Discrete Mathematics: Lecture 27

- Last time:
  - Chap 10.1: Graphs and graph models
  - Chap 10.2: Graph terminology and special types of graphs
  - Chap 10.3: Representing graphs and graph isomorphisms
- Today:
  - Chap 10.4: Connectivity
  - Chap 10.5: Euler and Hamilton paths

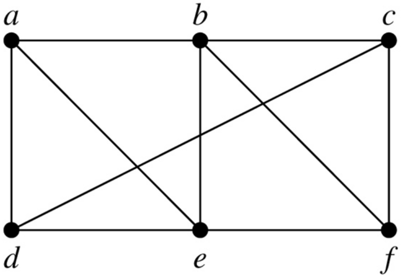
Review of last time

- Undirected and directed graphs
- Terminology for graphs
- Handshaking theorem
- Special types of graphs: complete graphs, cycles, wheels, cubes
- Bipartie graphs
- Subgraphs, union of graphs
- Representing graphs: adjacency lists, adjacency matrices, incidence matrices
- Isomorphism of graphs, graph invariant

Definition of paths

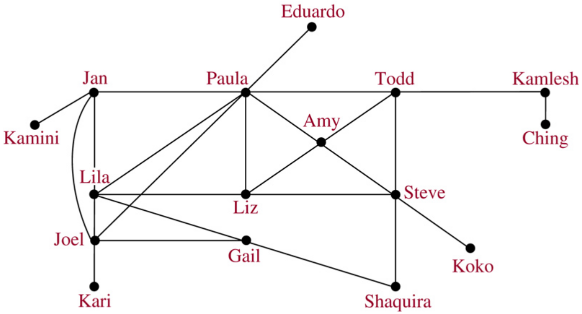
- Let n be a nonnegative integer and G an undirected (resp. directed) graph.
- A path (路径) of length n from u to v in G is a sequence of n edges $e_1, \ldots, e_n$ of G such that e_1 is associated with $\{x_0, x_1\}$ (resp. (x_0, x_1)), ..., e_n is associated with $\{x_{n-1}, x_n\}$ (resp. (x_{n-1}, x_n)), where $x_0 = u$ and $x_n = v$.
- The path is said to pass through the vertices $x_1, \ldots, x_{n-1}$ or traverse (遍历) the edges $e_1, \ldots, e_n$.
- A path is simple if it does not contain the same edge more than once.
- When the graph is simple, we denote this path by its vertex sequence $x_0, x_1, \ldots, x_n$.
- The path is a circuit (回路) if it begins and ends at the same vertex, that is, if $u = v$, and has length > 0.

Example



- a, d, c, f, e is a simple path
- b, c, f, e, b is a circuit
- a, b, e, d, a, b is a path which is not simple

Paths in acquaintanceship graphs



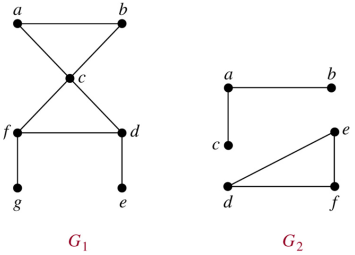
- There is a path of length 5 between Kamini and Ching.
- Many social scientists have conjectured that almost every pair of people in the world are linked by a path of length ≤ 6.
- Example

Paths in collaboration graphs

- Collaboration graphs model joint authorship of academic papers.
- Vertices represent people, and edges link two people if they have jointly written a paper.
- Paul Erdos is an extremely prolific mathematician.
- The Erdos number of a mathematician m is the length of the shortest path between m and Paul Erdos.
- Example

Connectedness in undirected graphs

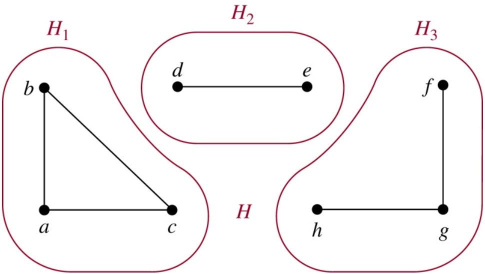
- When does a computer network has the property that every pair of computers can share information, if messages can be sent through one or more intermediate computers?
- Is there always a path between two vertices in the graph?
- Definition: An undirected graph is called connected (连通的) if there is a path between every pair of distinct vertices.



- Theorem: There is a simple path between every pair of distinct vertices of a connected undirected graph.

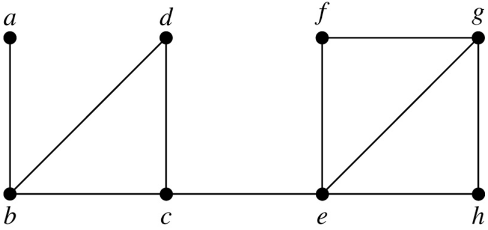
Connected components (连通分支)

- Definition: A connected component of a graph G is a maximal connected subgraph of G.
- A graph G that is not connected has two or more connected components that are disjoint and have G as their union.



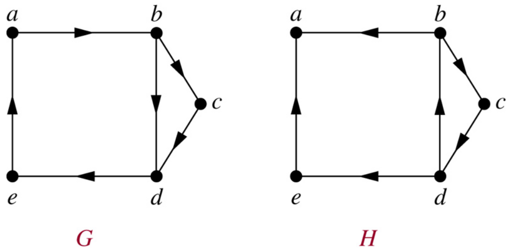
Cut vertices and edges

- A vertex is called a cut vertex (割点) if the removal of it and all edges incident with it produces a graph with more connected components than in the original graph.
- An edge whose removal produces a graph with more connected components than in the original graph is called a cut edge (割边) or bridge (桥).
- Find the cut vertices and edges in the following graph:



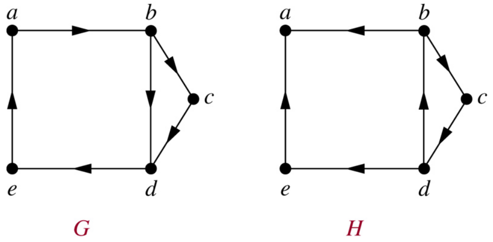
Connectedness in directed graphs

- Definition: A directed graph is strongly connected (强连通的) if there is a path from a to b and from b to a whenever a and b are vertices in the graph.
- Definition: A directed graph is weakly connected (弱连通的) if there is a path between any two vertices in the underlying undirected graph.
- Any strongly connected graph is also weakly connected.
- Example: Are G and H strongly / weakly connected?

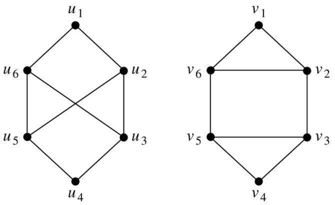


Strongly connected components (强连通分支)

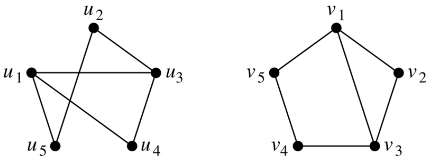
- Definition: A strongly connected component (or strong component) of a directed graph G is a maximal strongly connected subgraph of G.
- Example: What are the strong components of H?



A useful graph invariant is the existence of a simple circuit of a certain length.



G H

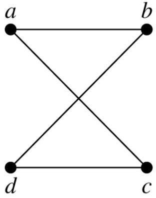


G H

Counting paths between vertices

- Theorem: Let G be a graph with adjacency matrix A wrt the ordering $v_1, v_2, \ldots, v_n$ (with directed or undirected edges, with multiple edges and loops allowed). The number of different paths of length r from v_i to v_j, where r is a positive integer, equals the (i, j)th entry of A^r.

- Example: How many paths of length 4 from a to d?



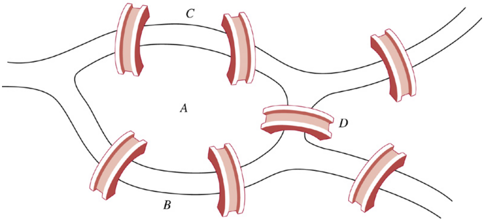
$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}, A^4 = \begin{pmatrix} 8 & 0 & 0 & 8 \\ 0 & 8 & 8 & 0 \\ 0 & 8 & 8 & 0 \\ 8 & 0 & 0 & 8 \end{pmatrix}$$

Introduction to Euler and Hamilton paths

- Can we travel along the edges of a graph returning to the start vertex by traversing each edge of the graph exactly once?
- Can we travel along the edges of a graph returning to the start vertex by visiting each vertex of the graph exactly once?
- The first question, asking for an Euler circuit, is easy.
- The second question, asking for a Hamilton circuit, is difficult.

The Konigsberg seven-bridge problem

- The town of Konigsberg, Prussia was divided into 4 sections by the branches of the Pregel river.
- Seven bridges connected these regions.
- Is it possible to start at some location, travel across all the bridges without crossing any bridge twice, and return to the same point?

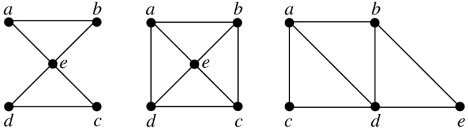


The Konigsberg seven-bridge problem

- The Swiss mathematician Euler solved this problem.
- Use the multigraph obtained when the 4 regions are represented by vertices and bridges by edges.
- Then the problem becomes: Is there a simple circuit in this multigraph that contains every edge?

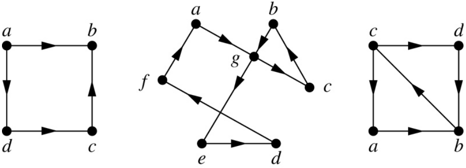
Euler paths and circuits

- Definition: An Euler circuit in a graph G is a simple circuit containing every edge of G. An Euler path in G is a simple path containing every edge of G.
- Example:



G_1 G_2 G_3

- Example:



H_1 H_2 H_3

Necessary and sufficient conditions for Euler circuits

Theorem: A connected multigraph with at least 2 vertices has an Euler circuit iff each of its vertices has even degree.

$\Rightarrow$:

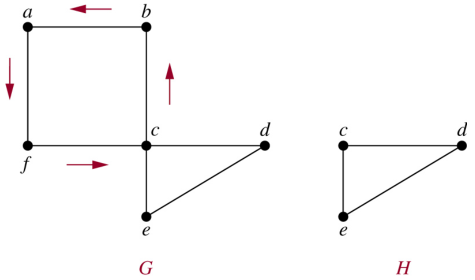
- Each time the circuit passes through a vertex v, it contributes 2 to $deg(v)$, since the circuit enters v via an edge, and leaves v via another edge
- When the circuit starts and ends at a vertex v, it contributes 2 to $deg(v)$

Necessary and sufficient conditions for Euler circuits

$\Leftarrow$:

- We start at an arbitrary vertex a, and build a simple path p until we cannot add another edge.
- p must end at a, since each vertex has an even degree.
- If p use all the edges, we get an Euler circuit.
- Otherwise, deleting the edges that are used and all vertices that become isolated when the edges are removed, we get a subgraph H.
- Every vertex in H has an even degree.
- Since G is connected, H share a vertex, say w, with p.
- Start at w, build a simple path p' in H as long as possible.
- p' must end at w. Splice p and p'.
- Continue this process until all edges have been used.

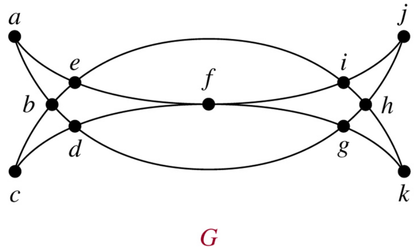
Examples



The Konigsberg bridge problem has no Euler circuit.

A puzzle

- Many puzzles ask us to draw a picture in a continuous motion without lifting a pencil.
- Such puzzles can be solved using Euler circuits and paths.
- Example: Can Mohammed's scimitars (弯刀) be drawn in this way where the drawing begins and ends at the same point?



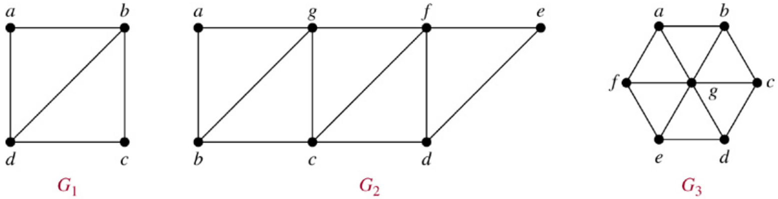
Necessary and sufficient conditions for Euler paths

Theorem: A connected multigraph has an Euler path but not an Euler circuit iff it has exactly two vertices of odd degree.

$\Leftarrow$:

- Let a and b be the vertices with odd degree.
- Add an edge between a and b.
- We get a graph where all vertices have even degree, so there is an Euler circuit.
- Remove the edge $\{a, b\}$ from the circuit, and we get an Euler path.

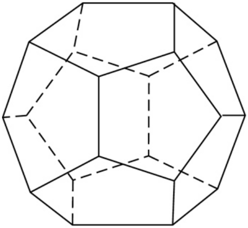
Examples



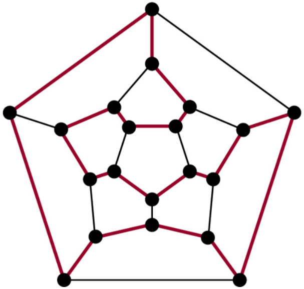
The konigsberg problem has no Euler path.

Hamilton's "A voyage round the world" puzzle

- A polyhedron (多面体) with 12 regular pentagons (五边形) as faces
- The 20 vertices were labeled with different cities in the world
- The object is to start at a city and travel along the edges of the polyhedron, visiting each of the other 19 cities exactly once, and back to the first city.

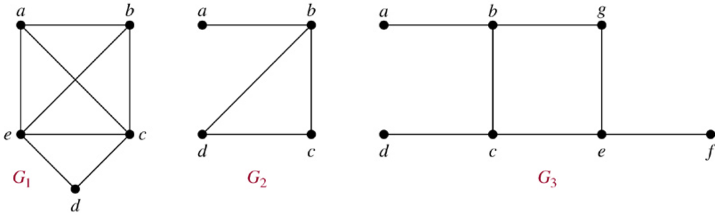


(a)



Hamilton paths and circuits

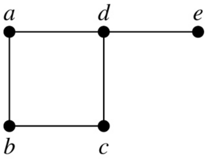
- Definition: A simple path in a graph G that passes through every vertex exactly once is called a Hamilton path, and a simple circuit in a graph G that passes through every vertex exactly once is called a Hamilton circuit.
- Examples



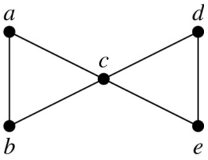
- Example: Show that K_n has a Hamilton path whenever $n \geq 3$.
  - Start at any vertex, repeatedly extend the path by adding an edge to a vertex not existing on the path, until we add an edge back to the first vertex

Determine if a graph has a Hamilton path or circuit

- There are no known simple sufficient and necessary conditions for the existence of Hamilton circuits.
- Many theorems give sufficient conditions for the existence of Hamilton circuits.
- Certain properties can be used to show that a graph has no Hamilton circuit, *e.g.*,
  - A graph with a pendant vertex cannot have a Hamilton circuit.
  - For any vertex, exactly two edges incident with the vertex are included in any Hamilton circuit.



G



H

Determine if a graph has a Hamilton path or circuit

- The more edges a graph has, the more likely it is to have a Hamilton circuit.
- There are sufficient conditions for the existence of Hamilton circuits that depend on the degree of vertices being sufficiently large.
- Dirac's Theorem: If G is a simple graph with n vertices with $n \geq 3$ such that $deg(v) \geq n/2$ for every vertex v of G, then G has a Hamilton circuit.
- Ore's Theorem: If G is a simple graph with n vertices with $n \geq 3$ such that $deg(u) + deg(v) \geq n$ for every pair of non-adjacent vertices u and v in G, then G has a Hamilton circuit.
- This is a NP-complete problem. The best algorithms known have exponential worst-time complexity.

Gray codes

- A Gray code is a labeling of the arcs of the circle such that adjacent arcs are labeled with bit strings that differ in exactly one bit.
- The assignment in (b) is a Gray code.
- We can find a Gray code by finding a Hamilton circuit in the n-cube Q_n.