THE GRAPHS OF THE q-CHARACTERS OF THE FUNDAMENTAL REPRESENTATIONS

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ABSTRACT. In this note, by using Frenkel-Mukhin algorithm, we record the directed colored graphs induced from the q-characters of the fundamental representations of types $A_3^{(1)}$, $C_3^{(1)}$, $C_4^{(1)}$, $B_3^{(1)}$, $D_4^{(1)}$ and $G_2^{(1)}$.

CONTENTS

1. UNTWISTED TYPES

1.1. **Notations.** Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra over $\mathbb{C}$ with an index set I. For simplicity, put

$$Y_{i,aq^k} = Y_{i,k},$$

where $i \in I$, $a \in \mathbb{C}^\times$ and $k \in \mathbb{Z}$. For $i \in I$ and $a \in \mathbb{C}^\times$, we set

$$A_{i,a} = Y_{i,aq_i^{-1}} Y_{i,aq_i} \prod_{a_{ji}=-1} Y_{j,a}^{-1} \prod_{a_{ji}=-2} Y_{j,aq}^{-1} Y_{j,aq^{-1}}^{-1} \prod_{a_{ji}=-3} Y_{j,aq^{-2}}^{-1} Y_{j,a}^{-1} Y_{j,aq^2}^{-1},$$

where $(a_{ij})_{i,j\in I}$ is the Cartan matrix of $\mathfrak{g}$. For two monomials $m, m' \in \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i\in I,\, a\in\mathbb{C}^\times}$, we use the following convention:

$$m \xrightarrow{i,k} m' \text{ if and only if } m' = m A_{i,aq^k}^{-1},$$

where $i \in I$ and $k \in \mathbb{Z}$.

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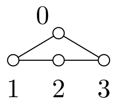
1.2. A. $A_3^{(1)}$



$\chi_q(L(Y_{2,0}))$, $\sharp = 6$

$\chi_q(L(Y_{1,0}))$, $\sharp = 4$

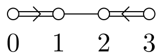
$$\begin{tikzcd}
Y_{1,0} \arrow[d, "{1,1}"] \\
Y_{2,1}Y_{1,2}^{-1} \arrow[d, "{2,2}"] \\
Y_{3,2}Y_{2,3}^{-1} \arrow[d, "{3,3}"] \\
Y_{3,4}^{-1}
\end{tikzcd}$$

$$\begin{tikzcd}
 & Y_{2,0} \arrow[d, "{2,1}"] & \\
 & Y_{1,1}Y_{3,1}Y_{2,2}^{-1} \arrow[dl, "{1,2}"'] \arrow[dr, "{3,2}"] & \\
Y_{3,1}Y_{1,3}^{-1} \arrow[dr, "{3,2}"'] & & Y_{1,1}Y_{3,3}^{-1} \arrow[dl, "{1,2}"] \\
 & Y_{2,2}Y_{1,3}^{-1}Y_{3,3}^{-1} \arrow[d, "{2,3}"'] & \\
 & Y_{2,4}^{-1} &
\end{tikzcd}$$

$\chi_q(L(Y_{3,0}))$, $\sharp = 4$

$$\begin{tikzcd}
Y_{3,0} \arrow[d, "{3,1}"] \\
Y_{2,1}Y_{3,2}^{-1} \arrow[d, "{2,2}"] \\
Y_{1,2}Y_{2,3}^{-1} \arrow[d, "{1,3}"] \\
Y_{1,4}^{-1}
\end{tikzcd}$$

1.3. C. $C_3^{(1)}$



$\chi_q(L(Y_{1,0}))$, $\sharp = 6$

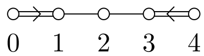
$$\begin{tikzcd}
Y_{1,0} \arrow[d, "1,1"'] \\
Y_{2,1}Y_{1,2}^{-1} \arrow[d, "2,2"'] \\
Y_{3,2}Y_{2,3}^{-1} \arrow[d, "3,4"'] \\
Y_{2,5}Y_{3,6}^{-1} \arrow[d, "2,6"'] \\
Y_{1,6}Y_{2,7}^{-1} \arrow[d, "1,7"'] \\
Y_{1,8}^{-1}
\end{tikzcd}$$

$\chi_q(L(Y_{2,0}))$, $\sharp = 14$

$$\begin{tikzcd}
& Y_{2,0} \arrow[d, "2,1"] & \\
& Y_{1,1}Y_{3,1}Y_{2,2}^{-1} \arrow[dl, "1,2"'] \arrow[dr, "3,3"] & \\
Y_{3,1}Y_{1,3}^{-1} \arrow[d, "3,3"'] & & Y_{1,1}Y_{2,4}Y_{3,5}^{-1} \arrow[dll, "1,2"'] \arrow[d, "2,5"] \\
Y_{2,2}Y_{2,4}Y_{1,3}^{-1}Y_{3,5}^{-1} \arrow[d, "2,5"'] & & Y_{1,1}Y_{1,5}Y_{2,6}^{-1} \arrow[dll, "1,2"'] \arrow[d, "1,6"] \\
Y_{1,5}Y_{2,2}Y_{1,3}^{-1}Y_{2,6}^{-1} \arrow[d, "2,3"'] \arrow[drr, "1,6"] & & Y_{1,1}Y_{1,7}^{-1} \arrow[d, "1,2"] \\
Y_{1,5}Y_{3,3}Y_{2,4}^{-1}Y_{2,6}^{-1} \arrow[d, "3,5"'] \arrow[drr, "1,6"] & & Y_{2,2}Y_{1,3}^{-1}Y_{1,7}^{-1} \arrow[d, "2,3"] \\
Y_{1,5}Y_{3,7}^{-1} \arrow[dr, "1,6"'] & & Y_{3,3}Y_{1,7}^{-1}Y_{2,4}^{-1} \arrow[dl, "3,5"] \\
& Y_{2,6}Y_{1,7}^{-1}Y_{3,7}^{-1} \arrow[d, "2,7"] & \\
& Y_{2,8}^{-1} &
\end{tikzcd}$$

$\chi_q(L(Y_{3,0}))$, $\sharp = 14$

$$\begin{tikzcd}
& Y_{3,0} \arrow[d, "3,2"] & \\
& Y_{2,1}Y_{2,3}Y_{3,4}^{-1} \arrow[d, "2,4"] & \\
& Y_{1,4}Y_{2,1}Y_{2,5}^{-1} \arrow[dl, "1,5"'] \arrow[dr, "2,2"] & \\
Y_{2,1}Y_{1,6}^{-1} \arrow[d, "2,2"'] & & Y_{1,2}Y_{1,4}Y_{3,2}Y_{2,3}^{-1}Y_{2,5}^{-1} \arrow[dll, "1,5"'] \arrow[d, "3,4"] \\
Y_{1,2}Y_{3,2}Y_{1,6}^{-1}Y_{2,3}^{-1} \arrow[d, "1,3"'] \arrow[drr, "3,4"] & & Y_{1,2}Y_{1,4}Y_{3,6}^{-1} \arrow[d, "1,5"] \\
Y_{3,2}Y_{1,4}^{-1}Y_{1,6}^{-1} \arrow[d, "3,4"'] & & Y_{1,2}Y_{2,5}Y_{1,6}^{-1}Y_{3,6}^{-1} \arrow[dll, "1,3"'] \arrow[d, "2,6"] \\
Y_{2,3}Y_{2,5}Y_{1,4}^{-1}Y_{1,6}^{-1}Y_{3,6}^{-1} \arrow[dr, "2,6"'] & & Y_{1,2}Y_{2,7}^{-1} \arrow[dl, "1,3"] \\
& Y_{2,3}Y_{1,4}^{-1}Y_{2,7}^{-1} \arrow[d, "2,4"] & \\
& Y_{3,4}Y_{2,5}^{-1}Y_{2,7}^{-1} \arrow[d, "3,6"] & \\
& Y_{3,8}^{-1} &
\end{tikzcd}$$

$C_4^{(1)}$ 

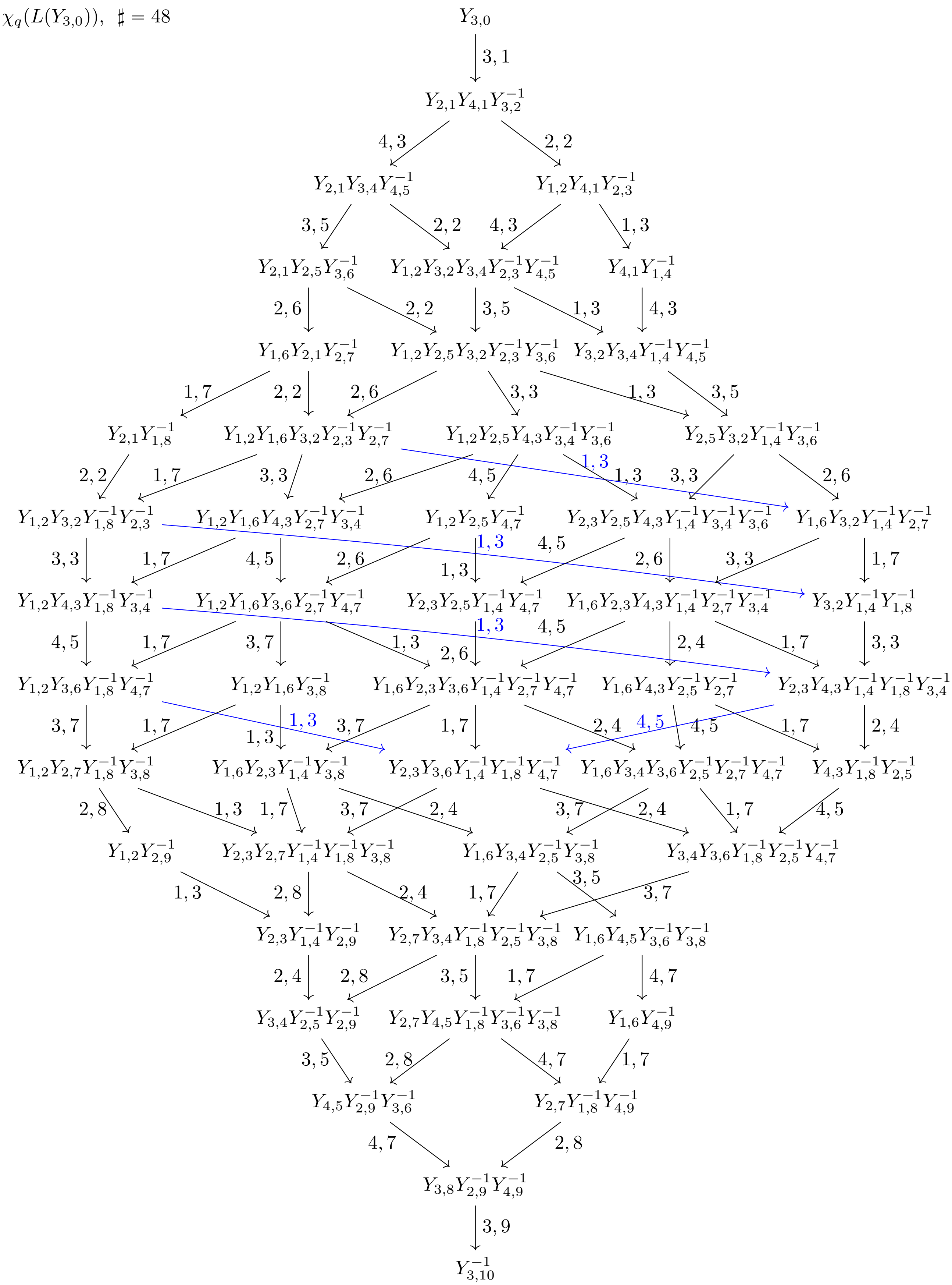
$\chi_q(L(Y_{1,0}))$, $\sharp = 8$

$$
\begin{tikzcd}
Y_{1,0} \arrow[d, "{1,1}"'] \\
Y_{2,1}Y_{1,2}^{-1} \arrow[d, "{2,2}"'] \\
Y_{3,2}Y_{2,3}^{-1} \arrow[d, "{3,3}"'] \\
Y_{4,3}Y_{3,4}^{-1} \arrow[d, "{4,5}"'] \\
Y_{3,6}Y_{4,7}^{-1} \arrow[d, "{3,7}"'] \\
Y_{2,7}Y_{3,8}^{-1} \arrow[d, "{2,8}"'] \\
Y_{1,8}Y_{2,9}^{-1} \arrow[d, "{1,9}"'] \\
Y_{1,10}^{-1}
\end{tikzcd}
$$

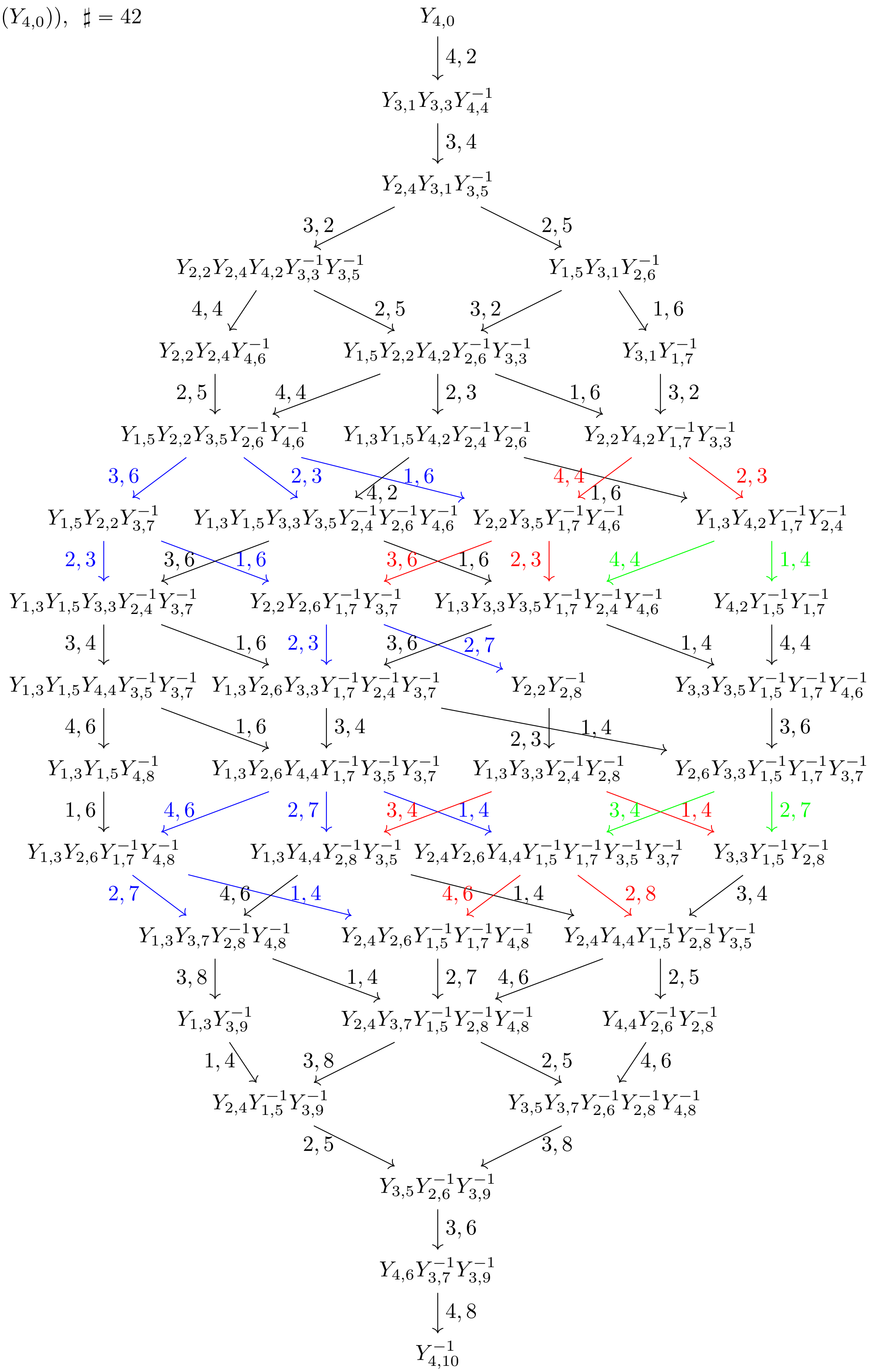
$\chi_q(L(Y_{2,0}))$, $\sharp = 27$

$$
\begin{tikzcd}
 & Y_{2,0} \arrow[d, "{2,1}"'] & \\
 & Y_{1,1}Y_{3,1}Y_{2,2}^{-1} \arrow[dl, "{1,2}"'] \arrow[dr, "{3,2}"] & \\
Y_{3,1}Y_{1,3}^{-1} \arrow[d, "{3,2}"'] & & Y_{1,1}Y_{4,2}Y_{3,3}^{-1} \arrow[dll, "{1,2}"'] \arrow[d, "{4,4}"] \\
Y_{2,2}Y_{4,2}Y_{1,3}^{-1}Y_{3,3}^{-1} \arrow[d, "{2,3}"'] \arrow[dr, "{4,4}"] & & Y_{1,1}Y_{3,5}Y_{4,6}^{-1} \arrow[dl, "{1,2}"'] \arrow[d, "{3,6}"] \\
Y_{4,2}Y_{2,4}^{-1} \arrow[d, "{4,4}"'] & Y_{2,2}Y_{3,5}Y_{1,3}^{-1}Y_{4,6}^{-1} \arrow[dl, "{2,3}"'] \arrow[d, "{3,6}"] & Y_{1,1}Y_{2,6}Y_{3,7}^{-1} \arrow[dl, "{1,2}"'] \arrow[d, "{2,7}"] \\
Y_{3,3}Y_{3,5}Y_{2,4}^{-1}Y_{4,6}^{-1} \arrow[d, "{3,6}"'] & Y_{2,2}Y_{2,6}Y_{1,3}^{-1}Y_{3,7}^{-1} \arrow[dl, "{2,3}"'] \arrow[d, "{2,7}"] & Y_{1,1}Y_{1,7}Y_{2,8}^{-1} \arrow[dl, "{1,2}"'] \arrow[d, "{1,8}"] \\
Y_{2,6}Y_{3,3}Y_{2,4}^{-1}Y_{3,7}^{-1} \arrow[d, "{3,4}"'] \arrow[dr, "{2,7}"] & Y_{1,7}Y_{2,2}Y_{1,3}^{-1}Y_{2,8}^{-1} \arrow[d, "{2,3}"'] \arrow[dr, "{1,8}"] & Y_{1,1}Y_{1,9}^{-1} \arrow[d, "{1,2}"] \\
Y_{2,6}Y_{4,4}Y_{3,5}^{-1}Y_{3,7}^{-1} \arrow[d, "{4,6}"'] \arrow[dr, "{2,7}"] & Y_{1,7}Y_{3,3}Y_{2,4}^{-1}Y_{2,8}^{-1} \arrow[d, "{3,4}"'] \arrow[dr, "{1,8}"] & Y_{2,2}Y_{1,3}^{-1}Y_{1,9}^{-1} \arrow[d, "{2,3}"] \\
Y_{2,6}Y_{4,8}^{-1} \arrow[d, "{2,7}"'] & Y_{1,7}Y_{4,4}Y_{2,8}^{-1}Y_{3,5}^{-1} \arrow[dl, "{4,6}"'] \arrow[dr, "{1,8}"] & Y_{3,3}Y_{1,9}^{-1}Y_{2,4}^{-1} \arrow[d, "{3,4}"] \\
Y_{1,7}Y_{3,7}Y_{2,8}^{-1}Y_{4,8}^{-1} \arrow[d, "{3,8}"'] \arrow[drr, "{1,8}"] & & Y_{4,4}Y_{1,9}^{-1}Y_{3,5}^{-1} \arrow[d, "{4,6}"] \\
Y_{1,7}Y_{3,9}^{-1} \arrow[dr, "{1,8}"'] & & Y_{3,7}Y_{1,9}^{-1}Y_{4,8}^{-1} \arrow[dl, "{3,8}"] \\
 & Y_{2,8}Y_{1,9}^{-1}Y_{3,9}^{-1} \arrow[d, "{2,9}"'] & \\
 & Y_{2,10}^{-1} &
\end{tikzcd}
$$

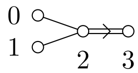
$\chi_q(L(Y_{3,0}))$, $\sharp = 48$



$\chi_q(L(Y_{4,0})),\ \sharp = 42$



1.4. B. $B_3^{(1)}$



$\chi_q(L(Y_{1,0}))$, $\sharp = 7$

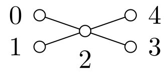
$$\begin{tikzcd}
Y_{1,0} \arrow[d, "{1,2}"'] \\
Y_{2,2}Y_{1,4}^{-1} \arrow[d, "{2,4}"'] \\
Y_{3,3}Y_{3,5}Y_{2,6}^{-1} \arrow[d, "{3,6}"'] \\
Y_{3,3}Y_{3,7}^{-1} \arrow[d, "{3,4}"'] \\
Y_{2,4}Y_{3,5}^{-1}Y_{3,7}^{-1} \arrow[d, "{2,6}"'] \\
Y_{1,6}Y_{2,8}^{-1} \arrow[d, "{1,8}"'] \\
Y_{1,10}^{-1}
\end{tikzcd}$$

$\chi_q(L(Y_{3,0}))$, $\sharp = 8$

$$\begin{tikzcd}
& Y_{3,0} \arrow[d, "{3,1}"'] & \\
& Y_{2,1}Y_{3,2}^{-1} \arrow[d, "{2,3}"'] & \\
& Y_{1,3}Y_{3,4}Y_{2,5}^{-1} \arrow[dl, "{1,5}"'] \arrow[dr, "{3,5}"] & \\
Y_{3,4}Y_{1,7}^{-1} \arrow[dr, "{3,5}"'] & & Y_{1,3}Y_{3,6}^{-1} \arrow[dl, "{1,5}"] \\
& Y_{2,5}Y_{1,7}^{-1}Y_{3,6}^{-1} \arrow[d, "{2,7}"'] & \\
& Y_{3,8}Y_{2,9}^{-1} \arrow[d, "{3,9}"'] & \\
& Y_{3,10}^{-1} &
\end{tikzcd}$$

$\chi_q(L(Y_{2,0})), \quad \sharp = 22$

$$
\begin{tikzcd}
 & & & Y_{2,0} \arrow[d, "{2,2}"] & & & \\
 & & & Y_{1,2}Y_{3,1}Y_{3,3}Y_{2,4}^{-1} \arrow[dll, "{3,4}"'] \arrow[drr, "{1,4}"] & & & \\
 & Y_{1,2}Y_{3,1}Y_{3,5}^{-1} \arrow[d, "{3,2}"'] \arrow[drrrr, "{1,4}"] & & & & Y_{3,1}Y_{3,3}Y_{1,6}^{-1} \arrow[d, "{3,4}"] & \\
 & Y_{1,2}Y_{2,2}Y_{3,3}^{-1}Y_{3,5}^{-1} \arrow[d, "{2,4}"'] \arrow[drr, "{1,4}"] & & & & Y_{2,4}Y_{3,1}Y_{1,6}^{-1}Y_{3,5}^{-1} \arrow[dll, "{3,2}"'] \arrow[d, "{2,6}"] & \\
 & Y_{1,2}Y_{1,4}Y_{2,6}^{-1} \arrow[dl, "{1,6}"'] \arrow[dr, "{1,4}"] & & Y_{2,2}Y_{2,4}Y_{1,6}^{-1}Y_{3,3}^{-1}Y_{3,5}^{-1} \arrow[dl, "{2,4}"'] \arrow[dr, "{2,6}"] & & Y_{3,1}Y_{3,7}Y_{2,8}^{-1} \arrow[dl, "{3,2}"'] \arrow[dr, "{3,8}"] & \\
Y_{1,2}Y_{1,8}^{-1} \arrow[dr, "{1,4}"'] & & Y_{1,4}Y_{2,4}Y_{1,6}^{-1}Y_{2,6}^{-1} \arrow[dl, "{1,6}"'] \arrow[dr, "{2,6}"] & & Y_{2,2}Y_{3,7}Y_{2,8}^{-1}Y_{3,3}^{-1} \arrow[dl, "{2,4}"'] \arrow[dr, "{3,8}"] & & Y_{3,1}Y_{3,9}^{-1} \arrow[dl, "{3,2}"] \\
 & Y_{2,4}Y_{1,6}^{-1}Y_{1,8}^{-1} \arrow[d, "{2,6}"'] & & Y_{1,4}Y_{3,5}Y_{3,7}Y_{2,6}^{-1}Y_{2,8}^{-1} \arrow[dll, "{1,6}"'] \arrow[drr, "{3,9}"] & & Y_{2,2}Y_{3,3}^{-1}Y_{3,9}^{-1} \arrow[d, "{2,4}"] & \\
 & Y_{3,5}Y_{3,7}Y_{1,8}^{-1}Y_{2,8}^{-1} \arrow[d, "{3,8}"'] & & & & Y_{1,4}Y_{3,5}Y_{2,6}^{-1}Y_{3,9}^{-1} \arrow[dllll, "{1,6}"'] \arrow[d, "{3,6}"] & \\
 & Y_{3,5}Y_{1,8}^{-1}Y_{3,9}^{-1} \arrow[drr, "{3,6}"'] & & & & Y_{1,4}Y_{3,7}^{-1}Y_{3,9}^{-1} \arrow[dll, "{1,6}"] & \\
 & & & Y_{2,6}Y_{1,8}^{-1}Y_{3,7}^{-1}Y_{3,9}^{-1} \arrow[d, "{2,8}"] & & & \\
 & & & Y_{2,10}^{-1} & & &
\end{tikzcd}
$$

1.5. D. $D_4^{(1)}$ 

$\chi_q(L(Y_{1,0}))$, $\sharp = 8$

$$\begin{tikzcd}
 & Y_{1,0} \arrow[d, "{1,1}"'] & \\
 & Y_{2,1}Y_{1,2}^{-1} \arrow[d, "{2,2}"'] & \\
 & Y_{3,2}Y_{4,2}Y_{2,3}^{-1} \arrow[dl, "{3,3}"'] \arrow[dr, "{4,3}"] & \\
Y_{4,2}Y_{3,4}^{-1} \arrow[dr, "{4,3}"'] & & Y_{3,2}Y_{4,4}^{-1} \arrow[dl, "{3,3}"] \\
 & Y_{2,3}Y_{3,4}^{-1}Y_{4,4}^{-1} \arrow[d, "{2,4}"'] & \\
 & Y_{1,4}Y_{2,5}^{-1} \arrow[d, "{1,5}"'] & \\
 & Y_{1,6}^{-1} &
\end{tikzcd}$$

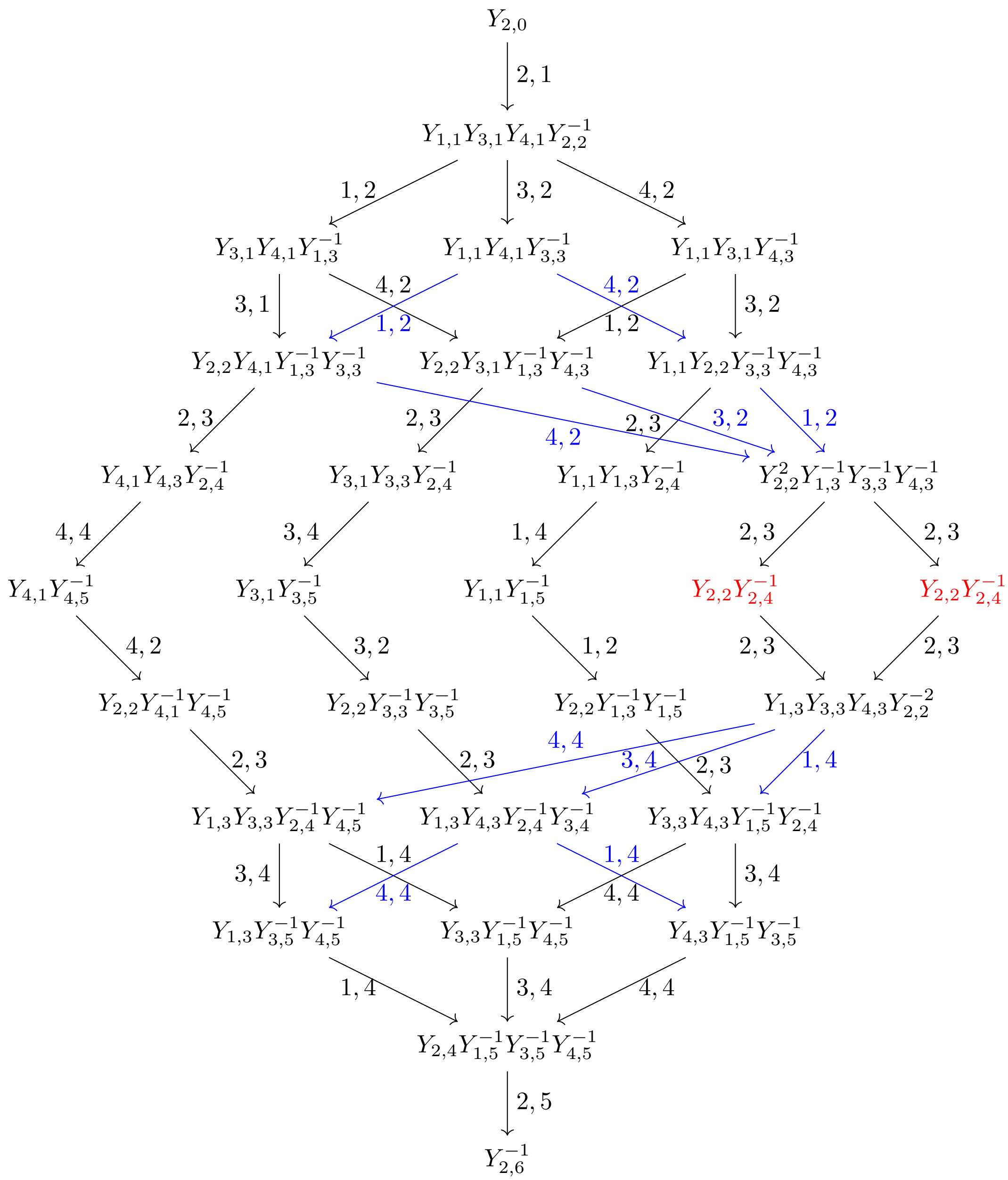
$\chi_q(L(Y_{3,0}))$, $\sharp = 8$

$$\begin{tikzcd}
 & Y_{3,0} \arrow[d, "{3,1}"'] & \\
 & Y_{2,1}Y_{3,2}^{-1} \arrow[d, "{2,2}"'] & \\
 & Y_{1,2}Y_{4,2}Y_{2,3}^{-1} \arrow[dl, "{1,3}"'] \arrow[dr, "{4,3}"] & \\
Y_{4,2}Y_{1,4}^{-1} \arrow[dr, "{4,3}"'] & & Y_{1,2}Y_{4,4}^{-1} \arrow[dl, "{1,3}"] \\
 & Y_{2,3}Y_{1,4}^{-1}Y_{4,4}^{-1} \arrow[d, "{2,4}"'] & \\
 & Y_{3,4}Y_{2,5}^{-1} \arrow[d, "{3,5}"'] & \\
 & Y_{3,6}^{-1} &
\end{tikzcd}$$

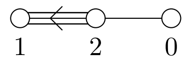
$\chi_q(L(Y_{4,0}))$, $\sharp = 8$

$$\begin{tikzcd}
 & Y_{4,0} \arrow[d, "{4,1}"'] & \\
 & Y_{2,1}Y_{4,2}^{-1} \arrow[d, "{2,2}"'] & \\
 & Y_{1,2}Y_{3,2}Y_{2,3}^{-1} \arrow[dl, "{1,3}"'] \arrow[dr, "{3,3}"] & \\
Y_{3,2}Y_{1,4}^{-1} \arrow[dr, "{3,3}"'] & & Y_{1,2}Y_{3,4}^{-1} \arrow[dl, "{1,3}"] \\
 & Y_{2,3}Y_{1,4}^{-1}Y_{3,4}^{-1} \arrow[d, "{2,4}"'] & \\
 & Y_{4,4}Y_{2,5}^{-1} \arrow[d, "{4,5}"'] & \\
 & Y_{4,6}^{-1} &
\end{tikzcd}$$

$\chi_q(L(Y_{2,0}))$, $\sharp = 29$



1.6. G.

$G_2^{(1)}$ 

$$\begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & -3 \\ -1 & -1 & 2 \end{pmatrix}$$

$\chi_q(L(Y_{2,0})),\ \ \sharp = 15$

$$\begin{tikzcd}
 & Y_{2,0} \arrow[d, "{2,3}"] & \\
 & Y_{1,1}Y_{1,3}Y_{1,5}Y_{2,6}^{-1} \arrow[d, "{1,6}"] & \\
 & Y_{1,1}Y_{1,3}Y_{1,7}^{-1} \arrow[d, "{1,4}"] & \\
 & Y_{1,1}Y_{2,4}Y_{1,5}^{-1}Y_{1,7}^{-1} \arrow[dl, "{2,7}"'] \arrow[dr, "{1,2}"] & \\
Y_{1,1}Y_{1,9}Y_{2,10}^{-1} \arrow[d, "{1,10}"'] \arrow[dr, "{1,2}"] & & Y_{2,2}Y_{2,4}Y_{1,3}^{-1}Y_{1,5}^{-1}Y_{1,7}^{-1} \arrow[dl, "{2,7}"'] \arrow[d, "{2,5}"] \\
Y_{1,1}Y_{1,11}^{-1} \arrow[d, "{1,2}"'] & Y_{1,9}Y_{2,2}Y_{1,3}^{-1}Y_{2,10}^{-1} \arrow[dl, "{1,10}"] \arrow[dr, "{2,5}"'] & Y_{2,4}Y_{2,8}^{-1} \arrow[d, "{2,7}"] \\
Y_{2,2}Y_{1,3}^{-1}Y_{1,11}^{-1} \arrow[dr, "{2,5}"'] & & Y_{1,5}Y_{1,7}Y_{1,9}Y_{2,8}^{-1}Y_{2,10}^{-1} \arrow[dl, "{1,10}"] \\
 & Y_{1,5}Y_{1,7}Y_{1,11}^{-1}Y_{2,8}^{-1} \arrow[d, "{1,8}"] & \\
 & Y_{1,5}Y_{1,9}^{-1}Y_{1,11}^{-1} \arrow[d, "{1,6}"] & \\
 & Y_{2,6}Y_{1,7}^{-1}Y_{1,9}^{-1}Y_{1,11}^{-1} \arrow[d, "{2,9}"] & \\
 & Y_{2,12}^{-1} &
\end{tikzcd}$$

$\chi_q(L(Y_{1,0})),\ \ \sharp = 7$

$$\begin{tikzcd}
Y_{1,0} \arrow[d, "{1,1}"'] \\
Y_{2,1}Y_{1,2}^{-1} \arrow[d, "{2,4}"'] \\
Y_{1,4}Y_{1,6}Y_{2,7}^{-1} \arrow[d, "{1,7}"'] \\
Y_{1,4}Y_{1,8}^{-1} \arrow[d, "{1,5}"'] \\
Y_{2,5}Y_{1,6}^{-1}Y_{1,8}^{-1} \arrow[d, "{2,8}"'] \\
Y_{1,10}Y_{2,11}^{-1} \arrow[d, "{1,11}"'] \\
Y_{1,12}^{-1}
\end{tikzcd}$$

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