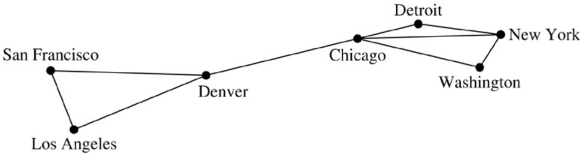
Discrete Mathematics: Lecture 10

Today:

- Chap 10.1: Graphs and graph models
- Chap 10.2: Graph terminology and special types of graphs
- Chap 10.3: Representing graphs and graph isomorphisms

Undirected graphs (无向图)

- Definition: A graph $G = (V, E)$ consists of V, a nonempty set of vertices (顶点) or nodes and E, a set of edges (边). Each edge has either one or two vertices associated with it, called its endpoints. An edge is said to connect its endpoints.
- A graph with an infinite vertex set is called an infinite graph; a graph with a finite vertex set is called a finite graph.
- Example: A computer network is made up of data centers and communication links between computers.



Undirected graphs

- A graph in which each edge connects two different vertices and where no two edges connect the same pair of vertices is called a simple graph (简单图).
- Graphs that may have multiple edges connecting the same vertices are called multigraphs (多重图).
- Edges that connect a vertex to itself are called loops (环).
- Graphs that may have loops and multiple edges are called pseudographs (伪图).

Directed graphs (有向图)

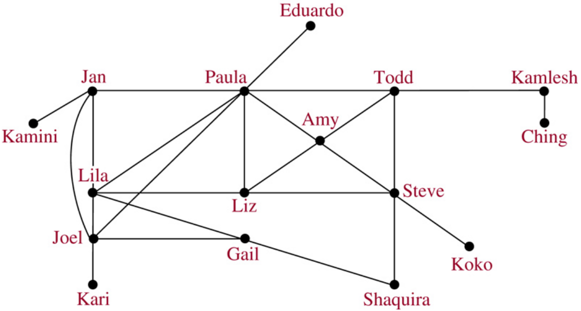
- Definition: A directed graph (or digraph) (V, E) consists of a nonempty set of vertices V and a set of directed edges (or arcs) E. Each directed edge is associated with an ordered pair of vertices. The directed edge associated with the ordered pair (u, v) is said to start at u and end at v.
- A graph with both directed and undirected edges is called a mixed graph.

Three key questions about the structure of a graph

- Are the edges directed or undirected?
- Do multiple edges connect the same pair of vertices?
- Are loops present?

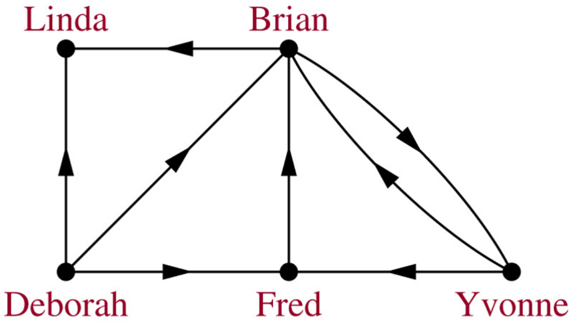
An example: Acquaintance graphs

A simple graph to represent whether two people know each other



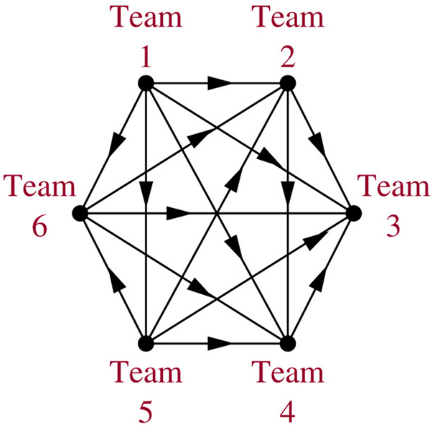
An example: Influence graphs

A directed graph to represent whether a person has influence on another one



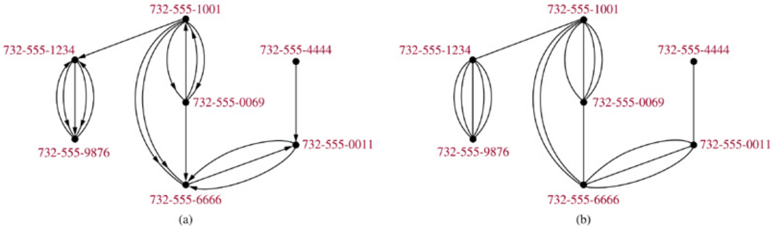
An example: Round-robin tournaments (循环锦标赛)

- Each team plays each other team exactly once
- (a, b) is an edge if team a beats team b



An example: call graphs

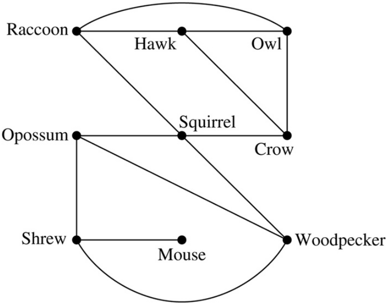
- Each phone call is represented by a directed edge
- There are multiple edges



Terminology for undirected graphs

- Definition: Two vertices u and v in an undirected graph G are called adjacent (相邻) or neighbors in G if u and v are endpoints of an edge of G. The edge is called incident (关联) with u and v.
- The degree (度) of a vertex v in an undirected graph, denoted by $deg(v)$, is the number of edges incident with it, except that a loop at a vertex contributes twice to the degree of that vertex.
- A vertex of degree zero is called isolated (孤立的)
- A vertex is pendant (悬挂的) if it has degree one

- Each species (物种) is represented by a vertex
- An undirected edge $\{u, v\}$ represents that u and v compete for food resources
- What does the degree of a vertex represent? Which vertices are pendant and which are isolated?



The handshaking theorem (握手定理)

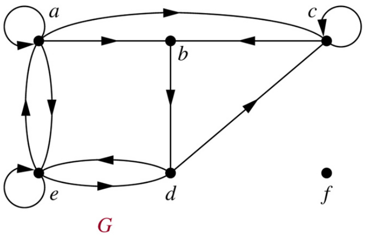
- The handshaking theorem: Let $G = (V, E)$ be an undirected graph. Then $\sum_{v \in V} deg(v) = 2|E|$.
- Note that this applies even if multiple edges and loops are present.
- Example: How many edges are there in a graph with 10 vertices each of degree 6?
- Theorem 2: An undirected graph has an even number of vertices of odd degree.

Terminology for directed graphs

- Definition: Let (u, v) be an edge of a directed graph G. We say that u is adjacent to v and v is adjacent from u. We call u the initial vertex (起点) of (u, v) and v the terminal or end vertex (终点) of (u, v).
- Definition: Let v be a vertex of a directed graph. The in-degree (入度) of v, denoted by $deg^-(v)$, is the number of edges with v as their terminal vertex. The out-degree (出度) of v, denoted by $deg^+(v)$, is the number of edges with v as their initial vertex.

Terminology for directed graphs

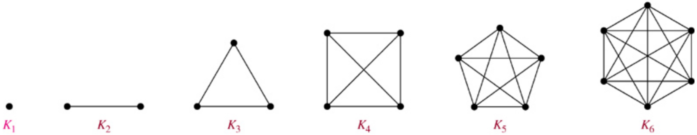
- Example: in- and out-degree of vertices in the following graph



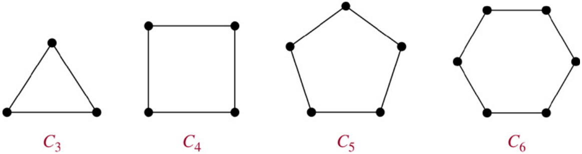
- Theorem 3: Let G be a digraph. Then $\sum_{v \in V} deg^{-}(v) = \sum_{v \in V} deg^{+}(v) = |E|$.

Some special simple graphs

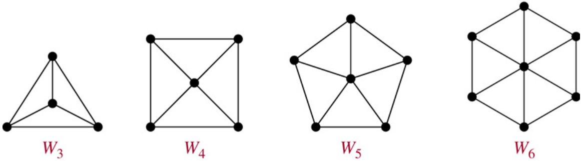
- The complete graph (完全图) on n vertices, denoted by K_n, is the simple graph containing an edge between any two distinct vertices.



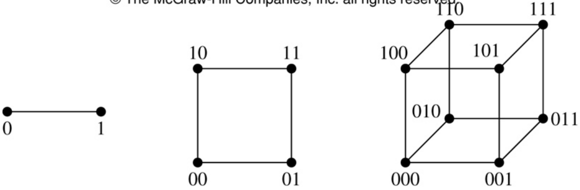
- The cycle (圈) C_n, $n \geq 3$, consists of n vertices $v_1, v_2, \ldots, v_n$ and edges $\{v_1, v_2\}$,$\{v_2, v_3\}, \ldots,$ $\{v_{n-1}, v_n\}$, and $\{v_n, v_1\}$.



- The wheel (轮图) W_n, $n \geq 3$, is obtained from C_n as follows: add an additional vertex, and connect this vertex to each of the n vertices of C_n

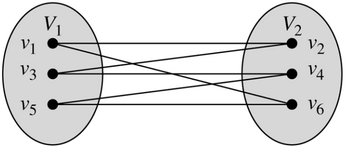


- The n-cube (方体图) Q_n: vertices labeled by the 2^n bit strings of length n, edges between vertices representing bit strings differ in exactly one position.



Bipartie graphs (二部图)

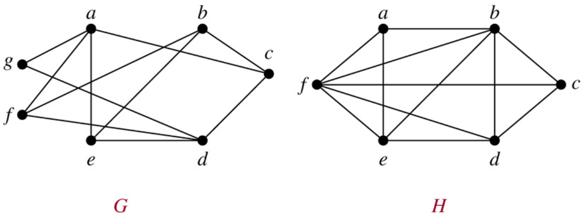
- Example: A graph representing marriages between men and women in a village: a person is represented by a vertex and a marriage by an edge
- Definition: A simple graph G is called bipartie if its vertex set V can be partitioned into two sets V_1 and V_2 such that every edge in the graph connects a vertex in V_1 and a vertex in V_2. We call the pair (V_1, V_2) a bipartition of V.
- Example: C_6 is bipartie.



- K_3 is not bipartie.

Bipartie graphs

- Example: Are the following graphs biparties?

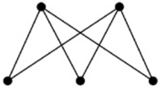


- Theorem 4: A simple graph is bipartie iff it is possible to assign one of two different colors to each vertex so that no two adjacent vertices are assigned the same color.

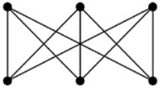
- Example: Apply Theorem 4 to the above example

Complete bipartie graphs

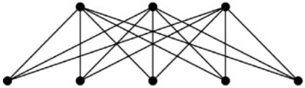
$K_{m,n}$: there is a partition of vertices into two subsets of m and n vertices, there is an edge between any vertex from the first subset and any vertex from the second subset



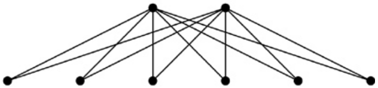
$K_{2,3}$



$K_{3,3}$

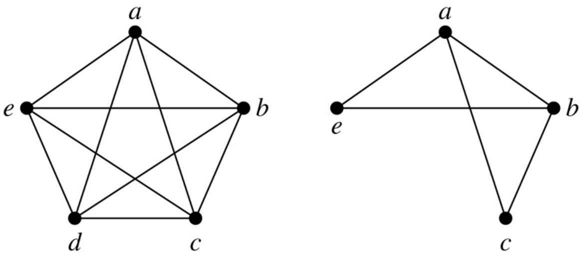


$K_{3,5}$



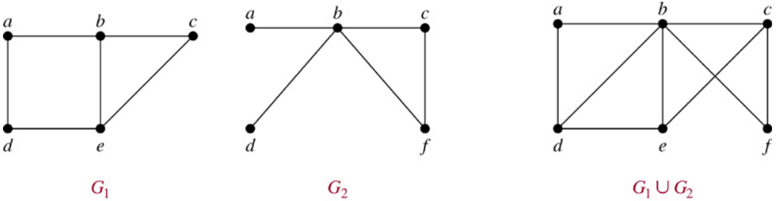
$K_{2,6}$

Definition: A subgraph (子图) of a graph $G = (V, E)$ is a graph $H = (W, F)$, where $W \subseteq V$, and $F \subseteq E$. A subgraph H of G is proper if $H \neq G$ (真子图).



Union of graphs

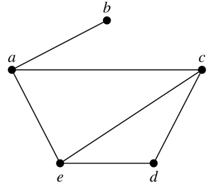
Definition: The union of two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the simple graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$. The union of G_1 and G_2 is denoted by $G_1 \cup G_2$.



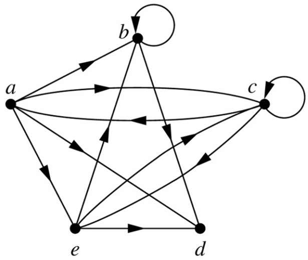
(a) (b)

Representing graphs

To represent graphs without multiple edges, use adjacency lists (邻接表), which specify the adjacent vertices for each vertex



Vertex	Adjacent vertices
a	b,c,e
b	a
c	a,d,e
d	c,e
e	a,c,d

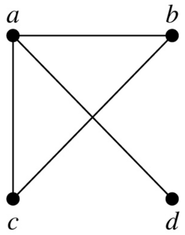


Initial Vertex	Terminal vertices
a	b,c,d,e
b	b,d
c	a,c,e
d	
e	b,c,d

Adjacency matrices for undirected graphs

Let $G = (V, E)$ be a simple graph where $|V| = n$. Suppose that vertices of G are listed as $v_1, v_2, \ldots, v_n$. The adjacency matrix (邻接矩阵) of G is $A_G = [a_{ij}]$, where

$$a_{ij} = \begin{cases} 1 & \text{if } \{v_i, v_j\} \text{ is an edge of } G \\ 0 & \text{otherwise} \end{cases}$$



We order the vertices as a, b, c, d.

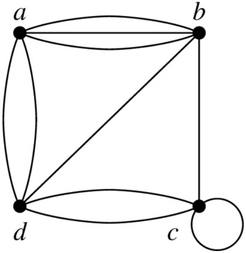
$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

based on the ordering for the vertices

symmetric, and 0 on the main diagonal

Adjacency matrices for undirected graphs

Can also be used to represent undirected graphs with loops and multiple edges



$$\begin{pmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{pmatrix}$$

We order the vertices as a, b, c, d.

Adjacency matrices for directed graphs

Let $G = (V, E)$ be a directed graph where $|V| = n$. Suppose that vertices of G are listed as v_1, v_2, ..., v_n. The adjacency matrix of G is $A_G = [a_{ij}]$, where

$$a_{ij} = \begin{cases} 1 & \text{if } (v_i, v_j) \text{ is an edge of } G \\ 0 & \text{otherwise} \end{cases}$$

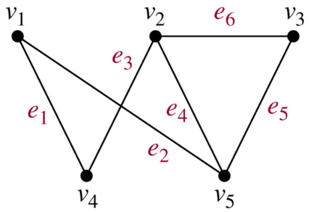
Trade-offs between adjacency lists and adjacency matrices

- When a simple graph is sparse (稀疏的), it is preferable to use adjacency lists
  - When each vertex has degree $\leq c$, the adjacency list has cn items, but the adjacency matric has n^2 items
- When a simple graph is dense, it is preferable to use adjacency matrices
  - To determine if an edge $\{v_i, v_j\}$ exists, the time complexity is $O(1)$ if using adjacency matrix, but $O(n)$ if use adjacency list

Incidence matrices (关联矩阵)

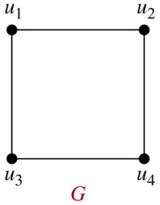
Let $G = (V, E)$ be a undirected graph. Suppose that v_1, v_2, ..., v_n are the vertices and e_1, e_2, ..., e_m are the edges. The incidence matrix of G is the $n \times m$ matrix $I_G = [a_{ij}]$, where

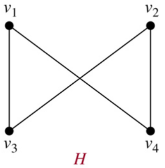
$$a_{ij} = \begin{cases} 1 & \text{if } e_j \text{ is incident with } v_i \\ 0 & \text{otherwise} \end{cases}$$



$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}$$

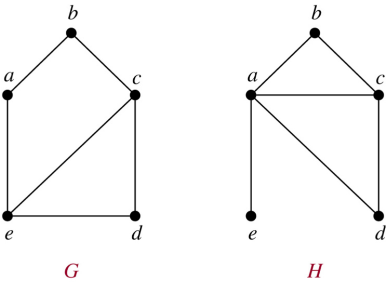
- Definition: The simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic (同构的) if there is a one-to-one and onto function f from V_1 to V_2 with the property that a and b are adjacent in G_1 iff $f(a)$ and $f(b)$ are adjacent in G_2, for all a and b in V_1. Such a function f is called an isomorphism.
- Remark: An isomorphism preserves the adjacency relationship.
- Isomorphism of simple graphs is an equivalence relation.
- Example: G and H are isomorphic





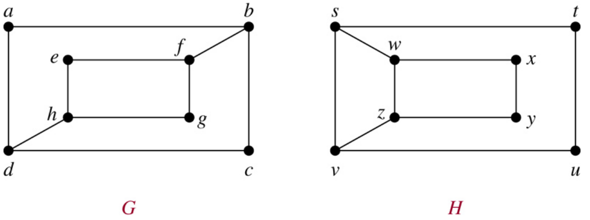
Isomorphism of graphs

- It is often difficult to determine if two graphs are isomorphic.
- But we can show that two graphs are not isomorphic if we can find a property that only one of the graphs has, but that is preserved by isomorphism.
- A property preserved by isomorphism of graphs is called a graph invariant (不变式).
- Example: G and H are not isomorphic



Isomorphism of graphs

- Example: G and H are not isomorphic



- Example: G and H are isomorphic