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AP EXAM 2020

Notes

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AP Calculus AB & BC Exam Review

StudyResources3.0–AP Calculus AB/BC (1-7: AB & 8-10: BC): https://t.me/apresources

0. Preparation

Before the exam begins make sure to have your calculator cleared and in Radians.

1. Limits and Continuity

<u>Relationship between the limit and one-sided limits</u>

$\lim_{x \to a} f(x) = L \;\Rightarrow\; \lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L$ $\qquad\qquad$ $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L \Rightarrow \lim_{x \to a} f(x) = L$

$\lim_{x \to a^+} f(x) \neq \lim_{x \to a^-} f(x) \Rightarrow \lim_{x \to a} f(x)$ = Does Not Exist

<u>Squeeze Theorem:</u>

If *h(x)* $\leq$ *f(x)* $\leq$ *g(x)* and $\lim_{x \to c} h(x) = \lim_{x \to c} g(x) = L,\ then\ \lim_{x \to c} f(x) = L$

Example:

Find the $\lim_{x \to 0} x^2 sin(1/x)$.

sin(1/x) is bounded between -1 and 1, so that's our inequality (it has the same range as the normal sine function because its amplitude is the same).

-1 $\leq$ sin(1/x) $\leq$ 1

We can then multiply everything by x^2 to make the middle look like the initial problem.

$-x^2 \leq x^2 \sin(1/x) \leq x^2$

Take the limits of the outer pieces, and you get 0 for both. So the limit of the middle function must also be 0.

<u>Continuity at a point:</u>

Check if a function is continuous at a certain point (<u>NEEDED FOR THE AP EXAM)</u>

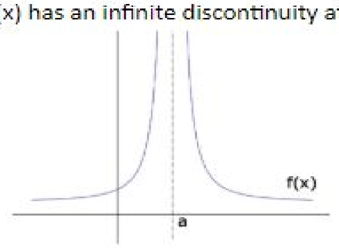
1. f(x) is defined
2. $\lim_{x \to c} f(x) = \lim_{x \to c+} f(x) = \lim_{x \to c-} f(x)$ & exists
3. $\lim_{x \to c} f(x) = f(c)$

<u>Terminology</u>: If a function f is not continuous at x = c, then we say f is discontinuous at c and c is a point of discontinuity

<u>Types of Discontinuity:</u>

Infinite Discontinuity:

1. Given equation f(x) and x = a, identify f(a).
2. If the division is by zero, the function does not exist at x = a. However, the $\frac{a}{0}$ form tells us that the function is becoming infinitely large as x approaches a.
3. Thus, if a vertical asymptote exists, then f(x) has an infinite discontinuity at x = a.



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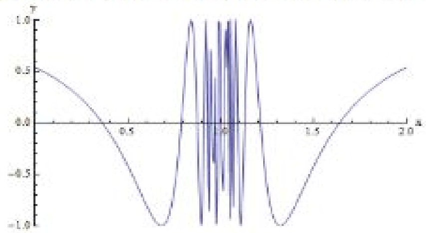
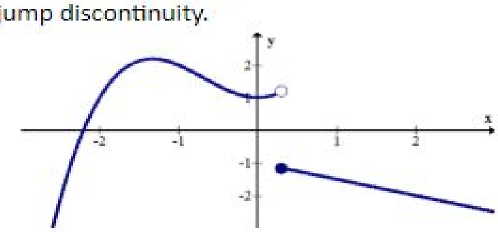
Oscillating Discontinuity:

1. Look for a variation of the function f(x) = sin $\frac{1}{x}$.
2. An oscillating discontinuity looks like it's approaching two values simultaneously.



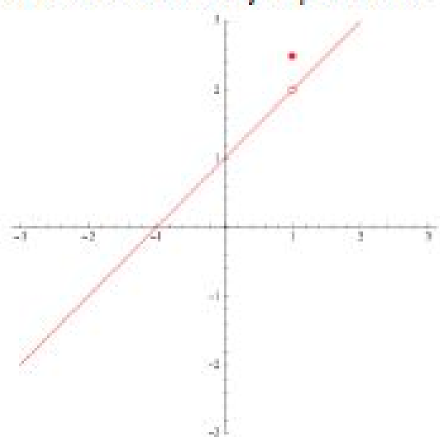
Jump Discontinuity:

1. Find the one-sided limits (left hand and right hand limits).
2. If they have different values, it is a jump discontinuity.



Removable Discontinuity:

1. If at x = a, the function has a limit, then it is a removable discontinuity.
2. Factor it out to find where the function has the discontinuity if possible.



Indeterminate Forms:

Note: these apply for ±∞ and ±0

- $0/0$
- ∞/∞
- $\infty - \infty$
- $0 * \infty$
- 0^0
- 1^∞
- ∞^0

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L'Hopital's Rule:

- $$\lim_{x \to a} \frac{g(x)}{h(x)} = \lim_{x \to a} \frac{g'(x)}{h'(x)}$$
    - Note: L'Hopital's only works with the following Indeterminate forms:
        - ±0/±0
        - ±∞/±∞
    - L'Hopital's rule may need to be utilized multiple times
    - Be sure to write "limit is indeterminate ∴ by L'Hopital's Rule:" before using the rule on FRQ

Limits you should know:

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1 \qquad \lim_{x \to 0} \frac{1-\cos(x)}{x} = 0 \qquad \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = e^x \qquad \lim_{n \to 0} (1 + xn)^{\frac{1}{n}} = e^x$$

Small Angle Approximation:

Definition: For small values of x (aka $\lim_{x\to 0}$), sin(x)=x

Example:

$$\lim_{x \to 0} \frac{\sin(nx)}{x} = \lim_{x \to 0} \frac{nx}{x} = n$$

Vertical Asymptotes:

- Find where the denominator equals zero. At that x-value you have a vertical asymptote.

Horizontal Asymptotes:

- Take the limit as x --> ∞ AND the limit as x --> -∞
- End Behavior/Infinite Limits:
    - Let N = highest growing term in the numerator and let D = highest growing term in the denominator
        - If N < D
            - the N becomes insignificant compared to it at large x, so there is a H.A. at 0
        - If N == D
            - then the HA is the ratio of their coefficients
        - If N > D
            - then the function increases without bound, so no HA :(

"Race to Infinity":

Here are the types of functions ranked by how quickly they approach infinity from fastest to slowest:

- x^x, x!, a^x, Polynomial (x^n) , ln(x), $\log_a x$, $x^{1/a}$

Limit definitions of "e" (this is where "e" comes from!)

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2. Differentiation: Definition and Fundamental Properties

A Derivative is just a slope. It can be defined as instantaneous (at a point) or average respectively:

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x \to a}$$

Derivative Rules:

Basic Derivatives Rules

Constant Rule: $\frac{d}{dx}(c) = 0$

Constant Multiple Rule: $\frac{d}{dx}[cf(x)] = cf'(x)$

Power Rule: $\frac{d}{dx}(x^n) = nx^{n-1}$

Sum Rule: $\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$

Difference Rule: $\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x)$

Product Rule: $\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$

Quotient Rule: $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$

Chain Rule: $\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$

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Common Derivatives:

Basic Properties/Formulas/Rules

$\frac{d}{dx}(cf(x)) = cf'(x)$, c is any constant. $(f(x) \pm g(x))' = f'(x) \pm g'(x)$

$\frac{d}{dx}(x^n) = nx^{n-1}$, n is any number. $\frac{d}{dx}(c) = 0$, c is any constant.

$(fg)' = f'g + fg'$ – **(Product Rule)** $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$ – **(Quotient Rule)**

$\frac{d}{dx}(f(g(x))) = f'(g(x))g'(x)$ **(Chain Rule)**

$\frac{d}{dx}\left(e^{g(x)}\right) = g'(x)e^{g(x)}$ $\frac{d}{dx}(\ln g(x)) = \frac{g'(x)}{g(x)}$

Common Derivatives

Polynomials

$\frac{d}{dx}(c) = 0$ $\frac{d}{dx}(x) = 1$ $\frac{d}{dx}(cx) = c$ $\frac{d}{dx}(x^n) = nx^{n-1}$ $\frac{d}{dx}(cx^n) = ncx^{n-1}$

Trig Functions

$\frac{d}{dx}(\sin x) = \cos x$ $\frac{d}{dx}(\cos x) = -\sin x$ $\frac{d}{dx}(\tan x) = \sec^2 x$

$\frac{d}{dx}(\sec x) = \sec x \tan x$ $\frac{d}{dx}(\csc x) = -\csc x \cot x$ $\frac{d}{dx}(\cot x) = -\csc^2 x$

Inverse Trig Functions

$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$ $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$ $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$ $\frac{d}{dx}(\csc^{-1} x) = -\frac{1}{|x|\sqrt{x^2-1}}$ $\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$

Exponential/Logarithm Functions

$\frac{d}{dx}(a^x) = a^x \ln(a)$ $\frac{d}{dx}(e^x) = e^x$

$\frac{d}{dx}(\ln(x)) = \frac{1}{x},\ x > 0$ $\frac{d}{dx}(\ln|x|) = \frac{1}{x},\ x \neq 0$ $\frac{d}{dx}(\log_a(x)) = \frac{1}{x \ln a},\ x > 0$

Logarithmic Differentiation:

$d/dx[x^x] = x^x (\ln|x| + 1)$

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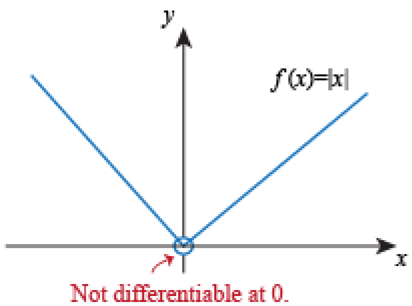
Derivatives can be defined through limits but they are intricately related to continuity:

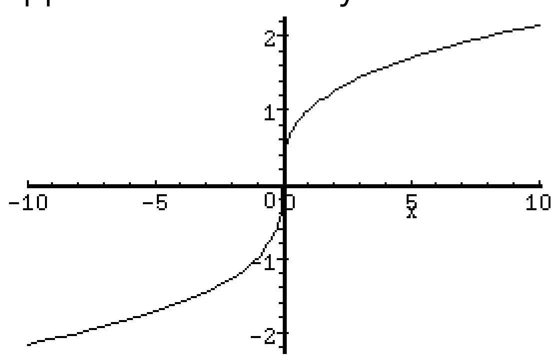
<u>Differentiability implies continuity!</u>

A function is not differentiable at a point when one of the following is true:

1. You are not continuous then you CANNOT be differentiable.
2. The graph has a sharp corner/cusp at that point - if one-sided derivatives (derivative definitions from left and right) aren't equal (commonly seen thru piecewise or abs-value graphs).



3. The tangent line at that point has a vertical/nonexistent slope (derivative definition approaches +/- infinity from both sides)



<u>Log Properties:</u>

Logarithmic Properties	
Product Rule	$\log_a(xy) = \log_a x + \log_a y$
Quotient Rule	$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$
Power Rule	$\log_a x^p = p\log_a x$
Change of Base Rule	$\log_a x = \frac{\log_b x}{\log_b a}$
Equality Rule	If $\log_a x = \log_a y$ then $x = y$

$$\log_a b = \frac{1}{\log_b a}$$

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3. Differentiation: Composite, Implicit, and Inverse Functions

The Chain Rule

If $y = f(u)$ is a differentiable function of u and $u = g(x)$ is a differentiable function of x, then $y = f(g(x))$ is a differentiable function of x and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

or, equivalently,

$$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x).$$

Implicit Form	**Explicit Form**
$xy = 1$	$y = \frac{1}{x} = x^{-1}$

implicit vs explicit forms

<u>Implicit differentiation:</u>

Note: implicit derivative of y^2 wrt x is 2y * dy/dx. Not just 2y!!!

a. $\frac{d}{dx}[x^3] = 3x^2$

Variables agree

b. $\frac{d}{dx}[\overbrace{y^3}^{u^n}] = \overbrace{3y^2}^{nu^{n-1}}\overbrace{\frac{dy}{dx}}^{u'}$

Variables disagree

c. $\frac{d}{dx}[x + 3y] = 1 + 3\frac{dy}{dx}$

d.
$$\begin{aligned}\frac{d}{dx}[xy^2] &= x\frac{d}{dx}[y^2] + y^2\frac{d}{dx}[x] \\ &= x\left(2y\frac{dy}{dx}\right) + y^2(1) \\ &= 2xy\frac{dy}{dx} + y^2\end{aligned}$$

Definition of Inverse Function

A function g is the **inverse function** of the function f when

$f(g(x)) = x$ for each x in the domain of g

and

$g(f(x)) = x$ for each x in the domain of f.

The function g is denoted by f^{-1} (read "f inverse").

The Derivative of an Inverse Function

Let f be a function that is differentiable on an interval I. If f has an inverse function g, then g is differentiable at any x for which $f'(g(x)) \neq 0$. Moreover,

$$g'(x) = \frac{1}{f'(g(x))}, \quad f'(g(x)) \neq 0.$$

This is derived from the fact that for inverse functions f and g, $f(g(x))=x$

$f(g(x))=x \Rightarrow f'(g(x))g'(x)=1 \Rightarrow g'(x)=1/f'(g(x))$

Alternatively defined as:

$$(f^{-1}(x))' = \frac{1}{f'(f^{-1}(x))}$$

<u>Inverse Differentiation Example:</u>

$f(x)=x^2$

$f'(x)=2x$

$f'(y)=2y$

$(f^{-1})'(x)=1/(2y)$

Flip: $x=y^2 \rightarrow y=\sqrt{x}$

$(f^{-1})'(x)=1/(2\sqrt{x})$

4. Contextual/Analytical Applications of Differentiation

- ### Extrema on an Interval
  - Local/Relative Extrema are U or n shaped max/min. Absolute extrema are *the one* max/min on the given [a, b]
    - Horizontal lines experience extrema at every point
  - Critical number: Any x-value which makes f ′(x) = 0 or f ′(x) = DNE
  - Relative extrema occurs only at critical numbers.
  - Guidelines for Finding Absolute Extrema on a Closed Interval
    - Find the critical numbers of f in (a,b).
    - Evaluate f at each critical number in (a,b).
    - Evaluate f at each endpoint of [a,b].
    - The least of these values is the absolute minimum. The greatest is the absolute maximum.
- The Extreme Value Theorem (EVT): If f is continuous on a closed interval [a,b], then f has both an absolute minimum and an absolute maximum on the interval.
- ### Intermediate Value Theorem
  - A function is said to have the **intermediate value property** if it never takes on two values without taking on all the values in between.

    A function y = f(x) that is continuous on a closed interval [a, b] takes on every value between f(a) and f(b). In other words, if y_0 is between f(a) and f(b), then y_0 = f(c) for some c in [a, b].
- ### The Mean Value Theorem (MVT) and Rolle's Theorem
  - Mean Value Theorem (MVT)
    - If f is continuous on the closed interval [a,b] and differentiable on the open interval (a,b), then there exists a number c in (a,b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.
    - Rolle's Theorem
      - Let f be continuous on the closed interval [a,b] and differentiable on the open interval (a,b). If f(a)=f(b), then there is at least one number c in (a,b) such that f ′(c)=0.
- ### Increasing and Decreasing Functions and the First Derivative Test
  - Guidelines for finding intervals on which a function is increasing or decreasing
    - Let f be continuous on the interval (a,b). To find the open intervals on which f is increasing or decreasing, use the following:
      - Locate the critical numbers of f in (a,b), and use these numbers to determine test intervals.
      - Determine the sign of f ′(x) at one test value in each of the intervals.
      - Use the following
        - $f'(x) > 0$ increasing.
        - $f'(x) < 0$ decreasing.
        - $f'(x) = 0$ or $f'(x)$ = DNE extrema of f(x).
          - If f′ changes from + to - then max
          - If f′ changes from - to + then min
- ### Concavity and the Second Derivative Test
  - Point of inflection- A point in which f(x) changes concavity.
    - Inflection points can be represented by f ′′(x)=0 or DNE AND a sign-change around x
  - Second derivative test
    - $f''(x) > 0$ concave up, relative minimum at (x, f(x))
    - $f''(x) < 0$ concave down, relative maximum at (x,f(x))
    - $f''(x) = 0$ could be an inflection point or local extrema, refer to the first derivative test.
- ### Summary of Curve Sketching (AB Only)
  - x-intercepts, y-intercepts
  - Symmetry
  - Domain and range

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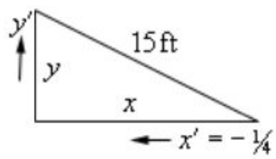
- Continuity, differentiability
- Vertical and horizontal asymptotes
- Relative Extrema
- Points of inflection
- Concavity
- Infinite limits at infinity/End behavior

○ Related Rates

Ex. A 15 foot ladder is resting against a wall. The bottom is initially 10 ft away and is being pushed towards the wall at $\frac{1}{4}$ ft/sec. How fast is the top moving after 12 sec?



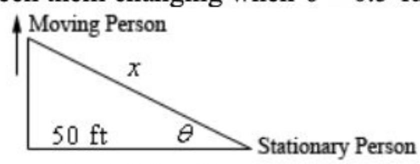
x' is negative because x is decreasing. Using Pythagorean Theorem and differentiating,

$x^2 + y^2 = 15^2 \quad \Rightarrow \quad 2x\,x' + 2y\,y' = 0$

After 12 sec we have $x = 10 - 12\left(\frac{1}{4}\right) = 7$ and so $y = \sqrt{15^2 - 7^2} = \sqrt{176}$. Plug in and solve for y'.

$$7\left(-\tfrac{1}{4}\right) + \sqrt{176}\;y' = 0 \;\Rightarrow\; y' = \frac{7}{4\sqrt{176}} \text{ ft/sec}$$

Ex. Two people are 50 ft apart when one starts walking north. The angle θ changes at 0.01 rad/min. At what rate is the distance between them changing when $\theta = 0.5$ rad?



We have $\theta' = 0.01$ rad/min. and want to find x'. We can use various trig fcns but easiest is,

$$\sec\theta = \frac{x}{50} \quad \Rightarrow \quad \sec\theta\tan\theta\;\theta' = \frac{x'}{50}$$

We know $\theta = 0.05$ so plug in θ' and solve.

$$\sec(0.5)\tan(0.5)(0.01) = \frac{x'}{50}$$

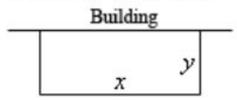
$$x' = 0.3112 \text{ ft/sec}$$

Remember to have calculator in radians!

○ Optimization Problems

- Guidelines for solving applied minimum and maximum problems
  - Identify all *given* quantities and all quantities to be *determined*. If possible, make a sketch.
  - Write a primary equation for the quantity that is to be maximized or minimized.
  - Reduce the primary equation to one having a *single independent variable.* This may involve the use of secondary equations relating the independent variables of the primary equation.
  - Determine the feasible domain of the primary equation. That is, determine the values for which the stated problem makes sense.
  - Determining the desired maximum or minimum value.

Ex. We're enclosing a rectangular field with 500 ft of fence material and one side of the field is a building. Determine dimensions that will maximize the enclosed area.



Maximize $A = xy$ subject to constraint of $x + 2y = 500$. Solve constraint for x and plug into area.

$$x = 500 - 2y \quad \Rightarrow \quad \begin{aligned} A &= y(500-2y) \\ &= 500y - 2y^2 \end{aligned}$$

Differentiate and find critical point(s).

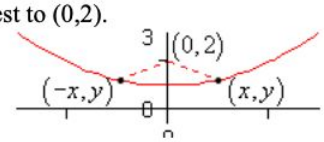
$$A' = 500 - 4y \quad \Rightarrow \quad y = 125$$

By 2nd deriv. test this is a rel. max. and so is the answer we're after. Finally, find x.

$$x = 500 - 2(125) = 250$$

The dimensions are then 250 x 125.

Ex. Determine point(s) on $y = x^2 + 1$ that are closest to (0,2).



Minimize $f = d^2 = (x-0)^2 + (y-2)^2$ and the constraint is $y = x^2 + 1$. Solve constraint for x^2 and plug into the function.

$$x^2 = y - 1 \Rightarrow f = x^2 + (y-2)^2$$
$$= y - 1 + (y-2)^2 = y^2 - 3y + 3$$

Differentiate and find critical point(s).

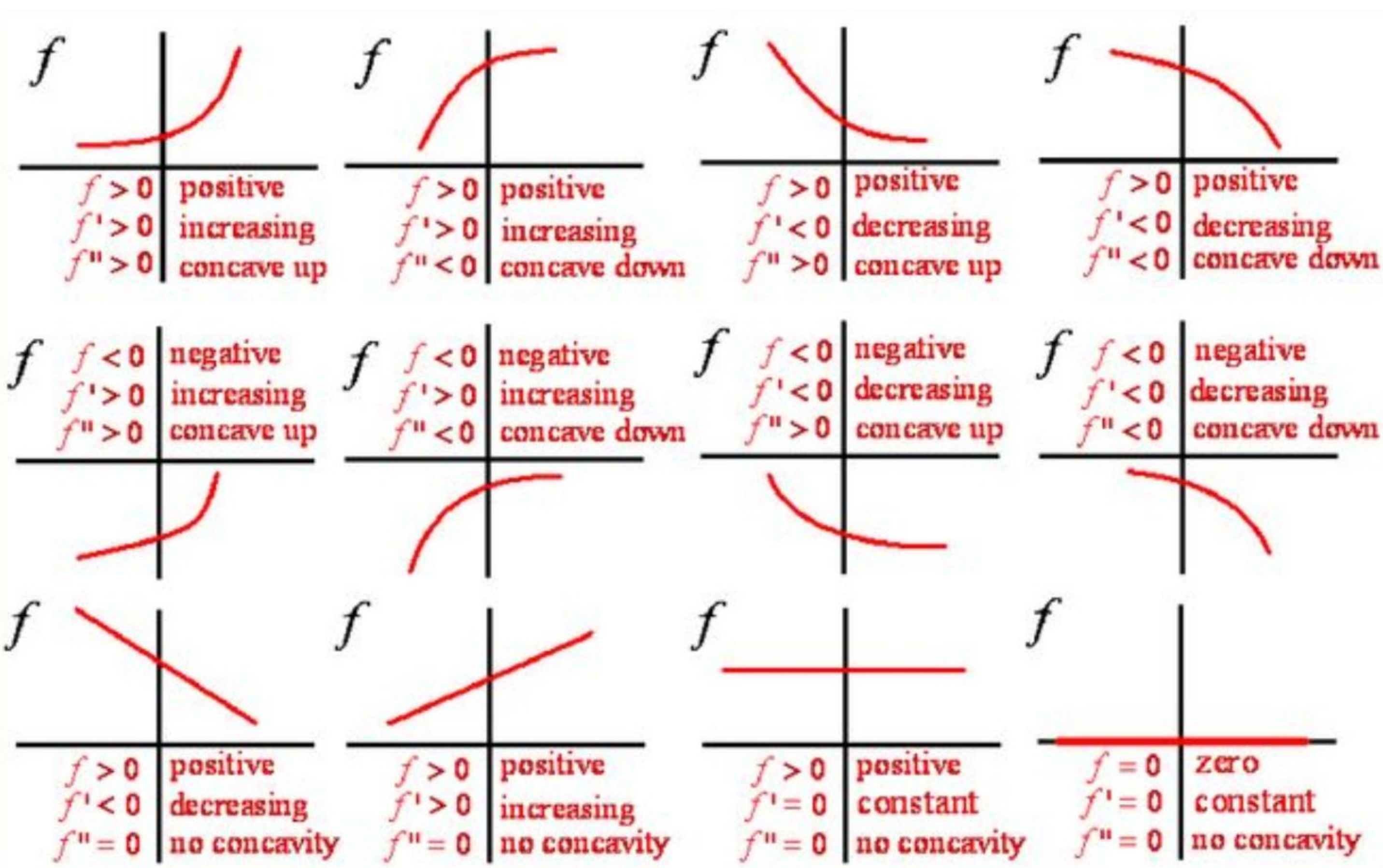
$$f' = 2y - 3 \quad \Rightarrow \quad y = \tfrac{3}{2}$$

By the 2nd derivative test this is a rel. min. and so all we need to do is find x value(s).

$$x^2 = \tfrac{3}{2} - 1 = \tfrac{1}{2} \quad \Rightarrow \quad x = \pm \tfrac{1}{\sqrt{2}}$$

The 2 points are then $\left(\frac{1}{\sqrt{2}}, \frac{3}{2}\right)$ and $\left(-\frac{1}{\sqrt{2}}, \frac{3}{2}\right)$.

- Linearization:
  - Tangent Line Approximation
    - Approximating the tangent line of f at c yields $y = f(c) + f'(c)(x-c)$



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6. Integration and Accumulation of Change

- Integration (AB/BC)
    - Antiderivatives and Indefinite Integration
        - Basic Integration Rules
            - Integration is the "inverse" of differentiation.
            - Differentiation is the "inverse" of integration.

Common Integrals

Polynomials

$$\int dx = x + c \qquad \int k\,dx = kx + c \qquad \int x^n dx = \frac{1}{n+1}x^{n+1} + c,\ n \neq -1$$

$$\int \frac{1}{x}dx = \ln|x| + c \qquad \int x^{-1}\,dx = \ln|x| + c \qquad \int x^{-n}dx = \frac{1}{-n+1}x^{-n+1} + c,\ n \neq 1$$

$$\int \frac{1}{ax+b}dx = \frac{1}{a}\ln|ax+b| + c \qquad \int x^{\frac{p}{q}}dx = \frac{1}{\frac{p}{q}+1}x^{\frac{p}{q}+1} + c = \frac{q}{p+q}x^{\frac{p+q}{q}} + c$$

Trig Functions

$$\int \cos u\,du = \sin u + c \qquad \int \sin u\,du = -\cos u + c \qquad \int \sec^2 u\,du = \tan u + c$$

$$\int \sec u \tan u\,du = \sec u + c \qquad \int \csc u \cot u\,du = -\csc u + c \qquad \int \csc^2 u\,du = -\cot u + c$$

$$\int \tan u\,du = \ln|\sec u| + c \qquad \int \cot u\,du = \ln|\sin u| + c$$

$$\int \sec u\,du = \ln|\sec u + \tan u| + c \qquad \int \sec^3 u\,du = \frac{1}{2}\left(\sec u \tan u + \ln|\sec u + \tan u|\right) + c$$

$$\int \csc u\,du = \ln|\csc u - \cot u| + c \qquad \int \csc^3 u\,du = \frac{1}{2}\left(-\csc u \cot u + \ln|\csc u - \cot u|\right) + c$$

Exponential/Logarithm Functions

$$\int \mathbf{e}^u\,du = \mathbf{e}^u + c \qquad \int a^u\,du = \frac{a^u}{\ln a} + c \qquad \int \ln u\,du = u\ln(u) - u + c$$

$$\int \mathbf{e}^{au}\sin(bu)\,du = \frac{\mathbf{e}^{au}}{a^2+b^2}\left(a\sin(bu) - b\cos(bu)\right) + c \qquad \int u\mathbf{e}^u du = (u-1)\mathbf{e}^u + c$$

$$\int \mathbf{e}^{au}\cos(bu)\,du = \frac{\mathbf{e}^{au}}{a^2+b^2}\left(a\cos(bu) + b\sin(bu)\right) + c \qquad \int \frac{1}{u\ln u}du = \ln|\ln u| + c$$

$$\int \sin^{-1}u\,du = u\sin^{-1}u + \sqrt{1-u^2} + c$$

$$\int \tan^{-1}u\,du = u\tan^{-1}u - \frac{1}{2}\ln\left(1+u^2\right) + c$$

$$\int \cos^{-1}u\,du = u\cos^{-1}u - \sqrt{1-u^2} + c$$

- Area, Riemann Sums and Definite Integrals

Value of n in the final term (may be ∞)

The Greek letter *sigma* means "sum."

Terms of the sum

$$a = \sum_{n=1}^{10} a_n = a_1 + a_2 + \ldots + a_9 + a_{10}$$

The index n labels each term. n = 1, 2, 3, ...

n ranges from 1 up to 10, counting by 1

- Sigma Notation
- Partial Summation formulas (AB only):

$$\sum_{i=1}^{n} k = kn \qquad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

AP Calculus AB & BC Exam Review

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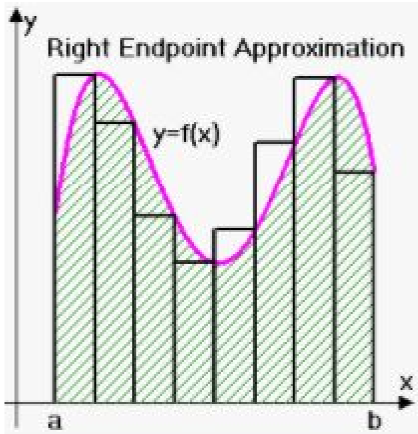
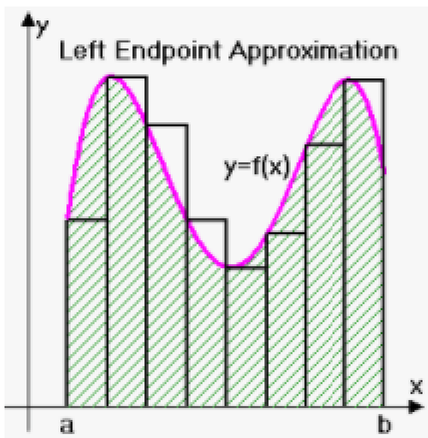
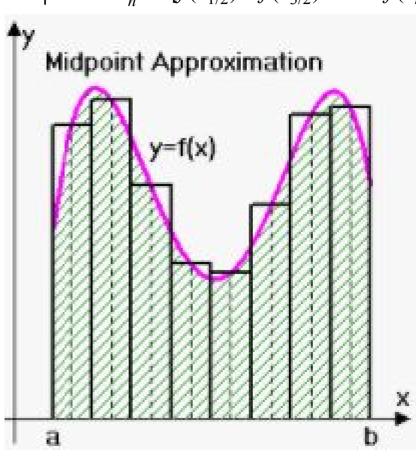
- Riemann Sum Formula
  - Right: $\frac{b-a}{n} * [f(x_1) + f(x_2) + \dots + f(x_n)]$



  - Left: $\frac{b-a}{n} * [f(x_0) + f(x_1) + \dots + f(x_{n-1})]$



  - Midpoint: $\frac{b-a}{n} * [f(x_{1/2}) + f(x_{3/2}) + \dots + f(x_{n-1/2})]$



- Numerical Integration
  - Trapezoidal Rule
    - $\int_a^b f(x)\,dx \approx \frac{b-a}{2n}[f(x_0) + 2f(x_1) + 2f(x_2) + \dots + 2f(x_{n-2}) + 2f(x_{n-1}) + f(x_n)]$

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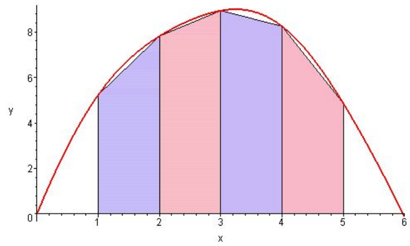

- The Trapezoidal sum is exactly the same as the average of the left and right endpoint Riemann sums

■ Definite Integral Notation

- $$\int_a^b f(t)dt = \lim_{n\to\infty} \sum_{k=1}^{n} f(a + k\Delta x) \cdot \Delta x \quad \text{where } \frac{b-a}{n} = \Delta x$$
- Implies the area under f, on the closed interval [a,b] bounded by the x-axis, and the vertical lines x=a, x=b.
- If a=b, the integral is equal to 0.
- 29. Which of the following expressions is equal to $\lim_{n\to\infty} \frac{1}{n}\left(\frac{1}{2+\frac{1}{n}} + \frac{1}{2+\frac{2}{n}} + \frac{1}{2+\frac{3}{n}} + \cdots + \frac{1}{2+\frac{n}{n}}\right)$?

  (A) $\int_1^2 \frac{1}{x}\, dx$

  (B) $\int_0^1 \frac{1}{2+x}\, dx$

  (C) $\int_0^2 \frac{1}{2+x}\, dx$

  (D) $\int_2^3 \frac{1}{2+x}\, dx$
- ^answer is B

■ Integral Properties:

- $\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$
- **Basic Properties/Formulas/Rules**

  $\int cf(x)dx = c\int f(x)dx$, c is a constant. $\quad \int f(x) \pm g(x)dx = \int f(x)dx \pm \int g(x)dx$

  $\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a)$ where $F(x) = \int f(x)dx$

  $\int_a^b cf(x)dx = c\int_a^b f(x)dx$, c is a constant. $\quad \int_a^b f(x) \pm g(x)dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

  $\int_a^a f(x)dx = 0 \quad \int_a^b f(x)dx = -\int_b^a f(x)dx$

  $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx \quad \int_a^b c\,dx = c(b-a)$

  If $f(x) \geq 0$ on $a \leq x \leq b$ then $\int_a^b f(x)dx \geq 0$

  If $f(x) \geq g(x)$ on $a \leq x \leq b$ then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$
- $\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$ where a≤c≤b

○ The Fundamental Theorem of Calculus

■ First Fundamental Theorem of Calculus (1 FTC)

- $\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)$
- $\frac{d}{dx}\left[\int_{a(x)}^{b(x)} f(t)\,dt\right] = f(b(x))b'(x) - f(a(x))a'(x)$

- Second Fundamental Theorem of Calculus (2nd FTC)
  - $\int_a^b f(x)\,dx = F(b) - F(a)$ where F is the antiderivative function
- Average Value of a Function on an Interval
  - $f_{avg}(x) = \frac{1}{b-a}\int_a^b f(x)\,dx$
- Mean Value Theorem for Integrals
  - If f is a continuous function on the closed, bounded interval [a, b], then there is at least one number c in (a, b) for which:
  - $\int_a^b f(x)\,dx = f(c)(b-a)$

- Misc functions
  - Displacement on [a,b]
    - $\int_a^b v(t)\,dt$
  - Total distance traveled on [a,b]
    - $\int_a^b |v(t)|\,dt$
  - Velocity
    - Velocity is the rate of change in position, so its definite integral will give us the net displacement of the moving object, while the derivative of the function will give the object's acceleration.
  - Acceleration
    - In calculus, acceleration is found as being the derivative of velocity, or the double derivative of the displacement of an object with respect to time.

Higher Order Derivatives

$y = f(x)$	Original Function (position /distance/ height)
$y' = f'(x) = \frac{dy}{dx}$	First Derivative (velocity)
$y'' = f''(x) = \frac{d^2y}{dx^2}$	Second Derivative (acceleration)

- Integration by Substitution (U-Substitution)
  - Guidelines for making a change of variables
    - Choose a substitution u=g(x). Usually, it is best to choose the *inner* part of a composite function, such as a quantity raised to a power.
    - Compute *du=g'(x)dx.*
    - Rewrite the integral in terms of the variable u.

- Find the resulting integral in terms of u.
- Replace u by g(x) to obtain an antiderivative in terms of x.
- Check your answer by differentiating.

■ Integration of even and odd functions

- If f is an even function, then $\int_{-a}^{a} f(t)\,dt = 2\int_{0}^{a} f(t)\,dt$
- If f is an odd function, then $\int_{-a}^{a} f(t)\,dt = 0$

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7. Differential Equations

Separation of Variables:

Find the general solution of

$$(x^2 + 4)\frac{dy}{dx} = xy.$$

Solution To begin, note that $y = 0$ is a solution. To find other solutions, assume that $y \neq 0$ and separate variables as shown.

$$(x^2 + 4)\,dy = xy\,dx \qquad \text{Differential form}$$

$$\frac{dy}{y} = \frac{x}{x^2 + 4}\,dx \qquad \text{Separate variables.}$$

Now, integrate to obtain

$$\int \frac{dy}{y} = \int \frac{x}{x^2 + 4}\,dx \qquad \text{Integrate.}$$

$$\ln|y| = \frac{1}{2}\ln(x^2 + 4) + C_1$$

$$\ln|y| = \ln\sqrt{x^2 + 4} + C_1$$

$$|y| = e^{\ln\sqrt{x^2+4}+C_1} \qquad \text{Exponentiate each side.}$$

$$|y| = e^{C_1}\sqrt{x^2 + 4} \qquad \text{Property of exponents}$$

$$y = \pm e^{C_1}\sqrt{x^2 + 4}.$$

Because $y = 0$ is also a solution, you can write the general solution as

$$y = C\sqrt{x^2 + 4}. \qquad \text{General solution}$$

If y is a differentiable function of t such that $y > 0$ and $dy/dt = ky$ for some constant k, then

$$y = Ce^{kt}$$

where C is the **initial value** of y, and k is the **proportionality constant. Exponential growth** occurs when $k > 0$, and **exponential decay** occurs when $k < 0$.

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Slope fields:

Sketch a slope field for the differential equation $y' = x - y$ for the points $(-1, 1)$, $(0, 1)$, and $(1, 1)$.

Solution The slope of the solution curve at any point (x, y) is

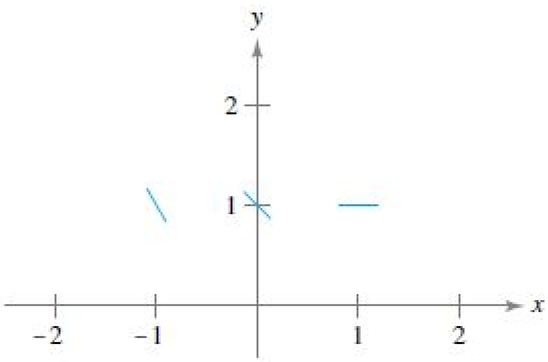
$F(x, y) = x - y.$ Slope at (x, y)

So, the slope at each point can be found as shown.

Slope at $(-1, 1)$: $y' = -1 - 1 = -2$

Slope at $(0, 1)$: $y' = 0 - 1 = -1$

Slope at $(1, 1)$: $y' = 1 - 1 = 0$



Law of natural growth:

Let $\frac{dP}{dt} = kP(t)$, where:

P =population (dependent variable)

k = constant of proportionality

t = time

- Solution is $P(t) = Ce^{kt}$
- $P'(t) = C(ke^{kt}) = k(Ce^{kt}) = kP(t)$
- $\frac{dP}{dt} \approx kP(t)$ if P is small (initial growth rate is proportional to P)

Newton's Law of Cooling:

$$\frac{dT}{dt} = k(T - T_S)$$
$$T(t) = T_S + (T_0 - T_S)e^{kt}$$

T: temperature of object at time t

k: proportionality constant

T_S: surrounding temperature (constant)

t: is the time after the object starts cooling

For AB Students: THE END

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AP Calculus BC ONLY starts here: Units 1-8 + 5 topics in Unit 10 (10.2, 10.5, 10.7, 10.8, and 10.11)

Logistic Differential Equation:

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{L}\right)$$

P = Population at time t

L = Limiting or Carrying Capacity
(max size of population)

k = constant of proportionality

Population experiences the fastest growth at half the carrying capacity.

- $\frac{dP}{dt} \approx kP(t)$ if L >> P
- $\frac{dP}{dt} < 0$ if $P > L$ (P decreases if it ever exceeds L)
- $P(t) = \frac{L}{1 + Ce^{-kt}}$; $C = \frac{L - P_0}{P}$
- $\lim_{t \to \infty} P(t) = L$
- $\lim_{P \to L} \frac{dP}{dt} = 0$

Euler's Method:

Euler's Method is a step-based method for approximating the solution to a *differential equation*
Point-Slope Form:
$y_2 - y_1 = m(x_2 - x_1)$
becomes:
$y_{new} = y_{old} + y'(x, y) \cdot \Delta x$

First, you must choose a small step size *Δx* (which is almost always given in the problem statement on the AP exam).

Then draw a table as follows:

The number of frogs living in a pond at time t is modeled by the function $y = F(t)$ that satisfies the logistic differential equation $\frac{dy}{dt} = \frac{1}{2500}y(1010 - y)$, where t is measured in weeks. The number of frogs in the pond at time $t = 0$ is $F(0) = b$, where b is a positive constant.

Let $b = 100$. Use Euler's method, starting at $t = 0$ with two steps of equal size, to approximate $F(4)$, the number of frogs in the pond at time $t = 4$ weeks. Show the work that leads to your answer.

Note: Since x increments by +2, Δx=2

x	y	dy/dx
0	100	36.4
2	y_{new} = 100 + 36.4*2 = 172.8	57.867264
4	y_{new} = 172.8 + 57.867264*2 = 288.534528 = 288.535	N/A

It's important to realize that Euler's method does not give exact answers — just good estimates. Unless the directions specify to use Euler's Method, do not use it!

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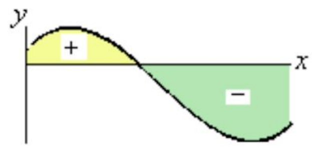
8. Applications of Integration

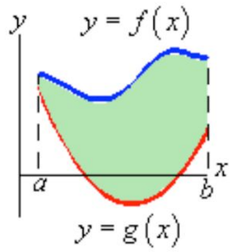
Net Area : $\int_a^b f(x)\,dx$ represents the net area between $f(x)$ and the x-axis with area above x-axis positive and area below x-axis negative.



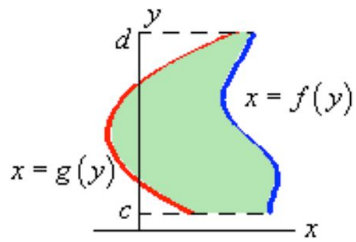
Area Between Curves : The general formulas for the two main cases for each are,

$$y=f(x) \Rightarrow A=\int_a^b [\text{upper function}]-[\text{lower function}]\,dx \quad \& \quad x=f(y) \Rightarrow A=\int_c^d [\text{right function}]-[\text{left function}]\,dy$$

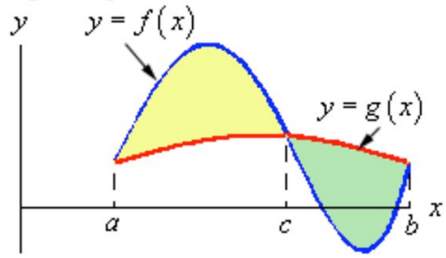
If the curves intersect then the area of each portion must be found individually. Here are some sketches of a couple possible situations and formulas for a couple of possible cases.



$$A=\int_a^b f(x)-g(x)\,dx$$



$$A=\int_c^d f(y)-g(y)\,dy$$

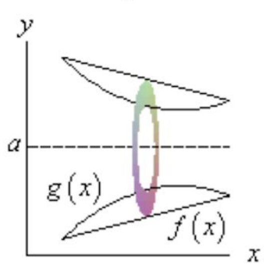


$$A=\int_a^c f(x)-g(x)\,dx+\int_c^b g(x)-f(x)\,dx$$

Volumes of Revolution : The two main formulas are $V=\int A(x)\,dx$ and $V=\int A(y)\,dy$. Here is some general information about each method of computing and some examples.

Rings	**Cylinders**
$A=\pi\left((\text{outer radius})^2-(\text{inner radius})^2\right)$	$A=2\pi(\text{radius})(\text{width / height})$
Limits: x/y of right/bot ring to x/y of left/top ring	Limits : x/y of inner cyl. to x/y of outer cyl.
Horz. Axis use $f(x)$, $g(x)$, $A(x)$ and dx. Vert. Axis use $f(y)$, $g(y)$, $A(y)$ and dy.	Horz. Axis use $f(y)$, $g(y)$, $A(y)$ and dy. Vert. Axis use $f(x)$, $g(x)$, $A(x)$ and dx.

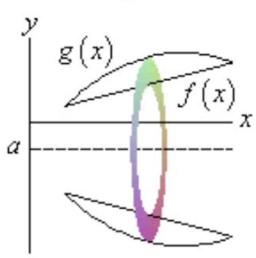
Ex. Axis : $y=a>0$



outer radius : $a-f(x)$

inner radius : $a-g(x)$

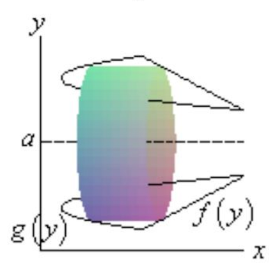
Ex. Axis : $y=a\le 0$



outer radius: $|a|+g(x)$

inner radius: $|a|+f(x)$

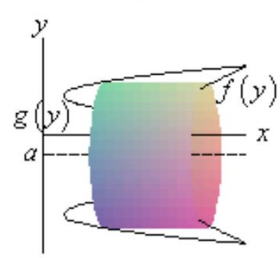
Ex. Axis : $y=a>0$



radius : $a-y$

width : $f(y)-g(y)$

Ex. Axis : $y=a\le 0$



radius : $|a|+y$

width : $f(y)-g(y)$

These are only a few cases for horizontal axis of rotation. If axis of rotation is the x-axis use the $y=a\le 0$ case with $a=0$. For vertical axis of rotation ($x=a>0$ and $x=a\le 0$) interchange x and y to get appropriate formulas.

Volumes of Solids with Known Cross Sections

Square: $1 \int_a^b (\text{top - bottom})^2 dA$

Equilateral Triangle: $\frac{\sqrt{3}}{4} \int_a^b (\text{top - bottom})^2 da$

Isosceles Triangle: $\frac{1}{4} \int_a^b (\text{top - bottom})^2 da$

Right Triangle: $\frac{1}{2} \int_a^b (\text{top - bottom})^2 da$

Semicircles: $\frac{\pi}{8} \int_a^b (\text{top - bottom})^2 da$

Rectangle: (constant height) $c \int_a^b (\text{top - bottom})^2 da$

Rectangle: (variable height) $\int_a^b c(\text{top - bottom})^2 da$

Around y=k (y=f(x)):

$$V = \pi \int_a^b (y-k)^2 \ dx$$

Around x=h (x=f(y)):

$$V = \pi \int_c^d [x-h]^2 \ dy$$

Surface Area:

$$SA = 2\pi \int_a^b y(x)\sqrt{1+(y'(x))^2} \ dx$$

or SA = 2π∫r(x)h(x)dx where h(x)=arc length

Arc Length:

$$L = \int_a^b \sqrt{1+(f'(x))^2} \ dx$$

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- Integration Techniques, L'Hopital's Rule, and Improper Integrals (BC)
  - Basic Integration Rules
    - Procedures for Fitting Integrands to Basic Integration Rules
      - Expand (numerator)
      - Separate numerator.
      - Complete the square.
      - Divide improper rational function.
      - Add and subtract terms in numerator.
      - Use trigonometric identities.
      - Multiply or divide by Pythagorean conjugate.
  - Long Division
    - When the rational function has a numerator of greater or equal degree than the denominator, we can use long division
      - When we divide, we want to achieve a polynomial term and a fractional term that we can work with
      - Example: Evaluate ∫(x^2 +1)/(x-2) dx
        - $\int \frac{(x^2+1)}{x-2}dx = \int \frac{(x-2)(x+2)+5}{x-2}dx = \int(x+2)dx + \int \frac{5}{x-2}\,dx = \frac{x^2}{2} + 2x + 5ln(x-2) + C$
        - If you could not understand the jump from the first to second term, please brush up on polynomial long division
        - For the first step, try to rewrite the numerator x^2 + 1 in terms of something in the denominator x - 2. We see that it is not immediately factorable, so we chip away constant values until we have something that is (kind of like the opposite of "completing the square"). We see that we can rewrite x^2 + 1 as x^2 - 4 + 5 where we see a difference of squares, which can be factored into (x-2)(x+2) giving us (x-2)(x+2) + 5 in the numerator.
          - You get better with this by getting intuition after practice algebra.
        - To go from the first to the second term, you must divide (x^2 + 1) by (x - 2); there are other methods to achieve the second term since it has a relatively simple polynomial as the numerator
  - Partial Fraction Decomposition
    - This should be used if you have a rational function where the denominator has a greater degree than the numerator
    - What you should aim for is to "decompose" the fraction into multiple fractions of degree 1
    - Example: Evaluate $\int \frac{3x+2}{x^3-x^2-2x}dx$
      - First, note that $x^3 - x^2 - 2x = (x)(x-2)(x+1)$
      - Therefore, our decomposed expression should be of the form $\frac{3x+2}{x^3-x^2-2x} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+1}$, where each value $A,\ B,\ C$ is a real number.
      - We begin finding A, B, and C by multiplying both sides by $x^3 - x^2 - 2x$ to get:
      - $3x + 2 = A(x-2)(x+1) + B(x)(x+1) + C(x)(x-2)$
      - $3x + 2 = A(x^2 - x - 2) + B(x^2 + x) + C(x^2 - 2x)$
      - $3x + 2 = (A + B + C)x^2 + (-A + B - 2C)x + (-2A)$
        - Note that all of the coefficients must match, so we get the following system:

- $(A + B + C) = 0$
- $(-A + B - 2C) = 3$
- $-2A = 2$

- From the system, we can immediately see that $A = -1$, so our system becomes:
  - $(B + C) = 1$
  - $(B - 2C) = 2$
- From here, it is trivial to find that $B = 4/3,\ C = -1/3$
- Going back to the original equation, we can now see that:
  - $\int \frac{3x+2}{x^3-x^2-2x}dx = \int((-1)\frac{1}{x} + (\frac{4}{3})\frac{1}{x-2} + (-\frac{1}{3})(\frac{1}{x+1})dx = -ln|x| + \frac{4}{3}ln|x-2| - \frac{1}{3}ln|x+1| + C$
  - And this thus ends our solution.

- Types of PFD:
  - Factor in denominator -> PFD Term
  - ax+b -> A/(ax+b)
  - $(ax+b)^k$ -> $A_1/(ax+b) + A_2/(ax+b)^2 + \ldots + A_k/(ax+b)^k$
  - ax^2+bx+c -> $(Ax+B)/(ax^2+bx+c)$
- Note that Partial Fraction Decomposition and Long Division are often used in conjunction.
  - Exercise: Show that $\int \frac{x^2+3x+1}{x^2-4}dx = x + \frac{11}{4}ln|x-2| + \frac{1}{4}ln|x+2| + C$

- Integration by Parts
  - Method of reducing an integral, whose integrand is the product of two functions of the variable integration, to make it easier to integrate.
  - Derived from the Product Rule for Differentiation (multiply both sides by dx, integrate, and isolate the integral of u w.r.t. v).

$$\int u\, dv = uv - \int v\, du$$

  - The trick is determining what u and dv should be to make the integral easier to solve. In general, u should simplify when differentiated and dv should remain manageable when integrated.
    - Follow the acronym LIPET for choosing u: Logarithmic first, Inverse Trigonometric next, then Polynomial, Exponential and Trigonometric.
  - When u is more complicated, you may have to apply the By Parts method more than once.
  - Sometimes with repeated use of By Parts, you will end up with the same integral as the one you are trying to solve for. In these cases, assign the unknown integral to some variable and solve.
  - Sometimes you may get the integral you started with from the ∫ v du part. If so, then add that to both sides and divide.
  - With bounds:

$$\int_a^b udv = uv|_a^b - \int_a^b vdu$$

- Trigonometric Substitution (not tested explicitly but has been in multiple past FRQs where they graded ONLY for the answer; no work. So not knowing how to do this IS ok)

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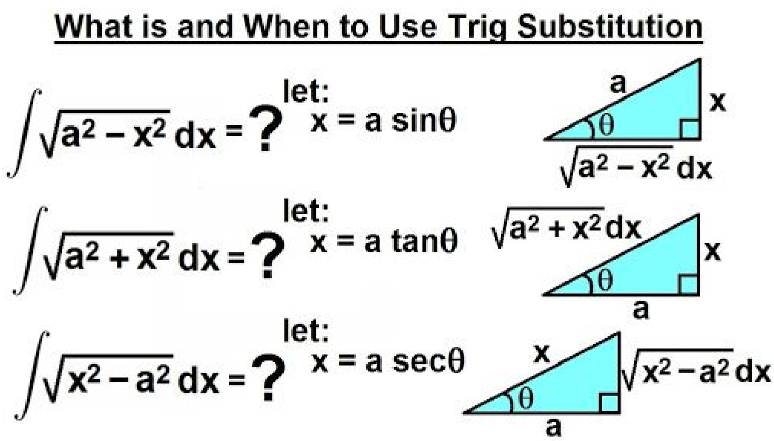
Trig Substitutions : If the integral contains the following root use the given substitution and formula to convert into an integral involving trig functions.

$\sqrt{a^2 - b^2x^2} \Rightarrow x = \frac{a}{b}\sin\theta$	$\sqrt{b^2x^2 - a^2} \Rightarrow x = \frac{a}{b}\sec\theta$	$\sqrt{a^2 + b^2x^2} \Rightarrow x = \frac{a}{b}\tan\theta$
$\cos^2\theta = 1 - \sin^2\theta$	$\tan^2\theta = \sec^2\theta - 1$	$\sec^2\theta = 1 + \tan^2\theta$

- Always remember to look at the denominator of a fraction to know which trig function to substitute



- Improper Integrals:
  - Definition of Proper Integrals with Infinite Integration Limits
    - 1. If f is continuous on the interval [a ∞), then
      - $\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx$.
    - 2. If f is continuous on the interval (-∞, b], then
      - $\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx$.
    - 3. If f is continuous on the interval (-∞, ∞), then
      - $\int_{-\infty}^{\infty} f(x)\,dx = \lim_{a\to-\infty}\int_a^c f(x)\,dx + \lim_{b\to\infty}\int_c^b f(x)\,dx$, where c is any real number.
    - In the first two cases, the improper integral converges if the limit exists--otherwise, the improper integral diverges. In the third case, the improper integral on the left diverges if either of the improper integrals on the right converges.
  - Definition of Improper Integrals with Infinite Discontinuities
    - 1. If f is continuous on the interval $[a, b)$ and has an infinite discontinuity at b, then
      - $\int_a^b f(x)\,dx = \lim_{c\to b^-}\int_a^c f(x)\,dx$.
    - 2. If f is continuous on the interval $(a, b]$ and has an infinite discontinuity at a, then
      - $\int_a^b f(x)\,dx = \lim_{c\to a^+}\int_c^b f(x)\,dx$.
    - 3. If f is continuous on the interval $[a, b]$,except for some c in (a,b) at which f has an infinite discontinuity, then
      - $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$.

- In the first two cases, the improper integral **converges** if the limit exists--otherwise, the improper integral **diverges**. In the third case, the improper integral on the left diverges if either of the improper integrals on the right diverges.

10.2 Working with Geometric Series

$$\sum_{n=1}^{\infty} a_1(r)^{n-1} \text{ converges if and only if } -1 < r < 1$$

$$\sum_{n=1}^{\infty} a_1(r)^{n-1} = \frac{a_1}{1-r}$$

Derived from taking $\lim_{n \to \infty}$ of S_n

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

10.5 Harmonic Series and p-Series

The p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots$$

converges for p > 1 and diverges for p ≤ 1

Special Case:

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

Harmonic series

The Harmonic series (p=1) diverges

Even though the $\lim_{n\to\infty}$ 1/n = 0, ∑1/n still diverges as the pieces still add up to non-negligible amounts which eventually add up infinity (it approaches 0 at too slow a rate)

10.7 Alternating Series Test for Convergence

Let $a_n > 0$. The alternating series

$$\sum_{n=1}^{\infty} (-1)^n a_n \quad \text{and} \quad \sum_{n=1}^{\infty} (-1)^{n+1} a_n$$

converge when these two conditions are met.

1. $\lim_{n \to \infty} a_n = 0$
2. $a_{n+1} \leq a_n$, for all n

<u>Alternating Series Remainder:</u>

Error = | S - S_n |

If a convergent alternating series satisfies the condition $a_{n+1} \leq a_n$, then the absolute value of the remainder R_N involved in approximating the sum S by S_N is less than (or equal to) the first neglected term. That is,

$$|S - S_N| = |R_N| \leq a_{N+1}.$$

Alternating Series Error ≤ abs value of First Term left off the series

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10.8 Ratio Test for Convergence

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$$

$$\text{If } L < 1\text{, then } \sum_{n=1}^{\infty} a_n \text{ is } \textit{absolutely convergent}$$

$$\text{If } L > 1 \text{ or } \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty \text{ then } \sum_{n=1}^{\infty} a_n \text{ is } \textit{divergent}$$

$$\text{If } L = 1\text{, then } \textit{inconclusive}$$

If asked to find an interval, REMEMBER to check endpoints!!

Example Problem:

Example 1: Find the radius of converge, then the interval of convergence, for $\sum_{n=1}^{\infty}(-1)^n\frac{n^2x^n}{2^n}$.

$$\sqrt[n]{\left|\frac{n^2x^n}{2^n}\right|} = \sqrt[n]{n^2}\,\frac{|x|}{2} \longrightarrow \frac{1}{2}|x|$$ (We used our very handy previous result: $\sqrt[n]{n^a} \to 1$ for any $a > 0$.)

By the Root Test, our series converges when $\frac{1}{2}|x| < 1$, i.e. when $|x| < 2$, so $R = 2$. Now we check the endpoints of the interval from -2 to 2.

When $x = -2$, we have $\sum_{n=1}^{\infty}(-1)^n\frac{n^2x^n}{2^n} = \sum_{n=1}^{\infty}(-1)^n\frac{n^2(-2)^n}{2^n} = \sum_{n=1}^{\infty}\frac{n^2(2)^n}{2^n} = \sum_{n=1}^{\infty}n^2$, which diverges by the Divergence Test.

When $x = 2$, we have $\sum_{n=1}^{\infty}(-1)^n\frac{n^2x^n}{2^n} = \sum_{n=1}^{\infty}(-1)^n\frac{n^2(2)^n}{2^n} = \sum_{n=1}^{\infty}(-1)^n n^2$ which also diverges by the Divergence Test.

$R = 2$ and our interval of convergence is $(-2, 2)$.

Note: When you are conditionally convergent, you are on the endpoints of the interval of convergence

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10.11 Finding Taylor Polynomial Approximations of Functions

Taylor Polynomial approximation at x=c:

$$P_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \ldots + \frac{f^n(c)}{n!}(x-c)^n$$

Kth Taylor Polynomial of f(x) centered at x=c:

$$\sum_{n=0}^{k} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

<u>Common MacLaurin Expansions</u>:

Function	Taylor Polynomial	Sigma Notation	Center
e^x	$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!}$	$\sum_{k=0}^{n} \frac{x^k}{k!}$	0
$\ln x$	$(x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots + \frac{(-1)^{n+1}(x-1)^n}{n}$	$\sum_{k=1}^{n} \frac{(-1)^{k+1}(x-1)^k}{k}$	1
$\sin x$	$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!}$	$\sum_{k=0}^{n} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$	0
$\cos x$	$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{(-1)^n x^{2n}}{(2n)!}$	$\sum_{k=0}^{n} \frac{(-1)^k x^{2k}}{(2k)!}$	0
$\frac{1}{1-x}$	$1 + x + x^2 + x^3 + \cdots + x^n$	$\sum_{k=0}^{n} x^k$	0

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Series Convergence/Divergence Flow Chart