AP CALCULUS REVIEW

TOPICS, METHODS AND TIPS FOR THE AP EXAM

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EFFECTIVE 2020

Suggested Music For Reading:

Schumann's Kinderszenen Op. 15

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Unit #00: Math Language Guide

Logic Symbols:

- $\exists$ - "There exists…"
- $\forall$ - "For all…"
- $\in$ - "Be an element of…"
- $\Rightarrow$ - "Implies…"
- $\ll$ - "Much less than…"
- $\gg$ - "Much greater than…"
- ~ - "Not…"
- $\therefore$ - "Therefore…"
- $\because$ - "Because… / Since…"
- s.t. - "Such that…"

Calculus Symbols:

∞ - "Infinity"

y' - "First derivative"

y'' - "Second derivative"

y''' - "Third derivative"

$y^{(n)}$- "nth derivative"

$\frac{dy}{dx}$- "First derivative"

$\frac{d^2y}{dx^2}$- "Second derivative"

$\frac{d^3y}{dx^3}$- "Third derivative"

$\frac{d^{(n)}y}{dx^{(n)}}$- "nth derivative"

$\int$ - "Integral"

ε - "Epsilon," a number very near zero, $\varepsilon \to 0$

e - "Euler's Number," 2.718281828…, $\lim_{x\to\infty}\left(1 + \frac{1}{x}\right)^x$

$\sum$ - "Sigma, the sum of…"

N - "Natural numbers," (0,1,2,3,4,5…)

Q - "Rational numbers," (Numbers that can be created by dividing two integers) **R** - "Real numbers," (All whole, rational, and irrational numbers)

Z - "Integer numbers," (…-3,-2,-1,0,1,2,3…)

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Unit #01: Limits and Continuity

What is a **limit**?

A limit is the value of a function as it approaches some value from one or both sides.

$\lim_{x\to a} f(x)$	"a" is some value "f(x)" is some function		$\lim_{x\to a^+} f(x)$	$\lim_{x\to a^-} f(x)$	$x \to a^+$ From the positive side $x \to a^-$ From the negative side

What is **continuity**?

Continuity is a region of a function with no abrupt changes or jumps.

Continuity @ $x = c$ for $f(x)$
: - $\lim_{x \to c} f(x) = L$
- $f(c)$ *exists*
- $\lim_{x \to c} f(x) = f(c)$

The **limit interchange property** permits us to switch any operation in and out of the limit. For example, the square of the limit of f(x) is equal to the limit of the square of f(x).

Methods of Solving

FIRST. Identify if you can plug in the limit value. For a quick answer, look at the variable with the highest power. We can prove this by using L'Hôpital's rule and by multiplying the numerator and denominator by the reciprocal of the variable with the highest power.

L'Hôpital's rule is applicable in an infinity over infinity senario. If applicable, we can take the derivative of both the numerator and denominator independently, put them together, and evaluate the limit again.

L'Hôpital's rule	Multiplying the reciprocal
$\lim_{x\to\infty} \frac{x^3 - 7x^2 + 9x - 3}{x^2 - 4x + 3}$ $\lim_{x\to\infty} \frac{3x^2 - 14x + 9}{2x - 4}$ $\lim_{x\to\infty} \frac{6x - 14}{2}$ $= \infty$	$\lim_{x\to\infty} \frac{x^3 - 7x^2 + 9x - 3}{x^2 - 4x + 3}$ $\lim_{x\to\infty} \frac{x^3 - 7x^2 + 9x - 3}{x^2 - 4x + 3} \times \frac{\frac{1}{x^3}}{\frac{1}{x^3}}$ $\lim_{x\to\infty} \frac{1 - \frac{7}{x} + \frac{9}{x^2} - \frac{3}{x^3}}{\frac{1}{x} - \frac{4}{x^2} + \frac{3}{x^3}} = \infty$

Factoring Common Denominator Multiplying by Conjugate

$\lim_{x\to3}\frac{x^2-2x-3}{x^2-4x+3}$ $\lim_{x\to3}\frac{(x-3)(x+1)}{(x-3)(x-1)}$ $\lim_{x\to3}\frac{x+1}{x-1}$ $\frac{4}{2}=2$	$\lim_{x\to4^+}\left(\frac{1}{x-4}-\frac{6}{x^2-2x-8}\right)$ $\lim_{x\to4^+}\left(\frac{1}{x-4}\times\frac{(x+2)}{(x+2)}-\frac{6}{(x-4)(x+2)}\right)$ $\lim_{x\to4^+}\frac{(x+2)-6}{(x-4)(x+2)}$ $\lim_{x\to4^+}\frac{(x-4)}{(x-4)(x+2)}$ $\lim_{x\to4^+}\frac{1}{(x+2)}$ $\frac{1}{6}=0.167$	$\lim_{x\to0}\frac{1-\cos(x)}{x}$ $\lim_{x\to0}\frac{1-\cos(x)}{x}\times\frac{1+\cos(x)}{1+\cos(x)}$ $\lim_{x\to0}\frac{1-\cos^2(x)}{x(1+\cos(x))}$ $\lim_{x\to0}\frac{1-\cos^2(x)}{x(1+\cos(x))}$ $\lim_{x\to0}\frac{\sin^2(x)}{x(1+\cos(x))}$ $\lim_{x\to0}\frac{\sin(x)}{x}\times\lim_{x\to0}\frac{\sin(x)}{1+\cos(x)}$ $1+0=1$
It's sometimes possible to cancel out a common factor. Nice.	It's sometimes possible to combine certain fractions by multiplying the less complex term by a special version of one, then solving.	It's sometimes possible to simplify a limit by multiplying by the numerator or denominator's conjugate, and applying a trigonometric identity.

U Substitution Special Cases

$\lim_{x\to2}\frac{\sin(3(x^2-4))}{x-2}$ $\lim_{x\to2}\frac{\sin(3(x-2)(x+2))}{(x-2)}$ $\lim_{x\to2}\frac{\sin(3(x-2)(x+2))}{(x-2)}\times\frac{3(x+2)}{3(x+2)}$ Let $u=3(x-2)(x+2)$ $x\to\infty\Rightarrow u\to\infty$	$\lim_{x\to0}\frac{\sin(x)}{x}=1$ $\lim_{x\to0}\frac{1-\cos(x)}{x}=0$ $\lim_{x\to0^+}(1+x)^{\frac{1}{x}}=e$

$\lim_{x\to 2} \frac{\sin(u)}{u} \times \frac{1}{3(x+2)}$ $\lim_{x\to 2} \frac{\sin(u)}{u} \times \lim_{x\to 2} \frac{1}{3(x+2)}$ $= 1 \times 12 = 12$	
It’s sometimes possible to simplify a limit by substituting u for a more complicated expression, to make it look like a special case, or to just make the limit more readable. Just remember that syntax!	Each of these limits have proofs, one of which is the example for “Multiplying by Conjugate.” We don’t have to constantly reinvent the wheel. Save everyone’s time, use the special cases.

Unit #02: Differentiation: Definition and Fundamental Properties

What is a **derivative**?
A derivative is the rate of change of a function. Technically described below.

$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$	“h” is some value “f(x)” is some function

To be differentientiable, a function must be smooth, and continuous with no cusps. This includes holes (removable discontinuities), function breaks (step discontinuities), and asymptotes. Differentiability implies continuity, not the other way around.

Basic Derivative Rules

// “c” = some constant // “u” and “v” = some functions // “n” = some number //

Constant Rule	Power Rule	Product Rule	Quotient Rule
$\frac{d}{dx}[cu] = cu'$	$\frac{d}{dx}[u^n] = nu^{n-1} \times u'$	$\frac{d}{dx}[uv] = uv' + vu'$	$\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{vu' - uv'}{v^2}$

Constants	Trivial Rule	Addition Rule	Absolute Value Rule
$\frac{d}{dx}[c] = 0$	$\frac{d}{dx}[x] = 1$	$\frac{d}{dx}[u \pm v] = u' \pm v'$	$\frac{d}{dx}[\lvert u \rvert] = \frac{u}{\lvert u \rvert} \times u'$

Log Rule	Natural Log Rule	e's Rule
$\frac{d}{dx}[\log_c u] = \frac{1}{ln(c) \times u} \times u'$	$\frac{d}{dx}[ln(u)] = \frac{1}{u} \times u'$	$\frac{d}{dx}[e^u] = e^u \times u'$

Complex Exponent Rule
$\frac{d}{dx}[c^u] = ln(c) \times c^u \times u'$

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Special Derivatives

Regular Trigonometric	Inverse Trigonometric
$\frac{d}{dx}[sin\ u] = (cos\ u) \times u'$ $\frac{d}{dx}[cos\ u] = (sin\ u) \times u'$ $\frac{d}{dx}[tan\ u] = (sec\ u)^2 \times u'$ $\frac{d}{dx}[cot\ u] = (csc\ u)^2 \times u'$ $\frac{d}{dx}[sec\ u] = (sec\ u\ tan\ u) \times u'$ $\frac{d}{dx}[csc\ u] = (csc\ u\ cot\ u) \times u'$	$\frac{d}{dx}[arcsin\ u] = \frac{1}{\sqrt{1-u^2}} \times u'$ $\frac{d}{dx}[arccos\ u] = \frac{1}{\sqrt{1-u^2}} \times u'$ $\frac{d}{dx}[arctan\ u] = \frac{1}{1+u^2} \times u'$ $\frac{d}{dx}[arccot\ u] = \frac{1}{1+u^2} \times u'$ $\frac{d}{dx}[arcsec\ u] = \frac{1}{\ {}^2} \times u'$

	$\frac{d}{dx}[arccsc\ u] = \frac{-1}{\lvert u\rvert\sqrt{u^2-1}} \times u'$

Unit #03: Differentiation: Composite, Implicit, and Inverse Functions

What is **composite differentiation**?

A fancy way of describing taking a derivative of a function in a function.

$$\frac{d}{dx}[g(f(x))] = g'(f(x)) \times f'(x)$$

What is **inverse differentiation**?

Taking the derivative of an inverse function. Described below.

$$[f^{-1}(b)]' = \frac{1}{f'(a)} \text{ for } (a, b) \text{ on } f(x)$$

What is **implicit differentiation**?

Taking the derivative with variables on both sides of the equation, leading to derivatives on both sides of the equation.

$$\frac{d}{dx}[f(x, y)]$$

Composite	Inverse	Implicit
$f(x) = 4x^2 - 1$ & $g(x) = 12x^3$ What is $f(g(x))'$?	$f(x) = \frac{1}{2}x^3 + 3x - 4$ $h(x) = f^{-1}(x)$	$\cos(x^2) = xe^y$ $-\sin(x^2) \times 2x = e^y + x(e^y) \times y'$

$f(g(x)) = 4(12x)^{3}_{2}\ 1$ $f(g(x)) = 576x^6\ 1$ $f(g(x))' = 3456x^5$	$f(2) = 14$ $Find : h'(14)$ $h'(x) = \frac{1}{f'(h(x))}$ $h'(14) = \frac{1}{f'(2)}$ $f'(x) = x\ 3\ {}_2 3^2 +$ $f'(2) = 9$ $h'(14) = {}_9 1$	$\frac{sin(x) \times 2x\ e}{x \times e^y}$ y ${}^2 + {}^2 =$ $x\ y\ xy\ 3x$ ${}^2 + {}^2 = \frac{d}{dx}$ $\frac{d}{dx}[x\ y\ xy]\ [3x]$ ${}^{2\prime} + + {}^2 + {}' =$ $x\ y\ 2xy\ y\ 2xyy\ 3\ {}^{2\prime} + {}' = {}^2$ $x\ y\ 2xyy\ 3\ 2xy\ y\ {}^2 + = {}^2$ $y'(x\ 2xy)\ 3\ 2xy\ y\ \underline{3\ 2xy\ y^2}$ $y' = x + 2xy_2$

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Unit #04: Contextual Applications of Differentiation

Optimization

$L = L\ L_1 + {}_2(L)\ 6\ x\ {}_1{}^2 = {}^2 + {}^2(L)\ (20)\ 15\ {}_2{}^2 = x^2 + {}^2$

${}^2 + {}^2 L_2 = \sqrt{(20\ x)\ 15}$

${}^2 + {}^2$

$L = total\ length\ L_1 = \sqrt{6}\ x$

$L = \sqrt{(6)}\ x$

${}^2 + {}^2 + \sqrt{(20\ x)\ 15}$

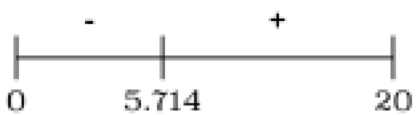
${}^2 + {}^2$

$\frac{1}{2}$

${}_2 \times 2 + \frac{1}{}$

$L' = x\ 2\ 20\ )\ )$

${}_2 \times (\times (\ x \times 1$



$2\sqrt{(6) + x}$

$2\sqrt{(20x) + 15}$

Try to set up an equation with what you're solving for on one side, and an expression with just one variable on the other. Sign test the derivative, and see where the possible minimums are (or maximums if the problem calls for it). The first time around, the trick is seeing how we can relate two distances with a single variable (20-x) in this case.

Related Rates

$V = \frac{1}{3}\pi r^2 h$

$V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h$

$V = \frac{1}{12}\pi h^3$

$\frac{dV}{dt} = \frac{1}{12} \times 3\pi h^2 \times \frac{dh}{dt}$

$\frac{\frac{dV}{dt}}{\frac{1}{12}\pi \times 3h^2} = \frac{dh}{dt}$

$\frac{30\ ft^3/min}{\frac{1}{12}\pi \times 3(10\ ft)^2} = \frac{dh}{dt}$

$0.382\ ft/min\ @\ 10\ ft = \frac{dh}{dt}$

Related rates is like optimization in that you have to rearrange the equation to eventually lead you to your givens. In this problem, we were given V' and h, so we had to develop an expression with V', h, and h' in order to get our final answer. We did that by first creating a descriptive equation, then taking the derivative in respect to a common variable to relate everything together.

Curve Sketching

Description of Process

At the function level			
Zeros	End Behavior	Asymptotes	Y Intercept
When f(x) = 0	Limits as x→∞ and x→-∞	When f(x) is undefined	f(0)
At the first derivative level		At the second derivative level	
Relative Extrema		Points of Inflection	
When f'(x) = 0	Graph looks like a mound	The concavity changes	When f''(x) = 0
Remember to include all the critical values in the sign tests, when the function is undefined and at the zeros. Also- remember to include endpoints.			

Example

$f(x) = x^{5/3} + 5x^{2/3}$

$\lim_{x\to\infty} f(x) = \infty$

$\lim_{x\to\infty} f(x) = \infty$

Zero at $x = 0$, *multiplicity* 2. *Zero at* $x = 5$, *multiplicity* 1. $f(x) = x^{2/3}(x\ 5)$

$f'(x) = \frac{5}{3}x^{2/3} + \frac{10}{3}x^{1/3}$

$f'(x) = \frac{5}{3}x^{1/3}(x\ 2)$

$x = 0$, $x = 2$

$f''(x) = \frac{10}{9}x^{1/3} + \frac{10}{9}x^{4/3}$

$f''(x) = \frac{10}{9}x^{4/3}(x\ 1)$

$x = 0$, $x = 1$

Remember to pull out the smallest exponent if applicable. Pay attention to the multiplicities of the zeros, and values in your sign tests (make sure to put all critical values!).

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Newton’s Method

$x^3 = x^2$

Use Newton's Method to estimate the point of intersection of $f(x) = e$ and $g(x)$ with an initial guess at x .500. Apply three iterations and include a sketch. $_0 = 0$

$f(x)\frac{0}{} + x_0 =$

x

$f'(x)_0$

$x^3 = ^2$

$e\ x$

$x^3\ x^2 =$

$e\ 0$

$^2 \times e^{x_3} =$

$3x\ 2x\ 0$

$x^3\ 2$

$f1 = x\ \underline{e\ x}$

$2\ x^3$

$3x \times e\ \ 2x$

$f1(0.500) = 0.881$

$f1(Ans) = 0.788$

$f1(Ans) = 0.785$

(0.785, ?)

? = f(0.785) = $(0.785)^2$

? = 0.616

(0.785,0.616)

Newton’s method is an antiquated method for approximating a function’s x intercept by picking a random value, and using the function’s derivative to get closer and closer to the zero. While the method may fall apart in certain circumstances, it’s still a really neat concept.

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Unit #05: Analytical Applications of Differentiation

Theorems

Intermediate Value Theorem (IVT)

If the function f(x) is continuous on the interval [a,b], then for any number c between f(a) and f(b), there exists a value d in the open interval (a,b) such that f(d) = c.

Prove $5x^3$-6x-1 has at least 1 zero.
Since $5x^3$-6x-1 is a polynomial, it is continuous, therefore we can apply the IVT.
f(0)=-1, f(2)=27, and -1<0<27 therefore there exists a value in the element of (0,2) such that f(value)=0

Rolle's Theorem

If the function f(x) is continuous on the interval [a,b], the first derivative exists on the interval (a,b), and f(a) = f(b); then there exists a number x = c on (a,b) such that f'(c) = 0.

^{4} 2^2 2 2
Find all c values such that $f'(c) = 0$ for $f(x) = x\ x$ on (,)
^{3} 4 = 4^3 4 = 4 1 + 1 = 0 1
$f'(x) = 4x\ x\ 0\ x\ x\ 0\ x(x\ )(x\ )\ c\ ,\ 1,$

Mean Value Theorem

If the function f(x) is continuous on the interval [a,b], and the first derivative exists on the interval (a,b), $\frac{f(b)-f(a)}{b-a}$
then there exists a number x = c on (a,b) such that $f'(c) =$.

If the function f(x) is continuous on the interval [a,b], and the first derivative exists on the interval (a,b), $\frac{\int f(x)\,dx}{b-a}$
then there exists a number x = c on (a,b) such that $f(c) =$. Where f(c) is the average value of the function on the interval [a,b].

$\frac{f(4)-f(1)}{4-1}$
^{4} 4 ' =
For $f(x) = 5$, find all values of c in the open interval $(1,)$ *s.t.* $f\ (c)$
x
$\frac{2}{x}$ = $\frac{4}{x}$ 2^2 = 4 = 2 = 2 4
4
$f'(c) = 1\ f'(c) = 1\ x\ x \pm c$ *on* $(1,)$

What is a **differential**?
A differential is just a method of estimating a function without the function itself.

Euler's Method

$y_n = y_{n-1} + hf'(x_{n-1}, y_{n-1})$	Where h is some interval on the x axis, y_n is the new estimated value on the function, y_{n-1} is the old estimated value on the function.

Example

Consider the differential equation : $\frac{dy}{dx} = 3x\ y^2$

Let $y = g(x)$ *be a solution to the differential equation with the initial condition* $g(0) =$ 10. *Starting at* $x = 0$ *with a step size of* 1, *approximate* $g(2)$.

$y\ hf\ (x\ ,\ )\ _n = y_{n\ 1} + '\ _{n\ 1} y_{n\ 1}$

$y\ y\ 3x\ y\ _n = _{n\ 1} + _{n\ 1}\ 2\ _{n\ 1}$

y (10) 3(0) (10) 0 (1, 0) $_1$ = + 2 = 1 1

y (0) 3(1) (0) 13 (2, 3) $_2$ = 1 + 2 1 = 1

This method becomes more and more accurate the smaller the step size, since there is less overshooting with the derivative lines.

Unit #06: Integration and Accumulation of Change

What is an **integral**?

An integral is the antiderivative of a function.

$\int F(x) = f(x)$	"f(x)" is some function whose derivative is "F(x)"

What is a **bounded integral**?

A bounded integral is the area under the curve between the specified values. From the FTC, we can declare:

$\int_a^b F(x) = f(b)\ f(a)$	"f(x)" is some function whose derivative is "F(x)" "a" and "b" are x values

Basic Integration Rules

$\int_a^a f(x)\,dx = 0$	$\int dx = x + C$	$\int_a^b f(x)\,dx = \int_b^a f(x)\,dx$	$\int_a^b f(x)\,dx = \int_b^a f(x)\,dx$

$\int [k \times f(x)]\,dx = k \times \int f(x)\,dx$	Fundamental Integral -~- Operations -~-	$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

$\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$	$\int_a^b f(x)\,dx + \int_a^b g(x)\,dx = \int_a^b [f(x) + g(x)]\,dx$

$\int u^n\,dx = \frac{u^{n+1}}{n+1} + C$	$\int \frac{1}{x}\,dx = \ln\lvert x\rvert + C$	$\int e^x\,dx = e^x + C$	$\int k^x\,dx = \frac{k^x}{\ln(k)} + C$

$If\ f(x)\ is\ even: \int_a^a f(x)\,dx = 2\int_0^a f(x)\,dx$	$If\ f(x)\ is\ odd: \int_a^a f(x)\,dx = 0$

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Special Integrals

$\int sin\ (u)\ dx =\ \ cos\ (u) + C$ $\int cos\ (u)\ dx = sin\ (u) + C$ $\int tan\ (u)\ dx =\ \ ln\ \mid cos\ (u) \mid + C$ $\int csc\ (u)\ dx =\ \ ln\ \mid csc\ (u) + cot\ (u) \mid + C \int sec$ $(u)\ dx = ln\ \mid sec\ (u) + tan\ (u) \mid + C \int cot\ (u)$ $dx = ln\ \mid sin\ (u) \mid + C$ $\int sec^2(u)\ dx = tan\ (u) + C$ $\int csc^2(u)\ dx =\ \ cot\ (u) + C$ $\int sec\ (u)\ dx\ [sec\ (u)\ an\ (u)$ $^3 = {}_2\frac{1}{} \times t +$ $ln\ \mid sec\ (u) \times tan\ (u) \mid] + C$	$\int$-$du\ arcsin$ (-) C $_{2\ 2} = {}_a\frac{u}{} +$ $\frac{1}{}$ $\sqrt{a}\ \ u$ $_{2\ 2} = {}_a\frac{1}{} \times a\ {}_a\frac{\mid u \mid}{} +$ $\int$-$du\ rcsec$ (-) C $\frac{1}{}$ $u\sqrt{a}\ \ u$ $\int$-$du\ rctan$ (-) C $_{2\ 2} = {}_a\frac{1}{} \times a\ {}_a\frac{u}{} +$ $\frac{1}{}$ $a\ \ u$ $\int$-$du\ arcsin\ u\ C$ $\frac{1}{}$ $\sqrt{1-u^2} = +$ $\int\ du\ arccos\ u\ C$ $\frac{1}{}$ $\sqrt{1-u^2} = +$ $\int$-$du\ arctan\ u\ C$ $\frac{1}{}$ $1 + u^2 = +$ $\int\ du\ arccot\ u\ C$ $\frac{1}{}$ $1 + u^2 = +$ $\int$-$du\ arcsec\ u\ C$ $\frac{1}{}$ $_2 = +$ $\mid u \mid \sqrt{u-1}$ $\int\ du\ arccsc\ u\ C$ $\frac{1}{}$ $_2 = +$ $\mid u \mid \sqrt{u-1}$

What is an **improper integral**?

An improper integral is a bounded integral where either or both bounds are infinite, or the graph is undefined somewhere between its bounds: [a,b].

$\int_{2}^{5} \frac{1}{\sqrt{x-2}} dx$ $\lim_{b\to 2^+} \int_b^5 \frac{1}{\sqrt{x-2}} dx$ $u = x \;\; 2$, $du = dx$ $\lim_{b\to 2^+} \int_b^5 u^{1/2} du$ $\lim_{b\to 2^+} 2\sqrt{x \;\; 2}\,\Big\vert_b^5$ $= 2\sqrt{3} \;\; 0$	We solve these types of problems by splitting up the larger integral into integrals such that the undefined point is at one of the bounds, then use limit notation to solve the integral. In the example to the left, the integral already had the undefined point at its lower bound, so splitting up the integral was not needed.

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Methods of Solving

What is **U Substitution**?

Definition

$\int f(x) \times f'(x)\, dx$ $u = f(x)$ $du = f'(x)\, dx$ $\int u\, du$	Where the function f(x)'s derivative is f'(x). We MUST make sure that either the whole integral is in terms of a single variable. In this case, u or x.

Example

$\int \frac{x+5}{\sqrt{9 \;\; (x \;\; 3)^2}} dx$

$\frac{1}{3}\int \frac{x+5}{\sqrt{1 \;\; (\frac{x \;\; 3}{3})^2}} dx$

$u = \frac{x \;\; 3}{3}$

$du = \frac{1}{3} dx$

$3u + 3 = x$

$\int \frac{3u+8}{\sqrt{1 \;\; (u)^2}} du$

$3\int \frac{u}{\sqrt{1 \;\; (u)^2}} du \;\; 8 \int \frac{1}{\sqrt{1 \;\; (u)^2}} du$

$w = 1 \;\; u^2$

$dw = \;\; 2u \; du$

$du \;\; 8 sin \; (u) \; C \;\; \frac{3}{2}\int \frac{1}{\sqrt{w}} \;\; \frac{1}{\frac{1}{1}+1} + 1$

$3\sqrt{w} + C_2 \;\; 8 sin \; (u) \; C$

$3\sqrt{1 \;\; (\frac{x \;\; 3}{3})^2} + C_2 \;\; 8 sin \; (\frac{x \;\; 3}{3}) \; C$

U substitution is extraordinarily powerful in most of the integral situations we encounter. By substituting u for some kind of algebraic expression, we can simplify the problem, we just need to make sure that the entire integral is in terms of a single variable. In the problem above, it was necessary to substitute u in the beginning, rearrange the first u-sub equation to be able to substitute for the x in the numerator, and perform the same u-sub steps in the end with a w instead of a u.. I prefer to pronounce “w” “double u” in these circumstances. It’s hilarious.

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What is **By Parts Integration**?

The Long Way

$uv \int v\, du = \int u\, dv$	Where the function y = uv.

Rapid Repeating

u			dv	
↓	↘ +		↓	Where the function y = uv. You take the antiderivative down the right side, and take the derivative down the left side. Continue until the final term in the u column is zero. You can split up any product in the u and dv terms, and multiply it across the center line. Denoted by a dashed line, you should never multiply across it. This method is more clear in the example form below.
↓	↘ -	↘ +	↓	
↓	↘ +	↘ -	↓	
↓		↘ +	↓	

How do you know which function you should choose to be u?

	ı =	**L**og	**I**nverse	**A**lgebraic	**T**rigonometric	**E**xponential	It's recommended that we follow this chart from left to right until we reach the function type present in the integral.

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Examples

Rapid Repeating	The Long Way

<table>
<tr>
<td>$\int x^5 e^{x^2} dx$

u dv
$x^5 \quad e^{x^2}$
$x^4 \searrow^{+} xe^{x^2}$
$4x^3 \searrow^{+} \frac{1}{2}e^{x^2}$
$4x^2 \searrow^{-} \frac{1}{2}xe^{x^2}$
$8x \searrow^{-} \frac{1}{4}e^{x^2}$
$8 \searrow^{+} \frac{1}{4}xe^{x^2}$
$0 \searrow^{+} \frac{1}{8}e^{x^2}$

$x^4 \times \frac{1}{2} e^{x^2} \; x^2 \times e^{x^2} + e^{x^2}$</td>
<td>$\int x^2 e^x dx$

$u = x^2 \quad v = e^x$
$\downarrow \uparrow$
$du = 2x\,dx \quad dv = e^x dx$

$x^2 e^x \; \int 2xe^x dx$
$u = 2x \quad v = e^x$
$\downarrow \uparrow$
$du = 2\,dx \quad dv = e^x dx$

$x^2 e^x \; [2xe^x \; \int 2e^x dx]$
$x^2e^x \; 2xe^x + 2e^x + C$</td>
</tr>
<tr>
<td>The dashed line marks the reassociation. The sloped arrows show the terms that are multiplied together. The signs are alternating downward, starting with +. Never multiply across a reassociation line.</td>
<td>Rapid repeating integration is often more efficient, but this method has its place in more complex calculations. Remember to use the FTC on the whole second expression if the integral is bounded, not just the other integral.</td>
</tr>
</table>

Cyclical By Parts

<table>
<tr>
<td>$\int e^{2x} \sin(x)\, dx$

u dv
$\sin(x) \searrow^{+} e^{2x}$
$\cos(x) \searrow^{-} \searrow^{+} \frac{1}{2}e^{2x}$
$\sin(x) \searrow^{-} \frac{1}{4}e^{2x}$

$\frac{1}{2}e^{2x} \times \sin(x) \; \frac{1}{4}e^{2x} \times \cos(x)$</td>
<td>The sloped arrows designate what is multiplied together.

The signs are alternating downward, starting with +.

In cyclical questions, we stop taking the derivative on the left when the last term is similar to the first term.

In this case: -sin(x) and sin(x).</td>
</tr>
</table>

/

Trigonometric Substitution

$\int e\,dx$

$^{x}\sqrt{1}\ \ e^{2x}$

$_{x}\sqrt{1\ \ (e\ )x}_{2}$

$\int e\,dx$

$sin\ \theta = \sqrt{1\ \ (e\ )x}_{2}$

$cos\ \theta = e^{x}$

$sin\ \theta\ d\theta = e\ dx^{x}$

$_{2} = \int\!-_{2}$

$\int sin\ \theta\ d\theta\ d\theta$

$\underline{1\ \ cos\ 2\theta}$

$sin\ 2\theta\ \theta\ C\ _{4}\frac{1}{}\ \ _{2}\frac{1}{} +$

$sin\ \theta\ os\ \theta\ \theta\ C\ _{2}\frac{1}{} \times c\ \ _{2}\frac{1}{} +$

$(\ )\ cos\ (e\ )\ C\ _{2}\frac{1}{}\sqrt{1\ \ (e\ )x}_{2} \times e^{x}\ \ _{2}\underline{1}\ _{1}\ ^{x} +$

Trigonometric substitution is handy when dealing with annoying and/or complicated trapped functions inside of an integral. In this case, the square root made this integral particularly nasty, yet it was DEFEATED…by trig sub.

Partial Fractions

$\underline{px \pm q}$

$\int\!-dx\ dx$

$(x \pm a)(x \pm b) = \int\underline{A}$

$(x \pm a) + \frac{B}{(x \pm b)}$

$\underline{px \pm q}$

$\int\!-dx\ dx$

$(x \pm a)^{2} = \int\underline{A}$

$(x \pm a) + \frac{B}{(x \pm a)^{2}}$

$\underline{px^{2} + qx + r}$

Where a ≠ b, and A, B, and C are constants. p, q, r, a, b, and c are all constants as well. The various rules are listed to the left. Cover all your bases.

<table>
<tr>
<td>

$$\int \frac{px^2+qx+r}{(x \pm a)(x \pm b)(x \pm c)}\,dx = \int \frac{A}{(x \pm a)} + \frac{B}{(x \pm b)} + \frac{C}{(x \pm c)}\,dx$$

$$\int \frac{px^2+qx+r}{(x \pm a)^2 (x \pm b)}\,dx = \int \frac{A}{(x \pm a)} + \frac{B}{(x \pm a)^2} + \frac{C}{(x \pm b)}\,dx$$

$$\int \frac{px^2+qx+r}{(x \pm a)(x^2 + bx + c)}\,dx = \int \frac{A}{(x \pm a)} + \frac{Bx + C}{(x^2 + bx + c)}\,dx$$

</td>
<td></td>
</tr>
<tr>
<td>

$$\int \frac{px+q}{(x \pm a)(x \pm b)}\,dx = \int \frac{A}{(x \pm a)} + \frac{B}{(x \pm b)}\,dx$$

$$A = \frac{p(\pm a) + q}{((\pm a) \pm b)} \quad B = \frac{p(\pm b) + q}{((\pm b) \pm a)}$$

</td>
<td>Heaviside's Method dictates that we should input an x value (that makes one of the factors on the denominator zero) into the whole fraction, and that the result will be the numerator to that factor's denominator in the final result.</td>
</tr>
</table>

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Example

$$\int \frac{5x^2 + 20x + 6}{x^3 + 2x^2 + x}\,dx$$

$$\frac{5x^2 + 20x + 6}{x(x+1)^2} = \frac{A}{x} + \frac{B}{(x+1)} + \frac{Cx + D}{(x+1)^2}$$

$$^2 + 2 + 6 = + 1^2 + B + 1 + (+ D$$

$5x\ 0x\ A(x\ )\ x(x\ )\ Cx\ )x$
$^{2} + 2 + 6 =^{2} + 2 + A + (^{2} + B + (^{2} + D$
$5x\ 0x\ (Ax\ Ax\ )\ Bx\ x)\ Cx\ x)$
$^{2} + 2 + 6 = + B + C\ ^{2} + (+ B + D + ($
$5x\ 0x\ (A\ )x\ 2A\ )x\ A)$
$5 = A + B + C\ 20 = 2A + B + D\ 6 = A$

$8 = B + D\ \ 1 = B + C\ C = \ 1\ \ B$
$D = 8\ \ B$
$(A, B, C, D) = (6, B, 1\ \ B, 8\ \ B)$
$Let\ B = 0. \rightarrow (6, 0, 1, 8)$

A B C D Ans 1 1 1 0 5 2 1 0

1 20

1 0 0 0 6 rref(MATRIX) →

1 0 0 0 6 0 1 0 1 8 0 0 1 -1

-9

$^{2}= \frac{(6)}{x} + \frac{(0)}{}$

$\frac{(1)x + (8)}{}$

$\frac{5x + 20x + 6}{x(x+1)^2}$

$(x + 1) + (x + 1)^2$

$\frac{(6)}{} + \frac{(0)}{}$

$(1)\underline{x + (8)}$

$\int \text{-} dx$

$x \qquad (x + 1) + (x + 1)^2$

$\frac{(6)}{x} + \int \frac{8\ \ x}{(x+1)^2}$

$\int \text{-} dx\ dx$

$u = x + 1\ du = dx\ u\ \ 1 = x$

$\frac{8\ (u1)}{(u)^2}$

$6ln\ |x| + \int \text{-} du$

$6ln\ |x| + \int \text{-} dx\ du$

$u^{2}\ \ \int \text{-}_{u} \underline{1}$

2

$6\ln\lvert x\rvert \quad 9(\frac{1}{x+1})\ \ln\lvert x+1\rvert + C$
This problem is split between the matrix way and the algebraic way. In the algebraic way, in the end, make sure to put all the numerator constants in terms of a single quantity, then pick the most convenient number. In this case, zero is convenient because it eliminates one of the fractions we have to deal with. In the matrix way, we can get more equations, which can also be solved. In the case of many equations, the matrix method would be much more convenient than the algebraic way.

020

Weierstrass

$u = \tan(\frac{x}{2})$ $dx = \frac{2(du)}{1+u^2}$ $\sin(x) = \frac{2u}{1+u^2}$ $\cos(x) = \frac{1\ u^2}{1+u^2}$ $\sec(x) = \frac{1+u^2}{1\ u^2}$ $\tan(x) = \frac{2u}{1\ u^2}$ $\cot(x) = \frac{1\ u^2}{2u}$ $\csc(x) = \frac{1+u^2}{2u}$	For $\frac{1}{f(x)}$ where f(x) is a function that is at least part trigonometric.

Example

$$\int \frac{1}{2-3\sin(\theta)}\,d\theta$$

$$\int \frac{1}{2-3\left(\frac{2t}{1+t^2}\right)} \times \left(\frac{2}{1+t^2}\right) dt$$

$$\int \frac{2}{2+2t^2-6t}\,dt$$

$$\int \frac{1}{t^2-3t+1}\,dt$$

$$\int \frac{1}{t^2-3t+\frac{9}{4}-\frac{9}{4}+1}\,dt$$

$$\int \frac{1}{\left(t-\frac{3}{2}\right)^2-\frac{5}{4}}\,dt$$

$$\int \frac{1}{\left(t-\frac{3}{2}-\frac{\sqrt5}{2}\right)\left(t-\frac{3}{2}+\frac{\sqrt5}{2}\right)}\,dt$$

$$\int \frac{\frac{1}{\sqrt5}}{\left(t-\frac{3+\sqrt5}{2}\right)} + \frac{-\frac{1}{\sqrt5}}{\left(t-\frac{3-\sqrt5}{2}\right)}\,dt$$

$$\frac{1}{\sqrt5}\int \frac{1}{\left(t-\frac{3+\sqrt5}{2}\right)} - \frac{1}{\left(t-\frac{3-\sqrt5}{2}\right)}\,dt$$

$$\frac{1}{\sqrt5}\left[\ln\left|\frac{2t}{2}-\frac{3+\sqrt5}{2}\right| - \ln\left|\frac{2t}{2}-\frac{3-\sqrt5}{2}\right|\right] + C$$

$\sqrt{5} \times l \,\|\, 2t \;\; 3 + \sqrt{5}$ $\underline{2t \;\; 3 \;\; \sqrt{5}} \,\| + C$ 1 n $\sqrt{5} \times l \,\|\|\, 2tan(-) \;\; 3 + {}_{2}^{\theta} \sqrt{5}$ $2tan(-) \;\; 3 \;\; {}_{2}^{\theta} \sqrt{5} \,\|\| + C$ 1 n
While Weierstrass may seem complicated and hard to spell, this method provides a very algebraic way to solve fairly complicated and otherwise difficult to solve integrals. Woo-hoo!

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Unit #07: Differential Equations

What is a **differential equation**?

A differential equation is an equation with a derivative in it. The **general solution** to a differential equation has an unknown constant in it (a "C" for example). The **specific solution** does not have any unknowns.

Special Differential Equation Definitions

First Order Linear Non-Separable Differential Equation

$\frac{dy}{dx} + p(x)y = q(x)$ $v(x) = e^{\int p(x)\, dx}$ $y = \frac{1}{v(x)} \int v(x) \times q(x)\, dx$	In order to find the general solution for this differential equation, use the method outlined to the right. To find the specific solution, an initial condition is required.

Bernoulli Differential Equation

$\frac{dy}{dx} + P(x)y = Q(x)y^{n}$ $z = y^{1-n}$ $\frac{dz}{dx} = (1-n)y^{-n}y'$	The process is more direct, and it's outlined below.

$1n = 1$ $e^{\int(1\ n)p(x)} \int v \times q$ $y\ (x)\ (x)\ dx$	

Homogeneous Differential Equation

$If\ f(tx, ty) = t^n f(x, y)$ $y = vx\quad dy = vdx + xdv$	If the first equation is true, then its possible to substitute the following equations into the function for one variable, to achieve a simpler result.

Methods of Solving

General Problem Solving

$$\frac{dy}{dx} = x^2 y$$

$$\frac{1}{y}\,dy = x^2\,dx$$

$$\int \frac{1}{y}\,dy = \int x^2\,dx$$

$$\ln|y| = \frac{1}{3}x^3 + C_1$$

$$y = e^{\frac{1}{3}x^3 + C_2}$$

/

First Order Linear Non-Separable Differential Equation

$(y + 1)cos(x)dx \;\; dy = 0$

$(y + 1)cos(x) = {}_{dx}dy$

$\frac{dy}{} \;\; c = c$

$os(x)y \; os(x)$

dx

$p(x) = cos(x)$

$q(x) = cos(x)$

$v(x) = e^{\int cos(x)\, dx} = e^{sin(x)}$

$e^{sin(x)} \int e^{sin(x)}$

$\frac{1}{}$

$y = cos(x)\; dx$

$u = \;\; sin(x)\; du = \;\; cos(x)\; dx$

$sin(x) \int \; e^u$

$e \;\; du$

$sin(x) \;\; e^{sin(x)} \; +$

$e \; (\; C)$

$1 + Ce^{sin(x)}$

After you arrange the function into the general form for all First Order Linear Non-Separable Equations, we just follow the steps, simplify the final integral and get the general solution.

Bernoulli Differential Equation

$x^2 \; 3$

$y' + xy = xe \; y$

$z = y^{1 \;\; n} = y^{1 \;\; (3)} = y^4$

$z' = 4y \; y^{3\prime}$

$3\prime + x = x^{x_2} \; 3$

$4y \; [y \; y \; e \; y \;]$

$3\prime + 4xy^4 = 4^{x_2}$

$4y \; y \; xe \;]$

$z' + 4xz = 4xe^{x_2}$

$v(x) = e^{\int 4x\, dx} = e^{2x_2}$

$e^{2x} \;\; \int e^{2x_2} \times \; 4^{x_2}$

$\frac{1}{}$

$z = xe \; dx$

$z = e \; xe \; dx \; {}^{2x_2} \times \int 4^{x_2}$

$^2 = 2$
$u = x\ du\ x\ dx$
$2x^2 \times (^{x_2} +$
$z = e\ 2e\ C)$
$^4 = 2e^{x_2} + C^{2x_2}$
$y\ e$
$y = \sqrt[3]{2e\ e}$
$x^2 + C^{\ 2x_2}$

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Using the Bernoulli Differential equation method, it is possible to manipulate this differential equation to a first order non-separable format, then proceed to solve it using that method. A bit of a long-haul, but in the end, worth it.

Homogeneous Differential Equation

$(2x + 3y)dx\ \ xdy = 0$
$(2(tx) + 3(ty))dx\ \ (tx)dy = 0$
$2tx\ dx + 3ty\ dx\ \ tx\ dy = 0$
$t[2x\ dx + 3y\ dx\ \ x\ dy] = 0$
$t \times f(x, y) = f(tx, ty)$

$(2x + 3(vx))dx\ \ x(vdx + xdv) = 0$
$^2 =$
$2xdx + 3vxdx\ \ xvdx\ \ x\ dv\ 0$
$2dx + 3vdx\ \ vdx\ \ xdv = 0$
$2dx + 3vdx\ \ vdx = xdv$
$(2 + 3v\ \ v)dx = (x)dv$
$(2 + 2v)dx = (x)dv$
$\underline{1} = \underline{1}$
$dx\ dv$
x
$2(1 + v)$

$\int dx\ dv$
$\underline{1} = \int \underline{1}$
x
$2(1 + v)$
$ln\ |x| = ln\ {}_2^1|1 + v| + C_1$
$|\,|2\,|\,| = |1 + v| + C_1$
$ln\ x\ ln$

$x^2 = 1 + v + C_2$ ^{2} 1 $C_2 =$ $x\ v$ x^2 1 $C_2 =$ $_{xy}$ ^{2} 1 $C_2 =$ $x(x\ )\ y$
It’s necessary to prove that a function is homogeneous before applying this method, although it is possible to eyeball it. After that, simply solve the differential equation.

Unit #08: Applications of Integration

Finding the Area Underneath and Between Curves

$A = \int^{b} (x)\ dx$ R $_a$	Area underneath the function R(x) on the interval [a,b].
$A = \int^{b} (\ R(x)\ (x)\ )\ dx$ r $_a$	Area between the function R(x) and r(x) on the interval [a,b].

Finding the Volume of a Solid

^{b} $[R(x)]^2$ $V = \pi\int\ dx$ $_a$	Disc Method. R(x) is the outer function.
^{b} $^{2\ 2}$ $V = \pi\int\ (\ [R(x)]\ [r(x)]\ )\ dx\ _a$	Washer Method. R(x) is the outer function. r(x) is the inner function.
$V = 2\pi\int^{b} (x)h(x)\ dx$ r $_a$	Shell Method. r(x) is the radius function. h(x) is the height function.

Finding the Arc Length of a Function

$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$	Where L is the arc length of f(x) from [a,b].

Finding the Surface Area of a Solid

$SA = 2\pi \int_a^b r(x) \times \sqrt{1 + [f'(x)]^2}\, dx$	Where SA is the surface area of f(x) rotated around the x axis with radius r(x) from [a,b].

Examples

Area Between and Area Underneath

Find the area between $R(x) = x$ and $r(x) = x^2$ on $[0, 1]$. $A = \int_0^1 (x - x^2)\, dx$ $A = \int_0^1 (x)\, dx - \int_0^1 (x^2)\, dx$ $A = [\frac{1}{2}x^2]\Big\vert_0^1 - [\frac{1}{3}x^3]\Big\vert_0^1$ $A = (\frac{1}{2} - 0) - (\frac{1}{3} - 0)$ $A = \frac{1}{6}$

/

The area between the functions can be split into two integrals. By subtracting the area under the inner function from the outer function, what's left is the area between the functions. This method is simply an application of the FTC.

Disc Method Washer Method

Find the volume of $R(x) = \frac{1}{x}$ *on* $[1, 4]$ *revolved around the x axis* $V = \pi\int_{1}^{4} [\frac{1}{x}]^2\, dx$ $V = \pi\int_{1}^{4} \frac{1}{x^2} dx$ $V = \pi \times [-\frac{1}{x}]\Big\rvert_{1}^{4}$ $V = \pi[(-\frac{1}{4}) - (-1)]$ $V = \pi\frac{3}{4}$	*Find the volume of the area between* $R(x) = \sqrt{x}$ *and* $r(x) = x^2$ *on* $[0, 1]$ *around the x axis* $V = \pi\int_{0}^{1} [(\sqrt{x})^2 - (x^2)^2]\, dx$ $V = \pi\int_{0}^{1} [x - x^4]\, dx$ $V = \pi\int_{0}^{1} x\, dx - \pi\int_{0}^{1} x^4\, dx$ $V = \pi[\frac{1}{2}x^2]\Big\rvert_{0}^{1} - \pi[\frac{1}{5}x^5]\Big\rvert_{0}^{1}$ $V = \pi[\frac{1}{2}(1)^2 - \frac{1}{5}(1)^5]$ $V = 0.3\pi$
The area underneath the function revolved circularly around the x axis using the formula $A = \pi r^2$ with another dimension.	This is essentially the area in between revolved circularly around the x axis, just like before, but with another revolved area being subtracted from the first.

Shell Method

$^3 + x + 1\ 2 = 1 = 5$

$f(x) = x$ *on* $[1,]$ *above* y *revolved around* x

$$V = 2\pi \int_1^2 (5\ )(x\ )\ dx$$

$$x^3 + x + 1 \quad 1$$

$$V = 2\pi \int_1^2 (5x\ x\ )\ dx$$

$$3 \quad x^4 + 5 \quad x^2$$

$$\underline{5}\,4 \quad \underline{1}\,5 \quad + \quad \underline{5}\,2 \quad 3\underline{1}\,3 \quad 2\ 1$$

$$5 \qquad 2$$

$$V = 2\pi(-x\ x\ x\ x\ )\Big|_4$$

$$\underline{5}^4 \quad 5\underline{1}^5 + 2\underline{5}^2 \quad 3\underline{1}^3 \quad (4\underline{5}^4 \quad 5\underline{1}^5 + 2\underline{5}^2 \quad 3\underline{1}\ 3$$

$$V = 2\pi[\ (-(2)\ (2)\ (2)\ (2)\ )\ (1)\ (1)\ (1)\ (1)\ )\ ]$$

$$4$$

$$V = 17.7$$

By using the shell method, every question can be a dx question, which is sometimes helpful when it comes to simplifying situations. It is important to define your radius and height equations correctly. The shifters are sometimes tricky, in this case, [1,2] were the bounds, the line y=1 made it necessary to add a -1 to the height function, and the center at x=5 made it necessary to write the radius equation from that perspective. It's also important to write the whole integral in terms of a single variable, in this case, x.

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Arc Length

$^2\,2$

Find the Arc Length of $f(x) = 4\ \ x$ *on* $[0,]$

$$L = \int_0^2 \sqrt{1 + [\ 2x]_2}\ dx$$

$$L = \int_0^2 \sqrt{1 + 4x^2}\ dx$$

$tan(\theta) = 2x$

$^2 = 2$

$sec\ (\theta)\ d\theta\ dx$

$\underline{1}^{\,2} = d$

$sec\ (\theta)\ d\theta\ x$

2

^{1}

$tan\ (4)$

$L = \int (\theta)\, d\theta$

sec^3

0

$L = (4\ n\ \frac{1}{4} \sqrt{17} + l\ 4| + \sqrt{17}|$

In this example, the changing of the bounds is important when traversing from x to ��. In terms of the formula, the taking of the derivative and subsequent insertion into the equation is fairly simple.

Surface Area

Find the Surface Area of a sphere in general form

$x^2 + y^2 = r^2$

$y = \sqrt{r^2 - x^2}$

$y' = \frac{x}{\sqrt{r^2 - x^2}}$

r

$\sqrt{r^2 - x^2} \times \sqrt{1 + (\ )}$

2

$SA = 2 \times 2\pi \int dx$

$\frac{x}{\sqrt{r^2 - x^2}}$

0

r

$SA = 4\pi \int dx$

$\sqrt{r^2 - x^2 + x^2}$

0

r

$SA = 4\pi \int dx$

r

0

$SA = 4\pi r^2$

In this example, one fourth of a circle (as located in the first quadrant) is rotated 360^{O} around the x axis to form a hemisphere, when we multiply this by two, it becomes the whole sphere. Don't forget to differentiate the dx in that last integral! Sneaky sneaky.

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Unit #09: Parametric Equations, Polar Coordinates, and Vector-Valued Functions

What are **parametric equations**?

Parametric equations are a set of equations which relate to an independent variable. There are special operations and advantages to using parametric equations to model growth from a more complex perspective.

Parametric equations and operations are given by:

$x = f(t)$ $y = g(t)$ $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$ $when$ $\frac{dx}{dt} \neq 0$ $\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}}$ $when$ $\frac{dx}{dt} \neq 0$	Where f(t) and g(t) are functions of t.

$\langle x(t), y(t)\rangle \rightarrow$ Position $\langle x'(t), y'(t)\rangle \rightarrow$ Velocity $\langle x''(t), y''(t)\rangle \rightarrow$ Acceleration $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \rightarrow$ Speed $\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \rightarrow$ Arc Length	Where x(t) and y(t) are functions of t.

Example

Given $x = e^{\sqrt{t}}$ and $y = \ln(t^3)$

Find $\frac{dy}{dx}$ at $t = 1$ without eliminating the parameter.

$\frac{dy}{dt} = 3\left(\frac{1}{t}\right)$, $\frac{dx}{dt} = e^{\sqrt{t}} \times \frac{1}{2\sqrt{t}}$

$\frac{dy/dt}{dx/dt} = \frac{1\ 3}{e \times \frac{1}{2}}$

$\frac{dy/dt}{dx/dt} = \frac{1\ 3}{e \times \frac{1}{2}} = 1$

.47

$_{dt}{}^{dx}$ $\frac{dy}{dx} = 1$ *Find at t by first eliminating the parameter.* $^2 = t^{\sqrt{t}} = 1 : = e$ $ln(x) = \sqrt{t}\ (ln(x))\ e\ @\ t\ x$

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$^2\ 3^2$ $y = (ln(x))\ ln((ln(x))\ )$ $y' = 2ln(x) \times \frac{1}{x}\ 6 \times \frac{1}{ln(x) \times x}$ $y' = 2ln(e) \times y\ (e)\ .47\ \frac{1}{e}\ 6 \times \frac{1}{ln(e) \times e}' = 1$
We can solve this question by using parametric rules, or by eliminating the parameter. It's easy to identify how the parametric rules simplify the problem greatly in this circumstance.

What are **polar coordinates and polar functions**?

A polar coordinate is a method of describing a location using an angle and a radius from the pole. A polar function then, is a method of describing a locus of points in terms of a changing radius with respect to a changing angle.

$x = rcos(\theta)\ y = rsin(\theta)\ x^2 + y^2 = r^2\ A = d\theta\ \frac{1}{2}\int_a^b(r(\theta)^2$ $L = \int_a^b \sqrt{(r(\theta))_2 + (r'(\theta))^2}\ d\theta$	Where A is area, L is length, a and b are specified angles that act as bounds for the area and length integrals. In this method, we can find the area of an angular sector of the curve, in other words, the area bound by two angles and the pole.

Example

$^2 + y^2 = 2 \quad 1^2 + y^2 = 1$ *Let R be the region bounded by the graphs of x , (x)* *and the x axis, closed to the origin.* *Set up an expression involving integral(s) with respect to x that represents* *R.* 1 $_0\sqrt{1 \quad (x \quad 1)_2} + \int^{\sqrt{2}}_1 \sqrt{2 \quad x^2}$ $A = \int dx \, dx$ *Set up an expression involving integral(s) with respect to y that represents* *R.* 1 $A = \int \,) \, dx$ $_0\sqrt{2 \quad y_2} \quad (\sqrt{1 \quad y_2} + 1$ *Set up an expression involving integral(s) with respect to* θ *that represents* *R.* $r = \sqrt{2}$ $r = 2cos(\theta)$ $\sqrt{2} = 2cos(\theta)_4{}^{\pi} = \theta$ $A = d\theta \, d\theta \, _2{}^1 \int^{\pi/4}$ $(\sqrt{2})^2 + {}_2{}^1 \int^{\pi/2}$ $(2cos(\theta))^2$ $^{\pi}/4$ 0
These three ways to calculate the same expression are a perfect example of the breadth of our knowledge.

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Unit #10: Infinite Sequences and Series

Special Series and Classifications

Arithmetic and Geometric Sequences and Series

	Sequence	Series	Where a_n is the nth term in the sequence, S_n is the sum of the first n terms in that series, a is $_1$ the first term in the sequence or series, d is the common difference in the sequence, and r is the common multiple of the sequence or series.
Arithmetic	$a\ (n\ )_n = a_1 + d$ 1	$S\ (a\ )_n = {}_2{}^n{}_1 \ a_n$	
Geometric	$a\ (r)_n = a_1 n \ 1$	$\underline{1 \ r^n}$ $S\ (\ )_n = a_1 \, 1 \ r$	

An **alternating** sequence/series rapidly changes between positive and negative terms.
An **oscillating** sequence/series rapidly changes between two specific terms.

	equence Alternating
1 1 + 1 1 + 1	$1 \ \frac{1}{2} + \frac{1}{3} \ \frac{1}{4} + \frac{1}{5} \ \frac{1}{6}$

A **lewis sequence** adds the previous two terms to make the newest term. The **fibonacci sequence** is a famous example of a lewis sequence with the first two terms being 1 and 1.

Lewis Sequence	Fibonacci Sequence
3, 7, 10, 17, 27, 44...	1, 1, 2, 3, 5, 8, 13...

A **telescopic series** is a series where most of the terms cancel with themselves. If the series looks like a partial fractions question with subtraction...it's probably telescopic.

Telescopic Series	Example
$\sum_{n=1}^{\infty} a_n = (a_1 \ a_2) + (a_2 \ a_3) + (a_3 \ a_4) + \ldots$	$\sum_{n=1}^{\infty} (\frac{1}{n} \ \frac{1}{n+1})$

A **p series** is a series which can be written in the following form. A **harmonic series** is a p series where p = 1.

P Series	Harmonic Series
$\sum_{n=1}^{\infty} (\frac{1}{n^p})$	$\sum_{n=1}^{\infty} (\frac{1}{n})$

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Tests of Series Behavior

Identification of Series Type

Arithmetic Series → → → by definition, divergent.

Geometric Series → → → when $|r| > 1$, divergent; when $|r| < 1$, convergent.

P Series → → → when $p \leq 1$, divergent; when $p > 1$, convergent towards $\frac{1}{p \; 1}$.

nth Term Test

If $\lim_{n\to\infty} a_n \neq 0$: *divergent.*

Integral Test

If f is positive, continuous and decreasing for all values $n \geq 1$, *and* $a_n = f(n)$, *then if* $\int_1^\infty f(x)\,dx$ *diverges* / *converges*, a_n *will behave the same way.*

Limit Comparison Test

Given $\sum_{n=1}^{\infty} a_n$ *is known to either converge or diverge, and* $\sum_{n=1}^{\infty} b_n$ *is unknown*:

If $\lim_{n\to\infty} \frac{a_n}{b_n} = L$ *and* $L > 0$, *then* $\sum_{n=1}^{\infty} b_n$ *will behave as* $\sum_{n=1}^{\infty} a_n$.

Direct Comparison Test

If $0 \leq a_n \leq b_n$, *then if* $\sum_{n=1}^{\infty} a_n$ *diverges*, $\sum_{n=1}^{\infty} b_n$ *diverges.*

If $0 \le a_n \le b_n$*, then if* $\sum_{n=1}^{\infty} a_n$ *diverges,* $\sum_{n=1}^{\infty} b_n$ *diverges.*

Alternating Series Test

If $\sum_{n=1}^{\infty} a_n$ *is alternating,* $\lim_{n\to\infty} a_n = 0$*, and for all* $n \ge 1$*,* $|a_{n+1}| < |a_n|$*, then* $\sum_{n=1}^{\infty} a_n$ *converges.*

Ratio Test

$\lim_{n\to\infty}$ *If ratio* $\left|\frac{a_{n+1}}{a_n}\right| < 1$*, convergent. If ratio* > 1*, divergent. If ratio* $= 1$*, inconclusive.*

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Examples

nth Term Test

$\sum_{n=1}^{\infty} 2^n$

$\lim_{n\to\infty} 2^n$ approaches infinity.

Since $\infty \neq 0$, by the nth term test, $\sum_{n=1}^{\infty} 2^n$ diverges.

Sometimes the limit is more complicated, but the idea is pretty clear. Remember to cite which test was used for the final conclusion.

Integral Test

$$\sum_{n=1}^{\infty} \frac{n+2}{n+1}$$

$$f(x) = \frac{n+2}{n+1}$$

$$f'(x) = \frac{1(n+1) - (n+2)}{(n+1)^2}$$

$\Rightarrow$ *f is pos, cont, and decr.*

$$\int_1^{\infty} \frac{x+2}{x+1}\,dx$$

$x + 1 = u \quad dx = du$

$$\int_2^{\infty} \frac{u+1}{u}\,du$$

$$\int_2^{\infty} 1 + \frac{1}{u}\,du$$

$$x + ln\,|x+1|\,\Big|_2^{\infty}$$

$= \infty$

By the integral test, $\sum_{n=1}^{\infty} \frac{n+2}{n+1}$ diverges.

Remember to prove that the derivative is always decreasing, or else this test is invalid. Remember to cite which test was used for the final conclusion.

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Limit Comparison Test

$$\sum_{n=1}^{\infty} \frac{(n)}{(n)^2+1}$$

$\sum_{n=1}^{\infty} \frac{1}{n}$ is a p series where $p = 1$, so the series is convergent.

$$\sum_{n=1}^{\infty} \frac{\frac{1}{n}}{\frac{(n)}{(n)^2+1}}$$

$$\sum_{n=1}^{\infty} \frac{1}{n} \times \frac{n^2}{n+1} = 1 > 0$$

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ converges, by the limit comparison test, $\sum_{n=1}^{\infty} \frac{(n)}{(n)^2+1}$ converges.

Be very clear with each part of the reasoning here, mixing up even one thing could throw the whole logic train off. Remember to cite which test was used for the final conclusion.

Direct Comparison Test

$$\sum_{n=1}^{\infty} \frac{(4)^n}{(5)^n+3}$$

$$\frac{(4)^n}{(5)^n} > \frac{(4)^n}{(5)^n+3}$$

$$\sum_{n=1}^{\infty} \frac{(4)^n}{(5)^n} = \sum_{n=1}^{\infty} \left(\frac{4}{5}\right)^n$$

$(-)\ 5\underline{4}^{n}\ r < 1,$

Since $\sum$ is a geometric series where it converges.

$n = 1$

∞

$(5)^{\cancel{n}} \sum^{\infty}$

$(4)^n$

$(4)^n$

Since $\sum$ converges, by the direct comparison test, converges. n

$(5) + 3$

$n = 1$

$n = 1$

Be very clear with each part of the reasoning here, mixing up even one thing could throw the whole logic train off. Remember to cite which test was used for the final conclusion.

Alternating Series Test

$\sum^{\infty}$

$(1)^{n+1}$

$n = 1 \cancel{n+1}$

Alternating? Yes.

$\underline{1}$

lim ? Yes.

$n + 1 = 0$

$n \to \infty$

Decreasing? Yes.

$f(x) = (n + 1)^1$

$f'(x) = (n + 1)^2$

By the alternating series test, this series is convergent.

When considering the sequence in the limit and derivative forms, ignore the $(-1)^{n+1}$. Remember to cite which test was used for the final conclusion.

Ratio Test

$$\sum_{n=1}^{\infty} \frac{2^n}{n!}$$

$$\lim_{n\to\infty} \left| \frac{(2)^{n+1}}{(n+1)!} \times \frac{n!}{2^n} \right|$$

$$\lim_{n\to\infty} \left| \frac{(2)^n (2)}{(n+1)n!} \times \frac{n!}{2^n} \right|$$

$$\lim_{n\to\infty} \left| \frac{(2)}{(n+1)} \right| = 0$$

$0 < 1$, *so by the ratio test*, $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ *is convergent*.

Remember to cite the ratio test as the method for determining the final conclusion.

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Telescoping Series

$$\sum_{n=1}^{\infty} ln \left| \frac{n}{n+2} \right|$$

$ln\,|n| \;\; ln\,|n+2|$

$ln\,|1| \;\; ln\,|3| + ln\,|2| \;\; ln\,|4| + ln\,|3| \;\; ln\,|5| + ln\,|4| \;\; ln\,|6| + \ldots\, ln\,|n| \;\; ln\,|n+2|$

Left with : $ln\,|n+2|$, *which approaches infinity*, *so....divergent*!

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} \; \frac{1}{n+1} \right)$$

$$\frac{1}{n} \; \frac{1}{n+1}$$

$$\frac{1}{} + \left(\frac{1}{2} \; \frac{1}{3}\right) + \left(\frac{1}{3} \; \frac{1}{4}\right) + \left(\frac{1}{4} \; \frac{1}{5}\right) + \ldots \frac{1}{n} \; \frac{1}{n+1}$$

$(1\ \ ))))\ ..(-)$ $\quad 2$ $n+1$ *Left with* : $\frac{1}{n+1}$, *which approaches zero*, *so....convergent at* 1, *since the first term wasn't cancelled.*
You can see that many of the terms cancel with themselves, and we can say that only one term is left, in the first case, the series is divergent, since the final term doesn't cancel and approaches infinity. In the second case, the series is convergent, the final term approaches zero, and we're left with the first term that doesn't cancel.

Power Series

What is a **power series**?
A power series is a method of modeling a non-polynomial function with a polynomial.

A **Taylor series** is a power series centered at some value "a". A **Maclauren series** is a Taylor series centered at zero.

Power series are applicable on everything smooth on an open domain (not endpoints).

$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x - c)^n$	Where $f^{(n)}(c)$ is the nth derivative, and c is the center.

What is an **interval of convergence**?
An interval of convergence is the domain at which a power series accurately models the desired function.

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Power Series Example

Determine the power series of $f(x) = cos(x)$

Function General Polynomial Evaluated at zero $f(x) = cos(x)$ $P(x) = a\ x\ x\ x\ x\ ..\ _0 +$ $a_1 + a_2^{\ 2} + a_3^{\ 3} + a_4^{\ 4} + .$ | $cos(0) = a_0 f'(x) = sin(x)$ $P'(x) = a\ a\ x\ x\ a\ x\ ..\ _1 + 2\ _2 + 3a_3^{\ 2} +$ $4\ _4^{\ 3} + .$ | $sin(0) = a_1 f''(x) = cos(x)$ $P''(x) = 2a\ x\ 2a\ x\ ..\ _2 + 6a_3 + 1\ _4^{\ 2} + .$ | $cos(0) =$ $2a_2 f'''(x) = sin(x)$ $P'''(x) = 6a\ 4a\ x\ ..\ _3 + 2\ _4 + .$ | $sin(0) = 6a_3\ ^{(4)} = c\ P\ (x)\ 4a\ ..$
$^{(4)} = 2\ _4 + .$ | $cos(0) = 24a_4$

$f\ (x)\ os(x)$

$a_0 = \frac{1}{0!} a_1 = 0\ a_2 = \frac{1}{2!} a_3 = 0\ a_4 = \frac{1}{4!} cos(x) = 1\ \ x\ x\ x\ ..\ _{2!}^{\ 2} + \frac{1}{4!}^{4}\ \ _6\frac{1}{}^{6} + .$

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$$cos(x) = \sum_{n=0}^{\infty} \frac{(1)^n (x)^{2n}}{(2n)!}$$

To find the power series of a function, take successive derivatives, plug in your center, then set them equal to the factorial expressions to the right, and put all the terms back into your general equation.

You can find the trend and write it using sigma notation as well.

When the power series isn't centered at zero, just shift the general polynomial to the predetermined center and instead of evaluating at zero, evaluate at that center.

Interval of Convergence Example

Determine the integral of convergence of the power series $\sum_{n=0}^{\infty} (x\ \ 4\ \ 3)^n$

$$\lim_{n\to\infty} \left| \frac{4(x\ \ 3)^{n+1}}{4(x\ \ 3)^n} \right|$$

$\lim_{n\to\infty} |x\ \ 3| = x\ \ 3$

$1 < x\ \ 3 < 1$

$2 < x < 4$

$$\sum_{n=0}^{\infty} (x\ \ 4\ \ 3)^n = 4 + (x\ \ 4\ \ 3) + (x\ \ 4\ \ 3)^2 + ..$$

@ $x = 2 : 4\ \ 4 + 4\ \ 4 + ... \rightarrow$ Divergent

@ $x = 4 : 4 + 4 + 4 + 4 + ... \rightarrow$ Divergent

Interval of Convergence: (2,4) Radius: 1 Center: 3
Since the bounds were both divergent, the interval's bounds are not inclusive. The original inequality's bounds were determined by the bounds of the ratio test, to determine the behavior of the series.

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Special Power Series

$cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x)^{2n}}{(2n)!}$ $sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x)^{2n+1}}{(2n+1)!}$ $e^x = \sum_{n=0}^{\infty} \frac{(x)^n}{n!}$ $ln(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} (x-1)^n}{n}$ $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$	Remember that it's possible to substitute any expression for x, so long as every x is substituted.

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AP Expectations (Updated 2020)

What's not on the AP (in no particular order)

1. Newton's Method

2. Shell Method

3. Surface Area

4. Non-Separable Differential Equations

5. nth Term Test

6. Integral Test

7. Direct Comparison Test

8. Limit Comparison Test

9. Absolute/Conditional Convergence

10. Alternating Series Error

11. Weirestrass

12. Parametrics and Polars

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Kleinberg’s Notes

General Helpful Advice

1. Almost every limit can be done with **L'hopital's Rule**

2. Know **all** the derivative rules

3. Know **all** the integral rules

4. Always remember your **dx's and** +**C's**

5. Don't forget to justify! See if the problem asks for it.
    a. When justifying, keep it short and sweet: **what, when, why**.

6. Sign test **any time** you're asked for a max or min. **Sign test all crit values.** 7.

Over/under estimation and concavity **must** be done through the second derivative. 8.

Always put what you're finding. In other words, **show work**, not just the answer.

9. If the test says to use a particular method, you don't have to prove it's applicable, but if you decide to use it yourself, **prove it applicable.**

10. Remember your identities and theorems. Concise sheets below.
    a. IDENTITIES:
    b. THEOREMS:

11. Remember that **Desmos is your friend**. It's advantageous to graph things, to check your thinking.

12. If ever confronted with something you don't know, bring it back to the

basics. 13. Stay calm. **Stay awesome.**

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A Farewell

I've sincerely enjoyed learning math with all of you this year. Each and every one of us contributed to an impeccable intellectual, cheerful, and personal atmosphere.

Every member of this class has been nothing short of incredible. We've come a long way. I thank each and every one of you for being yourselves. It's been a great and often underappreciated privilege just to get to know you all, and I have no doubt that each and every one of you will go on to live full and successful lives. The determination and perseverance we've shown is a testament to our strength and character, and let's just say there's plenty to go around.

I couldn't possibly share what an impact this year has had on me in any message of reasonable length, so I leave you all with these two quotes, because one didn't seem like enough.

"How lucky I am to have something that makes saying goodbye so hard." -Winnie the Pooh

"So long as the memory of certain beloved friends lives in my heart, I shall say that life is good." -Helen Keller.

You will always have a friend in me.

Best Wishes,
George Mitrev

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