习题10-1 中山大學 本科生考试草稿纸 2012/5/15

P.212.1、利用柯西收敛定理证明:

(1) 级数 $\sum_{n=1}^{\infty} \frac{\cos n}{n(n+1)}$ 收敛;

证明: $a_n = \frac{\cos n}{n(n+1)}$, $S_n = a_1 + a_2 + \cdots + a_n$, $S_m = a_1 + a_2 + \cdots + a_m$

$\forall \varepsilon > 0$, $|S_n - S_m| = |a_{m+1} + a_{m+2} + \cdots + a_n| = \left|\frac{\cos(m+1)}{(m+1)(m+2)} + \frac{\cos(m+2)}{(m+2)(m+3)} + \cdots + \frac{\cos n}{n(n+1)}\right|$

$\leq \left|\frac{1}{(m+1)(m+2)} + \frac{1}{(m+2)(m+3)} + \cdots + \frac{1}{n(n+1)}\right|$

$n > m$

$\frac{1}{m} > \frac{1}{n}$

$\frac{1}{m+1} > \frac{1}{n+1}$

$= \left|\left(\frac{1}{m+1} - \frac{1}{m+2}\right) + \left(\frac{1}{m+2} - \frac{1}{m+3}\right) + \cdots\cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right)\right|$

$= \left|\left(\frac{1}{m+1} - \frac{1}{n+1}\right)\right| < \frac{1}{m+1} < \frac{1}{N+1} < \varepsilon \quad (n \geq N, m > N)$

$N+1 > \frac{1}{\varepsilon}$, $N > \frac{1}{\varepsilon} - 1$, 取 $N = [\frac{1}{\varepsilon} - 1] + 1$

即, 对 $\forall \varepsilon > 0$, 取 $N = [\frac{1}{\varepsilon} - 1] + 1$, 则当 $n \geq N$, $m \geq N$ 时

有 $|S_n - S_m| < \varepsilon$, 由柯西收敛定理, 给定的级数收敛。

(2) 级数 $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ 发散;

证: $|S_n - S_m| = \left|\frac{1}{\sqrt{m+1}} + \frac{1}{\sqrt{m+2}} + \cdots\cdots + \frac{1}{\sqrt{n}}\right|$, 对某个$N$, 当 $m \geq N$, $n \geq N$

$> \left|\frac{1}{(\sqrt{m+1})^2} + \frac{1}{m+2} + \cdots + \frac{1}{n}\right|$

$> \frac{n-m}{n} \xlongequal{\text{取} n=2m} \frac{2m-m}{2m} = \frac{1}{2}$

所以, 对 $\forall \varepsilon > 0$, 取 $\varepsilon = \frac{1}{2}$, $\exists N$, 当 $m \geq N$, $n \geq N$ 时 $|S_n - S_m| > \frac{1}{2} = \varepsilon$

由柯西收敛定理, 给定级数发散。