-36c^2\cos(3x+3ct),$

$\dfrac{\partial u}{\partial x}=e^{x+ct}-12\sin(3x+3ct),\dfrac{\partial^2 u}{\partial x^2}=e^{x+ct}-36\cos(3x+3ct),$

故$\dfrac{\partial^2 u}{\partial t^2}=c^2\dfrac{\partial^2 u}{\partial x^2}$.

8.设$u=u(x,y)$及$v=v(x,y)$在D内又连续的二阶偏导数,且满足方程组

$\dfrac{\partial u}{\partial x}=\dfrac{\partial u}{\partial y},\dfrac{\partial u}{\partial y}=-\dfrac{\partial v}{\partial x}$.证明$u$及$v$在$D$内满足平面Laplace方程$\Delta u=\Delta v=0$,

其中$\Delta=\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}$.

证$\dfrac{\partial^2 u}{\partial x^2}=\dfrac{\partial}{\partial x}\dfrac{\partial v}{\partial y}=\dfrac{\partial^2 v}{\partial x\partial y},\dfrac{\partial^2 u}{\partial y^2}=-\dfrac{\partial}{\partial y}\dfrac{\partial v}{\partial x}=-\dfrac{\partial^2 v}{\partial y\partial x}=-\dfrac{\partial^2 v}{\partial x\partial y}$($\dfrac{\partial^2 v}{\partial y\partial x}$和$\dfrac{\partial^2 v}{\partial x\partial y}$连续),

故$\Delta u=0$.类似证$\Delta v=0$.

9.已知函数$z(x,y)$满足$\dfrac{\partial z}{\partial x}=-\sin y+\dfrac{1}{1-xy}$以及$z(0,y)=2\sin y+y^2$. 试求z的表达式.

解$z=\int\left(-\sin y+\dfrac{1}{1-xy}\right)dx=-x\sin y-\dfrac{1}{y}\ln(1-xy)+C,$

$z(0,y)=C=2\sin y+y^2, z(x,y)=-x\sin y-\dfrac{1}{y}\ln|1-xy|+2\sin y+y^2$

$=(2-x)\sin y+y^2-\dfrac{1}{y}\ln|1-xy|.$

10.求下列函数的全微分:

(1)$z=e^{y/x}$.

$dz=e^{y/x}d\dfrac{y}{x}=e^{y/x}\dfrac{xdy-ydx}{x^2}.$

(2)$z=\dfrac{x+y}{x-y}.dz=\dfrac{(dx+dy)(x-y)-(x+y)(dx-dy)}{(x-y)^2}=\dfrac{(-2y)dx+(2x)dy)}{(x-y)^2}.$

(3)$z=\arctan\dfrac{y}{x}+\arctan\dfrac{x}{y}=\arctan\dfrac{y}{x}+\operatorname{arccot}\dfrac{y}{x}=\dfrac{\pi}{2}, dz=0.$

(4)$u=\sqrt{x^2+y^2+z^2}, du=\dfrac{d(x^2+y^2+z^2)}{\sqrt{x^2+y^2+z^2}}=\dfrac{2(xd(x+ydy+zdz)}{\sqrt{x^2+y^2+z^2}}.$

11.已知函数$z(x,y)$的全微分

$dz=(4x^3+10xy^3-3y^4)dx+(15x^2y^2-12xy^3+5y^4)dy$,求$f(x,y)$的表达式.

解 $\dfrac{\partial z}{\partial x}=4x^3+10xy^3-3y^4,\dfrac{\partial z}{\partial y}=15x^2y^2-12xy^3+5y^4.$

$z=\int(4x^3+10xy^3-3y^4)dx=x^4+5x^2y^3-3xy^4+C(y),$

$\dfrac{\partial z}{\partial y}=15x^2y^2-12xy^3+C'(y)=15x^2y^2-12xy^3+5y^4,$

$C'(y)=5y^4,C(y)=y^5+C.f(x,y)=x^4+5x^2y^3-3xy^4+y^5+C.$

12.已知函数$z=f(x,y)$的全微分

$dz=\left(x-\dfrac{y}{x^2+y^2}\right)dx+(y+\dfrac{x}{x^2+y^2})dy$,求$z(x,y)$的表达式.

解$dz=\left(x-\dfrac{y}{x^2+y^2}\right)dx+(y+\dfrac{x}{x^2+y^2})dy$

$=xdx+ydy+\dfrac{xdy-ydx}{x^2+y^2}$

$=\dfrac{1}{2}d(x^2+y^2)+\dfrac{\dfrac{xdy-ydx}{x^2}}{1+\left|\dfrac{y}{x}\right|^2}=\dfrac{1}{2}d(x^2+y^2)+\dfrac{d\dfrac{y}{x}}{1+\left|\dfrac{y}{x}\right|^2}=\dfrac{1}{2}d(x^2+y^2)+d\arctan\dfrac{y}{x}$

$=d\left(\dfrac{1}{2}(x^2+y^2)+\arctan\dfrac{y}{x}\right).$

$z=\dfrac{1}{2}(x^2+y^2)+\arctan\dfrac{y}{x}+C.$

13.$z=f(x,y)D:\{(x-x_0)^2+(y-y_0)^2<R^2\}\dfrac{\partial f}{\partial x}=0,\dfrac{\partial f}{\partial y}=0$.证明: $f(x,y)$

在区域上恒等于常数.

证$\forall(x,y)\in D,$

$f(x,y)-f(x_0,y_0)=[f(x,y)-f(x_0,y)]+[f(x_0,y)-f(x_0,y_0)]$

$=f_x(\xi,y)(x-x_0)+f_y(x_0,\eta)(y-y_0)=0.f(x,y)=f(x_0,y_0),(x,y)\in D.$

14.证明:函数$f(x,y)=\sqrt{|xy|}$在点$(0,0)$处连续, $f_x(0,0)$, $f_y(0,0)$存在,但$f(x,y)$在$(0,0)$处不可微.

证$|f(x,y)|=\sqrt{|xy|}\to 0=f(0,0)((x,y)\to(0,0))$, $f(x,y)=\sqrt{|xy|}$在点$(0,0)$处连续.

$f_x(0,0)=0,\ f_y(0,0)=0$.若$f(x,y)$在$(0,0)$处可微，将有

$f(x,y)=o(\sqrt{x^2+y^2})(\sqrt{x^2+y^2}\to 0)$,特别应有

$f(x,x)=|x|=o(\sqrt{2}|x|)(x\to 0),$

但此式显然不成立.

15.设$P(x,y)dx+Q(x,y)dy$在区域D中是某个函数$u(x,y)$之全微分,且$P,Q\in C^1(D)$.

证明$\dfrac{\partial P}{\partial y}=\dfrac{\partial Q}{\partial x}$.

证由假设$du=P(x,y)dx+Q(x,y)dy.\dfrac{\partial u}{\partial x}=P,\dfrac{\partial u}{\partial y}=Q.$

$$\frac{\partial P}{\partial y}=\frac{\partial}{\partial y}\frac{\partial u}{\partial x}=\frac{\partial^2 u}{\partial y\partial x},\frac{\partial Q}{\partial x}=\frac{\partial}{\partial x}\frac{\partial u}{\partial y}=\frac{\partial^2 u}{\partial x\partial y},$$

由$P,Q\in C^1(D)$得$\dfrac{\partial P}{\partial y},\dfrac{\partial Q}{\partial x}\in C(D)$,故$\dfrac{\partial^2 u}{\partial y\partial x}=\dfrac{\partial^2 u}{\partial x\partial y}$,即$\dfrac{\partial P}{\partial y}=\dfrac{\partial Q}{\partial x}$.

16.设函数$f(x,y)=\begin{cases}\dfrac{(x^2-y^2)xy}{x^2+y^2},(x,y)\neq(0,0)\\ 0 \qquad\qquad (x,y)=(0,0).\end{cases}$

(1)计算$f_x(0,y)(y\neq 0)$;

(2)根据偏导数定义证明$f_x(0,0)=0$;

(3)在上述结果的基础上证明$f_{xy}(0,0)=-1$;

(4)重复上述步骤于$f_y(x,0)$,并证明$f_{yx}(0,0)=1$.

证(1)设$y\neq 0$,则$f_x(x,y)=\dfrac{[2x^2y+(x^2-y^2)y](x^2+y^2)-2x(x^2-y^2)xy}{(x^2+y^2)^2}$,

$f_x(0,y)=\dfrac{-y^5}{y^4}=-y.$

(2) $f(x,0)=0,\ f_x(0,0)=0.$

(3) $f_{xy}(0,0)=(-y)'|_{y=0}=-1.$

(4)设$x\neq 0$,则$f_y(x,y)=\dfrac{[-2xy^2+(x^2-y^2)x](x^2+y^2)-2y(x^2-y^2)xy}{(x^2+y^2)^2}$,

$f_y(x,0)=x.f(0,y)=0,\ f_y(0,0)=0.f_{yx}(0,0)=x'|_{x=0}=1.$

17.设$z=x\ln(xy)$,求$\dfrac{\partial^3 z}{\partial x^3}\dfrac{\partial^3 z}{\partial x\partial y^2}$.

解$\dfrac{\partial z}{\partial x}=\ln(xy)+x\dfrac{y}{xy}=\ln(xy)+1,\dfrac{\partial^2 z}{\partial x^2}=\dfrac{1}{x},\dfrac{\partial^3 z}{\partial x^3}=-\dfrac{1}{x^2},$

$\dfrac{\partial^2 z}{\partial x\partial y}=\dfrac{1}{y},\dfrac{\partial^3 z}{\partial x\partial y^2}=-\dfrac{1}{y^2}.$

习题 6.5

在下面的习题中,出现的函数$f(u,v)$或$F(u)$一律假定有连续的一阶偏导数或导数.

1.求下列复合函数的偏导数或导数:

(1)$z=\sqrt{u^2+v^2}$,其中$u=xy,v=y^2$.

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}=\frac{uy}{\sqrt{u^2+v^2}}=\frac{xy^2}{\sqrt{x^2y^2+y^4}}=\frac{x|y|}{\sqrt{x^2+y^2}}$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}=\frac{ux}{\sqrt{u^2+v^2}}+\frac{2vy}{\sqrt{u^2+v^2}}$$

$$=\frac{x^2y}{\sqrt{x^2y^2+y^4}}+\frac{2y^3}{\sqrt{x^2y^2+y^4}}=\frac{x^2\operatorname{sgn}y}{\sqrt{x^2+y^2}}+\frac{2y^2\operatorname{sgn}y}{\sqrt{x^2+y^2}}=\frac{(x^2+2y^2)\operatorname{sgn}y}{\sqrt{x^2+y^2}}.$$

(2)$z=\dfrac{u^2}{v}$,其中$u=ye^x,v=x\ln y$.

$$\frac{\partial z}{\partial x}=\frac{2u}{v}\times ye^x-\frac{u^2}{v^2}\ln y=\frac{2ye^x}{x\ln y}\times ye^x-\frac{(ye^x)^2}{(x\ln y)^2}\ln y=\frac{2y^2e^{2x}}{x\ln y}-\frac{(ye^x)^2}{x^2\ln y}=\frac{(2x-1)y^2e^{2x}}{x^2\ln y}$$

$$\frac{\partial z}{\partial y}=\frac{2ue^x}{v}-\frac{u^2x}{v^2y}=\frac{2ye^{2x}}{x\ln y}-\frac{y^2e^{2x}x}{x^2(\ln^2y)y}=\frac{ye^{2x}(2\ln y-1)}{x(\ln^2y)}.$$

(3)$z=f(u,v)$,其中$u=\sqrt{xy},v=x+y$.

$$\frac{\partial z}{\partial x}=f_u(u,v)\frac{y}{2\sqrt{xy}}+f_v(u,v),\frac{\partial z}{\partial y}=f_u(u,v)\frac{x}{2\sqrt{xy}}+f_v(u,v).$$

(4)$z=f\left(xy,\dfrac{x}{y}\right).\dfrac{\partial z}{\partial x}=f_1'\cdot y+f_2'\cdot\dfrac{1}{y},\dfrac{\partial z}{\partial y}=f_1'\cdot x-f_2'\cdot\dfrac{x}{y^2}.$

(5)$z=f(x^2-y^2,e^{xy}),\dfrac{\partial z}{\partial x}=f_1'\cdot 2x+f_2'\cdot e^{xy}y,\dfrac{\partial z}{\partial y}=f_1'\cdot(-2y)+f_2'\cdot e^{xy}x.$

2.设$u=f(x+y+z,x^2+y^2+z^2)$,求$\Delta u=\dfrac{\partial^2u}{\partial x^2}+\dfrac{\partial^2u}{\partial y^2}+\dfrac{\partial^2u}{\partial z^2}$.

$$\frac{\partial u}{\partial x}=f_1'+2xf_2',$$

$$\frac{\partial^2u}{\partial x^2}=f_{11}''+2xf_{12}''+2f_2'+2x(f_{21}''+2xf_{22}'')=f_{11}''+4xf_{12}''+4x^2f_{22}''+2f_2',$$

$$\frac{\partial^2u}{\partial y^2}=f''+4yf_{12}''+4y^2f_{22}''+2f_2',\frac{\partial^2u}{\partial z^2}=f_{11}''+4zf_{12}''+4z^2f_{22}''+2f_2',$$

$$\Delta u=\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}+\frac{\partial^2u}{\partial z^2}=3f_{11}''+4(x+y+z)f_{12}''+4(x^2+y^2+z^2)f_{22}''+6f_2'.$$

4.设$z=x^nf(\dfrac{y}{x^2})$其中函数f可微.证明z满足下列方程

$$x\frac{\partial z}{\partial x}+2y\frac{\partial z}{\partial x}=nz.$$

证 $\frac{\partial z}{\partial x}=nx^{n-1}f(\frac{y}{x^2})+x^n f'(\frac{y}{x^2})\left(-\frac{2y}{x^3}\right),\frac{\partial z}{\partial y}=x^n f'(\frac{y}{x^2})\left(\frac{1}{x^2}\right).$

$$x\frac{\partial z}{\partial x}+2y\frac{\partial z}{\partial x}=x\left(nx^{n-1}f(\frac{y}{x^2})+x^n f'(\frac{y}{x^2})\left(-\frac{2y}{x^3}\right)\right)+2yx^n f'(\frac{y}{x^2})\left(\frac{1}{x^2}\right)$$

$$=nx^n f(\frac{y}{x^2})=nz.$$

5.设 $z=\frac{y}{F(x^2-y^2)}$，试证明 $\frac{1}{x}\frac{\partial z}{\partial x}+\frac{1}{y}\frac{\partial z}{\partial y}=\frac{z}{y^2}.$

证 $\frac{\partial z}{\partial x}=-\frac{2xyF'(x^2-y^2)}{(F(x^2-y^2))^2},\frac{\partial z}{\partial x}=\frac{F(x^2-y^2)+2y^2F'(x^2-y^2)}{(F(x^2-y^2))^2}.$

$$\frac{1}{x}\frac{\partial z}{\partial x}+\frac{1}{y}\frac{\partial z}{\partial y}=-\frac{2yF'(x^2-y^2)}{(F(x^2-y^2))^2}+\frac{F(x^2-y^2)+2y^2F'(x^2-y^2)}{y(F(x^2-y^2))^2}$$

$$=\frac{1}{yF(x^2-y^2)}=\frac{1}{y^2}\square\frac{y}{F(x^2-y^2)}=\frac{z}{y^2}.$$

6.设函数 $u(x,y)$ 有二阶连续偏导数且满足Laplace方程 $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0.$

证明，作变量替换 $x=e^s\cos t, y=e^s\sin t$ 后，u 依然满足关于 s,t 的Laplace方程

$$\frac{\partial^2 u}{\partial s^2}+\frac{\partial^2 u}{\partial t^2}=0.$$

证 $\frac{\partial u}{\partial s}=\frac{\partial u}{\partial x}e^s\cos t+\frac{\partial u}{\partial y}e^s\sin t,$

$$\frac{\partial^2 u}{\partial s^2}=e^s\cos t\left(\frac{\partial^2 u}{\partial x^2}e^s\cos t+\frac{\partial^2 u}{\partial x\partial y}e^s\sin t\right)+\frac{\partial u}{\partial x}e^s\cos t+e^s\sin t\left(\frac{\partial^2 u}{\partial x\partial y}e^s\cos t+\frac{\partial^2 u}{\partial y^2}e^s\sin t\right)$$

$$+e^s\sin t\frac{\partial u}{\partial y},$$

$$\frac{\partial u}{\partial t}=-\frac{\partial u}{\partial x}e^s\sin t+\frac{\partial u}{\partial y}e^s\cos t,$$

$$\frac{\partial^2 u}{\partial t^2}=-e^s\sin t\left(-\frac{\partial^2 u}{\partial x^2}e^s\sin t+\frac{\partial^2 u}{\partial x\partial y}e^s\cos t\right)-\frac{\partial u}{\partial x}e^s\cos t+e^s\cos t\left(-\frac{\partial^2 u}{\partial x\partial y}e^s\sin t+\frac{\partial^2 u}{\partial y^2}e^s\cos t\right)$$

$$-e^s\sin t\frac{\partial u}{\partial y}.$$

$$\frac{\partial^2 u}{\partial s^2}+\frac{\partial^2 u}{\partial t^2}=e^s\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0.$$

7.验证下列各式：

(1) $u=F(x^2+y^2)$ 则，$y\frac{\partial u}{\partial x}-x\frac{\partial u}{\partial y}=0;$

(2) $u=F(x-ct)$，c 为常数，则 $\frac{\partial u}{\partial t}+c\frac{\partial u}{\partial x}=0.$

7.验证下列各式:

(1)$u=F(x^2+y^2)$则,$y\frac{\partial u}{\partial x}-x\frac{\partial u}{\partial y}=0$;

(2)$u=F(x-ct)$,c为常数,则$\frac{\partial u}{\partial t}+c\frac{\partial u}{\partial x}=0$.

证(1)$\frac{\partial u}{\partial x}=F'(x^2+y^2)2x,\frac{\partial u}{\partial y}=F'(x^2+y^2)2y$,

$y\frac{\partial u}{\partial x}-x\frac{\partial u}{\partial y}=yF'(x^2+y^2)2x-xF'(x^2+y^2)2y=0.$

(2)$\frac{\partial u}{\partial t}=F'(x-ct)(-c),\frac{\partial u}{\partial x}=F'(x-ct)$,

$\frac{\partial u}{\partial t}+c\frac{\partial u}{\partial x}=F'(x-ct)(-c)+cF'(x-ct)=0.$

8.若$f(x,y,z)$满足关系式$f(tx,ty,tz)=t^n f(x,y,z)$,其中t为任意实数,则称f为n次齐次函数.证明,任意一个可微的n次齐次函数满足下列方程

$xf_x+yf_y+zf_z=nz.$

证$f(tx,ty,tz)=t^n f(x,y,z)$,对t求导,

$f_1'(tx,ty,tz)x+f_2'(tx,ty,tz)y+f_3'(tx,ty,tz)z=nt^{n-1}f(x,y,z)$,

令$t=1$得$xf_x+yf_y+zf_z=nz.$

9.设$z=f(x,y)$在一个平面区域D中有定义.假定D有这样的性质,对于其中任意一点(x_0,y_0),区域D与直线$y=y_0$之交是一个区间.又设$z=f(x,y)$在区域D内有连续的一阶偏导数,若$f(x,y)$对x的偏导数恒为零,也即$\frac{\partial f(x,y)}{\partial x}=0,\forall(x,y)\in D.$

证明:$f(x,y)$可以表示成y的函数,也即存在一个函数$F(y)$,使得

$f(x,y)=F(y),\forall(x,y)\in D.$

证设$(x,y)\in D,(x_0,y)\in D,x_0<x$. 由Lagrange中值公式,

$f(x,y)-f(x_0,y)=\frac{\partial f(\xi,y)}{\partial x}(x-x_0)=0.$

即$f(x,y)$的值不依赖x,只依赖y,其值记为$F(y)$,则有$f(x,y)=F(y),\forall(x,y)\in D.$

10.设$z=f(x,y)$在全平面上有定义,且有连续的一阶偏导数,满足方程

$xf_x(x,y)+yf_y(x,y)=0$.证明:存在一个函数$F(\theta)$,

使得$f(r\cos\theta,r\sin\theta)=F(\theta)$.

证$x=r\cos\theta,y=r\sin\theta.\frac{\partial z}{\partial r}=\frac{\partial z}{\partial x}(r\cos\theta)+\frac{\partial z}{\partial y}(r\sin\theta)$

$=\frac{\partial z}{\partial x}(x)+\frac{\partial z}{\partial y}(y)=0.$

由上题,存在一个函数$G(r)$,使得$f(r\cos\theta,r\sin\theta)=F(\theta)$.

11.设$z=f(x,y)$在全平面上有定义,且有连续的一阶偏导数,满足方程
$yf_x(x,y)-xf_y(x,y)=0$.证明:存在一个函数$G(r)$,
使得$f(r\cos\theta,r\sin\theta)=G(r)$.

证$x=r\cos\theta, y=r\sin\theta$.$\dfrac{\partial z}{\partial\theta}=\dfrac{\partial z}{\partial x}(-r\sin\theta)+\dfrac{\partial z}{\partial y}(r\cos\theta)$

$=\dfrac{\partial z}{\partial x}(-y)+\dfrac{\partial z}{\partial y}(x)=0$.

由9题,存在一个函数$G(r)$,使得$f(r\cos\theta,r\sin\theta)=G(r)$.

习题 6.6

1.求函数$f(x,y)=x^2-xy+y^2$在点$P_0(2+\sqrt{3},1+2\sqrt{3})$处沿极角为$\theta$的方向$l$的方向导数. 并问$\theta$取何值时,对应的方向导数(1)达到最大值;(2)达到最小值;(3)等于0.

解(1)$\nabla f(x,y)=(2x-y,-x+2y),\nabla f(2+\sqrt{3},1+2\sqrt{3})=(3,3\sqrt{3})=3(1,\sqrt{3})$.

$$\frac{\partial f}{\partial l}=6(\frac{1}{2}\cos\theta+\frac{\sqrt{3}}{2}\sin\theta)=6(\cos\frac{\pi}{3}\cos\theta+\sin\frac{\pi}{3}\sin\theta)=6\cos(\frac{\pi}{3}-\theta).$$

$\theta=\frac{\pi}{3}$.(2)$\theta=\frac{4\pi}{3}$.(3)$\theta=\frac{5\pi}{6},\frac{11\pi}{6}$.

2.求函数$f(x,y)=x^3-3x^2y+3xy^2+2$在点$P_0(3,1)$处沿从P_0到$P(6,5)$方向的方向导数.

解$\nabla f(x,y)=(3x^2-6xy+3y^2,-3x^2+6xy)$,

$$\nabla f(3,1)=(12,-9)=3(4,-3)=3\sqrt{5}\left(\frac{4}{\sqrt{5}},-\frac{3}{\sqrt{5}}\right).$$

$$l=(6,5)-(3,1)=(3,4)=5\left(\frac{3}{\sqrt{5}},-\frac{4}{\sqrt{5}}\right),\frac{\partial f}{\partial l}(3,1)=3\sqrt{5}\left(\frac{4}{\sqrt{5}},-\frac{3}{\sqrt{5}}\right)\cdot\left(\frac{3}{\sqrt{5}},-\frac{4}{\sqrt{5}}\right)=0.$$

3.求函数$f(x,y)=\ln(x+y)$在点$(1,2)$沿抛物线$y=2x^2$在该点的切线方向的方向导数.

解$y'=4x$,切线斜率k=4, 方向$l=\pm(\frac{1}{\sqrt{17}},\frac{4}{\sqrt{17}})$. $\nabla f(x,y)=(\frac{1}{x+y},\frac{1}{x+y})$,

$$\nabla f(1,2)=(\frac{1}{3},\frac{1}{3}).\frac{\partial f}{\partial l}=\pm\left(\frac{1}{3\sqrt{17}}+\frac{4}{3\sqrt{17}}\right)=\pm\frac{5}{3\sqrt{17}}.$$

4.求函数$u(x,y,z)=xy+yz+zx$在点$P_0(2,1,3)$沿着与各坐标轴构成等角的方向的方向导数.

解设方向l与各坐标轴构成等角α,$3\cos^2\alpha=1,\cos\alpha=\pm\frac{1}{\sqrt{3}}$.

$\nabla u(x,y,z)=(y+z,x+z,x+y),\nabla u(2,1,3)=(4,5,3)$.

$$\frac{\partial u}{\partial l}=\pm\frac{12}{\sqrt{3}}=\pm4\sqrt{3}.$$

5.求$z=f(x,y)=x^2+2xy+y^2$在点$(1,2)$处的梯度.

解$\nabla f(x,y)=(2x+2y,2x+2y)=2(x+y.x+y)$,

$\nabla f(1,2)==2(3,3)=6(1,1)$.

6.求$z=f(x,y)=\arctan\frac{y}{x}$在点$(x_0,y_0)$的梯度,并求沿向量$(x_0,y_0)$的方向导数.

$$解\nabla f(x,y)=\left(\frac{-\frac{y}{x^2}}{1+\left(\frac{y}{x}\right)^2},\frac{\frac{1}{x}}{1+\left(\frac{y}{x}\right)^2}\right)=\left(-\frac{y}{x^2+y^2},\frac{x}{x^2+y^2}\right),$$

$$\nabla f(x_0,y_0)=\left(-\frac{y_0}{x_0^{\ 2}+y_0^{\ 2}},\frac{x_0}{x_0^{\ 2}+y_0^{\ 2}}\right),$$

$$\frac{\partial f}{\partial l}(x_0,y_0)=-\frac{y_0}{x_0^{\ 2}+y_0^{\ 2}}\frac{x_0}{\sqrt{x_0^{\ 2}+y_0^{\ 2}}}+\frac{x_0}{x_0^{\ 2}+y_0^{\ 2}}\frac{y_0}{\sqrt{x_0^{\ 2}+y_0^{\ 2}}}=0.$$

7.求函数$z=f(x,y)=\ln\frac{y}{x}$分别在点$A\left(\frac{1}{3},\frac{1}{10}\right)$及点$B\left(1,\frac{1}{6}\right)$处的两个梯度之间的夹角余弦.

解 $z=\ln|y|-\ln|x|.\nabla f(x,y)=\left(-\frac{1}{x},\frac{1}{y}\right),\nabla f\left(\frac{1}{3},\frac{1}{10}\right)=(-3,10),$

$\nabla f\left(1,\frac{1}{6}\right)=(-1,6).$

$$<\nabla f\left(\frac{1}{3},\frac{1}{10}\right),\nabla f\left(1,\frac{1}{6}\right)>=\frac{(-3,10)\cdot(-1,6)}{\sqrt{109}\sqrt{37}}=\frac{63}{\sqrt{109}\sqrt{37}}.$$

8.求函数$f(x,y)=x(x-2y)+x^2y^2$在点$(1,1)$处沿方向$(\cos\alpha,\cos\beta)$的方向导数,并求出最大的与最小的方向导数,它们各沿什么方向?

解$\nabla f(x,y)=(2x-2y+2xy^2,-2x+2x^2y),\nabla f(1,1)=(2,0).$

$\frac{\partial f}{\partial l}(1,1)=2\cos\alpha.$

最大的与方向导数:2,最小的方向导数:-2,分别沿方向x轴方向和负x轴方向.

9.证明函数$f(x,y)=\frac{y}{x^2}$在椭圆周$x^2+2y^2=1$上任一点处沿椭圆周法方向的方向导数等于0.

证$\nabla f(x,y)=\left(-\frac{2y}{x^3},\frac{1}{x^2}\right)$.椭圆周法方向n(x, y)=(2x, 4y).

$$\text{方向导数}=\frac{1}{|n|}\left(-\frac{2y}{x^3}\times 2x+\frac{1}{x^2}\times 4y\right)=\frac{2(-2xy+2xy)}{|n|x^3}=0.$$

习题 6.7

1.求函数$f(x,y)=xy-y$在点(1,1)的二阶Taylor多项式.

解$f(x,y)=xy-y=(x-1+1)(y-1+1)-(y-1)-1$

$=(x-1)+(x-1)(y-1).$

2.在点(0,0)的邻域内,将下列函数按带Peano型余项展开成Taylor公式(到二阶):

$$(1)f(x,y)=\frac{\cos x}{\cos y}=\frac{1-\frac{x^2}{2}+o(x^2)}{1-\frac{y^2}{2}+o(y^2)}=\left(1-\frac{x^2}{2}+o(x^2)\right)\left(1+\frac{y^2}{2}+o(y^2)\right)$$

$$=1-\frac{x^2}{2}+\frac{y^2}{2}+o(x^2+y^2)(\sqrt{x^2+y^2}\to 0).$$

$$(2)\ f(x,y)=\ln(1+x+y)=x+y-\frac{1}{2}(x+y)^2+o(x^2+y^2)$$

$$=x+y-\frac{1}{2}(x^2+2xy+y^2)+o(x^2+y^2)(\sqrt{x^2+y^2}\to 0).$$

$$(3)\ f(x,y)=\sqrt{1-x^2-y^2}=1-\frac{1}{2}(x^2-y^2)+o(x^2+y^2)(\sqrt{x^2+y^2}\to 0).$$

$$(4)\ f(x,y)=\sin(x^2+y^2)=x^2+y^2o(x^2+y^2)(\sqrt{x^2+y^2}\to 0).$$

3.在点(0,0)的邻域内,将函数$f(x,y)=\ln(1+x+y)$按Lagrange余项展开成Taylor公式(到一阶).

解 $\ln(1+x)=x-\dfrac{1}{2(1+\theta x)^2}x^2.$

$$\ln(1+x+y)=x+y-\frac{1}{2(1+\theta x+\theta y)^2}(x+y)^2.$$

4.利用Taylor公式证明:当$|x|,|y|,|z|$充分小时,有近似公式

$\cos(x+y+z)-\cos x\cos y\cos z\approx -(xy+yx+zx).$

证由于$\cos(x+y+z)-\cos x\cos y\cos z$

$$=1-\frac{1}{2}(x+y+z)^2+o(\rho^2)-\left(1-\frac{x^2}{2}+o(\rho^2)\right)\left(1-\frac{y^2}{2}+o(\rho^2)\right)\left(1-\frac{z^2}{2}+o(\rho^2)\right)$$

$=-(xy+yx+zx)+o(\rho^2)(\rho\to 0).$

故当$|x|,|y|,|z|$充分小时,有近似公式

$\cos(x+y+z)-\cos x\cos y\cos z\approx -(xy+yx+zx).$

5.设D是单位圆,即$D=\{(x,y)\,|\,x^2+y^2<1\}$,又设函数$f(x,y)$在$D$内有连续的偏导数且满足$xf_x(x,y)+yf_y(x,y)=0,(x,y)\in D$.证明: $f(x,y)$在D内是一常数.

证$f(x,y)-f(0,0)=f_x(\theta x,\theta y)x+f_y(\theta x,\theta y)y$

$$=\frac{1}{\theta}[f_x(\theta x,\theta y)\theta x+f_y(\theta x,\theta y)\theta y]=0.$$

$f(x,y)=f(0,0),(x,y)\in D.$

习题 6.8

在本节习题中所涉及的函数f或F都是有连续一阶偏导数的函数.

1.求由下列方程确定的隐函数$z=z(x,y)$的所有一阶偏导数:

(1)$x^3z+z^3x-2yz=0$.

$$3x^2z+x^3\frac{\partial z}{\partial x}+3z^2\frac{\partial z}{\partial x}x+z^3-2y\frac{\partial z}{\partial x}=0,$$

$$x^3\frac{\partial z}{\partial y}+3z^2\frac{\partial z}{\partial y}x-2z-2y\frac{\partial z}{\partial y}=0.$$

$$=-\frac{3x^2z+z^3}{x^3+3xz^2-2y},\frac{\partial z}{\partial x}=\frac{2z}{x^3+3xz^2-2y}.$$

(2)$yz-\ln z=x+y$.

$$y\frac{\partial z}{\partial x}-\frac{1}{z}\frac{\partial z}{\partial x}=1,z+y\frac{\partial z}{\partial y}-\frac{1}{z}\frac{\partial z}{\partial y}=1.$$

$$\frac{\partial z}{\partial x}=\frac{1}{y-\frac{1}{z}}=\frac{z}{yz-1},\frac{\partial z}{\partial y}=\frac{1-z}{y-\frac{1}{z}}=\frac{z-z^2}{yz-1}.$$

(3)$x+z-\varepsilon\sin z=y(0<\varepsilon<1)$.

$$1+(1-\varepsilon\cos z)\frac{\partial z}{\partial x}=0,(1-\varepsilon\cos z)\frac{\partial z}{\partial y}=1.$$

$$\frac{\partial z}{\partial x}=-\frac{1}{1-\varepsilon\cos z},\frac{\partial z}{\partial y}=\frac{1}{1-\varepsilon\cos z}.$$

(4)$z^x=y^z$.

$$z^x\ln z+xz^{x-1}\frac{\partial z}{\partial x}=y^z\ln y\frac{\partial z}{\partial x},xz^{x-1}\frac{\partial z}{\partial y}=zy^{z-1}+y^z\ln y\frac{\partial z}{\partial y}.$$

$$\frac{\partial z}{\partial x}=-\frac{z^x\ln z}{xz^{x-1}-y^z\ln y}=-\frac{z^x\ln z}{xz^{x-1}-z^x\ln y}=-\frac{z\ln z}{x-z\ln y},$$

$$\frac{\partial z}{\partial y}=\frac{zy^{z-1}}{xz^{x-1}-y^z\ln y}=\frac{zy^z}{xyz^{x-1}-y^zy\ln y}=\frac{zz^x}{xyz^{x-1}-z^xy\ln y}=\frac{z^2}{xy-zy\ln y}.$$

(5)$x\cos y+y\cos z+z\cos x=1$.

$$\cos y-z\sin x+(-y\sin z+\cos x)\frac{\partial z}{\partial x}=0,\frac{\partial z}{\partial x}=\frac{\cos y+z\sin x}{y\sin x-\cos x}.$$

$$-x\sin y+\cos z+(-y\sin z+\cos x)\frac{\partial z}{\partial y}=0,\frac{\partial z}{\partial y}=\frac{x\sin y-\cos z}{\cos x-y\sin z}.$$

2.设由方程$f(xy^2,x+y)=0$确定隐函数为$y=y(x)$,求$\dfrac{dy}{dx}$.

解$f_1'(xy^2,x+y)(y^2+2xyy')+f_2'(xy^2,x+y)(1+y')=0$,

$$\frac{dy}{dx}=-\frac{y^2f_1'(xy^2,x+y)+f_2'(xy^2,x+y)}{2xyf_1'(xy^2,x+y)+f_2'(xy^2,x+y)}.$$

3.设$z+\cos xy=e^z$,求$\dfrac{\partial z}{\partial x}$及$\dfrac{\partial^2 z}{\partial x^2}$.

解$z+\cos xy-e^z=0.-y\sin xy+(1-e^z)\dfrac{\partial z}{\partial x}=0,\dfrac{\partial z}{\partial x}=\dfrac{y\sin xy}{1-e^z}$.

$$\frac{\partial^2 z}{\partial x^2}=\frac{y^2(1-e^z)\cos xy-y\sin xy(-e^z\dfrac{\partial z}{\partial x})}{(1-e^z)^2}$$

$$=\frac{y^2(1-e^z)\cos xy-y\sin xy(-e^z\dfrac{y\sin xy}{1-e^z})}{(1-e^z)^2}$$

$$=y^2\frac{(1-e^z)^2\cos xy+e^z\sin^2 xy}{(1-e^z)^3}.$$

4.设$F(x,x+y,x+y+z)=0$,求$\dfrac{\partial z}{\partial x},\dfrac{\partial z}{\partial y}$.

解$F_1'+F_2'+F_3'(1+\dfrac{\partial z}{\partial x})=0,\dfrac{\partial z}{\partial x}=-\dfrac{F_1'+F_2'+F_3'}{F_3'}$.

$F_2'+F_3'(1+\dfrac{\partial z}{\partial y})=0,\dfrac{\partial z}{\partial y}=-\dfrac{F_2'+F_3'}{F_3'}$.

另解$\mathrm{d}F(x,x+y,x+y+z)=0$,

$F_1'dx+F_2'(dx+dy)+F_3'(dx+dy+dz)=0$,

$dz=-\dfrac{F_1'+F_2'+F_3'}{F_3'}dx-\dfrac{F_2'+F_3'}{F_3'}dy,\dfrac{\partial z}{\partial x}=-\dfrac{F_1'+F_2'+F_3'}{F_3'},\dfrac{\partial z}{\partial y}=-\dfrac{F_2'+F_3'}{F_3'}$.

5.设$z=z(x,y)$是方程$F(x,y,z)=0$确定的隐函数,利用一阶微分形式的不变型,

证明$dz=-\dfrac{F_x}{F_z}dx-\dfrac{F_y}{F_z}dy(F_z\neq 0)$,并求

$F(x^2+y^2+z^2,xy-z^2)=0$确定的隐函数$z=z(x,y)$的8微分dz.

证$dF(x,y,z)=F_xdx+F_ydy+F_zdz=0,dz=-\dfrac{F_x}{F_z}dx-\dfrac{F_y}{F_z}dy(F_z\neq 0)$.

记$dF(x^2+y^2+z^2,xy-z^2)=0$.

$F_1'(2xdx+2ydy+2zdz)+F_2'(ydx+xdy-2zdz)=0$,

$(2xF_1'+yF_2')dx+(2yF_1'+xF_2')dy+(2zF_1'-2zF_2')dz=0$,

$$dz=\frac{(2xF_1'+yF_2')dx+(2yF_1'+xF_2')dy}{2z(F_2'-F_1')}.$$

6.证明球坐标变换的Jacobi行列式$J=r^2\sin\varphi$.

证$\begin{cases}x=r\sin\varphi\cos\theta\\ y=r\sin\varphi\sin\theta\\ z=r\cos\varphi\end{cases}$

证$dx=\sin\varphi\cos\theta dr+r\cos\varphi\cos\theta d\varphi-r\sin\varphi\sin\theta d\theta,$

$dy=\sin\varphi\sin\theta dr+r\cos\varphi\sin\theta d\varphi+r\sin\varphi\cos\theta d\theta,$

$dz=\cos\varphi dr-r\sin\varphi d\varphi.$

$$J=\begin{vmatrix}\sin\varphi\cos\theta & r\cos\varphi\cos\theta & -r\sin\varphi\sin\theta\\ \sin\varphi\sin\theta & r\cos\varphi\sin\theta & r\sin\varphi\cos\theta\\ \cos\varphi & -r\sin\varphi & 0\end{vmatrix}$$

$=r^2\cos^2\varphi\sin\varphi+r^2\sin^3\varphi=r^2\sin\varphi.$

7.设由$x=u+v,y=u^2+v^2$，$z=u^3+v^3$确定函数$z=z(x,y)$，求当

$x=0,y=u=\dfrac{1}{2},v=-\dfrac{1}{2}$时，$\dfrac{\partial z}{\partial x}$与$\dfrac{\partial z}{\partial y}$的值.

解$\mathrm{dz}=3\mathrm{u}^2du+3\mathrm{v}^2dv.$

$$\begin{cases}du+dv=dx\\ 2udu+2vdv=dy\end{cases}\quad du=\frac{2vdx-dy}{2v-2u},dv=\frac{dy-2udx}{2v-2u}.$$

$x=0,y=u=\dfrac{1}{2},v=-\dfrac{1}{2}$

$$du=\frac{-dx-dy}{-2},dv=\frac{dy-dx}{-2},$$

$$dz=\frac{3}{4}\times\frac{1}{2}(dx+dy)+\frac{3}{4}\times\frac{1}{2}(dx-dy)=\frac{3}{4}dx,$$

$$\frac{\partial z}{\partial x}=\frac{3}{4},\frac{\partial z}{\partial y}=0.$$

8.设$\begin{cases} xu+yv=0, \\ uv-xy=5. \end{cases}$

求当$x=1, y=-1, u=v=2$时$\dfrac{\partial^2 u}{\partial x^2}$与$\dfrac{\partial^2 u}{\partial x\partial y}$的值.

解$\begin{cases} udx+xdu+vdy+ydv=0, \\ vdu+udv-ydx-xdy=0. \end{cases}$

$$\begin{cases} xdu+ydv=-udx-vdy, \\ vdu+udv=ydx+xdy. \end{cases}$$

$$du=\frac{u(-udx-vdy)-y(ydx+xdy)}{xu-yv}=\frac{(-u^2-y^2)dx+(-uv-xy)dy}{xu-yv}$$

$$\frac{\partial u}{\partial x}=-\frac{u^2+y^2}{xu-yv}, \frac{\partial u}{\partial y}=-\frac{uv+xy}{xu-yv}.$$

$$dv=\frac{x(ydx+xdy)+v(udx+vdy)}{xu-yv}=\frac{(xy+uv)dx+(x^2+v^2)dy}{xu-yv},$$

$$\frac{\partial v}{\partial x}=\frac{(xy+uv)}{xu-yv}.$$

$$\frac{\partial^2 u}{\partial x^2}=-\frac{\partial}{\partial x}\frac{u^2+y^2}{xu-yv}=-\frac{\left(2u\dfrac{\partial u}{\partial x}\right)(xu-yv)-(u^2+y^2)\left(u+x\dfrac{\partial u}{\partial x}-y\dfrac{\partial v}{\partial x}\right)}{(xu-yv)^2}.$$

$$\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial}{\partial x}\frac{\partial u}{\partial y}=-\frac{\partial}{\partial x}\frac{uv+xy}{xu-yv}$$

$$=-\frac{\left(\dfrac{\partial u}{\partial x}v+\dfrac{\partial v}{\partial x}u+y\right)(xu-yv)-\left(u+x\dfrac{\partial u}{\partial x}-y\dfrac{\partial v}{\partial x}\right)(uv+xy)}{(xu-yv)^2}.$$

当$x=1, y=-1, u=v=2$时,

$$\frac{\partial u}{\partial x}=-\frac{u^2+y^2}{xu-yv}=-\frac{5}{4}, \frac{\partial v}{\partial x}=\frac{(xy+uv)}{xu-yv}=\frac{3}{4}$$

$$\frac{\partial^2 u}{\partial x^2}=-\frac{\left(2u\dfrac{\partial u}{\partial x}\right)(xu-yv)-(u^2+y^2)\left(u+x\dfrac{\partial u}{\partial x}-y\dfrac{\partial v}{\partial x}\right)}{(xu-yv)^2}$$

$$=-\frac{4\times(-5)-5\times(-\frac{5}{4}+\frac{3}{4})}{16}=\frac{55}{32}.$$

$$\frac{\partial^2 u}{\partial x\partial y}=--\frac{\left(\dfrac{\partial u}{\partial x}v+\dfrac{\partial v}{\partial x}u+y\right)(xu-yv)-\left(u+x\dfrac{\partial u}{\partial x}-y\dfrac{\partial v}{\partial x}\right)(uv+xy)}{(xu-yv)^2}=\frac{25}{32}.$$

9.设$x^2+y^2=\frac{1}{2}z^2, x+y+z=2$,求当$x=1, y=-1, z=2$时$\frac{dx}{dz}$与$\frac{dy}{dz}$的值.

解$\begin{cases}2xdx+2ydy=zdz\\dx+dy+dz=0\end{cases}$

$$\begin{cases}2xdx+2ydy=zdz\\dx+dy=-dz\end{cases}$$

$$dx=\frac{z+2y}{2-2y}dz, dy=\frac{-2x-z}{2-2y}dz,$$

$$\frac{dx}{dz}=\frac{z+2y}{2-2y}, \frac{dy}{dz}=\frac{-2x-z}{2-2y}.$$

当$x=1, y=-1, z=2$时

$$\frac{dx}{dz}=\frac{z+2y}{2-2y}=\frac{0}{4}=0, \frac{dy}{dz}=\frac{-2x-z}{2-2y}=\frac{-4}{4}=-1.$$

10.设$x=\cos\varphi\cos\theta, y=\cos\varphi\sin\theta, z=\sin\varphi$,求$\frac{\partial z}{\partial x}$.

解$\begin{cases}-\sin\varphi\cos\theta d\varphi-\cos\varphi\sin\theta d\theta=dx\\-\sin\varphi\sin\theta d\varphi+\cos\varphi\cos\theta d\theta=dy\\\cos\varphi d\varphi=dz\end{cases}$

由前两个方程解出

$$d\varphi=\frac{\begin{vmatrix}dx & -\cos\varphi\sin\theta\\dy & \cos\varphi\cos\theta\end{vmatrix}}{\begin{vmatrix}-\sin\varphi\cos\theta & -\cos\varphi\sin\theta\\-\sin\varphi\sin\theta & \cos\varphi\cos\theta\end{vmatrix}}=-\frac{\cos\varphi\cos\theta dx+\cos\varphi\sin\theta dy}{\sin\varphi\cos\varphi}$$

$$=-\frac{\cos\theta}{\sin\varphi}dx-\frac{\sin\theta}{\sin\varphi}dy,$$

$$dz=\cos\varphi d\varphi=-\frac{\cos\varphi\cos\theta}{\sin\varphi}dx-\frac{\cos\varphi\sin\theta}{\sin\varphi}dy$$

$$\frac{\partial z}{\partial x}=-\frac{\cos\varphi\cos\theta}{\sin\varphi}=-\frac{x}{z}.$$

另解$x^2+y^2+z^2=1, 2xdx+2zdz=0,$

$$\frac{\partial z}{\partial x}=-\frac{x}{z}.$$

再解$z=\pm\sqrt{1-x^2-y^2-z^2}$,

$$\frac{\partial z}{\partial x}=\frac{-x}{\pm\sqrt{1-x^2-y^2-z^2}}=-\frac{x}{z}.$$

11.设$u=u(x,y)$及$v=v(x,y)$有连续一阶偏导数, 又设$x=x(\xi,\eta)$及$y=y(\xi,\eta)$也有连续一阶偏导数, 且使复合函数$u=u(x(\xi,\eta),y(\xi,\eta))$及$v=v(x(\xi,\eta),y(\xi,\eta))$有定义. 证明

$$\frac{D(u,v)}{D(\xi,\eta)}=\frac{D(u,v)}{D(x,y)}\frac{D(x,y)}{D(\xi,\eta)}.$$

证 $\dfrac{\partial u}{\partial \xi}=\dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial \xi}+\dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial \xi},\dfrac{\partial u}{\partial \eta}=\dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial \eta}+\dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial \eta},$

$$\frac{\partial v}{\partial \xi}=\frac{\partial v}{\partial x}\frac{\partial x}{\partial \xi}+\frac{\partial v}{\partial y}\frac{\partial y}{\partial \xi},\frac{\partial v}{\partial \eta}=\frac{\partial v}{\partial x}\frac{\partial x}{\partial \eta}+\frac{\partial v}{\partial y}\frac{\partial y}{\partial \eta},$$

$$\frac{D(u,v)}{D(\xi,\eta)}=\begin{vmatrix}\frac{\partial u}{\partial \xi} & \frac{\partial u}{\partial \eta}\\ \frac{\partial v}{\partial \xi} & \frac{\partial v}{\partial \eta}\end{vmatrix}=\begin{vmatrix}\frac{\partial u}{\partial x}\frac{\partial x}{\partial \xi}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial \xi} & \frac{\partial u}{\partial x}\frac{\partial x}{\partial \eta}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial \eta}\\ \frac{\partial v}{\partial x}\frac{\partial x}{\partial \xi}+\frac{\partial v}{\partial y}\frac{\partial y}{\partial \xi} & \frac{\partial v}{\partial x}\frac{\partial x}{\partial \eta}+\frac{\partial v}{\partial y}\frac{\partial y}{\partial \eta}\end{vmatrix}$$

$$\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}\begin{vmatrix}\frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta}\\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta}\end{vmatrix}=\frac{D(u,v)}{D(x,y)}\frac{D(x,y)}{D(\xi,\eta)}.$$

习题 6.9

1.求下列函数的极值:

$(1) z = x^2(x-1)^2 + y^2.$

$\frac{\partial z}{\partial x} = 2x(x-1)^2 + 2(x-1)x^2$

$= x(x-1)(2x-2+2x) = x(x-1)(4x-2) = 0,$

$x = 0, \frac{1}{2}, 1.$

$\frac{\partial z}{\partial y} = 2y = 0, y = 0.$

三个稳定点$(0,0), (\frac{1}{2}, 0), (1,0). 2x(x-1)^2 + 2(x-1)x^2$

$A = \frac{\partial^2 z}{\partial x^2} = 2(x-1)^2 + 4x(x-1) + 2x^2 + 4x(x-1) = 2(x-1)^2 + 8x(x-1) + 2x^2,$

$C = \frac{\partial^2 z}{\partial y^2} = 2, B = \frac{\partial^2 z}{\partial x \partial y} = 0.$

$(0,0), A = 2 > 0, B = 0, C = 2, AC - B^2 = 4,$ 极小值点， 极小值$z(0,0) = 1.$

$(\frac{1}{2}, 0), A = -1, B = 0, C = 2, AC - B^2 = -2,$ 非极小值点.

$(1,0), A = 2, B = 0, C = 2, AC - B^2 = 4 > 0,$ 极小值点.极小值$z(1,0) = 0.$

$(2) z = 2xy - 5x^2 - 2y^2 + 4x + 4y - 1.$

$\frac{\partial z}{\partial x} = 2y - 10x + 4 = 2(y - 5x + 2),$

$\frac{\partial z}{\partial y} = 2x - 4y + 4 = 2(x - 2y + 2).$

$\begin{cases} -5x + y = -2 \\ x - 2y = -2 \end{cases} x = \frac{2}{3}, y = \frac{4}{3}.$

稳定点$(\frac{2}{3}, \frac{4}{3}).$

$A = \frac{\partial^2 z}{\partial x^2} = -10 < 0, C = \frac{\partial^2 z}{\partial y^2} = -4, B = \frac{\partial^2 z}{\partial x \partial y} = 2.$

$AC - B^2 = 36 > 0, (\frac{2}{3}, \frac{4}{3})$极大值点.

极大值$= z(\frac{2}{3}, \frac{4}{3}) = 3.$

(3)$z=6x^2-2x^3+3y^2+6xy+1.$

$\frac{\partial z}{\partial x}=12x-6x^2+6y=6(2x-x^2+y)$

$\frac{\partial z}{\partial y}=6y+6x=6(x+y)$

$\begin{cases}2x-x^2+y=0\\x+y=0\end{cases}$ $x=0,1,$相应地$y=0,-1.$

稳定点$(0,0),(1,-1).$

在点$(0,0),A=\frac{\partial^2 z}{\partial x^2}=12-12x=12>0,C=\frac{\partial^2 z}{\partial y^2}=6,B=\frac{\partial^2 z}{\partial x\partial y}=6.$

$AC-B^2=66>0,(0,0)$极小值点，极小值z(0, 0)=1.

在点$(1,-1)$，$A=\frac{\partial^2 z}{\partial x^2}=12-12x=0,C=\frac{\partial^2 z}{\partial y^2}=6,B=\frac{\partial^2 z}{\partial x\partial y}=6.$

$AC-B^2=-36<0.z$不取极值.

(4)$z=4xy-x^4-y^4+5.$

$\frac{\partial z}{\partial x}=4y-4x^3=4(y-x^3),$

$\frac{\partial z}{\partial y}=4x-4y^3=4(x-y^3).$

$\begin{cases}y-x^3=0\\x-y^3=0\end{cases}$ $x=0,\pm1,$相应地$y=0,\pm1.$稳定点$(0,0),(1,1),(-1,-1).$

在点$(0,0),A=\frac{\partial^2 z}{\partial x^2}=-12x^2=0,C=\frac{\partial^2 z}{\partial y^2}=-12y^2=0,B=\frac{\partial^2 z}{\partial x\partial y}=4.$

$AC-B^2=-16<0,(0,0)$不是极值点.

在点$(1,1)$，$A=\frac{\partial^2 z}{\partial x^2}=-12<0,C=\frac{\partial^2 z}{\partial y^2}=-12,B=\frac{\partial^2 z}{\partial x\partial y}=4.$

$AC-B^2=128>0.z$取极大值$z(1,1)=7.$

在点$(-1,-1)$，$A=\frac{\partial^2 z}{\partial x^2}=-12<0,C=\frac{\partial^2 z}{\partial y^2}=-12,B=\frac{\partial^2 z}{\partial x\partial y}=4.$

$AC-B^2=128>0.z$取极大值$z(-1,-1)=7.$

$(5) z=x^3y^2(6-x-y)(x>0,y>0).$

$\frac{\partial z}{\partial x}=3x^2y^2(6-x-y)-x^3y^2=x^2y^2(18-3x-3y-x)=x^2y^2(18-4x-3y)$

$\frac{\partial z}{\partial y}=2x^3y(6-x-y)-x^3y^2=x^3y(12-2x-2y-y)=x^3y(12-2x-3y).$

$\begin{cases}4x+3y=18\\2x+3y=12\end{cases}$在$\{(x,y)\mid x>0,y>0\}$的稳定点$(x,y)=(3,2).$

在稳定点$(3,2)$, $A=\frac{\partial^2 z}{\partial x^2}=2xy^2(18-4x-3y)-4x^2y^2=-144,$

$C=\frac{\partial^2 z}{\partial y^2}=x^3(12-2x-3y)-3x^3y=-162,$

$B=\frac{\partial^2 z}{\partial x\partial y}=2x^2y(18-4x-3y)-3x^2y^2=-108.$

$AC-B^2=144\times162-108^2=11664>0$, $(3,2)$极大值点, 极大值$z(3,2)=108.$

2.确定下列函数在所给条件下的最大值及最小值:

(1)$z = x^2 + y^2$,当$\dfrac{x}{2}+\dfrac{y}{3}=1$时.

解 由于$\sqrt{x^2+y^2} \to +\infty$时，$z \to +\infty$,又$z = x^2 + y^2$是连续函数，故在平面$\dfrac{x}{2}+\dfrac{y}{3}=1$

上取极小值. $z =$代入法.$z = x^2 + (3(1-\dfrac{x}{2}))^2 = x^2 + \dfrac{9}{4}(2-x)^2 = \dfrac{13}{4}x^2 - 9x + 9$

$= \dfrac{1}{4}(13x^2 - 36x + 36) = f(x), x \in (-\infty, +\infty).$

$f'(x) = \dfrac{1}{4}(26x - 36) = 0, x_0 = \dfrac{18}{13}, y_0 = 3(1-\dfrac{9}{13}) = \dfrac{12}{13}.$

$f''(x) = \dfrac{13}{2} > 0.$ $\dfrac{18}{13}$是唯一极值点,且是极小值点, 故是最小值点.

最小值$f(\dfrac{18}{13}) = \dfrac{36}{13}.$

对二次函数f用配方法当然得到同一结果.

再解Lagrange乘子法. 考虑Lagrange函数

$F(x,y,\lambda) = x^2 + y^2 + \lambda\left(\dfrac{x}{2}+\dfrac{y}{3}-1\right).$

$$\begin{cases} 2x + \dfrac{\lambda}{2} = 0, \\ 2y + \dfrac{\lambda}{3} = 0, \\ \dfrac{x}{2}+\dfrac{y}{3}-1 = 0. \end{cases} \quad x = -\frac{\lambda}{4}, y = -\frac{\lambda}{6}, -\frac{\lambda}{8}-\frac{\lambda}{18}-1 = 0, \lambda_0 = -\frac{72}{13}.$$

得到满足条件的唯一点$x_0 = \dfrac{18}{13}, y_0 = \dfrac{12}{13}$.$z(x_0, y_0)$是最小值.

3.在某一行星表面要安装一个无线电望远镜,为了减少干扰,要将望远镜装在磁场最弱的位置.设该行星为一球体,半径为6个单位.若以球心为坐标原点建立坐标系$Oxyz$,则行星表面上点(x,y,z)处的磁场强度为$H(x,y,z) = 6x - y^2 + xz + 60$.问,应将望远镜安装在何处?

解 球面方程:$x^2 + y^2 + z^2 = 36$.$F(x,y,z,\lambda) = H(x,y,z) + \lambda(x^2+y^2+z^2-36).$

$$\begin{cases} \dfrac{\partial H}{\partial x} = 6 + z + 2\lambda x = 0 & (1) \\ \dfrac{\partial H}{\partial y} = -2y + 2\lambda y = 2y(\lambda - 1) = 0 & (2) \\ \dfrac{\partial H}{\partial z} = x + 2\lambda z = 0 & (3) \\ x^2 + y^2 + z^2 = 36 & (4) \end{cases}$$

由(2), $y=0$或$\lambda=1$.

设$y=0$, 则有$\begin{cases}6+z+2\lambda x=0\\ x+2\lambda z=0\\ x^2+y^2+z^2=36\end{cases}$

解之得$(\pm5,0,3),(0.0,-6)$, 相应H值为105,15和60.

设$\lambda=1$, 则

$$\begin{cases}6+z+2x=0\\ x+2z=0\\ x^2+y^2+z^2=36\end{cases}$$

解之得$(-4,\pm4,2)$, 相应H值为12.各条件极值比较得 $(x,y,z)=(-4,\pm4,2)$时

H取最小值12.

4.已知三角形的周长为$2p$,问怎样的三角形绕自己的一边旋转所得的体积最大?

解 设三角形底边上的高为x,垂足分底边的长度为y,z.设三角形饶底边旋转,

旋转体体积

$$V=\frac{\pi}{3}x^2(y+z),\ y+z+\sqrt{x^2+y^2}+\sqrt{x^2+z^2}=2p,\ x\ge0,\ y\ge0,\ z\ge0.$$

V在有界闭集上取最大值.

$$L(x,y,z,\lambda)=x^2(y+z)+\lambda(y+z+\sqrt{x^2+y^2}+\sqrt{x^2+z^2}-2p),$$

$$\begin{cases}2x(y+z)+\lambda\left(\dfrac{x}{\sqrt{x^2+y^2}}+\dfrac{x}{\sqrt{x^2+z^2}}\right)=0, & (1)\\ x^2+\lambda\left(1+\dfrac{y}{\sqrt{x^2+y^2}}\right)=0, & (2)\\ x^2+\lambda\left(1+\dfrac{z}{\sqrt{x^2+z^2}}\right)=0. & (3)\\ y+z+\sqrt{x^2+y^2}+\sqrt{x^2+z^2}-2p=0. & (4)\end{cases}$$

$$(2)-(3)\Rightarrow\lambda\left(\frac{y}{\sqrt{x^2+y^2}}-\frac{z}{\sqrt{x^2+z^2}}\right)=0.$$

若$\lambda=0$, 将有$x=0$, 不可能. 故$\dfrac{y}{\sqrt{x^2+y^2}}-\dfrac{z}{\sqrt{x^2+z^2}}=0.$

由于$y>0,z>0$, 易得$y=z$.

$$\begin{cases}2xy+\dfrac{\lambda x}{\sqrt{x^2+y^2}}=0,\\ x^2+\lambda\left(1+\dfrac{y}{\sqrt{x^2+y^2}}\right)=0,\\ y+\sqrt{x^2+y^2}=p.\end{cases}\quad\begin{cases}2y+\dfrac{\lambda}{\sqrt{x^2+y^2}}=0,\\ x^2+\lambda\left(1+\dfrac{y}{\sqrt{x^2+y^2}}\right)=0,\\ y+\sqrt{x^2+y^2}=p.\end{cases}$$

解之得$y=z=\dfrac{p}{4}$,底边长$=\dfrac{p}{2}$,两腰长$=\dfrac{1}{2}(2p-\dfrac{p}{2})=\dfrac{3p}{4}$.

5.在两平面有$y+2z=12$及$x+y=6$的交线上求到原点距离最近的点.

解$u=x^2+y^2+z^2$,

$$z=6-\frac{y}{2},\ x=6-y,\ u=(6-y)^2+y^2+\left(6-\frac{y}{2}\right)^2=\frac{9}{4}y^2-18y+72.$$

$z'=\dfrac{9}{2}y-18=0,\ z''=\dfrac{9}{2}.y_0=4$是唯一极值点,且是极小值点,故是最小值点.

$x_0=2,z_0=4$.所求的点为$(2,4,4)$.

6.求椭球面$x^2+y^2+\dfrac{z^2}{4}=1$与平面$x+y+z=0$的交线上到坐标原点的最大距离与最小距离.

解$L(x,y,z,\lambda,\mu)=x^2+y^2+z^2+\lambda(x^2+y^2+\dfrac{z^2}{4}-1)+\mu(x+y+z)$.

$$(*)\begin{cases} L_x=2x+2\lambda x+\mu=0,\\ L_y=2y+2\lambda y+\mu=0,\\ L_z=2z+\dfrac{1}{2}\lambda z+\mu=0,\\ x^2+y^2+\dfrac{z^2}{4}=1,\\ x+y+z=0. \end{cases}$$

由前三个方程得

$$(**)\begin{cases} 2x(1+\lambda)=2z+\dfrac{1}{2}\lambda z,\\ 2y(1+\lambda)=2z+\dfrac{1}{2}\lambda z. \end{cases}$$

下面分两种情况求解.

(1)$\lambda=-1$.由方程组(**)得$z=0$,再由(*)的后两个方程得$(\dfrac{1}{\sqrt{2}},-\dfrac{1}{\sqrt{2}},0)$,

$(-\dfrac{1}{\sqrt{2}},\dfrac{1}{\sqrt{2}},0)$.这两点与原点距离为1.

(2)$\lambda\neq-1$..由方程组(**)得$x=y$,再由(*)的后两个方程得$(\dfrac{1}{\sqrt{3}},\dfrac{1}{\sqrt{3}},-\dfrac{2}{\sqrt{3}})$,

$(-\dfrac{1}{\sqrt{3}},-\dfrac{1}{\sqrt{3}},\dfrac{2}{\sqrt{3}})$.这两点与原点距离为2.

在$(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}},0)$和$(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0)$有最小距离1, 在$(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},-\frac{2}{\sqrt{3}})$和$(-\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}},\frac{2}{\sqrt{3}})$有最大距离2.

7.在已知圆锥体内做一内接长方体,长方体的底面在圆锥体的底面上, 求使体积最大的那个长方体的边长.

解 设圆锥体高为H,底半径为R.取其底面为xy平面,底面中心为坐标原点.设内接长方体底面边长为 2x, 2y, 高为z, 则长方体体积

$V=4xyz, x\ge 0, y\ge 0, z\ge 0$ 满足圆锥面方程 $(H-z)^2=\frac{H^2}{R^2}(x^2+y^2)$.

$$L(x,y,z,\lambda)=xyz+\lambda\left((H-z)^2-\frac{H^2}{R^2}(x^2+y^2)\right).$$

$$(*)\begin{cases} L_x=yz-2\lambda\frac{H^2}{R^2}x=0, \\ L_y=xz-2\lambda\frac{H^2}{R^2}y=0, \\ L_z=xy-2\lambda(H-z)=0, \\ (H-z)^2=\frac{H^2}{R^2}(x^2+y^2). \end{cases}$$

由(*)的前两个方程易得$x=y$.由(*)的前三个方程易得$x^2=y^2=\frac{R^2}{H^2}z(H-z)$.

再与第四个方程联立得$(H-z)^2=2z(H-z), z=\frac{H}{3}, x=y=\frac{\sqrt{2}}{3}R$.

8.当n个正数$x_1,\cdots,x_n$的和等于常数l时,求它们的乘积的最大值.并证明:n个正数$a_1,\cdots,a_n$的几何平均值不超过算术平均值,即$\sqrt[n]{a_1\cdots a_n}\le\frac{a_1+\cdots+a_n}{n}$.

解$f(x_1,\cdots,x_n)=x_1\cdots x_n, x_1+\cdots+x_n=l$.

$F(x_1,\cdots,x_n,\lambda)=x_1\cdots x_n+\lambda(x_1+\cdots+x_n-l)$,

$$\begin{cases}F_{x_1}=x_2x_3\cdots x_n+\lambda=0\\ F_{x_2}=x_1x_3\cdots x_n+\lambda=0\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ F_{x_n}=x_1x_2\cdots x_{n-1}+\lambda=0.\end{cases}$$

$$\begin{cases}x_1x_2x_3\cdots x_n+\lambda x_1=0\\ x_1x_2x_3\cdots x_n+\lambda x_2=0\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ x_1x_2\cdots x_{n-1}x_n+\lambda x_n=0.\end{cases}\quad \lambda x_1=\lambda x_2\cdots=\lambda x_n.$$

若$\lambda=0$,将有$x_1x_2\cdots x_{n-1}x_n=0$,不会是最大值.若$\lambda\ne0$,则有$x_1=x_2\cdots=x_n=\frac{l}{n}$.

$$x_1x_2\cdots x_{n-1}x_n=\left(\frac{l}{n}\right)^n,\ \sqrt[n]{x_1x_2\cdots x_{n-1}x_n}=\left(\frac{l}{n}\right)=\frac{x_1+\cdots+x_n}{n}.$$

10.求函数$f(x,y)=\frac{1}{2}(x^n+y^n)(n>1$是常数$,x\ge0,y\ge0,)$在条件$x+y=A(A>0)$下的最小值,并由此证明

$$\frac{1}{2}(x^n+y^n)\ge\left(\frac{x+y}{2}\right)^n\ (x>0,y>0).$$

9.在椭圆$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$上哪些点处,其切线与坐标轴构成的三角形面积最大?

解 $\dfrac{2x}{a^2}+\dfrac{2yy'}{b^2}=0$,切线斜率:$y'=-\dfrac{b^2x}{a^2y}$,切线的点$(X,Y)$满足方程:

$Y-y=-\dfrac{b^2x}{a^2y}(X-x).Y_0=0,X_0=x+\dfrac{a^2y^2}{b^2x}.X_1=0,Y_1=y+\dfrac{b^2x^2}{a^2y}.$

三角形面积$f(x,y)=\left(x+\dfrac{a^2y^2}{b^2x}\right)\left(y+\dfrac{b^2x^2}{a^2y}\right)=\dfrac{a^2b^2}{xy}$,$(x,y)$满足

$x>0,y>0,\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1.$

由于$x\to 0 y\to 0$时$f(x,y)\to+\infty$,故f在所述条件下取极小值.

令$L(x,y,\lambda)=\dfrac{a^2b^2}{xy}+\lambda\left(\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-1\right)$,

$$\begin{cases} L_x=-\dfrac{a^2b^2}{x^2y}+2\dfrac{\lambda x}{a^2}=0, \\ L_y=-\dfrac{a^2b^2}{xy^2}+2\dfrac{\lambda y}{b^2}=0, \\ \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1. \end{cases} \begin{cases} -\dfrac{a^2b^2}{x^2y^2}+2\dfrac{\lambda x}{a^2y}=0 \\ -\dfrac{a^2b^2}{x^2y^2}+2\dfrac{\lambda y}{b^2x}=0 \end{cases} \dfrac{\lambda x}{a^2y}=\dfrac{\lambda y}{b^2x}$$

易见$\lambda\neq 0$,故$\dfrac{x}{a^2y}=\dfrac{y}{b^2x},\dfrac{y}{x}=\dfrac{b}{a},y=\dfrac{b}{a}x$,代入椭圆方程得

$\dfrac{x^2}{a^2}+\dfrac{x^2}{a^2}=1,x=\dfrac{a}{\sqrt{2}},y=\dfrac{b}{\sqrt{2}}.$

在第一象限,$(x,y)=(\dfrac{a}{\sqrt{2}},\dfrac{b}{\sqrt{2}})$时,该点切线与坐标轴构成的三角形面积

最小. 由对称性,$(-\dfrac{a}{\sqrt{2}},-\dfrac{b}{\sqrt{2}}),(\dfrac{a}{\sqrt{2}},-\dfrac{b}{\sqrt{2}}),(-\dfrac{a}{\sqrt{2}},\dfrac{b}{\sqrt{2}})$也满足要求.

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