4.16. 证明连续和离散二维傅里叶变换都是平移和旋转不变的。
解：连续傅里叶变换平移不变性：

$$\Gamma\left(f(x,y)e^{j2\pi(u_0x+v_0y)}\right)$$
$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,y)e^{j2\pi(u_0x+v_0y)}e^{-j2\pi(ux+vy)}$$
$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,y)e^{-j2\pi((u-u_0)x+(v-v_0)y)}\,dxdy$$
$$=F(u-u_0,v-v_0)$$

$$F(u,v)e^{-j2\pi(x_0u+y_0v)}$$
$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,y)e^{-j2\pi((x+x_0)u+(y+y_0)v)}\,\mathrm{dudv}$$
$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x-x_0,y-y_0)e^{-j2\pi(xu+yv)}\,\mathrm{dudv}$$
$$=\Gamma(f(x-x_0,y-y_o))$$

离散傅里叶变换平移不变性：

$$\Gamma\left(f(x,y)e^{j2\pi\left(\frac{u_0x}{M}+\frac{v_0y}{N}\right)}\right)$$
$$=\sum_{x=0}^{M-1}\sum_{y=0}^{N-1}f(x,y)e^{j2\pi\left(\frac{u_0x}{M}+\frac{v_0y}{N}\right)}e^{-j2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)}$$
$$=\sum_{x=0}^{M-1}\sum_{y=0}^{N-1}f(x,y)e^{-j2\pi\left(\frac{(u-u_0)x}{M}+\frac{(v-v_0)y}{N}\right)}$$
$$=F(u-u_0,v-v_0)$$

$$F(u,v)e^{-j2\pi\left(\frac{x_0u}{M}+\frac{y_0v}{N}\right)}$$
$$=\sum_{x=0}^{M-1}\sum_{y-0}^{N-1}f(x,y)e^{-j2\pi\left(\frac{(x+x_0)u}{M}+\frac{(y+y_0)v}{N}\right)}$$
$$\sum_{x=x_0}^{M+x_0-1}\sum_{y=y_0}^{N+y_0-1}f(x-x_0,y-y_o)e^{-j2\pi\left(\frac{xu}{M}+\frac{yv}{N}\right)}$$
$$=\Gamma(f(x-x_0,y-y_o))$$

旋转不变性：

$$\Gamma(f(r,\theta+\theta_0))$$
$$=\int_{-\pi}^{\pi}\int_{-\infty}^{\infty}f(r,\theta+\theta_0)\,e^{-j2\pi\mu r\omega(cos(\theta+\theta_0)cos\phi+sin(\theta+\theta_0)sin\phi)}\,rdrd\theta$$

$$= \int_{-\pi}^{\pi} \int_{-\infty}^{\infty} f(r,\theta + \theta_0) e^{-j2\pi\mu r\omega \cos(\theta + \theta_0 - \phi)} \, r dr d\theta$$

$$= F(\omega,\phi + \theta_0)$$