# 第五章总练习题

1. 设$\boldsymbol{a},\boldsymbol{b}$ 为两个非零向量,指出下列等式成立的充分必要条件:

(1) $|\boldsymbol{a}+\boldsymbol{b}|=|\boldsymbol{a}-\boldsymbol{b}|$;(2) $|\boldsymbol{a}+\boldsymbol{b}|=|\boldsymbol{a}|-|\boldsymbol{b}|$;(3)$\boldsymbol{a}+\boldsymbol{b}$与$\boldsymbol{a}-\boldsymbol{b}$共线.

解(1) $|\boldsymbol{a}+\boldsymbol{b}|=|\boldsymbol{a}-\boldsymbol{b}| \Leftrightarrow |\boldsymbol{a}+\boldsymbol{b}|^2=|\boldsymbol{a}-\boldsymbol{b}|^2 \Leftrightarrow |\boldsymbol{a}|^2+|\boldsymbol{b}|^2+2\boldsymbol{a}\cdot\boldsymbol{b}=|\boldsymbol{a}|^2+|\boldsymbol{b}|^2+2\boldsymbol{a}\cdot\boldsymbol{b}$

$\Leftrightarrow \boldsymbol{a}\cdot\boldsymbol{b}=0 \Leftrightarrow \boldsymbol{a},\boldsymbol{b}$正交.

(2) $|\boldsymbol{a}+\boldsymbol{b}|=|\boldsymbol{a}|-|\boldsymbol{b}| \Leftrightarrow |\boldsymbol{a}+\boldsymbol{b}|^2=(|\boldsymbol{a}|-|\boldsymbol{b}|)^2 \Leftrightarrow |\boldsymbol{a}|^2+|\boldsymbol{b}|^2+2\boldsymbol{a}\cdot\boldsymbol{b} \Leftrightarrow$

$|\boldsymbol{a}|^2+|\boldsymbol{b}|^2-2|\boldsymbol{a}||\boldsymbol{b}| \Leftrightarrow \boldsymbol{a}\cdot\boldsymbol{b}=|\boldsymbol{a}||\boldsymbol{b}| \Leftrightarrow |\boldsymbol{a}||\boldsymbol{b}|\cos<\boldsymbol{a},\boldsymbol{b}>$

$=-|\boldsymbol{a}||\boldsymbol{b}| \Leftrightarrow \cos<\boldsymbol{a},\boldsymbol{b}>=-1 \Leftrightarrow \boldsymbol{a},\boldsymbol{b}$共线且方向相反.

(3)$\boldsymbol{a}+\boldsymbol{b}$与$\boldsymbol{a}-\boldsymbol{b}$共线 $\Leftrightarrow (\boldsymbol{a}+\boldsymbol{b})\times(\boldsymbol{a}-\boldsymbol{b})=0 \Leftrightarrow \boldsymbol{b}\times\boldsymbol{a}-\boldsymbol{a}\times\boldsymbol{b}=0 \Leftrightarrow \boldsymbol{a}\times\boldsymbol{b}=0$

$\Leftrightarrow \boldsymbol{a},\boldsymbol{b}$共线.

2.设$\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}$为非零向量,判断下列等式是否成立:

(1)$(\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c}=\boldsymbol{a}(\boldsymbol{b}\cdot\boldsymbol{c})$;(2)$(\boldsymbol{a}\cdot\boldsymbol{b})^2=\boldsymbol{a}^2\boldsymbol{b}^2$;(3)$\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})=(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c}$.

解(1)不成立. 例如:$(\boldsymbol{i}\cdot\boldsymbol{i})\boldsymbol{j}=\boldsymbol{j}\neq\boldsymbol{i}(\boldsymbol{i}\cdot\boldsymbol{j})=\boldsymbol{0}$.

(2)不成立.例如:$(\boldsymbol{i}\cdot\boldsymbol{j})^2=0\neq\boldsymbol{i}^2\boldsymbol{j}^2=1$.

(3)成立.$\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})$和$(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c}$都是$\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}$的有向体积,且定向相同.

3.设$\boldsymbol{a},\boldsymbol{b}$为非零向量, 且$7\boldsymbol{a}-5\boldsymbol{b}$与$\boldsymbol{a}+3\boldsymbol{b}$正交, 与$\boldsymbol{a}-4\boldsymbol{b}$与$7\boldsymbol{a}-2\boldsymbol{b}$正交, 求$\boldsymbol{a}^2-\boldsymbol{b}^2$.

解 $(7\boldsymbol{a}-5\boldsymbol{b})\cdot(\boldsymbol{a}+3\boldsymbol{b})=0,(\boldsymbol{a}-4\boldsymbol{b})\cdot(7\boldsymbol{a}-2\boldsymbol{b})=0$.

$$\begin{cases}7\boldsymbol{a}^2-15\boldsymbol{b}^2+16\boldsymbol{a}\cdot\boldsymbol{b}=0 & (1)\\ 7\boldsymbol{a}^2+8\boldsymbol{b}^2-30\boldsymbol{a}\cdot\boldsymbol{b}=0 & (2)\end{cases}$$

$(1)\times15+(2)\times8$

$161(\boldsymbol{a}^2-\boldsymbol{b}^2)=0,\boldsymbol{a}^2-\boldsymbol{b}^2=0$.

4.利用向量运算,证明下列几何命题:射影定理.考虑直角三角形$ABC$,其中$\angle A$为直角, $AD$是斜边上的高,则$\overline{AD}^2=\overline{BD}\cdot\overline{CD},\overline{AB}^2=\overline{BD}\cdot\overline{BC},\overline{AC}^2=\overline{CD}\cdot\overline{CB}$.

证$\overrightarrow{AB}=\overrightarrow{AD}+\overrightarrow{DB},\overrightarrow{AC}=\overrightarrow{AD}+\overrightarrow{DC}$,

$0=\overrightarrow{AB}\cdot\overrightarrow{AC}=(\overrightarrow{AD}+\overrightarrow{DB})\cdot(\overrightarrow{AD}+\overrightarrow{DC})=\overrightarrow{AD}^2+\overrightarrow{AD}\cdot\overrightarrow{DC}+\overrightarrow{DB}\cdot\overrightarrow{AD}+\overrightarrow{DB}\cdot\overrightarrow{DC}$

$=\overrightarrow{AD}^2+\overrightarrow{DB}\cdot\overrightarrow{DC},\overline{AD}^2=-\overrightarrow{DB}\cdot\overrightarrow{DC}=\overrightarrow{BD}\cdot\overrightarrow{DC}=\overline{BD}\times\overline{DC}(\overrightarrow{BD},\overrightarrow{DC}$同向).

$\overline{AB}^2=\overline{AD}^2+\overline{BD}^2=\overline{BD}\cdot\overline{CD}+\overline{BD}^2=\overline{BD}(\overline{CD}+\overline{BD})=\overline{BD}\cdot\overline{BC})$.

$\overline{AC}^2=\overline{AD}^2+\overline{CD}^2=\overline{BD}\cdot\overline{CD}+\overline{CD}^2=\overline{CD}(\overline{BD}+\overline{CD})=\overline{CD}\cdot\overline{BC}$.

5.已知三点$A,B,C$的坐标分别为$(1,0,0),(1,1,0),(1,1,1)$.若$ACDBD$是一平行四边形，求点$D$的坐标.

解$A=(1,0,0),B=(1,1,0),C=(1,1,1).\overrightarrow{AC}=(0,1,1),\overrightarrow{AB}=(0,1,0),\overrightarrow{AD}=\overrightarrow{AB}+\overrightarrow{AC}=(0,2,1)$,

$\overrightarrow{OD}=\overrightarrow{OA}+\overrightarrow{AD}=(1,0,0)+(0,2,1)=(1,2,1)$.点$D$的坐标$(1,2,1)$.

6.设$\boldsymbol{a},\boldsymbol{b}$为非零向量, 证明$(\boldsymbol{a}\times\boldsymbol{b})^2=\boldsymbol{a}^2\boldsymbol{b}^2-(\boldsymbol{a}\cdot\boldsymbol{b})^2$.

证$(\boldsymbol{a}\times\boldsymbol{b})^2=|\boldsymbol{a}|^2|\boldsymbol{b}|^2\sin^2<\boldsymbol{a},\boldsymbol{b}>=|\boldsymbol{a}|^2|\boldsymbol{b}|^2(1-\cos^2<\boldsymbol{a},\boldsymbol{b}>)$

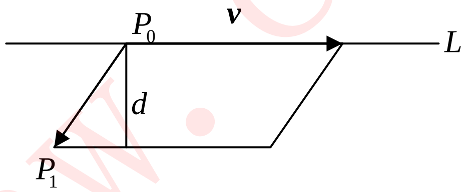
$=|\boldsymbol{a}|^2|\boldsymbol{b}|^2-|\boldsymbol{a}|^2|\boldsymbol{b}|^2\cos^2<\boldsymbol{a},\boldsymbol{b}>=\boldsymbol{a}^2\boldsymbol{b}^2-(\boldsymbol{a}\cdot\boldsymbol{b})^2$.

7.设有两直线$L_1:\dfrac{x-1}{-1}=\dfrac{y}{2}=\dfrac{z+1}{1},L_2:\dfrac{x+2}{0}=\dfrac{y-1}{1}=\dfrac{z-2}{-2}$,求平行于$L_1,L_2$且与它们等距的平面方程.

解$\boldsymbol{n}=\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\-1&2&1\\0&1&-2\end{vmatrix}=(-5,-2,-1)$,所求平面过点$A=(-1/2,1/2,1/2)$,

所求平面:$-5(x+1/2)-2(y-1/2)-(z-1/2)=0,5x+2y+z+1=0$.

8.设直线$L$通过点$P_0$且其方向向量为$\boldsymbol{v}$,证明$L$外一点$P_1$到$L$的距离$d$可表为$d=\dfrac{|\overrightarrow{P_0P_1}\times\boldsymbol{v}|}{|\boldsymbol{v}|}$.




证平行四边形$P_0P_1AB$的面积

$=d\times|\boldsymbol{v}|=\|\overrightarrow{P_0P_1}\times\boldsymbol{v}\|$.

9.设两直线$L_1,L_2$分别通过点$P_1P_2$,且它们的方向向量为$\boldsymbol{v}_1,\boldsymbol{v}_2$.证明$L_1$与$L_2$共面的充分必要条件为$\overrightarrow{P_1P_2}\cdot(\boldsymbol{v}_1\times\boldsymbol{v}_2)=0$.

证$L_1$与$L_2$共面$\Leftrightarrow\overrightarrow{P_1P_2},\boldsymbol{v}_1,\boldsymbol{v}_2$共面$\Leftrightarrow\overrightarrow{P_1P_2}\cdot(\boldsymbol{v}_1\times\boldsymbol{v}_2)=0$.

10.设两直线$L_1,L_2$分别通过点$P_1,P_2$,且它们的方向向量为$\boldsymbol{v}_1,\boldsymbol{v}_2$.$L_1$与$L_2$之间的距离定义为$d=\min\limits_{\substack{Q_1\in L_1\\Q_2\in L}}|\overrightarrow{Q_1Q_2}|$证明:(1)当$L_1$与$L_2$平行时,它们之间的距离可表示为$d=\dfrac{\overrightarrow{P_1P_2}\times\boldsymbol{v}_1}{|\boldsymbol{v}_1|}$

(2)当$L_1$与$L_2$为异面直线时, 它们之间的距离可表示为$d=\dfrac{|\overrightarrow{P_1P_2}\cdot(\boldsymbol{v}_1\times\boldsymbol{v}_2)|}{|\boldsymbol{v}_1\times\boldsymbol{v}_2|}$.

证(1)当$L_1$与$L_2$平行时, 它们之间的距离为$L_1$上任意一点到$L_2$的距离,由第8题,

$d=\dfrac{\overrightarrow{P_1P_2}\times\boldsymbol{v}_1}{|\boldsymbol{v}_1|}$.

(2)$\dfrac{\overrightarrow{P_1P_2}\cdot(\boldsymbol{v}_1\times\boldsymbol{v}_2)}{|\boldsymbol{v}_1\times\boldsymbol{v}_2|}=\overrightarrow{P_1P_2}\cdot(\boldsymbol{v}_1\times\boldsymbol{v}_2)^\circ$是$\overrightarrow{P_1P_2}$在$L_1$与$L_2$的公垂线方向的单位向量上的投影,

故其长度$|\overrightarrow{P_1P_2}\cdot(\boldsymbol{v}_1\times\boldsymbol{v}_2)^\circ|=\dfrac{|\overrightarrow{P_1P_2}\cdot(\boldsymbol{v}_1\times\boldsymbol{v}_2)|}{|\boldsymbol{v}_1\times\boldsymbol{v}_2|}$是异面直线$L_1$与$L_2$之间的距离.

11.设直线L的方程为$L:\begin{cases}A_1x+B_1y+C_1z+D_1=0\\A_2x+B_2y+C_2z+D_2=0\end{cases}$

证明:(1)对于任意两个不全为零的常数$\lambda_1,\lambda_2$, 方程

$\lambda_1(A_1x+B_1y+C_1z+D_1)+\lambda_2(A_2x+B_2y+C_2z+D_2)=0$

表示一个通过直线$L$的平面;

(2)任意给定一个通过直线$L$的平面$\pi$,必存在两个不全为零的实数$\lambda_1,\lambda_2$, 使平面$\pi$的方程为$\lambda_1(A_1x+B_1y+C_1z+D_1)+\lambda_2(A_2x+B_2y+C_2z+D_2)=0.$

证(1)向量$(A_1,B_1,C_1)$与$(A_2,B_2,C_2)$不共线,故对于两个不全为零的常数$\lambda_1,\lambda_2$,

$\lambda_1(A_1x+B_1y+C_1z+D_1)+\lambda_2(A_2x+B_2y+C_2z+D_2)=0$的主系数

$\lambda_1(A_1,B_1,C_1)+\lambda_2(A_2,B_2,C_2)\neq(0,0,0)$,是一个平面的方程, 并且 $L$上点的坐标

$(x,y,z)$满足$\begin{cases}A_1x+B_1y+C_1z+D_1=0\\A_2x+B_2y+C_2z+D_2=0\end{cases}$,故满足

$\lambda_1(A_1x+B_1y+C_1z+D_1)+\lambda_2(A_2x+B_2y+C_2z+D_2)=0.$

(2)设平面$\pi$通过直线$L$,其方程为

$A(x-x_0)+B(y-y_0)+C(z-z_0)=Ax+By+Cz+D=0.$

$(x_0,y_0,z_0)$在$L$上.三个向量$(A,B,C)$ $(A_1,B_1,C_1)$与$(A_2,B_2,C_2)$均垂直于$L$的方向向量,故共面,又$(A_1,B_1,C_1)$与$(A_2,B_2,C_2)$都是非零向量,故存在两个不全为零的常数$\lambda_1,\lambda_2$,使得

$(A,B,C)=\lambda_1(A_1,B_1,C_1)+\lambda_2(A_2,B_2,C_2).$

$D=-Ax_0-By_0-Cz_0=-(\lambda_1A_1+\lambda_2A_2)x_0-(\lambda_1B_1+\lambda_2B_2)y_0-(\lambda_1C_1+\lambda_2C_2)z_0$

$=-\lambda_1(A_1x_0+B_1y_0+C_1z_0)-\lambda_2(A_2x_0+B_2y_0+C_2z_0)=\lambda_1D_1+\lambda_2D_2.$

故$\pi$表示为$\lambda_1(A_1x+B_1y+C_1z+D_1)+\lambda_2(A_2x+B_2y+C_2z+D_2)=0.$

12.试求通过直线$L_1:\begin{cases}x-2z-4=0\\3y-z+8=0\end{cases}$且与直线$L_2:x-1=y+1=z-3$平行的平面方程.

解根据11题的结论,所求平面方程有形式

$\lambda_1(x-2z-4)+\lambda_2(3y-z+8)=0,\lambda_1x+3\lambda_2y+(-2\lambda_1-\lambda_2)z-4\lambda_1+8\lambda_2=0.$

由于平面与$L_2$平行,$(\lambda_1,3\lambda_2,-2\lambda_1-\lambda_2)\cdot(1,1,1)=0,\lambda_1+3\lambda_2-2\lambda_1-\lambda_2=0,-\lambda_1+2\lambda_2=0.$

令$\lambda_1=2,\lambda_2=1$,得所求平面方程$2x+3y-5z=0.$

13.已知曲面S的方程为$S:x^2+\dfrac{y^2}{4}+\dfrac{z^2}{2}=1$,平面$\pi$的方程为$\pi:2x+y+2z+6=0.$

(1)求曲面$S$的平行于$\pi$的切平面方程;

(2)在曲面$S$上求到平面$\pi$距离为最短及最长的点,并求最短及最长的距离.

解 (1)$S$的法向量$(2x,\dfrac{y}{2},z).4x+\dfrac{y}{2}+2z=0.$

$2x(X-x)+\dfrac{y}{2}(Y-y)+z(Z-z)=0$

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**解** (1)$S$上的点记为$(x,y,z)$.$S$的法向量$(2x,\frac{y}{2},z)$.

切平面与$\pi$平行,则法向量对应坐标成比例:$\frac{2x}{2}=\frac{y/2}{1}=\frac{z}{2}$.$z=2x, y=z$.

与曲面方程联立:$x^2+x^2+2x^2=1, x=\pm\frac{1}{2}, y=\pm1, z=\pm1$.

切平面方程:$2x(X-x)+\frac{y}{2}(Y-y)+z(Z-z)=0$,

利用曲面方程得$2xX+\frac{y}{2}Y+zZ=2$.$\pm X\pm\frac{1}{2}Y\pm Z=2$.

平面$\pi$过点$A=(-3,0,0)$.$\overrightarrow{P_1A}=$

点$P_1=(\frac{1}{2},1,1)$,$\overrightarrow{P_1A}=(-\frac{7}{2},-1,-1)$,

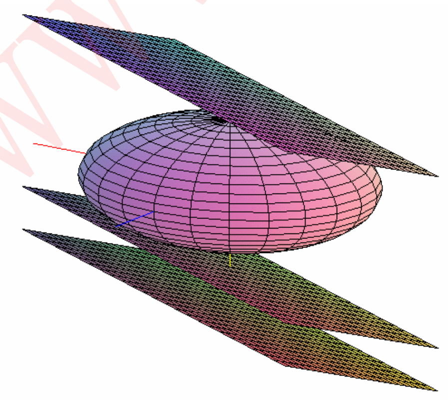
$P_1$到平面$\pi$的距离$d_1=\frac{|\overrightarrow{P_1A}\cdot\boldsymbol{n}|}{|\boldsymbol{n}|}=\frac{\left|(-\frac{7}{2},-1,-1)\cdot(2,1,2)\right|}{3}=\frac{10}{3}$.

点$P_2=(-\frac{1}{2},-1,-1)$,$\overrightarrow{P_2A}=(-\frac{5}{2},1,1)$,

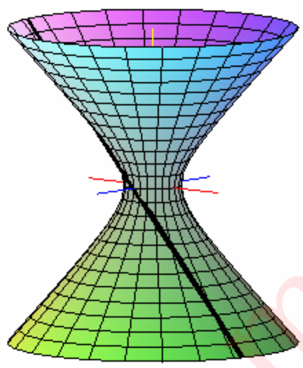
$P_2$到平面$\pi$的距离$d_2=\frac{|\overrightarrow{P_2A}\cdot\boldsymbol{n}|}{|\boldsymbol{n}|}=\frac{\left|(-\frac{5}{2},1,1)\cdot(2,1,2)\right|}{3}=\frac{2}{3}$.

在曲面$S$上到平面$\pi$距离为最短及最长的点分别是$(-\frac{1}{2},-1,-1)$和$(\frac{1}{2},1,1)$,

并求最短及最长的距离分别是$\frac{2}{3}$和$\frac{10}{3}$.



14.直线$\dfrac{x}{1}=\dfrac{y-1}{0}=\dfrac{z}{1}$绕z轴旋转一周,求所得旋转曲面的方程.



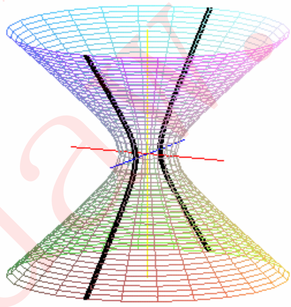
解直线参数方程$\begin{cases} x=z \\ y=1 & -\infty<z<+\infty. \\ z=z \end{cases}$

直线$\dfrac{x}{1}=\dfrac{y-1}{0}=\dfrac{z}{1}$绕z轴旋转,对于固定的z,

旋转曲面上的点组成一个圆,其半径为$\sqrt{1+z^2}$,

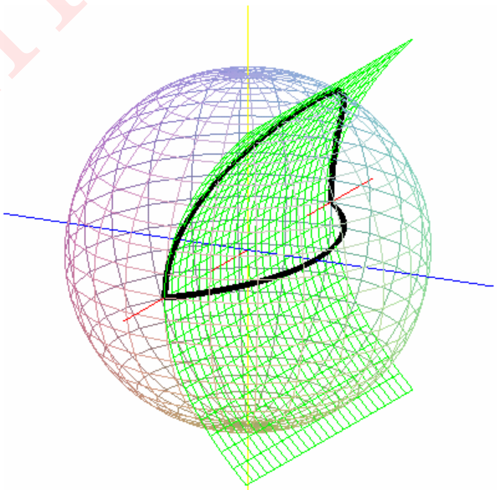
故旋转曲面的方程$\begin{cases} x=\sqrt{1+z^2}\cos\theta \\ y=\sqrt{1+z^2}\sin\theta & -\infty<z<+\infty, 0\le\theta\le 2\pi. \\ z=z \end{cases}$

15.求双曲线$\begin{cases} \dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}=1(b,c>0), \\ x=0 \end{cases}$绕z轴旋转一周所得曲面的方程.



解$\dfrac{x^2+y^2}{b^2}-\dfrac{z^2}{c^2}=1.$

16.求曲线$\begin{cases} x^2+y^2+z^2=1 \\ z^2=2y \end{cases}$在$Oxy$平面上的投影曲线的方程.



解$x^2+y^2+2y=1, x^2+(y+1)^2=2.$

## 习题 5.1

1.设$ABCD$为一平行四边形,$\overrightarrow{AB}=\boldsymbol{a},\overrightarrow{AD}=\boldsymbol{b}$.试用$\boldsymbol{a},\boldsymbol{b}$表示$\overrightarrow{AC},\overrightarrow{DB},\overrightarrow{MA}$($M$为平行四边形对角线的交点).

解$\overrightarrow{AC}=\boldsymbol{a}+\boldsymbol{b},\overrightarrow{DB}=\boldsymbol{a}-\boldsymbol{b},\overrightarrow{MA}=-\overrightarrow{AM}=-\frac{1}{2}\overrightarrow{AC}=-\frac{1}{2}(\boldsymbol{a}+\boldsymbol{b})$.

2.设$M$为线段$\overline{AB}$的中点,$O$为空间中的任意一点,证明

$$\overrightarrow{OM}=\frac{1}{2}(\overrightarrow{OA}+\overrightarrow{OB}).$$

证$\overrightarrow{OM}=\overrightarrow{OA}+\overrightarrow{AM}=\overrightarrow{OA}+\frac{1}{2}\overrightarrow{AB}=\overrightarrow{OA}+\frac{1}{2}\left(\overrightarrow{OB}-\overrightarrow{OA}\right)$

$$=\frac{1}{2}(\overrightarrow{OA}+\overrightarrow{OB}).$$

3.设$M$为三角形$ABC$的重心,$O$为空间中任意一点,

证明$\overrightarrow{OM}=\frac{1}{3}(\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC})$.

证$\overrightarrow{OM}=\overrightarrow{OA}+\overrightarrow{AM}=\overrightarrow{OA}+\frac{2}{3}\overrightarrow{AD}=\overrightarrow{OA}+\frac{2}{3}\times\frac{1}{2}(\overrightarrow{AB}+\overrightarrow{AC})$

$=\overrightarrow{OA}+\frac{1}{3}(\overrightarrow{AB}+\overrightarrow{AC})$,

$\overrightarrow{OM}=\overrightarrow{OB}+\frac{1}{3}(\overrightarrow{BA}+\overrightarrow{BC}),\overrightarrow{OM}=\overrightarrow{OC}+\frac{1}{3}(\overrightarrow{CA}+\overrightarrow{CB})$.

$3\overrightarrow{OM}=\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC},\overrightarrow{OM}=\frac{1}{3}(\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC})$.

4.设平行四边形$ABCD$的对角线交点为$M$,$O$为空间中的任意一点,证明$\overrightarrow{OM}=\frac{1}{4}(\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}+\overrightarrow{OD})$.

证$\overrightarrow{OM}=OA+AM=OA+\frac{1}{2}(AB+AD)$,

$\overrightarrow{OM}=OB+\frac{1}{2}(BA+AD),\overrightarrow{OM}=OC+\frac{1}{2}(BA+DA)$,

$\overrightarrow{OM}=OD+\frac{1}{2}(AB+DA)$.

$4\overrightarrow{OM}=\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}+\overrightarrow{OD},\overrightarrow{OM}=\frac{1}{4}(\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}+\overrightarrow{OD})$.

5.对于任意三个向量$\boldsymbol{a},\boldsymbol{b}$与$\boldsymbol{c}$,判断下列各式是否成立?

(1)$(\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c}=(\boldsymbol{b}\cdot\boldsymbol{c})\boldsymbol{a}$;

(2)$(\boldsymbol{a}\cdot\boldsymbol{b})^2=\boldsymbol{a}^2\cdot\boldsymbol{b}^2$;

(3)$\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})=(\boldsymbol{c}\times\boldsymbol{a})\cdot\boldsymbol{b}$.

解(1)不成立.例如:$\boldsymbol{a}=\boldsymbol{b}=\boldsymbol{i},\ \boldsymbol{c}=\boldsymbol{j}.(\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c}=\boldsymbol{j},(\boldsymbol{b}\cdot\boldsymbol{c})\boldsymbol{a}=\boldsymbol{0}$.

(2)不成立.例如:$\boldsymbol{a}=\boldsymbol{i},\ \boldsymbol{b}=\boldsymbol{j},(\boldsymbol{a}\cdot\boldsymbol{b})^2=0,\boldsymbol{a}^2\cdot\boldsymbol{b}^2=1$.

(3)成立,都是$\boldsymbol{a},\boldsymbol{b}$与$\boldsymbol{c}$组成的平行六面体的有向体积.

6.利用向量证明三角形两边中点的连线平行于第三边,

并且等于第三边长度之半.

证 $DE=DA+AE=\dfrac{1}{2}BA+\dfrac{1}{2}AC$

$=\dfrac{1}{2}(BA+AC)=\dfrac{1}{2}BC.$

7.利用向量证明:

(1)菱形的对角线互相垂直,且平分顶角;(2)勾股弦定理.

证(1)$\overrightarrow{AC}\cdot\overrightarrow{BD}=(\overrightarrow{AB}+\overrightarrow{BC})\cdot(\overrightarrow{BC}+\overrightarrow{CD})$

$=(\overrightarrow{AB}+\overrightarrow{BC})\cdot(\overrightarrow{BC}-\overrightarrow{CD})=|BC|^2-|CD|^2=0.$

$$\cos\alpha=\frac{AB\cdot AC}{|AB||AC|}\frac{AB\cdot(AB+AD)}{|AB||AC|}=\frac{AB\cdot AB+AB\cdot AD}{|AB||AC|}=\frac{a^2+AB\cdot AD}{|a|AC|},$$

$$\cos\beta=\frac{AD\cdot(AB+AD)}{|AB||AC|}=\frac{AD\cdot AB+AD\cdot AD}{|AB||AC|}=\frac{a^2+AB\cdot AD}{|a|AC|}=\cos\alpha.$$

$\alpha$与$\beta$都是锐角,故$\alpha=\beta$.

(2)$|AC|^2=\overrightarrow{AC}\cdot\overrightarrow{AC}=(\overrightarrow{AB}+\overrightarrow{BC})\cdot(\overrightarrow{AB}+\overrightarrow{BC})$

$=|\overrightarrow{AB}|^2+|\overrightarrow{BC}|^2+2\overrightarrow{AB}\cdot\overrightarrow{BC}=|\overrightarrow{AB}|^2+|\overrightarrow{BC}|^2$ .

8.证明恒等式$(\boldsymbol{a}\times\boldsymbol{b})^2+(\boldsymbol{a}\cdot\boldsymbol{b})^2=|\boldsymbol{a}|^2\cdot|\boldsymbol{b}|^2$ .

证$(\boldsymbol{a}\times\boldsymbol{b})^2+(\boldsymbol{a}\cdot\boldsymbol{b})^2=|\boldsymbol{a}|^2\cdot|\boldsymbol{b}|^2\cos^2\alpha+|\boldsymbol{a}|^2\cdot|\boldsymbol{b}|^2\sin^2\alpha$

$=|\boldsymbol{a}|^2\cdot|\boldsymbol{b}|^2(\cos^2\alpha+\sin^2\alpha)=|\boldsymbol{a}|^2\cdot|\boldsymbol{b}|^2$ .

9.试用向量$\overrightarrow{AB}$与$\overrightarrow{AC}$表示三角形$ABC$的面积.

解$\Delta ABC$的面积$=\dfrac{1}{2}\cdot ABDC$的面积$=\dfrac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|$.

10.给定向量$\boldsymbol{a}$,记$\boldsymbol{a}\cdot\boldsymbol{a}$为$\boldsymbol{a}^2$,即$\boldsymbol{a}^2=\boldsymbol{a}\cdot\boldsymbol{a}$.现设$\boldsymbol{a},\boldsymbol{b}$为任意向量,证明:

$(\boldsymbol{a}+\boldsymbol{b})^2+(\boldsymbol{a}-\boldsymbol{b})^2=2(\boldsymbol{a}^2+\boldsymbol{b}^2)$.

证$(\boldsymbol{a}+\boldsymbol{b})^2+(\boldsymbol{a}-\boldsymbol{b})^2=(\boldsymbol{a}+\boldsymbol{b})\cdot(\boldsymbol{a}+\boldsymbol{b})+(\boldsymbol{a}-\boldsymbol{b})\cdot(\boldsymbol{a}-\boldsymbol{b})$

$=\boldsymbol{a}\cdot\boldsymbol{a}+\boldsymbol{b}\cdot\boldsymbol{b}+2\boldsymbol{a}\cdot\boldsymbol{b}+\boldsymbol{a}\cdot\boldsymbol{a}+\boldsymbol{b}\cdot\boldsymbol{b}-2\boldsymbol{a}\cdot\boldsymbol{b}=2(\boldsymbol{a}^2+\boldsymbol{b}^2)$.

11.对于任意向量$\boldsymbol{a},\boldsymbol{b}$,证明:$(\boldsymbol{a}\times\boldsymbol{b})^2\le\boldsymbol{a}^2\boldsymbol{b}^2$.问:等号成立的充分必要条件是什么?

证$(\boldsymbol{a}\times\boldsymbol{b})^2=|\boldsymbol{a}\times\boldsymbol{b}|^2=(|\boldsymbol{a}||\boldsymbol{b}|\sin\alpha)^2=|\boldsymbol{a}|^2|\boldsymbol{b}|^2\sin^2\alpha\le|\boldsymbol{a}|^2|\boldsymbol{b}|^2=\boldsymbol{a}^2\boldsymbol{b}^2$.

等号成立的充分必要条件是$\boldsymbol{a},\boldsymbol{b}$正交.

# 习题 5.2

1.写出点$(x,y,z)$分别到$x$轴,$y$轴,$z$轴,$Oxy$平面,$Oyz$平面以及原点的距离.

解$d_x=\sqrt{y^2+z^2}, d_y=\sqrt{x^2+z^2}, d_z=\sqrt{x^2+y^2}, d_{xy}=|z|, d_{yz}=|x|, d_O=\sqrt{x^2+y^2+z^2}$.

2.已知三点$A=(-1,2,1), B=(3,0,1), C=(2,1,2)$,求$\overrightarrow{AB},\overrightarrow{BA},\overrightarrow{AC},\overrightarrow{BC}$的坐标与模.

解$\overrightarrow{AB}=(3,0,1)-(-1,2,1)=(4,-2,0), |\overrightarrow{AB}|=\sqrt{20}=2\sqrt{5}$,

$|\overrightarrow{BA}=-\overrightarrow{AB}=-(4,-2,0)=(-4,2,0)=-2\sqrt{5}$,

$\overrightarrow{AC}=(2,1,2)-(-1,2,1)=(3,-1,1), |\overrightarrow{AC}|=\sqrt{11}$,

$\overrightarrow{BC}=(2,1,2)-(3,0,1)-=(-1,1,1), |\overrightarrow{BC}|=\sqrt{3}$.

3.$\boldsymbol{a}=(3,-2,2), \boldsymbol{b}=(1,3,2), \boldsymbol{c}=(8,6,-2)$,

$3\boldsymbol{a}-2\boldsymbol{b}+\frac{1}{2}\boldsymbol{c}=(9,-6,6)+(-2,-6,-4)+(4,3,-1)=(11,-9,1)$.

4.设$\boldsymbol{a}=(2,5,1), \boldsymbol{b}=(1,-2,7)$,分别求出沿$\boldsymbol{a}$和$\boldsymbol{b}$方向的单位向量,并求常数$k$,使$k\boldsymbol{a}+\boldsymbol{b}$与$xy$平面平行.

解$\boldsymbol{a}^\circ=\frac{1}{\sqrt{30}}(2,5,1), \boldsymbol{b}^\circ=\frac{1}{3\sqrt{6}}(1,-2,7)$.

$k\boldsymbol{a}+\boldsymbol{b}=(2k,5k,k)+(1,-2,7)=(2k+1,5k,-2,k+7), k+7=0, k=-7$.

5.设$A,B$两点的坐标分别为$(x_1,y_1,z_1)$和$(x_2,y_2,z_2)$,求$A,B$连线中点$C$的坐标.

解$\overrightarrow{OC}=\frac{1}{2}(\overrightarrow{OA}+\overrightarrow{OB})=\frac{1}{2}((x_1,y_1,z_1)+(x_2,y_2,z_2))=\frac{1}{2}(x_1+x_2,y_1+y_2,z_1+z_2)$.

6.设$\boldsymbol{a}=(1,-2,3), \boldsymbol{b}=(5,2,-1)$,求

(1)$2\boldsymbol{a}\cdot 3\boldsymbol{b}$ (2)$\boldsymbol{a}\cdot\boldsymbol{i}$ (3)$\cos<\boldsymbol{a},\boldsymbol{b}>$.

解(1)$2\boldsymbol{a}\cdot 3\boldsymbol{b}=6\boldsymbol{a}\cdot\boldsymbol{b}=6\times(-2)=-12$.

(2)$\boldsymbol{a}\cdot\boldsymbol{i}=1$.

(3)$\cos<\boldsymbol{a},\boldsymbol{b}>=\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}=\frac{-2}{\sqrt{14}\sqrt{30}}=-\frac{1}{\sqrt{105}}$,

7.设$|\boldsymbol{a}|=1, |\boldsymbol{b}|=3, |\boldsymbol{c}|=2, |\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}|=\sqrt{17+6\sqrt{3}}$且$\boldsymbol{a}\perp\boldsymbol{c}$ $<\boldsymbol{a},\boldsymbol{b}>=\pi/3$,求$<\boldsymbol{b},\boldsymbol{c}>=$?

解$17+6\sqrt{3}=|\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}|^2=(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c})\cdot(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c})$

$=|\boldsymbol{a}|^2+|\boldsymbol{b}|^2+|\boldsymbol{c}|^2+2(\boldsymbol{a}\cdot\boldsymbol{b}+\boldsymbol{b}\cdot\boldsymbol{c}+\boldsymbol{a}\cdot\boldsymbol{c})=$

$=1+9+4+2(3\times\frac{1}{2}+\boldsymbol{b}\cdot\boldsymbol{c})$,

$\boldsymbol{b}\cdot\boldsymbol{c}=3\sqrt{3}, \cos<\boldsymbol{b},\boldsymbol{c}>=\frac{\boldsymbol{b}\cdot\boldsymbol{c}}{|\boldsymbol{b}||\boldsymbol{c}|}=\frac{3\sqrt{3}}{3\times 2}=\frac{\sqrt{3}}{2}. <\boldsymbol{b},\boldsymbol{c}>=\frac{\pi}{6}$.

8.设$|\boldsymbol{a}|=2,|\boldsymbol{b}|=6$,试求常数$k$,使$\boldsymbol{a}+k\boldsymbol{b}\perp\boldsymbol{a}-k\boldsymbol{b}$.

解$(\boldsymbol{a}+k\boldsymbol{b})\cdot(\boldsymbol{a}-k\boldsymbol{b})=|\boldsymbol{a}|^2-k^2|\boldsymbol{b}|^2=4-36k^2=0,k=\pm1/3.$

9.$\boldsymbol{a}=(1,-2,1),\boldsymbol{b}=(1,-1,3),\boldsymbol{c}=(2,5,-3)$

(1)$\boldsymbol{a}\times\boldsymbol{b}$ (2)$\boldsymbol{c}\times\boldsymbol{j}$ (3)$(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c}$ (4)$(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c}$ (5)$\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})$.

解(1)$\boldsymbol{a}\times\boldsymbol{b}=\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\1&-2&1\\1&-1&3\end{vmatrix}=(-5,-2,1),$

(2)$\boldsymbol{c}\times\boldsymbol{j}=\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\2&5&-3\\0&1&0\end{vmatrix}=(3,0,2).$

(3)$(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c}=\begin{vmatrix}1&-2&1\\1&-1&3\\2&5&-3\end{vmatrix}=-23.$(4)$(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c}=\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\-5&-2&1\\2&5&-3\end{vmatrix}=(1,-13,-21).$

(5)$\boldsymbol{b}\times\boldsymbol{c}=\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\1&-1&3\\2&5&-3\end{vmatrix}=(-12,9,7),\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})=\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\1&-2&1\\-12&9&7\end{vmatrix}=(-23,-19,-15).$

10.在平行四边形$ABCD$中,$\overrightarrow{AB}=(2,1,0)\overrightarrow{AD}=(0,-1,2)$,求两对角线的夹角$<\overrightarrow{AC},\overrightarrow{BD}>$.

解$\overrightarrow{AC}=\overrightarrow{AB}+\overrightarrow{AD}=(2,1,0)+(0,-1,2)=(2,0,2),$

$\overrightarrow{BD}=\overrightarrow{AD}-\overrightarrow{AB}=(0,-1,2)-(2,1,0)=(-2,-2,2).$

$\cos<\overrightarrow{AC},\overrightarrow{BD}>=\dfrac{\overrightarrow{AC}\cdot\overrightarrow{BD}}{|\overrightarrow{AC}||\overrightarrow{BD}|}=\dfrac{0}{|\overrightarrow{AC}||\overrightarrow{BD}|}=0,<\overrightarrow{AC},\overrightarrow{BD}>=\dfrac{\pi}{2}.$

解二$|\overrightarrow{AB}|=|\overrightarrow{AD}|=\sqrt{5}$,平行四边形$ABCD$为菱形,故两对角线的夹角$<\overrightarrow{AC},\overrightarrow{BD}>=\dfrac{\pi}{2}.$

11.已知三点$A(3,4,1),B(2,3,0),C(3,5,1)$,求三角形$ABC$的面积.

解$\overrightarrow{AB}=(-1,-1,-1)=-(1,1,1),\overrightarrow{AC}=(0,1,0),\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\1&1&1\\0&1&0\end{vmatrix}=(-1,0,1),$

三角形$ABC$的面积$=\dfrac{1}{2}\times\sqrt{2}.$

12.证明向量$\boldsymbol{a}=(3,4,5),\boldsymbol{b}=(1,2,2)$和$\boldsymbol{c}=(9,14,16)$是共面的.

证因为$(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c})=\begin{vmatrix}3 & 4 & 5\\1 & 2 & 2\\9 & 14 & 16\end{vmatrix}=0$,故$\boldsymbol{a},\boldsymbol{b}$和$\boldsymbol{c}$是共面的.

13.已知$|\boldsymbol{a}|=1,|\boldsymbol{b}|=5,\boldsymbol{a}\cdot\boldsymbol{b}=-3$,求$|\boldsymbol{a}\times\boldsymbol{b}|$.

解$\cos<\boldsymbol{a},\boldsymbol{b}>=\dfrac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}=\dfrac{-3}{5},\sin<\boldsymbol{a},\boldsymbol{b}>=\dfrac{4}{5},|\boldsymbol{a}\times\boldsymbol{b}|=|\boldsymbol{a}||\boldsymbol{b}|\sin<\boldsymbol{a},\boldsymbol{b}>=1\times5\times\dfrac{4}{5}=4.$

14.设向量$\boldsymbol{a}$的方向余弦$\cos\alpha,\cos\beta,\cos\gamma$,在下列各情况下,指出$\boldsymbol{a}$的方向特征.

(1) $\cos\alpha=0,\cos\beta\neq 0,\cos\gamma\neq 0$;

(2) $\cos\alpha=\cos\beta=0,\cos\gamma\neq 0$;

(3) $\cos\alpha=\cos\beta=\cos\gamma$.

解(1)$\boldsymbol{a}$与$x$轴垂直.

(2)$\boldsymbol{a}$是沿$z$轴的的向量.

(3) $\boldsymbol{a}$与三个轴的夹角相等, 都是$\arccos\frac{1}{\sqrt{3}}$或$\pi-\arccos\frac{1}{\sqrt{3}}$.

15.设$|\boldsymbol{a}|=\sqrt{2}$, $\boldsymbol{a}$的三个方向角满足$\alpha=\beta=\frac{1}{2}\gamma$,求$\boldsymbol{a}$的坐标.

解$2\cos^2\alpha+\cos^2 2\alpha=1, 2\cos^2\alpha+(2\cos^2\alpha-1)^2=1$.

$\cos^2\alpha=x, 2x+(2x-1)^2=1, 4x^2-2x+1=1, 2x(2x-1)=0, x=0, x=\frac{1}{2}$.

$\cos^2\alpha=0,\alpha=\frac{\pi}{2},\boldsymbol{a}=(0,0,-\sqrt{2})$.

$\cos^2\alpha=\frac{1}{2},\cos\alpha=\pm\frac{1}{\sqrt{2}},\alpha=\frac{\pi}{4},\frac{3\pi}{4}.\boldsymbol{a}=(1,1,0)$.

16.设$\boldsymbol{a},\boldsymbol{b}$为两非零向量,且$(7\boldsymbol{a}-5\boldsymbol{b})\perp(\boldsymbol{a}+3\boldsymbol{b}),(\boldsymbol{a}-4\boldsymbol{b})\perp(7\boldsymbol{a}-2\boldsymbol{b})$,

求$\cos<\boldsymbol{a},\boldsymbol{b}>$.

解$(7\boldsymbol{a}-5\boldsymbol{b})\cdot(\boldsymbol{a}+3\boldsymbol{b})=0, 7|\boldsymbol{a}|^2-15|\boldsymbol{b}|^2+16|\boldsymbol{a}||\boldsymbol{b}|\cos<\boldsymbol{a},\boldsymbol{b}>=0$,

$(\boldsymbol{a}-4\boldsymbol{b})\cdot(7\boldsymbol{a}-2\boldsymbol{b})=0, 7|\boldsymbol{a}|^2+8|\boldsymbol{b}|^2-30|\boldsymbol{a}||\boldsymbol{b}|\cos<\boldsymbol{a},\boldsymbol{b}>=0$.

$$\begin{cases}-15\frac{|\boldsymbol{b}|^2}{|\boldsymbol{a}|^2}+16\frac{|\boldsymbol{b}|}{|\boldsymbol{a}|}\cos<\boldsymbol{a},\boldsymbol{b}>=-7,\\ 8\frac{|\boldsymbol{b}|^2}{|\boldsymbol{a}|^2}-30\frac{|\boldsymbol{b}|}{|\boldsymbol{a}|}\cos<\boldsymbol{a},\boldsymbol{b}>=-7.\end{cases}$$

$$\frac{|\boldsymbol{b}|^2}{|\boldsymbol{a}|^2}=\frac{\begin{vmatrix}-7 & 16\\ -7 & -30\end{vmatrix}}{\begin{vmatrix}-15 & 16\\ 8 & -30\end{vmatrix}}=1,\frac{|\boldsymbol{b}|}{|\boldsymbol{a}|}=1$$

$$\cos<\boldsymbol{a},\boldsymbol{b}>=\frac{\begin{vmatrix}-15 & -7\\ 8 & -7\end{vmatrix}}{\begin{vmatrix}-15 & 16\\ 8 & -30\end{vmatrix}}=\frac{1}{2}.$$

## 习题 5.3

1.指出下列平面位置的特点:

(1)$5x-3z+1=0$(2)$x+2y-7z=0$(3)$y+5=0$(4)$2y-9z=0$(5)$x-y-5=0$(6)$x=0$.

解(1)平行于$y$轴.(2)过原点.(3)平行于$Oxz$平面.

(4)过$x$轴.(5)平行于$z$轴.(6)$Oyz$平面.

2.求下列各平面的方程:

(1)平行于$y$轴且通过点$(1,-5,1)$和$(3,2,-2)$;

(2)平行于$Oxz$平面且通过点$(5,2,-8)$;

(3)垂直于平面$x-4y+5z=1$且通过点$(-2,7,3)$及$(0,0,0)$;

(4)垂直于$Oyz$平面且通过点$(5,-4,3)$及$(-2,1,8)$.

解(1)$\boldsymbol{a}=(0,1,0),\boldsymbol{b}=(2,7,-3),\boldsymbol{n}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ 0 & 1 & 0\\ 2 & 7 & -3\end{vmatrix}=(-3,0,-2).$

$-3(x-1)-2(z-1)=0, 3x+2z-5=0.$

(2)$y=2$.

(3)$\boldsymbol{a}=(1,-4,5),\boldsymbol{b}=(-2,7,3),\boldsymbol{n}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ 1 & -4 & 5\\ -2 & 7 & 3\end{vmatrix}=(-47,-13,-1).$

$47x+13y+1=0.$

(4)$\boldsymbol{a}=(1,0,0),\boldsymbol{b}=(-7,5,5),\boldsymbol{n}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ 1 & 0 & 0\\ -7 & 5 & 5\end{vmatrix}=(0,-5,5)=5(0,-1,1).$

$-(y+4)+(z-3)=0, y-z+7=0.$

3.求通过点$A(2,4,8), B(-3,1,5)$及$C(6,-2,7)$的平面方程.

解$\boldsymbol{a}=(-5,-3,-3),\boldsymbol{b}=(4,-6,-1)$.

$\boldsymbol{n}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ -5 & -3 & -3\\ 4 & -6 & -1\end{vmatrix}=(-15,-17,42),$

$-15(x-2)-17(y-4)+42(z-8)=0, 15x+17y-42z+238=0.$

4.设一平面在各坐标轴上的截距都不等于零并相等，且过点(5, −7, 4)，求此平面的方程.

解　$\frac{\mathrm{x}}{\mathrm{a}}+\frac{y}{a}+\frac{z}{a}=1, \frac{5}{\mathrm{a}}+\frac{-7}{a}+\frac{4}{a}=1, a=2, x+y+z-2=0.$

5.已知两点$A(2,-1,-2)$及$B(8,7,5)$,求过$B$且与线段$AB$垂直的平面.

解$\boldsymbol{n}=(6,8,7).6(x-8)+8(y-7)+7(z-5)=0, 6x+8y+7z-139=0.$

6.求过点$(2,0,-3)$且与$2x-2y+4z+7=0,3x+y-2z+5=0$垂直的平面方程.

解$\boldsymbol{n}=\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\2&-2&4\\3&1&-2\end{vmatrix}=(0,16,8)=8(0,2,1).2y+(z+3)=0,y+z+3=0.$

7.求通过$x$轴且与平面$9x-4y-2z+3=0$垂直的平面方程.

解$By+Cz=0,-4B-2C=0$,取$B=1,C=-2,y-2z=0.$

8.求通过直线$l_1:\begin{cases}x+2z-4=0\\3y-z+8=0\end{cases}$且与直线$l_2:\begin{cases}x-y-4=0\\y-z-6=0\end{cases}$平行的平面方程.

解$\boldsymbol{a}=\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\1&0&2\\0&3&-1\end{vmatrix}=(-6,1,3),\boldsymbol{b}=\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\1&-1&0\\0&1&-1\end{vmatrix}=(1,1,1),$

$\boldsymbol{n}=\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\-6&1&3\\1&1&1\end{vmatrix}=(-2,9,-7)$.用$z_0=0$代入$l_1$的方程,得$x_0=4,y_0=-8/3.$

$-2(x-4)+9(y+8/3)-7(z)=0,-2x+9y-7z+32=0.$

9.求直线$l_1:\dfrac{x+3}{3}=\dfrac{y+1}{2}=\dfrac{z-2}{4}$与直线$l_2:\begin{cases}x=3t+8\\y=t+1\\z=2t+6\end{cases}$的交点坐标,

并求通过此两直线的平面方程.

解求两条直线交点坐标:

$\dfrac{3t+8+3}{3}=\dfrac{t+1+1}{2}=\dfrac{2t+6-2}{4},t+\dfrac{11}{3}=\dfrac{t}{2}+1=\dfrac{t}{2}+1,t=-\dfrac{16}{3},$

$x_0=-8,y_0=-\dfrac{13}{3},z_0=-\dfrac{14}{3}$,交点$(-8,-\dfrac{13}{3},-\dfrac{14}{3})$.

$\boldsymbol{n}=\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\3&2&4\\3&1&2\end{vmatrix}=(0,6,-3)=3(0,2,-1).2(y+1)-(z-2)=0,2y-z+4=0.$

10.求通过两直线$l_1:\dfrac{x-1}{2}=\dfrac{y+1}{-1}=\dfrac{z+1}{1}$和$l_2:\dfrac{x+2}{-4}=\dfrac{y-2}{2}=\dfrac{z}{-2}$的平面方程.

解 两直线平行. 平面过点$(1,-1,-1)$和$(-2,2,0)$.$\boldsymbol{n}=\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\2&-1&1\\-3&3&1\end{vmatrix}=(-4,-5,3).$

$-4(x-1)-5(y+1)+3(z+1)=0,-4x-5y+3z+2=0.$

11.证明两直线$l_1:\dfrac{x-1}{-1}=\dfrac{y}{2}=\dfrac{z+1}{1}$和$l_2:\dfrac{x+2}{0}=\dfrac{y-1}{1}=\dfrac{z-2}{-2}$是异面直线.

证首先,两直线的方向向量$(-1,2,1)$ 和 $(0,1,-2)$不平行.

$l_2\begin{cases}x=-2\\ y=1+t\\ z=2-2t\end{cases}\dfrac{-2-1}{-1}=\dfrac{1+t}{2}=\dfrac{-2t+3}{1},t=5,t=0,$矛盾.故两直线无公共点.

两直线不平行,又无交点,故是异面直线.

12. 将下列直线方程化为标准方程及参数方程:

(1)$\begin{cases}2x+y-z+1=0\\ 3x-y+2z-8=0;\end{cases}$(2)$\begin{cases}x-3z+5=0\\ y-2z+8=0.\end{cases}$

解(1)$\boldsymbol{n}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ 2 & 1 & -1\\ 3 & -1 & 2\end{vmatrix}=(1,-7,-5).$

(1)中令$x_0=0,\begin{cases}y-z+1=0\\ -y+2z-8=0;\end{cases}$解之得$y_0=6,z_0=7.$

标准方程$\dfrac{x}{1}=\dfrac{y-6}{-7}=\dfrac{z-7}{-5}.$

参数方程:$\begin{cases}x=t\\ y=6-7t,-\infty<t<+\infty.\\ z=7-5t\end{cases}$

(2)(1)$\boldsymbol{n}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ 1 & 0 & -3\\ 0 & 1 & -2\end{vmatrix}=(3,2,1).$

(2)中令$z_0=0$,直接得$x_0=-5,y_0=-8.$

标准方程$\dfrac{x+5}{3}=\dfrac{y+8}{2}=\dfrac{z}{1}.$

参数方程:$\begin{cases}x=-5+3t\\ y=-8+2t,-\infty<t<+\infty.\\ z=t\end{cases}$

13.求通过点$(3,2,-5)$及x轴的平面与平面$3x-y-7z+9=0$的交线方程.

解地第一个平面的法向量$\boldsymbol{n}=\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 0 & 0 \\ 3 & 2 & -5 \end{vmatrix}=(0,5,2)$,

平面方程$5y+2z=0$.

直线方程$\begin{cases} 5y+2z=0 \\ 3x-y-7z+9=0. \end{cases}$

直线的方向向量$\boldsymbol{a}=\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 0 & 5 & 2 \\ 3 & -1 & -7 \end{vmatrix}=(-33,6,-15)=3(-11,2,-5)$.

$z_0=0,\begin{cases} 5y=0 \\ 3x-y+9=0. \end{cases} y_0=0, x_0=-3.$

直线方程:$\dfrac{x+3}{-11}=\dfrac{y}{2}=\dfrac{z}{-5}$.

14.当$D$为何值时,直线$\begin{cases} 3x-y+2z-6=0 \\ x+4y-z+D=0 \end{cases}$与$Oz$轴相交?

解直线$\begin{cases} 3x-y+2z-6=0 \\ x+4y-z+D=0 \end{cases}$与$Oz$轴相交$\Leftrightarrow$存在$(0,0,z_0)$在此直线上,

$\Leftrightarrow \begin{cases} 2z_0-6=0 \\ -z_0+D=0 \end{cases} \Leftrightarrow D=z_0=3.$

15.试求通过直线$l_1:\begin{cases} x-2z-4=0 \\ 3y-z+8=0 \end{cases}$并与直线$l_2:\begin{cases} x-y-4=0 \\ z-y+6=0 \end{cases}$平行的平面方程.

解$l_1$的方向向量$\boldsymbol{a}=\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 0 & -2 \\ 0 & 3 & -1 \end{vmatrix}=(6,1,3)$.

$l_2$的方向向量$\boldsymbol{b}=\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & -1 & 0 \\ 0 & -1 & 1 \end{vmatrix}=(-1,-1,-1)=-(1,1,1)$.

平面的法向量$\boldsymbol{n}=\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 6 & 1 & 3 \\ 1 & 1 & 1 \end{vmatrix}=(-2,-3,5)$.

在的方程中令$z_0=0$得$x_0=4, y_0=-\dfrac{8}{3}$.

所求平面方程:$-2(x-4)-3(y+\dfrac{8}{3})+5z=0$,即$2x+3y-5z=0$.

16.求点$(1,2,3)$到直线$\dfrac{x}{1}=\dfrac{y-4}{-3}=\dfrac{z-3}{-2}$的距离.

解过点$(1,2,3)$垂直于直线的平面:

$(x-1)-3(y-2)-2(z-3)=0.$

直线参数方程:$\begin{cases} x=t \\ y=4-3t \\ z=3-2t \end{cases}.$

代入平面方程得对应交点的参数:

$(t-1)-3(4-3t-2)-2(3-2t-3)=0, t_0=\dfrac{1}{2},$

直线与平面交点为$(\dfrac{1}{2},\dfrac{5}{2},2).$

所求距离$d=\sqrt{(1-\dfrac{1}{2})^2+(2-\dfrac{5}{2})^2+(3-2)^2}=\dfrac{\sqrt{6}}{2}.$

17.求点$(2,1,3)$到平面$2x-2y+z-3=0$的距离与投影.

解过点$(2,1,3)$垂直于平面$2x-2y+z-3=0$的直线方程的参数方程:

$\begin{cases} x=2+2t \\ y=1-2t \\ z=3+t \end{cases}, -\infty<t<+\infty.$代入平面方程

$2(2+2t)-2(1-2t)+(3+t)-3=0. t_0=-\dfrac{2}{9}.$

$x_0=\dfrac{14}{9}, y_0=\dfrac{13}{9}, z_0=\dfrac{25}{9}.$

点$(2,1,3)$在平面$2x-2y+z-3=0$上的投影为$\left(\dfrac{14}{9},\dfrac{13}{9},\dfrac{25}{9}\right).$

点$(2,1,3)$在平面$2x-2y+z-3=0$的距离为

$\sqrt{(2-\dfrac{14}{9})^2+(1-\dfrac{13}{9})^2+(3-\dfrac{25}{9})^2}=\dfrac{2}{3}.$

18.求两平行直线$\frac{x-1}{1}=\frac{y+1}{-2}=\frac{z}{3}$与$\frac{x}{1}=\frac{y+1}{-2}=\frac{z-1}{3}$的距离.

解所求的就是点$(1,-1,0)$到直线$\frac{x}{1}=\frac{y+1}{-2}=\frac{z-1}{3}$的距离.

作法与16题雷同. 过点$(1,-1,0)$垂直于直线$\frac{x}{1}=\frac{y+1}{-2}=\frac{z-1}{3}$的平面:

$$(x-1)-2(y+1)+3z=0.$$

直线的参数方程$\begin{cases}x=t\\y=-1-2t\\z=1+3t\end{cases}$,代入平面方程

$$(t-1)-2(-2t)+3(1+3t)=0, t_0=-\frac{1}{7}.$$

直线与平面交点$(-\frac{1}{7},-\frac{5}{7},\frac{4}{7})$.

所求距离$d=\sqrt{(1+\frac{1}{7})^2+(-1+\frac{5}{7})^2+(0-\frac{4}{7})^2}=2\sqrt{\frac{3}{7}}$.

19.求过点$A(2,1,3)$并与直线$l_1:\frac{x+1}{3}=\frac{y-1}{2}=\frac{z}{-1}$垂直且相交的直线方程.

解过点$A$垂直于直线$l_1$的平面方程$3(x-2)+2(y-1)-(z-3)=0$.

直线$l_1$的参数方程$\begin{cases}x=-1+3t\\y=1+2t\\z=-t\end{cases}$

代入平面方程求交点对应的参数他$t$:

$$3(-3+3t)+2(2t)-(-t-3)=0, t_0=\frac{3}{7}.$$

交点$B(\frac{2}{7},\frac{13}{7},-\frac{3}{7})$.

连结点$A$, $B$的直线的方向向量

$$\overrightarrow{AB}=(\frac{2}{7}-2,\frac{13}{7}-1,-\frac{3}{7}-3)=(-\frac{12}{7},\frac{6}{7},-\frac{24}{7})=-\frac{6}{7}(2,-1,4).$$

所求直线方程:$\frac{x-2}{2}=\frac{y-1}{-1}=\frac{z-3}{4}$.

20.求两平行平面$3x+6y-2z-7=0$与$3x+6y-2z+14=0$之间的距离.

解点$A(0,0,-\frac{7}{2})$在第一张平面上.

过$A$垂直于第二张平面的直线的参数方程:$\begin{cases} x=3t \\ y=6t \\ z=-7/2-2t \end{cases}$

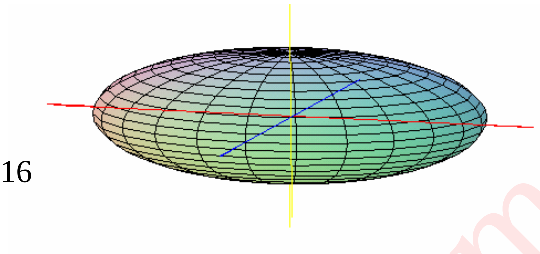
求直线与第二张平面的交点:$3(3t)+6(6t)-2(-7/2-2t)+14=0$,

$t_0=-\frac{3}{7},(-\frac{9}{7},-\frac{18}{7},-\frac{37}{14})$.

所求距离$=\sqrt{(\frac{9}{7})^2+(\frac{18}{7})^2+(\frac{6}{7})^2}=3$.

## 习题 5.4

1.求椭球面$2x^2+3y^2+4z^2-4x-6y+16z+16=0$的中心的坐标及三个半轴之长度.



解$2x^2+3y^2+4z^2-4x-6y+16z+16=0$,

$$2x^2+3y^2+4z^2-4x-6y+16z+17$$
$$=2(x-1)^2-2+3(y-1)^2-3+4(z+2)^2-16+16$$
$$=2(x-1)^2+3(y-1)^2+4(z+2)^2-5=0.$$
$$\frac{(x-1)^2}{\sqrt{\frac{5}{2}}^2}+\frac{(y-1)^2}{\sqrt{\frac{5}{3}}^2}+\frac{(z+2)^2}{\left(\frac{\sqrt{5}}{2}\right)^2}=1,$$

中心坐标:$(1,1,-2)$,半轴:$\sqrt{\frac{5}{2}},\sqrt{\frac{5}{3}},\frac{\sqrt{5}}{2}$.

2.说出下列曲面的名称,并画出略图:

(1)$8x^2+11y^2+24z^2=1$;椭球面.

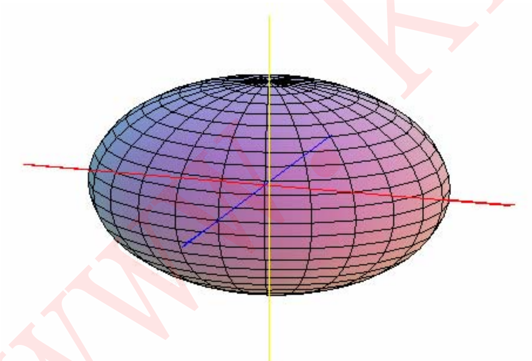
(2)$4x^2-9y^2-14z^2=-25$;单叶双曲面.

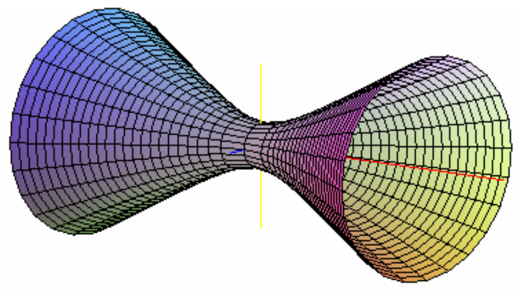
(3)$2x^2+9y^2-16z^2=-9$;双叶双曲面.

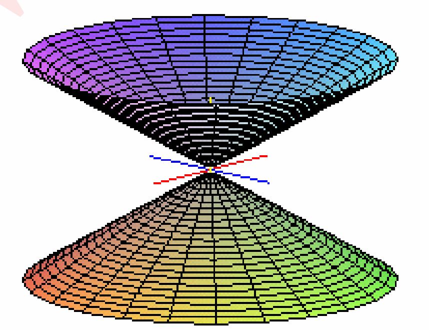
(4)$x^2-y^2=2x$;双曲柱面.

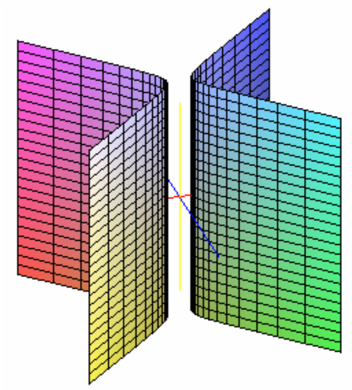
(5)$2y^2+z^2=x$;椭圆抛物面.

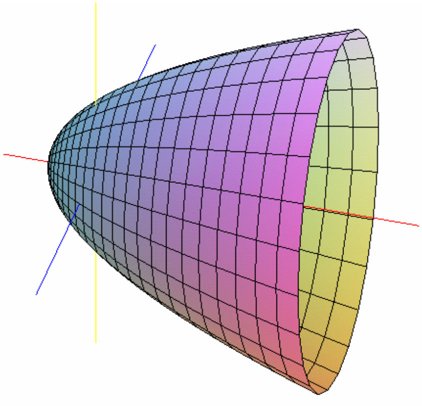
(6)$z=xy$.双曲抛物面.

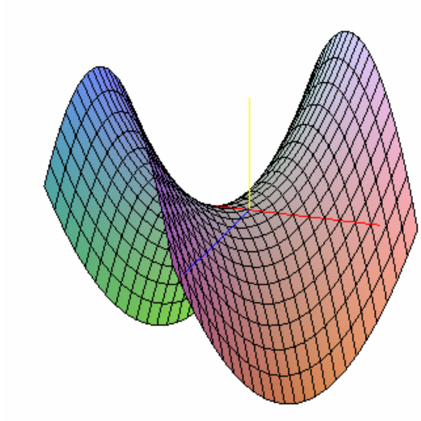












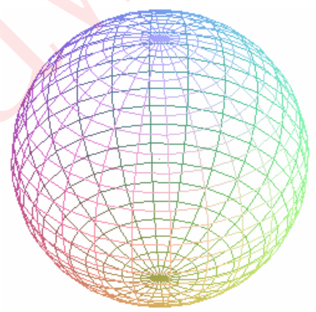
3.求下列曲面的参数方程:

$$(1)(x-1)^2+(y+1)^2+(z-3)^2=R^2;$$

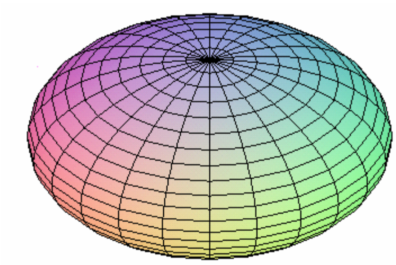
$$(2)x^2+\frac{y^2}{9}+\frac{z^2}{4}=1;(3)\frac{x^2}{4}+\frac{y^2}{9}-\frac{z^2}{16}=1;$$

$$(4)z=\frac{x^2}{a^2}-\frac{y^2}{b^2};(5)z=\frac{z^2}{a^2}+\frac{y^2}{b^2}.$$

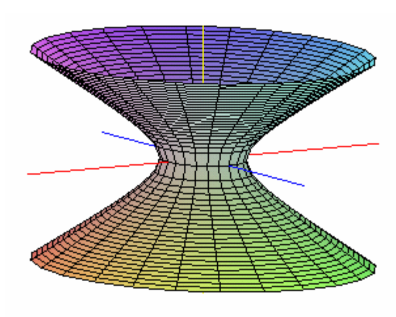
解(1) $\begin{cases} x=1+R\sin\varphi\cos\theta \\ y=-1+R\sin\varphi\sin\theta \; 0\le\varphi\le\pi, 0\le\theta<2\pi; \\ z=3+R\cos\varphi \end{cases}$



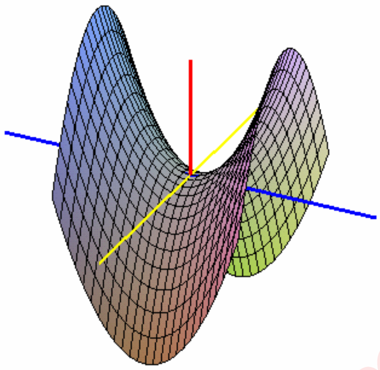
(2) $\begin{cases} x=\sin\varphi\cos\theta \\ y=3\sin\varphi\sin\theta \; 0\le\varphi\le\pi, 0\le\theta<2\pi; \\ z=2\cos\varphi \end{cases}$

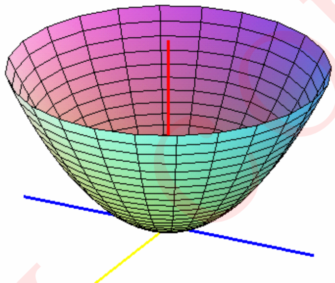


3) $\begin{cases} x=2\cosh\varphi\cos\theta \\ y=3\cosh\varphi\sin\theta \;\; 0-\theta<\varphi<+\infty, 0\le\theta<2\pi; \\ z=4\sinh\varphi \end{cases}$



$$(4)\begin{cases} x = r\cos\theta \\ y = r\sin\theta \quad 0 \le r < +\infty, 0 \le \theta \le 2\pi \\ z = r^2\cos\theta \end{cases}$$





$$(5)\begin{cases} x = ar\cos\theta \\ y = br\sin\theta \; 0 \le r < +\infty, 0 \le \theta \le 2\pi \\ z = r^2 \end{cases}$$

## 习题 5.5

1. 求下列曲线在指定点$P_0$的切线方程和法平面方程：

(1)$x=t,y=t^2,z=t^3,P_0=(1,1,1)$;

(2)曲面$z=x^2$与$y=x$的交线,$P_0=(2,2,4)$;

(3)柱面$x^2+y^2=R^2(R>0)$与平面$z=x+y$的交线$P_0=(R,0,R)$.

解 (1) $x'=1,y'=2t,z'=3t^2,\boldsymbol{t}=(1,2,3)$,切线方程:$\dfrac{x-1}{1}=\dfrac{y-1}{2}=\dfrac{z-1}{3}$,

法平面方程:$(x-1)+2(y-1)+3(z-1)=0,x+2y+3z-6=0.$

(2)$x=x,y=x,z=x^2,x'=1,y'=1,z'=2x,\boldsymbol{t}=(1,1,4)$.切线方程:$\dfrac{x-2}{1}=\dfrac{y-2}{1}=\dfrac{z-4}{4}$,

法平面方程:$(x-2)+(y-2)+4(z-4)=0,x+y+4z-20=0.$

$$(3)\boldsymbol{n}_1=(2x,2y,0)=(2R,0,0),\boldsymbol{n}_1=(1,1,-1),\boldsymbol{a}=\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\2R&0&0\\1&1&-1\end{vmatrix}=(0,2R,2R)=2R(0,1,1),$$

切线方程:$\dfrac{x-R}{0}=\dfrac{y}{1}=\dfrac{z-R}{1}$,法平面方程:$y+z-R=0.$

2.求出螺旋线$\begin{cases}x=R\cos t\\y=R\sin t\\z=bt\end{cases}(R>0,b>0,0\le t\le 2\pi)$在任意一

点处的切线的

方向余弦，并证明切线与$z$轴之夹角为常数.

解$(x',y',z')=(-R\sin t,R\cos t,b)$,

$$\boldsymbol{t}=(\cos\alpha,\cos\beta,\cos\gamma)=\frac{1}{\sqrt{R^2+b^2}}(-R\sin t,R\cos t,b),$$

$$\cos<\boldsymbol{t},\boldsymbol{k}>=\frac{b}{\sqrt{R^2+b^2}}=常数.0<<\boldsymbol{t},\boldsymbol{k}><\pi,<\boldsymbol{t},\boldsymbol{k}>=常数.$$

3.设$\boldsymbol{a}=\boldsymbol{a}(t)$与$\boldsymbol{b}=\boldsymbol{b}(t)$是两个可导的向量函数,$\alpha<t<\beta$.证明

$$\frac{d}{dt}\boldsymbol{a}(t)\cdot\boldsymbol{b}(t)=\boldsymbol{a}'(t)\cdot\boldsymbol{b}(t)+\boldsymbol{a}(t)\cdot\boldsymbol{b}'(t).$$

证 设$\boldsymbol{a}(t)=(a_1(t),a_2(t),a_3(t)),\boldsymbol{b}(t)=(b_1(t),b_2(t),b_3(t))$,

$$\boldsymbol{a}(t)\cdot\boldsymbol{b}(t)=a_1(t)b_1(t)+a_2(t)b_2(t)+a_3(t)b_3(t),$$

$$\frac{d}{dt}\boldsymbol{a}(t)\cdot\boldsymbol{b}(t)=\frac{d}{dt}[a_1(t)b_1(t)+a_2(t)b_2(t)+a_3(t)b_3(t)]$$

$$=a_1'(t)b_1(t)+a_1(t)b_1'(t)+a_2'(t)b_2(t)+a_2(t)b_2'(t)+a_3'(t)b_3(t)+a_3(t)b_3'(t)$$

$$=[a_1'(t)b_1(t)+a_2'(t)b_2(t)+a_3'(t)b_3(t)]+[a_1(t)b_1'(t)+a_2(t)b_2'(t)+a_3(t)b_3'(t)]$$

$$=\boldsymbol{a}'(t)\cdot\boldsymbol{b}(t)+\boldsymbol{a}(t)\cdot\boldsymbol{b}'(t).$$

4.设$\boldsymbol{r}=\boldsymbol{r}(t)(\alpha<t<\beta)$是一条光滑曲线,切$|\boldsymbol{r}(t)|=C$(常数).证明$\boldsymbol{r}(t)$与切线垂直,即$\boldsymbol{r}(t)\cdot\boldsymbol{r}'(t)=0$.

证 $\boldsymbol{r}(t)\cdot\boldsymbol{r}(t)=C^2,\frac{d}{dt}\boldsymbol{r}(t)\cdot\boldsymbol{r}(t)=\frac{d}{dt}C^2,\boldsymbol{r}'(t)\cdot\boldsymbol{r}(t)+\boldsymbol{r}(t)\cdot\boldsymbol{r}'(t)=0,2\boldsymbol{r}(t)\cdot\boldsymbol{r}'(t)=0,$

$\boldsymbol{r}(t)\cdot\boldsymbol{r}'(t)=0.$