Discrete Mathematics: Lecture 14

- Today:
  - Chap 5.1: Mathematical induction
  - Chap 5.2: Strong induction and well-ordering
- Next time:
  - Chap 5.3: Recursive definitions and structural induction

Review of last time

- Inverse modulo m, solving linear congruence
- The Chinese remainder theorem
- Format's little theorem

Principle of mathematical induction

To prove that $P(n)$ is true for all $n \in \mathbf{N}$, $n \geq n_0$, where $P(n)$ is a propositional function, we complete two steps:

- Basis step: We prove that $P(n_0)$ is true.
- Inductive step: We show that the conditional statement $P(k) \to P(k+1)$ is true for all $k \in \mathbf{N}$, $k \geq n_0$. Here $P(k)$ is called the inductive hypothesis.

$$[P(n_0) \wedge \forall k \geq n_0 (P(k) \to P(k+1))] \to \forall n \geq n_0 P(n)$$

Why it works? $P(n_0)$, $P(n_0+1)$, $P(n_0+2)$, $\ldots$

Examples

- Sums of geometric progression:

$$\sum_{j=0}^{n} ar^j = a + ar + ar^2 + \ldots + ar^n = \frac{ar^{n+1} - a}{r - 1} \text{ when } n \neq 1$$

- $n < 2^n$ for all $n > 0$
- $2^n < n!$ for all $n \geq 4$
- An inequality for harmonic (调和) numbers: Let $j \geq 1$. Define $H_j = 1 + \frac{1}{2} + \ldots \frac{1}{j}$. Show that $H_{2^n} \geq 1 + \frac{n}{2}$.

More examples

- $3 \mid n^3 - n$ for all $n > 0$
- $57 \mid 7^{n+2} + 8^{2n+1}$ for all $n \geq 0$
- If S is a finite set with n elements, then S has 2^n subsets

Tilings

Let n be a positive integer. Show that every $2^n \times 2^n$ checkerboard with one square removed can be tiled using right triominos (三联骨牌).

Odd pie fights

- An odd number of people stand in a yard.
- The distances between any two people are mutually distinct.
- At the same time, each person throws a pie at his nearest neighbor, hitting this person.
- Use induction to show that there is at least one survivor, that is, at least one person is not hit by a pie.

A proof with errors

Claim: every set of lines in the plane, no two of which are parallel, meet in a common point.

Proof: Let $P(n)$ be the statement that every set of n lines in the plane, no two of which are parallel, meet in a common point. We show that $P(n)$ holds for all $n \in \mathbf{N}$, $n \geq 2$.

- Basis: $P(2)$ holds.
- Inductive step: Let $k \geq 2$. Assume that $P(k)$ holds. We prove that $P(k+1)$ holds.
- Consider a set of $k+1$ lines. By inductive hypothesis, the first k of them meet in a common point p_1, and the last k of them meet in a common point p_2.
- We show that $p_1 = p_2$. Otherwise, all lines containing both of them must be the same line, contradictory that the $k+1$ lines are distinct.
- Hence the $k+1$ lines meet in a same point.

Chap 4.2: Strong induction

To prove that $P(n)$ is true for all $n \in \mathbf{N}$, $n \geq n_0$, where $P(n)$ is a propositional function, we complete two steps:

- Basis step: We prove that $P(n_0)$ is true.
- Inductive step: We show that the conditional statement

$$P(n_0) \wedge P(n_0+1) \wedge \ldots \wedge P(k) \rightarrow P(k+1)$$

  is true for all $k \in \mathbf{N}$, $k \geq n_0$. Here the inductive hypothesis is $P(n_0) \wedge P(n_0+1) \wedge \ldots \wedge P(k)$

Also called the second principle of mathematical induction

Also called complete induction (versus simple induction)

Equivalence between simple induction and strong induction

- the principle of complete induction $\Rightarrow$ the principle of simple induction
- the principle of simple induction $\Rightarrow$ the principle of complete induction
  - Let $Q(n)$ be $P(1) \wedge P(2) \wedge \ldots \wedge P(n)$.

Example proofs

- Show that if n is an integer greater than 1, then n can be written as the product of primes.
- Consider a game where two players take turns removing any positive number of matches from one of two piles of matches. The player who removes the last match wins the game. Show that if the two piles contain the same number of matches initially, the second player has a winning strategy.
- Prove that every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps.
  - proof by simple induction
  - proof by complete induction

The well-ordering property

Every nonempty set of nonnegative integers has a least element.

Example proofs of using the property

- If a is an integer and d is a positive integer, then there are unique integers q and r with $0 \leq r < d$ and $a = dq + r$.
- In a round-robin tournament every player plays every other player exactly once and each match has a winner and a loser. We say that the players $p_1, p_2, \ldots, p_m$ form a cycle if p_1 beats p_2, p_1 beats p_2, ..., and p_m beats p_1. Show that if there is a cycle of length m $(m \geq 3)$, there must be a cycle of 3.

Equivalence to simple induction

- the well-ordering property $\Rightarrow$ the principle of simple induction
  - Let S be the set of $n \in \mathbf{N}$ such that $P(n)$ does not hold
- the principle of simple induction $\Rightarrow$ the well-ordering property
  - Assume to the contrary
  - Let $P(n)$: for all $i \leq n$, $i \notin S$
  - We prove that $P(n)$ holds for all $n \in \mathbf{N}$