IE 440

PROCESS IMPROVEMENT THROUGH PLANNED EXPERIMENTATION

IE 406
Simulation

Basic Probability

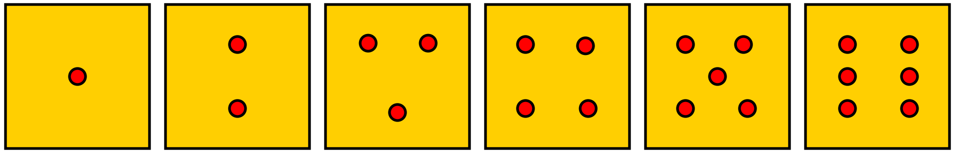
Dr. Xueping Li
University of Tennessee

Chapter Topics

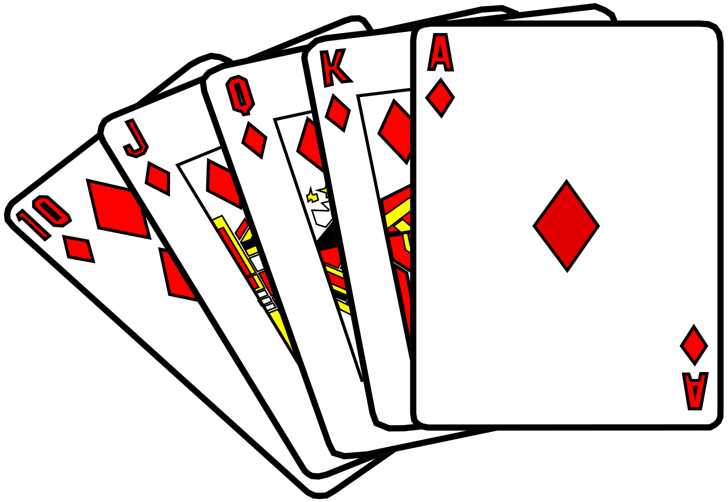
- Basic Probability Concepts
  - Sample spaces and events, simple probability, joint probability
- Conditional Probability
  - Statistical independence, marginal probability
- Bayes’ Theorem
- Counting Rules

Sample Spaces

- Collection of All Possible Outcomes
  - E.g., All 6 faces of a die:



  - E.g., All 52 cards of a bridge deck:



Events

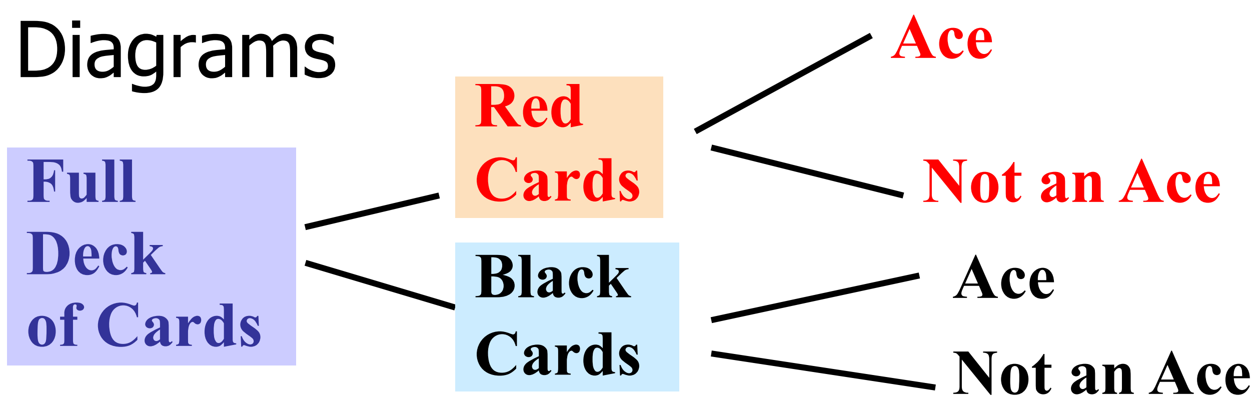
- Simple Event
  - Outcome from a sample space with 1 characteristic
  - E.g., a Red Card from a deck of cards
- Joint Event
  - Involves 2 outcomes simultaneously
  - E.g., an Ace which is also a Red Card from a deck of cards

Visualizing Events

- Contingency Tables

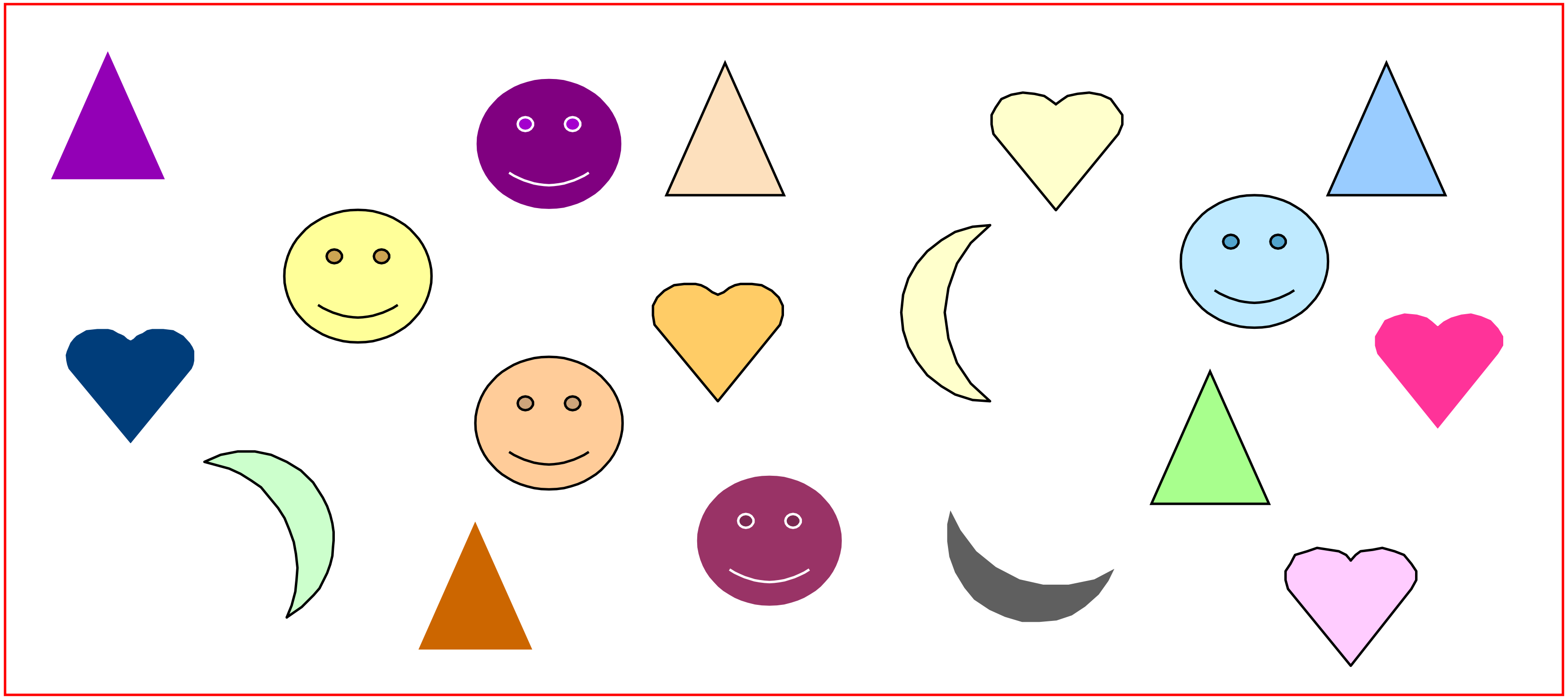
	Ace	Not Ace	Total
Black	2	24	26
Red	2	24	26
Total	4	48	52

- Tree Diagrams



Simple Events

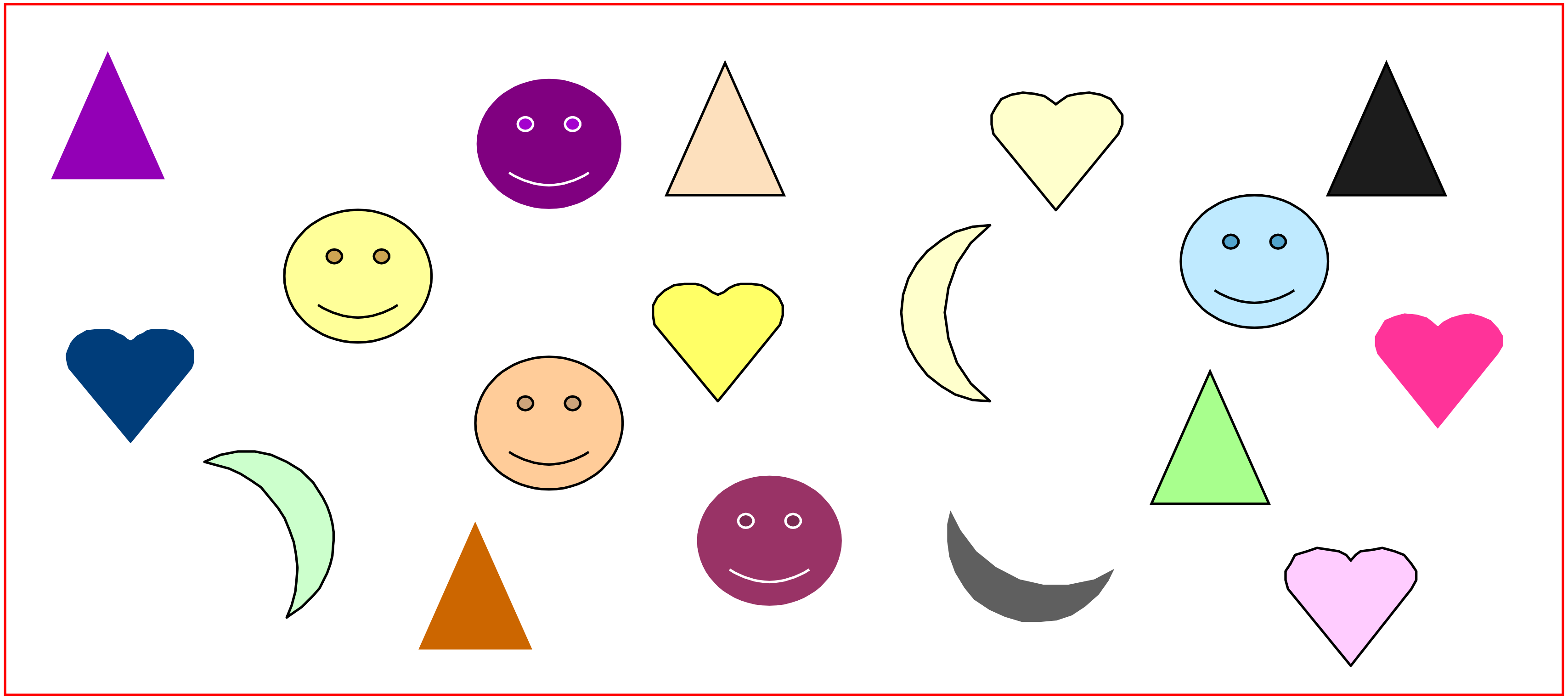
The Event of a Happy Face



There are 5 happy faces in this collection of 18 objects.

Joint Events

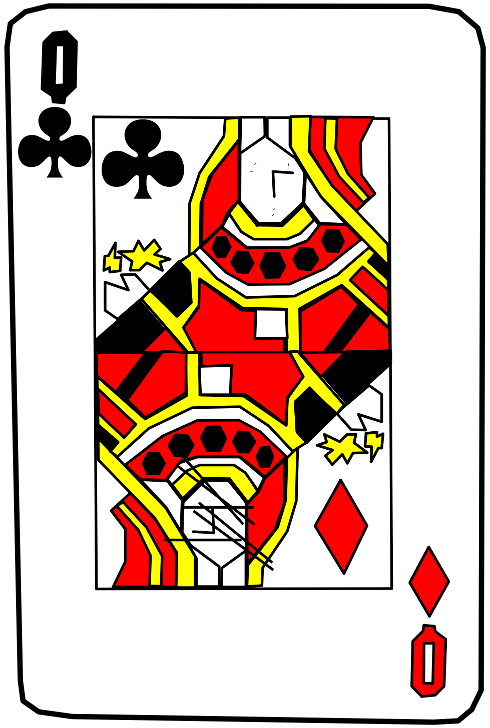
The Event of a Happy Face *AND* Yellow



1 Happy Face which is Yellow

Special Events

Impossible Event



- Impossible Event
  - Impossible event
  - E.g., Club & Diamond on 1 card draw
- Complement of Event
  - For event A, all events not in A
  - Denoted as A′
  - E.g., A: Queen of Diamond
    A′: All cards in a deck that are not Queen of Diamond

Special Events

(continued)

- Mutually Exclusive Events
  - Two events cannot occur together
  - E.g., A: Queen of Diamond; B: Queen of Club
    - Events A and B are mutually exclusive
- Collectively Exhaustive Events
  - One of the events must occur
  - The set of events covers the whole sample space
  - E.g., A: All the Aces; B: All the Black Cards; C: All the Diamonds; D: All the Hearts
    - Events A, B, C and D are collectively exhaustive
    - Events B, C and D are also collectively exhaustive

Contingency Table

A Deck of 52 Cards

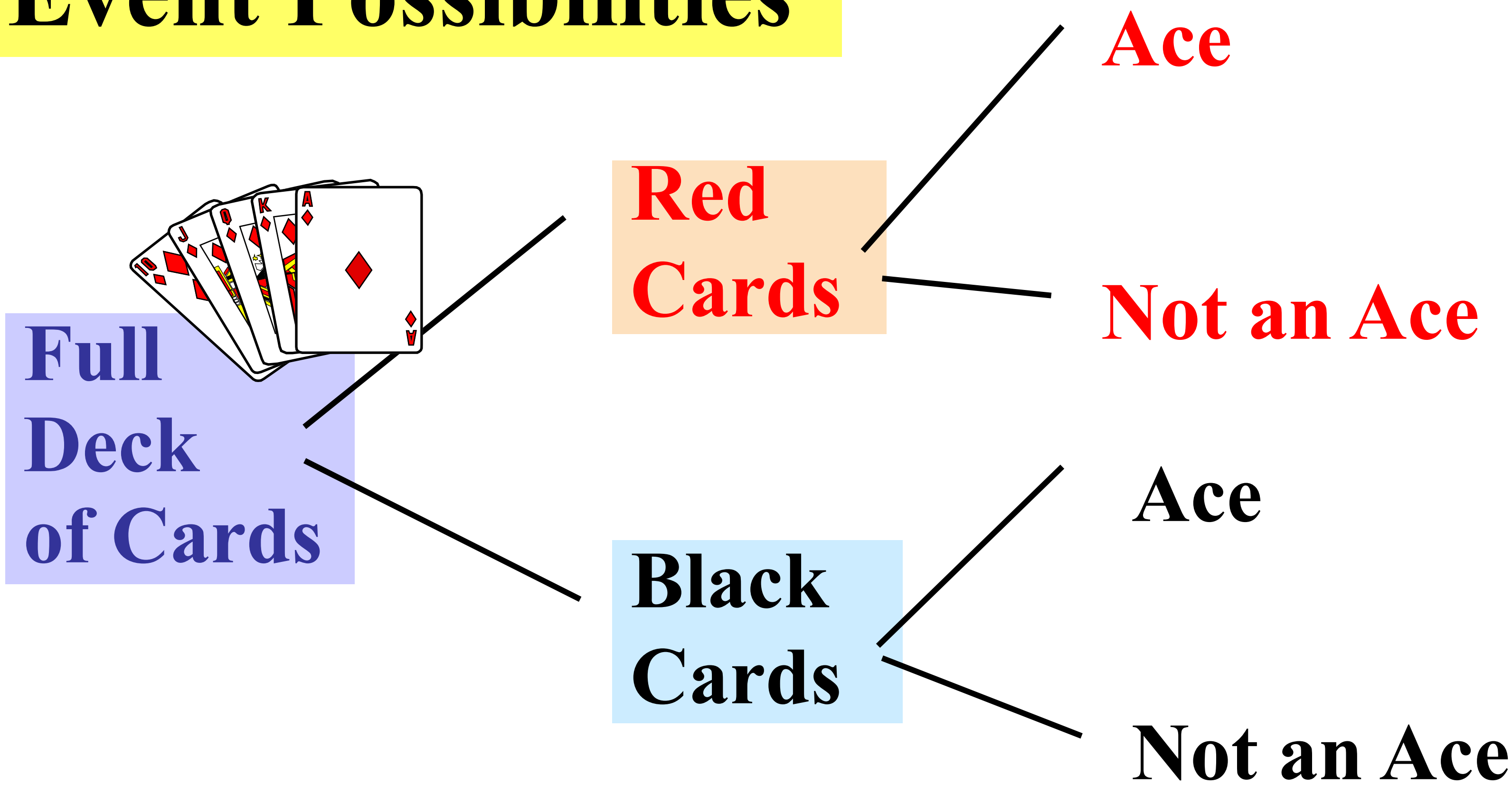
Red Ace

	Ace	Not an Ace	
Red	2	24	26
Black	2	24	26
Total	4	48	52

Sample Space

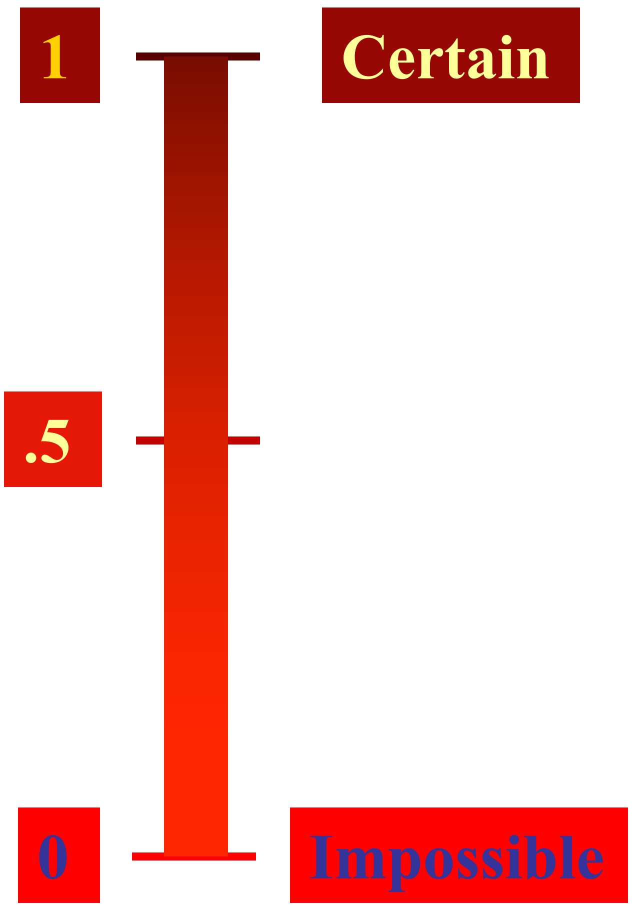
Tree Diagram

Event Possibilities



Probability

- Probability is the Numerical Measure of the Likelihood that an Event Will Occur
- Value is between 0 and 1
- Sum of the Probabilities of All Mutually Exclusive and Collective Exhaustive Events is 1

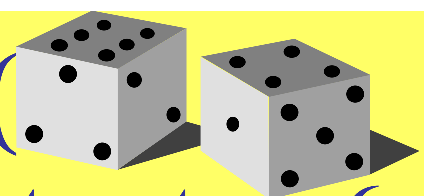


Computing Probabilities

- The Probability of an Event E:

$$P(E) = \frac{\text{number of event outcomes}}{\text{total number of possible outcomes in the sample space}}$$

$$= \frac{X}{T}$$

E.g., P( **) = 2/36**

(There are 2 ways to get one 6 and the other 4)

- Each of the Outcomes in the Sample Space is Equally Likely to Occur

Computing Joint Probability

- The Probability of a Joint Event, A and B:

$$P(A \text{ and } B)$$

$$= \frac{\text{number of outcomes from both A and B}}{\text{total number of possible outcomes in sample space}}$$

$$\text{E.g. } P(\text{Red Card and Ace})$$

$$= \frac{\text{2 Red Aces}}{\text{52 Total Number of Cards}} = \frac{1}{26}$$

Joint Probability Using Contingency Table

Event	Event B_1	B_2	Total
A_1	$P(A_1 \text{ and } B_1)$	$P(A_1 \text{ and } B_2)$	$P(A_1)$
A_2	$P(A_2 \text{ and } B_1)$	$P(A_2 \text{ and } B_2)$	$P(A_2)$
Total	$P(B_1)$	$P(B_2)$	1

Joint Probability

Marginal (Simple) Probability

Computing Compound Probability

- Probability of a Compound Event, A or B:

$$P(A \text{ or } B) = \frac{\text{number of outcomes from either A or B or both}}{\text{total number of outcomes in sample space}}$$

$$\text{E.g.} \quad P(\text{Red Card or Ace})$$

$$= \frac{\text{4 Aces + 26 Red Cards - 2 Red Aces}}{\text{52 total number of cards}}$$

$$= \frac{28}{52} = \frac{7}{13}$$

Compound Probability (Addition Rule)

$P(A_1 \text{ or } B_1) = P(A_1) + P(B_1) - P(A_1 \text{ and } B_1)$

Event	Event: B_1	Event: B_2	Total
A_1	$P(A_1 \text{ and } B_1)$	$P(A_1 \text{ and } B_2)$	$P(A_1)$
A_2	$P(A_2 \text{ and } B_1)$	$P(A_2 \text{ and } B_2)$	$P(A_2)$
Total	$P(B_1)$	$P(B_2)$	1

For Mutually Exclusive Events: P(A or B) = P(A) + P(B)

Computing Conditional Probability

- The Probability of Event A Given that Event B Has Occurred:

$$P(A \mid B) = \frac{P(A \text{ and } B)}{P(B)}$$

E.g.

$$P(\text{Red Card given that it is an Ace})$$

$$= \frac{2 \text{ Red Aces}}{4 \text{ Aces}} = \frac{1}{2}$$

Conditional Probability Using Contingency Table

Type	Color		Total
	Red	Black	
Ace	2	2	4
Non-Ace	24	24	48
Total	26	26	52

Revised Sample Space

$$P(\text{Ace} \mid \text{Red}) = \frac{P(\text{Ace and Red})}{P(\text{Red})} = \frac{2/52}{26/52} = \frac{2}{26}$$

Conditional Probability and Statistical Independence

- Conditional Probability:

$$P(A \mid B) = \frac{P(A \text{ and } B)}{P(B)}$$

- Multiplication Rule:

$$P(A \text{ and } B) = P(A \mid B)\ P(B)$$
$$= P(B \mid A)\ P(A)$$

Conditional Probability and Statistical Independence

(continued)

- Events A and B are Independent if

$$P(A \mid B) = P(A)$$

$$\text{or} \quad P(B \mid A) = P(B)$$

$$\text{or} \quad P(A \text{ and } B) = P(A)P(B)$$

- Events A and B are Independent When the Probability of One Event, A, is Not Affected by Another Event, B

Bayes' Theorem

$$P(B_i \mid A) = \frac{P(A \mid B_i)P(B_i)}{P(A \mid B_1)P(B_1) + \bullet\bullet\bullet + P(A \mid B_k)P(B_k)}$$

$$= \frac{P(B_i \text{ and } A)}{P(A)}$$

Same Event

Adding up the parts of A in all the B's

Bayes' Theorem Using Contingency Table

50% of borrowers repaid their loans. Out of those who repaid, 40% had a college degree. 10% of those who defaulted had a college degree. What is the probability that a randomly selected borrower who has a college degree will repay the loan?

$$P(R) = .50 \qquad P(C \mid R) = .4 \qquad P(C \mid \overline{R}) = .10$$

$$P(R \mid C) = ?$$

Bayes' Theorem Using Contingency Table

(continued)

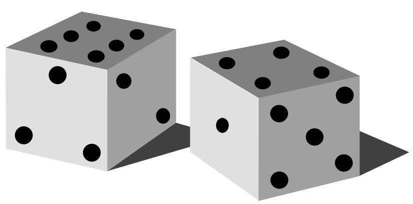
	Repay	$\overline{\text{Repay}}$	Total
College	.2	.05	.25
$\overline{\text{College}}$	.3	.45	.75
Total	.5	.5	1.0

$$P(R \mid C) = \frac{P(C \mid R)\,P(R)}{P(C \mid R)\,P(R) + P(C \mid \bar{R})\,P(\bar{R})}$$

$$= \frac{(.4)(.5)}{(.4)(.5)+(.1)(.5)} = \frac{.2}{.25} = .8$$

Counting Rule 1

- If any one of *k* different mutually exclusive and collectively exhaustive events can occur on each of the *n* trials, the number of possible outcomes is equal to k^n.
  - E.g., A six-sided die is rolled 5 times, the number of possible outcomes is $6^5 = 7776$.



Counting Rule 2

- If there are k_1 events on the first trial, k_2 events on the second trial, …, and k_n events on the n th trial, then the number of possible outcomes is $(k_1)(k_2)\bullet\bullet\bullet(k_n)$.
  - E.g., There are 3 choices of beverages and 2 choices of burgers. The total possible ways to choose a beverage and a burger are (3)(2) = 6.

Counting Rule 3

- The number of ways that n objects can be arranged in order is $n! = n(n - 1)\bullet\bullet\bullet(1)$.
  - $n!$ is called *n factorial*
  - 0! is 1
  - E.g., The number of ways that 4 students can be lined up is 4! = (4)(3)(2)(1)=24.

Counting Rule 4: Permutations

- The number of ways of arranging *X* objects selected from *n* objects in order is

$$\frac{n!}{(n-X)!}$$

  - The order is important.
  - E.g., The number of different ways that 5 music chairs can be occupied by 6 children are

$$\frac{n!}{(n-X)!} = \frac{6!}{(6-5)!} = 720$$

Counting Rule 5: Combinations

- The number of ways of selecting X objects out of n objects, irrespective of order, is equal to

$$\frac{n!}{X!(n-X)!}$$

  - The order is irrelevant.
  - E.g., The number of ways that 5 children can be selected from a group of 6 is

$$\frac{n!}{X!(n-X)!} = \frac{6!}{5!(6-5)!} = 6$$

Chapter Summary

- Discussed Basic Probability Concepts
  - Sample spaces and events, simple probability, and joint probability
- Defined Conditional Probability
  - Statistical independence, marginal probability
- Discussed Bayes' Theorem
- Described the Various Counting Rules