# Curve and Surface Modeling 2

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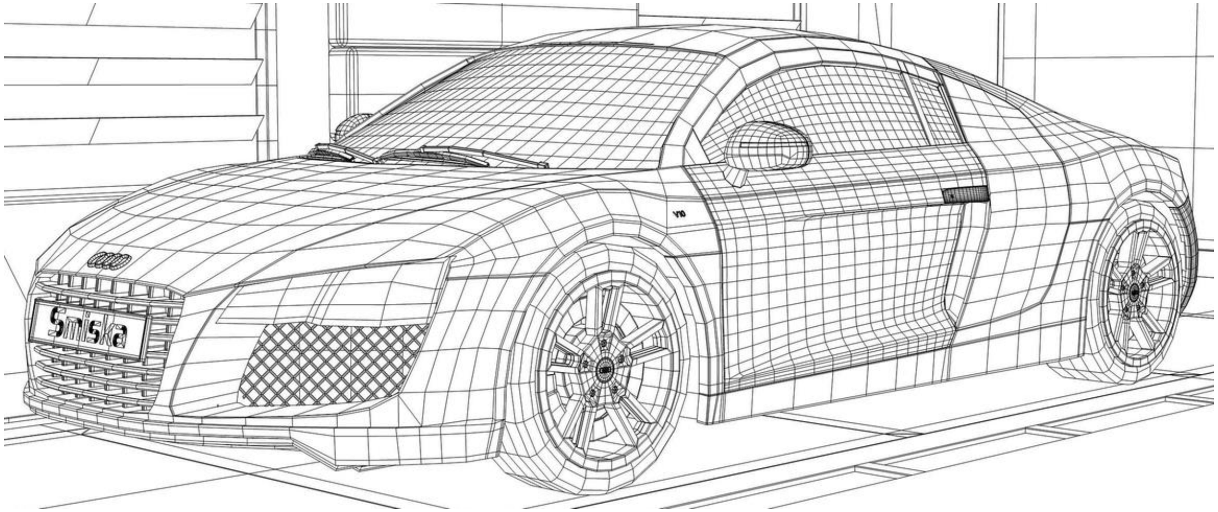
# Outline

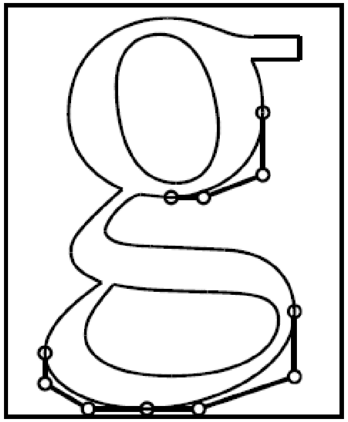
- Interpolation and Approximation
- Curve Modeling
  - Parametric curve
  - **Bézier curve**
- Surface Modeling
  - Bézier surface

# Bézier curve

- A Bézier curve is a parametric curve frequently used in computer graphics and related fields.
- Bézier曲线是由法国雷诺汽车公司的Pierre Bézier在1971年提出的一种构造样条曲线和曲面的方法。
- 这种方法能够比较直观地表示给定的条件与曲线形状的关系，使用户可以方便地通过修改参数来改变曲线的形状和阶次。

Pierre Bézier
An engineer at Renault



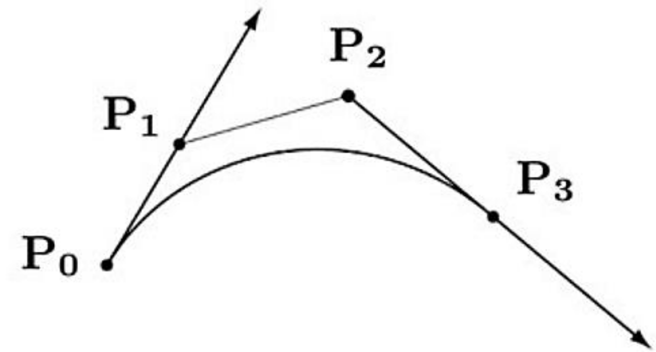
# The definition of Bézier curve

- Bézier curve本质上是由**调和函数（Harmonic functions）**根据**控制点（Control points）**插值生成。其参数方程如下：

$$Q(t) = \sum_{i=0}^{n} P_i B_{i,n}(t), \quad t \in [0,1]$$

- 上式为$n$次多项式，具有 $n+1$项。其中，$P_i(i=0,1\ldots n)$表示特征多边形的$n+1$个顶点向量； $B_{i,n}(t)$为**伯恩斯坦（Bernstein）基函数**，其多项式表示为：

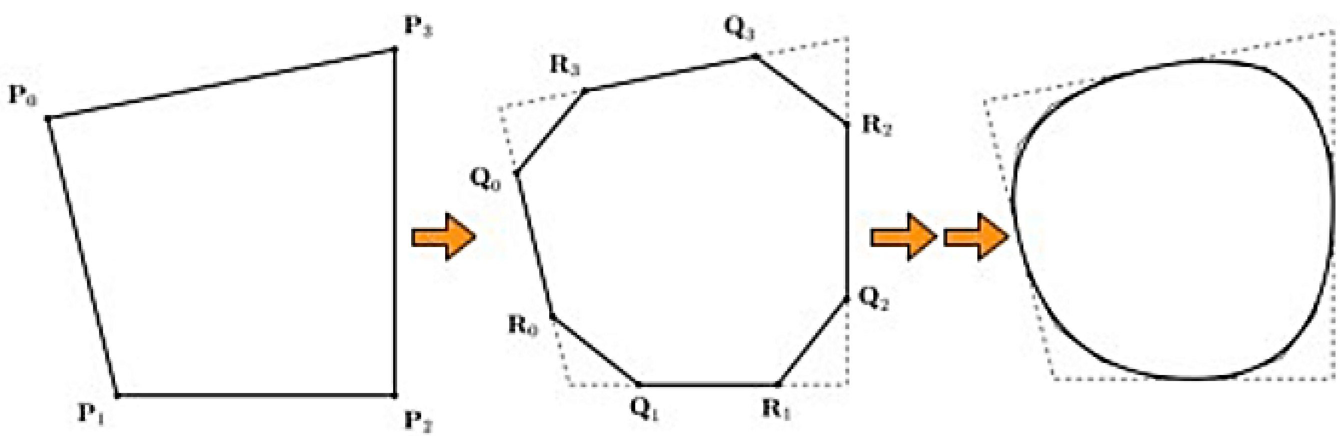
$$B_{i,n}(t) = \frac{n!}{i!(n-i)!} t^i (1-t)^{n-i}, \quad i=0, 1\ldots n$$



# Limit Curve

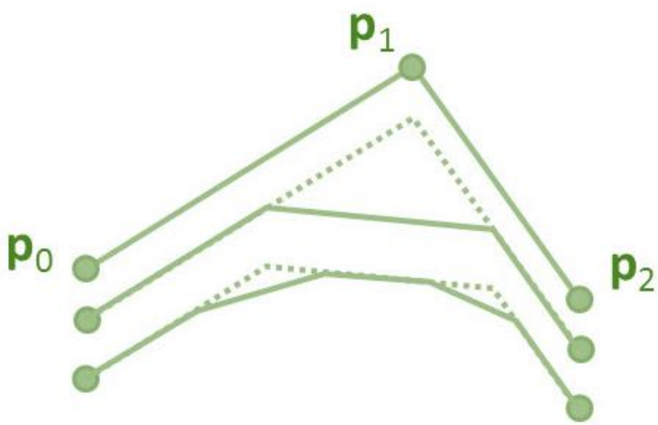
- Repeatedly cutting corners forms the “limit curve”

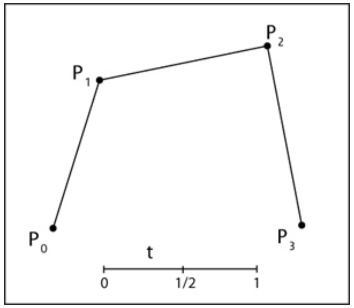
- Properties of the curve
- 1. Interpolates midpoints
- 2. Tangent preserved at midpoint

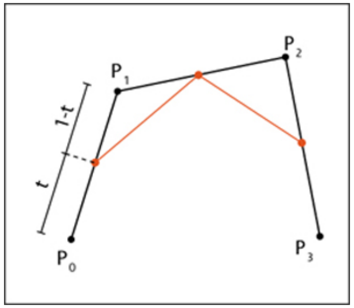


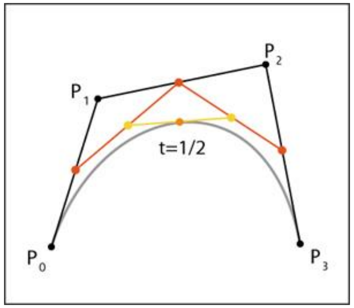
# Intuition for Bezier curves

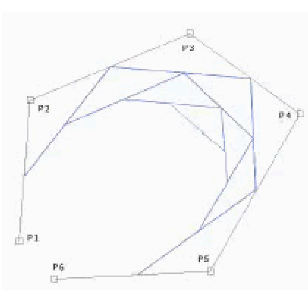
- Keep on cutting corners to make a “smoother” curve
- In the limit, the curve becomes smooth

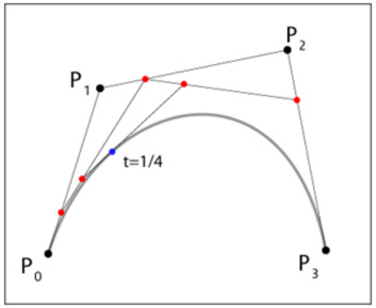


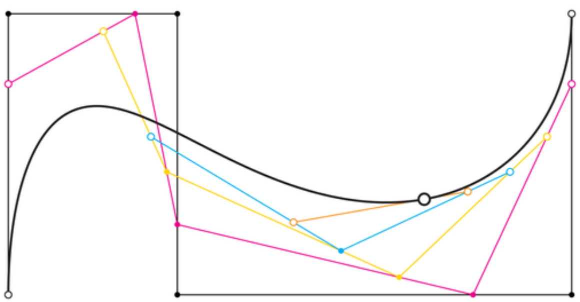



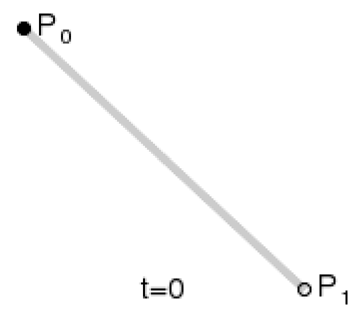


# Linear Bézier curve

- Linear polynomial (一次多项式) has two control points, the matrix representation is in the following:

$$
\begin{aligned}
Q(t) &= \sum_{i}^{1} P_i B_{i,1}(t) = P_0 B_{0,1}(t) + P_1 B_{1,1}(t) \\
&= (1-t)P_0 + tP_1 \qquad , \quad t \in [0,1] \\
&= [t \quad 1]\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} P_0 \\ P_1 \end{bmatrix}
\end{aligned}
$$



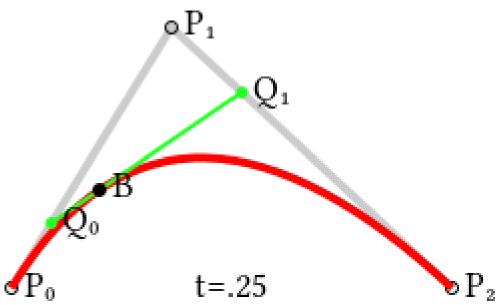

- Actually, it is a line.

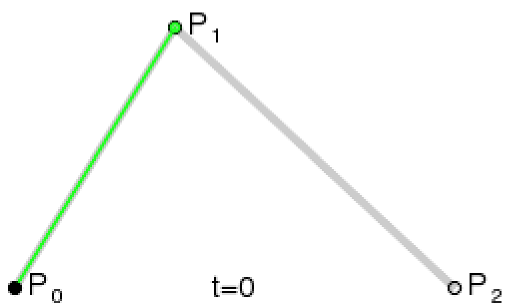
# Quadratic Bézier curve

- Quadratic polynomial (二次多项式) has 3 control points, the math formula is as follows:

$$
\begin{aligned}
Q(t) &= \sum_{i}^{2} P_i B_{i,2}(t) = P_0 B_{0,2}(t) + P_1 B_{1,2}(t) + P_2 B_{2,2}(t) \\
&= (1-t)^2 P_0 + 2t(1-t)P_1 + t^2 P_2 \qquad , \quad t \in [0,1] \\
&= (P_2 - 2P_1 + P_0)t^2 + 2(P_1 - P_0)t + P_0
\end{aligned}
$$

- 二次Bézier curve为抛物线，矩阵形式为： $Q(t) = [t^2 \quad t \quad 1] \begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \end{bmatrix}$ ， $t \in [0,1]$



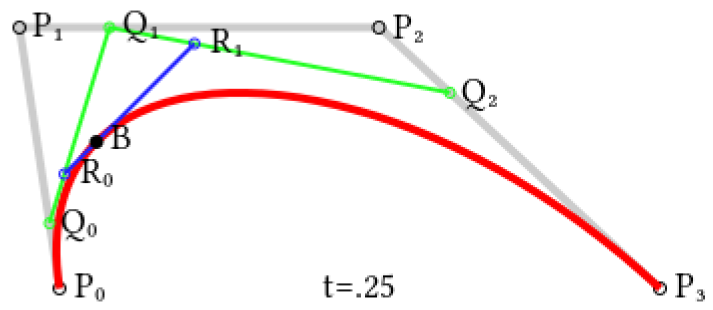
# Cubic Bézier curve

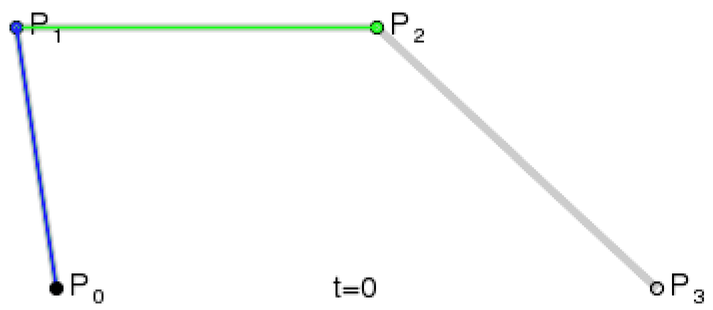
- Cubic polynomial (二次多项式) has 4 control points, the math formula is as follows:

$$Q(t)=\sum_{i}^{3} P_i B_{i,3}(t)=P_0B_{0,3}(t)+P_1B_{1,3}(t)+P_2B_{2,3}(t)+P_3B_{3,3}(t)\,,\quad t\in[0,1]$$
$$=(1-t)^3P_0+3t(1-t)^2P_1+3t^2(1-t)P_2+t^3P_3$$

- The matrix representation:

$$Q(t)=[t^3 \quad t^2 \quad t \quad 1]\begin{bmatrix}-1 & 3 & -3 & 1\\ 3 & -6 & 3 & 0\\ -3 & 3 & 0 & 0\\ 1 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}P_0\\ P_1\\ P_2\\ P_3\end{bmatrix},\quad t\in[0,1]$$

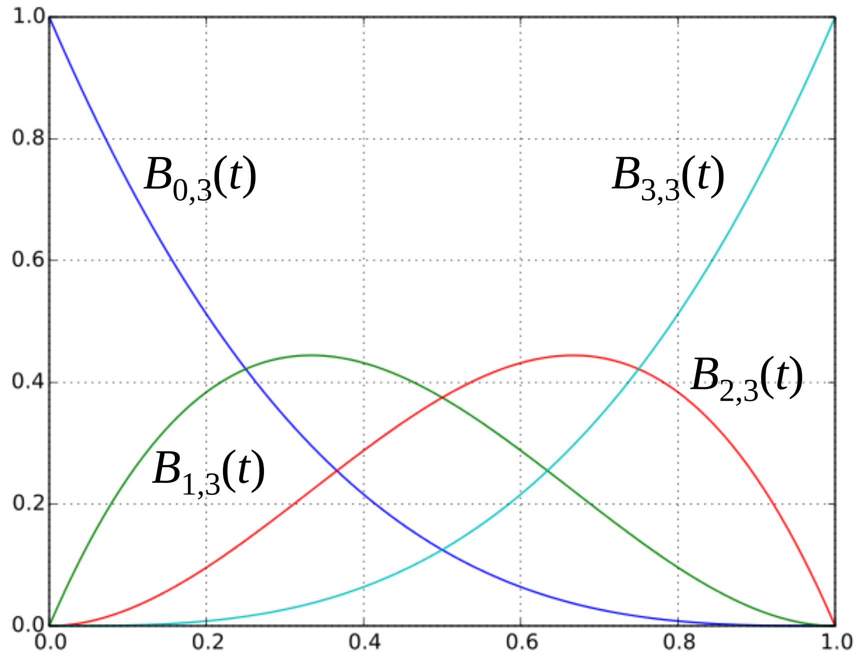


# Bernstein Basis Functions

- 根据Bernstein多项式构成了三次Bézier曲线的一组基，或称为三次Bézier曲线的调和函数，即：

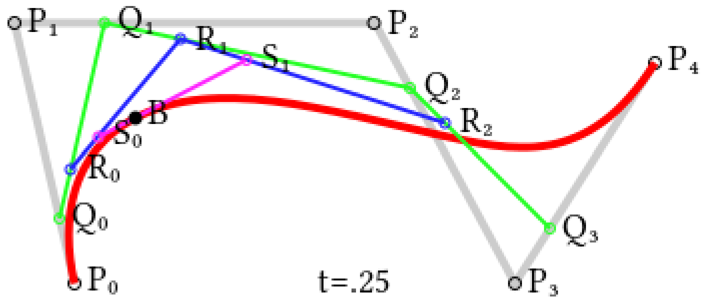
$$\begin{cases} B_{0,3}(t) = (1-t)^3 \\ B_{1,3}(t) = 3t(1-t)^2 \\ B_{2,3}(t) = 3t^2(1-t) \\ B_{3,3}(t) = t^3 \end{cases}$$

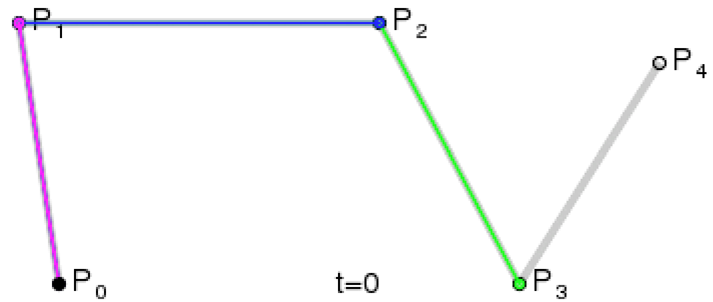



The basis functions of cubic Bézier curve on the range t in [0,1]

# High-order Bézier curve

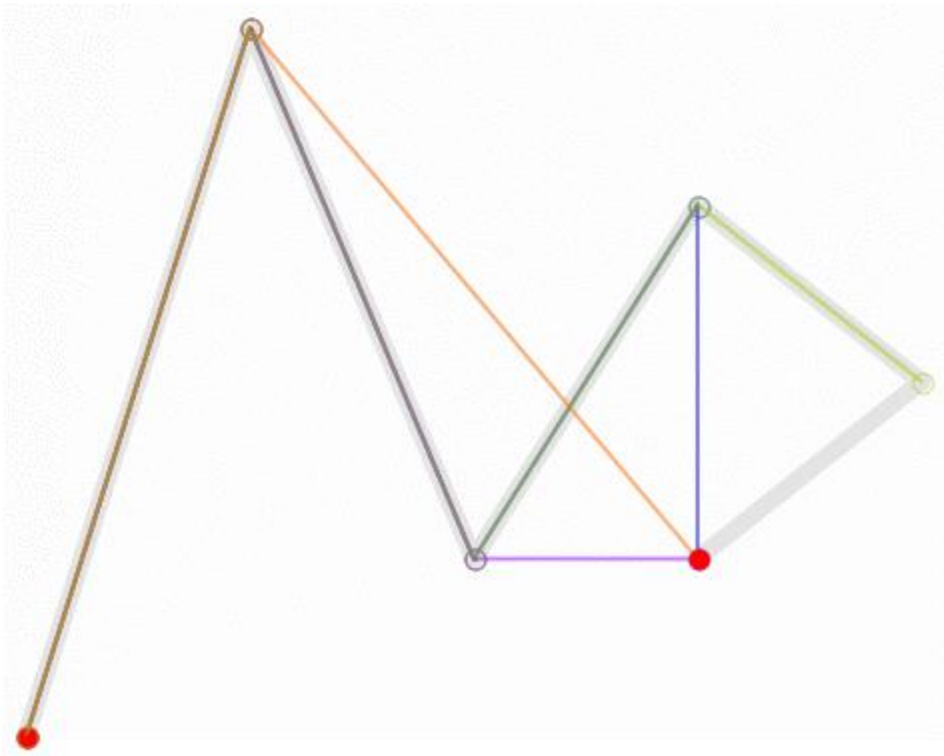
- For fourth-order curves one can construct intermediate points $\mathbf{Q}_0$, $\mathbf{Q}_1$, $\mathbf{Q}_2$ & $\mathbf{Q}_3$ that describe linear Bézier curves, points $\mathbf{R}_0$, $\mathbf{R}_1$ & $\mathbf{R}_2$ that describe quadratic Bézier curves, and points $\mathbf{S}_0$ & $\mathbf{S}_1$ that describe cubic Bézier curves:



# High-order Bézier curve

- For fifth-order curves, one can construct similar intermediate points.



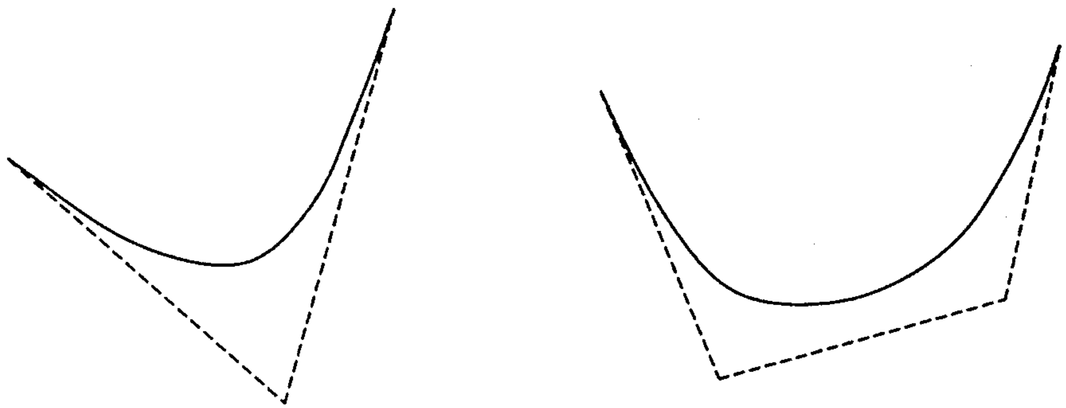
# The properties of Bézier curve

- 1. 端点性质：当$t = 0$和$t = 1$时，有：

$$Q(0)=\sum_{i=0}^{n} P_i B_{i,n}(0) = P_0 B_{0,n}(0) + P_1 B_{1,n}(0) + \cdots + P_n B_{n,n}(0) = P_0$$

$$Q(1)=\sum_{i=0}^{n} P_i B_{i,n}(1) = P_0 B_{0,n}(1) + P_1 B_{1,n}(1) + \cdots + P_n B_{n,n}(1) = P_n$$

- 这说明，Bézier曲线通过特征多边形的起点和终点。



# The properties of Bézier curve

- 2. 对称性：由于$B_{i,n}(t) = B_{n-i,n}(1-t)$，如果将控制点的顺序颠倒过来，记$P_i^* = P_{n-i}$,则根据Bézier曲线的定义可推出：

$$
\begin{aligned}
Q^*(t) &= \sum_{i=0}^{n} P_i^* B_{i,n}(t) = \sum_{i=0}^{n} P_{n-i} B_{i,n}(t) = \sum_{i=n}^{0} P_k B_{n-k,n}(t) \\
&= \sum_{i=n}^{0} P_k B_{k,n}(1-t) \\
&= Q(1-t)
\end{aligned}
$$

- 这说明，只要保持特征多边形的顶点位置不变，但顺序颠倒，所得的新的Bézier曲线形状不变，只是参数变化的方向相反。

# The properties of Bézier curve

- 凸包性：由于当$t \in [0,1]$时，Bernstein多项式之和为：

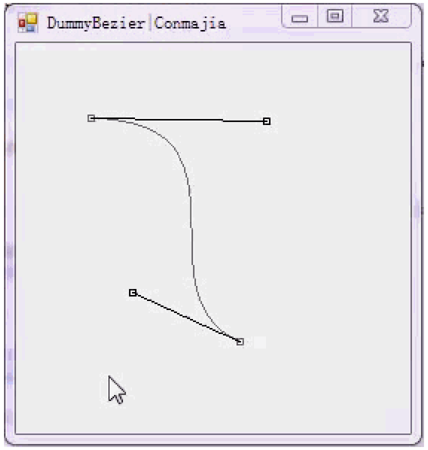
$$\sum_{i=0}^{n} B_{i,n}(t) = \sum_{i=0}^{n} \frac{n!}{i!(n-i)!} t^i (1-t)^{n-i} = [(1-t)+t] \equiv 1$$

- 且有 $B_{i,n}(t) = \frac{n!}{i!(n-i)!} t^i (1-t)^{n-i} \geqslant 0$

- 则说明$B_{i,n}(t)$构成了Bézier曲线的一组权函数，所以Bézier曲线一定落在其控制多边形的凸包之中。

# The properties of Bézier curve

- 几何不变形（Geometric Invariant）：指Bézier曲线的形状不随坐标变换而变化的特性。 Bézier曲线的形状只与各控制顶点的相对位置有关。
- 因此，在对Bézier曲线进行几何变换时，不需要对曲线上的所有点都进行处理，只需要先对控制顶点进行几何变换，然后重新绘制曲线就可以。

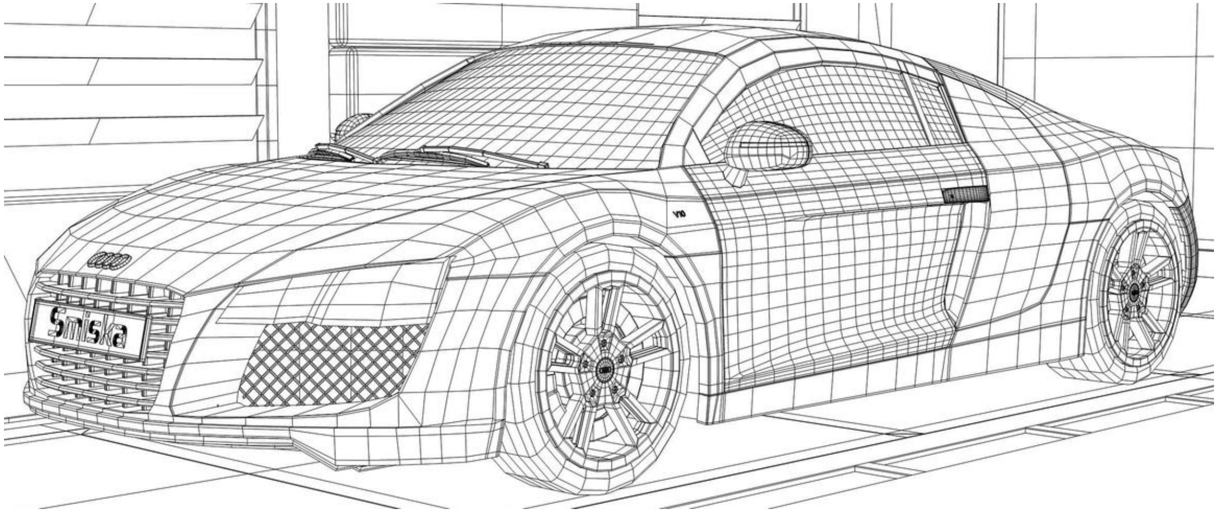


# Implementation – Cubic Bézier curve

```
//绘制由p0，p1，p2，p3确定的Bezier曲线
//参数区间[0，1]被离散为count份
void BezierCurve(Point p0,Point p1,Point p2,Point p3,int count)
{
    double t = 0.0;
    dt = 1.0 / count;
    moveto(p1.x,p1.y);                    //设置起点
    for(int i=0; i<count+1; i++)
    {
        double F1,F2,F3,F4,x,y;           //调和函数
        double u = 1.0 - t ;
        F1 = u * u * u ;
        F2 = 3 * t * u * u;
        F3 = 3 * t * t * u;
        F4 = t * t * t;
        x = p0.x * F1 + p1.x * F2 + p2.x * F3 + p3.x * F4;
        y = p0.y * F1 + p1.y * F2 + p2.y * F3 + p3.y * F4;
        lineto(x,y);
          t+=dt;
    }
}
```

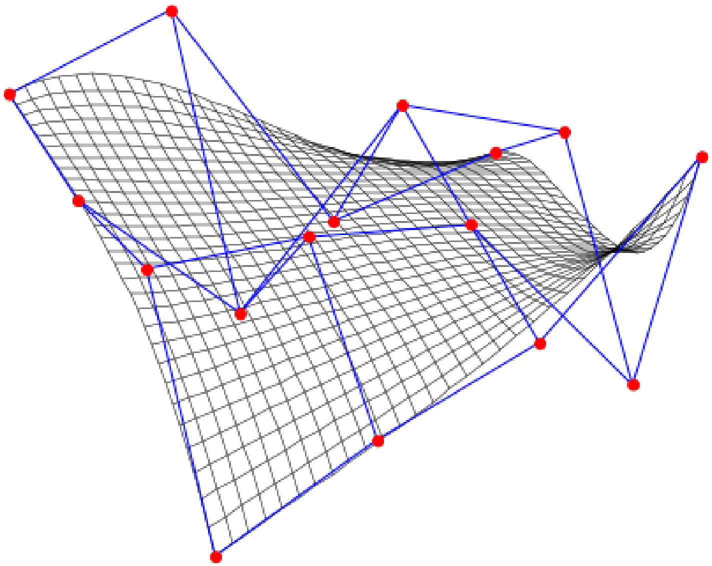
# Outline

- Interpolation and Approximation
- Curve Modeling
  - Parametric curve
  - Bézier curve
- **Surface Modeling**
  - **Bézier surface**

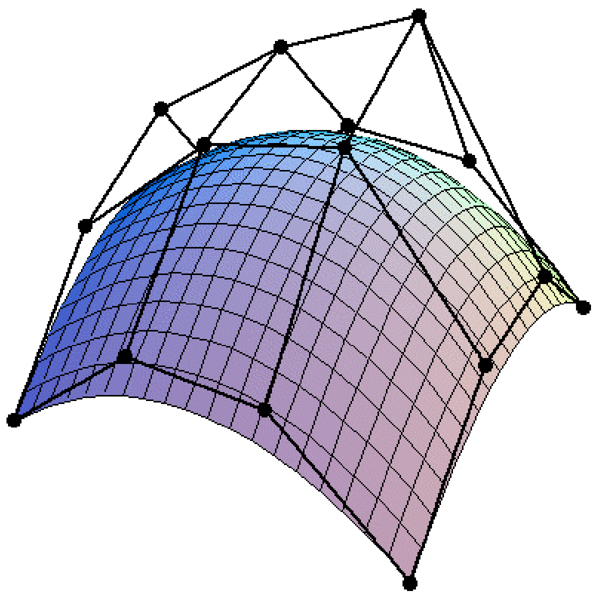
# Bézier surface

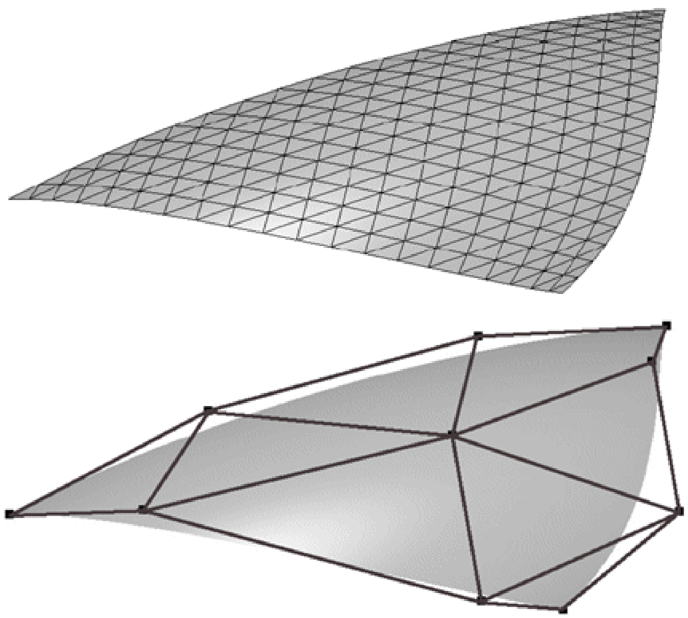
- Bézier surfaces are a species of mathematical spline used in computer graphics, computer-aided design, and finite element modeling.
- As with the Bézier curve, a Bézier surface is defined by a set of control points.



# Bézier surface

- 曲面表示方法与参数区域的选择有着密切的关系，若选择矩形参数区域，一般采用张量积或布尔和形式来构造曲面；若选择三角形（即单纯形）参数区域，则要采用直接升阶构造方法来表示曲面。





# The definition of Bézier surface in rectangular domain

- 当选定矩形参数区域$[0,1]\times[0,1]$后，则采用张量积（tensor product）方法把 Bézier curve 推广成 Bézier surface。
- 给定了$(n+1)(m+1)$个空间上顶点$P_{ij}(i=0,1,\dots,n;j=0,1,\dots,m)$，则称$n\times m$次参数曲面为$n\times m$次Bézier曲面，有：

$$P(u,v)=\sum_{i=0}^{n}\sum_{j=0}^{m}B_{i,n}(u)B_{j,m}(v)P_{ij}\qquad (u,v\in[0,1])$$

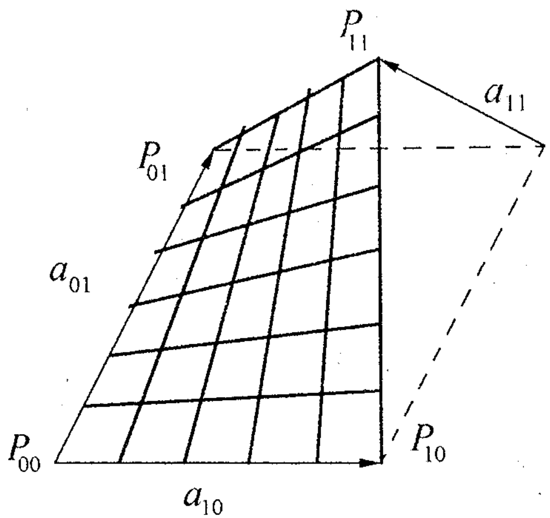
- 这里$B_{i,n}(u)$和$B_{j,m}(u)$为Bernstein基函数，依次用线段连接顶点$P_{ij}(i=0,1,\dots,n;j=0,1,\dots,m)$中相邻两顶点所形成的空间网格，称为Bézier曲面的特征网格。

# Bilinear Bézier surface

- 当$n = m = 1$时，有bilinear Bézier surface (双一次曲面)：

$$P(u, v) = (1-u)(1-v)P_{00} + (1-u)vP_{01} + u(1-v)P_{10} + uvP_{11} \quad (u, v \in [0, 1])$$

- 这是一块双曲抛物面（或成为马鞍面），它的边界由4条直线段组成，实际上这是一块直纹曲面。

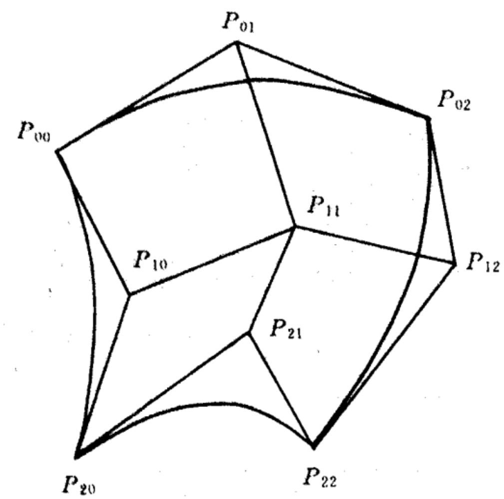


# Biquadratic Bézier surface

- 当$n$=$m$=2时，有biquadratic Bézier surface (双二次曲面)：

$$P(u,v)=\sum_{i=0}^{2}\sum_{j=0}^{2}B_{i,2}(u)\,B_{j,2}(v)\,P_{ij}\qquad(u,v\in[0,1])$$

- 该曲面的4条边界曲线都是抛物线，实际上其特征网格上的9个顶点，其中有8个边界顶点来确定4条边界曲线。
- 只有一个顶点$P_{11}$可以用来控制曲面的形状，并且对边界的曲线不会有影响，因此曲面的凹凸可以通过控制$P_{11}$来直观控制。



# Bicubic Bézier surface

- 当$n$=$m$=3时，有bicubic Bézier surface (双三次曲面)：

$$P(u,v)=\sum_{i=0}^{3}\sum_{j=0}^{3}B_{i,3}(u)B_{j,3}(v)P_{ij}\qquad(u,v\in[0,1])$$

- 有矩阵表示：

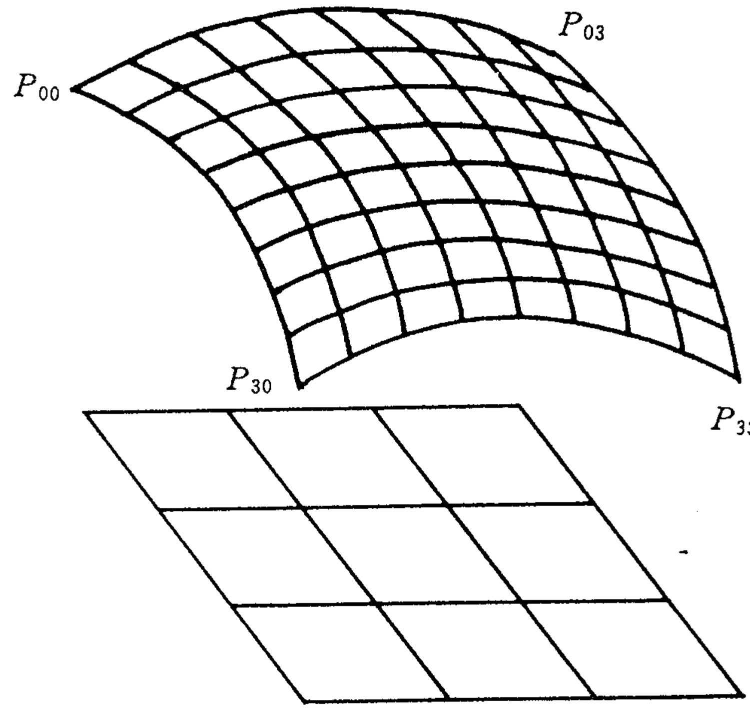
$$P(u,v)=[B_{0,3}(u),B_{1,3}(u),B_{2,3}(u),B_{3,3}(u)]\times\begin{bmatrix}P_{00}&P_{01}&P_{02}&P_{03}\\P_{10}&P_{11}&P_{12}&P_{13}\\P_{20}&P_{21}&P_{22}&P_{23}\\P_{30}&P_{31}&P_{32}&P_{33}\end{bmatrix}\begin{bmatrix}B_{0,3}(v)\\B_{1,3}(v)\\B_{2,3}(v)\\B_{3,3}(v)\end{bmatrix}\qquad(u,v\in[0,1])$$

- 进一步简化：$P(u,v)=UBPB^TV^T$
- 其中：

$$U^T=\begin{bmatrix}1\\u\\u^2\\u^3\end{bmatrix},\quad B=\begin{bmatrix}1&0&0&0\\-3&3&0&0\\3&-6&3&0\\-1&3&-3&1\end{bmatrix},\quad P=\begin{bmatrix}P_{00}&P_{01}&P_{02}&P_{03}\\P_{10}&P_{11}&P_{12}&P_{13}\\P_{20}&P_{21}&P_{22}&P_{23}\\P_{30}&P_{31}&P_{32}&P_{33}\end{bmatrix}$$

# Bicubic Bézier surface

- 曲面的4条边界都是三次Bézier曲线，由周边12个特征网格的顶点来确定。可通过调整内部4个顶点$P_{11}$，$P_{12}$，$P_{21}$，$P_{22}$的位置来控制曲面内部的形状。

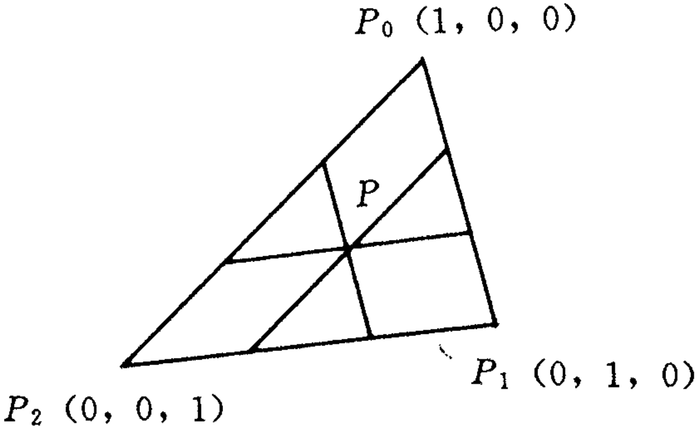


# Bézier surface in triangular domain

- 当采用三角形参数区域$\{(u,v,w)|u,v,w\geq 0,\ u+v+w=1\}$后，需要选择B-网的方法来构造Bézier 曲面。
- 对于不共线的三个顶点$P_0$，$P_1$，$P_2$就可以形成一个三角形，因此三角形组成的平面上任意一点$P$可以表示成：

$$P(u,\ v,\ w)=uP_0+vP_1+wP_2 \qquad (u+v+w=1)$$

- 其中：$(u,\ v,\ w)$ 称为$P(u,\ v,\ w)$关于$P_0$，$P_1$，$P_2$的重心坐标。
- 当$0\leq u+v+w\leq 1$时，$P(u,\ v,\ w)$位于$P_0$，$P_1$，$P_2$组成的三角形之内。



# The definition of Bézier surface in triangular domain

- 定义：参数曲面为三角域上的$n$次Bézier曲面：

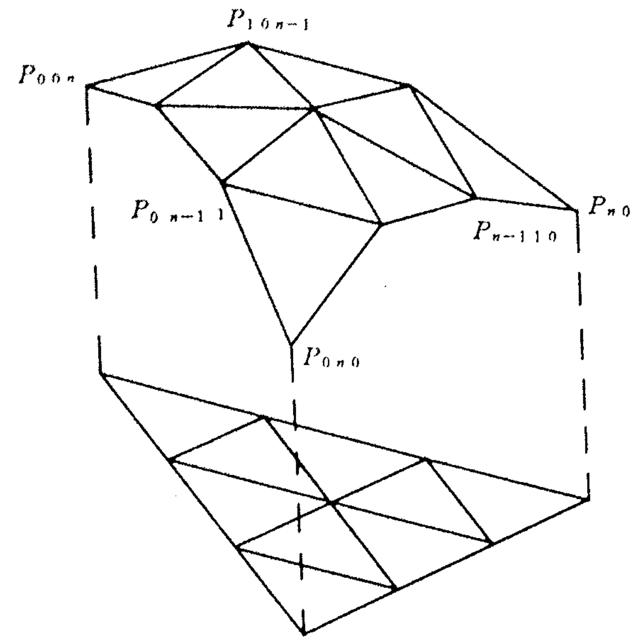
$$P(u, v, w) = \sum_{i+j+j=n} P_{ijk} B_{ijk}^{n}(u, v, w)$$

$$(u, v, w \geqslant 0;\ u + v + w = 1)$$

- 其中：

$$B_{ijk}^{n}(u, v, w) = \frac{n!}{i!\ j!\ k!} u^i v^j w^k$$

$$(i + j + k = n;\ u + v + w = 1)$$

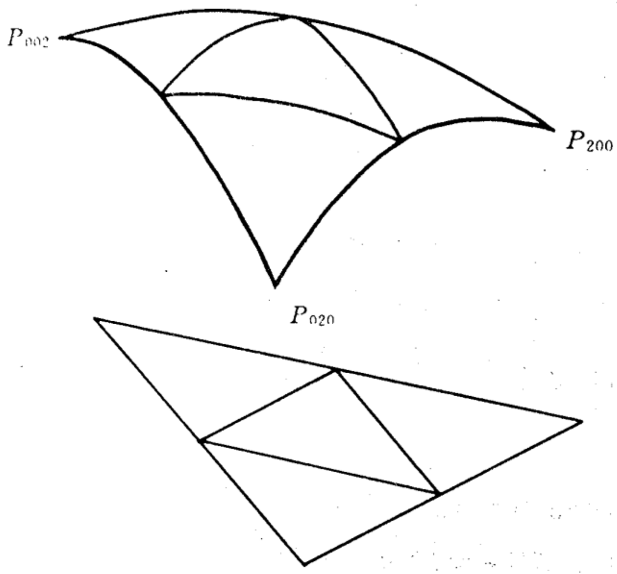


# Quadratic Bézier surface

- 当n=2时，可以直接写出二次Bézier曲面

$$
\begin{aligned}
P(u,v,w) &= \sum_{i+j+k=n} P_{ijk} B_{ijk}^{2}(u,v,w) \\
&= u^2 P_{200} + v^2 P_{020} + w^2 P_{002} + 2uvP_{110} + 2uwP_{101} + 2vwP_{011}
\end{aligned}
$$

- 从上式中可以看出，二次Bézier曲面完全由6个边界顶点确定，因此，其形状完全由边界曲线所控制。

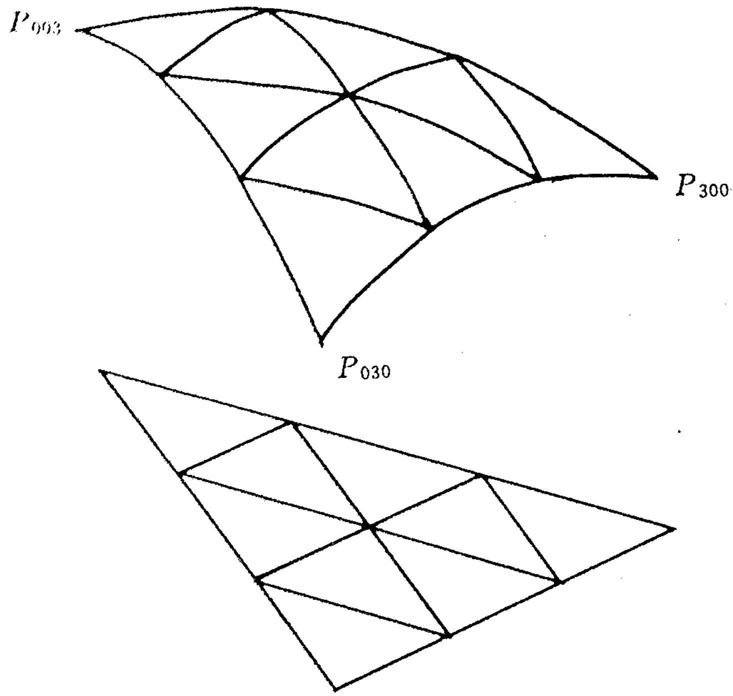


# Cubic Bézier surface

- 三次Bézier曲面为：

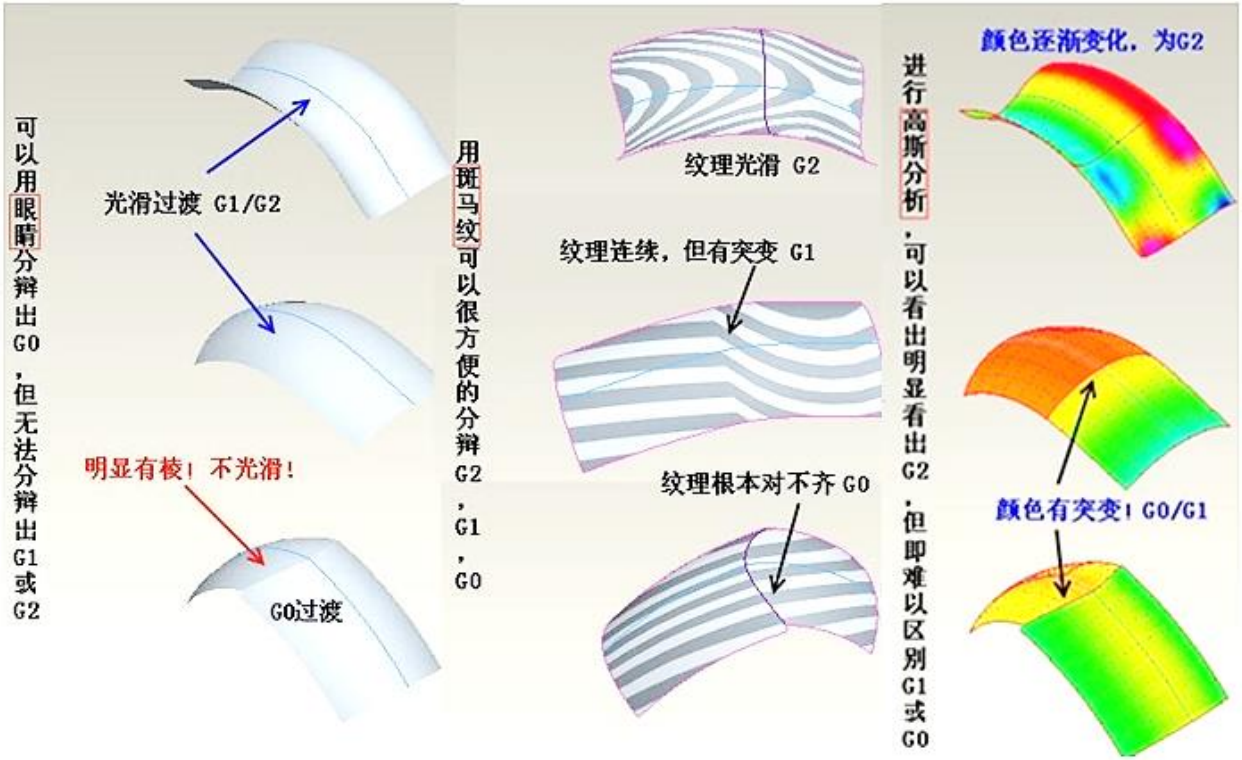
$$P(u, v, w) = \sum_{i+j+k=3} P_{ijk} B_{ijk}^3 (u, v, w)$$
$$= u^3 P_{300} + v^3 P_{030} + w^3 P_{003} + 3u^2 v P_{210} + 3uv^2 P_{120}$$
$$+ 3u^2 w P_{201} + 3uw^2 P_{102} + 3v^2 w P_{021} + 3vw^2 P_{012} + 6uvw P_{111}$$

- 若选定了三次Bézier曲面的9个边界顶点，其内部形状就可由一个控制顶点$P_{111}$来交互改变。



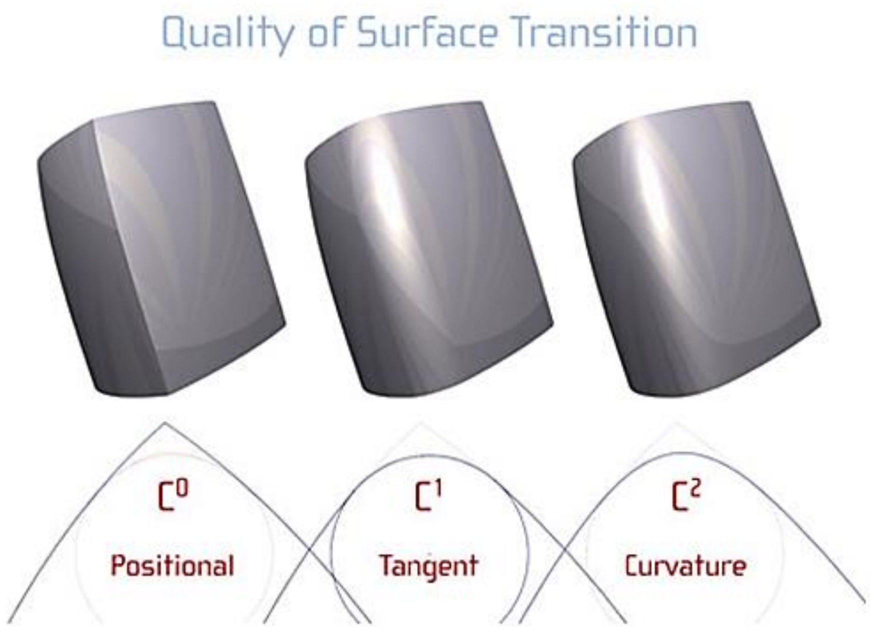
# The continuity of the curve and surface

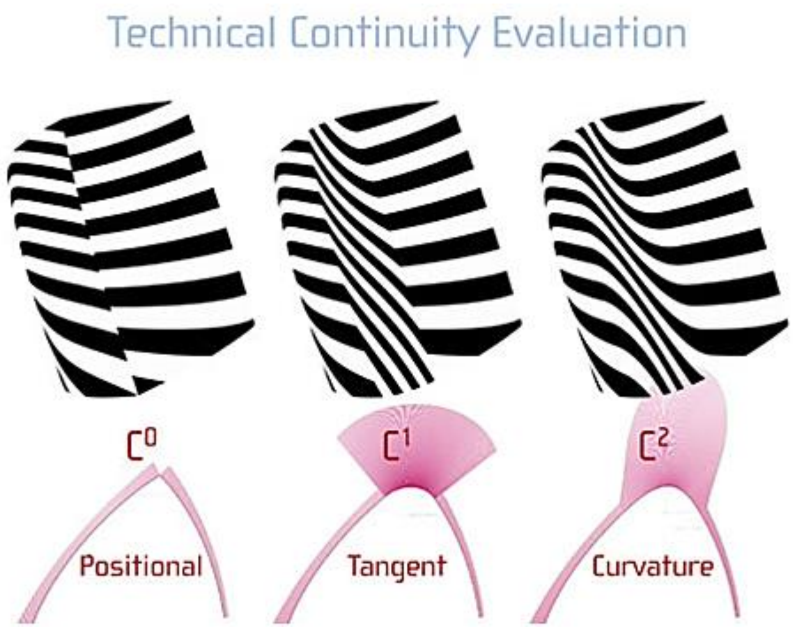
- $G^n$是两个几何对象间的实际连续程度，用于表示实际物理连续性；而$C^n$ 是实际物理连续性的数学表达，几何连续性$G^n$没有数学连续性$C^n$严格。 $G^n$的连续性是独立于表示（参数化）的。



# The continuity of the curve and surface

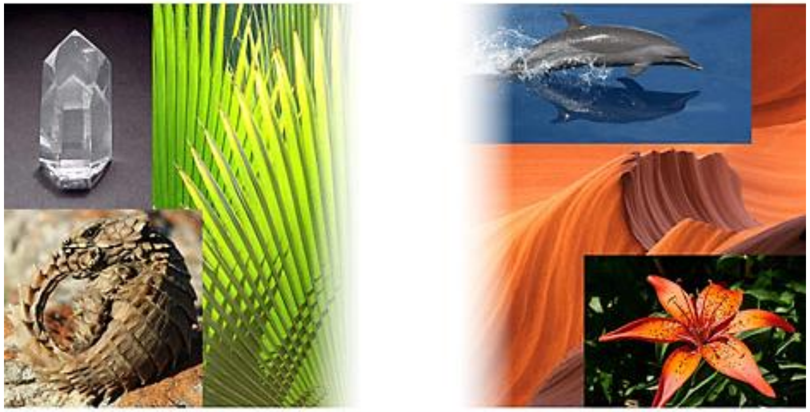
- $C^0$ –意味着两个相邻段间存在一个公共点（即两个段相连）。
- $C^1$ –对应于曲线方程的1阶导数，即意味着有一个公共点，并且多项式的一阶导数（即切向矢量）是相同的。
- $C^2$ –对应于曲线方程的2阶导数，意味着一阶导数和二阶导数都相同。



# The continuity in real world

Continuity in Nature



Positional Tangent Curvature

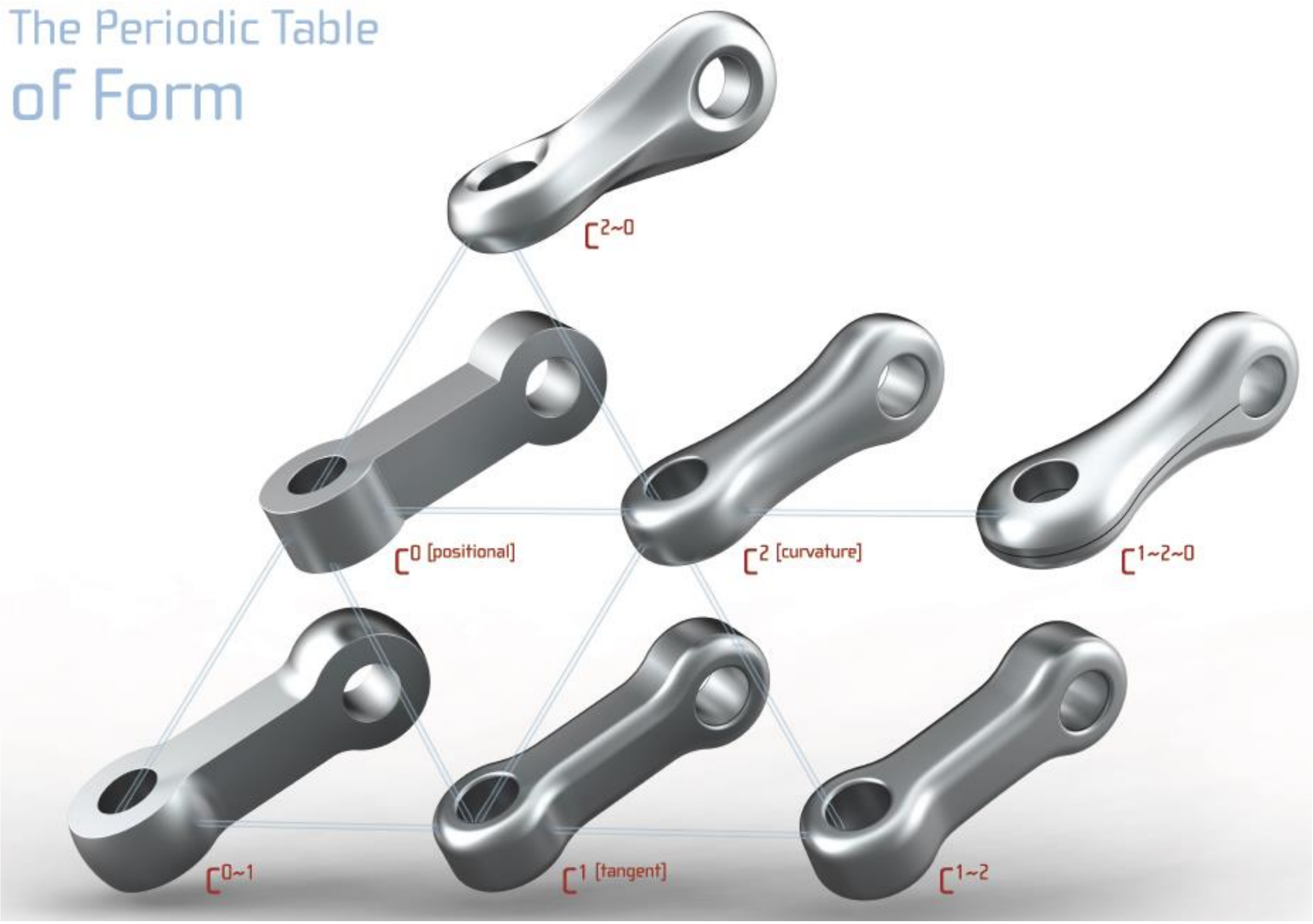
Continuity in Transportation Design

Positional Tangent Curvature

Continuity in Product Design

Positional Tangent Curvature

# Simplified cases

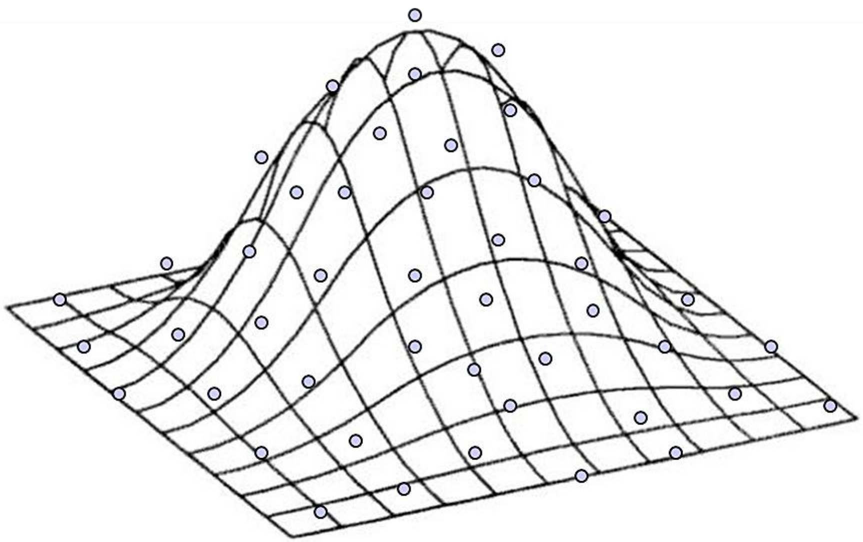


# Least square surface

- If you have a data set $(x_1, y_1, z_1)$, $(x_2, y_2, z_2)$ , ... , $(x_n, y_n, zn)$ and the best surface $z = f(x, y)$ should be with the property as follows

Minimum Least Square error

$$E = \sum_{i=1}^{n} (f(x_i, y_i) - z_i)^2$$



# Non-uniform rational B-spline (NURBS) surface

- Non-uniform rational basis spline (NURBS) is a mathematical model commonly used in computer graphics for generating and representing curves and surfaces.