Discrete Mathematics: Lecture 8

- Last time:
  - Chap 9.1: Relations and their properties
  - Chap 9.2: n-ary relations
  - Chap 9.3: Representing relations
- Today:
  - Chap 9.4: Closure of relations
- Assignment 3 due in two weeks

Review of last time

- Properties of relations: reflexive, symmetric, antisymmetric, transitive
- Composing relations, power of relations
- R is transitive iff $R^n \subseteq R$ for $n \geq 1$
- n-ary relations
- Representing relations by zero-one matrices and digraphs

Closure of relations (关系的闭包)

- Let R be a relation on a set A. R may or may not have some property P, such as reflexivity, symmetry, or transitivity.
- If there is a relation S with property P containing R such that S is a subset of every relation with property P containing R, then S is called the closure of R wrt P.

Reflexive and symmetric closures

- The reflexive closure of R equals $R \cup \Delta$, where $\Delta = \{(a,a) \mid a \in A\}$ is the diagonal relation on A.
- Example: Let $R = \{(a,b) \mid a, b \in \mathbf{Z}, a < b\}$. What is the reflexive closure of R?
- The symmetric closure of R equals $R \cup R^{-1}$, where $R^{-1} = \{(b,a) \mid (a,b) \in R\}$ is the inverse relation of R.
- Example: What is the symmetric closure of R?

Paths in directed graphs

- Definition: A path (路径) from a to b in the directed graph G is a sequence of edges $(x_0, x_1), (x_1, x_2), \ldots, (x_{n-1}, x_n)$ in G, where $x_0 = a$ and $x_n = b$. The path is denoted by $x_0, x_1, \ldots, x_{n-1}, x_n$ and has length n.
- We view the empty set of edges as a path from a to a. A path of length $n \geq 1$ that begins and ends at the same vertex is called a cycle (回路).
- The term path also applies to relations. There is a path from a to b in R if there exists a sequence of elements $x_1, x_2, \ldots, x_{n-1}$ such that $(a, x_1) \in R$, $(x_1, x_2) \in R$, ..., and $(x_{n-1}, b) \in R$.
- Theorem 1: Let R be a relation on a set A. There is a path of length n, where n is a positive integer, from a to b iff $(a, b) \in R^n$.

Transitive closures

- Definition: Let R be a relation on a set A. The connectivity relation R^* consists of the pairs (a, b) such that there is a path of length ≥ 1 from a to b in R. That is, $R^* = \bigcup_{n=1}^{\infty} R^n$.
- Example: Let R be the relation on the set of all subway stops in New York City that contains (a, b) if it is possible to travel from stop a to stop b without changing trains. What is R^n where $n \geq 1$? What is R^*?
- Theorem 2: The transitive closure of a relation R equals R^*.
- Lemma 1: Let A be a set with n elements, and let R be a relation on A. If there is a path of length ≥ 1 in R from a to b, then there is such a path with length $\leq n$.
- Theorem 3: Let R be a relation on a set of n elements. Then $R^* = R \cup R^2 \cup \ldots \cup R^n$. Hence $M_{R^*} = M_R \vee M_R^{[2]} \vee \ldots \vee M_R^{[n]}$.

A procedure for computing the transitive closure

procedure *transitive closure* ($\mathbf{M}_R$: zero–one $n \times n$ matrix)
$\mathbf{A} := \mathbf{M}_R$
$\mathbf{B} := \mathbf{A}$
for $i := 2$ **to** n
    $\mathbf{A} := \mathbf{A} \odot \mathbf{M}_R$
    $\mathbf{B} := \mathbf{B} \vee \mathbf{A}$
return $\mathbf{B}$\{$\mathbf{B}$ is the zero–one matrix for R^*\}

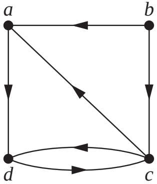
- $2n^3(n-1)$ bit operations
- A more efficient algorithm: Warshall's algorithm
- $2n^3$ bit operations
- Described by Stephen Warshall in 1960, and by Bernard Roy in 1959, hence also called Roy-Warshall algorithm

Warshall's algorithm

- Suppose that R is a relation on a set with n elements
- Let $v_1, v_2, \ldots, v_n$ be an arbitrary listing of these n elements
- If $a, x_1, \ldots, x_{m-1}, b$ is a path, its interior vertices are $x_1, \ldots, x_{m-1}$
- Warshall's algorithm constructs a sequence of zero-one matrices $W_0, W_1, \ldots, W_n$, where $W_0 = M_R$
- $W_k = [w_{ij}^{(k)}]$, where $w_{ij}^{(k)} = 1$ iff there is a path from v_i to v_j s.t. all the interior vertices are in the set $\{v_1, v_2, \ldots, v_k\}$
- Note that $W_n = M_{R^*}$

An example

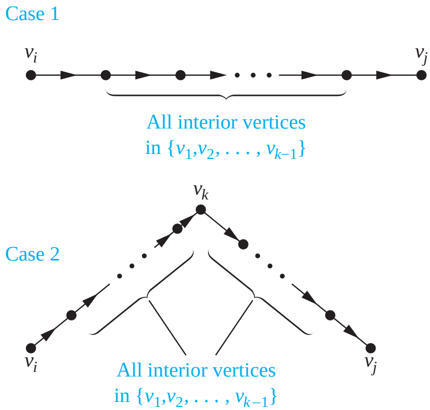
Let $v_1 = a, v_2 = b, v_3 = c, v_4 = d$.



$$\mathbf{W}_0 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \mathbf{W}_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

Computing W_k from W_{k-1}

There are two cases where there is a path from v_i to v_j with no vertices other than $v_1, v_2, \ldots, v_k$ as interior vertices



Warshall algorithm

Lemma:

Let $\mathbf{W}_k = [w_{ij}^{[k]}]$ be the zero–one matrix that has a 1 in its (i, j)th position if and only if there is a path from v_i to v_j with interior vertices from the set $\{v_1, v_2, \ldots, v_k\}$. Then

$$w_{ij}^{[k]} = w_{ij}^{[k-1]} \vee (w_{ik}^{[k-1]} \wedge w_{kj}^{[k-1]}),$$

whenever i, j, and k are positive integers not exceeding n.

procedure *Warshall* ($\mathbf{M}_R$: $n \times n$ zero–one matrix)
$\mathbf{W} := \mathbf{M}_R$
for $k := 1$ **to** n
    for $i := 1$ **to** n
        for $j := 1$ **to** n
        $w_{ij} := w_{ij} \vee (w_{ik} \wedge w_{kj})$
return $\mathbf{W}\{\mathbf{W} = [w_{ij}]$ is $\mathbf{M}_{R^*}\}$

- $2n^3$ bit operations