Calculus for Artificial Intelligence

1. Power Rule

For a function $f(x, y) = x^n$ or $f(x, y) = y^n$:

$$\frac{\partial}{\partial x}(x^n) = nx^{n-1}$$
$$\frac{\partial}{\partial y}(y^n) = ny^{n-1}$$

Example:

$$f(x, y) = x^3$$
$$\frac{\partial f}{\partial x} = 3x^2$$

$$g(x, y) = y^4$$
$$\frac{\partial g}{\partial y} = 4y^3$$

2. Constant Multiple Rule

For a function $f(x, y) = c \cdot g(x, y)$, where c is a constant:

$$\frac{\partial}{\partial x}(c \cdot g(x, y)) = c \cdot \frac{\partial g(x,y)}{\partial x}$$
$$\frac{\partial}{\partial y}(c \cdot g(x, y)) = c \cdot \frac{\partial g(x,y)}{\partial y}$$

Example:

$$f(x, y) = 5x^2$$
$$\frac{\partial f}{\partial x} = 5 \cdot 2x = 10x$$
$$g(x, y) = 7y^3$$
$$\frac{\partial g}{\partial y} = 7 \cdot 3y^2 = 21y^2$$

3. Sum Rule

For a function $f(x, y) = g(x, y) + h(x, y)$:

$$\frac{\partial}{\partial x}(g(x, y) + h(x, y)) = \frac{\partial g(x,y)}{\partial x} + \frac{\partial h(x,y)}{\partial x}$$

$$\frac{\partial}{\partial y}(g(x, y) + h(x, y)) = \frac{\partial g(x,y)}{\partial y} + \frac{\partial h(x,y)}{\partial y}$$

Example:

$$f(x, y) = x^2 + y^2$$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial x}(y^2) = 2x + 0 = 2x$$

$$g(x, y) = 3x^3 + 4y$$

$$\frac{\partial g}{\partial y} = \frac{\partial}{\partial y}(3x^3) + \frac{\partial}{\partial y}(4y) = 0 + 4 = 4$$

4. Product Rule

For a function $f(x, y) = g(x, y) \cdot h(x, y)$:

$$\frac{\partial}{\partial x}(g(x,y) \cdot h(x,y)) = g(x,y) \cdot \frac{\partial h(x,y)}{\partial x} + h(x,y) \cdot \frac{\partial g(x,y)}{\partial x}$$

$$\frac{\partial}{\partial y}(g(x,y) \cdot h(x,y)) = g(x,y) \cdot \frac{\partial h(x,y)}{\partial y} + h(x,y) \cdot \frac{\partial g(x,y)}{\partial y}$$

Simplified,

$$\frac{\partial f}{\partial x} = u \cdot \frac{\partial v}{\partial x} + v \cdot \frac{\partial u}{\partial x}$$

$$\frac{\partial f}{\partial x} = u \cdot \left(\frac{\partial}{\partial x} v\right) + v \cdot \left(\frac{\partial}{\partial x} u\right)$$

Example:

$$f(x, y) = (x^2) \cdot (y^3)$$

$$\frac{\partial f}{\partial x} = (x^2) \cdot \frac{\partial (y^3)}{\partial x} + (y^3) \cdot \frac{\partial (x^2)}{\partial x} = (x^2) \cdot 0 + (y^3) \cdot (2x) = 2xy^3$$

$$g(x, y) = (3x) \cdot (4y^2)$$

$$\frac{\partial g}{\partial y} = (3x) \cdot \frac{\partial (4y^2)}{\partial y} + (4y^2) \cdot \frac{\partial (3x)}{\partial y} = (3x) \cdot (8y) + (4y^2) \cdot 0 = 24xy$$

Partial Derivative with Respect to x

First, we compute the partial derivatives of $u(x, y) = x^2 + y^2$ and $v(x, y) = x^2y + y^2x$ with respect to x:

$$u(x, y) = x^2 + y^2$$
$$v(x, y) = x^2y + y^2x$$

$$\frac{\partial u}{\partial x} = 2x$$
$$\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}(x^2y + y^2x) = 2xy + y^2$$

$$\frac{\partial f}{\partial x} = u \cdot \frac{\partial v}{\partial x} + v \cdot \frac{\partial u}{\partial x}$$
$$\frac{\partial f}{\partial x} = (x^2 + y^2)(2xy + y^2) + (x^2y + y^2x)(2x)$$

Simplifying this:

$$\frac{\partial f}{\partial x} = (x^2 + y^2)(2xy + y^2) + 2x(x^2y + y^2x)$$
$$\frac{\partial f}{\partial x} = 2x^3y + x^2y^2 + 2xy^3 + y^4 + 2x^3y + 2xy^3$$
$$\frac{\partial f}{\partial x} = 4x^3y + x^2y^2 + 4xy^3 + y^4$$

Partial Derivative with Respect to y

Now, we compute the partial derivatives of $u(x, y)$ and $v(x, y)$ with respect to y:

$$\frac{\partial u}{\partial y} = 2y$$
$$\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}(x^2y + y^2x) = x^2 + 2yx$$

Applying the product rule:

$$\frac{\partial f}{\partial y} = u \cdot \frac{\partial v}{\partial y} + v \cdot \frac{\partial u}{\partial y}$$
$$\frac{\partial f}{\partial y} = (x^2 + y^2)(x^2 + 2yx) + (x^2y + y^2x)(2y)$$

Simplifying this:

$$\frac{\partial f}{\partial y} = (x^2 + y^2)(x^2 + 2yx) + 2y(x^2y + y^2x)$$
$$\frac{\partial f}{\partial y} = x^4 + 2x^3y + x^2y^2 + 2yx^2 + 2y^2x^2 + 2y^3x$$
$$\frac{\partial f}{\partial y} = x^4 + 2x^3y + x^2y^2 + 2x^2y + 2x^2y^2 + 2y^3x$$
$$\frac{\partial f}{\partial y} = x^4 + 2x^3y + 3x^2y^2 + 2yx^2 + 2y^3x$$

5. Quotient Rule

For a function $f(x, y) = \frac{g(x,y)}{h(x,y)}$:

$$\frac{\partial}{\partial x}\left(\frac{g(x,y)}{h(x,y)}\right) = \frac{h(x,y)\cdot\frac{\partial g(x,y)}{\partial x} - g(x,y)\cdot\frac{\partial h(x,y)}{\partial x}}{[h(x,y)]^2}$$

$$\frac{\partial}{\partial y}\left(\frac{g(x,y)}{h(x,y)}\right) = \frac{h(x,y)\cdot\frac{\partial g(x,y)}{\partial y} - g(x,y)\cdot\frac{\partial h(x,y)}{\partial y}}{[h(x,y)]^2}$$

Example:

$$f(x, y) = \frac{x^2}{y}$$

$$\frac{\partial f}{\partial x} = \frac{y\cdot\frac{\partial (x^2)}{\partial x} - x^2\cdot\frac{\partial y}{\partial x}}{y^2} = \frac{y\cdot(2x) - x^2\cdot 0}{y^2} = \frac{2xy}{y^2} = \frac{2x}{y}$$

$$g(x, y) = \frac{y^2}{x}$$

$$\frac{\partial g}{\partial y} = \frac{x\cdot\frac{\partial (y^2)}{\partial y} - y^2\cdot\frac{\partial x}{\partial y}}{x^2} = \frac{x\cdot(2y) - y^2\cdot 0}{x^2} = \frac{2xy}{x^2} = \frac{2y}{x}$$

Simplified,

$$f(x, y) = \frac{u(x,y)}{v(x,y)}$$

$$\frac{\partial f}{\partial x} = \frac{v\cdot\frac{\partial u}{\partial x} - u\cdot\frac{\partial v}{\partial x}}{v^2}$$

$$\frac{\partial f}{\partial x} = \frac{v\cdot\left(\frac{\partial}{\partial x}u\right) - u\cdot\left(\frac{\partial}{\partial x}v\right)}{v^2}$$

6. Chain Rule

For a function $z = f(g(x,y), h(x,y))$:

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial g} \cdot \frac{\partial g}{\partial x} + \frac{\partial z}{\partial h} \cdot \frac{\partial h}{\partial x}$$
$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial g} \cdot \frac{\partial g}{\partial y} + \frac{\partial z}{\partial h} \cdot \frac{\partial h}{\partial y}$$

Example:

$$f(x,y) = e^{x^2+y^2}$$

Let $u = x^2 + y^2$, then $f = e^u$.

$$\frac{\partial u}{\partial x} = 2x$$
$$\frac{\partial f}{\partial u} = e^u$$
$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} = e^{x^2+y^2} \cdot 2x = 2xe^{x^2+y^2}$$

Calculus for Deep Learning

Some Important Calculus Formulas

Derivative	**Integral (Antiderivative)**
$\frac{d}{dx}n = 0$	$\int 0\ dx = C$
$\frac{d}{dx}x = 1$	$\int 1\ dx = x + C$
$\frac{d}{dx}x^n = nx^{n-1}$	$\int x^n\ dx = \frac{x^{n+1}}{n+1} + C$
$\frac{d}{dx}e^x = e^x$	$\int e^x\ dx = e^x + C$
$\frac{d}{dx}\ln x = \frac{1}{x}$	$\int \frac{1}{x}\ dx = \ln x + C$
$\frac{d}{dx}n^x = n^x \ln x$	$\int n^x\ dx = \frac{n^x}{\ln n} + C$
$\frac{d}{dx}\sin x = \cos x$	$\int \cos x\ dx = \sin x + C$
$\frac{d}{dx}\cos x = -\sin x$	$\int \sin x\ dx = -\cos x + C$

$\frac{d}{dx}\tan x = \sec^2 x$	$\int \sec^2 x\ dx = \tan x + C$
$\frac{d}{dx}\cot x = -\csc^2 x$	$\int \csc^2 x\ dx = -\cot x + C$
$\frac{d}{dx}\sec x = \sec x \tan x$	$\int \tan x \sec x\ dx = \sec x + C$
$\frac{d}{dx}\csc x = -\csc x \cot x$	$\int \cot x \csc x\ dx = -\csc x + C$
$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}}\ dx = \arcsin x + C$
$\frac{d}{dx}\arccos x = -\frac{1}{\sqrt{1-x^2}}$	$\int -\frac{1}{\sqrt{1-x^2}}\ dx = \arccos x + C$
$\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$	$\int \frac{1}{1+x^2}\ dx = \arctan x + C$
$\frac{d}{dx}\operatorname{arc}\cot x = -\frac{1}{1+x^2}$	$\int -\frac{1}{1+x^2}\ dx = \operatorname{arc}\cot x + C$
$\frac{d}{dx}\operatorname{arc}\sec x = \frac{1}{x\sqrt{x^2-1}}$	$\int \frac{1}{x\sqrt{x^2-1}}\ dx = \operatorname{arc}\sec x + C$
$\frac{d}{dx}\operatorname{arc}\csc x = -\frac{1}{x\sqrt{x^2-1}}$	$\int -\frac{1}{x\sqrt{x^2-1}}\ dx = \operatorname{arc}\csc x + C$

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