线性代数

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1.7 Linear Independence

Definition

- Linear Independence(线性独立/无关)

  An indexed set of vectors $\{v_1, \ldots, v_p\}$ in R^n is said to be linearly independent if the vector equation

  $$x_1\boldsymbol{v_1} + x_2\boldsymbol{v_2} + \cdots + x_p\boldsymbol{v_p} = \mathbf{0}$$

  has only the trivial solution.

- Linear Dependence(线性相关)

  The set $\{v_1, \ldots, v_p\}$ is said to be linearly dependent if there exist weights $c_1, \ldots, c_p$, not all zero, such that

  $$c_1\boldsymbol{v_1} + c_2\boldsymbol{v_2} + \cdots + c_p\boldsymbol{v_p} = \mathbf{0}$$

Example 1

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix},\ v_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix},\ v_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

a. Determine if the set $\{v_1, v_2, v_3\}$ is linearly independent.

b. If possible, find a linear dependence relation among v_1, v_2, v_3

Example 1

a. Row reduce the augmented matrix

$$\begin{bmatrix} 1 & 4 & 2 & 0 \\ 2 & 5 & 1 & 0 \\ 3 & 6 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 2 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

a. Row reduce the augmented matrix

$$\begin{bmatrix} 1 & 4 & 2 & 0 \\ 2 & 5 & 1 & 0 \\ 3 & 6 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 2 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Clearly, x_1 and x_2 are basic variables, and x_3 is free.

Each nonzero value of x_3 determines a nontrivial solution.

Hence v_1, v_2, v_3 are linearly dependent.

Example 1

b. completely row reduce the augmented matrix:

$$\begin{bmatrix}1 & 4 & 2 & 0\\ 2 & 5 & 1 & 0\\ 3 & 6 & 0 & 0\end{bmatrix} \sim \begin{bmatrix}1 & 4 & 2 & 0\\ 0 & -3 & -3 & 0\\ 0 & 0 & 0 & 0\end{bmatrix} \sim \begin{bmatrix}1 & 0 & -2 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix} \quad \begin{aligned} x_1 \quad - 2x_3 &= 0\\ x_2 + x_3 &= 0\\ 0 &= 0\end{aligned}$$

Thus, x_1=2x_3, x_2=-x_3, and x_3 is free.

b. completely row reduce the augmented matrix:

$$\begin{bmatrix}1 & 4 & 2 & 0\\ 2 & 5 & 1 & 0\\ 3 & 6 & 0 & 0\end{bmatrix} \sim \begin{bmatrix}1 & 4 & 2 & 0\\ 0 & -3 & -3 & 0\\ 0 & 0 & 0 & 0\end{bmatrix} \sim \begin{bmatrix}1 & 0 & -2 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix} \quad \begin{aligned} x_1 \quad - 2x_3 &= 0\\ x_2 + x_3 &= 0\\ 0 &= 0\end{aligned}$$

Thus, x_1=2x_3, x_2=-x_3, and x_3 is free.

Choose x_3=5, Then x_1=10 and x_2=-5.

So one possible linear dependence relations among v_1, v_2, v_3 is

$10\boldsymbol{v_1} - 5\boldsymbol{v_2} + 5\boldsymbol{v_3} = 0$

2. Linear Independence of Matrix Columns

Suppose a matrix $A = [a_1 \cdots a_n]$ instead of a set vectors. Then the matrix equation Ax=0 can be written as

$$x_1\boldsymbol{a_1} + x_2\boldsymbol{a_2} + \cdots + x_n\boldsymbol{a_n} = \mathbf{0}$$

注意:

The columns of a matrix A are linearly independent if and only if the equation $\boldsymbol{Ax} = \mathbf{0}$ has only the trivial solution.

Example 3

Determine if the columns of the following matrix are linearly independent.

$$A = \begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & -1 \\ 5 & 8 & 0 \end{bmatrix}$$

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Clearly, Ax=0 has only the trivial solution, thus the columns of A are linearly independent.

3. Linear Independence of Sets of One or Tow Vectors

- A set of two vectors $\{v_1, v_2\}$ is linearly dependent if at least one of the vectors is a multiple of the other.
- The set is linearly independent if and only if neither of the vectors is a multiple of the other.

Sometimes we can determine linear independence of a set with minimal effort.

1. A Set of One Vector

Consider the set containing one nonzero vector:$\{v_1\}$

The only solution to $x_1\boldsymbol{v_1} = 0$ is x_1 $\boldsymbol{=}$ $\underline{\ \mathbf{0}\ }$.

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So $\{v_1\}$ is linearly independent when $\boldsymbol{v_1} \neq \mathbf{0}$.

2. *A Set of Two Vectors*

EXAMPLE Let

$$\mathbf{u}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

a. Determine if $\{\mathbf{u}_1, \mathbf{u}_2\}$ is a linearly dependent set or a linearly independent set.

b. Determine if $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a linearly dependent set or a linearly independent set.

Solution: (a) Notice that $\mathbf{u}_2 = \underline{\quad 2 \quad}\,\mathbf{u}_1$. Therefore

$$\underline{\quad 2 \quad}\,\mathbf{u}_1 + \underline{\quad -1 \quad}\,\mathbf{u}_2 = 0$$

This means that $\{\mathbf{u}_1, \mathbf{u}_2\}$ is a linearly __dependent__ set.

(b) Suppose

$$c\mathbf{v}_1 + d\mathbf{v}_2 = \mathbf{0}.$$

Then $\mathbf{v}_1 = \frac{-d}{c}\mathbf{v}_2$ if $c \neq 0$. But this is impossible since $\mathbf{v}_1$ is not a multiple of $\mathbf{v}_2$ which means $c =$ 0.

Similarly, $\mathbf{v}_2 = \frac{-c}{d}\mathbf{v}_1$ if $d \neq 0$.

But this is impossible since $\mathbf{v}_2$ is not a multiple of $\mathbf{v}_1$ and so $d = 0$.

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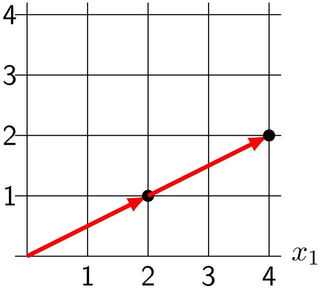
注意:

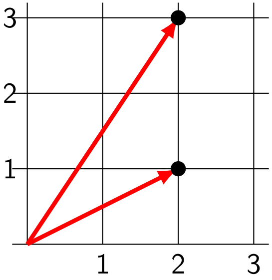
A set of two vectors is linearly dependent if at least one vector is a multiple of the other.

解:

(a)$\{u_1, u_2\}$ is linear dependent



(b)$\{v_1, v_2\}$ is linear independent



Theorem (7: Characterizarion of Linearly Dependent Sets)

An indexed set $S = \{v_1, \ldots, v_p\}$ *of two or more vectors is linearly dependent if and only if at least one of the vectors in* S *is a linear combination of the others. In fact, if* S *is linearly dependent and* $v_1 \neq 0$*, then some* v_j *is a linear combination of the preceding vectors,* $v_1, \ldots, v_{j-1}$.

PROOF OF THEOREM 7 (Characterization of Linearly Dependent Sets).

If some $\boldsymbol{v_j}$ in S equals a linear combination of the other vectors, then $\boldsymbol{v_j}$ can be subtracted from both sides of the equation, producing a linear dependence relation with a nonzero weight(-1) on $\boldsymbol{v_j}$. [For instance, if $\boldsymbol{v_1} = c_2\boldsymbol{v_2} + c_3\boldsymbol{v_3}$, then $\mathbf{0} = (-1)\boldsymbol{v_1} + c_2\boldsymbol{v_2} + c_3\boldsymbol{v_3} + 0\boldsymbol{v_4} + \cdots + 0\boldsymbol{v_p}$.] Thus S is linearly dependent.

Conversely, suppose S is linearly dependent. If $\boldsymbol{v_1}$ is zero, then it is a (trivial)linear combination of the other vectors in S. Otherwise, $\boldsymbol{v_1} \neq \mathbf{0}$, and there exist weights $c_1, \ldots, c_p$, not all zeros, such that

$$c_1\boldsymbol{v_1} + c_2\boldsymbol{v_2} + \cdots + c_p\boldsymbol{v_p} = \mathbf{0}$$

Let j be the largest subscript for which $c_j \neq 0$. If $j = 1$, then $c_1\boldsymbol{v_1} = \mathbf{0}$, which is impossible because $\boldsymbol{v_1} \neq \mathbf{0}$. So $j > 1$, and

$$c_1\boldsymbol{v_1} + \cdots + c_j\boldsymbol{v_j} + 0\boldsymbol{v_{j+1}} + \cdots + 0\boldsymbol{v_p} = \mathbf{0}$$

$$c_j\boldsymbol{v_j} = -c_1\boldsymbol{v_1} - \cdots - c_{j-1}\boldsymbol{v_{j-1}}$$

$$\boldsymbol{v_j} = (-\frac{c_1}{c_j})\boldsymbol{v_1} + \cdots + (-\frac{c_{j-1}}{c_j})\boldsymbol{v_{j-1}}$$

Example：

Let $\boldsymbol{u} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$, $\boldsymbol{v} = \begin{bmatrix} 1 \\ 6 \\ 0 \end{bmatrix}$,

Describe the set spanned by u and v, and explain why a vector w is in Span $\{u, v\}$ if and only if $\{u, v, w\}$ is linearly dependently

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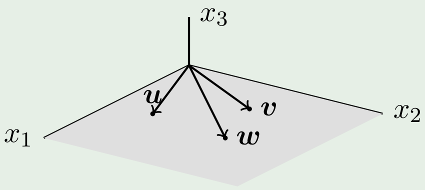
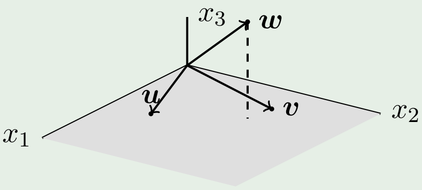
Describe the set spanned by u and v, and explain why a vector w is in Span $\{u, v\}$ if and only if $\{u, v, w\}$ is linearly dependently

解:

u and v are linearly independently, so they span a plane in R^3.Span $\{u, v\}$ is the $x_1 - x_2$ plane (x_3=0). If w is a linear combination of u and v, then $\{u, v, w\}$ is linearly dependently. Conversely, suppose that $\{u, v, w\}$ linearly dependently, by Theorem 7, some vector in $\{u, v, w\}$ is a linearly combination of the preceding vectors. That vector must be w. since v is not a multiple of u. So w is in Span $\{u, v\}$.

Example 6





Linearly dependent,
w in Span {u ,v}

Linearly independent,
w not in Span {u ,v}

Theorem (8)

If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. That is, any set $\{v_1, \ldots, v_p\}$ *in* R^n *is linearly dependent if* $p > n$.

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Outline of Proof:

$A = [v_1 \ v_2 \ \cdots \ v_p]$ is $n \times p$

Suppose $p > n$.

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Outline of Proof:

$A = [v_1 \; v_2 \; \cdots \; v_p]$ is $n \times p$

Suppose $p > n$.

$\Rightarrow$ $\boldsymbol{Ax} = \mathbf{0}$ has more variables than equations

$\Rightarrow$ $\boldsymbol{Ax} = \mathbf{0}$ has nontrivial solutions

$\Rightarrow$ columns of $\boldsymbol{A}$ are linearly dependent

$\square$

Theorem (9)

If a set $S = \{v_1, \ldots, v_p\}$ *in* R^n *contains the zero vector, then the set is linearly dependent.*

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Proof.

Renumber the vectors so that $\boldsymbol{v_1} = \underline{\ \mathbf{0}\ }$. Then

$$\underline{\ 1\ }\,\boldsymbol{v_1} + \underline{\ 0\ }\,\boldsymbol{v_2} + \cdots + \underline{\ 0\ }\,\boldsymbol{v_p} = \mathbf{0}$$

which shows that S is linearly _dependent_. □

Example

Determine by inspection if the given set is linearly dependent

a. $\begin{bmatrix}1\\7\\6\end{bmatrix}, \begin{bmatrix}2\\0\\9\end{bmatrix}, \begin{bmatrix}3\\1\\5\end{bmatrix}, \begin{bmatrix}4\\1\\8\end{bmatrix}$ $b.$ $\begin{bmatrix}2\\3\\5\end{bmatrix}, \begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\8\end{bmatrix}$ $c.$ $\begin{bmatrix}-2\\4\\6\\10\end{bmatrix}, \begin{bmatrix}3\\-6\\-9\\15\end{bmatrix}$

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解:

a. $\because$ The set contains 4 vectors, each has 3 entries.

$\therefore$ Dependent

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b. $\because$ The zero vector is in the set
$\therefore$ Dependent

c. $\because$ Neither is a multiple of the other
$\therefore$ Independent

§ 1.8 Linear Transformations

- Transformations
- Matrix Transformations
- Linear Transformations

$Ax = b$

Matrix $\boldsymbol{A}$ is an object acting on $\boldsymbol{x}$ by multiplication to produce a new vector $\boldsymbol{Ax}$ or $\boldsymbol{b}$.

1.Transformations

$$\boldsymbol{Ax} = \boldsymbol{b} \qquad \underset{\substack{\uparrow \\ \boldsymbol{A}}}{\begin{bmatrix} 4 & -3 & 1 & 3 \\ 2 & 0 & 5 & 1 \end{bmatrix}} \underset{\substack{\uparrow \\ \boldsymbol{x}}}{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}} = \underset{\substack{\uparrow \\ \boldsymbol{b}}}{\begin{bmatrix} 5 \\ 8 \end{bmatrix}}$$

$$\boldsymbol{Au} = \boldsymbol{0} \qquad \underset{\substack{\uparrow \\ \boldsymbol{A}}}{\begin{bmatrix} 4 & -3 & 1 & 3 \\ 2 & 0 & 5 & 1 \end{bmatrix}} \underset{\substack{\uparrow \\ \boldsymbol{u}}}{\begin{bmatrix} 1 \\ 4 \\ -1 \\ 3 \end{bmatrix}} = \underset{\substack{\uparrow \\ \boldsymbol{0}}}{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}$$

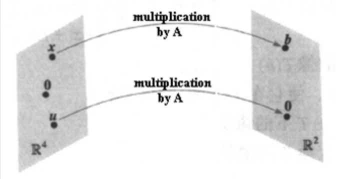
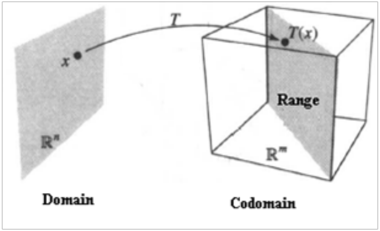


Figure: Transforming vectors

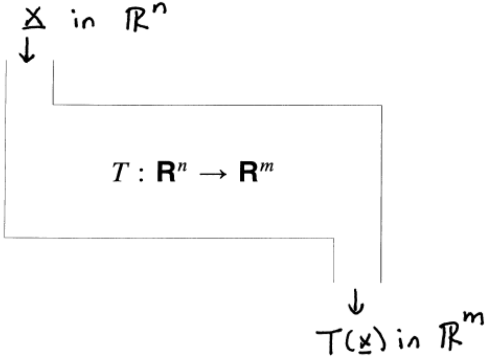
Transformation T



- R^n domain of T(定义域)
- R^m codomain of T(余定义域)
- T: $R^n \to R^m$
- Image of $\boldsymbol{x}$ $T(x)$ in R^m（像）
- Range of T Set of all images $T(x)$ range of T(值域)

Matrix Transformation

A **transformation** T from $\mathbf{R}^n$ to $\mathbf{R}^m$ is a rule that assigns to each vector $\mathbf{x}$ in $\mathbf{R}^n$ a vector $T(\mathbf{x})$ in $\mathbf{R}^m$.



Example

Let $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{bmatrix}$. Define a transformation $T : R^2 \to R^3$ by $T(x) = Ax$.

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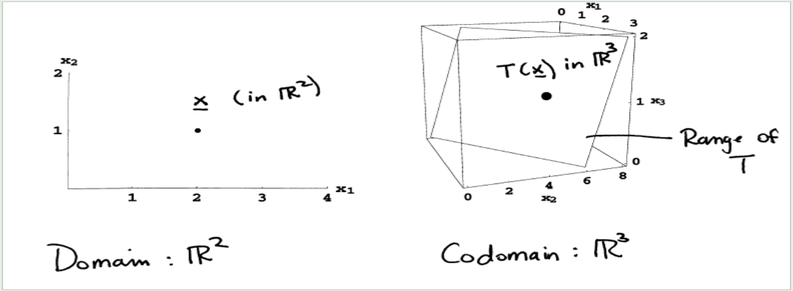
Then if $\boldsymbol{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\boldsymbol{T(x) = Ax} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$



Example

Let $A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$, $\boldsymbol{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$, $\boldsymbol{c} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$, $\quad T : R^2 \to R^3$

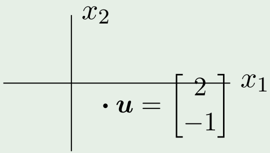
$$\boldsymbol{T}(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x} = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - 3x_2 \\ 3x_1 + 5x_2 \\ -x_1 + 7x_2 \end{bmatrix}$$

a. Find $\boldsymbol{T}(\boldsymbol{u})$, the image of $\boldsymbol{u}$ under the transformation $\boldsymbol{T}$.

b. Find an $\boldsymbol{x}$ in R^2 whose image under $\boldsymbol{T}$ is $\boldsymbol{b}$.

c. Is there more than one $\boldsymbol{x}$ whose image under $\boldsymbol{T}$ is $\boldsymbol{b}$?

d. Determine if $\boldsymbol{c}$ is in the range of the transformation $\boldsymbol{T}$.



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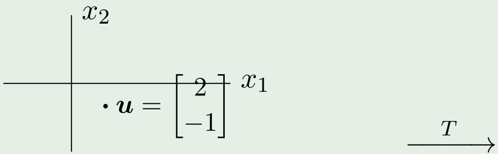

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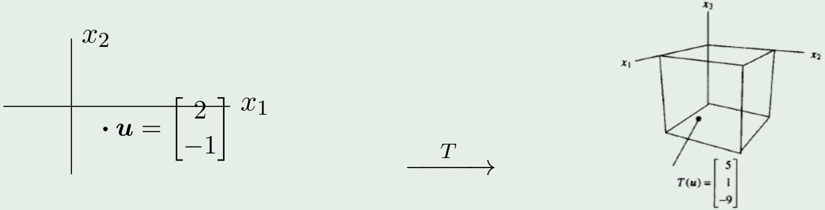

解:

a. Compute $$\boldsymbol{T}(\boldsymbol{u}) = \boldsymbol{A}\boldsymbol{u} = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -9 \end{bmatrix}$$

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b. Solve $\boldsymbol{T}(\boldsymbol{x}) = \boldsymbol{b}$ for $\boldsymbol{x}$.

$$\begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix} \qquad (1)$$

Hence,

$$\begin{bmatrix} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 14 & -7 \\ 0 & 4 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & -0.5 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1.5 \\ 0 & 1 & -0.5 \\ 0 & 0 & 0 \end{bmatrix} (2)$$

$$x_1 = 1.5, x_2 = -0.5, x = \begin{bmatrix} 1.5 \\ -0.5 \end{bmatrix}$$

c. Any $\boldsymbol{x}$ whose image under T is $\boldsymbol{b}$ must satisfy (1). From (2), it is clear that equation (1) has a unique solution. So there is exactly one $\boldsymbol{x}$ whose image is $\boldsymbol{b}$.

d. $$\begin{bmatrix} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 14 & -7 \\ 0 & 4 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & 2 \\ 0 & 14 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -35 \end{bmatrix}$$

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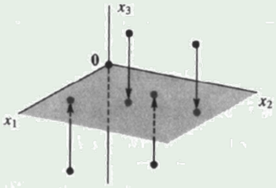
The system is inconsistent. So c is not in the range T.

Matrix transformation have many applications - including *computer graphics*.

E.g. Projection Transformation（投影变换）

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



Shear transformation（错切变换）

$$A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \qquad T : R^2 \to R^2 \qquad \boldsymbol{T}(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x}$$

E.g.

The image of the point $\boldsymbol{u} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ is $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$

$\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ is $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \end{bmatrix}$

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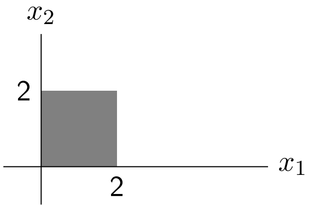

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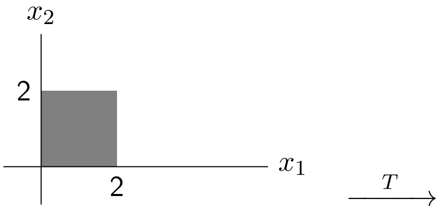

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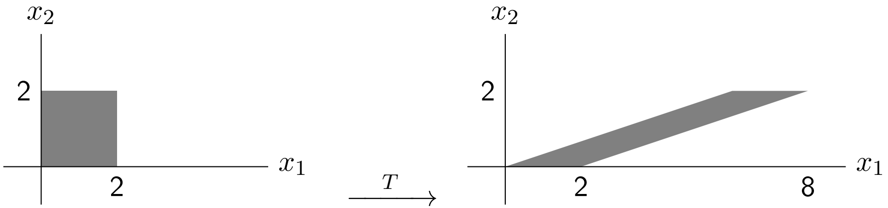
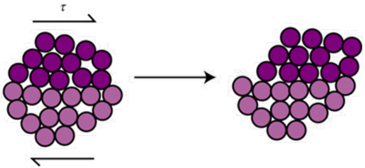
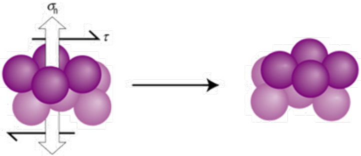
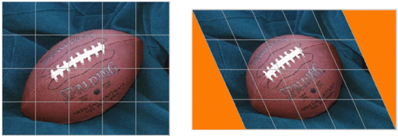
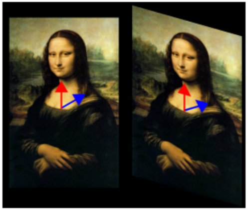


§ 1.8 Linear Transformations









Definition

A transformation T is linear if:

(a) $\boldsymbol{T}(\boldsymbol{u}+\boldsymbol{v})=\boldsymbol{T}(\boldsymbol{u})+\boldsymbol{T}(\boldsymbol{v})$ for all u, v in the domain of T;

(b) $\boldsymbol{T}(\mathrm{c}\boldsymbol{u})=\mathrm{c}\boldsymbol{T}(\boldsymbol{u})$ for all u and all scalars c.

Every Matrix transformation is a linear transformation.

RESULT

If T is a linear transformation, then

$$\boldsymbol{T}(\mathbf{0}) = \mathbf{0} \qquad \text{and} \qquad \boldsymbol{T}(\mathrm{c}\boldsymbol{u}+\mathrm{d}\boldsymbol{v}) = \mathrm{c}\boldsymbol{T}(\boldsymbol{u}) + \mathrm{d}\boldsymbol{T}(\boldsymbol{v}).$$

RESULT

If T is a linear transformation, then

$$\boldsymbol{T}(\mathbf{0}) = \mathbf{0} \qquad \text{and} \qquad \boldsymbol{T}(\mathrm{c}\boldsymbol{u}+\mathrm{d}\boldsymbol{v}) = \mathrm{c}\boldsymbol{T}(\boldsymbol{u}) + \mathrm{d}\boldsymbol{T}(\boldsymbol{v}).$$

Proof.

$$\boldsymbol{T}(\mathbf{0}) = \boldsymbol{T}(0\boldsymbol{u}) = 0\boldsymbol{T}(\boldsymbol{u}) = \mathbf{0}.$$

$$\boldsymbol{T}(\mathrm{c}\boldsymbol{u} + \mathrm{d}\boldsymbol{v}) = \boldsymbol{T}(\mathrm{c}\boldsymbol{u}) + \boldsymbol{T}(\mathrm{d}\boldsymbol{v})$$

$$= \mathrm{c}\boldsymbol{T}(\boldsymbol{u}) + \mathrm{d}\boldsymbol{T}(\boldsymbol{v}).$$

□

Example :

Define a linear transformation $T : R^2 \to R^2$ by

$$\boldsymbol{T}(\boldsymbol{x}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}$$

Find the image under T of $\boldsymbol{u} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \boldsymbol{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad \boldsymbol{u} + \boldsymbol{v} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$

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Find the image under T of $\boldsymbol{u} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \boldsymbol{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad \boldsymbol{u} + \boldsymbol{v} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$

解:

$$\boldsymbol{T}(\boldsymbol{u}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix} \qquad \boldsymbol{T}(\boldsymbol{v}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix},$$

$$\boldsymbol{T}(\boldsymbol{u} + \boldsymbol{v}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 6 \\ 4 \end{bmatrix} = \begin{bmatrix} -4 \\ 6 \end{bmatrix}$$

Example

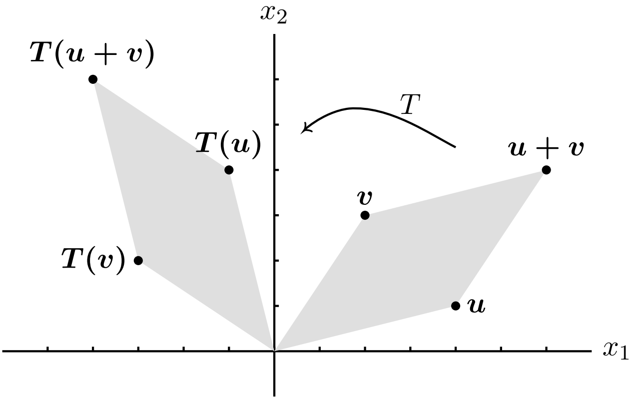


Figure: A rotation transformation

Example

Let $\boldsymbol{e_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \boldsymbol{e_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \boldsymbol{y_1} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ and $\boldsymbol{y_2} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. Suppose $T : R^2 \to R^3$ is a linear transformation which maps $\boldsymbol{e_1}$ into $\boldsymbol{y_1}$ and $\boldsymbol{e_2}$ into $\boldsymbol{y_2}$. Find the images of $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

解:

First, note that

$$\boldsymbol{T}(\boldsymbol{e_1}) = \boldsymbol{y_1} \quad and \quad \boldsymbol{T}(\boldsymbol{e_2}) = \boldsymbol{y_2}.$$

Also

$$3\boldsymbol{e_1} + 2\boldsymbol{e_2} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

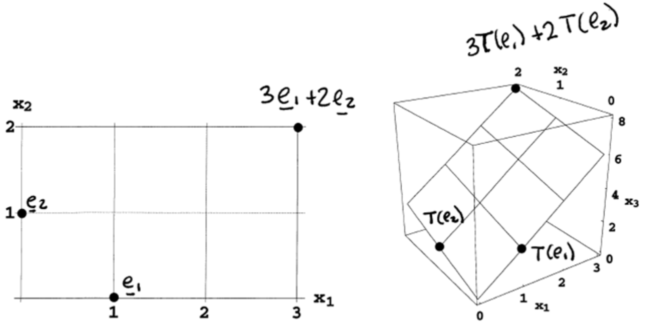
Then

$$\boldsymbol{T}(\begin{bmatrix} \mathbf{3} \\ \mathbf{2} \end{bmatrix}) = \boldsymbol{T}(\mathbf{3}\boldsymbol{e_1} + \mathbf{2}\boldsymbol{e_2}) =$$

$$3\boldsymbol{T}(\boldsymbol{e_1}) + 2\boldsymbol{T}(\boldsymbol{e_2}) =$$

$$3\boldsymbol{y_1} + 2\boldsymbol{y_2} =$$

$$\begin{bmatrix} 3 \\ 0 \\ 6 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 8 \end{bmatrix}$$



$$T(3\mathbf{e}_1 + 2\mathbf{e}_2) = 3T(\mathbf{e}_1) + 2T(\mathbf{e}_2)$$

Also

$$\boldsymbol{T}\left(\begin{bmatrix}\boldsymbol{x_1}\\ \boldsymbol{x_2}\end{bmatrix}\right) = \boldsymbol{T}(\boldsymbol{x_1 e_1} + \boldsymbol{x_2 e_2}) =$$

$$x_1\boldsymbol{T}(\boldsymbol{e_1}) + x_2\boldsymbol{T}(\boldsymbol{e_2}) =$$

$$x_1\begin{bmatrix}1\\0\\2\end{bmatrix} + x_2\begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}x_1\\x_2\\2x_1+x_2\end{bmatrix}$$