Discrete Mathematics: Lecture 12

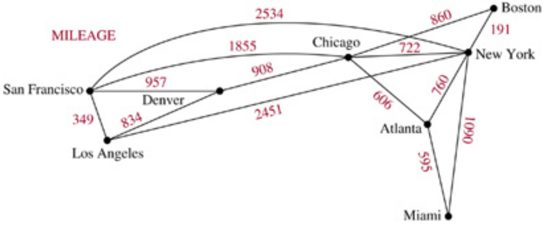
- Last time:
  - Chap 10.4: Connectivity
  - Chap 10.5: Euler and Hamilton paths
- Today:
  - Chap 10.6: Shortest path problems
  - Chap 10.7: Planar graphs

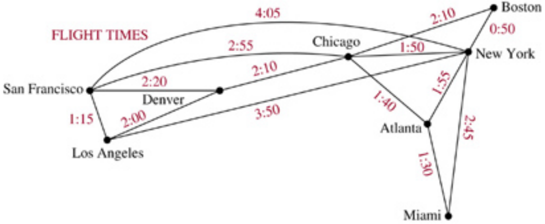
Review of last time

- Paths, simple paths, circuits
- Connected graphs, connected components, cut vertices and edges
- Counting paths between vertices
- Euler paths and circuits, necessary and sufficient conditions
- Hamilton paths and circuits, necessary/sufficient conditions

Weighted graphs (带权图)

Many problems can be modeled using weighted graphs, *i.e.*, graphs with a number assigned to each edge.



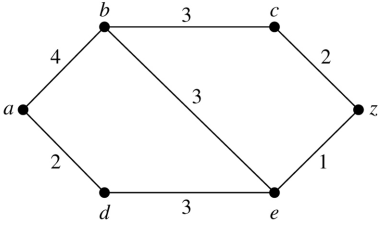


The shortest path problem (最短路径问题)

- The length of a path in a weighted graph is the sum of the weights of the edges of the graph.
- The shortest path problem: what's a shortest path between two given vertices?
- *e.g.*, what is a shortest path in air distance between Boston and Los Angeles?
- *e.g.*, what combination of flights has the smallest total flight time between the two cities?
- *e.g.*, what's the cheapest fare between the two cities?

Example 1

Find a shortest path between a and z:



- The closest vertex to a
- The second closest vertex
- The third closest vertex
- The fourth closest vertex

Dijkstra's algorithm: the general idea

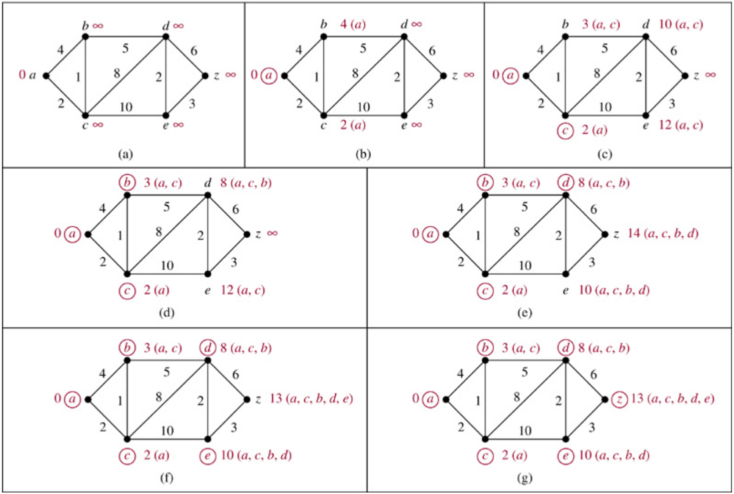
- Proceeds by finding a shortest path from a to a first vertex, a second vertex, and so on, until z is reached
- Maintain a set S of vertices, with $\varnothing$ as its initial value
- Each vertex w is labeled with the length of a shortest path from a to w that contains only vertices in S
- At each iteration, add to S the vertex u not in S with a minimal label, and update the labels
- To update the label of v not in S, if $L(u) + w(u,v) < L(v)$, then $L(v) := L(u) + w(u,v)$

ALGORITHM 1 Dijkstra's Algorithm.

procedure *Dijkstra*(G: weighted connected simple graph, with
    all weights positive)
{G has vertices $a = v_0, v_1, \ldots, v_n = z$ and lengths $w(v_i, v_j)$
    where $w(v_i, v_j) = \infty$ if $\{v_i, v_j\}$ is not an edge in G}
for $i := 1$ **to** n
    $L(v_i) := \infty$
$L(a) := 0$
$S := \emptyset$
{the labels are now initialized so that the label of a is 0 and all
    other labels are ∞, and S is the empty set}
while $z \notin S$
    $u :=$ a vertex not in S with $L(u)$ minimal
    $S := S \cup \{u\}$
    for all vertices v not in S
        if $L(u) + w(u, v) < L(v)$ **then** $L(v) := L(u) + w(u, v)$
        {this adds a vertex to S with minimal label and updates the
        labels of vertices not in S}
return $L(z)$ {$L(z) =$ length of a shortest path from a to z}

Example 2

Find a shortest path between a and z:



Correctness of Dijkstra's algorithm

Theorem 1: Dijkstra's algorithm finds the length of a shortest path between two vertices in a connected simple undirected weighted graph.
We prove by induction on k that at the kth iteration,

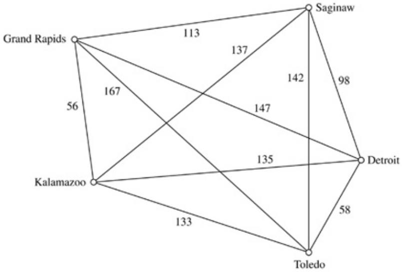
- the label of every vertex v in S is the length of a shortest path from a to this vertex
- the label of every vertex not in S is the length of a shortest path from a to this vertex that contains only (besides the vertex itself) vertices in S

The computational complexity of Dijkstra's algorithm

Theorem 2: Dijkstra's algorithm uses $O(n^2)$ operations (additions and comparisons), where n is the number of vertices.

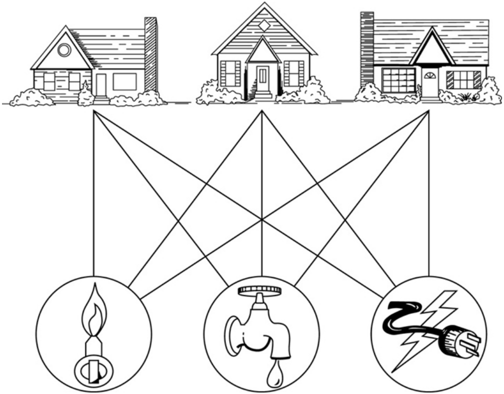
The traveling salesperson problem (旅行商问题)

- A traveling salesperson wants to visit each of n cities exactly once and return to the same city
- In which order should he visit these cities to travel the minimum total distance?
- Find a Hamilton circuit with minimum total weight in a weighted, complete, undirected graph
- No algorithm with polynomial worst-case time complexity is known



A motivating example

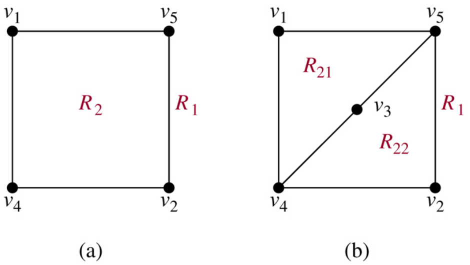
Is it possible to join three houses and utilities so that none of the connections cross?



Can $K_{3,3}$ be drawn in the plane so that no two of its edges cross?

Planar graphs (平面图)

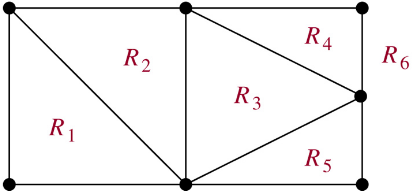
- Definition: A graph is called planar if it can be drawn in the plane without any edges crossing (where a crossing of edges is the intersection of the lines or arcs representing them at a point other than their common endpoint). Such a drawing is called a planar representation of the graph.
- *e.g.*, K_4 and Q_3 are plannar.
- *e.g.*, $K_{3,3}$ is not plannar.



Applications of planar graphs

- Design of electronic circuits
  - model a circuit with a graph: components – vertices, connections – edges
  - we can print a circuit on a single board with no connections crossing if the graph is plannar
  - if not, we must turn to more expensive options
- Design of road networks
  - model with a graph: cities – vertices, highways – edges
  - we can build a road network without using underpasses or overpasses if the graph is plannar

Euler's formula

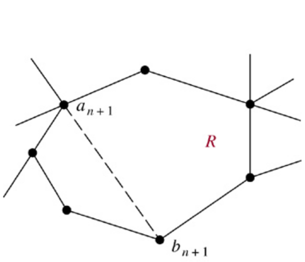


Theorem 1: Let G be a connected planar simple graph with e edges and v vertices. Let r be the number of regions in a planar representation of G. Then $r = e - v + 2$.

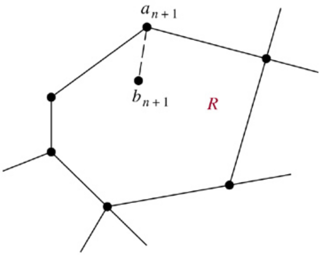
Example: Suppose that a connected planar simple graph has 20 vertices, each of degree 3. Into how many regions does a representation of this planar graph split the plane?

Proof of Euler's formula

- We specify a planar representation of G.
- Arbitrarily pick an edge of G to obtain G_1.
- Obtain G_n from G_{n-1}: arbitrarily add an edge incident with a vertex in G_{n-1}.
- G is obtained after e edges are added.
- Let r_n, e_n, v_n be the number of regions, edges, and vertices of the planar representation of G_n induced by the planar representation of G.
- We prove by induction on n that $r_n = e_n - v_n + 2$.



(a)



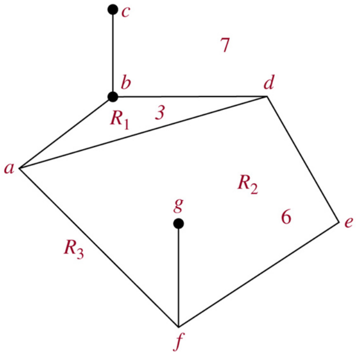
(b)

Corollaries

- Corollary 1: If G is a connected planar simple graph with e edges and v vertices, where $v \geq 3$, then $e \leq 3v - 6$.
- Example: Show that K_5 is nonplanar.
- Corollary 2: If G is a connected planar simple graph, then G has a vertex of degree ≤ 5.

Degree of a region

- The degree of a region R, denoted by $deg(R)$, is the number of edges on the boundary of the region.
- When an edge occurs twice on the boundary (so that it is traced out twice when the boundary is traced out), it contributes 2 to the degree.



Proof of Corollary 1

- Let the graph drawn in the plane divide the plane into r regions
- Since the graph is simple and $v \geq 3$, $deg(R) \geq 3$ for each region R
- The sum of $deg(R)$ is equal to $2e$, since each edge contributes 2 to the sum

Corollaries

- Corollary 3: If a connected planar simple graph has e edges and v vertices with $v \geq 3$ and no circuits of length 3, then $e \leq 2v - 4$.
- Proof similar to that of Corollary 1 except that when there is no circuit of length 3, the degree of each region is ≥ 4
- Example: Show that $K_{3,3}$ is nonplanar.