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# THE NP-COMPLETENESS OF EDGE-COLORING*

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**Abstract.** We show that it is NP-complete to determine the chromatic index of an arbitrary graph. The problem remains NP-complete even for cubic graphs.

**Key words.** computational complexity, NP-complete problems, chromatic index, edge-coloring

**1. Introduction.** The chromatic index of a graph is the number of colors required to color the edges of the graph in such a way that no two adjacent edges have the same color. By Vizing's theorem [1], the chromatic index is either $d$ or $d+1$, where $d$ is the maximum vertex degree.

We prove the conjecture (Garey and Johnson [2, p. 286]) that it is NP-complete to determine the chromatic index of an arbitrary graph. In fact, we prove the stronger result that it is NP-complete to determine whether the chromatic index of a cubic graph is 3 or 4. Thus this problem probably has no polynomial time algorithm.

The terminology and results of NP-completeness are given in [2]. It is clear that the chromatic index problem is in the class NP. To prove that the problem is NP-complete, we exhibit a polynomial reduction from the known NP-complete problem 3SAT which is defined as follows. A set of clauses $C=\{C_1, C_2, \cdots, C_r\}$ in variables $u_1, u_2, \cdots, u_s$ is given, each clause $C_i$ consisting of three literals $l_{i,1}, l_{i,2}, l_{i,3}$, where a literal $l_{i,j}$ is either a variable $u_k$ or its negation $\bar{u}_k$. The problem is to determine whether $C$ is satisfiable, that is, whether there is a truth assignment to the variables which simultaneously satisfies all the clauses in $C$. A clause is satisfied if one or more of its literals has value "true".

**2. The parity condition.** We will use the following lemma given in Isaacs [3].

LEMMA. *Let $G$ be a cubic, 3-edge-colored graph and $V' \subseteq V(G)$ a set of vertices of $G$. Let $E' \subseteq E(G)$ be the set of edges of $G$ which connect $V'$ to the remainder of the graph. If the number of edges of color $i$ in $E'$ is $k_i$ $(i=1, 2, 3)$, then*

$$k_1 \equiv k_2 \equiv k_3 \pmod 2.$$

*Proof.* If $E_{12}$ is the set of edges of $G$ which are colored with color 1 or 2, then $E_{12}$ consists of a collection of cycles. Thus $E_{12}$ meets $E'$ in an even number of edges, and so $k_1+k_2 \equiv 0 \pmod 2$ which gives $k_1 \equiv k_2 \pmod 2$. Similarly $k_2 \equiv k_3 \pmod 2$. □

**3. The components used in the construction.** Given an instance $C$ of the problem 3SAT, we will show how to construct a cubic graph $G$ which is 3-edge-colorable if and only if $C$ is satisfiable. The graph $G$ will be put together from pieces or "components" which carry out specific tasks. Information will be carried between components by pairs of edges. In a 3-edge-coloring of $G$, such a pair of edges is said to represent the value $T$ ("true") if the edges have the same color, and to represent $F$ ("false") if the edges have distinct colors.

The inverting component is shown with its symbol in Fig. 1. It was used by Loupekine (see [4]) to construct a large family of cubic graphs with chromatic index 4. Using the parity condition above, it may be checked that if this component is 3-edge-colored, one of the pairs of connecting edges marked $a, b$ or $c, d$ must have equal colors and the remaining 3 edges must have distinct colors. There is no further

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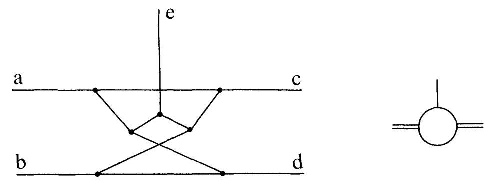


FIG. 1. *The inverting component and its symbolic representation.*

restriction on the possible colors of the five connecting edges. Regarding the pair of edges $a, b$ as the input and the pair $c, d$ as the output, the component changes a representation of $T$ to one of $F$ and vice versa.

The truth or falsity of each variable $u_i$ will be represented by a variable-setting component such as that shown in Fig. 2. The component shown has 4 pairs of output edges, but in general the component representing $u_i$ should have as many output pairs as there are appearances of $u_i$ or $\bar{u}_i$ among the clauses of $C$. It may be checked that in any 3-edge-coloring of a variable-setting component, all the output pairs must represent the same value.

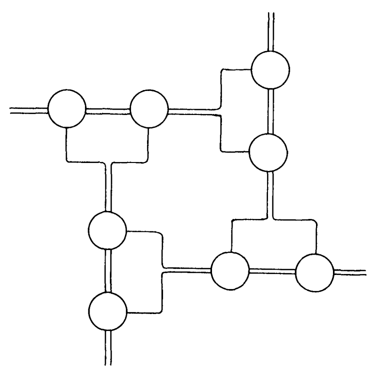

FIG. 2. *The variable-setting component made from* 8 *inverting components and having* 4 *output pairs of edges. More generally it is made from* $2n$ *inverting components and has* $n$ *output pairs.*

The truth of each clause $c_j$ will be tested by a satisfaction-testing component as shown in Fig. 3. This component can be 3-edge-colored if and only if the three input pairs of edges do not all represent $F$. The remaining connecting edges will be discussed later.

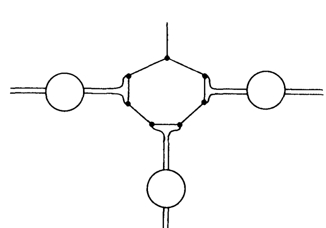

FIG. 3. *The satisfaction-testing component.*

**4. The main theorem.** We are now in a position to prove the following theorem.

THEOREM. *It is* NP-*complete to determine whether the chromatic index of a cubic graph is* 3 *or* 4.

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*Proof.* The problem is clearly in the class NP. We exhibit a polynomial reduction from the problem 3SAT. Consider an instance $C$ of 3SAT and construct from it a graph $G$ as follows.

For each variable $u_i$ take a variable-setting component $U_i$ with one output pair of edges associated with each appearance of $u_i$ or $\bar{u}_i$ among the clauses of $C$. Take also a satisfaction-testing component $C_j$ for each clause $c_j$. Suppose literal $l_{j,k}$ in clause $c_j$ is the variable $u_i$. Then identify the $k$th input pair of $C_j$ with the associated output pair of $U_i$. If, on the other hand, $l_{j,k}$ is $\bar{u}_i$, then insert an inverting component between the $k$th input pair of $C_j$ and the associated output pair of $U_i$. The resulting graph $H$ still has some connecting edges unaccounted for. The cubic graph $G$ is formed from two copies of $H$ by identifying the remaining connecting edges in corresponding pairs.

The graph $G$ has a 3-edge-coloring if and only if the collection $C$ of clauses is satisfiable, as can be verified using the properties of the components developed above. Moreover, the graph $G$ can be produced from $C$ using a polynomial time algorithm, so we have the result. $\square$

**5. Comments.** The above theorem may give some insight into the difficulty in classifying graphs according to their chromatic index. At any rate, it probably excludes the possibility of a polynomially checkable criterion, and it indicates that the restriction to cubic graphs is no easier.

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