Applications of entailments: de Groot duality

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De Groot duality

A topological space is **stably compact** if it is sober, locally compact, and finite intersections of compact saturated subsets are compact.

De Groot dual X^{d} of a stably compact space X is a set X equipped with the cocompact topology (i.e. the topology generated by the complements of compact saturated subsets). The space X^{d} is stably compact and $(X^{\mathsf{d}})^{\mathsf{d}} = X$.

Goubault-Larrecq (2010) showed that the de Groot duality induces a family of dualities on various powerdomain constructions:

- The dual of the Smyth powerdomain is the Hoare powerdomain of the dual, i.e. $\mathrm{P_U}(X)^{\mathsf{d}} \cong \mathrm{P_L}(X^{\mathsf{d}})$, and *vice versa*.
- The Plotkin powerdomain construction commutes with duality, i.e. $\mathrm{P_V}(X)^{\mathsf{d}} \cong \mathrm{P_V}(X^{\mathsf{d}})$.
- So does the probabilistic powerdomain construction.
- $\ldots$

Aim

We give a point-free (and constructive) account of de Groot duality.

A locale X is **stably compact** if the frame $\Omega(X)$ is a continuous lattice and the set

$$\twoheaduparrow x \stackrel{\text{def}}{=} \{x' \in X \mid x \ll x'\}$$

is a filter for each $x \in X$.

Stably compact locales are the Stone dual of stably compact spaces through the equivalence $\mathbf{SoberSpa} \cong \mathbf{SpatialLoc}$.

The de Groot dual $X^{\triangle}$ of a stably compact locale X is the frame of Scott open filters on $\Omega(X)$ (cf. Escardó 2000).

I tried to reconstruct the dualities due to Goubault-Larrecq in the point-free setting, and got a couple of results, e.g. $\mathrm{P_U}(X)^{\mathsf{d}} \cong \mathrm{P_L}(X^{\mathsf{d}})$. But I got stuck . . .

On the Duality of Compact vs. Open

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ABSTRACT: It is a pleasant fact that Stone-duality may be described very smoo
restricted to the category of compact spectral spaces: The Stone-duals
metic algebraic lattices, may be replaced by their sublattices of
ing infinitary operations.

We present a similar approach to describe the
he requirement of having a bas
he lattices by co

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Proposition

A locale is stably compact if and only if it is a retract of a spectral locale (i.e. the frame of ideals of a distributive lattice).

Proof.

A stably compact locale X is a retract of the frame of ideals $\mathbf{Idl}(X)$. $\square$

Since every idempotent splits in the category of locales:

Corollary

The category of stably compact locales is equivalent to the splitting of idempotents $\mathbf{Split}(\mathbf{Spec})$ of the category $\mathbf{Spec}$ of spectral locales.

- An object of $\mathbf{Split}(\mathbf{Spec})$ is an idempotent (i.e. $f\colon X \to X$ s.t. $f \circ f = f$) in $\mathbf{Spec}$.
- A morphism $g\colon (f\colon X \to X) \to (f'\colon X' \to X')$ in $\mathbf{Split}(\mathbf{Spec})$ is a continuous map $g\colon X \to X'$ in $\mathbf{Spec}$ such that $f' \circ g = g = g \circ f$.

Spectral locales

A relation $r \subseteq D \times D'$ between distributive lattices D and D' is **approximable** if

1. $ra \stackrel{\text{def}}{=} \{b \in D' \mid a\ r\ b\}$ is a filter for each $a \in D$,
2. $r^{-}b \stackrel{\text{def}}{=} \{a \in D \mid a\ r\ b\}$ is an ideal of D for each $b \in D'$,
3. $a\ r\ 0' \implies a = 0$,
4. $a\ r\ b \vee' c \implies (\exists b', c' \in D)\, a \leq b' \vee c'\ \&\ b'\ r\ b\ \&\ c'\ r\ c$.

Distributive lattices and approximable relations form a category $\mathbf{DL_{AP}}$ with identities $\leq_D$ and relational compositions.

Proposition

The category $\mathbf{DL_{AP}}$ is equivalent to the category of spectral locales.

Strong proximity lattices (Jung & Sünderhauf)

A **strong proximity lattice** is an object of $\mathbf{Split}(\mathbf{DL_{AP}})$, i.e. a distributive lattice D equipped with an idempotent relation $\prec$ such that

1. $\downarrow a \stackrel{\text{def}}{=} \{b \in D \mid b \prec a\}$ is an ideal
2. $\uparrow a \stackrel{\text{def}}{=} \{b \in D \mid b \succ a\}$ is a filter
3. $a \prec 0 \implies a = 0$
4. $a \prec b \vee c \implies (\exists b' \prec b)\,(\exists c' \prec c)\, a \leq b' \vee c'$
5. $1 \prec a \implies a = 1$
6. $a \wedge b \prec c \implies (\exists a' \succ a)\,(\exists b' \succ b)\, a' \wedge b' \leq c$.

Morphisms of strong proximity lattices are **approximable** relations.

Remark A strong proximity lattice $(D, \prec)$ represents a stably compact locale X such that

$$\Omega(X) \cong \text{Rounded ideals of } (D, \prec).$$

An ideal $I \subseteq D$ is **rounded** if $a \in I \iff (\exists b \succ a)\, b \in I$.

Continuous entailment relations (Coquand & Zhang)

An **entailment relation** on a set S is a binary relation $\vdash$ on the finite subsets of S such that

$$\frac{a \in S}{a \vdash a} \qquad \frac{A \vdash B}{A', A \vdash B, B'} \qquad \frac{A \vdash B, a \quad a, A \vdash B}{A \vdash B}$$

where "," denotes a union.

Remark An entailment relation $(S, \vdash)$ presents a distributive lattice generated with generators S and relations $\bigwedge A \leq \bigvee B$ for $A \vdash B$.

An entailment relation $(S, \vdash)$ is **continuous** if it is equipped with an idempotent relation $\prec$ on S such that

$$(\exists C)\, A \prec_U C \vdash B \iff (\exists D)\, A \vdash D \prec_L B$$

where

$$A \prec_U B \overset{\text{def}}{\iff} (\forall b \in B)\, (\exists a \in A)\, a \prec b$$

$$A \prec_L B \overset{\text{def}}{\iff} (\forall a \in A)\, (\exists b \in B)\, a \prec b.$$

Proposition

The category of continuous entailment relations is equivalent to that of strong proximity lattices.

Proof.

- If $(D, \prec)$ is a strong proximity lattice, then $(D, \vdash_D)$ defined by

$$A \vdash_D B \overset{\text{def}}{\iff} \bigwedge A \leq_D \bigvee B$$

  together with $\prec$ is a continuous entailment relation.

- If $(S, \vdash, \prec)$ is a continuous entailment relation, then the lattice D_S generated by $(S, \vdash)$ together with the relation $\ll$ on D_S defined by

$$\bigvee_{i<N} \bigwedge A_i \ll \bigwedge_{j<M} \bigvee B_j \overset{\text{def}}{\iff} \forall i < N \forall j < M \exists C \left[A_i \prec_U C \vdash B_j\right]$$

  is a strong proximity lattice. $\square$

Generated continuous entailment relations

Let R be a set of pairs of finite subsets of a set S (called R a set of **axioms**). An entailment relation $(S, \vdash)$ is **generated** by R if it is the smallest entailment relation on S that contains R, i.e. $\vdash$ is generated by the following rules:

$$\frac{(A,B) \in R}{A \vdash B} \qquad \frac{a \in S}{a \vdash a} \qquad \frac{A \vdash B}{A', A \vdash B, B'} \qquad \frac{A \vdash B, a \quad a, A \vdash B}{A \vdash B}$$

Lemma

Let $(S, \vdash)$ be the entailment relation generated by a set R of axioms. Then the dual $\dashv$ is generated by $R^{\mathrm{op}} \overset{\mathrm{def}}{=} \{(B,A) \mid (A,B) \in R\}$.

Proposition

Let $(S, \vdash)$ be the entailment relation generated by a set R of axioms, and let $\prec$ be an idempotent relation on S. Then $(S, \vdash, \prec)$ is a continuous entailment if and only if for each A and B

1. $A \prec_U C \;\&\; (C,D) \in R \implies (\exists D')\, A \vdash D' \prec_L D$
2. $(C,D) \in R \;\&\; D \prec_L B \implies (\exists C')\, C \prec_U C' \vdash B$

Continuous entailment relation as the space its models

Definition

A **model** of a continuous entailment relation $(S, \vdash, \prec)$ is a subset $\alpha \subseteq S$ such that

1. $A \subseteq \alpha \implies (\exists b \in B)\, b \in \alpha$ for each $A \vdash B$,
2. $a \in \alpha \implies (\exists b \prec a)\, b \in \alpha$.

Example

If X is a locale presented by a strong proximity lattice $(D, \prec)$, the **Scott topology** $\Sigma(X)$ can be defined as the space of its rounded ideals.

$\Sigma(X)$ can be presented by an entailment relation on $\boxtimes D \overset{\text{def}}{=} \{\boxtimes a \mid a \in D\}$ generated by

$$\emptyset \vdash \boxtimes 0 \qquad \boxtimes\, a, \boxtimes b \vdash \boxtimes(a \vee b) \qquad \boxtimes\, a \vdash \boxtimes b \quad (a \geq b)$$

together with the idempotent relation $\boxtimes a \prec_{\boxtimes} \boxtimes b \overset{\text{def}}{\iff} a \succ b$.

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Definition

- The **dual** D^{d} of a strong proximity lattice $(D, \prec)$ is $\left(D^{\mathsf{d}}, \succ\right)$, where D^{d} is the dual lattice of D.
- The **dual** S^{d} of a continuous entailment relation $(S, \vdash, \prec)$ is $(S, \dashv, \succ)$.

Proposition

The equivalence between continuous entailment relations and strong proximity relations commutes with the dualities.

Question

If X is the stably compact locale presented by $(D, \prec)$, does $(D^{\mathsf{d}}, \succ)$ present the de Groot dual of X?

De Groot duality

The de Groot dual $X^\triangle$ of a stably compact locale X is the frame of Scott open filters on $\Omega(X)$.

Definition

Let X be a stably compact locale. The **upper powerlocale** $\mathrm{P}_{\mathrm{U}}(X)$ is a locale whose points (i.e. models) are Scott open filters on $\Omega(X)$.

The de Groot dual of X can be characterized by $\mathrm{P}_{\mathrm{U}}(X) \cong \Sigma(X^{\mathsf{d}})$.

De Groot duality in strong proximity lattices

Proposition

Given a strong proximity lattice $(D, \prec)$, let X and Y be stably compact locales presented by $(D, \prec)$ and $(D^{\mathsf{d}}, \succ)$ respectively. Then $\mathrm{P_U}(X) \cong \Sigma(Y)$.

Proof. The upper powerlocale $\mathrm{P_U}(X)$ is presented by an entailment relation on $\Box D \overset{\text{def}}{=} \{\Box a \mid a \in D\}$ generated by

$$\emptyset \vdash \Box 1 \qquad \Box a, \Box b \vdash \Box(a \wedge b) \qquad \Box a \vdash \Box b \quad (a \leq b)$$

together with the idempotent relation $\Box a \prec_{\Box} \Box b \overset{\text{def}}{\iff} a \prec b$.

The Scott topology $\Sigma(Y)$ is presented by an entailment relation on $\boxtimes D \overset{\text{def}}{=} \{\boxtimes a \mid a \in D\}$ generated by

$$\emptyset \vdash \boxtimes 1 \qquad \boxtimes a, \boxtimes b \vdash \boxtimes(a \wedge b) \qquad \boxtimes a \vdash \boxtimes b \quad (a \leq b)$$

together with the idempotent relation $\boxtimes a \prec_{\boxtimes} \boxtimes b \overset{\text{def}}{\iff} a \prec b$. $\square$

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The lower and upper powerlocales

Let X be a stably compact locale presented by a strong proximity lattice $(D, \prec)$.

- The **lower powerlocale** $\mathrm{P_L}(X)$ is presented by an entailment relation on $\Diamond D \overset{\text{def}}{=} \{\Diamond a \mid a \in D\}$ generated by

$$\Diamond 0 \vdash \emptyset \qquad \Diamond(a \vee b) \vdash \Diamond a, \Diamond b \qquad \Diamond a \vdash \Diamond b \quad (a \leq b)$$

  together with the idempotent relation $\Diamond a \prec_\Diamond \Diamond b \overset{\text{def}}{\iff} a \prec b$.

- The **upper powerlocale** $\mathrm{P_U}(X)$ is presented by an entailment relation on $\Box D \overset{\text{def}}{=} \{\Box a \mid a \in D\}$ generated by

$$\emptyset \vdash \Box 1 \qquad \Box a, \Box b \vdash \Box(a \wedge b) \qquad \Box a \vdash \Box b \quad (a \leq b)$$

  together with the idempotent relation $\Box a \prec_\Box \Box b \overset{\text{def}}{\iff} a \prec b$.

Proposition

If X is stably compact then $\mathrm{P_L}(X)^{\mathsf{d}} \cong \mathrm{P_U}(X^{\mathsf{d}})$ and $\mathrm{P_U}(X)^{\mathsf{d}} \cong \mathrm{P_L}(X^{\mathsf{d}})$.

Proof.

Suppose X is presented by a strong proximity lattice $(D, \prec)$.

- $\mathrm{P_L}(X)$ is presented by an entailment relation generated by

  $$\Diamond 0 \vdash \emptyset \qquad \Diamond(a \vee b) \vdash \Diamond a, \Diamond b \qquad \Diamond a \vdash \Diamond b \quad (a \leq b)$$

  with the idempotent relation $\Diamond a \prec_{\Diamond} \Diamond b \overset{\text{def}}{\iff} a \prec b$.
- $\mathrm{P_L}(X)^{\mathsf{d}}$ is presented by an entailment relation generated by

  $$\emptyset \vdash \Diamond 0 \qquad \Diamond a, \Diamond b \vdash \Diamond(a \vee b) \qquad \Diamond b \vdash \Diamond a \quad (a \leq b)$$

  with the idempotent relation $\Diamond a \prec_{\Diamond} \Diamond b \overset{\text{def}}{\iff} a \succ b$.
- This is just a presentation of $\mathrm{P_U}(X^{\mathsf{d}})$. □

The Vietoris powerlocales

Let X be a stably compact locale presented by a strong proximity lattice $(D, \succ)$.
The **Vietoris powerlocale** $\mathrm{P_V}(X)$ is presented by an entailment relation on $\Diamond D \cup \Box D$ generated by

$$\begin{array}{ccc}
\Diamond 0 \vdash \emptyset & \Diamond(a \vee b) \vdash \Diamond a, \Diamond b & \Diamond a \vdash \Diamond b \quad (a \leq b) \\
\emptyset \vdash \Box 1 & \Box a, \Box b \vdash \Box(a \wedge b) & \Box a \vdash \Box b \quad (a \leq b)
\end{array}$$

$$\Box a, \Diamond b \vdash \Diamond(a \wedge b) \qquad \Box(a \vee b) \vdash \Box a, \Diamond b$$

The idempotent relation associated with $\mathrm{P_V}(X)$ is $\prec_{\Diamond} \cup \prec_{\Box}$.

Proposition

If X is stably compact then $\mathrm{P_V}(X)^{\mathsf{d}} \cong \mathrm{P_V}(X^{\mathsf{d}})$.

The space of valuations

A **probability valuation** on a locale X is a Scott continuous map $\mu\colon \Omega(X) \to \overrightarrow{[0,1]}$ to the lower reals $\overrightarrow{[0,1]}$ satisfying $\mu(0) = 0$, $\mu(1) = 1$, and the modular law: $\mu(x) + \mu(y) = \mu(x \wedge y) + \mu(x \vee y)$. Let $\mathfrak{V}(X)$ be the locale whose points are valuations on X.

A **covaluation** on X is a Scott continuous map $\nu\colon \Omega(X) \to \overleftarrow{[0,1]}$ to the upper reals $\overleftarrow{[0,1]}$ satisfying $\nu(1) = 0$, $\nu(0) = 1$, and the modular law. Let $\mathfrak{C}(X)$ be a locale whose points are covaluations on X.

The space of valuations

Proposition

If X is a locale presented by a strong proximity lattice $(D, \prec)$, then the space of valuations $\mathfrak{V}(X)$ is presented by an entailment relation on $\{\langle p, a\rangle \mid p \in \mathbb{Q} \;\&\; a \in D\}$ generated by the axioms

$$
\begin{array}{llll}
\emptyset \vdash \langle p, 0\rangle & (p < 0) & \langle p, 0\rangle \vdash \emptyset & (0 < p) \\
\emptyset \vdash \langle p, 1\rangle & (p < 1) & \langle p, 1\rangle \vdash \emptyset & (1 < p) \\
\langle p, a\rangle \vdash \langle q, b\rangle & & & (p \geq q \;\&\; a \leq b) \\
\langle p, a\rangle, \langle q, b\rangle \dashv\vdash \langle r, a \wedge b\rangle, \langle s, a \vee b\rangle & & & (p + q = r + s)
\end{array}
$$

with an idempotent relation $\langle p, a\rangle \prec_{\mathfrak{V}} \langle q, b\rangle \overset{\text{def}}{\iff} p > q \;\&\; a \prec b$.

Note. Each generator $\langle p, a\rangle$ expresses $p < \mu(a)$.

Proposition

If X is a stably compact locale, then $\mathfrak{V}(X)^{\mathsf{d}} \cong \mathfrak{C}(X^{\mathsf{d}})$.

Summary and future work

Summary

By representing a stably compact locale by a continuous entailment relation, one may be able to read off what its de Groot dual is.

Future work

Can strong proximity lattices (or continuous entailment relations) handle combination of non-determinism and probability powerdomain constructions?