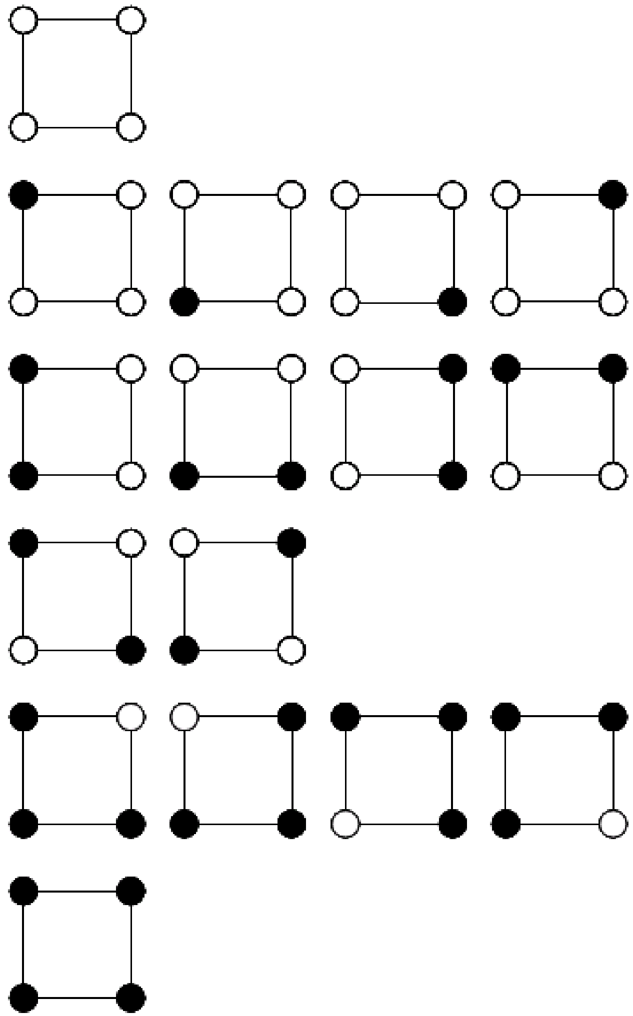
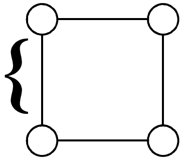
# 组合数学
# 第14章 Pólya计数法

# 主要内容

- 1. Burnside引理
- 2. Pólya计数法
- 3. 刚体变换群
- 关键词: 群, 不变, 循环



# 等价关系

$X = \{$  $\}$ 染色对象

$C = \{\, c_0, c_1, \ldots, c_{15} \,\}$ 全体着色方案

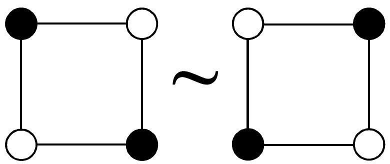
$G = \{\, \rho_4^0, \rho_4^1, \rho_4^2, \rho_4^3 \,\}$ 旋转变换, $\circ$, $*$

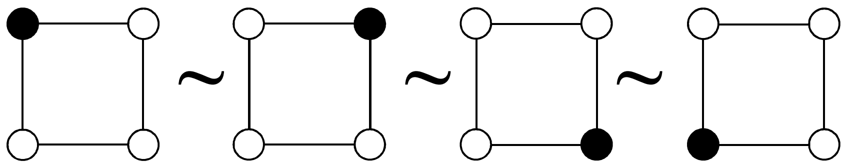
群$(\,G, \circ\,)$ : (1) 封闭; (2) 么元; (3) 逆元.

$*$ : $\forall f,g \in G,\ c \in C,\ \ f*(g*c) = (f \circ g)*c.$

等价 : 若 $c,d \in C$ 且 $\exists f \in G$ 使得$f*c=d$,

则 称$c$等价于$d$, 记为$c \sim d$. 例:





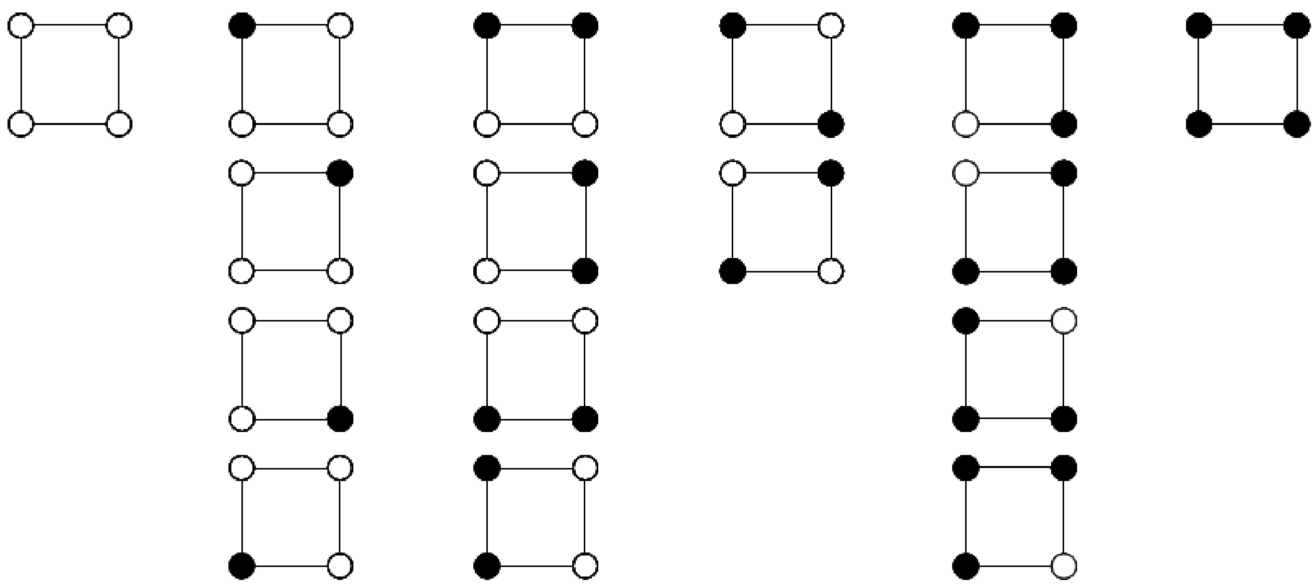
# 等价类

$\forall c \in C$, $[c] = \{ d \in C : d \sim c \}$ 是c的等价类.

$= \{ f^*c : f \in G \}$

定义 $N(G,C)$ = C中G产生的等价类个数.

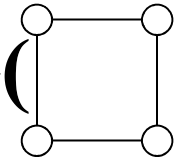
注：$\|[c]\|$ = 与c等价的方案数.

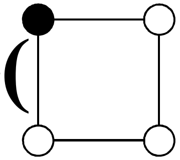
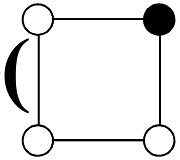
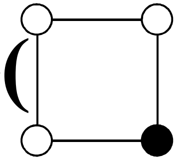
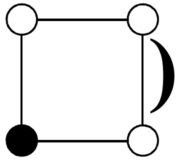


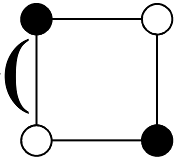
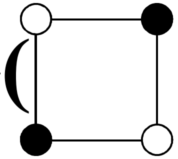
# 保持c不变的变换集

定义: $\forall c \in C$, 保持c不变的变换集合定义为

$$G(c) = \{ f \in G : f^*c = c \}$$

G(  ) $= \{ \rho_4^0, \rho_4^1, \rho_4^2, \rho_4^3 \}$,

G(  ) = G(  ) = G(  ) = G(  ) $= \{ \rho_4^0 \}$

G(  ) = G(  ) $= \{ \rho_4^0, \rho_4^2 \}$

引理: 对任意$c \in C$, $|G(c)| = \dfrac{|G|}{|[c]|}$

对比例子验证公式

# 等价类个数

引理: 对任意c∈C, $|G(c)| = \frac{|G|}{|[c]|}$

取1个等价类[c]: $\sum_{d\in[c]} |G(d)| = |[c]| \times \frac{|G|}{|[c]|} = |G|$

取N(G,C)个等价类: $\sum_{d\in C} |G(d)| = N(G,C)\cdot |G|$

# 群G(c)

引理: 对任意$c\in C$, $|G(c)| = \dfrac{|G|}{|[c]|}$

证明: $[c] = \{ f{*}c : f\in G \}$

$= \{ d_0, d_1, \ldots, d_k \}$, 令$d_0=c$.

令 $G_i = \{ f\in G : f{*}c=d_i \}$, $i=0,1,\ldots,k$, 则有

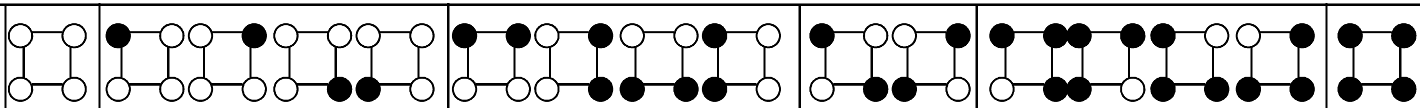
(1) $G_0 = G(c)$;

(2) $G = G_0\cup G_1\cup\ldots\cup G_k$是不交并;

(3) $|G_i| = |G(c)|$. 取$f\in G_i$, $G(c)\xrightarrow{f} G_i$是单满射

$$\sum_{d\in[c]} |G(d)| = |G| \qquad \sum_{d\in C} |G(d)| = N(G,C)\cdot |G|$$

# 不变 { (f,c) : f*c=c }



| | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | | | | | | | | \|C(f)\| |
| $\rho_4^0$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 16 |
| $\rho_4^1$ | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 2 |
| $\rho_4^2$ | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 4 |
| $\rho_4^3$ | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 2 |
| \|G(c)\| | 4 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 2 | 2 | 1 | 1 | 1 | 1 | 4 | 24 |

$\forall c\in C$, $G(c)=\{f\in G: f^*c=c\}$ 保持c不变的变换集

$\forall f\in G$, $C(f)=\{c\in C: f^*c=c\}$ f作用下不变的染色集

$$\sum_{f\in G}|C(f)|=\sum_{c\in C}|G(c)|=N(G,C)\cdot|G|$$

# Burnside定理

**定义：** $C(f) = \{ c \in C : f^*c = c \}$

**Burnside 定理** 设 G是X的一个变换群，

C是X的一个着色集，

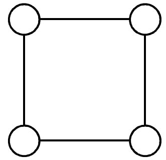
且对$\forall c \in C, f \in G, f^*c \in C$，

则有

$$N(G,C) = \frac{1}{|G|} \sum_{f \in G} |C(f)|$$

# Burnside定理举例

用两种颜色染色(可以旋转)



$G = \{ \rho_4^0, \rho_4^1, \rho_4^2, \rho_4^3 \},$

$C(\rho_4^0)=2^4$

$C(\rho_4^1)=2$

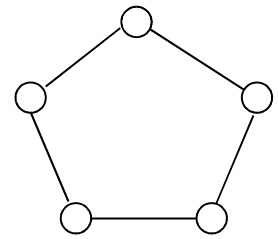
$C(\rho_4^2)=4$

$C(\rho_4^3)=2$

$N(G,C)=(16+2+2+4)/4=6$

# Burnside定理举例

用两种颜色染色(可以旋转)



$G=\{\rho_5^0, \rho_5^1, \rho_5^2, \rho_5^3, \rho_5^4\}$,

$C(\rho_5^0)=2^5$

$C(\rho_5^1)=2$

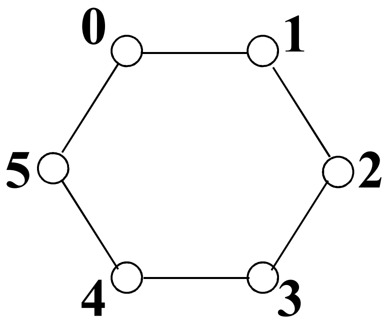
$C(\rho_5^2)=2$

$C(\rho_5^3)=2$

$C(\rho_5^4)=2$

$N(G,C)=(32+2+2+2+2)/5=8$

# 循环节与Polya计数法




例: 用2色染正六边形的顶点,可旋转.

观察顶点的变化: $\rho_6^{\ 1} = (012345)$

$\rho_6^{\ 2} = (024)(135)$ $\rho_6^{\ 3} = (03)(14)(25)$

$$N(G,C) = \frac{2^6 + 2^1 + 2^2 + 2^3 + 2^2 + 2^1}{6} = 14$$

定理: 设对X用p种颜色染色, 且f是X上的变换, 以#f记f的循环节个数, 则有

$$|C(f)| = p^{\#f}.$$

命题: $\#\rho_n^{\ k} = \gcd(k,n)$.

# 圆排列

从p色珠子中取n个均匀镶嵌圆环上, 求方案数.

$$G = \{ \rho_n^k : k=0,1,\ldots,n-1 \}$$

- 对于$\rho_n^k$, 其循环节数$d=\gcd(k,n)$
- 对于$d|n$, $0\sim n-1$中有$\phi(n/d)$个k满足$\gcd(k,n)=d$

由Polya计数法, 圆排列个数为

$$\frac{1}{n}\sum_{d|n}\phi\left(\frac{n}{d}\right)p^d$$

# 手镯刚体变换群

给手镯均匀点缀**n**个珠子:

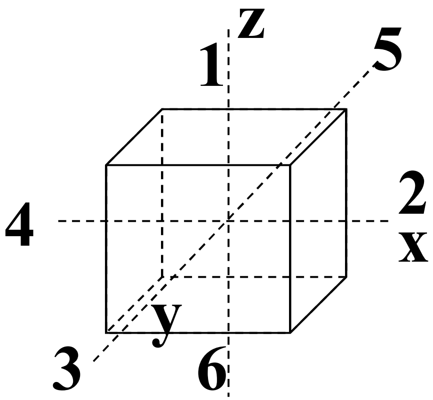
**G={ $\rho_n^0$, $\rho_n^1$, …, $\rho_n^{n-1}$,** **n**个旋转

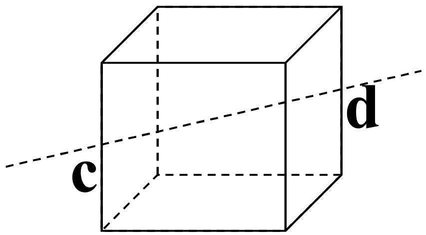
**$\tau_1$, $\tau_2$, …, $\tau_n$, }** **n**个翻转

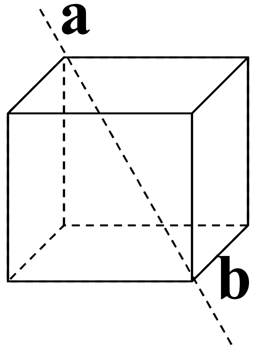
**n**是奇数的公式 $$\frac{1}{2n}\left(\sum_{d|n}\phi\left(\frac{n}{d}\right)p^d+n\times p^{\frac{n+1}{2}}\right)$$

**n**是偶数的公式 $$\frac{1}{2n}\left(\sum_{d|n}\phi\left(\frac{n}{d}\right)p^d+\frac{n}{2}\times p^{\frac{n}{2}+1}+\frac{n}{2}\times p^{\frac{n}{2}}\right)$$

# 正六面体刚体变换群




**24个变换:**

**不动置换: (1)(2)(3)(4)(5)(6), 1个;**

**绕x轴转90°: (2)(1365)(4), 3×2个;**

**绕x轴转180°: (2)(35)(16)(4), 3个;**

**绕cd轴旋转180°: (16)(25)(34), 6个;**

**绕ab轴旋转120°: (154)(263), 2×4个.**

例. 正六面体贴正方形大头贴

$$N(G,C)=\frac{4^6+6\times4^3+8\times4^2}{24}=192$$

# 应用举例

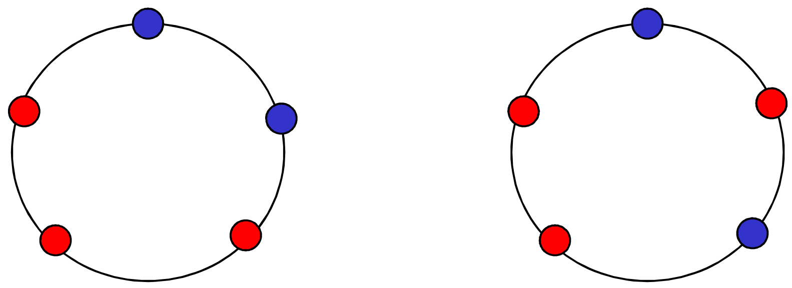
**ex25. 用3红色2蓝色珠子均匀点缀手镯,求方案数.**

**解:** $G = \{\rho_5^0, \ldots, \rho_5^4, \tau_1, \ldots, \tau_5\}$

$|C(\rho_5^0)| = C(5,3)$, $|C(\rho_5^i)| = 0$, $i = 1,2,3,4$

$|C(\tau_j)| = C(2,1)$, $j = 1,2,3,4,5$

$N(G,C) = (10 + 2\times 5)/10 = 2$



# 本章小结

**Burnside**引理

**Polya**计数法

刚体变换群