初等数论
第七章 连分数

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1. 有限连分数的定义

形如:

$$x_0+\cfrac{1}{x_1+\cfrac{1}{x_2+\cfrac{1}{x_3+\cfrac{1}{x_4+\cfrac{1}{x_5+\cfrac{1}{x_6}}}}}}$$

的数，其中$x_1\sim x_6$都是正实数，将这个数称为是一个6阶连分数(6条横线)，它的值自然是一个实数.

特别，如果$x_1\sim x_6$都是正整数，x_0是整数的话，这个分数被称为6阶简单连分数，比如

$$-3+\cfrac{1}{2+\cfrac{1}{4+\cfrac{1}{3+\cfrac{1}{5+\cfrac{1}{9+\cfrac{1}{100}}}}}}$$

可以将上述连分数记为: $[x_0,x_1,x_2,x_3,x_4,x_5,x_6]$，比如$[-3,2,4,3,5,9,100]$

- 一般地，形如：

$$x_0+\cfrac{1}{x_1+\cfrac{1}{x_2+\cfrac{1}{x_3+\cfrac{1}{x_4+\cfrac{1}{\cdots+\cfrac{1}{x_{n-1}+\cfrac{1}{x_n}}}}}}}$$

的数($x_i(i\leq 0)\in\mathbb{R}, x_j(j\leq 1)>0$), 称为$n$阶有限连分数, 为书写方便, 记为$[x_0,x_1,x_2,\ldots,x_n]$. 如果这里面的数字都是整数, 这个连分数被称为有限简单连分数.

将$[x_0,x_1,x_2,\ldots,x_k]$称为是$[x_0,x_1,x_2,\ldots,x_n]$的第k个渐近分数, 比如:

$$-3+\frac{1}{2},\quad -3+\cfrac{1}{2+\cfrac{1}{4}},\quad -3+\cfrac{1}{2+\cfrac{1}{4+\cfrac{1}{3}}}$$

都是

$$-3+\cfrac{1}{2+\cfrac{1}{4+\cfrac{1}{3+\cfrac{1}{5+\cfrac{1}{9+\cfrac{1}{100}}}}}}$$

的渐近分数(值越来越精确).

2. 有限连分数的性质

以下性质中, 均假设$(x_i(i \le 0) \in \mathbb{R}, x_j(j \le 1) > 0)$
比如:

$$-3+\cfrac{1}{2+\cfrac{1}{4+\cfrac{1}{3+\cfrac{1}{5+\frac{1}{9+\frac{1}{100}}}}}} = -3+\cfrac{1}{2+\cfrac{1}{4+\cfrac{1}{\left(3+\cfrac{1}{5+\frac{1}{9+\frac{1}{100}}}\right)}}}$$

i.e., $[-3,2,4,3,5,9,100]=[-3,2,4,\bigstar]$, where, $\bigstar=[3,5,9,100]$
所以有: $[-3,2,4,3,5,9,100]=[-3,2,4,[3,5,9,100]]$

2.1 所以根据我们的记号, 可以很容易的看到:

$$[x_0,x_1,x_2,\ldots,x_{n-1},x_n,x_{n+1},\ldots,x_{n+r}]=[x_0,x_1,x_2,\ldots,x_{n-1},[x_n,x_{n+1},\ldots,x_{n+r}]]$$

事实上, 又可以看到:

$$[3,5,9,100]=3+\frac{1}{\left(5+\frac{1}{9+\frac{1}{100}}\right)}$$

所以又有:

$$[-3,2,4,3,5,9,100]=[-3,2,4,[3,5,9,100]]=[-3,2,4,3+\frac{1}{[5,9,100]}]$$

2.2 **即,一般地我们有:**

$$\begin{aligned}&[x_0,x_1,x_2,\ldots,x_{n-1},x_n,x_{n+1},\ldots,x_{n+r}]\\=&[x_0,x_1,x_2,\ldots,x_{n-1},[x_n,x_{n+1},\ldots,x_{n+r}]]\\=&[x_0,x_1,x_2,\ldots,x_{n-1},x_n+\frac{1}{[x_{n+1},\ldots,x_{n+r}]}]\end{aligned}$$

另外,我们还可以注意到($\eta > 0$):

$$-3+\frac{1}{2+\frac{1}{4}} < -3+\frac{1}{2+\frac{1}{4+\eta}}$$

$$i.e., \qquad x_0+\frac{1}{x_1+\frac{1}{x_2}} < x_0+\frac{1}{x_1+\frac{1}{x_2+\eta}}$$

类似地:

$$-3+\frac{1}{2+\frac{1}{4+\frac{1}{3+\frac{1}{5}}}} < -3+\frac{1}{2+\frac{1}{4+\frac{1}{3+\frac{1}{5+\eta}}}}$$

$$i.e., \qquad x_0+\frac{1}{x_1+\frac{1}{x_2+\frac{1}{x_3+\frac{1}{x_4}}}} < x_0+\frac{1}{x_1+\frac{1}{x_2+\frac{1}{x_3+\frac{1}{x_4+\eta}}}}$$

2.3 更一般地我们有:

$$[x_0, x_1, x_2, \ldots, x_{2k-1}, x_{2k}] < [x_0, x_1, x_2, \ldots, x_{2k-1}, x_{2k}+\eta]$$

另外,我们还可以注意到($\eta > 0$):

$$-3+\cfrac{1}{2+\cfrac{1}{4+\frac{1}{3}}} > -3+\cfrac{1}{2+\cfrac{1}{4+\frac{1}{3\eta}}}$$

$$i.e.,\qquad x_0+\cfrac{1}{x_1+\cfrac{1}{x_2+\frac{1}{x_3}}} > x_0+\cfrac{1}{x_1+\cfrac{1}{x_2+\frac{1}{x_3+\eta}}}$$

类似地:

$$-3+\cfrac{1}{2+\cfrac{1}{4+\frac{1}{3+\frac{1}{5+\frac{1}{9}}}}} > -3+\cfrac{1}{2+\cfrac{1}{4+\frac{1}{3+\frac{1}{5+\frac{1}{9+\eta}}}}}$$

$$i.e.,\qquad x_0+\cfrac{1}{x_1+\cfrac{1}{x_2+\frac{1}{x_3+\frac{1}{x_4+\frac{1}{x_5}}}}} > x_0+\cfrac{1}{x_1+\cfrac{1}{x_2+\frac{1}{x_3+\frac{1}{x_4+\frac{1}{x_5+\eta}}}}}$$

2.4 **更一般地我们有**:

$$[x_0,x_1,x_2,\ldots,x_{2k},x_{2k+1}] > [x_0,x_1,x_2,\ldots,x_{2k},x_{2k+1}+\eta]$$

2.5 注意到这里的η可以是任意正实数:

$$[x_0, x_1, x_2, \ldots, x_{2k-1}, x_{2k}] < [x_0, x_1, x_2, \ldots, x_{2k-1}, x_{2k} + \eta]$$

所以有

$$[x_0, x_1, \ldots, x_{2k-1}, x_{2k}] < [x_0, x_1, \ldots, x_{2k-1}, x_{2k} + \frac{1}{x_{2k+1}}] = [x_0, x_1, \ldots, x_{2k}, x_{2k+1}]$$

类似地:

$$[x_0, x_1, \ldots, x_{2k-1}, x_{2k}] < [x_0, x_1, \ldots, x_{2k-1}, x_{2k} + \frac{1}{x_{2k+1} + \frac{1}{x_{2k+2}}}]$$
$$= [x_0, x_1, \ldots, x_{2k}, x_{2k+1}, x_{2k+2}]$$

更一般的有:

$$[x_0, x_1, \ldots, x_{2k-1}, x_{2k}] < [x_0, x_1, \ldots, x_{2k}, x_{2k+1}, x_{2k+2}, \ldots, x_{2k+r}]$$

(where $r \geq 1$). 记$\theta_n = [x_0, x_1, \ldots, x_n]$, 则有

$$\theta_{2k} < \theta_{2k+r} \quad (r \geq 1)$$

2.6 注意到这里的η可以是任意正实数:

$$[x_0, x_1, x_2, \ldots, x_{2k}, x_{2k+1}] > [x_0, x_1, x_2, \ldots, x_{2k}, x_{2k+1} + \eta]$$

所以有

$$[x_0, x_1, \ldots, x_{2k}, x_{2k+1}] > [x_0, x_1, \ldots, x_{2k}, x_{2k+1} + \frac{1}{x_{2k+2}}] = [x_0, x_1, \ldots, x_{2k}, x_{2k+1}]$$

类似地:

$$[x_0, x_1, \ldots, x_{2k}, x_{2k+1}] > [x_0, x_1, \ldots, x_{2k}, x_{2k+1} + \frac{1}{x_{2k+2} + \frac{1}{x_{2k+3}}}]$$
$$= [x_0, x_1, x_2, \ldots, x_{2k+1}, x_{2k+2}, x_{2k+3}]$$

更一般的有:

$$[x_0, x_1, \ldots, x_{2k}, x_{2k+1}] > [x_0, x_1, \ldots, x_{2k+1}, x_{2k+2}, x_{2k+3}, \ldots, x_{2k+1+r}]$$

(where $r \geq 1$), 记$\theta_n = [x_0, x_1, \ldots, x_n]$, 则有

$$\theta_{2k+1} > \theta_{2k+1+r} \quad (r \geq 1)$$

2.7 对:

$$\theta_{2k} < \theta_{2k+r} \quad (r \geq 1):$$

$k = 0: \theta_0 < \theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6, \theta_7, \theta_8, \ldots\ldots$

$k = 1: \theta_2 < \theta_3, \theta_4, \theta_5, \theta_6, \theta_7, \theta_8, \theta_9, \theta_{10} \ldots\ldots$

$k = 2: \theta_4 < \theta_5, \theta_6, \theta_7, \theta_8, \theta_9, \theta_{10} \ldots\ldots$

$\ldots\ldots$

即:

$$\theta_0 < \theta_1$$

$$\theta_2 < \theta_3$$

$$\theta_4 < \theta_5$$

$$\theta_6 < \theta_7$$

$$\theta_8 < \theta_9$$

$$\theta_{10} < \theta_{11}$$

$$\ldots\ldots$$

2.7 对:

$$\theta_{2k} < \theta_{2k+r} \quad (r \geq 1):$$

$k = 0 : \theta_0 < \theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6, \theta_7, \theta_8, \ldots\ldots$

$k = 1 : \theta_2 < \theta_3, \theta_4, \theta_5, \theta_6, \theta_7, \theta_8, \theta_9, \theta_{10} \ldots\ldots$

$k = 2 : \theta_4 < \theta_5, \theta_6, \theta_7, \theta_8, \theta_9, \theta_{10} \ldots\ldots$

$\ldots\ldots$

即:

$$\begin{array}{c} \theta_0 \\ \wedge \\ \theta_2 \\ \wedge \\ \theta_4 \\ \wedge \\ \theta_6 \\ \wedge \\ \theta_8 \\ \wedge \\ \theta_{10} \end{array}$$

2.7 对:

$$\theta_{2k} < \theta_{2k+r} \quad (r \geq 1):$$

$k=0: \theta_0 < \theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6, \theta_7, \theta_8, \ldots\ldots$

$k=1: \theta_2 < \theta_3, \theta_4, \theta_5, \theta_6, \theta_7, \theta_8, \theta_9, \theta_{10} \ldots\ldots$

$k=2: \theta_4 < \theta_5, \theta_6, \theta_7, \theta_8, \theta_9, \theta_{10} \ldots\ldots$

$\ldots\ldots$

即:

$$\begin{array}{c} \theta_0 < \theta_1 \\ \wedge \\ \theta_2 < \theta_3 \\ \wedge \\ \theta_4 < \theta_5 \\ \wedge \\ \theta_6 < \theta_7 \\ \wedge \\ \theta_8 < \theta_9 \\ \wedge \end{array}$$

2.7 对:

$$\theta_{2k+1} > \theta_{2k+1+r} \quad (r \geq 1):$$

$k = 0 : \theta_1 > \theta_2, \theta_3, \theta_4, \theta_5, \theta_6, \theta_7, \theta_8, \ldots\ldots$

$k = 1 : \theta_3 > \theta_4, \theta_5, \theta_6, \theta_7, \theta_8, \theta_9, \theta_{10} \ldots\ldots$

$k = 2 : \theta_5 > \theta_6, \theta_7, \theta_8, \theta_9, \theta_{10} \ldots\ldots$

$\ldots\ldots$

即:

$$\theta_1 > \theta_2$$

$$\theta_3 > \theta_4$$

$$\theta_5 > \theta_6$$

$$\theta_7 > \theta_8$$

$$\theta_9 > \theta_{10}$$

$$\theta_{11} > \theta_{12}$$

$$\ldots\ldots$$

2.7 对:

$$\theta_{2k+1} > \theta_{2k+1+r} \quad (r \geq 1):$$

$k = 0 : \theta_1 > \theta_2, \theta_3, \theta_4, \theta_5, \theta_6, \theta_7, \theta_8, \ldots\ldots$

$k = 1 : \theta_3 > \theta_4, \theta_5, \theta_6, \theta_7, \theta_8, \theta_9, \theta_{10} \ldots\ldots$

$k = 2 : \theta_5 > \theta_6, \theta_7, \theta_8, \theta_9, \theta_{10} \ldots\ldots$

$\ldots\ldots$

即:

$$\begin{array}{c} \theta_1 \\ \vee \\ \theta_3 \\ \vee \\ \theta_5 \\ \vee \\ \theta_7 \\ \vee \\ \theta_9 \\ \vee \\ \theta_{11} \end{array}$$

2.7 对:

$$\theta_{2k+1} > \theta_{2k+1+r} \quad (r \geq 1):$$

$k = 0 : \theta_1 > \theta_2, \theta_3, \theta_4, \theta_5, \theta_6, \theta_7, \theta_8, \ldots\ldots$

$k = 1 : \theta_3 > \theta_4, \theta_5, \theta_6, \theta_7, \theta_8, \theta_9, \theta_{10} \ldots\ldots$

$k = 2 : \theta_5 > \theta_6, \theta_7, \theta_8, \theta_9, \theta_{10} \ldots\ldots$

$\ldots\ldots$

即:

$$\begin{array}{c} \theta_1 > \theta_2 \\ \vee \\ \theta_3 > \theta_4 \\ \vee \\ \theta_5 > \theta_6 \\ \vee \\ \theta_7 > \theta_8 \\ \vee \\ \theta_9 > \theta_{10} \\ \vee \end{array}$$

$$\theta_{2k} < \theta_{2k+r}, \theta_{2k+1} > \theta_{2k+1+r} \quad (r \geq 1):$$

$$\begin{array}{ccc}
\theta_0 < \theta_1 \\
\wedge \\
\theta_2 < \theta_3 \\
\wedge \\
\theta_4 < \theta_5 \\
\wedge \\
\theta_6 < \theta_7 \\
\wedge \\
\theta_8 < \theta_9 \\
\wedge \\
\theta_{10} < \theta_{11} \\
\wedge \\
\cdots\cdots
\end{array}
\qquad
\begin{array}{c}
\theta_1 > \theta_2 \\
\vee \\
\theta_3 > \theta_4 \\
\vee \\
\theta_5 > \theta_6 \\
\vee \\
\theta_7 > \theta_8 \\
\vee \\
\theta_9 > \theta_{10} \\
\vee \\
\theta_{11} > \theta_{12} \\
\vee \\
\cdots\cdots
\end{array}
\qquad
\begin{array}{ccc}
\theta_0 & < & \theta_1 \\
\wedge & \swarrow & \vee \\
\theta_2 & < & \theta_3 \\
\wedge & & \\
\theta_4 & < & \theta_5 \\
\wedge & & \\
\theta_6 & < & \theta_7 \\
\wedge & & \\
\theta_8 & < & \theta_9 \\
\wedge & & \\
\theta_{10} & < & \theta_{11} \\
\wedge & & \\
 & \cdots\cdots &
\end{array}$$

$$\theta_{2k} < \theta_{2k+r}, \theta_{2k+1} > \theta_{2k+1+r} \quad (r \geq 1):$$

$$
\begin{array}{c}
\theta_0 < \theta_1 \\
\wedge \qquad\quad \\
\theta_2 < \theta_3 \\
\wedge \qquad\quad \\
\theta_4 < \theta_5 \\
\wedge \qquad\quad \\
\theta_6 < \theta_7 \\
\wedge \qquad\quad \\
\theta_8 < \theta_9 \\
\wedge \qquad\quad \\
\theta_{10} < \theta_{11} \\
\wedge \qquad\quad \\
\cdots\cdots
\end{array}
\qquad
\begin{array}{c}
\theta_1 > \theta_2 \\
\vee \qquad\quad \\
\theta_3 > \theta_4 \\
\vee \qquad\quad \\
\theta_5 > \theta_6 \\
\vee \qquad\quad \\
\theta_7 > \theta_8 \\
\vee \qquad\quad \\
\theta_9 > \theta_{10} \\
\vee \qquad\quad \\
\theta_{11} > \theta_{12} \\
\vee \qquad\quad \\
\cdots\cdots
\end{array}
\qquad
\begin{array}{ccc}
\theta_0 & < & \theta_1 \\
\wedge & \angle & \vee \\
\theta_2 & < & \theta_3 \\
\wedge & \angle & \vee \\
\theta_4 & < & \theta_5 \\
\wedge & & \\
\theta_6 & < & \theta_7 \\
\wedge & & \\
\theta_8 & < & \theta_9 \\
\wedge & & \\
\theta_{10} & < & \theta_{11} \\
\wedge & & \\
 & \cdots\cdots &
\end{array}
$$

$$\theta_{2k} < \theta_{2k+r}, \theta_{2k+1} > \theta_{2k+1+r} \quad (r \geq 1):$$

$$\begin{array}{c}
\theta_0 < \theta_1 \\
\wedge \\
\theta_2 < \theta_3 \\
\wedge \\
\theta_4 < \theta_5 \\
\wedge \\
\theta_6 < \theta_7 \\
\wedge \\
\theta_8 < \theta_9 \\
\wedge \\
\theta_{10} < \theta_{11} \\
\wedge \\
\cdots\cdots
\end{array}
\qquad
\begin{array}{c}
\theta_1 > \theta_2 \\
\vee \\
\theta_3 > \theta_4 \\
\vee \\
\theta_5 > \theta_6 \\
\vee \\
\theta_7 > \theta_8 \\
\vee \\
\theta_9 > \theta_{10} \\
\vee \\
\theta_{11} > \theta_{12} \\
\vee \\
\cdots\cdots
\end{array}
\qquad
\begin{array}{ccc}
\theta_0 & < & \theta_1 \\
\wedge & \swarrow & \vee \\
\theta_2 & < & \theta_3 \\
\wedge & \swarrow & \vee \\
\theta_4 & < & \theta_5 \\
\wedge & \swarrow & \vee \\
\theta_6 & < & \theta_7 \\
\wedge & & \\
\theta_8 & < & \theta_9 \\
\wedge & & \\
\theta_{10} & < & \theta_{11} \\
\wedge & & \\
\cdots\cdots & &
\end{array}$$

$$\theta_{2k} < \theta_{2k+r}, \theta_{2k+1} > \theta_{2k+1+r} \quad (r \geq 1):$$

$$\begin{array}{ccc}
\theta_0 & < & \theta_1 \\
\wedge & & \\
\theta_2 & < & \theta_3 \\
\wedge & & \\
\theta_4 & < & \theta_5 \\
\wedge & & \\
\theta_6 & < & \theta_7 \\
\wedge & & \\
\theta_8 & < & \theta_9 \\
\wedge & & \\
\theta_{10} & < & \theta_{11} \\
\wedge & & \\
& \cdots\cdots &
\end{array}
\qquad
\begin{array}{ccc}
\theta_1 & > & \theta_2 \\
\vee & & \\
\theta_3 & > & \theta_4 \\
\vee & & \\
\theta_5 & > & \theta_6 \\
\vee & & \\
\theta_7 & > & \theta_8 \\
\vee & & \\
\theta_9 & > & \theta_{10} \\
\vee & & \\
\theta_{11} & > & \theta_{12} \\
\vee & & \\
& \cdots\cdots &
\end{array}
\qquad
\begin{array}{ccc}
\theta_0 & < & \theta_1 \\
\wedge & \angle & \vee \\
\theta_2 & < & \theta_3 \\
\wedge & \angle & \vee \\
\theta_4 & < & \theta_5 \\
\wedge & \angle & \vee \\
\theta_6 & < & \theta_7 \\
\wedge & \angle & \vee \\
\theta_8 & < & \theta_9 \\
\wedge & & \\
\theta_{10} & < & \theta_{11} \\
\wedge & & \\
& \cdots\cdots &
\end{array}$$

$$\theta_{2k} < \theta_{2k+r}, \theta_{2k+1} > \theta_{2k+1+r} \quad (r \geq 1):$$

$$\begin{array}{ccc}
\theta_0 & < & \theta_1 \\
\wedge & & \\
\theta_2 & < & \theta_3 \\
\wedge & & \\
\theta_4 & < & \theta_5 \\
\wedge & & \\
\theta_6 & < & \theta_7 \\
\wedge & & \\
\theta_8 & < & \theta_9 \\
\wedge & & \\
\theta_{10} & < & \theta_{11} \\
\wedge & & \\
 & \cdots\cdots &
\end{array}
\qquad
\begin{array}{ccc}
\theta_1 & > & \theta_2 \\
\vee & & \\
\theta_3 & > & \theta_4 \\
\vee & & \\
\theta_5 & > & \theta_6 \\
\vee & & \\
\theta_7 & > & \theta_8 \\
\vee & & \\
\theta_9 & > & \theta_{10} \\
\vee & & \\
\theta_{11} & > & \theta_{12} \\
\vee & & \\
 & \cdots\cdots &
\end{array}
\qquad
\begin{array}{ccc}
\theta_0 & < & \theta_1 \\
\wedge & \angle & \vee \\
\theta_2 & < & \theta_3 \\
\wedge & \angle & \vee \\
\theta_4 & < & \theta_5 \\
\wedge & \angle & \vee \\
\theta_6 & < & \theta_7 \\
\wedge & \angle & \vee \\
\theta_8 & < & \theta_9 \\
\wedge & \angle & \vee \\
\theta_{10} & < & \theta_{11} \\
\wedge & & \\
 & \cdots\cdots &
\end{array}$$

$$\theta_{2k} < \theta_{2k+r}, \theta_{2k+1} > \theta_{2k+1+r} \quad (r \geq 1):$$

$$\begin{array}{c} \theta_0 < \theta_1 \\ \wedge \\ \theta_2 < \theta_3 \\ \wedge \\ \theta_4 < \theta_5 \\ \wedge \\ \theta_6 < \theta_7 \\ \wedge \\ \theta_8 < \theta_9 \\ \wedge \\ \theta_{10} < \theta_{11} \\ \wedge \\ \cdots\cdots \end{array} \qquad \begin{array}{c} \theta_1 > \theta_2 \\ \vee \\ \theta_3 > \theta_4 \\ \vee \\ \theta_5 > \theta_6 \\ \vee \\ \theta_7 > \theta_8 \\ \vee \\ \theta_9 > \theta_{10} \\ \vee \\ \theta_{11} > \theta_{12} \\ \vee \\ \cdots\cdots \end{array} \qquad \begin{array}{ccc} \theta_0 & < & \theta_1 \\ \wedge & \angle & \vee \\ \theta_2 & < & \theta_3 \\ \wedge & \angle & \vee \\ \theta_4 & < & \theta_5 \\ \wedge & \angle & \vee \\ \theta_6 & < & \theta_7 \\ \wedge & \angle & \vee \\ \theta_8 & < & \theta_9 \\ \wedge & \angle & \vee \\ \theta_{10} & < & \theta_{11} \\ \wedge & \angle & \vee \\ & \cdots\cdots & \end{array}$$

$\theta_{2k} < \theta_{2k+r}, \theta_{2k+1} > \theta_{2k+1+r} \quad (r \geq 1) \Longrightarrow$ 任意奇数下标的θ >任意偶数下标的θ

$\theta_{2k} < \theta_{2k+r}, \theta_{2k+1} > \theta_{2k+1+r} \quad (r \geq 1) \Longrightarrow$ 任意下标的θ的值都在$[\theta_0, \theta_1]$区间内.

$$\begin{array}{c} \theta_0 < \theta_1 \\ \wedge \\ \theta_2 < \theta_3 \\ \wedge \\ \theta_4 < \theta_5 \\ \wedge \\ \theta_6 < \theta_7 \\ \wedge \\ \theta_8 < \theta_9 \\ \wedge \\ \theta_{10} < \theta_{11} \\ \wedge \\ \cdots\cdots \end{array} \qquad \begin{array}{c} \theta_1 > \theta_2 \\ \vee \\ \theta_3 > \theta_4 \\ \vee \\ \theta_5 > \theta_6 \\ \vee \\ \theta_7 > \theta_8 \\ \vee \\ \theta_9 > \theta_{10} \\ \vee \\ \theta_{11} > \theta_{12} \\ \vee \\ \cdots\cdots \end{array} \qquad \begin{array}{c} \theta_0 < \theta_1 \\ \wedge \angle \ \vee \\ \theta_2 < \theta_3 \\ \wedge \angle \ \vee \\ \theta_4 < \theta_5 \\ \wedge \angle \ \vee \\ \theta_6 < \theta_7 \\ \wedge \angle \ \vee \\ \theta_8 < \theta_9 \\ \wedge \angle \ \vee \\ \theta_{10} < \theta_{11} \\ \wedge \angle \ \vee \\ \cdots\cdots \end{array}$$

2.8 $[x_0, x_1, \ldots, x_n] = \dfrac{P_n}{Q_n}$，其中$P_n = x_nP_{n-1} + P_{n-2}, Q_n = x_nQ_{n-1} + Q_{n-2}$，
$P_{-2} = 0, P_{-1} = 1, Q_{-2} = 1, Q_{-1} = 0$
对n使用数学归纳法: $n = 0$的时候直接检验:

$$P_0 = x_0P_{-1} + 0 = x_0, Q_0 = x_0 \cdot 0 + Q_{-2} \Longrightarrow \frac{P_0}{Q_0} = x_0$$

现在假设$n = k$时结论成立，即$[x_0, x_1, \ldots, x_k] = \dfrac{P_k}{Q_k}$，我们需要说明$n = k + 1$时结论也成立:

$$[x_0, x_1, \ldots, x_k, x_{k+1}] = [x_0, x_1, \ldots, x_k + \frac{1}{x_{k+1}}]$$

此时右边可以使用归纳假设:

$$[x_0, x_1, \ldots, x_k + \frac{1}{x_{k+1}}] = \frac{(x_k + \frac{1}{x_{k+1}})P_{k-1} + P_{k-2}}{(x_k + \frac{1}{x_{k+1}})Q_{k-1} + Q_{k-2}}$$

$$= \frac{(x_kx_{k+1} + 1)P_{k-1} + x_{k+1}P_{k-2}}{(x_kx_{k+1} + 1)Q_{k-1} + x_{k+1}Q_{k-2}} = \frac{x_{k+1}(x_kP_{k-1} + P_{k-2}) + P_{k-1}}{x_{k+1}(x_kQ_{k-1} + Q_{k-2}) + Q_{k-1}}$$

$$= \frac{x_{k+1}P_k + P_{k-1}}{x_{k+1}Q_k + Q_{k-1}} = \frac{P_{k+1}}{Q_{k+1}}$$

这个结论也给出了一种求连分数的方法。

求有限连分数$[1,1,1,1,1,1,1,1,1,1]$的各个渐进分数

利用前述结论，$[1,1,1,1,1,1,1,1,1,1] = \frac{P_9}{Q_9}$

而P_9和Q_9可以递推出来:

$$x_0 = 1, x_1 = 1, x_2 = 1, \ldots, x_9 = 1, P_{-2} = 0, P_{-1} = 1, Q_{-2} = 1, Q_{-1} = 0$$

$$P_n = P_{n-1} + P_{n-2}, Q_n = Q_{n-1} + Q_{n-2}$$

n	0	1	2	3	4	5	6	7	8	9
x_n	1	1	1	1	1	1	1	1	1	1
P_n	1	2	3	5	8	13	21	34	55	89
Q_n	1	1	2	3	5	8	13	21	34	55

从而可以写出各个渐进分数$\frac{P_n}{Q_n}$.

2.8.1. $P_n = x_n P_{n-1} + P_{n-2}$, $Q_n = x_n Q_{n-1} + Q_{n-2}$, $P_{-2} = 0$, $P_{-1} = 1$, $Q_{-2} = 1$, $Q_{-1} = 0$, 则, $P_n Q_{n-1} - P_{n-1} Q_n = (-1)^{n+1}, (n \geq -1)$

对n使用数学归纳法:

$n = -1$时, 直接验算: $P_{-1}Q_{-2} - P_{-2}Q_{-1} = 1 - 0 = 1 = (-1)^0$

假设$n = k$时成立, 即$P_k Q_{k-1} - P_{k-1} Q_k = (-1)^{k+1}$, 需要说明$n = k + 1$时结论也成立: 即有$P_{k+1}Q_k - P_k Q_{k+1} = (-1)^{k+2}$

事实上,由

$$\begin{cases} P_{k+1} = x_{k+1}P_k + P_{k-1} \\ Q_{k+1} = x_{k+1}Q_k + Q_{k-1} \end{cases} \Longrightarrow \begin{cases} x_{k+1} = \frac{P_{k+1} - P_{k-1}}{P_k} \\ x_{k+1} = \frac{Q_{k+1} - Q_{k-1}}{Q_k} \end{cases}$$

$$\Longrightarrow \frac{P_{k+1} - P_{k-1}}{P_k} = \frac{Q_{k+1} - Q_{k-1}}{Q_k}$$

$$\Longrightarrow P_k(Q_{k+1} - Q_{k-1}) = Q_k(P_{k+1} - P_{k-1})$$

$$\Longrightarrow P_k Q_{k+1} - P_k Q_{k-1} = Q_k P_{k+1} - Q_k P_{k-1}$$

$$\Longrightarrow Q_k P_{k+1} - P_k Q_{k+1} = -(P_k Q_{k-1} - Q_k P_{k-1}) = -(-1)^{k+1} = (-1)^k + 2 \qquad \diamond$$

2.8.2. 这样:

$$\theta_n - \theta_{n-1} = \frac{P_n}{Q_n} - \frac{P_{n-1}}{Q_{n-1}} = \frac{P_n Q_{n-1} - Q_n P_{n-1}}{Q_n Q_{n-1}} = \frac{(-1)^{n+1}}{Q_n Q_{n-1}} \quad (n \geq -1)$$

2.8.3. $P_n = x_n P_{n-1} + P_{n-2}$, $Q_n = x_n Q_{n-1} + Q_{n-2}$, $P_{-2} = 0$, $P_{-1} = 1$, $Q_{-2} = 1$, $Q_{-1} = 0$, 则, $P_n Q_{n-2} - P_{n-2} Q_n = (-1)^n x_n, (n \geq 0)$

在$n \geq 0$时,

$$\begin{cases} P_n = x_n P_{n-1} + P_{n-2} \\ Q_n = x_n Q_{n-1} + Q_{n-2} \end{cases} \implies \begin{cases} P_n Q_{n-2} = x_n P_{n-1} Q_{n-2} + P_{n-2} Q_{n-2} \\ Q_n P_{n-2} = x_n Q_{n-1} P_{n-2} + Q_{n-2} P_{n-2} \end{cases}$$

$$\implies P_n Q_{n-2} - Q_n P_{n-2} = x_n P_{n-1} Q_{n-2} + P_{n-2} Q_{n-2} - x_n Q_{n-1} P_{n-2} - Q_{n-2} P_{n-2}$$

$$\implies P_n Q_{n-2} - Q_n P_{n-2} = x_n (P_{n-1} Q_{n-2} - Q_{n-1} P_{n-2})$$

$$\implies P_n Q_{n-2} - Q_n P_{n-2} = x_n (-1)^n \qquad \diamond$$

2.8.4. 这样:

$$\theta_n - \theta_{n-2} = \frac{P_n}{Q_n} - \frac{P_{n-2}}{Q_{n-2}} = \frac{P_n Q_{n-2} - P_{n-2} Q_n}{Q_n Q_{n-2}} = \frac{(-1)^n x_n}{Q_n Q_{n-2}} \quad (n \geq -1)$$

3. 有理分数的有限简单连分数表示

对于一个分子, 分母很大的分数, 比如$\frac{103993}{33102}$, 可以用连分数找到一个分子,分母较小的数来近似它,

$$\begin{aligned}\frac{103993}{33102} &= 3+\frac{4687}{33102} = 3+\frac{1}{\frac{33102}{4687}} = 3+\frac{1}{7+\frac{293}{4687}}\\ &= 3+\frac{1}{7+\frac{1}{\frac{4687}{293}}} = 3+\frac{1}{7+\frac{1}{15+\frac{292}{293}}}\\ &= 3+\frac{1}{7+\frac{1}{15+\frac{1}{1+\frac{1}{292}}}}\end{aligned}$$

扔掉这些"分数"中小于1的数可以以此得到近似值:

$$3, \frac{22}{7}, \frac{333}{106}, \frac{355}{113}, \ldots$$

上例的做法具有一般性: 对于一个不是整数的有理分数$\frac{u_2}{u_1}(u_1 \geq 2)$, 要得到它的有限简单连分数表示, 可以使用辗转相除法,

$$\begin{cases} u_0 = b_0 u_1 + u_2, (0 < u_2 < u_1) \\ u_1 = b_1 u_2 + u_3, (0 < u_3 < u_2) \\ u_2 = b_2 u_3 + u_4, (0 < u_4 < u_3) \\ u_3 = b_3 u_4 + u_5, (0 < u_5 < u_4) \\ \quad \cdots\cdots \\ u_{s-1} = b_{s-1} u_s + u_{s+1}, (0 < u_{s+1} < u_s) \\ u_s = b_s u_{s+1} \end{cases}$$

这样就得到了$\frac{u_2}{u_1}(u_1 \geq 2)$的简单连分数表示$[b_0, b_1, b_2, b_3, \ldots, b_{s-1}, b_s](b_s > 1)$,
另外, 我们知道$[x_0, x_1, \ldots, x_{n-1}, x_n, x_{n+1}] = [x_0, x_1, \ldots, x_{n-1}, x_n + \frac{1}{x_{n+1}}]$
而$[b_0, b_1, b_2, b_3, \ldots, b_{s-1}, b_s] = [b_0, b_1, b_2, b_3, \ldots, b_{s-1}, b_s - 1 + \frac{1}{1}]$
所以$[b_0, b_1, b_2, b_3, \ldots, b_{s-1}, b_s] = [b_0, b_1, b_2, b_3, \ldots, b_{s-1}, b_s - 1, 1]$
对$\frac{u_2}{u_1}(u_1 \geq 2)$再也没有其他的连分数表示形式了.

表示的唯一性 给定两个有限简单连分数$[a_0, a_1, \ldots, a_n](a_n > 1)$, $[b_0, b_1, \ldots, b_s]$ $(b_s > 1)$, 如果$[a_0, a_1, \ldots, a_n] = [b_0, b_1, \ldots, b_s]$, 则必有$s = n, a_j = b_j (j = 0, 1, \ldots, s)$

证明: 对n使用数学归纳法

当$n = 0$时, 如果$s \geq 1$的话,

$$a_0 = [b_0, b_1, \ldots, b_s] = [b_0, [b_1, \ldots, b_s]] = b_0 + \frac{1}{[b_1, \ldots, b_s]}$$

因为$b_s > 1, \therefore [b_1, \ldots, b_s] > 1$, 所以上式不可能成立(左边是整数,右边是分数).
这样$s = 0$, 从而$a_0 = b_0$.
假设$n = k$时结论成立, 当$n = k + 1$时,

$$[a_0, a_1, \ldots, a_k, a_{k+1}] = a_0 + \frac{1}{[a_1, \ldots, a_{k+1}]}, \quad [b_0, \ldots, b_s] = b_0 + \frac{1}{[b_1, \ldots, b_s]}$$

又$a_{k+1} > 1 \Longrightarrow [a_1, \ldots, a_{k+1}] > 1, b_s > 1 \Longrightarrow [b_1, \ldots, b_s] > 1$, 从而

$$[a_0, \ldots, a_k, a_{k+1}] = [b_0, \ldots, b_s] \Longrightarrow a_0 + \frac{1}{[a_1, \ldots, a_{k+1}]} = b_0 + \frac{1}{[b_1, \ldots, b_s]}$$

$$\Longrightarrow \begin{cases} a_0 = b_0 \\ \dfrac{1}{[a_1, \ldots, a_{k+1}]} = \dfrac{1}{[b_1, \ldots, b_s]} \Longrightarrow [a_1, \ldots, a_{k+1}] = [b_1, \ldots, b_s] \end{cases}$$

此时可以使用归纳假设, 得到$a_1 = b_1, \ldots$ $\diamond$

4.1. 定义

形如:

$$x_0+\cfrac{1}{x_1+\cfrac{1}{x_2+\cfrac{1}{\dots+\cfrac{1}{x_{n-1+\frac{1}{\dots\dots}}}}}}$$

的数$(x_1,x_2,x_3,\dots>0)$, 称为无限连分数, 记为$[x_0,x_1,x_2,\dots,x_{n-1},\dots\dots]$. 如果这里面的数字都是整数, 这个连分数被称为无限简单连分数.

将$[x_0,x_1,x_2,\dots,x_k](k\geq 0)$称为是$[x_0,x_1,x_2,\dots,x_n,\dots]$的第$k$个渐近分数.

如果有

$$\lim_{k\to\infty}[x_0,x_1,x_2,\dots,x_k]=\theta$$

则称无限连分数$[x_0,x_1,x_2,\dots,x_n,\dots]$是收敛的, θ即为其值, 记作$[x_0,x_1,x_2,\dots,x_n,\dots]=\theta$.

令$\theta_k=[x_0,x_1,x_2,\dots,x_k]$, 这样$\theta_0,\theta_1,\dots$都是$\theta$的渐进分数.

如果不存在极限, 则称无限连分数$[x_0,x_1,x_2,\dots,x_n,\dots]$是发散的.

4.2. 性质

无限简单连分数一定是收敛的，也就是说$\theta_k = [x_0, \dots, x_k]$，则一定存在极限

$$\lim_{k\to\infty} \theta_k = \theta$$

$$s.t., \theta_0 < \theta_2 < \theta_4 < \dots < \theta < \dots < \theta_{2s-1} < \dots \theta_3 < \theta_1$$

事实上，下标为奇数的渐进分数序列是有下界θ_0的严格递减数列，所以我们知道它必定有极限，即$\lim_{n\to\infty} \theta_{2n-1} = \theta''$

下标为偶数的渐近分数序列是有上届θ_1的严格递增数列，所以我们知道它必定有极限，即$\lim_{n\to\infty} \theta_{2n} = \theta'$，从而有

$$\theta_1 > \theta_3 > \theta_5 > \theta_7 > \dots > \theta'' \qquad \theta' > \dots > \theta_6 > \theta_4 > \theta_2 > \theta_0$$

我们知道任意下标为奇数渐近分数值都>任意下标为偶数的渐近分数值，所以有

$$\theta_1 > \theta_3 > \theta_5 > \theta_7 > \dots > \theta'' \geq \theta' > \dots > \theta_6 > \theta_4 > \theta_2 > \theta_0$$

从而

$$0 \leq \theta'' - \theta' \leq \theta_{2k-1} - \theta_{2k} = \frac{1}{Q_{2k-1}Q_{2k}}$$

对无限简单连分数来说，可以看到$1 = Q_0 \leq x_1 = Q_1 < Q_2 < Q_3 < Q_4 < \dots < Q_k < \dots$, $Q_k \longrightarrow \infty (k \to \infty)$ 所以$\theta' = \theta''$ ◇

由此可以看到, $\theta = [x_0, x_1, x_2, \ldots]$必定处于它的两个渐近分数$\theta_k$和$\theta_{k+1}$之间, 而且这个$\theta$肯定是无理数(即,无限简单连分数的值是无理数), 当然肯定是实数.
否则, 设它是有理数, $\theta = \frac{u}{v}$ 则由

$$\theta_1 > \theta_3 > \theta_5 > \theta_7 > \ldots > \theta'' \geq \theta' > \ldots > \theta_6 > \theta_4 > \theta_2 > \theta_0$$

知

$$0 < |\theta - \theta_k| < |\theta_{k+1} - \theta_k| = \frac{1}{Q_{k+1}Q_k}$$

从而

$$0 < \left|\frac{u}{v} - \frac{P_k}{Q_k}\right| < \frac{1}{Q_{k+1}Q_k}$$

从而

$$0 < \left|\frac{uQ_k - vP_k}{vQ_k}\right| < \frac{1}{Q_kQ_{k+1}}$$

即有

$$0 < \left|\frac{uQ_k - vP_k}{v}\right| = \frac{|uQ_k - vP_k|}{|v|} < \frac{1}{Q_{k+1}}$$

自然有$|uQ_k - vP_k|$一定是整数, 即$|uQ_k - vP_k| \geq 1$, 所以应该有$Q_{k+1} < |v|$,
换句话说, $\{Q_k\}$是有界的, 不可能. $\diamond$

5. 循环连分数

5.1. **定义**: 设实数θ是无限简单连分数$[x_0, x_1, \ldots]$, 如果从某个下标(比如m)开始,之后的数字都是一段一段的循环, 即存在整数$k \geq 1$, 对于所有的大于等于m的下标来说都有$x_{n+k} = x_n$, 这个连分数(即θ)就叫做循环简单连分数, 简称循环连分数

比如

$$[-100, 2, 3, 4, 2, 3, 4, 2, 3, 4, 2, 3, 4, \ldots]$$

$$0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, \ldots$$

是循环连分数, 这里$m = 1, k = 3$, 即从x_1开始每3位循环, 可以把它记为$[-100, \overline{2, 3, 4}]$.

一般情况下就是

$$[x_0, x_1, \ldots, x_{m-1}, \overline{x_m, x_{m+1}, \ldots, x_{m+k-1}}]$$

注: 这种记法不是唯一的, 比如$[-100, 2, 3, 4, 2, 3, 4, 2, 3, 4, 2, 3, 4, \ldots]$也可以记为$[-100, 2, \overline{3, 4, 2}]$, 也可以是$[-100, 2, 3, \overline{4, 2, 3}]$

再如$[1, 2, 2, 2, 2, \ldots]$也是一个循环连分数, 即$[1, \overline{2}]$.

如果一个循环连分数中$m = 0$, 则这个循环连分数称为纯循环连分数,

比如$[2, 5, 3, 2, 5, 3, 2, 5, 3, \ldots] = [\overline{2, 5, 3}]$, $[5, 3, 2, 5, 3, 2, 5, 3, 2, \ldots] = [\overline{5, 3, 2}]$, $[1, 1, 1, 1, \ldots] = [\overline{1}]$都是纯循环连分数.

5.2. 性质

注1: 一个复数α称为是一个二次无理数(二次代数数)指的是α是某个整系数二次方程

$$ax^2+bx+c=0$$

的根, 其中判别式b^2-4ac不是平方数.

这就是说α是

$$\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

中的一个.

当二次无理数α是实数时, 就称之为实二次无理数.

显然, 二次无理数α是实的 $\iff$ $b^2-4ac>0$

注2: α是二次无理数的充要条件是存在非平方数的整数d, 及有理数$r, s(s \neq 0)$使得$\alpha = r + s\sqrt{d}$

证明:
"$\Longrightarrow$:" 设α是二次无理数, 即α是某个整系数二次方程

$$ax^2 + bx + c = 0$$

的根, 从而α是

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

中的一个, 从而取

$$d = b^2 - 4ac, r = \frac{b}{2a}$$

$$s = \frac{1}{2a} \quad \text{or} \quad -\frac{1}{2a}$$

"$\Longleftarrow$:" 设$\alpha = r + s\sqrt{d}$, 其中非平方数的整数d, 及有理数$r, s(s \neq 0)$则α满足二次方程

$$[x - (r + s\sqrt{d})][x - (r - s\sqrt{d})] = 0$$

即

$$x^2 - 2rx + (r^2 - ds^2) = 0$$

令

$$r = \frac{h}{l}, s = \frac{k}{l}$$

其中h, k, l均为整数, $l > 0, k \neq 0$, 代入方程, 得

$$l^2x^2 - 2lhx + (h^2 - dk^2) = 0$$

它的判别式为

$$(2lh)^2 - 4l^2(h^2 - dk^2) = (2lk)^2 d$$

由于d不是平方数, 所以这个判别式不是平方数, 所以α是二次无理数. $\diamond$

注3: α是二次无理数的充要条件是存在非平方数的整数$d>0$, 及有理数$r,s(s\neq 0)$使得$\alpha=r+s\sqrt{d}$

注4: 整数d不是平方数,r,s为有理数, 则形如$\alpha=r+s\sqrt{d}$的数的和,差,积,商仍是这种形式的数.

定理

θ是循环简单连分数 $\Longleftrightarrow$ θ是实二次无理数.

"$\Longrightarrow$:" 设θ是纯循环简单连分数(从而它的值一定是无理数)

$$
\begin{aligned}
\theta =&[a_0, a_1, \ldots, a_{k-1}, a_0, a_1, \ldots, a_{k-1}, a_0, a_1, \ldots, a_{k-1}, \ldots](a_0, a_1, \ldots > 0) \\
=&[\overline{a_0, a_1, \ldots, a_{k-1}}] = [a_0, a_1, \ldots, a_{k-1}, \overline{a_0, a_1, \ldots, a_{k-1}}] \\
=&[a_0, a_1, \ldots, a_{k-1}, \theta] \\
=&\frac{P_k}{Q_k} = \frac{\theta P_{k-1} + P_{k-2}}{\theta Q_{k-1} + Q_{k-2}} \\
\Longrightarrow&\theta \cdot (\theta Q_{k-1} + Q_{k-2}) - (\theta P_{k-1} + P_{k-2}) = 0 \\
\Longrightarrow&Q_{k-1}\theta^2 + (Q_{k-2} - P_{k-1})\theta - P_{k-2} = 0
\end{aligned}
$$

即θ满足这样的整系数二次方程, 而θ是无理数, 所以这个方程的判别式必定不是平方数, 所以, 这个θ是实二次无理数.

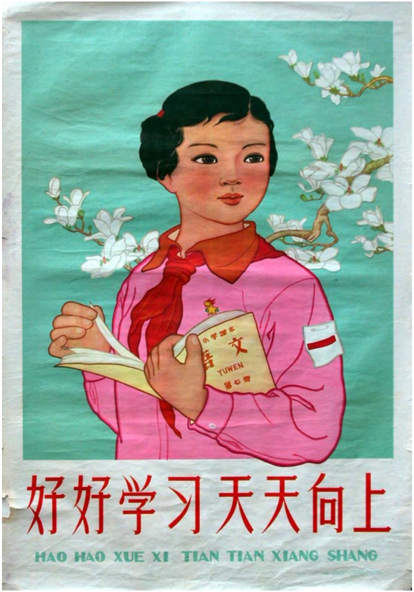
如果θ不是纯循环简单连分数,

$$
\begin{aligned}
\theta =&[a_0, a_1, \ldots, a_{m-1}, a_m, a_{m+1}, \ldots, a_n, a_m, a_{m+1}, \ldots, a_n, \ldots] \\
=&[a_0, a_1, \ldots, a_{m-1}, a_m, a_{m+1}, \ldots, a_n, \overline{a_m, a_{m+1}, \ldots, a_n}] \\
=&[a_0, a_1, \ldots, a_{m-1}, \theta_0] \\
=&\frac{P_m}{Q_m} = \frac{\theta_0 P_{m-1} + P_{m-2}}{\theta_0 Q_{m-1} + Q_{m-2}}
\end{aligned}
$$

由于θ_0是实二次无理数, 从而存在非平方数的整数$d > 0$, 及有理数$r, s(s \neq 0)$使得$\theta_0 = r + s\sqrt{d}$

由于$P_{m-1}, P_{m-2}, Q_{m-1}, Q_{m-2}$都是整数, 所以$\theta$就是对$\theta_0$作的和差积商运算, 结果还是非平方的整数$d > 0$, 及有理数$r', s'(s' \neq 0)$使得$\theta = r' + s'\sqrt{d}$, 从而$\theta$还是实二次无理数.

"$\Longleftarrow$"(不证) $\diamond$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