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高等计算流体力学课程家庭作业题 (A2)

问题 1 把有量纲二维定常抛物化 N-S 方程组:

$$u\frac{\partial\rho}{\partial x}+v\frac{\partial\rho}{\partial y}+\rho\Big(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\Big)=0 \tag{1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+\frac{1}{\rho}\frac{\partial p}{\partial x}=\frac{\mu}{\rho}\frac{\partial^2 u}{\partial y^2} \tag{2}$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+\frac{1}{\rho}\frac{\partial p}{\partial y}=\frac{4}{3}\frac{\mu}{\rho}\frac{\partial^2 v}{\partial y^2} \tag{3}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+(\gamma-1)T\Big(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\Big)=\frac{\gamma-1}{\rho R}N+\frac{\gamma-1}{\rho R}\frac{\partial}{\partial y}\Big(k\frac{\partial T}{\partial y}\Big) \tag{4}$$

转化成无量纲形式. 其中 $N=\mu[(\frac{\partial u}{\partial y})^2+\frac{4}{3}(\frac{\partial v}{\partial y})^2]$.

引入参考量: 流场中物体特征长度 L_*, 自由来流在 x 方向的速度 u_*, 自由来流的密度 ρ_* 和温度 T_*, 自由来流等熵声速 a_*, 标准状态下重力加速度 g_*. 由此可得定义如下各量纲为一的流动量:

$$\overline{x}=\frac{x}{L_*},\ \ \overline{y}=\frac{y}{L_*},\ \ \overline{u}=\frac{u}{u_*},\ \ \overline{v}=\frac{v}{u_*},\ \ \overline{\rho}=\frac{\rho}{\rho_*},\ \ \overline{p}=\frac{p}{\rho_* u_*^2},\ \ \overline{\mu}=\frac{\mu}{\mu_*},\ \ \overline{T}=\frac{T}{T_*}$$

因此有

$$x=\overline{x}L_*,\ y=\overline{y}L_*,\ u=\overline{u}u_*,\ v=\overline{v}u_*,\ \rho=\overline{\rho}\rho_*,\ p=\overline{p}\rho_*u_*^2,\ \mu=\overline{\mu}\mu_*,\ T=\overline{T}T_* \tag{5}$$

下面分别对题中四式无量纲化:

- 将式 (5) 中的各式代入 (1) 得

$$\overline{u}u_*\frac{\partial\overline{\rho}\rho_*}{\partial\overline{x}L_*}+\overline{v}u_*\frac{\partial\overline{\rho}\rho_*}{\partial\overline{y}L_*}+\overline{\rho}\rho_*\Big(\frac{\partial\overline{u}u_*}{\partial\overline{x}L_*}+\frac{\partial\overline{v}u_*}{\partial\overline{y}L_*}\Big)=0$$

  整理得

$$\frac{u_*\rho_*}{L_*}\Big[\overline{u}\frac{\partial\overline{\rho}}{\partial\overline{x}}+\overline{v}\frac{\partial\overline{\rho}}{\partial\overline{y}}+\overline{\rho}\Big(\frac{\partial\overline{u}}{\partial\overline{x}}+\frac{\partial\overline{v}}{\partial\overline{y}}\Big)\Big]=0$$

  由于 $u_*\rho_*/L_*\neq 0$, 因此可得到 (1) 式的无量纲形式

$$\overline{u}\frac{\partial\overline{\rho}}{\partial\overline{x}}+\overline{v}\frac{\partial\overline{\rho}}{\partial\overline{y}}+\overline{\rho}\Big(\frac{\partial\overline{u}}{\partial\overline{x}}+\frac{\partial\overline{v}}{\partial\overline{y}}\Big)=0$$

- 将式 (5) 中的各式代入 (2) 得

$$\bar{u}u_*\frac{\partial\bar{u}u_*}{\partial\bar{x}L_*}+\bar{v}u_*\frac{\partial\bar{v}u_*}{\partial\bar{y}L_*}+\frac{1}{\bar{\rho}\rho_*}\frac{\partial\bar{p}\rho_*u_*^2}{\partial\bar{x}L_*}=\frac{\bar{\mu}\mu_*}{\bar{\rho}\rho_*}\frac{\partial^2\bar{u}u_*}{\partial(\bar{y}L_*)^2}$$

整理得

$$\frac{u_*^2}{L_*}\left[\bar{u}\frac{\partial\bar{u}}{\partial\bar{x}}+\bar{v}\frac{\partial\bar{v}}{\partial\bar{y}}+\frac{1}{\bar{\rho}}\frac{\partial\bar{p}}{\partial\bar{x}}\right]=\frac{\bar{\mu}\mu_*u_*}{\bar{\rho}\rho_*L_*^2}\frac{\partial^2\bar{u}}{\partial\bar{y}^2}\Rightarrow\bar{u}\frac{\partial\bar{u}}{\partial\bar{x}}+\bar{v}\frac{\partial\bar{v}}{\partial\bar{y}}+\frac{1}{\bar{\rho}}\frac{\partial\bar{p}}{\partial\bar{x}}=\frac{\bar{\mu}\mu_*}{\bar{\rho}\rho_*u_*L_*}\frac{\partial^2\bar{u}}{\partial\bar{y}^2}$$

其中 $\frac{\mu_*}{\rho_*u_*L_*}=\frac{1}{Re}$, Re 为雷诺数. 代入上式便可得到 (2) 式的无量纲形式

$$\bar{u}\frac{\partial\bar{u}}{\partial\bar{x}}+\bar{v}\frac{\partial\bar{v}}{\partial\bar{y}}+\frac{1}{\bar{\rho}}\frac{\partial\bar{p}}{\partial\bar{x}}=\frac{\bar{\mu}}{\bar{\rho}\mathrm{Re}}\frac{\partial^2\bar{u}}{\partial\bar{y}^2}$$

- 与 (2) 式的无量纲过程类似, 可得到 (3) 式的无量纲形式

$$\bar{u}\frac{\partial\bar{v}}{\partial\bar{x}}+\bar{v}\frac{\partial\bar{v}}{\partial\bar{y}}+\frac{1}{\bar{\rho}}\frac{\partial\bar{p}}{\partial\bar{y}}=\frac{4\bar{\mu}}{3\bar{\rho}\mathrm{Re}}\frac{\partial^2\bar{v}}{\partial\bar{y}^2}$$

- 将式 (5) 中的各式代入 (4) 中并整理得

$$\frac{u_*T_*}{L_*}\left[\bar{u}\frac{\partial\bar{T}}{\partial\bar{x}}+\bar{v}\frac{\partial\bar{T}}{\partial\bar{y}}+(\gamma-1)\bar{T}\left(\frac{\partial\bar{u}}{\partial\bar{x}}+\frac{\partial\bar{v}}{\partial\bar{y}}\right)\right]=\frac{\mu_*u_*^2}{\rho_*L_*^2}\frac{\gamma-1}{\bar{\rho}R}\bar{N}+\frac{k_*T_*}{\rho_*L_*^2}\frac{\gamma-1}{\bar{\rho}R}\frac{\partial}{\partial\bar{y}}\left(\bar{k}\frac{\partial\bar{T}}{\partial\bar{y}}\right)$$

其中 $\bar{N}=\bar{\mu}[(\frac{\partial\bar{u}}{\partial\bar{x}})^2+\frac{4}{3}(\frac{\partial\bar{v}}{\partial\bar{y}})^2]$. 化简上式得

$$\bar{u}\frac{\partial\bar{T}}{\partial\bar{x}}+\bar{v}\frac{\partial\bar{T}}{\partial\bar{y}}+(\gamma-1)\bar{T}\left(\frac{\partial\bar{u}}{\partial\bar{x}}+\frac{\partial\bar{v}}{\partial\bar{y}}\right)=\underbrace{\frac{\mu_*u_*}{\rho_*L_*T_*}\frac{\gamma-1}{\bar{\rho}R}}_{(a)}\bar{N}+\underbrace{\frac{k_*}{\rho_*L_*u_*}\frac{\gamma-1}{\bar{\rho}R}}_{(b)}\frac{\partial}{\partial\bar{y}}\left(\bar{k}\frac{\partial\bar{T}}{\partial\bar{y}}\right)$$

对于 (a), 注意到其中 $R=\frac{p}{\rho T}=\frac{\bar{p}}{\bar{\rho}\bar{T}}\frac{u_*^2}{T_*}$, 因此可令 $R=\bar{R}u_*^2/T_*$. 上式中的 (a) 项可化为

$$\frac{\mu_*u_*}{\rho_*L_*T_*}\frac{\gamma-1}{\bar{\rho}R}=\frac{\mu_*u_*T_*}{\rho_*L_*T_*}\frac{\gamma-1}{\bar{\rho}\bar{R}u_*^2}=\frac{\mu_*}{\rho_*L_*u_*}\frac{\gamma-1}{\bar{\rho}\bar{R}}=\frac{\gamma-1}{\mathrm{Re}\bar{\rho}\bar{R}}$$

对于 (b) 项, 注意到 $R=C_p-C_v$, $\gamma=C_p/C_v$, $\gamma-1=(C_p-C_v)/C_v$. 因此 (b) 项可化为

$$\begin{aligned}\frac{k_*}{\rho_*L_*u_*}\frac{\gamma-1}{\bar{\rho}R}&=\frac{k_*}{\rho_*L_*u_*}\frac{(C_p-C_v)/C_v}{\bar{\rho}(C_p-C_v)}=\frac{k_*}{\rho_*L_*u_*}\frac{1}{C_v\bar{\rho}}\\&=\frac{k_*}{\rho_*L_*u_*}\frac{C_p}{C_pC_v\bar{\rho}}=\frac{\mu_*}{\rho_*u_*L_*}\frac{k_*}{\mu_*C_p}\frac{\gamma}{\bar{\rho}}\\&=\frac{1}{\mathrm{Re}}\frac{1}{\mathrm{Pr}}\frac{\gamma}{\bar{\rho}}\end{aligned}$$

因此, 最终可得 (4) 式的无量纲形式

$$\bar{u}\frac{\partial\bar{T}}{\partial\bar{x}}+\bar{v}\frac{\partial\bar{T}}{\partial\bar{y}}+(\gamma-1)\bar{T}\left(\frac{\partial\bar{u}}{\partial\bar{x}}+\frac{\partial\bar{v}}{\partial\bar{y}}\right)=\frac{\gamma-1}{\bar{\rho}\ \mathrm{Re}\ \bar{R}}\bar{N}+\frac{\gamma}{\bar{\rho}\ \mathrm{Pr}\ \mathrm{Re}}\frac{\partial}{\partial\bar{y}}\left(\bar{k}\frac{\partial\bar{T}}{\partial\bar{y}}\right)$$

其中 $\bar{N}=\bar{\mu}[(\frac{\partial\bar{u}}{\partial\bar{x}})^2+\frac{4}{3}(\frac{\partial\bar{v}}{\partial\bar{y}})^2]$.

为了方便, 常常把量纲为一的量上的上画线符号略去, 最终得到二维定常抛物化 N-S 方程组的无量纲形式:

$$u\frac{\partial\rho}{\partial x}+v\frac{\partial\rho}{\partial y}+\rho\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=0$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial y}+\frac{1}{\rho}\frac{\partial p}{\partial x}=\frac{\mu}{\rho\mathrm{Re}}\frac{\partial^2 u}{\partial y^2}$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+\frac{1}{\rho}\frac{\partial p}{\partial y}=\frac{4\mu}{3\rho\mathrm{Re}}\frac{\partial^2 v}{\partial y^2}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+(\gamma-1)T\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=\frac{\gamma-1}{\rho\ \mathrm{Re}\ R}N+\frac{\gamma}{\rho\ \mathrm{Pr}\ \mathrm{Re}}\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)$$

其中 $N=\mu[(\frac{\partial u}{\partial x})^2+\frac{4}{3}(\frac{\partial v}{\partial y})^2]$. ■

问题 2 一维非定常粘性流动 N-S 方程组

$$\mathbf{u_t}+\mathbf{f_x}=\mathbf{s_x}$$

其中

$$\mathbf{u}=\begin{bmatrix}\rho\\ \rho u\\ E\end{bmatrix},\quad \mathbf{f}=\begin{bmatrix}\rho u\\ \rho u^2+p\\ u(E+p)\end{bmatrix},\quad \mathbf{s}=\begin{bmatrix}0\\ \frac{4}{3}\frac{\mu}{\mathrm{Re}}\frac{\partial u}{\partial x}\\ \frac{4}{3}\frac{\mu u}{\mathrm{Re}}\frac{\partial u}{\partial x}+\frac{c_pK}{\mathrm{Pr}\ \mathrm{Re}}\frac{\partial T}{\partial x}\end{bmatrix}$$

这里的状态方程为:

$$p=(\gamma-1)\rho e=(\gamma-1)\left(E-\frac{1}{2}\rho u^2\right)$$

求出该方程组的特征根, 并分析它的数学性质和类型.

由状态方程可得

$$E=c_vT+\frac{1}{2}\rho u^2,\ u(E+p)=u\left(c_pT+\frac{1}{2}\rho u^2\right)$$

因此上述守恒型方程组可改写成非守恒型方程组, 并消去能量和压强消项:

$$\frac{\partial\rho}{\partial t}+u\frac{\partial\rho}{\partial x}+\rho\frac{\partial u}{\partial x}=0$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\frac{(c_p-c_v)}{\rho}\frac{\partial T}{\partial x}=\frac{4}{3}\frac{\mu}{\rho\mathrm{Re}}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$$

$$\begin{aligned}c_v\frac{\partial T}{\partial t}+\rho u\frac{\partial u}{\partial t}+\frac{1}{2}u^2\frac{\partial\rho}{\partial t}+c_pu\frac{\partial T}{\partial x}+c_pT\frac{\partial u}{\partial x}+\frac{1}{2}u^3\frac{\partial\rho}{\partial x}+\frac{3}{2}\rho u^2\frac{\partial u}{\partial x}&\\ =\frac{4}{3}\frac{\mu}{\mathrm{Re}}\left(\frac{\partial u}{\partial x}\right)^2+\frac{4}{3}\frac{\mu}{\mathrm{Re}}u\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)+\frac{c_p}{\mathrm{Pr}}\frac{K}{\mathrm{Re}}\frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right)&\end{aligned}$$

对上述非守恒型方程组进行适当变换, 令

$$\frac{\partial u}{\partial x}=u_x,\quad \frac{\partial T}{\partial x}=T_x$$

可简化为下式

$$\mathbf{A}\frac{\partial \mathbf{Z}}{\partial t}+\mathbf{B}\frac{\partial \mathbf{z}}{\partial x}=\mathbf{H}$$

其中

$$\mathbf{Z}=[\rho\ u\ T\ u_x\ T_x]^{\mathrm{T}}$$

$$A=\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&c_v&0&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}\qquad B=\begin{bmatrix}u&0&0&0&0\\0&0&\frac{(c_p-c_v)}{\rho}&\frac{-4\mu}{3\rho\mathrm{Re}}&0\\0&0&0&0&\frac{-c_pK}{\mathrm{PrRe}}\\0&1&0&0&0\\0&0&1&0&0\end{bmatrix}$$

$$H=\begin{bmatrix}-\rho u_x\\-uu_x\\\frac{4}{3}\frac{\mu}{\mathrm{Re}}(u_x^2+u_{xx})-\frac{3}{2}\rho u^2u_x-\frac{1}{2}u^3\rho_x-c_p(Tu_x+uT_x)-\rho uu_t-\frac{1}{2}u^2\rho_t\\u_x\\T_x\end{bmatrix}$$

系数矩阵的特征方程为

$$\det(\sigma_1\mathbf{A}+\sigma_2\mathbf{B})=0\Rightarrow(u\sigma_2+\sigma_1)\sigma_2^4=0$$

它的特征根为:

$$\lambda_1=u,\quad \lambda_{2,3,4,5}=0$$

除实根 λ_1 外, 其它均是重根. 由此可知, 该方程组是抛物型的, 但也具有双曲型性质. 从总体上说, 它是抛物型的. ■

问题 3 对流方程 $\frac{\partial u}{\partial t}+a\frac{\partial u}{\partial x}=0$ $(a>0)$ 的前半隐差分格式 $(r=\frac{a\Delta t}{\Delta x})$ 为:

$$u_j^{n+1}=u_j^n-\frac{1}{2}r\left[\left(u_j^n-u_{j-1}^n\right)+\left(u_{j+1}^{n+1}-u_j^{n+1}\right)\right]$$

分析它的精度和稳定性.

精度分析: 将 u_j^{n+1}, u_{j-1}^n, u_{j+1}^{n+1} 分别在 $u=u_j^n$ 展开成泰勒级数

$$\begin{aligned}
u_j^{n+1}&=u_j^n+\frac{\partial u}{\partial t}\Delta t+\frac{1}{2}\frac{\partial^2 u}{\partial t^2}\Delta t^2+\frac{1}{6}\frac{\partial^3 u}{\partial t^3}\Delta t^3+\cdots\\
u_{j-1}^n&=u_j^n-\frac{\partial u}{\partial x}\Delta x+\frac{1}{2}\frac{\partial^2 u}{\partial x^2}\Delta x^2-\frac{1}{6}\frac{\partial^3 u}{\partial x^3}\Delta x^3+\cdots\\
u_{j+1}^{n+1}&=u_j^n+\left(\Delta x\frac{\partial}{\partial x}+\Delta t\frac{\partial}{\partial t}\right)u+\frac{1}{2}\left(\Delta x\frac{\partial}{\partial x}+\Delta t\frac{\partial}{\partial t}\right)^2u+\frac{1}{6}\left(\Delta x\frac{\partial}{\partial x}+\Delta t\frac{\partial}{\partial t}\right)^3u+\cdots
\end{aligned}$$

因此有

$$\frac{u_{j+1}^{n+1}-u_j^{n+1}}{\Delta x}=\frac{\partial u}{\partial x}+\frac{1}{2}\frac{\partial^2 u}{\partial x^2}\Delta x+\frac{\partial^2 u}{\partial x\partial t}\Delta t+\frac{1}{6}\frac{\partial^3 u}{\partial x^3}\Delta x^2+\frac{1}{2}\frac{\partial^3 u}{\partial x\partial t^2}\Delta t^2+\frac{1}{2}\frac{\partial^3 u}{\partial x^2\partial t}\Delta x\Delta t+\cdots$$

将以上四式代入到差分方程得

$$
\begin{aligned}
&\frac{u_j^{n+1}-u_j^n}{\Delta t}+\frac{a}{2}\left[\frac{u_j^n-u_{j-1}^n}{\Delta x}+\frac{u_{j+1}^{n+1}-u_j^{n+1}}{\Delta x}\right]\\
&=\frac{\partial u}{\partial t}+\frac{1}{2}\frac{\partial^2 u}{\partial t^2}\Delta t+\frac{1}{6}\frac{\partial^3 u}{\partial t^3}\Delta t^2+O(\Delta t^3)+\frac{a}{2}\left(\frac{\partial u}{\partial x}-\frac{1}{2}\frac{\partial^2 u}{\partial x^2}\Delta x+\frac{1}{6}\frac{\partial^3 u}{\partial x^3}\Delta x^2+O(\Delta x^3)\right)\\
&\qquad+\frac{a}{2}\left(\frac{\partial u}{\partial x}+\frac{1}{2}\frac{\partial^2 u}{\partial x^2}\Delta x+\frac{\partial^2 u}{\partial x\partial t}\Delta t+\frac{1}{6}\frac{\partial^3 u}{\partial x^3}\Delta x^2+\frac{1}{2}\frac{\partial^3 u}{\partial x\partial t^2}\Delta t^2+O(\Delta x\Delta t)\right)\\
&=\frac{\partial u}{\partial t}+\frac{1}{2}\left(\frac{\partial^2 u}{\partial t^2}+a\frac{\partial^2 u}{\partial x\partial t}\right)\Delta t+a\frac{\partial u}{\partial x}+O(\Delta x^2,\Delta t^2,\Delta x\Delta t)\\
&=\frac{\partial u}{\partial t}+a\frac{\partial u}{\partial x}+O(\Delta x^2,\Delta t^2,\Delta t\Delta x)
\end{aligned}
$$

因此该差分格式在时间和空间上都是二阶精度.

稳定性分析: 对差分方程中的各项作 Fourier 展开 $u_j^n=\sum_{k=1}^{\infty}\hat{u}_j^n e^{ijk\Delta x}$ 并代入原差分方程得到它的每个分量的误差方程:

$$
\hat{u}_j^{n+1}e^{ijk}=\hat{u}_j^n e^{ijk\Delta x}-\frac{1}{2}r\left[\left(\hat{u}_j^n e^{ijk\Delta x}-\hat{u}_{j-1}^n e^{i(j-1)k\Delta x}\right)+\left(\hat{u}_{j+1}^{n+1}e^{i(j+1)k\Delta x}-\hat{u}_j^{n+1}e^{ik\Delta x}\right)\right]
$$

放大因子为 $G=\hat{u}_j^{n+1}/\hat{u}_j^n$, 则上式可化为

$$
G=1-\frac{1}{2}r\left[\left(1-e^{-ik\Delta x}\right)+G\left(e^{ik\Delta x}-1\right)\right]\implies|G|=\left|\frac{2-r+re^{-ik\Delta x}}{2-r+re^{+ik\Delta x}}\right|=\left|\frac{\bar{A}}{A}\right|\equiv 1
$$

因此该差分格式是弱稳定的. ■

问题 4 分析对流方程的 Warming-Beam 差分格式的精度和稳定性.

$$
\begin{aligned}
\overline{u_j^{n+1}}&=u_j^n-r\left(u_j^n-u_{j-1}^n\right)\\
u_j^{n+1}&=\frac{1}{2}\left(u_j^n+\overline{u_j^{n+1}}\right)-\frac{1}{2}r\left[\left(u_j^n-2u_{j-1}^n+u_{j-2}^n\right)+\left(\overline{u_j^{n+1}}-\overline{u_{j-1}^{n+1}}\right)\right]
\end{aligned}
$$

精度分析: 将 u_j^{n+1}, u_{j-1}^n, u_{j-2}^n 分别在 $u=u_j^n$ 展开成泰勒级数

$$
u_j^{n+1}=u_j^n+\frac{\partial u}{\partial t}\Delta t+\frac{1}{2}\frac{\partial^2 u}{\partial t^2}\Delta t^2+\frac{1}{6}\frac{\partial^3 u}{\partial t^3}\Delta t^3+O(\Delta t^4) \tag{6}
$$

$$
u_{j-1}^n=u_j^n-\frac{\partial u}{\partial x}\Delta x+\frac{1}{2}\frac{\partial^2 u}{\partial x^2}\Delta x^2-\frac{1}{6}\frac{\partial^3 u}{\partial x^3}\Delta x^3+O(\Delta x^4) \tag{7}
$$

$$
u_{j-2}^n=u_j^n-\frac{\partial u}{\partial x}2\Delta x+\frac{1}{2}\frac{\partial^2 u}{\partial x^2}(2\Delta x)^2-\frac{1}{6}\frac{\partial^3 u}{\partial x^3}(2\Delta x)^3+O(\Delta x^4) \tag{8}
$$

将题中第一式代入第二式得

$$
\begin{aligned}
u_j^{n+1}&=\frac{1}{2}\left[u_j^n+u_j^n-r\left(u_j^n-u_{j-1}^n\right)\right]-\frac{1}{2}r\left[\left(u_j^n-2u_{j-1}^n+u_{j-2}^n\right)+\right.\\
&\qquad\left.+u_j^n-r\left(u_j^n-u_{j-1}^n\right)-u_{j-1}^n+r\left(u_{j-1}^n-u_{j-2}^n\right)\right]\\
&=u_j^n-r(u_j^n-u_{j-1}^n)-\frac{1}{2}r(1-r)\left(u_j^n-2u_{j-1}^n+u_{j-2}^n\right)
\end{aligned} \tag{9}
$$

其中 $r = a\frac{\Delta t}{\Delta x}$, 将 u_j^{n+1}, u_{j-1}^n, u_{j-2}^n 的泰勒展开代入上式, 并注意到$\frac{\partial^2 u}{\partial t^2} - a^2\frac{\partial^2 u}{\partial x^2} = 0$:

$$
\begin{aligned}
&\frac{u_j^{n+1}-u_j^n}{\Delta t} + a\frac{u_j^n - u_{j-1}^n}{\Delta x} + \frac{a}{2}(\Delta x - a\Delta t)\frac{u_j^n - 2u_{j-1}^n + u_{j-2}^n}{\Delta x\Delta x} \\
= &\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial^2 u}{\partial t^2}\Delta t + \frac{1}{6}\frac{\partial^3 u}{\partial t^3}\Delta t^2 + O(\Delta t^3) + a\frac{\partial u}{\partial x} - \frac{a}{2}\frac{\partial^2 u}{\partial x^2}\Delta x + \frac{a}{6}\frac{\partial^3 u}{\partial x^3}\Delta x^2 + O(\Delta x^3) + \\
&\quad + \frac{a}{2}\left(\frac{\partial^2 u}{\partial x^2}\Delta x - \frac{\partial^3 u}{\partial x^3}\Delta x^2 + O(\Delta x^3)\right) - \frac{a^2}{2}\left(\frac{\partial^2 u}{\partial x^2}\Delta t - \frac{\partial^3 u}{\partial x^3}\Delta x\Delta t + O(\Delta x^2\Delta t)\right) \\
= &\frac{\partial u}{\partial t} + \frac{1}{6}\frac{\partial^3 u}{\partial t^3}\Delta t^2 + a\frac{\partial u}{\partial x} - \frac{a}{3}\frac{\partial^3 u}{\partial x^3}\Delta x^2 + \frac{a^2}{2}\frac{\partial^3 u}{\partial x^3}\Delta x\Delta t + O(\Delta t^3, \Delta x^3, \Delta x^2\Delta t) \\
= &\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} + O(\Delta t^2, \Delta x^2, \Delta t\Delta x)
\end{aligned}
$$

因此该差分格式在时间和空间上都是二阶精度.

稳定性分析: 对差分方程中的各项作 Fourier 展开 $u_j^n = \sum_{k=1}^{\infty}\hat{u}_j^n e^{ijk\Delta x}$ 并代入式 (9) 得到它的每个分量的误差方程:

$$
\begin{aligned}
\hat{u}_j^{n+1}e^{ijk\Delta x} = \hat{u}_j^n e^{ijk\Delta x} &- r\left(\hat{u}_j^n e^{ijk\Delta x} - \hat{u}_{j-1}^n e^{i(j-1)k\Delta x}\right) \\
&- \frac{1}{2}r(1-r)\left(\hat{u}_j^n e^{ijk\Delta x} - 2\hat{u}_{j-1}^n e^{i(j-1)k\Delta x} + \hat{u}_{j-2}^n e^{i(j-2)k\Delta x}\right)
\end{aligned}
$$

放大因子为 $G = \hat{u}_j^{n+1}/\hat{u}_j^n$, 则上式可化为

$$
\begin{aligned}
G &= 1 - r(1 - e^{-ijk\Delta x}) - \frac{1}{2}r(1-r)\left(1 - 2e^{-ijk\Delta x} + e^{-2ik\Delta x}\right) \\
&= 1 - r(1 - e^{-ijk\Delta x}) - \frac{1}{2}r(1-r)\left(1 - e^{-ijk\Delta x}\right)^2 \\
&= 1 - rz - \frac{1}{2}r(1-r)z^2
\end{aligned}
$$

其中 $z = 1 - e^{-ijk\Delta x} = 1 - \cos\theta + i\sin\theta$. 注意到 $z\bar{z} = z + \bar{z} = 2(1-\cos\theta) = 2s$, 其中 $0 \le s \le 2$. 则有

$$
\begin{aligned}
|G|^2 = G\bar{G} &= \left[1 - rz - \frac{1}{2}r(1-r)z^2\right]\left[1 - r\bar{z} - \frac{1}{2}r(1-r)\bar{z}^2\right] \\
&= 1 - rz - \frac{1}{2}r(1-r)z^2 \qquad - r\bar{z} + r^2 z\bar{z} + \frac{1}{2}r^2(1-r)z^2\bar{z} \\
&\qquad\qquad - \frac{1}{2}r(1-r)\bar{z}^2 + \frac{1}{2}r^2(1-r)\bar{z}^2 z + \frac{1}{4}r^2(1-r)^2(z\bar{z})^2 \\
&= 1 - r(z+\bar{z}) - \frac{1}{2}r(1-r)(z^2+\bar{z}^2) + r^2 z\bar{z} + \frac{1}{2}r^2(1-r)z\bar{z}(z+\bar{z}) + \frac{1}{4}r^2(1-r)^2(z\bar{z})^2 \\
&= 1 - 2rs - 2r(1-r)(s^2 - s) + 2r^2 s + 2r^2(1-r)s^2 + r^2(1-r)^2 s^2 \\
&= 1 - 2r(1 - (1-r) - r)s + r(1-r)(-2 + 3r - r^2)s^2 \\
&= 1 - r(1-r)^2(2-r)s^2
\end{aligned}
$$

显然当 $0 < r = a\frac{\Delta t}{\Delta x} \le 2$ 时 $|G| \le 1$, 此时差分格式是稳定的. ∎

问题 5 分析对流方程的紧致差分格式的精度和稳定性:

$$u_j^{n+1} = u_j^n - rF_j^n$$

$$-\frac{1}{3}F_{j+1}^n + \frac{2}{3}F_j^n + \frac{2}{3}F_{j-1}^n = -\frac{1}{2}\left(u_{j+1}^n - u_j^n\right) + \frac{3}{2}\left(u_j^n - u_{j-1}^n\right)$$

精度分析: 将 u_{j+1}^{n+1}, u_{j-1}^{n+1}, u_{j+1}^n, u_j^{n+1}, u_{j-1}^n 分别在 u_j^n 处展开成泰勒级数

$$\begin{aligned}
u_{j+1}^{n+1} &= u_j^n + \left(\Delta t\frac{\partial}{\partial t} + \Delta x\frac{\partial}{\partial x}\right)u + \frac{1}{2}\left(\Delta t\frac{\partial}{\partial t} + \Delta x\frac{\partial}{\partial x}\right)^2 u + \frac{1}{6}\left(\Delta t\frac{\partial}{\partial t} + \Delta x\frac{\partial}{\partial x}\right)^3 u + \cdots \\
u_{j-1}^{n+1} &= u_j^n + \left(\Delta t\frac{\partial}{\partial t} - \Delta x\frac{\partial}{\partial x}\right)u + \frac{1}{2}\left(\Delta t\frac{\partial}{\partial t} - \Delta x\frac{\partial}{\partial x}\right)^2 u + \frac{1}{6}\left(\Delta t\frac{\partial}{\partial t} - \Delta x\frac{\partial}{\partial x}\right)^3 u + \cdots \\
u_{j+1}^n &= u_j^n + \frac{\partial u}{\partial x}\Delta x + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}\Delta x^2 + \frac{1}{6}\frac{\partial^3 u}{\partial x^3}\Delta x^3 + O(\Delta x^4) \\
u_{j-1}^n &= u_j^n - \frac{\partial u}{\partial x}\Delta x + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}\Delta x^2 - \frac{1}{6}\frac{\partial^3 u}{\partial x^3}\Delta x^3 + O(\Delta x^4) \\
u_j^{n+1} &= u_j^n + \frac{\partial u}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 u}{\partial t^2}\Delta t^2 + \frac{1}{6}\frac{\partial^3 u}{\partial t^3}\Delta t^3 + O(\Delta t^4)
\end{aligned}$$

因此有

$$\begin{aligned}
\frac{u_{j+1}^{n+1} - u_{j+1}^n}{\Delta t} &= \frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial^2 u}{\partial t^2}\Delta t + \frac{\partial^2 u}{\partial x\partial t}\Delta x + \frac{1}{6}\frac{\partial^3 u}{\partial t^3}\Delta t^2 + \frac{1}{2}\frac{\partial^3 u}{\partial t\partial x^2}\Delta x^2 + \frac{1}{2}\frac{\partial^3 u}{\partial t^2\partial x}\Delta t\Delta x + \cdots \\
\frac{u_{j-1}^{n+1} - u_{j-1}^n}{\Delta t} &= \frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial^2 u}{\partial t^2}\Delta t - \frac{\partial^2 u}{\partial x\partial t}\Delta x + \frac{1}{6}\frac{\partial^3 u}{\partial t^3}\Delta t^2 + \frac{1}{2}\frac{\partial^3 u}{\partial t\partial x^2}\Delta x^2 - \frac{1}{2}\frac{\partial^3 u}{\partial t^2\partial x}\Delta t\Delta x + \cdots \\
\frac{u_j^{n+1} - u_j^n}{\Delta t} &= \frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial^2 u}{\partial t^2}\Delta t + \frac{1}{6}\frac{\partial^3 u}{\partial t^3}\Delta t^2 + \cdots \\
\frac{u_{j+1}^n - u_j^n}{\Delta x} &= \frac{\partial u}{\partial x} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}\Delta x + \frac{1}{6}\frac{\partial^3 u}{\partial x^3}\Delta x^2 + \cdots \\
\frac{u_j^n - u_{j-1}^n}{\Delta x} &= \frac{\partial u}{\partial x} - \frac{1}{2}\frac{\partial^2 u}{\partial x^2}\Delta x + \frac{1}{6}\frac{\partial^3 u}{\partial x^3}\Delta x^2 + \cdots
\end{aligned}$$

将题中的差分格式合并成如下差分方程

$$\frac{1}{3}\frac{u_{j+1}^{n+1} - u_{j+1}^n}{\Delta t} - \frac{2}{3}\frac{u_j^{n+1} - u_j^n}{\Delta t} - \frac{2}{3}\frac{u_{j-1}^{n+1} - u_{j-1}^n}{\Delta t} = -\frac{1}{2}a\frac{u_{j+1}^n - u_j^n}{\Delta x} + \frac{3}{2}a\frac{u_j^n - u_{j-1}^n}{\Delta x}$$

上式的左边和右边分别如下 (仅保留了二阶项)

$$\begin{aligned}
\text{左边} &= +\frac{1}{3}\left[\frac{\partial u}{\partial t}+\frac{1}{2}\frac{\partial^2 u}{\partial t^2}\Delta t+\frac{\partial^2 u}{\partial x\partial t}\Delta x+\frac{1}{6}\frac{\partial^3 u}{\partial t^3}\Delta t^2+\frac{1}{2}\frac{\partial^3 u}{\partial t\partial x^2}\Delta x^2+\frac{1}{2}\frac{\partial^3 u}{\partial t^2\partial x}\Delta x\Delta t\right]\\
&\quad -\frac{2}{3}\left[\frac{\partial u}{\partial t}+\frac{1}{2}\frac{\partial^2 u}{\partial t^2}\Delta t+\frac{1}{6}\frac{\partial^3 u}{\partial t^3}\Delta t^2\right]\\
&\quad -\frac{2}{3}\left[\frac{\partial u}{\partial t}+\frac{1}{2}\frac{\partial^2 u}{\partial t^2}\Delta t-\frac{\partial^2 u}{\partial x\partial t}\Delta x+\frac{1}{6}\frac{\partial^3 u}{\partial t^3}\Delta t^2+\frac{1}{2}\frac{\partial^3 u}{\partial t\partial x^2}\Delta x^2-\frac{1}{2}\frac{\partial^3 u}{\partial t^2\partial x}\Delta x\Delta t\right]\\
&= -\frac{\partial u}{\partial t}-\frac{1}{2}\frac{\partial^2 u}{\partial t^2}\Delta t+\frac{\partial^2 u}{\partial x\partial t}\Delta x-\frac{1}{6}\frac{\partial^3 u}{\partial t^3}\Delta t^2-\frac{1}{6}\frac{\partial^3 u}{\partial t\partial x^2}\Delta x^2+\frac{1}{2}\frac{\partial^3 u}{\partial t^2\partial x}\Delta x\Delta t\\
\text{右边} &= -\frac{1}{2}a\left[\frac{\partial u}{\partial x}+\frac{1}{2}\frac{\partial^2 u}{\partial x^2}\Delta x+\frac{1}{6}\frac{\partial^3 u}{\partial x^3}\Delta x^2\right]+\frac{3}{2}a\left[\frac{\partial u}{\partial x}-\frac{1}{2}\frac{\partial^2 u}{\partial x^2}\Delta x+\frac{1}{6}\frac{\partial^3 u}{\partial x^3}\Delta x^2\right]\\
&= a\frac{\partial u}{\partial x}-a\frac{\partial^2 u}{\partial x^2}\Delta x+a\frac{1}{6}\frac{\partial^3 u}{\partial x^3}\Delta x^2
\end{aligned}$$

因此有

$$\begin{aligned}
\text{右边 -左边} &= a\frac{\partial u}{\partial x}-a\frac{\partial^2 u}{\partial x^2}\Delta x+\frac{1}{6}a\frac{\partial^3 u}{\partial x^3}\Delta x^2+\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x\partial t}\Delta x+\frac{1}{6}\frac{\partial^3 u}{\partial t\partial x^2}\Delta x^2+O(\Delta t,\Delta x\Delta t)\\
&= \frac{\partial u}{\partial t}+a\frac{\partial u}{\partial x}-\left(a\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial x\partial t}\right)\Delta x+\frac{1}{6}\left(a\frac{\partial^3 u}{\partial x^3}+\frac{\partial^3 u}{\partial t\partial x^2}\right)\Delta x^2+O(\Delta t,\Delta x\Delta t)\\
&= \frac{\partial u}{\partial t}+a\frac{\partial u}{\partial x}+O(\Delta t,\Delta x\Delta t)
\end{aligned}$$

因此该差分格式在时间上是一阶精度, 空间上二阶精度.

稳定性分析: 对差分方程中的各项作 Fourier 展开 $u_j^n=\sum_{k=1}^{\infty}\hat{u}_j^n e^{ijk\Delta x}$ 并代入式差分方程得到它的每个分量的误差方程:

$$\begin{aligned}
\frac{1}{3}(G-1)e^{ik\Delta x}-\frac{2}{3}(G-1)-\frac{2}{3}(G-1)e^{-ik\Delta x} &= -\frac{1}{2}r(e^{ik\Delta x}-1)+\frac{3}{2}r(1-e^{-ik\Delta x})\\
&\Downarrow\\
2(G-1)\left(e^{ik\Delta x}-2-2e^{-ik\Delta x}\right) &= 3r(1-e^{ik\Delta x})+9r(1-e^{-ik\Delta x})\\
&\Downarrow\\
2(G-1)\left[-\left(1-e^{ik\Delta x}\right)+2\left(1-e^{-ik\Delta x}\right)-3\right] &= 3r(1-e^{ik\Delta x})+9r(1-e^{-ik\Delta x})
\end{aligned}$$

其中 $G=\hat{u}_j^{n+1}/\hat{u}_j^n$ 为放大因子. 令 $1-e^{-ik\Delta x}=z$, 则上式可化为

$$G=\frac{(3r-2)z+(9r+4)\bar{z}-6}{2(2\bar{z}-z-3)}=\frac{az+b\bar{z}-6}{4\bar{z}-2z-6}$$

其中 $\bar{z}$ 为 z 的复共轭, $a=3r-2$, $b=9r+4$. 令 $z\bar{z}=z+\bar{z}=2(1-\cos\theta)=s(0\le$

$s \le 4$) 则有

$$
\begin{aligned}
|G|^2 = G\bar{G} &= \frac{(az+b\bar{z}-6)(a\bar{z}+bz-6)}{(4\bar{z}-2z-6)(4z-2\bar{z}-6)} \\
&= \frac{(a^2+b^2)z\hat{z}+ab(\hat{z}+z)^2-2abz\hat{z}-6(a+b)(\hat{z}+z)+36}{36z\hat{z}-8(\hat{z}+z)^2-12(\hat{z}+z)+36} \\
&= \frac{(a-b)^2s+abs^2-6(a+b)s+36}{24s-8s^2+36}
\end{aligned}
$$

显然上式分母大于 0. 为比较分子与分母的大小, 将分子分母作差:

$$
\begin{aligned}
\text{分子 -分母} &= \Big((a-b)^2s+abs^2-6(a+b)s+36\Big)-(24s-8s^2+36) \\
&= (a-b)^2s+abs^2-6(a+b)s-24s+8s^2 \\
&= s\Big((a-b)^2+(ab+8)s-6(a+b)-24\Big) \\
&= s\Big(36(r+1)^2+(27r^2-6r)s-12(6r+1)-24\Big) \\
&= s\Big(36r^2+72r+36+27r^2s-6rs-72r-12-24\Big) \\
&= 3sr(12r+9rs-2s)
\end{aligned}
$$

因此 $|G| \le 1$ 成立的条件为

$$12r+9rs-2s<0 \ \Rightarrow\ r<\frac{2s}{12+9s}=\frac{2}{12/s+9}$$

由于 $0 \le s \le 4$, 因此上述条件在 $r>0$ 时不一定满足, 因此该差分格式是不稳定的. ∎

问题 6 分析热传导方程 $\frac{\partial u}{\partial t}-\beta\frac{\partial^2 u}{\partial x^2}=0$, $\beta>0$ 如下差分格式 ($\sigma=\frac{\beta\Delta t}{\Delta x^2}$) 的稳定性:

$$u_j^{n+1}=u_j^{n-1}+\frac{2}{3}\sigma\left[\delta_x^2u_j^{n+1}+\delta_x^2u_j^n+\delta_x^2u_j^{n-1}\right]$$

其中 $\delta_x^2u_j^n=(u_{j+1}^n-2u_j^n+u_{j-1}^n)$.

将 $\delta_x^2u_j^n=(u_{j+1}^n-2u_j^n+u_{j-1}^n)$ 代入原题中的差分方程得

$$u_j^{n+1}=u_j^{n-1}+\frac{2}{3}\sigma\sum_{k=n-1}^{n+1}(u_{j+1}^k-2u_j^k+u_{j-1}^k)$$

对差分方程中的各项作 Fourier 展开 $u_j^n=\sum_{k=1}^{\infty}\hat{u}_j^n\mathrm{e}^{\mathrm{i}jk\Delta x}$ 并代入上式得到它的每个分量的误差方程:

$$
\begin{aligned}
\hat{u}_j^{n+1} &= \hat{u}_j^{n-1}+\frac{2}{3}\sigma(\mathrm{e}^{\mathrm{i}k\Delta x}-2+\mathrm{e}^{-\mathrm{i}k\Delta x})\left[\hat{u}_j^{n+1}+\hat{u}_j^n+\hat{u}_j^{n-1}\right] \\
&= \hat{u}_j^{n-1}+\frac{4}{3}\sigma(\cos\theta-1)\left[\hat{u}_j^{n+1}+\hat{u}_j^n+\hat{u}_j^{n-1}\right] \\
&= A\hat{u}_j^{n+1}+A\hat{u}_j^n+(1+A)\hat{u}_j^{n-1}
\end{aligned}
$$

其中 $A = \frac{4}{3}\sigma(\cos\theta - 1)$. 这是一个三层差分方程, 为此, 引入新变量 v, 并令 $v_j^{n+1} = u_j^n$, $\mathbf{u} = [u, v]^T$, 则上式可化为

$$\hat{\mathbf{u}}_j^{n+1} = \begin{bmatrix} \hat{u}_j^{n+1} \\ \hat{v}_j^{n+1} \end{bmatrix} = \begin{bmatrix} \frac{A}{1-A} & \frac{1+A}{1-A} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \hat{u}_j^n \\ \hat{v}_j^n \end{bmatrix}$$

特征方程为

$$\begin{vmatrix} \frac{A}{1-A} - \lambda & \frac{1+A}{1-A} \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + \frac{A}{A-1}\lambda + \frac{A+1}{A-1} = 0$$

特征根为

$$\begin{aligned} \lambda_{1,2} &= \frac{-\frac{A}{A-1} \pm \sqrt{\frac{A^2}{(A-1)^2} - \frac{4(A+1)(A-1)}{(A-1)^2}}}{2} = \frac{A \pm \sqrt{A^2 - 4(A^2-1)}}{2(1-A)} \\ &= \frac{A \pm \sqrt{4 - 3A^2}}{2(1-A)} = \frac{4\sigma(\cos\theta - 1) \pm 2\sqrt{9 - 12\sigma^2(\cos\theta-1)^2}}{6 - 8\sigma(\cos\theta - 1)} \end{aligned}$$

下面分两种情况讨论

- 当 $9 - 12\sigma^2(\cos\theta - 1)^2 \geq 0$ 时. 特征方程有两个实根. 有绝对值最大值:

$$\begin{aligned} |\lambda_{\max}| &= \left| \frac{4\sigma(\cos\theta - 1) - 2\sqrt{9 - 12\sigma^2(\cos\theta-1)^2}}{6 - 8\sigma(\cos\theta - 1)} \right| \\ &\leq \left| \frac{4\sigma(1 - \cos\theta) + 2\sqrt{9}}{6 + 8\sigma(1 - \cos\theta)} \right| = \frac{4\sigma(1-\cos\theta) + 6}{8\sigma(1-\cos\theta) + 6} \\ &\leq 1 \end{aligned}$$

  此时差分格式恒稳定.

- 当 $9 - 12\sigma^2(\cos\theta - 1)^2 < 0$. 特征方程有两个复根. 有绝对值最大值:

$$\begin{aligned} |\lambda_{\max}| &= \left[\frac{16\sigma^2(\cos\theta-1)^2 - 36 + 48\sigma^2(\cos\theta-1)^2}{\big(6 + 8\sigma(1-\cos\theta)\big)^2} \right]^{1/2} \\ &= \left[\frac{60\sigma^2(1-\cos\theta)^2 - 36}{\big(6 + 8\sigma(1-\cos\theta)\big)^2} \right]^{1/2} \leq \left[\frac{64\sigma^2(1-\cos\theta)^2 - 36}{\big(6 + 8\sigma(1-\cos\theta)\big)^2} \right]^{1/2} \\ &\leq \sqrt{\frac{8\sigma(1-\cos\theta) - 6}{8\sigma(1-\cos\theta) + 6}} \\ &< 1 \end{aligned}$$

  此时差分格式恒稳定.

综上所述, 对于任何 $\sigma = \beta\Delta t/\Delta x^2 > 0$, 该差分格式恒稳定. ∎

问题 7 分析波动方程 $\frac{\partial^2 u}{\partial t^2} - a^2\frac{\partial^2 u}{\partial x^2} = 0$, $a > 0$ 下列差分格式 ($r = \frac{a\Delta t}{\Delta x}$) 的稳定性:

$$
\begin{aligned}
u_j^{n+1} &= \frac{1}{2}\left(u_{j+1}^n + u_{j-1}^n\right) - \frac{1}{2}r\left(v_{j+1}^n - v_{j-1}^n\right) \\
v_j^{n+1} &= \frac{1}{2}\left(v_{j+1}^n + v_{j-1}^n\right) - \frac{1}{2}r\left(u_{j+1}^n - u_{j-1}^n\right)
\end{aligned}
$$

对差分方程中的各项作 Fourier 展开 $u_j^n = \sum_{k=1}^{\infty} \hat{u}_j^n \mathrm{e}^{\mathrm{i}jk\Delta x}$ 并代入上式得到它的每个分量误差方程的矩阵形式:

$$
\begin{bmatrix} \hat{u}_j^{n+1} \\ \hat{v}_j^{n+1} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\left(\mathrm{e}^{\mathrm{i}k\Delta x} + \mathrm{e}^{-\mathrm{i}k\Delta x}\right) & -\frac{1}{2}r\left(\mathrm{e}^{\mathrm{i}k\Delta x} - \mathrm{e}^{-\mathrm{i}k\Delta x}\right) \\ -\frac{1}{2}r\left(\mathrm{e}^{\mathrm{i}k\Delta x} - \mathrm{e}^{-\mathrm{i}k\Delta x}\right) & \frac{1}{2}\left(\mathrm{e}^{\mathrm{i}k\Delta x} + \mathrm{e}^{-\mathrm{i}k\Delta x}\right) \end{bmatrix} \begin{bmatrix} \hat{u}_j^n \\ \hat{v}_j^n \end{bmatrix}
$$

上述方程的特征方程为

$$
\begin{vmatrix} \frac{1}{2}\left(\mathrm{e}^{\mathrm{i}k\Delta x} + \mathrm{e}^{-\mathrm{i}k\Delta x}\right) - \lambda & -\frac{1}{2}r\left(\mathrm{e}^{\mathrm{i}k\Delta x} - \mathrm{e}^{-\mathrm{i}k\Delta x}\right) \\ -\frac{1}{2}r\left(\mathrm{e}^{\mathrm{i}k\Delta x} - \mathrm{e}^{-\mathrm{i}k\Delta x}\right) & \frac{1}{2}\left(\mathrm{e}^{\mathrm{i}k\Delta x} + \mathrm{e}^{-\mathrm{i}k\Delta x}\right) - \lambda \end{vmatrix} = 0
$$

即

$$
\left[\cos\theta - \lambda\right]^2 - \frac{1}{4}r^2\left(2\mathrm{i}\sin\theta\right)^2 = \lambda^2 - 2\lambda\cos\theta + r^2\sin^2\theta + \cos^2\theta = 0
$$

特征根为

$$
\lambda_{1,2} = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4(r^2\sin^2\theta + \cos^2\theta)}}{2} = \cos\theta \pm \mathrm{i}\sin\theta r
$$

绝对值最大值为

$$
|\lambda_{\max}| = \cos^2\theta + r\sin^2\theta = 1 + (r-1)\sin^2\theta
$$

显然在 $0 < r < 1$ 的条件下, $|\lambda_{\max}| \leq 1$ 成立, 此时该差分格式是稳定的. ■

问题 8 用 Taylor 分析法求出对流方程的 L-W 差分格式:

$$
u_j^{n+1} = u_j^n - \frac{1}{2}r\left(u_{j+1}^n - u_{j-1}^n\right) + \frac{1}{2}r^2\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right)
$$

的修正方程, 并求出它的耗散项和色散项的表达式.

将 u_j^{n+1}, u_{j-1}^n, u_{j-2}^n 分别在 $u = u_j^n$ 展开成泰勒级数

$$
\begin{aligned}
u_j^{n+1} &= u_j^n + \frac{\partial u}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 u}{\partial t^2}\Delta t^2 + \frac{1}{6}\frac{\partial^3 u}{\partial t^3}\Delta t^3 + \frac{1}{24}\frac{\partial^4 u}{\partial t^4}\Delta t^4 + O(\Delta t^5) \\
u_{j+1}^n &= u_j^n + \frac{\partial u}{\partial x}\Delta x + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}\Delta x^2 + \frac{1}{6}\frac{\partial^3 u}{\partial x^3}\Delta x^3 + \frac{1}{24}\frac{\partial^4 u}{\partial x^4}\Delta x^4 + O(\Delta x^5) \\
u_{j-1}^n &= u_j^n - \frac{\partial u}{\partial x}\Delta x + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}\Delta x^2 - \frac{1}{6}\frac{\partial^3 u}{\partial x^3}\Delta x^3 + \frac{1}{24}\frac{\partial^4 u}{\partial x^4}\Delta x^4 + O(\Delta x^5)
\end{aligned}
$$

将以上泰勒级数代入差分方程

$$\frac{\partial u}{\partial t}\Delta t+\frac{1}{2}\frac{\partial^2 u}{\partial t^2}\Delta t^2+\frac{1}{6}\frac{\partial^3 u}{\partial t^3}\Delta t^3+\frac{1}{24}\frac{\partial^4 u}{\partial t^4}\Delta t^4+O(\Delta t^5)=-\frac{1}{2}r\left[2\frac{\partial u}{\partial x}\Delta x+\frac{1}{3}\frac{\partial^3 u}{\partial x^3}\Delta x^3+O(\Delta x^5)\right]+\frac{1}{2}r^2\left[\frac{\partial^2 u}{\partial x^2}\Delta x^2+\frac{1}{12}\frac{\partial^4 u}{\partial x^4}\Delta x^4+O(\Delta x^6)\right]$$

将 $r=a\Delta t/\Delta x$ 代入上式并整理得

$$0=u_t+au_x+\frac{1}{2}u_{tt}\Delta t-\frac{1}{2}a^2u_{xx}\Delta t+\frac{1}{6}u_{ttt}\Delta t^2+\frac{1}{6}au_{xxx}\Delta x^2+\frac{1}{24}u_{tttt}\Delta t^3-\frac{1}{24}a^2u_{xxxx}\Delta x^2\Delta t+\cdots$$

采用图表法消元, 消元后可得到下表.

表 1: 图表消元法列表

(a)		u_t	u_x	u_{tt}	u_{xt}	u_{xx}	u_{ttt}	u_{xtt}	u_{xxt}	u_{xxx}	u_{tttt}	u_{xttt}	u_{xxtt}	u_{xxxt}	u_{xxxx}
(b)		1	a	$\frac{\Delta t}{2}$		$-\frac{a^2}{2}\Delta t$	$\frac{\Delta t^2}{6}$			$\frac{a}{6}\Delta x^2$	$\frac{1}{24}\Delta t^3$				$-\frac{a^2}{24}\Delta x^2\Delta t$
(c)	$-\frac{\Delta t}{2}\frac{\partial}{\partial t}(b)$			$-\frac{\Delta t}{2}$	$-\frac{a}{2}\Delta t$		$-\frac{\Delta t^2}{4}$		$\frac{a^2}{4}\Delta t^2$		$-\frac{1}{12}\Delta t^3$			$-\frac{a}{12}\Delta x^2\Delta t$	
(d)	$a\frac{\Delta t}{2}\frac{\partial}{\partial x}(b)$				$\frac{a}{2}\Delta t$	$\frac{a^2}{2}\Delta t$		$\frac{a}{4}\Delta t^2$		$-\frac{a^3}{4}\Delta t^2$		$\frac{a}{12}\Delta t^3$			$\frac{a^2}{12}\Delta x^2\Delta t$
(e)	$\frac{\Delta t^2}{12}\frac{\partial^2}{\partial t^2}(b)$						$\frac{\Delta t^2}{12}$	$\frac{a}{12}\Delta t^2$			$\frac{1}{24}\Delta t^3$		$-\frac{a^2}{24}\Delta t^3$		
(f)	$-a\frac{\Delta t^2}{3}\frac{\partial^2}{\partial t\partial x}(b)$							$-\frac{a}{3}\Delta t^2$	$-a^2\frac{\Delta t^2}{3}$			$-\frac{a}{6}\Delta t^3$		$\frac{a^3}{6}\Delta t^3$	
(g)	$a^2\frac{\Delta t^2}{12}\frac{\partial^2}{\partial x^2}(b)$								$a^2\frac{\Delta t^2}{12}$	$\frac{a^3}{12}\Delta t^2$			$\frac{a^2}{24}\Delta t^3$		$-\frac{a^4}{24}\Delta t^3$
(h)	$\frac{a}{12}\Delta t^3\frac{\partial^3}{\partial t^2\partial x}(b)$											$\frac{a}{12}\Delta t^3$	$\frac{a^2}{12}\Delta t^3$		
(i)	$-\frac{a^2}{12}\Delta t^3\frac{\partial^3}{\partial x^2\partial t}(b)$												$-\frac{a^2}{12}\Delta t^3$	$-\frac{a^3}{12}\Delta t^3$	
(j)	$(\frac{a}{12}\Delta x^2\Delta t-\frac{a^3}{12}\Delta t^3)\frac{\partial^3}{\partial x^3}(b)$													$\frac{a}{12}\Delta x^2\Delta t-\frac{a^3}{12}\Delta t^3$	$\frac{a^2}{12}\Delta x^2\Delta t-\frac{a^4}{12}\Delta t^3$
										$\mathbf{\frac{a}{6}\Delta x^2-a^3\frac{\Delta t^2}{6}}$					$\mathbf{\frac{a^2}{8}\Delta x^2\Delta t-\frac{a^4}{8}\Delta t^3}$

将表1中的 (b) 至 (h) 相加, 得到差分方程的修正方程 (色散项和耗散项已标出):

$$u_t+au_x=-\left(\frac{a}{6}\Delta x^2-a^3\frac{\Delta t^2}{6}\right)u_{xxx}-\left(\frac{a^2}{8}\Delta x^2\Delta t-\frac{a^4}{8}\Delta t^3\right)u_{xxxx}+\cdots=\underbrace{\frac{a\Delta x^2}{6}\left(r^2-1\right)u_{xxx}}_{\text{色散项}}+\underbrace{\frac{a\Delta x^3}{8}r\left(r^2-1\right)u_{xxxx}}_{\text{耗散项}}+\cdots$$

■

附录

计算流体力学 2012 年考试重点

01. 试分析一维非定常等熵流方程组 $\begin{cases} \dfrac{\partial \rho}{\partial t} + u\dfrac{\partial \rho}{\partial x} + \rho\dfrac{\partial u}{\partial x} = 0 \\ \dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + \dfrac{a^2}{\rho}\dfrac{\partial \rho}{\partial x} = 0 \end{cases}$ 数学性质和类型.

令 $\mathbf{z} = \begin{bmatrix} \rho \\ u \end{bmatrix}$ 则一维非定常等熵流方程组可写成

$$\frac{\partial \mathbf{z}}{\partial t} + \begin{bmatrix} u & \rho \\ a^2/\rho & u \end{bmatrix} \frac{\partial \mathbf{z}}{\partial x} = 0$$

系数行列式特征方程为

$$\begin{vmatrix} u - \lambda & \rho \\ a^2/\rho & u - \lambda \end{vmatrix} = 0 \to (u - \lambda)^2 - a^2 = 0 \to \lambda = u \pm a$$

为双曲型方程, 特征值为 $u \pm a$.

02. 对流方程, 热传导方程和对流 -扩散方程的主要数学物理性质.
对流方程的数学物理性质:

- 它是双曲方程. 反映的流动特性可以用特征线来描述, 流场中扰动沿特征线传播, 在特征线上满足一定相容关系.
- 它反映了流动中扰动传播的速度是有限的. 对于线性方程, 当 $a > 0$ 时, 扰动沿正方向; 当 $a > 0$ 时, 沿负方向.
- 扰动波幅不衰减, 波动不变化. 无耗散和色散.
- 通解存在依赖区和影响区. 扰动传播范围有限, 不影响全流场.

扩散方程 (热传导方程) 的数学物理性质:

- 它是抛物型方程, 有耗散无色散. 描述具有耗散机制的流动现象.
- 任何局部扰动都会影响全流场.
- 通解依赖于初始时刻粒子密度分布 $\phi(\xi)$, 只要初值在 $|x| < \infty$ 内连续有界, 则通解唯一, 连续的存在.
- 它存在极值原理, 即如果初值是有界的, 且满足 $m \leq \phi(x) \leq M$, 则通解也一定有界, 且满足 $m \leq u(x, t) \leq M$

对流 -扩散方程的数学物理性质:

- 它是双曲 -抛物型方程. 解单值连续, 且永远存在.
- 具有波动特性, 扰动沿特征线传播, 且传播速度是有限的.

- 具有黏性流动特性, 有耗散无色散.

03. 什么是差分格式相容性, 收敛性, 稳定性

- 相容性: 对于足够光滑的函数, 若 $\Delta x \to 0$, $\Delta t \to 0$, 差分方程截断误并 R_j^n 对每一点 (x_j, t_n) 都趋近于 0, 则该差分方程 $(L\Delta u)_j^n = 0$ 逼近微分方程 $L\Delta u = 0$, 差分方程与微分方程是相容的.
- 收敛性: 节点 (x_p, t_p) 为偏微分方程求解区域 Ω 内任意一点, 当 $x \to x_p$, $t \to t_p$ 时, 差分方程数值解 u_j^n 逼近于微分方程精确解, 即 $e_j^n = u - u_j^n = 0$, 则差分方程收敛于该偏微分方程.
- 稳定性: 在某一时刻 t_n, 差分方程的计算误差为 ε_j^n, 若在 t^{n+1} 时刻满足:

$$||\varepsilon_j^{n+1}|| \le k||\varepsilon_j^n||$$

  则该差分方程是稳定的.

04. 写出 Lax 等价定理和 Von Neumann 准则基本内容.

- Lax 等价定理: 对于适定和线性的偏微分方程的初值问题, 若逼近它的差分方程与它是相容的, 则差分方程的稳定性是保证差分方程收敛性的充分必要条件.
- Von Neumann 准则: 差分方程稳定性必要条件是当 $\Delta t \le \Delta t_0$ 时, 对于所有的波数 k 有:

  (1) $||G(\Delta t, \Delta x, k)|| \le 1 + K_1\Delta t$
  (2) $\rho(G) = \max_j |\lambda_j(\Delta t, \Delta x, k)| \le 1 + K_2\Delta t$

  其中 $\lambda_j(\Delta t, \Delta x, k)$ 为差分方程放大矩阵的特征值.

05. 简要说明差分方程耗散性和色散性的主要特征和判别公式.

- 耗散效应: 差分方程计算激波时激波被拉宽, 幅度减小, 出现抹平和光滑现象.
- 色散效应: 激波上下游出现高频振荡.
- 修正方程截断误差的偶阶导数项为耗散项, 奇阶为色散项.
- Founier 分析法: 偏微分方程放大因子 $G_e = |G_e|\mathrm{e}^{\varphi_e}$, 差分方程放大因子 $G = |G|\mathrm{e}^{\varphi}$. 则: $\frac{|G|}{|G_e|} > 1$ 负耗散, 不稳定; $\frac{|G|}{|G_e|} < 1$ 正耗散, 稳定; $\frac{\varphi}{\varphi_e} > 1$ 正色散, 相位是超前的; $\frac{\varphi}{\varphi_e} < 1$ 负色散, 相位是滞后的.

06. 写出非线性发展方程间断形式和商条件, 并说明熵条件几何特性和物理特性.

- 间断形式: 在 ξ 处有间断 $x_2 < \xi(t) < x_1$, 则有 $f^+ - f^- = \frac{d\xi}{dt}(u^+ - u^-)$.
- 熵条件: $u(x,t)$ 是分片连续可微函数, 在连续区域内满足非线性演化方程 $u_t + f_x = 0 (t > 0, -\infty < x < +\infty)$ 和初始条件 $u(x,0) = \phi(x)$, 间断线上满足

$$\frac{f(u^-) - f(u)}{u^- - u} \ge \frac{f(u^+) - f(u^-)}{u^+ - u^-} \ge \frac{f(u^+) - f(u)}{u^+ - u}$$

- 熵条件几何特性: 对于曲线 $y = f(u)$, $u^+ < u < u^-$. 由熵条件不等式左边得: 在 (u^+, u^-) 内, $f(u)$ 位于 (u^+, f^+) 和 (u^-, f^-) 两点连线下方. 由熵条件不等式左右得: 在 (u^-, u^+) 内, $f(u)$ 位于 (u^+, f^+) 和 (u^-, f^-) 两点连线上方.
- 熵条件物理特性:

07. 什么是守恒方程和守恒差分格式是什么? 它们的相容条件是什么.

- 守恒方程: $u_t + f_x = 0$, 初始条件 $u(x,0) = \phi(x)$.
- 守恒差分格式: $u_j^{n+1} = u_j^n - r(h_{j+1/2}^n - h_{j-1/2}^n)$, 其中 $h_{j+1/2}^n = h^n(u_{j-l+1}^n, u_{j-l+2}^n, \cdots, u_{j+l}^n)$, $r = \Delta t/\Delta x$.
- 相容条件: $h(w, w, \cdots, w) = f(w)$, 其中 w 是守恒型方程中的一个参数.

08. 什么是单调差分格式和保单调格式? 说明 Godunov 线性单调格式形式和性质.

- 单调差分格式: 用差分方程解 $u_t + au_x = 0$, $u(x,0) = \phi(x)$ 时, 若初始函数是单调非增 (非减), 差分格式的解也是单调非增 (非减).
- 保单调格式: 若非线性差分格式 $u_j^{n+1} = G(u_{j-l}^n, \cdots, u_{j+l}^n)$ 在 n 时刻是单调非增 (非减), 在 $n+1$ 时刻 u_j^{n+1} 也是单调非增 (非减).
- Godunov 线性单调格式形式及性质:

$$u_j^{n+1} = \sum_k \alpha_k u_{j+k}^n$$

  定理: 对于初始条件为 $u(x,0) = \phi(x)$ 的线性对流方程 $u_t + au_x = 0$:

  1. 差分格式 $u_j^{n+1} = \sum_k \alpha_k u_{j+k}^n$ 是单调格式的充要条件是所有的 $\alpha_k \geq 0$.
  2. 若 $\sum_k \alpha_k = 1$, 则单调差分格式是稳定的. 它的数值解收敛于物理解.
  3. 单调差分格式是一阶精度，并且解也保持单调性.

09. 说明有限体积算法基本思想, 控制单元类型, 离散格式应遵循的四个原则.

- 基本思想: 首先把计算区域近似离散成有限个互不重叠的网格. 围绕每个网格点取一系列互不重叠的控制体单元, 在每个控制体单元中只包含一个节点. 并把待求流动量设置在网格节点上, 然后利用流动量守恒律对每个控制体单元进行积分, 导出一组离散格式. 对它进行求解, 得到流动的数值解.
- 控制单元类型: 二维有三解形, 四边形或多边形; 三维有四面体, 锥形体, 楔体, 八面体等.
- 四个原则:
  1. 界面一致性. 相邻控制单元界面上流通量离散格式相同, 确保守恒律.
  2. 正系数原则. 离散格式中各项系数为正或同号, 确保稳定性和物理解.

3. 负斜率原则. 源项线性化时, 系数小于零, 以确保误差不会越来越大.
4. 系数和原则. 对无源流动, 中心节点系数为相邻节点系数和.$a_p = \sum a_{nb}$.

10. 说明有限体积算法和有限差分算法关系.

对于一维流动问题有限体积算法和有限差分算法完全等价; 对于多统流动问题, 只要控制体单元取矩形 (2D), 长方体 (3D), 它们基本上也等价. 但如果在多维问题中控制体单元不取矩形 (2D), 长方体 (3D), 它们并不完全等价.