Discrete Mathematics: Lecture 6

- Last time:
  - Chap 8.3: Divide-and-conquer algorithms and recurrence relations
- Today:
  - Chap 8.5: Inclusion-exclusion
  - Chap 8.6: Applications of inclusion-exclusion
- Assignment 2 due next week

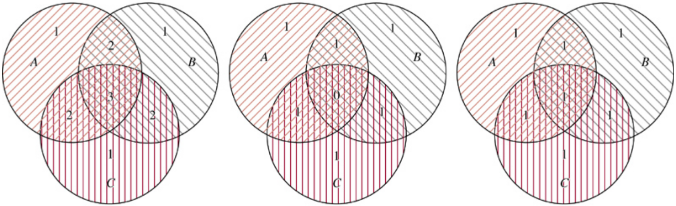
Review of last time

- Divide-and-conquer algorithms
- Divide-and-conquer recurrence relations
- Master Theorem

Union of two sets

- $|A \cup B| = |A| + |B| - |A \cap B|$
- How many positive integers not exceeding 1000 are divisible by 7 or 11?
- 1807 freshmen, 453 takes CS, 567 takes Math, 299 takes both CS and Math. How many students take neither CS or Math?

Union of three sets



(a) Count of elements by $|A|+|B|+|C|$

(b) Count of elements by $|A|+|B|+|C|-|A\cap B|-|A\cap C|-|B\cap C|$

(c) Count of elements by $|A|+|B|+|C|-|A\cap B|-|A\cap C|-|B\cap C|+|A\cap B\cap C|$

- $|A\cup B\cup C| = |A|+|B|+|C|-|A\cap B|-|A\cap C|-|B\cap C|+|A\cap B\cap C|$
- Example: Spanish: 1232, French: 879, Russian: 114; Spanish and French: 103, Spanish and Russian: 23, French and Russian: 14; Spanish, French, or Russian: 2092. How many students have taken all three languages?

The principle of inclusion-exclusion

Theorem: Let $A_1, A_2, \ldots, A_n$ be finite sets. Then

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = \sum_{1 \le i \le n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j|$$

$$+ \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| - \ldots + (-1)^{n+1} |A_1 \cap A_2 \cap \ldots \cap A_n|$$

There is one term for each non-empty subset of $\{1, 2, \ldots, n\}$, hence there are $2^n - 1$ terms.

Proof: We show that each element a of $A_1 \cup A_2 \cup \ldots \cup A_n$ is counted exactly once.
Suppose a belongs to exactly r sets of $A_1, A_2, \ldots, A_n$.
Then the number of times a is counted is:
$C(r,1) - C(r,2) + \ldots + (-1)^{r+1} C(r,r)$

Another form of inclusion-exclusion

- Can be used to solve problems asking for the number of elements in a set A that have none of n properties $P_1, P_2, \ldots, P_n$.
- Let A_i be the subset containing elements that satisfy P_i.
- Let $N(P_{i_1} P_{i_2} \ldots P_{i_k})$ denote $|A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}|$.
- Let N denote $|A|$, and $N(P_1' P_2' \ldots P_n')$ denote the number of elements with none of the properties $P_1, P_2, \ldots, P_n$
- Then $N(P_1' P_2' \ldots P_n') = N - |A_1 \cup A_2 \cup \ldots \cup A_n| =$

$$N - \sum_{1 \le i \le n} N(P_i) + \sum_{1 \le i < j \le n} N(P_i P_j)$$

$$- \sum_{1 \le i < j < k \le n} N(P_i P_j P_k) + \ldots + (-1)^n N(P_1 P_2 \ldots P_n)$$

Examples

- The number of solutions of $x_1 + x_2 + x_3 = 11$, where x_1, x_2, and x_3 are non-negative integers with $x_1 \leq 3$, $x_2 \leq 4$, and $x_3 \leq 6$.
- The number of primes not exceeding 100
- The number of onto functions from a set with 6 elements to a set with 3 elements
- A derangement is a permutation of objects that leaves no objects in its original position. For example, 21453 is a derangement of 12345, but 21543 is not. The number of derangements of a set with 4 elements