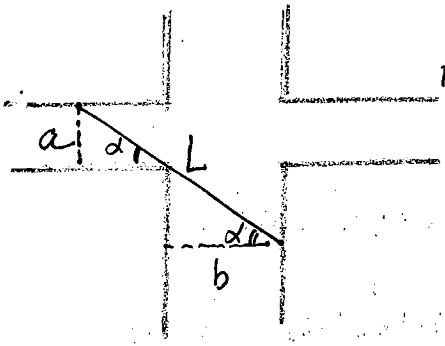
P.209.6. 两条宽分别为 a 与 b 的河垂直相交。若一船能从一河转入另一条河，问其长度是多少？ 2011/7/20 —96.



解：设船的长度为 L，则 $L=\frac{a}{\sin\alpha}+\frac{b}{\cos\alpha}=a\cdot\csc\alpha+b\sec\alpha$.

$$L'(\alpha)=-\frac{a\cdot\cos\alpha}{\sin^2\alpha}-\frac{b(-\sin\alpha)}{\cos^2\alpha}=\frac{-a\cos^3\alpha+b\sin^3\alpha}{\sin^2\alpha\cdot\cos^2\alpha}$$

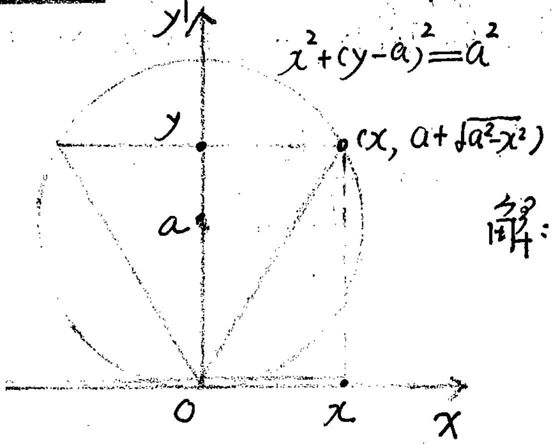
令 $L'(\alpha)=0$，得 $b\sin^3\alpha=a\cdot\cos^3\alpha$，$\tan^3\alpha=\frac{a}{b}$

$\tan\alpha=\left(\frac{a}{b}\right)^{\frac{1}{3}}$

$\cot\alpha=\left(\frac{b}{a}\right)^{\frac{1}{3}}$

$$\begin{aligned}L&=a\cdot\csc\alpha+b\sec\alpha\\&=a\cdot\sqrt{1+\cot^2\alpha}+b\cdot\sqrt{1+\tan^2\alpha}\\&=a\cdot\sqrt{1+\left(\frac{b}{a}\right)^{\frac{2}{3}}}+b\cdot\sqrt{1+\left(\frac{a}{b}\right)^{\frac{2}{3}}}\\&=a\cdot\frac{\sqrt{a^{\frac{2}{3}}+b^{\frac{2}{3}}}}{a^{\frac{1}{3}}}+b\cdot\frac{\sqrt{b^{\frac{2}{3}}+a^{\frac{2}{3}}}}{b^{\frac{1}{3}}}=\sqrt{a^{\frac{2}{3}}+b^{\frac{2}{3}}}\cdot\left(a^{\frac{2}{3}}+b^{\frac{2}{3}}\right)=\left(a^{\frac{2}{3}}+b^{\frac{2}{3}}\right)^{\frac{3}{2}}.\end{aligned}$$

P.209.7. 在半径为 a 的球内作一内接圆锥体，要使锥体体积最大，问高及底半径应是多少？



解： $V(x)=\frac{1}{3}\pi x^2\left(a+\sqrt{a^2-x^2}\right)$

$$\begin{aligned}V'(x)&=\frac{2\pi}{3}x\left(a+\sqrt{a^2-x^2}\right)+\frac{1}{3}\pi x^2\cdot\left(\frac{-2x}{2\sqrt{a^2-x^2}}\right)\\&=\frac{2\pi x}{3}\left(a+\sqrt{a^2-x^2}\right)-\frac{1}{3}\pi x^3\cdot\frac{1}{\sqrt{a^2-x^2}}\\&=\frac{\pi x}{3}\left[2\left(a+\sqrt{a^2-x^2}\right)-\frac{x^2}{\sqrt{a^2-x^2}}\right]\end{aligned}$$

令 $V'(x)=0$，则 $2\left(a+\sqrt{a^2-x^2}\right)=\frac{x^2}{\sqrt{a^2-x^2}}$

$$2a\cdot\sqrt{a^2-x^2}+2(a^2-x^2)=x^2$$

$$2a\sqrt{a^2-x^2}=3x^2-2a^2$$

$$4a^2(a^2-x^2)=9x^4-12a^2x^2+4a^4$$

$9x^4=8a^2x^2$，$9x^2=8a^2$，$x=\frac{2\sqrt{2}a}{3}$.

高 $y=a+\sqrt{a^2-\frac{8}{9}a^2}=a+\frac{a}{3}=\frac{4a}{3}$.