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Editorial

The 2010 Modeling Contests

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Background

Based on Ben Fusaro's suggestion for an "applied Putnam" contest, in 1985 COMAP introduced the Mathematical Contest in Modeling (MCM)®. Since then, this *Journal* has devoted an issue each year to the Outstanding contest papers. Even after substantial editing, that issue has sometimes run to more than three times the size of an ordinary issue. From 2005 through 2009, some papers appeared in electronic form only.

The 2,254 MCM teams in 2010 was almost double the number in 2008.

Also, since the introduction in 1999 of the Interdisciplinary Contest in Modeling (ICM)® (which has separate funding and sponsorship from the MCM), the *Journal* has devoted a second of its four annual issues to Outstanding papers from that contest.

A New Designation for Papers

It has become increasingly difficult to identify just a handful of Outstanding papers for each problem. After 14 Outstanding MCM teams in 2007, there have been 9 in each year since, despite more teams competing.

The judges have been overwhelmed by increasing numbers of Meritorious papers from which to select the truly Outstanding. As a result, this year there is a new designation of Finalist teams, between Outstanding and Meritorious. It recognizes the less than 1% of papers that reached the final (seventh) round of judging but were not selected as Outstanding. Each Finalist paper displayed some modeling that distinguished it from the rest of the Meritorious papers. We think that the Finalist papers deserve special

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recognition, and the mathematical professional societies are investigating ways to recognize the Finalist papers.

Just One Contest Issue Each Year

Taking up two of the four *Journal* issues each year, and sometimes two-thirds of the pages, the amount of material from the two contests has come to overbalance the other content of the *Journal*.

The Executive Publisher, Sol Garfunkel, and I have decided to return more of the *Journal* to its original purpose, as set out 30 years ago, to:

> acquaint readers with a wide variety of professional applications of the mathematical sciences, and provide a forum for discussions of new directions in mathematical education.
>
> [Finney and Garfunkel 1980, 2–3]

Henceforth, we plan to devote just a single issue of the *Journal* each year to the two contests combined. That issue—this issue—will appear during the summer and contain

- reports on both contests, including the problem statements and names of the Outstanding teams and their members;
- authors', judges', and practitioners' commentaries (as available) on the problems and the Outstanding papers; and
- just one Outstanding paper from each problem.

Available separately from COMAP on a CD-ROM very soon after the contests (as in 2009 and again this year) will be:

- full original versions of all of the Outstanding papers, and
- full results for all teams.

Your Role

The ever-increasing engagement of students in the contests has been astonishing; the steps above help us to cope with this success.

There will now be more room in the *Journal* for material on mathematical modeling, applications of mathematics, and ideas and perspectives on mathematics education at the collegiate level—articles, UMAP Modules, Minimodules, ILAP Modules, guest editorials. We look forward to your contribution.

Reference

Finney, Ross L., and Solomon Garfunkel. 1980. UMAP and *The UMAP Journal*. *The UMAP Journal* 0: 1–4.

Modeling Forum

Results of the 2010 Mathematical Contest in Modeling

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Introduction

A total of 2,254 teams of undergraduates from hundreds of institutions and departments in 14 countries, spent a weekend in February working on applied mathematics problems in the 26th Mathematical Contest in Modeling (MCM)®.

The 2010 MCM began at 8:00 P.M. EST on Thursday, February 18, and ended at 8:00 P.M. EST on Monday, February 22. During that time, teams of up to three undergraduates researched, modeled, and submitted a solution to one of two open-ended modeling problems. Students registered, obtained contest materials, downloaded the problem and data, and entered completion data through COMAP's MCM Website. After a weekend of hard work, solution papers were sent to COMAP on Monday. Two of the top papers appear in this issue of *The UMAP Journal*, together with commentaries.

In addition to this special issue of *The UMAP Journal*, COMAP has made available a special supplementary *2010 MCM-ICM CD-ROM* containing the press releases for the two contests, the results, the problems, and original versions of the Outstanding papers. Information about ordering the CD-ROM is at `http://www.comap.com/product/cdrom/index.html` or from (800) 772–6627.

Results and winning papers from the first 25 contests were published in special issues of *Mathematical Modeling* (1985–1987) and *The UMAP Journal* (1985–2009). The 1994 volume of *Tools for Teaching*, commemorating the tenth anniversary of the contest, contains the 20 problems used in the first 10 years of the contest and a winning paper for each year. That volume and the special

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MCM issues of the *Journal* for the last few years are available from COMAP. The 1994 volume is also available on COMAP's special *Modeling Resource* CD-ROM. Also available is *The MCM at 21* CD-ROM, which contains the 20 problems from the second 10 years of the contest, a winning paper from each year, and advice from advisors of Outstanding teams. These CD-ROMs can be ordered from COMAP at `http://www.comap.com/product/cdrom/index.html`.

This year, the two MCM problems represented significant challenges:

- Problem A, "The Sweet Spot," asked teams to explain why the spot on a baseball bat where maximum power is transferred to the ball is not at the end of the bat and to determine whether "corking" a bat (hollowing it out and replacing the hardwood with cork) enhances the "sweet spot" effect.
- Problem B, "Criminology," asked teams to develop geographical profiling to aid police in finding serial criminals.

In addition to the MCM, COMAP also sponsors the Interdisciplinary Contest in Modeling (ICM)® and the High School Mathematical Contest in Modeling (HiMCM)®:

- The ICM runs concurrently with MCM and for the next several years will offer a modeling problem involving an environmental topic. Results of this year's ICM are on the COMAP Website at `http://www.comap.com/undergraduate/contests`. The contest report, an Outstanding paper, and commentaries appear in this issue.
- The HiMCM offers high school students a modeling opportunity similar to the MCM. Further details about the HiMCM are at `http://www.comap.com/highschool/contests`.

2010 MCM Statistics

- 2,254 teams participated
- 15 high school teams (<1%)
- 358 U.S. teams (21%)
- 1,890 foreign teams (79%), from Australia, Canada, China, Finland, Germany, Indonesia, Ireland, Jamaica, Malaysia, Pakistan, Singapore, South Africa, United Kingdom
- 9 Outstanding Winners (<0.5%)
- 12 Finalists (0.5%)
- 431 Meritorious Winners (19%)
- 542 Honorable Mentions (24%)
- 1,245 Successful Participants (55%)

Problem A: The Sweet Spot

Explain the "sweet spot" on a baseball bat. Every hitter knows that there is a spot on the fat part of a baseball bat where maximum power is transferred to the ball when hit. Why isn't this spot at the end of the bat? A simple explanation based on torque might seem to identify the end of the bat as the sweet spot, but this is known to be empirically incorrect. Develop a model that helps explain this empirical finding.

Some players believe that "corking" a bat (hollowing out a cylinder in the head of the bat and filling it with cork or rubber, then replacing a wood cap) enhances the "sweet spot" effect. Augment your model to confirm or deny this effect. Does this explain why Major League Baseball prohibits "corking"?

Does the material out of which the bat is constructed matter? That is, does this model predict different behavior for wood (usually ash) or metal (usually aluminum) bats? Is this why Major League Baseball prohibits metal bats?

Problem B: Criminology

In 1981, Peter Sutcliffe was convicted of 13 murders and subjecting a number of other people to vicious attacks. One of the methods used to narrow the search for Mr. Sutcliffe was to find a "center of mass" of the locations of the attacks. In the end, the suspect happened to live in the same town predicted by this technique. Since that time, a number of more sophisticated techniques have been developed to determine the "geographical profile" of a suspected serial criminal based on the locations of the crimes.

Your team has been asked by a local police agency to develop a method to aid in their investigations of serial criminals. The approach that you develop should make use of at least two different schemes to generate a geographical profile. You should develop a technique to combine the results of the different schemes and generate a useful prediction for law enforcement officers. The prediction should provide some kind of estimate or guidance about possible locations of the next crime based on the time and locations of the past crime scenes. If you make use of any other evidence in your estimate, you must provide specific details about how you incorporate the extra information. Your method should also provide some kind of estimate about how reliable the estimate will be in a given situation, including appropriate warnings.

In addition to the required one-page summary, your report should include an additional two-page executive summary. The executive summary should provide a broad overview of the potential issues. It should provide an overview of your approach and describe situations when it is an appropriate tool and situations in which it is not an appropriate tool. The executive summary will be read by a chief of police and should include technical details appropriate to the intended audience.

The Results

The solution papers were coded at COMAP headquarters so that names and affiliations of the authors would be unknown to the judges. Each paper was then read preliminarily by two "triage" judges at either Appalachian State University (Sweet Spot Problem) or at the National Security Agency (Criminology Problem). At the triage stage, the summary and overall organization are the basis for judging a paper. If the judges' scores diverged for a paper, the judges conferred; if they still did not agree, a third judge evaluated the paper.

Additional Regional Judging sites were created at the U.S. Military Academy and at the Naval Postgraduate School to support the growing number of contest submissions.

Final judging took place at the Naval Postgraduate School, Monterey, CA. The judges classified the papers as follows:

	Outstanding	Finalist	Meritorious	Honorable Mention	Successful Participation	Total
Sweet Spot Problem	4	5	180	217	533	939
Criminology Problem	5	7	251	325	712	1300
	9	12	431	542	1245	2239

We list here the 9 teams that the judges designated as Outstanding; the list of all participating schools, advisors, and results is at the COMAP Website.

Outstanding Teams

Institution and Advisor	Team Members

Sweet Spot Problem

Institution and Advisor	Team Members
"An Optimal Model of 'Sweet Spot' Effect" Huazhong University of Science and Technology Wuhan, Hubei, China Liang Gao	Zhe Xiong Qipei Mei Fei Han
"The Sweet Spot: A Wave Model of Baseball Bats" Princeton University Princeton, NJ Robert Calderbank	Yang Mou Peter Diao Rajib Quabili

"Brody Power Model: An Analysis of Baseball's 'Sweet Spot'"
U.S. Military Academy — David Covell
West Point, NY — Ben Garlick
Elizabeth Schott — Chandler Williams

"An Identification of 'Sweet Spot'"
Zhejiang University — Cong Zhao
Hangzhou, China — Yuguang Yang
Xinxin Xu — Zuogong Yue

Criminology Papers

"Predicting a Serial Criminal's Next Crime Location Using Geographic Profiling"
Bucknell University — Bryan Ward
Lewisburg, PA — Ryan Ward
Nathan C. Ryan — Dan Cavallaro

"Following the Trail of Data"
Rensselaer Polytechnic Institute — Yonatan Naamad
Troy, NY — Joseph H. Gibney
Peter R. Kramer — Emily P. Meissen

"From Kills to Kilometers: Using Centrographic Techniques and Rational Choice Theory for Geographical Profiling of Serial Killers"
Tufts University — Daniel Brady
Medford, MA — Liam Clegg
Scott MacLachlan — Victor Minden

"Centroids, Clusters, and Crime: Anchoring the Geographic Profile of Serial Criminals"
University of Colorado—Boulder — Anil S. Damle
Boulder, CO — Colin G. West
Anne M. Dougherty — Eric J. Benzel

"Tracking Serial Criminals with a Road Metric"
University of Washington — Ian Zemke
Seattle, WA — Mark Bun
James Allen Morrow — Jerry Li

Awards and Contributions

Each participating MCM advisor and team member received a certificate signed by the Contest Director and the appropriate Head Judge.

INFORMS, the Institute for Operations Research and the Management Sciences, recognized the teams from Princeton University (Sweet Spot Problem) and Tufts University (Criminology Problem) as INFORMS Outstanding teams and provided the following recognition:

- a letter of congratulations from the current president of INFORMS to each team member and to the faculty advisor;
- a check in the amount of $300 to each team member;
- a bronze plaque for display at the team's institution, commemorating team members' achievement;
- individual certificates for team members and faculty advisor as a personal commemoration of this achievement; and
- a one-year student membership in INFORMS for each team member, which includes their choice of a professional journal plus the *OR/MS Today* periodical and the INFORMS newsletter.

The Society for Industrial and Applied Mathematics (SIAM) designated one Outstanding team from each problem as a SIAM Winner. The teams were from Huazhong University of Science and Technology (Sweet Spot Problem) and Rensselaer Polytechnic Institute (Criminology Problem). Each of the team members was awarded a $300 cash prize, and the teams received partial expenses to present their results in a special Minisymposium at the SIAM Annual Meeting in Pittsburgh, PA in July. Their schools were given a framed hand-lettered certificate in gold leaf.

The Mathematical Association of America (MAA) designated one Outstanding North American team from each problem as an MAA Winner. The teams were from the U.S. Military Academy (Sweet Spot Problem) and the University of Colorado—Boulder (Criminology Problem). With partial travel support from the MAA, the teams presented their solution at a special session of the MAA Mathfest in Pittsburgh, PA in August. Each team member was presented a certificate by an official of the MAA Committee on Undergraduate Student Activities and Chapters.

Ben Fusaro Award

One Meritorious or Outstanding paper was selected for each problem for the Ben Fusaro Award, named for the Founding Director of the MCM and awarded for the seventh time this year. It recognizes an especially creative approach; details concerning the award, its judging, and Ben Fusaro are in

Vol. 25 (3) (2004): 195–196. The Ben Fusaro Award winners were Princeton University (Sweet Spot Problem) and Duke University (Criminology Problem). A commentary on the latter appears in this issue.

Judging

Director
Frank R. Giordano, Naval Postgraduate School, Monterey, CA

Associate Director
William P. Fox, Dept. of Defense Analysis, Naval Postgraduate School, Monterey, CA

Sweet Spot Problem

Head Judge
Marvin S. Keener, Executive Vice-President, Oklahoma State University, Stillwater, OK

Associate Judges
William C. Bauldry, Chair, Dept. of Mathematical Sciences, Appalachian State University, Boone, NC (Head Triage Judge)
Patrick J. Driscoll, Dept. of Systems Engineering, U.S. Military Academy, West Point, NY (INFORMS Judge)
J. Douglas Faires, Youngstown State University, Youngstown, OH
Ben Fusaro, Dept. of Mathematics, Florida State University, Tallahassee, FL (SIAM Judge)
Michael Jaye, Dept. of Mathematical Sciences, Naval Postgraduate School, Monterey, CA
John L. Scharf, Mathematics Dept., Carroll College, Helena, MT (MAA Judge)
Michael Tortorella, Dept. of Industrial and Systems Engineering, Rutgers University, Piscataway, NJ (Problem Author)
Richard Douglas West, Francis Marion University, Florence, SC

Criminology Problem

Head Judge
Maynard Thompson, Mathematics Dept., University of Indiana, Bloomington, IN

Associate Judges
Peter Anspach, National Security Agency, Ft. Meade, MD (Head Triage Judge)
Kelly Black, Mathematics Dept., Union College, Schenectady, NY

Jim Case (SIAM Judge)
William P. Fox, Dept. of Defense Analysis, Naval Postgraduate School, Monterey, CA
Frank R. Giordano, Naval Postgraduate School, Monterey, CA
Veena Mendiratta, Lucent Technologies, Naperville, IL
David H. Olwell, Naval Postgraduate School, Monterey, CA
Michael O'Leary, Towson State University, Towson, MD (Problem Author)
Kathleen M. Shannon, Dept. of Mathematics and Computer Science, Salisbury University, Salisbury, MD (MAA Judge)
Dan Solow, Case Western Reserve University, Cleveland, OH (INFORMS Judge)
Marie Vanisko, Dept. of Mathematics, Carroll College, Helena, MT (Ben Fusaro Award Judge)

Regional Judging Session at U.S. Military Academy

Head Judges
Patrick J. Driscoll, Dept. of Systems Engineering, United States Military Academy (USMA), West Point, NY

Associate Judges
Tim Elkins, Dept. of Systems Engineering, USMA
Darrall Henderson, Sphere Consulting, LLC
Steve Horton, Dept. of Mathematical Sciences, USMA
Tom Meyer, Dept. of Mathematical Sciences, USMA
Scott Nestler, Dept. of Mathematical Sciences, USMA

Regional Judging Session at Naval Postgraduate School

Head Judges
William P. Fox, Dept. of Defense Analysis, Naval Postgraduate School, Monterey, CA
Frank R. Giordano, Naval Postgraduate School, Monterey, CA

Associate Judges
Matt Boensel, Robert Burks, Peter Gustaitis, Michael Jaye, and Greg Mislick
—all from the Naval Postgraduate School, Monterey, CA

Triage Session for Sweet Spot Problem

Head Triage Judge
William C. Bauldry, Chair, Dept. of Mathematical Sciences, Appalachian State University, Boone, NC

Associate Judges
Jeffry Hirst, Greg Rhoads, and Kevin Shirley
—all from Dept. of Mathematical Sciences, Appalachian State University, Boone, NC

Triage Session for Criminology Problem

Head Triage Judge
Peter Anspach, National Security Agency (NSA), Ft. Meade, MD

Associate Judges
Jim Case
Other judges from inside and outside NSA, who wish not to be named.

Sources of the Problems

The Sweet Spot Problem was contributed by Michael Tortorella (Rutgers University), and the Criminology Problem by Michael O'Leary (Towson University) and Kelly Black (Clarkson University).

Acknowledgments

Major funding for the MCM is provided by the National Security Agency (NSA) and by COMAP. Additional support is provided by the Institute for Operations Research and the Management Sciences (INFORMS), the Society for Industrial and Applied Mathematics (SIAM), and the Mathematical Association of America (MAA). We are indebted to these organizations for providing judges and prizes.

We also thank for their involvement and support the MCM judges and MCM Board members for their valuable and unflagging efforts, as well as

- **Two Sigma Investments.** (This group of experienced, analytical, and technical financial professionals based in New York builds and operates sophisticated quantitative trading strategies for domestic and international markets. The firm is successfully managing several billion dollars using highly-automated trading technologies. For more information about Two Sigma, please visit `http://www.twosigma.com`.)
- **Jane Street Capital, LLC.** (This proprietary trading firm operates around the clock and around the globe. "We bring a deep understanding of markets, a scientific approach, and innovative technology to bear on the problem of trading profitably in the world's highly competitive financial markets, focusing primarily on equities and equity derivatives. Founded in 2000, Jane Street employes over 200 people in offices in new York, London, and Tokyo. Our entrepreneurial culture is driven by our talented team of traders and programmers." For more information about Jane Street Capital, please visit `http://www.janestreet.com`.)

Cautions

To the reader of research journals:

Usually a published paper has been presented to an audience, shown to colleagues, rewritten, checked by referees, revised, and edited by a journal editor. Each paper here is the result of undergraduates working on a problem over a weekend. Editing (and usually substantial cutting) has taken place; minor errors have been corrected, wording altered for clarity or economy, and style adjusted to that of *The UMAP Journal*. The student authors have proofed the results. Please peruse their efforts in that context.

To the potential MCM Advisor:

It might be overpowering to encounter such output from a weekend of work by a small team of undergraduates, but these solution papers are highly atypical. A team that prepares and participates will have an enriching learning experience, independent of what any other team does.

COMAP's Mathematical Contest in Modeling and Interdisciplinary Contest in Modeling are the only international modeling contests in which students work in teams. Centering its educational philosophy on mathematical modeling, COMAP uses mathematical tools to explore real-world problems. It serves the educational community as well as the world of work by preparing students to become better-informed and better-prepared citizens.

Editor's Note

The complete roster of participating teams and results has become too long to reproduce in the printed copy of the *Journal*. It can now be found at the COMAP Website, in separate files for each problem:

`http://www.comap.com/undergraduate/contests/mcm/contests/2010/results/2010_MCM_Problem_A.pdf`
`http://www.comap.com/undergraduate/contests/mcm/contests/2010/results/2010_MCM_Problem_B.pdf`

The Sweet Spot: A Wave Model of Baseball Bats

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Peter Diao
Yang Mou

Princeton University
Princeton, NJ

Advisor: Robert Calderbank

Abstract

We determine the sweet spot on a baseball bat. We capture the essential physics of the ball–bat impact by taking the ball to be a lossy spring and the bat to be an Euler-Bernoulli beam. To impart some intuition about the model, we begin by presenting a rigid-body model. Next, we use our full model to reconcile various correct and incorrect claims about the sweet spot found in the literature. Finally, we discuss the sweet spot and the performances of corked and aluminum bats, with a particular emphasis on hoop modes.

Introduction

Although a hitter might expect a model of the bat–baseball collision to yield insight into how the bat breaks, how the bat imparts spin on the ball, how best to swing the bat, and so on, we model only the sweet spot.

There are at least two notions of where the sweet spot should be—an impact location on the bat that either

- minimizes the discomfort to the hands, or
- maximizes the outgoing velocity of the ball.

We focus exclusively on the second definition.

The velocity of the ball leaving the bat is determined by

- the initial velocity and rotation of the ball,
- the initial velocity and rotation of the bat,

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- the relative position and orientation of the bat and ball, and
- the force over time that the hitter's hands applies on the handle.

We assume that the ball is not rotating and that its velocity at impact is perpendicular to the length of the bat. We assume that everything occurs in a single plane, and we will argue that the hands' interaction is negligible. In the frame of reference of the center of mass of the bat, the initial conditions are completely specified by

- the angular velocity of the bat,
- the velocity of the ball, and
- the position of impact along the bat.

The location of the sweet spot depends not on just the bat alone but also on the pitch and on the swing.

The simplest model for the physics involved has the sweet spot at the *center of percussion* [Brody 1986], the impact location that minimizes discomfort to the hand. The model assumes the ball to be a rigid body for which there are *conjugate points*: An impact at one will exactly balance the angular recoil and linear recoil at the other. By gripping at one and impacting at the other (the center of percussion), the hands experience minimal shock and the ball exits with high velocity. The center of percussion depends heavily on the moment of inertia and the location of the hands. We cannot accept this model because it both erroneously equates the two definitions of sweet spot and furthermore assumes incorrectly that the bat is a rigid body.

Another model predicts the sweet spot to be between nodes of the two lowest natural frequencies of the bat [Nathan 2000]. Given a free bat allowed to oscillate, its oscillations can be decomposed into fundamental modes of various frequencies. Different geometries and materials have different natural frequencies of oscillation. The resulting wave shapes suggest how to excite those modes (e.g., plucking a string at the node of a vibrational mode will not excite that mode). It is ambiguous which definition of sweet spot this model uses. Using the first definition, it would focus on the uncomfortable excitations of vibrational modes: Choosing the impact location to be near nodes of important frequencies, a minimum of uncomfortable vibrations will result. Using the second definition, the worry is that energy sent into vibrations of the bat will be lost. This model assumes that the most important energies to model are those lost to vibration.

This model raises many questions. Which frequencies get excited and why? The Fourier transform of an impulse in general contains infinitely many modes. Furthermore, wood is a viscoelastic material that quickly dissipates its energies. Is the notion of an oscillating bat even relevant to modeling a bat? How valid is the condition that the bat is free? Ought the system be coupled with hands on the handle, or the arm's bone structure, or possibly even the ball? What types of oscillations are relevant? A cylindrical structure can support numerous different types of modes beyond the transverse modes usually assumed by this model [Graff 1975].

Following the center-of-percussion line of reasoning, how do we model the recoil of the bat? Following the vibrational-nodes line of reasoning, how do we model the vibrations of the bat? In the general theory of impact mechanics [Goldsmith 1960], these two effects are the main ones (assuming that the bat does not break or deform permanently). Brody [1986] ignores vibrations, Cross [1999] ignores bat rotation but studies the propagation of the impulse coupled with the ball, and Nathan [2000] emphasizes vibrational modes. Our approach reconciles the tension among these approaches while emphasizing the crucial role played by the *time-scale* of the collision.

Our main goal is to understand the sweet spot. A secondary goal is to understand the differences between the sweet spots of different bat types. Although marketers of bats often emphasize the sweet spot, there are other relevant factors: ease of swing, tendency of the bat to break, psychological effects, and so on. We will argue that it doesn't matter to the collision whether the batter's hands are gripping the handle firmly or if the batter follows through on the swing; these circumstances have no bearing on the technique required to swing the bat or how the bat's properties affect it.

Our paper is organized as follows. First, we present the Brody rigid-body model, illuminating the recoil effects of impact. Next we present a full computational model based on wave propagation in an Euler-Bernoulli beam coupled with the ball modeled as a lossy spring. We compare this model with others and explore the local nature of impact, the interaction of recoil and vibrations, and robustness to parameter changes. We adjust the parameters of the model to comment on the sweet spots of corked bats and aluminum bats. Finally, we investigate the effect of hoop frequencies on aluminum bats.

A Simple Example

We begin by considering only the rigid recoil effects of the bat–ball collision, much as in Brody [1986]. For simplicity, we assume that the bat is perfectly rigid. Because the collision happens on such a short time-scale (around 1 ms), we treat the bat as a free body. That is to say, we are not concerned with the batter's hands exerting force on the bat that may be transferred to the ball.

The bat has mass M and moment of inertia I about its center of mass. From the reference frame of the center of mass of the bat just before the collision, the ball has initial velocity v_i in the positive x-direction while the bat has initial angular velocity ω_i. In our setup, v_i and ω_i have opposite signs when the batter is swinging at the ball as in **Figure 1**, in which arrows point in the positive directions for the corresponding parameters.

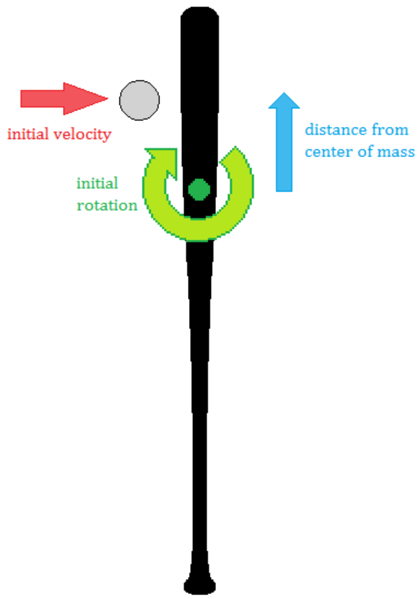


Figure 1. The collision.

The ball collides with the bat at a distance l from the center of mass of the bat. We assume that the collision is head-on and view the event such that all the y-component velocities are zero at the moment of the collision. After the collision, the ball has a final velocity v_f and the bat has a final linear velocity V_f and an angular velocity ω_f at the center of mass.

When the ball hits the bat, the ball briefly compresses and decompresses, converting kinetic energy to potential energy and back. However, some energy is lost in the process, that is, the collision is inelastic. The ratio of the relative speeds of the bat and the ball before and after the collision is known as the *coefficient of restitution*, customarily designated by e: $e = 0$ represents a perfectly inelastic collision, and $e = 1$ means a perfectly elastic one. In this basic model, we make two simplifying assumptions:

- e is constant along the length of the bat, and
- e is constant for all v_i.

Given our pre-collision conditions, we can write:

Conservation of linear momentum:

$$MV_f = m(v_i - v_f)$$

Conservation of angular momentum:

$$I(\omega_f - \omega_i) = ml(v_i - v_f),$$

Definition of the coefficient of restitution:

$$e(v_i - \omega_i l) = -v_f + V_f + w_f l.$$

Solving for v_f gives

$$v_f = \frac{-v_i(e - \frac{m}{M^*}) + \omega_i l(1+e)}{1 + \frac{m}{M^*}},$$

where

$$M^* = \frac{M}{1 + \frac{Ml^2}{I}}$$

is the effective mass of the bat.

For calibration purposes, we use the following data, which are typical of a regulation bat connecting with a fastball in Major League Baseball. The results are plotted in **Figure 2**.

m	0.145 kg	5.1 oz
M	0.83 kg	29 oz
L	0.84 m	33 in
I	$0.039 \text{ kg} \cdot \text{m}^2$	
v_i	67 m/s	150 mph
ω_i	-60 rad/s	
e	0.55	

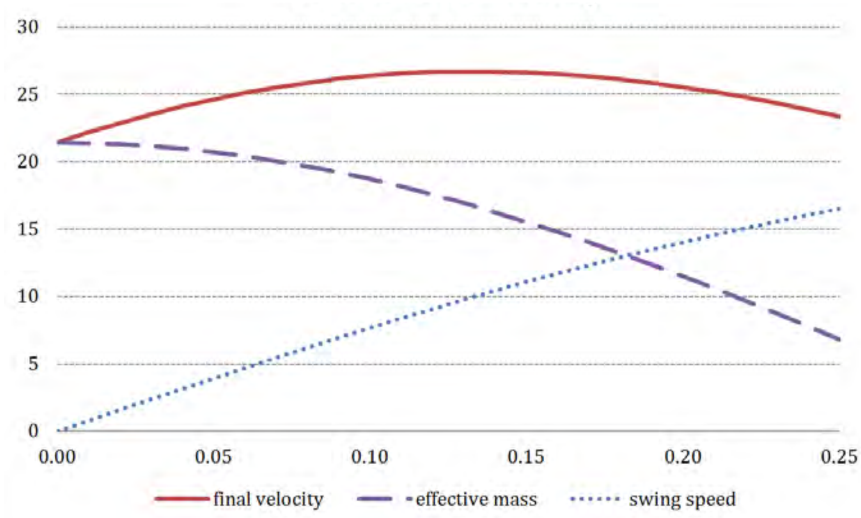


Figure 2. Final velocity v_f (solid arc at top), swing speed $\omega_i l$ (dotted rising line), and effective mass (dashed falling curve) as a function of distance l (in meters) from center of mass.

The maximum exit velocity is 27 m/s, and the sweet spot is 13 cm from the center of mass. Missing the sweet spot by up to 5 cm results in at most 1 m/s difference from the maximum velocity, implying a relatively wide sweet spot.

From this example, we see that the sweet spot is determined by a multitude of factors, including the length, mass, and shape of the baseball bat; the mass of the baseball; and the coefficient of restitution between bat and ball. Furthermore, the sweet spot is not uniquely determined by the bat and ball: It depends also on the incoming baseball speed and the batter's swing speed.

Figure 2 also shows intuitively why the sweet spot is located somewhere between the center of mass and the end of the barrel. As the point of collision moves outward along the bat, the effective mass of the bat goes up, so that a greater fraction of the initial kinetic energy is put into the bat's rotation. At the same time, the rotation in the bat means that the barrel of the bat is moving faster than the center of mass (or handle). These two effects work in opposite directions to give a unique sweet spot that's not at either endpoint.

However, this model tells only part of the story. Indeed, some of our starting assumptions contradict each other:

- We treated the bat as a free body because the collision time was so short. In essence, during the 1 ms of the collision, the ball "sees" only the local geometry of the bat, not the batter's hands on the handle. On the other hand, we assumed that the bat was perfectly rigid—but that means that the ball "sees" the entire bat.
- We also assumed that e is constant along the length of the bat and for different collision velocities. Experimental evidence [Adair 1994] suggests that neither issue can be ignored for an accurate prediction of the location of the sweet spot.

We need a more sophisticated model to address these shortcomings.

Our Model

We draw from Brody's rigid-body model but more so from Cross [1999]. One could describe our work as an adaptation of Cross's work to actual baseball bats. Nathan [2000] attempted such an adaptation but was misled by incorrect intuition about the role of vibrations. We describe his approach and error as a way to explain Cross's work and to motivate our work.

Previous Models

Brody's rigid-body model correctly predicts the existence of a sweet spot not at the end of the bat. That model suffers from the fact that the bat is not a rigid body and experiences vibrations. One way to account for vibrations is to model the bat as a flexible object. Beam theories (of varying degrees of accuracy and complication) can model a flexible bat. Van Zandt [1992] was the first to carry out such an analysis, modeling the beam as a *Timoshenko beam*, a fourth-order theory that takes into account both shear forces and tensile stresses. The equations are complicated and we will not need them. Van Zandt's model assumes the ball to be uncoupled from the beam and simply takes the impulse of the ball as a given. The resulting vibrations of the bat are used to predict the velocity of the beam at the impact point (by summing the Brody velocity with the velocity of the displacement at the impact point due to vibrations) and thence the exit velocity of the ball from the equations of the coefficient of restitution [van Zandt 1992].

Cross [1999] modeled the interaction of the impact of a ball with an aluminum beam, using the less-elaborate Euler-Bernoulli equations to model the propagation of waves. In addition, he provided equations to model the dynamic coupling of the ball to the beam during the impact. After discretizing the beam spatially, he assumed that the ball acts as a lossy spring coupled to the single component of the region of impact.

Cross's work was motivated by both tennis rackets and baseball bats, which differ importantly in the *time-scale* of impact. The baseball bat's collision lasts only about 1 ms, during which the propagation speed of the wave is very important. In this local view of the impact, the importance of the baseball's coupling with the bat is increased.

Cross argues that the actual vibrational modes and node points are largely irrelevant because the interaction is localized on the bat. The boundary conditions matter only if vibrations reflect off the boundaries; an impact not close enough to the barrel end of the bat will not be affected by the boundary there. In particular, a pulse reflected from a free boundary returns with the same sign (deflected away from the ball, decreasing the force on the ball, decreasing the exit velocity), but a pulse reflected from a fixed boundary returns with the opposite sign (deflected towards the ball, pushing it back, increasing the exit velocity). Away from the boundary, we expect the exit velocity to be uniform along a non-rotating bat. Cross's model predicts all of these effects, and he experimentally verified them. In our model, we expect similar phenomena, plus the narrowing of the barrel near the handle to act somewhat like a boundary.

Nathan's model also attempted to combine the best features of Van Zandt and Cross [Nathan 2000]. His theory used the full Timoshenko theory for the beam and the Cross model for the ball. He even acknowledged the local nature of impact. So where do we diverge from him? His error stems from an overemphasis on trying to separate out the ball's interaction with each separate vibrational mode.

The first sign of inconsistency comes when he uses the "orthogonality of the eigenstates" to determine how much a given impulse excites each mode. The eigenstates are *not* orthogonal. Many theories yield symmetric matrices that need to be diagonalized, yielding the eigenstates; but Timoshenko's theory does not, due to the presence of odd-order derivatives in its equations. Nathan's story plays out beautifully if only the eigenstates were actually orthogonal; but we have numerically calculated the eigenstates, and they are not even approximately orthogonal. He uses the orthogonality to draw important conclusions:

- The location of the nodes of the vibrational modes are important.
- High-frequency effects can be completely ignored.

We disagree with both of these.

The correct derivation starts with the following equation of motion, where k is the position of impact, y_i is the displacement and F_i is the external force on the ith segment of the bat, and H_{ij} is an asymmetric matrix:

$$y_k''(t) = H_{kj}y_j(t) + F_k(t).$$

We write the solutions as $y_k(t) = \Phi_{kn}a_n(t)$, where the rows of Φ_{kn} are eigenmodes with eigenvalues $-\omega_n^2$. Explicitly, $H_{jk}\Phi_{kn} = -\omega_n^2\Phi_{jn}$, and Φ_{kn}

indicates the kth component of the nth eigenmode. Then we write the equation of motion:

$$\Phi_{kn} a_n''(t) + \Phi_{kn}\omega_n^2 a_n(t) = F_k = \Phi_{kn}\Phi_{nj}^{-1} F_j,$$
$$a_n''(t) + \omega_n^2 a_n(t) = \Phi_{nk}^{-1} F_k.$$

In the last step, we used the fact that the eigenmodes form a complete basis.

Nathan's paper uses on the right-hand side simply $\Phi_{kn}F_k$ scaled by a normalization constant. At first glance, this seems like a minor technical detail, but the physics here is important. We calculate that the $\Phi_{nk}^{-1}F_k$ terms stay fairly large for even high values of n, corresponding to the high-frequency modes (k is just the position of the impact). This means that there are significant high-frequency components, at least at first. In fact, the high-frequency modes are necessary for the impulse to propagate slowly as a wave packet. In Nathan's model, only the lowest standing modes are excited; so the entire bat starts vibrating as soon as the ball hits. This contradicts his earlier belief in localized collision (which we agree with), that the collision is over so quickly that the ball "sees" only part of the bat. Nathan also claims that the sweet spot is related to the nodes of the lowest mode, which contradicts locality: The location of the lowest-order nodes depends on the geometry of the entire bat, including the boundary conditions at the handle.

While the inconsistencies in the Nathan model may cancel out, we build our model on a more rigorous footing. For simplicity, we use the Euler-Bernoulli equations rather than the full Timoshenko equations. The difference is that the former ignore shear forces. This should be acceptable; Nathan points out that his model is largely insensitive to the shear modulus. We solve the differential equations directly after discretizing in space rather than decomposing into modes. In these ways, we are following the work of Cross [1999].

On the other hand, our model extends Cross's work in several key ways:

- We examine parameters much closer to those relevant to baseball. Cross's models focused on tennis, featuring an aluminum beam of width 0.6 cm being hit with a ball of 42 g at around 1 m/s. For baseball, we have an aluminum or wood bat of radius width 6 cm being hit with a ball of 145 g traveling at 40 m/s (which involves 5,000 times as much impact energy).
- We allow for a varying cross-section, an important feature of a real bat.
- We allow the bat to have some initial angular velocity. This will let us scrutinize the rigid-body model prediction that higher angular velocities lead to the maximum power point moving farther up the barrel.

To reiterate, the main features of our model are

- an emphasis on the ball coupling with the bat,

- finite speed of wave propagation in a short time-scale, and
- adaptation to realistic bats.

These are natural outgrowths of the approaches in the literature.

Mathematics of Our Model

Our equations are a discretized version of the Euler-Bernoulli equations:

$$\rho \frac{\partial^2 y(z,t)}{\partial t^2} = F(z,t) + \frac{\partial^2}{\partial z^2}\left(YI\,\frac{\partial^2 y(z,t)}{\partial z^2}\right),$$

where

ρ is the mass density,

$y(z,t)$ is the displacement,

$F(z,t)$ is the external force (in our case, applied by the ball),

Y is the Young's modulus of the material (a constant), and

I is the second moment of area ($\pi R^4/4$ for a solid disc).

We discretize z in steps of Δ. The only force is from the ball, in the negative direction to the kth segment. Our discretized equation is:

$$\rho A\Delta \frac{d^2 y_i}{dt^2} = -\delta_{ik}F(t) - \frac{Y}{\Delta^3}\Big[I_{i-1}(y_{i-2} - 2y_{i-1} + y_i) \\ -2I_i(y_{i-1} - 2y_i + y_{i+1}) + I_{i+1}(y_i - 2y_{i+1} + y_{i+2})\Big].$$

Our dynamic variables are y_1 through y_N. For a fixed left end, we pretend that $y_{-1} = y_0 = 0$. For a free left end, we pretend that

$$y_1 - y_0 = y_0 - y_{-1} = y_{-1} - y_{-2}.$$

The conditions on the right end are analogous. These are the same conditions that Cross uses.

Finally, we have an additional variable for the ball's position (relative to some zero point) $w(t)$. Initially, $w(t)$ is positive and $w'(t)$ is negative, so the ball is moving from the positive direction towards the negative. Let $u(t) = w(t) - y_k(t)$. This variable represents the compression of the ball, and we replace $F(t)$ with $F\big(u(t), u'(t)\big)$. Initially, $u(t) = 0$ and $u'(t) = -v_{\text{ball}}$. The force between the ball and the bat takes the form of hysteresis curves such as the ones shown in **Figure 3**.

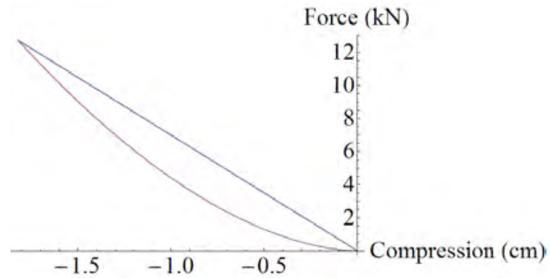


Figure 3. A hysteresis curve used in our modeling, with maximum compression 1.5 cm.

The higher curve is taken when $u'(t) < 0$ (compression) and the lower curve when $u'(t) > 0$ (expansion). When $u(t) > 0$, the force is zero. The equation of motion for the ball is then

$$w''(t) = u''(t) + y_k''(t) = F\big(u(t), u'(t)\big).$$

We have eliminated the variable w.

We have yet to specify the function $F\big(u(t), u'(t)\big)$. As can be seen in videos [Baseball Research Center n.d.], the ball compresses significantly (often more than 1 cm) in a collision. The compression and decompression is lossy. We could model this loss by subtracting a fraction of the ball's energy after the collision; that approach is good enough for many purposes, but we instead follow Nathan and use a nonlinear spring with hysteresis.

Since $W = \int F dx$, the total energy lost is the area between the two curves in **Figure 3**. A problem with creating hysteresis curves is that one does not know the maximum compression (i.e., where to start drawing the bottom curve) until after solving the equations of motion. In practice, we solve the equation in two steps.

The main assumptions of our model derive from the main assumptions of each equation:

- The first is the exact form of the hysteresis curve of the ball. Cross [1999] argues that the exact form of the curve is not very important as long as the duration of impact, magnitude of impulse, maximum compression of the ball, and energy loss are roughly correct.
- Both the Timoshenko and Euler-Bernoulli theories ignore azimuthal and longitudinal waves. This is a fundamental assumption built into all of the approaches in the literature. Assuming that the impact of the ball is transverse and the ball does not rotate, the assumption is justified.

The assumptions of our models are the same as those in the literature, so they are confirmed by the literature's experiments.

Simulation and Analysis

Simulation Results

Our model's two main features are wave propagation in the bat and nonlinear compression/decompression of the ball. The latter is illustrated by the asymmetry of the plot in **Figure 4a**. This plot also reveals the time-scale of the collision: The ball leaves the bat 1.4 ms after impact. During and after collision, shock waves propagate through the bat.

In this example, the bat was struck 60 cm from the handle. What does the collision look like at 10 cm from the handle? **Figure 4b** shows the answer: The other end of the bat does not feel anything until about 0.4 ms and does not feel significant forces until about 1.0 ms. By the time that portion of the bat swings back (almost 2.0 ms), the ball has already left contact with the bat. This illuminates an important point: We are concerned only with forces on the ball that act within the 1.4 ms time-frame of the collision. Any waves taking longer to return to the impact location do not affect exit velocity.

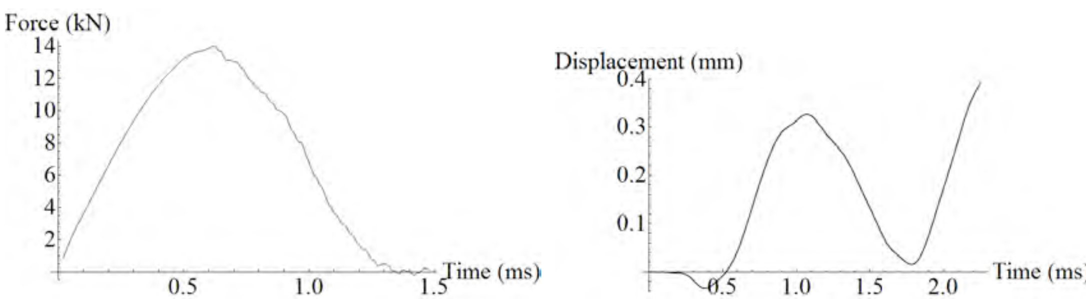


Figure 4.

a. Left: The force between the ball and the bat as a function of time; the impulse lasts 1.4 ms.
b. Right: The waveform of $y_{10}(t)$ when the bat is struck at 60 cm. The impulse reaches this chunk at around 0.4 ms but does not start moving significantly until later.

Having demonstrated the basic features of our model, we now replicate some of Cross's results but with baseball-like parameters. In **Figure 5a**, we show that the effects of fixed- vs. free-boundary conditions are in agreement with Cross's model.

As we expected, fixed boundaries enhance the exit velocity and free boundaries reduce them. From this result, we see the effect of the shape of the bat. The handle does indeed act like a free boundary. The distance between the boundaries is too small to get a flat zone in the exit velocity vs. position curve. If we extend the barrel by 26 cm, a flat zone develops (**Figure 5b**; notice the change in axes). Intuitively, this flat zone exists because the ball "sees" only the local geometry of the bat and the boundaries are too far away to have a substantial effect.

From now on, we use an 84-cm bat free on both ends, where position zero denotes the handle end. In this base case, the sweet spot is at 70 cm. We

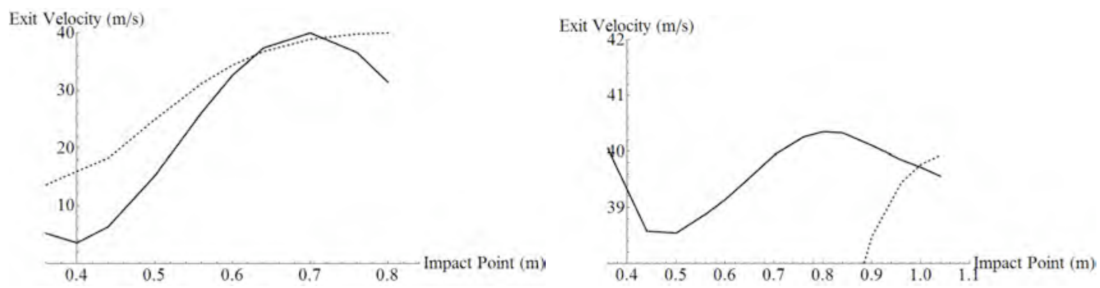


Figure 5.

a. Left: Exit velocity vs. impact position for a free boundary (solid line) and for a fixed boundary (dashed line), with barrel end fixed but handle end free, for an 84-cm bat
b. Right: The same graph for a free 110-cm bat.

investigate the dependence of the exit speed on the initial angular velocity. According to rigid-body models, the sweet spot is exactly at the center of mass if the bat has no angular velocity. In **Figure 6**, we present the results of changing the angular velocity. Our results contrast greatly with the simple example presented earlier. While the angular-rotation effect is still there, the effective mass plays only a negligible role in determining the exit speed. In other words, the bat is not a rigid body because the entire bat does not react instantly. The dominating effect is from the boundaries: the end of the barrel and where the barrel tapers off. These free ends cause a significant drop in exit velocity. Increasing the angular velocity of the bat increases the exit velocity, in part just because the impact velocity is greater (by a factor of ω_i times the distance from the center of mass of the bat).

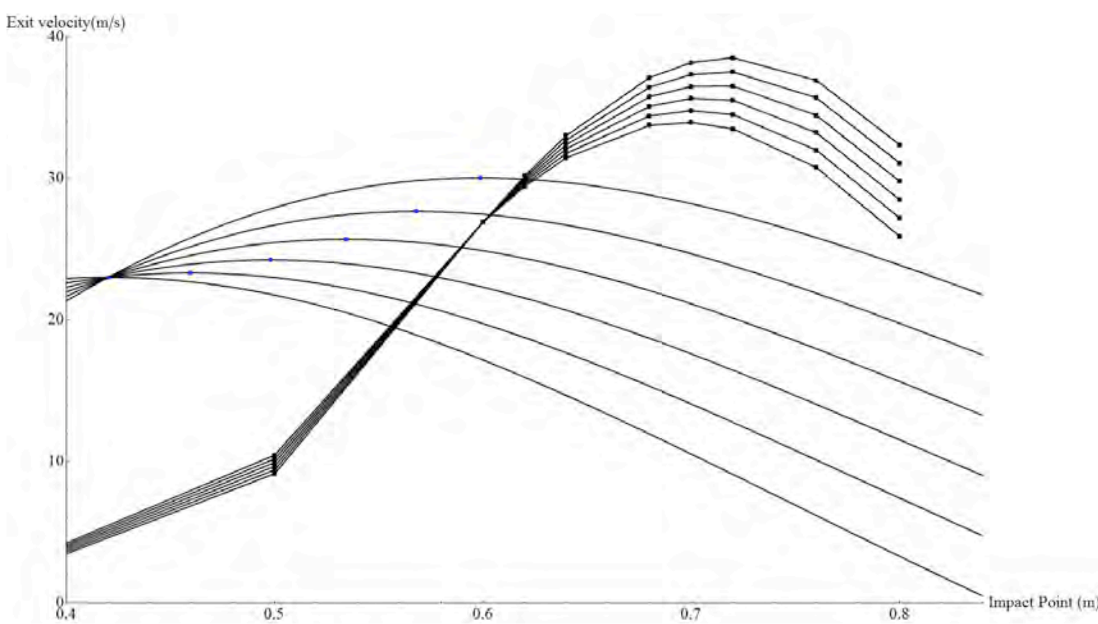


Figure 6. Exit velocity vs. impact position at various initial angular velocities of the bat. Our model predicts the solid curves, while the dashed lines represent the simple model. The dots are at the points where Brody's solution is maximized.

In **Figure 7a**, we show that near the sweet spot (at 0.7 m), increasing angular velocity actually decreases the excess exit velocity (relative to the impact velocity). We should expect this, since at higher impact velocity, more energy is lost to the ball's compression and decompression. To confirm this result, we also recreate the plot in **Figure 7b** but without the hysteresis curve—in which case this effect disappears. This example is one of the few places where the hysteresis curve makes a difference, confirming experimental evidence [Adair 1994; Nathan 2003] that the coefficient of restitution decreases with increasing impact velocity.

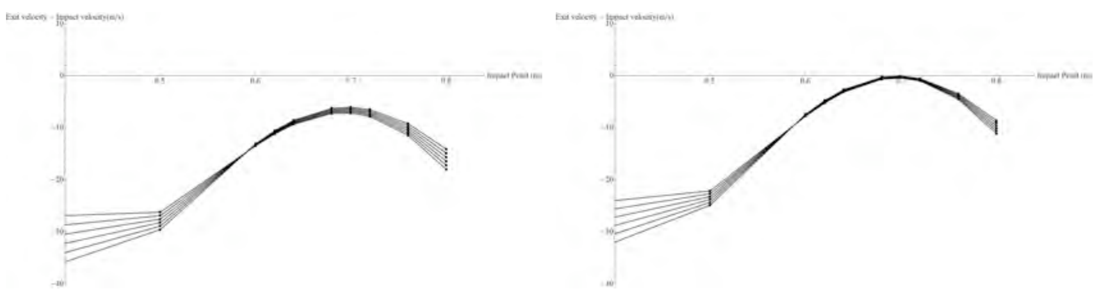


Figure 7.

Exit velocity minus impact velocity vs. impact position, for initial angular velocities of the bat.
a. Left: Near the center of mass, higher angular velocity gives higher excess exit velocity, but towards the sweet spot the lines cross and higher angular velocity gives lower excess exit velocity.
b. Right: The same plot without a hysteresis curve. The effect disappears.

The results for angular velocity contrast with the simple model. As evident from **Figure 8**, the rigid-body model greatly overestimates this effect for large angular velocities.

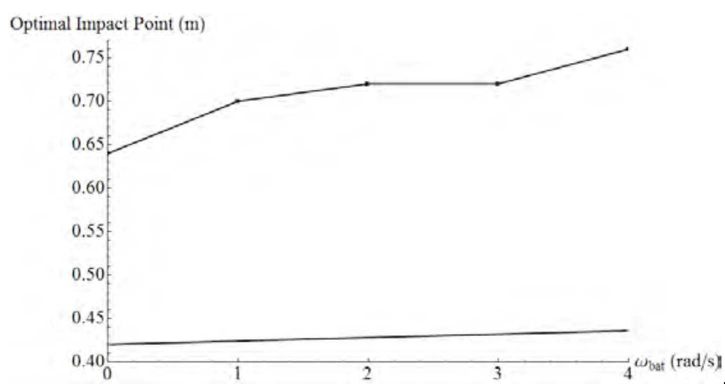


Figure 8. Optimal impact position vs. angular velocity. The straight line is the rigid-body prediction, while the points are our model's prediction.

Parameter Space Study

There are various adjustable parameters in our model. For the bat, we use density $\rho = 649\,\text{kg/m}^3$ and Young's modulus $Y = 1.814 \times 10^{10}\,\text{N/m}^2$.

Figure 9. The profile of our bat.

These values, as well as our bat profile (**Figure 9)**, were used by Nathan as typical values for a wooden bat. While these numbers are in good agreement with other sources, we will see that these numbers are fairly special. As a result of our bat profile, the mass is 0.831 kg and the moment of inertia around the center of mass (at 59.3 cm from the handle of our 84 cm bat) is 0.039 kg·m^2. We let the 145-g ball's initial velocity be 40 m/s, and set up our hysteresis curve so that the compression phase is linear with spring constant 7×10^5 N/m.

- We vary the density of the bat and see that the density value occupies a narrow region that gives peaked exit-velocity curves (see **Figure 10**).

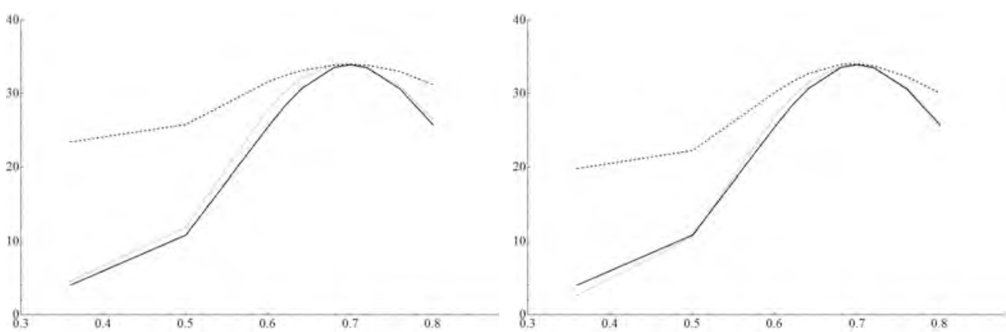


Figure 10.

Exit velocity vs. impact position for various densities. The solid line is the original $\rho = 649\text{kg/m}^3$.
a. Left: Dotted is $\rho = 700$, dashed is $\rho = 1000$. **b.** Right: Dotted is $\rho = 640$, dashed is $\rho = 500$.

- We also vary the Young's modulus and shape of bat to similar effect (see **Figure 11**). The fact that varying any of Nathan's parameters makes the resulting exit velocity vs. location plot less peaked means that baseball bats are specially designed to have the shape shown in **Figure 9** (or else the parameters were picked in a special way).
- Finally, we vary y, the speed of the ball (see **Figure 12**). The exit velocity simply scales with the input velocity, as expected.

Alternatives to Wooden Bats

Having checked the stability of our model for small parameter changes, we now change the parameters drastically, so as to model corked and aluminum bats.

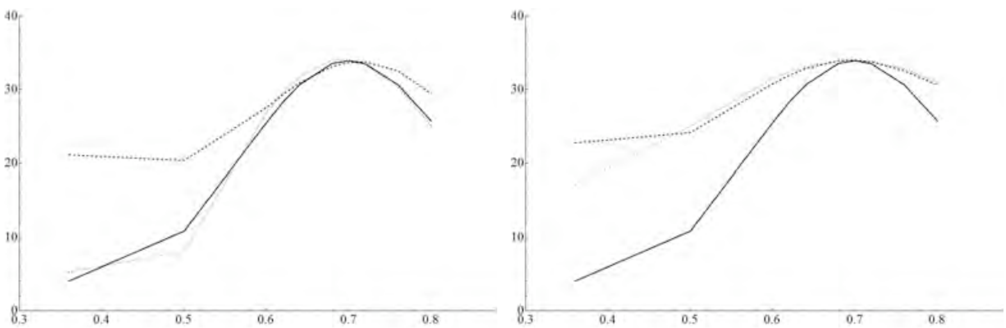


Figure 11.

a. Left: Varying the value of Y. Solid is $Y = 1.1814 \times 10^{10} \mathrm{N/m}^2$; dashed is 1.25 times as much, while dotted is 0.8 times.
b. Right: Varying the shape of the bat. Solid is the original shape; dashed has a thicker handle region, while dotted has a narrower handle region.

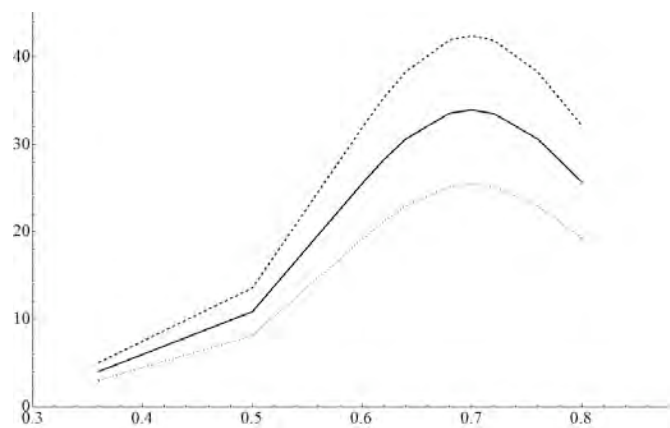


Figure 12. Varying the speed of the ball. Solid is the original 40 m/s, dashed is 50 m/s, while dotted is 30 m/s.

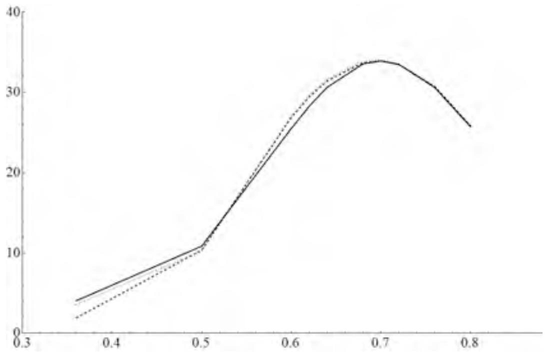
Corked Bat

We model a corked bat as a wood bat with the barrel hollowed out, leaving a shell 1 cm or 1.5 cm thick. The result is shown in **Figure 13a**. The exit velocities are higher, but this difference is too small to be taken seriously. This result agrees with the literature: The only advantages of a corked bat are the changes in mass and in moment of inertia.

Aluminum Bat

We model an aluminum bat as a 0.3 cm-thick shell with a density of 2700 kg/m^3 and Young's modulus 6.9×10^{10} N/m^2. The aluminum bat performs much better than the wood bat (**Figure 13b**). It has the same sweet spot (70 cm) and similar sweet-spot performance, but the exit velocity falls off more gradually away from the sweet spot.

To gain more insight, we animated the displacement of the bat vs. time; we present two frames of the animation in **Figure 14**. The aluminum bat is



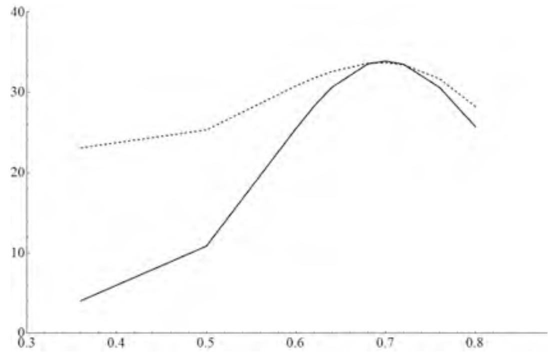
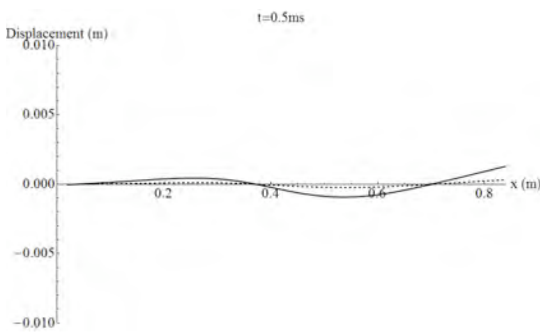


Figure 13. Exit velocity vs. distance of point of collision on the bat from the handle end.

a. Corked bat. **b.** Aluminum bat.

displaced less (absorbing less energy). More importantly, in the right-hand diagram of **Figure 14**, the curve for the wood bat is still moving down and left, while the aluminum bat's curve is moving left and pushing the ball back up. The pulse in the aluminum bat travels faster and returns in time to give energy back to the ball. By the time the pulse for the wood bat returns to the impact location, the ball has already left the bat.



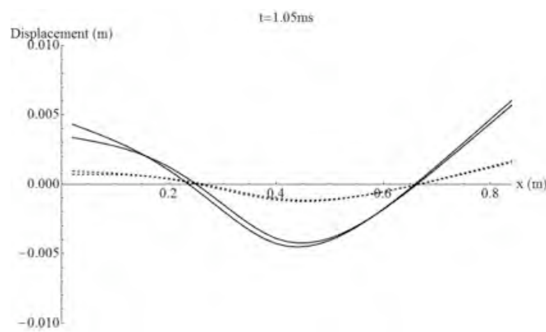


Figure 14. Plots of the displacement of an aluminum bat (dashed) and wood bat (solid) being hit by a ball 60 cm from the handle end. The diagram on the right shows two frames superimposed ($t = 1.05$ ms and $t = 1.10$ ms) so as to show the motion. The rigid translation and rotation has been removed from the diagrams.

In the literature, the performance of aluminum bats is often attributed to a "trampoline effect," in which the bat compresses on impact and then springs back before the end of the collision [Russell 2003]. This effect would improve aluminum-bat performance further. The trampoline effect involves exciting so-called "hoop modes," modes with an azimuthal dependence, which our model cannot simulate directly. For an aluminum bat, one could conceivably use wave equations for a cylindrical sheet (adjusting for the changing radius) and then solve the resulting partial differential equations in three variables. Analysis of such a complex system of equations is beyond the scope of this paper.

Instead, we artificially insert a hoop mode by hanging a mass from a spring at the spot of the bat where the ball hits. We expect the important modes to be the ones with periods near the collision time (1.4 ms, corresponding to 714 Hz). We find that this mode does affect the sweet spot, although the exact change does not seem to follow a simple relationship with the frequency. Our results, as shown in **Figure 15**, show that hoop modes around 700 Hz do enhance the exit velocity. They not only make the sweet spot wider but also shift it slightly toward the barrel end of the bat.

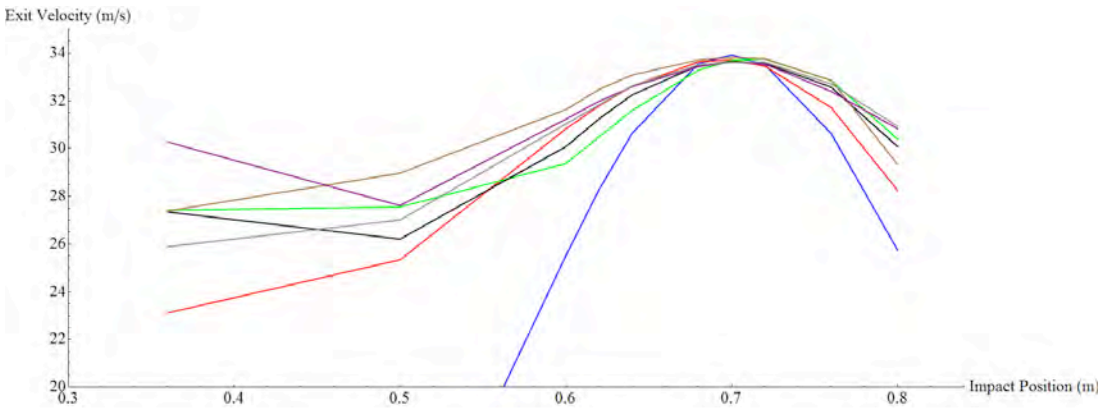


Figure 15. Exit velocity vs. impact position at different hoop frequencies. The lines from bottom to top at the left edge (color) are: (blue, starts off the chart) wood bat, (next higher, red) no hoop mode, (gray) 2000 Hz, (black) 500 Hz, (green) 300 Hz, (brown) 800 Hz, and (purple) 1250 Hz.

Conclusion

We model a ball–bat collision by using Euler-Bernoulli equations for the bat and hysteresis curves for the baseball. By doing so, we reconcile the literature by emphasizing the role of the time-scale of the collision and how the ball “sees” only a local region of the bat because of the finite speed of wave propagation. As a result, the sweet spot is farther out in our model than the rigid-body recoil model predicts.

We vary the input parameters and show that the effects are in line with intuition and key results in previous experimental work.

Finally, we show that aluminum bats have wider sweet spots than wooden bats.

We offer several suggestions for improvements and extensions:

- The ball is assumed to be non-rotating with head-on impact; rotating balls and off-center collisions excite torsional modes in the bat that we ignore and make the problem nonplanar.
- We neglect shear forces in the bat.
- Our analysis of hoop modes is rather cursory.

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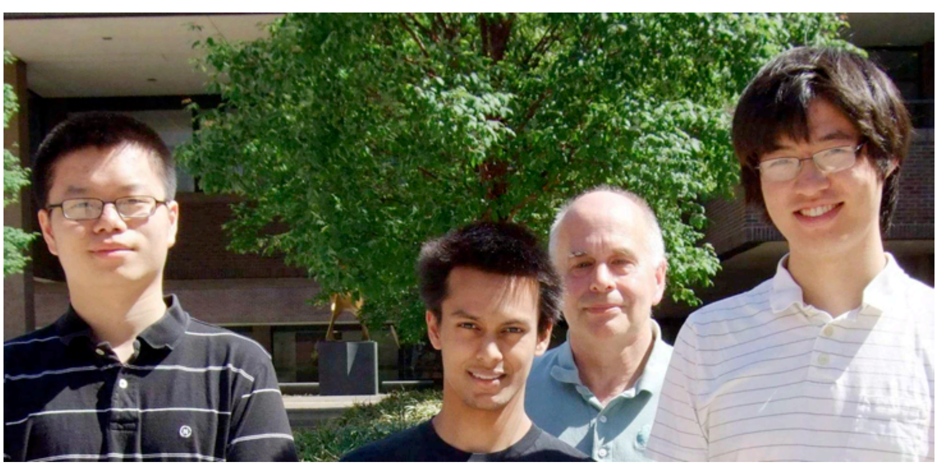

Team members Yang Mou, Rajib Quabili, and Peter Diao,
with advisor Robert Calderbank.

Judges' Commentary: The Outstanding Sweet Spot Papers

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Introduction

Apparently the march of technology in Major League Baseball (MLB) is more of a crawl. The basic tools of baseball have not changed or been substantially modified for a long time. It would seem that the business goals of MLB are being adequately met with tools that are decades—if not centuries—old.

In particular, the baseball bat is pretty much the same implement that it was when Abner Doubleday walked the earth. It is not often that a tool persists basically unchanged without some improvement being brought to bear. Some began to wonder what the properties of such a remarkable tool might possess.

A Few Words About the Problem

Like most problems in the Mathematical Contest in Modeling (MCM)®, this problem was deliberately designed to be open-ended. In particular, the key phrase "sweet spot" in the statement of the problem was not defined. This was fortunate because teams brought many definitions forward and this produced a richer experience not only for the teams but also for the judges. Some of the useful interpretations of "sweet spot" included:

- the spot where a batted ball would travel farthest,
- the spot where the sensation of vibration in the batter's hands is minimized,
- the center of percussion,
- the location that produces the greatest batted ball speed, and
- the location where maximum energy is transferred to the ball.

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Several other definitions or interpretations are easily found through even a cursory literature search. Teams that did not discover this were generally eliminated in triage.

This observation compels us to consider also the relationships among problem statements, the Internet, and competing teams. It is extremely difficult, if not impossible, to imagine a problem that would be suitable for the MCM and for which there has been no prior art. Truly original problems, ones at which the MCM teams are the first to have a go, must be rare. Sometimes, we see a situation in which the proposed problem—while in its most general form is familiar—may be novel as an application to a particular situation which has received scant prior attention.

An example of this kind of problem is the Tollbooth Problem of the 2005 MCM.[1] While it would have been nearly impossible to find prior art applied specifically to the situation presented (namely, to barrier tolls on the Garden State Parkway in New Jersey), standard methods of queueing theory and dimensional analysis could be brought to bear.

In general, the Internet provides teams with a powerful resource to help find what, if anything, has been done on a topic before. This is a recent development that was not in play even a decade ago. Teams, coaches, and judges need to find a fair way of coping with this changed situation:

- On one end of the spectrum, it is not reasonable for a team to simply copy what they find on the Internet and submit this as their solution. No one learns any modeling from this.
- At the other end of the spectrum, teams may develop entirely new models that do not resemble anything found online. While this may be desirable, it is probably unusual.

Most submitted papers will fall somewhere between these extremes. The challenge for everyone is to make the MCM a learning experience for the teams and an enriching one for the judges in the face of this new technology. A general discussion of this issue is beyond the scope of this article; so suffice it to say that for this particular problem, the presence on the Internet of substantial material on solving the problem was appropriately treated by the winning teams. Teams who simply copied material from sources without adding any value of their own were not considered winning teams.

Interpretation Is Important

As always, interpretation is a key to success in modeling problems. Teams must recognize that in addition to their usual semantic or prose usage, key words in the problem statement must also be given a mathematical meaning in

[1] EDITOR'S NOTE: Dr. Tortorella was the author of both the Tollbooth Problem, as well as the Sweet Spot Problem that he comments on here as a contest judge.

the context of a model. Successful papers began by providing definitions of at least two possible interpretations of "sweet spot." Once that is accomplished, it begins to be possible to talk in quantitative terms about how to determine such a sweet spot (or spots).

Modeling

Whatever model is chosen, it is necessary to produce an expression relating the sweet spot (SS) to physical parameters of the batter–bat–ball system. For instance, the Zhejiang University team investigated the SS as the location on the bat where the batted ball speed is greatest upon leaving the bat. Then the team developed a relationship between this definition and

- impact location,
- ball mass,
- ball initial speed,
- the moment of inertia of bat,
- the swinging bat speed, and
- the coefficient of restitution (COR) of the ball.

This team made good use of clear illustrations to help the reader grasp the work involved.

The Huazhong University of Science and Technology team made use of a weighted average of two SS criteria and found, not surprisingly, that the resulting location of SS is a compromise between batter comfort and batted-ball departure speed. This is a nice example of how a team amplified results available on the Internet to generate new insights. The Princeton University team defined SS as the location on the bat that imparts maximum outgoing velocity to the batted ball.

An interesting comment on the choice of SS criteria is that most teams did not explicitly connect their choice to the strategy of the game. That is, the criteria for the SS should be related in some explicit way to the result that the batter is trying to achieve, namely, to score runs. From this point of view, criteria such as "maximum batter comfort" are perhaps secondary desirable features but are probably not the most important ones in the short term. It may be more suitable to choose criteria such as maximum batted-ball departure velocity, maximum location controllability, or something that is directly related to producing runs. Most teams accepted their criteria as being implicitly connected with results of the game, but few if any discussed this point—clearly a key point!—at all.

The Outstanding teams were able to develop clear equations, based on the dynamics of the batter–bat–ball system, for the location of the SS. Most teams followed this approach, but the Outstanding papers were especially clearly reasoned and made good use of illustrations to help clarify points for the reader.

The contest weekend is a busy weekend; but those papers that took the time to pay attention to helping the reader with good organization, clear writing, and attractive presentation in their report received more favorable reviews. Of course, these desirable features cannot make up for a weak solution; but lack of such features can easily cover up a good solution and make it harder to discern. This is not a trivial concern, because triage reads are very fast and it would be distressing if a triage judge were to pass over a worthwhile paper because its presentation made it hard for the judge to discern its solution quality.

Some teams, including the Huazhong University of Science and Technology team, expressed their results very precisely (for example, the SS is 20.15 cm from the end of the bat). This may be more than is required, partly because of the limited precision of real-world measuring instruments, but also because teams should be aware that stating a result in such a fashion compels a sensitivity analysis for this quantity. The Outstanding teams determined that even though a location for the SS could be calculated, the point of impact of the ball with the bat could vary somewhat from the SS without too much change in the value of the objective function (e.g., the batted-ball departure velocity). The Huazhong University of Science and Technology team, as well as several other teams, defined a "Sweet Zone" to capture the notions that

- different SS criteria lead to different locations on the bat, and
- most of the objective functions employed are not very sensitive to the specific location of the bat-ball impact.

Conclusion

Studying these Outstanding papers offers good lessons in preparing entries for the MCM. Here are a few:

- **Make your paper easy to read.** That means at the very least:
  - number the pages and the equations,
  - check your spelling and grammar,
  - provide a table of contents, and
  - double-space the text (or at least use a font size large enough for easy readability).

  All four Outstanding papers did a good job with this.
- **Good organization will not make up for poor results, but poor organization can easily overwhelm good results**—and make them hard to dig out. It can help to organize the paper into sections corresponding to the requirements in the problem statement and into subsections corresponding to parts of the problem. The teams from the U.S. Military Academy and Princeton University did an especially good job with this.

- **Define all terms that a reader might find ambiguous.** In particular, any term used in the model that also has a common prose meaning should be carefully considered.
- **Complete all the requirements of the problem.** If the problem statement says that certain broad topics are required, begin by making an outline based on those requirements. Typical examples are statement and discussion of assumptions, strengths and weaknesses of model, and sensitivity analysis.
- **Read the problem statement carefully, looking for key words implying actions:** "design," "analyze," "compare," and other imperative verbs. These are keys to the work that you need to do and to the sections that your paper ought to contain.
- **When you do "strengths and weaknesses" or sensitivity analysis, go back to your list of assumptions and make sure that each one is addressed.** This is your own built-in checklist aiding completeness; use it.
- **Your summary should state the results that you obtained, not just what you did.** Keeping the reader in suspense is a good technique in a novel, but it simply frustrates judges who typically read dozens of papers in a weekend. The Princeton University paper has an excellent summary: crisp, concise, and thorough.
- **Use high-quality references.** Papers in peer-reviewed journals, books, and government Websites are preferred to individuals' Websites. Note also that it is not sufficient to copy, summarize, or otherwise recast existing literature; judges want to see *your* ideas. It's okay to build on the literature, but there must be an obvious contribution from the team.
- **Verify as much as you can.** For example, the physical characteristics of baseballs and baseball bats are readily verifiable. Make whatever sanity checks are possible: Is your answer for the departing ball's speed faster than the speed of light? If so, should it be?
- **Finally, an Outstanding paper usually does more than is asked.** For example, the team from the U.S. Military Academy (and many other teams that lacked other qualities needed to be Outstanding) studied two different models for the problem and compared the results from each approach; the reasonably good agreement that they obtained showed that either
  - they were on the right track; or
  - they were victims of very bad luck, in that both of the methods gave nearly the same bad answers!

About the Author

Mike Tortorella is Visiting Professor at RUTCOR, the Rutgers Center for Operations Research at Rutgers, the State University of New Jersey, and Managing Director of Assured Networks, LLC. He retired from Bell Laboratories as a Distinguished Member of the Technical Staff after 26 years of service. He holds the Ph.D. degree in mathematics from Purdue University. His current interests include stochastic flow networks, network resiliency and critical infrastructure protection, and stochastic processes in reliability and performance modeling. Mike has been a judge at the MCM since 1993 and particularly enjoys the MCM problems that have a practical flavor of mathematical analysis of social policy. Mike enjoys amateur radio, playing the piano, and cycling.

Centroids, Clusters, and Crime: Anchoring the Geographic Profiles of Serial Criminals

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Abstract

A particularly challenging problem in crime prediction is modeling the behavior of a serial killer. Since finding associations between the victims is difficult, we predict where the criminal will strike next, instead of whom. Such predicting of a criminal's spatial patterns is called *geographic profiling*.

Research shows that most violent serial criminals tend to commit crimes in a radial band around a central point: home, workplace, or other area of significance to the criminal's activities (for example, a part of town where prostitutes abound). These "anchor points" provide the basis for our model.

We assume that the entire domain of analysis is a potential crime spot, movement of the criminal is uninhibited, and the area in question is large enough to contain all possible strike points. We consider the domain a metric space on which predictive algorithms create spatial likelihoods. Additionally, we assume that the offender is a "violent" serial criminal, since research suggests that serial burglars and arsonists are less likely to follow spatial patterns.

There are substantial differences between one anchor point and several. We treat the single-anchor-point case first, taking the spatial coordinates of the criminal's last strikes and the sequence of the crimes as inputs. Estimating the point to be the centroid of the previous crimes, we generate a "likelihood crater," where height corresponds to the likelihood of a future crime at that location. For the multiple-anchor-point case, we use a cluster-finding and sorting method: We identify groupings in the data and build a likelihood crater around the centroid of each. Each cluster is given weight according to recency and number of points. We test single point vs. multiple points by

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using the previous crimes to predict the most recent one and comparing with its actual location.

We extract seven datasets from published research. We use four of the datasets in developing our model and examining its response to changes in sequence, geographic concentration, and total number of points. Then we evaluate our models by running blind on the remaining three datasets.

The results show a clear superiority for multiple anchor points.

Introduction

The literature on geographic patterns in serial crimes shows a strong patterning around an anchor point—a location of daily familiarity for the criminal. We build prediction schemes based on this underlying theory and produce a surface of likelihood values and a robust metric.

The first scheme finds a single anchor point using a center-of-mass method; the second scheme assumes two to four anchor points and uses a cluster-finding algorithm to sort and group points. Both schemes use a statistical technique that we call *cratering* to predict future crime locations.

Background

The arrest in 1981 (and subsequent conviction) of Peter Sutcliffe as the "Yorkshire Ripper" marked a victory for Stuart Kind, a forensic biologist whose application of mathematical principles had successfully predicted where the Yorkshire Ripper lived.

Today, information-intensive models can be constructed using heat-map techniques to identify the hot spots for a specific type of crime, or to derive associations between the rate of criminal activity and attributes of a location (such as lighting, urbanization, etc.) [Boba 2005].

"Geographically profiling" the crimes of a single criminal has focused on locating the criminal's anchor points—locations (such as a home, workplace, or a relative's house) at which he spends substantial amounts of time and to which he returns regularly between crimes.

Canter and Larkin [1993] proposed that a serial criminal's home (or other anchor point) tends to be contained within a circle whose diameter is the line segment between the two farthest-apart crime locations; and this is true in the vast majority of cases [Kocsis and Irwin 1997]. Canter et al. [2000] found that for serial murders, generalizations of such techniques on average reduce the area to be searched by nearly a factor of 10.

By contrast, forecasting *where* a criminal will strike next has not been explored deeply [Rossmo 1999]. Paulsen and Robinson [2009] observe that for many U.S. police departments there are substantial practical, ethical, and legal issues involved in collecting the data for a detailed mapping of criminal tendencies, with the result that only 16% of them employ a computerized mapping technique.

Our treatment of the problem will employ anchor-point-finding algorithm. We generate likelihood surfaces that act as a prioritization scheme for regions to monitor, patrol, or search.

Assumptions

Domain is Approximately Urban

We use the word "urban" to denote features of an urbanized area that simplify our treatment: The entire domain is a potential crime spot, the movement of the criminal is completely unconstrained, and the area is large enough to contain all possible strike points. It is important to note, however, that even for serial crime committed in suburbs, villages, or spread between towns, the urbanization condition holds on the subset of the map in which crimes are regularly committed. To see this, consider the three urbanization conditions separately:

- **Entire domain is a potential crime spot.** Every neighborhood contains a possible crime location. Such an assumption is made by nearly all geographic profiling techniques [Canter et al. 2000; Rossmo 1999]

  It is obvious that every domain will violate these conditions to some extent: All but the most inventive serial killers, for example, will not commit a crime in the middle of a lake, or in the uninhabited farmland between small towns. Nevertheless, this observation simply requires that the output of the model be interpreted intelligently. In other words, while we assume for simplicity that the entire map is a potential target, police officers interpreting the results can easily ignore any predictions we make which fall into an obvious "dead zone."

- **Criminal's movement is unconstrained.** Because of the difficulty of finding real-world distance data, we invoke the "Manhattan assumption": There are enough streets and sidewalks in a sufficiently grid-like pattern that movements along real-world movement routes is the same as "straight-line" movement in a space discretized into city blocks [Rossmo 1999]. Kent [2006] demonstrated that across several types of serial crime, the Euclidean and Manhattan distances are essentially interchangeable in predicting anchor points.

- **Domain contains all possible strike points.** This condition says that the two conditions above hold on a sufficiently large area.

Taken together, these three conditions describe the region of interest as a metric space in which

- The subset of potential targets is dense,

- the metric is the L^2 norm, and
- the space is "complete": Sequences of crimes do not lead to predictions of crimes outside the space.

Violent Serial Crimes by a Single Offender

- **Focus on violent crimes.** Geographic profiling is most successful for murders and rapes, with the average anchor-point prediction algorithm being 30% less effective for criminals who are serial burglars or arsonists [Canter et al. 2000; Rossmo 1999].
- **Serial crimes.** We take serial killing (or violent crime) as involving "three or more people over a period of 30 or more days, with a significant cooling-off period between" [Holmes and Holmes 1998].
- **Single offender.**

Spatial Focus

Use of temporal data is problematic. Time data can be inaccurate. Also, while research has found cyclical patterns within the time between crimes, these patterns don't associate directly to predicting the next geographic location. What is useful is general trends in spatial movement over an ordering of the locations. We hence ignore specific time data in crime sets except for ordering of the crime sequence.

Developing a Serial Crime Test Set

Existing Crime Sets

Researchers have compiled databases of serial crimes for their own use: Rossmo's FBI and SFU databases [Rossmo 1999], LeBeau's San Diego Rape Case dataset [LeBeau 1992], and Canter's Baltimore crime set [Canter et al. 2000]. Each of these databases was developed with specific methods of integrity and specific source locations. These proprietary databases are not available to us, so we are faced with two options: simulate serial criminal data or find an indirect way of using the private data.

The Problem with Simulation

Simulation might seem like an attractive solution to the lack of data. However, utterly random crime-site generation would contradict the underlying assumption of a spatial pattern to serial crimes, while generating sites according to an underlying distribution would prejudge the pattern! Actual data must be used if there is to be any confidence in the model.

An Alternative: Pixel Point Analysis

Instead, we "mine" the available data, in Rossmo [1995] and in the spatial analysis of journey-to-crime patterns in serial rape cases in LeBeau [1992]. LeBeau depicts the data as scatterplots, which we rasterize and re-render with scaling. **Figure 1** shows an example of this process, which we applied to seven criminals' data (four killers, three rapists). The rape sequences have an explicit ordering, while the murder sequences are unordered.

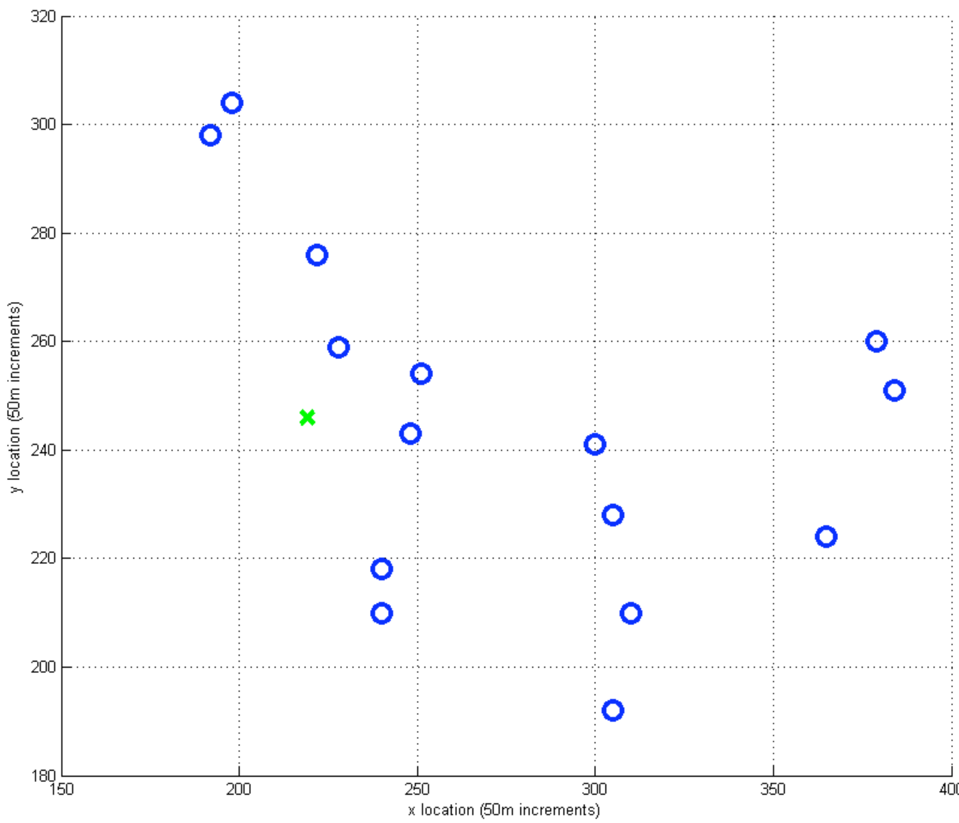


Figure 1. Re-rendering of scatterplot in LeBeau [1992] for Offender B.

Metrics of Success

A successful model must outperform random predictions.

The Effectiveness Multiplier

We assume that police effectiveness is proportional to resources allocated, and that the resources allocated at a location are proportional to the likelihood given by the model. We say that one model outperforms another if it recommends allocating more police resources to where the next crime is actually committed.

We assess how much one model outperforms another by the *effectiveness multiplier* κ, the ratio of resources allocated at the crime point under each model. Alternatively, κ is the ratio of the percentage of total department resources allocated to the point. Since the total resources are the same in both models, we can evaluate κ simply as

$$\kappa = \frac{Z_1(CrimePoint)}{Z_2(CrimePoint)}$$

where Z_i is the likelihood function of model i and $CrimePoint$ is the actual location of the next crime.

A randomly guessing algorithm will have a uniform distribution over locations and hence a flat likelihood plot. We compare our model to such a random guess by computing a *standard effectiveness multiplier* κ_s:

$$\kappa_s = \frac{Z_{\text{our model}}(CrimePoint)}{Z_{\text{flat}}(CrimePoint)}$$

A value of 1 would indicate that the model was no better than a random guess, and a value less than 1 would indicate that the model *misled* the police.

Robustness of the Metric

We also want to compare the success of our algorithm across multiple datasets. It is legitimate to compare κ_s values between two datasets only if they have the same ratio of the killer's active region to the total area. The size of the killer's active region cannot be precisely known; however, we employ a standard technique to make this condition approximately true. According to Canter and Larkin [1993] and Paulsen [2005], in more than 90% of cases all future crimes fall within a square whose side length is the maximum distance between previous crime points and whose center is the centroid of the data. For each of our datasets, we construct such a square and then multiply its side length by 3, thereby creating an overall search area nearly 9 times as large as the criminal's active area. This ratio is nearly constant for all datasets, allowing us to compare effectiveness multipliers.

Two Schemes for Spatial Prediction

Journey-to-crime research for violent serial crimes strongly suggests that serial crime is patterned around a criminal's home, workplace, or other place of daily activity [Godwin and Rosen 2005; Holmes and Holmes 1998; Kocsis and Irwin 1997; Rossmo 1999; Snook et al. 2005]; so researchers have developed and evaluated methods of finding such a crime centroid and investigated it as an anchor point in the criminal's activity. In most research, this anchor point is the serial criminal's home. This method has been tested and found to reduce the necessary search area by a factor of 10.

We develop two schemes, one for a single anchor point and the other for multiple anchor points.

Single Anchor Point: Centroid Method

Figure 2 shows our algorithm to predict likely crime locations using a single anchor point.

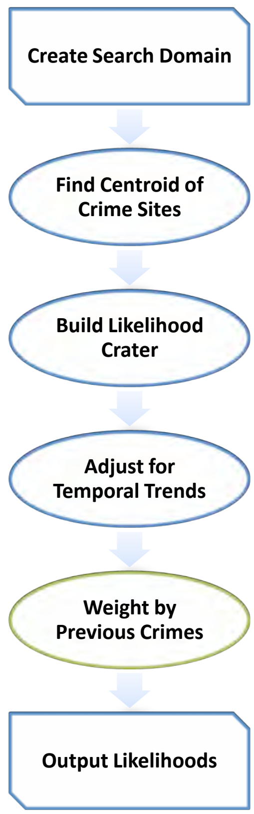


Figure 2. Flow chart for centroid method.

Algorithm

Create Search Domain

We construct the smallest square that contains every previous crime, then scale up each dimension by a factor of 3. This ensures that all of our fundamental assumptions about the underlying domain are satisfied, and the consistent scale factor of 3 allows us to compare the algorithm between datasets.

Find Centroid of Crime Sites

The anchor point is the average of the n crime coordinates (x_i, y_i).

Building a Likelihood Crater

We predict future crime locations using the "journey-to-crime" model, which says that the criminal's spatial pattern of crime around an anchor point does not change. A rough first prediction might be to draw a large shape (circle, square, polygon, etc.) around this anchor point based on the largest distance from a crime point to the anchor point; such a method is incredibly ineffective compared to the largest-circle guess that we described earlier [Paulsen 2005].

We instead use a cratering technique first described by Rossmo [1999]. The two-dimensional crime points x_i are mapped to their radius from the anchor point a_i, that is, we have $f : x_i \to r_i$, where $f(x_i) = \|x_i - a_i\|_2$ (a shifted modulus). The set r_i is then used to generate a crater around the anchor point.

There are two dominating theories for the pattern serial crimes follow around an anchor point:

- **There is a buffer zone around the anchor point.** The criminal commits crimes in an annulus centered at the anchor point [Kocsis and Irwin 1997]. This theory is often modeled using the positive portion of a Gaussian curve with parameters the mean and the variance of the $\{r_i\}$.
- **Crimes follow a decaying exponential pattern from the anchor point.**

Both theories have been substantiated by journey-to-crime research. We seek a distribution that would model either theory, depending on the pattern in the crime sequence. For this we turn to the flexibility of the gamma distribution, which offers a "shifted-Gaussian"-like behavior when points lie farther away but a curve similar to a negative exponential when the parameters are small.

Define the random variable X_i to be the distance between the ith crime point and the anchor point r. We let each X_i have a gamma distribution with parameters k and θ: $X_i \sim \Gamma(k, \theta)$, with probability density function (pdf)

$$f(x; k, \theta) = \frac{\theta^k}{\Gamma(k)}\, x^{k-1} e^{-\theta x}.$$

We assume independence of the X_i and use the maximum likelihood estimates of the parameters k and θ as calculated by the *gamfit* function in MatLab.

We then build the crater of likely crime locations using the resulting distribution. For every point in the search region, we evaluate the pdf. We then normalize so that the volume under the likelihood surface is exactly 1.

Applying this method to the set of crime locations of Peter Sutcliffe, the "Yorkshire Ripper," we get the heat map of **Figure 3**.

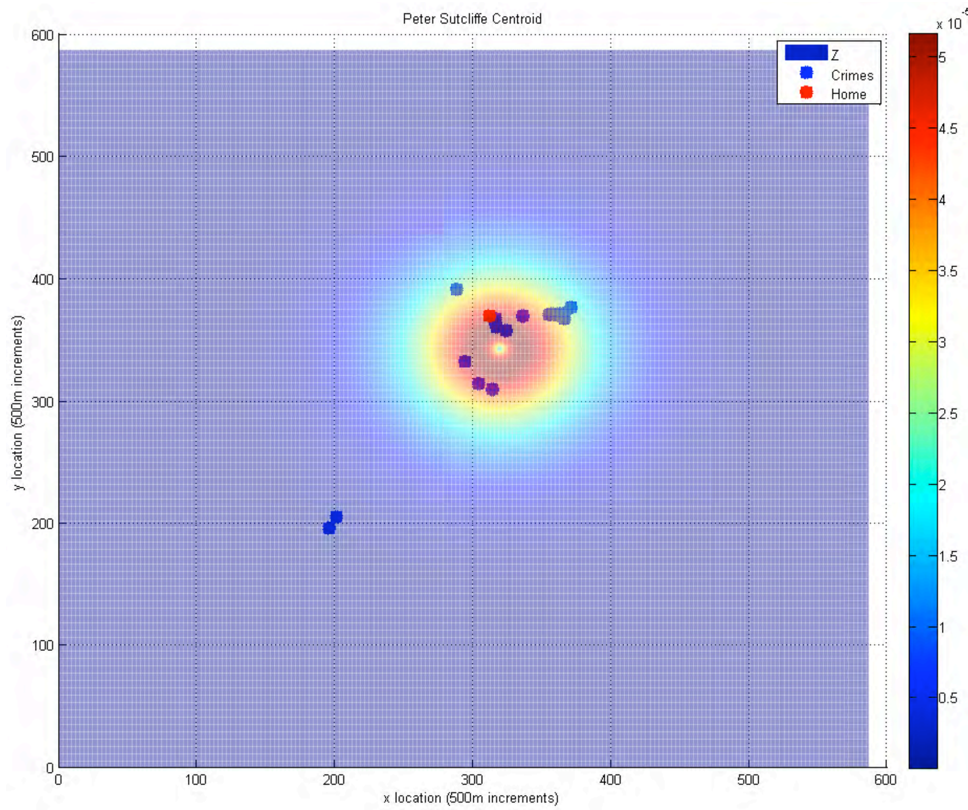


Figure 3. Heat map showing cratering technique applied to the crime sequence of Peter Sutcliffe.

Adjust for Temporal Trends

We would like our prediction to account for any radial trend in time (the criminal becoming more bold and committing a crime closer or further from home): An outward or inward trend in r_i may suggest that the next crime will follow this trend [Kocsis and Irwin 1997]. We let $\tilde{X} = X + \overline{\triangle r}$, where $\triangle r = r_n - r_{n-1}$. The new random variable $\tilde{X}$ gives our intended temporal adjustment in expected value:

$$E[\tilde{X}] = E[X + \overline{\triangle r}] = E[X] + \overline{\triangle r}.$$

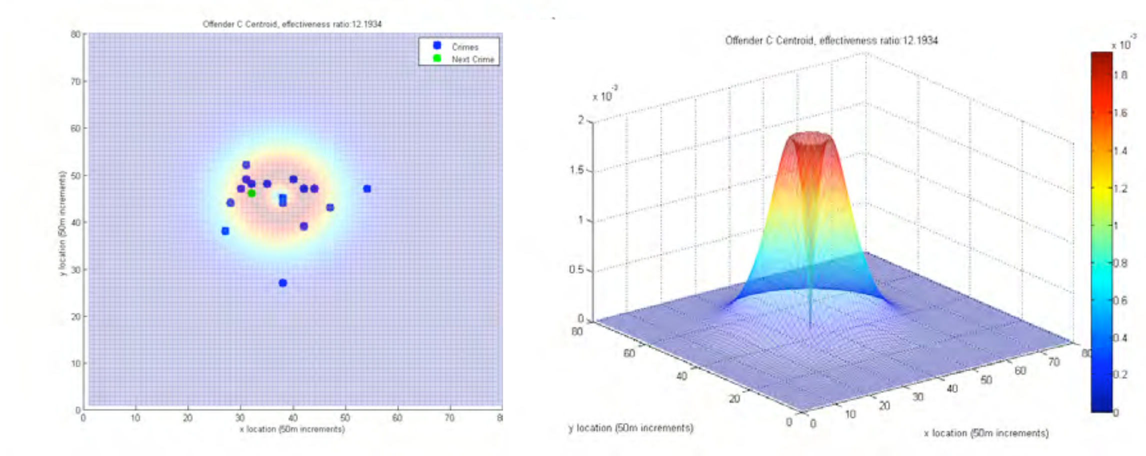
Results and Analysis

To evaluate our method, we feed it data from three serial-rape sprees.

In each test case, we remove the data point for the final crime and produce a likelihood surface $Z(x, y)$. We then estimate the location of the final crime and compute the standard effectiveness multiplier κ_s.

Offender C

Our first test dataset, for Offender C, is a comparative success (**Figure 4**). With $\kappa_s \approx 12$, it is a full order of magnitude better to distribute police resources using this model instead of distributing them uniformly.



a. Heat map. **b.** Surface plot.

Figure 4. Offender C predictions of location of final crime, from centroid model of previous crimes.

The next-crime estimate falls satisfyingly near the isoline of maximum height; but there 120 grid squares are rated greater or equal in likelihood, meaning 0.3 km^2 must be patrolled at the same or greater intensity. This area is small in an absolute sense; but it is comparatively large given that the vast majority of the crimes in this case were committed within an area of 1 km^2.

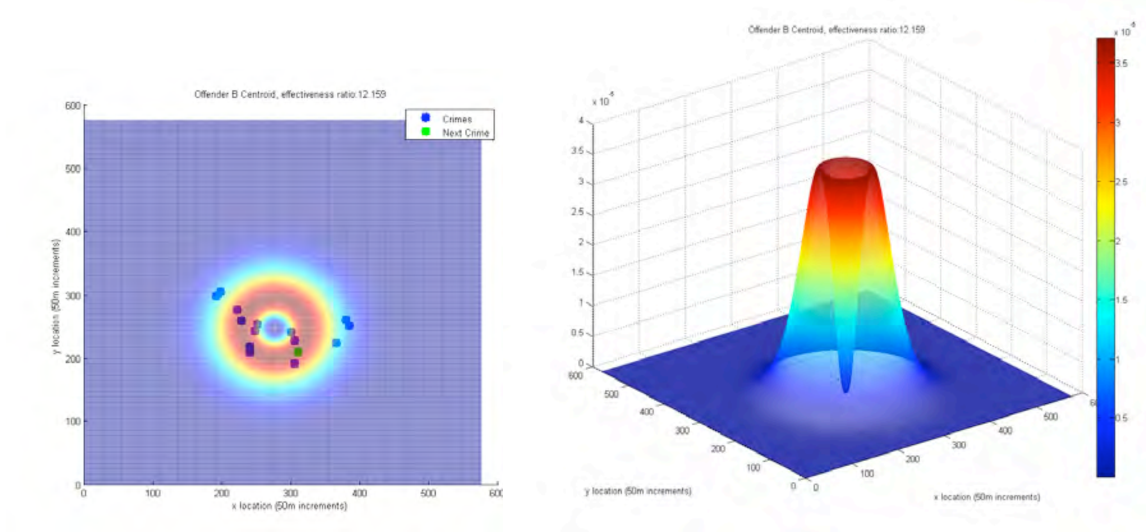
With $\overline{\Delta r} = -0.276$, the temporal corrections in this distribution are negligible; our projection is simply a radially symmetric fit to the geographic dispersion of previous crimes. The surface plot also shows the steepness of the "inside" of the crater, as the geographic distribution apparently suggests a small buffer zone around the centroid.

Offender B

Our second test dataset, for Offender B (see **Figure 5**), is similarly successful, with $\kappa_s \approx 12$.

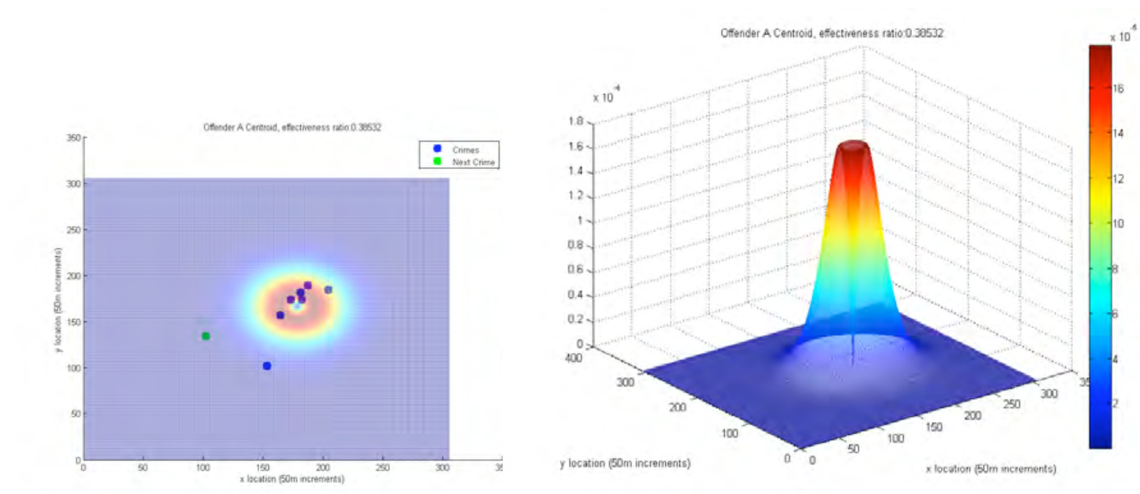
Offender A

For Offender A, we find a clear example of how our model can fail. The last crime (see **Figure 6**) is one of two substantial outliers, and in fact, with a standard effectiveness multiplier of $\kappa_s \approx 0.4$, our model is less helpful than random guessing. However, the majority of previous crimes are still well described by the model, so the assumption of an anchor point somewhere within the crater region does not appear to have been a poor one. Some scheme, whim, or outside influence caused the criminal to deviate from his previous pattern.



a. Heat map. **b.** Surface plot.

Figure 5. Offender B predictions of location of final crime, from centroid model of previous crimes.



a. Heat map. **b.** Surface plot.

Figure 6. Offender A predictions of location of final crime, from centroid model of previous crimes.

Multiple Anchor Points: Cluster Method

Our second method explicitly assumes at least two anchor points (for example, a home and a workplace) and treats each as the centroid of its own local cluster of crimes. This method requires determining an appropriate number of clusters, which we derive from the locations of the previous crimes.

Algorithm

The basis of this algorithm is a hierarchical clustering scheme [Jain, Murty, and Flynn 1999]. Once clusters are found, the previous algorithm is applied at each cluster centroid.

Finding Clusters in Crime Sequences

We force a minimum of 2 clusters and a maximum of 4. The clustering algorithm is accomplished in a 3-step process.

1. Compute the distances between all crime locations, using the Euclidean distance.
2. Organize the distances into a hierarchical cluster tree, represented by a *dendrogram*. The cluster tree of data points $P_1, \ldots, P_N$ is built up by first assuming that each data point is its own cluster. The dendrogram for Offender B is shown in **Figure 7**.

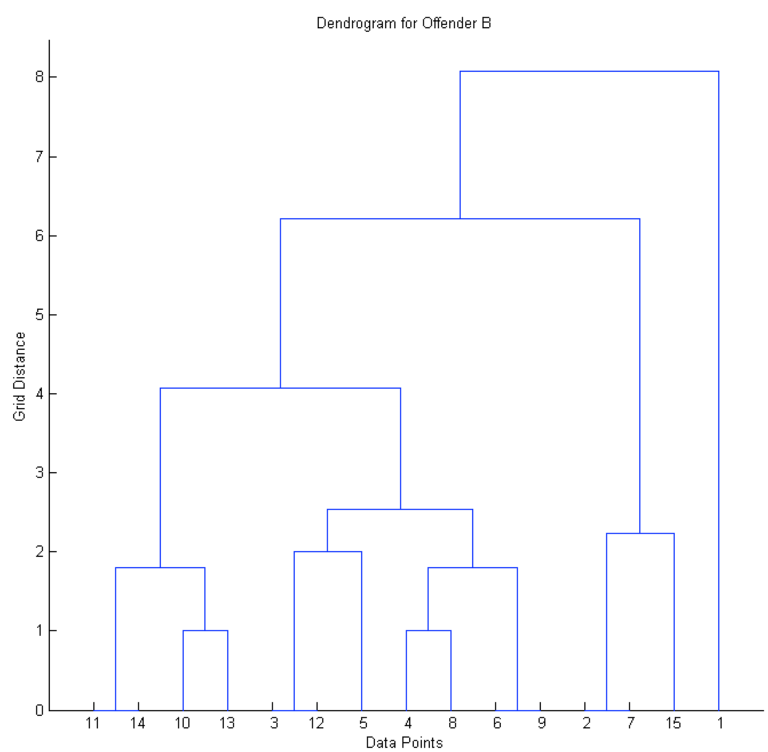


Figure 7. Dendrogram for Offender B.

3. Merge the two clusters that are the closest (in distance between their centroids), and continue such merging until the desired number of clusters is reached. These cluster merges are plotted as the horizontal lines in the dendrogram, and their height is based on the distance between merged clusters at the time of merging.

To determine the optimal number of clusters, we use the notion of *silhouettes* [Rousseeuw 1987]. We denote by $a(P_i)$ the average distance from P_i to all other points in its cluster and by $b(P_i, k)$ the average distance from P_i to points in a different cluster C_k. Then the silhouette of P_i is

$$s(P_i) = \frac{\left[\min_{k|P_i \notin C_k} b(P_i, k)\right] - a(P_i)}{\max\left(a(P_i), \min_{k|P_i \notin C_k} b(P_i, k).\right)} \tag{1}$$

The silhouette s can take values in $[-1, 1]$: The closer $s(P_i)$ is to 1, the better P_i fits into its current cluster; and the closer $s(P_i)$ is to -1, the worse it fits within its current cluster.

To optimize the number of clusters, we compute the clusterings for 2, 3, and 4 clusters. Then for each number of clusters, we compute the average silhouette value across every point that is not the only point in a cluster. (We ignore silhouette values at single-point clusters because otherwise such clusters influence the average in an undesirable way.) We then find the maximum of the three average silhouette values. For Offender B, we found average silhouette values of 0.52, 0.50, and 0.69 for 2, 3, and 4 clusters. So in this case, we go for four clusters. The cluster groupings computed by the algorithm are shown in **Figure 8**. Because the average silhouette value tends to increase with the number of clusters, we cap the possible number of clusters at 4.

Cluster Loop Algorithm

We compute the likelihood surface for the centroid of each cluster.

If a cluster contains a single point, we do not assume that this cluster represents an anchor point; instead, we treat this point as an outlier. We use a Gaussian distribution centered at the point as the likelihood surface, with mean the expected value of the gamma distribution placed over every anchor point of a cluster that has more than one point.

Combining Cluster Predictions: Temporal and Size Weighting

Using the separate likelihood surfaces computed for each cluster, we create our final surface as a normalized linear combination of the individual surfaces, using weights for the number of points in the cluster (to weight

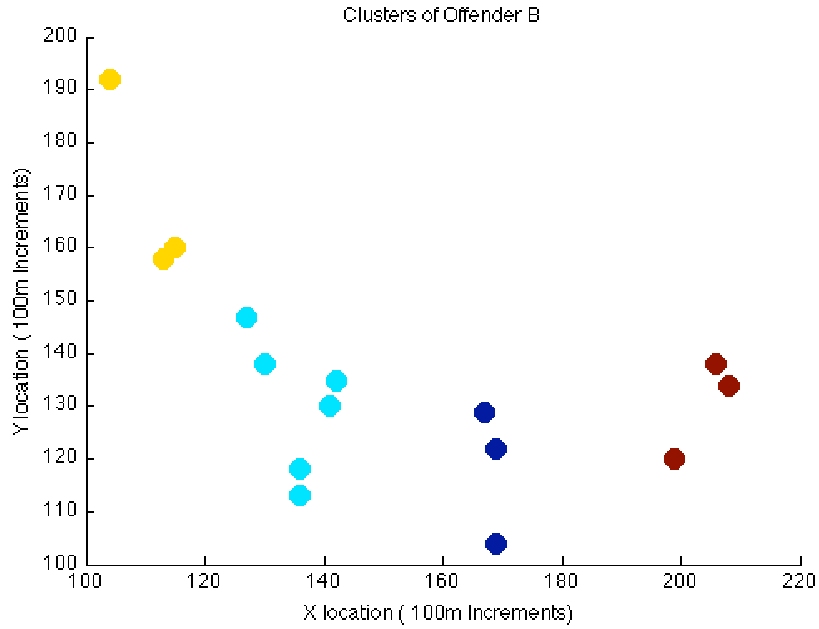


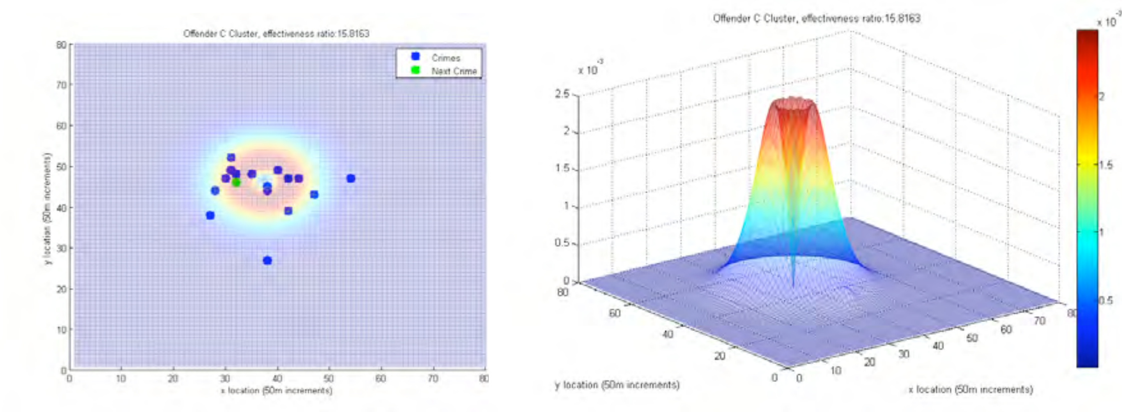
Figure 8. Offender B crime points sorted into 4 clusters; the clusters are colored differently and separated by virtual vertical lines into 3, 6, 3, and 3 locations.

more-common locations) and for the average temporal index of the events in the cluster (to weight more-recent clusters).

Results and Analysis

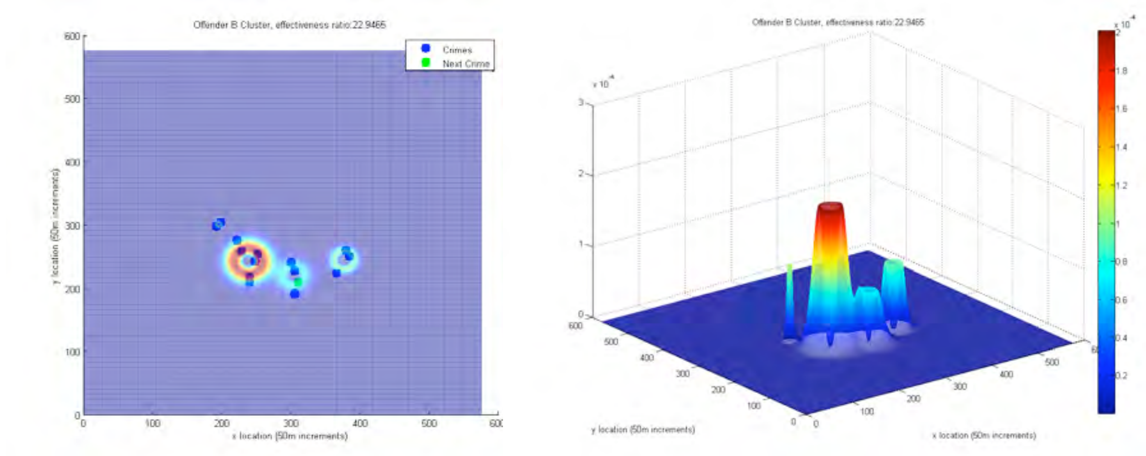
The three test datasets conveniently display the cluster method's superior adaptability.

- **Offender C:** The highly-localized nature of the data points in **Figure 9** means there is little difference between the centroid-method results and the cluster-method results. The only difference is that the cluster method identifies the point directly below the centroid as a cluster of a single point (an outlier) and therefore excludes it from the computation of the larger cluster's centroid (a slight Gaussian contribution from this point's "own" cluster can be seen in the surface plot). This has the effect of slightly reducing the variance and therefore narrowing the fit function; consequently, the standard effectiveness multiplier rises slightly, from about 12 to almost 16.
- **Offender B:** By contrast, **Figure 10** shows the cluster method operating at the other edge of its range, as the silhouette-optimization routine produces four clusters. It might appear that the centroid method outperforms the cluster algorithm for this dataset; after all, the actual crime point no longer appears in the band of maximum likelihood. This is true and intentional, since the model weights the largest cluster most strongly



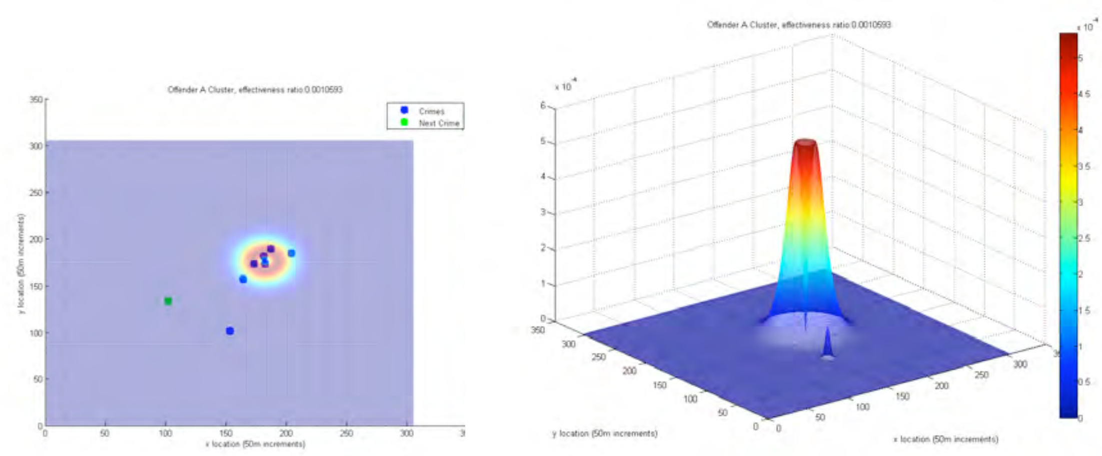
a. Heat map. **b.** Surface plot.

Figure 9. Offender C predictions of location of final crime, from cluster model of previous crimes.



a. Heat map. **b.** Surface plot.

Figure 10. Offender B predictions of location of final crime, from cluster model of previous crimes.



a. Heat map. **b.** Surface plot.

Figure 11. Offender A predictions of location of final crime, from cluster model of previous crimes.

and the "freshest" cluster next-most. Nevertheless, while not accurately predicting that the offender returns to an earlier activity zones, the cluster method *still* outperforms the centroid method with a $\kappa_s \approx 23$. This is because the craters generated by the cluster method are sharper and taller for this dataset, so fewer resources are "wasted" at high-likelihood areas where no crime is committed.

- **Offender A:** Unsurprisingly, the cluster model fares no better than the centroid method (**Figure 11**). Since the outlier points are excluded from the centroid calculation for the larger cluster, the model bets even more aggressively on this cluster, with a resulting $\kappa_s \approx 0$.

Combining the Schemes

Based on further work not shown in this briefer version of our contest paper, we developed a combined method to decide for a given situation whether the centroid method or the cluster method is better, based on calculating running means of the effectiveness multipliers of the two methods as we predict each of the crime locations from the previous ones.

Our combined method would use the clustering algorithm for all three Offenders A, B, and C. One would have to have a a highly-cohesive single cluster of data with no outliers in order for the centroid method to ever prevail. This is as it should be, since even when there appears to be only one true anchor point, the cluster method can reject up to three statistical outliers before computing the centroid, which capability should in general improve the fits and consequently improve the results.

The Model vs. Random Guess, Intuitive Cops, etc.

Although our scant datasets do not allow anything conclusive, our model is a strong candidate over other alternatives, for several reasons:

- The predictions are based on the assumption of trends in serial crime behavior which *have* been tested on large sets of real-world data [Canter et al. 2000; Kocsis and Irwin 1997; Paulsen and Robinson 2009].
- Similar mathematical techniques are used in the anchor-point estimation schemes currently employed, which consistently outperform random guesses when tested across data samples.
- The model is successful in the two of the three real-life datasets on which we tested it.
- Several crimes in each dataset can be predicted well, even in the dataset where our model fails to predict the final, outlier crime point—i.e., in a dataset of 16 crimes, we are also often able to predict the 15th crime

using the preceding 14, the 14th using the preceding 13, etc., all with more success than a random guess.

We do not claim that the model will do better than a police department in assigning resources based on knowledge of the area, a sense of patterning in the offender's crimes, and any "gut feelings" developed about the offender's psychology, based on previous experience in law enforcement.

Additionally, research suggests that with a little informal training, any layperson can perform nearly as well as anchor-point-prediction algorithms in guessing a criminal's home location [Snook, Taylor, and Bennell 2004]. As a result, one might well expect an "intuitive cop" to outperform the model overall, by approximating its mathematical strengths through intelligent estimation plus bringing a breadth of knowledge and experience to bear.

Executive Summary

Overview: Strengths and Weaknesses of the Model

We present a model of where a violent serial criminal will strike next, based on the locations and times of previous crimes. Our algorithm creates a color-coded map of the area surrounding the criminal's previous strikes, with the color at each point indicating the likelihood of a strike there. The model has several key strengths:

- **The model contains no arbitrary parameters**. In other words, most aspects of the model are determined simply by trends observed in datasets about many serial criminals.
- **The model can estimate the level of confidence in its predictions.** Our model first checks how well it would have predicted the criminal's previous crimes, in order to provide an estimate of how well it can predict future crimes.
- **The model understands that police have limited resources**. In particular, the confidence level described above becomes large if we are making good predictions but will shrink again if our predicted areas become so large as to become unhelpful.

At the same time, our model contains some fundamental limitations:

- **The model is applicable only to violent serial criminals**. We claim applicability only for serial killers and rapists, since our research shows that serial burglars and arsonists are more unpredictable and influenced by non-geographic factors.
- **The model cannot predict when a criminal will strike**. While we consider the order of previous crimes in order to predict locations, we do *not* predict a strike *time*.

- **The model cannot make use of underlying map data**. To maintain generality, we do not make any assumptions about the underlying physical geography. A human being must interpret the output (for example, choosing to ignore a prediction in the middle of a lake).
- **The model has not been validated on a large set of empirical data**. Sizable sets of data on serial criminals are not widely available.

In addition, the standard warnings that would apply to any geographic profiling scheme apply: The output should not be treated as a single prediction but rather as a tool to help prioritize areas of focus. It is designed to do well on average but may fail in outlier cases. And to implement it reliably, it must be paired with a human assessment. Please note also that models for predicting the offender's "home base" are much more well researched and in general will be more accurate than any algorithm claiming to predict the next strike point. A police department should choose on a case-by-case basis which model type to use.

Internal Workings of the Model

Inputs

Our model requires the coordinate locations of a serial criminal's previous offenses, as well as the order in which these crimes were committed.

Assumptions

1. The criminal will tend to strike at locations around one or more anchor points (often, the criminal's home).
2. Around this anchor point, there may be a "buffer zone" within which he will not strike.
3. If the criminal has multiple anchor points, the regions around those from which he has struck most often or most recently are more likely.

Method

The algorithm implements two different models, and then decides which is better.

- The first method assumes that the criminal has a single anchor point and builds likelihood regions around the anchor point based on the distribution of his past crimes.
- The second method assumes multiple anchor points, calculates the best number of anchor points to use, determines likelihood around each point individually, and weights the area around each by the criminal's apparent preferences.

Finally, the algorithm tests *both* models to see how well they would have predicted the previous crimes and uses the model with the better track record.

Summary and Recommendations

While our model needs more real-world testing, its strong theoretical basis, self-evaluation scheme, and awareness of practical considerations make it a good option for a police department looking to forecast a criminal's next strike. Combining its results with the intuition of a human being will maximize its utility.

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Advisor Anne Dougherty with team members Anil Damle, Colin West, and Eric Benzel.

Judges' Commentary: The Outstanding Geographic Profiling Papers

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Introduction

The stated problem this year dealt with the issue of geographical profiling in the investigation of serial criminals. International interest in this topic has led to numerous publications, many of which present mathematical models for analyzing the problems involved. Although it was entirely appropriate and expected that teams working on this problem would review the literature on the subject and learn from their review, teams that simply presented published schemes as their mathematical models fell far short of what was expected. The judges looked for sparks of creativity and carefully explained mathematical model building with sensitivity analysis that went beyond what is found in the literature. This factor is what added value to a paper.

Documentation and Graphs

We observed a noticeable improvement in how references were identified and in the specific precision in documenting them within the papers. Considering the numerous online resources available, proper documentation was an especially important factor in this year's problem.

Despite the improvement, many papers contained charts and graphs from Web sources with no documentation. All graphs and tables need labels and/or legends, and they should provide information about what is referred to in the paper. The best papers used graphs to help clarify their results and documented trustworthy resources whenever used.

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Assumptions

In many cases, teams made tacit assumptions about the criminals being considered but did not state or justify critical mathematical assumptions that were later used implicitly. Assumptions concerning probability distributions, anchor points, distances, units, mathematical procedures, and how to measure results were generally not discussed or justified.

Since this is a modeling contest, a lot of weight is put on whether or not the model could be used, with modification, in the real world. Also, clear writing and exposition is essential to motivate and explain assumptions and to derive and test models based on those assumptions.

Summary

The summary is of critical importance, especially in early judging. It should motivate the reader and be polished with a good synopsis of key results. For this problem, teams were asked to add to their one-page summary (which can have some technical details) also a two-page executive summary appropriate for the Chief of Police. Many teams seemed to assume that the Chief of Police would have impressive mathematical credentials.

The Problem and Its Analysis

Teams were asked to develop at least two different schemes for generating geographical profiles and then to develop a technique for combining the results of the different schemes in such a way as to generate a useful prediction for law enforcement officers. Although the papers designated as Meritorious generally developed interesting schemes, very few papers did an adequate job of testing their results and doing sensitivity analysis.

Most papers dealt with issues associated with the serial criminal's home base, usually referred to as the anchor point, and the buffer zone around that point within which the criminal is unlikely to commit crimes. Locations were identified using latitude and longitude and sometimes a time factor. Weights were frequently assigned to data points, sometimes taking more recent crimes into account more heavily and sometimes incorporating qualitative factors into the scheme. Teams used various metrics in describing "distances" between the anchor point and crime locations. Papers that rose to the top used well-defined metrics that were clearly explained. One cannot measure the reliability or validity of a model without clearly defined metrics.

Many teams mentioned that there was not a lot of data with which they could validate their model, although they did find some specific location information that included from 13 to 20 crimes in a given series. Some teams used as their only example the Sutcliffe case cited in the problem. In almost all cases, teams used their model to predict the location of the final crime based on all of the previous locations for that criminal. They could easily have had many more data points with which to validate their models. For example, if 13 crime locations were available, they could have used the first n locations to predict the location of crime $n+1$, for each $n = 7, \ldots, 12$. The judges agreed that this problem did not lend itself to validation by simulation, as many other problems do.

In describing the reliability of predicted results for proposed models, it was sometimes difficult to determine precisely how teams had arrived at their results. Since the literature is full of models and even computer models, it would have been worthy if teams had solved a problem via one of these methods and used that as a baseline to compare the results of original models that they proposed. Not a single team did this to the judge's satisfaction. Judges do not generally look for computer code, but they definitely look for precise algorithms that produce results based on a given model.

Concluding Remarks

Mathematical modeling is an art. It is an art that requires considerable skill and practice in order to develop proficiency. The big problems that we face now and in the future will be solved in large part by those with the talent, the insight, and the will to model these real-world problems and continuously refine those models. Surely the issue of solving crimes involving serial killers is an important challenge that we face.

The judges are very proud of all participants in this Mathematical Contest in Modeling and we commend you for your hard work and dedication.

About the Author

Marie Vanisko is a Mathematics Professor Emerita from Carroll College in Helena, Montana, where she taught for more than 30 years. She was also a Visiting Professor at the U.S. Military Academy at West Point and taught for five years at California State University Stanislaus. In both California and Montana, she directed MAA Tensor Foundation grants on mathematical modeling for high school girls. She also directs a mathematical modeling project for Montana high school and college mathematics and science teachers through the Montana Learning Center at Canyon Ferry, where she chairs the Board of Directors. She has served as a judge for both the MCM and HiMCM.

Judges' Commentary: The Fusaro Award for the Geographic Profiling Problem

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Introduction

MCM Founding Director Fusaro attributes the competition's popularity in part to the challenge of working on practical problems. "Students generally like a challenge and probably are attracted by the opportunity, for perhaps the first time in their mathematical lives, to work as a team on a realistic applied problem," he says. The most important aspect of the MCM is the impact that it has on its participants and, as Fusaro puts it, "the confidence that this experience engenders."

The Ben Fusaro Award for the 2010 Geographic Profiling Problem went to a team from Duke University in Durham, NC. This solution paper was among the top Meritorious papers that this year received the designation of Finalist. It exemplified some outstanding characteristics:

- It presented a high-quality application of the complete modeling process.
- It demonstrated noteworthy originality and creativity in the modeling effort to solve the problem as given.
- It was well-written, in a clear expository style, making it a pleasure to read.

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Criminology and Geographic Profiling

Each team was asked to develop a method to aid in the investigations of serial criminals. The team was to develop an approach that makes use of at least two different schemes and then combine those schemes to generate a geographic profile that would be a useful prediction for law enforcement officers. The prediction was to provide some kind of estimate or guidance about possible locations of the next crime. based on the time and locations of the past crimes. In addition to the required one-page summary, teams had to write a two-page less-technical executive summary for the Chief of Police.

In doing Web searches on this topic, teams found many publications and many proposed models. While it was important to review the literature, to receive an Outstanding or Meritorious designation, teams had to address all the issues raised and come up with a solution that demonstrated their own creativity.

The Duke University Paper

Abstract (One-Page Summary)

The Duke team did an excellent job with their abstract. In one page, they motivated the reader and provided the reader with a good sense of what the team had accomplished. It gave an overview of everything from the assumptions, to the modeling and how it was done, to the testing of their models, and finally, to the analysis of the accuracy of their results and limitations of their models. It was well-written and a great example of what an abstract should be.

Executive Summary (for the Police Chief)

The executive summary too was well-written and gave an overview of the team's approach, acknowledging limitations of their models. However, it was a little too vague in providing a precise idea of exactly what information would need to be collected and how to go about implementing the proposed models. Because the executive summary is a critical part of the requirements, this was part of what kept the Duke paper from being designated as Outstanding.

Assumptions

The team began with a brief survey of previous research on geographic profiling and used the information that they had gathered to decide what assumptions seemed appropriate. As a result, their list of assumptions was well-founded. The team exemplified one of the most important aspects in mathematical modeling by demonstrating precisely how their assumptions were used in the development of their models and how the assumptions enabled them to determine parameters in their models.

The Models

The team's first model involved a geographic method that took into account not only the location of crimes but also population densities, crime rates, and selected psychological characteristics. They used a bivariate Gaussian probability function and numerous parameters associated with previous crime locations. They did a very good job of showing how their assumptions and previous crime scenes led to the computation of these parameters and then using these parameters to estimate the probability function to be used in their model.

The team's second model involved a risk-intensity method and made use of the geographic method but extended it to make different projections with different probabilities associated with each of those projections.

Testing the Models

The Duke team was among the top papers, not only because of their well-based models, but because they tested their models—not with just one serial crime case, but with many cases. Their parameters allowed them to consider crimes other than murder, and they were able to examine how good their models were in several real-life situations. By analyzing their results, they were able to comment on the sensitivity and robustness of their models. This was something that very few papers were able to do, and a very important step in the modeling process.

Recognizing Limitations of the Model

Recognizing the limitations of a model is an important last step in the completion of the modeling process. The teams recognized that their models would fail if their assumptions did not hold—for example, if the criminal did not have a predictable pattern of movement.

References and Bibliography

The list of references consulted and used was sufficient, but specific documentation of where those references were used appeared only for a few at the start of the paper. Precise documentation of references used is important throughout the paper.

Conclusion

The careful exposition in the development of the mathematical models made this paper one that the judges felt was worthy of the Finalist designation. The team members are to be congratulated on their analysis, their clarity, and for using the mathematics that they knew to create and justify their own model for the problem.

About the Authors

Marie Vanisko is a Mathematics Professor Emerita from Carroll College in Helena, Montana, where she taught for more than 30 years. She was also a Visiting Professor at the U.S. Military Academy at West Point and taught for five years at California State University Stanislaus. In both California and Montana, she directed MAA Tensor Foundation grants on mathematical modeling for high school girls. She also directs a mathematical modeling project for Montana high school and college mathematics and science teachers through the Montana Learning Center at Canyon Ferry, where she chairs the Board of Directors. She has served as a judge for both the MCM and HiMCM.

Peter Anspach was born and raised in the Chicago area. He graduated from Amherst College, then went on to get a Ph.D. in Mathematics from the University of Chicago. After a post-doc at the University of Oklahoma, he joined the National Security Agency to work as a mathematician.

Modeling Forum

Results of the 2010 Interdisciplinary Contest in Modeling

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Introduction

A total of 356 teams from four countries spent a weekend in February working in the 12th Interdisciplinary Contest in Modeling (ICM)®. This year's contest began on Thursday, Feb. 18, and ended on Monday, Feb. 22. During that time, teams of up to three undergraduate or high school students researched, modeled, analyzed, solved, wrote, and submitted their solutions to an open-ended interdisciplinary modeling problem involving marine ecology. After the weekend of challenging and productive work, the solution papers were sent to COMAP for judging. One of the top papers, which were judged to be Outstanding by the expert panel of judges, appears in this issue of *The UMAP Journal*.

COMAP's Interdisciplinary Contest in Modeling (ICM), along with it sibling, the Mathematical Contest in Modeling (MCM)®, involves students working in teams to model and analyze an open problem. Centering its educational philosophy on mathematical modeling, COMAP supports the use of mathematical tools to explore real-world problems. It serves society by developing students as problem solvers in order to become better informed and prepared as citizens, contributors, consumers, workers, and community leaders. The ICM and MCM are examples of COMAP's efforts in working towards its goals.

This year's problem once again involved environmental sciences. The Great Pacific Ocean Garbage Patch Problem (or Marine Pollution Problem) was challenging in its demand for teams to utilize many aspects of science, mathemat-

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ics, and analysis in their modeling and problem solving. The problem required teams to understand the complexity of marine ecology and oceanography and to model that complexity to understand the effects of plastics building up in the Pacific Ocean Gyre. To accomplish their tasks, the teams had to consider many difficult and complex issues. The problem also included the requirements of the ICM to use thorough analysis, research, creativity, and effective communication. The author of the problem was marine biology researcher Miriam Goldstein of the Scripps Institute of Oceanography.

All members of the 356 competing teams are to be congratulated for their excellent work and dedication to modeling and problem solving. The judges remarked that this year's problem was especially challenging and demanding in many aspects of modeling and problem solving.

Next year, we will continue the environmental science theme for the contest problem. Teams preparing for the 2011 contest should consider reviewing interdisciplinary topics in the area of environmental issues.

The Problem Statement: The Great Pacific Ocean Garbage Patch

Your ICM submission should consist of a 1 page Summary Sheet and a 10 page report/solution for a total of 11 pages.

Recently, there has been considerable news coverage of the "Great Pacific Ocean Garbage Patch." See the following:

`http://www.nytimes.com/2009/11/10/science/10patch.html?em`
`http://www.sciencefriday.com/program/archives/200907314`
`http://www.reuters.com/article/idUSTRE57R05E20090828`

Based on recent scientific expeditions into the Pacific Ocean Gyre (a convergence zone where debris is accumulating), a wide variety of technical and scientific problems associated with this debris mass are coming to light. While dumping waste into the ocean is not a new activity, what is new is the scientific community's realization that much of the debris (plastics, in particular) is accumulating in high densities over a large area of the Pacific Ocean. The scientific community also is learning that this debris creates many potential threats to marine ecology, and as such, to human well-being. Those who study this accumulation often describe it as plastic soup or confetti. See

`http://news.nationalgeographic.com/news/2009/09/photogalleries/pacific-garbage-patch-pictures/`

This year's ICM problem addresses the complex issues stemming from the presence of ocean debris through the use of interdisciplinary modeling to help researchers (and ultimately, government policy-makers) to understand the severity, range, and potential global impact of the situation. As the modeling advisors to the expedition, your job is to focus on ***one*** element of this debris problem and model and analyze its behavior, determining its potential effect on marine ecology and what government policies and practices should be implemented to ameliorate its negative effects. Be sure to consider needs for future scientific research and the economic aspects of the problem, and then write a report to your expedition leader to summarize your findings and proposals for solutions and needed policies and practices. Some of the possible issues/questions you could investigate with your model include:

1. What are the potential short- and long-term effects of the plastic on the ocean environment? What kind of monitoring is required to track the impact on the marine ecosystem? Be sure to account for temporal and spatial variability. What are the associated resourcing requirements?

2. How can the extent, density, and distribution of the plastic in the gyre be best understood and described? What kind of monitoring plan is required to track the growth/decay/movement of the plastics, and what kind of resourcing is required to implement that plan?

3. What is the nature or mechanism of the photodegradation of the plastic and its composition as it enters the ocean and accumulates in the gyre? (We are amazed, for example, to find that the particles of degraded plastic tend to reach a similar size.)

4. Where does the plastic come from and what steps can be taken to control or reduce the risks associated with this situation? What are the economic costs and the economic benefits of controlling or ending the situation, and how do they compare? How much plastic is manufactured, discarded, and recycled? How much of that is likely to go into the ocean? How much of that is likely to float?

5. Could similar situations develop in other areas in the oceans? What should we monitor and how? What is happening in the North Atlantic Gyre and in the Alaskan Gyre? Use your model to estimate the plastic density in the future in the southern gyres (South Atlantic, South Pacific).

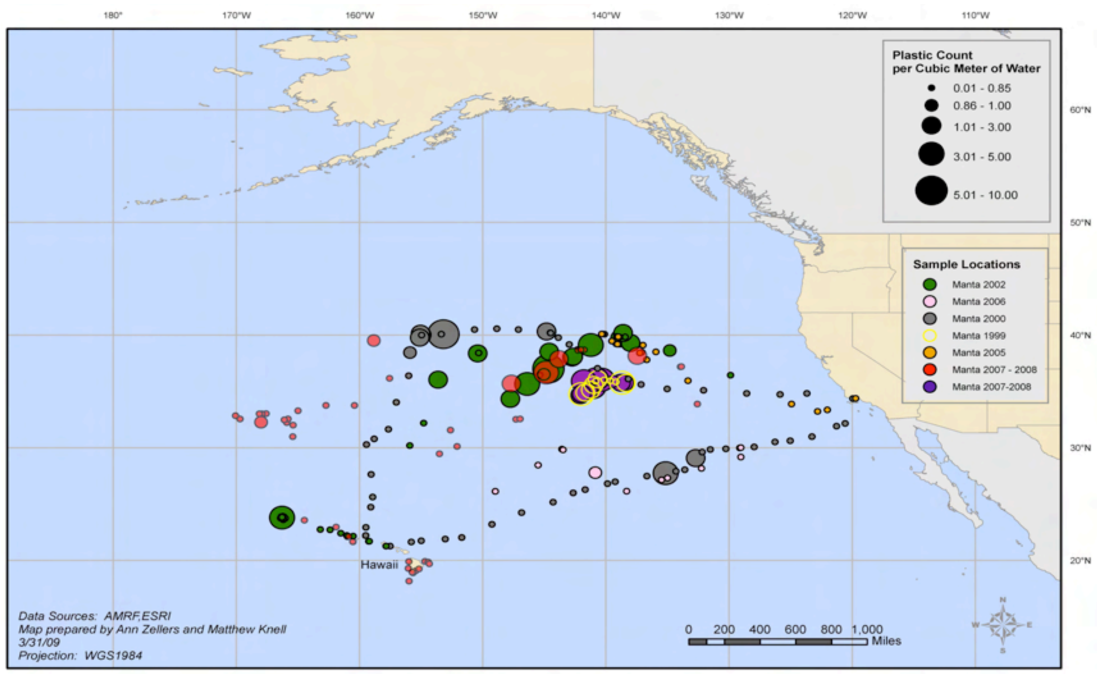
6. What is the immediate impact of banning polystyrene takeout containers? (See `http://bit.ly/5koJHB`.) What is the impact over 10–50 years?

7. Your team can address any other scientific/technological issue associated with this situation, as long as modeling is an important component of your investigation and analysis.

To clarify your task: Focus on ***one*** critical aspect of this problem and model the behavior of the important matters or phenomena. Quantify the quantities that are of greatest present or future interest to the one aspect you choose to model and analyze. Your ICM report should be in the form of a ***10-page*** team report to an expedition leader who has asked you to help her identify the relevant behaviors of the matters and phenomena under consideration, provide the analysis for impact of the behavior of those matters or phenomena, and advise her on the government's potential to act on the problem to improve this situation before it worsens.

The following source materials [provided with the problem statement] contain some helpful data:

- the figure below;
- Moore et al. [2001];
- Rei and Tanimura [2007].



Count Densities of Plastic Debris from Ocean Surface Samples
North Pacific Gyre 1999 - 2008

The References below suggest some papers to inform your model formulation and obtain more data.

Note: As a reminder, it is best to stick to the scientific literature, not the media coverage, for your facts. The mainstream media coverage of this issue has been misleading in many cases. For further explanation, see:

`http://seaplexscience.com/2009/11/13/millions-billions-trillions-of-scientific-errors-in-the-nyt/`.

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The Results

The 356 solution papers were coded at COMAP headquarters so that names and affiliations of the authors were unknown to the judges. Each paper was then read preliminarily by "triage" judges at the U.S. Military Academy at West Point, NY. At the triage stage, the summary, the model description, and overall organization are the primary elements in judging a paper. Final judging by a team of modelers, analysts, and subject-matter experts took place in late March. The judges classified the 356 submitted papers as follows:

	Outstanding	Finalist	Meritorious	Honorable Mention	Successful Participant	Total
Marine Pollution	4	6	45	120	181	356

One of the papers that the judges designated as Outstanding appear in this special issue of *The UMAP Journal*, together with a commentary by the judges. We list those Outstanding teams below.

Outstanding Teams

Institution and Advisor	**Team Members**
"A New Method for Pollution Abatement: Different Solutions to Different Types"	
Beijing Jiaotong University	Yuxi Li
Beijing, China	Lei Lu
Bingtuan Wang	Fandong Meng
"Shedding Light on Marine Pollution"	
Carroll College	Britttany Harris
Helena, MT	Chase Peaselee
Philip Rose	Kyle Perkins
"Quantitative Marine Debris Impacts and Evaluation of Ocean System"	
Hangzhou Dianzi University	Chenguang Fu
Hangzhou, China	Longbin Shen
Zheyong Qiu	Yawei Wen
"Size-Classified Plastic Concentration in the Ocean"	
Lawrence University	Fangzhou Qiu
Appleton, WI	Lu Yu
Stefan L. Debbert	Jian Gong

Awards and Contributions

Each participating ICM advisor and team member received a certificate signed by the Contest Directors and the Head Judge. Additional awards were presented to the team from Carroll College (Montana) by the Institute for Operations Research and the Management Sciences (INFORMS).

Judging

Contest Directors

Chris Arney, Dept. of Mathematical Sciences, U.S. Military Academy, West Point, NY

Joseph Myers, Computing Sciences Division, Army Research Office, Research Triangle Park, NC

Associate Director
Rodney Sturdivant, Dept. of Mathematical Sciences,
U.S. Military Academy, West Point, NY

Judges
John Kobza, Dept. of Industrial Engineering, Texas Tech University,
Lubbock, TX
Miriam Goldberg, Scripps Institute of Oceanography, La Jolla, CA
Giora Proskurowski, Woods Hole Oceanographic Institution, Woods Hole, MA
Kathleen Snook, COMAP Consultant, Bedford, MA
Frank Wattenberg, Dept. of Mathematical Sciences, U.S. Military Academy,
West Point, NY

Triage Judges
Dept. of Mathematical Sciences, U.S. Military Academy, West Point, NY:

Darryl Ahner, Chris Arney, Amanda Beecher, Fr. Gabriel Costa, Chris Eastburg, Andy Glen, Andy Hall, Tina Hartley, Michael Harvey, Alex Heidenberg, Craig Lennon, Chris Marks, Elizabeth Moseman, Elisha Peterson, Donovan Phillips, Elizabeth Russell, Libby Schott, Rodney Sturdivant, Edward Swim, Csilla Szabo, JoAnna Crixell Whitener, and Shaw Yoshitani

Acknowledgments

We thank:

- INFORMS, the Institute for Operations Research and the Management Sciences, for its support in judging and providing prizes for the INFORMS winning team;
- IBM for its support for the contest;
- all the ICM judges and ICM Board members for their valuable and unflagging efforts; and
- the staff of the U.S. Military Academy, West Point, NY, for hosting the triage and final judgings.

Cautions

To the reader of research journals:

Usually a published paper has been presented to an audience, shown to colleagues, rewritten, checked by referees, revised, and edited by a journal editor. Each of the team papers here is the result of undergraduates working on a problem over a weekend; allowing substantial revision by the authors could give a false impression of accomplishment. So these papers are essentially au naturel. Light editing has taken place: minor errors have been corrected, wording has been altered for clarity or economy, style has been adjusted to that of *The UMAP Journal*, and the papers have been edited for length. Please peruse these student efforts in that context.

To the potential ICM Advisor:

It might be overpowering to encounter such output from a weekend of work by a small team of undergraduates, but these solution papers are highly atypical. A team that prepares and participates will have an enriching learning experience, independent of what any other team does.

Shedding Light on Marine Pollution

Brittany Harris
Chase Peaslee
Kyle Perkins

Carroll College
Helena, MT

Advisor: Philip Rose

Abstract

Massive amounts of plastic waste have been accumulating in the Great Pacific Garbage Gyre and are posing a threat to the marine environment. Since little is known about the degradation of plastics in this marine setting, we adopt the goal of modeling the photodegradation of a common plastic, polyethylene, in seawater. Plastic in the ocean is exposed to ultraviolet (UV) light from the sun, which causes photodegradation, a natural source of plastic decomposition.

We develop two models to describe the rate of photodegradation of polyethylene floating in seawater, a Low-Transmittance of Light Model (LTM) and a High-Transmittance of Light Model (HTM). Using the constant rate of UV irradiance and the average bond dissociation energy of carbon-carbon single bonds (C–C), we calculate the mass lost per unit of time.

The results from our models are realistic. The HTM predicts that a rectangular prism of polyethylene $1 \times 1 \times 2$ cm weighing 1.87 g will lose 1.27 g of mass in one year; the LTM predicts that a hollow sphere with thickness 0.0315 cm, radius 5 cm, and weight 9.145 g, partially submerged in low-transmittance water, will lose 0.190 g of mass in the first year. The design of our models allows us to model other shapes, adjust the intensity of UV light, and realistically predict the photodegradation of polyethylene.

Introduction

The accumulation of plastic debris in our oceans is quickly coming to light as one of the most prevalent and devastating threats to the marine environment. The "Great Pacific Ocean Garbage Patch" is one of many areas of wind-current convergence where massive amounts of debris collect and stew. The "garbage" is not primarily in the form of bottles and bags,

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but rather as tiny particles referred to as *neustonic plastics*. These neustonic plastics are the products of degradation of post-consumer and industrial wastes and may pose great risk for marine life. The nature of the degradation of plastics has thus become an important element in the study of this environmental catastrophe.

We focus specifically on the photolytic degradation of polyethylene plastic accumulating in the gyre. We consider

- the UV radiation reaching the surface of the ocean;
- the energy required to break the bonds in polyethylene; and
- physical considerations concerning buoyancy, mass, and surface area of the plastic particles.

Description of the Problem

We consider the degradation of floating polyethylene fragments by photolytic degradation. The fragments are considered to be hollow spheres partially filled with seawater, to represent common post-consumer waste containers. The fragments are partially submerged in water with either low or high transmittance of light. High-transmittance water can use the entire effective surface area of the fragment to model degradation, while for fragments in low-transmittance water only the portion of the fragment above water will be susceptible to photolytic degradation [Ivanhoff, Jerlov, and Waterman 1961].

Ultraviolet light is assumed to hit the fragment orthogonally to the plane of the ocean, thereby exposing a two-dimensional surface area of effective area c. We relate c to the radius r of the fragment; the mass m of the fragment depends on both. The goal is to model these relationships over time to describe the loss of mass experienced by a polyethylene fragment.

Photolytic Degradation of Polyethylene

Polyethylene is a polymer consisting of long chains of the monomer ethylene [Carey and Sundberg 2007]. There are two types of bonds present in polyethylene: carbon-carbon single bonds (C–C) and carbon-hydrogen single bonds (C–H) [Leeming 1973]. Polyethylene has the structure

$$-(\mathrm{CH_2}-\mathrm{CH_2})_n-$$

where n is the number of monomers in the chain.

Photodegradation is a process by which chemical bonds are broken when struck by light [Carey and Sundberg 2007; Okabe 1978]. The light must carry enough energy to cleave a bond, which can be estimated using the average bond dissociation energy [Leeming 1973]. The equation

$$E = \frac{hc}{\lambda},$$

where h is Planck's constant and c is the speed of light, can be used to find the minimum wavelength λ of light that carries enough energy to break the bond [Skoog, Holler, and Crouch 2007]. For example, the energy to cleave a C–C single bond is 5.778×10^{-19} kg $\cdot$ m^2/s^2. Using $h = 6.626 \times 10^{-34}$ kg $\cdot$ m^2/s and $c = 3 \times 10^8$ m/s, we find λ = 344 nm, a wavelength in the ultraviolet.

Thus, when polyethylene is exposed to ultraviolet light (UV) with a wavelength of 344 nm, C–C single bonds are cleaved and free radicals are formed that react quickly with O_2 to form peroxy radicals. Then either the peroxy radicals continue a chain reaction of radical formation or else two free radicals react to terminate the chain reaction [Carey and Sundberg 2007; McNaught and Wilkinson 2007; Trozzolo and Winslow 1967]. The pathway of free radical chain reactions and termination reactions can be seen in **Figure 1**.

Photo oxidative reaction mechanism:

$$\begin{aligned} \text{R–H} + UV &\longrightarrow \text{R}\cdot \\ \text{R} \cdot + \text{O}_2 &\longrightarrow \text{R–O–O}\cdot \\ \text{R–O–O} \cdot + \text{R} \cdot \text{H} &\longrightarrow \text{R–O–O–H} + \text{R}\cdot \end{aligned}$$

Photo oxidation termination reactions:

$$\begin{aligned} \text{R} \cdot + \text{R}\cdot &\longrightarrow \text{R–R} \\ \text{R–O–O} \cdot + \text{R}\cdot &\longrightarrow \text{R–O–O–R} \end{aligned}$$

Figure 1. Reactions of peroxy radicals.

The cleavage of C–C single bonds breaks the polyethylene into fragments and the polyethylene loses mass. The rate of degradation can be estimated by assuming that every time a C–C bond is cleaved by UV light, a monomer is removed from the original mass of polyethylene. The rate at which the C–C bonds can be cleaved depends on the amount of UV light emitted by the sun, which is 0.0005 watts/m^2 = J/s$\cdot$m^2 [Karam 2005]. Since the energy to break a C–C bond is 5.778×10^{-19} J and there are Avogadro's number (6.022×10^{23}) molecules in a mole, we can find via unit conversion the rate of photodegradation of polyethylene:

$$\frac{1 \text{ C–C bond}}{5.778 \times 10^{-19}\text{ J}} \times \frac{0.0005\text{ J}}{1\text{ s} \cdot 1\text{ cm}^2} \times \frac{1\text{ mole polyethylene}}{6.022 \times 10^{23}\text{ C–C bonds}} \times \frac{28\text{ g monomer polyethylene}}{1\text{ mole polyethylene}} = \frac{4.02 \times 10^{-8}\text{ g monomer polyethylene}}{\text{s} \cdot \text{cm}^2}.$$

General Assumptions

- Since mechanical degradation due to torque on plastic is minimal due to the small size of plastic particles [Tipler 2004], and colliding plastic particles are rare due to low particle density [Moore, Lattin, and Zellers 2005], in our model we neglect mechanical degradation.
- Polyethylene particles float in seawater, since medium-density polyethylene's density is 0.937 g/ml [Chevron Phillips Chemical Company n.d.] and average density of 35 ppt saline seawater at 15°C is 1.0255 g/ml. We neglect water currents.
- The source of UV light is a constant average at sea level in the Pacific Northwest [Karam 2005].
- Polyethylene in the model does not contain UV stabilizers and is medium density.
- Polyethylene is composed of ethylene monomers and the average bond dissociation energy for C–C single bonds is used to predict the energy needed to cleave the C–C bonds [Leeming 1973].
- Only the portions of the plastic fragments that are perpendicular to the UV light are subject to photolytic degradation.
- Only the effective surface area above water can receive UV light.
- The photolytic cleavage of C–C bond in the model is a fast forward reaction ($K_{rxn} \gg 1$) and the reverse reaction is very slow. Immediately after the bond is cleaved, the free radical forms and is quenched by any of the termination reactions that also have a $K_{rxnb} \gg 1$. We also assumed 100% quantum efficiency of these reactions.

High-Transmittance Model

This model considers the degradation of a square prism of polyethylene on a flat surface on land or in water with a very high transmittance of light. One of the faces of the prism faces directly perpendicular to the UV light source. Let K_1 the rate of degradation from the calculation on p. 168, 4.02×10^{-8} (g monomer polyethylene)/s·cm^2. Then

$$\int_0^y \int_0^x \int_0^t K_1 \, dt \, dx \, dy = K_1 txy,$$

due to the simple geometry. The result is a simple linear model of degradation based on time and area.

Consider a square prism with the dimensions of 1 cm×1 cm×2 cm, with an area 1 cm×1 cm exposed to UV radiation. It has an initial mass of 1.87 g [Chevron Phillips Chemical Company n.d.]. After one year of UV exposure, about two-thirds—1.27 g of polyethylene—is lost as a result of photolytic degradation.

Low-Transmittance Model

A primary factor in describing the amount of UV light that a fragment of plastic absorbs is the effective surface area perpendicular to direct sunlight. Since the North Pacific Gyre is a collection of floating debris, we use Archimedes' principle to relate the buoyancy of a piece of plastic to its effective surface area. For simplicity, we neglect the effect of air in the container on buoyancy. Archimedes' principle states that "the buoyant force on a submerged object is equal to the weight of the fluid that is displaced by that object" [Hodanbosi 1996]. In addition, since the plastic is in kinetic equilibrium in the vertical direction, its buoyant force must be equal in magnitude to its weight:

$$F_{\text{buoyant}} = W_{\text{plastic}} = M_{\text{plastic}}\, g = M_{\text{water displaced}}\, g = V_{\text{plastic submerged}}\, d_{\text{water}}\, g,$$

where M is mass, W is weight, V is volume, d is density, and g is acceleration due to gravity. Thus, we have

$$M_{\text{plastic}} = V_{\text{plastic submerged}}\, d_{\text{water}},$$

whose right-hand side can be calculated from the triple integral

$$\int_x^\pi \int_0^{2\pi} \int_{r-h}^h d \cdot \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi,$$

where h is the thickness of the plastic and x is the angle from the zenith to the point on the sphere's surface where the sphere contacts the water level (see **Figure 2**). Setting this integral equal to the total mass of the plastic, we can find x. Trigonometry relates the radius r of the sphere and the angle x to the effective surface area c of the sphere exposed perpendicularly to UV rays, as indicated in **Figure 2**.

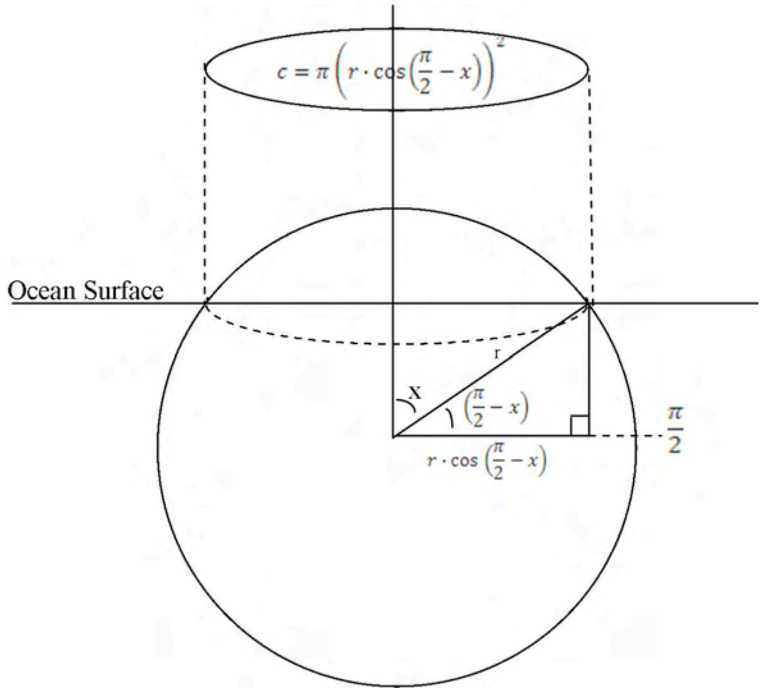

Figure 2. Geometry of a sphere at the ocean's surface.

As shown in **Figure 2**, the effective solar radius is

$$r_e = r \cos\left(\frac{\pi}{2} - x\right).$$

By subtracting the volumes of two concentric spheres with $\Delta r = h$, then multiplying by the density l of the plastic, we get

$$M = \left[\left(\frac{4}{3}\pi r^3\right) - \left(\frac{4}{3}\pi(r-h)^3\right)\right] l.$$

We solve for r:

$$r = \frac{\pi h^2 \pm \sqrt{\pi^2 h^4 - \pi h\left(\frac{4}{3}\pi h^3 - \frac{m}{l}\right)}}{2\pi h}.$$

We define a constant C based on the relationship between the mass (in grams) of plastic and the total bond energies (in joules) within that mass, based on the molar mass and the average bond energy of polyethylene:

$$C \equiv 1\text{ J} \times \frac{1\text{ mole}}{348000\text{ J}} \times \frac{28\text{ g}}{1\text{ mole}} = 8.046 \times 10^{-5}\text{ g/J}.$$

Ultraviolet light is the source of energy in this model. A true empirical value for the amount U (J/cm^2·yr) would need to be measured on site. The product UC has units of g/cm^2·yr. To solve for a total change in grams over a specific time and area, we integrate the term with respect to time and then with respect to area:

$$\int_0^t \int_0^{2\pi} \int_0^{r_e} UC \cdot r\,dr\,d\theta\,dt = \Delta m = UCr_e^2\pi t.$$

Subtracting from the initial mass, we have

$$M_{\text{final}} = m - UVCr_e^2\pi t.$$

Using the Low-Transmittance Model, consider a hollow sphere with thickness 0.0315 cm, radius 5.00 cm, and initial mass 9.145 g. After one year of UV exposure, the loss in mass is 0.190 g of polyethylene. The rate of degradation can be visualized in **Figure 3**.

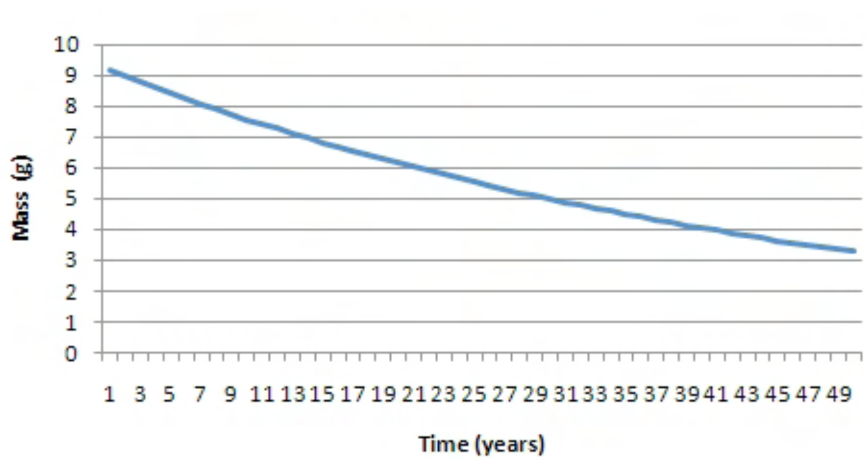

Figure 3. Degradation of mass of polyethylene sphere over time.

Comparisons and Limitations

We develop two models to determine mass lost to photodegradation from a piece of polyethylene plastic exposed to UV light. The High-Transmittance Model models the polyethylene as a rectangular prism; the Low-Transmittance Model models the exposed effective surface area of a hollow sphere of a particular thickness, partially submerged. Depending on the conditions of the water (density and transmittance) and the shape of the object, the models can be modified to describe photodegradation of polyethylene in many other shapes in high- or low- transmittance water.

To increase the accuracy of our models, a few main points need additional research and refinement:

- The value U used for irradiance of UV light in the North Pacific Ocean needs to be verified.
- Mechanical degradation will also take place and should be included.
- The models do not describe the fact that particles tend to converge on a similar size around 3–5 mm [Yamashita and Tanimura 2007].
- Many polyethylene products contain UV stabilizers that increase the longevity of the plastic by inhibiting the free-radical chain reaction [Carey and Sundberg 2007].
- Polyethylene can vary in density. Our model uses medium-density polyethylene (MDPE).
- Plastics are not just on the ocean surface but also at depths up to 100 ft.
- Polyethylene, although very common, is not the only plastic in the North Pacific Gyre.

Discussion of Impacts

Our models describe the rate at which UV light breaks down polyethylene. The process is slow, and there is inconclusive evidence as to whether plastics ever degrade entirely in the Gyre. Plastics are thus a prevalent long-term environmental antagonist. Possible ecologic effects of the accumulation of massive amounts of plastic in the Pacific Ocean Gyre include ingestion of plastic particles by marine life, the disturbance of the transmittance of light below the surface of the water (which may affect many organisms' ability to synthesize energy from photosynthesis), and the distribution of hydrophobic pollutants. Our model relates to the ingestion of plastic particles by marine life because it predicts the mass of fragments at a given time and marine organisms may confuse plastic fragments that are similar in size to their normal food source.

Contributing to the growing problem of plastic pollution in the ocean is the lack of governmental regulation on pollution by cruise ships. During a one-week trip, a typical cruise ship produces 50 tons of garbage. Regulations are tricky though, because international waters do not have well-defined environmental authority structures, and monitoring is minimal [State Environmental Resource Center 2010]. Stronger regulations and monitoring systems are required to decrease the impact of pollution by cruise ships.

Land-based sources contribute up to 80% of marine debris, 65% of which is from post-consumer plastics that were improperly disposed of [Algalita Marine Research Foundation 2009]. This means that the plastics are littered, not just that they are not recycled. Many states have laws against littering, but monitoring efforts need to be improved. Education and monitoring programs may be expensive, but the cost would likely be small when compared to the potential for environmental protection.

Conclusion

We propose two realistic models for the photodegradation of polyethylene. The first model is for a solid chunk of polyethylene either on land or in water with 100% transmittance of light. The second model is more complex and considers a partially submerged hollow sphere of polyethylene that is degraded only over the effective surface area. Our models can accurately describe degradation of partially degraded or intact plastic products, since the initial physical properties (size, mass, etc.) of polyethylene can be varied in both models. The ease of customization and thorough consideration of realistic variables make our models suitable for use.

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Team members Brittany Harris, Kyle Perkins, and Chase Peaslee.

Author's Commentary: The Marine Pollution Problem

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Introduction

Lightweight, inexpensive, durable, and infinitely variable, plastic defines the modern age. However, the very qualities that make plastic indispensable make it an environmental problem.

Today, plastic waste is found throughout the world's oceans, from the coast to the depths to the center of the open sea far from land. The best-known accumulation of trash is the "Great Pacific Garbage Patch," located in the North Pacific Central Gyre (NPCG), a vast swathe of ocean that stretches between the west coast of North America and the east coast of Asia.

Bordered by four major currents, the NPCG slowly rotates clockwise, pulling water in towards the center. Plastic debris from North America and Asia that does not sink or degrade becomes trapped in the NPCG. While larger pieces of plastic such as fishing nets and disposable drink bottles are found in the NPCG, most of the plastic debris is small. This is because as plastic items are exposed to ultraviolet light, they become brittle and are broken into smaller and smaller pieces by the movement of the ocean. This process is known as photodegradation.

The environmental impacts of small pieces of plastic debris are poorly understood. Larger pieces of debris, such as lost fishing gear, can entangle and drown oceanic animals such as seals and turtles. Seabirds and turtles eat plastic debris, and turtle death has been linked to intestinal blockage from plastic bags. However, effects on the organisms at the base of the food chain, such as phytoplankton, zooplankton, and small fishes, remain less studied but may be more

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significant due to the high proportion of plastic debris that is less than 3 mm in diameter. Small particles of plastic are readily ingested by filter-feeding and deposit-feeding invertebrates, and plastic resin pellets accumulate high levels of persistent organic pollutants such as PCBs and DDT. Plastic debris also serves as a "raft" for benthic invertebrates such as barnacles, and has already been responsible for at least one exotic species introduction in the Atlantic.

The "Great Pacific Garbage Patch" has captured the public imagination, leading to a great deal of coverage in the popular media. While detectable amounts of plastic debris were documented in the NPCG as early as 1972, the public awareness of this issue is due in large part to Capt. Charles Moore and the nonprofit organization he founded to combat marine debris, the Algalita Marine Research Foundation. However, relatively little is known about the extent and environmental effects of the plastic debris. Since a robust scientific understanding of the problem is necessary to seeking a solution, the "Great Pacific Garbage Patch" was the topic for this year's problem in the Interdisciplinary Contest in Modeling (ICM)®.

Formulation and Intent of the Problem

The goal of this year's ICM problem was for student teams to model one aspect of the marine debris issue in the NPCG. Because the issue encompasses physical oceanography, ecology, and waste management, there were many potential issues to choose from. Teams were asked to focus on one critical aspect of the problem of oceanic marine debris, and to model the important behavior and phenomena. The end result was to be in the form of a 10-page report to the leader of an expedition setting off to study marine debris.

Suggested tasks included:

- Create a monitoring plan, with the option of including other oceanic gyres such as the North Atlantic Gyre and South Pacific Gyre.
- Characterize the extent, distribution, and density of debris.
- Describe the photodegradation of debris.
- Model the impact of banning polystyrene takeout containers.
- Pursue any relevant topic of the team's choosing that included modeling.

Models were evaluated based on the team's understanding of the nature of the problem, their use of realistic parameters, and their approach to describing either the existing problem or a proposed solution.

This year's ICM problem is based on ongoing research by a variety of organizations and scientists, particularly the Algalita Marine Research Foundation, the Sea Education Association, Project Kaisei, and Scripps Institution of Oceanography. Most work has focused on understanding the abundance and distribution of plastic particles in the NPCG. Future work will focus on the impacts and mitigation of marine debris.

About the Author

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Judges' Commentary: The Outstanding Marine Pollution Papers

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Introduction

The Interdisciplinary Contest in Modeling (ICM)® is an opportunity for teams of students to tackle challenging real-world problems that require a wide breadth of understanding in multiple academic subjects. This year's problem focused on the recently much publicized "Great Pacific Ocean Garbage Patch." Scientific expeditions into the North Pacific Central Gyre (a convergence zone where debris is accumulating) have led to a number of interesting scientific and technical problems. (Hereafter we refer to it simply as "the Gyre.")

Eight judges gathered to select the most successful entries of this challenging competition out of an impressive set of submissions.

The Problem

The primary goal of this year's ICM was to model and analyze one issue associated with the debris problem. Specifically, teams were asked to address several elements with their model:

1. Determine the problem's potential effect on marine ecology.
2. Address government policies and practices that should be implemented to ameliorate the negative effects.
3. Consider needs for future scientific research.
4. Consider the economic aspects of the problem.

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Several examples of issues that the teams might consider were also provided. For the first time, the ICM problem submissions were limited to a maximum length of 10 pages.

Overall, the judges were impressed both by the strength of many of the submissions and by the variety of issues they chose to model. In many cases, very different modeling approaches were used to address the same issue; as a result, this year's problem led to the greatest variety in submissions to the ICM in memory.

Judges' Criteria

The framework used to evaluate submissions is described below, and it remained very similar to the criteria used in 2009. However, the 10-page limit for the submissions had an impact on the importance of the final criterion concerning communication. Teams that dramatically exceeded the limit were not considered for the Outstanding paper category.

Executive Summary

It was important that a team succinctly and clearly explain the highlights of its submission. The executive summary needed to include the issue that the team chose to address and the modeling approach(es) used for analysis. Further, the summary needed to answer the most pressing questions posed in the problem statement, namely, the effect on the marine ecology, economic aspects of the issue, and how to ameliorate the problem. Truly outstanding papers were those that communicated their approach and recommendations in well-connected and concise prose.

Domain Knowledge and Science

The problem this year was particularly challenging for students in terms of the science. To address the requirements effectively, teams needed first to establish an ecological frame of reference. Many teams were able to do this reasonably well; teams that excelled clearly did a great deal of research. Often, what distinguished the top teams was the ability not just to describe the garbage patch in a single section of the paper, but also to integrate this domain knowledge throughout the modeling process.

A second important facet of the problem was the ability to understand economic issues associated with the chosen problem. Many teams created reasonable models but unfortunately never tied them to the economic discussion.

Modeling and Assumptions

For teams that chose to focus on describing and understanding the distribution of plastic in the Gyre, simulation was a popular approach to the problem. Differential equations were probably the most prevalent models used (in a wide variety of ways). Often, the models appeared appropriate but lacked any discussion of important assumptions. Additionally, some papers lacked a reasonable discussion of model development. Finally, the very best papers not only formulated the models well but also were able to use the models to produce meaningful results to address the problem and to make recommendations.

Solution/Recommendation

Perhaps the most distinct difference between the best papers and others was the ability to utilize the team's models to develop or propose an actual solution to the problem. For example, a team might effectively model the distribution of plastic in the Gyre in one section of the paper. A completely independent section would then provide recommendations for remediating the plastic problem but without ever making use of the model or the model results.

Analysis/Reflection

Successful papers utilized the models developed in early sections of the paper to draw conclusions about the important issues in addressing problems with the garbage patch and addressed how assumptions made in the model could impact the solution and recommendation. In the best papers, trade-offs were discussed and—in truly exceptional cases—some sensitivity analysis was conducted to identify potential issues with the solutions presented.

Communication

The challenges of the modeling in this problem and the page limit may have contributed to the difficulty that many teams had in clearly explaining their solutions. Papers that were clearly exposited distinguished themselves significantly, emphasizing that *it is not only good science that is important, but also the presentation of the ideas*. In some cases, teams spent all their space describing the modeling and never presented important results, conclusions, or recommendations. On the other hand, some teams never really explained their models, making it difficult to judge the validity of their results. Balancing the need to present enough work to fully answer the question, while keeping to the 10-page limit, was clearly a challenge in this year's contest.

Discussion of the Outstanding Papers

The judges were most impressed by papers that offered unique and innovative ideas. Three of the four Outstanding papers this year took very novel approaches to the problem and issues. The fourth paper was representative of what many teams opted to do but was more clearly articulated and the modeling more complete than others attempting the same approach.

- The Beijing Jiaotong University submission "A New Method for Pollution Abatement: Different Solutions to Different Types" was unique in looking at the pollution problem from a risk-analysis perspective. Using multi-attribute decision theory, this team developed a model to rank the types of debris in the Gyre by their level of "risk." The modeling was complete and well explained. The team also then used the results of the model to propose a strategy for reducing the debris problem. The judges were a bit troubled by the conclusions of the paper—considering types of debris as significantly different may not be realistic—but the results followed from the assumptions in the Moore et al. [2001] paper provided with the problem statement.

- The paper from Lawrence University, "Size-Classified Plastic Concentration in the Ocean," was perhaps the most clearly written and thorough among the Outstanding papers. The team developed a model to classify the plastics in the Gyre. Their models looked at many factors, physical and chemical, to determine size and concentration of the debris. In addition to the very thoroughly explained modeling efforts, the paper ends with sections discussing some of the limitations of the model and then some very specific conclusions and recommendations that stem directly from the model itself.

- The Hangzhou Dianzi University submission, "Quantitative Marine Debris Impacts and Evaluation of Ocean System," became known among the judges as the "monk seal" paper. This team took a unique approach to the problem by studying the impacts of ocean debris on a single species, the Hawaiian monk seal. A "grey model" and time-series approach was utilized to consider trends for the monk seal, and then an analytical hierarchy process (AHP) used to try to quantify impacts of debris. The paper was not the strongest in terms of how well the team explained and presented their results, but the clever approach to the problem appealed to the judges.

- The final Outstanding paper, "Shedding Light on Marine Pollution" by the team from Carroll College, considered the specific issue of photodegradation of polyethylene floating in seawater. The team developed models for this process and very clearly articulated their approach and assumptions. This paper was among the best at presenting the modeling efforts and also noteworthy for the science (namely, chemistry) utilized in the process. The judges would have liked to see a bit more in the conclusions to explain the importance of the modeling results and ties to policy, but they were very impressed by the focus and clarity of this paper.

Conclusion and Recommendations for Future Participants

The judges really enjoyed reading the submissions for this year's ICM contest. All teams deserve congratulations for the tremendous work done in a very short period of time and on a very difficult problem. The judging was, as a result, both pleasurable and challenging.

One issue worthy of mention that arises each year is that of proper scholarship in utilizing other sources in writing a paper. In researching the science for these complicated problems, teams naturally use information and ideas taken from a variety of resources. This is acceptable as long as those ideas are clearly documented in the paper. Copying the exact words from other papers should be minimized; but, if done, the words need to be placed in quotation marks, so that it is clear that it is not original to the authors.

Several key points from this year's contest judging emerged in determining the very best submissions. These are thoughts that may be useful to future ICM competitors.

- Many teams failed to select a single issue to model and analyze, instead trying to address all of the ideas for issues proposed in the problem statement. In some cases, these teams appeared to have done a remarkable amount of excellent modeling. However, it was simply impossible for them to present all this work in such a short report or to do justice to such a wide array of problems in such a short time period. The teams that were most successful clearly shaped the problem that they would address. When presented with a problem with a very large scope, narrowing the focus is critical.
- Judges were impressed with those who took a unique perspective on the problem. That could be either a different modeling approach (perhaps using a particular science, such as chemistry) or considering a different aspect of the problem (one example was a team that looked at how the plastic gets into the ocean). Original thought, as long as it was grounded in solid research, was cherished.
- Finally, a well-written and integrated report that reads well from start to finish is critical. The sections of the report should follow naturally and not appear as completely separate sections or ideas. The conclusions and recommendations, in particular, should be clearly tied to the modeling work presented.

Reference

Moore, C.J., S.L. Moore, M.K. Leecaster, and S.B. Weisberg. 2001. A comparison of plastic and plankton in the North Pacific central gyre. *Marine Pollution Bulletin* 42: 1297–1300.

About the Author

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