Discrete Mathematics: Lecture 2

- Last time:
  - Chap 6.1: The basics of counting
  - Chap 6.2: The pigeonhole principle
- Today:
  - Chap 6.3: Permutations and combinations
  - Chap 6.4: Binomial coefficients and identities
- Assignment 1 due in two weeks

Review of last time

- Basic counting principles: the product, sum, substraction and division rules
- Tree diagrams
- The pigeonhole principle and its generalized version

Permutations (排列)

- Definition: A permutation of a set of distinct objects is an ordered arrangement of these objects. An ordered arrangement of r elements of a set is called an r-permutation.
- Example: Let $S = \{1, 2, 3\}$. The sequence 3,1,2 is a permutation of S. The sequence 3,1 is a 2-permutation of S.
- Theorem: Let $P(n, r)$ denote the number of r-permutations of a set with n elements. Then $P(n, r) = n(n-1)\ldots(n-r+1) = \frac{n!}{(n-r)!}$.
- Note: $P(n, 0) = 1$
- Example: How many ways are there to select a first-prize winner, a second-prize winner, and a third-prize winner from 100 different people who have entered a contest.
- Example: How many permutations of the letters ABCDEFGH contain the string ABC?

Combinations (组合)

- Definition: An unordered selection of r elements of a set is called an r-combination.
- Theorem: Let $C(n,r)$ denote the number of r-combinations of a set with n elements, where n and r are nonnegative integers with $r \le n$. Then

$$C(n,r) = \frac{n!}{r!(n-r)!}$$

- Corollary: Let n and r be nonnegative integers with $r \le n$. Then $C(n,r) = C(n,n-r)$.
- Example: How many poker hands of 5 cards can be dealt from a standard deck of 52 cards? How many ways are there to select 47 cards from a standard deck of 52 cards?
- How many bit strings of length n contain exactly r 1s?

Combinatorial proofs (组合证明)

- Definition: A combinatorial proof of an identity is a proof that uses counting arguments to prove that both sides of the identity count the same objects but in different ways.
- Example application: $C(n, r) = C(n, n - r)$

The binomial theorem

- A binomial expression is the sum of two terms, such as $x + y$.
- Theorem: Let x and y be variables, and let n be a nonnegative integer. Then $(x+y)^n = \sum_{j=0}^{n} C(n,j)x^{n-j}y^j$.
  Proof:
  - The terms in the product $(x+y)(x+y)\dots(x+y)$ are of the form $x^{n-j}y^j$
  - To obtain a term $x^{n-j}y^j$, we choose j ys from the n sums.
  - Thus the coefficient of $x^{n-j}y^j$ is $C(n,j)$.
- Example: $(x+y)^4$
- Example: what is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x-3y)^{25}$

Corollaries

- Corollary 1: $\sum_{j=0}^{n} C(n,j) = 2^n$
    - combinatorial proof
- Corollary 2: $\sum_{j=0}^{n} (-1)^j C(n,j) = 0$
- Corollary 3: $\sum_{j=0}^{n} 2^j C(n,j) = 3^n$

Theorem: Let n and k be positive integers with $n \geq k$. Then
$C(n+1,k) = C(n,k-1) + C(n,k)$

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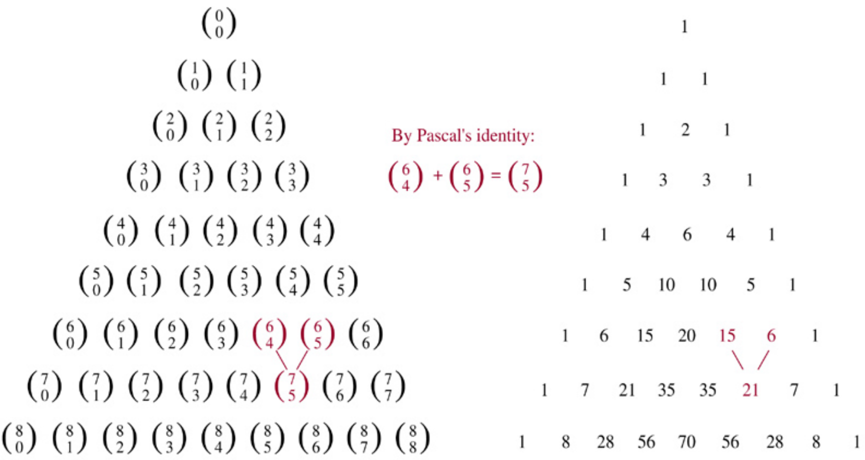

Some other identities

- Vandermonde's identity: Let m, n and r be nonnegative integers with $r \leq m, n$. Then $C(m+n, r) = \sum_{k=0}^{r} C(m, r-k)C(n, k)$
- Let n and r be nonnegative integers with $r \leq n$. Then $C(n+1, r+1) = \sum_{j=r}^{n} C(j, r)$