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Source: *Operations Research*, Vol. 45, No. 6 (Nov. - Dec., 1997), pp. 931-939
Published by: INFORMS
Stable URL: http://www.jstor.org/stable/172076
Accessed: 13/03/2014 08:16

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OPTIMALITY OF (s, S) POLICIES IN INVENTORY MODELS WITH MARKOVIAN DEMAND

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(Received September 1993; revisions received November 1994, February 1995, November 1995; accepted December 1995)

This paper is concerned with a generalization of classical inventory models (with fixed ordering costs) that exhibit (s, S) policies. In our model, the distribution of demands in successive periods is dependent on a Markov chain. The model includes the case of cyclic or seasonal demand. The model is further extended to incorporate some other realistic features such as no ordering periods and storage and service level constraints. Both finite and infinite horizon nonstationary problems are considered. We show that (s, S) policies are also optimal for the generalized model as well as its extensions.

One of the most important developments in the inventory theory has been to show that (s, S) policies are optimal for a class of dynamic inventory models with random periodic demands and fixed ordering costs. Under an (s, S) policy, if the inventory level at the beginning of a period is less than the reorder point s, then a sufficient quantity must be ordered to achieve an inventory level S, the order-up-to level, upon replenishment. However, in working with some real-life inventory problems, we have observed that some of the assumptions required for inventory models exhibiting (s, S) policies are too restrictive. It is our purpose, therefore, to relax these assumptions toward more realism and still demonstrate the optimality of (s, S) policies.

The nature of the demand process is an important factor that affects the type of optimal policy in a stochastic inventory model. With possible exceptions of Karlin and Fabens (1959) and Iglehart and Karlin (1962), classical inventory models have assumed demand in each period to be a random variable independent of demands in other periods and of environmental factors other than time. However, as elaborated recently in Song and Zipkin (1993), many randomly changing environmental factors, such as fluctuating economic conditions and uncertain market conditions in different stages of a product life-cycle, can have a major effect on demand. For such situations, the Markov chain approach provides a natural and flexible alternative for modeling the demand process. In such an approach, environmental factors are represented by the *demand state* or the *state-of-the-world* of a Markov process, and demand in a period is a random variable with a distribution function dependent on the demand state in that period. Furthermore, the demand state can also affect other parameters of the inventory system such as the cost functions.

Another feature that is not usually treated in the classical inventory models but is often observed in real life is the presence of various constraints on ordering decisions and inventory levels. For example, there may be periods, such as weekends and holidays, during which deliveries cannot take place. Also, the maximum inventory that can be accommodated is often limited by finite storage space. On the other hand, one may wish to keep the amount of inventory above a certain level to reduce the chance of a stock-out and ensure a satisfactory service to customers.

While some of these features are dealt with in the literature in a piecemeal fashion, we shall formulate a sufficiently general model that has models with one or more of these features as special cases and that retains the optimal policy to be of (s, S) type. Thus, our model considers more general demands, costs, and constraints than most of the fixed-cost inventory models in the literature.

The plan of the paper is as follows. The next section contains a review of relevant models and how our model relates to them. In Section 2, we develop a general finite horizon inventory model with a Markovian demand process. In Section 3, we state the dynamic programming equations for the problem and the results on the uniqueness of the solution and the existence of an optimal feedback or Markov policy. In Section 4, we derive some properties of K-convex functions, which represent important extensions of the existing results. These properties allow us to show more generally and simply that the optimal policy for the finite horizon model is still of (s, S) type, with s and S dependent on the demand state and the time remaining. The analysis of models incorporating no-ordering periods and those with the shelf capacity and service level constraints is presented in Section 5. The nonstationary infinite horizon version of the model is examined in Section 6. The cyclic demand case is treated in Section 7. Section 8 concludes the paper.

Subject classifications: Inventory models with Markovian demand. Optimality of (s, S) policies. Models with fixed ordering costs.
Area of review: STOCHASTIC PROCESSES & THEIR APPLICATIONS.

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1. REVIEW OF LITERATURE AND ITS RELATION TO OUR MODEL

Classical papers on the optimality of (s, S) policies in dynamic inventory models with stochastic demands and fixed setup costs are those of Arrow et al. (1951), Dvoretzky et al. (1953), Karlin (1958), Scarf (1960), Iglehart (1963), and Veinott (1966). Scarf develops the concept of K-convexity and uses it to show that (s, S) policies are optimal for finite horizon inventory problems with fixed ordering costs. That a stationary (s, S) policy is optimal for the stationary infinite horizon problem is proved by Iglehart (1963). Furthermore, Bensoussan et al. (1983) provide a rigorous formulation of the problem with nonstationary but stochastically independent demand. They also deal with the issue of the existence of optimal feedback policies along with a proof of the optimality of an (s, S)-type policy in the nonstationary finite as well as infinite horizon cases. Kumar (1992) has attempted to extend the classical inventory model by incorporating service level and storage capacity constraints, but without a rigorous proof.

The effect of a randomly changing environment in inventory models with fixed costs received only limited attention in the earlier literature. Karlin and Fabens (1959) introduced a Markovian demand model similar to ours. They indicate that given the Markovian demand structure in their model, it appears reasonable to postulate an inventory policy of (s, S) type with a different set of critical numbers for each demand state. But they considered the analysis to be complex, and concentrated instead on optimizing only over the restricted class of ordering policies each characterized by a single pair of critical numbers s and S irrespective of the demand state.

Recently and independently, Song and Zipkin (1993) have presented a continuous-time, discrete-state formulation with a Markov-modulated Poisson demand and with linear costs of inventory and backlogging. They show that the optimal policy is of state-dependent (s, S) type when the ordering cost consists of both a fixed cost and a linear cost. An algorithm for computing the optimal policy is also developed using a modified value iteration approach.

The basic model presented in the next section extends the classical Karlin and Fabens model in two significant ways. It generalizes the cost functions that are involved and it optimizes over the natural class of all history-dependent ordering policies. The model and the methods used here are essentially more general than those of Song and Zipkin (1993) in that we consider general demands (Remark 4.5), state-dependent convex inventory/backlog costs without the restrictive assumption relating backlog and purchase costs (see Remark 4.2), and extended properties of K-convex functions (Remark 4.4). The constrained models discussed in Section 5 generalize Kumar (1992) with respect to both demands and costs. The nonstationary infinite horizon model extends Bensoussan et al. (1983) to allow for Markovian demands and more general asymptotic behavior on the shortage cost as the shortage becomes large (Remark 4.2).

2. FORMULATION OF THE MODEL

In order to specify the discrete time inventory problem under consideration, we introduce the following notation and basic assumptions:

$\langle 0, N\rangle = \{0, 1, 2, \ldots, N\}$, the horizon of the inventory problem;
$(\Omega, \mathcal{F}, P)$ = the probability space;
$I = \{1, 2, \ldots, L\}$, a finite collection of possible demand states;
i_k = the demand state in period k;
$\{i_k\}$ = a Markov chain with the $(L \times L)$-transition matrix $P = \{p_{ij}\}$;
ξ_k = the demand in period k, $\xi_k \geq 0$, ξ_k dependent on i_k;
$\phi_{i,k}(\cdot)$ = the conditional density function of ξ_k when $i_k = i$, $E\{\xi_k | i_k = i\} \leq M < \infty$;
$\Phi_{i,k}(\cdot)$ = the distribution function corresponding to $\phi_{i,k}$;
u_k = the nonnegative order quantity in period k;
x_k = the surplus (inventory/backlog) level at the *beginning* of period k;
$c_k(i, u)$ = the cost of ordering $u \geq 0$ units in period k when $i_k = i$;
$f_k(i, x)$ = the surplus cost when $i_k = i$ and $x_k = x$, $f_k(i, x) \geq 0$ and $f_k(i, 0) \equiv 0$;
$\delta(z)$ = 0 when $z \leq 0$ and 1 when $z > 0$.

We suppose that orders are placed at the beginning of a period, delivered instantaneously, and followed by the period's demand. Unsatisfied demands are fully backlogged. Furthermore,

$$c_k(i, u) = K_k^i \delta(u) + c_k^i u, \tag{2.1}$$

where the fixed ordering costs $K_k^i \geq 0$ and the variable costs $c_k^i \geq 0$, and the surplus cost functions $f_k(i, \cdot)$ are convex and asymptotically linear, i.e.,

$$f_k(i, x) \leq C(1 + |x|) \tag{2.2}$$

for some $C > 0$.

The objective function to be minimized is the expected value of all the costs incurred during the interval $\langle n, N\rangle$ with $i_n = i$ and $x_n = x$:

$$J_n(i, x; U) = E\left\{ \sum_{k=n}^{N} [c_k(i_k, u_k) + f_k(i_k, x_k)] \right\}, \tag{2.3}$$

where $U = (u_n, \ldots, u_{N-1})$ is a *history-dependent* or *nonanticipative* admissible decision (order quantities) for the problem. The inventory balance equations are given by

$$x_{k+1} = x_k + u_k - \xi_k, \; k \in \langle n, N - 1\rangle. \tag{2.4}$$

Finally, we define the value function for the problem over $\langle n, N\rangle$ with $i_n = i$ and $x_n = x$ to be

$$v_n(i, x) = \inf_{U \in \mathcal{U}} J_n(i, x; U), \tag{2.5}$$

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where $\mathcal{U}$ denotes the class of all admissible decisions. Note that the existence of an optimal policy is not required to define the value function. Of course, once the existence is established, the "inf" in (2.5) can be replaced by "min".

3. DYNAMIC PROGRAMMING AND OPTIMAL FEEDBACK POLICY

In this section we give the dynamic programming equations satisfied by the value function. We then provide a *verification theorem* that states the cost associated with the feedback or Markov policy obtained from the solution of the dynamic programming equations equals the value function of the problem on $\langle 0, N\rangle$. The proofs of these results require some higher mathematics, and they are available in Beyer, Sethi, and Taksar (1998); see also Cheng (1996), Beyer and Sethi (1997), and Bertsekas and Shreve (1976).

Let B_0 denote the class of all continuous functions from $I \times R$ into R^+ and the pointwise limits of sequences of these functions (see Feller 1971). Note that it includes piecewise-continuous functions. Let B_1 be the space of functions in B_0 that are of linear growth, i.e., for any $b \in B_1$, $0 \leqslant b(i, x) \leqslant C_b(1 + |x|)$ for some $C_b > 0$. Let C_1 be the subspace of functions in B_1 that are uniformly continuous with respect to $x \in R$. For any $b \in B_1$, we define the notation

$$F_{n+1}(b)(i, y) = \sum_{j=1}^{L} p_{ij} \int_0^\infty b(j, y - z)\phi_{i,n}(z)\, dz. \tag{3.1}$$

Using the principle of optimality, we can write the following dynamic programming equations for the value function:

$$\begin{cases} v_n(i, x) = f_n(i, x) + \inf\limits_{u\geqslant 0} \{c_n(i, u) \\ \qquad\qquad + E[v_{n+1}(i_{n+1}, x \\ \qquad\qquad + u - \xi_n)|i_n = i\} \\ \quad = f_n(i, x) + \inf\limits_{u\geqslant 0} \{c_n(i, u) \\ \qquad\qquad + F_{n+1}(v_{n+1})(i, x + u), \\ \qquad\qquad n \in \langle 0, N-1\rangle, \\ v_N(i, x) = f_N(i, x). \end{cases} \tag{3.2}$$

We can now state our existence results in the following two theorems.

Theorem 3.1. *The dynamic programming Equations* (3.2) *define a sequence of functions in C_1. Moreover, there exists a function $\hat{u}_n(i, x)$ in B_0, which provides the infimum in* (3.2) *for any x.*

To solve the problem of minimizing $J_0(i, x, U)$, we use $\hat{u}_n(i, x)$ of Theorem 3.1 to define

$$\begin{cases} \hat{u}_k = \hat{u}_k(i_k, \hat{x}_k), \quad k \in \langle 0, N-1\rangle \text{ with } i_0 = i, \\ \hat{x}_{k+1} = \hat{x}_k + \hat{u}_k - \xi_k, \quad k \in \langle 0, N-1\rangle \text{ with } \hat{x}_0 = x. \end{cases} \tag{3.3}$$

Theorem 3.2. (*Verification Theorem.*) *Set $\hat{U} = (\hat{u}_0, \hat{u}_1, \ldots, \hat{u}_{N-1})$. Then $\hat{U}$ is an optimal decision for the problem $J_0(i, x, U)$. Moreover,*

$$v_0(i, x) = \min_{U\in\mathcal{U}} J_0(i, x, U). \tag{3.4}$$

Taken together, Theorems 3.1 and 3.2 establish the existence of an optimal feedback policy. This means that there exists a policy in the class of all admissible (or history-dependent) policies, whose objective function value equals the value function defined by (2.5), and there is a Markov (or feedback) policy which gives the same objective function value.

Remark 3.1. The results corresponding to Theorems 3.1 and 3.2 hold under more general cost functions than those specified by (2.1) and (2.2); see Beyer, Sethi, and Taksar (1998, forthcoming).

4. OPTIMALITY OF (*s*, *S*) ORDERING POLICIES

We make additional assumptions under which the optimal feedback policy $\hat{u}_n(i, x)$ turns out to be an (s, S)-type policy. For $n \in \langle 0, N-1\rangle$ and $i \in I$, let

$$K_n^i \geqslant \bar{K}_{n+1}^i \equiv \sum_{j=1}^{L} p_{ij} K_{n+1}^j \geqslant 0, \tag{4.1}$$

and

$$c_n^i x + F_{n+1}(f_{n+1})(i, x) \to +\infty \quad \text{as } x \to \infty. \tag{4.2}$$

Remark 4.1. Condition (4.1) means that the fixed cost of ordering in a given period with demand state i should be no less than the expected fixed cost of ordering in the next period. The condition is a generalization of the similar conditions used in the standard models. It includes the cases of the constant ordering costs ($K_n^i = K$, $\forall i, n$) and the nonincreasing ordering costs ($K_n^i \geqslant K_{n+1}^j$, $\forall i, j, n$). The latter case may arise on account of the learning curve effect associated with fixed ordering costs over time. Moreover, when all the future costs are calculated in terms of their present values, even if the undiscounted fixed cost may increase over time, Condition (4.1) still holds as long as the rate of increase of the fixed cost over time is less than or equal to the discount rate.

Remark 4.2. Condition (4.2) means that either the unit ordering cost $c_n^i > 0$ or the expected holding cost $F_{n+1}(f_{n+1})(i, x) \to +\infty$ as $x \to \infty$, or both. Condition (4.2) is borne out of practical considerations and is not very restrictive. In addition, it rules out such unrealistic trivial cases as the one with $c_n^i = 0$ and $f_n(i, x) = 0$, $x \geqslant 0$, for each i and n, which implies ordering an infinite amount whenever an order is placed. The condition generalizes the usual assumptions made by Scarf (1960) and others that the unit inventory carrying cost $h > 0$. Furthermore, we need not to impose a condition like (4.2) on the backlog side assumed in Bensoussan et al. (1983) because of an essential asymmetry between the inventory side and the backlog side. Whereas we can order any number of units

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to decrease backlog or build inventory, it is not possible to sell anything more than the demand to decrease inventory or increase backlog. If it were possible, then the condition like (4.2) as $x \to -\infty$ would be needed to make backlog more expensive than the revenue obtained by sale of units, asymptotically. In the special case of stationary linear backlog costs, this would imply $p > c$ (or $p > \alpha c$ if costs are discounted at the rate α, $0 < \alpha \leqslant 1$), where p is the unit backlog cost. But since revenue-producing sales are not allowed, we are able to dispense with the condition like (4.2) on the backlog side or the standard assumption $p > c$ (or $p > \alpha c$) as in Scarf (1960) and others or the strong assumption $p > \alpha c^i$ for each i as in Song and Zipkin (1993).

Remark 4.3. In the standard model with $L = 1$, Veinott (1966) gives an alternative proof to the one by Scarf (1960) based on K-convexity. For this, he does not need a condition like (4.2), but requires other assumptions instead.

Definition 4.1. A function $g: R \to R$ is said to be *K-convex*, $K \geqslant 0$, if it satisfies the property

$$K + g(z + y) \geqslant g(y) + z\,\frac{g(y) - g(y - b)}{b},$$

$$\forall\, z \geqslant 0,\ b > 0,\ y. \tag{4.3}$$

Definition 4.2. A function g: $R \to D$, where D is a convex subset $D \subset R$, is K-convex if (4.3) holds whenever $y + z$, y, and $y - b$ are in D.

Required well-known results on K-convex functions or their extensions are collected in the following two propositions (cf. Bertsekas 1978 or Bensoussan et al. 1983).

Proposition 4.1. (i) *If g: $R \to R$ is K-convex, it is L-convex for any $L \geqslant K$. In particular, if g is convex, i.e., 0-convex, it is also K-convex for any $K \geqslant 0$.*

(ii) *If g_1 is K-convex and g_2 is L-convex, then for α, $\beta \geqslant 0$, $\alpha g_1 + \beta g_2$ is $(\alpha K + \beta L)$-convex.*

(iii) *If g* is K-convex, and ξ is a random variable such that $\mathrm{E}|g(x - \xi)| < \infty$, *then $Eg(x - \xi)$ is also K-convex.*

(iv) *Restriction of g on any convex set $D \subset R$ is K-convex.*

Proof. Proposition 4.1 (i)–(iii) is proved in Bertsekas (1978). The proof of (iv) is straightforward. □

Proposition 4.2. *Let g: $R \to R$ be a K-convex lower semicontinuous (l.s.c.) function such that $g(x) \to +\infty$ as $x \to +\infty$. Let A and B, $A \leqslant B$, be two extended real numbers with the understanding that the closed interval $[A, B]$ becomes open at A (or B) if $A = -\infty$ (or $B = \infty$). Let the notation $g(-\infty)$ denote the extended real number $\liminf_{x \to -\infty} g(x)$. Let*

$$g^* = \inf_{A \leqslant x \leqslant B} g(x) > -\infty. \tag{4.4}$$

Define the extended real numbers S and s, $S \geqslant s \geqslant -\infty$ as follows:

$$S = \min\{x \in R \cup \{-\infty\} | g(x) = g^*, A \leqslant x \leqslant B\}, \tag{4.5}$$

$$s = \min\{x \in R \cup \{-\infty\} | g(x) \leqslant K + g(S), A \leqslant x \leqslant S\}. \tag{4.6}$$

Then

(i) *$g(x) \geqslant g(S)$, $\forall x \in [A, B]$;*

(ii) *$g(x) \leqslant g(y) + K$, $\forall x, y$ with $s \leqslant x \leqslant y \leqslant B$;*

(iii)

$$h(x) \equiv \inf_{y \geqslant x, A \leqslant y \leqslant B} [K\delta(y - x) + g(y)]$$

$$= \begin{Bmatrix} K + g(S) & \text{for } x < s, \\ g(x), & \text{for } s \leqslant x \leqslant B \end{Bmatrix},$$

and h: $(-\infty, B] \to R$ is l.s.c.; moreover, if g is continuous, $A = -\infty$, and $B = \infty$, then h: $R \to R$ is continuous;

(iv) *h is K-convex on $(-\infty, B]$.*

Moreover, if $s > A$, then

(v) *$g(s) = K + g(S)$;*

(vi) *$g(x)$ is strictly decreasing on $(A, s]$.*

Proof. When $s > A$, it is easy to see from (4.6) that (v) holds and that $g(x) > g(s)$ for $x \in (A, s)$. Furthermore for $A < x_1 < x_2 < s$, K-convexity implies

$$K + g(S) \geqslant g(x_2) + \frac{S - x_2}{x_2 - x_1}[g(x_2) - g(x_1)].$$

According to (4.6), $g(x_2) > K + g(S)$ when $A < x_2 < s$. Then, it is easy to conclude that $g(x_2) < g(x_1)$. This proves (vi).

As for (i), it follows directly from (4.4) and (4.5). Property (ii) holds trivially for $x = y$, holds for $x = S$ in view of (i), and holds for $x = s$ since $g(s) \leqslant K + g(S) \leqslant K + g(y)$ from (4.6) and (i). We need now to examine two other possibilities: (a) $S < x < y \leqslant B$ and (b) $s < x < S$, $x < y \leqslant B$.

In case (a), let $z = S$ if $S > -\infty$ and $z \in (S, x)$ if $S = -\infty$. By K-convexity of g, we have

$$K + g(y) \geqslant g(x) + \frac{y - x}{x - z}[g(x) - g(z)].$$

Use (i) to conclude $K + g(y) \geqslant g(x)$ if $S > -\infty$. If $S = -\infty$, let $z \to -\infty$. Since $\liminf g(z) = g^* < \infty$, we can once again conclude $K + g(y) \geqslant g(x)$.

In case (b), if $s = -\infty$, let $z \in (-\infty, x)$. Then $K + g(S) \geqslant g(x) + [(S - x)/(x - z)][g(x) - g(z)]$. Let $z \to -\infty$. Since $\liminf g(z) \leqslant K + g^* < \infty$, we can use (i) to conclude $g(x) \leqslant K + g(S) \leqslant K + g(y)$. If $s > -\infty$, then by K-convexity of g and (v),

$$K + g(S) \geqslant g(x) + \frac{S - x}{x - s}[g(x) - g(s)] \geqslant g(x)$$

$$+ \frac{S - x}{x - s}[g(x) - g(S) - K],$$

which leads to

$$\left(1 + \frac{S - x}{x - s}\right)[K + g(S)] \geqslant \left(1 + \frac{S - x}{x - s}\right)g(x).$$

Dividing both sides of the above inequality by $[1 + (S - x)/(x - s)]$ and using (i), we obtain $g(x) \leqslant K + g(S) \leqslant K + g(y)$, which completes the proof of (ii).

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To prove (iii), it follows easily from (i) and (ii) that $h(x)$ equals the right-hand side. Moreover, since g is l.s.c. and $g(s) \leqslant K + g(S)$, h: $R \to R$ is also l.s.c. The last part of (iii) is obvious.

Next, if $s = -\infty$, then $h(x) = g(x)$, $x \in (-\infty, B]$, and (iv) follows from Proposition 4.1(iv). Now let $s > -\infty$. Therefore, $S > -\infty$. Note from (4.6) that $s \geqslant A$. To show (iv), we need to verify

$$K + h(y+z) - \left[h(y) + z \frac{h(y) - h(y-b)}{b} \right] \geqslant 0,$$

in the following four cases: (a) $s \leqslant y - b < y \leqslant y + z \leqslant B$, (b) $y - b < y \leqslant y + z < s$, (c) $y - b < s \leqslant y \leqslant y + z \leqslant B$, and (d) $y - b < y < s \leqslant y + z \leqslant B$. The proof in cases (a) and (b) is obvious from the definition of $h(x)$.

In case (c), the definition of $h(x)$ and the fact that $g(s) \leqslant g(S) + K$ imply

$$K + h(y+z) - \left[h(y) + z \frac{h(y) - h(y-b)}{b} \right]$$
$$= K + g(y+z) - \left[g(y) + z \frac{g(y) - g(S) - K}{b} \right]$$
$$\geqslant K + g(y+z) - \left[g(y) + z \frac{g(y) - g(s)}{b} \right]. \tag{4.7}$$

Clearly if $g(y) \leqslant g(s)$, then the right-hand side of (4.7) is nonnegative in view of (ii). If $g(y) > g(s)$, then $y > s$ and $b > y - s$, and we can use K-convexity of g to conclude

$$K + g(y+z) - \left[g(y) + z \frac{g(y) - g(s)}{b} \right]$$
$$> K + g(y+z) - \left[g(y) + z \frac{g(y) - g(s)}{y - s} \right] \geqslant 0.$$

In case (d), we have from the definition of $h(x)$ and (i) that

$$K + h(y+z) - \left[h(y) + z \frac{h(y) - h(y-b)}{b} \right]$$
$$= K + g(y+z) - [g(S) + K] = g(y+z)$$
$$- g(S) \geqslant 0. \quad \square$$

Remark 4.4. To our knowledge, Definition 4.2 of K-convex functions defined on an interval of the real line rather than on the whole real line is new. Proposition 4.2 is an important extension of similar results found in the literature (see Lemma (d) in Bertsekas 1978, p. 85). The extension allows us to prove easily the optimality of (s, S)-type policy without imposing a condition like (4.2) for $x \to -\infty$ and with capacity constraints discussed later in Section 5. Note that Proposition 4.2 allows the possibility of $s = -\infty$ or $s = S = -\infty$; such an (s, S) pair simply means that it is optimal not to order.

We can now derive the following result.

Theorem 4.1. *Assume* (4.1) *and* (4.2) *in addition to the assumptions made in Section* 2. *Then there exists a sequence of numbers* s_n^i, S_n^i, $n \in \langle 0, N-1 \rangle$, $i \in I$, *with* $s_n^i \leqslant S_n^i$, *such that the optimal feedback policy is*

$$\hat{u}_n(i, x) = (S_n^i - x)\delta(s_n^i - x). \tag{4.8}$$

Proof. The dynamic programming Equations (3.2) can be written as follows:

$$\begin{cases} v_n(i, x) = f_n(i, x) - c_n^i x + h_n(i, x), \\ \qquad\qquad \text{for } n \in \langle 0, N-1 \rangle,\ i \in I, \\ v_N(i, x) = f_N(i, x), \\ \qquad\qquad i \in I, \end{cases} \tag{4.9}$$

where

$$h_n(i, x) = \inf_{y \geqslant x} [K_n^i \delta(y - x) + z_n(i, y)], \text{ and} \tag{4.10}$$

$$z_n(i, y) = c_n^i y + F_{n+1}(v_{n+1})(i, y). \tag{4.11}$$

From (2.1) and (3.2), we have $v_n(i, x) \geqslant f_n(i, x)$, $\forall n \in \langle 0, N-1 \rangle$. From Theorem 3.1, we know that $v_n \in C_1$. These along with (4.2) ensure for $n \in \langle 0, N-1 \rangle$ and $i \in I$, that $z_n(i, y) \to +\infty$ as $y \to \infty$, and $z_n(i, y)$ is uniformly continuous.

In order to apply Proposition 4.2 to obtain (4.8), we need only to prove that $z_n(i, x)$ is K_n^i-convex. According to Proposition 4.1, it is sufficient to show that $v_{n+1}(i, x)$ is K_{n+1}^i-convex. This is done by induction. First, $v_N(i, x)$ is convex by definition and, therefore, K-convex for any $K \geqslant 0$. Let us now assume that for a given $n \leqslant N - 1$ and i, $v_{n+1}(i, x)$ is K_{n+1}^i-convex. By Proposition 4.1 and Assumption (4.1), it is easy to see that $z_n(i, x)$ is $\bar{K}_{n+1}^i$-convex, hence also K_n^i-convex. Then, Proposition 4.2 implies that $h_n(i, x)$ is K_n^i-convex. Therefore, $v_n(i, x)$ is K_n^i-convex. This completes the induction argument.

Thus, it follows that $z_n(i, x)$ is K_n^i-convex for each n and i. Since $z_n(i, y) \to +\infty$ when $y \to \infty$, we apply Proposition 4.2 to obtain the desired s_n^i and S_n^i. According to Theorem 3.2, the (s, S)-type policy defined in (4.8) is optimal. $\square$

Remark 4.5. Theorem 4.1 can be extended easily to allow for a constant lead time in the delivery of orders. The usual approach is to replace the surplus level by the so-called *surplus position*. It can also be generalized to Markovian demands with discrete components and countably many states.

5. CONSTRAINED MODELS

In this section we incorporate some additional constraints that arise often in practice. We show that (s, S) policies continue to remain optimal for the extended models.

5.1. An (s, S) Model with No-ordering Periods

Consider the special situation in which ordering is not possible in certain periods (e.g., suppliers do not accept orders on weekends), we shall show that the following theorem holds in such a situation.

Theorem 5.1. *In the dynamic inventory problem with some no-ordering periods, the optimal policy is still of* (s, S) *type for any period except when the ordering is not allowed.*

Proof. To stay with our earlier notation, it is no loss of generality to continue assuming the setup cost to be K_m^i in

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a no-ordering period m with the demand state i; clearly, setup costs are of no use in no-ordering periods. The definition (4.10) is revised as

$$h_n(i, x) = \begin{cases} \inf_{y \geq x} [K_n^i \delta(y - x) + z_n(i, y)], \\ \quad \text{if ordering is allowed in period } n, \\ z_n(i, x), \\ \quad \text{if ordering is not allowed in period } n, \end{cases} \tag{5.1}$$

and $z_n(i, y)$ is defined as before in (4.11). Using the same induction argument as in the proof of Theorem 4.1, we can show that $h_n(i, x)$ and $v_n(i, x)$ are K_n^i-convex if ordering is allowed in period n. If ordering is disallowed in period n, then $h_n(i, x) = z_n(i, x)$, which is $\bar{K}_{n+1}^i$-convex, and therefore, also K_n^i-convex. In both cases, therefore, $v_n(i, x)$ is K_n^i-convex. □

Remark 5.1. Theorem 5.1 can be easily generalized to allow for supply uncertainty as in Parlar et al. (forthcoming). One needs to replace u_k in (2.4) by $a_k u_k$, where $Pr\{a_k = 1 | i_k = i\} = q_k^i$ and $Pr\{a_k = 0 | i_k = i\} = 1 - q_k^i$, and to modify (3.2) appropriately.

5.2. An (s, S) Model with Storage and Service Constraints

Let $B < \infty$ denote an upper bound on the inventory level. Moreover, to guarantee a reasonable measure of service, we shall follow Kumar (1992) and introduce a chance constraint requiring that the probability of the ending inventory falling below a certain penetration level, say C_{min}, in any given period does not exceed a given β, $0 < \beta < 1$. Thus, $Pr\{x_{k+1} \leq C_{min}\} \leq \beta$, $k \in \langle 0, N-1 \rangle$. Given the demand state i in period k, it is easy to convert the above condition into $x_k + u_k \geq A_k^i \equiv C_{min} + \Phi_{i,k}^{-1}(1 - \beta)$, where $\Phi_{i,k}^{-1}(\cdot)$ is defined as $\Phi_{i,k}^{-1}(z) = \inf(x | \Phi_{i,k}(x) \geq z)$.

The dynamic programming equations can be written as (4.11), where $z_n(i, y)$ is as in (4.11) and

$$h_n(i, x) = \inf_{y \geq x, A_n^i \leq y \leq B} [K_n^i \delta(y - x) + z_n(i, y)], \tag{5.2}$$

provided $A_n^i \leq B$, $n \in \langle 0, N-1 \rangle$, $i \in I$; if not, then there is no feasible solution, and $h_n(i, x) = \inf \emptyset \equiv \infty$. This time, since y is bounded by $B < \infty$, Theorem 3.1 can be relaxed as follows.

Theorem 5.2. *The dynamic programming Equations* (4.9) *with* (5.2) *define a sequence of l.s.c. functions on* $(-\infty, B]$. *Moreover, there exists a function* $\hat{u}_n(i, x)$ *in* B_0, *which attains the infimum in* (4.9) *for any* $x \in (-\infty, B]$.

With $\hat{u}_n(i, x)$ of Theorem 5.2, it is possible to prove Theorem 3.2 as the verification theorem also for the constrained case. We now show that the optimal policy is of (s, S) type.

Theorem 5.3. *There exists a sequence of numbers* s_n^i, S_n^i, $n \in \langle 0, N-1 \rangle$, $i \in I$, *with* $s_n^i \leq S_n^i$, $A_n^i \leq s_n^i$, *and* $B \geq S_n^i$, *such that the feedback policy* $\hat{u}_n(i, x) = (S_n^i - x)\delta(s_n^i - x)$ *is optimal for the model with capacity and service constraints defined above.*

Proof. First note that Proposition 4.2 holds when g is l.s.c. and K-convex on $(-\infty, B]$, $B < \infty$. Also, by Proposition 4.1 (iii) and (iv), one can see that $Eg(x - \xi)$ is K-convex on $(-\infty, B]$ since $\xi \geq 0$. Because g is l.s.c., it is easily seen that $Eg(x - \xi)$ is l.s.c. on $(-\infty, B]$. Furthermore, by Theorem 5.2, v_n is l.s.c. on $(-\infty, B]$. With these observations in mind, the proof of Theorem 4.1 can be easily modified to complete the proof. □

Remark 5.2. A constant integer lead time $\tau \geq 1$ can also be included in this model with the surplus level replaced by the surplus position and with the lower bound A_k^i properly redefined in terms of the distribution of the total demand during the lead time.

6. THE NONSTATIONARY INFINITE HORIZON PROBLEM

We now consider an infinite horizon version of the problem formulated in Section 2. By letting $N = \infty$ and $U = (u_n, u_{n+1}, \ldots)$, the extended real-valued objective function of the problem becomes

$$J_n(i, x; U) = \sum_{k=n}^{\infty} \alpha^{k-n} E[c_k(i_k, u_k) + f_k(i_k, x_k)], \tag{6.1}$$

where α is a given discount factor, $0 < \alpha \leq 1$. The dynamic programming equations are:

$$\begin{aligned} v_n(i, x) &= f_n(i, x) + \inf_{u \geq 0} \{c_n(i, u) \\ &\quad + \alpha F_{n+1}(v_{n+1})(i, x + u)\}, \qquad (6.2) \\ &\qquad n = 0, 1, 2, \ldots. \end{aligned}$$

In what follows, we shall show that there exists a solution of (6.2) in class C_1, which is the value function of the infinite horizon problem; see also Remark 6.1. Moreover, the decision that attains the infimum in (6.2) is an optimal feedback policy. Our method is that of successive approximation of the infinite horizon problem by longer and longer finite horizon problems.

Let us, therefore, examine the finite horizon approximation $J_{n,k}(i, x; U)$ of (6.1), which is obtained by the first k-period truncation of the infinite horizon problem of minimizing $J_n(i, x; U)$, i.e.,

$$J_{n,k}(i, x; U) = \sum_{l=n}^{n+k-1} E[c_l(i_l, u_l) + f_l(i_l, x_l)]\alpha^{l-n}. \tag{6.3}$$

Let $v_{n,k}(i, x)$ be the value function of the truncated problem, i.e.,

$$v_{n,k}(i, x) = \inf_{U \in \mathcal{U}} J_{n,k}(i, x; U). \tag{6.4}$$

Since (6.4) is a finite horizon problem on the interval $\langle n, n+k \rangle$, we may apply Theorems 3.1 and 3.2 and obtain its value function by solving the dynamic programming equations

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$$\begin{cases} v_{n,k+1}(i,x) = f_n(i,x) \\ \qquad + \inf_{u \geq 0} \{c_n(i,u) + \alpha F_{n+1}(v_{n+1,k})(i, x+u)\}, \\ v_{n+k,0}(i,x) = 0. \end{cases} \quad (6.5)$$

Moreover, $v_{n,0}(i, x) = 0$, $v_{n,k} \in C_1$, and the infimum in (6.4) is attained.

It is not difficult to see that the value function $v_{n,k}$ increases in k. In order to take its limit as $k \to \infty$, we need to establish an upper bound on $v_{n,k}$. One possible upper bound on $\inf_{U \in \mathcal{U}} J_n(i, x; u)$ can be obtained by computing the objective function value associated with a policy of ordering nothing ever. With the notation $\mathbf{0} = \{0, 0, \ldots\}$, let us write

$$w_n(i,x) = J_n(i,x;\mathbf{0}) = f_n(i,x) + E\left\{ \sum_{k=n+1}^{\infty} \alpha^{k-n} f_k\left(i_k, x - \sum_{j=1}^{k-1} \xi_j\right) \middle| i_n = i \right\}. \quad (6.6)$$

In a way similar to Bensoussan et al. (1983, p. 299–300), it is easy to see that given (2.2), $w_n(i, x)$ is well defined and is in C_1. Furthermore in class C_1, w_n is the unique solution of

$$w_n(i,x) = f_n(i,x) + \alpha F_{n+1}(w_{n+1})(i,x). \quad (6.7)$$

We can state the following result for the infinite horizon problem; see appendix for its proof.

Theorem 6.1. *Assume* (2.1) *and* (2.2). *Then we have*

$$0 = v_{n,0} \leq v_{n,1} \leq \cdots \leq v_{n,k} \leq w_n, \quad (6.8)$$

and

$$v_{n,k} \uparrow v_n, \textit{ a solution of (6.2) in } B_1. \quad (6.9)$$

Furthermore, $v_n \in C_1$ *and we can obtain* $\hat{U} = \{\hat{u}_n, \hat{u}_{n+1}, \ldots\}$ *for which the infimum in* (6.2) *is attained. Moreover,* $\hat{U}$ *is an optimal feedback policy, i.e.,*

$$v_n(i,x) = \min_{U \in \mathcal{U}} J_n(i,x;U) = J_n(i,x;\hat{U}). \quad (6.10)$$

Remark 6.1. We should indicate that Theorem 6.1 does not imply that there is a unique solution of the dynamic programming equations (6.2). There may well be other solutions. Moreover, one can show that the value function is the minimal positive solution of (6.2). It is also possible to obtain a uniqueness proof under additional assumptions.

With Theorem 6.1 in hand, we can now prove the optimality of an (s, S) policy for the nonstationary infinite horizon problem.

Theorem 6.2. *Assume* (2.1), (2.2), *and* (4.2) *hold for the infinite horizon problem. Then, there exists a sequence of numbers* s_n^i, S_n^i, $n = 0, 1, \ldots$, *with* $s_n^i \leq S_n^i$ *for each* $i \in I$, *such that the optimal feedback policy is* $\hat{u}_n(i, x) = (S_n^i - x)\delta(s_n^i - x)$.

Proof. Let v_n denote the value function. Define the functions z_n and h_n as in Section 4. We know that $z_n(i, x) \to \infty$ as $x \to +\infty$ and $z_n(i, x) \in C_1$ for all n and $i \in I$.

We now prove that v_n is K_n^i-convex. Using the same induction as in Section 4, we can show that $v_{n,k}(i, x)$ defined in (6.4) is K_n^i-convex. This induction is possible since we know that $v_{n,k}(i, x)$ satisfies the dynamic programming equations (6.5). It is clear from the definition of K-convexity and from taking the limit as $k \to \infty$, that the value function $v_n(i, x)$ is also K_n^i-convex.

From Theorem 6.1, we know that $v_n \in C_1$ and that v_n satisfies the dynamic programming equations (6.2). Therefore, we can obtain an optimal feedback policy $\hat{U} = \{\hat{u}_n, \hat{u}_{n+1}, \ldots\}$ that attains the infimum in (6.2). Because v_n is K_n^i-convex, $\hat{u}_n$ can be expressed as in Theorem 6.2. □

7. THE CYCLICAL DEMAND MODEL

Cyclic or seasonal demand often arises in practice. Such a demand represents a special case of the Markovian demand, where the number of demand states L is given by the cycle length, and

$$p_{ij} = \begin{cases} 1, & \text{if } j = i+1,\ i = 1, \ldots, L-1, \text{ or } i = L,\ j = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore, we assume that the cost functions and density functions are all time invariant. The result is considerably simplified optimal policy, i.e., only L pairs of (s_n, S_n) need to be computed.

We can state the following corollary to Theorem 6.2.

Corollary 7.1. *In the infinite horizon inventory problem with the demand cycle of L periods, let* n_1 *and* n_2 $(n_1 < n_2)$ *be any two periods such that* $n_2 = n_1 + m \cdot L$, $m = 1, 2, \ldots$. *Then, we have* $s_{n_1} = s_{n_2}$ *and* $S_{n_1} = S_{n_2}$.

8. CONCLUDING REMARKS

This paper develops various more realistic extensions of the classical dynamic inventory model with stochastic demands. The models consider demands that are dependent on a finite state Markov chain including demands that are cyclic. Some constraints commonly encountered in practice, namely no-ordering periods, finite storage capacities, and service levels, are also treated. Both finite and infinite horizon cases are studied. It is shown that all these models, not unlike the classical model, exhibit the optimality of (s, S) policies.

In some real-life situations, the demand unsatisfied is often lost instead of backlogged as assumed in this paper and frequently in the literature on (s, S) models. An extension of the (s, S)-type results presented in this paper to the lost sales case is given in Cheng and Sethi (forthcoming).

APPENDIX

PROOF OF THEOREM 6.1

By definition, $v_{n,0} = 0$. Let $\tilde{U}_{n,k} = \{\tilde{u}_n, \tilde{u}_{n+1}, \ldots, \tilde{u}_{n+k}\}$ be a minimizer of (6.3). Thus,

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$$v_{n,k}(i, x) = J_{n,k}(i, x; \tilde{U}_{n,k}) \geq J_{n,k-1}(i, x; \tilde{U}_{n,k})$$
$$\geq \min_{U \in \mathcal{U}} J_{n,k-1}(i, x; U) = v_{n,k-1}(i, x).$$

It is also obvious from (6.3) and (6.6) that $v_{n,k}(i, x) \leq J_{n,k}(i, x; \mathbf{0}) \leq w_n(i, x)$. This proves (6.8). Since $v_{n,k} \in C_1$, we have

$$v_{n,k}(i, x) \uparrow v_n(i, x) \leq w_n(i, x), \tag{A.1}$$

with $v_n(i, x)$ l.s.c., and hence in B_1. Next, we show that v_n satisfies the dynamic programming equations (6.2). Observe from (6.5) and (6.8) that for each k, we have

$$v_{n,k}(i, x) \leq f_n(i, x) + \inf_{u \geq 0} \{c_n(i, u) + \alpha F_{n+1}(v_{n+1,k})(i, x + u)\}.$$

Thus, in view of (A.1), we obtain

$$v_n(i, x) \leq f_n(i, x) + \inf_{u \geq 0} \{c_n(i, u) + \alpha F_{n+1}(v_{n+1})(i, x + u)\}. \tag{A.2}$$

In order to obtain the reverse inequality, let $\hat{u}_{n,k}$ attain the infimum on the right-hand side of (6.5). From Assumptions (2.1) and (2.2), we obtain that

$$c_n^i \hat{u}_{n,k}(i, x) \leq \alpha F_{n+1}(v_{n+1,k})(i, x) \leq \alpha(1 + M)\|w_n\|(1 + |x|),$$

where $\|\cdot\|$ is the norm defined on B_1, i.e., for $b \in B_1$,

$$\|b\| = \max_i \sup_x \frac{b(i, x)}{1 + |x|}.$$

This provides us with the bound

$$0 \leq \hat{u}_{n,k}(i, x) \leq M_n(1 + |x|). \tag{A.3}$$

Let $k < l$, so that we can deduce from (6.5):

$$\begin{cases} v_{n,l+1}(i, x) = f_n(i, x) + c_n(i, \hat{u}_{n,l}(x)) \\ \qquad + \alpha F_{n+1}(v_{n+1,l})(i, x + \hat{u}_{n,l}(x)) \\ \qquad \geq f_n(i, x) + c_n(i, \hat{u}_{n,l}(x)) \\ \qquad + \alpha F_{n+1}(v_{n+1,k})(i, x + \hat{u}_{n,l}(x)) \end{cases} \tag{A.4}$$

Fix k and let $l \to \infty$. In view of (A.3), we can, for any given n, i and x, extract a subsequence $\hat{u}_{n,l'}(i, x)$ such that $\hat{u}_{n,l'}(i, x) \to \bar{u}_n(i, x)$. Since $v_{n+1,k}$ is uniformly continuous and c_n is l.s.c., we can pass to the limit on the right-hand side of (A.4). We obtain (noting that the left-hand side converges as well)

$$\begin{aligned} v_n(i, x) &\geq f_n(i, x) \\ &\quad + c_n(i, \bar{u}_n(x)) + \alpha F_{n+1}(v_{n+1,k}) \\ &\quad \cdot (i, x + \bar{u}_n(x)) \geq f_n(i, x) \\ &\quad + \inf_{u \geq 0} \{c_n(i, u) \\ &\qquad + \alpha F_{n+1}(v_{n+1,k})(i, x + u)\}. \end{aligned}$$

This along with (A.2), (A.1) and the fact that $v_n(i, x) \in B_1$ proves (6.9).

Next we prove that $v_n \in C_1$. Let us consider problem (6.3) again. By definition (6.1),

$$\begin{aligned} &J_n(i, x'; U) - J_n(i, x; U) \\ &= \sum_{l=n}^{\infty} \alpha^{l-n} E\Big[f_l\Big(i_l, x' + \sum_{j=n}^{l-1} u_j - \sum_{j=n+1}^{l} \xi_l\Big) \\ &\qquad - f_l(i_l, x + \sum_{j=n}^{l-1} u_j - \sum_{j=n+1}^{l} \xi_l) \Big]. \end{aligned}$$

From Assumption (2.2), we have

$$|J_{n,k}(i, x'; U) - J_{n,k}(i, x; U)| \leq \sum_{l=n}^{\infty} \alpha^{l-n} C|x' - x|$$
$$= C|x' - x|/(1 - \alpha),$$

which implies $|v_{n,k}(i, x') - v_{n,k}(i, x)| \leq C|x' - x|/(1 - \alpha)$. By taking the limit as $k \to \infty$, we have $|v_n(i, x') - v_n(i, x)| \leq C|x' - x|/(1 - \alpha)$, from which it follows that $v_n \in C_1$. Therefore, there exists a function $\hat{u}_n(i, x)$ in B_0 such that

$$c_n(i, \hat{u}_n(i, x)) + \alpha F_{n+1}(v_{n+1})(i, x + \hat{u}_n(i, x))$$
$$= \inf_{u \geq 0} \{c_n(i, u) + \alpha F_{n+1}(v_{n+1})(i, x + u)\}.$$

Hence, we have

$$v_n(i, x) = J_n(i, x; \hat{U}) \geq \inf_{U \in \mathcal{U}} J_n(i, x; U).$$

But for any arbitrary admissible control U, we also know that $v_n(i, x) \leq J_n(i, x; U)$. Therefore, we conclude that

$$v_n(i, x) = J_n(i, x; \hat{U}) = \min_{U \in \mathcal{U}} J_n(i, x; U). \qquad \square$$

ACKNOWLEDGMENT

This research was supported in part by NSERC grant A4619 and Canadian Centre for Marketing Information Technologies. The authors would like to thank Vinay Kanetkar, Dmitry Krass, Ernst Presman, Dirk Beyer, four anonymous referees, and two anonymous associate editors for their comments on earlier versions of this paper.

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