EXHIBITION BOOKLET

# LaLaLab

## THE MATHEMATICS OF MUSIC

*“Mathematics and music, the most sharply contrasted fields of scientific activity which can be found, and yet related, supporting each other, as if to show forth the secret connection which ties together all the activities of our mind, and which leads us to surmise that the manifestations of the artist’s genius are but the unconscious expressions of a mysteriously acting rationality.”*

Hermann von Helmholtz (1821 - 1894)
in *Vorträge und Reden* (1884, 1896), Vol 1, p 122.

Music and Mathematics share many similarities as fields of study. Both disciplines study abstract objects, have complex structures and manipulation rules, a well defined notation, and are absolutely precise in their results. Working on them requires practice, creativity and an analytical mind. It is no surprise that Mathematics and Music are closely related.

But their relationship goes far beyond the skills needed for its study. Mathematics is deeply infused in all aspects of Music, from the physics of sound to the crafting of instruments, from rhythmic patterns to tonal harmony, from classical to electronic music. Mathematics supports music and our understanding of art the same way as it supports physics and our understanding of the world.

Hearing music with mathematical ears brings to the music lover deeper comprehension, appreciation for details, and a greater enjoyment of the art; and to professionals the ability to compose and tools to express their creativity.

IMAGINARY, 2019

La La Lab - The Mathematics of Music / Exhibition Booklet
IMAGINARY, `www.imaginary.org`, info@imaginary.org
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## LA LA LAB - THE MATHEMATICS OF MUSIC

Conceived in collaboration with experts in current research of music and mathematics, La La Lab combines a laboratory format with interactive exhibits. Therefore we can reveal the stunning connections between mathematics and music, pushing the boundaries of musical creativity and mathematical knowledge.

La La Lab is a modular lab exhibition experience. This booklet invites you to dive deeper. You will find exhibits and accompanying information, links to discover additional features and recommended readings. Please take a copy with you or download it on our web page. This exhibition is open source and available under free licenses. You can find all software, images, texts, 3d data at: `lalalab.imaginary.org`

IMAGINARY is a non-profit company for interactive and open, artistic and collaborative communication of modern mathematics to the general public. Created in 2007 at the Mathematische Forschungsinstitut Oberwolfach, a Leibniz institute, IMAGINARY has received several awards for its contribution to science communication. Since 2008, more than 340 exhibition activities have been organized in more than 60 countries and in 30 languages, attracting millions of visitors.

# LA LA LAB CREDITS

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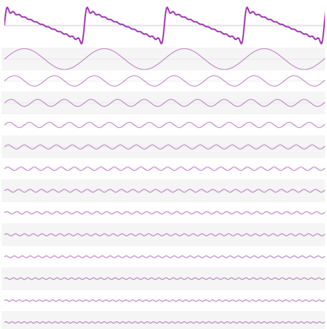
# THE SPECTRUM OF SOUND

Sound is a vibration of some body transmitted through the air and perceived by your ears. If you move up and down your hand once per second, we say that your hand vibrates at a frequency of 1 Hertz (Hz). If you move it up and down twice per second, the frequency is 2 Hz. If you could do it about 20 times per second (20 Hz), your eardrums would start sensing the vibration transmitted through the air. Probably you can do it with your hand, but you can use a string on a guitar, the membrane of a drum, the air inside a flute, your vocal folds, the body of a bell... Typically a human can hear vibrations with a frequency between 20 and 20 000 Hz. The higher the frequency, the higher the pitch of the sound.

However, if you use a guitar string or a flute as vibrating bodies, the sound you will perceive will be very different in each of the two cases. This is because we almost never hear a "pure" frequency, any natural sound is a mixture of vibrations at different frequencies and varying intensities. A guitar string tuned at a middle A note, will vibrate at 440 Hz. This is the fundamental frequency, but the string will inevitably vibrate also at 880 Hz (the double), 1320 Hz (the triple), and with all the multiple frequencies (called harmonic frequencies), but with less intensity each time. A flute will also vibrate at all these frequencies, but with different intensities. It is the varying intensities of the harmonic frequencies that makes our ear distinguish between guitar and flute. The set of frequencies with their intensities is called the spectrum of the sound, and it defines what we call timbre in music.



A sawtooth wave as a sum of sine waves.

What would be a "pure" sound, just one frequency and no "contamination" from others? Mathematically, we can describe this sound as a sinusoidal wave. It is the sound that would make a perfectly elastic spring that could vibrate without losing any energy. We can't build such a spring, but we can make the membrane of a speaker vibrate at this pure frequency using electronics. A surprising, yet totally useful result in mathematics, is that every periodic function of period $2\pi$ can be approximated by summing up a sequence of sine waves if they are scaled and shifted appropriately. In a formula:

$$g(t) \approx A_0 + \sum_{i=1}^{N} A_i \cdot \sin(it + \varphi_i).$$

This implies that by adding appropriate sine waves we can simulate any static sound, the only information needed is the set of intensities $A_i$ and the phase shifts $\varphi_i$. This has deep implications. On the one hand it allows us to decompose of a complex signal into many simpler ones in a very structured way. Thus, it makes things much simpler to analyze and understand. On a more abstract level it connects different spaces: the time space where signals are described by values (like air pressure) changing over time and the frequency space where signals are composed from simpler ones. Depending on the situation it might be way easier to address a certain problem in one space or the other.

Its applications are far reaching and cover many diverse areas besides sound and music, such as communication technology, quantum theory, coding theory, statistics, electrical circuit design, and an endless series of other topics.

On the first screen of the exhibit, you can try to generate a wave by adding harmonic frequencies.Use just one frequency (sine) to hear a "mathematically pure" sound. Adjust the intensity of each of the harmonic frequencies, or click on the square or sawtooth waves, to hear different timbres.

Heuristically, a string instrument has more intense harmonics, and wind instruments have less intense harmonics. But anyway, we are still far from a full synthesizer. To simulate a real instrument, we have to take into account that the sound is not static and eternal, it starts at some point, increases in intensity, then it decays and fades out. So, frequencies and their intensities change through time. Instruments such as bells, xylophones and most percussions do not have a spectrum formed by multiples of a fundamental frequency, but other frequencies depending on the geometric shape of the vibrating body. Therefore, it is quite difficult to synthesize a real instrument.

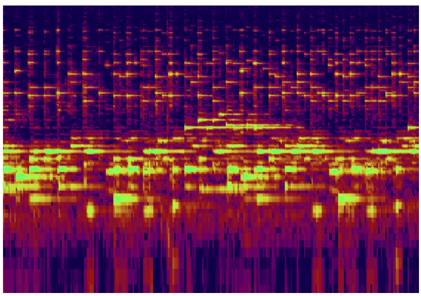
However, there is an extraordinary mathematical tool that allows us to reverse the process. Instead of adding up frequencies to obtain a composed signal, the Fourier transform is a procedure that allows to take an arbitrary signal (recorded with a microphone, for instance) and to decompose it into its fundamental pieces: the frequencies and their intensities that you would need to use to generate the signal using pure frequencies as building blocks.

Let $g(t)$ be a signal represented as a function, so that for each time t it returns the intensity of the sound (this intensity may be the air pressure, or the voltage in the line wire of a microphone). The Fourier transform of a signal $g(t)$ is a new function $\hat{g}(t)$ that for each frequency $f$ returns the intensity of a sine wave with that frequency required to synthesize the signal $g(t)$. It is achieved with this formula:

$$\hat{g}(f) = \int_{-\infty}^{\infty} g(t)\, e^{2\pi i f t}\, dt$$

Note that there are complex values inside the integral, so that the Fourier transform is a complex-valued function. This is because there are two pieces of information needed to synthesize a signal: the amplitude $A_i$ and the phase shifts $\varphi_i$.

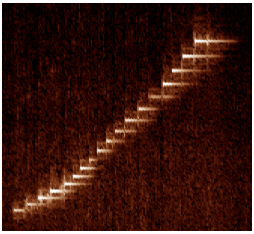
The Fourier transform gives both the amplitude and phase of the signal as modulus and argument of the complex value. Usually, for audio analysis often only the modulus is used to obtain a graph with peaks at the most prominent frequencies, which is sufficient to reconstruct the original signal with accuracy.



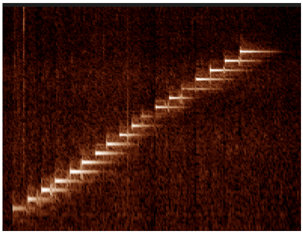
Sonogram of part of Pachelbel's canon.

For practical applications, there is a useful algorithm called the Fast Fourier Transform (FFT), that substitutes the abstract mathematical formula. This allows us to make extremely fast computations and analyze signals in real time. Today many electronic ways of working with music are based on FFT. It forms the first step of extracting a score from a piece of played music. It can be used for instance for the automatic transcription of Jazz soli. It is also the basis of modifying music. The common practice of pitch adjustment in pop music (auto-tune) can be basically described as an "analyze - correct - synthesis" procedure. Special sound effects like letting an instrument speak like a human voice rely on automatic sound analysis of the involved spectra. Also automatic music recognition services like Shazam heavily rely on the fingerprint of a tune resembled by its sonogram.

The second screen of the exhibit displays a spectrum analyzer connected to the microphone input. On the left you have the spectrum, i.e. the instantaneous analysis of frequencies (Fourier transform) of the playing signal. At the right the spectrum leaves a trace to visualize the history of the input signal (sonogram).

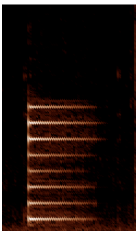


Chromatic scale linear.

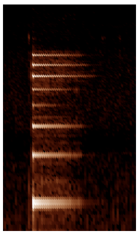


Chromatic scale logarithmic.

Linear scale is best suited for sounds and overtones (frequencies $f$, $2f$, $3f$, $4f$, $5f$, ...), while logarithmic scale is best suited for musical tones (frequencies $f$, $af$, $a^2f$, $a^2f$, $a^2f$, ...). Use the microphone to analyze your voice or the available instruments. Sing something, make noises, talk, or use the synthesizer to generate a tone. Put the microphone next to the speaker and analyze the sound produced. Try to identify the frequencies, their intensities, when they appear, when they fade out.



Overtones linear.



Overtones logarithmic.

**Authors of this exhibit:**

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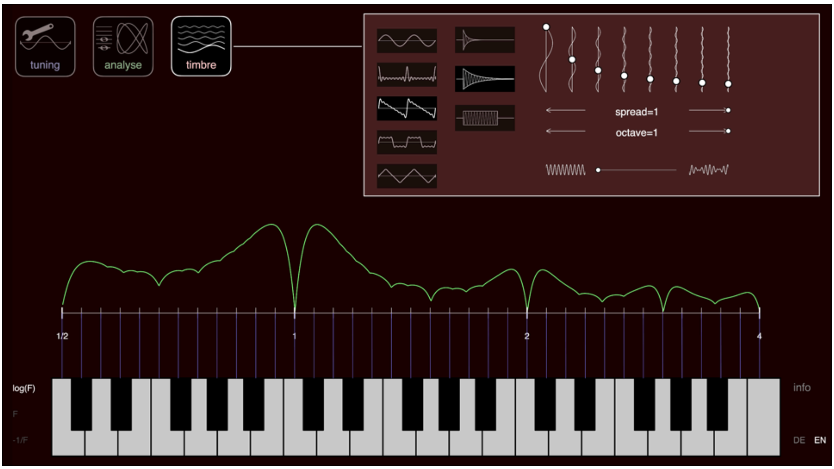
**Text:** Daniel Ramos (IMAGINARY).

# SCALE LAB

We can hear a wide range of frequencies. Usually we don't use all the audible frequencies to make music, we select a discrete set of them to determine the pitch of each tone, and we call this set of frequencies a scale. Depending on the choice of the scale, we will be able to play some combinations. Interestingly, humans find that some pairs of frequencies sound "good" together (consonant), while others sound "bad" (dissonant). This is a physiological and psychological phenomenon still under debate and being actively researched today. This preference has determined the choice of scales and the practice of music since the beginning of its history.

In this exhibit, we explore the creation of scales and their properties. Along the gray horizontal line all the frequencies are represented. The markers represent frequencies selected for the particular scale, and they are connected to the keyboard by the blue lines that allows us to play them.




## THE TOOLS

A set of three toolboxes helps us influence the characteristics of a tone (timbre), the selection of the pitches for the scale (tuning) and to analyze properties of tones and combinations of tones (analyze).

### Tuning

You can choose Western scales (with different tunings), two oriental scales (Raga, Gamelan), and two tools to generate scales (one with a step-generator and another with free pointing).

### Analyze

The analyze toolbox contains the implements that help you to understand the scale and/or tuning you selected.

### Waveform

This is the signal you hear, the movement of the speaker's membrane (and your eardrum) over time. Choose a sine as timbre and press one key to hear a pure frequency. Play two or more to see the combined waves (addition). Choose a different timbre (composed of many frequencies) to see its waveform and hear the difference.

### Lissajous

This curve is drawn by a moving point with an x-coordinate given by the oscillation determined by the first key stroke, and a y-coordinate given by the second key stroke. This works better with a pure sine waveform as timbre, but it also creates very interesting curves for more complicated timbres. Simple ratios between the two frequencies give simpler figures, which are also the more "consonant".

### Ratio

You can see the proportions between the (fundamental) frequencies of the notes played. Check the different scales to see how simpler ratios are related to consonant intervals.

### Dissonance curve

Real instruments don't produce a single frequency, instead many of them simultaneously (spectrum). Thus consonance or dissonance depends on the interaction of the two groups of frequencies. This curve (based on a concept from Helmholtz) attempts to measure dissonance (badness) of every tone with respect to the fixed middle C tone, and it varies when the timbre is changed. Note that dissonance is at a minimum for an octave jump or fifth and major third jumps.

### Timbre

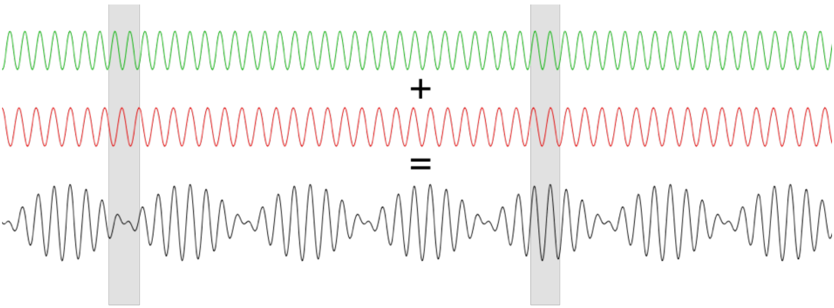
A musical tone produced by an instrument is not usually a single pure frequency, but a mixture of different frequencies with different intensities (spectrum). This toolbox allows the selection of some predefined spectra as well as defining one yourself by adjusting the sliders. All scales, tuning settings, and analysis performed with other toolboxes is strongly affected by the timbre of the instrument you choose to play.

## BACKGROUNDS AND EXPERIMENTS

The following pages describe concepts and experiments that can be experienced using the Scale Lab exhibit.

## Beating

If two frequencies are close, an exciting and fundamental acoustic phenomenon occurs: the sound seems to periodically increase and decrease in volume… and it actually does. There are two ways to explain this effect. One is more qualitative: assum e you have two frequencies one of 100 vibrations per second and one of 101 vibrations per second. At every moment of time these two waves will sum up. If the trough of one wave meets the peak of the other the two signals will cancel out. If a peak meets another peak the intensity is double that of the single signal.




Mountain meets valley, signal gets silent.

Mountain meets mountain, signal gets loud.

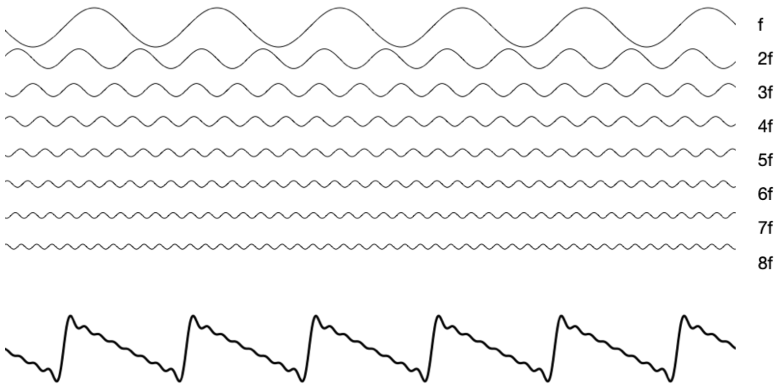
The second way to think about it is more quantitative. There is a nice trigonometric formula that describes what happens if two sine waves of different frequencies $f_1$ and $f_2$ are added:

$$\sin(f_1 \cdot t) + \sin(f_2 \cdot t) = 2 \sin\left(\frac{f_1 + f_2}{2} \cdot t\right) \cdot \cos\left(\frac{f_1 - f_2}{2} \cdot t\right).$$

The resulting wave can be considered as a signal vibrating in the average frequency (the sin part of the right hand equation) and modulated by a low frequency signal depending on the difference of the frequencies (the cos part of the right hand equation). If the frequencies are close together this will be heard as low frequencies change in intensity.

**Experiment:** Go to timbre and select a pure sine wave. Then use multi-touch on the central horizontal frequency line to create two nearby tones by using two fingers. Listen to the beating. The closer the frequencies are the slower the beating. If you move them far apart at some point the beating gets so fast that you perceive it as a rough dissonance. Moving them even further apart they will be perceived as two different tones. Analyze this using Lissajous or wave to actually see the beating.

**Music:** Slow beating sounds like a bit like a vibrato, it makes a tone sound richer and a bit more organic. This is the reason why in several instruments beating is taken as part of the sound generation process. In a piano the high frequency notes usually




This illustrates how the summation of overtones with intensity 1, 1/2, 1/3, 1/4, ... results in a sawtooth-like periodic function.

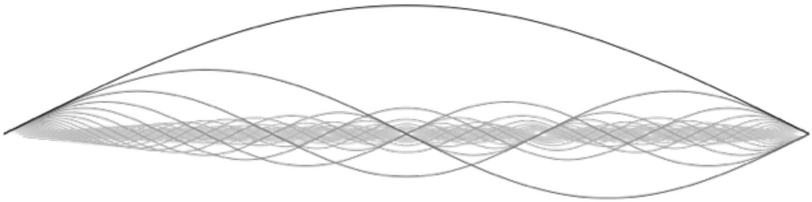
have two or three strings. These strings are slightly detuned with respect to each other, resulting in an organic and rich tone. Also a Musette register on an accordion has two slightly detuned vibrating metal tongues generating a vibrato sound.

### Sound from Overtones

The timbre settings of Scale Lab allow for much more than just simple sine waves, you can combine sine waves to generate a complex sound. The sliders in the timbre box allow you to add multiples of the base frequency. If the spread slider is set to 1 these are really the integer multiples $f$, $2f$, $3f$, $4f$, $5f$, $6f$, $7f$, $8f$. of the base frequency. Fourier synthesis implies that any periodic signal can be generated by a weighted sum of sine waves of the above type. You would need infinite summands to replicate a perfect signal, but the eight frequencies offered by Scale Lab are already pretty precise.

**Experiment:** Go to timbre and experiment with the different presets for wave forms (buttons on the left). Move the individual sliders of the timbre overtones to monitor how the sound changes. Use the “analyze tool” to see the shapes of the waves you generated.

**Music:** As a matter of fact, the sound of many instruments (in particular strings, brass or woodwind instruments is nicely approximated by decomposing the sound into a composition of overtones. This has to do with the specific physics of the instruments.



The image shows the partial waves of an ideal vibrating string. They correspond exactly to the overtones, resulting in a sound that can be very well modeled by this approach.

## Spread partials

The world is not an ideal laboratory. Physics is dirty and sounds are not perfect sums of overtones. As a matter of fact, most instruments create sounds where frequencies of the higher vibrations do not perfectly match overtones. For instance the fact that a string is not infinitely thin results in a certain spread of the overtones. The thicker the vibrating material gets the more extreme is the deviation form the ideal overtone series. The spectrum of a church bell is substantially different from an overtone series. This is not a bad thing. It makes sounds richer, more individual, or more characteristic. Below you see what happens if the overtone spectrum is replaced by a spread spectrum where the frequencies follow the law

$$1^a f, 2^a f, 3^a f, 4^a f, 5^a f, 6^a f, 7^a f, 8^a f$$

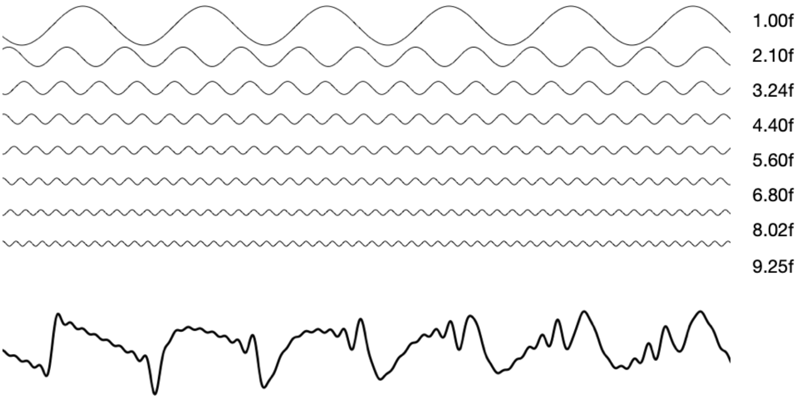
for a small exponent $a$. Observe that the resulting sum of the partials (this is the correct name here) shows ever changing behavior in the resulting sound curve and is no longer periodic. The resulting sound is rich and organic.

**Experiment:** In the Scale Lab Timbre Box there is a slider called spread right under the overtone sliders. Play with the slider to experiment with spread and compressed spectra. Listen how the sound changes even if you only add a very slight spread. Analyze the resulting sound with the wave tool to see how the wave forms change. How do intervals sound? What is the difference between a small and a large spread? Between a spreading and a compression of the spectrum? There is a whole world to be discovered.

## Dissonance curves

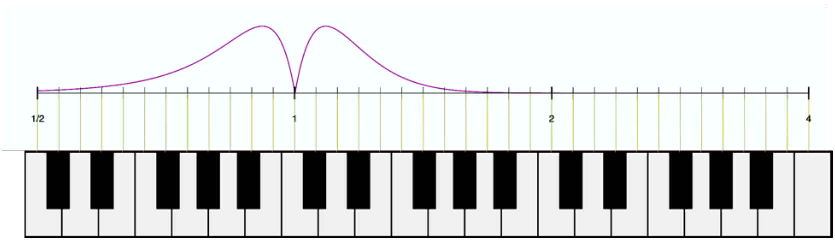
As soon as two tones (with a certain spectrum of partials) are played together our brain starts to categorize the intervals in a range A between pleasant sounding (consonant) and not-so-nice sounding (dissonant). One can think of the entire art of music as the art of creating the right tension between consonance (to please people) and dissonance (to make it interesting). Different epochs and cultures have different opinions at what is good style, but the global topic remains.

Already around 1870 Hermann von Helmholtz developed a mathematical theory of consonance and dissonance. The idea behind it is brilliant and simple. Since whenever two tones are played together many partials sound at the same time, one has to sum

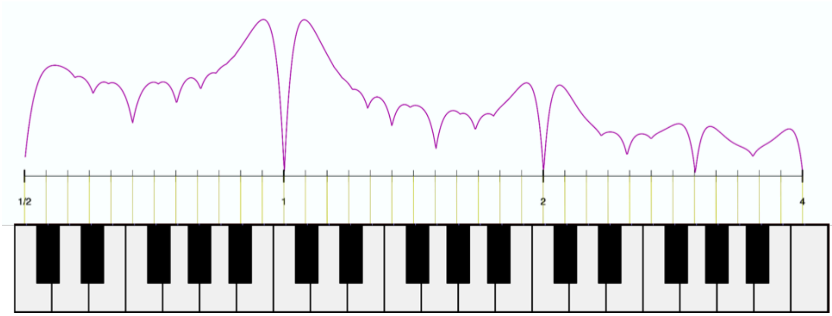



There is another interpretation of the richness of sound that arises. Since the partials do not perfectly match the overtones as soon as intervals are played, there is a substantial and quite irregular beating of the higher frequencies.

up the dissonances between all pairs of involved sine waves to get the dissonance of the entire sound. Therefore, he reduces the question of dissonance of complex sounds to the question of dissonance between sine waves. (One might ask if this is a reasonable assumption but at least it is a good starting point). However, what is the dissonance between two sine waves? Here we need empirical data. It turns out that if sine waves with more or less of identical frequency are not perceived as dissonant. A small, low frequency beat even slightly pleases. However, as soon as the beating becomes too fast it is perceived as rough and dissonant. This effect is particularly strong for beats between 10 and 25 beats per second, after that the effect decreases again. If the tones are too far apart they are just heard as two sine waves. Below you see a dissonance curve of a sine wave referring to a middle C.




Things get interesting as soon as the spectra become complicated. The next picture shows the dissonance curve resulting from a rich overtone spectrum. Dissonances between overtones of the two tones generate characteristic maxima and minima of that curves. Notice that for the usual perfect overtones you get minima at the octave, the fifth, the quarter and the third.

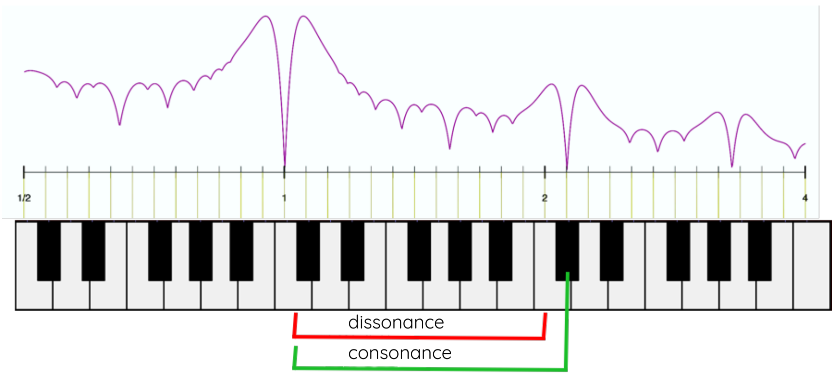



**Experiment:** Start the dissonance curve analyzer and play intervals with the central C. Play with different timbres and experiment with different intervals, Does the prediction meet you personal perception?

### Dissonance and spread partials

What happens if we study the dissonance behavior of a spectrum with spread partials? The dissonance behavior is very much determined by the dissonance between the overtones or partials. Consequently, as soon as the partials are "out of place" you get a new dissonance behavior.

The following image shows what happens to a slightly spread spectrum ($a = 1.08$):




For a musician used to our classical tone system the result might be a little bit shocking. Suddenly intervals that usually sound consonant (like the octave) sound heavily dissonant. While others that usually sound very dissonant (like the small ninth between C and c♯) sound as sweet as sugar.

Therefore, instruments that generate a spread or compressed spectrum require a different kind of scale. Or to put it differently, our Western tonal system is heavily influenced by the fact that we have strings, brass and woodwinds.

There is an amazing cure to fix the dissonant sounding octaves: Spread the frequencies of your scale as well! After that an octave is no longer an octave, but it sounds like one.

**Experiment:** The following experiments may be eye-opening. Start the dissonance curve analyzer and play intervals with the central C. Experiment with the spread of the spectrum and with the spread of the octave. Spreading both by the same amount gets you close to the usual tonal experience.

**Music:** In some instances, it may be good practice to spread scales. If you design the bells for a church tower adapting the scale to a spectrum is a good idea. Even an ordinary piano sounds more brilliant if the octaves are slightly spread. However, a piano with slightly spread scales will make it difficult to play along with other instruments. Music is full of compromises.

### Western Tuning Systems

We have seen that the timbre and the frequencies of the partials heavily influence the dissonance behavior of intervals. As a consequence, the selection of pitches (the scale) used on an instrument should depend on the sound characteristics of the instrument. Western culture instruments (harps, pianos, flutes, trumpets, have a close to perfect spectrum in the sense that the partials are very close to the theoretical overtones. Therefore, frequency ratios of 2 : 1 (octave), 3 : 2 (fifth), 4 : 3 (fourth), 4 : 5 (major third) and 6 : 5 (minor third) result in relatively consonant intervals (in this order with decreasing strength).

It turns out that fifths, fourths (Gregorian music), then later thirds (baroque) and even later more dissonant intervals (impressionists, expressionists) entered the musical language in the Western culture.

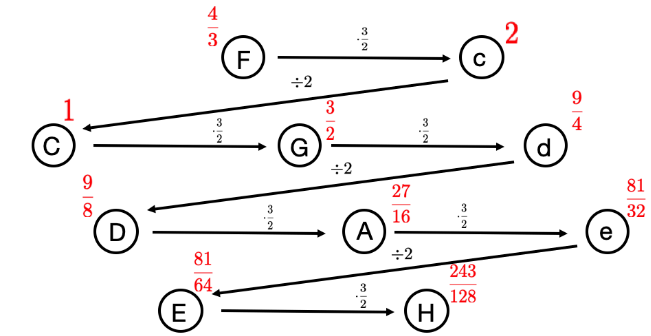
What is a good selection of pitches for such music? What are good scales? Answering these questions in depth could easily fill a one year university class. We will only scratch the surface.

12-Tone system: Current Western music culture systems of twelve tones within an octave are predominant. The reason for this is in a sense mathematical. Stacking 12 perfect fifths nearly hits stacking seven octaves. Or in formulas:

$$\left(\frac{3}{2}\right)^{12} = 129.7463\ldots \approx 2^7 = 128.$$

The 12 steps are the first time this stacking procedure comes close to an octave relation. If one wants a tonal system that is focused around octaves and fifth, then 12 tones are a good start.

Unfortunately, this relation is not perfect and this is where the entire story of tuning systems begins: one has to make compromises and different epochs resolved this tension in different ways.




Pythagorean tuning

Pythagorean Tuning: Let us start with Pythagorean tuning, which focuses around the idea of having as many perfect fifths as possible. Focus on the seven tones of our C major scale and see what happens. We label the tones by frequency ratios with respect to C and divide by two (drop an octave) every time we become greater than two. The following diagram shows the frequency ratios of notes in the Pythagorean scale and how they are derived.

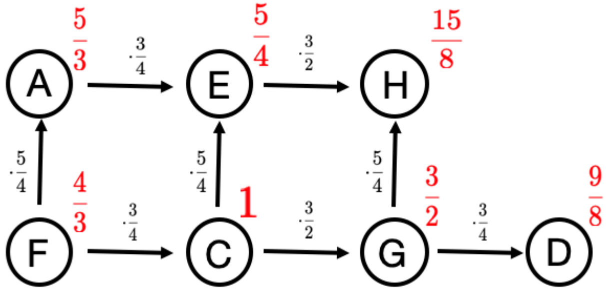
All fifths and octaves in this tuning are perfect. However, notice that the note E gets assigned a frequency value of 81/64. This is not a perfectly tuned major third, the perfect third would be 5/4 = 80/64.

Thus the Pythagorean third is off by a factor of 81/80 from the perfectly tuned third. Most likely the Pythagoreans would not have cared a lot since thirds only became an important interval in music over 1500 years later. As mentioned above not all the fifths can be perfect. If 11 fifths are perfectly tuned the last one must sound really terrible.

Just Tuning: Let us now consider a tuning that focuses on fifths and thirds. It is called just tuning. For the seven notes in the C major scale it uses the following tuning pattern.

You can play many perfect thirds and fifth with this tuning, but you loose some others. As a matter of fact, this little grid is a small portion of the Tonnetz.

Equal tempered tuning: If you extend both the Pythagorean and the just tuning to all 12 halftones. Some intervals will be perfect, but others won't sound as perfect. The




Just tuning

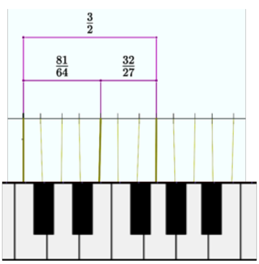
modern approach taken to tuning is to avoid this by equally spreading the error over all intervals. (This might be considered as a kind of democratic approach to tuning). In equal tempered tuning an octave is simply split into 12 equal steps. Each halftone step multiplies the frequency by a factor of $\sqrt[12]{2}$ , which reaches a perfect octave after 12 steps. Using this method, all fifths carry the same error. The same happens for chords and for all other intervals. Let us compare the values of equal tempered tuning to perfect tunings,

$$\text{for fifth: } (\sqrt[12]{2})^7 = 1.498\ldots \approx 1.5 = \frac{3}{2},$$

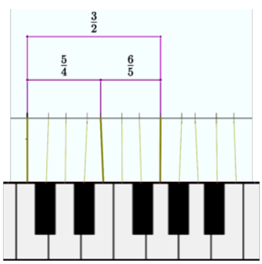
$$\text{for thirds: } (\sqrt[12]{2})^4 = 1.26\ldots \approx 1.25 = \frac{5}{4}.$$

Note that these approximations are very good.

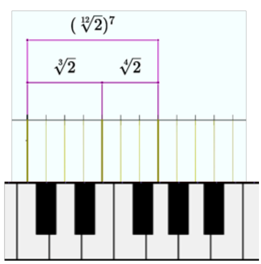
**Experiment:** Open the tuning box and experiment with different Western tunings. The colors shown in the Tonnetz indicate how close an interval is to a perfect one. It is also very instructive to analyze the different tunings with the ratio tool. It shows the ratios of intervals if they are played on the keyboard. The picture below shows the frequency ratios for a major C chord.




Pythagorean tuning



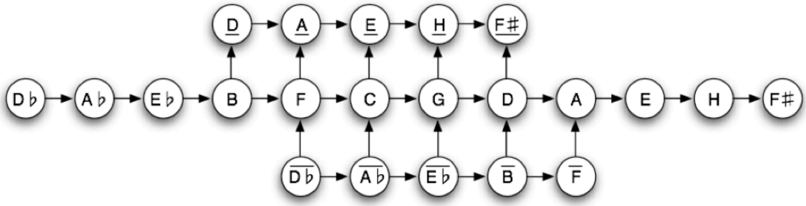

Just tuning




Equal tempered tuning

## Indian Raga Scales

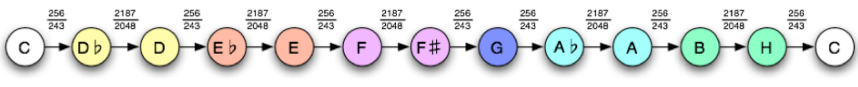
There is another way of playing many perfect fifths and thirds: Add more tones to the scale! The tuning system for Indian Raga music make utilizes this approach. In short, these tuning systems can be characterized by the following tuning scheme.




Every horizontal arrow indicates (up to octave jumps) a perfect fifth and every vertical arrow a perfect third. Unlike Western music, Raga music has more an oral than a written tradition. Hence, it is a little difficult to nail down a concrete tuning system but the one described above consisting of 22 tones is accepted among people who analyzed Indian music. What is so special and interesting about this tuning system and why are there exactly 22 tones?

First of all, it should be noted that Indian Ragas always refer to a given base tone. Usually a drone interval is constantly played in the background, the usual one is a C-G sound. Observe how the middle between the C and G is the symmetry center of the above diagram.

The central row of the diagram is the Pythagorean scale with 11 perfect fifths. Now we saturate it by adding perfect thirds, which is done in a very clever way. To see this we reorder the notes of the Pythagorean scale to form a chain of halftones. We get:

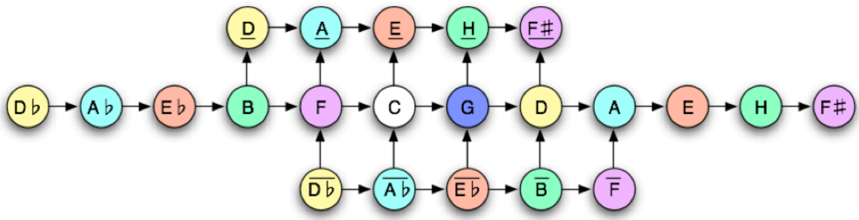



The ratios between the tones represent the frequency ratios that come from the Pythagorean tuning.

You can see that, almost magically, only two kinds of fractions occur. The coloring in the diagram marks the bigger of the two fractions, thus we have big halftone steps and small halftone steps. Two tones that are related by a big step have the same color. Now one can split the interval of 2187/2048 in an elegant way by introducing two new notes.


In the example above you additionally get a higher version of the D♭ and a lower version of the D. These are exactly the two versions added in the above tone system.

We can now do a similar split for all other big halftone steps. This results in the system with 22 tones, shown in the following colored version of this tone system:




Notice that the C and G play a special role in this system as the only notes that are not involved in a split interval.

The tones of the Raga tuning system are called Shrutis. As in the Western tuning systems, often a melody does not make use of all tones at the same time. Instead, certain notes are selected to form smaller scales based on the bigger tone system. Also the selection of sub scales (usually of four to seven tones) follows certain rules. But we will not go into those details here.

**Experiment:** Open the tuning box and experiment with the sounds of the Raga tunings. You can start the drone tone and select among plenty of commonly used sub scales of the 22 tone system. It might also be interesting to measure frequency ratios between the tones.

**Author of this exhibit:** Jürgen Richter-Gebert (Technical University of Munich)

**Acknowledgements:** Patrick Wilson and Aaron Montag (Sound Engine). Based on CindyJS.org

**Text:** Jürgen Richter-Gebert (TU Munich)

# TONNETZ

The musical scale we use in Western music tradition is a choice of tones, that is, a selection of frequencies that we use to tune an instrument to make music.

How do we name the notes? How do we represent these notes graphically? Which notation do we use to describe them on a score or to build an instrument? The first obvious choice would be to order them by pitch and name them sequentially. However, this doesn't reflect their relationships or their respective roles in musical language.

By playing together a first note and a second one with double the frequency (ratio 1:2), the result is so consonant and the two tones mix together so well that we have the impression that they are the "same" note. In fact we even call them by the same name. In this case, the interval between two such notes is called an "octave". In modern tuning (the equal tempered system of the piano), an octave is divided into 12 equal parts and these equal parts correspond to the 12 notes of the chromatic scale.

But we don't use all these 12 notes equally, some are used more often, at least historically, when the notation was developed in the Middle Age. This is why we have the "main" notes in a scale representing the white keys of the piano (C, D, E, F, G, A, B), called the diatonic scale, and "secondary" notes that we describe as alterations of a main note with flat and sharp (such as C♯ or E♭) are represented by the black keys.

Thus there are seven main notes in an octave, and the eighth has the same name as the first one (hence the term octave).

Ordering the notes by pitch, like in the piano keyboard, may be reasonable; but it does not reflect all the relationships between notes.

Because of the octave relation, it makes sense to represent notes in a circle, the so called Chromatic circle. The notes are

C, C♯, D, D♯, E, F, F♯, G, G♯, A, A♯, B

and after the last note of the scale (B), the first note (C) follows. In modern tuning, the intervals are all equal (called semitone) and sharp and flat coincide as in C♯ = D♭. This relation is called the enharmonic identification.

Another fundamental relation is the Fifth (or more precisely the perfect Fifth). This is the interval between C and G or between F and C, and it corresponds to adding 7 semitones to an initial note. It is called a fifth because in the 7 notes scale, if C is the first note, G is the fifth one. After the octave, it is considered as the most consonant interval, as one can experience by playing the two notes together. Thus, it makes sense to have them close as in the Circle of fifths, which is built by listing the notes in steps of 7 semitones:

C, G, D, A, E, B, G♭, D♭, A♭, E♭, B♭, F

Same as before, the F is followed by the C.

This representation is very useful to musicians because the notes that match well are closer together. This also explains why the white keys of the piano, represented by the notes A,B,C,D,E,F,G, are the "main" ones (with respect to the "secondary" ones): if you start at F, you can repeat the fifth interval six times and this will give you a chain of seven notes corresponding precisely to the diatonic scale. The note "C" is the first one in Latin notation (Do, Re, Mi, Fa, Sol, La, Si), in what is called the "major mode", but if you start the scale in A, like in the English notation, you obtain the "minor mode".

Two other very consonant relations are the Major Third (adding 4 semitones) and the Minor Third (adding 3 semitones). For instance between C and E there is a Major Third (E is the 3rd note on the scale). Between C and E♭, or between D and F there is a Minor Third (they are only 3 semitones apart).

When notes are played together, we obtain what is called a chord. The most common chords are triads (when 3 notes are played together) and the basic recipe to build a Major chord is to pick one note x (tonic) then add its Major Third and Fifth. This gives you three notes (x, x+4, x+7) that sound very pleasant and bright. For instance C - E - G is the C-Major chord. Analogously, Minor chords are created by the tonic x then adding its Minor third and Fifth, to obtain (x, x+3, x+7), that sound more sad and dark. For instance, C - E♭ - G is the C-Minor chord.

There is a graphical representation of notes that reflects some of these relations. It goes back to mathematician Leonhard Euler, and it is known today as the Tonnetz (German for "network of tones"). The classical tonnetz (labeled here as 3,4,5) consists of a triangular grid where each vertex is associated to a note (up to an octave). There are three lines or directions in the triangular grid. In one direction notes are rising in fifths, in another in major thirds and in the final one in minor thirds. This is possible because raising 7 semitones in one direction corresponds to raising 4 semitones and 3 semitones in the two other edges of each triangle. The three notes form a triangle in the Chromatic circle with arc-sides 3, 4 and 5, which is the label we use for this tonnetz.

With this diagram, all pairs of notes separated by a fifth, major third or minor third are adjacent. Furthermore, all major and minor chords are represented as the triangular faces of this diagram.

There are other types of triads, depending on the intervals between their notes. For instance Augmented chords are (x, x+4, x+8), which form a triangle with sides 4,4,4 in the chromatic circle (or in the fifth circle). We can build a tonnetz that represents these chords using these intervals as increasing steps in each direction. Any three numbers a,b,c, that add up to 12 represent a triangle on the chromatic circle and can be represented in a tonnetz. Using combinatorics, there are 12 possible ways of choosing 3 numbers a,b,c between 1 and 12, so that a+b+c=12. These are the 12 possible tonnetze.

On the diagram we can also see the dual graphs of the tonnetz. A dual graph is built by replacing each face by a vertex, each vertex by a face, and connected vertices by adjacent faces. This diagram has the same information as the original one, but now each triad is a vertex of an hexagonal tiling and each note is a face (an hexagon).

The tonnetz representation contains a lot of musical information. For instance, adjacent triangles represent triads that have two tones in common, and thus it is a natural chord progression in composition to go from one to another.

Finally, these graphical representations can be used to transform a piece. The most basic transformation is pitch shifting, just adding a fixed amount of semitones to all notes in a music piece. This is usual to adapt a piece for a singer's range of voice. Geometrically it corresponds to a translation in the piano, but the white-black keys do not match. Better it is to consider a rotation on the chromatic circle, then all the notes are just rotated, or a translation in the tonnetz.

An example of such a transformation would be a rotation on the Circle of fifths. That is just jumping 7 semitones at once (since a fifth is equal to 7 semitones). Again, if we have the piece represented on a tonnetz, that corresponds to a translation of the net in any of the directions.

A more interesting transformation is a rotation of the tonnetz. Consider one piece represented in the tonnetz. Fix one note (for instance the first note on the piece), and then switch all the rest of the notes by their counterpart after rotating 180 degrees on the tonnetz. This transformation is called a "negative harmony" and transforms every major chord into minor chords and vice-versa. You can hear an example on the exhibit, by selecting the classical 3,4,5 tonnetz.

**Authors of this exhibit:** Moreno Andreatta and Corentin Guichaoua (SMIR Project). Adapted by Philipp Legner.

**Acknowledgements:** Supported by CNRS/IRCAM/Sorbonne University, USIAS (University of Strasbourg Institute for Advanced Study), IRMA/University of Strasbourg.

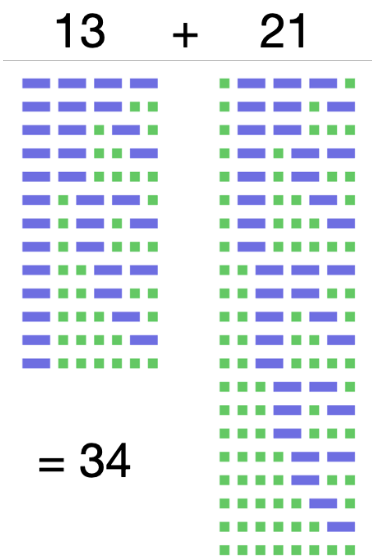
**Text:** Daniel Ramos (IMAGINARY).

# BEAT BOX

Along with melody and harmony, rhythm is one of the most important components of music. The exhibit “Beat Box” explores various ways to create and analyze rhythms by mathematical methods.

## FIBONACCI RHYTHM / HEMACHENDRA (FIBONACCI) NUMBERS AND 2-1 RHYTHMS

It only seldom happens in science that important concepts are named after the person who dealt with them first. This is also the case for the famous Fibonacci numbers. Fibonacci mentioned them in the context of a calculation exercise in his book Liber Abaci published in 1202. As a matter of fact already much earlier in 1050 these numbers were studied by the Indian scholar Hemachendra in a very interesting context related to rhythm.




Fibonacci rhythm

Assume you want to fill a rhythm of n beats with patterns of either length 1 or of length 2. For instance an 8-beat rhythm could be filled by 1-1-1-1-1-1-1-1 or 2-2-2-2 or more complicated patterns like 2-1-2-1-2 or 2-1-2-2-1. How many possibilities are there? Such questions arise both in poetry when talking about verses and syllables, or in music when talking about rhythms built from quarters and halves. It turns out that the number of actual possibilities is a Fibonacci (or better Hemachendra ) number. A number from the sequence

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144,...

where (after starting with 1, 1) each number is the sum of the preceding two. In fact there are exactly 34 ways to fill an eight beat pattern with 1s and 2s.

**The Music:** To avoid repetitive patterns in music it is sometimes very good to be aware of all possibilities that satisfy a given requirement. The question of filling in beats with longs and shorts for instance has nice applications in the art of tabla playing. Here often the shorts and longs are not literally one note but fixed drumming patterns that fill either one or two beats. Using only two fixed patterns but being flexible with the macro rhythmic level of how to combine them creates a stylistic coherence while at the same time it creates richness.

**The Math:** Why do Fibonacci numbers pop up in this context? Let us see how we can reduce the problem of filling 8 beats to the problems of filling 6 and 7 beats. Each eight beat rhythm either starts with a short or with a long note. If it starts with a long note there are as many ways to complete this as there are 7 beat patterns. If we start with a long note then we have 6 beats to fill and there are as many possibilities as there are for 6 beats. Hence the number of 8 beat patterns equals the sum of the number of 7 beat and 6 beat patterns. Et voilà... the Fibonacci recursion.

**The Exhibit:** The exhibit is mainly an automatic tabla rhythm machine based on the above observations. Listen and enjoy! The sound samples for the fast Rhythm were, by the way, taken from a talk of Manjul Bhargava, a Fields Medalist, who also gave very inspiring talks on that topic, where he himself plays the tabla.

## CLAPPING / STEVE REICH'S CLAPPING MUSIC

Want a challenge? Here it is! Try to clap along with one of the two voices in our Clapping exhibit. Or even better, find a partner and together clap both voices. For every position of the green point in the exhibit the challenge is different.

The piece you hear in the exhibit is an amazing example of how a simple idea can create interesting and complex musical patterns. Inspired by the clapping of Flamenco dancers, Steve Reich composed this amazing little piece of music. It consists of a simple 12-beat clapping pattern:

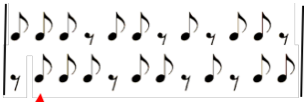
3 claps - pause - 2 claps - pause - 1 clap - pause - 2 claps - pause ... repeat

The complexity is achieved by playing exactly the same rhythm with two players but shifting the beginning of the second player by n-beats for a chosen number n=0 to n=11. There are twelve possible such shifts each generating a different rhythm. Can you clap all of them with your partner?

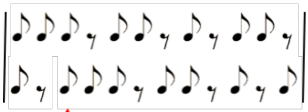
**The Music:** A Minimal Music (mainly composed by Steve Reich) attempts to create interesting music by minimalistic progression of score. Shifting a rhythmical pattern by just one beat might be considered as a typical stylistic element of Minimal Music. In fact the clapping piece needs a tremendous concentration as soon as both parts are played together. Hence although it is minimal it is by no means easy.

**The Math:** How many rhythmical patterns are suitable for creating a piece like Clapping Music. Let's stick to a 12 beat sequence. First of all there are altogether $2^{12} = 4096$ possibilities to have either clap or pause at one beat.

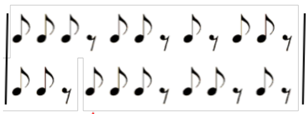
We might require that the pattern itself has no repetitions since this would cause identical music under more than one shift. This kills 76 of the patterns. Of the remaining 4020 patterns we only need one shifted version of each. This dived the total number by 12 and leaves us with only 335 patterns.



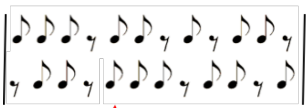
shift 1



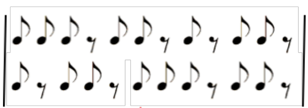
shift 2



shift 3



shift 4



shift 5

Most of them would sound pretty boring since there are either too many or too few claps played. Restricting to patterns with at least 4 and at most 8 claps leaves us with 287 possibilities. Narrowing down further and requiring that like in Reichs piece no consecutive pauses occur leaves us with no more than eleven patterns (shown below). Reich's choice (marked in red) is one of only two without repeating the number of claps between two pauses.

```
(1,1,0,1,0,1,1,0,1,0,1,0)
(1,1,0,1,1,0,1,0,1,0,1,0)
(1,1,1,0,1,0,1,0,1,0,1,0)
(1,1,1,0,1,0,1,1,0,1,1,0)
(1,1,1,0,1,1,0,1,0,1,1,0)
(1,1,1,0,1,1,0,1,1,0,1,0)
(1,1,1,0,1,1,1,0,1,0,1,0)
(1,1,1,1,0,1,0,1,0,1,1,0)
(1,1,1,1,0,1,0,1,1,0,1,0)
(1,1,1,1,0,1,1,0,1,0,1,0)
(1,1,1,1,1,0,1,0,1,0,1,0)
```

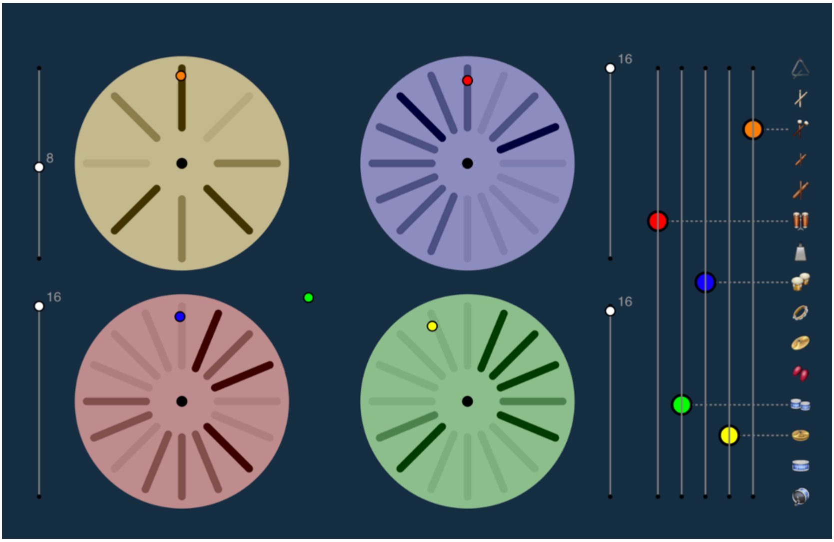
**The Exhibit:** Go and clap along. Try to start slowly and increase your speed. Select the different shifts by moving the green point on the circle.

## SEQUENCER

Have you ever stepped in to a printing hall of a big newspaper? Where the machines spin round and round and create a sound pattern with each rotation. Or just listened to a sewing machine? Whenever some mechanical process produces a repeating pattern our brain is trained to search for structure in it. Forming rhythm. Forming music. The beat machine in our exhibition is a sequencer that lets you experiment with cyclic patterns. Play and explore.

**The Exhibit:** Perhaps a little usage description is useful here. Consider the four big wheels as turntables equipped with bars. Whenever the bars hit an instrument it creates a sound. The instruments are resembled by the five small colored points in the central area. You can move them around freely. If they get hit they make a sound. Which instrument is resembled by each point can be chosen by the colored sliders at the far right. The further out an instrument is on a wheel, the louder it is played.

When the machine is at rest, the number of bars on each disk can be changed by moving the sliders. By this, it is possible to create very uneven but still cyclic repetition patterns. For instance you can set one of the disks to 8 bars per turn, the other to 11 and the other two to 15. This sounds wild but still very rhythmic. As a matter of fact, African music cultures are by far more used to such uneven rhythm distributions than Western ones.

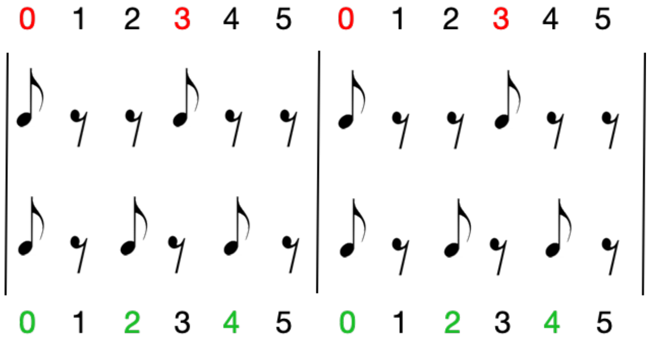



In the rest state you can also click on the bars on the disk and alter them between three states, loud-medium-off. By this you can create a great variety of known and of totally new rhythms.

## N OVER M RHYTHMS

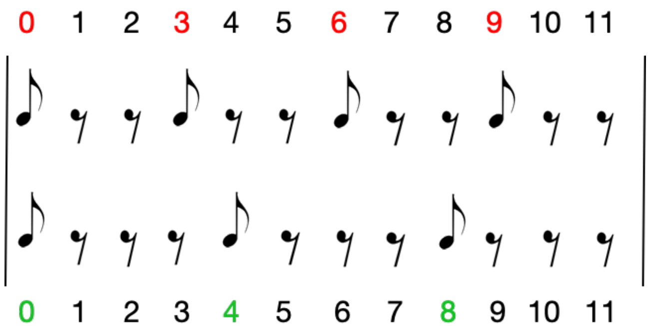
We are very much used to clap simple regular rhythms like a two-step march 1-2-1-2-1-2 or a waltz in a three fold pattern 1-2-3-1-2-3. It is perhaps one of the first exercises when learning a percussion instrument to be able to perform these kinds of rhythms. But things become really difficult if we want to clap two rhythms simultaneously. Combining a regular rhythm with n beats in a bar with another rhythm, that has m beats in the same time is called an n over m rhythm. The speed ratio between the two rhythms is then n/m. The picture below illustrates a 2 over 3 rhythm. In each bar the top voice has exactly two evenly distributed notes while the lower voice has three evenly distributed notes.

**The Math:** Perhaps the easiest way to learn simple n over m rhythms is to merge them into a common framework of constant ticks fine enough to incorporate both rhythms. Mathematically this asks for the least common multiple of the two numbers n and m. Thus a 2 over 3 rhythm can be embedded in a regular rhythm of 6 ticks, a 3 over 4 rhythm can be embedded in 12 ticks and a 7 over 5 rhythm can be embedded into a rhythm with 35 ticks. If one counts the ticks by starting from zero the m-beat rhythm is played on the multiples of n and the n-beat rhythm is played on the multiples of m.




**The Music:** Playing the n over m rhythm in the above way is a good way to learn it. However it has one big disadvantage. One does not "feel" the rhythm as being composed from two distinct regular ones.

Here is a "pro tip" for learning these rhythms. First learn to clap them with two hands by using the above method (it will take a while). After feeling secure in clapping these rhythms play them and "split your mind" and focus to only one hand and what it does. And then switch the mind between the two hands. At some point you will be able to feel both rhythms independently and simultaneously in your mind.



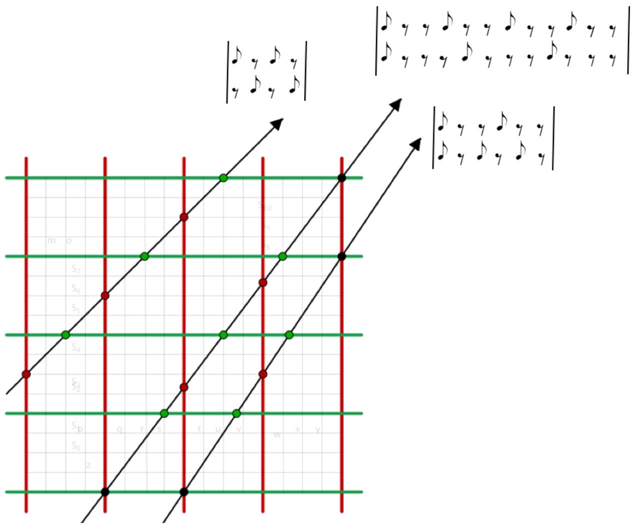

**The Exhibit:** The exhibit is a clap along exhibit. You can hear and experience n over m rhythms and you can also try to learn clapping them. It might take a little practice but once you get it you won't un-learn.

## GRID RHYTHMS

Can you hear the rhythm of a regular ornamental drawing? Have you ever rushed with a trolley over a regular pavement busy to get your train or airplane? Did you listen to the sound it created? In fact already for the simplest geometric patterns (like a square grid) moving a point over it in constant speed creates interesting rhythmical patterns. If you associate different voices to the horizontal and vertical lines of a grid then moving a point in any direction with constant speed over the grid creates a regular rhythmic pattern in each voice. However the two patterns have different speed and are shifted in phase, leaving lots of space for musical experience.

**The Music:** Getting inspirations for interesting musical patterns is an important part of the work of creative musicians. Those inspirations can come from very different sources.

We strongly recommend to listen to the YouTube Video “Stoiber on Drums” in which a percussionist performs along with Edmund Stoibers speech about a fast train connection from Munich Center to the Airport trying to hit an instrument every time there is a syllable in the speech.

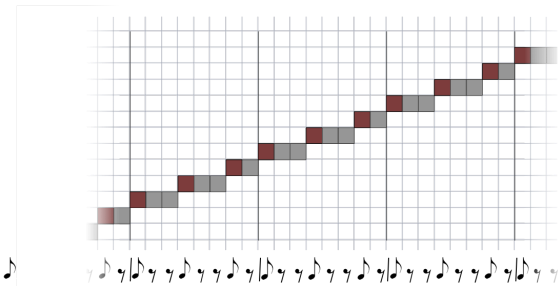


**The Math:** The square grid creates an overlay of two regular beat patterns. As a matter of fact except for the regularity of each pattern everything else is free in this context: both speeds and the phase between them. The examples in the figure show a few instances, among them a 2 over 3 and a 3 over 4 rhythm. If the slope of the line is irrational (not a fraction) then the rhythmic pattern generated will even never repeat.

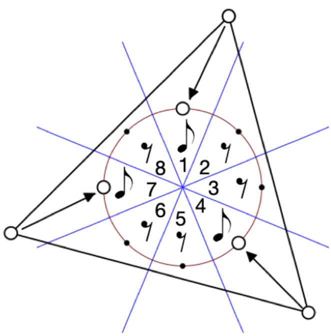
**The Exhibit:** By adjusting the slope of the movement you can experience all possibilities that are accessible through that approach of rhythm generation. You can adjust the direction of the grid motion. Pressing one of the synchronize buttons sets the grid to the indicated position with respect to the measuring point. This allows you to get a bit more control over the rhythm.

## EUCLIDEAN RHYTHM / LINE STEP RHYTHMS

How to distribute 3 drum hits in a measure of 8 beats? And what does this have to do with low resolution computer screens? Here is a nice connection. Imagine you want to draw a line with slope 3/8 on a low resolution grid by a pixelized line. The picture below indicates the line as a sequence of highlighted pixels. The steps in that image form a regular and rhythmic pattern. The slope of 3/8 implies that while you go 8 units to the right you must go also 3 units up. Thus the pattern repeats after 8 steps. If you hit a drum every time you step up, you get a nice pattern of 3 hits distributed over 8 beats.



**The Math:** This generation method for a rhythm has a remarkable property. It distributes the 3 hits as equal as possible among the 8 beats. The reason for that is that the pixelized drawing of the line approximates the line as good as possible. Another way to think about this generation method is to consider an equilateral triangle and then to choose points from an octagon with same centre that are as close as possible to the corners of the triangle. Clearly the method can easily be generalized to any number of hits and beats.




**The Music:** The equal distribution indicated by this method creates very interesting rhythms. As a matter of fact they can be found across all different cultures and styles from Balkan folklore, via Latin American and Japanese rhythms, to Dave Brubeck's Take Five and Unsquare dance.

**The Exhibit:** In the exhibit you can create two such rhythms in parallel and vary the numbers that generate the rhythms. It is a great device to create and exercise quite complicated and elaborate rhythm patterns.

**Author of this exhibit:** Jürgen Richter-Gebert (Technical University of Munich).

**Acknowledgements:** Patrick Wilson and Aaron Montag (Sound Engine). Based on CindyJS.org.

**Text:** Jürgen Richter-Gebert (TU Munich).

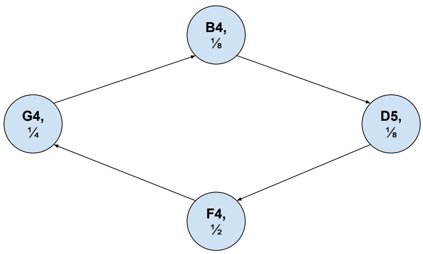
# GRAPH COMPOSER

A graph is a mathematical object consisting of a set of vertices and edges that connect them, you may know a graph under the name network. Each vertex can have several attributes or values depending on what is being modeled. The edges can have direction, in which case we talk about a directed graph.

Graph Composer offers the possibility to compose music by walking along paths on a graph. In Graph Composer, each vertex is associated with a note and its duration over time. The edges connecting the vertices define a path along which one travels over time, meaning notes are played along the sequence of visited vertices. The sound obtained at each vertex takes on different shapes based on the decorator of the vertex, such as a pure note, a chord or arpeggio.

For example, the following score:

can be drawn as the following directed graph:

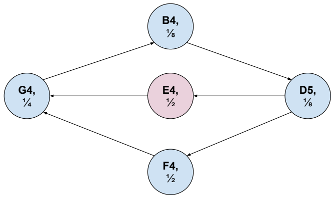


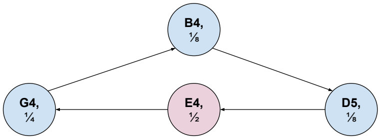

Paths on the graph represent sequences of notes. If the path has a loop, as in the example above, there is a repetition of a fragment. In the example, the graph has a unique possible path, but when there is more than one edge leaving a vertex, multiple paths are possible.

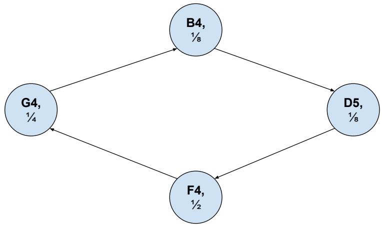
For example, the following graph has two possible paths:

In this case, the program needs to make a choice in which way to follow the paths in the graph. Graph Composer chooses randomly between the two options (with equal probability), which creates what is called random walks through the graph. Since the music produced by each path may result in different lengths and therefor has a different overall duration for the music played, the listener will perceive this as some kind of “un-rhythm”.

The model used by Graph Composer is used in Computational Musicology, a field of study where mathematics and computer science are applied to music. In research,




musical compositions are modeled as graphs to compare and extract different mathematical features; for example, the number of times that a transition from one note to another is repeated, or the number of times that three (or more) different notes appear in the same sequence, amongst others.

Graph Composer walks paths randomly along the graph from the root vertex through the directed edges until it finds a vertex with no outward edges; then it returns to the root vertex. When the program is started, you can only see the root vertex (the only one that plays with no incoming edges). You can drag and add more vertices, change their notes, duration, and decoration. All of this can be done without the need to stop the music from playing. The complexity of the graph by adding many vertices and edges can quickly increase. Can you create patterns in your graph that create pleasant musical motifs? Happy listening.

**Authors of the exhibit:** Pedro Arthur, Vitor Guerra Rolla, and José Ezequiel Soto Sánchez (Instituto de Matemática Pura e Aplicada, Brazil).

**Text:** by the authors.

# AI JAM

Machines can generate and play music, but how good are they at making music together with humans? Could they be part of a Jam Session? Could an Artificial Intelligence eventually replace a missing musician during a band rehearsal?

Even leaving aside questions of musical quality, there are many issues that arise when attempting to compose music using machine learning. One of them, is the generation of a melody. Humans are usually good at producing long-term structures, but machine learning algorithms are not, so the music they produce might sound good on a note-to-note basis, but its structure will seem random and wandering after some bars. AI Jam tackles this problem by explicitly training its Recurrent Neural Network[1] (RNN) with musical structure in mind. When feeding it with thousands of musical pieces, the input was specially labeled whenever a bar was repeating the one immediately preceding it, or the one before that. This is called a Lookback[2] RNN by its developers, the Google Magenta team.

A second technique used is that of Attention. In this case, whenever the RNN produces some output, it looks back at the previous n outputs (with n being configurable), weights them using a calculated attention mask and adds the result. The result is basically the previous n outputs combined, but each given a different amount of emphasis. It is then combined with the output of the current step. In this way, each step produces an output that is related to the previous ones.

By using both mechanisms together, the algorithm can not only produce short fragments of notes that sound good by themselves, but that also form a nice sounding melody when played after a fragment played by a human.

**Author of the exhibit:** Sebastian Uribe (Exhibit conception and project management), Eric Londaits (Software development), Christian Stussak (Additional software development and OS configuration), and Daniel Weiss (Case design and construction) for IMAGINARY.

**Original software (based on):** Yotam Mann, the Magenta and Creative Lab teams at Google.

[1] www.en.wikipedia.org/wiki/Recurrent_neural_network

[2] www.magenta.tensorflow.org/2016/07/15/lookback-rnn-attention-rnn

# CON ESPRESSIONE!

A piece of music can be described with mathematical accuracy on a score, with symbols perfectly defined in a well-founded theory. However, music is above all an art, and as such it conveys emotions, feelings, and human sensations. How is it possible to express these feelings, given such a strict notation? How can a performer make a piece come alive? What lies beneath the score sheet and the execution of a piece?

A human performer does not play all the notes of a chord at the same time, nor do they keep a strict tempo; some notes start a few milliseconds earlier, or are released a few milliseconds later, some are played louder than others, some appear quicker, etc. All these nuances allow a performing musician to express themselves and imprint some feelings and emotions onto the performance. A musician is not a machine that perfectly reproduces the score, and these “imperfections” or deviations are what make the music alive, something that humans do naturally but a machine could never do... or could it?

This exhibit allows you to explore the difference between a mechanical reproduction of a piece and a more “human” interpretation produced by an Artificial Intelligence that was trained to behave as a musician. The visitor takes the role of a music conductor, controlling overall tempo and loudness of a piano performance. Via a camera sensor, the hand of the visitor is tracked in space. The up-down position determines the loudness (volume) of the music, and the left-right position determines the tempo (the speed) of the music. Initially, this is achieved by directly adapting the loudness and tempo of the overall piece according to the position of the hand, but even if the machine obeys you to set these values, the music feels automatic and soul-less. This is because with your hand movements, you can only control overall tempo and loudness, but not the fine details of a performance (such as how to play the individual notes, or how to stress the melody line).

Then, a slider allows you to activate the Artificial Intelligence. The higher the value, the more freedom has the machine to choose small deviations from the prescribed parameters. The machine adjusts the tempo and loudness to be slightly different from what you conduct, to make the music more lively and less “mechanical”. It also introduces changes in the dynamic spread, micro-timing, and articulation.

- The loudness is the volume, i.e. the amount of amplification of the sound of each note. Raising the loudness is the most obvious way to stress a note.
- Dynamic spread relates to loudness differences between notes simultaneously played (e.g., in a chord). This is important to make the melody line come out clearly and to change the overall “sound” of a chord.
- Musical tempo is defined as the rate at which musical events or beats are played. Music performers continually change the tempo, speeding up or slowing down, to express the “ebb and flow” of the music. This is what makes music sound natural to us (and we may not even be conscious of all these tempo fluctuations).
- Microtiming refers to the moment that a note plays with respect to its supposed onset. For example, if a chord consists of several notes that are supposed to be played together, one can advance one note over another by a few milliseconds, so that not all of them are perfectly synchronized. This is inevitable in real-life performance, and it makes the piece more warm, human and expressive.

- Articulation here refers to the duration of a note with respect to its supposed duration according to the score. Notes can be played a bit longer or shorter than the composer described in the score, tying them together or separating them, which helps to stress or diffuse some notes amongst the others. In musical language, this is described with terms as legato and staccato.

Each performer has their own experience, understanding of a piece, and expressive intentions, and communicating these in a performance requires control over musical parameters at many levels – from precise note-level details like articulation or micro-timing to high-level, long-term tempo and the shaping of its dynamics. The computer program behind this exhibit was trained with hundreds of real-life performances of music pieces to analyze and learn how these parameters are used and controlled in real-life interpretations by human pianists. Experimental results show that computers are already very good at learning the low-level, detailed decisions, but still have problems understanding the larger-scale form and dramatic structure of music, and the high-level shaping this requires. Thus, the exhibit explores and demonstrates a compromise: you control overall loudness and tempo with your hand, at a high level, based on your understanding of the music, and the computer adds its own local details and deviations. In this way, the resulting performance is the product of a true cooperation between a human (you) and a computer (AI).

**Authors of the exhibit:** Gerhard Widmer, Florian Henkel, Carlos Eduardo Cancino Chacón, Stefan Balke (Institute of Computational Perception, Johannes Kepler University Linz, Austria Austrian Research Institute for Artificial Intelligence (OFAI), Vienna, Austria), and Christian Stussak, Eric LONDAITS (IMAGINARY).

**Text:** Daniel Ramos (IMAGINARY).

**Acknowledgments:** This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement number 670035).

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  `www.cp.jku.at/research/papers/Cancino_Dissertation_2018.pdf`
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  `www.frontiersin.org/articles/10.3389/fdigh.2018.00025/full`

# NSYNTH

Can you mix and blend sounds? What is the morphed sound of a guitar and a piano? And that of a car engine sound and a flute?

A synthesizer is a device that generates sound, tuned so it can be used as an instrument. Classical analog synthesizers use oscillators and filters, while modern digital synthesizers can work with stored samples, but it is always necessary to describe the timbre with detailed parameters to adjust and tune the final sound. Traditionally, this is done manually by analyzing the sound of real instruments (spectrum, waveform), or by an expert that "sculpts" the waveform by hand.

Finding a sound "in between" two other sounds is not easy, since parameters are highly non-linear. If you take the average of the waveforms (or spectrum) of two instruments, mathematically this would mean adding them, all you obtain is the sound of the two instruments played at the same time, not a new instrument. What you would need instead is features that describe qualities of both instruments, such as how "bright", "multi-phonic", or "percussive" a sound is. Then, what you are looking for is a sound that has the average "brightness", "multi-phonicity" or "percussiveness" of your two original sounds.

NSynth is a synthesizer that uses Artificial Intelligence to interpolate and mix sounds. A Deep Neural Network is trained with thousands of sample sounds to extract 16 features (dimensions) per each time-step of 32 ms (a total of 2000 parameters for samples of 4 seconds). With this process, each sound is encoded with these dimensions as parameters, and conversely these parameters allow to generate sounds. The correspondence is not perfect, the sound you can produce from certain parameters is not exactly the same as the original one, however the sounds are close enough and sound similar. Most importantly, you can now mix these parameters linearly with each other to generate new sounds which really average the qualities of the source sounds.

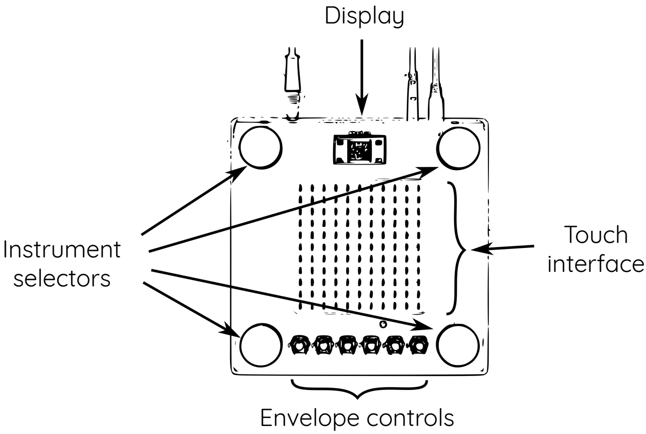
The NSynth device on the exhibition contains the trained neural network and can interpolate between four selected sounds (using the four big knobs). By touching a point on the square touch sensor, the synthesizer adjusts the parameters to be proportional to the coordinates on the square. The four original sounds are associated with the corners, and all other points correspond to newly created sounds.

**Instrument selectors** - These instruments are assigned to the corners of the touch interface.

**Display** - Shows the state of the instrument and additional information about the controls that you are interacting with.

**Envelope controls** - Used to further customize the audio output by the device:

- "Position" sets the initial position of the wave, (cut out the attack of a waveform, or start from the tail).
- "Attack" controls the time taken for initial run-up of level from nil to peak.




- “Decay” controls the time taken for the subsequent run down from the attack level to the designated sustain level.
- “Sustain” sets the level during the main sequence of the sound’s duration, until the key is released.
- “Release” controls the time taken for the level to decay from the sustain level to zero after the key is released.
- “Volume” adjusts the overall output volume of the device.

**Touch interface** - This is a capacitive sensor, like the mouse pad on a laptop, which is used to explore the world of new sounds that NSynth has generated between your chosen source audio.

**Author of the exhibit:** NSynth is an open-software synthesizer based on machine learning developed by Google’s Magenta project. NSynth Super is an open-hardware interface developed by Google Creative Lab.

**Text:** Daniel Ramos (IMAGINARY).

**References:**


- *NSynth: Neural Audio Synthesis*. Website at Google’s magenta project. `www.magenta.tensorflow.org/nsynth`.
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- *NSynth Super: An experimental physical interface for the NSynth algorithm*. Website at Google’s Creative Lab. `www.nsynthsuper.withgoogle.com`.

# LISSAJOUS-BASED EXHIBITS

Lissajous figures (named after the 19th century French mathematician Jules Antoine Lissajous) are curves determined by the intersection of two perpendicular oscillating movements. Mathematically, their coordinates $x, y$ in the plane are given for each time $t$ by the formulas

$$\begin{cases} x &= \sin(a \cdot t) \\ y &= \sin(b \cdot t + \varphi) \end{cases}$$

where $t$ is the time parameter (angle), $a$ and $b$ are the two frequencies, and $\varphi$ is the phase shift between one and the other. For different values of the time $t$, the equations trace a curve inside a square. Its shape depends essentially on the numerical ratio of the two frequencies. If $a = b$, or $a/b = 1$, then the figure closes after one period and the result is a line or an ellipse. If $a/b = 3/2$, the curve closes after $3 \cdot 2 = 6$ cycles and is quite clean. In general, ratios of small numbers make simple curves. On the contrary, if the values are in more complex ratios such as $a/b = 23/22$, the curve closes after many cycles ($22 \cdot 23 = 506$ cycles) and the figure is more dense and messy. Thus, Lissajous figures are a tool to visualize when two numbers are in a simple ratio.

Here our oscillations are sound, and the frequency of the sound is perceived as pitch in our ears. In music, a fundamental idea tracing back to Pythagoras, is that sounds with frequencies in small number ratios sound consonant together, and frequencies with more complicated number ratios are less consonant. This is why Lissajous figures are used to visualize consonance or dissonance. Classical consonant intervals are the octave (ratio 2:1), the Fifth (ratio 3:2) and the Major Third (5:4).

It is also possible to have three-dimensional Lissajous curves, just analogous to the flat ones, but with three perpendicular vibrations instead of two. Three notes playing together form a triad chord, which are the base for music composition. In the just intonation tuning, major chords are notes in ratios 4:5:6, minor chords in ratios 10:12:15, and so on.

Remarkably, these arithmetic relations produce in us feelings as “happy” for major chords, “sad” for minors, “tense”, “cozy”, etc. although a real mastery is needed to compose a piece that successfully conveys emotions.

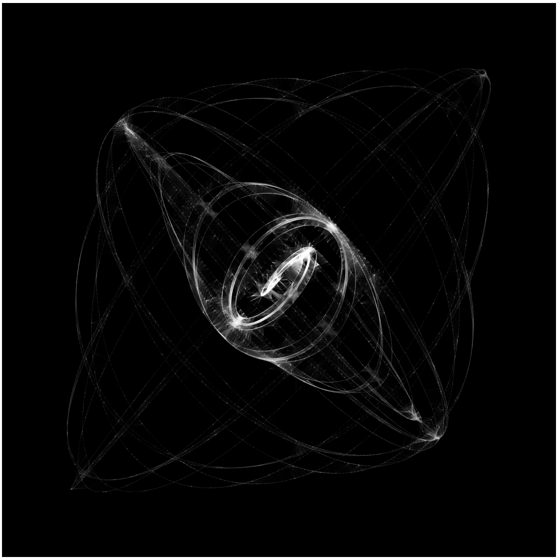
# LISSAJOUS GALLERY

The six pictures here are an artistic representation of Lissajous figures. To create them, the artist Ryan Cashman starts with a curve in space defined by two sinusoidal waves, that is, a Lissajous figure, but also modulated by a second oscillation. With the chosen frequency ratios, the curve drifts from itself passing close but not exactly over itself. The program calculates points at regular intervals along the curve, creating vertices. The vertices are connected in order to create the base curve of the shape. Then additional lines are drawn between any pair of vertices located close enough in the space, to form a web like representation of a surface. The frequencies chosen for each pictures are those from musical chords in the standard equal tempered tuning.

Mathematically, the $x$ (horizontal) and $y$ (vertical) position of each vertex is calculated with the formula

$$\begin{cases} x &= \sin(f_X \cdot t + \varphi) \cdot \cos(m_X \cdot t) \\ y &= \sin(f_Y \cdot t + \varphi) \cdot \cos(m_Y \cdot t) \end{cases}$$

where additionally we have $m_X$ and $m_Y$ as modulation coefficients.



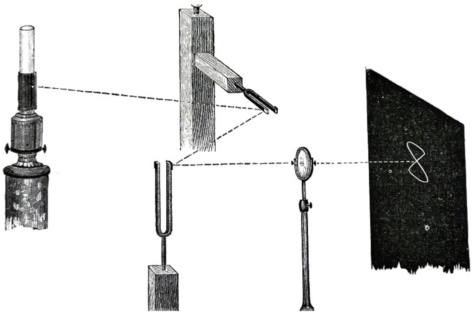
Major second.

**Author of the gallery:** Ryan Cashman.

**Text:** Ryan Cashman.

## THE HARMONIC SERIES

When Jules Antoine Lissajous invented the device that produces the curves named after him, it was with the purpose of standardizing musical sound. In his original design, two tuning forks are placed perpendicularly, each with a mirror attached to its tip. A beam of light is directed to the first mirror, reflected to the second one and finally to a screen. If the frequencies of the tuning forks are related by simple integer ratios, which make them harmonious musically, the figures are harmonious visually as well.



The pieces in this series by artists Manuela Donoso and Luisa Pereira re-create and extend Lissajous' device using contemporary technology, bringing it from the realm of the functional into the realm of interactive art. Their devices invite us to use our voices to interact with sound vibrations visually, and develop a deeper, intuitive understanding of the interplay between noise, consonance, dissonance, and harmony.

### DEVICE #1

Microphones, amplifiers, speakers, mirrors, laser.

This device adds interactivity to Lissajous' original apparatus by replacing Lissajous tuning forks with two speakers, and the light source with a laser pointer. Each speaker is connected to a microphone, and the two mirrors attached to their membranes vibrate in perpendicular directions. As the speakers vibrate with the voices of visitors, the projected figures evolve: percussive sounds generate chaotic figures, dissonant intervals generate messy figures, consonant intervals generate harmonious figures. Whistling will result in sharper curves than singing the same notes, hinting at the variety and richness of timbre in music.

### DEVICE #2

Microphone, amplifier, A/D converter, micro-controller, synthesizers, holographic display.

This device adds a third dimension to Lissajous figures, allowing for the visualization of triadic chords – that is, chords composed by three notes. The first and second vibrations are notes produced by synthesizers; visitors can change their pitches by moving sliders up and down. The third vibration is input by visitors through a microphone: they can sing varying pitches, talk, speak, or experiment with any sound they like. The custom software plots these three vibrations onto 3D space over time: the first synthesizer determines the x, the voice the y, and the second synthesizer the z position of each point in the resulting 3D figure. The figure is then encoded to be rendered onto a holographic display ("The Looking Glass"), creating an ever-evolving sculpture of light.

## TRIAD SCULPTURES

3D stereolithography prints.

Three-dimensional Lissajous figures represent three different triads: Major (with frequency ratios of $4 : 5 : 6$), Minor ($10 : 12 : 15$), and Diminished (approximately $20 : 24 : 29$). Observe that in both the Major and the Minor cases, when we take pairs of frequencies they are in ratios $4 : 5$, $5 : 6$, and $2 : 3$, but the Lissajous figures (and the chords) are not the same. You can use one of the bright light focus to cast a shadow. The number of lobes on each side of the flat Lissajous figures give you the ratios.

These figures can be reproduced by having Device #2 receive pitches at these ratios, by singing and shifting the synthesizer sliders.

**Authors of the series:** Manuela Donoso and Luisa Pereira.
Production (Device #1): Manuela Donoso and Lukas Reck for IMAGINARY.
Programming (Device #2): Luisa Pereira and Ricardo Dodds.

**Sponsored:** 3D holographic display by The Looking Glass
`https://lookingglassfactory.com`.

**Text:** Manuela Donoso and Luisa Pereira.

**References:**
`www.theharmonicseries.net`

# PINK TROMBONE

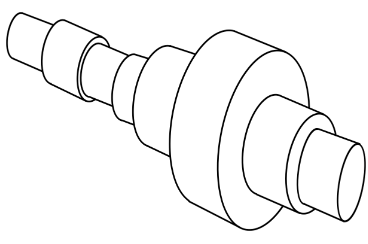
Voice is the oldest and most complex musical instrument. The human voice organs are unique amongst all the animals and allow us to communicate and express ideas and feelings through sophisticated sounds that we can create and control instinctively.

As an instrument, voice is extremely rich and complex. Some sound characteristics are fixed and particular of each individual, while others can be changed to produce sounds that we can recognize as letters and words. Singing requires the sound to be not only recognizable, but exactly the one intended for the particular piece of music, which demands much more practice and skill. The film station shows a video of the cross section of a singer taken with an MRI (magnetic resonance imaging) machine while singing.

Here we present an interactive simplified model that allows to replicate with certain fidelity the human voice. This model consist of two components: sound production and sound articulation.

The sound is produced in the glottis (throat). This organ has some membranes or vocal folds that we can make vibrate and produce sound when we expel air. With the tension we put on this muscle (vocal cords) and the pressure of the air we expel we can control the pitch of the sound and also the intensity, from a whisper to a shout. Using these parameters the model generates an initial wave form.

The sound wave coming from the glottis enters the vocal tract, where we articulate the sound. Articulation is to modify the voice sound to make a particular speech sound (or phone), that is, the sounds of letters that we use to speak. We can think of a simplified vocal tract as a tube, where the sound wave travels at certain speed (the speed of sound) only in one dimension (forward or backwards). If the tube were completely smooth and uniform, the wave would travel forward uninterrupted until the lips. However, the tube is not smooth nor uniform, and the sound wave bounces back and forth. The main characteristic of the tract is the diameter of the cross section at each point, that we can change with the position of the tongue and lips. Changes in the diameter of the tube make the wave bounce. In this model, the voice tract is a tube of 17 cm divided into 44 sections, each one with a constant diameter.



The difficulty of propagating a wave through a tube is measured by its impedance $Z$, which is inversely proportional to its area. When the wave passes from a section tube of impedance $Z_i$ to one of $Z_{i+1}$, the reflection coefficient is

$$r = \frac{Z_i - Z_{i+1}}{Z_i + Z_{i+1}}$$

and a fraction $r$ of the wave intensity is reflected backwards, and $(1-r)$ is transmitted forward. The model keeps track of two waves, one moving forward and another moving backwards. By computing the wave intensity (air pressure) at each section at each time step (the time-step is the time required to travel one section of the tube at the speed of sound), we can work out the pressure wave coming from the 'lips', which is the sound we hear. The model is improved by including a second tube (the nose), that attaches to the main tube (mouth) at the palatal velum.

On the screen, the voicebox control allows you to choose the pitch (horizontally) and loudness (vertically) of the sound produced at the glottis.

Clicking on the tongue control, oral cavity, and nasal cavity allows you to modify the vocal tract. The crossing lines represent the diameter of the tract at each section.

**Author of the exhibit:** Neil Thapen (Institute of Mathematics / Academy of Sciences of the Czech Republic). Adaptations by Eric Londaits (IMAGINARY).

**Text:** Daniel Ramos (IMAGINARY).

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- Julius O. Smith III, *Physical audio signal processing for virtual musical instruments and audio effects*.
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  `www.phon.ox.ac.uk/jcoleman/kelly-lochbaum.htm`
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- Pink trombone (original version): `www.dood.al/pinktrombone`

# MIND AND MUSIC JUKEBOX

What are mathematicians and computer scientists favorite songs? What music inspires scientists? What music relaxes them? What music makes them dance?

The Mind and Music Jukebox presents the music of mathematics, or to be more precise, the music of mathematicians and other great minds! We asked mathematicians and researchers from the Women of Mathematics throughout Europe exhibition project and the recipients of the most prestigious awards in mathematics and computer science, namely the Abel Prize, the ACM A.M. Turing Award, the ACM Prize in Computing, the Fields Medal and the Nevanlinna Prize, to fill the jukebox with their beloved music pieces.

What music is your favorite? What music should be the background track of a mathematics class? What music makes you think?

**Idea and Implementation:** Andreas Daniel Matt and Bianca Violet for IMAGINARY.

**References:** Pictures for this exhibit are from the book *Masters of Abstraction* photographed by Peter Badge, and from the exhibition *Women in Mathematics throughout Europe*, photographed by Noel Tovia Matoff.

`www.heidelberg-laureate-forum.org/masters-of-abstraction`

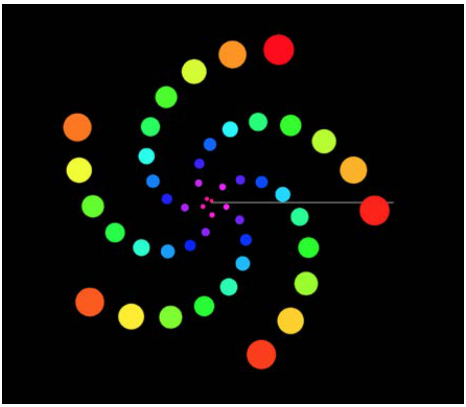
`www.womeninmath.net`

# WHITNEY MUSIC BOX

This mesmerizing applet is inspired by the animations of John Whitney (1917-1995), a filmmaker that explored and experimented with animation techniques governed by mathematical and mechanical rules, before the digital era. In the 1950s, Whitney experimented with pendulums, gears, mechanical artifacts, and early analog computers combined with cameras to produce short films of lines and dots of light, being a pioneer of video as a plastic form of abstract art. In the 1960s he embraced digital computers and devoted his career to computer graphics art, with shorts and films as Permutations (1968), Arabesque (1976), or Moon Drum (1991).

In 1980, Whitney published the book Digital Harmony, where he describes his experiments on visualizing music with computer graphics. In particular, he explains what he called "incremental drift", a kind of algorithm to set the position of particles, where each particle is "drifted" from the previous one by a simple rule; a kind of differential rule that by integration gives rise to a global behavior that was originally unexpected. Whitney gave in his book an example of this spiral incremental drift.

In 2006, the programmer and Disney animator Jim Bumgardner took Whitney's ideas and created an interactive applet adding synchronized sound to Whitney's spirals. Bumgardner coined the term "Whitney's music box" and the applet has become popular on the Internet since.



Mathematically the applet can be described fairly easily. Each point moves along a concentric circle, and moves at an angular speed proportional to its radius. Or said in other words: every time the innermost dot goes once around the circle, the next one goes twice, the next one goes three times, and so forth.

The overall reference for rotation speed is given by the controlling wheel/knob. Each point plays a particular tone (assigned in the chromatic scale) when the point crosses the horizontal line. This rotations produce changing spiral patterns and chords when several tones play at once. The speed of rotation imposes a rhythmic cadence of tones that reflect the geometry of the pattern, evoking the consonance and dissonance of the notes in the scale.

This music box enables to visualize harmonic resonance and musical harmony in a pleasant way. Playful and challenging in the same time, you might find it hard to predict what will the sound and patterns resemble. It is a tool for admiration and a bit of self-hypnotic recreation.

**Author of the exhibit:** Eric Londaits (IMAGINARY). Inspired by Jürgen Richter-Gebert's implementation.

**Text:** Eric Londaits and Daniel Ramos (IMAGINARY).

**References:**


- `https://krazydad.com/blog/2006/04/23/visual-harmony`
- `https://boingboing.net/2016/04/11/john-whitney-music-box-a-psyc.html`
- `https://jbum.com/papers/whitney_paper.pdf`
- John Whitney. *Digital Harmony: On the Complementarity of Music and Visual Art*. Byte Books / McGraw-Hill (1980). `https://archive.org/details/DigitalHarmony_201611`

## MUSICAL BENCH

The Musical Bench is an exhibit which makes music when people touch, kiss, or hold hands. It uses a micro-controller to detect changes in resistance, via the metal armrests, and plays high or low notes depending on how much current flows through you and a friend.

A simple resistive voltage divider sensor circuit is used for input. The harder a person presses, or the more surface is in contact with the touch pads, or the moister a person's hands, the lower the touch resistance will be. Low resistances are mapped to high notes, so the harder a person presses, the higher the note. A light touch gets the low notes. With this information the chip arpeggiates the chromatic scale.

**Built by:** Tobias Hermann for IMAGINARY.

**Based on an idea by:** Exploratorium / San Francisco.

**Reference:** Check out the webpage and instructions to build your own Musical Bench. `https://www.exploratorium.edu/tinkering/projects/musical-bench`.

# HANDS-ON TABLE

In the exhibition you will find a hands-on table with the invitation to start exploring and tinkering. If you want to continue later at home or in school, please check the web page of the Exploratorium / San Francisco to find further instructions and resources: https://www.exploratorium.edu/snacks. Science snacks are well tested activities with cheap and available materials on a broad variety of topics. Of course we recommend to search for “music” and “math”.

**Experiments by:** Exploratorium / San Francisco.

**Selection of activities:** IMAGINARY.

# LA LA CINEMA

The film station is a collection of short videos focusing on the relation between music and mathematics. The program includes films, videos and animations that have been created for educational, corporate and artistic purposes, or just for fun. The total length of all films is about one hour (1:05 h). We thank all the authors for giving us permission to screen their films within the La La Lab exhibition.

The following films are shown:

### Peace for Triple Piano (4:15)

Vi Hart in collaboration with Henry Segerman, with additional help from Sabetta Matsumoto.
This is a spherical video in a mathematically triplified space with symmetry in space-time.

### Making of Peace for Triple Piano (9:42)

Vi Hart in collaboration with Henry Segerman, with additional help from Sabetta Matsumoto.
This video explains the concepts, as well as the math, movie, and piano magic used to create the previous film (Peace for Triple Piano).

### J.S. Bach - Crab Canon on a Möbius Strip (3:07)

Jos Leys - www.josleys.com
The manuscript depicts a single musical sequence that is to be played front to back and back to front.

La La Relation: Compare this video with the Bach canon in the exhibit 'Show me Music'.

### Improvising a canon at the fifth above (4:22)

Singers: Peter Schubert, Schulich School of Music, McGill University, Montreal, and Dawn Bailey
Production: Tuscan Bean Soup, Montreal
Producer: George Massenburg
Editor: Michelle Hugill
Concept and strategy: Shelley Stein-Sacks
Post-production sound editing: David Rafferty
Peter Schubert and Dawn Bailey show how to improvise a canon, Renaissance style.

La La Relation: Compare this video with the Bach canon in the exhibit 'Show me Music'.

### Algebraic Vibrations (2:48)

Bianca Violet, Stephan Klaus

In this film, different characteristic patterns of a drum vibration are approximated and effectively simulated using algebraic surfaces.

La La Relation: This video is related to the exhibit ‘Fingerprint of Sound’.

### Why It’s Impossible to Tune a Piano (4:19)

Henry Reich - minutephysics

This video explains, why it is mathematically impossible to tune a piano consistently across all keys using harmonics.

La La Relation: This video is related to the exhibit ‘Scale Lab’.

### Dance of the Line Riders (2:13)

Animation: Mark Robbins - DoodleChaos
Music: Dance of the Sugar Plum Fairy Kevin MacLeod (incompetech.com) CC BY 3.0

The author synchronized the song “Dance of the Sugar Plum Fairy” by Tchaikovsky to a line rider track (`www.linerider.com`). He drew everything by hand.

### CYMATICS: Science Vs. Music (5:52)

Nigel John Stanford
Music: From the album Solar Echoes

This video features audio visualized by science experiments - including the Chladni Plate, Ruben’s Tube, Tesla Coil and Ferro Fluid. All of the experiments are real.

La La Relation: This video is related to the exhibit ‘Fingerprint of Sound’.

### Four Clarinets (3:49)

Animation: Jeffrey Ventrella
Music: Four Clarinets by Robby Elfman, performed by musicians of the USC Thornton School of Music.

The video shows an artistic animation of the musical score created by a custom made algorithm, together with some real time adjustments that the author performed while listening to the music.

### Gerhard Widmer on Expressive Music Performance (5:04)

Video director: Ethan Vincent
Production: Austrian Science Fund (FWF) `www.fwf.ac.at/en/`
Filmed at the Bösendorfer Piano Factory, Wiener Neustadt, Austria.

Artificial Intelligence & Music researcher Gerhard Widmer talks about his computer-based research on expressive piano performance, the role of the Bösendorfer computer-monitored concert grand piano CEUS in this process, and then controls (tempo and dynamics) of a Chopin performance on the Bösendorfer CEUS with his hand, using a MIDI Theremin as a control device.

La La Relation: This video is related to the exhibit 'Con Espressione!'.

**Dynamic real-time MRI Bruder Jakob/Frère Jacques (0:39)**

From the DVD-ROM "The Voice", © Helbling Verlag GmbH, Esslingen / Freiburger Institut fur Musikermedizin (FIM)

In this video, Magnetic resonance imaging (MRI) is used to show detailed images of the inside of the body while singing.

La La Relation: This video is related to the exhibit 'Pink Trombone'.

**Muse - Take A Bow (4:35)**

Animation: Louis Bigo
Music: "Take A Bow" from the album "Black Holes and Revelations" composed by Matthew Bellamy performed by Muse.

This video presents the harmonic analysis of the song Take A Bow. Chords are represented within the pitch space named as Tonnetz, which displays musical pitches along axis associated with the intervals of fifth (horizontal), minor and major thirds (diagonals).

La La Relation: This video is related to the exhibit 'Tonnetz'.

**La Sera (ZhiZhu) (4:11)**

Animation:Gilles Baroin
Music and vocals: Moreno Andreatta
Lyrics: Mario Luzi

The spider web symbolizes the tonal center, basic harmonic movements are illustrated: falling fifths and relative relation. In the composition for La Sera, Moreno Andreatta uses both Tonal attraction principia and an Hamiltonian path.

La La Relation: This video is related to the exhibit 'Tonnetz'.

**The Science Behind the Arts: The Maths Behind Music (3:51)**

University of Surrey

What determines the frequency of a vibrating string? What is a musical interval?

La La Relation: This video is related to the exhibit 'Scale Lab'.

**JSBach333 canone permutativo al triangolo (5:46)**

Production, animation: Ulrich Seidel - seidel.graphics
Composer and musical director: Thomas M. J. Schäfer
Musicians of Fellbacher Kammerorchester: Regine Rosin (Violine), Daniel Egger (Viola), Cora Wacker (Violoncello)

This is an animation of one of the submitted canons to the Bach333 canon competition contest 2018: `https://seidel.graphics/bach333en/`. The tonal and modern canon refers to the name, the oeuvre and the contrapuntal execution of J. S. Bach.

La La Relation: Compare this video with the Bach canon in the exhibit 'Show me Music'.

* * *

Outside the official La La Cinema program, there are a few films we recommend to watch. They are not screened within the exhibition either because they are two long or we do not have the rights to screen them publicly.

**Music And Measure Theory (13:12)**

Grant Sanderson - 3blue1brown

**Visual Fourier Transform (20:56)**

Grant Sanderson - 3blue1brown

**Musician Explains One Concept in 5 Levels of Difficulty (15:41)**

Jacob Collier & Herbie Hancock | WIRED

**Visualizing the Notes as Ratios (11:11)**

Why These Notes - Adventures in Music Theory

**Poetry, Daisies and Cobras: Math class with Manjul Bhargava (11:42)**

NDTV

**A different way to visualize rhythm (5:22)**

John Varney, TEDed

**Quadruple Major/Minor Canon (2:30)**

PlayTheMind

**Fractal Fugues (self-similar counterpoint) (3:01)**

Jeffrey Ventrella

**Illustrated Music (youtube channel)**

Tom Johnson

**Debussy, Arabesque #1, Piano Solo (5:04)**

Stephen Malinowski

**Singing in the MRI with Tyley Ross - Making the Voice Visible (4:14)**

Tyley Ross

**MatheMusic4D (youtube channel)**

Gilles Baroin

**1ucasvb (youtube channel)**

Lucas Vieira

**Film selection and curation:** Bianca Violet (IMAGINARY).

**Acknowledgments:** Thanks to all film authors for their contributions.

# SILENT AREA

Visitors who need an acoustic break or want to deepen their La La Lab experience are welcome to the silent area in the exhibition space. Here you can discover the following:

You can sit in our library and browse books and articles on the field of Mathematics and Music, a rich domain of current research with dozens of modern books on the subject. The domain got a major impulse with the Society for Mathematics and Computation in Music, founded in 2006. You can find here its Journal of Mathematics and Music, and the proceedings of its conferences. Also a rich selection of music-related articles from the Bridges conferences on art and mathematics, and some other selected articles and books on the subject.

Did you always want to know how certain sequences of numbers sound when transformed into sounds? Choose the “The Sound of Sequences” app and find out more.

Use the “Note compass” to explore diatonic scales, to understand why the piano has white and black keys, and to hear the difference of styles they convey.

Move through the “Space of Pentatonic Scales” and create your own impressionistic style music by playing pieces by classical composers with only five notes.

**With contributions from:** IMAGINARY, Neil Sloan, Jürgen Richter-Gebert, Aaron Montag, Thomas Noll, various researchers and musicians.

**Special thanks to:** OEIS Foundation, Springer Publishing House and Taylor & Francis Publishing House.

# NOTE COMPASS

Why is the music scale irregular? Why are there black and white keys in the piano, distributed in such a peculiar pattern? Why do we have a scale with tones and semitones? More deeply, how are notes grouped into families? What role do they play within such families? And more philosophically, what turns a sequence of notes into music? The answers may be buried in musical tradition and notation, which may appear obscure and confusing. This Note Compass can help us to orient ourselves and to find the logic behind it.

Underlying all the Western music tradition is the idea that the available notes – when examined from low to high – repeat cyclically. Aside from acoustic considerations, the octave is the period of repetition of notes, named as such because we traditionally choose to have seven notes on each cycle (so the eighth is again the first, in circle). We call such a periodic selection of notes a scale.

A fundamental property of the scale in Western tradition is that it is generated by a single interval (up to octave windings around the circle). This generating interval is a favored distance between some notes that appears repeatedly, and depending on its size it may result that the seven notes are not evenly distributed on the cycle[3]. To perform the scale generation we start with a first note and add the generating interval to get the second note, then we repeat this procedure until we get 7 notes[4]. This generating interval is called perfect fifth, because classically it is the distance between the first and fifth notes in the scale (four steps).

The generation property as such is independent of acoustics and does not prescribe the actual size of the generating fifth. But music tradition has favored a consonant interval. In the twelve-note system, which is a robust and practical simplification of the traditional notation system, the approach is to take a fifth interval of 7/12 of the size of the octave[5]. This means that the intervals between our seven notes come in multiples of the basic unit of 1/12 of the octave.

Graphically, we can think of the octave cycle as a circle, and the basic units of 1/12 length as the vertices of a dodecagon. The interval of the fifth is a jump of 7 basic units. If we join all the twelve fifth intervals (separated by 7/12 of the circle), we obtain a 12-pointed star. Our scale is then a selection of 7 out of the 12 points on the star, joined by the edges in a chain of fifths.

Thereby we will see that some points of the scale are adjacent (at distance of 1 unit) and others leave a gap of distance 2 (skipping over points which are not in the scale). By their arithmetic properties, it is impossible to distribute 7 out of 12 points evenly on the dodecagon, but the notes of our scale are distributed as evenly as possible. We find that there are always five major steps (2 units, also called tone) and two minor steps (1 unit, called semitone). The minor steps are separated by groups of two and

[3]We cannot know for sure whether the scale was historically invented on the basis of this property, because it is logically equivalent to other musically relevant properties.

[4]In order to experiment with other generating intervals and scales with less or more notes, see the 'Scale theory' section in the Scale Lab exhibit.

[5]An octave is acoustically associated to the interval between a frequency f and its double 2f. Normalizing the size of the octave to the log-frequency 1 = log2(2), the just tuned fifth of frequency ratio 3:2 has size log2(3/2)=0.58496 which is quite close to 7/12=0.58333, within a 0.16% of difference.

three major steps, respectively. This scale is called the diatonic scale. If we account for the beginning and ending of the scale we end up with one of the following patterns (called modes), that receive Greek names:

| | |
|---|---|
| 2 , 2 , 1 , 2 , 2 , 2 , 1 | Ionian (major) |
| 2 , 1 , 2 , 2 , 2 , 1 , 2 | Dorian |
| 1 , 2 , 2 , 2 , 1 , 2 , 2 | Phrygian |
| 2 , 2 , 2 , 1 , 2 , 2 , 1 | Lydian |
| 2 , 2 , 1 , 2 , 2 , 1 , 2 | Mixolydian |
| 2 , 1 , 2 , 2 , 1 , 2 , 2 | Aeolian (minor) |
| 1 , 2 , 2 , 1 , 2 , 2 , 2 | Locrian |

Note that all the modes are in cyclic permutation of each other, that is, taking each time the first item and placing it on the end. With this model, to create any concrete mode you only need to choose one of the 12 points (tonic note), and one of the 7 step patterns that you want to use. This gives you $12 \cdot 7 = 84$ possibilities. It is important to observe that the mode is not just the set of 7 notes, it is also their order and starting point that matters.

The Note Compass allows you to visualize the 84 different possible diatonic modes. The instrument consists of two concentric disks. The foreground rotating 12-pointed star represents the 12 notes in an octave. The background plate is divided into sectors that allow to select the notes. Notes on the star that lie on a colored sector are part of the mode, notes that lie on a gray sector are not. The multicolor keyboard of the tone compass app has only the keys for the notes in the scale, and not the others.

How do these arithmetic and combinatorial properties relate to music? First of all, in order the play the notes you need to assign a pitch (or pitch class) to each of the 12 vertices of the star. The precise pitch is a tuning or intonation issue that we leave aside here, but we just remark that pitch is just one attribute of the performance of a note.

Secondly, the notes of a mode behave as a family, and interact with one another and provide a unique character to each family member. The first note in a mode is called tonic, and it serves as an anchor for the entire family. When a musical piece is said to be in C-major, then it is the note C which serves as the tonic. In a piece in C-minor the same note serves as the tonic, but it has a different character. And of course, the same note C can serve in different roles and can have different characters in other modes. Thus, its order position inside the mode and the relation to the major and minor steps are also other attributes of a note that refer to its function inside the logic of the composition.

In view of these attributes, there are three ways to refer to a note:

1. Latin note names A, B, C, D, E, F, G (possibly with sharps or flats attached to them) indicate the pitch heights to be played. It is fixed as soon as your instrument is tuned. For instance, in modern equal temperament tuning, the middle A is 440 Hz. Note names appear in the twelve needles of the turnable star-shaped dodecagon.
2. Arabic numerals 1, 2, 3, 4, 5, 6, 7, (8 = 1) designate their position (scale degree) in the scale in ascending order. Scale degree 1 refers to the tonic note musically

governing the collection of the seven scale degrees. Each numeral corresponds to one of the "rainbow"-colored segments in the outer disk of the tone compass (1 = red, 2 = orange, 3 = yellow, 4 = light green, 5 = dark green, 6 = dark blue, 7 = violet). In any configuration of the compass there are precisely seven of the 12 note-name-needles pointing into these seven colored segments and thereby to the seven scale degrees 1, 2, 3, 4, 5, 6, 7.

3. Syllables do, re, mi, fa, so, la, ti designate the different characters of the seven notes, arising from the specific locations with respect to the two minor steps in the step interval pattern. The minor steps (1 semitone) are always located between mi-fa and between ti-do. One may think of a mode as a little society of notes, where a unique character may be attributed to each note. On the Note compass, the seven syllables appear in the little subdivisions of each colored segment in the (counter-clockwise) order fa - do - so - re - la - mi - ti. In any configuration of the compass the seven active needles (pointing into colored segments) also point precisely to one of the seven syllables each.

Note that the latin note names are

C, C♯/D♭, D, D♯/E♭, E, F, F♯/G♭, G, G♯/A♭, A, A♯/B♭, B.

The notation is essentially based on the diatonic scale, and it is anchored in one particular family of modes whose notes are named by the single letters A, B, C, D, E, F, G. They correspond to the white keys of the piano. All other notes have accidentals ♯ (sharps) or ♭ (flats) attached to their names. A♯ is meant to be higher than A, B♭ is meant to be lower than B. The amount of alteration is the difference between the major and the minor step, and in the twelve-note system, A♯ and B♭ are identified and located on the same needle of the star.

The seven notes of any diatonic mode are graphically represented on seven successive height degrees of the musical staff. Their note names always run through all the seven latin names A, B, C, D, E, F, G (possibly supplemented with either sharps ♯ or flats ♭ so that we don't have pairs like A and A♯ in the same mode). The locations of the major and minor steps are graphically not explicitly shown, musicians know them implicitly from the clefs.

This is a tribute to historical tradition, to the fact that all modes are instances and, at the same time, alterations of one single scale (that today we call C-Major).

Let us see an example of the Note Compass working. Start with the most usual scale: C-Major (or C-Ionian). Point the needle with the C pitch note to the red sector pointing to syllabe do. We have this familiar concordance:

C -> (1, do), D -> (2, re), E -> (3, mi), F -> (4, fa) , G -> (5,so), A -> (6, la), B -> (7, ti)

Observe that the active needles are (from the 1st to 7th sectors) arranged in steps 2,2,1,2,2,2,1 (Ionian mode). The minor step (1 semitone) is when two consecutive needles are active.

Now move the star one elementary rotation in the counter-clockwise direction. The B-needle pointing to (7, ti) leaves the violet-colored segment and points into the gray segment between the violet and the red one. It is no longer an active note of the chosen mode. Instead the B♭-needle enters from a previously gray position between the dark blue and violet segments into the violet segment and points at the syllable fa. The minor step (ti-do) between B and C is transformed into a major step (fa-so) between B♭ and C and the major step (la-ti) between A and B is transformed into a minor step (mi-fa) between A and B♭. The resulting step interval pattern is 2,2,1,2,2,1,2 (Mixolydian mode).

It is crucial that with each elementary rotation only one needle gets de-selected and one (adjacent to it) gets selected, thus only one note is altered in the scale. This fact comes from the particular irregularity of the step pattern of the diatonic scale. Mathematically, it is a cyclic permutation of the step pattern that can also be obtained by switching adjacent steps (a transposition). Here is an illustration of this peculiarity: If we rotate the name of our exhibition lalalab four letters to the left, we obtain lablala. If we exchange its last two letters, we obtain lalalba, which obviously is different from lablala. Applying, however, these two transformations to the Ionian step interval pattern 2212221, we obtain 2212212 in both cases.

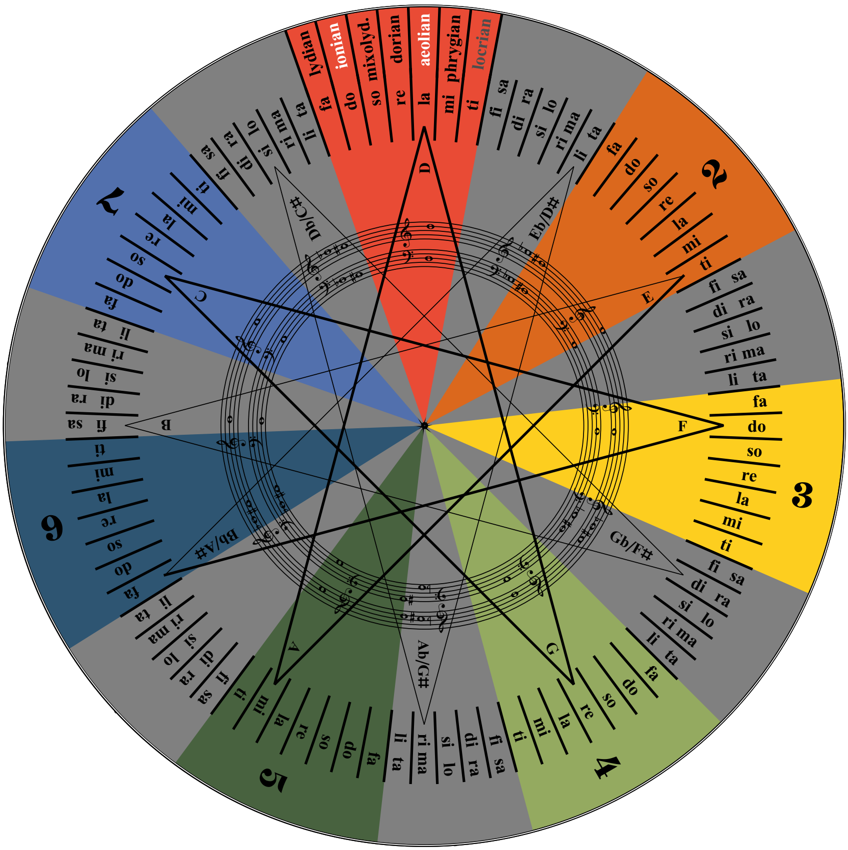
This property is reflected in how musicians use the key signature in the staff to notate the mode, namely by adding just one flat or sharp at a time.

Finally, a few remarks on the nomenclature in different European traditions, that can be a source of confusion: Firstly, in German-speaking countries, the notes B and Bb are called H and B, respectively. This comes etymologically from the b quadratum and the b rotundum and it has stuck into tradition. Secondly, in most Romance and Slavic languages, the syllables do, re, mi fa, sol, la, si have been hijacked to replace the roman letters C, D, E, F, G, A, B in their role of note names. This use can be traced back to around 1600 in France, although the six syllables ut, re, mi, fa, sol, la appear much earlier, around 1000 AD with Guido of Arezzo.

Despite of differences in naming and teaching traditions musicians are aware of the importance of all three specifications of a note (pitch height, scale degree, modal character). The tone compass conveys the nature of their interdependence. For the music lover it might be enlightening to understand how music notation and practice subtly encapsulate tradition, perception and arithmetic insight.

**Authors of the exhibit:** Thomas Noll (conception) and Daniel Ramos for IMAGINARY (implementation).

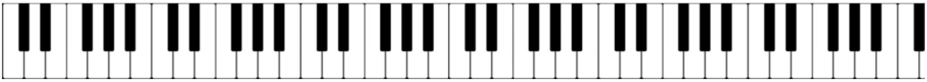
**Text:** Thomas Noll and Daniel Ramos.




Note Compass displaying D-Aeolian. The thicker lines on the star mark the chain of fifth intervals in the scale.

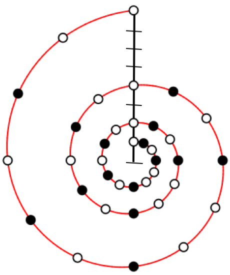
# SHOW ME MUSIC

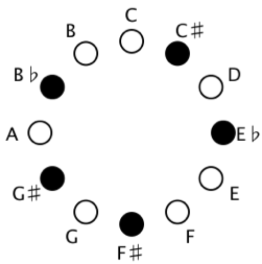
Dive into a multitude of topics that visualize the complex interrelations of melody, harmony and mathematics. Each of the animations looks at a certain musical piece or pattern from a special mathematical viewpoint. Aspects of symmetry, both in time and space help to understand musical ideas.



## SPIRAL TONE SPACE

The keys on a piano form a pretty regular pattern. Within every group of seven white keys the pattern of black and white keys repeats exactly. If you play the piano and shift your fingers up to the eighth white note on the right the music will sound almost exactly the same... just performed one octave higher. A nice way to express this fact mathematically is to arrange the tones in a spiral such that each full turn corresponds exactly to one octave.





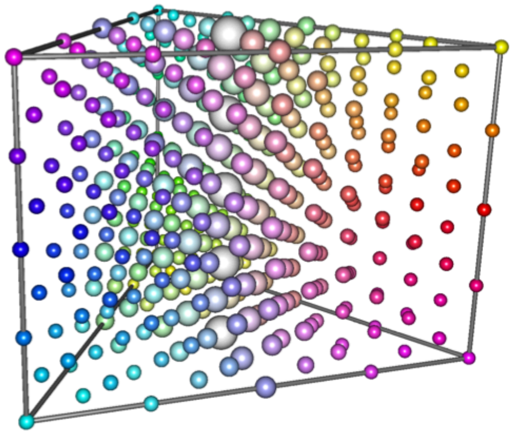

**The Music:** The concept of an octave is close to omnipresent in music. Even throughout different cultures. Playing a tune one octave higher means essentially doubling all frequencies, and by this it is closely related to the overtone series. A piano usually covers a bit more than 7 octaves. The human singing voice covers usually around two octaves.

**The Maths:** The picture above illustrates a sequence of three octaves mapped to a logarithmic spiral. Low tones are closer to the center of the spiral. It starts with a C and spirals clockwise to higher and higher notes. In order to keep track, the black and white pattern of the piano keys is used. The scaling of the spiral is chosen in a way that one full term corresponds to an octave and that the distance of any point to the center is proportional to its frequency. Thus along the vertical black ray you see four different "C" notes. Each of the twelve rays corresponds to one of the tones in our twelve-tone-system. Depending on the context it is often also reasonable not to distinguish between the same notes in different octaves, this leads to so called circular tone space.

**The Exhibit:** The exhibit visualizes tones of well known tunes in the spiral tone space. Scaling parameters of the spiral are chosen slightly differently to squeeze in more octaves. Predominant notes (in circular tone space) are automatically detected and the corresponding tone class is highlighted. By this you can follow the dominating notes in a tune. Enjoy hearing and seeing music. Even watching well known tunes this way will reveal several hidden patterns and make the composition more transparent.

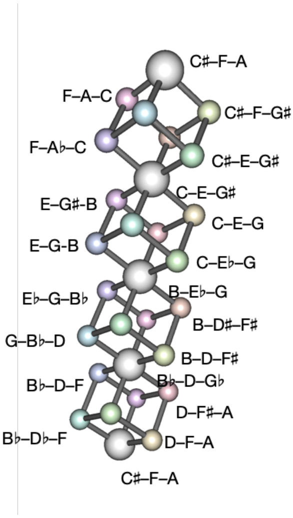
## SPACE OF THREE-NOTE CHORDS

Sometimes surprising structures emerge if one studies objects from a standpoint that does not only look at one object but at many of them and their mutual relations. For instance, considering all possibilities to play a chord consisting of three notes is a surprisingly rich geometric object. For this let us ignore the octave information (and therefore for instance treat all C notes equally). The picture shows that space. Each point corresponds to exactly one possible chord. It turns out that this space forms a triangular prism filled with all together 364 possible chords. Each such point corresponds to a selection of three notes. The three vertical edges correspond to situations where all three notes are identical (maximally close together notes). The central axis of white dots corresponds to chords such as C-E-G♯ (stacked thirds, with maximally spread notes).



Chords that are close to each other in that space differ by a halftone step. Making small changes from one chord to another (by halftone step movements) means wandering around in that structure. Fixing two notes of a chord and changing the other in halftone steps results in a linear path in the inside of the prism. The quadrangular sides of the prism structure act like walls or mirrors at which the path bounces of. The top and the bottom of the prism have to be considered identified with a 120°twist.

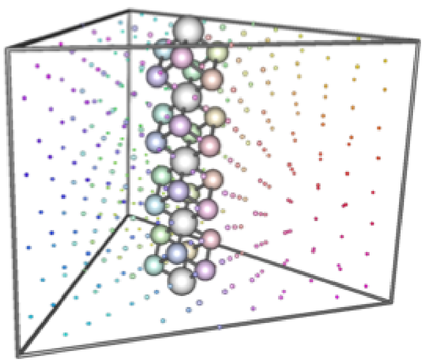
Hence leaving the structure on the top wormholes you and brings you back to the bottom of the figure.

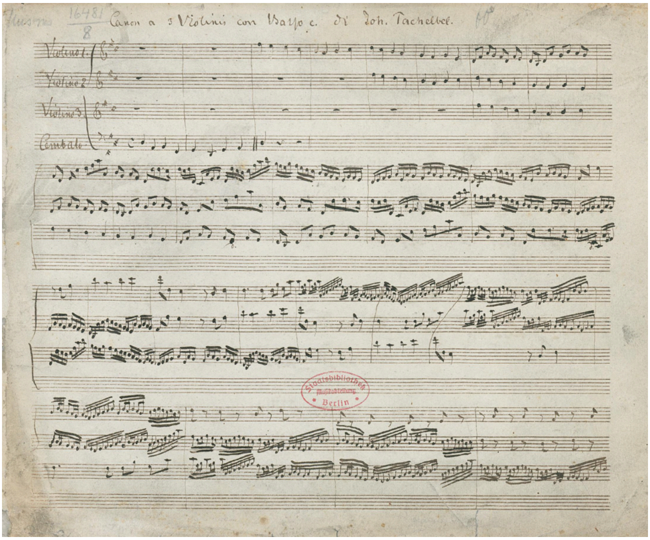



**The Music:** This 3D structure encodes a lot of classical music theory. For instance at a distance of 1 to the central axis you can find all major and minor chords. In the picture below several chords are labeled. Classical chord progressions typically correspond to nice walks through this structure.

**The Math** : This image has many deep relations to symmetry theory. The fact that the quadrangular faces act like mirrors and the top and bottom are identified makes this prism to be a fundamental cell of a 3-dimensional symmetric object in which infinitely many of such prisms fill the entire space. Just like a three dimensional symmetric wallpaper pattern. Considering the space of chords in such a general way is a relatively new field of research mainly driven by Princeton professor Dmitri Tymoczko.

**The Exhibit:** The exhibit allows you to navigate in the space of such chords. The big red ball indicates the current position. A random arpeggio with the selected chord is played. Move the tones, listen to the chords and create your own musical progressions!






## PACHELBELS CANON IN D AND PACHELBALLS

Pachelbel's Canon in D is perhaps one of the most well known pieces of classical music. Despite the fact that it had already been composed around 1690 (most probably before Bach's works) it sounds amazingly fresh and modern. It follows a very simple but very impressive chord scheme that even today forms the basis of many well known Pop, Folk, Country, and Jazz Pieces as diverse as April Lavigne's Skater Boy, Bob Marley's No Woman No Cry or The Beatle's Let it be.

**The Math:** A canon is repetition and repetition is symmetry. In a certain sense a canon is symmetry in time. The same pattern repeats after a certain amount of time has elapsed. As a matter of fact, the Canon of Pachelbel is slightly different from many other canons since the musical pattern of each voice does not repeat after a while.

**The Music:** The fact that the single voices do not repeat allows Pachelbel to create a progression of density in the canon (unlike many other canons). Pachelbel composes with many tricky twists both rhythmically and melodically. Patterns that form the leading voice at one moment become the accompanying voice a few moments later.

**The Exhibit:** Two of our exhibits are based on Pachelbel's Canon. One of them visualizes the entire piece and emphasizes the fact that all three voices play exactly the same score with a shift in time, the other exhibit (Pachelballs) is a game. Based on Pachelbel's chord pattern you can create soothing physics driven chimes.

## MOZART’S MUSICAL DICE GAME

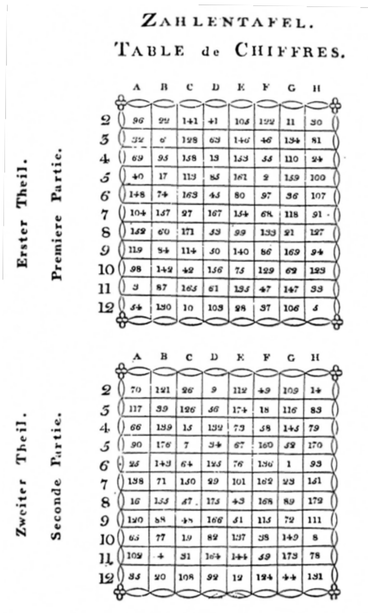
Some years after Mozarts death a musical dice game was published that is attributed to Mozart and most probably actually is his invention. In modern terms it might be called a randomized music generator. The game allows for the creation of a well sounding minuet that (with close to certain probability) was never played before. For this, Mozart presents a score with numbered measures. Along with the score comes a table that for each measure in the final waltz has a column of possible measures that can be used as a fill-in. Which fill-in is actually used for each measure is determined by throwing two dice, summing up their values and selecting the corresponding row for each measure. Amazingly, every waltz created in this way has a very authentic sounding Mozart feel to it.

**The Music:** How did he do it? At a first glance, many people are amazed by well sounding music generated by a simple random procedure. At a second glance, this is not very surprising. The main musical content on a conceptual level is determined more by the underlying harmony progression than by the concrete melodic lines of a tune. For a given harmony there are many melodic ways to express the same idea. What Mozart does is to simply offer 11 “equivalent” choices for each measure. It is like replacing the sentence “Wow, the bunny looks nice” by “Yeah, the rabbit is beautiful”. This is the reason why each of the minuets created in this game sounds more or less the same.

**The Math:** The first and most obvious question is how many possible pieces of music, can there be created from Mozart’s Dice Game? The waltz consists of two parts. Each of them consisting of eight bars. For each bar there are 11 choices which, with a little elementary combinatorics, makes an astronomic number of $11^{16} = 45949729863572161$ possibilities. Listening to all of them would require around 60 billion years. Trust me, you would be pretty bored by then.

There is also a slightly subtler mathematical point to the game. The distribution of chosen measures will not be an equal distribution. Since for each bar the chosen possibilities are determined by the sum of two dice there is a bias toward the numbers in the middle rows. There are all in all $6 \times 6 = 36$ possible outcomes of throwing the dice. Only one of them, 1+1 = 2, creates the number two, but six of them create the number 7 = 1+6 = 2+5 = 3+4 = 4+3 = 5+2 = 6+1.

**The Exhibit:** You want your very individual never heard before Mozart piece. Just press the play button and enjoy.



ZAHLENTAFEL.

TABLE de CHIFFRES.

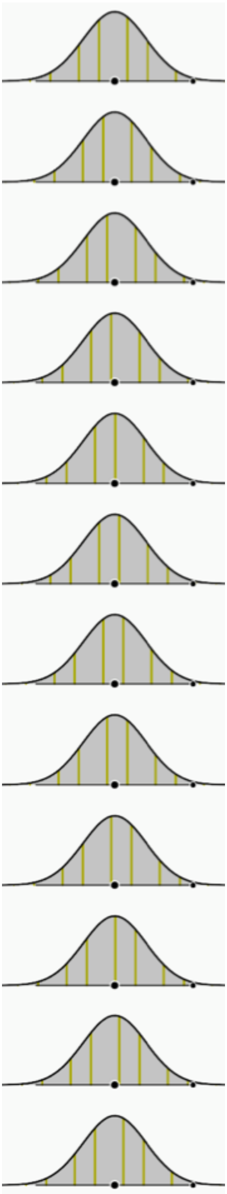
Erster Theil. Premiere Partie.

| | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| 2 | 96 | 22 | 141 | 41 | 105 | 122 | 11 | 30 |
| 3 | 32 | 6 | 128 | 63 | 146 | 46 | 134 | 81 |
| 4 | 69 | 95 | 158 | 13 | 153 | 55 | 110 | 24 |
| 5 | 40 | 17 | 113 | 85 | 161 | 2 | 159 | 100 |
| 6 | 148 | 74 | 163 | 45 | 80 | 97 | 36 | 107 |
| 7 | 104 | 157 | 27 | 167 | 154 | 68 | 118 | 91 |
| 8 | 152 | 60 | 171 | 53 | 99 | 133 | 21 | 127 |
| 9 | 119 | 84 | 114 | 50 | 140 | 86 | 169 | 94 |
| 10 | 98 | 142 | 42 | 156 | 75 | 129 | 62 | 123 |
| 11 | 3 | 87 | 165 | 61 | 135 | 47 | 147 | 33 |
| 12 | 54 | 130 | 10 | 103 | 28 | 37 | 106 | 5 |

Zweiter Theil. Seconde Partie.

| | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| 2 | 70 | 121 | 26 | 9 | 112 | 49 | 109 | 14 |
| 3 | 117 | 39 | 126 | 56 | 174 | 18 | 116 | 83 |
| 4 | 66 | 139 | 15 | 132 | 73 | 58 | 145 | 79 |
| 5 | 90 | 176 | 7 | 34 | 67 | 160 | 52 | 170 |
| 6 | 25 | 143 | 64 | 125 | 76 | 136 | 1 | 93 |
| 7 | 138 | 71 | 150 | 29 | 101 | 162 | 23 | 151 |
| 8 | 16 | 155 | 57 | 175 | 43 | 168 | 89 | 172 |
| 9 | 120 | 88 | 48 | 166 | 51 | 115 | 72 | 111 |
| 10 | 65 | 77 | 19 | 82 | 137 | 38 | 149 | 8 |
| 11 | 102 | 4 | 31 | 164 | 144 | 59 | 173 | 78 |
| 12 | 35 | 20 | 108 | 92 | 12 | 124 | 44 | 131 |

## SHEPARD TONE



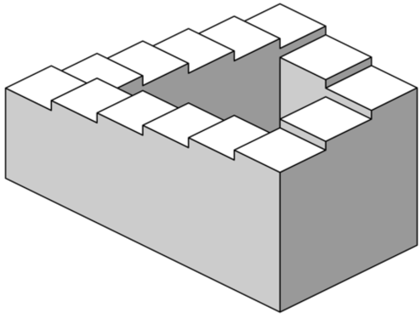
Perception is a strange thing. Sometimes we hear or see things that are not really there. There is a phenomenal acoustic illusion that creates the effect of an ever rising or ever falling tone. Similar in concept to one of these Escher staircases that seem to ascend in every step, but actually just go in circles.

The illusion is generated in the following way: One defines a frequency window; ideally with smooth boundaries. This can be done by for example using a Gaussian bell curve. Within this window all overtones of a certain spectrum belonging to a tone are played. The amplitude of the different partial tones is determined by the window. Now the spectrum is shifted within the window (say in the direction of getting deeper). While deep overtones are faded out gradually new high notes enter the window. On average, the frequency remains the same. But each step sounds like a descending tone. This edict is called a Shepard tone.

**The Music:** It is extremely difficult to play a convincing Shepard tone effect on a classical non-electronic instrument. It requires incredible control of the volume and intensity of the notes played. Nevertheless, it was used by modern composers in several musical pieces: Ligety's Piano etudes, or in the tension rich soundtrack by Hans Zimmer to the film Dunkirk, the end of Pink Floyd's Echoes or even in the soundtrack of the Super Mario Game when the protagonist is running an endless stair.

**The Math:** A Creating Shepard tones requires a little fine tuning. Tones must enter and leave the spectrum almost without getting noticed. A good way to model this is to use a bell shaped Gaussian distribution that controls the intensity of the tones played. With this, even a very narrow range of frequencies produces a good effect. However, the broader and more overtone-rich the chosen spectrum is, the more convincing the effect. The picture shows the sonogram of a Shepard tone.

**The Exhibit:** With our exhibit you can tune various parameters of the Shepard tone by simply changing the shape and position of the Gaussian curve involved. Explore how much freedom there is when creating this acoustic illusion.



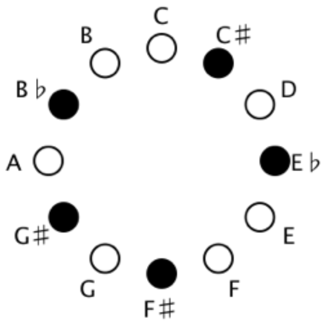
**Author of this exhibit:** Jürgen Richter-Gebert (Technical University of Munich)

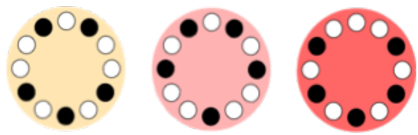
**Acknowledgements:** Patrick Wilson and Aaron Montag (Sound Engine). Based on CindyJS.org

**Text:** Jürgen Richter-Gebert (TU Munich)

# THE SPACE OF PENTATONIC SCALES

Our usual tonal system consists of 12 halftone steps C-C♯-D-E♭... Each tone occurs exactly once in every octave. A nice way of visualizing this collection of tones is to arrange them in a circle and neglect the octave information. The image on the left shows the twelve notes with the usual distribution of black and white keys of a piano. The white circles form the material of the well-known C-major scale. But just playing the black notes also sounds interesting. They form a so called pentatonic scale: a collection of five notes within an octave. Forming melodies with them sounds a bit like modal music from the Far East. Rotating the pattern cyclically creates 12 similar sounding pentatonic scales.

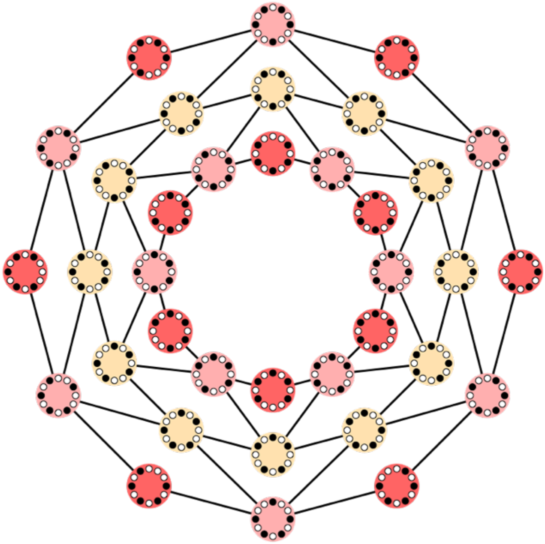


There are plenty of other ways to select exactly 5 notes out of the 12 notes in our tonal system. However, if we want to restrict ourselves to more or less equally distributed selections we can in addition require that no two "neighboring" notes are selected. Under these conditions there are, depending on rotation, exactly two more such arrangements of five tones. Each of them can occur in 12 different rotations (see picture).

**The Music:** The pentatonic scales above have a kind of free floating sound that has no specific appeal of major or minor. Some of them sound very peaceful (the yellow one), some of them are full of tension. These scales were extensively used by the composers of the impressionistic epoch such as Debussy. They are used to create very colorful sounds invoking vivid pictures of other cultures. Debussy, for example became interested in pentatonic scales after hearing Gamelan Music from Java and Bali.

**The Math:** The picture shows the space of all the above-mentioned halftone free pentatonic scales. Each of the three types occurs in 12 rotational positions. In the picture the scales are arranged in such a way that two of them are connected by an arc if they differ exactly by one halftone step. You see the "space of pentatonic scales".



**The Exhibit:** In the Exhibit, a piece by a classical composer (Mozart, Bach, Debussy, Liszt, and others) can be chosen for playing. But the piece is not played according to notation. You choose a pentatonic scale and each note of the piece is replaced by the nearest tone on the scale. By tapping the rosettes, you can walk around and move from one pentatonic scale to another. Through this, you can create your own pieces of impressionistic music.

**Author of this exhibit:** Jürgen Richter-Gebert (Technical University of Munich)

**Acknowledgements:** Patrick Wilson and Aaron Montag (Sound Engine). Based on CindyJS.org

**Text:** Jürgen Richter-Gebert (TU Munich)

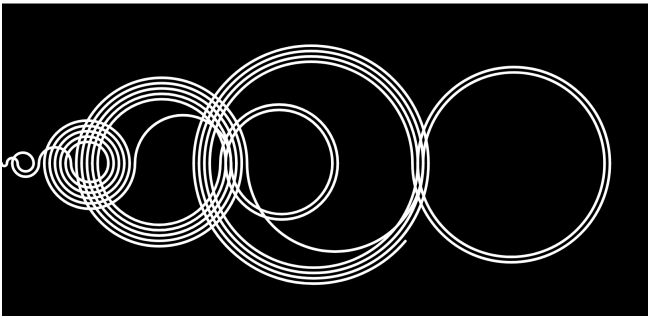
**Music:** Bach. Prelude and Fugue in C major BWV 846 / Mozart, Sonata No. 16 C major (Sonata facile), KV 545 / Liszt, Hungarian Rhapsody No 9 / Chopin, Prelude No 4 in e minor. Recorded by Bernd Krueger, Germany. Gershwin. Rhapsody in Blue / Debussy, Arabesque 1 / Debussy, Images – Reflets dans l'eau. Performed by Katsuhiro Oguri, Japan.

# THE SOUND OF SEQUENCES

The On-line Encyclopedia of Integer Sequences (or OEIS) is an online database of number sequences taken from all branches of scientific investigation. It contains classical sequences such as the list of prime numbers or the sequence of Fibonacci numbers; or less known sequences taken from the solutions to mathematics problems, such as the "number of planar graphs with n vertices". The database was started in 1964 by mathematician Neil J. A. Sloane as a tool for identifying and understanding sequences. Today it includes more than 300 000 sequences.

Sequences can be analyzed mathematically, visualized as function graphs, or, as in this case, mapped onto sounds that can be heard. Besides revealing features of the sequence that are not obvious from just looking at the numbers, the resulting pieces of "music" are interesting in their own right, as we hope you agree from listening to the examples in this exhibit.

To convert a sequence of numbers to music – to "sonify" it – we use a simple algorithm. We map the sequence to the notes on a grand piano. This piano has 88 keys, and we number them 0, 1, ..., 87. We take the terms of the sequence, and add or subtract multiples of 88 until we get a number in the range 0 to 87. In technical words, we read the sequence "modulo 88".



**Authors of the exhibit:** Neil J. A. Sloane (data) and Eric Londaits for IMAGINARY (user interface).

**Text:** Daniel Ramos (IMAGINARY).

**Reference:** The On-line Encyclopedia of Integer Sequences. `www.oeis.org`.

## Exhibits page index:


| | |
|---|---|
| The Spectrum of Sound | 5 |
| Scale Lab | 8 |
| Tonnetz | 20 |
| Beat Box | 23 |
| Graph Composer | 31 |
| AI Jam | 33 |
| Con Espressione! | 34 |
| NSynth | 36 |
| Lissajous-based exhibits | 38 |
| Lissajous gallery | 39 |
| The Harmonic Series | 40 |
| Pink Trombone | 42 |
| Mind and Music Jukebox | 44 |
| Whitney Music Box | 45 |
| Musical Bench | 47 |
| Hands-on Table | 48 |
| La La Cinema | 49 |
| Silent area | 54 |
| Note compass | 55 |
| Show me music | 60 |
| The Space of Pentatonic Scales | 67 |
| The Sound of sequences | 69 |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