Higher-dimensional categories: induction on extensivity

Thomas Cottrell[1]

Soichiro Fujii[2]

John Power[3]

[1]Department of Mathematical Sciences, University of Bath

[2]Department of Computer Science, University of Tokyo

[3]Department of Computer Science, University of Bath

7 July, 2018

Overview

Leinster's method for defining weak n-categories:

Overview

Leinster's method for defining weak n-categories:

- Start with the free strict n-category monad $T^{(n)}$, which is cartesian.

Overview

Leinster's method for defining weak n-categories:

- Start with the free strict n-category monad $T^{(n)}$, which is cartesian.
- For a cartesian monad T on $\mathcal{C}$, define T-operads as monoids in $\mathcal{C}/T1$.

Overview

Leinster's method for defining weak n-categories:

- Start with the free strict n-category monad $T^{(n)}$, which is cartesian.
- For a cartesian monad T on $\mathcal{C}$, define T-operads as monoids in $\mathcal{C}/T1$.
- Define a notion of *contraction* on $T^{(n)}$-operads.

Overview

Leinster's method for defining weak n-categories:

- Start with the free strict n-category monad $T^{(n)}$, which is cartesian.
- For a cartesian monad T on $\mathcal{C}$, define T-operads as monoids in $\mathcal{C}/T1$.
- Define a notion of *contraction* on $T^{(n)}$-operads.
- A weak n-category is an algebra for the initial $T^{(n)}$-operad with contraction.

Overview

Leinster's method for defining weak n-categories:

- Start with the free strict n-category monad $T^{(n)}$, which is cartesian.
- For a cartesian monad T on $\mathcal{C}$, define T-operads as monoids in $\mathcal{C}/T1$.
- Define a notion of *contraction* on $T^{(n)}$-operads.
- A weak n-category is an algebra for the initial $T^{(n)}$-operad with contraction.

Aims:

Overview

Leinster's method for defining weak *n*-categories:

- Start with the free strict *n*-category monad $T^{(n)}$, which is cartesian.
- For a cartesian monad T on $\mathcal{C}$, define T-operads as monoids in $\mathcal{C}/T1$.
- Define a notion of *contraction* on $T^{(n)}$-operads.
- A weak *n*-category is an algebra for the initial $T^{(n)}$-operad with contraction.

Aims:

- Enrich this to define weak *n*-dimensional $\mathcal{V}$-categories.

Overview

Leinster's method for defining weak n-categories:

- Start with the free strict n-category monad $T^{(n)}$, which is cartesian.
- For a cartesian monad T on $\mathcal{C}$, define T-operads as monoids in $\mathcal{C}/T1$.
- Define a notion of *contraction* on $T^{(n)}$-operads.
- A weak n-category is an algebra for the initial $T^{(n)}$-operad with contraction.

Aims:

- Enrich this to define weak n-dimensional $\mathcal{V}$-categories.
- Build dimensions through iterated enrichment.

Extensivity

Definition

Extensivity

Definition

A category $\mathcal{V}$ (with small coproducts) is *extensive* if,

Extensivity

Definition

A category $\mathcal{V}$ (with small coproducts) is *extensive* if, for any set I and family of objects $(X_i)_{i \in I}$,

Extensivity

Definition

A category $\mathcal{V}$ (with small coproducts) is *extensive* if, for any set I and family of objects $(X_i)_{i\in I}$, the functor

$$\begin{array}{rcl} \coprod \colon \prod_{i\in I}(\mathcal{V}/X_i) & \longrightarrow & \mathcal{V}/(\coprod_{i\in I} X_i) \\ \left(\begin{array}{c} A_i \\ \downarrow{\scriptstyle f_i} \\ X_i \end{array}\right)_{i\in I} & \longmapsto & \begin{array}{c} \coprod_{i\in I} A_i \\ \downarrow{\scriptstyle \coprod_{i\in I} f_i} \\ \coprod_{i\in I} X_i \end{array} \end{array}$$

Extensivity

Definition

A category $\mathcal{V}$ (with small coproducts) is *extensive* if, for any set I and family of objects $(X_i)_{i \in I}$, the functor

$$
\begin{array}{rccc}
\coprod\colon & \prod_{i \in I} (\mathcal{V}/X_i) & \longrightarrow & \mathcal{V}/(\coprod_{i \in I} X_i) \\
& \left(\begin{array}{c} A_i \\ \big\downarrow {\scriptstyle f_i} \\ X_i \end{array} \right)_{i \in I} & \longmapsto & \begin{array}{c} \coprod_{i \in I} A_i \\ \big\downarrow {\scriptstyle \coprod_{i \in I} f_i} \\ \coprod_{i \in I} X_i \end{array}
\end{array}
$$

is an equivalence of categories.

Extensivity

Definition

A category $\mathcal{V}$ (with small coproducts) is *extensive* if, for any set I and family of objects $(X_i)_{i\in I}$, the functor

$$\coprod \colon \prod_{i\in I}(\mathcal{V}/X_i) \longrightarrow \mathcal{V}/(\coprod_{i\in I} X_i)$$

$$\left(\begin{array}{c} A_i \\ \Big\downarrow f_i \\ X_i \end{array}\right)_{i\in I} \longmapsto \begin{array}{c} \coprod_{i\in I} A_i \\ \Big\downarrow \coprod_{i\in I} f_i \\ \coprod_{i\in I} X_i \end{array}$$

is an equivalence of categories.

Examples:

Extensivity

Definition

A category $\mathcal{V}$ (with small coproducts) is *extensive* if, for any set I and family of objects $(X_i)_{i\in I}$, the functor

$$\begin{array}{ccc} \coprod \colon \prod_{i\in I}(\mathcal{V}/X_i) & \longrightarrow & \mathcal{V}/(\coprod_{i\in I} X_i) \\ \left(\begin{array}{c} A_i \\ {\scriptstyle f_i}\big\downarrow \\ X_i \end{array}\right)_{i\in I} & \longmapsto & \begin{array}{c} \coprod_{i\in I} A_i \\ \big\downarrow {\scriptstyle \coprod_{i\in I} f_i} \\ \coprod_{i\in I} X_i \end{array} \end{array}$$

is an equivalence of categories.

Examples: **Set**,

Extensivity

Definition

A category $\mathcal{V}$ (with small coproducts) is *extensive* if, for any set I and family of objects $(X_i)_{i \in I}$, the functor

$$\coprod : \prod_{i \in I} (\mathcal{V}/X_i) \longrightarrow \mathcal{V}/(\coprod_{i \in I} X_i)$$

$$\begin{pmatrix} A_i \\ \downarrow f_i \\ X_i \end{pmatrix}_{i \in I} \longmapsto \begin{array}{c} \coprod_{i \in I} A_i \\ \downarrow \coprod_{i \in I} f_i \\ \coprod_{i \in I} X_i \end{array}$$

is an equivalence of categories.

Examples: **Set**, ω-**Cpo**,

Extensivity

Definition

A category $\mathcal{V}$ (with small coproducts) is *extensive* if, for any set I and family of objects $(X_i)_{i \in I}$, the functor

$$\coprod : \prod_{i \in I} (\mathcal{V}/X_i) \longrightarrow \mathcal{V}/(\coprod_{i \in I} X_i)$$

$$\left(\begin{array}{c} A_i \\ \Big\downarrow{\scriptstyle f_i} \\ X_i \end{array} \right)_{i \in I} \longmapsto \begin{array}{c} \coprod_{i \in I} A_i \\ \Big\downarrow{\scriptstyle \coprod_{i \in I} f_i} \\ \coprod_{i \in I} X_i \end{array}$$

is an equivalence of categories.

Examples: **Set**, ω-**Cpo**, **Cat**,

Extensivity

Definition

A category $\mathcal{V}$ (with small coproducts) is *extensive* if, for any set I and family of objects $(X_i)_{i \in I}$, the functor

$$\coprod : \prod_{i \in I} (\mathcal{V}/X_i) \longrightarrow \mathcal{V}/(\coprod_{i \in I} X_i)$$

$$\begin{pmatrix} A_i \\ \big\downarrow f_i \\ X_i \end{pmatrix}_{i \in I} \longmapsto \begin{array}{c} \coprod_{i \in I} A_i \\ \big\downarrow \coprod_{i \in I} f_i \\ \coprod_{i \in I} X_i \end{array}$$

is an equivalence of categories.

Examples: **Set**, ω-**Cpo**, **Cat**, $\mathcal{V}$-**Cat** and $\mathcal{V}$-**Gph** (for extensive $\mathcal{V}$).

$\mathcal{V}$-**Cat**$^{(n)}$ and $\mathcal{V}$-**Gph**$^{(n)}$

Definition

$\mathcal{V}$-$\mathbf{Cat}^{(n)}$ and $\mathcal{V}$-$\mathbf{Gph}^{(n)}$

Definition

Let $\mathcal{V}$ be a category with finite products.

$\mathcal{V}$-**Cat**$^{(n)}$ and $\mathcal{V}$-**Gph**$^{(n)}$

Definition

Let $\mathcal{V}$ be a category with finite products.

For each natural number n, $\mathcal{V}$-**Cat**$^{(n)}$ is defined by:

$\mathcal{V}$-**Cat**$^{(n)}$ and $\mathcal{V}$-**Gph**$^{(n)}$

Definition

Let $\mathcal{V}$ be a category with finite products.

For each natural number n, $\mathcal{V}$-**Cat**$^{(n)}$ is defined by:

$$\mathcal{V}\text{-}\mathbf{Cat}^{(0)} = \mathcal{V}; \qquad \mathcal{V}\text{-}\mathbf{Cat}^{(n+1)} = (\mathcal{V}\text{-}\mathbf{Cat}^{(n)})\text{-}\mathbf{Cat},$$

$\mathcal{V}$-$\mathbf{Cat}^{(n)}$ and $\mathcal{V}$-$\mathbf{Gph}^{(n)}$

Definition

Let $\mathcal{V}$ be a category with finite products.

For each natural number n, $\mathcal{V}$-$\mathbf{Cat}^{(n)}$ is defined by:

$$\mathcal{V}\text{-}\mathbf{Cat}^{(0)} = \mathcal{V}; \qquad \mathcal{V}\text{-}\mathbf{Cat}^{(n+1)} = (\mathcal{V}\text{-}\mathbf{Cat}^{(n)})\text{-}\mathbf{Cat},$$

and $\mathcal{V}$-$\mathbf{Gph}^{(n)}$ is defined by:

$\mathcal{V}$-$\mathbf{Cat}^{(n)}$ and $\mathcal{V}$-$\mathbf{Gph}^{(n)}$

Definition

Let $\mathcal{V}$ be a category with finite products.

For each natural number n, $\mathcal{V}$-$\mathbf{Cat}^{(n)}$ is defined by:

$$\mathcal{V}\text{-}\mathbf{Cat}^{(0)} = \mathcal{V}; \qquad \mathcal{V}\text{-}\mathbf{Cat}^{(n+1)} = (\mathcal{V}\text{-}\mathbf{Cat}^{(n)})\text{-}\mathbf{Cat},$$

and $\mathcal{V}$-$\mathbf{Gph}^{(n)}$ is defined by:

$$\mathcal{V}\text{-}\mathbf{Gph}^{(0)} = \mathcal{V}; \qquad \mathcal{V}\text{-}\mathbf{Gph}^{(n+1)} = (\mathcal{V}\text{-}\mathbf{Gph}^{(n)})\text{-}\mathbf{Gph}.$$

$\mathcal{V}$-$\mathbf{Cat}^{(n)}$ and $\mathcal{V}$-$\mathbf{Gph}^{(n)}$

Definition

Let $\mathcal{V}$ be a category with finite products.

For each natural number n, $\mathcal{V}$-$\mathbf{Cat}^{(n)}$ is defined by:

$$\mathcal{V}\text{-}\mathbf{Cat}^{(0)} = \mathcal{V}; \qquad \mathcal{V}\text{-}\mathbf{Cat}^{(n+1)} = (\mathcal{V}\text{-}\mathbf{Cat}^{(n)})\text{-}\mathbf{Cat},$$

and $\mathcal{V}$-$\mathbf{Gph}^{(n)}$ is defined by:

$$\mathcal{V}\text{-}\mathbf{Gph}^{(0)} = \mathcal{V}; \qquad \mathcal{V}\text{-}\mathbf{Gph}^{(n+1)} = (\mathcal{V}\text{-}\mathbf{Gph}^{(n)})\text{-}\mathbf{Gph}.$$

When $\mathcal{V} = \mathbf{Set}$, $\mathbf{Set}\text{-}\mathbf{Cat}^{(n)} = n\text{-}\mathbf{Cat}$, the category of strict n-categories.

Extensivity and enrichment

Proposition

If $\mathcal{V}$ is extensive and finitely complete,

Extensivity and enrichment

Proposition

*If $\mathcal{V}$ is extensive and finitely complete, then $\mathcal{V}$-**Gph** and $\mathcal{V}$-**Cat** are also extensive and finitely complete.*

Extensivity and enrichment

Proposition

*If $\mathcal{V}$ is extensive and finitely complete, then $\mathcal{V}$-**Gph** and $\mathcal{V}$-**Cat** are also extensive and finitely complete.*

Corollary

If $\mathcal{V}$ is extensive and finitely complete, then $\mathcal{V}\text{-}\mathbf{Gph}^{(n)}$ and $\mathcal{V}\text{-}\mathbf{Cat}^{(n)}$ are also extensive and finitely complete.

Cartesian monads

Definition

A monad (T, η, μ) on $\mathcal{C}$ is *cartesian* if

Cartesian monads

Definition

A monad (T, η, μ) on $\mathcal{C}$ is *cartesian* if

- $\mathcal{C}$ has all pullbacks,

Cartesian monads

Definition

A monad (T, η, μ) on $\mathcal{C}$ is *cartesian* if

- $\mathcal{C}$ has all pullbacks,
- T preserves pullbacks,

Cartesian monads

Definition

A monad (T, η, μ) on $\mathcal{C}$ is *cartesian* if

- $\mathcal{C}$ has all pullbacks,
- T preserves pullbacks,
- all the naturality squares for η and μ are pullback squares.

Cartesian monads

Definition

A monad (T, η, μ) on $\mathcal{C}$ is *cartesian* if

- $\mathcal{C}$ has all pullbacks,
- T preserves pullbacks,
- all the naturality squares for η and μ are pullback squares.

Proposition

Let $\mathcal{V}$ be extensive and finitely complete. For each n, there is an adjunction

Cartesian monads

Definition

A monad (T, η, μ) on $\mathcal{C}$ is *cartesian* if

- $\mathcal{C}$ has all pullbacks,
- T preserves pullbacks,
- all the naturality squares for η and μ are pullback squares.

Proposition

Let $\mathcal{V}$ be extensive and finitely complete. For each n, there is an adjunction

$$\mathcal{V}\text{-}\mathbf{Gph}^{(n)} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{V}\text{-}\mathbf{Cat}^{(n)}$$

Cartesian monads

Definition

A monad (T, η, μ) on $\mathcal{C}$ is *cartesian* if

- $\mathcal{C}$ has all pullbacks,
- T preserves pullbacks,
- all the naturality squares for η and μ are pullback squares.

Proposition

Let $\mathcal{V}$ be extensive and finitely complete. For each n, there is an adjunction

$$\mathcal{V}\text{-}\mathbf{Gph}^{(n)} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{V}\text{-}\mathbf{Cat}^{(n)}$$

and the induced monad $T^{(n)}$ is cartesian.

T-operads

For a cartesian monad T on a finitely complete category $\mathcal{C}$, $\mathcal{C}/T1$ can be given a monoidal structure.

T-operads

For a cartesian monad T on a finitely complete category $\mathcal{C}$, $\mathcal{C}/T1$ can be given a monoidal structure.

Definition

A *T-operad* is a monoid in $\mathcal{C}/T1$.

T-operads

For a cartesian monad T on a finitely complete category $\mathcal{C}$, $\mathcal{C}/T1$ can be given a monoidal structure.

Definition

A *T-operad* is a monoid in $\mathcal{C}/T1$.

Given a T-operad $m\colon M \to T1$, an *algebra for* (M, m) consists of an object X of $\mathcal{C}$ together with an action of (M, m) on X.

T-operads

For a cartesian monad T on a finitely complete category $\mathcal{C}$, $\mathcal{C}/T1$ can be given a monoidal structure.

Definition

A *T-operad* is a monoid in $\mathcal{C}/T1$.

Given a T-operad $m \colon M \to T1$, an *algebra for* (M, m) consists of an object X of $\mathcal{C}$ together with an action of (M, m) on X.

- Classical operads: $\mathcal{C} = \mathbf{Set}$, $T =$ free monoid.

T-operads

For a cartesian monad T on a finitely complete category $\mathcal{C}$, $\mathcal{C}/T1$ can be given a monoidal structure.

Definition

A *T-operad* is a monoid in $\mathcal{C}/T1$.

Given a T-operad $m\colon M \to T1$, an *algebra for* (M, m) consists of an object X of $\mathcal{C}$ together with an action of (M, m) on X.

- Classical operads: $\mathcal{C} = \mathbf{Set}$, $T =$ free monoid.
- For weak n-dimensional $\mathcal{V}$-categories, use $\mathcal{C} = \mathcal{V}\text{-}\mathbf{Gph}^{(n)}$, $T = T^{(n)}$.

Contractions

Contractions

A *contraction* on $m\colon M \to T^{(n)}1$ consists of

Contractions

A *contraction* on $m\colon M \to T^{(n)}1$ consists of a lifting

$$\begin{tikzcd} \partial_j \arrow[r, "h"] \arrow[d, "f_j"'] & M \arrow[d, "m"] \\ C_j \arrow[r, "k"'] \arrow[ur, dashed] & T^{(n)}1 \end{tikzcd}$$

for every such commuting square, where f_j is a "cell boundary inclusion".

Contractions

A *contraction* on $m\colon M \to T^{(n)}1$ consists of a lifting

$$\begin{tikzcd} \partial_j \arrow[r, "h"] \arrow[d, "f_j"'] & M \arrow[d, "m"] \\ C_j \arrow[r, "k"'] \arrow[ur, dashed] & T^{(n)}1 \end{tikzcd}$$

for every such commuting square, where f_j is a "cell boundary inclusion".

Definition

Let $\mathcal{V}$ be extensive, finitely complete, and locally presentable.

Contractions

A *contraction* on $m\colon M \to T^{(n)}1$ consists of a lifting

$$\begin{tikzcd} \partial_j \arrow[r, "h"] \arrow[d, "f_j"'] & M \arrow[d, "m"] \\ C_j \arrow[r, "k"'] \arrow[ur, dashed] & T^{(n)}1 \end{tikzcd}$$

for every such commuting square, where f_j is a "cell boundary inclusion".

Definition

Let $\mathcal{V}$ be extensive, finitely complete, and locally presentable.

A *weak n-dimensional $\mathcal{V}$-category* is an algebra for the initial $T^{(n)}$-operad with contraction.

Contractions

A *contraction* on $m \colon M \to T^{(n)}1$ consists of a lifting

$$\begin{tikzcd} \partial_j \arrow[r, "h"] \arrow[d, "f_j"'] & M \arrow[d, "m"] \\ C_j \arrow[r, "k"'] \arrow[ur, dashed] & T^{(n)}1 \end{tikzcd}$$

for every such commuting square, where f_j is a "cell boundary inclusion".

Definition

Let $\mathcal{V}$ be extensive, finitely complete, and locally presentable.

A *weak n-dimensional $\mathcal{V}$-category* is an algebra for the initial $T^{(n)}$-operad with contraction.

For $\mathcal{V} = \mathbf{Set}$, this agrees with Leinster's definition of weak n-category.