中 山 大 学 软 件 学 院 **2009** 级 软 件 工 程 专 业 (2009 秋季学期)

《线性代数》期末试题答案 (B)

(考 试 形 式： 闭 卷　　考 试 时 间： 2 小 时)

警 示

《中山大学授予学士学位工作细则》第六条

考 试 作 弊 不 授 予 学 士 学 位

方向：________________　　姓名：________　　学号：__________

1. Fill the blank (4 titles * 4 points/title = 16 points)

(1) The matrics A and B below are row equivalent.

$$A=\begin{bmatrix}-2 & -5 & 8 & 0 & -17\\ 1 & 3 & -5 & 1 & 5\\ 3 & 11 & -19 & 7 & 1\\ 1 & 7 & -13 & 5 & -3\end{bmatrix};\quad B=\begin{bmatrix}1 & 3 & -5 & 1 & 5\\ 0 & 1 & -2 & 2 & -7\\ 0 & 0 & 0 & -4 & 20\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}$$

A basis for $Row\ A$ is

$$\underline{\{(1\ \ 3\ \ -5\ \ 1\ \ 5),(0\ \ 1\ \ -2\ \ 2\ \ -7),(0\ \ 0\ \ 0\ \ -4\ \ 20)\}}.$$

(2) If $A=\begin{bmatrix}1 & -1 & -3 & 0\\ 0 & 1 & 5 & 4\\ -1 & 2 & 8 & 5\\ 3 & -1 & -2 & 3\end{bmatrix}$, then $\det A=$ **_3_**.

(3) Find matrix $A=\underline{\begin{bmatrix}0 & 2 & 3\\ 1 & 1 & -2\\ 4 & 1 & 0\\ 3 & -1 & -1\end{bmatrix}}$ such that $ColA=\left\{\begin{bmatrix}2s+3t\\ r+s-2t\\ 4r+s\\ 3r-s-t\end{bmatrix}: r,s,t\in R\right\}$.

(4) If α_1 and α_2 are orthonomal vectors，and $x=\alpha_1+5\alpha_2$， $y=4\alpha_1-3\alpha_2$， then

$x\cdot y=$ **_-11_**。

2. Mark each statement True or False, and descript your reasons (3titles * 8 points/title = 24 points)

(1) If $AB=C$ and C has 5 columns, then A has 5 columns.　　**(False)**

(2) All polynomials of degree at most 5 consists of a vector space.　　**(True)**

(3) Let $A \in R^{n\times n}$ and $\det(A)=a$, then $\det(3A)=3a$. **(False)**

(4) If $A^T = A$ and if vectors u and v satisfy $Au=3u$ and $Av=4v$, then $u\cdot v=0$. **(True)**

(5) Let $A \in R^{5\times 4}$ and $rankA=3$, then $\dim NulA=3$. **(False)**

(6) $A \in R^{n\times n}$ is invertible if and only if all columns of A are linear independent. **(True)**

(7) $Nul\ A=\{0\}$ if and only if the linear transformation $x \mapsto Ax$ is one to one. **(True)**

(8) If $A \in R^{n\times n}$ is diagonalizable, then A has n distinct eigenvalues **(False)**

3. Calculation issues (40 points)

(1) Make a change of variable, $x=Py$, that transforms the following quadratic form into one with no cross-product term. Give P and the new quadratic form. (12 points).

$$3x_1^2+6x_2^2+3x_3^2-4x_1x_2+8x_1x_3+4x_2x_3$$

Solution: The matrix of the given quadratic form is $A=\begin{bmatrix}3 & -2 & 4\\ -2 & 6 & 2\\ 4 & 2 & 3\end{bmatrix}$, whose eigenvales are 7, 7, -2. The transformation matrix P is formed by the orthonomal eigenvectors corresponding to these eigenvalues. That is

$$P=\begin{bmatrix}1/\sqrt{2} & -1/\sqrt{18} & -2/3\\ 0 & 4/\sqrt{18} & -1/3\\ 1/\sqrt{2} & 1/\sqrt{18} & 2/3\end{bmatrix}$$

The new quadratic form with no cross-product term is $7x_1^2+7x_2^2-2x_3^2$.

(2) Let the system equations be $\begin{cases}(2-\lambda)x_1+2x_2-2x_3=1\\ 2x_1+(5-\lambda)x_2-4x_3=2\\ -2x_1-4x_2+(5-\lambda)x_3=-\lambda-1\end{cases}$. Find the approriate values for λ to make the system have at most one solution, no solution, and infinite solutions, respectivly. When the system has infinite solutions, write them in parametric vector form. (12 points).

Solution: The system has at most one solution as $\lambda \neq 1$ and $\lambda \neq 10$, no solution as $\lambda = 10$, and infinite solutions as $\lambda = 1$. The infinite solutions of the system can be written as $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $c_1, c_2 \in R$.

(3) Find a least-squares solution of Ax = b, and compute the least-squares error. (10 points).

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 3 & 3 \end{bmatrix}, \mathrm{b} = \begin{bmatrix} 3 \\ 5 \\ 7 \\ -3 \end{bmatrix}$$

Solution: The least-squares solution of Ax = b and the least-squares error are given as $\hat{x} = \begin{bmatrix} 10 \\ -6 \\ 2 \end{bmatrix}$ and $\|b - A\hat{x}\| = \sqrt{(-1)^2 + (-1)^2 + 1^2 + 1^2} = 2$, respectively.

(4) Let $\mathfrak{B} = \{\mathrm{b}_1, \mathrm{b}_2, \mathrm{b}_3\}$ be a basis for a vector space V, and let T: V--->V be a linear transformation with the property tha $T(\mathrm{b}_1) = \mathrm{b}_1 + \mathrm{b}_3, T(\mathrm{b}_2) = 2\mathrm{b}_1 - \mathrm{b}_3$, and. $T(\mathrm{b}_3) = 3\mathrm{b}_1 + 4\mathrm{b}_2 + 5\mathrm{b}_3$ .Find the matrix of T relative to the basis (6 points).

Solution:

$T(\mathrm{b}_1) = \mathrm{b}_2 + \mathrm{b}_3, \quad T(\mathrm{b}_2) = 2\mathrm{b}_1 - \mathrm{b}_3, \quad T(\mathrm{b}_3) = 3\mathrm{b}_1 + 4\mathrm{b}_2 + 5\mathrm{b}_3$

Set up coordinate vectors:

$$[T(\mathrm{b}_1)]_{\mathfrak{B}} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad [T(\mathrm{b}_2)]_{\mathfrak{B}} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}, \quad [T(\mathrm{b}_3)]_{\mathfrak{B}} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}. \quad \text{Set up} \quad [T]_{\mathfrak{B}} = \begin{bmatrix} 0 & 2 & 3 \\ 1 & 0 & 4 \\ 1 & -1 & 5 \end{bmatrix}$$

4. Prove issues (20 points)

(1) Let $\{\xi_1 \quad \xi_2 \quad \xi_3\}$ be a basis for R^3, and $\alpha_1 = \xi_1 + \xi_2 - 2\xi_3$, $\alpha_2 = \xi_1 - \xi_2 - \xi_3$, $\alpha_3 = \xi_1 + \xi_3$, $\beta = 6\xi_1 - \xi_2 - \xi_3$. Prove that $\{\alpha_1 \quad \alpha_2 \quad \alpha_3\}$ is also a basis for R^3, and find the coordinate vector of β relative to $\{\alpha_1 \quad \alpha_2 \quad \alpha_3\}$. (10 points)

Proof: We have $\begin{pmatrix}\alpha_1 & \alpha_2 & \alpha_3\end{pmatrix} = \begin{pmatrix}\xi_1 & \xi_2 & \xi_3\end{pmatrix}\begin{pmatrix}1 & 1 & 1 \\ 1 & -1 & 0 \\ -2 & -1 & 1\end{pmatrix} = \begin{pmatrix}\xi_1 & \xi_2 & \xi_3\end{pmatrix}A$.

ξ_1, ξ_2, ξ_3 are linear independent, and $|A| = -5 \neq 0$, therefore $\alpha_1, \alpha_2, \alpha_3$ are also linear independent, and consist of a basis of R^3. Let $\beta = k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3$. We obtain the following linear system equations

$\begin{cases} k_1 + k_2 + k_3 = 6 \\ k_1 - k_2 = -1 \\ -2k_1 - k_2 + k_3 = -1 \end{cases}$. A solution to the equations is $k_1 = 1$, $k_2 = 2$, $k_3 = 3$. Thus, the coordinate vector of β relative to $\{\alpha_1 \quad \alpha_2 \quad \alpha_3\}$ is $\begin{pmatrix}1 \\ 2 \\ 3\end{pmatrix}$.

(2) Suppose A is a $m \times n$ matrix such that the matrix $A^T A$ is invertible. Let b be any vector in R^n. Show that the Linear system $Ax = b$ has at most one solution. (10 points)

Proof:

Suppose Ax = b has two solutions. Let these solytions be x1 and x2. Thus Ax1=b and Ax2= b.Therefore Ax1 = Ax2. Mutiplying by A^T gives A^TAx1 = A^TAx2. So $(A^TA)^{-1}A^T$Ax1 = $(A^TA)^{-1}A^T$Ax2. So Ix1 = Ix2. Since $(A^TA)^{-1}A^TA$ = I is the identity matrix. So x1 = x2, as desired.