Discrete Mathematics: Lecture 2

- Today
  - Chap 1.4: Predicates and Quantifiers
  - Chap 1.5: Nested Quantifiers
- Assignment 1: due when we finish Chap 1
- Next time
  - Chap 1.6: Rules of Inference

Review of last time

- Proving logical equivalences using existing ones
- Propositional satisfiability
- Modeling Sudoku as a satisfiability problem
- CNF and DNF
- Every propositional formula is logically equivalent to one in CNF (resp. DNF)

Predicates and quantifiers

Propositional logic cannot adequately express the meaning of statements in mathematics and in natural language.

- Every computer connecting to the university network is functioning properly. Thus MATH3 is functioning properly.
- CS2 is under attack by an intruder. Thus there is a computer on the university network that is under attack by an intruder.

a more powerful logic: predicate logic, or predicate calculus

Predicates （谓词）

- Statements involving variables, such as
  - $x > 3$, $x = y + 3$, $x + y = z$
  - computer x is functioning properly
- These statements do not have truth values when the values of the variables are not specified.
- We denote "$x > 3$" by $P(x)$, where P denotes the predicate "is greater than 3". We can consider P as a propositional function

Examples

- $P(x) : x > 3$, truth values of $P(4)$ and $P(2)$
- $A(x)$: computer x is under attack by an intruder. Suppose only CS2 and MATH1 are under attack. truth values of $A(CS1)$ and $A(CS2)$
- $Q(x, y) : x = y + 3$, truth values of $Q(1, 2)$ and $Q(3, 0)$

In general, $P(x_1, \ldots, x_n)$, an n-place predicate, or an n-ary predicate

Universal quantifier （全称量词）

- **Definition:** The universal quantification of $P(x)$, denoted by $\forall x P(x)$, read "for all x $P(x)$" or "for every x $P(x)$", is the statement "$P(x)$ is true for all values of x in the domain".
- The domain （论域） is called the domain of discourse or the universe of discourse
- $\forall$ is called the universal quantifier
- An element for which $P(x)$ is false is called a counterexample of $\forall x P(x)$
- **Example:** $P(x) : x + 1 > x$. What is the truth value of $\forall x P(x)$ when the domain consists of all real numbers

Universal quantifier

- An implicit assumption is that all domains of discourse are not empty
- If the domain is empty, then $\forall x P(x)$ is true for all predicates P, since there is no element x in the domain for which $P(x)$ is false
- Other expressions for universal quantification: all of, for each, given any, for any, for arbitrary
- **Example:** $Q(x) : x < 2$. What is the truth value of $\forall x Q(x)$ when the domain consists of all real numbers
- **Example:** $P(x) : x^2 > 0$. What is the truth value of $\forall x P(x)$ when the domain consists of all integers

Universal quantifier

- When all the elements in the domain can be listed, say $x_1, x_2, \ldots, x_n$, then $\forall x P(x)$ is the same as the conjunction $P(x_1) \wedge P(x_2) \wedge \ldots \wedge P(x_n)$
- **Example:** $P(x) : x^2 < 10$. What is the truth value of $\forall x P(x)$ when the domain consists of the positive integers not exceeding 4
- The truth value of $\forall x P(x)$ depends on the domain
- **Example:** $P(x) : x^2 \geq x$. What is the truth value of $\forall x P(x)$ when the domain consists of all real numbers (resp. integers)?

Existential quantifier（存在量词）

- **Definition:** The existential quantification of $P(x)$, denoted by $\exists x P(x)$, read "there exists x $P(x)$", is the statement "there exists an x in the domain such that $P(x)$".
- Expression for existential quantification: for some, for at least one, there is
- **Example:** $P(x) : x > 3$. What is the truth value of $\exists x P(x)$ when the domain consists of all real numbers
- $\exists x P(x)$ is false iff $P(x)$ is false for every element x of the domain
- **Example:** $Q(x) : x = x + 1$. What is the truth value of $\exists x Q(x)$ when the domain consists of all real numbers
- If the domain is empty, $\exists x P(x)$ is false, since there is no element of the domain for which $P(x)$ is true

Existential quantifier

- When all the elements in the domain can be listed, say $x_1, x_2, \ldots, x_n$, then $\exists x P(x)$ is the same as the disjunction $P(x_1) \vee P(x_2) \vee \ldots \vee P(x_n)$
- **Example:** $P(x) : x^2 > 10$. What is the truth value of $\exists x P(x)$ when the domain consists of the positive integers not exceeding 4

Other quantifiers

- the uniqueness quantifier, $\exists! x P(x)$ or $\exists_1 x P(x)$, there exists a unique x such that $P(x)$ is true
- Example: $P(x) : x^2 \leq x$, $\exists! x P(x)$, the domain consists of all real numbers (resp. integers)
- Quantifiers with restricted domains
- Example: the domain consists of all real numbers, $\forall x \leq 0 (x^2 > 0), \forall y \neq 0 (y^3 \neq 0), \exists z > 0 (z^2 = 2)$

Translating from English into logical expressions

An important task in mathematics and many other disciplines

- "Every student in this class has studied calculus"
- $\Rightarrow$ "For every student in this class, that student has studied calculus"
- $\Rightarrow$ "For every student x in this class, x has studied calculus"
- $\Rightarrow \forall x C(x)$, where $C(x)$: x has studied calculus, and the domain consists of students in the class

Translating from English into logical expressions

- Different domains of discourse and predicates can be used.
- Our decision depends on the subsequent reasoning we want.
- If we change the domain of discourse to all people,
- "For every person x, if x is a student in this class, then x has studied calculus"
- $\forall x(S(x) \to C(x))$
- We may be interested in subjects besides calculus
- $Q(x, y)$: student x has studied subject y

Translating from English into logical expressions

- "Some student in this class has visited Mexico"
- $\Rightarrow$ "There is a student x in this class having the property that x has visited Mexico"
- "Every student in this class has visited either Canada or Mexico"
- $\Rightarrow$ "For every student x in this class, x has the property that x has visited either Canada or Mexico"

Example arguments

An argument consists of several premises and a conclusion.

- All lions are fierce.
  Some lions do not drink coffee.
  Some fierce creatures do not drink coffee.

- All hummingbirds are richly colored.
  No large birds live on honey.
  Birds that do not live on honey are dull in color.
  Hummingbirds are small.

Nested Quantifiers

- Two quantifiers are nested if one is within the scope of the other, *e.g.*, $\forall x \exists y(x + y = 0)$
- Assume the domain consists of all real numbers
- $\forall x \forall y(x + y = y + x)$, $\forall x \exists y(x + y = 0)$, $\forall x \forall y \forall z((x + y) + z = x + (y + z))$
- $\forall x \forall y((x > 0) \wedge (y < 0) \rightarrow (xy < 0))$

Thinking quantification as loops

- To decide whether $\forall x \forall y P(x,y)$ is true,
  - we loop through the values for x, and for each x, we loop through the values for y
  - if $P(x,y)$ is true for all values for x and y, $\forall x \forall y P(x,y)$ is true
  - if we ever hit a value x for which we hit a value y for which $P(x,y)$ is false, $\forall x \forall y P(x,y)$ is false
- To decide whether $\forall x \exists y P(x,y)$ is true
  - we loop through the values for x, and for each x, we loop through the values for y until we find a y for which $P(x,y)$ is true
  - if for every x we hit such a y, $\forall x \exists y P(x,y)$ is true
  - if for some x we never hit such a y, $\forall x \exists y P(x,y)$ is false

The order of quantifiers

The order of quantifiers is important, unless all are universal quantifiers or all are existential quantifiers

- $P(x, y) : x + y = y + x$, the domain consists of all real numbers, truth values of $\forall x \forall y P(x, y)$ and $\forall y \forall x P(x, y)$
- $Q(x, y) : x + y = 0$, the domain consists of all real numbers, truth values of $\forall x \exists y Q(x, y)$ and $\exists y \forall x Q(x, y)$
  - $\forall x \exists y Q(x, y)$: y can depend on x, $\exists y \forall x Q(x, y)$: y is independent of x
- $Q(x, y, z) : x + y = z$, the domain consists of all real numbers, truth values of $\forall x \forall y \exists z Q(x, y, z)$ and $\exists z \forall x \forall y Q(x, y, z)$

Translating from mathematical statements

- The sum of two positive integers is always positive
- Every real number except 0 has a multiplicative inverse
- The concept of limit: $\lim_{x \to a} f(x) = L$
  - for every real number $\epsilon > 0$, there exists a real number $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta$

Translating from logical expressions

- $\forall x(C(x) \vee \exists y(C(y) \wedge F(x,y)))$, $C(x)$: x has a computer, $F(x,y)$: x and y are friends, the domain consists of all students in your school
- $\exists x \forall y \forall z((F(x,y) \wedge F(x,z) \wedge (y \neq z)) \to \neg F(y,z))$, $F(a,b)$: a and b are friends, the domain consists of all students in your school

Translating from English sentences

- the domain consists of all people
- if a person is a female and is a parent, then this person is someone's mother
  - for every person x, if x is female and x is a parent, then there exists a person y such that x is y's mother
- everyone has exactly one best friend
  - for every person x, there exists a person y such that y is x's best friend, and for every person z, if z is not y, then z is not x's best friend
- there is a woman who has taken a flight on every airline in the world
  - $P(w, f)$: w has taken f, $Q(f, a)$: f is a flight on a
  - the domains for w, f, a consists of all women, flights, airlines, respectively
  - $\exists w \forall a \exists f (P(w, f) \wedge Q(f, a))$

Precedence of quantifiers (量词的优先级)

- $\forall$ and $\exists$ have higher precedence than all logical operators from propositional calculus
- Example: $\forall x P(x) \vee Q(x)$

Binding Variables

- When a quantifier is used on the variable x, we say that this occurrence of the variable is bound (约束出现).
- An occurrence of a variable that is not bound by a quantifier is said to be free (自由出现).
- The part of a logical expression to which a quantifier is applied is called the scope of the quantifier (量词的辖域)
- Example: $\exists x(x + y = 1)$, bound/free variables
- Example: $\exists x(P(x) \wedge Q(x)) \vee \forall x R(x)$, bound/free variables, scope of quantifiers

Logical equivalences involving quantifiers

- Definition: Two statements involving predicates and quantifiers are logically equivalent (逻辑等价) iff they have the same truth value no matter which predicates are substituted into these statements and which domain of discourse is used.
- Notation: $S \equiv T$, or $S \Leftrightarrow T$
- Example: $\forall x(P(x) \wedge Q(x)) \equiv \forall x P(x) \wedge \forall x Q(x)$
- Question: Do we have $(\forall x A \vee \forall x B) \equiv \forall x(A \vee B)$?
- How about $\exists x(P(x) \vee Q(x)) \equiv \exists x P(x) \vee \exists x Q(x)$?
- How about $\exists x(A \wedge B) \equiv (\exists x A \wedge \exists x B)$?

Null quantification (1)

When x does not occur as a free variable in A

- $\forall x P(x) \vee A \equiv \forall x (P(x) \vee A)$
- $\exists x P(x) \vee A \equiv \exists x (P(x) \vee A)$
- $\forall x P(x) \wedge A \equiv \forall x (P(x) \wedge A)$
- $\exists x P(x) \wedge A \equiv \exists x (P(x) \wedge A)$

De Morgan's laws for quantifiers

- $\neg \exists x P(x) \equiv \forall x \neg P(x)$
- $\neg \forall x P(x) \equiv \exists x \neg P(x)$
- Relationship to De Morgan's laws in propositional logic when the domain has n elements $x_1, x_2, \ldots, x_n$

Proving logical equivalence

- Using existing logical equivalence
- Replacement theorem: If B is a subformula of A and $B \Leftrightarrow B'$, let A' be the result of replacing B in A by B', then $A \Leftrightarrow A'$
- Example: $\neg \forall x (P(x) \rightarrow Q(x)) \Leftrightarrow \exists x (P(x) \wedge \neg Q(x))$

Null quantification (2)

When x does not occur as a free variable in A

- $\forall x P(x) \to A \equiv \exists x (P(x) \to A)$
- $\exists x P(x) \to A \equiv \forall x (P(x) \to A)$
- $A \to \forall x P(x) \equiv \forall x (A \to P(x))$
- $A \to \exists x P(x) \equiv \forall x (A \to P(x))$

Negating quantified expressions

- Find the negation of the statement “Every student in your class has taken a course in calculus”
- Find the negation of the statement “There is a student in your class who has taken a course in calculus”
- The negation of “There is an honest politician”
- Note that the English statement “All politicians are not honest” means ”not all politicians are honest”
- The negation of “All Americans eat cheeseburgers”
- The negations of $\forall x(x^2 > x)$ and $\exists x(x^2 = 2)$

Negating Nested Quantifiers

We negate nested quantifiers by successively applying the rules for negating a single quantifier.

- Negate $\forall x \exists y(xy = 1)$ so that no negation precedes a quantifier
- There does not exist a woman who has taken a flight on every airline in the world
- Express the fact that $\lim_{x \to a} f(x)$ does not exist