中山大學 本科生考试草稿纸 2017/7-25.

警示 《中山大学授予学士学位工作细则》第七条："考试作弊者不授予学士学位。"

P.165.20 试证：双纽线：$r^2=2a^2\cos 2\theta \quad (a>0)$ 的全长可表为 $L=4\sqrt{2}a\int_0^1\frac{dx}{\sqrt{1-x^4}}$.

证：见 P.165.8 图形，$r=\sqrt{2}a\cdot\sqrt{\cos 2\theta}$

$$r'(\theta)=\sqrt{2}a\cdot\frac{-2\sin 2\theta}{2\sqrt{\cos 2\theta}}=-\sqrt{2}a\cdot\frac{\sin 2\theta}{\sqrt{\cos 2\theta}}$$

$$\sqrt{r^2+r'^2}=\sqrt{2a^2\cos 2\theta+2a^2\frac{\sin^2 2\theta}{\cos 2\theta}}=\sqrt{2}a\cdot\frac{\sqrt{\sin^2 2\theta+\cos^2 2\theta}}{\sqrt{\cos 2\theta}}=\frac{\sqrt{2}a}{\sqrt{\cos 2\theta}}$$

$$L=2\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\sqrt{2}a\cdot\frac{d\theta}{\sqrt{\cos 2\theta}}=4\sqrt{2}a\int_0^{\frac{\pi}{4}}\frac{d\theta}{\sqrt{\cos 2\theta}}=2\sqrt{2}a\int_0^{\frac{\pi}{4}}\frac{1}{\sqrt{\cos 2\theta}}d(2\theta)$$

$$=2\sqrt{2}a\cdot\int_0^{\frac{\pi}{2}}\frac{dt}{\sqrt{\cos t}}$$

令 $\sqrt{\cos t}=x$，则 $\cos t=x^2$

$t=0,\ x=1;\quad t=\frac{\pi}{2},\ x=0.$

$t=\arccos x^2,\quad dt=-\frac{2x\,dx}{\sqrt{1-x^4}}$

$$=2\sqrt{2}a\cdot\int_1^0\frac{1}{x}\cdot\frac{-2x}{\sqrt{1-x^4}}dx$$

$$=4\sqrt{2}a\int_0^1\frac{dx}{\sqrt{1-x^4}}.$$

P.165.21 求抛物线：$y=1+\frac{x^2}{4}\ (0\le x\le 2)$ 绕x轴旋转所得的旋转体的侧面积。

解：$y'=\frac{x}{2}$，$\sqrt{1+y'^2}=\sqrt{1+\frac{x^2}{4}}$，令 $\frac{x}{2}=\tan u$，则 $dx=2\sec^2 u\,du$

$$F=2\pi\int_0^2 y\cdot\sqrt{1+y'^2}dx=2\pi\int_0^2(1+\frac{x^2}{4})\cdot\sqrt{1+\frac{x^2}{4}}dx=2\pi\int_0^2(\sqrt{1+\frac{x^2}{4}})^3dx$$

$$=2\pi\int_0^{\frac{\pi}{4}}\sec^3 u\cdot 2\sec^2 u\,du=4\pi\int_0^{\frac{\pi}{4}}\frac{du}{\cos^5 u}$$

$$=4\pi\left[\frac{\sin u}{(5-1)\cos^4 u}\Big|_0^{\frac{\pi}{4}}+\frac{5-2}{5-1}\int_0^{\frac{\pi}{4}}\frac{du}{\cos^3 u}\right]$$

$$=4\pi\left[\frac{\frac{1}{\sqrt{2}}}{4\cdot\frac{1}{4}}+\frac{3}{4}\left(\frac{\sin u}{2\cos^2 u}\Big|_0^{\frac{\pi}{4}}+\frac{1}{2}\int_0^{\frac{\pi}{4}}\frac{du}{\cos u}\right)\right]$$

$$=4\pi\left[\frac{1}{\sqrt{2}}+\frac{3}{4}\left(\frac{1}{2}\cdot\frac{2}{\sqrt{2}}+\frac{1}{2}\ln(\sec u+\tan u)\Big|_0^{\frac{\pi}{4}}\right)\right]$$

$$=4\pi\left[\frac{1}{\sqrt{2}}+\frac{3}{4\sqrt{2}}+\frac{3}{8}\ln(\sqrt{2}+1)\right]=4\pi\left(\frac{4\sqrt{2}+3\sqrt{2}}{8}+\frac{3}{8}\ln(1+\sqrt{2})\right)$$

$$=\frac{\pi}{2}\left[7\sqrt{2}+3\ln(1+\sqrt{2})\right]$$