Discrete Mathematics: Lecture 3

- Last time:
  - Chap 6.3: Permutations and combinations
  - Chap 6.4: Binomial coefficients and identities
- Today:
  - Chap 6.5: Generalized permutations and combinations
  - Chap 6.6: Generating permutations and combinations
- Assignment 1 due next week

Review of last time

- Permutations and combinations
- Combinatorial proofs
- The binomial theorem
- Pascal's identity

Permutations with repetition

- Example: how many strings of length r can be formed from the English alphabet?
- Theorem: The number of r-permutations of a set of n objects with repetition allowed is n^r.

Combinations with repetition

Theorem: The number of r-combinations of a set of n objects with repetition allowed is $C(n+r-1, r)$.

Proof:

- Each r-combination can be represented by a list of $n-1$ bars and r stars
- The $n-1$ bars are used to mark off the n cells
- Example: $n = 4$, $r = 6$: **|*||***
- Each such list corresponds to a way of choosing r positions from $n+r-1$ positions

Examples

- How many ways are there to select 5 bills from a cash box containing \$1,2,5,10,20,50,100 bills? Assume that the order the bills are chosen does no matter, that the bills of each denomination are indistinguishable, and there are at least 5 bills of each type.
- How many solutions does the equation $x_1 + x_2 + x_3 = 11$ have, where x_1, x_2, and x_3 are nonnegative integers? How about with the constraints $x_1 \geq 1, x_2 \geq 2, x_3 \geq 3$?

One more example

What is the value of k at the end of the program:

$k := 0$
for $i_1 := 1$ to n
    for $i_2 := 1$ to i_1
⋮
        for $i_m := 1$ to i_{m-1}
            $k := k + 1$

TABLE 1 Combinations and Permutations with and without Repetition.

Type	*Repetition Allowed?*	*Formula*
r-permutations	No	$\frac{n!}{(n-r)!}$
r-combinations	No	$\frac{n!}{r!\,(n-r)!}$
r-permutations	Yes	n^r
r-combinations	Yes	$\frac{(n+r-1)!}{r!\,(n-1)!}$

Permutations with indistinguishable objects

- Example: How many different strings can be made by reordering the letters of the word SUCCESS?
- Theorem: The number of different permutations of n objects, where there are n_i indistinguishable objects of type i, $i = 1, \ldots, k$, is

$$\frac{n!}{n_1! n_2! \ldots n_k!}$$

Distributing objects into boxes

- Many counting problems can be solved by enumerating the ways objects can be placed into boxes.
- The objects can be either distinguishable / labeled or indistinguishable / unlabeled.
- The boxes can be either distinguishable or indistinguishable.
- There are closed formulas for counting the ways to distribute objects into distinguishable boxes.
- But there are no closed formulas for counting the ways to distribute objects into indistinguishable boxes.

Distinguishable objects into distinguishable boxes

- Example: How many ways are there to distribute hands of 5 cards to each of 4 players from the standard deck of 52 cards?
- Theorem: The number of ways to distribute n distinguishable objects into k distinguishable boxes so that n_i objects are placed into box i, $i = 1, \ldots, k$, equals

$$\frac{n!}{n_1! n_2! \ldots n_k!}$$

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  - Proof: There is a one-to-one correspondence between permutations of n_i objects of type i, $i = 1, \ldots, k$, and ways to distribute objects into k boxes so that n_i objects are placed into box i, $i = 1, \ldots, k$

Indistinguishable objects into distinguishable boxes

- Example: How many ways are there to place 10 indistinguishable balls into 8 distinguishable bins?
- Theorem: The number of ways to distribute r indistinguishable objects into n distinguishable boxes is $C(n+r-1, n-1)$.

Indistinguishable objects into distinguishable boxes

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- Theorem: The number of ways to distribute r indistinguishable objects into n distinguishable boxes is $C(n+r-1, n-1)$.
  - Proof: There is a one-to-one correspondence between r-combinations of a set of n objects with repetition and ways to distribute r indistinguishable objects into n distinguishable boxes

Distributing objects into indistinguishable boxes

- How many ways are there to put 4 different employees into 3 indistinguishable offices, when each office can contain any number of employees?
- How many ways are there to pack 6 copies of the same book into 4 identical boxes, where a box can contain as many as 6 books?

Motivating problems

1. A salesperson must visit 6 different cities. In which order should these cities be visited to minimize total travel time?
2. Given a set of 6 positive integers, find a subset of them that has 100 as their sum, if such a subset exists.
3. A lab has 95 employees. Each of them has one or more skills. Choose 12 employees with a particular set of 25 skills.

Generating permutations

- Any set with n elements can be placed in one-to-one correspondence with $\{1, 2, \ldots, n\}$.
- We can generate the permutations of any set of n elements by generating those of $\{1, 2, \ldots, n\}$.
- An algorithm for generating permutations of $\{1, 2, \ldots, n\}$ based on lexicographic order
- $a_1 a_2 \ldots a_n$ precedes $b_1 b_2 \ldots b_n$ if for some k with $1 \le k \le n$, $a_1 = b_1$, ..., $a_{k-1} = b_{k-1}$, and $a_k < b_k$
- The first permutation of $\{1, 2, \ldots, n\}$ is $1, 2, \ldots, n$
- The last permutation of $\{1, 2, \ldots, n\}$ is $n, n-1, \ldots, 1$

Generating the next permutation of $a_1 a_2 \ldots a_n$

What is the next permutation in lexicographic order after 362541?

1. Find the largest j such that $a_j < a_{j+1}$
2. Let a_k be the least among $a_{j+1}, \ldots, a_n$ such that $a_k > a_j$
3. Put a_k at the jth position
4. After a_k, list in increasing order the rest of the integers $a_j, \ldots, a_n$

ALGORITHM 1 Generating the Next Permutation in Lexicographic Order.

procedure *next permutation*($a_1 a_2 \ldots a_n$: permutation of
$\{1, 2, \ldots, n\}$ not equal to $n \ n-1 \ \ldots \ 2 \ 1$)
$j := n - 1$
while $a_j > a_{j+1}$
  $j := j - 1$
$\{j$ is the largest subscript with $a_j < a_{j+1}\}$
$k := n$
while $a_j > a_k$
  $k := k - 1$
$\{a_k$ is the smallest integer greater than a_j to the right of $a_j\}$
interchange a_j and a_k
$r := n$
$s := j + 1$
while $r > s$
  interchange a_r and a_s
  $r := r - 1$
  $s := s + 1$
{this puts the tail end of the permutation after the jth position in increasing order}
$\{a_1 a_2 \ldots a_n$ is now the next permutation$\}$

Generating combinations

- Since a combination is a subset, use the correspondence between subsets of $\{a_1, a_2, \ldots, a_n\}$ and bit strings of length n.
- List all the bit strings of length n in order of their increasing size as integers
- To find the next binary expansion, locate the first position from the right that is not a 1, make it a 1, change all the following 0s to 1s

ALGORITHM 2 Generating the Next Larger Bit String.

procedure *next bit string*($b_{n-1}\, b_{n-2} \ldots b_1 b_0$: bit string not equal to $11 \ldots 11$)
$i := 0$
while $b_i = 1$
  $b_i := 0$
  $i := i + 1$
$b_i := 1$
$\{b_{n-1}\, b_{n-2} \ldots b_1 b_0$ is now the next bit string$\}$

Example: Find the next bit string after 10 0010 0111?

Generating r-combinations

- An r-combination can be represented by a sequence containing the elements in the subset in increasing order
- The r-combinations can be listed using lexicographic order on these sequences
- The first r-combination is $\{1, 2, \dots, r\}$
- The last r-combination is $\{n - r + 1, n - r + 2, \dots, n\}$

ALGORITHM 3 Generating the Next r-Combination in Lexicographic Order.

procedure *next r-combination*($\{a_1, a_2, \ldots, a_r\}$: proper subset of
$\{1, 2, \ldots, n\}$ not equal to $\{n-r+1, \ldots, n\}$ with
$a_1 < a_2 < \cdots < a_r$)
$i := r$
while $a_i = n - r + i$
  $i := i - 1$
$a_i := a_i + 1$
for $j := i + 1$ **to** r
  $a_j := a_i + j - i$
$\{\{a_1, a_2, \ldots, a_r\}$ is now the next combination$\}$

Example: Find the next larger 4-combination of the set $1, 2, 3, 4, 5, 6$ after $\{1, 2, 5, 6\}$.