P.200.4 利用泰勒级数求下列极限. 2011—$\frac{19}{5}$—90.

(1) $\lim\limits_{x\to0}\dfrac{1-x^2-e^{-x^2}}{x\cdot\sin^3(2x)}$

由于 $e^{-x^2}=1-x^2+\dfrac{x^4}{2!}+o(x^4)$

所以 $1-x^2-e^{-x^2}=-\dfrac{x^4}{2!}-o(x^4)$, $(x\to0)$

又由于 $\sin^3(2x)\sim(2x)^3=8x^3$, $(x\to0)$

所以 $\sin^3(2x)=8x^3+o(x^3)$

$$=\lim_{x\to0}\frac{-\frac{x^4}{2!}+o(x^4)}{x(8x^3+o(x^3))}$$

$$=\lim_{x\to0}\frac{-\frac{x^4}{2!}+o(x^4)}{8x^4+o(x^4)}$$

$$=\lim_{x\to0}\frac{-\frac{1}{2}+\frac{o(x^4)}{x^4}}{8+\frac{o(x^4)}{x^4}}$$

$$=-\frac{1}{2}\times\frac{1}{8}=-\frac{1}{16}.$$

(2) $\lim\limits_{x\to0}\left(\dfrac{1}{x}-\dfrac{1}{e^x-1}\right)$

$e^x=1+x+\dfrac{x^2}{2!}+o(x^2)$ $(x\to0)$

$e^x=1+x+o(x)$

$e^x-1=x+o(x)$

$e^x-1-x=\dfrac{x^2}{2!}+o(x^2)$

$$=\lim_{x\to0}\frac{e^x-1-x}{x(e^x-1)}$$

$$=\lim_{x\to0}\frac{\frac{x^2}{2!}+o(x^2)}{x\cdot[x+o(x)]}$$

$$=\lim_{x\to0}\frac{\frac{x^2}{2!}+o(x^2)}{x^2+o(x^2)}=\lim_{x\to0}\frac{\frac{1}{2!}+\frac{o(x^2)}{x^2}}{1+\frac{o(x^2)}{x^2}}=\frac{1}{2}.$$

(3) $\lim\limits_{x\to0}\left(\dfrac{1}{x}-\dfrac{\cos x}{\sin x}\right)\cdot\dfrac{1}{\sin x}$;

$\sin x=x-\dfrac{x^3}{6}+o(x^3)$

$x\cdot\cos x=x\left(1-\dfrac{x^2}{2}+o(x^3)\right)=x-\dfrac{x^3}{2}+o(x^4)$

$\sin x-x\cdot\cos x=\dfrac{x^3}{3}+o(x^3)$

$\sin^2x\sim x^2$, $\sin^2x=x^2+o(x^2)$.

$$=\lim_{x\to0}\frac{\sin x-x\cdot\cos x}{x\cdot\sin^2x}$$

$$=\lim_{x\to0}\frac{\frac{x^3}{3}+o(x^3)}{x^3+o(x^3)}$$

$$=\lim_{x\to0}\frac{\frac{1}{3}+\frac{o(x^3)}{x^3}}{1+\frac{o(x^3)}{x^3}}$$

$$=\frac{1}{3}.$$