



Computer Graphics

Curve and Surface Modeling 1

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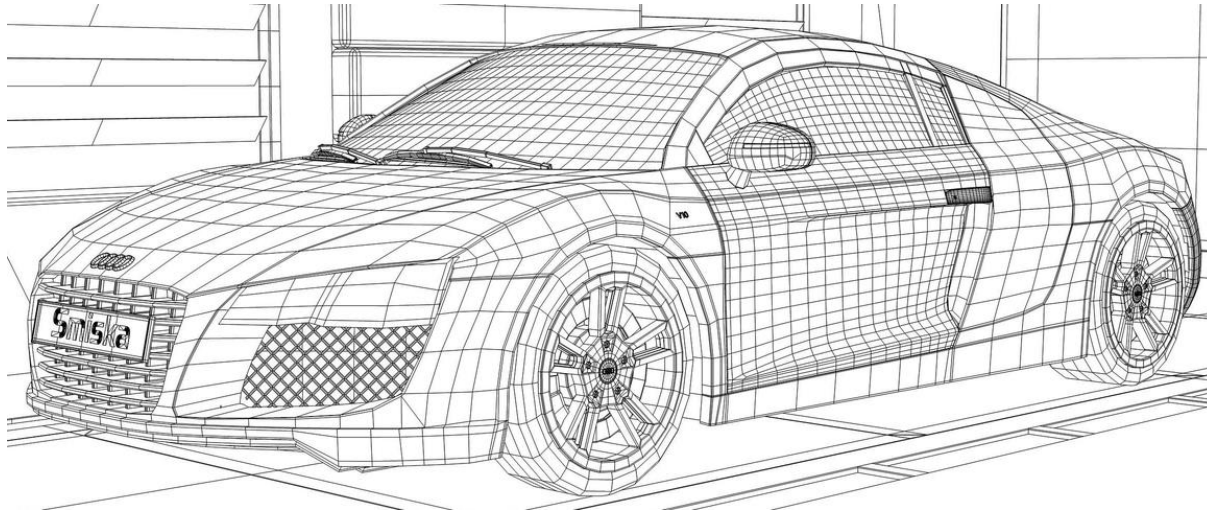
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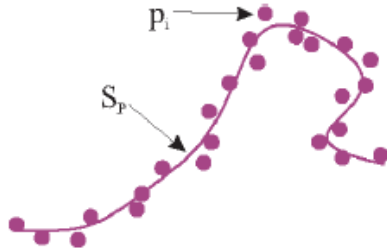
Outline

- Interpolation and Approximation
- Curve Modeling
 - Parametric curve
 - Bézier curve
- Surface Modeling
 - Bézier surface

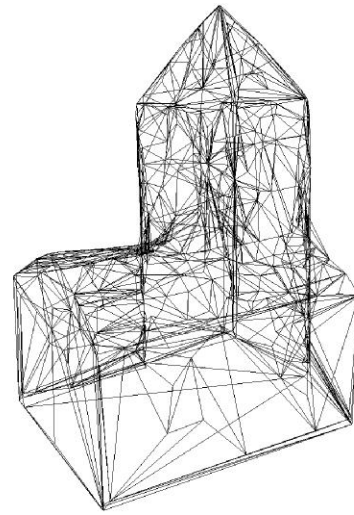
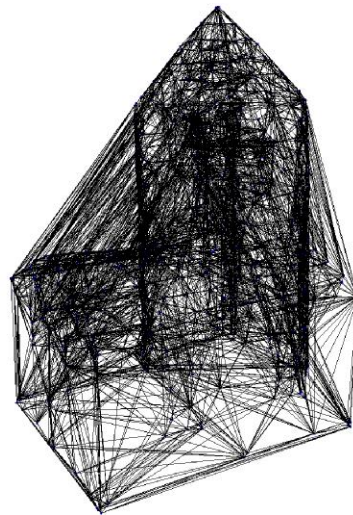
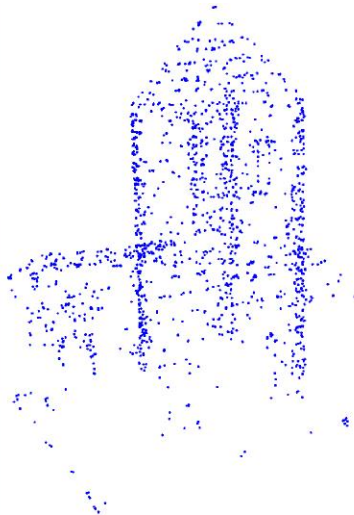


Introduction

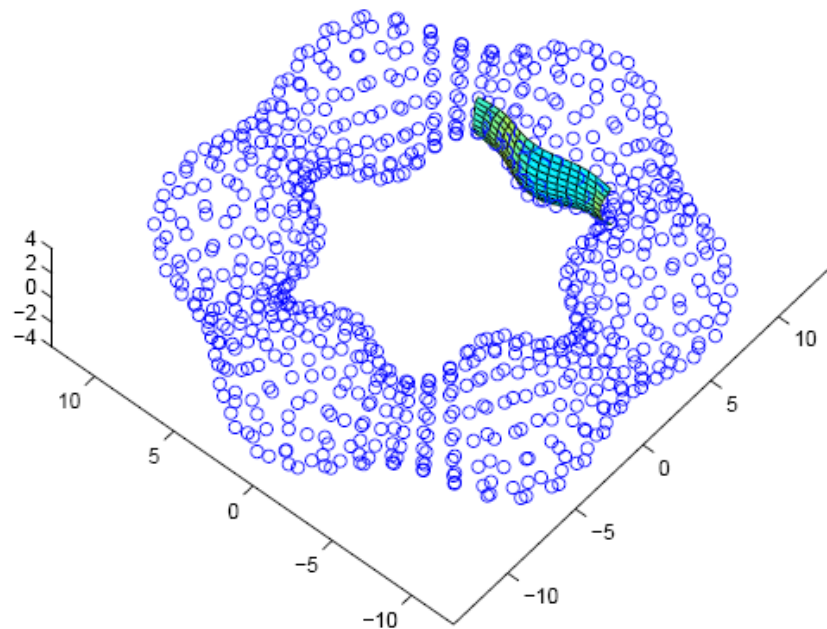
- Raw data is very popular in many experimental study and usually it need fitting before it can be understand well.



Introduction



Introduction



Surface fitting to 3D points

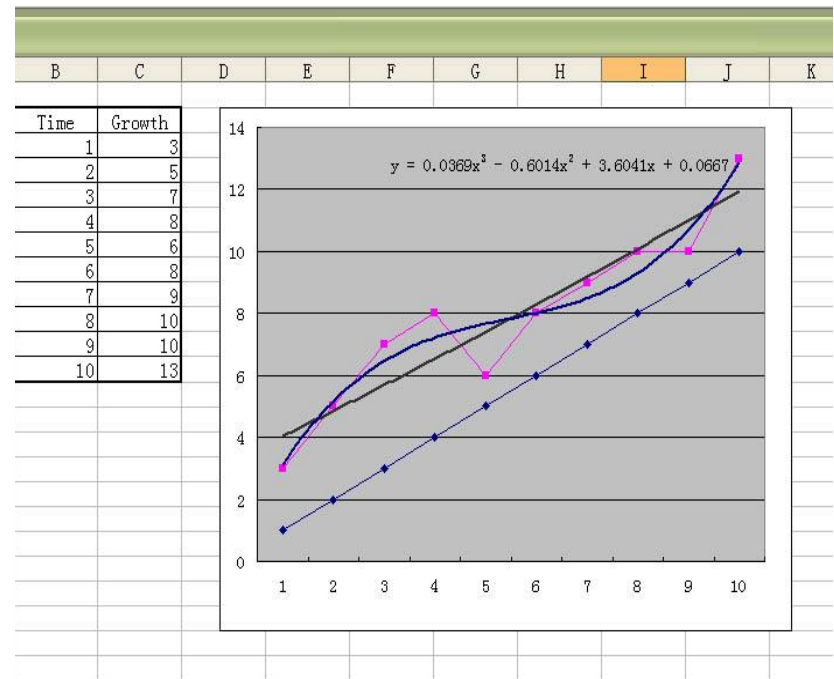
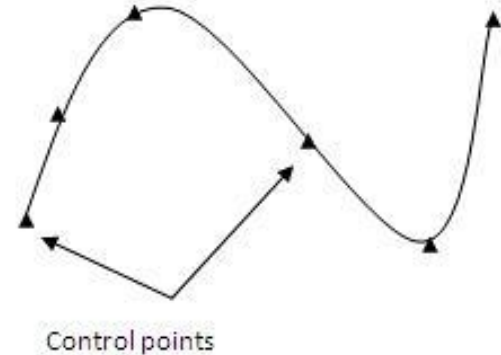


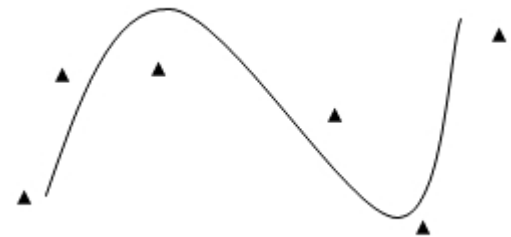
Chart by Microsoft Excel

Interpolation and approximation

- Interpolation: When the curve passes through all the control points then it is called as **Interpolation**.



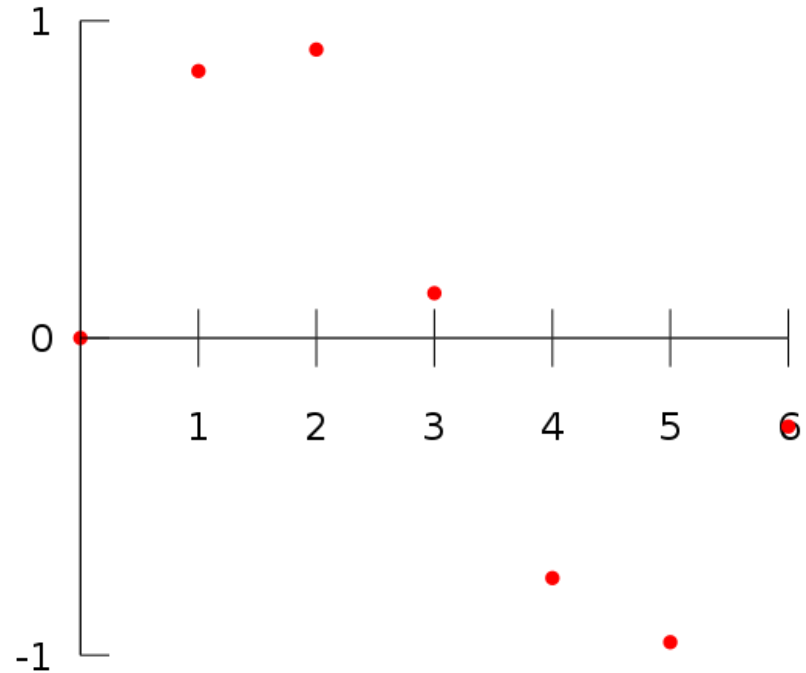
- Approximation: When the curve does not pass through the control points then it is called as **Approximation**.



Interpolation case

- For example, suppose we have a table like this, which gives some values of an unknown function f .

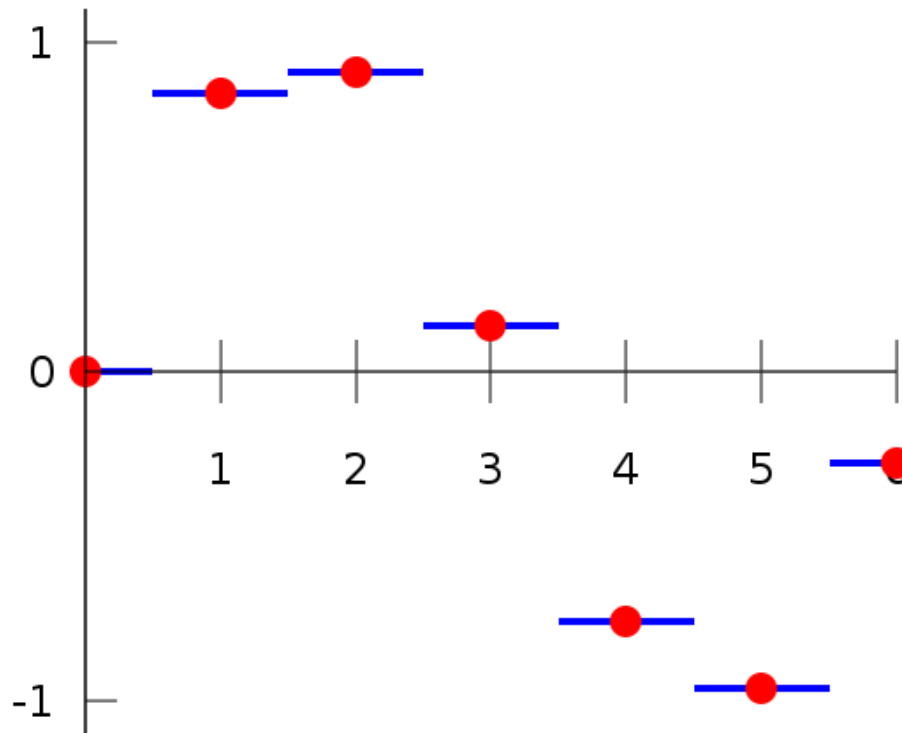
x	$f(x)$
0	0
1	0.8415
2	0.9093
3	0.1411
4	-0.7568
5	-0.9589
6	-0.2794



Interpolation provides a means of estimating the function at intermediate points, such as $x = 2.5$.

Piecewise constant interpolation

- The simplest interpolation method is to locate the nearest data value, and assign the same value.



Linear interpolation

- Generally, linear interpolation takes two data points, say (x_a, y_a) and (x_b, y_b) , and the interpolant is given by:

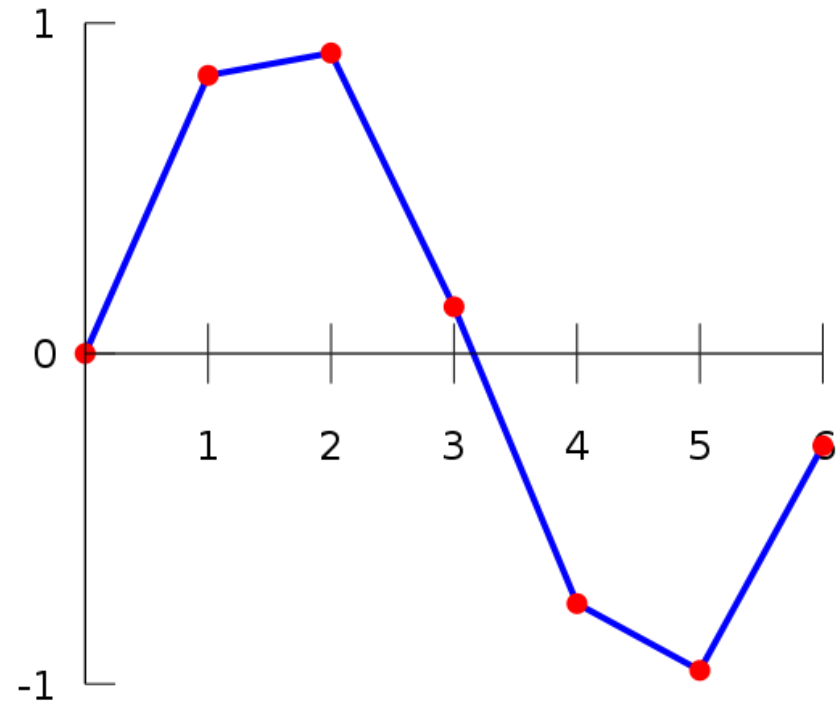
$$y = y_a + (y_b - y_a) \frac{x - x_a}{x_b - x_a} \text{ at the point } (x, y)$$

$$\frac{y - y_a}{y_b - y_a} = \frac{x - x_a}{x_b - x_a}$$

$$\frac{y - y_a}{y_b - y_a} = \frac{x - x_a}{x_b - x_a}$$

$$\frac{y - y_a}{x - x_a} = \frac{y_b - y_a}{x_b - x_a}$$

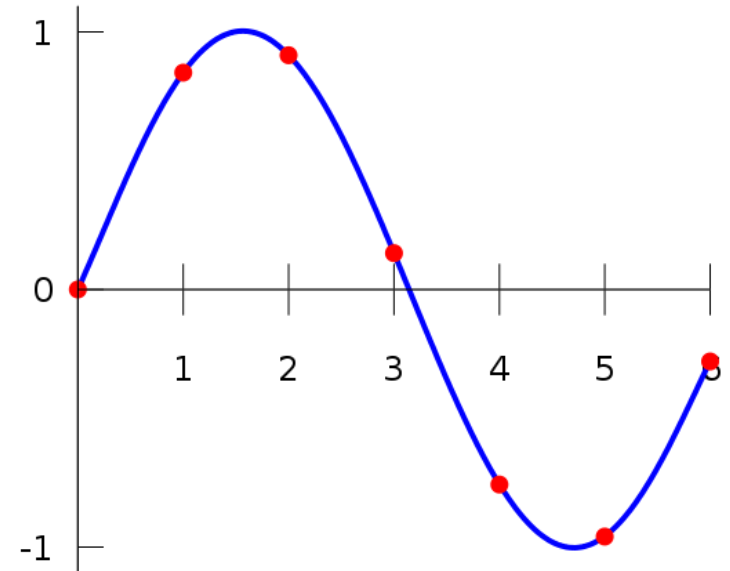
$$\frac{y - y_a}{x - x_a} = \frac{y_b - y_a}{x_b - x_a}$$



Polynomial interpolation

- Polynomial interpolation is a generalization of linear interpolation. Note that the linear interpolation is a linear function. We now replace this interpolation with a polynomial of higher degree.
- The following sixth degree polynomial goes through all the seven points:

$$\begin{aligned}f(x) = & -0.0001521x^6 + 0.003130x^5 \\ & +0.07321x^4 - 0.3577x^3 \\ & +0.2255x^2 + 0.9038x\end{aligned}$$



Approximation – Least squares fitting

- Linear least squares
- A fitting model is a linear one when the model comprises a linear combination of the parameters, i.e.,

$$f(x, \beta) = \sum_{j=1}^m \beta_j \phi_j(x),$$

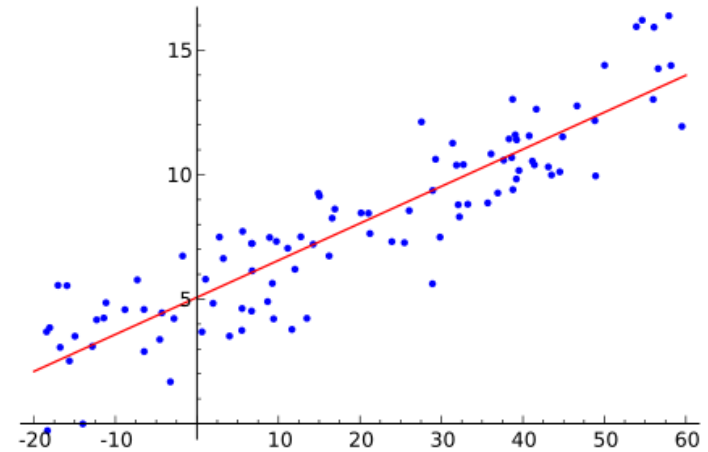
where the function ϕ_j is a function of x .

- Letting

$$X_{ij} = \frac{\partial f(x_i, \beta)}{\partial \beta_j} = \phi_j(x_i),$$

- we can then see that in that case the least square estimate (or estimator, in the context of a random sample), β is given by

$$\hat{\beta} = (X^T X)^{-1} X^T \mathbf{y}.$$



Least squares fitting example

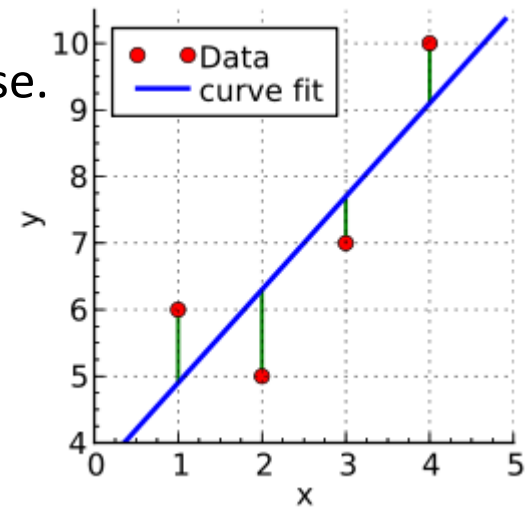
- As a result of an experiment, four (x, y) data points were obtained, (1, 6), (2, 5), (3, 7), and (4, 10).
- We hope to find a line $y = \beta_1 + \beta_2 x$ that best fits these four points. In other words, we would like to find the numbers β_1 and β_2 that approximately solve the over-determined linear system

$$\begin{aligned}\beta_1 + 1\beta_2 &= 6, & \beta_1 + 2\beta_2 &= 5, \\ \beta_1 + 3\beta_2 &= 7, & \beta_1 + 4\beta_2 &= 10.\end{aligned}$$

of four equations in two unknowns in some "best" sense.

- A **residual** is defined as the difference between the actual value of the dependent variable and the value predicted by the model.

$$r_i = y_i - f(x_i, \beta).$$



Least squares fitting example

- The "error", at each point, between the curve fit and the data is the difference between the right- and left-hand sides of the equations above. The least squares approach to solving this problem is to try to make the sum of the squares of these errors as small as possible; that is, to find the minimum of the function

$$\begin{aligned} S(\beta_1, \beta_2) &= [6 - (\beta_1 + 1\beta_2)]^2 + [5 - (\beta_1 + 2\beta_2)]^2 \\ &\quad + [7 - (\beta_1 + 3\beta_2)]^2 + [10 - (\beta_1 + 4\beta_2)]^2 \\ &= 4\beta_1^2 + 30\beta_2^2 + 20\beta_1\beta_2 - 56\beta_1 - 154\beta_2 + 210 \end{aligned}$$

- The minimum is determined by calculating the partial derivatives of $S(\beta_1, \beta_2)$ with respect to β_1 and β_2 and setting them to zero

$$\frac{\partial S}{\partial \beta_1} = 0 = 8\beta_1 + 20\beta_2 - 56, \quad \frac{\partial S}{\partial \beta_2} = 0 = 20\beta_1 + 60\beta_2 - 154.$$

- This results in a system of two equations in two unknowns, called the normal equations, which give, when solved $\beta_1 = 3.5, \beta_2 = 1.4$,
and the equation $y = 3.5 + 1.4x$ of the line of best fit.



Approximation – Quadratic least squares fitting

- Importantly, in "linear least squares", we are not restricted to using a line as the model as in the above example. For instance, we could have chosen the restricted quadratic model $y = \beta_1 x^2$. This model is still linear in the β_1 parameter, so we can still perform the same analysis, constructing a system of equations from the data

points: $6 = \beta_1(1)^2, \quad 5 = \beta_1(2)^2$

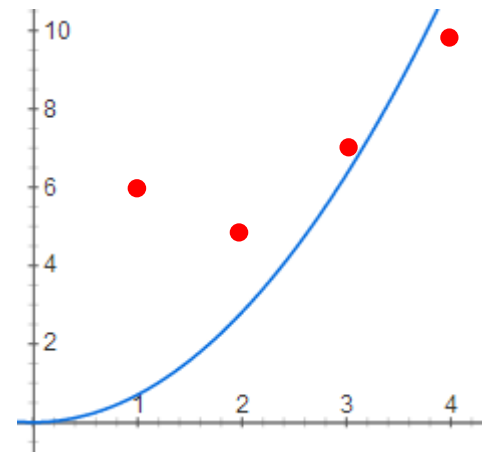
$$7 = \beta_1(3)^2, \quad 10 = \beta_1(4)^2$$

- The partial derivatives with respect to the parameters (this time there is only one) are again computed and set to 0:

$$\frac{\partial S}{\partial \beta_1} = 0 = 708\beta_1 - 498$$

and solved $\beta_1 = 0.703x^2$

- leading to the resulting best fit model $y = 0.703x^2$



General - Least Square Methods

- How to draw a curve approximately fitting to raw data?
 - Raw data usually has noise. The values of dependent variables vary even though all the independent variables are constant. Therefore, the estimation of the trend the dependent variables is needed. This process is called **regression** or **curve fitting**.
 - The estimated equation (matrix) satisfy the raw data. However, the equation is not usually unique, and the equation or curve with a minimal deviation from all data points is desirable.
 - This desirable **best-fitting equation** can be obtained by least square method which uses the minimal sum of the deviations squared (偏差的平方和) from a given set of data.



Least square formulation

- If you have a data set $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ and the best curve $f(x)$ should be with the property as follows

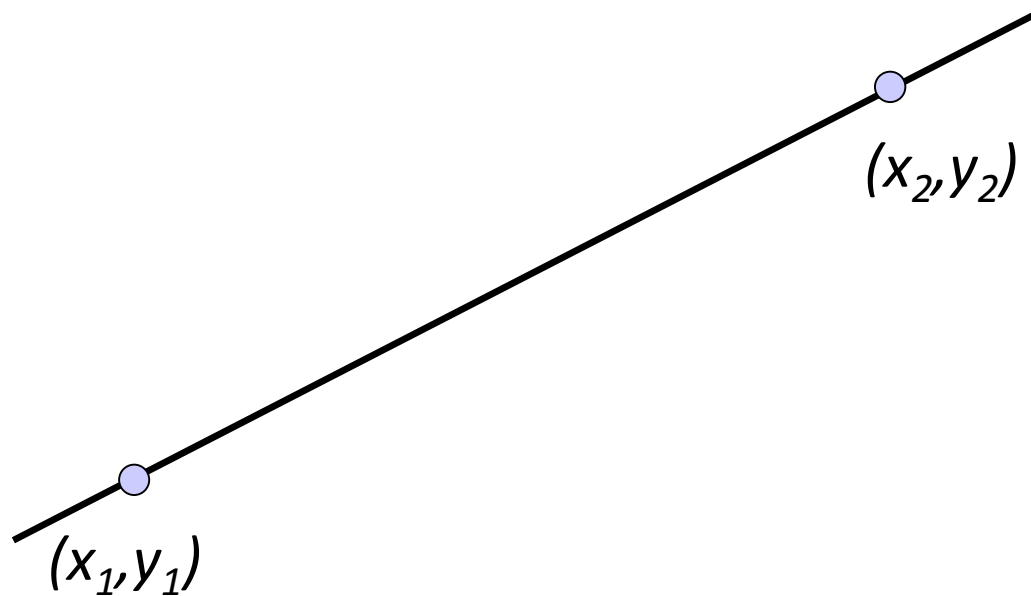
Minimum Least
Square error

$$E = \sum_{i=1}^n (f(x_i) - y_i)^2$$



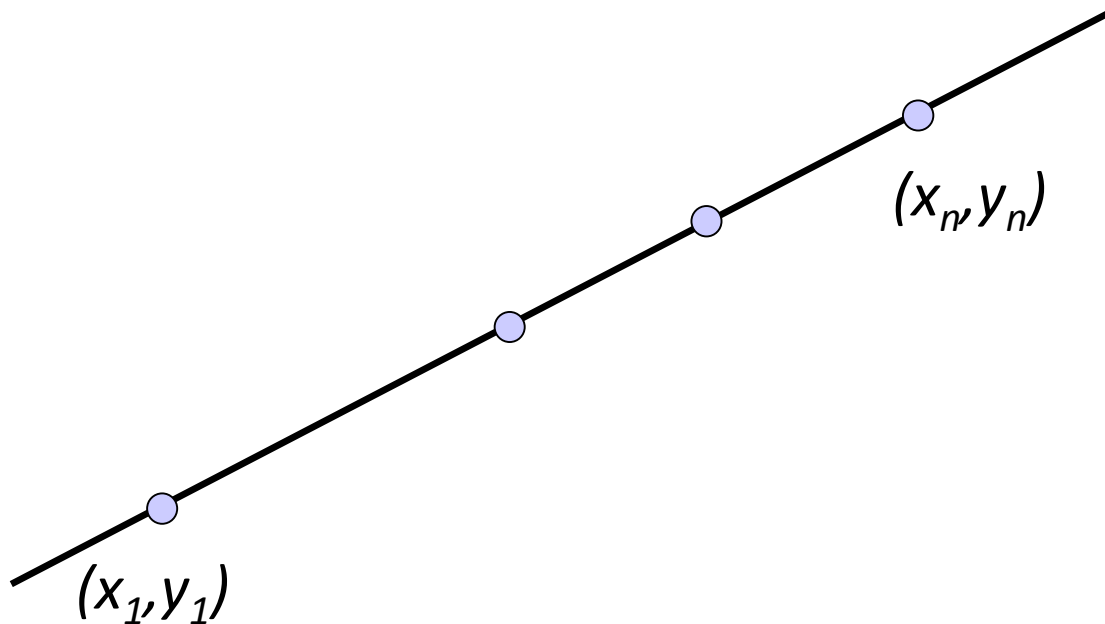
Least square line

- When $n = 2$, $E = 0$



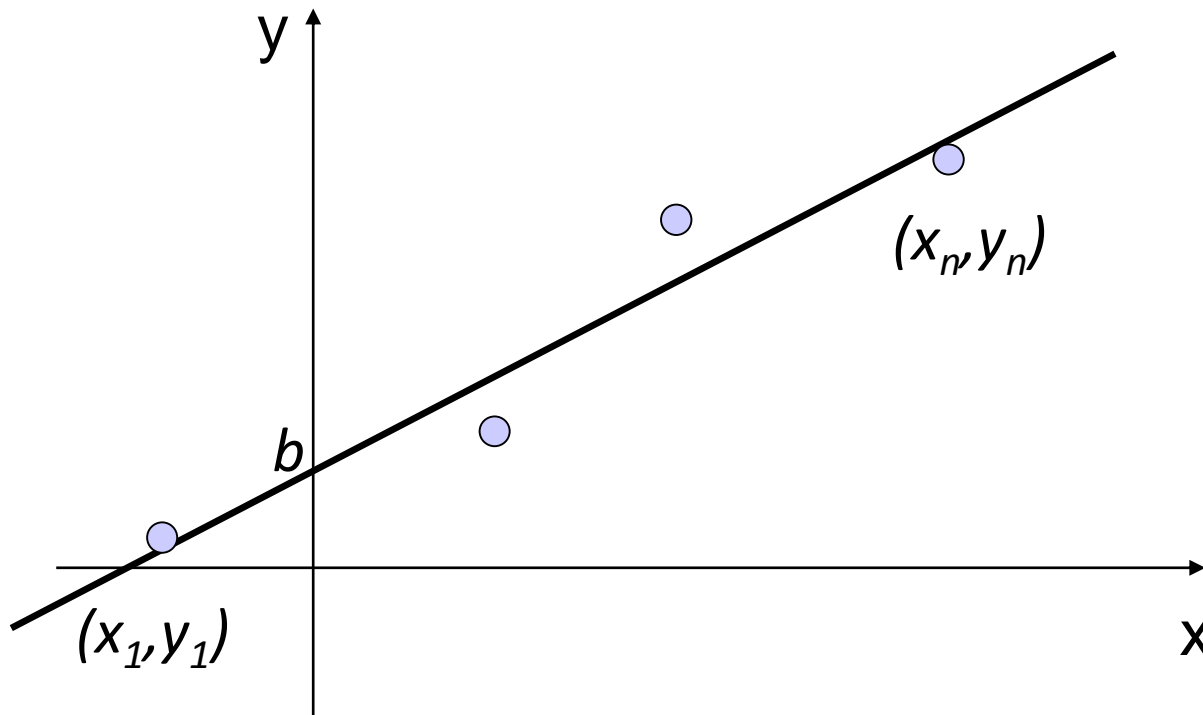
Least square line

- When $n > 2$, if $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ are **collinear**,
 $E = 0$



Least square line

- Line equation $y = kx + b$



Least square line

- How to get k and b ?

$$\begin{cases} kx_1 + b = y_1 \\ kx_2 + b = y_2 \\ \dots \\ kx_n + b = y_n \end{cases} \quad \text{or} \quad \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & 1 \\ x_n & 1 \end{bmatrix} \begin{bmatrix} k \\ b \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

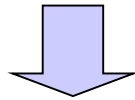
Mostly an approximation solution can exist, when the rank of the coefficient matrix is 2, which is the column number.



Least square line

- How to get k and b ?

$$\begin{bmatrix} x_1 & x_2 & \cdots & x_n \\ 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix} \begin{bmatrix} k \\ b \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \\ 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$



$$\begin{bmatrix} \sum_{i=1}^n x_i^2 & \sum_{i=1}^n x_i \\ \sum_{i=1}^n x_i & n \end{bmatrix} \begin{bmatrix} k \\ b \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^n x_i y_i \\ \sum_{i=1}^n y_i \end{bmatrix}$$

Least square line

- How to get k and b ?

$$\begin{bmatrix} \sum_{i=1}^n x_i^2 & \sum_{i=1}^n x_i \\ \sum_{i=1}^n x_i & n \end{bmatrix} \begin{bmatrix} k \\ b \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^n x_i y_i \\ \sum_{i=1}^n y_i \end{bmatrix}$$

The unique solution of this system k and b can satisfy the following condition and a least square line is obtained.

$$\text{Minimum } E = \sum_{i=1}^n (f(x_i) - y_i)^2$$



Least square line

- When the coefficient matrix is singular, for example $x_1 = x_2 = \dots = x_n$, k and b cannot be solved directly



~~$$y = kx + b$$~~

~~$$x = ky + b$$~~

Or even general parametric form

$$\begin{cases} x = x(t) = a_0 + a_1 t \\ y = y(t) = b_0 + b_1 t \end{cases}$$

Least square line

- For parametric definition, the least square line problem is now to find a_0, a_1, b_0, b_1 satisfying

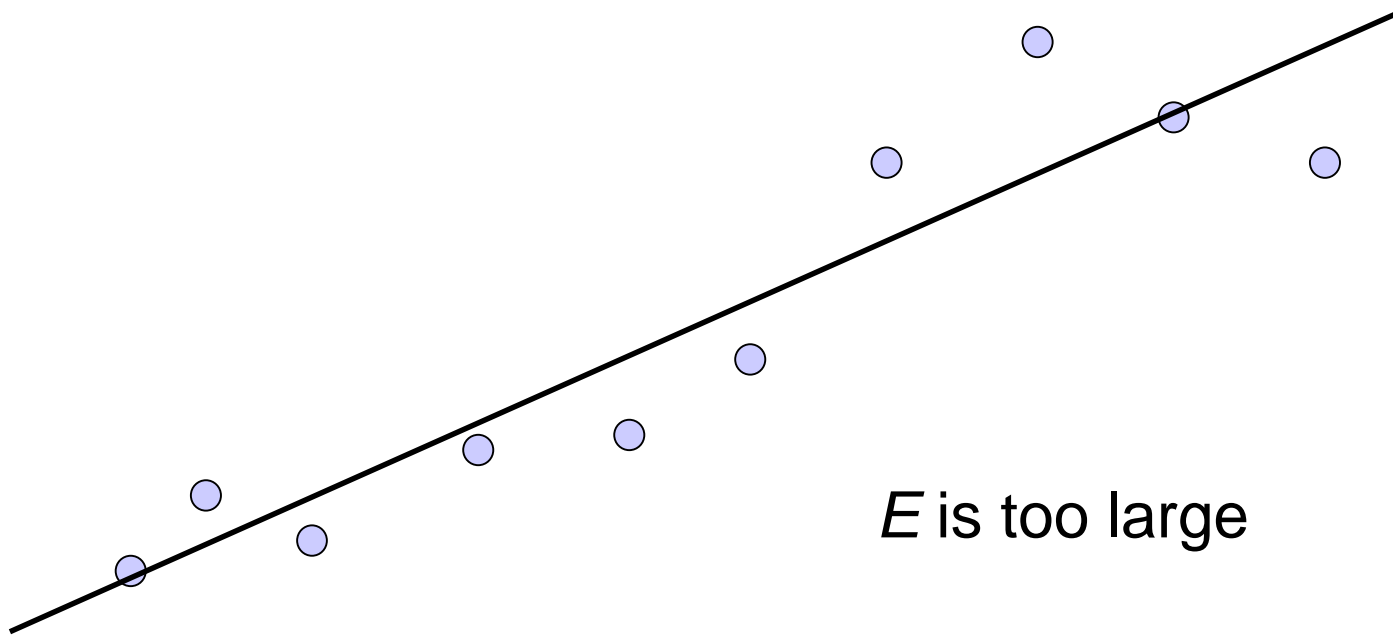
$$\text{Minimum } E = \sum_{i=1}^n \left[(x(t_i) - x_i)^2 + (y(t_i) - y_i)^2 \right]$$

- Ways to choose t_1, t_2, \dots, t_n will affect the result.



Least square curve

- Why we need least square curve? When raw data is too complicated, least square line is not good enough.



Least square curve

- In more general, $f(x) \in P_k$ can be a polynomial of degree k

$$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_kx^k$$

- The problem becomes now to find a_0, a_1, \cdots, a_k satisfying the following

$$\text{Minimum } E = \sum_{i=1}^n (f(x_i) - y_i)^2$$



Least square curve

- Systems to be solved are

$$a_0 + a_1x_1 + a_2x_1^2 + \cdots + a_kx_1^k = y_1$$

$$a_0 + a_1x_2 + a_2x_2^2 + \cdots + a_kx_2^k = y_2$$

...

$$a_0 + a_1x_n + a_2x_n^2 + \cdots + a_kx_n^k = y_n$$

And

$$\begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^k \\ 1 & x_2 & x_2^2 & \cdots & x_2^k \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^k \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_k \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$



The coefficient matrix \mathbf{M} : $n \times (k + 1)$



Least square curve

- When $n > k$ and rank of M is $k + 1$, we solve the following system to get the least square curve

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_1 & x_2 & x_3 & \cdots & x_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_1^k & x_2^k & x_3^k & \cdots & x_n^k \end{bmatrix} \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^k \\ 1 & x_2 & x_2^2 & \cdots & x_2^k \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^k \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_k \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_1 & x_2 & x_3 & \cdots & x_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_1^k & x_2^k & x_3^k & \cdots & x_n^k \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

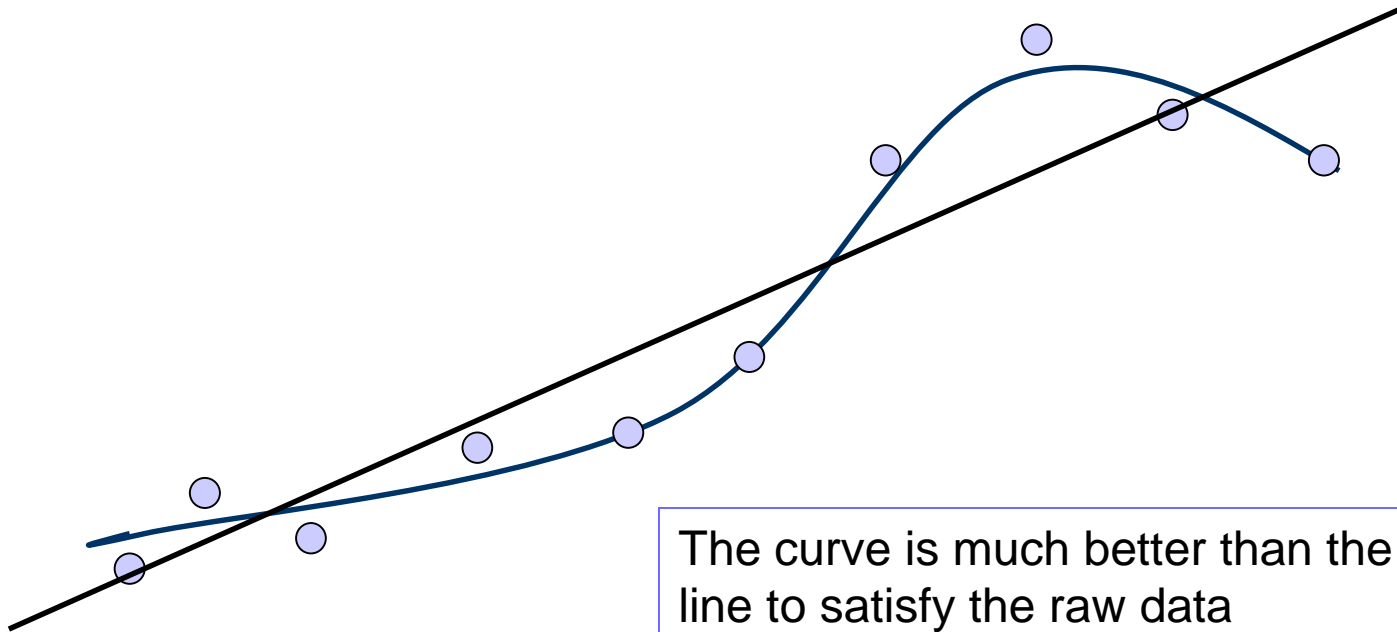
$$(M^T M)X = M^T D$$



The coefficient matrix $M^T M$: $(k + 1) \times (k + 1)$

Least square curve

- If $x_1 \neq x_2 \neq \dots \neq x_n$ and $n > k$, we can always find the unique solution of the system, and it will be the least square solution to the original system.



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- Interpolation and Approximation
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- Surface Modeling
 - Bézier surface

