《高等数学》第七章习题解答

习题7.1

3. 设函数$f(x,y)$在有界闭区域D上连续, $g(x,y)$在D上非负, 且$g(x,y)$与$f(x,y)g(x,y)$在D上可积. 证明: 在D中存在一点(x_0,y_0)使$\iint\limits_D f(x,y)g(x,y)d\sigma = f(x_0,y_0)\iint\limits_D g(x,y)d\sigma$.

证. 设m, M为f在D上的最小, 最大值. 则$mg(x,y) \le f(x,y)g(x,y) \le Mg(x,y)$.
因此$\iint\limits_D mg(x,y)d\sigma \le \iint\limits_D f(x,y)g(x,y)d\sigma \le \iint\limits_D Mg(x,y)d\sigma$.
若$\iint\limits_D g(x,y)d\sigma = 0$, 则$\iint\limits_D f(x,y)g(x,y)d\sigma = 0$, 可任取一点$(x_0,y_0) \in D$使命题成立. 否则有$m \le \frac{\iint\limits_D f(x,y)g(x,y)d\sigma}{\iint\limits_D g(x,y)d\sigma} \le M$. 由介值定理, 存在$(x_0,y_0) \in D$使得
$f(x_0,y_0) = \frac{\iint\limits_D f(x,y)g(x,y)d\sigma}{\iint\limits_D g(x,y)d\sigma}$, 即$\iint\limits_D f(x,y)g(x,y)d\sigma = f(x_0,y_0)\iint\limits_D g(x,y)d\sigma$.

4. 设函数$f(x,y)$在有界闭区域D上连续, 非负, 且$\iint\limits_D f(x,y)dxdy = 0$. 证明$f(x,y) = 0$, 当$(x,y) \in D$时.

证. 因为f非负, 若f不处处为零, 则f在某点$P \in D$处大于0. 又因f连续, 因此在P的一个邻域内f的值大于$\frac{1}{2}f(P)$. 于是$\iint\limits_D f(x,y)dxdy > 0$, 矛盾.

习题7.2

计算下列二重积分.

3. $\iint\limits_D ydxdy$, 其中D由$y=0$及$y=\sin x$ $(0 \le x \le \pi)$所围.

$I = \int_0^\pi dx \int_0^{\sin x} dy y = \int_0^\pi dx \frac{\sin^2 x}{2} = \frac{\pi}{4}$.

4. $\iint\limits_D xy^2dxdy$, 其中D由$x=1$, $y^2=4x$所围.

$I = \int_{-2}^2 dy \int_{y^2/4}^1 dx xy^2 = \int_{-2}^2 dy \frac{1}{2}(1-\frac{y^4}{16})y^2 = \frac{32}{21}$.

5. $\iint\limits_D e^{\frac{x}{y}}dxdy$, 其中$D$由$y^2=x$, $x=0$, $y=1$所围.

$I = \int_0^1 dy \int_0^{y^2} dx e^{\frac{x}{y}} = \int_0^1 dy y e^y = 1$.

6. $\int_0^1 dy \int_{y^{\frac{1}{3}}}^1 \sqrt{1-x^4}dx = \int_0^1 dx \int_0^{x^3} dy\sqrt{1-x^4} = \int_0^1 dx x^3\sqrt{1-x^4} = \frac{1}{6}$.

7. $\iint\limits_D (x^2+y)dxdy$, 其中$D$由$y=x^2$, $x=y^2$所围.

$I = \int_0^1 dx \int_{x^2}^{\sqrt{x}} dy(x^2+y) = \int_0^1 dx(\frac{1}{2}x + x^{\frac{5}{2}} - \frac{3}{2}x^4) = \frac{33}{140}$.

8. $\int_0^\pi dx \int_x^\pi \frac{\sin y}{y}dy = \int_0^\pi dy \int_0^y dx \frac{\sin y}{y} = \int_0^\pi dy \sin y = 2$.

9. $\int_0^2 dx \int_x^2 2y\sin(xy)dy = \int_0^2 dy \int_0^2 dx 2y\sin(xy) = \int_0^2 dy 2(1-\cos 2y) = 4 - \sin 4$.

10. $\iint\limits_D y^2\sqrt{1-x^2}dxdy$, $D=\{(x,y)\mid x^2+y^2\le 1\}$.

$I=4\int_0^1 dx\int_0^{\sqrt{1-x^2}} dyy^2\sqrt{1-x^2}=4\int_0^1 dx\frac{1}{3}(1-x^2)^2=\frac{32}{45}$.

11. $\iint\limits_D (|x|+y)dxdy$, $D=\{(x,y)\mid |x|+|y|\le 1\}$.

$I=\iint\limits_D |x|dxdy+\iint\limits_D ydxdy=4\int_0^1 dx\int_0^{1-x} dyx+0=4\int_0^1 dxx(1-x)=\frac{2}{3}$.

12. $\iint\limits_D (x+y)dxdy$, 其中D为由$x^2+y^2=1$, $x^2+y^2=2y$所围区域的中间一块.

$I=\iint\limits_D xdxdy+\iint\limits_D ydxdy=0+2\int_0^{\frac{\sqrt{3}}{2}} dx\int_{1-\sqrt{1-x^2}}^{\sqrt{1-x^2}} ydy=2\int_0^{\frac{\sqrt{3}}{2}} dx(\sqrt{1-x^2}-\frac{1}{2})=\frac{\pi}{3}-\frac{\sqrt{3}}{4}$.

利用极坐标计算下列累次积分或二重积分.

13. $\int_0^1 dx\int_0^{\sqrt{1-x^2}}(x^2+y^2)dy=\int_0^{\frac{\pi}{2}} d\theta\int_0^1 r^2rdr=\frac{\pi}{8}$.

14. $\int_{-1}^0 dx\int_{-\sqrt{1-x^2}}^0 \frac{2}{1+\sqrt{x^2+y^2}}dy=\int_\pi^{\frac{3\pi}{2}} d\theta\int_0^1 \frac{2}{1+r}rdr=\pi(1-\ln 2)$.

15. $\int_0^2 dx\int_0^{\sqrt{1-(x-1)^2}} 3xydy=\int_0^{\frac{\pi}{2}} d\theta\int_0^{2\cos\theta} 3r\cos\theta r\sin\theta rdr=\int_0^{\frac{\pi}{2}} d\theta 12\cos^5\theta\sin\theta=2$.

16. $\int_0^R dx\int_0^{\sqrt{R^2-x^2}}\ln(1+x^2+y^2)dy=\int_0^{\frac{\pi}{2}} d\theta\int_0^R \ln(1+r^2)rdr=\frac{\pi}{4}[(1+R^2)\ln(1+R^2)-R^2]$.

17. $\iint\limits_D \frac{1}{x^2}dxdy$, D是由$y=\alpha x$, $y=\beta x$ $(\frac{\pi}{2}>\beta>\alpha>0)$, $x^2+y^2=a^2$, $x^2+y^2=b^2$ $(b>a>0)$所围的在第一象限的部分.

$I=\int_{\arctan\alpha}^{\arctan\beta} d\theta\int_a^b \frac{1}{(r\cos\theta)^2}rdr=(\beta-\alpha)\ln\frac{b}{a}$.

18. $\iint\limits_D rd\sigma$, 其中D是由心脏线$r=a(1+\cos\theta)$ 与圆周$r=a$ $(a>0)$所围的不包含极点的区域.

$I=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta\int_a^{a(1+\cos\theta)} rrdr=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta a^3(\cos\theta+\cos^2\theta+\frac{1}{3}\cos^3\theta)=(\frac{22}{9}+\frac{\pi}{2})a^3$.

19. 利用二重积分的几何意义证明: 由射线$\theta=\alpha$, $r=\beta$与曲线$r=r(\theta)$ $(\alpha\le\theta\le\beta)$所围区域$D$的面积可表示成$\frac{1}{2}\int_\alpha^\beta [r(\theta)^2]d\theta$.

证. $S=\iint\limits_D dxdy=\int_\alpha^\beta d\theta\int_0^{r(\theta)} rdr=\frac{1}{2}\int_\alpha^\beta [r(\theta)^2]d\theta$.

20. 求心脏线$r=a(1+\cos\theta)$ $(a>0, 0\le\theta<2\pi)$所围区域之面积.

解. $S=\int_0^{2\pi} d\theta\int_0^{a(1+\cos\theta)} rdr=\int_0^{2\pi} d\theta\frac{1}{2}a^2(1+\cos\theta)^2=\frac{3}{2}\pi a^2$.

计算下列二重积分.

21. $\iint\limits_D (2x^2-xy-y^2)dxdy$, 其中$D$由$y=-2x+4$, $y=-2x+7$, $y=x-2$, $y=x+1$所围.

解. 设$u=2x+y$, $v=x-y$. 则$x=\frac{u+v}{3}$, $y=\frac{u-2v}{3}$, $\frac{D(x,y)}{D(u,v)}=-\frac{1}{3}$.
因此$I=\int_4^7 du\int_{-1}^2 (uv)\frac{1}{3}dv=\frac{33}{4}$.

22. $\iint\limits_D(\sqrt{\frac{y}{x}}+\sqrt{xy})dxdy$, 其中$D$由$xy=1$, $xy=9$, $y=x$与$y=4x$所围.

解. 设$u=\sqrt{\frac{y}{x}}$, $v=\sqrt{xy}$. 则$x=\frac{v}{u}$, $y=uv$, $\frac{D(x,y)}{D(u,v)}=-\frac{2v}{u}$.
因此$I=\int_1^2 du\int_1^3(u+v)\frac{2v}{u}dv=8+\frac{52}{3}\ln 2$.

23. $\iint\limits_D ydxdy$, 其中D为圆域$x^2+y^2\le x+y$.

解1. 设$x=\frac{1}{2}+r\cos\theta$, $y=\frac{1}{2}+r\sin\theta$. 则$\frac{D(x,y)}{D(u,v)}=r$.
因此$I=\int_0^{\frac{1}{\sqrt{2}}}dr\int_0^{2\pi}(\frac{1}{2}+r\sin\theta)rd\theta=\frac{\pi}{4}$.
解2. 设$x=u+\frac{1}{2}$, $y=v+\frac{1}{2}$. 则$\frac{D(x,y)}{D(u,v)}=1$. 设D'为圆域$u^2+v^2\le\frac{1}{2}$,
得$I=\iint_{D'}(v+\frac{1}{2})dudv=0+\frac{1}{2}\cdot\frac{\pi}{2}=\frac{\pi}{4}$.

24. $\iint\limits_D(x^2+y^2)dxdy$, 其中$D$为椭圆域$\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1$.

解. 设$x=ar\cos\theta$, $y=br\sin\theta$. 则$\frac{D(x,y)}{D(u,v)}=abr$.
因此$I=\int_0^1 dr\int_0^{2\pi}(a^2r^2\cos^2\theta+b^2r^2\sin^2\theta)abrd\theta=\int_0^1 dr\pi(a^2r^2+b^2r^2)abr=\frac{\pi}{4}(a^2+b^2)ab$.

26. 设$a>0$, 并令$I(a)=\int_0^a e^{-x^2}dx$, $J(a)=\iint\limits_{D_a}e^{-x^2-y^2}dxdy$, 其中$D_a=\{(x,y)\mid x^2+y^2\le a^2, x\ge 0, y\ge 0\}$. 证明
(1) $[I(a)]^2=\iint\limits_{R_a}e^{-x^2-y^2}dxdy$, 其中$R_a=\{(x,y)\mid 0\le x\le a, 0\le y\le a\}$;
(2) $J(a)\le[I(a)]^2\le J(\sqrt{2}a)$;
(3) 利用本节例10的结果推出$\lim\limits_{a\to+\infty}\int_0^a e^{-x^2}dx=\frac{\sqrt{\pi}}{2}$.

证. (1) $[I(a)]^2=\int_0^a e^{-x^2}dx\int_0^a e^{-y^2}dy=\iint\limits_{R_a}e^{-x^2-y^2}dxdy$.

(2) $D_a\subset R_a\subset D_{\sqrt{2}a}$, 所以$\iint\limits_{D_a}e^{-x^2-y^2}dxdy\le\iint\limits_{R_a}e^{-x^2-y^2}dxdy\le\iint\limits_{D_{\sqrt{2}a}}e^{-x^2-y^2}dxdy$.

(3) $J(a)=\int_0^{\frac{\pi}{2}}d\theta\int_0^a e^{-r^2}rdr=\frac{\pi}{4}(1-e^{-a^2})$. 因此$\lim\limits_{a\to+\infty}J(a)=\lim\limits_{a\to+\infty}J(\sqrt{2}a)=\frac{\pi}{4}$. 由夹逼定理, 得$\lim\limits_{a\to+\infty}I(a)=\frac{\sqrt{\pi}}{2}$.

习题7.3

计算下列三重积分.

1. $\iiint\limits_\Omega(z+z^2)dV$, 其中$\Omega$为单位球$x^2+y^2+z^2\le 1$.

$I=\iiint\limits_\Omega zdV+\iiint\limits_\Omega z^2dV=0+\int_0^{2\pi}d\theta\int_0^\pi d\varphi\int_0^1 drr^2\sin\varphi(r\cos\varphi)^2=\frac{4\pi}{15}$.

2. $\iiint\limits_{\Omega} x^2y^2zdV$, 其中$\Omega$是由$2z=x^2+y^2$, $z=2$所围成的区域.

$I = \int_0^{2\pi} d\theta \int_0^2 rdr \int_{\frac{r^2}{2}}^2 dz(r\cos\theta)^2(r\sin\theta)^2 z = \int_0^{2\pi}(\sin^2\theta\cos^2\theta)d\theta \cdot \int_0^2 r^5(2-\frac{r^4}{8})dr = \frac{\pi}{4}\cdot\frac{128}{15} = \frac{32\pi}{15}$.

3. $\iiint\limits_{\Omega} x^2\sin xdxdydz$, 其中$\Omega$为由平面$z=0$, $y+z=1$及柱面$y=x^2$所围的区域.

区域Ω关于Oyz平面对称, 被积函数是关于x的奇函数, 因此积分为0.

4. $\iiint\limits_{\Omega} zdxdydz$, 其中$\Omega$由$x^2+y^2=4$, $z=x^2+y^2$及$z=0$所围.

$I = \int_0^{2\pi} d\theta \int_0^2 rdr \int_0^{r^2} dzz = 2\pi\cdot\frac{16}{3} = \frac{32\pi}{3}$.

5. $\iiint\limits_{\Omega}(x^2-y^2-z^2)dV$, $\Omega: x^2+y^2+z^2\le a^2$.

$\iiint\limits_{\Omega} z^2dV = \int_0^{2\pi} d\theta\int_0^{\pi} d\varphi\int_0^a r^2\sin\varphi dr(r\cos\varphi)^2 = \frac{4\pi}{15}a^5$. 同理$\iiint\limits_{\Omega} x^2dV = \iiint\limits_{\Omega} y^2dV = \frac{4\pi}{15}a^5$. 因此$I = -\frac{4\pi}{15}a^5$.

6. $\iiint\limits_{\Omega}(x^2+y^2)dV$, $\Omega: 3\sqrt{x^2+y^2}\le z\le 3$.

$I = \int_0^{2\pi} d\theta\int_0^1 rdr\int_{3r}^3 dzr^2 = 2\pi\int_0^1 3(1-r)r^3dr = \frac{3\pi}{10}$.

7. $\iiint\limits_{\Omega}(y^2+z^2)dV$, $\Omega: 0\le a^2\le x^2+y^2+z^2\le b^2$.

取球坐标系$x=r\cos\varphi$, $y=r\sin\varphi\cos\theta$, $z=r\sin\varphi\sin\theta$.
得$I = \int_0^{2\pi} d\theta\int_0^{\pi} d\varphi\int_a^b r^2\sin\varphi dr(r\sin\varphi)^2 = 2\pi\cdot\frac{4}{3}\cdot\frac{b^5-a^5}{5} = \frac{8\pi}{15}(b^5-a^5)$.

8. $\iiint\limits_{\Omega}(x^2+z^2)dV$, $\Omega: x^2+y^2\le z\le 1$.

$I = \int_0^{2\pi} d\theta\int_0^1 rdr\int_{r^2}^1 dz(r^2\cos^2\theta+z^2) = \frac{\pi}{12}+\frac{\pi}{4} = \frac{\pi}{3}$.

9. $\iiint\limits_{\Omega} z^2dV$, $\Omega: x^2+y^2+z^2\le R^2, x^2+y^2\le Rx$.

$I = 4\int_0^{\frac{\pi}{2}} d\theta\int_0^{R\cos\theta} 2rdr\int_0^{\sqrt{R^2-r^2}} dzz^2 = 4\int_0^{\frac{\pi}{2}} d\theta\frac{1}{15}R^5(1-\sin^5\theta) = \frac{2}{15}(\pi-\frac{16}{15})R^5$.

10. $\iiint\limits_{\Omega}(1+xy+yz+zx)dV$, 其中$\Omega$为由曲面$x^2+y^2=2z$及$x^2+y^2+z^2=8$所围$z\ge 0$的部分.

由对称性$I = \iiint\limits_{\Omega} 1dV = \int_0^{2\pi} d\theta\int_0^2 rdr\int_{\frac{r^2}{2}}^{\sqrt{8-r^2}} dz = 2\pi\frac{16\sqrt{2}-14}{3}$.

11. $\iiint\limits_{\Omega}(x^2+y^2)dV$, Ω由$z=\sqrt{R^2-x^2-y^2}$与$z=\sqrt{x^2+y^2}$所围.

$I = \int_0^{2\pi} d\theta\int_0^{\frac{\pi}{4}} d\varphi\int_0^R r^2\sin\varphi drr^2\sin^2\varphi = 2\pi(\frac{2}{15}-\frac{\sqrt{2}}{12})R^5$.

12. $\iiint\limits_{\Omega}\sqrt{x^2+y^2+z^2}dV$, Ω由$z=x^2+y^2+z^2=z$所围.

$I=\int_0^{2\pi}d\theta\int_0^{\frac{\pi}{2}}d\varphi\int_0^{\cos\varphi}r^2\sin\varphi drr=\frac{\pi}{10}$.

13. $\iiint\limits_{\Omega}z^2dV$, $\Omega:\sqrt{3(x^2+y^2)}\le z\le\sqrt{1-x^2-y^2}$.

$I=\int_0^{2\pi}d\theta\int_0^{\frac{\pi}{6}}d\varphi\int_0^1 r^2\sin\varphi drr^2\cos^2\varphi=\frac{2\pi}{15}(1-\frac{3\sqrt{3}}{8})$.

14. $\iiint\limits_{\Omega}\frac{zdV}{\sqrt{x^2+y^2+z^2}}$, Ω由$x^2+y^2+z^2=2az$所围.

$I=\int_0^{2\pi}d\theta\int_0^{\frac{\pi}{2}}d\varphi\int_0^{2a\cos\varphi}r^2\sin\varphi dr\cos\varphi=\frac{16\pi}{15}$.

15. $\iiint\limits_{\Omega}\frac{2xy+1}{x^2+y^2+z^2}dV$, Ω为由$x^2+y^2+z^2=2a^2$与$az=x^2+y^2$所围$z\ge0$的部分.

由对称性$I=\iiint\limits_{\Omega}\frac{1}{x^2+y^2+z^2}dV=\int_0^{2\pi}d\theta\int_0^{\frac{\pi}{4}}d\varphi\int_0^{\sqrt{2}a}r^2\sin\varphi dr\frac{1}{r^2}$

$+\int_0^{2\pi}d\theta\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}d\varphi\int_0^{\frac{a\cos\varphi}{\sin^2\varphi}}r^2\sin\varphi dr\frac{1}{r^2}=2\pi a(\sqrt{2}-1+\ln\sqrt{2})$.

16. $\iiint\limits_{\Omega}\frac{dV}{\sqrt{x^2+y^2+(z-2)^2}}$, $\Omega:x^2+y^2+z^2\le1$.

$I=\int_0^{2\pi}d\theta\int_0^1 dr\int_0^{\pi}d\varphi r^2\sin\varphi\frac{1}{\sqrt{r^2-4r\cos\varphi+4}}=2\pi\int_0^1 drr^2\frac{|r+2|-|r-2|}{r}=\frac{2}{3}\pi$.

17. $\iiint\limits_{\Omega}(x^3+\sin y+z)dV$, Ω由$x^2+y^2+z^2\le2az$, $\sqrt{x^2+y^2}\le z$所围.

由对称性$I=\iiint\limits_{\Omega}zdV=\int_0^{2\pi}d\theta\int_0^{\frac{\pi}{4}}d\varphi\int_0^{2a\cos\varphi}r^2\sin\varphi drr\cos\varphi=\frac{7}{6}\pi a^4$.

18. $\iiint\limits_{\Omega}(x^2y+3xyz)dV$, $\Omega:1\le x\le2$, $0\le xy\le2$, $0\le z\le1$.

设$u=x,v=xy,w=z$. 则$x=u,y=\frac{v}{u},z=w$, $\frac{D(x,y,z)}{D(u,v,w)}=\frac{1}{u}$.
$I=\int_1^2du\int_0^2dv\int_0^1dw(uv+3vw)\frac{1}{u}dudvdw=2+3\ln2$.

19. $\iiint\limits_{\Omega}(x+1)(y+1)dV$, $\Omega:\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le1$.

由对称性$I=\iiint\limits_{\Omega}(xy+x+y+1)dV=\iiint\limits_{\Omega}1dV=\int_0^{2\pi}d\theta\int_0^{\pi}d\varphi\int_0^1 abcr^2\sin\varphi dr=$
$\frac{4}{3}\pi abc$.

20. $\iiint\limits_{\Omega}(x+y+z)dV$, $\Omega:(x-x_0)^2+(y-y_0)^2+(z-z_0)^2\le a^2$.

设$x=x_0+u$, $y=y_0+v$, $z=z_0+w$, $\Omega':u^2+v^2+w^2\le a^2$. 则$\frac{D(x,y,z)}{D(u,v,w)}=1$.
$I=\iiint\limits_{\Omega'}(x_0+y_0+z_0+u+v+w)dudvdw=\iiint\limits_{\Omega'}(x_0+y_0+z_0)dudvdw=$
$\frac{4}{3}\pi a^3(x_0+y_0+z_0)$

21. 分别用柱坐标和球坐标, 把三重积分$I=\iiint\limits_{\Omega}f(\sqrt{x^2+y^2+z^2})dV$表成累次积分, 其中$\Omega$为球体$x^2+y^2+z^2\le z$在锥面$z=\sqrt{3x^2+3y^2}$上方的部分.

解. 柱坐标: $I=\int_0^{2\pi}d\theta\int_0^{\frac{\sqrt{3}}{4}}rdr\int_{\sqrt{3x^2+3y^2}}^{\frac{1+\sqrt{1-4r^2}}{2}}f(\sqrt{r^2+z^2})dz$.

球坐标: $I=\int_0^{2\pi}d\theta\int_0^{\frac{\pi}{6}}d\varphi\int_0^{\cos\varphi}f(r)r^2\sin\varphi dr$.

22. 化累次积分$I=\int_0^a dx\int_0^x dy\int_0^y dzf(z)dz$为定积分.

解. 设Ω为区域$0\le z\le y\le x\le a$. 则$I=\iiint\limits_{\Omega}f(z)dxdydz=\int_0^a dz\int_z^a dy\int_y^a dxf(z)=\frac{1}{2}\int_0^a f(z)(z-a)^2dz$.

习题7.4

1. 求由上半球面$z=\sqrt{3a^2-x^2-y^2}$及旋转抛物面$x^2+y^2=2az$所围立体的表面积$(a>0)$.

解. $S=\iint\limits_{x^2+y^2\le 2a^2}(\sqrt{\frac{3a^2}{3a^2-x^2-y^2}}+\sqrt{1+\frac{x^2}{a^2}+\frac{y^2}{a^2}})dxdy$
$=\int_0^{2\pi}d\theta\int_0^{\sqrt{2}a}rdr(\frac{\sqrt{3}a}{\sqrt{3a^2-r^2}}+\sqrt{1+\frac{r^2}{a^2}})=\frac{16}{3}\pi a^2$.

2. 求锥面$z=\sqrt{x^2+y^2}$被柱面$z^2=2x$所割下部分的曲面面积.

解. $S=\iint\limits_{x^2+y^2\le 2x}\sqrt{2}dxdy=\sqrt{2}\pi$

3. 求由三个圆柱面$x^2+y^2=R^2$, $x^2+z^2=R^2$, $y^2+z^2=R^2$所围立体的表面积.

解. 设$D=\{(x,y)\mid 0\le y\le x\le\frac{R}{\sqrt{2}}\}$, 柱面$x^2+z^2=R^2$投影到$D$上的第一卦限部分的面积为$\iint\limits_{D}\frac{R}{\sqrt{R^2-x^2}}dxdy=(1-\frac{1}{\sqrt{2}})R^2$. 因此总的表面积为$48(1-\frac{1}{\sqrt{2}})R^2$.

4. 求由三个圆柱面$x^2+y^2=R^2$, $x^2+z^2=R^2$, $y^2+z^2=R^2$所围立体的体积.

解. 设$D=\{(x,y)\mid x^2+y^2\le R^2, 0\le y\le x\}$, 投影到$D$上的第一卦限部分的立体体积为$\iint\limits_{D}\sqrt{R^2-x^2}dxdy=\int_0^{\frac{\pi}{4}}d\theta\int_0^R R\sin\theta rdr=\frac{1}{2}(1-\frac{1}{\sqrt{2}})R^3$. 因此总体积为$8(1-\frac{1}{\sqrt{2}})R^3$.

第七章总练习题

4. 求下列累次积分.

(1) $\int_0^1 dy\int_{2y}^2 4\cos x^2dx=\int_0^2 dx\int_0^{\frac{x}{2}}4\cos x^2dy=\int_0^2 2x\cos x^2dx=\sin 4$.

(2) $\int_0^8 dx\int_{\sqrt[3]{x}}^2\frac{dy}{1+y^4}=\int_0^2 dy\int_0^{y^3}\frac{dx}{1+y^4}=\int_0^2 dy\frac{y^3dy}{1+y^4}=\frac{\ln 17}{4}$.

11. 求圆$x^2+y^2\le a^2$上所有的点到原点的平均距离.

解. $d=\frac{1}{\pi a^2}\iint\limits_{x^2+y^2\le a^2}\sqrt{x^2+y^2}dxdy=\frac{2}{3}a$.

21. 设闭曲面S在球坐标下的方程为$\rho=2\sin\varphi$. 求S所围立体的体积.

解. $V=\int_0^{2\pi}d\theta\int_0^{\pi}d\varphi\int_0^{2\sin\varphi}r^2\sin\varphi dr=2\pi^2$.