习题 9.2

1.(1) $\frac{y}{1+y^2}dy = \frac{dx}{x(1+x^2)}$. $\frac{1}{2}\ln(1+y^2) = \int(\frac{1}{x} - \frac{x}{1+x^2})dx = \ln|x| - \frac{1}{2}\ln(1+x^2)+C$

$\ln(1+y^2) = \ln x^2 - \ln(1+x^2)+C \Rightarrow (1+x^2)(1+y^2) = e^C x^2 = C^* x^2$. $C^*>0$.

(3) $\frac{1}{\sqrt{1-y^2}}dy = \frac{1}{\sqrt{1+x^2}}dx$. $\arcsin y = \ln(\underbrace{x+\sqrt{1+x^2}}_{>0})+C = \ln e^C(x+\sqrt{1+x^2}) = \ln C^*(x+\sqrt{1+x^2})$ $C^*>0$.

(7) $\frac{dy}{dx} = -\frac{2x^2+y^2}{2xy+3y^2} = -\frac{2+u^2}{2u+3u^2}$. $u=\frac{y}{x}$

$u' = \frac{h(u)-u}{x} = \frac{1}{x}(-\frac{2+3u^2+3u^3}{2u+3u^2})$, $\frac{2u+3u^2}{2+3u^2+3u^3}du = -\frac{1}{x}dx$

$\frac{1}{3}\ln|2+3u^2+3u^3| = -\ln|x|+C$. $|2+3u^2+3u^3| = C|x|^{-3}$. $C>0$. $2+3u^2+3u^3 = Cx^{-3}$. C任意

(8) $y'=(x+y+2)^2$, $z=x+y+2$, $z'=1+y'=1+z^2$. $\frac{dz}{1+z^2}=dx$. $\arctan z = x+C$

(10) $\frac{dy}{dx} = \frac{-(x-2y+5)}{2x-y+4}$. $\Delta = \begin{vmatrix}-1&2\\2&-1\end{vmatrix}\neq0$. $(x_0,y_0)=(-1,2)$, $u=x_0+1$. $v=y_0-2$

$\frac{dv}{du} = \frac{-u+2v}{2u-v} = \frac{-1+2\frac{v}{u}}{2-\frac{v}{u}}$ 令 $\frac{v}{u} = z$. $z' = \frac{1}{u}(h(z)-z) = \frac{1}{u}\cdot\frac{-1+2z-2z+z^2}{2-z} = \frac{1}{u}\cdot\frac{z^2-1}{2-z}$

$z=\pm1$ 是特解. $\frac{(2-z)dz}{z^2-1} = \frac{1}{u}du \Rightarrow [-\frac{1}{1+z} - \frac{1}{1-z} - \frac{2z}{2(z^2-1)}]dz = \frac{1}{u}du$

$-\ln|1+z| + \ln|1-z| - \frac{1}{2}\ln|z^2-1| = \ln|u|+C$

$-\ln(1+z)^2 + \ln(1-z)^2 - \ln|z^2-1| = \ln u^2 + C$

$|z^2-1| = C\cdot\frac{(1-z)^2}{(1+z)^2}\cdot\frac{1}{u^2}$. $C>0$. $\Rightarrow$ $z^2-1 = C\cdot\frac{(1-z)^2}{(1+z)^2}\cdot\frac{1}{u^2}$ $C\neq0$.

又 $z=\pm1$ 是特解. $z^2-1 = C\frac{(1-z)^2}{(1+z)^2}\cdot\frac{1}{u^2}$. $C\in R$. $\frac{(z+1)^3}{z-1} = C\frac{1}{u^2}$. $(u+v)^3 = C(u-v)$

8. $\frac{dR}{dt} = -kR$. $k>0$. $R=Ce^{-kt}$

令 $t=0$, $R_0=C$ $t=1600$, $R_{1600}=R_0e^{-1600k} = \frac{1}{2}R_0$. $e^{-1600k}=\frac{1}{2}$. $k=\frac{\ln 2}{1600}$

$R_0=1(g)$. $t=1$. $R_1=e^{-k}$. $R_0-R_1 = 1-e^{-k} = 1-e^{-\frac{\ln 2}{1600}} = 1-2^{-\frac{1}{1600}} \approx 0.00043(g)$

9. $u=tx$. $\int_0^x \varphi(u)\cdot\frac{1}{x}du = n\varphi(x)$. $\int_0^x\varphi(u)du = nx\varphi(x)$. 两边求导: $\varphi(x) = n\varphi(x)+nx\varphi'(x)$

$\frac{d\varphi}{\varphi} = \frac{1-n}{nx}dx$. $\ln|\varphi| = \frac{1-n}{n}\ln|x|+C$. $|\varphi| = |x|^{\frac{1-n}{n}}\cdot C_1$. $C_1>0 \Rightarrow \varphi = |x|^{\frac{1-n}{n}}\cdot C_1$. $C\in R$

13.(1) 令 $z=y'$. $x^2z'=z^2$. $\frac{dz}{z^2} = \frac{dx}{x^2}$. $-\frac{1}{z} = -\frac{1}{x} + C_1$. $z=\frac{x}{1+C_1x}$

当 $C_1=0$. $z=x\Rightarrow y'=x$. $y=\frac{x^2}{2}+C_2$. 当 $C_1\neq0$. $y'=\frac{x}{1+C_1x} = \frac{1}{C_1} - \frac{1}{C_1(1+C_1x)}$

$y=\frac{x}{C_1} - \frac{1}{C_1^2}\ln|1+C_1x|+C_2 \Rightarrow C_1x - C_1^2y = \ln|1+C_1x|+C_2$. 另还有特解 $z\equiv0\Rightarrow y\equiv C$

(2) 令 $p=y'$. $p^2+2y\cdot p\cdot p'=0$. $p\equiv0\Rightarrow y\equiv C$.

$\frac{dp}{p} = -\frac{dy}{2y}$. $\ln|p| = -\frac{1}{2}\ln|y|+C$ $|p|=|y|^{-\frac{1}{2}}C$. $C>0$, $p=C|y|^{-\frac{1}{2}}$ C任意.

$y'=C|y|^{-\frac{1}{2}}$. $|y|^{\frac{1}{2}}dy = Cdx$. $\frac{2}{3}|y|^{\frac{3}{2}}\cdot\mathrm{sgn}(y) = Cx+C_1$. $\frac{2}{3}|y|^{\frac{3}{2}} = Cx+C_1$

$|y|^{\frac{3}{2}} = Cx+C_1 = C(x+C_1)$ $|y| = C^{\frac{2}{3}}(x+C_1)^{\frac{2}{3}} = C_2(x+C_1)^{\frac{2}{3}}$. $C_2\geq0$.

$y=C_2(x+C_1)^{\frac{2}{3}}$. $C_1, C_2\in R$

14. (3). $\frac{\partial P}{\partial y}=e^y$. $\frac{\partial Q}{\partial x}=e^y$. $u(x,y)=xe^y+\varphi(y)$. $\frac{\partial u}{\partial y}=xe^y+\varphi'(y)$

$\varphi'(y)=-2y$. $\varphi(y)=-y^2+C$. $u=xe^y-y^2+C=C'$. $xe^y-y^2\equiv C$.

(6) $\frac{\partial P}{\partial y}=-b$. $\frac{\partial Q}{\partial x}=b$ 不是.

(7). $\frac{\partial P}{\partial y}=e^x+2y$. $\frac{\partial Q}{\partial x}=e^x+2y$. $u=ye^x+2e^x+xy^2+\varphi(y)$, $\frac{\partial u}{\partial y}=e^x+2xy+\varphi'(y)$

$\varphi(y)\equiv 0$. $ye^x+2e^x+xy^2\equiv C$

15. (1) $dx-dy=\frac{1}{(x+y)^2}(dx+dy)$. $d(x-y)=-d\frac{1}{x+y}$. $x-y=-\frac{1}{x+y}+C$

(4) $(x^2+y^2)dx+ydx-xdy=0$. $(x^2+y^2)dx+d\left(\frac{x}{y}\right)\cdot y^2=0$

$dx+d\left(\frac{x}{y}\right)\cdot\frac{1}{1+\left(\frac{x}{y}\right)^2}=0$. $dx+d\left(\frac{x}{y}\right)\cdot d\arctan(u)=0$.

$x+\arctan\frac{x}{y}\equiv C$

16. (3). $y(x+1)dx+x(y+1)dy=0$. $xy\neq 0$.

$\frac{\partial P}{\partial y}=x+1$. $\frac{\partial Q}{\partial x}=y+1$

两边同除 xy. $\left(1+\frac{1}{x}\right)dx+\left(1+\frac{1}{y}\right)dy=0$. $x+\ln|x|+y+\ln|y|=C$

(6) $e^x dx+(e^x\cot y+2y\cos y)dy=0$.

$\frac{\partial M}{\partial y}=0$. $\frac{\partial N}{\partial x}=e^x\cot y$. $\frac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M}=\cot y$. $\mu=e^{\ln\sin y}=\sin y$

$\sin y\cdot e^x dx+\sin y\,(e^x\cot y+2y\cos y)dy=0$.

$u=\sin y\cdot e^x+\varphi(y)$. $\frac{\partial u}{\partial y}=e^x\cos y+\varphi'(y)=e^x\cos y+y\sin 2y$

$\varphi'(y)=y\sin 2y$. $\varphi(y)=-\frac{1}{2}y\cos 2y+\frac{1}{4}\sin 2y$

或观察法：两边乘 $\sin y$.

$e^x\sin y\,dx+e^x\cos y\,dy+2y\sin y\cos y\,dy=0$

$d(\sin y\,e^x)+y\sin 2y\,dy=0$

$d(\sin y\,e^x)+d\left(-\frac{1}{2}y\cos 2y+\frac{1}{4}\sin 2y\right)$

习题9.4

2. 设 φ_1, φ_2 是 $y''+q(x)y=0$ 的两个解.

$\varphi_1''=-q(x)\varphi_1$. $\varphi_2''=-q(x)\varphi_2$.

$W(x)=\varphi_1\varphi_2'-\varphi_1'\varphi_2$. $W'(x)=\varphi_1'\varphi_2'+\varphi_1\varphi_2''-\varphi_1''\varphi_2-\varphi_1'\varphi_2'$

$=\varphi_1\varphi_2''-\varphi_1''\varphi_2=-q(x)\varphi_1\varphi_2+q(x)\varphi_1\varphi_2=0.$

$W(x)\equiv C$.

3. 设 φ_1, φ_2 是齐次方程 $y''+p(x)y'+q(x)y=0$ 的线性无关解.

则 $\varphi(x)$ 可表示为 $C_1\varphi_1(x)+C_2\varphi_2(x)$, C_1, C_2 为某固定常数.(通解包含一切解)

若 $\exists x_0$, $\varphi(x_0)=0$. $\Rightarrow C_1\varphi_1(x_0)+C_2\varphi_2(x_0)=0$.

若 $\varphi'(x_0)=0 \Rightarrow C_1\varphi_1'(x_0)+C_2\varphi_2'(x_0)=0$

由于 $\varphi(x)$ 是非零解. C_1, C_2 一定不同时为零. 则上述方程组有非零解.

$\Rightarrow \begin{vmatrix}\varphi_1(x_0) & \varphi_2(x_0)\\ \varphi_1'(x_0) & \varphi_2'(x_0)\end{vmatrix}=0$. 总言之. $\exists x_0$. $W(x_0)=0$ 与 φ_1, φ_2 线性无关矛盾.

4. 若存在 x_0 是 φ_1, φ_2 的公共零点, 即 $\varphi_1(x_0)=0$ $\varphi_2(x_0)=0$.

$$\begin{vmatrix}\varphi_1(x_0) & \varphi_2(x_0)\\ \varphi_1'(x_0) & \varphi_2'(x_0)\end{vmatrix}=\begin{vmatrix}0 & 0\\ \varphi_1'(x_0) & \varphi_2'(x_0)\end{vmatrix}=0. \text{ 也即存在 } x_0. W(x_0)=0$$

与 φ_1, φ_2 线性无关矛盾.

习题 9.6

1. $y''+3y'+2y=\frac{1}{e^x+1}$

$y''+3y'+2y=0 \Rightarrow \lambda^2+3\lambda+2=0$, $\lambda=-1$, $\lambda=-2$ $\quad C_1e^{-x}+C_2e^{-2x}$

$$\begin{cases} C_1'e^{-x}+C_2'e^{-2x}=0 \\ -C_1'e^{-x}-2C_2'e^{-2x}=\frac{1}{e^x+1} \end{cases} \Rightarrow \begin{cases} C_1'(x)=\frac{e^x}{e^x+1} \\ C_2'(x)=-\frac{e^{2x}}{e^x+1} \end{cases} \Rightarrow \begin{cases} C_1(x)=\ln(1+e^x)+C_1 \\ \end{cases}$$

$C_2'(x)=-e^x+\frac{e^x}{1+e^x}$, $\quad C_2(x)=-e^x+\ln(1+e^x)+C_2$

$y=e^{-x}\ln(1+e^x)+C_1e^{-x}-e^{-x}+e^{-2x}\ln(1+e^x)+C_2e^{-2x}$

$=(e^{-x}+e^{-2x})\ln(1+e^x)+\underbrace{(C_1-1)}_{C_1\ (仍记为)}e^{-x}+C_2e^{-2x}$

2. $y''+y=\frac{1}{\sin x}$.

$\lambda^2+1=0$, $\lambda=\pm i$. $\quad C_1\cos x+C_2\sin x$

$$\begin{cases} C_1'\cos x+C_2'\sin x=0 \\ -C_1'\sin x+C_2'\cos x=\frac{1}{\sin x} \end{cases} \Rightarrow \begin{cases} C_1'=-1 \\ C_2'=\operatorname{ctg} x \end{cases} \Rightarrow \begin{cases} C_1(x)=-x+C_1 \\ C_2(x)=\ln|\sin x|+C_2 \end{cases}$$

$y=(-x+C_1)\cos x+(\ln|\sin x|+C_2)\sin x=-x\cos x+\sin x\ln|\sin x|+C_1\cos x+C_2\sin x$

3. $y''+4y=2\tan x$

$\lambda^2+4=0$, $\lambda=\pm 2i$. $\quad C_1\cos 2x+C_2\sin 2x$

$$\begin{cases} C_1'\cos 2x+C_2'\sin 2x=0 \\ -2C_1'\sin 2x+2C_2'\cos 2x=2\tan x \end{cases} \Rightarrow \begin{cases} C_1'=-2\sin^2 x=-1+\cos 2x \\ C_2'=\tan x\cos 2x=\sin 2x-\tan x \end{cases} \Rightarrow \begin{cases} C_1(x)=-x+\frac{1}{2}\sin 2x+C_1 \\ C_2(x)=-\frac{1}{2}\cos 2x+\ln|\cos x|+C_2 \end{cases}$$

$y=(-x+\frac{1}{2}\sin 2x+C_1)\cos 2x+(-\frac{1}{2}\cos 2x+\ln|\cos x|+C_2)\sin 2x$

$=-x\cos 2x+\sin 2x\ln|\cos x|+C_1\cos 2x+C_2\sin 2x$

4. $y''+y=2\sec^3 x$. $\quad C_1\cos x+C_2\sin x$

$$\begin{cases} C_1'\cos x+C_2'\sin x=0 \\ -C_1'\sin x+C_2'\cos x=2\sec^3 x \end{cases} \Rightarrow \begin{cases} C_1'=-2\sec^3 x\cdot\sin x=\frac{-2\sin x}{\cos^3 x} \\ C_2'=2\sec^3 x\cdot\cos x=2\sec^2 x \end{cases} \Rightarrow \begin{cases} C_1(x)=-\sec^2 x+C_1 \\ C_2(x)=2\tan x+C_2 \end{cases}$$

$y=(-\sec^2 x+C_1)\cos x+(2\tan x+C_2)\sin x=-\sec x+2\tan x\sin x+C_1\cos x+C_2\sin x$

$=C_1\cos x+C_2\sin x-\frac{1-2\sin^2 x}{\cos x}=C_1\cos x+C_2\sin x-\frac{\cos 2x}{\cos x}$

5. $x^2y''-4xy'+6y=0$

$x=e^t$. $\quad y_x'=y_t'\cdot\frac{1}{x}$. $\quad y_x''=(y_t''-y_t')\frac{1}{x^2}$

$(t=\ln|x|)$ $\quad \frac{dt}{dx}=\frac{1}{x}$

$e^t=|x|$

$y_t''-y_t'-4y_t'+6y_t=0$. $\lambda^2-5\lambda+6=0$. $\lambda=2$. $\lambda=3$

$e^{2t}, e^{3t} \Rightarrow C_1x^2+C_2x^3$

6. $x^2y''-xy'-3y=0$. $y_t''-y_t'-y_t'-3y=0$. $\lambda^2-2\lambda-3=0$ $\lambda=-1, \lambda=3$

$C_1\frac{1}{x}+C_2x^3$

7. $x^3y'''+xy'-y=0$. $y_x'''=[(y_t'''-y_t'')\frac{1}{x^2}-2(y_t''-y_t')\frac{1}{x^3}x]\cdot\frac{1}{x}=(y_t'''-3y_t''+2y_t')\frac{1}{x^3}$

$y_t'''-3y_t''+2y_t'+y_t'-y=0 \Rightarrow y_t'''-3y_t''+3y_t'-y=0$ $\lambda^3-3\lambda^2+3\lambda-1=0$, $(\lambda-1)^3=0$

$C_1e^t+C_2te^t+C_3t^2e^t \Rightarrow x(C_1+C_2\ln|x|+C_3(\ln|x|)^2)$

8. $x^2y''+xy'+4y=10$

特解: $y=\frac{5}{2}$

$y_t''-y_t'+y_t'+4y=10$. $y_t''+4y=10$. $\lambda^2+4=0$. $\lambda=\pm 2i$

$C_1\cos 2t+C_2\sin 2t+\frac{5}{2} \Rightarrow C_1\cos(\ln x^2)+C_2\sin(\ln x^2)+\frac{5}{2}$

习题 9.5

1. (1) $y''-3y'+2y=0$. $\lambda^2-3\lambda+2=0$. $\lambda=1, \lambda=2$. $C_1e^x+C_2e^{2x}$

(3) $y''+6y'+9y=0$ $\lambda^2+6\lambda+9=0$. $\lambda=-3$. $C_1e^{-3x}+C_2xe^{-3x}$

(5) $y''-y'+2y=0$ $\lambda^2-\lambda+2=0$. $\lambda=\frac{1\pm\sqrt{7}i}{2}$. $C_1e^{\frac{x}{2}}\sin\frac{\sqrt{7}}{2}x+C_2e^{\frac{x}{2}}\cos\frac{\sqrt{7}}{2}x$

3. (7) $y''-y=2e^x-x^2$. $\lambda^2-1=0$. $\lambda=\pm 1$. $C_1e^x+C_2e^{-x}$.

$y=Axe^x$: $y''=2Ae^x+Axe^x$. $2Ae^x=2e^x \Rightarrow A=1$. $y_1=xe^x$

$y=ax^2+bx+c$. $y''=2a$. $2a-ax^2-bx-c=-x^2$. $y_2=x^2+2$

$y=C_1e^x+C_2e^{-x}+xe^x+x^2+2$

(8) $y''+y'=\sin 4x-2\sin 2x$. $\lambda^2+\lambda=0$. $\lambda=0, -1$ $C_1+C_2e^{-x}$

$y=A\sin 4x+B\cos 4x$. $y'=4A\cos 4x-4B\sin 4x$. $y''=-16A\sin 4x-16B\cos 4x$

$$\begin{cases}-16A-4B=1\\ 4A-16B=0\end{cases} \quad \begin{cases}A=-\frac{1}{17}\\ B=-\frac{1}{68}\end{cases} \Rightarrow y_1=-\frac{1}{17}\sin 4x-\frac{1}{68}\cos 4x$$

$y=A\sin 2x+B\cos 2x$ $y'=2A\cos 2x-2B\sin 2x$. $y''=-4A\sin 2x-4B\cos 2x$

$$\begin{cases}-4A-2B=-2\\ 2A-4B=0\end{cases} \Rightarrow \begin{cases}A=\frac{2}{5}\\ B=\frac{1}{5}\end{cases} \Rightarrow y_2=\frac{2}{5}\sin 2x+\frac{1}{5}\cos 2x$$

$y=C_1+C_2e^{-x}-\frac{1}{17}\sin 4x-\frac{1}{68}\cos 4x+\frac{2}{5}\sin 2x+\frac{1}{5}\cos 2x$

4. (2) $y''+y'=x+2$, $\lambda^2+\lambda=0$, $\lambda=0,-1$. $x(ax+b)$

(4) $y''-y=e^x(x^2-1)$, $\lambda^2-1=0$, $\lambda=\pm 1$. $x(ax^2+bx+c)e^x$

(5) $y'''+3y''+3y'+y=e^{-x}(x-5)$

$\lambda^3+3\lambda^2+3\lambda+1=0$, $(\lambda+1)^3$ $\lambda=-1$, $x^3(ax+b)e^{-x}$