习题 3.1

求下列不定积分：

$$1.\int\sqrt{1+2x}dx=\frac{1}{2}\int\sqrt{1+2x}d(1+2x)=\frac{1}{3}(1+2x)^{3/2}+C.$$

$$2.\int\frac{3x}{(x^2+1)^2}dx=\frac{1}{2}\int\frac{3}{(x^2+1)^2}d(x^2+1)=-\frac{3}{2(x^2+1)}+C.$$

$$3.\int x\sqrt{2x^2+7}dx=\frac{1}{4}\int\sqrt{2x^2+7}d(2x^2+7)=\frac{1}{6}(2x^2+7)^{3/2}+C.$$

$$4.\int(2x^{3/2}+1)^{2/3}\sqrt{x}dx=\frac{2}{3}\int(2x^{3/2}+1)^{2/3}dx^{3/2}$$

$$=\frac{2}{3}\cdot\frac{1}{2}\int(2x^{3/2}+1)^{2/3}d(2x^{3/2}+1)=\frac{1}{5}(2x^{3/2}+1)^{5/3}+C.$$

$$5.\int\frac{e^{1/x}}{x^2}dx=-\int e^{1/x}d(1/x)=-e^{1/x}+C.$$

$$6.\int\frac{dx}{(2-x)^{100}}=-\int\frac{d(2-x)}{(2-x)^{100}}=\frac{1}{99(2-x)^{99}}+C.$$

$$7.\int\frac{dx}{3+5x^2}=\frac{1}{3}\int\frac{dx}{1+[(5/3)x]^2}=\frac{1}{3}\cdot\sqrt{\frac{3}{5}}\int\frac{d\sqrt{5/3}x}{1+[\sqrt{5/3}x]^2}=\frac{1}{\sqrt{15}}\arctan\sqrt{\frac{5}{3}}x+C.$$

$$8.\int\frac{dx}{\sqrt{7-3x^2}}=\int\frac{dx}{\sqrt{7}\sqrt{1-3/7x^2}}=\frac{1}{\sqrt{7}}\cdot\sqrt{\frac{7}{3}}\int\frac{d\sqrt{3/7}x}{\sqrt{7}\sqrt{1-\sqrt{3/7}x^2}}=\frac{1}{\sqrt{3}}\arcsin\sqrt{\frac{3}{7}}x+C.$$

$$9.\int\frac{dx}{\sqrt{x}(1+x)}=2\int\frac{d\sqrt{x}}{(1+x)}=2\arctan\sqrt{x}+C.$$

$$10.\int\frac{e^x}{2+e^{2x}}dx=\int\frac{1}{2+\left(e^x\right)^2}de^x=\frac{1}{\sqrt{2}}\arctan e^x+C.$$

$$11.\int\frac{dx}{\sqrt{e^{-2x}-1}}=\int\frac{de^x}{\sqrt{1-(e^x)^2}}=\arcsin e^x+C.$$

$$12.\int\frac{dx}{e^x-e^{-x}}=\int\frac{de^x}{e^{2x}-1}=\int\frac{du}{(u-1)(u+1)}=\frac{1}{2}\int\left(\frac{1}{u-1}-\frac{1}{u+1}\right)du$$

$$=\frac{1}{2}\ln\frac{u-1}{u+1}+C=\frac{1}{2}\ln\frac{e^x-1}{e^x+1}+C.$$

$$13.\int\frac{\ln\ln x}{x\ln x}dx=\int\frac{\ln\ln x}{\ln x}d\ln x=\int\ln\ln xd\ln\ln x=\frac{1}{2}(\ln\ln x)^2+C.$$

$$14.\int\frac{dx}{1+\cos x}=\int\frac{dx}{2\sin^2\frac{x}{2}}=\int\frac{d\frac{x}{2}}{\sin^2\frac{x}{2}}=-\cot^2\frac{x}{2}+C.$$

$$15.\int\frac{dx}{1-\sin x}=\int\frac{d\left(x+\frac{\pi}{2}\right)}{1+\cos\left(x+\frac{\pi}{2}\right)}=-\cot^2\left(\frac{x}{2}+\frac{\pi}{4}\right)+C.$$

$$16.\int\frac{x^{14}}{(x^5+1)^4}dx=\frac{1}{5}\int\frac{x^{10}}{(x^5+1)^4}dx^5=\frac{1}{5}\int\frac{u^2}{(u+1)^4}du(u=x^5)$$

$$=\frac{1}{5}\int\frac{u^2-1+1}{(u+1)^4}du=\frac{1}{5}\int\frac{(v-1)^2}{v^4}dv(v=u+1)$$

$$=\frac{1}{5}\int\frac{v^2-2v+1}{v^4}dv=\frac{1}{5}\int\left(v^{-2}-2v^{-3}+v^{-4}\right)dv$$

$$=\frac{1}{5}\left(-v^{-1}+v^{-2}-\frac{1}{3}v^{-3}\right)+C=\frac{1}{5}\left(-(x^5+1)^{-1}+(x^5+1)^{-2}-\frac{1}{3}(x^5+1)^{-3}\right)+C.$$

$$17.\int\frac{x^{2n-1}}{x^n-1}dx=\frac{1}{n}\int\frac{x^n}{x^n-1}dx^n=\frac{1}{n}\int\frac{u}{u-1}du(u=x^n)$$

$$=\frac{1}{n}\int\left(1+\frac{1}{u-1}\right)du=\frac{1}{n}(u+\ln|u-1|)+C=\frac{1}{n}(x^n+\ln|x^n-1|)+C.$$

$$18.\int\frac{dx}{x(x^5+2)}=\int\frac{x^4dx}{x^5(x^5+2)}=\frac{1}{5}\int\frac{du}{u(u+2)}(u=x^5)$$

$$=\frac{1}{5}\square\frac{1}{2}\int\left(\frac{1}{u}-\frac{1}{u+2}\right)du=\frac{1}{10}\left(\ln|u|-\ln|u+2|\right)+C=\frac{1}{10}\ln\left|\frac{u}{u+2}\right|+C.$$

$$19.\int\frac{\ln(x+1)-\ln x}{x(x+1)}dx=\int(\ln(x+1)-\ln x)\left(\frac{1}{x}-\frac{1}{x+1}\right)dx$$

$$=\int(\ln(x+1)-\ln x)d(\ln x-\ln(x+1)=-\int(\ln(x+1)-\ln x)d(\ln(x+1)-\ln x)$$

$$=-\frac{1}{2}\ln^2\frac{x+1}{x}+C.$$

$$20.\int\frac{e^{\arctan x}+x\ln(1+x^2)}{1+x^2}dx=\int\frac{e^{\arctan x}}{1+x^2}dx+\int\frac{x\ln(1+x^2)}{1+x^2}dx$$

$$=\int e^{\arctan x}d\arctan x+\frac{1}{2}\int\ln(1+x^2)d\ln(1+x^2)$$

$$=e^{\arctan x}+\frac{1}{4}\ln^2(1+x^2)+C.$$

$$21.\int \sin 2x\cos 2x dx=\frac{1}{2}\int \sin 2x d\sin 2x=\frac{1}{4}\sin^2 2x+C.$$

$$22.\int \sin^2\frac{x}{2}\cos\frac{x}{2}dx=2\int \sin^2\frac{x}{2}d\sin\frac{x}{2}=\frac{2}{3}\sin^3\frac{x}{2}+C.$$

$$23.\int \sin 5x\sin 6x dx=\frac{1}{2}\int(\cos x-\cos 11x)dx=\frac{1}{2}\left(\sin x-\frac{1}{11}\sin 11x\right)+C.$$

$$24.\int\frac{2x-1}{\sqrt{1-x^2}}dx=\int\frac{2x}{\sqrt{1-x^2}}dx-\int\frac{1}{\sqrt{1-x^2}}dx$$

$$=-\int\frac{d(1-x^2)}{\sqrt{1-x^2}}-\arcsin x+C=-2\sqrt{1-x^2}-\arcsin x+C.$$

$$25.\int\frac{x^3+x}{\sqrt{1-x^2}}dx=\int\frac{x^3}{\sqrt{1-x^2}}dx+\int\frac{x}{\sqrt{1-x^2}}dx$$

$$=\frac{1}{2}\int\frac{x^2}{\sqrt{1-x^2}}dx^2-\frac{1}{2}\int\frac{d(1-x^2)}{\sqrt{1-x^2}}$$

$$=\frac{1}{2}\int\sqrt{1-x^2}d(1-x^2)+\frac{1}{2}\int\frac{1}{\sqrt{1-x^2}}dx^2-\sqrt{1-x^2}$$

$$=\frac{1}{3}(1-x^2)^{3/2}-2\sqrt{1-x^2}+C.$$

$$26.\int\frac{dx}{(a^2-x^2)^{3/2}}(a>0)$$

$$x=a\sin t, t\in(-\pi/2,\pi/2), dx=a\cos t dt,$$

$$(a^2-x^2)^{3/2}=a^3\cos^3 t,$$

$$\int\frac{dx}{(a^2-x^2)^{3/2}}=\int\frac{dt}{a^2\cos^2 t}dx=\frac{1}{a^2}\tan t+C$$

$$=\frac{1}{a^2}\frac{x/a}{\sqrt{1-(x/a)^2}}+C=\frac{x}{a^2\sqrt{a^2-x^2}}+C.$$

$x<0$时, 令$x=-y, y>0,$

$$\int\frac{\sqrt{x^2-a^2}}{x}dx=\int\frac{\sqrt{y^2-a^2}}{y}dy=\sqrt{y^2-a^2}-a\arccos\frac{a}{y}+C$$

$$=\sqrt{x^2-a^2}-a\arccos\frac{a}{-x}+C=\sqrt{x^2-a^2}-\left(\pi-a\arccos\frac{a}{x}\right)+C$$

$$=\sqrt{x^2-a^2}+a\arccos\frac{a}{x}+C'.$$

$$27.\int\frac{\sqrt{x^2-a^2}}{x}dx(a>0).x>0\text{时},\text{令}x=a\sec t,t\in(0,\pi/2).$$

$$dx=a\tan t\sec t dt,\sqrt{x^2-a^2}=a\tan t,$$

$$\int\frac{\sqrt{x^2-a^2}}{x}dx=a\int\tan^2 tdt=a\int(\sec^2 t-1)dt=a(\tan t-t)+C$$

$$=a(\sqrt{\sec^2 t-1}-\arccos\frac{a}{x})+C=a(\sqrt{\left(\frac{x}{a}\right)^2-1}-\arccos\frac{a}{x})+C$$

$$=\sqrt{x^2-a^2}-a\arccos\frac{a}{x}+C.$$

$$28.\int\frac{x^2}{\sqrt{a^2-x^2}}dx=-\int\sqrt{a^2-x^2}dx+a^2\int\frac{dx}{\sqrt{a^2-x^2}}$$

$$=-\frac{a^2}{2}\arcsin\frac{x}{a}-\frac{1}{2}x\sqrt{a^2-x^2}+a^2\arcsin\frac{x}{a}+C$$

$$=\frac{a^2}{2}\arcsin\frac{x}{a}-\frac{1}{2}x\sqrt{a^2-x^2}+C.$$

$$29.\int\frac{dx}{\sqrt{1+e^{3x}}}=\int\frac{e^{-3x/2}dx}{\sqrt{1+e^{-3x}}}=-\frac{2}{3}\int\frac{de^{-3x/2}}{\sqrt{1+e^{-3x}}}=-\frac{2}{3}\ln(e^{-3x/2}+\sqrt{1+e^{-3x}})+C$$

$$=-\frac{2}{3}\ln(1+\sqrt{1+e^{3x}})+x+C=-\frac{2}{3}\ln\frac{(\sqrt{1+e^{3x}}+1)(\sqrt{1+e^{3x}}-1)}{\sqrt{1+e^{3x}}-1}+x+C$$

$$=\frac{2}{3}\ln(\sqrt{1+e^{3x}}-1)-x+C.$$

$$30.\int\frac{x^3}{\sqrt{1+x^8}}dx=\frac{1}{4}\int\frac{dx^4}{\sqrt{1+x^8}}=\frac{1}{4}\int\frac{du}{\sqrt{1+u^2}}(u=x^4)$$

$$=\frac{1}{4}\ln(u+\sqrt{1+u^2})+C=\frac{1}{4}\ln(x^4+\sqrt{1+x^8})+C.$$

$$31.\int\frac{dx}{x^6\sqrt{1+x^2}}=\int\frac{dx}{x^7\sqrt{1+x^{-2}}}=-\frac{1}{2}\int\frac{dx^{-2}}{x^4\sqrt{1+x^{-2}}}=-\frac{1}{2}\int\frac{u^2du}{\sqrt{1+u}}(u=\frac{1}{x^2})$$

$$=-\frac{1}{2}\int\frac{(v-1)^2}{v^{1/2}}dv=-\frac{1}{2}\int\frac{v^2-2v+1}{v^{1/2}}dv(v=1+u)$$

$$=-\frac{1}{2}\int(v^{3/2}-2v^{1/2}+v^{-1/2})dx$$

$$=-\frac{1}{2}\left(\frac{2}{5}v^{\frac{5}{2}}-2\cdot\frac{2}{3}v^{\frac{3}{2}}+2\cdot v^{\frac{1}{2}}\right)$$

$$=-\frac{1}{5}\left(1+\frac{1}{x^2}\right)^{\frac{5}{2}}+\frac{2}{3}\left(1+\frac{1}{x^2}\right)^{\frac{3}{2}}-\left(1+\frac{1}{x^2}\right)^{\frac{1}{2}}+C$$

$$=-\frac{\sqrt{1+x^2}^5}{5x^5}+\frac{\sqrt{1+x^2}^3}{3x^3}-\frac{\sqrt{1+x^2}}{x}+C.$$

$$32.\int\frac{e^{2x}}{\sqrt[3]{1+e^x}}dx=\int\frac{e^x}{\sqrt[3]{1+e^x}}de^x=\int\frac{u}{\sqrt[3]{1+u}}du(u=e^x)(\sqrt[3]{u+1}=v,u=v^3-1)$$

$$=\int\frac{u}{\sqrt[3]{1+u}}du=\int\frac{v^3-1}{v}3v^2dv=3\int(v^4-v)dv=3\left(\frac{v^5}{5}-\frac{v^2}{2}\right)+C$$

$$=\frac{3}{5}(e^x+1)^{5/3}-\frac{3}{2}(e^x+1)^{2/3}+C.$$

$$33.\int\frac{dx}{\sqrt{3+x-x^2}}=\int\frac{dx}{\sqrt{3-\left(x-\frac{1}{2}\right)^2+\frac{1}{4}}}=\int\frac{d\left(x-\frac{1}{2}\right)}{\sqrt{\frac{13}{4}-\left(x-\frac{1}{2}\right)^2}}$$

$$=\arcsin\frac{x-\frac{1}{2}}{\frac{\sqrt{13}}{2}}+C=\arcsin\frac{2x-1}{\sqrt{13}}+C.$$

$$34.\int\sqrt{7+x-x^2}\,dx=\int\sqrt{7-\left(x-\frac{1}{2}\right)^2+\frac{1}{4}}dx=\int\sqrt{\frac{29}{4}-\left(x-\frac{1}{2}\right)^2}\,d\left(x-\frac{1}{2}\right)$$

$$=\frac{1}{2}\left(x-\frac{1}{2}\right)\sqrt{\frac{29}{4}-\left(x-\frac{1}{2}\right)^2}+\frac{29}{8}\arcsin\frac{x-\frac{1}{2}}{\frac{\sqrt{29}}{2}}+C$$

$$=\frac{2x-1}{4}\sqrt{7+x-x^2}+\frac{29}{8}\arcsin\frac{2x-1}{\sqrt{29}}+C.$$

$$35.\int\frac{dx}{1+\sqrt{x-1}},1+\sqrt{x-1}=u,x=1+(u-1)^2,dx=2(u-1)du,$$

$$\int\frac{dx}{1+\sqrt{x-1}},1+\sqrt{x-1}=u,x=1+(u-1)^2,dx=2(u-1)du,$$

$$\int\frac{dx}{1+\sqrt{x-1}}==\int\frac{2(u-1)du}{u}=2(u-\ln u)+C=2(1+\sqrt{x-1})-\ln(1+\sqrt{x-1}))+C$$

$$=2\sqrt{x-1}-\ln(1+\sqrt{x-1}))+C'.$$

习题 3.2

求下列不定积分：

$$1.\int x\ln xdx=\frac{1}{2}\int \ln xdx^2=\frac{x^2}{2}\ln x-\frac{1}{2}\int x^2 d\ln x$$

$$=\frac{x^2}{2}\ln x-\frac{1}{2}\int x^2\cdot\frac{1}{x}dx=\frac{x^2}{2}\ln x-\frac{1}{2}\int xdx=\frac{x^2}{2}\ln x-\frac{x^2}{4}+C.$$

$$2.\int x^2e^{ax}dx=\frac{1}{a}\int x^2de^{ax}=\frac{1}{a}x^2e^{ax}-\frac{1}{a}\int e^{ax}dx^2=\frac{1}{a}x^2e^{ax}-\frac{2}{a}\int xe^{ax}dx$$

$$=\frac{1}{a}x^2e^{ax}-\frac{2}{a^2}\int xde^{ax}=\frac{1}{a}x^2e^{ax}-\frac{2x}{a^2}e^{ax}+\frac{2}{a^2}\int e^{ax}dx$$

$$=\frac{1}{a}x^2e^{ax}-\frac{2}{a^2}\int xde^{ax}=\frac{1}{a}x^2e^{ax}-\frac{2x}{a^2}e^{ax}+\frac{2}{a^3}e^{ax}+C$$

$$=e^{ax}\left(\frac{1}{a}x^2-\frac{2x}{a^2}+\frac{2}{a^3}\right)+C.$$

$$3.\int x\sin 2xdx=-\frac{1}{2}\int xd\cos 2x=-\frac{1}{2}x\cos 2x+\frac{1}{2}\int \cos 2xdx$$

$$=-\frac{1}{2}x\cos 2x+\frac{1}{4}\sin 2x+C.$$

$$4.\int \arcsin xdx=x\arcsin x-\int xd\arcsin x=x\arcsin x-\int\frac{xdx}{\sqrt{1-x^2}}$$

$$=x\arcsin x+\frac{1}{2}\int\frac{d(1-x^2)}{\sqrt{1-x^2}}=x\arcsin x+\sqrt{1-x^2}+C.$$

$$5.\int \arctan xdx=x\arctan x-\int xd\arctan x=x\arctan x-\int\frac{xdx}{1+x^2}$$

$$=x\arctan x-\frac{1}{2}\int\frac{d(1+x^2)}{1+x^2}=x\arctan x-\frac{1}{2}\ln(1+x^2)+C.$$

$$6.I=\int e^{2x}\cos 3xdx=\frac{1}{2}\int\cos 3xde^{2x}=\frac{1}{2}e^{2x}\cos 3x-\frac{1}{2}\int e^{2x}d\cos 3x$$

$$=\frac{1}{2}e^{2x}\cos 3x+\frac{3}{2}\int e^{2x}\sin 3xdx=\frac{1}{2}e^{2x}\cos 3x+\frac{3}{4}\int\sin 3xde^{2x}$$

$$=\frac{1}{2}e^{2x}\cos 3x+\frac{3}{4}\left(e^{2x}\sin 3x-3\int e^{2x}\cos 3xdx\right)$$

$$=\frac{1}{2}e^{2x}\cos 3x+\frac{3}{4}e^{2x}\sin 3x-\frac{9}{4}I,$$

$$I=\frac{4}{13}\left(\frac{1}{2}\cos 3x+\frac{3}{4}\sin 3x\right)e^{2x}+C=\frac{1}{13}\left(2\cos 3x+3\sin 3x\right)e^{2x}+C.$$

$$7.I=\int\frac{\sin 3x}{e^x}dx=-\int\sin 3xde^{-x}=-e^{-x}\sin 3x+3\int e^{-x}\cos 3xdx$$

$$=-e^{-x}\sin 3x-3\int\cos 3xde^{-x}=-e^{-x}\sin 3x-3\left(e^{-x}\cos 3x+3\int e^{-x}\sin 3xdx\right)$$

$$= -e^{-x}\sin 3x - 3(e^{-x}\cos 3x + 3I),$$

$$I = \frac{1}{10}\left(-e^{-x}\sin 3x - 3e^{-x}\cos 3x\right) + C = -\frac{e^{-x}}{10}(\sin 3x + 3\cos 3x) + C.$$

$$8.I = \int e^{ax}\sin bxdx = \frac{1}{a}\int \sin bxde^{ax} = \frac{1}{a}e^{ax}\sin bx - \frac{b}{a}\int e^{ax}\cos bxdx$$

$$= \frac{1}{a}e^{ax}\sin bx - \frac{b}{a^2}\int \cos bxde^{ax}$$

$$= \frac{1}{a}e^{ax}\sin bx - \frac{b}{a^2}\left(e^{ax}\cos bx + b\int e^{ax}\sin bxdx\right)$$

$$= \frac{1}{a}e^{ax}\sin bx - \frac{b}{a^2}\left(e^{ax}\cos bx + bI\right).$$

$$I = \frac{1}{1+\frac{b^2}{a^2}}\left(\frac{1}{a}e^{ax}\sin bx - \frac{b}{a^2}e^{ax}\cos bx\right),$$

$$I = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + C.$$

$$9.I = \int \sqrt{1+9x^2}\,dx = x\sqrt{1+9x^2} - \int xd\sqrt{1+9x^2}$$

$$= x\sqrt{1+9x^2} - \int \frac{x \cdot 18xdx}{2\sqrt{1+9x^2}}$$

$$= x\sqrt{1+9x^2} - \left(\int \sqrt{1+9x^2}\,dx - \int \frac{dx}{\sqrt{1+9x^2}}\right)$$

$$= x\sqrt{1+9x^2} - \left(I - \int \frac{dx}{\sqrt{1+9x^2}}\right),$$

$$I = \frac{1}{2}x\sqrt{1+9x^2} + \frac{1}{2}\cdot\frac{1}{3}\ln(3x + \sqrt{1+9x^2}) + C$$

$$= \frac{1}{2}x\sqrt{1+9x^2} + \frac{1}{6}\ln(3x + \sqrt{1+9x^2}) + C.$$

$$10.\int x\cosh xdx = \int xd\sinh x = x\sinh x - \int \sinh xdx$$

$$= x\sinh x - \cosh x + C.$$

$$11.\int \ln(x + \sqrt{1+x^2})dx = x\ln(x+\sqrt{1+x^2}) - \int xd\ln(x+\sqrt{1+x^2})$$

$$= x\ln(x+\sqrt{1+x^2}) - \int \frac{xdx}{\sqrt{1+x^2}} = x\ln(x+\sqrt{1+x^2}) - \sqrt{1+x^2} + C.$$

$$12.\int (\arccos x)^2 dx = x(\arccos x)^2 + 2\int \frac{x\arccos x}{\sqrt{1-x^2}}dx$$

$$= x(\arccos x)^2 - 2\int \arccos xd\sqrt{1-x^2}$$

$$= x(\arccos x)^2 - 2\left(\sqrt{1-x^2}\arccos x + \int 1dx\right)$$

$$= x(\arccos x)^2 - 2\sqrt{1-x^2}\arccos x - 2x + C.$$

$$13.\int \frac{x\arccos x dx}{(1-x^2)^2} = \frac{1}{2}\int \arccos x d\frac{1}{1-x^2}$$

$$= \frac{\arccos x}{2(1-x^2)} + \frac{1}{2}\int \frac{dx}{(1-x^2)\sqrt{1-x^2}}$$

$$= \frac{\arccos x}{2(1-x^2)} + \frac{1}{2}\frac{x}{\sqrt{1-x^2}} + C.$$

$$14.\int \arctan\sqrt{x}dx = x\arctan\sqrt{x} - \int \frac{xdx}{2(1+x)\sqrt{x}}$$

$$= x\arctan\sqrt{x} - \frac{1}{2}\int \frac{\sqrt{x}dx}{1+x}.\sqrt{x} = u, x = u^2, dx = 2udu$$

$$\int \frac{\sqrt{x}dx}{1+x} = \int \frac{u2udu}{1+u^2} = 2(u - \arctan u) + C,$$

$$\int \arctan\sqrt{x}dx = x\arctan\sqrt{x} - \frac{1}{2}\cdot 2(\sqrt{x} - \arctan\sqrt{x}) + C$$

$$= x\arctan\sqrt{x} - (\sqrt{x} - \arctan\sqrt{x}) + C$$

$$= (x+1)\arctan\sqrt{x} - \sqrt{x} + C.$$

$$15.\int \frac{\arcsin x}{x^2}dx = -\int \arcsin x d\left(\frac{1}{x}\right) = -\frac{\arcsin x}{x} + \int \frac{dx}{x\sqrt{1-x^2}}$$

$$= -\frac{\arcsin x}{x} + \int \frac{dx}{x^2\sqrt{1/x^2-1}}(x>0)$$

$$= -\frac{\arcsin x}{x} - \int \frac{d(1/x)}{\sqrt{1/x^2-1}} = -\frac{\arcsin x}{x} - \ln|1/x + \sqrt{1/x^2-1}| + C$$

$$= -\frac{\arcsin x}{x} + \ln(1-\sqrt{1-x^2}) - \ln x + C$$

$$= -\frac{\arcsin x}{x} + \ln(1-\sqrt{1-x^2}) - \ln|x| + C(x\neq 0)(\text{原函数为偶函数}).$$

$$16.\int x^3(\ln x)^2dx = \frac{1}{4}\int (\ln x)^2 dx^4 = \frac{x^4(\ln x)^2}{4} - \frac{1}{4}\int \frac{x^4\cdot 2\ln x dx}{x}$$

$$= \frac{x^4(\ln x)^2}{4} - \frac{1}{2}\int x^3\ln x dx = \frac{x^4(\ln x)^2}{4} - \frac{1}{8}\int \ln x dx^4$$

$$= \frac{x^4(\ln x)^2}{4} - \frac{x^4}{8}\ln x + \frac{1}{2}\int x^3 dx = \frac{x^4(\ln x)^2}{4} - \frac{x^4}{8}\ln x + \frac{1}{8}x^4 + C.$$

$$17.\int \frac{x\arctan x dx}{(1+x^2)^{5/2}} = \frac{1}{2}\int \frac{\arctan x d(1+x^2)}{(1+x^2)^{5/2}} = \frac{1}{2}\cdot\left(-\frac{2}{3}\right)\int \arctan x d(1+x^2)^{-3/2}$$

$$=-\frac{\arctan x}{3(1+x^2)^{3/2}}+\frac{1}{3}\int\frac{dx}{(1+x^2)^{5/2}}.x=\tan u,u\in(-\pi/2,\pi/2).dx=\sec^2 udu,$$

$$\int\frac{dx}{(1+x^2)^{5/2}}=\int\cos^3 udu=\int(1-\sin^2 u)d\sin u=$$

$$=\sin u-\frac{1}{3}\sin^3 u+C=\frac{x}{\sqrt{1+x^2}}-\frac{1}{3}\left(\frac{x}{\sqrt{1+x^2}}\right)^3+C,$$

$$\int\frac{x\arctan xdx}{(1+x^2)^{5/2}}=-\frac{\arctan x}{3(1+x^2)^{3/2}}+\frac{1}{3}\left(\frac{x}{\sqrt{1+x^2}}-\frac{1}{3}\left(\frac{x}{\sqrt{1+x^2}}\right)^3\right)+C$$

$$=-\frac{\arctan x}{3(1+x^2)^{3/2}}+\frac{1}{3}\frac{x}{\sqrt{1+x^2}}-\frac{1}{9}\frac{x^3}{(1+x^2)^{3/2}}+C.$$

18. $\int x\ln(x+\sqrt{1+x^2})dx=\frac{1}{2}\int\ln(x+\sqrt{1+x^2})dx^2$

$$=\frac{1}{2}x^2\ln(x+\sqrt{1+x^2})-\frac{1}{2}\int\frac{x^2dx}{\sqrt{1+x^2}}$$

$$=\frac{1}{2}x^2\ln(x+\sqrt{1+x^2})-\frac{1}{2}\int\frac{(x^2+1)-1dx}{\sqrt{1+x^2}}$$

$$=\frac{1}{2}x^2\ln(x+\sqrt{1+x^2})-\frac{1}{2}\int\sqrt{1+x^2}dx+\frac{1}{2}\int\frac{dx}{\sqrt{1+x^2}}$$

$$=\frac{1}{2}x^2\ln(x+\sqrt{1+x^2})-\frac{1}{2}\left(\frac{x\sqrt{1+x^2}}{2}+\frac{\ln(x+\sqrt{1+x^2})}{2}\right)+\frac{1}{2}\ln(x+\sqrt{1+x^2})+C$$

$$=\frac{1}{2}x^2\ln(x+\sqrt{1+x^2})-\frac{1}{4}x\sqrt{1+x^2}+\frac{1}{4}\ln(x+\sqrt{1+x^2})+C.$$

习题 3.3

求下列不定积分：

1. $\int\frac{x-1}{x^2+6x+8}dx=\int\frac{x-1}{(x+2)(x+4)}dx,$

$$\frac{x-1}{(x+2)(x+4)}=\frac{A}{x+2}+\frac{B}{x+4},$$

$$A=\frac{-2-1}{-2+4}=-\frac{3}{2},B=\frac{-4-1}{-4+2}=\frac{5}{2},$$

$$\frac{x-1}{(x+2)(x+4)}=\frac{-3/2}{x+2}+\frac{5/2}{x+4},$$

$$\int\frac{x-1}{x^2+6x+8}dx=-\frac{3}{2}\ln|x+2|+\frac{5}{2}\ln|x+4|+C.$$

$$2.I=\int\frac{3x^4+x^2+1}{x^2+x-6}dx.$$

$$\frac{3x^4+x^2+1}{x^2+x-6}=3x^2-3x+22+\frac{-40x+133}{x^2+x-6},$$

$$\frac{-40x+133}{x^2+x-6}=\frac{-40x+133}{(x+3)(x-2)}=\frac{A}{x+3}+\frac{B}{x-2},$$

$$A=\frac{-40\cdot(-3)+133}{-3-2}=-\frac{253}{5},B=\frac{-40\cdot 2+133}{2+3}=\frac{53}{5}.$$

$$I=x^3-\frac{3x^2}{2}+22x-\frac{253}{5}\ln|x+3|+\frac{53}{5}\ln|x-2|+C.$$

$$3.I=\int\frac{2x^2-5}{x^4-5x^2+6}dx$$

$$\frac{2x^2-5}{x^4-5x^2+6}=\frac{2u-5}{u^2-5u+6}(u=x^2)$$

$$=\frac{2u-5}{(u-2)(u-3)}=\frac{A}{u-2}+\frac{B}{u-3},$$

$$A=\frac{2\cdot 2-5}{2-3}=1,B=\frac{2\cdot 3-5}{3-2}=1.$$

$$\frac{2x^2-5}{x^4-5x^2+6}=\frac{1}{x^2-\sqrt{2}^2}+\frac{1}{x^2-\sqrt{3}^2},$$

$$I=\frac{1}{2\sqrt{2}}\ln\frac{x-\sqrt{2}}{x+\sqrt{2}}+\frac{1}{2\sqrt{3}}\ln\frac{x-\sqrt{3}}{x+\sqrt{3}}+C.$$

$$4.I=\int\frac{dx}{(x-1)^2(x-2)}.$$

$$\frac{1}{(x-1)^2(x-2)}=\frac{1}{x-2}\left(\frac{1}{x-2}-\frac{1}{x-1}\right)$$

$$=\frac{1}{(x-2)^2}-\left(\frac{1}{x-2}-\frac{1}{x-1}\right),$$

$$I=-\frac{1}{x-2}+\ln\left|\frac{x-1}{x-2}\right|+C.$$

$$5. I=\int\frac{x^2}{1-x^4}dx.$$

$$\frac{x^2}{1-x^4}=\frac{x^2}{(1-x^2)(1+x^2)}=\frac{1}{2}\cdot\frac{(1+x^2)-(1-x^2)}{(1-x^2)(1+x^2)}$$

$$=\frac{1}{2}\left(\frac{1}{1-x^2}-\frac{1}{1+x^2}\right),$$

$$I=\frac{1}{4}\ln\frac{1+x}{1-x}-\frac{1}{2}\arctan x+C.$$

$$6. I=\int\frac{dx}{x^3+1}.$$

$$\frac{1}{x^3+1}=\frac{1}{(x+1)(x^2-x+1)}=\frac{A}{x+1}+\frac{Bx+C}{x^2-x+1},$$

$$A=\frac{1}{1^2+1+1}=\frac{1}{3},$$

$$1=\frac{x^2-x+1}{3}+(x+1)(Bx+C)=(B+\frac{1}{3})x^2+(B+C-\frac{1}{3})x+C+\frac{1}{3},$$

$$C+\frac{1}{3}=1, C=\frac{2}{3}, B+\frac{1}{3}=0, B=-\frac{1}{3}.$$

$$\frac{1}{x^3+1}=\frac{1}{3(x+1)}+\frac{-x+2}{3(x^2-x+1)}=\frac{1}{3(x+1)}-\frac{x-2}{3(x^2-x+1)}$$

$$=\frac{1}{3(x+1)}-\frac{2x-4}{6(x^2-x+1)}=\frac{1}{3(x+1)}-\frac{1}{6}\cdot\frac{(2x-1)-3}{(x^2-x+1)}.$$

$$=\frac{1}{3(x+1)}-\frac{1}{6}\cdot\frac{2x-1}{(x^2-x+1)}+\frac{1}{2}\cdot\frac{1}{\left(x-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2},$$

$$I=\frac{1}{3}\ln|x+1|-\frac{1}{6}\ln(x^2-x+1)+\frac{1}{\sqrt{3}}\arctan\frac{2x-1}{\sqrt{3}}+C.$$

$$7.I=\int\frac{dx}{1+x^4}.\frac{1}{1+x^4}=\frac{1}{(1+2x^2+x^4)-2x^2}=\frac{1}{(x^2+1)^2-2x^2}$$

$$=\frac{1}{(x^2+\sqrt{2}x+1)(x^2-\sqrt{2}x+1)}=\frac{Ax+B}{x^2+\sqrt{2}x+1}+\frac{Cx+D}{x^2-\sqrt{2}x+1},$$

$$1=(Ax+B)(x^2-\sqrt{2}x+1)+(Cx+D)(x^2+\sqrt{2}x+1),$$

$$1=(A+C)x^3+(B-\sqrt{2}A+D+\sqrt{2}C)x^2+(A-\sqrt{2}B+C+\sqrt{2}D)x+B+D.$$

$$\begin{cases}A+C=0\\ B-\sqrt{2}A+D+\sqrt{2}C=0,\\ A-\sqrt{2}B+C+\sqrt{2}D=0,\\ B+D=1.\end{cases}$$

$$A=\frac{1}{2\sqrt{2}},B=\frac{1}{2},C=-\frac{1}{2\sqrt{2}},D=\frac{1}{2}.$$

$$\frac{1}{1+x^4}=\frac{\frac{1}{2\sqrt{2}}x+\frac{1}{2}}{x^2+\sqrt{2}x+1}+\frac{-\frac{1}{2\sqrt{2}}x+\frac{1}{2}}{x^2-\sqrt{2}x+1}$$

$$=\frac{1}{2\sqrt{2}}\left(\frac{x+\sqrt{2}}{x^2+\sqrt{2}x+1}+\frac{-x+\sqrt{2}}{x^2-\sqrt{2}x+1}\right)$$

$$=\frac{1}{4\sqrt{2}}\left(\frac{2x+2\sqrt{2}}{x^2+\sqrt{2}x+1}-\frac{2x-2\sqrt{2}}{x^2-\sqrt{2}x+1}\right)$$

$$=\frac{1}{4\sqrt{2}}\left(\frac{(2x+\sqrt{2})+\sqrt{2}}{x^2+\sqrt{2}x+1}-\frac{(2x-\sqrt{2})-\sqrt{2}}{x^2-\sqrt{2}x+1}\right)$$

$$=\frac{1}{4\sqrt{2}}\left(\frac{(2x+\sqrt{2})}{x^2+\sqrt{2}x+1}-\frac{(2x-\sqrt{2})}{x^2-\sqrt{2}x+1}\right)+\frac{1}{4}\square\frac{1}{(x+\frac{1}{\sqrt{2}})^2+\left(\frac{1}{\sqrt{2}}\right)^2}$$

$$+\frac{1}{4}\square\frac{1}{(x-\frac{1}{\sqrt{2}})^2+\left(\frac{1}{\sqrt{2}}\right)^2}.$$

$$I=\frac{1}{4\sqrt{2}}\ln\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}+\frac{\sqrt{2}}{4}\left(\arctan(\sqrt{2}x+1)+\arctan(\sqrt{2}x-1)\right)+C.$$

$$8.I=\int\frac{x^3+x^2+2}{(x^2+2)^2}dx.$$

$$\frac{x^3+x^2+2}{(x^2+2)^2}=\frac{x(x^2+2)}{(x^2+2)^2}+\frac{x^2-2x+2}{(x^2+2)^2}$$

$$=\frac{x}{(x^2+2)}+\frac{1}{(x^2+2)}-\frac{2x}{(x^2+2)^2}.$$

$$I=\frac{1}{2}\ln(x^2+2)+\frac{1}{\sqrt{2}}\arctan\frac{x}{\sqrt{2}}+\frac{1}{x^2+2}+C.$$

$$9.\int\frac{e^x dx}{e^{2x}+3e^x+2}=\int\frac{de^x}{e^{2x}+3e^x+2}=\int\frac{du}{u^2+3u+2}=$$

$$=\int\frac{du}{(u+1)(u+2)}=\int\left(\frac{1}{u+1}-\frac{1}{u+2}\right)du=\ln\frac{u+1}{u+2}+C=\ln\frac{e^x+1}{e^x+2}+C.$$

$$10.\int\frac{\cos xdx}{\sin^2 x+\sin x-6}=\int\frac{d\sin x}{\sin^2 x+\sin x-6}=\int\frac{du}{u^2+u-6}(u=\sin x)=$$

$$\int\frac{du}{(u+3)(u-2)}=\frac{1}{5}\int\left(\frac{1}{u-2}-\frac{1}{u+3}\right)du=\ln\left|\frac{u-2}{u+3}\right|+C=\ln\left|\frac{\sin x-2}{\sin x+3}\right|+C.$$

$$11.\int\frac{x^3dx}{x^4+x^2+2}=\frac{1}{2}\int\frac{x^2dx^2}{x^4+x^2+2}=\frac{1}{2}\int\frac{udu}{u^2+u+2}$$

$$=\frac{1}{4}\int\frac{2udu}{u^2+u+2}=\frac{1}{4}\int\frac{(2u+1)-1}{u^2+u+2}du=$$

$$=\frac{1}{4}\int\frac{d(u^2+u+2)}{u^2+u+2}du-\frac{1}{4}\int\frac{1}{\left(u+\frac{1}{2}\right)^2+\frac{7}{4}}du$$

$$=\frac{1}{4}\ln(u^2+u+2)-\frac{1}{2\sqrt{7}}\arctan\frac{2u+1}{\sqrt{7}}+C$$

$$=\frac{1}{4}\ln(x^4+x^2+2)-\frac{1}{2\sqrt{7}}\arctan\frac{2x^2+1}{\sqrt{7}}+C.$$

$$12.I=\int\frac{dx}{(x+2)(x^2-2x+2)}.$$

$$\frac{1}{(x+2)(x^2-2x+2)}=\frac{A}{x+2}+\frac{Bx+C}{x^2-2x+2}$$

$$A=\frac{1}{(-2)^2-2(-2)+2}=\frac{1}{10}.$$

$$\frac{1}{(x+2)(x^2-2x+2)}-\frac{1}{10(x+2)}=\frac{Bx+C}{x^2-2x+2}$$

$$\frac{10-(x^2-2x+2)}{10(x+2)(x^2-2x+2)}=\frac{Bx+C}{x^2-2x+2}$$

$$\frac{-(x^2-2x-8)}{10(x+2)(x^2-2x+2)}=\frac{Bx+C}{x^2-2x+2}$$

$$\frac{-(x+2)(x-4)}{10(x+2)(x^2-2x+2)}=\frac{Bx+C}{x^2-2x+2}$$

$$\frac{-(x-4)}{10(x^2-2x+2)}=\frac{Bx+C}{x^2-2x+2},B=-\frac{1}{10},C=\frac{2}{5}.$$

$$I=\frac{1}{10}\ln|x+2|-\frac{1}{10}\int\frac{x-4}{x^2-2x+2}dx$$

$$=\frac{1}{10}\ln|x+2|-\frac{1}{10}\int\frac{2x-8}{x^2-2x+2}dx$$
$$=\frac{1}{10}\ln|x+2|-\frac{1}{20}\int\frac{(2x-2)-6}{x^2-2x+2}dx$$
$$=\frac{1}{10}\ln|x+2|-\frac{1}{20}\ln(x^2-2x+2)+\frac{3}{10}\int\frac{dx}{(x-1)^2+1}$$
$$=\frac{1}{10}\ln|x+2|-\frac{1}{20}\ln(x^2-2x+2)+\frac{3}{10}\arctan(x-1)+C$$

$$13.I=\int\frac{dx}{2+\sin x}.\tan\frac{x}{2}=u,x=2\arctan u,dx=\frac{2du}{1+u^2},\sin x=\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}=\frac{2u}{1+u^2}.$$

$$I=\int\frac{\frac{2du}{1+u^2}}{2+\frac{2u}{1+u^2}}=\int\frac{1}{u^2+u+1}du=\int\frac{1}{\left(u+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}du$$

$$=\frac{2}{\sqrt{3}}\arctan\frac{2u+1}{\sqrt{3}}+C=\frac{2}{\sqrt{3}}\arctan\frac{2\tan\frac{x}{2}+1}{\sqrt{3}}+C.$$

$$14.I=\int\frac{dx}{1+\sin x+\cos x}.\tan\frac{x}{2}=u,x=2\arctan u,dx=\frac{2du}{1+u^2},$$
$$\sin x=\frac{2u}{1+u^2},\cos x=\frac{1-u^2}{1+u^2}.$$
$$I=\int\frac{\frac{2du}{1+u^2}}{1+\frac{2u}{1+u^2}+\frac{1-u^2}{1+u^2}}=2\int\frac{1}{1+u^2+2u+1-u^2}du=\int\frac{1}{u+1}du$$
$$=\ln|u+1|+C=\ln|\tan\frac{x}{2}+1|+C.$$

$$15.\int\cot^4 xdx$$
$$=\int\cot^2 x(\csc^2 x-1)dx$$
$$=\int\cot^2 x\csc^2 xdx-\int\cot^2 xdx$$
$$=-\int\cot^2 xd\cot x-\int(\csc^2 x-1)dx$$
$$=-\frac{1}{3}\cot^3 x+\cot x+x+C.$$

$$16.\int\sec^4 xdx=\int(1+\tan^2 x)d\tan x=\tan x+\frac{1}{3}\tan^3 x+C.$$

$$17.I=\int\frac{\cos xdx}{5-3\cos x}=-\frac{1}{3}\int\frac{-3\cos xdx}{5-3\cos x}=-\frac{1}{3}\int\frac{(-3\cos x+5)-5dx}{5-3\cos x}$$

$$=-\frac{x}{3}+\frac{5}{3}\int\frac{dx}{5-3\cos x}.$$

$$\tan\frac{x}{2}=u,dx=\frac{2du}{1+u^2},\cos x=\frac{1-u^2}{1+u^2},$$

$$I=-\frac{x}{3}+\frac{5}{3}\int\frac{\frac{2du}{1+u^2}}{5-\frac{3(1-u^2)}{1+u^2}}=-\frac{x}{3}+\frac{5}{3}\int\frac{2du}{5(1+u^2)-3(1-u^2)}$$

$$=-\frac{x}{3}+\frac{5}{3}\int\frac{2du}{8u^2)+2}=-\frac{x}{3}+\frac{5}{3}\int\frac{du}{4u^2+1}=-\frac{x}{3}+\frac{5}{3}\cdot\frac{1}{2}\int\frac{d2u}{4u^2+1}$$

$$=-\frac{x}{3}+\frac{5}{6}\arctan 2u+C=-\frac{x}{3}+\frac{5}{6}\arctan\left(2\tan\frac{x}{2}\right)+C.$$

$$18.I=\int\frac{\cos^3 xdx}{\sin x+\cos x}=\int\frac{\cos^2 xdx}{1+\tan x}=\int\frac{dx}{(1+\tan x)(1+\tan^2 x)}.$$

$$\tan x=u,x=\arctan u,dx=\frac{du}{1+u^2},$$

$$I=\int\frac{\frac{du}{1+u^2}}{(1+u)(1+u^2)}=\int\frac{du}{(1+u)(1+u^2)^2},$$

$$\frac{1}{(1+u)(1+u^2)^2}=\frac{1}{2(1+u^2)}\left(\frac{1}{1+u}+\frac{1-u}{1+u^2}\right)$$

$$=\frac{1}{4}\left(\frac{1}{1+u}+\frac{1-u}{1+u^2}\right)+\frac{1-u}{2(1+u^2)^2},$$

$$I=\frac{1}{4}\ln|1+\tan x|+\frac{1}{4}\arctan u-\frac{1}{8}\ln(1+u^2)+\frac{1}{4(1+u^2)}+\frac{1}{2}\left(\frac{1}{2}\arctan u+\frac{u}{2(1+u^2)}\right)+C$$

$$=\frac{1}{4}\ln|1+\tan x|+\frac{x}{2}+\frac{1}{4}\ln|\cos u|++\frac{1}{4}\cos^2 x+\frac{1}{4}\tan x\cos^2 x+C.$$

$$19.\int\sin^5 x\cos^2 xdx=-\int\sin^4 x\cos^2 xd\cos x=-\int(1-u^2)^2u^2du$$

$$=-\int(u^2-2u^4+u^6)dx=-\frac{1}{3}u^3+\frac{2}{5}u^5-\frac{1}{7}u^7+C$$

$$=-\frac{1}{3}(\cos x)^3+\frac{2}{5}(\cos x)^5-\frac{1}{7}(\cos x)^7+C.$$

$$20.\int\sin^6 xdx=\int\left(\frac{1-\cos 2x}{2}\right)^3dx$$

$$=\frac{1}{8}\int(1-3\cos 2x+3\cos^2 2x-\cos^3 2x)dx$$

$$=\frac{x}{8}-\frac{3}{16}\sin 2x+\frac{3}{16}\int(1+\cos 4x)dx-\frac{1}{16}\int(1-\sin^2 2x)d\sin 2x$$

$$=\frac{x}{8}-\frac{3}{16}\sin 2x+\frac{3}{16}\left(x+\frac{1}{4}\sin 4x\right)-\frac{1}{16}\left(\sin 2x-\frac{1}{3}\sin^3 2x\right)+C$$

$$=+C.$$

21. $$\int\sin^2 x\cos^4 xdx=\frac{1}{4}\int\sin^2 2x\cos^2 xdx=\frac{1}{4}\int\left(\frac{\sin 3x+\sin x}{2}\right)^2 dx$$

$$=\frac{1}{16}\int\left(\sin^2 3x+\sin^2 x+2\sin 3x\sin x\right)dx$$

$$=\frac{1}{16}\int\left(\frac{1-\cos 6x}{2}+\frac{1-\cos 2x}{2}+\cos 2x-\cos 4x\right)dx$$

$$=\frac{1}{16}\left(x+\frac{1}{4}\sin 2x-\frac{1}{4}\sin 4x-\frac{1}{12}\sin 6x\right)+C.$$

另解: $$\int\sin^2 x\cos^4 xdx=\int\frac{1-\cos 2x}{2}\cdot\left(\frac{1+\cos 2x}{2}\right)^2 dx$$

$$=\frac{1}{8}\int(1+\cos^2 2x+2\cos 2x)(1-\cos 2x)dx$$

$$=\frac{1}{8}\int\left(1+\cos^2 2x+2\cos 2x-\cos 2x-\cos^3 2x-2\cos^2 2x\right)dx$$

$$=\frac{1}{8}\int\left(1+\cos 2x-\cos^2 2x-\cos^3 2x\right)dx$$

$$=\frac{1}{8}\left(x+\frac{1}{2}\sin 2x\right)-\frac{1}{16}\int(1+\cos 4x)dx-\frac{1}{16}\int(1-\sin^2 2x)d\sin 2x$$

$$=\frac{1}{8}\left(x+\frac{1}{2}\sin 2x\right)-\frac{1}{16}\left(x+\frac{1}{4}\sin 4x\right)-\frac{1}{16}\left(\sin 2x-\frac{1}{3}\sin^3 2x\right)+C$$

$$=\frac{1}{16}x-\frac{1}{64}\sin 4x+\frac{1}{48}\sin^3 2x+C.$$

$$22.I=\int\frac{dx}{\sin x+2\cos x}.\tan\frac{x}{2}=u,x=2\arctan u,dx=\frac{2du}{1+u^2}.$$

$$I=\int\frac{\frac{2du}{1+u^2}}{\frac{2u}{1+u^2}+\frac{2(1-u^2)}{1+u^2}}=\int\frac{2du}{-2u^2+2u+2}=-\int\frac{du}{u^2-u-1}=-\int\frac{du}{\left(u-\frac{1}{2}\right)^2-\left(\frac{\sqrt{5}}{2}\right)^2}=$$

$$=\frac{1}{\sqrt{5}}\ln\left|\frac{u-\frac{1}{2}+\frac{\sqrt{5}}{2}}{u-\frac{1}{2}-\frac{\sqrt{5}}{2}}\right|+C=\ln\left|\frac{2u+\sqrt{5}-1}{2u-\sqrt{5}-1}\right|+C.$$

$$23.\int\frac{\sin x\cos x}{\sin^2 x+\cos^4 x}dx=$$

$$=\int\frac{\tan x}{\tan^2 x(1+\tan^2 x)+1}d\tan x=\int\frac{u}{u^2(1+u^2)+1}du(u=\tan x)$$

$$=\frac{1}{2}\int\frac{du^2}{u^2(1+u^2)+1}=\frac{1}{2}\int\frac{dv}{v(1+v)+1}(v=u^2)$$

$$=\frac{1}{2}\int\frac{dv}{\left(v+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}=\frac{1}{2}\cdot\frac{2}{\sqrt{3}}\arctan\frac{2v+1}{\sqrt{3}}+C$$

$$=\frac{1}{\sqrt{3}}\arctan\frac{2\tan^2 x+1}{\sqrt{3}}+C.$$

另解：$I=\frac{1}{2}\int\frac{d\sin^2 x}{\sin^2 x+(1-\sin^2 x)^2}=\frac{1}{2}\int\frac{dw}{w+(1-w)^2}(w=\sin^2 x)$

$$=\frac{1}{2}\int\frac{dw}{(w-\frac{1}{2})^2+\left(\frac{\sqrt{3}}{2}\right)^2}=\frac{1}{\sqrt{3}}\arctan\frac{2w-1}{\sqrt{3}}+C=\arctan\frac{2\sin^2 x-1}{\sqrt{3}}+C.$$

$$24.\int\frac{dx}{\sin^4 x}=-\int(1+\cot^2 x)d\cot x=-\cot x-\frac{1}{3}\cot^3 x+C.$$

$$25.\int\sqrt{\frac{1-x}{1+x}}dx=\int\frac{1-x}{\sqrt{1-x^2}}dx=\arcsin x+\sqrt{1-x^2}+C.$$

$$26.I=\int\frac{1-\sqrt{x-1}}{1+\sqrt[3]{x-1}}dx.\sqrt[6]{x-1}=u,x=1+u^6,dx=6u^5du,$$

$$I=6\int\frac{(1-u^3)u^5du}{1+u^2}=6\int\frac{u^5-u^8}{1+u^2}du=-6\int(u^6-u^4-u^3+u^2+u+1+\frac{-u+1}{1+u^2})dx$$

$$=-6\left(\frac{1}{7}u^7-\frac{1}{5}u^5-\frac{1}{4}u^4+\frac{1}{3}u^3+\frac{1}{2}u^2+u-\frac{1}{2}\ln(1+u^2)+\arctan u\right)+C.$$

$$27.\int\frac{\sqrt{x+1}+\sqrt{x-1}}{\sqrt{x+1}-\sqrt{x-1}}dx=\int\frac{\left(\sqrt{x+1}+\sqrt{x-1}\right)^2}{\left(\sqrt{x+1}-\sqrt{x-1}\right)\left(\sqrt{x+1}+\sqrt{x-1}\right)}dx$$

$$=\int\frac{2x+2\sqrt{x^2-1}}{2}dx=\frac{1}{2}x^2+\frac{1}{2}x\sqrt{x^2-1}-\frac{1}{2}\ln(x+\sqrt{x^2-1})+C.$$

$$28.I=\int\frac{dx}{\sqrt[3]{(x+1)^2(x-1)^4}}=\int\frac{dx}{(x^2-1)\sqrt[3]{\frac{x-1}{x+1}}}.\sqrt[3]{\frac{x-1}{x+1}}=u,\frac{x-1}{x+1}=u^3,$$

$$x-1=(x+1)u^3,x=\frac{1+u^3}{1-u^3}=-1+\frac{2}{1-u^3},dx=\frac{6u^2du}{(1-u^3)^2},$$

$$I=\int\frac{\frac{6u^2du}{(1-u^3)^2}}{\left(\left(\frac{1+u^3}{1-u^3}\right)^2-1\right)u}=6\int\frac{u}{2(2u^3)}du=\frac{3}{2}\left(-\frac{1}{u}\right)+C=-\frac{3}{2}\sqrt[3]{\frac{x+1}{x-1}}+C.$$

$$29.\int\frac{xdx}{\sqrt{x^2-x+3}}=\frac{1}{2}\int\frac{2xdx}{\sqrt{x^2-x+3}}=\frac{1}{2}\int\frac{2x-1+1dx}{\sqrt{x^2-x+3}}=$$

$$=\frac{1}{2}\int\frac{d(x^2-x+3)}{\sqrt{x^2-x+3}}+\frac{1}{2}\int\frac{dx}{\sqrt{\left(x-\frac{1}{2}\right)^2+\left(\frac{\sqrt{11}}{2}\right)^2}}=$$

$$=\sqrt{x^2-x+3}+\frac{1}{2}\ln\left(x-\frac{1}{2}+\sqrt{x^2-x+3}\right)+C.$$

$$30.I=\int\frac{x}{(1+x^{1/3})^{1/2}}dx.(1+x^{1/3})^{1/2}=u,x=(u^2-1)^3,dx=3(u^2-1)^2(2u)du,$$

$$I=6\int\frac{(u^2-1)^3(u^2-1)^2(u)du}{u}=6\int(u^6-3u^4+3u^2-1)(u^4-2u^2+1)du$$

$$=6\int(u^{10}-5u^8+10u^6-10u^4+5u^2-1)du$$

$$=6\left(\frac{1}{11}u^{11}-\frac{5}{9}u^9+\frac{10}{7}u^7-2u^5+\frac{5}{3}u^3-u\right)+C.$$

$$31.I=\int\frac{\sqrt{x}dx}{\sqrt[4]{x^3}+1}.\sqrt[4]{x}=u,x=u^4,dx=4u^3du.$$

$$I=\int\frac{u^2 4u^3du}{u^3+1}=4\int\frac{u^5}{u^3+1}dx=4\int\frac{(u^5+u^2)-u^2}{u^3+1}du$$

$$=4\int\left(u^2-\frac{u^2}{u^3+1}\right)du=\frac{4}{3}u^3-\frac{4}{3}\ln(u^3+1)+C=\frac{4}{3}\sqrt[4]{x}^3-\frac{4}{3}\ln(\sqrt[4]{x}^3+1)+C.$$

$$32.\int\frac{2x+3}{\sqrt{x^2+x}}dx=\int\frac{(2x+1)+2}{\sqrt{x^2+x}}dx=\int\frac{1}{\sqrt{x^2+x}}d(x^2+x)+2\int\frac{1}{\sqrt{x^2+x}}dx$$

$$=2\sqrt{x^2+x}+2\int\frac{1}{\sqrt{\left(x+\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2}}dx$$

$$=2\sqrt{x^2+x}+2\ln\left|x+\frac{1}{2}+\sqrt{x^2+x}\right|+C.$$

$$33.\int\frac{2+x}{\sqrt{4x^2-4x+5}}dx=\frac{1}{8}\int\frac{16+8x}{\sqrt{4x^2-4x+5}}dx$$

$$=\frac{1}{8}\int\frac{8x-4+20}{\sqrt{4x^2-4x+5}}dx=\frac{1}{8}\int\frac{d(4x^2-4x+5)}{\sqrt{4x^2-4x+5}}dx+\frac{5}{2}\int\frac{dx}{\sqrt{4x^2-4x+5}}$$

$$=\frac{1}{4}\sqrt{4x^2-4x+5}+\frac{5}{4}\int\frac{dx}{\sqrt{x^2-x+5/4}}$$

$$=\frac{1}{4}\sqrt{4x^2-4x+5}+\frac{5}{4}\int\frac{dx}{\sqrt{\left(x-\frac{1}{2}\right)^2+1}}$$

$$=\frac{1}{4}\sqrt{4x^2-4x+5}+\frac{5}{4}\ln\left(x-\frac{1}{2}+\sqrt{x^2-x+5/4}\right)+C$$

$$=\frac{1}{4}\sqrt{4x^2-4x+5}+\frac{5}{4}\ln\left(2x-1+\sqrt{4x^2-4x+5}\right)+C'.$$

$$34.\int\sqrt{5-2x+x^2}dx=\int\sqrt{2^2+(x-1)^2}dx$$
$$=\frac{(x-1)}{2}\sqrt{5-2x+x^2}+2\ln(\sqrt{5-2x+x^2})+C.$$

习题 3.4

求下列各定积分:

$$1.I=\int_{-1}^{1}\frac{xdx}{\sqrt{5-4x}}.\sqrt{5-4x}=u,-1\to 3,1\to 1.5-4x=u^2,x=\frac{1}{4}(5-u^2),dx=-\frac{1}{2}udu,$$
$$I=\int_3^1\frac{\frac{1}{4}(5-u^2)}{u}\cdot\left(-\frac{1}{2}udu\right)=\frac{1}{8}\int_1^3(5-u^2)dx=\frac{1}{8}\left(5u-\frac{1}{3}u^3\right)\Big|_1^3=\frac{1}{6}.$$

$$2.\int_0^{\ln 2}xe^{-x}dx=-\int_0^{\ln 2}xde^{-x}=-xe^{-x}\Big|_0^{\ln 2}+\int_0^{\ln 2}e^{-x}dx=-\frac{\ln 2}{2}-e^{-x}\Big|_0^{\ln 2}=\frac{1}{2}(1-\ln 2).$$

$$3.\int_0^1x^2\sqrt{1-x^2}dx=\int_0^{\pi/2}\sin^2 t\cos^2 tdt(x=\sin t)$$
$$=\int_0^{\pi/2}\sin^2 t(1-\sin^2 t)dt=I_2-I_4=\left(\frac{1}{2}-\frac{3\cdot 1}{4\cdot 2}\right)\frac{\pi}{2}=\frac{\pi}{16}.$$

$$4.\int_0^{\pi}x\sin xdx=-\int_0^{\pi}xd\cos x=-x\cos x\Big|_0^{\pi}+\int_0^{\pi}\cos xdx=\pi+\sin x\Big|_0^{\pi}=\pi.$$

$$5.\int_0^4\sqrt{x^2+9}dx=\left(\frac{x}{2}\sqrt{x^2+9}+\frac{9}{2}\ln(x+\sqrt{x^2+9})\right)\Big|_0^4=10+\frac{9}{2}\ln 3.$$

$$6.\int_0^{\frac{1}{2}}\frac{x^2}{\sqrt{1-x^2}}dx=\int_0^{\frac{\pi}{6}}\sin^2 tdt=\frac{1}{2}\int_0^{\frac{\pi}{6}}(1-\cos 2t)dt=\frac{1}{2}(t-\frac{1}{2}\sin 2t)\Big|_0^{\frac{\pi}{6}}=\frac{1}{2}\left(\frac{\pi}{6}-\frac{\sqrt{3}}{4}\right).$$

$$7.\int_0^1\sqrt{4-x^2}dx=\left(\frac{x}{2}\sqrt{4-x^2}+2\arcsin\frac{x}{2}\right)\Big|_0^1=\frac{\sqrt{3}}{2}+\frac{\pi}{3}.$$

$$8.\int_0^3x\sqrt[3]{1-x^2}dx=\frac{1}{2}\int_0^3\sqrt[3]{1-x^2}dx^2=\frac{1}{2}\int_0^9\sqrt[3]{1-u}du=-\frac{3}{8}(1-u)^{\frac{4}{3}}\Big|_0^9=-\frac{45}{8}.$$

$$9.\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sqrt{\cos x-\cos^3 x}dx=2\int_0^{\frac{\pi}{2}}\sqrt{\cos x-\cos^3 x}dx=2\int_0^{\frac{\pi}{2}}\sqrt{\cos x}\sin xdx$$
$$=-2\int_0^{\frac{\pi}{2}}\sqrt{\cos x}d\cos x=-\frac{4}{3}\cos^{\frac{3}{2}}x\Big|_0^{\frac{\pi}{2}}=\frac{4}{3}.$$

$$10.\int_0^{\frac{\pi}{2}}\cos^n 2xdx=\frac{1}{2}\int_0^{\frac{\pi}{2}}\cos^n 2xd2x=\frac{1}{2}\int_0^{\pi}\cos^n udu=\frac{1}{2}\int_{-\pi/2}^{\pi/2}\cos^n(t+\frac{\pi}{2})dt$$
$$=\frac{(-1)^n}{2}\int_{-\pi/2}^{\pi/2}\sin^n(t)dt=\begin{cases}0,n=2k-1;\\ \int_0^{\pi/2}\sin^n(t)dt=\frac{(n-1)!!}{n!!}\cdot\frac{\pi}{2}.\end{cases}$$

$$11.\int_0^a (a^2-x^2)^{\frac{n}{2}}dx(x=a\sin t)=\int_0^{\frac{\pi}{2}}\cos^{n+1}tdt=\begin{cases}\dfrac{n!!}{(n+1)!!} & n\text{是偶数};\\ \dfrac{n!!}{(n+1)!!}\dfrac{\pi}{2} & n\text{是奇数}.\end{cases}$$

$$12.\int_0^{\pi/2}\sin^{11}xdx=\frac{10!!}{11!!}=\frac{156}{693}.$$

$$13.\int_0^{\pi}\sin^6\frac{x}{2}dx=2\int_0^{\pi/2}\sin^6udu=2\cdot\frac{5\cdot 3}{6\cdot 4\cdot 2}\cdot\frac{\pi}{2}=\frac{5\pi}{16}.$$

$$14.\int_0^{\pi}(x\sin x)^2dx=\frac{1}{2}\int_0^{\pi}x^2(1-\cos 2x)dx=\frac{1}{2}\cdot\frac{1}{3}x^3\Big|_0^{\pi}-\frac{1}{4}\int_0^{\pi}x^2d\sin 2x$$

$$=\frac{\pi^3}{6}-\frac{1}{4}x^2\sin 2x\Big|_0^{\pi}+\frac{1}{2}\int_0^{\pi}x\sin 2xdx$$

$$=\frac{\pi^3}{6}-\frac{1}{4}\int_0^{\pi}xd\cos 2x=\frac{\pi^3}{6}-\frac{1}{4}x\cos 2x\Big|_0^{\pi}+\frac{1}{4}\int_0^{\pi}\cos 2xdx$$

$$=\frac{\pi^3}{6}-\frac{\pi}{4}+\frac{1}{8}\sin 2x\Big|_0^{\pi}=\frac{\pi^3}{6}-\frac{\pi}{4}.$$

$$15.\int_0^{\pi/4}\tan^4xdx=\int_0^{\pi/4}\tan^2x(\sec^2x-1)dx$$

$$=\int_0^{\pi/4}\tan^2x\sec^2xdx-\int_0^{\pi/4}\tan^2xdx$$

$$=\int_0^{\pi/4}\tan^2xd\tan x-\int_0^{\pi/4}(\sec^2x-1)dx$$

$$=\frac{1}{3}\tan^3x\Big|_0^{\pi/4}-\tan x\Big|_0^{\pi/4}+\frac{\pi}{4}=\frac{1}{3}-1+\frac{\pi}{4}=-\frac{2}{3}+\frac{\pi}{4}.$$

$$16.\int_0^1\arcsin xdx=x\arcsin x\Big|_0^1-\int_0^1xd\arcsin x$$

$$=\frac{\pi}{2}-\int_0^1\frac{x}{\sqrt{1-x^2}}dx=\frac{\pi}{2}+\sqrt{1-x^2}\Big|_0^1=\frac{\pi}{2}-1.$$

$$17.\int_0^{\pi}\ln(x+\sqrt{x^2+a^2})dx=x\ln(x+\sqrt{x^2+a^2})\Big|_0^{\pi}-\int_0^{\pi}xd\ln(x+\sqrt{x^2+a^2})$$

$$=\pi\ln(\pi+\sqrt{\pi^2+a^2})-\int_0^{\pi}\frac{x}{\sqrt{x^2+a^2}}dx=\pi\ln(\pi+\sqrt{\pi^2+a^2})-\sqrt{x^2+a^2}\Big|_0^{\pi}$$

$$=\pi\ln(\pi+\sqrt{\pi^2+a^2})-\sqrt{\pi^2+a^2}+|a|.$$

18.设$f(x)$在$[a,b]$连续.证明$\int_a^bf(x)dx=(b-a)\int_0^1f(a+(b-a)x)dx.$

证 令$x=a+(b-a)t$,则$0\to a,1\to b,dx=(b-a)dt$,故

$$\int_a^bf(x)dx=(b-a)\int_0^1f(a+(b-a)t)dt=(b-a)\int_0^1f(a+(b-a)x)dx.$$

19.证明$\int_0^ax^3f(x^2)dx=\frac{1}{2}\int_0^{a^2}xf(x)dx.$

证令$x^2=t$,则$x=0$时,$t=0$,$x=a$时,$t=a^2$故

$$\int_0^ax^3f(x^2)dx=\frac{1}{2}\int_0^ax^2f(x^2)dx^2=\frac{1}{2}\int_0^{a^2}tf(t)dt=\frac{1}{2}\int_0^{a^2}xf(x)dx.$$

20.证明$\int_0^1 x^m(1-x)^n dx=\int_0^1 x^n(1-x)^m dx.$

证令$x=1-t$, 则$x=0$时, $x=1$时, $t=0.dx=-dt$, 故

$$\int_0^1 x^m(1-x)^n dx=-\int_1^0(1-t)^m t^n dt=\int_0^1(1-t)^m t^n dt=\int_0^1 x^n(1-x)^m dx.$$

21.利用分部积分公式证明,若$f(x)$连续,则

$$\int_0^x\int_0^t f(x)dxdt=\int_0^x f(t)(x-t)dx.$$

证$\int_a^x\int_0^t f(x)dxdt=t\int_0^t f(x)dx\Big|_0^x-\int_0^x t\left(\int_0^t f(x)dx\right)' dt$

$=\int_0^x xf(x)dx-\int_0^x tf(t)dt=\int_0^x xf(t)dt-\int_0^x tf(t)dt$

$=\int_0^x f(t)(x-t)dt.$

22.利用换元积分法证明$\int_0^\pi xf(\sin x)dx=\pi\int_0^{\pi/2} f(\sin x)dx.$

证$x=\pi-t, x=0$ 时, $t=\pi, x=\pi$时, $dx=-dt$, 故

$\int_0^\pi xf(\sin x)dx=-\int_\pi^0(\pi-t)f(\sin(\pi-t))dt$

$=\int_0^\pi(\pi-t)f(\sin t)dt=\pi\int_0^\pi f(\sin t)dt-\int_0^\pi tf(\sin t)dt$

$=\pi\int_0^\pi f(\sin t)dt-\int_0^\pi xf(\sin x)dx.$

$2\int_0^\pi xf(\sin x)dx=\pi\int_0^\pi f(\sin t)dt,$

$\int_0^\pi xf(\sin x)dx=\frac{1}{2}\pi\int_0^\pi f(\sin t)dt$

$=\frac{1}{2}\pi\int_0^{\pi/2} f(\sin t)dt+\frac{1}{2}\pi\int_{\pi/2}^\pi f(\sin t)dt$

令$u=\pi-t$,则$t=\pi/2$时, $u=\pi/2, t=\pi$时, $u=0, du=-dt,$

$\int_{\pi/2}^\pi f(\sin t)dt=-\int_{\pi/2}^0 f(\sin(\pi-u))du=\int_0^{\pi/2} f(\sin u)du,$

$\int_0^\pi xf(\sin x)dx=\pi\int_0^{\pi/2} f(\sin x)dx.$

23.利用上题结果求$\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}dx.$

解$\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}dx=\int_0^{\pi/2}\frac{\sin x}{1+\cos^2 x}dx=-\int_0^{\pi/2}\frac{d\cos x}{1+\cos^2 x}$

$=-\arctan\cos x|_0^{\pi/2}=\frac{\pi}{4}.$

24.设函数$f(x)$在$(-\infty,+\infty)$上连续,以T为周期,证明:

(1)函数$F(x)=\frac{x}{T}\int_0^T f(x)dx-\int_0^x f(t)dt$也以$T$为周期;

(2) $\lim\limits_{x\to+\infty}\frac{1}{x}\int_0^x f(t)dt=\frac{1}{T}\int_0^T f(x)dx.$

证(1)$F(x+T)=\frac{x+T}{T}\int_0^T f(x)dx-\int_0^{x+T} f(t)dt$

$=\frac{x}{T}\int_0^T f(x)dx+\int_0^T f(x)dx-\left(\int_0^x f(t)dt+\int_x^{x+T} f(t)dt\right)$

$=\frac{x}{T}\int_0^T f(x)dx+\int_0^T f(x)dx-\left(\int_0^x f(t)dt+\int_0^T f(t)dt\right)$

$=\frac{x}{T}\int_0^T f(x)dx-\int_0^x f(t)dt=F(x).$

(2)$\frac{1}{x}\int_0^x f(t)dt-\frac{1}{T}\int_0^T f(x)dx$

$=-\frac{1}{x}\left(\frac{x}{T}\int_0^T f(x)dx-\int_0^x f(t)dt\right)=-\frac{F(x)}{x}.$

$F(x)$在$(-\infty,+\infty)$上连续,以T为周期,故有界,

$\lim\limits_{x\to+\infty}\left(\frac{1}{x}\int_0^x f(t)dt-\frac{1}{T}\int_0^T f(x)dx\right)=\lim\limits_{x\to+\infty}\frac{F(x)}{x}=0.$

于是$\lim\limits_{x\to+\infty}\frac{1}{x}\int_0^x f(t)dt=\frac{1}{T}\int_0^T f(x)dx.$

25.设$f(x)$是以T为周期的连续函数,$f(x_0)\neq 0$,且$\int_0^T f(x)dx=0$,证明:

$f(x)$在区间(x_0,x_0+T)内至少有两个根.

证为明确起见,设$f(x_0)>0$.如果f在(x_0,x_0+T)没有根,则由连续函数的中间值定理,f在(x_0,x_0+T)恒正,设其最小值为m.则$m>0$,

$\int_{x_0}^{x_0+T} f(x)dx\geq\int_{x_0}^{x_0+T} mdx=mT>0$.由周期性和假设$\int_{x_0}^{x_0+T} f(x)dx=\int_0^T f(x)dx=0$,

矛盾.故f在(x_0,x_0+T)至少有一个根x_1.若f在(x_0,x_0+T)再无其它根,由于

$f(x_0+T)=f(x_0)>0$, f在(x_0,x_1)和(x_1,x_0+T)恒正,

$\int_{x_0}^{x_0+T} f(x)dx=\int_{x_0}^{x_1} f(x)dx+\int_{x_1}^{x_0+T} f(x)dx>0$,矛盾.故$f$在$(x_0,x_1)$或$(x_1,x_0+T)$至少还有一个根根,即$f(x)$在区间$(x_0,x_0+T)$内至少有两个根.

26.求定积分

$$\int_0^{2m\pi}\frac{dx}{\sin^4 x+\cos^4 x}dx$$

其中m为正整数.

解被积函数以2π为周期,故$\int_0^{2m\pi}\frac{dx}{\sin^4 x+\cos^4 x}dx=m\int_0^{2\pi}\frac{dx}{\sin^4 x+\cos^4 x}dx.$

$$\int_0^{2\pi}\frac{dx}{\sin^4 x+\cos^4 x}dx=\int_0^{2\pi}\frac{dx}{\left(\sin^2 x+\cos^2 x\right)^2-2\sin^2 x\cos^2 x}dx$$

$$=\int_0^{2\pi}\frac{dx}{1-\frac{1}{2}\sin^2 2x}dx=4\int_0^{\pi/2}\frac{dx}{1-\frac{1}{2}\sin^2 2x}dx(\sin^2 2x\text{周期为}\frac{\pi}{2})$$

$$=8\int_0^{\pi/2}\frac{dx}{2-\sin^2 2x}dx=-4\int_0^{\pi/2}\frac{d\cot 2x}{2\csc^2 2x-1}$$

$$=-4\int_0^{\pi/2}\frac{d\cot 2x}{2\cot^2 2x+1}=4\int_{-\infty}^{+\infty}\frac{du}{1+2u^2}=\frac{4}{\sqrt{2}}\arctan\sqrt{2}u\,|_{-\infty}^{+\infty}=2\sqrt{2}\pi.$$

$$\int_0^{2m\pi}\frac{dx}{\sin^4 x+\cos^4 x}dx=2m\sqrt{2}\pi.$$

第三章总练习题

1.为什么用Newton-Leibniz公式于下列积分会得到不正确结果?

$(1)\int_{-1}^{1}\frac{d}{dx}\left(e^{\frac{1}{x}}\right)dx.\frac{d}{dx}\left(e^{\frac{1}{x}}\right)=-\left(e^{\frac{1}{x}}\right)\frac{1}{x^2}[-1,1]$无界,从而不可积.

$(2)\int_0^{2\pi}\frac{d\tan x}{2+\tan^2 x}dx.u=\tan x$在$(0,2\pi)$的一些点不可导.

2.证明奇连续函数的原函数为偶函数,而偶连续函数的原函数之一为奇函数.

证设奇连续函数f的原函数为F,现在证明F是偶函数.

$F'(x)=f(x).(F(-x)-F(x))'=-F'(-x)-F'(x)=-f(-x)-f(x)=0,$

$F(-x)-F(x)=C,C=F(-0)-F(0)=0.F(-x)-F(x)=0.$

设偶连续函数f的原函数为F,现在证明F是奇函数.

$F'(x)=f(x).(F(-x)+F(x))'=-F'(-x)+F'(x)=-f(-x)+f(x)=0,$

$F(-x)-F(x)=C.$设$F(0)=0,$则$C=F(-0)-F(0)=0.F(-x)+F(x)=0.$

3.$f(x)\,f(x)=\begin{cases}\sin x, x\ge 0,\\ x^3,\quad x<0,\end{cases}$求定积分$\int_a^b f(x)dx=?$其中$a<0,b>0.$

解$\int_a^b f(x)dx=\int_a^0 f(x)dx+\int_0^b f(x)dx=\int_a^0 x^3dx+\int_0^b \sin xdx$

$=\left.\frac{x^4}{4}\right|_0^a-\cos x\,|_0^b=1+\frac{a^4}{4}-\cos b.$

4.求微商$\frac{d}{dx}\int_0^1\sin(x+t)dx.$

解$\frac{d}{dx}\int_0^1\sin(x+t)dx=\frac{d}{dx}\int_x^{x+1}\sin(u)du=\sin(x+1)-\sin(x).$

5.试证明$\lim\limits_{h\to 0}\int_0^1 f(x+ht)dx=f(x),$其中$f(x)$是实轴上的连续函数.

证$\lim\limits_{h\to 0}\frac{1}{h}\int_x^{x+h}f(x+ht)du=\left.\left(\int_0^u f(t)dt\right)'\right|_{u=x}=f(x).$

6.求极限$\lim\limits_{n\to\infty}\int_0^1(1-x^2)^n dx.$

解$\int_0^1(1-x^2)^n dx=\int_0^{\pi/2}\cos^{2n+1}tdt=I_{2n+1}=\frac{(2n)!!}{(2n+1)!!}.$

$(I_{2n+1})^2<\frac{2(2n)!!}{(2n+1)!!}\square\frac{(2n+1)!!}{(2n+2)!!}=\frac{1}{n+1},$

$0<I_{2n+1}<\frac{1}{\sqrt{n+1}}\to 0(n\to\infty),\lim\limits_{n\to\infty}\int_0^1(1-x^2)^n dx=0.$

7.$\int\frac{\sin x+\cos x}{2\sin x-3\cos x}dx.$

解令$\sin x+\cos x=A(2\sin x-3\cos x)+B(2\sin x-3\cos x)'$

$=A(2\sin x-3\cos x)+B(2\cos x+3\sin x)=(2A+3B)\sin x+(-3A+2B)\cos x,$

$$\begin{cases}2A+3B=1\\-3A+2B=1\end{cases}, A=-\frac{1}{13}, B=\frac{5}{13}.$$

$$\int\frac{\sin x+\cos x}{2\sin x-3\cos x}dx=$$

$$=\int\frac{A(2\sin x-3\cos x)+B(2\sin x-3\cos x)'}{2\sin x-3\cos x}dx$$

$$=Ax+B\ln|2\sin x-3\cos x|+C$$

$$=-\frac{1}{13}x+\frac{5}{13}\ln|2\sin x-3\cos x|+C.$$

8.通过适当的有理化或变量替换求下列积分：

(1) $\int\sqrt{e^x-2}dx.\sqrt{e^x-2}=u, x=\ln(2+u^2), dx=\frac{2udu}{2+u^2}.$

$$\int\sqrt{e^x-2}dx=2\int\frac{u^2du}{2+u^2}=2\left(u-2\int\frac{du}{2+u^2}\right)$$

$$=2\left(u-\sqrt{2}\arctan\frac{u}{\sqrt{2}}\right)+C=2\left(\sqrt{e^x-2}-\sqrt{2}\arctan\frac{\sqrt{e^x-2}}{\sqrt{2}}\right)+C.$$

(2) $\int\frac{xe^x}{\sqrt{e^x-2}}dx=\int\frac{x}{\sqrt{e^x-2}}d(e^x-2)=2\int xd\sqrt{e^x-2}$

$$=2x\sqrt{e^x-2}-2\int\sqrt{e^x-2}dx$$

$$=2x\sqrt{e^x-2}-4\left(\sqrt{e^x-2}-\sqrt{2}\arctan\frac{\sqrt{e^x-2}}{\sqrt{2}}\right)+C.$$

$$=2\sqrt{e^x-2}(x-2)+4\sqrt{2}\arctan\frac{\sqrt{e^x-2}}{\sqrt{2}}+C.$$

(3) $\int\sqrt{\frac{x}{1-x\sqrt{x}}}dx=\frac{2}{3}\int\frac{dx\sqrt{x}}{\sqrt{1-x\sqrt{x}}}=-\frac{2}{3}\times2\sqrt{1-x\sqrt{x}}+C$

$$=-\frac{4}{3}\sqrt{1-x\sqrt{x}}+C.$$

(4) $\int\frac{dx}{1+\sqrt{x}+\sqrt{1+x}}=\int\frac{(1+\sqrt{x}-\sqrt{1+x})dx}{(1+\sqrt{x}+\sqrt{1+x})(1+\sqrt{x}-\sqrt{1+x})}$

$$=\int\frac{(1+\sqrt{x}-\sqrt{1+x})dx}{2\sqrt{x}}=\frac{1}{2}\left(2\sqrt{x}+x-\sqrt{x(1+x)}+\ln(\sqrt{x}+\sqrt{1+x})\right)+C.$$

9.$\int\frac{dx}{\sin^4 x+\cos^4 x}=\int\frac{\sec^2 xd\tan x}{1+\tan^4 x}=\int\frac{(1+u^2)du}{1+u^4}.$

$$\frac{1+u^2}{1+u^4}=\frac{1+u^2}{(1+u^2+\sqrt{2}u)(1+u^2-\sqrt{2}u)}=\frac{1}{2}\left(\frac{1}{1+u^2+\sqrt{2}u}+\frac{1}{1+u^2-\sqrt{2}u}\right)$$

$$=\frac{1}{2}\left(\frac{1}{(u+\frac{1}{\sqrt{2}})^2+\left(\frac{1}{\sqrt{2}}\right)^2}+\frac{1}{(u-\frac{1}{\sqrt{2}})^2+\left(\frac{1}{\sqrt{2}}\right)^2}\right).$$

$$\int\frac{dx}{\sin^4 x+\cos^4 x}=\frac{1}{\sqrt{2}}\left(\arctan(\sqrt{2}u+1)+\arctan(\sqrt{2}u-1)\right)+C.$$

10.设函数$f(x)$在$(-\infty,+\infty)$上连续,以T为周期,令$g(x)=f(x)-\frac{1}{T}\int_0^T f(x)dx$,证明:

函数$h(x)=\int_0^x g(t)dt$也以T为周期.

证(此即习题3.4第24题)

11.设函数$f(x)$在区间$[a,b]$上连续,且$\int_a^b f(x)dx=0$.证明:在(a,b)内至少存在一点c,

使$f(c)=0$.

证若不然,$f(x)$在(a,b)没有 零点,由f的连续性和连续函数的中间值定理,

f在(a,b)不变号.不妨设$f(x)>0,x\in(a,b)$.取c,d满足,$a<c<d<b$,则f在$[c,d]$取最小值

$m>0$.于是

$$\int_a^b f(x)dx=\int_a^c f(x)dx+\int_c^d f(x)dx+\int_d^b f(x)dx\geq m(d-c)>0.$$

矛盾.

12.设函数f在区间$[a,b]$上连续,且$\int_a^b f^2(x)dx=0$,证明:$f(x)\equiv 0,x\in[a,b]$.

证若不然,存在c$\in$[a,b],f(c)$\neq$0.由f在c的连续性,存在区间 $[d,e]\subseteq[a,b]$,

$|f(x)|^2>\frac{|f(c)|^2}{2},x\in[d,e]$.

$$\int_a^b f^2(x)dx\geq\int_d^e f^2(x)dx>\frac{|f(c)|^2}{2}(d-e)>0.$$

矛盾.

13.设$f(x)$在$(-\infty,+\infty)$上可积,证明

(1)对于任意实数a,有$\int_0^a f(x)dx=\int_0^a f(a-x)dx$;

(2)$\int_0^\pi\frac{x\sin x}{1+\cos^2 x}dx=\frac{\pi^2}{4}$;

(3)$\int_0^{\pi/2}\frac{\sin^2 x}{\cos x+\sin x}dx=\frac{1}{\sqrt{2}}\ln(1+\sqrt{2})$.

证(1)$\int_0^a f(x)dx(x=a-t)=-\int_a^0 f(a-t)dt=\int_0^a f(a-t)dt=\int_0^a f(a-x)dx.$

$$(2) I=\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}dx=-\int_0^{\pi}\frac{(x-\pi)\sin x}{1+\cos^2 x}dx=\int_0^{\pi}\frac{(\pi-x)\sin x}{1+\cos^2 x}dx=\int_0^{\pi}\frac{\pi\sin x}{1+\cos^2 x}dx-I,$$

$$I=\frac{\pi}{2}\int_0^{\pi}\frac{\sin xdx}{1+\cos^2 x}=-\frac{\pi}{2}\int_0^{\pi}\frac{d\cos x}{1+\cos^2 x}=\int_0^1\frac{\pi du}{1+u^2}=\pi\arctan u\Big|_0^1=\frac{\pi^2}{4}.$$

$$(3) I=\int_0^{\pi/2}\frac{\sin^2 x}{\cos x+\sin x}dx=\int_0^{\pi/2}\frac{\sin^2(\pi/2-x)}{\cos(\pi/2-x)+\sin(\pi/2x)}dx$$

$$=\int_0^{\pi/2}\frac{\cos^2 x}{\cos x+\sin x}dx, 2I=\int_0^{\pi/2}\frac{\cos^2 x}{\cos x+\sin x}dx+\int_0^{\pi/2}\frac{\sin^2 x}{\cos x+\sin x}dx$$

$$=\int_0^{\pi/2}\frac{dx}{\cos x+\sin x}dx=\int_0^{\pi/2}\frac{dx}{\sqrt{2}\sin(x+\pi/4)}=\frac{1}{\sqrt{2}}\ln|\csc(x+\pi/4)-\cot(x+\pi/4)|\Big|_0^{\pi/2}$$

$$=\frac{1}{\sqrt{2}}\left[\ln\left(\frac{1}{\cos\frac{\pi}{4}}+1\right)\right]-\ln\left(\frac{1}{\sin\frac{\pi}{4}}-1\right)=\sqrt{2}\ln(\sqrt{2}+1),$$

$$I=\frac{1}{\sqrt{2}}\ln(\sqrt{2}+1).$$

14.一质点作直线运动, 其加速度$a(t)=(2t-3)\text{m/s}^2$.若$t=0$时$x=0$且$v=-4\text{m/s}$,求
(1)质点改变动方向的时刻;
(2)头5秒钟内质点所走的总路程.

解(1)$x''(t)=2t-3, x'=t^2-3t+C_1, -4=C_1, x'=t^2-3t-4, x=\frac{t^3}{3}-\frac{3}{2}t^2-4t+C_2,$

$0=C_2. x(t)=\frac{t^3}{3}-\frac{3}{2}t^2-4t. x'=t^2-3t-4=(t-4)(t+1)=0, t_0=4.$

$$s=x(5)-x(4)+|x(4)|=\left(\frac{t^3}{3}-\frac{3}{2}t^2-4t\right)\Big|_{t=5}-2\left(\frac{t^3}{3}-\frac{3}{2}t^2-4t\right)\Big|_{t=4}=\frac{43}{2}\text{m}.$$

15.一运动员跑完100m, 共用了10.2s, 在跑头25m时以等加速度进行, 然后保持等速运动跑完了剩余路程. 求跑头25m时的加速度.

解$v(t)=\begin{cases}at, & 0\le t\le t_0;\\ at_0, & t_0\le t\le 10.2.\end{cases}$

$$s(t)=\begin{cases}\frac{at^2}{2}, & 0\le t\le t_0;\\ at_0t+C & t_0\le t\le 10.2.\end{cases}$$

$$\begin{cases}at_0^2/2=at_0^2+C\\ at_0^2/2=25\\ 100=10.2at_0+C_2\end{cases}\quad a\approx 3\text{m/s}^2.$$

16.(1)利用积分的几何意义证明:

$$\frac{1}{n+1}<\ln\frac{n+1}{n}<\frac{1}{n}, n=1,2,\cdots$$

(2)令$x_n=1+\frac{1}{2}+\cdots+\frac{1}{n-1}-\ln n,$

$y_n=1+\frac{1}{2}+\cdots+\frac{1}{n-1}+\frac{1}{n}-\ln n,$

证明序列x_n单调上升,而序列y_n单调下降.

(3)证明极限$\lim\limits_{n\to\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{n-1}+\frac{1}{n}-\ln n\right)$存在(此极限称为Euler常数).

证 (1)$\frac{1}{n+1}=\int_n^{n+1}\frac{dx}{n+1}<\int_n^{n+1}\frac{dx}{x}=\ln x\big|_n^{n+1}$

$=\ln(n+1)-\ln n=\ln\frac{n+1}{n}<\int_{\mathrm{n}}^{\mathrm{n}+1}\frac{dx}{n}=\frac{1}{n}.$

(2)$x_{n+1}-x_n=\left(1+\frac{1}{2}+\cdots+\frac{1}{n}-\ln(n+1)\right)-\left(1+\frac{1}{2}+\cdots+\frac{1}{n-1}-\ln n\right)$

$=\frac{1}{n}-\ln\left(1+\frac{1}{n}\right)>0$(由(1)).

$y_{n+1}-y_n=\left(1+\frac{1}{2}+\cdots+\frac{1}{n}+\frac{1}{n+1}-\ln(n+1)\right)-\left(1+\frac{1}{2}+\cdots+\frac{1}{n}-\ln n\right)$

$=\frac{1}{n+1}-\ln\left(1+\frac{1}{n}\right)<0$(由(1)).

(3)$y_n>x_n>x_2=1-\ln 2>0(n>2)$.$y_n$单调下降有下界,故有极限$\lim\limits_{n\to\infty}y_n$.

17.证明:当$x>0$时,

$\int_x^1\frac{1}{1+t^2}dt=\int_1^{1/x}\frac{1}{1+t^2}dt.$

证$\int_x^1\frac{1}{1+t^2}dt(x=1/u)=\int_1^{1/x}\frac{1}{1+1/u^2}\times\frac{1}{u^2}dx=\int_1^{1/x}\frac{1}{1+t^2}dt.$

18.设$f(x)$在$(-\infty,+\infty)$上连续(书上为可积，欠妥),且对一切实数x,均有

$f(2-x)=-f(x)$.求实数$a\neq 2$,使$\int_a^2 f(x)dx=0.$

解(条件$f(2-x)=-f(x)$相当f关于$x=1$为奇函数$f(1+1-x)=-f(1+x-1)$)

$\int_0^2 f(x)dx=\int_0^2 f(2-u)du=-\int_0^2 f(u)du, \int_0^2 f(x)dx=0$.取$a=0$即可.

19.利用定积分的性质，证明不等式$\ln(1+x)\le\arctan x, 0\le x\le 1$.

证$\frac{1}{1+t}\le\frac{1}{1+t^2}, t\in[0,1]$,在$[0,x]$上积分得$\int_0^x\frac{dt}{1+t}\le\int_0^x\frac{dt}{1+t^2},$

$\ln(1+x)\le\arctan x, 0\le x\le 1.$

20.(1)设$f(x)$在$[0,a]$上可积，证明$\int_0^a\frac{f(x)dx}{f(x)+f(a-x)}dx=\frac{a}{2};$

(2)利用(1)中的公式求下列积分的值:

$$\int_0^2 \frac{x^2}{x^2-2x+2}dx;\int_0^{\pi/2}\frac{\sin x}{\sin x+\cos x}dx$$

证(1) $I=\int_0^a \frac{f(x)}{f(x)+f(a-x)}dx=\int_0^a \frac{f(a-u)}{f(u)+f(a-u)}du$

$$2I=\int_0^a \frac{f(x)}{f(x)+f(a-x)}dx+\int_0^a \frac{f(a-u)}{f(u)+f(a-u)}du$$

$$=\int_0^a \frac{f(x)}{f(x)+f(a-x)}dx+\int_0^a \frac{f(a-x)}{f(x)+f(a-x)}dx=\int_0^a 1dx=a, I=\frac{a}{2}.$$

解 (2) $\int_0^2 \frac{x^2}{x^2-2x+2}dx=2\int_0^2 \frac{x^2}{x^2+(2-x)^2}dx=2\times\frac{2}{2}=2.$

$$\int_0^{\pi/2}\frac{\sin x}{\sin x+\sin(\pi/2-x)}dx=\frac{\pi/2}{2}=\frac{\pi}{4}.$$

21.设 $f(x)=\int_{\sin x}^{\tan x}(1+xt^2)dt$ 求 $\frac{df(x)}{dx}$.

解 $f(x)=\int_{\sin x}^{\tan x}(1+xt^2)dt=\tan x-\sin x+x\int_{\sin x}^{\tan x}t^2dt,$

$$\frac{df(x)}{dx}=\sec^2 x-\cos x+x\tan^2 x\sec^2 x-x\sin^2 x\cos x+\int_{\sin x}^{\tan x}t^2dt$$

$$=\sec^2 x-\cos x+x\tan^2 x\sec^2 x-x\sin^2 x\cos x+\frac{t^3}{3}\Big|_{\sin x}^{\tan x}$$

$$==\sec^2 x-\cos x+x\tan^2 x\sec^2 x-x\sin^2 x\cos x+\frac{1}{3}\left(\tan^3 x-\sin^3 x\right)$$

$$=\sec^2 x(1+x\tan^2 x)-\cos x(1+x\sin^2 x)+\frac{1}{3}\left(\tan^3 x-\sin^3 x\right).$$

22.求定积分 $I=\int_0^{\pi/2}\cos^2 3\theta d\theta$ 的值.

解 $I=\int_0^{\pi/2}\cos^2 3\theta d\theta=\frac{1}{2}\int_0^{\pi/2}(1+\cos 6\theta)d\theta=\frac{\pi}{4}+\frac{1}{12}\sin 6\theta\Big|_0^{\pi/2}=\frac{\pi}{4}.$

23.求定积分 $I=\int_0^{2\pi}|\sin x-\cos x|dx$ 的值.

解 $\mathrm{I}=2\int_0^{\pi}|\sin x-\cos x|dx$

$$=2\left(\int_0^{\pi/2}|\sin x-\cos x|dx+\int_{\pi/2}^{\pi}|\sin x-\cos x|dx\right)$$

$$=2\left(\int_0^{\pi/2}|\sin x-\cos x|dx+\int_0^{\pi/2}|\sin(\frac{\pi}{2}+t)-\cos(\frac{\pi}{2}+t)|dx\right)$$

$$=2\left(\int_0^{\pi/2}|\sin x-\cos x|dx+\int_0^{\pi/2}|\cos t+\sin t|dx\right)$$

$$=2\left(\int_0^{\pi/4}(\cos x-\sin x)dx+\int_{\pi/4}^{\pi/2}(\sin x-\cos x)dx+\int_0^{\pi/2}(\cos t+\sin t)dx\right)$$

$$=2\left((\sin x+\cos x)\Big|_0^{\pi/4}+(-\cos x-\sin x)\Big|_{\pi/4}^{\pi/2}+(\sin x-\cos x)\Big|_0^{\pi/2}\right)=4\sqrt{2}.$$

24.设$0 < x_0 < x_1$,求定积分$I = \int_{x_0}^{x_1} \sqrt{(x-x_0)(x_1-x)}dx$的值.

解$I = \int_{x_0}^{x_1} \sqrt{(x-x_0)(x_1-x)}dx$

$= \int_{x_0}^{x_1} \sqrt{-x^2+(x_1+x_0)x-x_0x_1}dx$

$= \int_{x_0}^{x_1} \sqrt{-\left(x-\frac{x_1+x_0}{2}\right)^2+\frac{(x_1+x_0)^2}{4}-x_0x_1}dx$

$= \int_{x_0}^{x_1} \sqrt{-\left(x-\frac{x_1+x_0}{2}\right)^2+\frac{(x_1-x_0)^2}{4}}dx\left(u = x-\frac{x_1+x_0}{2}\right)$

$= \int_{-(x_1-x_0)/2}^{(x_1-x_0)/2} \sqrt{-(u)^2+\frac{(x_1-x_0)^2}{4}}du$

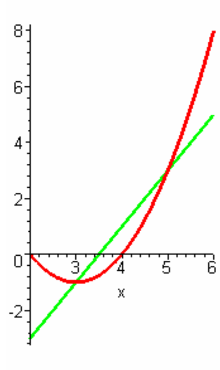
$= 2\int_0^a \sqrt{a^2-u^2}dx(a = \frac{x_1-x_0}{2})$

$= \left[u\sqrt{a^2-u^2}+a^2\arcsin\frac{u}{a}\right]\Big|_0^a$

$= \frac{\pi a^2}{2} = \frac{\pi(x_1-x_0)^2}{8}$.

25.求下列曲线所围图形的面积:

(1)$y = x^2-6x+8$与$y = 2x-7$.



解$\begin{cases} y = x^2-6x+8 \\ y = 2x-7 \end{cases}$ $2x-7 = x^2-6x+8$,

$x^2-8x+15 = 0, (x-3)(x-5) = 0$.

$x_1 = 3, x_2 = 5$.

$S = \int_3^5 (2x-7-(x^2-6x+8))dx = \int_3^5 (-x^2+8x-15)dx$

$= \left(-\frac{x^3}{3}+4x^2-15x\right)\Big|_3^5 = \frac{4}{3}$.

(2)$y = x^4+x^3+16x-4$与$y = x^4+6x^2+8x-4$.

解$\begin{cases} y = x^4+x^3+16x-4 \\ y = x^4+6x^2+8x-4 \end{cases}$ $x^3+16x-4 = 6x^2+8x-4, x^3-6x^2+8x = 0$,

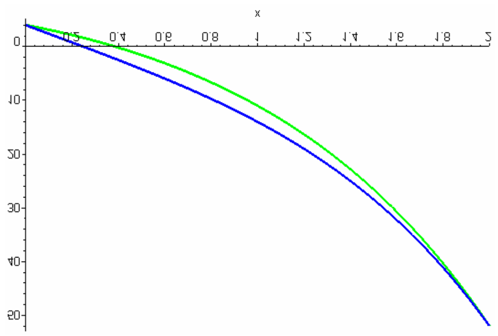
$x = 0, x^2-6x+8 = 0, (x-2)(x-4) = 0, x = 2, 4$.

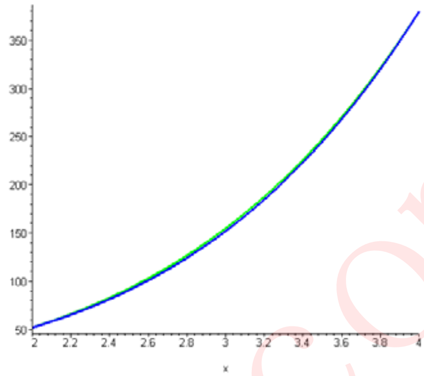
$S = \int_0^2 \{(x^4+x^3+16x-4)-(x^4+6x^2+8x-4)]dx$

$+\int_2^4 [(x^4+6x^2+8x-4)-(x^4+x^3+16x-4)]dx$

$$=\int_0^2 (x^3-6x^2+8x)dx+\int_2^4(-x^3+6x^2-8x)dx$$

$$=\left[\frac{x^4}{4}-2x^3+4x^2\right]\Bigg|_0^2+\left[-\frac{x^4}{4}+2x^3-4x^2\right]\Bigg|_2^4=8.$$





(3) $y^2=x-1$ 与 $y=x-3$.

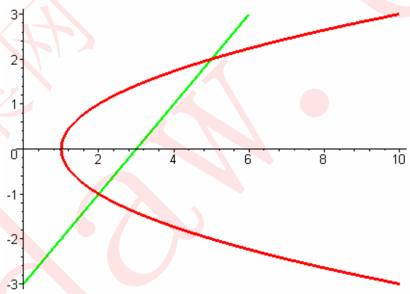
解 $(x-3)^2=x-1, x^2-7x+10=0,$

$(x-2)(x-5)=0,$

$x=2,5. y=-1,2.$

$$S=\int_{-1}^2[(y+3)-(1+y^2)]dx$$

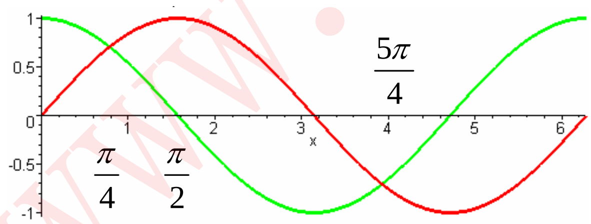
$$=\left(-\frac{y^3}{3}+\frac{y^2}{2}+2y\right)\Bigg|_{-1}^2=\frac{9}{2}.$$



(4) $y=\sin x, y=\cos x$ 与 $x=\pi/2$.

解 $S=\int_{\pi/4}^{\pi/2}(\sin x - \cos x)dx=(-\cos x-\sin x)\big|_{\pi/4}^{\pi/2}=\sqrt{2}-1;$

$$S=\int_{\pi/2}^{\pi 5/4}(\sin x-\cos x)dx=(-\cos x-\sin x)\big|_{\pi/2}^{5\pi/4}=\sqrt{2}+1.$$



26.设区域 σ 由曲线 $y=\cos x, y=1$ 及 $x=\pi/2$ 所围成,将 σ 绕 x 轴旋转一周,得一旋转体 V. 试用两种不同的积分表示体积 V,并且求 V 的值.

解 $V=\pi\int_0^{\pi/2}(1-\cos^2 x)dx=2\pi\int_0^1 y\left(\frac{\pi}{2}-\arccos y\right)dy=2\pi\int_0^1 y\arcsin y dy=$

$$V=\frac{\pi}{2}\int_0^{\pi/2}(1-\cos 2x)dx=\frac{\pi^2}{4}.$$

$$V=2\pi\int_0^1 y\arcsin y dy==\pi\int_0^1\arcsin y dy^2$$

$$=\pi\arcsin y(y^2)\Big|_0^1-\pi\int_0^1 y^2\times\frac{1}{\sqrt{1-y^2}}dx$$

$$=\frac{\pi^2}{2}-\pi\left[\frac{y}{2}\sqrt{1-y^2}+\frac{1}{2}\arcsin y\right]\Bigg|_0^1=\frac{\pi^2}{2}-\frac{\pi^2}{4}=\frac{\pi^2}{4}.$$

27.求下列定积分的值：

(1)$\displaystyle\int_{\sqrt{2}}^{2}\frac{du}{u\sqrt{u^2-1}}=\int_{\sqrt{2}}^{2}\frac{du}{u^2\sqrt{1-1/u^2}}=\int_{1/2}^{1/\sqrt{2}}\frac{dx}{\sqrt{1-x^2}}=\arcsin x\Big|_{1/2}^{1/\sqrt{2}}=\frac{\pi}{4}-\frac{\pi}{6}=\frac{\pi}{12}.$

(2)$\displaystyle\int_{-200}^{200}(91x^{21}-80x^{33}+5580x^{97}+1)dx=400.$

28.设$f(x)$在$[0,7]$上可积，且一直已知$\int_0^2 f(x)dx=5,\int_2^5 f(x)dx=6,\int_0^7 f(x)dx=3.$

(1)求$\int_0^5 f(x)dx$的值;

(2)求$\int_5^7 f(x)dx$的值.

(3)证明：在$(5,7)$内至少存在一点，使$f(x)<0.$

解(1)$\int_0^5 f(x)dx=\int_0^2 f(x)dx+\int_2^5 f(x)dx=5+6=11.$

(2)$\int_5^7 f(x)dx=\int_0^7 f(x)dx-\int_0^5 f(x)dx=3-11=-8.$

证(3)若不然，$f(x)\ge 0, x\in(5,7),$

$\int_5^7 f(x)dx\ge 0,$但是$\int_5^7 f(x)dx=-8<0,$矛盾.

29.设$f(x)=\sin x, h(x)=\dfrac{1}{x^2}, g(x)=\begin{cases}1, & -\pi\le x\le 2,\\ 2, & 2<x\le\pi.\end{cases}$试求下列定积分的值或表达式：

(1)$\int_{-\pi/2}^{\pi/2}f(x)g(x)dx$;(2)$\int_1^3 g(x)h(x)dx$;(3)$\int_{\pi/2}^{x}f(t)g(t)dx.$

解(1)$\int_{-\pi/2}^{\pi/2}f(x)g(x)dx=\int_{-\pi/2}^{\pi/2}\sin xdx=0.$

(2)$\int_1^3 g(x)h(x)dx=\int_1^2 g(x)h(x)dx+\int_2^3 g(x)h(x)dx$

$$=\int_1^2\frac{1}{x^2}dx+\int_2^3\frac{2}{x^2}dx=-\frac{1}{x}\Big|_1^2-\frac{2}{x}\Big|_2^3=\frac{5}{6}.$$

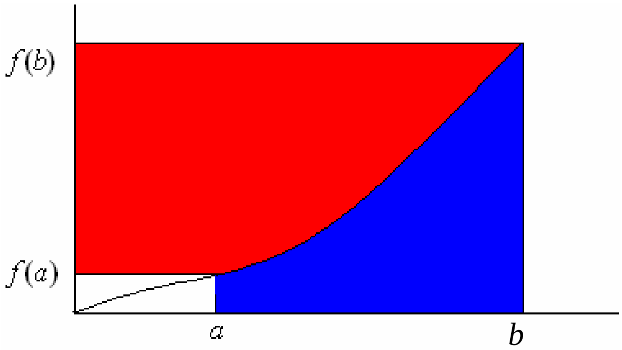
$$(3)\int_{\pi/2}^{x}f(t)g(t)dx=\begin{cases}\displaystyle\int_{\pi/2}^{x}\sin tdt=-\cos x, t-\pi\le x\le 2\\ \displaystyle\int_{\pi/2}^{2}\sin tdt+\int_2^x 2\sin tdt=\cos 2-2\cos x, 2<x\le\pi.\end{cases}$$

30设函数$f(x)$在区间$[a,b]$上连续，严格单调递增$(a>0)$, $g(y)$是$f(x)$的反函数，利用定积分的几何意义证明下列公式

$$\int_a^b f(x)dx=bf(b)-af(a)-\int_{f(a)}^{f(b)}g(y)dx.$$

并作图解释这一公式.

解



31.(1)设函数$\varphi(x)$在$[0,+\infty)$上连续且严格单调递增,又设当$x\to+\infty$时$\varphi(x)\to+\infty$且$\varphi(0)$=0. 证明:对于任意实数$a\geq 0, B\geq 0$, 下列不等式成立:

$$aB\leq\int_0^a\varphi(x)dx+\int_0^B\varphi^{-1}(x)dx$$

其中$\varphi^{-1}(x)$是$\int_0^a\varphi(x)$的反函数.

证 由30题, $\int_0^a\varphi(x)dx+\int_0^{\varphi(a)}\varphi^{-1}(x)dx=a\varphi(a)(*)$.

$B=0$时不等式显然成立. 设$B>0=\varphi(0)$,由于$x\to+\infty$时$\varphi(x)\to+\infty$,存在$a'>0$, $\varphi(a')>B$,φ在$[0,a']$连续,根据连续函数的中间值定理,存在$a_1>0,\varphi(a_1)=B$.

若$a_1=a$,则由(*)得$aB=\int_0^a\varphi(x)dx+\int_0^B\varphi^{-1}(x)dx$.

若$a_1>a$,则$\int_0^a\varphi(x)dx+\int_0^B\varphi^{-1}(x)dx$

$=\int_0^a\varphi(x)dx+\int_0^{\varphi(a)}\varphi^{-1}(x)dx+\int_{\varphi(a)}^B\varphi^{-1}(x)dx$

$=a\varphi(a)++\int_{\varphi(a)}^B\varphi^{-1}(x)dx$

$\geq a\varphi(a)+\varphi^{-1}(\varphi(a))(B-\varphi(a))=aB.$

若$a_1<a$,则$\int_0^a\varphi(x)dx+\int_0^B\varphi^{-1}(x)dx$

$=\int_0^a\varphi(x)dx+\int_0^{\varphi(a)}\varphi^{-1}(x)dx-\int_B^{\varphi(a)}\varphi^{-1}(x)dx$

$=a\varphi(a)-\int_B^{\varphi(a)}\varphi^{-1}(x)dx$

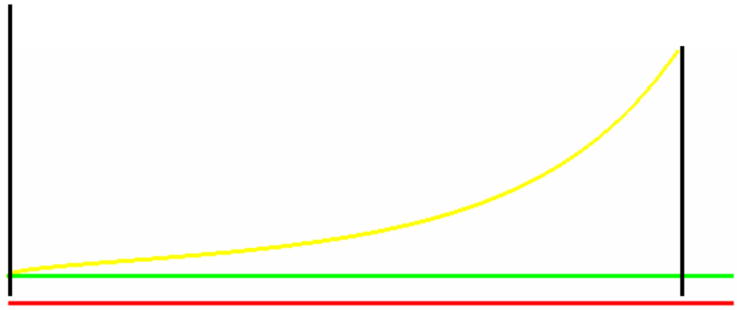
$\geq a\varphi(a)-\varphi^{-1}(\varphi(a))(\varphi(a)-B)=aB.$

(2)利用(1)中的不等式, 对于任意实数$a,b\geq 0,\ p,q\geq 1\frac{1}{p}+\frac{1}{q}=1$,证明下列Minkowski不等式$ab\leq\frac{a^p}{p}+\frac{b^q}{q}$.

证 不妨设$p>1$.在(1)中取$\varphi(x)=x^{p-1}$,则$\varphi^{-1}(x)=x^{1/(p-1)}$.

$$ab\leq\int_0^a x^p dx+\int_0^b x^{1/p}dx=\frac{a^p}{p}+\frac{b^{1/(p-1)+1}}{1/(p-1)+1}=\frac{a^p}{p}+\frac{b^{p/(p-1)}}{p/(p-1)}=\frac{a^p}{p}+\frac{b^q}{q}.$$

32.设$a>0$,求a的值,使由曲线$y=1+\sqrt{x}e^{x^2}$, $y=1$及$x=a$所围成的区域绕直线$y=1$旋转所得之旋转体的体积等于2π.

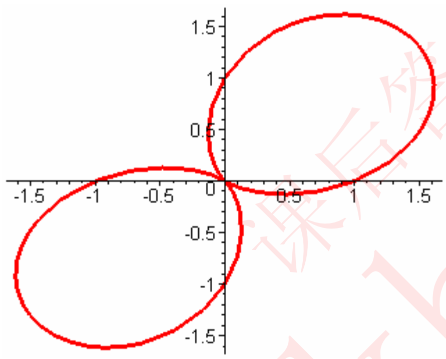


解$\pi\int_0^a (y-1)^2 dx = 2\pi . \int_0^a (\sqrt{x}e^{x^2})^2 dx = 2,$

$\int_0^a xe^{2x^2} dx = 2, \frac{1}{4}\int_0^a e^{2x^2} d2x^2 = 2, \frac{1}{4}\int_0^{2a^2} e^u du = 2, e^{2a^2} - 1 = 8, 2a^2 = \ln 9, a = \sqrt{\ln 3}.$

33.作由极坐标方程$r = 1 + \sin 2\theta$所确定的函数的图形, 并求它所围区域的面积.

解$S = \int_0^{\pi} (1+\sin 2\theta)^2 d\theta = \int_0^{\pi} (1 + 2\sin 2\theta + \frac{1-\cos 4\theta}{2}) d\theta = \frac{3\pi}{2}.$



习题 3.1

求下列不定积分：

$$1.\int\sqrt{1+2x}dx=\frac{1}{2}\int\sqrt{1+2x}d(1+2x)=\frac{1}{3}(1+2x)^{3/2}+C.$$

$$2.\int\frac{3x}{(x^2+1)^2}dx=\frac{1}{2}\int\frac{3}{(x^2+1)^2}d(x^2+1)=-\frac{3}{2(x^2+1)}+C.$$

$$3.\int x\sqrt{2x^2+7}dx=\frac{1}{4}\int\sqrt{2x^2+7}d(2x^2+7)=\frac{1}{6}(2x^2+7)^{3/2}+C.$$

$$4.\int(2x^{3/2}+1)^{2/3}\sqrt{x}dx=\frac{2}{3}\int(2x^{3/2}+1)^{2/3}dx^{3/2}$$

$$=\frac{2}{3}\cdot\frac{1}{2}\int(2x^{3/2}+1)^{2/3}d(2x^{3/2}+1)=\frac{1}{5}(2x^{3/2}+1)^{5/3}+C.$$

$$5.\int\frac{e^{1/x}}{x^2}dx=-\int e^{1/x}d(1/x)=-e^{1/x}+C.$$

$$6.\int\frac{dx}{(2-x)^{100}}=-\int\frac{d(2-x)}{(2-x)^{100}}=\frac{1}{99(2-x)^{99}}+C.$$

$$7.\int\frac{dx}{3+5x^2}=\frac{1}{3}\int\frac{dx}{1+[(5/3)x]^2}=\frac{1}{3}\cdot\sqrt{\frac{3}{5}}\int\frac{d\sqrt{5/3}x}{1+[\sqrt{5/3}x]^2}=\frac{1}{\sqrt{15}}\arctan\sqrt{\frac{5}{3}}x+C.$$

$$8.\int\frac{dx}{\sqrt{7-3x^2}}=\int\frac{dx}{\sqrt{7}\sqrt{1-3/7x^2}}=\frac{1}{\sqrt{7}}\cdot\sqrt{\frac{7}{3}}\int\frac{d\sqrt{3/7}x}{\sqrt{7}\sqrt{1-\sqrt{3/7}x^2}}=\frac{1}{\sqrt{3}}\arcsin\sqrt{\frac{3}{7}}x+C.$$

$$9.\int\frac{dx}{\sqrt{x}(1+x)}=2\int\frac{d\sqrt{x}}{(1+x)}=2\arctan\sqrt{x}+C.$$

$$10.\int\frac{e^x}{2+e^{2x}}dx=\int\frac{1}{2+\left(e^x\right)^2}de^x=\frac{1}{\sqrt{2}}\arctan e^x+C.$$

$$11.\int\frac{dx}{\sqrt{e^{-2x}-1}}=\int\frac{de^x}{\sqrt{1-(e^x)^2}}=\arcsin e^x+C.$$

$$12.\int\frac{dx}{e^x-e^{-x}}=\int\frac{de^x}{e^{2x}-1}=\int\frac{du}{(u-1)(u+1)}=\frac{1}{2}\int\left(\frac{1}{u-1}-\frac{1}{u+1}\right)du$$

$$=\frac{1}{2}\ln\frac{u-1}{u+1}+C=\frac{1}{2}\ln\frac{e^x-1}{e^x+1}+C.$$

$$13.\int\frac{\ln\ln x}{x\ln x}dx=\int\frac{\ln\ln x}{\ln x}d\ln x=\int\ln\ln x d\ln\ln x=\frac{1}{2}(\ln\ln x)^2+C.$$

$$14.\int\frac{dx}{1+\cos x}=\int\frac{dx}{2\sin^2\frac{x}{2}}=\int\frac{d\frac{x}{2}}{\sin^2\frac{x}{2}}=-\cot^2\frac{x}{2}+C.$$

$$15.\int\frac{dx}{1-\sin x}=\int\frac{d\left(x+\frac{\pi}{2}\right)}{1+\cos\left(x+\frac{\pi}{2}\right)}=-\cot^2\left(\frac{x}{2}+\frac{\pi}{4}\right)+C.$$

$$16.\int\frac{x^{14}}{(x^5+1)^4}dx=\frac{1}{5}\int\frac{x^{10}}{(x^5+1)^4}dx^5=\frac{1}{5}\int\frac{u^2}{(u+1)^4}du(u=x^5)$$

$$=\frac{1}{5}\int\frac{u^2-1+1}{(u+1)^4}du=\frac{1}{5}\int\frac{(v-1)^2}{v^4}dv(v=u+1)$$

$$=\frac{1}{5}\int\frac{v^2-2v+1}{v^4}dv=\frac{1}{5}\int\left(v^{-2}-2v^{-3}+v^{-4}\right)dv$$

$$=\frac{1}{5}\left(-v^{-1}+v^{-2}-\frac{1}{3}v^{-3}\right)+C=\frac{1}{5}\left(-(x^5+1)^{-1}+(x^5+1)^{-2}-\frac{1}{3}(x^5+1)^{-3}\right)+C.$$

$$17.\int\frac{x^{2n-1}}{x^n-1}dx=\frac{1}{n}\int\frac{x^n}{x^n-1}dx^n=\frac{1}{n}\int\frac{u}{u-1}du(u=x^n)$$

$$=\frac{1}{n}\int\left(1+\frac{1}{u-1}\right)du=\frac{1}{n}(u+\ln|u-1|)+C=\frac{1}{n}(x^n+\ln|x^n-1|)+C.$$

$$18.\int\frac{dx}{x(x^5+2)}=\int\frac{x^4dx}{x^5(x^5+2)}=\frac{1}{5}\int\frac{du}{u(u+2)}(u=x^5)$$

$$=\frac{1}{5}\frac{1}{2}\int\left(\frac{1}{u}-\frac{1}{u+2}\right)du=\frac{1}{10}(\ln|u|-\ln|u+2|)+C=\frac{1}{10}\ln\left|\frac{u}{u+2}\right|+C.$$

$$19.\int\frac{\ln(x+1)-\ln x}{x(x+1)}dx=\int(\ln(x+1)-\ln x)\left(\frac{1}{x}-\frac{1}{x+1}\right)dx$$

$$=\int(\ln(x+1)-\ln x)d(\ln x-\ln(x+1))=-\int(\ln(x+1)-\ln x)d(\ln(x+1)-\ln x)$$

$$=-\frac{1}{2}\ln^2\frac{x+1}{x}+C.$$

$$20.\int\frac{e^{\arctan x}+x\ln(1+x^2)}{1+x^2}dx=\int\frac{e^{\arctan x}}{1+x^2}dx+\int\frac{x\ln(1+x^2)}{1+x^2}dx$$

$$=\int e^{\arctan x}d\arctan x+\frac{1}{2}\int\ln(1+x^2)d\ln(1+x^2)$$

$$=e^{\arctan x}+\frac{1}{4}\ln^2(1+x^2)+C.$$

$$21.\int \sin 2x\cos 2xdx=\frac{1}{2}\int \sin 2xd\sin 2x=\frac{1}{4}\sin^2 2x+C.$$

$$22.\int \sin^2\frac{x}{2}\cos\frac{x}{2}dx=2\int \sin^2\frac{x}{2}d\sin\frac{x}{2}=\frac{2}{3}\sin^3\frac{x}{2}+C.$$

$$23.\int \sin 5x\sin 6xdx=\frac{1}{2}\int(\cos x-\cos 11x)dx=\frac{1}{2}\left(\sin x-\frac{1}{11}\sin 11x\right)+C.$$

$$24.\int\frac{2x-1}{\sqrt{1-x^2}}dx=\int\frac{2x}{\sqrt{1-x^2}}dx-\int\frac{1}{\sqrt{1-x^2}}dx$$

$$=-\int\frac{d(1-x^2)}{\sqrt{1-x^2}}-\arcsin x+C=-2\sqrt{1-x^2}-\arcsin x+C.$$

$$25.\int\frac{x^3+x}{\sqrt{1-x^2}}dx=\int\frac{x^3}{\sqrt{1-x^2}}dx+\int\frac{x}{\sqrt{1-x^2}}dx$$

$$=\frac{1}{2}\int\frac{x^2}{\sqrt{1-x^2}}dx^2-\frac{1}{2}\int\frac{d(1-x^2)}{\sqrt{1-x^2}}$$

$$=\frac{1}{2}\int\sqrt{1-x^2}d(1-x^2)+\frac{1}{2}\int\frac{1}{\sqrt{1-x^2}}dx^2-\sqrt{1-x^2}$$

$$=\frac{1}{3}(1-x^2)^{3/2}-2\sqrt{1-x^2}+C.$$

$$26.\int\frac{dx}{(a^2-x^2)^{3/2}}(a>0)$$

$$x=a\sin t,t\in(-\pi/2,\pi/2),dx=a\cos tdt,$$

$$(a^2-x^2)^{3/2}=a^3\cos^3 t,$$

$$\int\frac{dx}{(a^2-x^2)^{3/2}}=\int\frac{dt}{a^2\cos^2 t}dx=\frac{1}{a^2}\tan t+C$$

$$=\frac{1}{a^2}\frac{x/a}{\sqrt{1-(x/a)^2}}+C=\frac{x}{a^2\sqrt{a^2-x^2}}+C.$$

$x<0$时，令$x=-y,y>0,$

$$\int\frac{\sqrt{x^2-a^2}}{x}dx=\int\frac{\sqrt{y^2-a^2}}{y}dy=\sqrt{y^2-a^2}-a\arccos\frac{a}{y}+C$$

$$=\sqrt{x^2-a^2}-a\arccos\frac{a}{-x}+C=\sqrt{x^2-a^2}-\left(\pi-a\arccos\frac{a}{x}\right)+C$$

$$=\sqrt{x^2-a^2}+a\arccos\frac{a}{x}+C'.$$

$$27.\int\frac{\sqrt{x^2-a^2}}{x}dx(a>0).x>0\text{时},\text{令}x=a\sec t,t\in(0,\pi/2).$$

$$dx=a\tan t\sec t dt,\sqrt{x^2-a^2}=a\tan t,$$

$$\int\frac{\sqrt{x^2-a^2}}{x}dx=a\int\tan^2 t dt=a\int(\sec^2 t-1)dt=a(\tan t-t)+C$$

$$=a(\sqrt{\sec^2 t-1}-\arccos\frac{a}{x})+C=a(\sqrt{\left(\frac{x}{a}\right)^2-1}-\arccos\frac{a}{x})+C$$

$$=\sqrt{x^2-a^2}-a\arccos\frac{a}{x}+C.$$

$$28.\int\frac{x^2}{\sqrt{a^2-x^2}}dx=-\int\sqrt{a^2-x^2}dx+a^2\int\frac{dx}{\sqrt{a^2-x^2}}$$

$$=-\frac{a^2}{2}\arcsin\frac{x}{a}-\frac{1}{2}x\sqrt{a^2-x^2}+a^2\arcsin\frac{x}{a}+C$$

$$=\frac{a^2}{2}\arcsin\frac{x}{a}-\frac{1}{2}x\sqrt{a^2-x^2}+C.$$

$$29.\int\frac{dx}{\sqrt{1+e^{3x}}}=\int\frac{e^{-3x/2}dx}{\sqrt{1+e^{-3x}}}=-\frac{2}{3}\int\frac{de^{-3x/2}}{\sqrt{1+e^{-3x}}}=-\frac{2}{3}\ln(e^{-3x/2}+\sqrt{1+e^{-3x}})+C$$

$$=-\frac{2}{3}\ln(1+\sqrt{1+e^{3x}})+x+C=-\frac{2}{3}\ln\frac{(\sqrt{1+e^{3x}}+1)(\sqrt{1+e^{3x}}-1)}{\sqrt{1+e^{3x}}-1}+x+C$$

$$=\frac{2}{3}\ln(\sqrt{1+e^{3x}}-1)-x+C.$$

$$30.\int\frac{x^3}{\sqrt{1+x^8}}dx=\frac{1}{4}\int\frac{dx^4}{\sqrt{1+x^8}}=\frac{1}{4}\int\frac{du}{\sqrt{1+u^2}}(u=x^4)$$

$$=\frac{1}{4}\ln(u+\sqrt{1+u^2})+C=\frac{1}{4}\ln(x^4+\sqrt{1+x^8})+C.$$

$$31.\int\frac{dx}{x^6\sqrt{1+x^2}}=\int\frac{dx}{x^7\sqrt{1+x^{-2}}}=-\frac{1}{2}\int\frac{dx^{-2}}{x^4\sqrt{1+x^{-2}}}=-\frac{1}{2}\int\frac{u^2du}{\sqrt{1+u}}(u=\frac{1}{x^2})$$

$$=-\frac{1}{2}\int\frac{(v-1)^2}{v^{1/2}}dv=-\frac{1}{2}\int\frac{v^2-2v+1}{v^{1/2}}dv(v=1+u)$$

$$=-\frac{1}{2}\int(v^{3/2}-2v^{1/2}+v^{-1/2})dx$$

$$=-\frac{1}{2}\left(\frac{2}{5}v^{\frac{5}{2}}-2\cdot\frac{2}{3}v^{\frac{3}{2}}+2\cdot v^{\frac{1}{2}}\right)$$

$$=-\frac{1}{5}\left(1+\frac{1}{x^2}\right)^{\frac{5}{2}}+\frac{2}{3}\left(1+\frac{1}{x^2}\right)^{\frac{3}{2}}-\left(1+\frac{1}{x^2}\right)^{\frac{1}{2}}+C$$

$$=-\frac{\sqrt{1+x^2}^5}{5x^5}+\frac{\sqrt{1+x^2}^3}{3x^3}-\frac{\sqrt{1+x^2}}{x}+C.$$

$$32.\int\frac{e^{2x}}{\sqrt[3]{1+e^x}}dx=\int\frac{e^x}{\sqrt[3]{1+e^x}}de^x=\int\frac{u}{\sqrt[3]{1+u}}du(u=e^x)(\sqrt[3]{u+1}=v,u=v^3-1)$$

$$=\int\frac{u}{\sqrt[3]{1+u}}du=\int\frac{v^3-1}{v}3v^2dv=3\int(v^4-v)dv=3\left(\frac{v^5}{5}-\frac{v^2}{2}\right)+C$$

$$=\frac{3}{5}(e^x+1)^{5/3}-\frac{3}{2}(e^x+1)^{2/3}+C.$$

$$33.\int\frac{dx}{\sqrt{3+x-x^2}}=\int\frac{dx}{\sqrt{3-\left(x-\frac{1}{2}\right)^2+\frac{1}{4}}}=\int\frac{d\left(x-\frac{1}{2}\right)}{\sqrt{\frac{13}{4}-\left(x-\frac{1}{2}\right)^2}}$$

$$=\arcsin\frac{x-\frac{1}{2}}{\frac{\sqrt{13}}{2}}+C=\arcsin\frac{2x-1}{\sqrt{13}}+C.$$

$$34.\int\sqrt{7+x-x^2}dx=\int\sqrt{7-\left(x-\frac{1}{2}\right)^2+\frac{1}{4}}dx=\int\sqrt{\frac{29}{4}-\left(x-\frac{1}{2}\right)^2}d\left(x-\frac{1}{2}\right)$$

$$=\frac{1}{2}\left(x-\frac{1}{2}\right)\sqrt{\frac{29}{4}-\left(x-\frac{1}{2}\right)^2}+\frac{29}{8}\arcsin\frac{x-\frac{1}{2}}{\frac{\sqrt{29}}{2}}+C$$

$$=\frac{2x-1}{4}\sqrt{7+x-x^2}+\frac{29}{8}\arcsin\frac{2x-1}{\sqrt{29}}+C.$$

$$35.\int\frac{dx}{1+\sqrt{x-1}},1+\sqrt{x-1}=u,x=1+(u-1)^2,dx=2(u-1)du,$$

$$\int\frac{dx}{1+\sqrt{x-1}},1+\sqrt{x-1}=u,x=1+(u-1)^2,dx=2(u-1)du,$$

$$\int\frac{dx}{1+\sqrt{x-1}}==\int\frac{2(u-1)du}{u}=2(u-\ln u)+C=2(1+\sqrt{x-1})-\ln(1+\sqrt{x-1}))+C$$

$$=2\sqrt{x-1}-\ln(1+\sqrt{x-1}))+C'.$$

习题 3.2

求下列不定积分：

$$1.\int x\ln xdx=\frac{1}{2}\int \ln xdx^2=\frac{x^2}{2}\ln x-\frac{1}{2}\int x^2 d\ln x$$

$$=\frac{x^2}{2}\ln x-\frac{1}{2}\int x^2\cdot\frac{1}{x}dx=\frac{x^2}{2}\ln x-\frac{1}{2}\int xdx=\frac{x^2}{2}\ln x-\frac{x^2}{4}+C.$$

$$2.\int x^2e^{ax}dx=\frac{1}{a}\int x^2de^{ax}=\frac{1}{a}x^2e^{ax}-\frac{1}{a}\int e^{ax}dx^2=\frac{1}{a}x^2e^{ax}-\frac{2}{a}\int xe^{ax}dx$$

$$=\frac{1}{a}x^2e^{ax}-\frac{2}{a^2}\int xde^{ax}=\frac{1}{a}x^2e^{ax}-\frac{2x}{a^2}e^{ax}+\frac{2}{a^2}\int e^{ax}dx$$

$$=\frac{1}{a}x^2e^{ax}-\frac{2}{a^2}\int xde^{ax}=\frac{1}{a}x^2e^{ax}-\frac{2x}{a^2}e^{ax}+\frac{2}{a^3}e^{ax}+C$$

$$=e^{ax}\left(\frac{1}{a}x^2-\frac{2x}{a^2}+\frac{2}{a^3}\right)+C.$$

$$3.\int x\sin 2xdx=-\frac{1}{2}\int xd\cos 2x=-\frac{1}{2}x\cos 2x+\frac{1}{2}\int\cos 2xdx$$

$$=-\frac{1}{2}x\cos 2x+\frac{1}{4}\sin 2x+C.$$

$$4.\int\arcsin xdx=x\arcsin x-\int xd\arcsin x=x\arcsin x-\int\frac{xdx}{\sqrt{1-x^2}}$$

$$=x\arcsin x+\frac{1}{2}\int\frac{d(1-x^2)}{\sqrt{1-x^2}}=x\arcsin x+\sqrt{1-x^2}+C.$$

$$5.\int\arctan xdx=x\arctan x-\int xd\arctan x=x\arctan x-\int\frac{xdx}{1+x^2}$$

$$=x\arctan x-\frac{1}{2}\int\frac{d(1+x^2)}{1+x^2}=x\arctan x-\frac{1}{2}\ln(1+x^2)+C.$$

$$6.I=\int e^{2x}\cos 3xdx=\frac{1}{2}\int\cos 3xde^{2x}=\frac{1}{2}e^{2x}\cos 3x-\frac{1}{2}\int e^{2x}d\cos 3x$$

$$=\frac{1}{2}e^{2x}\cos 3x+\frac{3}{2}\int e^{2x}\sin 3xdx=\frac{1}{2}e^{2x}\cos 3x+\frac{3}{4}\int\sin 3xde^{2x}$$

$$=\frac{1}{2}e^{2x}\cos 3x+\frac{3}{4}\left(e^{2x}\sin 3x-3\int e^{2x}\cos 3xdx\right)$$

$$=\frac{1}{2}e^{2x}\cos 3x+\frac{3}{4}e^{2x}\sin 3x-\frac{9}{4}I,$$

$$I=\frac{4}{13}\left(\frac{1}{2}\cos 3x+\frac{3}{4}\sin 3x\right)e^{2x}+C=\frac{1}{13}(2\cos 3x+3\sin 3x)e^{2x}+C.$$

$$7.I=\int\frac{\sin 3x}{e^x}dx=-\int\sin 3xde^{-x}=-e^{-x}\sin 3x+3\int e^{-x}\cos 3xdx$$

$$=-e^{-x}\sin 3x-3\int\cos 3xde^{-x}=-e^{-x}\sin 3x-3\left(e^{-x}\cos 3x+3\int e^{-x}\sin 3xdx\right)$$

$$= -e^{-x}\sin 3x - 3(e^{-x}\cos 3x + 3I),$$

$$I = \frac{1}{10}\left(-e^{-x}\sin 3x - 3e^{-x}\cos 3x\right) + C = -\frac{e^{-x}}{10}(\sin 3x + 3\cos 3x) + C.$$

$$8.I = \int e^{ax}\sin bx dx = \frac{1}{a}\int \sin bx de^{ax} = \frac{1}{a}e^{ax}\sin bx - \frac{b}{a}\int e^{ax}\cos bx dx$$

$$= \frac{1}{a}e^{ax}\sin bx - \frac{b}{a^2}\int \cos bx de^{ax}$$

$$= \frac{1}{a}e^{ax}\sin bx - \frac{b}{a^2}\left(e^{ax}\cos bx + b\int e^{ax}\sin bx dx\right)$$

$$= \frac{1}{a}e^{ax}\sin bx - \frac{b}{a^2}\left(e^{ax}\cos bx + bI\right).$$

$$I = \frac{1}{1+\frac{b^2}{a^2}}\left(\frac{1}{a}e^{ax}\sin bx - \frac{b}{a^2}e^{ax}\cos bx\right),$$

$$I = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + C.$$

$$9.I = \int \sqrt{1+9x^2}dx = x\sqrt{1+9x^2} - \int xd\sqrt{1+9x^2}$$

$$= x\sqrt{1+9x^2} - \int \frac{x\Box 18xdx}{2\sqrt{1+9x^2}}$$

$$= x\sqrt{1+9x^2} - \left(\int \sqrt{1+9x^2}dx - \int \frac{dx}{\sqrt{1+9x^2}}\right)$$

$$= x\sqrt{1+9x^2} - \left(I - \int \frac{dx}{\sqrt{1+9x^2}}\right),$$

$$I = \frac{1}{2}x\sqrt{1+9x^2} + \frac{1}{2}\Box\frac{1}{3}\ln(3x + \sqrt{1+9x^2}) + C$$

$$= \frac{1}{2}x\sqrt{1+9x^2} + \frac{1}{6}\ln(3x + \sqrt{1+9x^2}) + C.$$

$$10.\int x\cosh x dx = \int xd\sinh x = x\sinh x - \int \sinh x dx$$

$$= x\sinh x - \cosh x + C.$$

$$11.\int \ln(x+\sqrt{1+x^2})dx = x\ln(x+\sqrt{1+x^2}) - \int xd\ln(x+\sqrt{1+x^2})$$

$$= x\ln(x+\sqrt{1+x^2}) - \int \frac{xdx}{\sqrt{1+x^2}} = x\ln(x+\sqrt{1+x^2}) - \sqrt{1+x^2} + C.$$

$$12.\int (\arccos x)^2 dx = x(\arccos x)^2 + 2\int \frac{x\arccos x}{\sqrt{1-x^2}}dx$$

$$= x(\arccos x)^2 - 2\int \arccos x d\sqrt{1-x^2}$$

$$= x(\arccos x)^2 - 2\left(\sqrt{1-x^2}\arccos x + \int 1dx\right)$$

$$=x(\arccos x)^2-2\sqrt{1-x^2}\arccos x-2x+C.$$

$$13.\int\frac{x\arccos x dx}{(1-x^2)^2}=\frac{1}{2}\int\arccos x d\frac{1}{1-x^2}$$

$$=\frac{\arccos x}{2(1-x^2)}+\frac{1}{2}\int\frac{dx}{(1-x^2)\sqrt{1-x^2}}$$

$$=\frac{\arccos x}{2(1-x^2)}+\frac{1}{2}\frac{x}{\sqrt{1-x^2}}+C.$$

$$14.\int\arctan\sqrt{x}dx=x\arctan\sqrt{x}-\int\frac{xdx}{2(1+x)\sqrt{x}}$$

$$=x\arctan\sqrt{x}-\frac{1}{2}\int\frac{\sqrt{x}dx}{1+x}.\sqrt{x}=u,x=u^2,dx=2udu$$

$$\int\frac{\sqrt{x}dx}{1+x}=\int\frac{u2udu}{1+u^2}=2\left(u-\arctan u\right)+C,$$

$$\int\arctan\sqrt{x}dx=x\arctan\sqrt{x}-\frac{1}{2}\cdot 2(\sqrt{x}-\arctan\sqrt{x})+C$$

$$=x\arctan\sqrt{x}-(\sqrt{x}-\arctan\sqrt{x})+C$$

$$=(x+1)\arctan\sqrt{x}-\sqrt{x}+C.$$

$$15.\int\frac{\arcsin x}{x^2}dx=-\int\arcsin x d\left(\frac{1}{x}\right)=-\frac{\arcsin x}{x}+\int\frac{dx}{x\sqrt{1-x^2}}$$

$$=-\frac{\arcsin x}{x}+\int\frac{dx}{x^2\sqrt{1/x^2-1}}(x>0)$$

$$=-\frac{\arcsin x}{x}-\int\frac{d(1/x)}{\sqrt{1/x^2-1}}=-\frac{\arcsin x}{x}-\ln|1/x+\sqrt{1/x^2-1}|+C$$

$$=-\frac{\arcsin x}{x}+\ln(1-\sqrt{1-x^2})-\ln x+C$$

$$=-\frac{\arcsin x}{x}+\ln(1-\sqrt{1-x^2})-\ln|x|+C(x\neq 0)(\text{原函数为偶函数}).$$

$$16.\int x^3(\ln x)^2dx=\frac{1}{4}\int(\ln x)^2dx^4=\frac{x^4(\ln x)^2}{4}-\frac{1}{4}\int\frac{x^4\cdot 2\ln xdx}{x}$$

$$=\frac{x^4(\ln x)^2}{4}-\frac{1}{2}\int x^3\ln xdx=\frac{x^4(\ln x)^2}{4}-\frac{1}{8}\int\ln xdx^4$$

$$=\frac{x^4(\ln x)^2}{4}-\frac{x^4}{8}\ln x+\frac{1}{2}\int x^3dx=\frac{x^4(\ln x)^2}{4}-\frac{x^4}{8}\ln x+\frac{1}{8}x^4+C.$$

$$17.\int\frac{x\arctan xdx}{(1+x^2)^{5/2}}=\frac{1}{2}\int\frac{\arctan xd(1+x^2)}{(1+x^2)^{5/2}}=\frac{1}{2}\cdot\left(-\frac{2}{3}\right)\int\arctan xd(1+x^2)^{-3/2}$$

$$=-\frac{\arctan x}{3(1+x^2)^{3/2}}+\frac{1}{3}\int\frac{dx}{(1+x^2)^{5/2}}.x=\tan u,u\in(-\pi/2,\pi/2).dx=\sec^2 udu,$$

$$\int\frac{dx}{(1+x^2)^{5/2}}=\int\cos^3 udu=\int(1-\sin^2 u)d\sin u=$$

$$=\sin u-\frac{1}{3}\sin^3 u+C=\frac{x}{\sqrt{1+x^2}}-\frac{1}{3}\left(\frac{x}{\sqrt{1+x^2}}\right)^3+C,$$

$$\int\frac{x\arctan xdx}{(1+x^2)^{5/2}}=-\frac{\arctan x}{3(1+x^2)^{3/2}}+\frac{1}{3}\left(\frac{x}{\sqrt{1+x^2}}-\frac{1}{3}\left(\frac{x}{\sqrt{1+x^2}}\right)^3\right)+C$$

$$=-\frac{\arctan x}{3(1+x^2)^{3/2}}+\frac{1}{3}\frac{x}{\sqrt{1+x^2}}-\frac{1}{9}\frac{x^3}{(1+x^2)^{3/2}}+C.$$

$$18.\int x\ln(x+\sqrt{1+x^2})dx=\frac{1}{2}\int\ln(x+\sqrt{1+x^2})dx^2$$

$$=\frac{1}{2}x^2\ln(x+\sqrt{1+x^2})-\frac{1}{2}\int\frac{x^2dx}{\sqrt{1+x^2}}$$

$$=\frac{1}{2}x^2\ln(x+\sqrt{1+x^2})-\frac{1}{2}\int\frac{(x^2+1)-1dx}{\sqrt{1+x^2}}$$

$$=\frac{1}{2}x^2\ln(x+\sqrt{1+x^2})-\frac{1}{2}\int\sqrt{1+x^2}dx+\frac{1}{2}\int\frac{dx}{\sqrt{1+x^2}}$$

$$=\frac{1}{2}x^2\ln(x+\sqrt{1+x^2})-\frac{1}{2}\left(\frac{x\sqrt{1+x^2}}{2}+\frac{\ln(x+\sqrt{1+x^2})}{2}\right)+\frac{1}{2}\ln(x+\sqrt{1+x^2})+C$$

$$=\frac{1}{2}x^2\ln(x+\sqrt{1+x^2})-\frac{1}{4}x\sqrt{1+x^2}+\frac{1}{4}\ln(x+\sqrt{1+x^2})+C.$$

习题 3.3

求下列不定积分：

$$1.\int\frac{x-1}{x^2+6x+8}dx=\int\frac{x-1}{(x+2)(x+4)}dx,$$

$$\frac{x-1}{(x+2)(x+4)}=\frac{A}{x+2}+\frac{B}{x+4},$$

$$A=\frac{-2-1}{-2+4}=-\frac{3}{2},B=\frac{-4-1}{-4+2}=\frac{5}{2},$$

$$\frac{x-1}{(x+2)(x+4)}=\frac{-3/2}{x+2}+\frac{5/2}{x+4},$$

$$\int\frac{x-1}{x^2+6x+8}dx=-\frac{3}{2}\ln|x+2|+\frac{5}{2}\ln|x+4|+C.$$

$$2.I=\int\frac{3x^4+x^2+1}{x^2+x-6}dx.$$

$$\frac{3x^4+x^2+1}{x^2+x-6}=3x^2-3x+22+\frac{-40x+133}{x^2+x-6},$$

$$\frac{-40x+133}{x^2+x-6}=\frac{-40x+133}{(x+3)(x-2)}=\frac{A}{x+3}+\frac{B}{x-2},$$

$$A=\frac{-40\cdot(-3)+133}{-3-2}=-\frac{253}{5},B=\frac{-40\cdot2+133}{2+3}=\frac{53}{5}.$$

$$I=x^3-\frac{3x^2}{2}+22x-\frac{253}{5}\ln|x+3|+\frac{53}{5}\ln|x-2|+C.$$

$$3.I=\int\frac{2x^2-5}{x^4-5x^2+6}dx$$

$$\frac{2x^2-5}{x^4-5x^2+6}=\frac{2u-5}{u^2-5u+6}(u=x^2)$$

$$=\frac{2u-5}{(u-2)(u-3)}=\frac{A}{u-2}+\frac{B}{u-3},$$

$$A=\frac{2\cdot2-5}{2-3}=1,B=\frac{2\cdot3-5}{3-2}=1.$$

$$\frac{2x^2-5}{x^4-5x^2+6}=\frac{1}{x^2-\sqrt{2}^2}+\frac{1}{x^2-\sqrt{3}^2},$$

$$I=\frac{1}{2\sqrt{2}}\ln\frac{x-\sqrt{2}}{x+\sqrt{2}}+\frac{1}{2\sqrt{3}}\ln\frac{x-\sqrt{3}}{x+\sqrt{3}}+C.$$

$$4.I=\int\frac{dx}{(x-1)^2(x-2)}.$$

$$\frac{1}{(x-1)^2(x-2)}=\frac{1}{x-2}\left(\frac{1}{x-2}-\frac{1}{x-1}\right)$$

$$=\frac{1}{(x-2)^2}-\left(\frac{1}{x-2}-\frac{1}{x-1}\right),$$

$$I=-\frac{1}{x-2}+\ln\left|\frac{x-1}{x-2}\right|+C.$$

$$5.I=\int\frac{x^2}{1-x^4}dx.$$

$$\frac{x^2}{1-x^4}=\frac{x^2}{(1-x^2)(1+x^2)}=\frac{1}{2}\cdot\frac{(1+x^2)-(1-x^2)}{(1-x^2)(1+x^2)}$$

$$=\frac{1}{2}\left(\frac{1}{1-x^2}-\frac{1}{1+x^2}\right),$$

$$I=\frac{1}{4}\ln\frac{1+x}{1-x}-\frac{1}{2}\arctan x+C.$$

$$6.I=\int\frac{dx}{x^3+1}.$$

$$\frac{1}{x^3+1}=\frac{1}{(x+1)(x^2-x+1)}=\frac{A}{x+1}+\frac{Bx+C}{x^2-x+1},$$

$$A=\frac{1}{1^2+1+1}=\frac{1}{3},$$

$$1=\frac{x^2-x+1}{3}+(x+1)(Bx+C)=(B+\frac{1}{3})x^2+(B+C-\frac{1}{3})x+C+\frac{1}{3},$$

$$C+\frac{1}{3}=1,C=\frac{2}{3},B+\frac{1}{3}=0,B=-\frac{1}{3}.$$

$$\frac{1}{x^3+1}=\frac{1}{3(x+1)}+\frac{-x+2}{3(x^2-x+1)}=\frac{1}{3(x+1)}-\frac{x-2}{3(x^2-x+1)}$$

$$=\frac{1}{3(x+1)}-\frac{2x-4}{6(x^2-x+1)}=\frac{1}{3(x+1)}-\frac{1}{6}\cdot\frac{(2x-1)-3}{(x^2-x+1)}.$$

$$=\frac{1}{3(x+1)}-\frac{1}{6}\cdot\frac{2x-1}{(x^2-x+1)}+\frac{1}{2}\cdot\frac{1}{\left(x-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2},$$

$$I=\frac{1}{3}\ln|x+1|-\frac{1}{6}\ln(x^2-x+1)+\frac{1}{\sqrt{3}}\arctan\frac{2x-1}{\sqrt{3}}+C.$$

$$7.I=\int \frac{dx}{1+x^4}.\frac{1}{1+x^4}=\frac{1}{(1+2x^2+x^4)-2x^2}=\frac{1}{(x^2+1)^2-2x^2}$$

$$=\frac{1}{(x^2+\sqrt{2}x+1)(x^2-\sqrt{2}x+1)}=\frac{Ax+B}{x^2+\sqrt{2}x+1}+\frac{Cx+D}{x^2-\sqrt{2}x+1},$$

$$1=(Ax+B)(x^2-\sqrt{2}x+1)+(Cx+D)(x^2+\sqrt{2}x+1),$$

$$1=(A+C)x^3+(B-\sqrt{2}A+D+\sqrt{2}C)x^2+(A-\sqrt{2}B+C+\sqrt{2}D)x+B+D.$$

$$\begin{cases} A+C=0 \\ B-\sqrt{2}A+D+\sqrt{2}C=0, \\ A-\sqrt{2}B+C+\sqrt{2}D=0, \\ B+D=1. \end{cases}$$

$$A=\frac{1}{2\sqrt{2}},B=\frac{1}{2},C=-\frac{1}{2\sqrt{2}},D=\frac{1}{2}.$$

$$\frac{1}{1+x^4}=\frac{\frac{1}{2\sqrt{2}}x+\frac{1}{2}}{x^2+\sqrt{2}x+1}+\frac{-\frac{1}{2\sqrt{2}}x+\frac{1}{2}}{x^2-\sqrt{2}x+1}$$

$$=\frac{1}{2\sqrt{2}}\left(\frac{x+\sqrt{2}}{x^2+\sqrt{2}x+1}+\frac{-x+\sqrt{2}}{x^2-\sqrt{2}x+1}\right)$$

$$=\frac{1}{4\sqrt{2}}\left(\frac{2x+2\sqrt{2}}{x^2+\sqrt{2}x+1}-\frac{2x-2\sqrt{2}}{x^2-\sqrt{2}x+1}\right)$$

$$=\frac{1}{4\sqrt{2}}\left(\frac{(2x+\sqrt{2})+\sqrt{2}}{x^2+\sqrt{2}x+1}-\frac{(2x-\sqrt{2})-\sqrt{2}}{x^2-\sqrt{2}x+1}\right)$$

$$=\frac{1}{4\sqrt{2}}\left(\frac{(2x+\sqrt{2})}{x^2+\sqrt{2}x+1}-\frac{(2x-\sqrt{2})}{x^2-\sqrt{2}x+1}\right)+\frac{1}{4}\square\frac{1}{(x+\frac{1}{\sqrt{2}})^2+\left(\frac{1}{\sqrt{2}}\right)^2}$$

$$+\frac{1}{4}\square\frac{1}{(x-\frac{1}{\sqrt{2}})^2+\left(\frac{1}{\sqrt{2}}\right)^2}.$$

$$I=\frac{1}{4\sqrt{2}}\ln\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}+\frac{\sqrt{2}}{4}\left(\arctan(\sqrt{2}x+1)+\arctan(\sqrt{2}x-1)\right)+C.$$

$$8.I=\int \frac{x^3+x^2+2}{(x^2+2)^2}dx.$$

$$\frac{x^3+x^2+2}{(x^2+2)^2}=\frac{x(x^2+2)}{(x^2+2)^2}+\frac{x^2-2x+2}{(x^2+2)^2}$$

$$=\frac{x}{(x^2+2)}+\frac{1}{(x^2+2)}-\frac{2x}{(x^2+2)^2}.$$

$$I=\frac{1}{2}\ln(x^2+2)+\frac{1}{\sqrt{2}}\arctan\frac{x}{\sqrt{2}}+\frac{1}{x^2+2}+C.$$

$$9.\int\frac{e^x dx}{e^{2x}+3e^x+2}=\int\frac{de^x}{e^{2x}+3e^x+2}=\int\frac{du}{u^2+3u+2}=$$
$$=\int\frac{du}{(u+1)(u+2)}=\int\left(\frac{1}{u+1}-\frac{1}{u+2}\right)du=\ln\frac{u+1}{u+2}+C=\ln\frac{e^x+1}{e^x+2}+C.$$

$$10.\int\frac{\cos xdx}{\sin^2 x+\sin x-6}=\int\frac{d\sin x}{\sin^2 x+\sin x-6}=\int\frac{du}{u^2+u-6}(u=\sin x)=$$
$$\int\frac{du}{(u+3)(u-2)}=\frac{1}{5}\int\left(\frac{1}{u-2}-\frac{1}{u+3}\right)du=\ln\left|\frac{u-2}{u+3}\right|+C=\ln\left|\frac{\sin x-2}{\sin x+3}\right|+C.$$

$$11.\int\frac{x^3dx}{x^4+x^2+2}=\frac{1}{2}\int\frac{x^2dx^2}{x^4+x^2+2}=\frac{1}{2}\int\frac{udu}{u^2+u+2}$$
$$=\frac{1}{4}\int\frac{2udu}{u^2+u+2}=\frac{1}{4}\int\frac{(2u+1)-1}{u^2+u+2}du=$$
$$=\frac{1}{4}\int\frac{d(u^2+u+2)}{u^2+u+2}du-\frac{1}{4}\int\frac{1}{\left(u+\frac{1}{2}\right)^2+\frac{7}{4}}du$$
$$=\frac{1}{4}\ln(u^2+u+2)-\frac{1}{2\sqrt{7}}\arctan\frac{2u+1}{\sqrt{7}}+C$$
$$=\frac{1}{4}\ln(x^4+x^2+2)-\frac{1}{2\sqrt{7}}\arctan\frac{2x^2+1}{\sqrt{7}}+C.$$

$$12.I=\int\frac{dx}{(x+2)(x^2-2x+2)}.$$
$$\frac{1}{(x+2)(x^2-2x+2)}=\frac{A}{x+2}+\frac{Bx+C}{x^2-2x+2}$$
$$A=\frac{1}{(-2)^2-2(-2)+2}=\frac{1}{10}.$$
$$\frac{1}{(x+2)(x^2-2x+2)}-\frac{1}{10(x+2)}=\frac{Bx+C}{x^2-2x+2}$$
$$\frac{10-(x^2-2x+2)}{10(x+2)(x^2-2x+2)}=\frac{Bx+C}{x^2-2x+2}$$
$$\frac{-(x^2-2x-8)}{10(x+2)(x^2-2x+2)}=\frac{Bx+C}{x^2-2x+2}$$
$$\frac{-(x+2)(x-4)}{10(x+2)(x^2-2x+2)}=\frac{Bx+C}{x^2-2x+2}$$
$$\frac{-(x-4)}{10(x^2-2x+2)}=\frac{Bx+C}{x^2-2x+2},B=-\frac{1}{10},C=\frac{2}{5}.$$
$$I=\frac{1}{10}\ln|x+2|-\frac{1}{10}\int\frac{x-4}{x^2-2x+2}dx$$

$$=\frac{1}{10}\ln|x+2|-\frac{1}{10}\int\frac{2x-8}{x^2-2x+2}dx$$
$$=\frac{1}{10}\ln|x+2|-\frac{1}{20}\int\frac{(2x-2)-6}{x^2-2x+2}dx$$
$$=\frac{1}{10}\ln|x+2|-\frac{1}{20}\ln(x^2-2x+2)+\frac{3}{10}\int\frac{dx}{(x-1)^2+1}$$
$$=\frac{1}{10}\ln|x+2|-\frac{1}{20}\ln(x^2-2x+2)+\frac{3}{10}\arctan(x-1)+C$$

13. $I=\int\frac{dx}{2+\sin x}.\tan\frac{x}{2}=u, x=2\arctan u, dx=\frac{2du}{1+u^2}, \sin x=\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}=\frac{2u}{1+u^2}.$

$$I=\int\frac{\frac{2du}{1+u^2}}{2+\frac{2u}{1+u^2}}=\int\frac{1}{u^2+u+1}du=\int\frac{1}{\left(u+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}du$$
$$=\frac{2}{\sqrt{3}}\arctan\frac{2u+1}{\sqrt{3}}+C=\frac{2}{\sqrt{3}}\arctan\frac{2\tan\frac{x}{2}+1}{\sqrt{3}}+C.$$

14. $I=\int\frac{dx}{1+\sin x+\cos x}.\tan\frac{x}{2}=u, x=2\arctan u, dx=\frac{2du}{1+u^2},$

$\sin x=\frac{2u}{1+u^2}, \cos x=\frac{1-u^2}{1+u^2}.$

$$I=\int\frac{\frac{2du}{1+u^2}}{1+\frac{2u}{1+u^2}+\frac{1-u^2}{1+u^2}}=2\int\frac{1}{1+u^2+2u+1-u^2}du=\int\frac{1}{u+1}du$$
$$=\ln|u+1|+C=\ln\left|\tan\frac{x}{2}+1\right|+C.$$

15. $\int\cot^4 xdx$

$$=\int\cot^2 x(\csc^2 x-1)dx$$
$$=\int\cot^2 x\csc^2 xdx-\int\cot^2 xdx$$
$$=-\int\cot^2 xd\cot x-\int(\csc^2 x-1)dx$$
$$=-\frac{1}{3}\cot^3 x+\cot x+x+C.$$

16. $\int\sec^4 xdx=\int(1+\tan^2 x)d\tan x=\tan x+\frac{1}{3}\tan^3 x+C.$

$$17. I=\int\frac{\cos xdx}{5-3\cos x}=-\frac{1}{3}\int\frac{-3\cos xdx}{5-3\cos x}=-\frac{1}{3}\int\frac{(-3\cos x+5)-5dx}{5-3\cos x}$$

$$=-\frac{x}{3}+\frac{5}{3}\int\frac{dx}{5-3\cos x}.$$

$$\tan\frac{x}{2}=u,dx=\frac{2du}{1+u^2},\cos x=\frac{1-u^2}{1+u^2},$$

$$I=-\frac{x}{3}+\frac{5}{3}\int\frac{\frac{2du}{1+u^2}}{5-\frac{3(1-u^2)}{1+u^2}}=-\frac{x}{3}+\frac{5}{3}\int\frac{2du}{5(1+u^2)-3(1-u^2)}$$

$$=-\frac{x}{3}+\frac{5}{3}\int\frac{2du}{8u^2)+2}=-\frac{x}{3}+\frac{5}{3}\int\frac{du}{4u^2+1}=-\frac{x}{3}+\frac{5}{3}\frac{1}{2}\int\frac{d2u}{4u^2+1}$$

$$=-\frac{x}{3}+\frac{5}{6}\arctan 2u+C=-\frac{x}{3}+\frac{5}{6}\arctan\left(2\tan\frac{x}{2}\right)+C.$$

$$18. I=\int\frac{\cos^3 xdx}{\sin x+\cos x}=\int\frac{\cos^2 xdx}{1+\tan x}=\int\frac{dx}{(1+\tan x)(1+\tan^2 x)}.$$

$$\tan x=u,x=\arctan u,dx=\frac{du}{1+u^2},$$

$$I=\int\frac{\frac{du}{1+u^2}}{(1+u)(1+u^2)}=\int\frac{du}{(1+u)(1+u^2)^2},$$

$$\frac{1}{(1+u)(1+u^2)^2}=\frac{1}{2(1+u^2)}\left(\frac{1}{1+u}+\frac{1-u}{1+u^2}\right)$$

$$=\frac{1}{4}\left(\frac{1}{1+u}+\frac{1-u}{1+u^2}\right)+\frac{1-u}{2(1+u^2)^2},$$

$$I=\frac{1}{4}\ln|1+\tan x|+\frac{1}{4}\arctan u-\frac{1}{8}\ln(1+u^2)+\frac{1}{4(1+u^2)}+\frac{1}{2}\left(\frac{1}{2}\arctan u+\frac{u}{2(1+u^2)}\right)+C$$

$$=\frac{1}{4}\ln|1+\tan x|+\frac{x}{2}+\frac{1}{4}\ln|\cos u|++\frac{1}{4}\cos^2 x+\frac{1}{4}\tan x\cos^2 x+C.$$

$$19.\int\sin^5 x\cos^2 xdx=-\int\sin^4 x\cos^2 xd\cos x=-\int(1-u^2)^2u^2du$$

$$=-\int(u^2-2u^4+u^6)dx=-\frac{1}{3}u^3+\frac{2}{5}u^5-\frac{1}{7}u^7+C$$

$$=-\frac{1}{3}(\cos x)^3+\frac{2}{5}(\cos x)^5-\frac{1}{7}(\cos x)^7+C.$$

$$20.\int\sin^6 xdx=\int\left(\frac{1-\cos 2x}{2}\right)^3dx$$

$$=\frac{1}{8}\int(1-3\cos 2x+3\cos^2 2x-\cos^3 2x)dx$$

$$=\frac{x}{8}-\frac{3}{16}\sin 2x+\frac{3}{16}\int(1+\cos 4x)dx-\frac{1}{16}\int(1-\sin^2 2x)d\sin 2x$$

$$=\frac{x}{8}-\frac{3}{16}\sin 2x+\frac{3}{16}\left(x+\frac{1}{4}\sin 4x\right)-\frac{1}{16}\left(\sin 2x-\frac{1}{3}\sin^3 2x\right)+C$$

$$=+C.$$

21. $\int\sin^2 x\cos^4 xdx=\frac{1}{4}\int\sin^2 2x\cos^2 xdx=\frac{1}{4}\int\left(\frac{\sin 3x+\sin x}{2}\right)^2 dx$

$$=\frac{1}{16}\int\left(\sin^2 3x+\sin^2 x+2\sin 3x\sin x\right)dx$$

$$=\frac{1}{16}\int\left(\frac{1-\cos 6x}{2}+\frac{1-\cos 2x}{2}+\cos 2x-\cos 4x\right)dx$$

$$=\frac{1}{16}\left(x+\frac{1}{4}\sin 2x-\frac{1}{4}\sin 4x-\frac{1}{12}\sin 6x\right)+C.$$

另解: $\int\sin^2 x\cos^4 xdx=\int\frac{1-\cos 2x}{2}\left(\frac{1+\cos 2x}{2}\right)^2 dx$

$$=\frac{1}{8}\int(1+\cos^2 2x+2\cos 2x)(1-\cos 2x)dx$$

$$=\frac{1}{8}\int\left(1+\cos^2 2x+2\cos 2x-\cos 2x-\cos^3 2x-2\cos^2 2x\right)dx$$

$$=\frac{1}{8}\int\left(1+\cos 2x-\cos^2 2x-\cos^3 2x\right)dx$$

$$=\frac{1}{8}\left(x+\frac{1}{2}\sin 2x\right)-\frac{1}{16}\int(1+\cos 4x)dx-\frac{1}{16}\int(1-\sin^2 2x)d\sin 2x$$

$$=\frac{1}{8}\left(x+\frac{1}{2}\sin 2x\right)-\frac{1}{16}\left(x+\frac{1}{4}\sin 4x\right)-\frac{1}{16}\left(\sin 2x-\frac{1}{3}\sin^3 2x\right)+C$$

$$=\frac{1}{16}x-\frac{1}{64}\sin 4x+\frac{1}{48}\sin^3 2x+C.$$

22. $I=\int\frac{dx}{\sin x+2\cos x}.\tan\frac{x}{2}=u,x=2\arctan u,dx=\frac{2du}{1+u^2}.$

$$I=\int\frac{\frac{2du}{1+u^2}}{\frac{2u}{1+u^2}+\frac{2(1-u^2)}{1+u^2}}=\int\frac{2du}{-2u^2+2u+2}=-\int\frac{du}{u^2-u-1}=-\int\frac{du}{\left(u-\frac{1}{2}\right)^2-\left(\frac{\sqrt{5}}{2}\right)^2}=$$

$$=\frac{1}{\sqrt{5}}\ln\left|\frac{u-\frac{1}{2}+\frac{\sqrt{5}}{2}}{u-\frac{1}{2}-\frac{\sqrt{5}}{2}}\right|+C=\ln\left|\frac{2u+\sqrt{5}-1}{2u-\sqrt{5}-1}\right|+C.$$

$$23.\int\frac{\sin x\cos x}{\sin^2 x+\cos^4 x}dx=$$

$$=\int\frac{\tan x}{\tan^2 x(1+\tan^2 x)+1}d\tan x=\int\frac{u}{u^2(1+u^2)+1}du(u=\tan x)$$

$$=\frac{1}{2}\int\frac{du^2}{u^2(1+u^2)+1}=\frac{1}{2}\int\frac{dv}{v(1+v)+1}(v=u^2)$$

$$=\frac{1}{2}\int\frac{dv}{\left(v+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}=\frac{1}{2}\frac{2}{\sqrt{3}}\arctan\frac{2v+1}{\sqrt{3}}+C$$

$$=\frac{1}{\sqrt{3}}\arctan\frac{2\tan^2 x+1}{\sqrt{3}}+C.$$

另解：$I=\frac{1}{2}\int\frac{d\sin^2 x}{\sin^2 x+(1-\sin^2 x)^2}=\frac{1}{2}\int\frac{dw}{w+(1-w)^2}(w=\sin^2 x)$

$$=\frac{1}{2}\int\frac{dw}{(w-\frac{1}{2})^2+\left(\frac{\sqrt{3}}{2}\right)^2}=\frac{1}{\sqrt{3}}\arctan\frac{2w-1}{\sqrt{3}}+C=\arctan\frac{2\sin^2 x-1}{\sqrt{3}}+C.$$

$$24.\int\frac{dx}{\sin^4 x}=-\int(1+\cot^2 x)d\cot x=-\cot x-\frac{1}{3}\cot^3 x+C.$$

$$25.\int\sqrt{\frac{1-x}{1+x}}dx=\int\frac{1-x}{\sqrt{1-x^2}}dx=\arcsin x+\sqrt{1-x^2}+C.$$

$$26.I=\int\frac{1-\sqrt{x-1}}{1+\sqrt[3]{x-1}}dx.\sqrt[6]{x-1}=u,x=1+u^6,dx=6u^5du,$$

$$I=6\int\frac{(1-u^3)u^5du}{1+u^2}=6\int\frac{u^5-u^8}{1+u^2}du=-6\int(u^6-u^4-u^3+u^2+u+1+\frac{-u+1}{1+u^2})dx$$

$$=-6\left(\frac{1}{7}u^7-\frac{1}{5}u^5-\frac{1}{4}u^4+\frac{1}{3}u^3+\frac{1}{2}u^2+u-\frac{1}{2}\ln(1+u^2)+\arctan u\right)+C.$$

$$27.\int\frac{\sqrt{x+1}+\sqrt{x-1}}{\sqrt{x+1}-\sqrt{x-1}}dx=\int\frac{\left(\sqrt{x+1}+\sqrt{x-1}\right)^2}{\left(\sqrt{x+1}-\sqrt{x-1}\right)\left(\sqrt{x+1}+\sqrt{x-1}\right)}dx$$

$$=\int\frac{2x+2\sqrt{x^2-1}}{2}dx=\frac{1}{2}x^2+\frac{1}{2}x\sqrt{x^2-1}-\frac{1}{2}\ln(x+\sqrt{x^2-1})+C.$$

$$28.I=\int\frac{dx}{\sqrt[3]{(x+1)^2(x-1)^4}}=\int\frac{dx}{(x^2-1)\sqrt[3]{\frac{x-1}{x+1}}}.\sqrt[3]{\frac{x-1}{x+1}}=u,\frac{x-1}{x+1}=u^3,$$

$$x-1=(x+1)u^3,x=\frac{1+u^3}{1-u^3}=-1+\frac{2}{1-u^3},dx=\frac{6u^2du}{(1-u^3)^2},$$

$$I=\int\frac{\dfrac{6u^2du}{(1-u^3)^2}}{\left(\left(\dfrac{1+u^3}{1-u^3}\right)^2-1\right)u}=6\int\frac{u}{2(2u^3)}du=\frac{3}{2}\left(-\frac{1}{u}\right)+C=-\frac{3}{2}\sqrt[3]{\frac{x+1}{x-1}}+C.$$

$$29.\int\frac{xdx}{\sqrt{x^2-x+3}}=\frac{1}{2}\int\frac{2xdx}{\sqrt{x^2-x+3}}=\frac{1}{2}\int\frac{2x-1+1dx}{\sqrt{x^2-x+3}}=$$

$$=\frac{1}{2}\int\frac{d(x^2-x+3)}{\sqrt{x^2-x+3}}+\frac{1}{2}\int\frac{dx}{\sqrt{\left(x-\dfrac{1}{2}\right)^2+\left(\dfrac{\sqrt{11}}{2}\right)^2}}=$$

$$=\sqrt{x^2-x+3}+\frac{1}{2}\ln\left(x-\frac{1}{2}+\sqrt{x^2-x+3}\right)+C.$$

$$30.I=\int\frac{x}{(1+x^{1/3})^{1/2}}dx.(1+x^{1/3})^{1/2}=u,x=(u^2-1)^3,dx=3(u^2-1)^2(2u)du,$$

$$I=6\int\frac{(u^2-1)^3(u^2-1)^2(u)du}{u}=6\int(u^6-3u^4+3u^2-1)(u^4-2u^2+1)du$$

$$=6\int(u^{10}-5u^8+10u^6-10u^4+5u^2-1)du$$

$$=6\left(\frac{1}{11}u^{11}-\frac{5}{9}u^9+\frac{10}{7}u^7-2u^5+\frac{5}{3}u^3-u\right)+C.$$

$$31.I=\int\frac{\sqrt{x}dx}{\sqrt[4]{x^3}+1}.\sqrt[4]{x}=u,x=u^4,dx=4u^3du.$$

$$I=\int\frac{u^2 4u^3du}{u^3+1}=4\int\frac{u^5}{u^3+1}dx=4\int\frac{(u^5+u^2)-u^2}{u^3+1}du$$

$$=4\int\left(u^2-\frac{u^2}{u^3+1}\right)du=\frac{4}{3}u^3-\frac{4}{3}\ln(u^3+1)+C=\frac{4}{3}\sqrt[4]{x}^3-\frac{4}{3}\ln(\sqrt[4]{x}^3+1)+C.$$

$$32.\int\frac{2x+3}{\sqrt{x^2+x}}dx=\int\frac{(2x+1)+2}{\sqrt{x^2+x}}dx=\int\frac{1}{\sqrt{x^2+x}}d(x^2+x)+2\int\frac{1}{\sqrt{x^2+x}}dx$$

$$=2\sqrt{x^2+x}+2\int\frac{1}{\sqrt{\left(x+\dfrac{1}{2}\right)^2-\left(\dfrac{1}{2}\right)^2}}dx$$

$$=2\sqrt{x^2+x}+2\ln\left|x+\frac{1}{2}+\sqrt{x^2+x}\right|+C.$$

$$33.\int\frac{2+x}{\sqrt{4x^2-4x+5}}dx=\frac{1}{8}\int\frac{16+8x}{\sqrt{4x^2-4x+5}}dx$$

$$=\frac{1}{8}\int\frac{8x-4+20}{\sqrt{4x^2-4x+5}}dx=\frac{1}{8}\int\frac{d(4x^2-4x+5)}{\sqrt{4x^2-4x+5}}dx+\frac{5}{2}\int\frac{dx}{\sqrt{4x^2-4x+5}}$$

$$=\frac{1}{4}\sqrt{4x^2-4x+5}+\frac{5}{4}\int\frac{dx}{\sqrt{x^2-x+5/4}}$$

$$=\frac{1}{4}\sqrt{4x^2-4x+5}+\frac{5}{4}\int\frac{dx}{\sqrt{\left(x-\frac{1}{2}\right)^2+1}}$$

$$=\frac{1}{4}\sqrt{4x^2-4x+5}+\frac{5}{4}\ln\left(x-\frac{1}{2}+\sqrt{x^2-x+5/4}\right)+C$$

$$=\frac{1}{4}\sqrt{4x^2-4x+5}+\frac{5}{4}\ln\left(2x-1+\sqrt{4x^2-4x+5}\right)+C'.$$

34. $\int\sqrt{5-2x+x^2}\,dx=\int\sqrt{2^2+(x-1)^2}\,dx$

$$=\frac{(x-1)}{2}\sqrt{5-2x+x^2}+2\ln(\sqrt{5-2x+x^2})+C.$$

习题 3.4

求下列各定积分:

$1. I=\int_{-1}^{1}\frac{xdx}{\sqrt{5-4x}}.\sqrt{5-4x}=u,-1\to 3,1\to 1.5-4x=u^2,x=\frac{1}{4}(5-u^2),dx=-\frac{1}{2}udu,$

$I=\int_{3}^{1}\frac{\frac{1}{4}(5-u^2)}{u}\cdot\left(-\frac{1}{2}udu\right)=\frac{1}{8}\int_{1}^{3}(5-u^2)dx=\frac{1}{8}\left(5u-\frac{1}{3}u^3\right)\Big|_1^3=\frac{1}{6}.$

$2.\int_0^{\ln 2}xe^{-x}dx=-\int_0^{\ln 2}xde^{-x}=-xe^{-x}\Big|_0^{\ln 2}+\int_0^{\ln 2}e^{-x}dx=-\frac{\ln 2}{2}-e^{-x}\Big|_0^{\ln 2}=\frac{1}{2}(1-\ln 2).$

$3.\int_0^1 x^2\sqrt{1-x^2}dx=\int_0^{\pi/2}\sin^2 t\cos^2 tdt(x=\sin t)$

$=\int_0^{\pi/2}\sin^2 t(1-\sin^2 t)dt=I_2-I_4=\left(\frac{1}{2}-\frac{3\cdot 1}{4\cdot 2}\right)\frac{\pi}{2}=\frac{\pi}{16}.$

$4.\int_0^{\pi}x\sin xdx=-\int_0^{\pi}xd\cos x=-x\cos x\Big|_0^{\pi}+\int_0^{\pi}\cos xdx=\pi+\sin x\Big|_0^{\pi}=\pi.$

$5.\int_0^4\sqrt{x^2+9}dx=\left(\frac{x}{2}\sqrt{x^2+9}+\frac{9}{2}\ln(x+\sqrt{x^2+9})\right)\Bigg|_0^4=10+\frac{9}{2}\ln 3.$

$6.\int_0^{\frac{1}{2}}\frac{x^2}{\sqrt{1-x^2}}dx=\int_0^{\frac{\pi}{6}}\sin^2 tdt=\frac{1}{2}\int_0^{\frac{\pi}{6}}(1-\cos 2t)dt=\frac{1}{2}(t-\frac{1}{2}\sin 2t)\Big|_0^{\frac{\pi}{6}}=\frac{1}{2}\left(\frac{\pi}{6}-\frac{\sqrt{3}}{4}\right).$

$7.\int_0^1\sqrt{4-x^2}dx=\left(\frac{x}{2}\sqrt{4-x^2}+2\arcsin\frac{x}{2}\right)\Bigg|_0^1=\frac{\sqrt{3}}{2}+\frac{\pi}{3}.$

$8.\int_0^3 x\sqrt[3]{1-x^2}dx=\frac{1}{2}\int_0^3\sqrt[3]{1-x^2}dx^2=\frac{1}{2}\int_0^9\sqrt[3]{1-u}du=-\frac{3}{8}(1-u)^{\frac{4}{3}}\Bigg|_0^9=-\frac{45}{8}.$

$9.\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sqrt{\cos x-\cos^3 x}dx=2\int_0^{\frac{\pi}{2}}\sqrt{\cos x-\cos^3 x}dx=2\int_0^{\frac{\pi}{2}}\sqrt{\cos x}\sin xdx$

$=-2\int_0^{\frac{\pi}{2}}\sqrt{\cos x}d\cos x=-\frac{4}{3}\cos^{\frac{3}{2}}x\Bigg|_0^{\frac{\pi}{2}}=\frac{4}{3}.$

$10.\int_0^{\frac{\pi}{2}}\cos^n 2xdx=\frac{1}{2}\int_0^{\frac{\pi}{2}}\cos^n 2xd2x=\frac{1}{2}\int_0^{\pi}\cos^n udu=\frac{1}{2}\int_{-\pi/2}^{\pi/2}\cos^n(t+\frac{\pi}{2})dt$

$=\frac{(-1)^n}{2}\int_{-\pi/2}^{\pi/2}\sin^n(t)dt=\begin{cases}0,n=2k-1;\\ \int_0^{\pi/2}\sin^n(t)dt=\frac{(n-1)!!}{n!!}\cdot\frac{\pi}{2}.\end{cases}$

$11.\int_0^a(a^2-x^2)^{\frac{n}{2}}dx(x=a\sin t)=\int_0^{\frac{\pi}{2}}\cos^{n+1}tdt=\begin{cases}\frac{n!!}{(n+1)!!} & n\text{是偶数};\\ \frac{n!!}{(n+1)!!}\frac{\pi}{2} & n\text{是奇数}.\end{cases}$

12. $\int_0^{\pi/2}\sin^{11}xdx=\frac{10!!}{11!!}=\frac{156}{693}.$

13. $\int_0^{\pi}\sin^6\frac{x}{2}dx=2\int_0^{\pi/2}\sin^6 udu=2\cdot\frac{5\cdot3}{6\cdot4\cdot2}\cdot\frac{\pi}{2}=\frac{5\pi}{16}.$

14. $\int_0^{\pi}(x\sin x)^2dx=\frac{1}{2}\int_0^{\pi}x^2(1-\cos 2x)dx=\frac{1}{2}\cdot\frac{1}{3}x^3\Big|_0^{\pi}-\frac{1}{4}\int_0^{\pi}x^2d\sin 2x$

$=\frac{\pi^3}{6}-\frac{1}{4}x^2\sin 2x\Big|_0^{\pi}+\frac{1}{2}\int_0^{\pi}x\sin 2xdx$

$=\frac{\pi^3}{6}-\frac{1}{4}\int_0^{\pi}xd\cos 2x=\frac{\pi^3}{6}-\frac{1}{4}x\cos 2x\Big|_0^{\pi}+\frac{1}{4}\int_0^{\pi}\cos 2xdx$

$=\frac{\pi^3}{6}-\frac{\pi}{4}+\frac{1}{8}\sin 2x\Big|_0^{\pi}=\frac{\pi^3}{6}-\frac{\pi}{4}.$

15. $\int_0^{\pi/4}\tan^4 xdx=\int_0^{\pi/4}\tan^2 x(\sec^2 x-1)dx$

$=\int_0^{\pi/4}\tan^2 x\sec^2 xdx-\int_0^{\pi/4}\tan^2 xdx$

$=\int_0^{\pi/4}\tan^2 xd\tan x-\int_0^{\pi/4}(\sec^2 x-1)dx$

$=\frac{1}{3}\tan^3 x\Big|_0^{\pi/4}-\tan x\Big|_0^{\pi/4}+\frac{\pi}{4}=\frac{1}{3}-1+\frac{\pi}{4}=-\frac{2}{3}+\frac{\pi}{4}.$

16. $\int_0^1\arcsin xdx=x\arcsin x\Big|_0^1-\int_0^1 xd\arcsin x$

$=\frac{\pi}{2}-\int_0^1\frac{x}{\sqrt{1-x^2}}dx=\frac{\pi}{2}+\sqrt{1-x^2}\Big|_0^1=\frac{\pi}{2}-1.$

17. $\int_0^{\pi}\ln(x+\sqrt{x^2+a^2})dx=x\ln(x+\sqrt{x^2+a^2})\Big|_0^{\pi}-\int_0^{\pi}xd\ln(x+\sqrt{x^2+a^2})$

$=\pi\ln(\pi+\sqrt{\pi^2+a^2})-\int_0^{\pi}\frac{x}{\sqrt{x^2+a^2}}dx=\pi\ln(\pi+\sqrt{\pi^2+a^2})-\sqrt{x^2+a^2}\Big|_0^{\pi}$

$=\pi\ln(\pi+\sqrt{\pi^2+a^2})-\sqrt{\pi^2+a^2}+|a|.$

18. 设$f(x)$在$[a,b]$连续.证明$\int_a^b f(x)dx=(b-a)\int_0^1 f(a+(b-a)x)dx.$

证 令$x=a+(b-a)t$,则$0\to a,1\to b,dx=(b-a)dt$,故

$\int_a^b f(x)dx=(b-a)\int_0^1 f(a+(b-a)t)dt=(b-a)\int_0^1 f(a+(b-a)x)dx.$

19. 证明$\int_0^a x^3f(x^2)dx=\frac{1}{2}\int_0^{a^2}xf(x)dx.$

证令$x^2=t$,则$x=0$时,$t=0,x=a$时,$t=a^2$故

$\int_0^a x^3f(x^2)dx=\frac{1}{2}\int_0^a x^2f(x^2)dx^2=\frac{1}{2}\int_0^{a^2}tf(t)dt=\frac{1}{2}\int_0^{a^2}xf(x)dt.$

20. 证明$\int_0^1 x^m(1-x)^ndx=\int_0^1 x^n(1-x)^mdx.$

证令$x=1-t$, 则$x=0$时,$x=1$时,$t=0.dx=-dt$,故

$\int_0^1 x^m(1-x)^n dx = -\int_1^0 (1-t)^m t^n dt = \int_0^1 (1-t)^m t^n dt = \int_0^1 x^n(1-x)^m dx.$

21.利用分部积分公式证明,若$f(x)$连续,则

$\int_0^x \int_0^t f(x)dxdt = \int_0^x f(t)(x-t)dx.$

证$\int_a^x \int_0^t f(x)dxdt = t\int_0^t f(x)dx\Big|_0^x - \int_0^x t\left(\int_0^t f(x)dx\right)' dt$

$= \int_0^x xf(x)dx - \int_0^x tf(t)dt = \int_0^x xf(t)dt - \int_0^x tf(t)dt$

$= \int_0^x f(t)(x-t)dt.$

22.利用换元积分法证明$\int_0^\pi xf(\sin x)dx = \pi\int_0^{\pi/2} f(\sin x)dx.$

证$x = \pi - t, x = 0$ 时$, t = \pi, x = \pi$时$, dx = -dt,$故

$\int_0^\pi xf(\sin x)dx = -\int_\pi^0 (\pi - t) f(\sin(\pi - t))dt$

$= \int_0^\pi (\pi - t) f(\sin t)dt = \pi\int_0^\pi f(\sin t)dt - \int_0^\pi tf(\sin t)dt$

$= \pi\int_0^\pi f(\sin t)dt - \int_0^\pi xf(\sin x)dx.$

$2\int_0^\pi xf(\sin x)dx = \pi\int_0^\pi f(\sin t)dt,$

$\int_0^\pi xf(\sin x)dx = \frac{1}{2}\pi\int_0^\pi f(\sin t)dt$

$= \frac{1}{2}\pi\int_0^{\pi/2} f(\sin t)dt + \frac{1}{2}\pi\int_{\pi/2}^\pi f(\sin t)dt$

令$u = \pi - t,$则$t = \pi/2$时$, u = \pi/2, t = \pi$时$, u = 0, du = -dt,$

$\int_{\pi/2}^\pi f(\sin t)dt = -\int_{\pi/2}^0 f(\sin(\pi - u))du = \int_0^{\pi/2} f(\sin u)du,$

$\int_0^\pi xf(\sin x)dx = \pi\int_0^{\pi/2} f(\sin x)dx.$

23.利用上题结果求$\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}dx.$

解$\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}dx=\int_0^{\pi/2}\frac{\sin x}{1+\cos^2 x}dx=-\int_0^{\pi/2}\frac{d\cos x}{1+\cos^2 x}$

$=-\arctan\cos x\big|_0^{\pi/2}=\frac{\pi}{4}.$

24.设函数$f(x)$在$(-\infty,+\infty)$上连续,以T为周期,证明:

(1)函数$F(x)=\frac{x}{T}\int_0^T f(x)dx-\int_0^x f(t)dt$也以$T$为周期;

(2) $\lim\limits_{x\to+\infty}\frac{1}{x}\int_0^x f(t)dt=\frac{1}{T}\int_0^T f(x)dx.$

证(1)$F(x+T)=\frac{x+T}{T}\int_0^T f(x)dx-\int_0^{x+T} f(t)dt$

$=\frac{x}{T}\int_0^T f(x)dx+\int_0^T f(x)dx-\left(\int_0^x f(t)dt+\int_x^{x+T} f(t)dt\right)$

$=\frac{x}{T}\int_0^T f(x)dx+\int_0^T f(x)dx-\left(\int_0^x f(t)dt+\int_0^T f(t)dt\right)$

$=\frac{x}{T}\int_0^T f(x)dx-\int_0^x f(t)dt=F(x).$

(2)$\frac{1}{x}\int_0^x f(t)dt-\frac{1}{T}\int_0^T f(x)dx$

$=-\frac{1}{x}\left(\frac{x}{T}\int_0^T f(x)dx-\int_0^x f(t)dt\right)=-\frac{F(x)}{x}.$

$F(x)$在$(-\infty,+\infty)$上连续,以T为周期,故有界,

$\lim\limits_{x\to+\infty}\left(\frac{1}{x}\int_0^x f(t)dt-\frac{1}{T}\int_0^T f(x)dx\right)=\lim\limits_{x\to+\infty}\frac{F(x)}{x}=0.$

于是$\lim\limits_{x\to+\infty}\frac{1}{x}\int_0^x f(t)dt=\frac{1}{T}\int_0^T f(x)dx.$

25.设$f(x)$是以T为周期的连续函数,$f(x_0)\neq 0$,且$\int_0^T f(x)dx=0$,证明:

$f(x)$在区间(x_0,x_0+T)内至少有两个根.

证为明确起见,设$f(x_0)>0$.如果f在(x_0,x_0+T)没有根,则由连续函数的

中间值定理,f在(x_0,x_0+T)恒正,设其最小值为m.则$m>0$,

$\int_{x_0}^{x_0+T} f(x)dx\geq\int_{x_0}^{x_0+T} mdx=mT>0$.由周期性和假设$\int_{x_0}^{x_0+T} f(x)dx=\int_0^T f(x)dx=0,$

矛盾.故f在(x_0,x_0+T)至少有一个根x_1.若f在(x_0,x_0+T)再无其它根,由于

$f(x_0+T)=f(x_0)>0$, f在(x_0,x_1)和(x_1,x_0+T)恒正,

$\int_{x_0}^{x_0+T} f(x)dx=\int_{x_0}^{x_1} f(x)dx+\int_{x_1}^{x_0+T} f(x)dx>0$,矛盾.故$f$在$(x_0,x_1)$或$(x_1,x_0+T)$至少

还有一个根根,即$f(x)$在区间(x_0,x_0+T)内至少有两个根.

26.求定积分

$$\int_0^{2m\pi}\frac{dx}{\sin^4 x+\cos^4 x}dx$$

其中m为正整数.

解被积函数以2π为周期,故$\int_0^{2m\pi}\frac{dx}{\sin^4 x+\cos^4 x}dx=m\int_0^{2\pi}\frac{dx}{\sin^4 x+\cos^4 x}dx.$

$$\int_0^{2\pi}\frac{dx}{\sin^4 x+\cos^4 x}dx=\int_0^{2\pi}\frac{dx}{\left(\sin^2 x+\cos^2 x\right)^2-2\sin^2 x\cos^2 x}dx$$

$$=\int_0^{2\pi}\frac{dx}{1-\frac{1}{2}\sin^2 2x}dx=4\int_0^{\pi/2}\frac{dx}{1-\frac{1}{2}\sin^2 2x}dx(\sin^2 2x\text{周期为}\frac{\pi}{2})$$

$$=8\int_0^{\pi/2}\frac{dx}{2-\sin^2 2x}dx=-4\int_0^{\pi/2}\frac{d\cot 2x}{2\csc^2 2x-1}$$

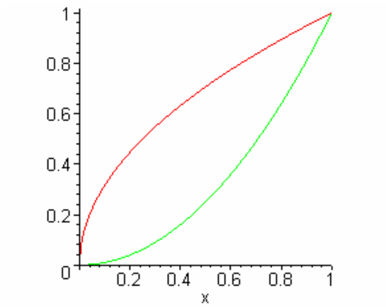
$$=-4\int_0^{\pi/2}\frac{d\cot 2x}{2\cot^2 2x+1}=4\int_{-\infty}^{+\infty}\frac{du}{1+2u^2}=\frac{4}{\sqrt{2}}\arctan\sqrt{2}u\,\Big|_{-\infty}^{+\infty}=2\sqrt{2}\pi.$$

$$\int_0^{2m\pi}\frac{dx}{\sin^4 x+\cos^4 x}dx=2m\sqrt{2}\pi.$$

习题 3.

求下列曲线所围成的的图形的面积:

1.$y=x^2$与$x=y^2$.

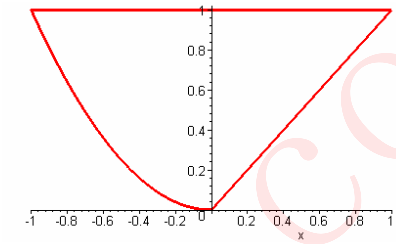


解求交点:$\begin{cases} y=x^2 \\ x=y^2 \end{cases}, x=x^4,$

$x(1-x)(1+x+x^2)=0, x=0, x=1.$

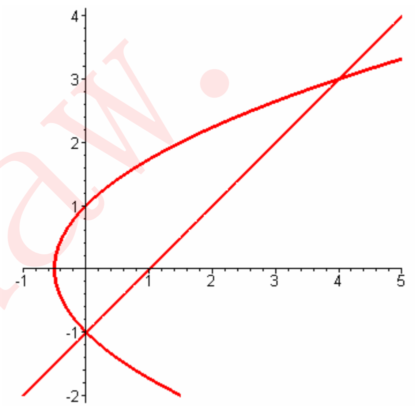
$$S=\int_0^1(\sqrt{x}-x^2)dx=\left(\frac{2}{3}x^{3/2}-\frac{1}{3}x^2\right)\Big|_0^1=\frac{1}{3}.$$

2.$y=x, y=1$与$y=\dfrac{x^2}{4}$.



解$S=\int_0^1(y+2\sqrt{y})dy=\left(\dfrac{y^2}{2}+y^{\frac{3}{2}}\right)\Big|_0^1=\dfrac{3}{2}.$

3.$y^2=2x+1$与$x-y=1$.



解$\begin{cases} y^2=2x+1 \\ x-y=1 \end{cases}(x-1)^2=2x+1,$

$x^2-4x=0, x=0, y=-1; x=4, y=3.$

$$S=\int_{-1}^3\left(y+1-\frac{1}{2}(y^2-1)\right)dx$$

$$=\left(\frac{3}{2}y+\frac{y^2}{2}-\frac{1}{6}y^3\right)\Big|_{-1}^3=\frac{16}{3}.$$

4.$y=0$与$\begin{cases} x=a(t-\sin t) \\ y=a(1-\cos t) \end{cases}$ $0\le t\le 2\pi$(a>0)

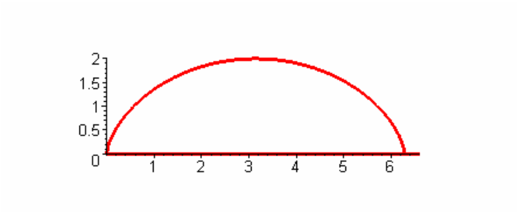
$$S=\int_0^{2\pi}a(1-\cos t)da(t-\sin t)$$

$$=a^2\int_0^{2\pi}(1-\cos t)^2dt$$

$$=4a^2\int_0^{2\pi}\sin^4\frac{t}{2}dt=8a^2\int_0^{\pi}\sin^4 udu$$

$$=16a^2\int_0^{\pi/2}\sin^4 udu=16a^2\frac{3}{4\cdot 2}\cdot\frac{\pi}{2}$$

$$=3\pi a^2.$$



5.$y=x^2-4$与$y=-x^2-2x$.

解$\begin{cases} y=x^2-4 \\ y=-x^2-2x \end{cases}$ $x^2-4=-x^2-2x,$

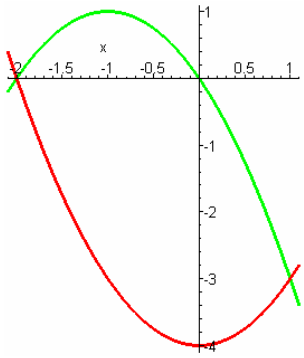
$2x^2+2x-4=0,(2x-2)(x+2)=0,$

$x_1=-2,x_2=1.$

$S=\int_{-2}^{1}(-x^2-2x-x^2+4)dx$

$=\int_{-2}^{1}(-2x^2-2x+4)dx$

$=\left(-\frac{2}{3}x^3-x^2+4x\right)\Big|_{-2}^{1}=9.$



$6.x^2+y^2=8$与$y=\frac{1}{2}x^2$(分上下两部分).

解$\begin{cases}x^2+y^2=8\\ y=\frac{1}{2}x^2\end{cases}$ $x^2+\frac{1}{4}x^4=8$

$x^4+4x^2-32=0,x^2=u$

$u^2+4u-32=0,(u+8)(u-4)=0$

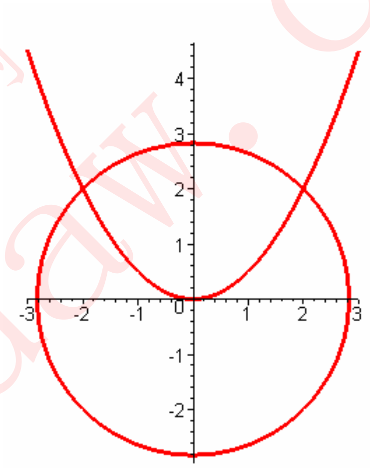
$u_2=-8$(舍)$u_2=4,x^2=4,x_1=-2,x_2=2$

$S_1=\int_{-2}^{2}\left(\sqrt{8-x^2}-\frac{1}{2}x^2\right)dx$

$=2\int_{0}^{2}\left(\sqrt{8-x^2}-\frac{1}{2}x^2\right)dx$

$=2\left(\frac{x}{2}\sqrt{8-x^2}+4\arcsin\frac{x}{2\sqrt{2}}\right)\Big|_{0}^{2}=2\pi+\frac{4}{3}$

$S_2=8\pi-\left(2\pi+\frac{4}{3}\right)=6\pi-\frac{4}{3}.$



$7.y=4-x^2$与$y=x+2.$

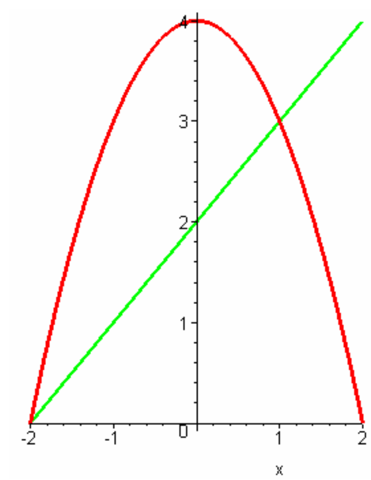
解$\begin{cases}y=4-x^2\\ y=x+2\end{cases}$ $4-x^2=x+2$

$x^2+x-2=0,(x+2)(x-1)=0,$

$x_1=-2,x_2=1.$

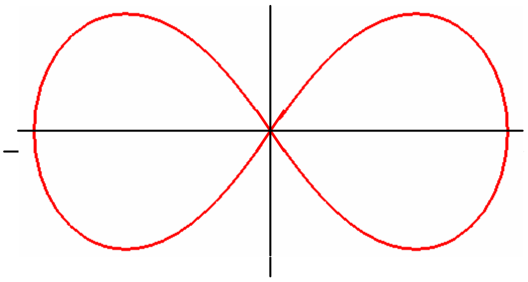
$S=\int_{-2}^{1}(4-x^2-x-2)dx=6-\int_{-2}^{1}(x^2+x)dx$

$=6-\left(\frac{x^3}{3}+\frac{x^2}{2}\right)\Big|_{-2}^{1}=\frac{9}{2}.$



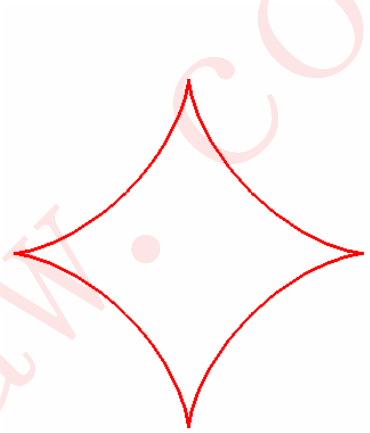
8.其求双纽线$r^2=a^2\cos 2\varphi (a>0)$所围图形的面积.

解$S=4\cdot\frac{1}{2}\int_0^{\pi/4} a^2\cos 2\varphi d\varphi = 2a^2\cdot\frac{1}{2}\sin 2\varphi\Big|_0^{\pi/4} = a^2.$



求下列曲线围成的平面图形绕轴旋转
所成旋转体的体积:

9.$x^{2/3}+y^{2/3}=a^{2/3}(a>0).$



解$\begin{cases} x=a\cos^3 t \\ y=a\sin^3 t \end{cases}, 0\le t\le 2\pi.$

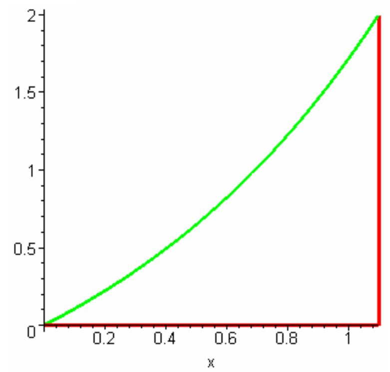
$V=2\pi\int_0^a y^2dx = 2\pi\int_0^{\pi/2} a^2\sin^6 t a\cdot 3\cos^2 t\sin t dt$

$=6\pi a^3\int_0^{\pi/2}\sin^7 t\cos^2 t dt$

$=6\pi a^3\int_0^{\pi/2}\sin^7 t(1-\sin^2 t)dt$

$=6\pi a^3\left(\frac{6\cdot 4\cdot 2}{7\cdot 5\cdot 3}\left(1-\frac{8}{9}\right)\right)=\frac{32}{105}\pi a^3.$

10.$y=e^x-1, x=\ln 3, y=e.$

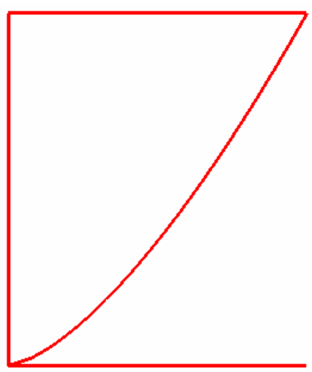


$V=\pi\int_0^{\ln 3}(e^x-1)^2dx = \pi\int_0^{\ln 3}(e^{2x}-2e^x+1)dx$

$=\pi\left(\frac{1}{2}e^{2x}-2e^x+x\right)\Big|_0^{\ln 3} = \pi\ln 3.$

求下列平面曲线围成的平面图形绕轴
旋转所成旋转体的体积:

11.$ay^2=x^3, x=0$及$y=b(a>0,b>0).$

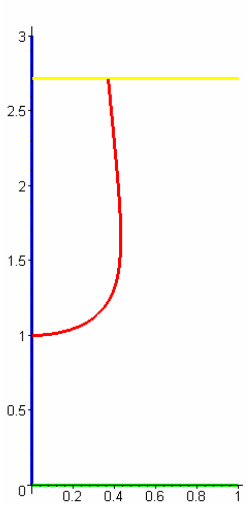


解$x=a^{1/3}y^{2/3},$

$V=\pi\int_0^b (a^{1/3}y^{2/3})^2dy$

$=\pi a^{2/3}\cdot\frac{3}{7}y^{7/3}\Big|_0^b=\frac{3}{7}\pi a^{2/3}b^{7/3}.$

12. $x=\dfrac{\sqrt{8\ln y}}{y}, x=0, y=e.$



解 $V=\pi\int_1^e \dfrac{8\ln y}{y^2}dy$

$=8\pi\left[-\int_1^e \ln y dy^{-1}\right]$

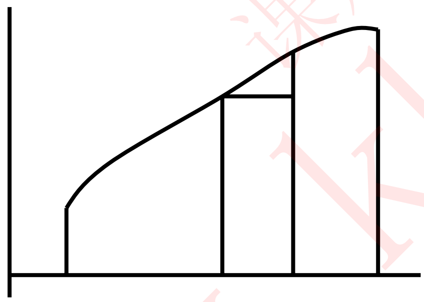
$=8\pi\left[-y^{-1}\ln y|_1^e+\int_1^e y^{-1}d\ln y\right]$

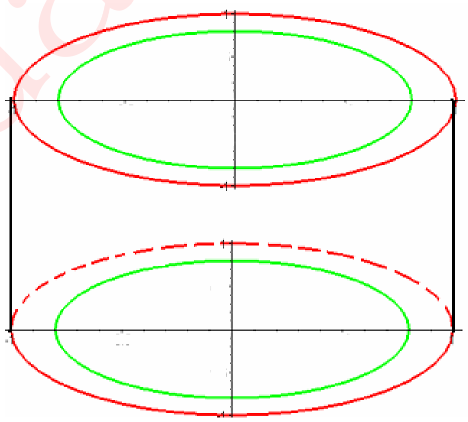
$=8\pi\left[-\dfrac{1}{e}+\int_1^e y^{-2}dy\right]$

$=8\pi\left[-\dfrac{1}{e}-y^{-1}|_1^e\right]=8\pi\left[1-\dfrac{2}{e}\right].$

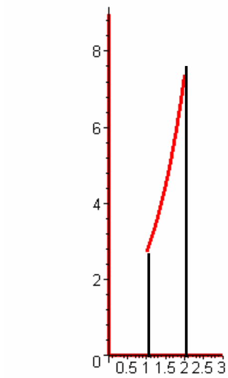
13. 设 $y=f(x)$ 在区间 $[a,b](a>0)$ 上连续且不取负值，试用微元法推导：由曲线 $y=f(x)$，直线 $x=a$，$x=b$ 及轴围成的平面图形绕 y 轴旋转所成立体的体积为 $V=2\pi\int_a^b xf(x)dx.$

解 厚度 dx 的圆筒的体积 $dV=2\pi xf(x)dx, V=2\pi\int_a^b xf(x)dx.$





14. 求曲线 $y=e^x, x=1, x=2$ 及 x 轴所围成的平面图形绕 y 轴旋转所成的立体的体积.



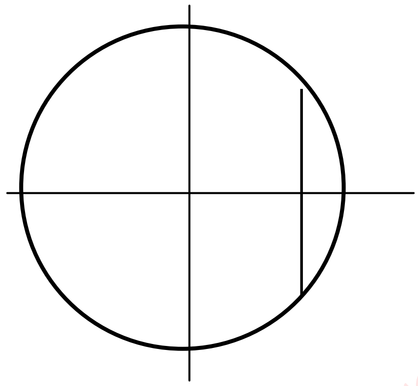
解 $V=2\pi\int_1^2 xe^x dx=2\pi\left[\int_1^2 xde^x\right]$

$=2\pi\left[xe^x|_1^2-\int_1^2 e^x dx\right]=2\pi\left[2e^2-e-e^x|_1^2\right]$

$=2\pi\left[2e^2-e-(e^2-e)\right]=2\pi e^2.$

15.证明：半径为a高为h的球缺的体积为

$$V=\pi h^2\left(a-\frac{h}{3}\right).$$

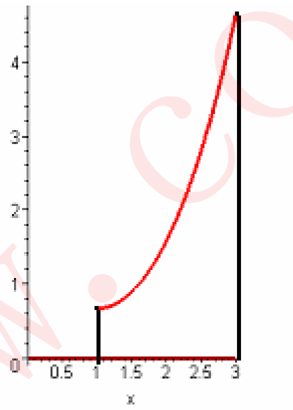


证$y=f(x)=\sqrt{a^2-x^2},a-h\le x\le a.$

$$V=\pi\int_{a-h}^{a}(a^2-x^2)dx=\pi\left[a^2h-\frac{1}{3}x^3\Big|_{a-h}^{a}\right]$$

$$=\pi\left[a^2h-\frac{1}{3}(a^3-(a-h)^3)\right]=\pi h^2\left(a-\frac{h}{3}\right)$$

16.求曲线$y=\dfrac{x^3}{6}+\dfrac{1}{2x}$在$x=1$到$x=3$之间的弧长.

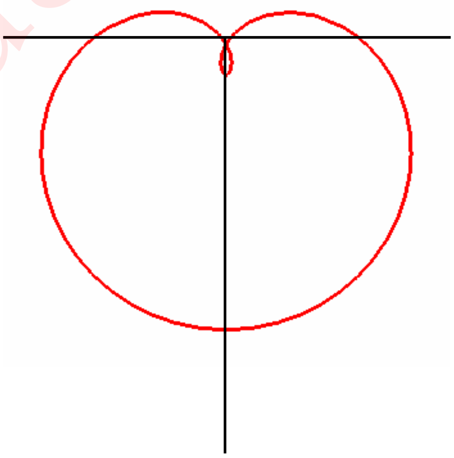


解$y'=\dfrac{x^2}{2}-\dfrac{1}{2x^2}.$

$$s=\int_1^3\sqrt{1+\left(\frac{x^2}{2}-\frac{1}{2x^2}\right)^2}\,dx$$

$$=\int_1^3\frac{x^4+1}{2x^2}dx=\left[\frac{x^3}{6}-\frac{1}{2x}\right]\Bigg|_1^3=\frac{14}{3}.$$

17.求曲线$r=a\sin^3\dfrac{\theta}{3}$的全长.

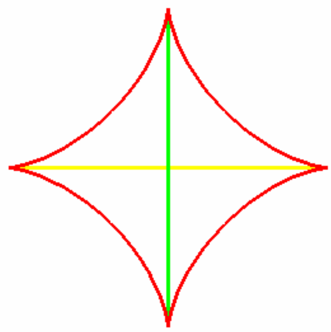


解$r'=a\sin^2\dfrac{\theta}{3}\cos\dfrac{\theta}{3},$

$$s=a\int_0^{3\pi}\sqrt{\sin^6\frac{\theta}{3}+\sin^4\frac{\theta}{3}\cos^2\frac{\theta}{3}}\,d\theta$$

$$=a\int_0^{3\pi}\sin^2\frac{\theta}{3}d\theta$$

$$=6a\int_0^{\pi/2}\sin^2\theta d\theta=6a\cdot\frac{1}{2}\cdot\frac{\pi}{2}=\frac{3}{2}\pi a.$$

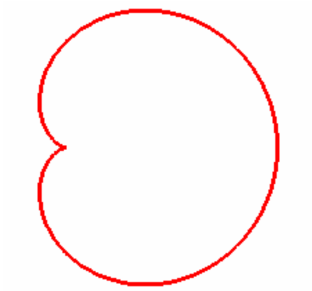


18.求向星形线$x=a\cos^3t,y=a\sin^3t$的弧长.

解$x'=3a\cos^2t(-\sin t),y'=3a\sin^2t\cos t$

$$s=4\cdot3a\int_0^{\pi/2}\sin t\cos tdt=12a\cdot\frac{1}{2}\sin^2t\Big|_0^{\pi/2}=6a.$$

19.求心脏线$r=a(1+\cos\theta)$的全长.



解$r'=a(-\sin\theta)$

$$s=2a\int_0^{\pi}\sqrt{(1+\cos\theta)^2+\sin^2\theta}\,d\theta$$

$$=4a\int_0^{\pi}\cos\frac{\theta}{2}dx=8a\sin\frac{\theta}{2}\Big|_0^{\pi}=8a.$$

20.试证双纽线$r^2=2a^2\cos 2\theta(a>0)$的全长$L$可表为$L=4\sqrt{2}a\int_0^1\frac{dx}{\sqrt{1-x^4}}$.20

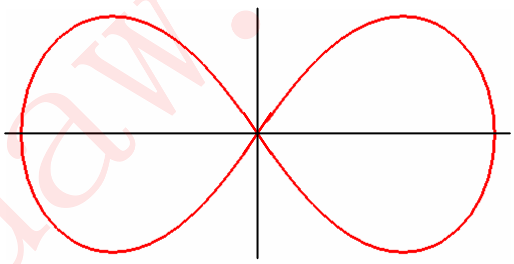
证$2rr'=-4a^2\sin 2\theta, r'=-2a^2\sin 2\theta/r$,

$$s=4\int_0^{\pi/4}\sqrt{2a^2\cos 2\theta+\frac{4a^4\sin^2 2\theta}{2a^2\cos 2\theta}}\,d\theta$$

$$=4\sqrt{2}a\int_0^{\pi/4}\frac{1}{\sqrt{\cos 2\theta}}d\theta$$

$$=4\sqrt{2}a\int_0^{\pi/4}\frac{d\theta}{\sqrt{\cos^2\theta-\sin^2\theta}}$$

$$=4\sqrt{2}a\int_0^{\pi/4}\frac{d\theta}{\sqrt{(\cos^2\theta-\sin^2\theta)(\cos^2\theta-\sin^2\theta)}}$$



$$=4\sqrt{2}a\int_0^{\pi/4}\frac{d\theta}{\sqrt{\cos^4\theta-\sin^4\theta}}$$

$$=4\sqrt{2}a\int_0^{\pi/4}\frac{d\tan\theta}{\sqrt{1-\tan^4\theta}}(\tan\theta=x)$$

$$=4\sqrt{2}a\int_0^1\frac{dx}{\sqrt{1-x^4}}.$$

21.求抛物线$y=1+\frac{x^2}{4}(0\le x\le 2)$绕$x$旋转所得的旋转体的侧面积.

解 $y'=\frac{x}{2}$.

$$S=2\pi\int_0^2\left(1+\frac{x^2}{4}\right)\sqrt{1+\left(\frac{x}{2}\right)^2}\,dx$$

$$=\frac{1}{4}\pi\int_0^2\sqrt{4+x^2}^{\,3}dx=4\pi\int_0^{\pi/4}\frac{dx}{\cos^5 x}$$

$$I_n=\int\sec^n xdx=\int\sec^{n-2}xd\tan x$$

$$=\sec^{n-2}x\tan x-\int(n-2)\sec^{n-2}x\tan^2 xdx$$

$$=\sec^{n-2}x\tan x-(n-2)I_n+(n-2)I_{n-2},$$

$$I_n=\frac{1}{n-1}\sec^{n-2}x\tan x+\frac{n-2}{n-1}I_{n-2}.$$

$$I_5=\frac{1}{4}\sec^3 x\tan x+\frac{3}{4}I_3=\frac{1}{4}\sec^3 x\tan x+\frac{3}{4}\left(\frac{1}{2}\sec x\tan x+\frac{1}{2}I_1\right)$$

$$=\frac{1}{4}\sec^3 x\tan x+\frac{3}{8}\sec x\tan x+\frac{3}{8}\ln(\tan x+\sec x)+C.$$

$$S=4\pi(\frac{1}{4}\sec^3 x\tan x+\frac{3}{8}\sec x\tan x+\frac{3}{8}\ln(\tan x+\sec x))\Big|_0^{\pi/4}$$

$$=\frac{\pi}{2}[7\sqrt{2}+3\ln(1+\sqrt{2})].$$

22.求$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1(0<b\le a)$分别绕长,短轴旋转而成的椭球面的面积.

解$\begin{cases}x=a\cos t\\ y=b\sin t\end{cases},0\le t\le 2\pi,x'=-a\sin t,y'=b\cos t.$

$$S_a=2\cdot 2\pi b\int_0^{\pi/2}\sqrt{a^2\sin^2 t+b^2\cos^2 t}\sin tdt=$$

$$=-4\pi b\int_0^{\pi/2}\sqrt{a^2-(a^2-b^2)\cos^2 t}d\cos t=$$

$$=4\pi b\int_0^1\sqrt{a^2-(a^2-b^2)u^2}du$$

$$=4\pi b\sqrt{a^2-b^2}\int_0^1\sqrt{\varepsilon^{-2}-u^2}du$$

$$=4\pi ab\frac{\sqrt{a^2-b^2}}{a}\left[\frac{u}{2}\sqrt{\varepsilon^{-2}-u^2}+\frac{\varepsilon^{-2}}{2}\arcsin\varepsilon u\right]\Bigg|_0^1$$

$$=2\pi ab\left(\sqrt{1-\varepsilon^2}+\frac{\arcsin\varepsilon}{\varepsilon}\right).$$

$$S_b=2\cdot 2\pi a\int_0^{\pi/2}\sqrt{a^2\sin^2 t+b^2\cos^2 t}\cos tdt$$

$$=4\pi a\int_0^{\pi/2}\sqrt{b^2+(a^2-b^2)\sin^2 t}d\sin t$$

$$=4\pi a\int_0^1\sqrt{b^2+(a^2-b^2)u^2}du$$

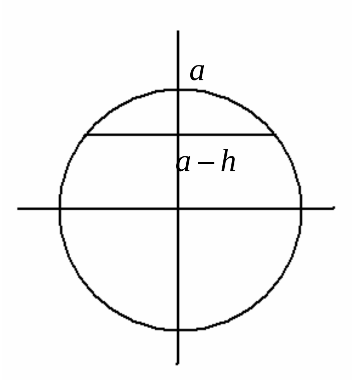
$$=4\pi a\sqrt{a^2-b^2}\int_0^1\sqrt{\frac{b^2}{a^2-b^2}+u^2}du$$

$$=4\pi a\sqrt{a^2-b^2}\left[\frac{u}{2}\sqrt{\frac{b^2}{a^2-b^2}+u^2}+\frac{b^2}{2(a^2-b^2)}\ln(u+\sqrt{\frac{b^2}{a^2-b^2}+u^2})\right]\Bigg|_0^1$$

$$=2\pi a^2+\frac{2\pi b^2}{\varepsilon}\ln\left[\frac{a}{b}(1+\varepsilon)\right].$$

23.计算圆弧$x^2+y^2=a^2(a-h\le y\le a,0<h<a)$绕$y$轴
旋转所得球冠的面积.

解 $\begin{cases} x = a\cos t \\ y = a\sin t \end{cases}$ $\arcsin\dfrac{a-h}{a} \le t \le \dfrac{\pi}{2}$.



$$S = 2\pi\int_{\arcsin\frac{a-h}{a}}^{\frac{\pi}{2}} x\sqrt{x'^2 + y'^2}\,dt$$

$$= 2\pi a^2\int_{\arcsin\frac{a-h}{a}}^{\frac{\pi}{2}} \cos t\,dt$$

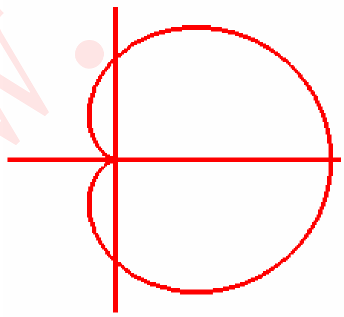
$$= \pi a^2 \left[\sin t\right]\Big|_{\arcsin\frac{a-h}{a}}^{\frac{\pi}{2}}$$

$$= 2\pi a^2\left[1 - \frac{a-h}{a}\right] = 2\pi ah.$$

24.求心脏线$r = a(1+\cos\theta)$绕极轴旋转所成的旋转体的侧面积.

解$r' = -a\sin\theta$.

$$S = 2\pi\int_0^{\pi} a(1+\cos\theta)\sin\theta\sqrt{a^2(1+\cos\theta)^2 + a^2\sin^2\theta}\,d\theta$$



$$= 2\pi a^2\sqrt{2}\int_0^{\pi}(1+\cos\theta)^{3/2}\sin\theta\,d\theta$$

$$= -2\pi a^2\sqrt{2}\int_0^{\pi}(1+\cos\theta)^{3/2}\,d\cos\theta$$

$$= 2\pi a^2\sqrt{2}\int_{-1}^{1}(1+x)^{3/2}dx$$

$$= 2\pi a^2\sqrt{2}\,\frac{2}{5}(1+x)^{5/2}\Big|_{-1}^{1}$$

$$= \frac{32}{5}\pi a^2.$$

25.有一细棒长10m已知距左端点x处的线密度是$\rho(x) = (7+0.2x)$kg/m求这细棒的质量.

解$m = \int_0^{10}(7+0.2x)dx = \left[7x + 0.1x^2\right]\Big|_0^{10} = 80(\text{kg})$.

26.求半径为a的均匀半圆周的重心坐标.

解由对称性, $x_0 = 0$. $\begin{cases} x = a\cos t \\ y = a\sin t \end{cases}, 0 \le t \le \pi$

$$y_0 = \frac{\int_0^{\pi} a\sin t\,dt}{\pi a} = \frac{a}{\pi}[-\cos t]\Big|_0^{\pi} = \frac{2a}{\pi}.$$

重心坐标$(0, \dfrac{2a}{\pi})$.

27.有一均匀细杆,长为l.质量为M.计算细杆绕距离一端$l/5$处的转动惯量.

解$\rho = M/l$. $J = \int_0^{l/5}\dfrac{M}{l}x^2dx + \int_0^{4l/5}\dfrac{M}{l}x^2dx$

$$= \frac{M}{l}\frac{x^3}{3}\Big|_0^{l/5} + \frac{M}{l}\frac{x^3}{3}\Big|_0^{4l/5} = \frac{13}{75}Ml^2.$$

28.设有一均匀圆盘,半径为a,质量为M,求它对于通过其圆心且与盘垂直的轴之转动惯量.

解$\rho = \dfrac{M}{\pi a^2}.dm = \dfrac{M}{\pi a^2} 2\pi x dx = \dfrac{2Mxdx}{a^2}.$

$$J = \int_0^a x^2 \frac{2Mxdx}{a^2} = \frac{2M}{a^2}\frac{x^4}{4}\Big|_0^a = \frac{1}{2}Ma^2.$$

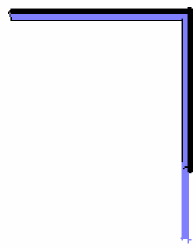
29.有一均匀的圆锥形陀螺,质量为M,底半径为a,高为h,试求此陀螺关于其对称轴的转动惯量.

解$y = \dfrac{a}{h}x, \rho = \dfrac{M}{\frac{1}{3}\pi a^2 h} = \dfrac{3M}{\pi a^2 h}, dm = \rho\pi\left(\dfrac{a}{h}x\right)^2 dx = \dfrac{3M}{h^3}x^2 dx$

$$dJ = \frac{1}{2}dm\left(\frac{a}{h}x\right)^2 = \frac{1}{2}\frac{3a^2M}{h^5}x^4 dx$$

$$J = \int_0^h \frac{1}{2}\frac{3a^2M}{h^5}x^4 dx = \frac{1}{2}\frac{3a^2M}{h^5}\frac{x^5}{5}\Big|_0^h = \frac{3}{10}Ma^2.$$

30.楼顶上有一绳索沿墙壁下垂,该绳索的密度为2kg/m.若绳索下垂部分长为5m, 求将下垂部分全部拉到楼顶所需做的功.



解$dW = 2 \cdot 9.8xdx.$

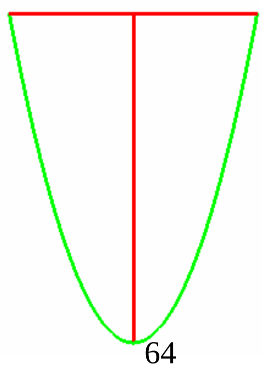
$$W = \int_0^5 2 \cdot 9.8xdx = 9.8x^2\Big|_0^5 = 25 \cdot 9.8(J).$$

31.设$y = f(x)$在$[a,b]$上连续,非负,将由$y = f(x)$ $x = a, x = b$及x轴围成的曲边梯形垂直放置于水中,使y轴与水平面相齐,求水对此曲边梯形的压力.

解$dS = f(x)dx, dF = \rho p dS = g\rho x f(x)dx,$

$$F = g\rho\int_a^b xf(x)dx.$$

32.一 水闸门的边界线为一抛物线,沿水平面的宽度为48m,最低处在水面下64m,求水对闸门的的压力.



解$y = 64 - ax^2, 0 = 64 - a \cdot 24^2, a = \dfrac{1}{9}, x = \pm 3\sqrt{(64-y)}.$

$F = 6g\rho\int_0^{64} y\sqrt{64-y}dy.\sqrt{64-y} = u, y = 64 - u^2,$

$y = 0$时$u = 8, y = 64$时$, u = 0.$

$$F = 6g\rho\int_0^8 (64-u^2)u(2u)dy$$

$$= 12g\rho\left[64 \cdot \frac{u^3}{3} - \frac{u^5}{5}\right]\Bigg|_0^8 = 52428.8g\rho.$$

习题 3.6

1.利用定积分近似计算π的值:

(1)证明公式$\frac{\pi}{4}=\int_0^1\frac{dx}{1+x^2}$;

(2)令$f(x)=\frac{1}{1+x^2}$,给出$|f'(x)|$在(0,1)的上界;

(3)在使用矩形法近似计算上述积分时,欲使公式误差小于5×10^{-5},应取矩形法中分点个数$n>$?

(4)用电脑计算π到小数点后4位.

解 (1)$\int_0^1\frac{dx}{1+x^2}=atc\tan x|_0^1=\frac{\pi}{4}$.

(2)$f'(x)=-\frac{2x}{(1+x^2)^2}$,$|f'(x)|\le 2$.

(3)$|R_n|\le\frac{1}{2n}\cdot 2=\frac{1}{n}<5\times10^{-5},n>\frac{1}{5\times10^{-5}}=2\times10^4$.

(4)n:=2.0*10^4:assume(m,integer):J:=4*sum(1/(1.0+(m/n)^2),m=1..n)/n;

J := 3.141542654 + 0. I

2.自河的一岸开始沿河的横截面方向,每隔5m测量一次水深,一直测到河对岸,依次得到如下21个数据(单位:m)

0, 0. 9, 1. 2, 3. 5, 2. 8, 4. 6, 8. 8, 7. 5, 9. 6, 12. 1, 13. 8,

20. 1, 18. 2, 15. 6, 11. 9, 9. 2, 7. 6, 5. 3, 4. 5, 2. 7, 0.

假定河宽为100m. 试用simpson法计算河床的横截面面积.

解n:=10:d:=[0,0.9,1.2,3.5,2.8,4.6,8.8,7.5,9.6,12.1,13.8,20.1,18.2,15.6,11.9,9.2, 7.6,5.3,4.5,2.7,0]:

S:=((100)/(6*n))*(d[1]+d[21]+2*sum(d[2*i+1],i=1..n－1)+4*sum(d[2*i],i=1..n));
(Maple程序)

$S := 804.6666667$