Numerical Analysis

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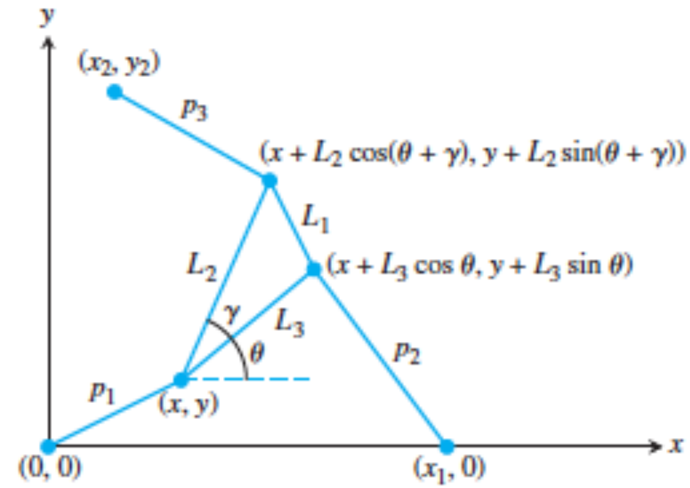
Outline

- Feb. 25
- March 4

Solving Equations S1



$$p_1^2 = x^2 + y^2$$
$$p_2^2 = (x + A_2)^2 + (y + B_2)^2$$
$$p_3^2 = (x + A_3)^2 + (y + B_3)^2.$$

In these equations,

$$A_2 = L_3\cos\theta - x_1$$
$$B_2 = L_3\sin\theta$$
$$A_3 = L_2\cos(\theta+\gamma) - x_2 = L_2[\cos\theta\cos\gamma - \sin\theta\sin\gamma] - x_2$$
$$B_3 = L_2\sin(\theta+\gamma) - y_2 = L_2[\cos\theta\sin\gamma + \sin\theta\cos\gamma] - y_2.$$

x_1, x_2, y_2, L_1, L_2, L_3 are fixed.

Goal: to find x, y, θ , given p_1, p_2, p_3 .

How to solve this problem?

Solving Equations S2

- The bisection method
- Fixed-point iteration
- Limits of accuracy
- Newton's method
- Root-finding without derivatives

The bisection method S1

- Definition:

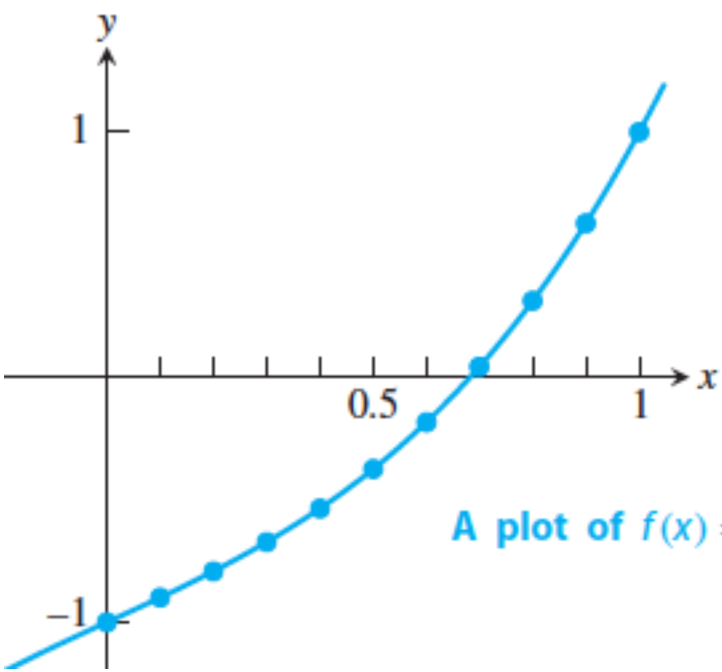
The function $f(x)$ has a **root** at $x = r$ if $f(r) = 0$.

- The first step to solving an equation is to verify that a root exists. One way to ensure this is to bracket the root: to find an interval **[**a,b**]** on the real line for which one of the pair {f(a),f(b)} is positive and the other is negative, i.e., $f(a)f(b) < 0$.

The bisection method S2

- Theorem:

Let f be a continuous function on $[a,b]$, satisfying $f(a)f(b)<0$. Then f has a root between a and b, that is, there exists a number r satisfying $a<r<b$ and $f(r)=0$. ■



A plot of $f(x)=x^3+x-1$. The function has a root between 0.6 and 0.7.

The bisection method S3

Bisection Method

Given initial interval $[a, b]$ such that $f(a)f(b) < 0$
while $(b - a)/2 > \text{TOL}$
  $c = (a + b)/2$
  if $f(c) = 0$, **stop**, **end**
  if $f(a)f(c) < 0$
    $b = c$
  else
    $a = c$
  end
end
The final interval $[a, b]$ contains a root.
The approximate root is $(a + b)/2$.

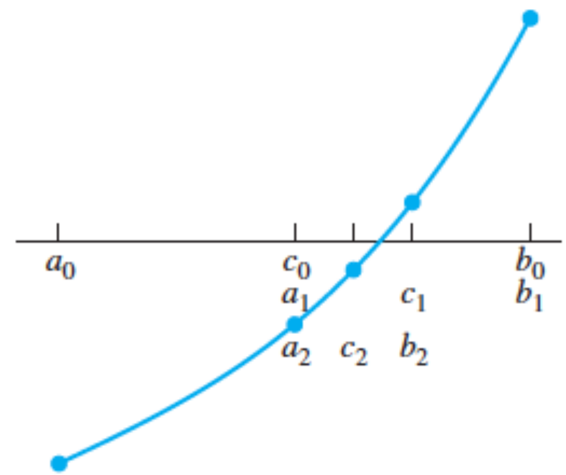


Figure 1.2 The Bisection Method. On the first step, the sign of $f(c_0)$ is checked. Since $f(c_0)f(b_0) < 0$, set $a_1 = c_0, b_1 = b_0$, and the interval is replaced by the right half $[a_1, b_1]$. On the second step, the subinterval is replaced by its left half $[a_2, b_2]$.

The bisection method S4

- EXAMPLE:

Find a root of the function $f(x) = x^3 + x - 1$ by using the Bisection Method on the interval $[0, 1]$.

We conclude from the table that the solution is bracketed between $a_9 \approx$ 0. 6816 and $c_9 \approx$0. 6826. The midpoint of that interval $c_{10} \approx$ 0. 6821 is our best guess for the root.
Although the problem was to find a root, what we have actually found is an interval [0. 6816, 0. 6826] that contains a root.

i	a_i	$f(a_i)$	c_i	$f(c_i)$	b_i	$f(b_i)$
0	0.0000	−	0.5000	−	1.0000	+
1	0.5000	−	0.7500	+	1.0000	+
2	0.5000	−	0.6250	−	0.7500	+
3	0.6250	−	0.6875	+	0.7500	+
4	0.6250	−	0.6562	−	0.6875	+
5	0.6562	−	0.6719	−	0.6875	+
6	0.6719	−	0.6797	−	0.6875	+
7	0.6797	−	0.6836	+	0.6875	+
8	0.6797	−	0.6816	−	0.6836	+
9	0.6816	−	0.6826	+	0.6836	+

[BisectExample.m]

The bisection method S5

If $[a, b]$ is the starting interval, then after n bisection steps, the interval $[a_n, b_n]$ has length $(b-a)/2^n$. Choosing the midpoint $x_c = (a_n + b_n)/2$ gives a best estimate of the solution r, which is within half the interval length of the true solution. Summarizing, after n steps of the Bisection Method, we find that

$$\text{Solution error} = |x_c - r| < \frac{b-a}{2^{n+1}} \tag{1.1}$$

and

$$\text{Function evaluations} = n + 2. \tag{1.2}$$

- Definition

A solution is **correct within p decimal places** if the error is less than 0.5×10^{-p}.

The bisection method S6

- EXAMPLE

Use the Bisection Method to find a root of $f(x) = \cos x - x$ in the interval [0, 1] to within six correct places.

First we decide how many steps of bisection are required. According to (1.1), the error after n steps is $(b-a)/2^{n+1} = 1/2^{n+1}$. From the definition of p decimal places, we require that

$$\frac{1}{2^{n+1}} < 0.5 \times 10^{-6}$$

$$n > \frac{6}{\log_{10} 2} \approx \frac{6}{0.301} = 19.9.$$

Therefore, $n = 20$ steps will be needed. Proceeding with the Bisection Method, the following table is produced:

The bisection method S7

k	a_k	$f(a_k)$	c_k	$f(c_k)$	b_k	$f(b_k)$
0	0.000000	+	0.500000	+	1.000000	−
1	0.500000	+	0.750000	−	1.000000	−
2	0.500000	+	0.625000	+	0.750000	−
3	0.625000	+	0.687500	+	0.750000	−
4	0.687500	+	0.718750	+	0.750000	−
5	0.718750	+	0.734375	+	0.750000	−
6	0.734375	+	0.742188	−	0.750000	−
7	0.734375	+	0.738281	+	0.742188	−
8	0.738281	+	0.740234	−	0.742188	−
9	0.738281	+	0.739258	−	0.740234	−
10	0.738281	+	0.738770	+	0.739258	−
11	0.738769	+	0.739014	+	0.739258	−
12	0.739013	+	0.739136	−	0.739258	−
13	0.739013	+	0.739075	+	0.739136	−
14	0.739074	+	0.739105	−	0.739136	−
15	0.739074	+	0.739090	−	0.739105	−
16	0.739074	+	0.739082	+	0.739090	−
17	0.739082	+	0.739086	−	0.739090	−
18	0.739082	+	0.739084	+	0.739086	−
19	0.739084	+	0.739085	−	0.739086	−
20	0.739084	+	0.739085	−	0.739085	−

From the table, the approximate root to six correct places is 0. 739085.

Coding assignment (page 30, Computer Problems 7)

Fixed-point iteration S1

- What is the fixed point of a function?

FixedpointExample.m

Definition:

The real number r is a **fixed point** of the function g if $g(r) = r$.

Fixed-Point Iteration

$$x_0 = \text{initial guess}$$
$$x_{i+1} = g(x_i) \textbf{ for } i = 0, 1, 2, \ldots$$

The sequence $\mathbf{x_i}$ may or may not converge as the number of steps goes to infinity.

$$g(r) = g\left(\lim_{i\to\infty} x_i\right) = \lim_{i\to\infty} g(x_i) = \lim_{i\to\infty} x_{i+1} = r.$$

Fixed-point iteration S2

- Can every equation f (x) = 0 be turned into a fixed-point problem g(x) = x ? Yes, and in many different ways. For instance, in the

$$x^3 + x - 1 = 0 \Rightarrow \quad x = 1 - x^3, \quad x = \sqrt[3]{1 - x}, \quad x = \frac{1 + 2x^3}{1 + 3x^2}$$

Fixed-point iteration S3

$$g(x) = 1 - x^3$$

i	x_i
0	0.50000000
1	0.87500000
2	0.33007813
3	0.96403747
4	0.10405419
5	0.99887338
6	0.00337606
7	0.99999996
8	0.00000012
9	1.00000000
10	0.00000000
11	1.00000000
12	0.00000000

$$g(x) = \sqrt[3]{1 - x}.$$

i	x_i
0	0.50000000
1	0.79370053
2	0.59088011
3	0.74236393
4	0.63631020
5	0.71380081
6	0.65900615
7	0.69863261
8	0.67044850
9	0.69072912
10	0.67625892
11	0.68664554
12	0.67922234

i	x_i
13	0.68454401
14	0.68073737
15	0.68346460
16	0.68151292
17	0.68291073
18	0.68191019
19	0.68262667
20	0.68211376
21	0.68248102
22	0.68221809
23	0.68240635
24	0.68227157
25	0.68236807

$$g(x) = \frac{1 + 2x^3}{1 + 3x^2}$$

i	x_i
0	0.50000000
1	0.71428571
2	0.68317972
3	0.68232842
4	0.68232780
5	0.68232780
6	0.68232780
7	0.68232780

Fixed-point iteration S4

cobweb diagram

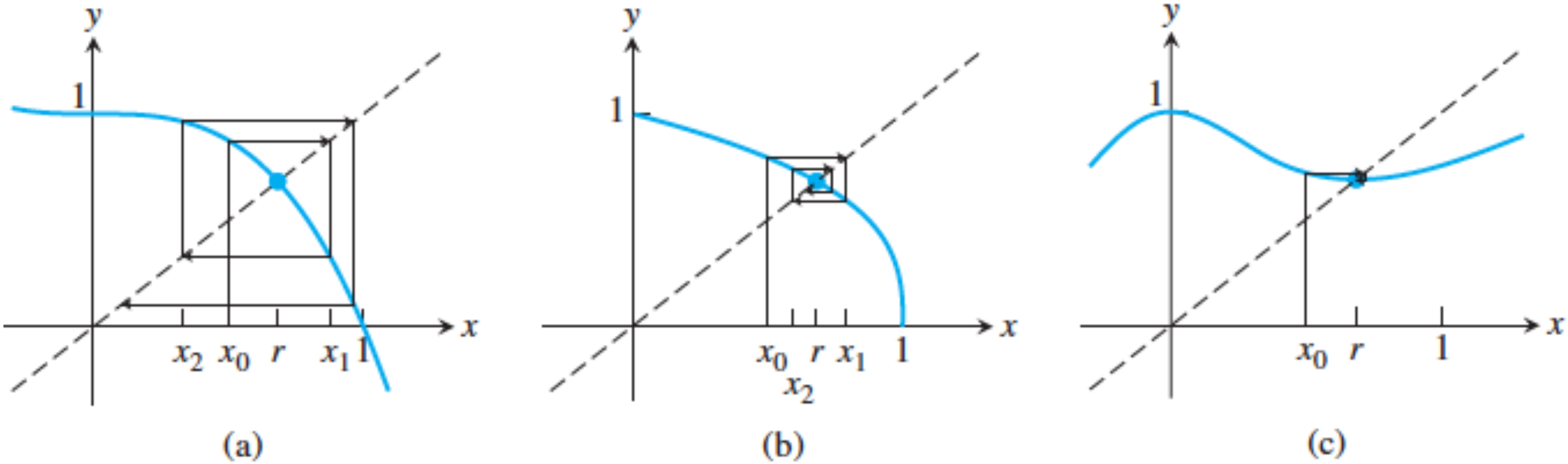


Figure 1.3 Geometric view of FPI. The fixed point is the intersection of $g(x)$ and the diagonal line. Three examples of $g(x)$ are shown together with the first few steps of FPI. (a) $g(x) = 1 - x^3$ (b) $g(x) = (1-x)^{1/3}$ (c) $g(x) = (1+2x^3)/(1+3x^2)$

Fixed-point iteration S5

$$g_1(x) = -\frac{3}{2}x + \frac{5}{2}$$

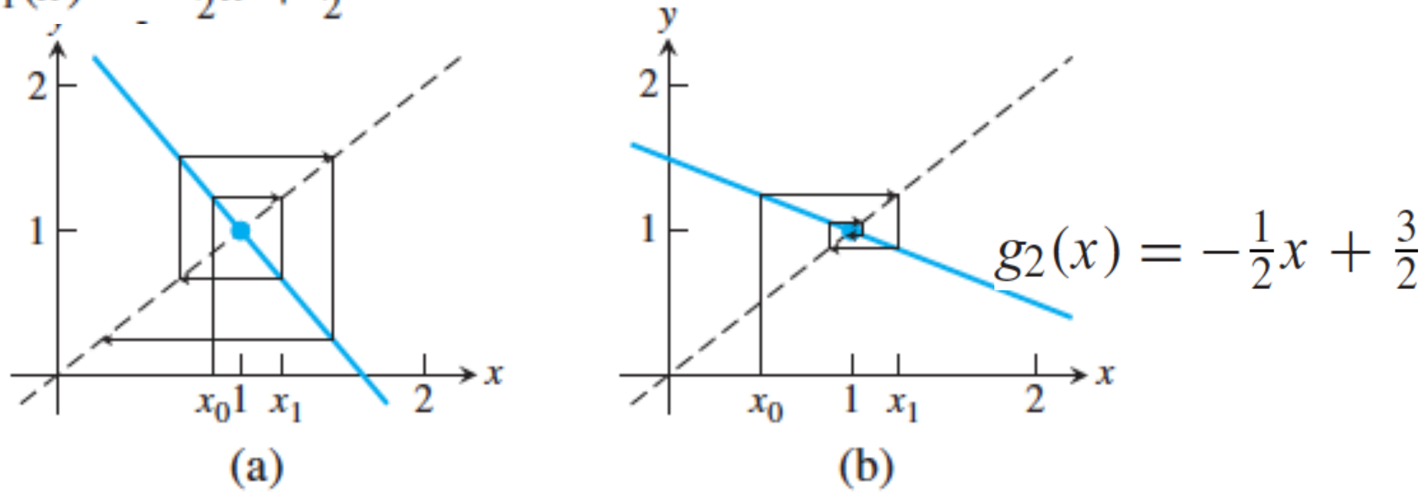


Figure 1.4 Cobweb diagram for linear functions. (a) If the linear function has slope greater than one in absolute value, nearby guesses move farther from the fixed point as FPI progresses, leading to failure of the method. (b) For slope less than one in absolute value, the reverse happens, and the fixed point is found.

Fixed-point iteration S6

- Definition:

Let e_i denote the error at step i of an iterative method. If

$$\lim_{i\to\infty} \frac{e_{i+1}}{e_i} = S < 1,$$

the method is said to obey **linear convergence** with rate S.

- Theorem:

Assume that g is continuously differentiable, that $g(r) = r$, and that $S = |g'(r)| < 1$. Then Fixed-Point Iteration converges linearly with rate S to the fixed point r for initial guesses sufficiently close to r. ■

Fixed-point iteration S7

- Definition:

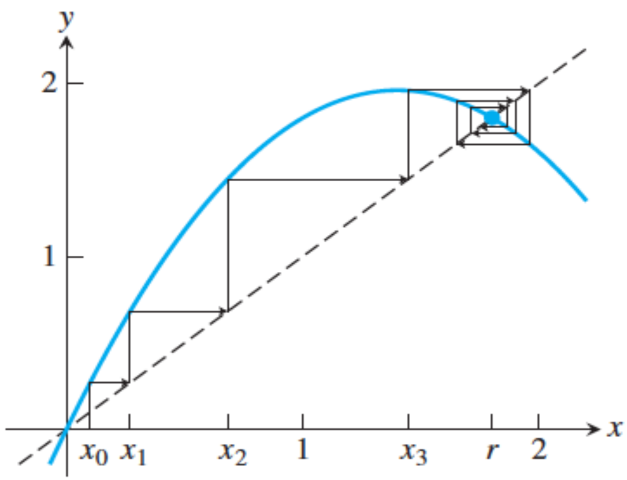
An iterative method is called **locally convergent** to r if the method converges to r for initial guesses sufficiently close to r. □

- According to the previous theorem, Fixed-Point Iteration is locally convergent if |g'(r)| < 1.

$g(x) = 1 - x^3$	$g(x) = \sqrt[3]{1-x}$	$g(x) = (1+2x^3)/(1+3x^2)$
$g'(x) = -3x^2$	$g'(x) = 1/3(1-x)^{-2/3}(-1)$	$g'(x) = \dfrac{6x(x^3+x-1)}{(1+3x^2)^2}$
$\lvert g'(r)\rvert \approx 1.3966 > 1$,	$\approx 0.716 < 1$	$= 0$

Fixed-point iteration S8

- EXAMPLE: Find the fixed points of $g(x) = 2.8x - x^2$.



$x_0 = 0.1000$

$x_1 = 0.2700$

$x_2 = 0.6831$

$x_3 = 1.4461$

$x_4 = 1.9579$

Figure 1.5 Cobweb diagram for Fixed-Point Iteration. Example 1.5 has two fixed points, 0 and 1.8. An iteration with starting guess 0.1 is shown. Only 1.8 will be converged to by FPI.

Fixed-point iteration S9

- EXAMPLE: Calculate $\sqrt{2}$ by using FPI.

  [FpiSqrt2Example.m]

1.000000000000000
1.500000000000000
1.416666666666667
1.414215686274510
1.414213562374690
1.414213562373095
1.414213562373095
1.414213562373095
1.414213562373095

$$x_{i+1} = \frac{x_i + \frac{2}{x_i}}{2}$$

$$g'(\sqrt{2}) = \frac{1}{2}\left(1 - \frac{2}{(\sqrt{2})^2}\right) = 0$$

So, the FPI will converge, and very fast.

Fixed-point iteration S10

- Stopping criteria

1. Absolute error stopping criterion

$$|x_{i+1} - x_i| < \mathrm{TOL}$$

2. Relative error stopping criterion

$$\frac{|x_{i+1} - x_i|}{|x_{i+1}|} < \mathrm{TOL}$$

3. Hybrid absolute/relative stopping criterion

$$\frac{|x_{i+1} - x_i|}{\max(|x_{i+1}|, \theta)} < \mathrm{TOL}$$

Coding assignment (page 43, Computer Problems 2)

Apply Fixed-Point Iteration to find the solution of each equation to eight correct decimal places. (a) $x^5 + x = 1$ (b) $\sin x = 6x + 5$ (c) $\ln x + x^2 = 3$

Newton’s method S1

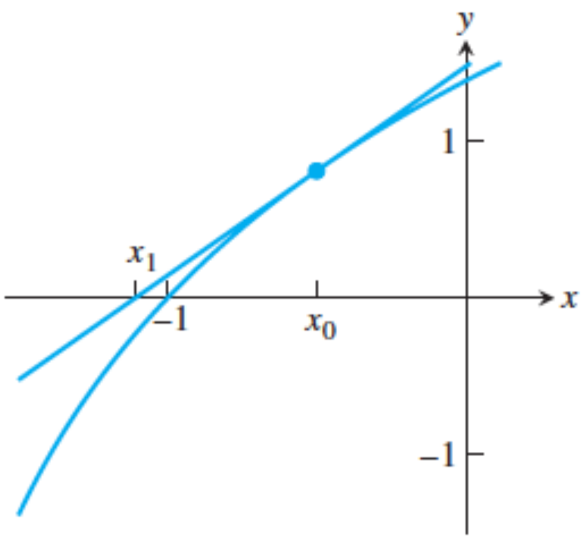


Figure 1.8 One step of Newton’s Method. Starting with x_0, the tangent line to the curve $y = f(x)$ is drawn. The intersection point with the x-axis is x_1, the next approximation to the root.

$$x_0 = \text{initial guess}$$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \quad \textbf{for} \quad i = 0, 1, 2, \ldots .$$

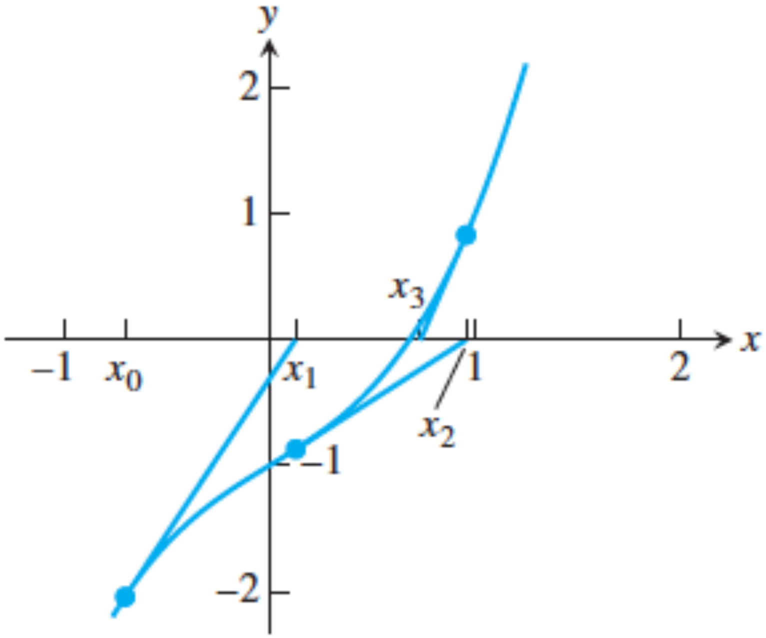
Newton's method S2

- EXAMPLE: Find the Newton's Method formula for the equation $x^3 + x - 1 = 0$.

Since $f'(x) = 3x^2 + 1$, the formula is given by

$$x_{i+1} = x_i - \frac{x_i^3 + x_i - 1}{3x_i^2 + 1} = \frac{2x_i^3 + 1}{3x_i^2 + 1}.$$

i	x_i	$e_i = \lvert x_i - r \rvert$	e_i / e_{i-1}^2
0	−0.70000000	1.38232780	
1	0.12712551	0.55520230	0.2906
2	0.95767812	0.27535032	0.8933
3	0.73482779	0.05249999	0.6924
4	0.68459177	0.00226397	0.8214
5	0.68233217	0.00000437	0.8527
6	0.68232780	0.00000000	0.8541
7	0.68232780	0.00000000	



Newton's method S3

- Definition:

Let e_i denote the error after step i of an iterative method. The iteration is **quadratically convergent** if

$$M = \lim_{i\to\infty} \frac{e_{i+1}}{e_i^2} < \infty. \quad \square$$

- Theorem:

Let f be twice continuously differentiable and $f(r) = 0$. If $f'(r) \neq 0$, then Newton's Method is locally and quadratically convergent to r. The error e_i at step i satisfies

$$\lim_{i\to\infty} \frac{e_{i+1}}{e_i^2} = M,$$

where

$$M = \frac{f''(r)}{2f'(r)}. \quad \blacksquare$$

Newton's method S4

- EXAMPLE: Use Newton's Method to find a root of $f(x) = x^2$.

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} = x_i - \frac{x_i^2}{2x_i} = \frac{x_i}{2}.$$

i	x_i	$e_i = \lvert x_i - r \rvert$	e_i / e_{i-1}
0	1.000	1.000	
1	0.500	0.500	0.500
2	0.250	0.250	0.500
3	0.125	0.125	0.500
$\vdots$	$\vdots$	$\vdots$	$\vdots$

- Similarly $f(x) = x^m$

The Newton formula is
$$x_{i+1} = x_i - \frac{x_i^m}{m x_i^{m-1}} = \frac{m-1}{m} x_i.$$

Newton's method S5

- Theorem:

Assume that the $(m+1)$-times continuously differentiable function f on $[a, b]$ has a multiplicity m root at r. Then Newton's Method is locally convergent to r, and the error e_i at step i satisfies

$$\lim_{i \to \infty} \frac{e_{i+1}}{e_i} = S, \tag{1.29}$$

where $S = (m-1)/m$. ■

- EXAMPLE:

Find the multiplicity of the root $r = 0$ of $f(x) = \sin x + x^2 \cos x - x^2 - x$, and estimate the number of steps of Newton's Method required to converge within six correct places (use $x_0 = 1$).

Newton's method S6

$$\left.\begin{aligned} f(x) &= \sin x + x^2\cos x - x^2 - x \\ f'(x) &= \cos x + 2x\cos x - x^2\sin x - 2x - 1 \\ f''(x) &= -\sin x + 2\cos x - 4x\sin x - x^2\cos x - 2 \end{aligned}\right\} \text{each evaluates to 0 at } r = 0$$

$$f'''(x) = -\cos x - 6\sin x - 6x\cos x + x^2\sin x \qquad f'''(0) = -1$$

so the root $r = 0$ is a triple root, meaning that the multiplicity is $m = 3$

Newton should converge linearly with $e_{i+1} \approx 2e_i/3$.

Therefore, a rough approximation to the number of steps needed to get the error within six decimal places, or smaller than 0.5×10^{-6}, can be found by solving

$$\left(\frac{2}{3}\right)^n < 0.5 \times 10^{-6}$$

$$n > \frac{\log_{10}(.5) - 6}{\log_{10}(2/3)} \approx 35.78.$$

Approximately 36 steps will be needed.

Newton's method S7

- Theorem:

If f is $(m+1)$-times continuously differentiable on $[a,b]$, which contains a root r of multiplicity $m > 1$, then **Modified Newton's Method**

$$x_{i+1} = x_i - \frac{m f(x_i)}{f'(x_i)} \tag{1.32}$$

converges locally and quadratically to r. ■

Returning to the previous example, we can apply Modified Newton's Method to achieve quadratic convergence. After five steps, convergence to the root r = 0 has taken place to about eight digits of accuracy.

i	x_i
0	1.00000000000000
1	0.16477071958224
2	0.01620733771144
3	0.00024654143774
4	0.00000006072272
5	−0.00000000633250

Newton's method S8

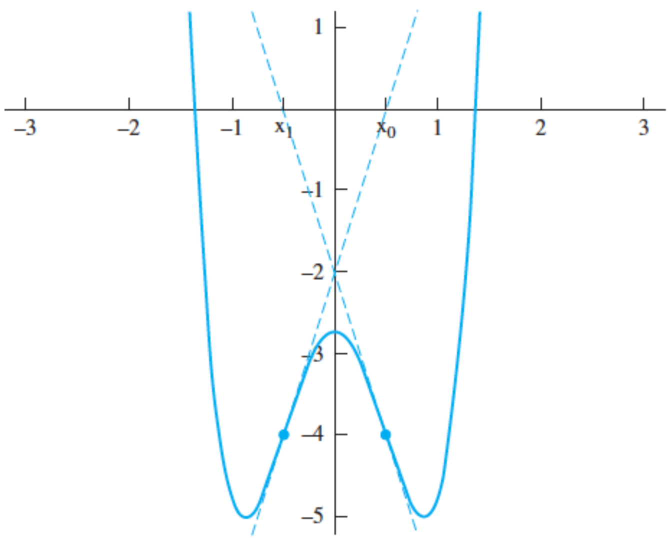
- EXAMPLE:

Apply Newton's Method to $f(x) = 4x^4 - 6x^2 - 11/4$ with starting guess $x_0 = 1/2$

The Newton formula is

$$x_{i+1} = x_i - \frac{4x_i^4 - 6x_i^2 - \frac{11}{4}}{16x_i^3 - 12x_i}.$$

Substitution gives $x_1 = -1/2$, and then $x_2 = 1/2$ again. Newton's Method alternates on this example between the two nonroots $1/2$ and $-1/2$, and fails to find a root.



Coding assignment (page 60, Computer Problems 13)

Root-finding without derivatives S1

- Why Newton's method converges at a faster rate than the bisection and fixed-point iteration methods?
- Because it uses the function's derivative, which may be unavailable in some cases.
- The Secant Method is a good substitute for Newton's Method in this case, which replaces the derivative with an approximation called the secant line, and converges almost as quickly.

Root-finding without derivatives S2

- The Secant Method is similar to the Newton's Method, but replaces the derivative by a difference quotient $\dfrac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}$

Secant Method

$$x_0, x_1 = \text{initial guesses}$$
$$x_{i+1} = x_i - \frac{f(x_i)(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})} \textbf{ for } i = 1, 2, 3, \ldots.$$

- Unlike Fixed-Point Iteration and Newton's Method, two starting guesses are needed to begin the Secant Method.

superlinear convergence

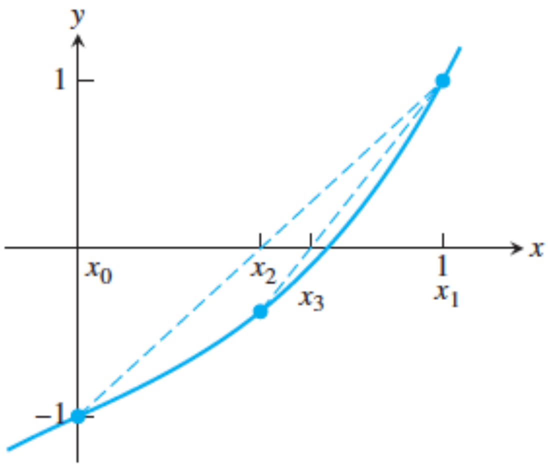
Root-finding without derivatives S3

- EXAMPLE:

Apply the Secant Method with starting guesses $x_0 = 0, x_1 = 1$ to find the root of $f(x) = x^3 + x - 1$.

The formula gives

$$x_{i+1} = x_i - \frac{(x_i^3 + x_i - 1)(x_i - x_{i-1})}{x_i^3 + x_i - (x_{i-1}^3 + x_{i-1})}.$$



i	x_i
0	0.00000000000000
1	1.00000000000000
2	0.50000000000000
3	0.63636363636364
4	0.69005235602094
5	0.68202041964819
6	0.68232578140989
7	0.68232780435903
8	0.68232780382802
9	0.68232780382802

Starting with $\mathbf{x}_0 = 0$ and $\mathbf{x}_1 = 1$, the Secant Method iterates are plotted along with the secant lines.

Coding assignment (page 66, Computer Problems 1)

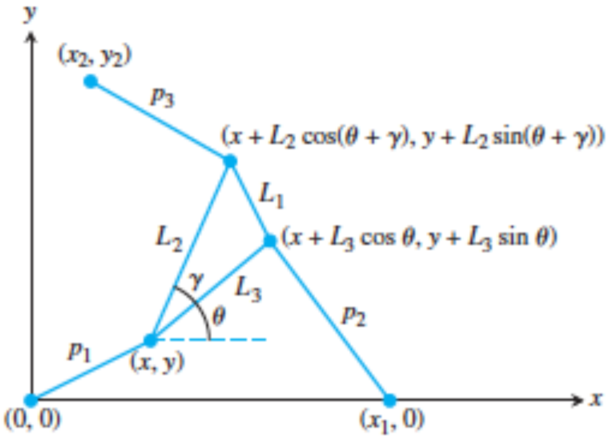
Comparison

- To find the root of $x^3 + x - 1$ with 4 correct digits, e.g., 0.6823.

Methods		Iteration number	Side information	Convergence
Bisection method		9	No	Linearly
FPI	$g(x) = \sqrt[3]{1-x}$	25	No	Linearly
	$g(x) = (1+2x^3)/(1+3x^2)$	3	No	Linearly
Newton's method		5	Derivative	Quadratically
Secant method		6	No	Superlinearly

How to apply numerical analysis? S1

- Reality check: **forward problem of the Stewart platform**



$$
\begin{aligned}
p_1^2 &= x^2 + y^2 \\
p_2^2 &= (x + A_2)^2 + (y + B_2)^2 \\
p_3^2 &= (x + A_3)^2 + (y + B_3)^2.
\end{aligned}
\tag{1.38}
$$

x_1, x_2, y_2, L_1, L_2, L_3 are fixed.

In these equations,

$$
\begin{aligned}
A_2 &= L_3\cos\theta - x_1 \\
B_2 &= L_3\sin\theta \\
A_3 &= L_2\cos(\theta + \gamma) - x_2 = L_2[\cos\theta\cos\gamma - \sin\theta\sin\gamma] - x_2 \\
B_3 &= L_2\sin(\theta + \gamma) - y_2 = L_2[\cos\theta\sin\gamma + \sin\theta\cos\gamma] - y_2.
\end{aligned}
$$

Goal: to find x, y, θ , given p_1, p_2, p_3 .

How to solve this problem?

How to apply numerical analysis? S2

- EXAMPLE:

Multiplying out the last two equations of (1.38) and using the first yields

$$p_2^2 = x^2 + y^2 + 2A_2x + 2B_2y + A_2^2 + B_2^2 = p_1^2 + 2A_2x + 2B_2y + A_2^2 + B_2^2$$
$$p_3^2 = x^2 + y^2 + 2A_3x + 2B_3y + A_3^2 + B_3^2 = p_1^2 + 2A_3x + 2B_3y + A_3^2 + B_3^2$$

which can be solved for x and y as

$$x = \frac{N_1}{D} = \frac{B_3(p_2^2 - p_1^2 - A_2^2 - B_2^2) - B_2(p_3^2 - p_1^2 - A_3^2 - B_3^2)}{2(A_2B_3 - B_2A_3)}$$
$$y = \frac{N_2}{D} = \frac{-A_3(p_2^2 - p_1^2 - A_2^2 - B_2^2) + A_2(p_3^2 - p_1^2 - A_3^2 - B_3^2)}{2(A_2B_3 - B_2A_3)}, \qquad (1.39)$$

as long as $D = 2(A_2B_3 - B_2A_3) \neq 0$.

How to apply numerical analysis? S3

Substituting these expressions for x and y into the first equation of (1.38), and multiplying through by D^2, yields one equation, namely,

$$f = N_1^2 + N_2^2 - p_1^2 D^2 = 0 \tag{1.40}$$

in the single unknown θ. (Recall that $p_1, p_2, p_3, L_1, L_2, L_3, \gamma, x_1, x_2, y_2$ are known.) If the roots of $f(\theta)$ can be found, the corresponding x- and y- values follow immediately from (1.39).

Coding assignment (page 68, Suggested activities 1)

Write a MATLAB function file for $f(\theta)$. The parameters $L_1, L_2, L_3, \gamma, x_1, x_2, y_2$ are fixed constants, and the strut lengths p_1, p_2, p_3 will be known for a given pose. Check Appendix B.5 if you are new to MATLAB function files. Here, for free, are the first and last lines:

```
function out=f(theta)
       :
       :
out=N1^2+N2^2-p1^2*D^2;
```

To test your code, set the parameters $L_1 = 2, L_2 = L_3 = \sqrt{2}, \gamma = \pi/2, p_1 = p_2 = p_3 = \sqrt{5}$ from Figure 1.15. Then, substituting $\theta = -\pi/4$ or $\theta = \pi/4$, corresponding to Figures 1.15(a, b), respectively, should make $f(\theta) = 0$.