Discrete Mathematics: Lecture 4

- Last time:
  - Chap 1.4: Predicates and Quantifiers
  - Chap 1.5: Nested Quantifiers
- Today:
  - Chap 1.6: Rules of Inference
- Next time:
  - Chap 1.7: Introduction to Proofs

Review of last time

- Predicates, universal and existential quantifiers
- Translating into logical expressions
  - identify predicates and quantifiers, decide on the domain to use
- The order of quantifiers is important
- Precedence of quantifiers
- Bound and free occurrences of variables

Logical equivalences involving quantifiers

- Definition: Two statements involving predicates and quantifiers are logically equivalent (逻辑等价) iff they have the same truth value no matter which predicates are substituted into these statements and which domain of discourse is used.
- Notation: $S \equiv T$, or $S \Leftrightarrow T$
- Example: $\forall x(P(x) \wedge Q(x)) \equiv \forall x P(x) \wedge \forall x Q(x)$
- Question: Do we have $(\forall x A \vee \forall x B) \equiv \forall x(A \vee B)$?
- How about $\exists x(P(x) \vee Q(x)) \equiv \exists x P(x) \vee \exists x Q(x)$?
- How about $\exists x(A \wedge B) \equiv (\exists x A \wedge \exists x B)$?

Proof that $\forall x(P(x) \land Q(x)) \equiv \forall x P(x) \land \forall x Q(x)$

- Consider an arbitrary domain and arbitrary instances of predicates P and Q.
- Then $\forall x(P(x) \land Q(x))$ is true iff
- $P(x) \land Q(x)$ is true for all values of x in the domain iff
- both $P(x)$ and $Q(x)$ are true for all values of x in the domain
- iff $P(x)$ is true for all values of x in the domain, and $Q(x)$ is true for all values of x in the domain iff
- $\forall x P(x)$ is true and $\forall x Q(x)$ is true iff
- $\forall x P(x) \land \forall x Q(x)$ is true

Null quantification (1)

When x does not occur as a free variable in A

- $\forall x P(x) \vee A \equiv \forall x (P(x) \vee A)$
- $\exists x P(x) \vee A \equiv \exists x (P(x) \vee A)$
- $\forall x P(x) \wedge A \equiv \forall x (P(x) \wedge A)$
- $\exists x P(x) \wedge A \equiv \exists x (P(x) \wedge A)$

De Morgan's laws for quantifiers

- $\neg \exists x P(x) \equiv \forall x \neg P(x)$
- $\neg \forall x P(x) \equiv \exists x \neg P(x)$
- Relationship to De Morgan's laws in propositional logic when the domain has n elements $x_1, x_2, \ldots, x_n$

Proving logical equivalence

- Using existing logical equivalence
- Replacement theorem: If B is a subformula of A and $B \Leftrightarrow B'$, let A' be the result of replacing B in A by B', then $A \Leftrightarrow A'$
- Example: $\neg\forall x(P(x) \to Q(x)) \Leftrightarrow \exists x(P(x) \land \neg Q(x))$

Null quantification (2)

When x does not occur as a free variable in A

- $\forall x P(x) \to A \equiv \exists x (P(x) \to A)$
- $\exists x P(x) \to A \equiv \forall x (P(x) \to A)$
- $A \to \forall x P(x) \equiv \forall x (A \to P(x))$
- $A \to \exists x P(x) \equiv \forall x (A \to P(x))$

Negating quantified expressions

- Find the negation of the statement "Every student in your class has taken a course in calculus"
- Find the negation of the statement "There is a student in your class who has taken a course in calculus"
- The negation of "There is an honest politician"
- Note that the English statement "All politicians are not honest" means "not all politicians are honest"
- The negation of "All Americans eat cheeseburgers"
- The negations of $\forall x(x^2 > x)$ and $\exists x(x^2 = 2)$

Negating Nested Quantifiers

We negate nested quantifiers by successively applying the rules for negating a single quantifier.

- Negate $\forall x \exists y (xy = 1)$ so that no negation precedes a quantifier
- There does not exist a woman who has taken a flight on every airline in the world
- Express the fact that $\lim_{x \to a} f(x)$ does not exist

- Proofs in mathematics are valid arguments that establish the truth of mathematical statements.
- By an argument (论证), we mean a sequence of statements that end with a conclusion (结论).
- By valid (有效的), we mean that the truth of the conclusion must follow from the truth of the premises (前提).
- To deduce new statements from statements we already have, we use rules of inference.

Valid Arguments in Propositional Logic

- Argument:
  - If you have a current password, then you can log onto the network.
  - You have a current password.
  - Therefore, you can log onto the network.
- Argument form (形式结构): $p \to q, p, \therefore q$
- $(p \to q) \land p \to q$ is a tautology (永真式)
- The argument form is valid since whenever all premises are true, the conclusion is true.
- The argument is valid since its form is valid.

Formal definition

- An argument in propositional logic is a sequence of propositions.
- All but the final proposition in the argument are called premises and the final proposition is called the conclusion.
- An argument is valid if the truth of all its premises implies the truth of the conclusion.
- An argument form is a sequence of propositional formulas
- An argument form with premises $A_1, A_2, \ldots, A_n$ and conclusion A is valid if $A_1 \wedge A_2 \wedge \ldots \wedge A_n \to A$ is a tautology.

Rules of Inference for Propositional Logic

- We can always use a truth table to show that an argument form is valid.
- However, this can be a tedious approach.
- Alternatively, we can first establish the validity of some relatively simple argument forms, called rules of inference.
- These rules of inference can be used to construct more complicated valid argument forms.

Modus Ponens / law of detachment (分离规则)

- $p \to q, p, \therefore q$
- If it snows today, then we will go skiing. It is snowing today. Therefore, we will go skiing.
- If $\sqrt{2} > \frac{3}{2}$, then $(\sqrt{2})^2 > (\frac{3}{2})^2$. We know that $\sqrt{2} > \frac{3}{2}$. Therefore, $(\sqrt{2})^2 = 2 > (\frac{3}{2})^2 = \frac{9}{4}$.

Rules of Inference

TABLE 1 **Rules of Inference.**

Rule of Inference	*Tautology*	*Name*
p $p \to q$ $\therefore q$	$(p \land (p \to q)) \to q$	Modus ponens
$\neg q$ $p \to q$ $\therefore \neg p$	$(\neg q \land (p \to q)) \to \neg p$	Modus tollens
$p \to q$ $q \to r$ $\therefore p \to r$	$((p \to q) \land (q \to r)) \to (p \to r)$	Hypothetical syllogism
$p \lor q$ $\neg p$ $\therefore q$	$((p \lor q) \land \neg p) \to q$	Disjunctive syllogism
p $\therefore p \lor q$	$p \to (p \lor q)$	Addition
$p \land q$ $\therefore p$	$(p \land q) \to p$	Simplification
p q $\therefore p \land q$	$((p) \land (q)) \to (p \land q)$	Conjunction
$p \lor q$ $\neg p \lor r$ $\therefore q \lor r$	$((p \lor q) \land (\neg p \lor r)) \to (q \lor r)$	Resolution

假言推理，假言易位，假言三段论，析取三段论，
附加律，化简律，合取律，归结律

Examples

- It is below freezing now. Therefore, it is either below freezing or raining now.
- It is below freezing and raining now. Therefore, it is below freezing now.
- If it rains today, then we will not have a barbecue today. If we do not have a barbecue today, then we will have a barbecue tomorrow. Therefore, if it rains today, then we will have a barbecue tomorrow.

Using Rules of Inference to Build Arguments

- It is not sunny this afternoon and it is colder than yesterday.
- We will go swimming only if it is sunny.
- If we do not go swimming, we will take a canoe trip.
- If we take a canoe trip, then we will be home by sunset.
- Therefore, we will be home by sunset.

Using Rules of Inference to Build Arguments

- If you send me an email message, then I will finish writing the program.
- If you do not send me an email message, then I will go to sleep early.
- If I go to sleep early, then I will wake up feeling refreshed.
- Therefore, if I do not finish writing the program, then I will wake up feeling refreshed.

Resolution

- $p \vee q, \neg p \vee r, \therefore q \vee r$, $q \vee r$ is called the resolvent
- Example: Jasmine is skiing or it is not snowing. It is snowing or Bart is playing hockey. Therefore, Jasmine is skiing or Bart is playing hockey.
- Resolution plays an important role in programming languages such as Prolog.
- Show that $p \wedge q \vee r$ and $r \rightarrow s$ imply $p \vee s$

Common fallacies (谬误)

- $p \to q, q, \therefore p$, fallacy of affirming the conclusion
- If you do every problem in this book, then you will learn discrete mathematics. You learned discrete mathematics. Therefore, you did every problem in this book.
- $p \to q, \neg p, \therefore \neg q$, fallacy of denying the hypothesis
- If you do every problem in this book, then you will learn discrete mathematics. You didn't do every problem in this book. Therefore, you didn't learn discrete mathematics.

Rules of Inference for Quantified Statements

- universal instantiation (实例化): $\forall x P(x), \therefore P(c)$, where c is a particular member of the domain
- universal generalization (一般化): $P(c)$ for an arbitrary c, $\therefore \forall x P(x)$
- existential instantiation: $\exists x P(x)$, $\therefore P(c)$ for some element c
- existential generalization: $P(c)$ for some element c, $\therefore \exists x P(x)$

Examples

- Everyone in this discrete mathematics class has taken a course in computer science.
  Marla is a student in this class.
  Therefore, Marla has taken a course in computer science.

- A student in this class has not read the book.
  Everyone in this class passed the first exam.
  Therefore, someone who passed the first exam has not read the book.

- All hummingbirds are richly colored.
  No large birds live on honey.
  Birds that do not live on honey are dull in color.
  Therefore, hummingbirds are small.

Combining Rules of Inference

- universal instantiation and modus ponens combined, called universal modus ponens: $\forall x(P(x) \rightarrow Q(x)), P(a), \therefore Q(a)$
- example: for all positive integers n, if n is greater than 4, then n^2 is less than 2^n. 100 is greater than 4. Hence, $100^2 < 2^{100}$.
- universal instantiation and modus tollens combined, called universal modus tollens: $\forall x(P(x) \rightarrow Q(x)), \neg Q(a), \therefore \neg P(a)$

A common mistake

Prove that $\{\forall x(P(x) \to Q(x)), \exists x P(x))\} \models \exists x Q(x)$

1. $\forall x(P(x) \to Q(x))$
2. $P(a) \to Q(a)$
3. $\exists x P(x)$
4. $P(a)$
5. $Q(a)$
6. $\exists x Q(x)$

Another example

Prove that $\{\exists x P(x) \wedge \exists x Q(x)\} \models \exists x (P(x) \wedge Q(x))$

1. $\exists x P(x)$
2. $P(a)$
3. $\exists x Q(x)$
4. $Q(a)$
5. $P(a) \wedge Q(a)$
6. $\exists x (P(x) \wedge Q(x))$