Discrete Mathematics: Lecture 8

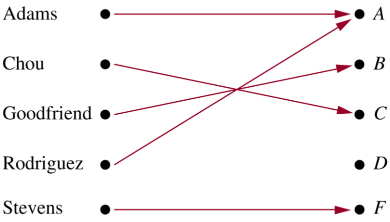
- Last time:
  - Chap 2.1: Sets
  - Chap 2.2: Set Operations
- Today:
  - Chap 2.3: Functions
  - Chap 2.4: Sequences and Summations
- Assignment 2 due in two weeks (Nov. 25)
- Next time:
  - Chap 2.5: Cardinality of Sets
  - Chap 2.6: Matrices

Review of last time

- Set, Venn diagram, subset, power set
- Cartesian product, relation
- Union, intersection, difference, complement
- Set identities

Functions

- Definition: Let A and B be nonempty sets. A function (函数) from A to B is an assignment of exactly one element of B to each element of A.
- Notation: $f(a) = b$, $f : A \to B$.
- Remark: Functions are sometimes also called mappings (映射) or transformations.
- Ways to specify functions: explicitly state the assignment, give a formula, or give a program



Function as a relation

- A relation from A to B is a subset of $A \times B$
- A function from A to B is a relation from A to B that contains one and only one ordered pair (a, b) for every element $a \in A$

Some terms

- If f is a function from A to B, we say that A is the domain (域) of f and B is the codomain (共域) of f.
- If $f(a) = b$, we say that b is the image (像) of a and a is a preimage (原像) of b.
- The range (值域) of f is the set of all images of elements of A.
- If $f : A \to B$, we say that f maps A to B.

Functions

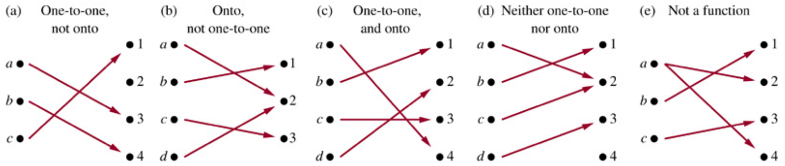
- Example: The domain, codomain and range of the grade function
- Example: $f : \mathbf{Z} \to \mathbf{Z}$ where $f(x) = x^2$
- Definition: $f_1 + f_2$, $f_1 f_2$
- Definition: Let $f : A \to B$ and $S \subseteq A$. The image of S under f is the set $f(S) = \{t \mid \exists s \in S(t = f(s))\}$ or $f(S) = \{f(s) \mid s \in S\}$
- Example: the grade function

One-to-one functions

- Definition: A function f is said to be one-to-one, or injective if $\forall a \forall b (f(a) = f(b) \to a = b)$. A function is said to be an injection (单射) if it is one-to-one.
- Examples: $f(x) = x^2$, $f(x) = x + 1$, the domain is $\mathbf{Z}$
- Definition: A function f whose domain and codomain are subsets of $\mathbf{R}$ is called increasing (递增的) if $\forall x \forall y (x < y \to f(x) \leq f(y))$, and strictly increasing if $\forall x \forall y (x < y \to f(x) < f(y))$.
- Examples: $f(x) = x^2$, $f(x) = x + 1$

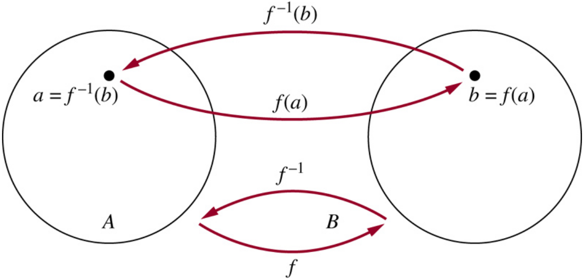
Onto functions

- Definition: A function f from A to B is called onto, or surjective if $\forall b \in B \exists a \in A (f(a) = b)$. A function is said to be an surjection (满射) if it is onto.
- Examples: $f(x) = x^2$, $f(x) = x + 1$, the domain is $\mathbf{Z}$
- Definition: A function f is a one-to-one correspondence (一一对应), or a bijection (双射) if it is both one-to-one and onto.



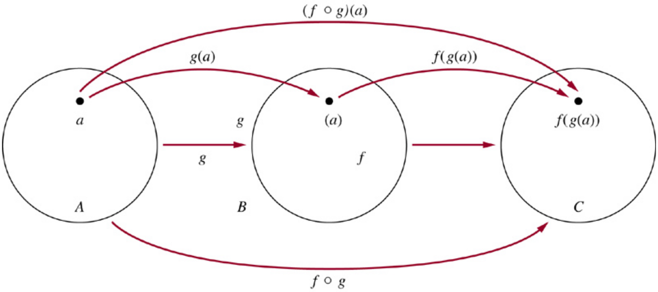
Inverse functions (逆函数)

- Definition: Let $f : A \to B$ be a bijection. The inverse function of f, denoted by f^{-1}, is defined by: $f^{-1}(b) = a$ when $f(a) = b$.
- We say a function is invertible if it has an inverse function.
- A function is invertible iff it is a bijection.



Composition of functions (函数的复合)

- Definition: Let $f : A \to B$ and $g : B \to C$. The composition of f and g, denoted by $f \circ g$, is defined by $(f \circ g)(a) = f(g(a))$.
- Example: $f(x) = 2x + 3$, $g(x) = 3x + 2$,
- Note: $f \circ g$ and $g \circ f$ may not be equal

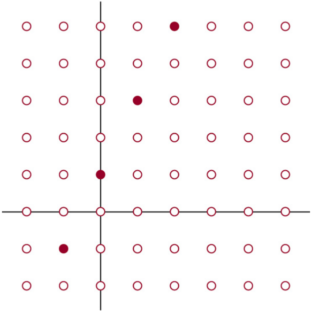


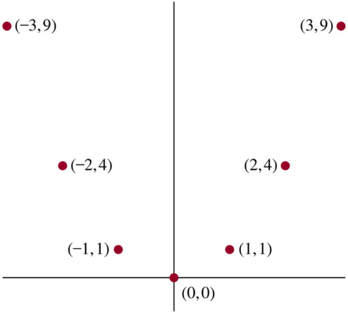
Composition of functions

- The identity function on A (恒等函数), denoted by ι_A, is defined by: $\iota_A(a) = a$
- Let f be a bijection from A to B
- Then $f^{-1} \circ f = \iota_A$, $f \circ f^{-1} = \iota_B$

The graphs of functions (函数的图像)

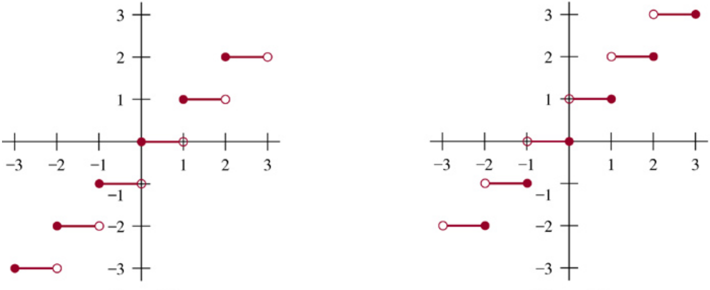
- Definition: The graph of $f : A \to B$ is the set of ordered pairs $\{(a,b) \mid a \in A \text{ and } f(a) = b\}$
- Examples: $f(x) = 2x + 1$, $f(x) = x^2$, the domain is $\mathbf{Z}$





The floor and ceiling functions

- Definition: The floor function assigns to the real number x the largest integer less than or equal to x. The value of the floor function at x is denoted by $\lfloor x \rfloor$.
- Definition: The ceiling function assigns to the real number x the smallest integer greater than or equal to x. The value of the ceiling function at x is denoted by $\lceil x \rceil$.



(a) $y = \lfloor x \rfloor$ (b) $y = \lceil x \rceil$

The floor and ceiling functions

- Example: Computers represent information using bits. Each byte (字节) is made of 8 bits (位). How many bytes are required to encode 100 bits?
- Example: In asynchronous transfer mode (ATM), data are organized into cells (单元) of 53 bytes. How many ATM cells can be transmitted in 1 minute over a connection that transmits data at the rate of 500 kilobits per second?

Useful properties of the floor and ceiling functions

- $\lfloor x \rfloor = n$ iff $n \leq x < n+1$
- $\lceil x \rceil = n$ iff $n-1 < x \leq n$
- $\lfloor x \rfloor = n$ iff $x-1 < n \leq x$
- $\lceil x \rceil = n$ iff $x \leq n < x+1$
- $x - 1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x+1$
- $\lfloor -x \rfloor = -\lceil x \rceil$, $\lceil -x \rceil = -\lfloor x \rfloor$
- $\lfloor x+n \rfloor = \lfloor x \rfloor + n$, $\lceil x+n \rceil = \lceil x \rceil + n$

Useful properties of the floor and ceiling functions

- Prove that $\lfloor x + n \rfloor = \lfloor x \rfloor + n$
- Prove that if $x \in \mathbf{R}$, then $\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{2} \rfloor$
- Prove or disprove that $\lceil x + y \rceil = \lceil x \rceil + \lceil y \rceil$, where $x, y \in \mathbf{R}$

Other functions

- $\log_b x$, $\log x$, $\ln x$: the natural logarithm (对数)
- $f(n) = n! = 1 \cdot 2 \ldots (n-1) \cdot n$: the factorial (阶乘) function
- The factorial function grows extremely rapidly:
  $f(20) > 2.433 \cdot 10^{18}$
- $n! \sim \sqrt{2\pi n}(n/e)^n$, where $f(n) \sim g(n)$ means
  $\lim_{n\to\infty} f(n)/g(n) = 1$

Partial functions (偏函数)

- A partial function f from a set A to a set B is an assignment to each element a in a subset of A of a unique element b in B
- The subset is called the domain of definition (定义域) of f
- We say that f is undefined (无定义的) for elements not in the domain of definition of f
- When the domain of definition of f equals A, we say that f is a total function (全函数)
- We write $f : A \to B$ to denote that f is a partial function from A to B
- Example: $f : \mathbf{Z} \to \mathbf{R}$ where $f(n) = \sqrt{n}$

Sequences

- Definition: A sequence (序列) is a function f from a subset of $\mathbf{Z}$ to a set S. We use a_n to represent $f(n)$. We call a_n a term of the sequence. We use $\{a_n\}$ to describe the sequence.
- Example: $\{a_n\}$ where $a_n = \frac{1}{n}$
- Definition: A geometric progression (几何级数) is a sequence of the form $a, ar, ar^2, \ldots, ar^n, \ldots$ where the initial term a and the common ratio (比率) r are real numbers.
- Examples: $\{(-1)^n\}$, $\{2 \cdot 5^n\}$,

Sequences

- Definition: An arithmetic progression (算术级数) is a sequence of the form $a, a + d, a + 2d, \ldots, a + nd, \ldots$ where the initial term a and the common difference d are real numbers.
- Examples: $\{-1 + 4n\}$, $\{7 - 3n\}$,
- Definition: Finite sequences are called strings (串). The string $a_1, a_2, \ldots, a_n$ is often denoted by $a_1 a_2 \ldots a_n$. The empty string is denoted by λ.

Recurrence relation (递推关系)

- Another way to specify a sequence: provides one or more initial terms and a rule for determining subsequent terms form those preceding them.
- Definition: A recurrence relation for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms, for all $n \geq n_0 \geq 0$. A sequence is a solution of a recurrence relation if its terms satisfies the relation.
- Example: consider $a_n = 2a_{n-1} - a_{n-2}$ for $n \geq 2$.
  Are the following its solutions? $a_n = 3n$, $a_n = 2^n$, $a_n = 5$.
- The initial conditions for a sequence specify the terms where the recurrence relation has not taken effect.
- The initial conditions and recurrence relation uniquely determines a sequence.

Examples

- The Fibonacci sequence: $f_0 = 0$, $f_1 = 1$, $f_n = f_{n-1} + f_{n-2}$, $n \geq 2$
- We say that we have solved a recurrence relation together with the initial conditions when we find an explicit formula, called a closed formula, for the terms of the sequence
- Example: $a_0 = 2$, $a_n = a_{n-1} + 3$, $n \geq 1$
- Technique: iteratively apply the recurrence relation
- Suppose that a person deposits \$10,000 in a savings account at a bank yielding 11% per year with interest compounded annually. How much will be in the account after 30 years?

Finding a formula for a sequence

When trying to deduce a possible formula or rule for the terms of a sequence from the initial terms, try to find a pattern in the terms. Questions to ask:

- Does the same value occur many times in a sequence?
- Are terms obtained from previous terms by adding the same amount or an amount that depends on the position in the sequence
- Are terms obtained from previous terms by multiplying by a particular amount
- Are terms obtained by combining previous terms in a certain way
- Are there cycles among the terms

Examples

- 1,2,2,3,3,3,4,4,4,4
  $a_n = l$ such that $1+2+\ldots+l-1 < n \leq 1+2+\ldots+l$
- 3,5,8,12,17,23,30,38,47
- 0,2,8,26,80,242,728,2186,6560,19682

Find a formula for a sequence

Compare with the terms of a well-known integer sequence

TABLE 1 **Some Useful Sequences.**

nth Term	*First 10 Terms*
n^2	1, 4, 9, 16, 25, 36, 49, 64, 81, 100, . . .
n^3	1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, . . .
n^4	1, 16, 81, 256, 625, 1296, 2401, 4096, 6561, 10000, . . .
2^n	2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, . . .
3^n	3, 9, 27, 81, 243, 729, 2187, 6561, 19683, 59049, . . .
$n!$	1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800, . . .

- 1,7,25,79,241,727,2185,6559,19681,59047
- 2, 16, 54, 128, 250, 432, 686
- 2,3,7,25,121,721,5041,40321

Summations

- To represent $a_m + a_{m+1} + \ldots + a_n$, we use

$$\sum_{j=m}^{n} a_j, \qquad \textstyle\sum_{j=m}^{n} a_j, \qquad \text{or } \textstyle\sum_{m \le j \le n} a_j$$

  where j – the index of the summation, m – lower limit, n – upper limit,

- Examples: $\sum_{j=1}^{100} \frac{1}{j}, \qquad \sum_{k=4}^{8} (-1)^k$
- Shift the index of a summation, *e.g.*, $\sum_{j=1}^{5} j^2$
- Double summations, *e.g.*, $\sum_{i=1}^{4} \sum_{j=1}^{3} ij$
- Summations of function values where the index runs over all values in a set: $\sum_{s \in S} f(s), \qquad e.g., \sum_{s \in \{1,3,5\}} s^2$

TABLE 2 Some Useful Summation Formulae.

Sum	*Closed Form*
$\sum_{k=0}^{n} ar^k \ (r \neq 0)$	$\dfrac{ar^{n+1} - a}{r - 1}, r \neq 1$
$\sum_{k=1}^{n} k$	$\dfrac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\dfrac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$\dfrac{n^2(n+1)^2}{4}$
$\sum_{k=0}^{\infty} x^k, \lvert x\rvert < 1$	$\dfrac{1}{1-x}$
$\sum_{k=1}^{\infty} kx^{k-1}, \lvert x\rvert < 1$	$\dfrac{1}{(1-x)^2}$

Proofs of Equations 1,5,6

Example: Find $\sum_{k=50}^{100} k^2$