P.165.17. 求曲线 $r = a\cdot\sin^3\frac{\theta}{3}$ 的全长。

解: $r'(\theta) = 3a\cdot\sin^2\frac{\theta}{3}\cdot\cos\frac{\theta}{3}\cdot\frac{1}{3} = \frac{3a}{3}\sin^2\frac{\theta}{3}\cdot\cos\frac{\theta}{3} = a\cdot\sin^2\frac{\theta}{3}\cdot\cos\frac{\theta}{3}$.

$$ds = \sqrt{r^2(\theta)+r'^2(\theta)}\,d\theta = \sqrt{a^2\cdot\sin^6\frac{\theta}{3} + a^2\sin^4\frac{\theta}{3}\cdot\cos^2\frac{\theta}{3}} = a\cdot\sin^2\frac{\theta}{3}\,d\theta$$

因为 $\sin^3\alpha$ 的周期为 2π，从而 $\sin^3\frac{\theta}{3}$ 的周期为 6π.

不难看出，当 $0\le\theta\le 3\pi$ 时，$r(\theta)\ge 0$

从而 $S = \int_0^{3\pi} ds = \int_0^{3\pi} a\cdot\sin^2\frac{\theta}{3}\,d\theta = 3a\int_0^{3\pi}\sin^2\frac{\theta}{3}\,d\frac{\theta}{3}$

$$= 3a\int_0^{\pi}\sin^2 t\,dt = 6a\int_0^{\frac{\pi}{2}}\sin^2 t\,dt = 6a\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{3\pi a}{2}.$$

P.165.18. 求星形线 $x = a\cdot\cos^3 t$, $y = a\cdot\sin^3 t$ 的周长。

解: $x'(t) = -3a\cdot\cos^2 t\cdot\sin t$

$y'(t) = 3a\sin^2 t\cdot\cos t$

$$ds = \sqrt{x'^2(t)+y'^2(t)}\,dt = \sqrt{9a^2\cos^4 t\cdot\sin^2 t + 9a^2\sin^4 t\cos^2 t}\,dt$$

$$= 3a|\sin t|\cdot|\cos t|\,dt$$

$$S = \int_0^{2\pi} ds = \int_0^{2\pi} 3a\cdot|\sin t|\,|\cos t|\,dt$$

$$= 4\times 3a\int_0^{\frac{\pi}{2}}\sin t\cdot\cos t\,dt = 12a\left[\frac{\sin^2 t}{2}\right]_0^{\frac{\pi}{2}} = 6a.$$

P.165.18. 求心脏线 $r = a(1+\cos\theta)$ 的全长。

解: $r'(\theta) = -a\sin\theta$, $ds = \sqrt{r^2(\theta)+r'^2(\theta)}\,d\theta = \sqrt{a^2(1+\cos\theta)^2+a^2\sin^2\theta}\,d\theta$

$$= \sqrt{2}a\cdot\sqrt{1+\cos\theta}\,d\theta$$

$$S = 2\int_0^{\pi} ds = 2\sqrt{2}a\int_0^{\pi}\sqrt{1+\cos\theta}\,d\theta = 2\sqrt{2}a\cdot\int_0^{\pi}\sqrt{2\cos^2\frac{\theta}{2}}\,d\theta$$

$$= 4a\cdot\int_0^{\pi}\cos\frac{\theta}{2}\,d\theta = 8a\left[\sin\frac{\theta}{2}\right]_0^{\pi} = 8a.$$