Discrete Mathematics: Lecture 1

- Today
  - Review of final exam
  - Chap 6.1: The basics of counting
  - Chap 6.2: The pigeonhole principle
- Announments
  - Moshe Vardi's talks next Thursday and Friday
  - Class next Friday moved to

Basic counting principle: the product rule (乘法原理)

- Suppose that a procedure can be broken down into a sequence of two tasks. If there are n_1 ways to do the first task and for each of them, there are n_2 ways to do the second task, then there are $n_1 n_2$ ways to do the procedure.
- $|A_1 \times A_2 \times \ldots \times A_m| = |A_1| \times |A_2| \times \ldots \times |A_m|$
- How many different license plates are available if each plate consists of a sequence of 3 letters followed by 3 digits?
- How many functions are there from a set with m elements to a set with n elements?
- How many one-to-one functions are there from a set with m elements to a set with n elements?
- How many subsets are there for a finite set?

Basic counting principle: the sum rule (加法原理)

- If a task can be done either in one of n_1 ways or in one of n_2 ways, where none of the set of n_1 ways is the same as any of the n_2 ways, then there are $n_1 + n_2$ ways to do the task.
- If $A_1, A_2, \ldots, A_m$ are disjoint sets, then $|A_1 \cup A_2 \cup \ldots \cup A_m| = |A_1| + |A_2| + \ldots + |A_m|$
- Example: A student can choose a computer project from one of 3 lists. The 3 lists contain 23, 15, and 19 possible projects, respectively. No project is on more than one list. How many choices does the student have?

Compare two programs with loops

What is the value of k at the end of each program:

$k := 0$
for $i_1 := 1$ to n_1
    for $i_2 := 1$ to n_2
$\vdots$
        for $i_m := 1$ to n_m
            $k := k + 1$

$k := 0$
for $i_1 := 1$ to n_1
    $k := k + 1$
for $i_2 := 1$ to n_2
    $k := k + 1$
$\vdots$
for $i_m := 1$ to n_m
    $k := k + 1$

Combining the product and sum rules

- In a version of BASIC, a variable name is a string of 1 or 2 alphanumeric chars, where uppercase and lowercase letters are not distinguished. Moreover, a variable name must begin with a letter and must be different from the 5 strings of chars reserved for programming use. How many different variable names are there?
- Each user on a computer system has a password, which is 6 to 8 characters long, where each character is an uppercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?

Counting internet addresses

- In the internet, each computer is assigned an internet address.
- In Version 4 of the Internet Protocol (IPv4), an address is a string of 32 bits.
- It begins with a network number (netid, 网络标识符).
- The netid is followed by a hostname (hostid, 主机名), which identifies a computer as a member of a particular network.
- Three forms of addresses are used.

Counting internet addresses

- Class A addresses, used for large networks, consist of 0, a 7-bit netid and a 24-bit hostid.
- Class B addresses, used for medium-sized networks, consist of 10, a 14-bit netid and a 16-bit hostid.
- Class C addresses, used for small networks, consist of 110, a 21-bit netid and a 8-bit hostid.
- There are restrictions on addresses.
- 1111111 cannot be used as a netid for a class A network.
- The hostids consisting of all 0s and all 1s cannot be used in any network.

The substraction rule (减法原理)

Also known as the inclusion-exclusion principle (容斥原理)

- If a task can be done in one of n_1 ways or in one of n_2 ways, but n_3 of the set of n_1 ways are the same as n_3 of the n_2 ways, then there are $n_1 + n_2 - n_3$ ways to do the task.
- $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$
- How many bit strings of length 8 either start with 1 or end with 00?

The division rule (除法原理)

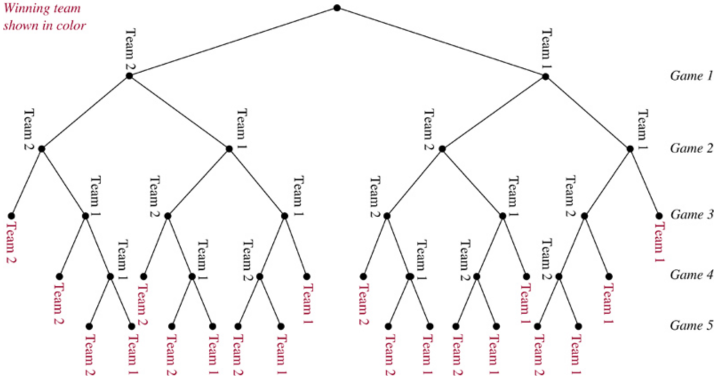
- There are n/d equivalent ways to do a task if it can be done in n ways, and for every way w, exactly d of the n ways are equivalent to way w.
- If the finite set A is the union of n pairwise disjoint subsets each with d elements, then $n = |A|/d$.
- How many different ways are there to seat four people around a circular table, where two seatings are considered the same when each person has the same left neighbor and the same right neighbor?

Tree diagrams

- Counting problems can be solved using tree diagrams (树图).
- We use a branch to represent a possible choice.
- The possible outcomes are represented by leaves.
- How many bit strings of length 4 do not have two consecutive (连续的) 1s?

Tree diagrams

A playoff (季后赛) between two teams consists of at most 5 games.
The first team that wins 3 games wins the playoff.
In how many different ways can the playoff occur? 20



The pigeonhole principle (鸽笼原理)

- Theorem: If k is a positive integer and $k+1$ or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.
- Corollary: A function f from a set with $k+1$ or more elements to a set with k elements is not one-to-one.
- In any group of 27 English words, there must be at least two that begin with the same letter.
- How many students must be in a class to guarantee that at least two students receive the same score on the final exam, if the exam is graded on a scale from 0 to 100 points.
- Show that for every integer n there is a multiple of n that has only 0s and 1s in its decimal expansion.

The generalized pigeonhole principle (广义鸽笼原理)

- Theorem: If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.
- A common type of problems ask for the minimum number of objects such that at least r of these objects must be in one of k boxes when these objects are distributed among the boxes.
  - Answer: $k(r-1)+1$

Examples

- Among 100 people there are at least $\lceil 100/12 \rceil = 9$ who were born in the same month.
- What is the minimum number of students required in a discrete mathematics class to be sure that at least 6 will receive the same grade, if there are 5 possible grades: A, B, C, D, and F?
- How many cards must be selected from a standard deck of 52 cards to guarantee that at least 3 cards of the same suit are chosen? How about 3 hearts?
- What is the least number of area codes needed to guarantee that the 25 million phones in a state can be assigned distinct 10-digit telephone numbers? Assume that telephone numbers are of the form NXX-NXX-XXXX, where the first NXX is the area code, N:2-9, X:0-9.

Some elegant applications of the pigeonhole principle

- During a month with 30 days, a baseball team plays at least one game a day, but ≤ 45 games. Show that there must be a period of some number of consecutive days during which the team must play exactly 14 games.
- Show that among any $n + 1$ positive integers not exceeding $2n$ there must be an integer that divides one of the other integers.
- Theorem: Every sequence of $n^2 + 1$ distinct real numbers contains a subsequence of length $n + 1$ that is either strictly increasing or strictly decreasing.
- Example: 8,11,9,1,4,6,12,10,5,7
- Assume that in a group of 6 people, each pair of individuals consists of two friends or two enemies. Show that there are either 3 mutual friends or 3 mutual enemies in the group.