

# The Censored Newsvendor Problem with Newsvendor Demand Distributions

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*Abstract:* We study the dynamic newsvendor problem with censored demand for both perishable and storable inventory models. We focus on the cases when demand is represented by a member of the newsvendor family of distributions. We show that the Weibull density is the only member of newsvendor distributions for which the optimal solution can be expressed in scalable form. Consequently, scalability yields sufficient **dimensionality** reduction so that when inventory is perishable, we can derive the optimal solution and cost in easily computable simple recursions for the Weibull demand and in explicit closed form for the exponential demand. Moreover, for the storable inventory problem with exponential demand, scalability is shown to lead to convexity and recursion for the cost function, thereby yielding efficient computation of the unique optimal solution.

*Key words:* inventory; stochastic demand; lost sales; scalability; optimal policies.

# 1 Introduction

In the single period Newsvendor Problem (NP), the decision maker must choose the quantity to order before the realization of demand. Any demand in excess of stock is lost while any stock in excess of demand perishes because it is either salvaged or disposed. In an optimal solution the target inventory is set so that the probability of meeting all demand is equal to the *newsvendor fractile*; the newsvendor fractile is the probability that equates the expected marginal cost of overstocking to the expected marginal cost of understocking.

In the dynamic newsvendor problem (DNP), at the start of each of  $N$  periods, the decision maker must choose the order quantity before the realization of demand in that period. Demand in excess of current stock is lost or backordered. If the stock in excess of demand perishes then DNP reduces to determining the target inventory level in a series of NPs, one for each period. However, this need not be the case when inventory is storable; this is because leftover stock is available for future use, thereby interlinking consecutive periods.

When leftover stock is available to fill future demand, leftover stock in each period is charged a holding cost and becomes part of the initial inventory of the next period. Initially formulated by Dvoretzky et al. (1952), the structure of the resulting dynamic program has been extensively studied thereafter; the work of Veinott (1965) is especially significant. This is because Veinott (1965) established that when all economic parameters are stationary, the optimal solution can be reduced to a myopic solution if inventory at the end of the planning horizon can be sold back to the supplier.

While it is the case that DNP can have optimal solutions that are myopic, this need not be the case when the parameters that describe the uncertainty in demand are not known precisely. This is because estimates of parameters are updated dynamically, resulting in non-stationarity of the demand process. Consequently, analytical and computational difficulties arise in finding optimal solutions because dimensions of the state space that describes the dynamic program can become quite large. However, significant progress towards **dimensionality** reduction and therefore, towards computational tractability, can be made when a Bayesian scheme is used to update the demand distribution. Specifically, in full information scenarios like those initially studied by Scarf (1959, 1960) and substantially generalized by Azoury (1985), the key idea is that the dynamic program can be made more tractable if the decision problem is *scalable*. Demand distributions yielding this requisite scalability include the Weibull and Gamma families and the exponential, which is a member of both.

While these *full information* scenarios arise most naturally when demand in excess of available inventory is backordered, this is not the case when demand in excess of inventory is lost so that *sales* are recorded and information on demand gets *censored*. As observed by Harpaz et al. (1982), the analysis of DNP becomes significantly more difficult when less than full information is available. While significant progress has been made on elucidating the structure of the optimal solution, as in Lariviere and Porteus (1999), Ding et al. (2002), Lu et al. (2006, 2008), Bensoussan et al. (2007, 2008, 2009a, 2009b), Bisi and Dada (2007), and Chen and Plambeck

(2008), computing optimal solutions remains a difficult problem in general. As noted by Chen and Plambeck (2008) and Chen (2009), scalability, since it leads to dimensional reducibility, may hold the key to computing optimal solutions exactly.

To examine this dimensional reducibility closely, we focus on problems whose demand distributions are scalable. Hence, we consider those cases where the demand is from the family of newsvendor distributions, which includes the Weibull distribution. This is because the family of newsvendor distributions, as noted by Braden and Freimer (1991), appears to be the only one whose updated distributions with censored information have conjugate priors under a Bayesian updating scheme. Subsequently, Lariviere and Porteus (1999) have shown that for the important case when the demand distribution is Weibull with a gamma prior, the finite-horizon DNP is scalable. In Theorem 1 and Corollary 2, we extend this result to show that Weibull is the only class of the newsvendor distributions for which optimal solutions are scalable. We further observe that these scalability results also extend to full demand information models, suggesting that the newsvendor family has limited potential for yielding scalable solutions.

In addition to yielding these theoretical properties of DNP, our analysis also simplifies the structure of the dynamic program to yield tractable computational schemes. In particular, when inventory is perishable, the resulting dynamic program is shown to have sufficient structure so that scalability yields tractable analytical solutions for the case of Weibull demand. Hence, it is possible to reduce finding the optimal solution to solving a series of one-step look ahead recursive equations that can be solved efficiently by backward induction for any finite horizon problem. Our method directly shows that each of these equations has a unique solution. The case of exponential demand is even simpler and yields explicit closed form expressions for the optimal order quantity and cost. This latter result is a refinement over Theorems 3(a) and (b) of Lariviere and Porteus (1999). In turn, taking limits appropriately results in a simple equation that easily yields the unique optimal solution for the infinite horizon discounted cost problem.

Although the DNP for the case of storable inventory is significantly more complex, we are also able to apply our approach to this case. We show that, analogous to the perishable case with Weibull demand, it is possible to reduce the problem to solving a series of one-step look ahead recursions when demand is exponential. After showing a structural result that relates the *standardized* cost with the prior distribution's shape parameter, we prove that for the exponential demand case, the cost function is convex so that the optimal solution can be computed uniquely. This provides theoretical justification for why the direct computation used by Chen (2009) finds the unique optimal solution. Finally, using the recursions we have conducted extensive computations to explore the structure of the optimal solution.

The paper is organized as follows. In §2 we describe the problem. In §3, for the perishable inventory model, we study the scalability property of the optimal solutions for newsvendor demand distributions. In this section, we also develop recursive equations and closed form solutions for the optimal order quantities and costs for the Weibull and exponential demands, respectively. In §4, we study the storable inventory model and discuss why optimal solutions can

be computed exactly and uniquely for the exponential demand case. Insights from numerical studies are presented in §5. We conclude the paper in §6. The proofs of algebraic nature are included in an Addendum; the remaining proofs except that of Theorem 1 are presented in the Appendix.

## 2 Problem Description and Related Properties

In our problem, the decision maker must choose the stocking quantity  $y_n$  at the beginning of each period  $n$ ,  $n = 1, 2, \dots, N$ . Then, the demand  $X_n$  is realized. We assume that the random demands  $X_n$ 's are generated such that given a value for the unknown parameter  $\theta$ , the conditional distributions of  $X_n$ 's are independent and identically distributed (*iid*) with a known probability density  $f(\cdot|\theta)$ ,  $\theta \in \Theta$ .

For each period denote the sales by  $s_n = \min(X_n, y_n)$ , where demand is exactly observed when sales are less than the stocking quantity, that is, when  $X_n < y_n$ ; and the demand is *censored* at the stocking quantity when sales equal  $y_n$ , that is, when  $X_n \geq y_n$ . The procurement cost for the newsvendor model in each period is a variable ordering cost of  $c$  per unit. Since we will formulate the problem in terms of minimizing cost, demand in excess of sales represents a loss of revenue (and other penalties) at the rate of  $p$  per unit. If the inventory is perishable, inventory in excess of demand, if any, is salvaged at a unit value of  $h$ . Otherwise, if the inventory is storable (non-perishable), a holding cost of  $h$  per unit is charged on leftover inventory at the end of each period. To rule out the trivial cases of ordering zero or holding infinite stock for speculative purposes, we assume  $h < c < p$  when inventory is perishable and  $c < p$  when inventory is storable.

Since the underlying demand parameter is unknown, we will use a Bayesian scheme to update its distribution over time. Let  $\hat{\pi}_{n+1}(\theta|x_n)$  be the posterior density in period  $n$  which equals  $\pi_{n+1}(\theta|x_n)$  if  $s_n = x_n < y_n$ , and  $\pi_{n+1}^c(\theta|y_n)$  if  $s_n = y_n$ , where

$$\pi_{n+1}(\theta|x_n) = \frac{f(x_n|\theta)\hat{\pi}_n(\theta)}{\int_{\Theta} f(x_n|\theta')\hat{\pi}_n(\theta')d\theta'} \quad \text{and} \quad \pi_{n+1}^c(\theta|y_n) = \frac{\int_{y_n}^{\infty} f(x|\theta)dx \hat{\pi}_n(\theta)}{\int_{\Theta} \int_{y_n}^{\infty} f(x|\theta')\hat{\pi}_n(\theta')dx d\theta'}.$$

Let us denote  $\psi_n(x|\hat{\pi}_n) = \int_{\Theta} f(x|\theta)\hat{\pi}_n(\theta|x_{n-1})d\theta$  and  $\Psi_n(x|\hat{\pi}_n) = \int_0^x \psi_n(s|\hat{\pi}_n)ds$  for the updated probability density and distribution function of  $X_n$ , respectively.

With censored data on demand, the compact analytical form of using conjugate priors does not exist in general. However, when demand is described by a member of the newsvendor family, as shown by Braden and Freimer (1991), the gamma distribution remains a conjugate prior and therefore, it may be possible to mimic the analytical tractability of the conjugate approach under full information. As defined by Braden and Freimer (1991), a random variable  $X$  is a member of the newsvendor class of distributions if its density is given by

$$f(x|\theta) = \theta d'(x)e^{-\theta d(x)}, \tag{1}$$

for some function  $d : (0, \infty) \rightarrow (0, \infty)$ , where the prime represents its derivative. For  $f(x|\theta)$  to be a valid density function on  $[0, \infty)$ , it is necessary and sufficient that

$$d'(x) \geq 0, \quad \lim_{x \rightarrow 0} d(x) = 0 \quad \text{and} \quad \lim_{x \rightarrow \infty} d(x) = \infty. \quad (2)$$

This can be established by considering the cumulative distribution function  $F(x|\theta) = 1 - e^{-\theta d(x)}$  which is required to be non-decreasing with  $\lim_{x \rightarrow 0} F(x|\theta) = 0$  and  $\lim_{x \rightarrow \infty} F(x|\theta) = 1$ . We restrict attention to choices of  $d(\cdot)$  for which  $d'(x) > 0$  for all  $x > 0$ , so that  $f(x|\theta)$  is positive on the whole of  $(0, \infty)$ . To ensure identifiability, we impose the condition  $d(1) = 1$ , since any newsvendor model with a given  $d(\cdot)$  can be transformed into a newsvendor model with  $d^*(1) = 1$  by rescaling the original  $d(x)$  function to  $d^*(x) = d(x)/d(1)$ , and applying the inverse scale to change  $\theta$  into  $\theta^* = d(1)\theta$ .

In (1), for  $d(x) = x^l$  with a known constant  $l > 0$ , we get the Weibull distribution and in particular, if  $l = 1$ , we get the exponential distribution. The prior density of  $\theta$  is given by a gamma density with shape and scale parameters  $a$  and  $S$ , respectively (denoted by  $\text{Gamma}(a, S)$  for later use). Setting  $a_1$  and  $S_1$  as the initial parameters in period 1, the sufficient statistics for  $a$  and  $S$  at the beginning of period  $n$  are given by  $a_n = a_1 + m_n$  and  $S_n = S_1 + \sum_{i=1}^{n-1} d(s_i)$ , where  $m_n$  denotes the number of exact demand observations by period  $n$  and  $s_i$  is the observed sales in period  $i$  (see Braden and Freimer 1991). Then the updated demand density and distribution function in period  $n$  are respectively given by

$$\psi_n(x|a_n, S_n) = \frac{a_n S_n^{a_n} d'(x)}{[S_n + d(x)]^{a_n+1}} \quad \text{and} \quad \Psi_n(x|a_n, S_n) = 1 - \left( \frac{S_n}{S_n + d(x)} \right)^{a_n}.$$

Now, following Lariviere and Porteus (1999), we call a decision variable *scalable* if it is of the form  $y_n = q(S_n)u(a_n)$ , so that  $y_n$  is separable into two terms, namely,  $q(S_n)$  that depends on the scale parameter  $S_n$  and  $u(a_n)$  that depends on the shape parameter  $a_n$  of period  $n$ . In this paper, we will examine when the optimal decision variable for the censored newsvendor problem with demand from the newsvendor family is scalable. We will further examine when  $u(a_n)$  has simple structure so that the optimal solution can either be expressed in closed/analytical form or be easily computed. To proceed, we will first discuss the perishable inventory model.

### 3 The Perishable Inventory Problem

In the perishable inventory problem, leftover inventory is salvaged at a value of  $h$  per unit at the end of each period. In this case, the single-period Bayesian expected cost with prior distribution  $\hat{\pi}_n$  and order quantity  $y_n$  is given by

$$M(\hat{\pi}_n, y_n) = cy_n - h \int_0^{y_n} (y_n - x) \psi_n(x|\hat{\pi}_n) dx + p \int_{y_n}^{\infty} (x - y_n) \psi_n(x|\hat{\pi}_n) dx.$$

Then, with the discount factor  $0 < \beta \leq 1$ , the optimality equations can be written as

$$V_n(\hat{\pi}_n) = \min_{y_n \geq 0} \left\{ M(\hat{\pi}_n, y_n) + \beta \int_0^{y_n} V_{n+1}(\pi_{n+1}(\cdot|x)) \psi_n(x|\hat{\pi}_n) dx \right. \\ \left. + \beta V_{n+1}(\pi_{n+1}^c(\cdot|y_n)) [1 - \Psi_n(y_n|\hat{\pi}_n)] \right\}, \quad (3)$$

for  $n = 1, 2, \dots, N$ , with the boundary condition  $V_{N+1}(\hat{\pi}_{N+1}) = 0$ , for all  $\hat{\pi}_{N+1}$ . We denote the optimal order quantity  $y_n^*$  in period  $n$  by  $y_n^{*c}$  if  $s_{n-1} = y_{n-1}$ , and  $y_n^{*e}$  if  $s_{n-1} < y_{n-1}$ .

To proceed with our analysis, we first show the following result on scalability. While the sufficiency part of the result is already proved by Lariviere and Porteus (1999), the necessity part is new and it shows that, when looking for scalable solutions, one need not look beyond the Weibull family which is a subclass of the newsvendor distributions.

**Theorem 1.** *If the demand distribution is from the newsvendor class, then the optimal order quantity is scalable if and only if the demand distribution is Weibull.*

Proof. (Necessity) It suffices to show that

$$d(xy) = d(x)d(y) \text{ for all } x, y > 0, \quad (4)$$

because the only function satisfying (2) and (4) is  $d(x) = x^l$  for some constant  $l > 0$  [To see this, let  $x = e^u$  and  $f(u) = d(e^u)$  for any real  $u$ . Then (4) is equivalent to  $d(e^{u+v}) = d(e^u)d(e^v)$  which is equivalent to  $f(u+v) = f(u)f(v)$  for all real  $u, v$ . The last identity is an equivalent characterization of the exponential function (see Rudin 1976, Chapter 8, Exercise 6). So  $f(u) = e^{lu}$  for some constant  $l$ . Thus,  $d(e^u) = e^{lu}$  for some constant  $l$ . But, since  $d(\cdot)$  is non-decreasing, we must have  $l > 0$ . Substituting  $x = e^u$ , we get  $d(x) = e^{l \log x} = x^l$ ].

Now, for the last period ( $n = N$ ), with prior  $\hat{\pi}_N = \text{Gamma}(a, S)$ , the optimal order quantity  $y_{N,a,S}^*$  is characterized by the equation

$$1 - \Psi_N(y_{N,a,S}^*|a, S) = \frac{c-h}{p-h}, \quad (5)$$

which leads to

$$\Psi_N(y_{N,a,1}^*|a, 1) = \Psi_N(y_{N,a,S}^*|a, S) = \Psi_N(q(S)y_{N,a,1}^*|a, S), \text{ for all } a > 0, \quad (6)$$

where the first equality follows from (5) and the second equality follows from scalability. Note that  $\Psi_N(y|a, 1) = 1 - (1 + d(y))^{-a}$ , and hence any  $y > 0$  can be identified with an  $y_{N,a,1}^*$  by appropriately choosing  $a > 0$ . Therefore, for any  $y > 0$ , from (6) we get

$$\Psi_N(q(S)y|a, S) = \Psi_N(y|a, 1), \quad (7)$$

for some  $a > 0$ . Now substitute  $\Psi_N(x|a, S) = 1 - (1 + d(x)/S)^{-a}$  in (7) to see

$$d(q(S)y) = Sd(y). \quad (8)$$

Setting  $y = 1$  in (8) gives  $d(q(S)) = S$ , hence  $q(S) = d^{-1}(S)$  so that  $q(\cdot)$  inherits the full  $(0, \infty)$  range of  $d(\cdot)$ . Taking  $x = q(S)$  in (8) then gives  $d(xy) = d(x)d(y)$ , as desired.

(Sufficiency) For the proof of the sufficiency part, see Theorem 2 of Lariviere and Porteus (1999) who proved the result for the storable (and hence, perishable) inventory problem. ■

In the above theorem, to prove the necessity part we only needed to analyze the last period's problem whose solution, for given values of  $a$  and  $S$ , is the same for both the censored as well as the observed demand models. For the sufficiency part, it is easy to see that Lariviere and Porteus (1999)'s proof also goes through when demands are fully observed. Therefore, Theorem 1 extends to full demand information models which is a simple but surprising result. In §4 we will show that Theorem 1 does also extend to storable inventory models.

In the following two subsections, we will address the models with Weibull demands. In particular, we will derive the optimal order quantity and cost for inventory models when demand distributions are Weibull and exponential which is a special case of the Weibull.

### 3.1 The Case of Weibull Demand Distribution

We first use Figure 1 to explain the underlying decision problem with perishable inventory. Suppose we are at node  $A$  and have made the order quantity decision  $y_{n,k}$  at the beginning of period  $n$  when the number of exact demand observations is  $k$ ,  $k = 0, 1, 2, \dots, n-1$ . Then, after the demand realizes for the period, one of the following two types of sample paths can be observed. Let  $c$  and  $e$  on the arrows represent the sample paths with censored and exact demands, respectively. If the demand in period  $n$  is censored, we denote the order quantity  $y_{n+1}^{*c}$  in period  $(n+1)$  by  $y_{n+1,k}$  because if we had  $k$  exact demand observations at the beginning of period  $n$ , then we will also have  $k$  exact demand observations at the beginning of period  $(n+1)$ . Analogously, if the demand in period  $n$  is exactly observed, we denote the order quantity  $y_{n+1}^{*e}$  in period  $(n+1)$  by  $y_{n+1,k+1}$ .

Now, using the superscript  $P$  for perishable, for the Weibull demand with prior  $\hat{\pi}_n = \text{Gamma}(a_n, S_n)$ , by scalability we can write the optimality equation (3) as

$$V_n^P(a_n, S_n) = S_n^{\frac{1}{l}} \min_{q_n \geq 0} G_n^P(q_n | a_n, 1),$$

where  $S_n = (S_1 + \sum_{i=1}^{n-1} s_i^l)$ , and  $G_n^P(q_n | a_n, 1)$  can be written as

$$\begin{aligned} G_n^P(q_n | a_n, 1) &= E_{\psi_n(\cdot | a_n, 1)} [cq_n - h \max(0, q_n - \xi_n) + p \max(0, \xi_n - q_n) \\ &\quad + \beta V_{n+1}^P(a_n, 1 + q_n) I(\xi_n \geq q_n) + \beta V_{n+1}^P(a_n + 1, 1 + \xi_n) I(\xi_n < q_n)] \\ &= E_{\psi_n(\cdot | a_n, 1)} [cq_n - h \max(0, q_n - \xi_n) + p \max(0, \xi_n - q_n) \\ &\quad + \beta(1 + q_n) V_{n+1}^P(a_n, 1) I(\xi_n \geq q_n) + \beta(1 + \xi_n) V_{n+1}^P(a_n + 1, 1) I(\xi_n < q_n)] \\ &= E_{\psi_n(\cdot | a_n, 1)} [cq_n - h \max(0, q_n - \xi_n) + p \max(0, \xi_n - q_n)] \end{aligned} \quad (9)$$

$$+\beta V_{n+1}^P(a_n, 1) \frac{1}{(1+q_n^l)^{a_n-\frac{1}{l}}} + \beta V_{n+1}^P(a_n+1, 1) \frac{a_n}{a_n-\frac{1}{l}} \left(1 - \frac{1}{(1+q_n^l)^{a_n-\frac{1}{l}}}\right).$$

The first three terms of (9) correspond to the single-period cost in period  $n$ . The fourth and fifth terms are the optimal discounted cost-to-go from period  $(n+1)$  onwards given that demands are respectively censored and exactly observed in period  $n$  resulting in  $c$  and  $e$  sample paths of Figure 1. Notice that, after using scalability of the optimal cost we are able to explicitly integrate out the last two terms. This allows us to derive the optimal order quantity and cost in simple analytical form for the case of Weibull demand as we will see in the following theorem. To guarantee finiteness of the mean of the updated distributions and optimal costs, we will assume  $a_1 > \frac{1}{l}$  whenever we discuss the Weibull distribution case in this paper.

**Theorem 2.** *If the demand distribution is Weibull with an unknown parameter  $\theta$  and the prior distribution on  $\theta$  is gamma with initial parameters  $a_1$  and  $S_1$ , then*

(a) *The optimal order quantity in period  $n$ ,  $n = 1, 2, \dots, N$ , is given by:*

$$y_{n,k}^{*P} = S_n^{\frac{1}{l}} q_{n,k}, \quad (10)$$

for  $k = 0, 1, 2, \dots, n-1$ , where  $q_{n,k}$  is uniquely obtained by solving

$$(c-h)(1+q^l)^{a_1+k-\frac{1}{l}+1} - (p-h)(1+q^l)^{1-\frac{1}{l}} + lq^{l-1}\tilde{v}_{n,k} = 0, \quad (11)$$

where  $\tilde{v}_{n,k} = \beta \left[ (a_1+k)v_{n+1,k+1} - (a_1+k-\frac{1}{l})v_{n+1,k} \right]$ .

(b) *The optimal cost in period  $n$ ,  $n = 1, 2, \dots, N$ , is given by:*

$$V_{n,k}^P(a_n, S_n) = S_n^{\frac{1}{l}} v_{n,k}, \quad (12)$$

where  $v_{n,k}$  is obtained from

$$\begin{aligned} v_{n,k} &= p\mu_{a_1+k} + (c-h)q_{n,k} - (p-h)\bar{H}_{a_1+k}(q_{n,k}) \\ &\quad + \beta \left[ v_{n+1,k}r_{n,k} + \frac{a_1+k}{a_1+k-\frac{1}{l}}v_{n+1,k+1}(1-r_{n,k}) \right], \end{aligned} \quad (13)$$

for  $k = 0, 1, 2, \dots, n-1$ , with the terminal conditions  $v_{N+1,k} = 0$ , for all  $k = 0, 1, 2, \dots, N$ , where

$$\begin{aligned} \mu_{a_1+k} &= (a_1+k)B\left(a_1+k-\frac{1}{l}, 1+\frac{1}{l}\right) \\ \bar{H}_{a_1+k}(q) &= \frac{1}{l}B\left(a_1+k-\frac{1}{l}, \frac{1}{l}\right) \left\{ 1 - F_B\left(\frac{1}{1+q^l} \middle| a_1+k-\frac{1}{l}, \frac{1}{l}\right) \right\} \\ r_{n,k} &= \frac{1}{(1+q_{n,k}^l)^{a_1+k-\frac{1}{l}}}, \end{aligned}$$



where  $B(p, q)$  denotes the beta function and  $F_B(x|p, q)$  denotes the CDF of the Beta( $p, q$ ) distribution.

**Stocking Factor Interpretation:** We provide an interpretation for the optimal order quantity  $y_{n,k}^{*P}$  given by (10). The same interpretation also holds for the optimal order quantity given by (14) for the exponential case. Since  $E(X_n) = (a_1 + k)B\left(a_1 + k - \frac{1}{l}, 1 + \frac{1}{l}\right) S_n^{\frac{1}{l}}$ , from (10) we can write  $y_{n,k}^{*P} = z_{n,k}^* E(X_n)$ , where  $z_{n,k}^* = \frac{q_{n,k}}{(a_1 + k)B\left(a_1 + k - \frac{1}{l}, 1 + \frac{1}{l}\right)}$ . As for the multiplicative demand case in Petruzzi and Dada (1999),  $z_{n,k}^*$  can be interpreted as a stocking factor with exogenous price. Notice that, for any  $N$ -period problem, the values of  $z_{n,k}^*$ 's can be pre-computed and assigned to all the nodes like those in Figure 1 before observing any sales data. Thus, to find the optimal order quantity at any node, all we need to do is multiply the pre-computed stocking factor with the updated mean of  $X_n$  that is obtained based on the sales data. Thus, for the perishable inventory model, scalability of optimal solution leads to pre-computed stocking factors that greatly simplifies computation.

Now, notice that (11) and (13) are only one-step look ahead recursive equations of polynomial form so that to solve for period  $n$  we only need the solutions for period  $(n + 1)$ . To find the optimal solutions, we first solve (11) for period  $N$ . With this solution, we compute (13) for period  $N$ . Using this computed cost, we first solve (11) and then compute (13) for period  $N - 1$ . Proceeding recursively this way we find the unique optimal solutions for all periods.

### 3.2 The Case of Exponential Demand Distribution

When demand is exponential, (11) and (13) become simpler and yield the following closed form solutions.

**Theorem 3.** *If the demand distribution is exponential with an unknown parameter  $\theta$  and the prior distribution on  $\theta$  is gamma with initial parameters  $a_1$  and  $S_1$ , then*

(a) *The optimal order quantity in period  $n$ ,  $n = 1, 2, \dots, N$ , is given by:*

$$y_{n,k}^{*P} = S_n(\alpha_{n,k} - 1), \quad (14)$$

where  $S_n = (S_1 + \sum_{i=1}^{n-1} s_i)$  and

$$\begin{aligned} \alpha_{n,k} = & \left[ \frac{1}{c-h} \left( p-h + \beta(c-h) \left[ \sum_{j=k}^{N-(n+1)+k} \{ (a_1 + j) \alpha_{n+1+j-k,j} \right. \right. \right. \\ & \left. \left. \left. - (a_1 + j + 1) \alpha_{n+1+j-k,j+1} \} + N - n \right] \right) \right]^{\frac{1}{a_1+k}}, \end{aligned} \quad (15)$$

for  $k = 0, 1, 2, \dots, n - 1$ .

(b) *The optimal cost in period  $n$ ,  $n = 1, 2, \dots, N$ , is given by:*

$$V_{n,k}^P(a_n, S_n) = S_n \gamma_{n,k}, \quad (16)$$

where

$$\begin{aligned} \gamma_{n,k} = & \frac{1}{a_1 + k - 1} \left[ c + (c - h)(a_1 + k)(\alpha_{n,k} - 1) \right. \\ & \left. + \beta \left( (N - n)h + (c - h) \sum_{j=k}^{N-(n+1)+k} \{ (a_1 + j + 1)\alpha_{n+1+j-k,j+1} - (a_1 + j) \} \right) \right] \end{aligned} \quad (17)$$

for  $k = 0, 1, 2, \dots, n - 1$ .

While the proof is presented in the Appendix, the main idea behind the proof is that the uniqueness and a linear structure (for  $l = 1$ ) combined with the scalability of the solutions help to fold back the closed form of the optimal policies and costs by backward induction. Note that in (14) we have expressed the optimal order quantity for the normalized system with unit scale parameter in the form  $(\alpha_{n,k} - 1)$  instead of  $q_{n,k}$  as in (10). This representation helps us in writing the solutions in closed form as given by (15) and (17). Notice that in (15) and (17), both  $\alpha_{n,k}$  and  $\gamma_{n,k}$  are written explicitly only in terms of the  $\alpha$  values that are already known from backward induction. These results are refinements over the recursive equations presented in Theorems 3(a) and (b) of Lariviere and Porteus (1999).

Using Theorem 3 we now develop recursive relations for  $\alpha_{n,k}$  and  $\gamma_{n,k}$  that simplify the computation of the optimal order quantities and costs for any finite horizon problem and help to derive the solutions in simple form for the infinite horizon case.

**Corollary 1.** For  $n = 1, 2, \dots, N - 1$ ,  $\alpha_{n,k}$  and  $\gamma_{n,k}$  satisfy:

$$\begin{aligned} (a) \quad \alpha_{n,k} = & \left[ \beta \{ (a_1 + k)\alpha_{n+1,k} - (a_1 + k + 1)\alpha_{n+1,k+1} + 1 \} + \alpha_{n+1,k+1}^{a_1+k+1} \right]^{\frac{1}{a_1+k}}, \quad k = 0, 1, 2, \dots, n-1 \\ & 1, \text{ with the terminal values } \alpha_{N,k} = \left( \frac{p-h}{c-h} \right)^{\frac{1}{a_1+k}}, \text{ for } k = 0, 1, 2, \dots, N-1, \text{ and} \\ (b) \quad \gamma_{n,k} = & \frac{1}{a_1+k-1} [c + (c - h)(a_1 + k)(\alpha_{n,k} - 1) + \beta(a_1 + k)\gamma_{n+1,k+1}], \quad k = 0, 1, 2, \dots, n-1, \\ & \text{with the terminal values } \gamma_{N+1,k} = 0, \text{ for all } k = 0, 1, 2, \dots, N. \end{aligned}$$

To provide some insightful illustration, we use the recursive characterization of Corollary 1 to solve the following problem.

**Example.** Let us consider a 6-period problem with  $\beta = 1$  and values of  $c, p$  and  $h$  such that  $\frac{p-h}{c-h} = 3$ . Let  $a_1 = 1.1$  and any  $S_1$  for the gamma prior. Using Corollary 1(a) we readily obtain the following values of  $\alpha_{n,k}$ 's:  $\alpha_{1,0} = 4.462384$ ;  $\alpha_{2,0} = 4.14382$ ,  $\alpha_{2,1} = 1.78303$ ;  $\alpha_{3,0} = 3.80212$ ,  $\alpha_{3,1} = 1.76701$ ,  $\alpha_{3,2} = 1.44691$ ;  $\alpha_{4,0} = 3.44342$ ,  $\alpha_{4,1} = 1.74655$ ,  $\alpha_{4,2} = 1.44129$ ,  $\alpha_{4,3} = 1.31380$ ;  $\alpha_{5,0} = 3.07693$ ,  $\alpha_{5,1} = 1.72043$ ,  $\alpha_{5,2} = 1.43423$ ,  $\alpha_{5,3} = 1.31089$ ,  $\alpha_{5,4} = 1.24217$ ;  $\alpha_{6,0} = 2.71485$ ,  $\alpha_{6,1} = 1.68733$ ,  $\alpha_{6,2} = 1.42531$ ,  $\alpha_{6,3} = 1.30729$ ,  $\alpha_{6,4} = 1.24038$ ,  $\alpha_{6,5} = 1.19734$ . Then using (14) we obtain the optimal order quantities. If  $c = 4$ ,  $h = 2$  and  $p = 8$ , then from Corollary 1(b) and (16) we obtain the optimal total expected cost as  $451.27601S_1$ .

The following observations can be made from the above example. Suppose we are in period  $n$ . Then, for all  $k < n$ , the difference  $\alpha_{\hat{n},k} - \alpha_{\hat{n}+1,k} > 0$  for all  $\hat{n} > n$ . Also, for any  $k < n$ , the

successive differences are contracting as  $\hat{n} \downarrow n$ , that is,  $(\alpha_{\hat{n},k} - \alpha_{\hat{n}+1,k}) < (\alpha_{\hat{n}+1,k} - \alpha_{\hat{n}+2,k})$ . This suggests that the sequence  $\{\alpha_{\hat{n},k}\}_{\hat{n} > n}$  may converge as  $\hat{n} \downarrow n$  and  $N$  increases. This is indeed the case for the infinite horizon problem (see Theorem 4). Also, observe that  $\alpha_{\hat{n},k} > \alpha_{\hat{n},k+1}$  for all  $k < n$ , which leads to a monotone property of the limits as can be seen in Theorem 5 for the infinite horizon case. Note that the above observations cannot be proved in general for a finite horizon problem because the inequalities may marginally fail to hold; however, for the infinite horizon problem, we can be definitive about these properties which we will see next by invoking Corollary 1(a).

**Theorem 4.** *For the exponential demand model, there exists a limit  $l_k (\geq 1)$  such that for any given period  $n$ , the sequence  $\{\alpha_{\hat{n},k}\}_{\hat{n} > n}$  converges to  $l_k$  as  $\hat{n} \downarrow n$  and  $N \rightarrow \infty$  for all  $k = 0, 1, 2, \dots, n-1$ .*

The proof expands the equation in Corollary 1(a) and shows inductively that the sequence  $\{\alpha_{\hat{n},k}\}_{\hat{n} > n}$  satisfies Cauchy's convergence criterion. With the help of Theorem 4 we are now ready to derive results for the optimal order quantity and cost for the infinite horizon problem with exponential demand.

**Theorem 5.** *If the demand distribution is exponential and the gamma prior has initial parameters  $a_1$  and  $S_1$ , then*

(a) *The optimal order quantity in period  $n$ ,  $n = 1, 2, \dots$ , is given by:  $y_{n,k}^{*\infty} = S_n(l_k - 1)$ , where  $S_n = (S_1 + \sum_{i=1}^{n-1} s_i)$ , and for  $k = 0, 1, 2, \dots, n-1$ ,  $l_k$  satisfies*

$$l_k^{a_1+k} - \beta(a_1+k)l_k = \frac{p-h}{c-h} - \beta(a_1+k). \quad (18)$$

Moreover,  $l_k$  is unique, bounded and decreases in  $k$ .

(b) *The optimal cost in period  $n$ ,  $n = 1, 2, \dots$  is given by:  $V_{n,k}^\infty(a_n, S_n) = S_n m_k$ , where  $m_k$  satisfies*

$$m_k = \frac{1}{a_1 + k - 1} \left[ \frac{c}{1 - \beta} + (c - h) \sum_{j=k}^{\infty} \beta^{j-k} (a_1 + j)(l_j - 1) \right], \quad (19)$$

for  $k = 0, 1, 2, \dots, n-1$ , and the above series converges.

Theorem 5 shows that the optimal order quantity and cost for the infinite horizon problem with exponential demand inherit the scalability and simple solution of the finite horizon counterparts. Now that we have analyzed the perishable inventory model, we study the storable case next.

## 4 The Storable Inventory Problem

In the storable inventory problem, the excess inventory from one period is carried forward to the next period and the newsvendor incurs a holding cost of  $h$  per unit of leftover inventory

at the end of each period. This leftover inventory can sometimes exceed the optimal order-up-to level for the next period when the underlying inventory problem is non-stationary. This significantly complicates the problem because now the  $e$  sample paths of Figure 1 are replaced by the  $e_L$  and  $e_H$  sample paths of Figure 2. The notation  $e_L$  represents the exactly observed demand sample paths with relatively low leftover inventory (i.e., leftover inventory is below the next period's order-up-to level) so that order is placed in the next period, and the notation  $e_H$  represents the exactly observed demand sample paths with relatively high leftover inventory (i.e., leftover inventory is greater than or equal to the next period's order-up-to level) so that no order is placed in the next period. The sample paths represented by  $e_H$  are the ones that make it difficult to compute the optimal solutions and costs for any storable inventory problem with time-varying order-up-to levels, which is the case here. Censoring of demand observations add more complications to it because  $e_H$  sample paths can occur after censoring had occurred in an earlier period and visa versa (see Figure 2). The combined effect of censored demand observations on  $c$  sample paths and excess inventory carryover on  $e_H$  sample paths in a storable inventory model makes it a difficult problem to solve.

Nevertheless, analogous to Theorem 1, we have the following result for the storable inventory problem.

**Corollary 2.** *If the demand distribution is from the newsvendor class, then the optimal order-up-to level is scalable if and only if the demand distribution is Weibull.*

Thus, similarly to the perishable inventory case, the dimensionality of the problem can be reduced for the storable model by using scalability. However, unlike the perishable model, scalability does not lead to pre-computed stocking factors for the storable inventory problem because on  $e_H$  sample paths, the stocking levels are random variables since these depend on the sample observations (e.g., see nodes  $B_1$  and  $C_1$  in Figure 2).

Now, proceeding as in the perishable inventory case, for the Weibull demand model with prior  $\hat{\pi}_n = \text{Gamma}(a_n, S_n)$ , with initial inventory  $z_n$  at the beginning of period  $n$ , we can write the optimality equation for the storable model as

$$V_n(z_n|a_n, S_n) = -cz_n + S_n^{\frac{1}{l}} \min_{q_n \geq z'_n} G_n(q_n|a_n, 1), \quad (20)$$

where  $z'_n = z_n/S_n^{\frac{1}{l}}$  and  $S_n = (S_1 + \sum_{i=1}^{n-1} s_i^l)$ , with  $G_n(q_n|a_n, 1)$  given by

$$\begin{aligned} & G_n(q_n|a_n, 1) \\ &= E_{\psi_n(\cdot|a_n, 1)} \left[ cq_n + h \max(0, q_n - \xi_n) + p \max(0, \xi_n - q_n) + \beta V_{n+1}(0|a_n, 1 + q_n^l) I(\xi_n \geq q_n) \right. \\ & \quad \left. + \beta V_{n+1}(q_n - \xi_n|a_n + 1, 1 + \xi_n^l) I(\xi_n < q_n) \right] \\ &= E_{\psi_n(\cdot|a_n, 1)} \left[ cq_n + h \max(0, q_n - \xi_n) + p \max(0, \xi_n - q_n) + \beta V_{n+1}(0|a_n, 1 + q_n^l) I(\xi_n \geq q_n) \right] \end{aligned}$$

$$\begin{aligned}
& + \beta \left\{ -c(q_n - \xi_n) + G_{n+1}(q_{n+1}^{*e} | a_n + 1, 1 + \xi_n^l) \right\} I(\xi_n < q_n, q_{n+1}^{*e} \geq q_n - \xi_n) \\
& + \beta \left\{ -c(q_n - \xi_n) + G_{n+1}(q_n - \xi_n | a_n + 1, 1 + \xi_n^l) \right\} I(\xi_n < q_n, q_{n+1}^{*e} < q_n - \xi_n) \Big] 1)
\end{aligned}$$

The first three terms of (21) correspond to the single-period cost in period  $n$ . The fourth term is the optimal discounted cost-to-go from period  $(n + 1)$  onwards given that demand is censored in period  $n$ . The fifth and sixth terms are the optimal discounted cost-to-go from period  $(n + 1)$  onwards given that demands are exactly observed in period  $n$  resulting in respectively the  $e_L$  and  $e_H$  sample paths of Figure 2.

We now establish the following result for the optimal cost of the Weibull demand model using a decision theoretic approach based on the concept of Bayes risk.

**Lemma 1.** *Suppose the demand distribution is Weibull with a gamma prior on the unknown parameter  $\theta$ . Let  $\tilde{q}_n(a) = \operatorname{argmin}_{q \geq 0} G_n(q | a, 1)$  and  $\tilde{v}_n(a) = G_n(\tilde{q}_n(a) | a, 1)$ . Then, for any period  $n$  and all  $a > \frac{1}{l}$ ,*

$$(a - \frac{1}{l})\tilde{v}_n(a) \geq a\tilde{v}_n(a + 1). \quad (22)$$

Lemma 1 has an interesting interpretation. Since  $aB\left(a - \frac{1}{l}, 1 + \frac{1}{l}\right)$  is the mean of the random variable  $\xi$  with density  $\psi(\xi | a, 1) = \frac{a l \xi^{l-1}}{(1 + \xi^l)^{a+1}}$ , (22) can be rewritten as

$$\frac{\tilde{v}_n(a)}{E_{\psi(\cdot | a, 1)}[\xi]} \geq \frac{\tilde{v}_n(a + 1)}{E_{\psi(\cdot | a+1, 1)}[\xi]}.$$

Therefore, Lemma 1 implies that for the storable inventory model with Weibull demand, the *standardized* cost (optimal cost per unit of mean demand) is higher with a lower value of the shape parameter  $a$ , that is, higher demand uncertainty. A special case of Lemma 1 for the perishable inventory model with exponential demand is presented in Theorem 3(c) of Lariviere and Porteus (1999). Now, for the case of exponential demand ( $l = 1$ ), using Lemma 1 we establish the convexity of the cost function so that the unique optimal order-up-to level can be computed using a recursion that we develop in the following theorem.

**Theorem 6.** *If the demand distribution is exponential with a gamma prior on the unknown parameter  $\theta$ , then for any period  $n$ ,  $n = 1, 2, \dots, N$ ,*

(a)  $G_n(q | a, 1)$  is strictly convex and twice differentiable in  $q$ , for all  $a > 1$ , and satisfies,

$$(b) G_n(q | a, 1) = (c + h - \beta c)y + \frac{1}{a-1} \left[ \frac{p + h - \beta c + \beta \{(a-1)\tilde{v}_{n+1}(a) - a\tilde{v}_{n+1}(a+1)\}}{(1+q)^{a-1}} - (h - \beta c) \right] + \beta T_{n,a}(q),$$

where, for  $q < \tilde{q}_{n+1}(a + 1)$ ,

$$T_{n,a}(q) = \frac{a}{a-1} \tilde{v}_{n+1}(a + 1),$$

and for  $q \geq \tilde{q}_{n+1}(a + 1)$ ,

$$T_{n,a}(q) = \frac{a}{(1+q)^{a-1}} \left[ \int_{\tilde{q}_{n+1}(a+1)}^q G_{n+1}(\xi | a + 1, 1) (1 + \xi)^{a-2} d\xi \right]$$

$$+ \frac{\tilde{v}_{n+1}(a+1)}{a-1} (1 + \tilde{q}_{n+1}(a+1))^{a-1} \Big].$$

(c) With initial parameters  $a_1$  and  $S_1$  for the gamma prior, the optimal order-up-to level and cost in period  $n$  are given by:  $y_{n,k}^* = S_n \hat{q}_{n,k}$  and  $V_{n,k}(z_n|a_n, S_n) = -cz_n + S_n \hat{v}_{n,k}$ , for  $k = 0, 1, 2, \dots, n-1$ , with  $S_n = (S_1 + \sum_{i=1}^{n-1} s_i)$ , where  $\hat{q}_{n,k}$  and  $\hat{v}_{n,k}$ , the order-up-to level and cost for the normalized system with unit scale parameter, can be solved optimally and are unique.

While Theorems 6(a) and (b) help to efficiently compute the order-up-to levels and costs for the storable inventory model with exponential demand for any finite horizon problem, Theorem 6(c) establishes that these computations do really yield exact optimal solutions. As a demonstration of Theorem 6(c), the explicit solutions for the first period of a two-period and a three-period problem are shown in the Addendum. Note that, the storable inventory results in our Theorem 6 extend the perishable inventory results of Theorems 3(a) and (b) in Lariviere and Porteus (1999). It remains to be seen what properties in addition to Lemma 1 can be established so that scalability will lead to easy computation of the optimal solution for the storable inventory problem with Weibull demand.

## 5 Numerical Studies

In this section we report our results from numerical computations. In our study, we consider the Weibull demand models with  $l = 2$  and  $4$  for the perishable inventory case and the exponential demand model for the storable inventory case. We compute the first-period optimal solutions and total optimal expected costs for various values of the parameters for the above models. Since the solutions of other periods depend on the sample observations and solutions of previous periods, those are not unique and hence we do not list them here. In our computation schemes, we vary the initial shape parameter  $a_1$  by setting  $a_1 = 2, 4$  and  $8$ , while keeping the first period mean demand  $E(X_1)$  fixed at the value of  $1$  (by adjusting the scale parameter  $S_1$  accordingly). This ensures comparability of the optimal solutions and costs over different cases as the prior uncertainty represented by the coefficient of variation  $\sqrt{1/a_1}$  decreases with  $a_1$ . We keep the purchase cost and salvage value fixed at  $c = 4$  and  $h = 2$  throughout the experiments and vary the shortage penalty by setting  $p = 8$  and  $40$  (or  $80$ ). To study the effects of the length of problem horizons on the optimal solutions and their convergence behavior, we vary  $N$  from  $3$  to  $800$  periods for the perishable model and from  $3$  to  $400$  periods for the storable model. Finally, we vary the discount factor by setting  $\beta = 0.8, 0.9$ , and  $1$  to examine its effect on the optimal solutions.

The results are shown in Tables 1 and 2 for the Weibull demand models with  $l = 2$  and  $4$  respectively for perishable inventory and in Tables 3 and 4 for the exponential demand model with storable inventory. The following observations can be made from the results of these tables:

1. For fixed values of  $p$ ,  $a_1$  and  $\beta$ , the first-period optimal solution increases with  $N$ . This shows that by stocking more in the first period of a longer horizon problem we acquire more information on the demand distribution which in turn helps to reduce costs in future periods. Moreover, the amount of increase in the first-period stocking quantity is greater for lower values of  $a_1$ . This suggests that the marginal expected value of incremental inventory is greater when prior uncertainty is higher.
2. With  $p$ ,  $N$  and  $\beta$  fixed, as the gamma prior's shape parameter  $a_1$  increases, the optimal expected cost always decreases; this is consistent with the result of Lemma 1. The explanation for why the optimal solutions increase with  $a_1$  in some cases and decrease with  $a_1$  in other cases will be given in item 6 below. Also, we observe that the difference between the optimal and myopic solution decreases with  $a_1$ .
3. For fixed  $a_1$ ,  $N$  and  $\beta$ , as the shortage penalty  $p$  increases, the first-period optimal solution increases. This shows that informational value of demand distribution is higher when  $p$  is higher.
4. In all the above cases, the informational benefit of a higher stocking quantity in the first-period increases as the present value of future cost increases, that is, the discount factor  $\beta$  increases.
5. Comparing the results in Tables 1 and 2 we observe that, for all cases of  $p$ ,  $a_1$ ,  $N$  and  $\beta$ , the optimal expected costs are always lower with  $l = 4$  than those with  $l = 2$ . This is due to lower demand variance when  $l = 4$ .
6. Finally, comparisons of the optimal solutions across different cases are quite complex and depend on all model parameters as well as their combined effects. From Tables 1 and 2 notice that for most cases of  $p$ ,  $a_1$ , and  $N$ , the optimal solutions are lower with  $l = 4$  than those with  $l = 2$ , except for the cases when  $p = 8$ ,  $a_1 = 2$ , and  $N = 3$  with  $\beta = 0.8$  and  $0.9$ . We also observe that as  $a_1$  increases, for each  $N$ , with  $p = 8$  and  $40$ , the optimal order-up-to levels increase in Table 3, but with  $p = 80$ , they decrease in Table 4. For the case with  $l = 2$  in Table 1, the order quantities increase when  $p = 8$  but decrease when  $p = 40$  for each  $N$ ; whereas for the case with  $l = 4$  in Table 2, the order quantities mostly decrease for both  $p = 8$  and  $40$  for each  $N$ , except for the cases when  $p = 8$ ,  $a_1 = 2$  and  $4$ , and  $N = 3$  with  $\beta = 0.8$  and  $0.9$ , and  $p = 8$ ,  $a_1 = 4$ ,  $N = 5$  with  $\beta = 0.8$ . This behavior can be explained via the properties of the *single crossing point* of two distribution functions. It is known from the analysis of Gerchak and Mossman (1992) (see also Porteus (2002), p. 283) that if two demand distribution functions have a single crossing point, then as we move from the distribution with higher standard deviation to the one with lower standard deviation, the stocking quantity increases if it was originally below the crossing point and decreases if it was originally above the crossing point.

## 6 Conclusions

In this paper we focus on the study of the dynamic newsvendor problem when the retailer's demand uncertainty is expressed through newsvendor distributions with an unknown parameter and excess demand is lost and unobserved (censored). We address both perishable and storable inventory models. The scope of scalability for the censored demand problem with newsvendor distributions is explored. Importantly, we show that the Weibull is the only member of the newsvendor family of distributions that yields scalability for the optimal solutions. We also demonstrate how scalability can be exploited to reduce the dimensionality of the problem so that easily computed optimal solutions are found. Several insights are developed directly from extensive computations of the optimal solutions for both perishable and storable inventory models. In the following three paragraphs we compare our results with some recent literature to delineate the contributions of this paper.

By establishing in Theorem 1 and Corollary 2, that the Weibull is the only distribution that has scalable solutions (for both censored and full demand information models), we generalize Theorem 2 of Lariviere and Porteus (1999). Previously, they had shown that Weibull was sufficient to yield scalability for the storable (and hence, perishable) inventory problem.

These structural results on scalability are shown to lead to efficient computation schemes for the perishable inventory model. While Lariviere and Porteus (1999) had previously found the recursions analogous to (11) and (13) for the exponential case, we show how to operationalize them into a simple one-step look ahead computation scheme for the Weibull case and explicit closed form solutions for the exponential case. We also generalize to the Weibull storable case a structural property found by Lariviere and Porteus (1999) for the perishable exponential case. This property which relates the standardized cost with the prior's shape parameter is used to establish the convexity of the cost function and therefore, uniqueness of the optimal solution for the exponential case. Subsequently, this is used to derive a recurrence relation that can compute the optimal solutions and costs even for very long horizon problems as we have shown in Tables 3 and 4.

While we develop exact computation schemes and structural results for the Weibull and exponential demand distributions, Lu et al. (2006) and Chen (2009) develop bounds and heuristics for optimal solutions for general demand distributions. While their methods are general, those do not necessarily provide insights into the special structure of the particular inventory models we study here. For example, in the case of the storable model with exponential demand, our results can be used to provide theoretical justification for why the direct computation used by Chen (2009) finds the unique optimal solution.

Our findings open up a few research questions. For example, will results similar to Theorem 6 hold for the storable inventory problem with Weibull demand? One can also ask, instead of using a Bayesian approach which requires a prior specification, how would the results change if one uses an operational statistics approach? In this approach, one needs to estimate the unknown parameter(s) *pointwise* while optimizing the objective function. Liyanage and Shanthikumar



(2005) and Chu et al. (2008) use this method for observed demand models with perishable inventory. While Bensoussan et al. (2008) study a censored demand model with perishable inventory using the maximum likelihood estimation method, they focus on the myopic solution for the finite horizon problem and the existence of the optimal policy for the infinite horizon problem. It remains to determine what structural results can be obtained for the optimal policies for such models? When are exact or tractable optimal solutions possible? When are these problems scalable? What role does scalability play in facilitating computation of optimal solutions? Answering such questions will further enhance our understanding of the censored newsvendor problems.

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## Appendix

**Proof of Theorem 2.** For the Weibull demand, after some simplification, we can write

$$\begin{aligned}
& E_{\psi_n(\cdot|a_n,1)} [cq_n - h \max(0, q_n - \xi_n) + p \max(0, \xi_n - q_n)] \\
&= cq_n - h[q_n - \bar{H}_{a_n}(q_n)] + p[\mu_{a_n} - \bar{H}_{a_n}(q_n)] \\
&= p\mu_{a_n} + (c - h)q_n - (p - h)\bar{H}_{a_n}(q_n),
\end{aligned} \tag{23}$$

where

$$\begin{aligned}
\mu_{a_n} &= \int_0^\infty \xi \psi_n(\xi|a_n, 1) d\xi = a_n B\left(a_n - \frac{1}{l}, 1 + \frac{1}{l}\right) \\
\bar{H}_{a_n}(q) &= \int_0^{q_n} (1 - \Psi_n(\xi|a_n, 1)) d\xi \\
&= \frac{1}{l} B\left(a_n - \frac{1}{l}, \frac{1}{l}\right) \left\{ 1 - F_B\left(\frac{1}{1 + q_n^l} \middle| a_n - \frac{1}{l}, \frac{1}{l}\right) \right\}.
\end{aligned}$$

Using (23) in (9) and setting  $\frac{dG_n^P(q_n|a_n, 1)}{dq_n} = 0$ , we get

$$\begin{aligned}
& (c - h) - (p - h)(1 + q_n^l)^{-a_n} \\
& + \beta l q_n^{l-1} (1 + q_n^l)^{-(a_n - \frac{1}{l} + 1)} \left[ a_n V_{n+1}^P(a_n + 1, 1) - (a_n - \frac{1}{l}) V_{n+1}^P(a_n, 1) \right] = 0.
\end{aligned} \tag{24}$$

With  $a_n = a_1 + k$  and  $v_{n+1,k} = V_{n+1}^P(a_1 + k, 1)$  for  $k = 0, 1, 2, \dots, n - 1$ , (11) follows from (24). Also, since  $v_{n,k} = G_n^P(q_{n,k}|a_1 + k, 1)$ , (13) follows from (9) and (23). ■

**Proof of Theorem 3.** We use backward induction for the proof. Let us start with the last period  $N$ . The number of exact observations at the start of period  $N$  is given by  $k$ , where  $k$  takes values in  $\{0, 1, \dots, N - 1\}$  depending on what sample path is followed to arrive at period  $N$ . Now, the optimal policy  $q_{N,k}$  for the normalized system with unit scale parameter and  $a_N = a_1 + k$  is obtained from  $\frac{dG_N^P(q_N|a_N, 1)}{dq_N} = 0$ , that is,  $(c - h) - (p - h)[1 - \Psi_N(q_N|a_N, 1)] = 0$ , which leads to

$$q_{N,k} = \left( \frac{p - h}{c - h} \right)^{\frac{1}{a_1 + k}} - 1 = \alpha_{N,k} - 1,$$

for  $k = 0, 1, 2, \dots, N - 1$ , where  $\alpha_{N,k} = \left( \frac{p - h}{c - h} \right)^{\frac{1}{a_1 + k}}$ . Also, the optimal cost in period  $N$  for the normalized system is given by

$$\begin{aligned}
G_{N,k}^P(q_{N,k}|a_N, 1) &= E_{\psi_N(\cdot|a_N, 1)} [cq_{N,k} - h \max(0, q_{N,k} - \xi_N) + p \max(0, \xi_N - q_{N,k})] \\
&= \frac{1}{a_1 + k - 1} [c + (c - h)(a_1 + k)(\alpha_{N,k} - 1)] \\
&= \gamma_{N,k},
\end{aligned}$$

for  $k = 0, 1, 2, \dots, N - 1$ . The above equations prove the theorem for period  $n = N$ .

Next, we will derive the optimal policies and costs for period  $n$ , where  $n = 1, 2, \dots, N - 1$ , assuming the form of the policies and costs for period  $(n + 1)$ . The number of exact observations at the beginning of period  $n$  is  $k$ , where  $k \in \{0, 1, 2, \dots, n - 1\}$ . Now, for exponential demand with  $l = 1$ , (9) becomes

$$\begin{aligned} G_n^P(q_n|a_n, 1) &= E_{\psi_n(\cdot|a_n, 1)} [cq_n - h \max(0, q_n - \xi_n) + p \max(0, \xi_n - q_n)] \\ &\quad + \beta V_{n+1}^P(a_n, 1) \frac{1}{(1 + q_n)^{a_n - 1}} + \beta V_{n+1}^P(a_n + 1, 1) \frac{a_n}{a_n - 1} \left(1 - \frac{1}{(1 + q_n)^{a_n - 1}}\right). \end{aligned} \quad (25)$$

Setting  $\frac{dG_n^P(q_n|a_n, 1)}{dq_n} = 0$ , we get

$$\left( p - h + (a_n - 1)\beta V_{n+1}^P(a_n, 1) - a_n\beta V_{n+1}^P(a_n + 1, 1) \right) \frac{1}{(1 + q_n)^{a_n}} = c - h. \quad (26)$$

With  $a_n = a_1 + k$  and  $\gamma_{n,k} = V_n^P(a_1 + k, 1)$  for  $k = 0, 1, 2, \dots, n - 1$ , from (26) we can write

$$\begin{aligned} (1 + q_{n,k})^{a_1 + k} &= \frac{1}{c - h} \left( p - h + (a_1 + k - 1)\beta\gamma_{n+1,k} - (a_1 + k)\beta\gamma_{n+1,k+1} \right) \\ &= \frac{1}{c - h} \left( p - h + \beta(c - h) \left[ \sum_{j=k}^{N-(n+1)+k} \{ (a_1 + j)\alpha_{n+1+j-k,j} \right. \right. \\ &\quad \left. \left. - (a_1 + j + 1)\alpha_{n+1+j-k,j+1} \} + N - n \right] \right), \end{aligned} \quad (27)$$

where the last equation is obtained by using the values of  $\gamma_{n+1,k}$  and  $\gamma_{n+1,k+1}$  from (17) due to the induction hypothesis and then doing some algebraic simplification. Now, from (27) we can write the optimal policy  $q_{n,k}$  for the normalized system with unit scale parameter as

$$q_{n,k} = \alpha_{n,k} - 1,$$

for  $k = 0, 1, 2, \dots, n - 1$ , where  $\alpha_{n,k}$  is as given in (15). This proves Theorem 3(a) for period  $n$ .

Now, from (25), the optimal cost in period  $n$  for the normalized system is given by

$$\begin{aligned} \gamma_{n,k} &= V_n^P(a_1 + k, 1) \\ &= E_{\psi_n(\cdot|a_1+k, 1)} [cq_{n,k} - h \max(0, q_{n,k} - \xi_n) + p \max(0, \xi_n - q_{n,k})] \\ &\quad + \beta\gamma_{n+1,k} \frac{1}{\alpha_{n,k}^{a_1+k-1}} + \beta\gamma_{n+1,k+1} \frac{a_1 + k}{a_1 + k - 1} \left(1 - \frac{1}{\alpha_{n,k}^{a_1+k-1}}\right), \end{aligned} \quad (28)$$

for  $k = 0, 1, 2, \dots, n - 1$ . We can simplify the first term on the right-hand-side of (28) as

$$\begin{aligned} &E_{\psi_n(\cdot|a_1+k, 1)} [cq_{n,k} - h \max(0, q_{n,k} - \xi_n) + p \max(0, \xi_n - q_{n,k})] \\ &= \frac{1}{a_1 + k - 1} \left[ h + (c - h)(\alpha_{n,k} - 1)(a_1 + k - 1) + (p - h) \frac{1}{\alpha_{n,k}^{a_1+k-1}} \right]. \end{aligned} \quad (29)$$

Using (29) and the values of  $\gamma_{n+1,k}$  and  $\gamma_{n+1,k+1}$  from (17) by the induction hypothesis, the expression for  $\gamma_{n,k}$  as given in (17) follows from (28). This completes the induction and the proof of the theorem. ■

**Proof of Corollary 2.** (Necessity) Note that the last period of the perishable inventory model is a special case of the storable inventory model when we restrict the initial inventory on-hand at the beginning of period  $N$  to zero. Therefore, the necessity part of the result follows immediately from that of Theorem 1.

(Sufficiency) For the proof of the sufficiency part, see Theorem 2 of Lariviere and Porteus (1999). ■

**Proof of Lemma 1.** It suffices to show this property for  $n = 1$ , since every period  $n$  can be thought of as the beginning of a new dynamic problem. Consider the case where one starts with zero inventory. We would recast this dynamic problem as a one stage decision problem where the decision space  $\mathcal{D}$  consists of functions  $\delta : R_+^N \rightarrow R_+^N$  of the form

$$\delta(\{\xi_n\}) = (\delta_1, \delta_2(\phi_1(\delta_1, \xi_1)), \dots, \delta_N(\{\phi_n(\delta_n, \xi_n)\}_{n < N})),$$

where

$$\phi_n(\delta_n, \xi_n) = (\min\{\delta_n, \xi_n\}, \max\{0, \delta_n - \xi_n\})$$

and the loss function is given by

$$L(\{\xi_n\}, \delta) = \sum_{n=1}^N [h \max(0, \delta_n(\{\phi_j(\delta_j, \xi_j)\}_{j < n}) - \xi_n) + p \max(0, \xi_n - \delta_n(\{\phi_j(\delta_j, \xi_j)\}_{j < n}))].$$

Now  $V_1(0|a, S)$  equals the minimum Bayes risk  $r(\hat{\delta}|a, S)$  where  $\hat{\delta}$  is the optimal Bayes strategy and

$$r(\delta|a, S) = \int R(\theta, \delta) \pi(\theta | a, S) d\theta$$

with

$$R(\theta, \delta) = E_{\otimes_{n=1}^N \psi(\xi_n|\theta)} L(\{\xi_n\}, \delta),$$

where the expectation is taken with respect to the product distribution  $\otimes_{n=1}^N \psi(\xi_n | \theta)$ .

Now consider the extended problem where an extra observation  $\xi_0$  is available before the decision in the first period is taken. The loss function remains the same as above but the expected risk function is now calculated as

$$R^\dagger(\theta, \delta) = E_{\otimes_{n=0}^N \psi(\xi_n|\theta)} L(\{\xi_n\}, \delta).$$

The optimum Bayes risk for this problem will be lower than that of the earlier – since the decision space now extends to  $\mathcal{D}^\dagger$  to include functions that may use  $\xi_0$ . Therefore,

$$\begin{aligned}
V_1(0|a, 1) &\geq \inf_{\delta \in \mathcal{D}^\dagger} \int R^\dagger(\theta, \delta) \pi(\theta|a, 1) d\theta \\
&= \inf_{\delta \in \mathcal{D}^\dagger} \int \int R(\theta, \delta) \pi(\theta|a+1, 1+\xi_0^l) \psi(\xi_0|a, 1) d\theta d\xi_0 \\
&\geq \int \left[ \inf_{\delta \in \mathcal{D}} \int R(\theta, \delta) \pi(\theta|a+1, 1+\xi_0^l) d\theta \right] \psi(\xi_0|a, 1) d\xi_0 \\
&= \int V_1(0|a+1, 1+\xi_0^l) \psi(\xi_0|a, 1) d\xi_0,
\end{aligned}$$

where the penultimate inequality above utilizes the fact that conditional on  $\xi_0$ , the extended decision space  $\mathcal{D}^\dagger$  collapses to the original space  $\mathcal{D}$ . Now using the scalability property of the model with Weibull demand we get

$$V_1(0|a, 1) \geq \int (1+\xi_0^l)^{\frac{1}{l}} V_1(0|a+1, 1) \psi(\xi_0|a, 1) d\xi_0 = \frac{a}{a-\frac{1}{l}} V_1(0|a+1, 1).$$

From this the result follows for any period  $n$  since  $V_n(0|a, 1) = \tilde{v}_n(a)$ . ■

**Proof of Theorem 6.** We will prove by backward induction on  $n$ . For  $n = N$ , the results trivially hold with  $G_{N+1} \equiv 0$  and  $\tilde{v}_{N+1} \equiv 0$ . To prove for any  $n$ , under the induction hypothesis that the results are true for  $(n+1)$ , first we note that for any  $a > 1$ ,

$$V_{n+1}(z|a, 1) = \begin{cases} -cz + \tilde{v}_{n+1}(a) & \text{if } z < \tilde{q}_{n+1}(a) \\ -cz + G_{n+1}(z|a, 1) & \text{if } z \geq \tilde{q}_{n+1}(a) \end{cases}$$

since, by strict convexity,  $G_{n+1}(z|a, 1)$  is strictly increasing to the right of  $\tilde{q}_{n+1}(a)$ . Therefore,

$$\begin{aligned}
G_n(q|a, 1) &= E_{\psi_n(\cdot|a, 1)} \left[ cq + h \max(0, q - \xi_n) + p \max(0, \xi_n - q) \right. \\
&\quad \left. + \beta V_{n+1}(0|a, 1+q) I(\xi_n \geq q) + \beta V_{n+1}(q - \xi_n|a+1, 1+\xi_n) I(\xi_n < q) \right] \\
&= E_{\psi_n(\cdot|a, 1)} \left[ cq + h \max(0, q - \xi_n) + p \max(0, \xi_n - q) \right. \\
&\quad \left. + \beta(1+q) V_{n+1}(0|a, 1) I(\xi_n \geq q) + \beta(1+\xi_n) V_{n+1}\left(\frac{q-\xi_n}{1+\xi_n}|a+1, 1\right) I(\xi_n < q) \right] \\
&= (c+h)q + \frac{1}{a-1} \left[ \frac{p+h}{(1+q)^{a-1}} - h \right] \\
&\quad + \beta \frac{\tilde{v}_{n+1}(a)}{(1+q)^{a-1}} + \beta \int_0^q (1+\xi) V_{n+1}\left(\frac{q-\xi}{1+\xi}|a+1, 1\right) \frac{a}{(1+\xi)^{a+1}} d\xi.
\end{aligned}$$

The last integral can be rewritten as

$$\begin{aligned}
& \beta \left[ \int_0^q I\left(\frac{q-\xi}{1+\xi} < \tilde{q}_{n+1}(a+1)\right) \left\{ -c \left(\frac{q-\xi}{1+\xi}\right) + \tilde{v}_{n+1}(a+1) \right\} \frac{a}{(1+\xi)^a} d\xi \right. \\
& \quad \left. + \int_0^q I\left(\frac{q-\xi}{1+\xi} \geq \tilde{q}_{n+1}(a+1)\right) \left\{ -c \left(\frac{q-\xi}{1+\xi}\right) + Q_{n+1}\left(\frac{q-\xi}{1+\xi} | a+1, 1\right) \right\} \frac{a}{(1+\xi)^a} d\xi \right] \\
= & \beta \left[ \tilde{v}_{n+1}(a+1) \int_0^q I\left(\xi > \frac{q - \tilde{q}_{n+1}(a+1)}{1 + \tilde{q}_{n+1}(a+1)}\right) \frac{a}{(1+\xi)^a} d\xi \right. \\
& \quad + \int_0^q I\left(\xi \leq \frac{q - \tilde{q}_{n+1}(a+1)}{1 + \tilde{q}_{n+1}(a+1)}\right) G_{n+1}\left(\frac{q-\xi}{1+\xi} | a+1, 1\right) \frac{a}{(1+\xi)^a} d\xi \\
& \quad \left. + \frac{c}{a-1} \left(1 - \frac{1}{(1+q)^{a-1}}\right) - cy \right].
\end{aligned}$$

Now, for  $q < \tilde{q}_{n+1}(a+1)$ , the first two terms with integrals equal

$$\tilde{v}_{n+1}(a+1) \int_0^q \frac{a}{(1+\xi)^a} d\xi = \frac{a\tilde{v}_{n+1}(a+1)}{a-1} \left[1 - \frac{1}{(1+q)^{a-1}}\right]$$

and for  $q \geq \tilde{q}_{n+1}(a+1)$ , those two terms equal

$$\begin{aligned}
& \frac{a\tilde{v}_{n+1}(a+1)}{a-1} \frac{1}{(1+q)^{a-1}} \left[ (1 + \tilde{q}_{n+1}(a+1))^{a-1} - 1 \right] \\
& \quad + \int_0^{\frac{q - \tilde{q}_{n+1}(a+1)}{1 + \tilde{q}_{n+1}(a+1)}} G_{n+1}\left(\frac{q-\xi}{1+\xi} | a+1, 1\right) \frac{a}{(1+\xi)^a} d\xi \\
= & \frac{a}{(1+q)^{a-1}} \left[ \frac{\tilde{v}_{n+1}(a+1)}{a-1} \left( (1 + \tilde{q}_{n+1}(a+1))^{a-1} - 1 \right) \right. \\
& \quad \left. + \int_{\tilde{q}_{n+1}(a+1)}^q G_{n+1}(\xi | a+1, 1) (1+\xi)^{a-2} d\xi \right].
\end{aligned}$$

Combining above terms part (b) of the theorem follows.

To prove the differentiability and strict convexity of  $G_n(q|a, 1)$  in  $q$ , note that

$$(c+h-\beta c)y + \frac{1}{a-1} \left[ \frac{p+h-\beta c + \beta\{(a-1)\tilde{v}_{n+1}(a) - a\tilde{v}_{n+1}(a+1)\}}{(1+q)^{a-1}} - (h-\beta c) \right]$$

is strictly convex as  $(a-1)\tilde{v}_{n+1}(a) - a\tilde{v}_{n+1}(a+1) \geq 0$  from Lemma 1. Therefore, it suffices to show that  $T_{n,a}$  is twice differentiable and convex. In the following we drop subscripts and simplify our notations to replace  $G_{n+1}(q|a+1, 1)$  by  $G(q)$ ,  $\tilde{q}_{n+1}(a+1)$  by  $\tilde{q}$ ,  $\tilde{v}_{n+1}(a+1)$  by  $\tilde{v}$  and  $(a-1)T_{n,a}(q)/a$  by  $T(q)$ .

Note that  $T(q)$  is continuous and has the following derivatives on the two sides of  $\tilde{q}$ :

$$T'(q) = \begin{cases} 0 & \text{if } q < \tilde{q} \\ \frac{(a-1)G(q)}{1+q} - \frac{a-1}{(1+q)^a} \left[ (a-1) \int_{\tilde{q}}^q G(\xi) (1+\xi)^{a-2} d\xi + \tilde{v}(1+\tilde{q})^{a-1} \right] & \text{if } q > \tilde{q}. \end{cases}$$

Hence  $T'(q)$  is continuous and  $T''(q)$  is zero for  $q < \tilde{q}$  and equals

$$\begin{aligned} T''(q) &= \frac{a-1}{1+q}G'(q) - \frac{a-1}{(1+q)^2}G(q) \\ &\quad + \frac{a(a-1)}{(1+q)^{a+1}} \left[ (a-1) \int_{\tilde{q}}^q G(\xi)(1+\xi)^{a-2}d\xi + \tilde{v}(1+\tilde{q})^{a-1} \right] - \frac{(a-1)^2}{(1+q)^2}G(q) \end{aligned}$$

for  $q > \tilde{q}$ . The two derivatives agree at  $\tilde{q}$  and hence  $T$  is twice differentiable. Integrating by parts, the integral in the above expression simplifies to

$$\begin{aligned} (a-1) \int_{\tilde{q}}^q G(\xi)(1+\xi)^{a-2}d\xi &= G(\xi)(1+\xi)^{a-1} \Big|_{\tilde{q}}^q - \int_{\tilde{q}}^q G'(\xi)(1+\xi)^{a-1}d\xi \\ &= G(q)(1+q)^{a-1} - \tilde{v}(1+\tilde{q})^{a-1} - \int_{\tilde{q}}^q G'(\xi)(1+\xi)^{a-1}d\xi. \end{aligned}$$

Therefore, for  $q > \tilde{q}$

$$T''(q) = (a-1) \left[ \frac{1}{1+q}G'(q) - \frac{a}{(1+q)^{a+1}} \int_{\tilde{q}}^q G'(\xi)(1+\xi)^{a-1}d\xi \right].$$

But by induction hypothesis,  $G$  is convex with minima at  $\tilde{q}$ . Hence  $G'(q) \geq G'(\xi)$  for  $q \geq \xi \geq \tilde{q}$ . Therefore,

$$\begin{aligned} T''(q) &\geq (a-1) \left[ \frac{1}{1+q}G'(q) - \frac{G'(q)}{(1+q)^{a+1}}(1+\xi)^a \Big|_{\tilde{q}}^q \right] \\ &= (a-1)G'(q) \frac{(1+\tilde{q})^a}{(1+q)^{a+1}}. \end{aligned}$$

So  $T''(q) > 0$  for  $q > \tilde{q}$  since  $G'(q) > 0$  for  $q > \tilde{q}$ . This completes the proof of part (a) of the theorem.

Now we will prove part (c) of the theorem. For the exponential demand ( $l = 1$ ), after simplifications (21) can be written as

$$\begin{aligned} &G_n(q_n|a_n, 1) \\ &= c \sum_{j=0}^{N-n} \beta^j \oint_{A_j} (q_n - \sum_{l=0}^{j-1} \xi_{n+l}) \tilde{\psi}_j(\tilde{\xi}_j) d\tilde{\xi}_j \\ &\quad + (h - \beta c) \sum_{j=0}^{N-n} \beta^j \oint_{A_j} \left[ \int_0^{q_n - \sum_{l=0}^{j-1} \xi_{n+l}} (q_n - \sum_{l=0}^{j-1} \xi_{n+l} - \xi_{n+j}) \psi_{n+j}(\xi_{n+j}|a_n + j, 1 + \sum_{l=0}^{j-1} \xi_{n+l}) d\xi_{n+j} \right] \tilde{\psi}_j(\tilde{\xi}_j) d\tilde{\xi}_j \\ &\quad + p \sum_{j=0}^{N-n} \beta^j \oint_{A_j} \left[ \int_{q_n - \sum_{l=0}^{j-1} \xi_{n+l}}^{\infty} (\xi_{n+j} - \sum_{l=0}^{j-1} \xi_{n+l} + q_n) \psi_{n+j}(\xi_{n+j}|a_n + j, 1 + \sum_{l=0}^{j-1} \xi_{n+l}) d\xi_{n+j} \right] \tilde{\psi}_j(\tilde{\xi}_j) d\tilde{\xi}_j \end{aligned}$$



$$\begin{aligned}
& + (1 + q_n) \sum_{j=0}^{N-n} \beta^{j+1} V_{n+j+1}(0|a_n + j, 1) \oint_{A_j} [1 - \Psi_{n+j}(q_n - \sum_{l=0}^{j-1} \xi_{n+l}|a_n + j, 1 + \sum_{l=0}^{j-1} \xi_{n+l})] \tilde{\psi}_j(\tilde{\xi}_j) d\tilde{\xi}_j \\
& + \sum_{j=0}^{N-n} \beta^{j+1} V_{n+j+1}(0|a_n + j + 1, 1) \oint_{A_j} \left[ \int_0^{q_n - \sum_{l=0}^{j-1} \xi_{n+l}} \frac{1+q_n}{1+\delta_{n+j+1, a_n+j+1}} - (1 + \sum_{l=0}^{j-1} \xi_{n+l}) \right] (1 + \sum_{l=0}^{j-1} \xi_{n+l} + \xi_{n+j}) \times \\
& \quad \psi_{n+j}(\xi_{n+j}|a_n + j, 1 + \sum_{l=0}^{j-1} \xi_{n+l}) d\xi_{n+j} \tilde{\psi}_j(\tilde{\xi}_j) d\tilde{\xi}_j, \tag{30}
\end{aligned}$$

where we denote

$$\begin{aligned}
\oint_{A_j} &= \prod_{k=0}^{j-1} \int_0^{\frac{1+q_n}{1+\delta_{n+k+1, a_n+k+1}} - (1 + \sum_{l=0}^{j-2} \xi_{n+l})}, \\
\tilde{\psi}_j(\tilde{\xi}_j) d\tilde{\xi}_j &= \prod_{k=0}^{j-1} \psi_{n+k}(\xi_{n+k}|a_n + k, 1 + \sum_{l=0}^{k-1} \xi_{n+l}) d\xi_{n+k},
\end{aligned}$$

for  $j = 0, 1, \dots, N - n$ , with the convention that  $\prod_{k=0}^{j-1} = \emptyset$  for  $j = 0$  and  $\sum_{l=0}^{j-2} = \emptyset$  for  $j = 0, 1$ .

Now, since the integrals in (30) can be integrated out, the result follows from (20) and (30), where  $\hat{q}_{n,k}$  solves  $\frac{dG_n(q_n|a_1+k,1)}{dq_n} = 0$ , and  $\hat{v}_{n,k} = G_n(\hat{q}_{n,k}|a_1 + k, 1)$ .  $\blacksquare$

## Addendum

**Proof of Corollary 1.** (a) We rewrite equation (15) as follows

$$\begin{aligned}
\alpha_{n,k} &= \left[ \frac{p-h}{c-h} + (a_1+k)\alpha_{n+1,k} - (a_1+k+1)\alpha_{n+1,k+1} \right. \\
&\quad \left. + \sum_{j=k+1}^{N-(n+1)+k} \{(a_1+j)\alpha_{n+1+j-k,j} - (a_1+j+1)\alpha_{n+1+j-k,j+1}\} + N - (n+1) + 1 \right]^{\frac{1}{a_1+k}} \\
&= \left[ (a_1+k)\alpha_{n+1,k} - (a_1+k+1)\alpha_{n+1,k+1} \right. \\
&\quad \left. + \frac{1}{c-h} \left( p-h + (c-h) \left[ \sum_{j=k+1}^{N-(n+2)+(k+1)} \{(a_1+j)\alpha_{n+2+j-(k+1),j} \right. \right. \right. \\
&\quad \left. \left. \left. - (a_1+j+1)\alpha_{n+2+j-(k+1),j+1}\} + N - (n+1) \right] \right) + 1 \right]^{\frac{1}{a_1+k}} \\
&= \left[ (a_1+k)\alpha_{n+1,k} - (a_1+k+1)\alpha_{n+1,k+1} + \alpha_{n+1,k+1}^{a_1+k+1} + 1 \right]^{\frac{1}{a_1+k}}.
\end{aligned}$$

(b) We rewrite equation (17) as follows

$$\begin{aligned}
\gamma_{n,k} &= \frac{1}{a_1+k-1} \left[ h + (N-n)h + (c-h) \{(a_1+k)\alpha_{n,k} - (a_1+k-1)\} \right. \\
&\quad \left. + (c-h) \sum_{j=k}^{N-(n+1)+k} \{(a_1+j+1)\alpha_{n+1+j-k,j+1} - (a_1+j)\} \right] \\
&= \frac{1}{a_1+k-1} \left[ c + (c-h)(a_1+k)(\alpha_{n,k} - 1) \right. \\
&\quad \left. + (N-n)h + (c-h) \sum_{j=k}^{N-(n+2)+(k+1)} \{(a_1+j+1)\alpha_{n+2+j-(k+1),j+1} - (a_1+j)\} \right] \\
&= \frac{1}{a_1+k-1} \left[ c + (c-h)(a_1+k)(\alpha_{n,k} - 1) + (a_1+k)\gamma_{n+1,k+1} \right].
\end{aligned}$$

■

**Proof of Theorem 4.** For the proof it suffices to show that for any  $k < n$ ,  $\{\alpha_{\hat{n},k}\}_{\hat{n} > n}$  is a Cauchy sequence in  $\hat{n}$  as  $N \rightarrow \infty$ . To this end, we first expand the recursion in Corollary 1(a) as follows:

$$\alpha_{1,0} = \left[ \beta \{(a_1)\alpha_{2,0} - (a_1+1)\alpha_{2,1} + 1\} + \alpha_{2,1}^{a_1+1} \right]^{\frac{1}{a_1}},$$

$$\begin{aligned}
\alpha_{2,0} &= \left[ \beta \{ (a_1) \alpha_{3,0} - (a_1 + 1) \alpha_{3,1} + 1 \} + \alpha_{3,1}^{a_1+1} \right]^{\frac{1}{a_1}}, \\
\alpha_{2,1} &= \left[ \beta \{ (a_1 + 1) \alpha_{3,1} - (a_1 + 2) \alpha_{3,2} + 1 \} + \alpha_{3,2}^{a_1+2} \right]^{\frac{1}{a_1+1}}, \\
&\vdots \\
\alpha_{N-1,0} &= \left[ \beta \{ (a_1) \alpha_{N,0} - (a_1 + 1) \alpha_{N,1} + 1 \} + \alpha_{N,1}^{a_1+1} \right]^{\frac{1}{a_1}}, \\
\alpha_{N-1,1} &= \left[ \beta \{ (a_1 + 1) \alpha_{N,1} - (a_1 + 2) \alpha_{N,2} + 1 \} + \alpha_{N,2}^{a_1+2} \right]^{\frac{1}{a_1+1}}, \\
&\vdots \\
\alpha_{N-1,N-2} &= \left[ \beta \{ (a_1 + N - 2) \alpha_{N,N-2} - (a_1 + N - 1) \alpha_{N,N-1} + 1 \} + \alpha_{N,N-1}^{a_1+N-1} \right]^{\frac{1}{a_1+N-2}},
\end{aligned}$$

with  $\alpha_{N,k} = \left( \frac{p-h}{c-h} \right)^{\frac{1}{a_1+k}}$  for all  $k = 0, 1, 2, \dots, N-1$ .

We will now use backward induction to show that  $\{\alpha_{\hat{n},k}\}_{\hat{n}>n}$  satisfies Cauchy's convergence criterion. First we show that it holds for  $k = N-2$ , that is, for any arbitrarily small  $\epsilon > 0$ ,  $(\alpha_{N-1,N-2} - \alpha_{N,N-2}) < \epsilon$  for large  $N$ . Notice that

$$\begin{aligned}
&\alpha_{N-1,N-2}^{a_1+N-2} - \alpha_{N,N-2}^{a_1+N-2} \\
&= \beta [(a_1 + N - 2) \alpha_{N,N-2} - (a_1 + N - 1) \alpha_{N,N-1} + 1] + \alpha_{N,N-1}^{a_1+N-1} - \alpha_{N,N-2}^{a_1+N-2} \\
&= \beta \left[ (a_1 + N - 2) \left( \frac{p-h}{c-h} \right)^{\frac{1}{a_1+N-2}} - (a_1 + N - 1) \left( \frac{p-h}{c-h} \right)^{\frac{1}{a_1+N-1}} + 1 \right] + \frac{p-h}{c-h} - \frac{p-h}{c-h} \\
&= \beta \left[ (a_1 + N - 1) \left( \frac{p-h}{c-h} \right)^{\frac{1}{a_1+N-1}} \left\{ \left( \frac{p-h}{c-h} \right)^{\frac{1}{(a_1+N-1)(a_1+N-2)}} - 1 \right\} \right] - \beta \left[ \left( \frac{p-h}{c-h} \right)^{\frac{1}{a_1+N-2}} - 1 \right] \\
&< \epsilon,
\end{aligned} \tag{31}$$

for sufficiently large  $N$ . Now, for real numbers  $a \geq b \geq 1$  and  $p \geq 1$ , we know  $a^p - b^p$  is an increasing function in  $p$ . Let  $\lfloor p \rfloor$  denote the largest positive integer less than or equal to  $p$ . Then we can show

$$\begin{aligned}
a^p - b^p &\geq a^{\lfloor p \rfloor} - b^{\lfloor p \rfloor} \\
&= (a - b) [a^{\lfloor p \rfloor - 1} + a^{\lfloor p \rfloor - 2} b + \dots + ab^{\lfloor p \rfloor - 2} + b^{\lfloor p \rfloor - 1}] \\
&\geq \lfloor p \rfloor (a - b),
\end{aligned}$$

which implies

$$(a - b) \leq \frac{a^p - b^p}{\lfloor p \rfloor}. \tag{32}$$

Using this inequality, from (31) we get  $(\alpha_{N-1,N-2} - \alpha_{N,N-2}) < \epsilon$ . Hence the result holds for  $k = N - 2$ . Similarly, we can also show that for sufficiently large  $N$ ,  $(\alpha_{N-1,N-3} - \alpha_{N,N-3}) < \epsilon$ , which we will need below for the induction over  $\hat{n}$ .

Now we assume that the result holds for  $k + 1$ . We want to prove the result for  $k$ , for  $k < n$ . For this we will show that for any arbitrarily small  $\epsilon > 0$ , there exists large  $N$  such that  $(\alpha_{\hat{n}_1,k} - \alpha_{\hat{n}_2,k}) < \epsilon$  for  $\hat{n}_2 \geq \hat{n}_1 > n$ . By the induction hypothesis (over both  $k$  and  $\hat{n}$ ), there exists sufficiently large  $N$  so that we have

$$(\alpha_{\hat{n}_1+1,k+1}^{a_1+k+1} - \alpha_{\hat{n}_2+1,k+1}^{a_1+k+1}) < \frac{\epsilon}{2} \quad \text{and} \quad (\alpha_{\hat{n}_1+1,k} - \alpha_{\hat{n}_2+1,k}) < \frac{\epsilon}{4}.$$

Now, from Corollary 1(a) we write

$$\begin{aligned} & \alpha_{\hat{n}_1,k}^{a_1+k} - \alpha_{\hat{n}_2,k}^{a_1+k} \\ &= \beta[(a_1 + k)(\alpha_{\hat{n}_1+1,k} - \alpha_{\hat{n}_2+1,k}) - (a_1 + k + 1)(\alpha_{\hat{n}_1+1,k+1} - \alpha_{\hat{n}_2+1,k+1})] + \alpha_{\hat{n}_1+1,k+1}^{a_1+k+1} - \alpha_{\hat{n}_2+1,k+1}^{a_1+k+1} \\ &< (a_1 + k)(\alpha_{\hat{n}_1+1,k} - \alpha_{\hat{n}_2+1,k}) + \alpha_{\hat{n}_1+1,k+1}^{a_1+k+1} - \alpha_{\hat{n}_2+1,k+1}^{a_1+k+1} \\ &< (a_1 + k)\frac{\epsilon}{4} + \frac{\epsilon}{2}, \end{aligned}$$

which, using inequality (32) leads to  $\alpha_{\hat{n}_1,k} - \alpha_{\hat{n}_2,k} < \frac{(a_1+k)\frac{\epsilon}{4} + \frac{\epsilon}{2}}{[a_1+k]} < \epsilon$ . Hence the Cauchy's convergence criterion holds for  $\{\alpha_{\hat{n},k}\}_{\hat{n} > n}$ .  $\blacksquare$

**Proof of Theorem 5.** (a) From the recursion in Corollary 1(a) we can successively write the following equations:

$$\begin{aligned} \alpha_{n,k}^{a_1+k} - \beta(a_1 + k)\alpha_{n+1,k} &= \alpha_{n+1,k+1}^{a_1+k+1} - \beta(a_1 + k + 1)\alpha_{n+1,k+1} + \beta, \\ \alpha_{n+1,k+1}^{a_1+k+1} - \beta(a_1 + k + 1)\alpha_{n+2,k+1} &= \alpha_{n+2,k+2}^{a_1+k+2} - \beta(a_1 + k + 2)\alpha_{n+2,k+2} + \beta, \\ &\vdots \\ \alpha_{N-1,k+N-n-1}^{a_1+k+N-n-1} - \beta(a_1 + k + N - n - 1)\alpha_{N,k+N-n-1} &= \alpha_{N,k+N-n}^{a_1+k+N-n} - \beta(a_1 + k + N - n)\alpha_{N,k+N-n} + \beta, \end{aligned}$$

where  $\alpha_{N,k+N-n} = \left(\frac{p-h}{c-h}\right)^{\frac{1}{a_1+k+N-n}}$ .

Summing over both sides of the system of equations we get

$$\begin{aligned} & \left(\alpha_{n,k}^{a_1+k} - \beta(a_1 + k)\alpha_{n+1,k}\right) + \left(\alpha_{n+1,k+1}^{a_1+k+1} - \beta(a_1 + k + 1)\alpha_{n+2,k+1}\right) + \dots \\ & \quad + \left(\alpha_{N-1,k+N-n-1}^{a_1+k+N-n-1} - \beta(a_1 + k + N - n - 1)\alpha_{N,k+N-n-1}\right) \\ &= \left(\alpha_{n+1,k+1}^{a_1+k+1} - \beta(a_1 + k + 1)\alpha_{n+1,k+1}\right) + \dots + \left(\alpha_{N-1,k+N-n-1}^{a_1+k+N-n-1} - \beta(a_1 + k + N - n - 1)\alpha_{N-1,k+N-n-1}\right) \\ & \quad + \alpha_{N,k+N-n}^{a_1+k+N-n} - \beta(a_1 + k + N - n)\alpha_{N,k+N-n} + (N - n)\beta. \end{aligned} \tag{33}$$

We first rewrite the last three terms on the right hand side of (33) as

$$\frac{p-h}{c-h} - \beta(a_1+k) \left( \frac{p-h}{c-h} \right)^{\frac{1}{a_1+k+N-n}} + (N-n)\beta \left[ 1 - \left( \frac{p-h}{c-h} \right)^{\frac{1}{a_1+k+N-n}} \right].$$

Then letting  $N \rightarrow \infty$ , using Theorem 4, and cancelling common terms, from (33) we get

$$l_k^{a_1+k} - \beta(a_1+k)l_k = \frac{p-h}{c-h} - \beta(a_1+k).$$

To prove that  $l_k$  is unique and bounded, let us denote the left hand side of equation (18) as  $f(l_k) = l_k^{a_1+k} - \beta(a_1+k)l_k$ . Since  $a_1 > 1$  and  $l_k \geq 1$ ,  $f(l_k)$  is a strictly increasing and convex function of  $l_k$ . Also, note that  $f(1) = 1 - \beta(a_1+k) < \frac{p-h}{c-h} - \beta(a_1+k)$ , the right hand side of (18). Hence,  $l_k$  is unique and bounded. Now, notice that the slope of  $f(l_k)$ , that is,  $f'(l_k) = (a_1+k) \left( l_k^{a_1+k-1} - \beta \right)$  is increasing in  $k$ . Hence,  $l_k$  decreases in  $k$ .

(b) From the recursion in Corollary 1(b) we write the following system of equations:

$$\begin{aligned} (a_1+k-1)\gamma_{n,k} &= c + (c-h)(a_1+k)(\alpha_{n,k}-1) + \beta(a_1+k)\gamma_{n+1,k+1} \\ \beta(a_1+k)\gamma_{n+1,k+1} &= \beta\{c + (c-h)(a_1+k+1)(\alpha_{n+1,k+1}-1)\} \\ &\quad + \beta^2(a_1+k+1)\gamma_{n+2,k+2} \\ &\vdots \\ \beta^{N-n-1}(a_1+k+N-n-2)\gamma_{N-1,k+N-n-1} &= \beta^{N-n-1}\{c + (c-h)(a_1+k+N-n-1)(\alpha_{N-1,k+N-n-1}-1)\} \\ &\quad + \beta^{N-n}(a_1+k+N-n-1)\gamma_{N,k+N-n}, \end{aligned}$$

with  $\gamma_{N,k+N-n} = \frac{1}{a_1+k+N-n-1} [c + (c-h)(a_1+k+N-n)(\alpha_{N,k+N-n}-1)]$ .

Summing over both sides of the above equations we get

$$\begin{aligned} (a_1+k-1)\gamma_{n,k} &= \frac{c(1-\beta^{N-n+1})}{1-\beta} + (c-h) \left\{ (a_1+k)(\alpha_{n,k}-1) + \beta(a_1+k+1)(\alpha_{n+1,k+1}-1) \right. \\ &\quad \left. + \dots + \beta^{N-n}(a_1+k+N-n)(\alpha_{N,k+N-n}-1) \right\}. \end{aligned}$$

By letting  $N \rightarrow \infty$  and invoking Theorem 4 we get

$$m_k = \lim_{N \rightarrow \infty} \gamma_{n,k} = \frac{1}{a_1+k-1} \left[ \frac{c}{1-\beta} + (c-h) \sum_{j=k}^{\infty} \beta^{j-k}(a_1+j)(l_j-1) \right].$$

Now, we will prove the convergence of the series. By part (a), since  $l_j \leq l_k$ , for all  $j > k$  and  $l_k$  is bounded, we can write

$$\sum_{j=k}^{\infty} \beta^{j-k}(a_1+j)(l_j-1) \leq (l_k-1) \sum_{j=k}^{\infty} \beta^{j-k}(a_1+j).$$

By the ratio test, we have

$$\lim_{q \rightarrow \infty} \frac{\beta^{q+1-k}(a_1 + q + 1)}{\beta^{q-k}(a_1 + q)} = \lim_{q \rightarrow \infty} \left(1 + \frac{1}{a_1 + q}\right) \beta = \beta < 1.$$

Hence, the series converges. This completes the proof of the theorem.  $\blacksquare$

**Two-Period and Three-Period Examples to Demonstrate Theorem 6(c).** For exponential demand, after some simplification, the first-period cost function for the normalized system in a two-period and a three-period problem can be written from (21) as

$$\begin{aligned} & G_1(q_1|a_1, 1) \\ = & cq_1 + (h - \beta c) \left( q_1 - \frac{1}{a_1 - 1} \right) + (p + h - \beta c) \frac{1}{a_1 - 1} \frac{1}{(1 + q_1)^{a_1 - 1}} \\ & + \beta V_2(0|a_1, 1) \frac{1}{(1 + q_1)^{a_1 - 1}} + \beta V_2(0|a_1 + 1, 1) \left( (1 + \delta_{2,a_1+1})^{a_1 - 1} - 1 \right) \frac{a_1}{a_1 - 1} \frac{1}{(1 + q_1)^{a_1 - 1}} \\ & + \beta \left\{ c \left[ q_1 - \frac{1}{a_1 - 1} + (1 + \delta_{2,a_1+1})^{a_1 - 1} \left( \frac{1}{a_1 - 1} - \delta_{2,a_1+1} \right) \frac{1}{(1 + q_1)^{a_1 - 1}} \right] \right. \\ & \quad + (h - \beta c) \left[ q_1 - \frac{2}{a_1 - 1} + (1 + \delta_{2,a_1+1})^{a_1 - 1} \left( \frac{2}{a_1 - 1} - \delta_{2,a_1+1} \right) \frac{1}{(1 + q_1)^{a_1 - 1}} \right] \\ & \quad \left. + (p + h - \beta c) \frac{1}{(1 + q_1)^{a_1}} \left( \frac{1 + q_1}{1 + \delta_{2,a_1+1}} - 1 \right) \right\} \tag{34} \\ & + \beta^2 V_3(0|a_1 + 1, 1) \frac{a_1}{(1 + q_1)^{a_1}} \left( \frac{1 + q_1}{1 + \delta_{2,a_1+1}} - 1 \right) \\ & + \beta^2 V_3(0|a_1 + 2, 1) \left( (1 + \delta_{3,a_1+2})^{a_1} - 1 \right) \frac{a_1 + 1}{(1 + q_1)^{a_1}} \left( \frac{1 + q_1}{1 + \delta_{2,a_1+1}} - 1 \right) \\ & + \beta^2 \left\{ c \left[ q_1 - \frac{2}{a_1 - 1} + (1 + \delta_{2,a_1+1})^{a_1 - 1} \left( \frac{2}{a_1 - 1} - \delta_{2,a_1+1} \right) \frac{1}{(1 + q_1)^{a_1 - 1}} \right. \right. \\ & \quad \left. \left. + (1 + \delta_{3,a_1+2})^{a_1} \left( \frac{1}{a_1} - \delta_{3,a_1+2} \right) \frac{a_1}{(1 + q_1)^{a_1}} \left( \frac{1 + q_1}{1 + \delta_{2,a_1+1}} - 1 \right) \right] \right. \\ & \quad + (h - \beta c) \left[ q_1 - \frac{3}{a_1 - 1} + (1 + \delta_{2,a_1+1})^{a_1 - 1} \left( \frac{3}{a_1 - 1} - \delta_{2,a_1+1} \right) \frac{1}{(1 + q_1)^{a_1 - 1}} \right. \\ & \quad \left. \left. + (1 + \delta_{3,a_1+2})^{a_1} \left( \frac{2}{a_1} - \delta_{3,a_1+2} \right) \frac{a_1}{(1 + q_1)^{a_1}} \left( \frac{1 + q_1}{1 + \delta_{2,a_1+1}} - 1 \right) \right] \right. \\ & \quad \left. + (p + h - \beta c) \left[ \frac{1}{(1 + \delta_{3,a_1+2})} \frac{a_1}{(1 + q_1)^{a_1}} \left( \frac{1 + q_1}{1 + \delta_{2,a_1+1}} - 1 \right) \right] \right\} \end{aligned}$$

$$- \frac{a_1}{(1+q_1)^{a_1+1}} \frac{1}{2} \left\{ \left( \frac{1+q_1}{1+\delta_{2,a_1+1}} \right)^2 - 1 \right\} \Bigg] \Bigg\}, \quad (35)$$

where  $\delta_{2,a_1+1}$  and  $\delta_{3,a_1+2}$  are the optimal policies in the second and the third period respectively (at nodes  $B_2$  and  $C_2$  in Figure 2) for the normalized systems with unit scale parameter. In the above, the expression upto (34) represents the cost for the two-period problem.

Now, the first-period optimal policy  $\hat{q}_{1,0}$  is obtained by solving  $\frac{dG_1(q_1|a_1,1)}{dq_1} = 0$ , and consequently, from the above equation, we obtain the optimal cost  $\hat{v}_{1,0} = G_1(\hat{q}_{1,0}|a_1,1)$ .