A logical view of complex analytic maps

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Stone Duality

- Classical Stone duality:
  Boolean Algebras^{op} $\cong$ **Stone Spaces**.
- Johnstone: **Sober spaces** $\cong$ **Spatial locals**.
- Smyth, Abramsky, Vickers:
  Open sets as observational properties of programs.
- Abramsky 1991: Domain Theory in Logical Form,
  Denotational Semantics $\cong$ Geometric Logic.

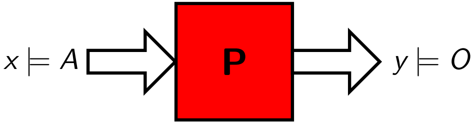
Geometric Logic

- Open sets of a topological space as propositions or semi-decidable properties.
- $\Omega(X)$ the lattice of open sets of X.
- Open set $a \in \Omega(X)$ defines a proposition P_a, with axioms:
  (I) If $a \subseteq b$ then $P_a \vdash P_b$.
  (II) If S is a family of open sets then $P_{\cup S} \vdash \bigvee_{a \in S} P_a$.
  (III) If S is a finite family of open sets then $\bigwedge_{a \in S} P_a \vdash P_{\cap S}$.
- $\wedge\emptyset = \textbf{true}$, $\vee\emptyset = \textbf{false}$ then $P_\emptyset \vdash \textbf{false}$ and $P_X \vdash \textbf{true}$.
- $x \in X$ is a model, $x \models P_a$ iff $x \in a$.

Predicate transformer

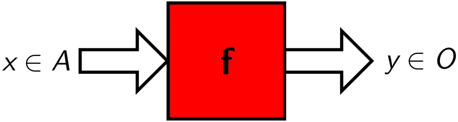
Dijkstra weakest precondition

- **P** a program.
- Weakest-precondition of **P** is a function mapping any postcondition O to a precondition A.
- $\mathrm{wp}(\mathbf{P}, O) = A$.



Differentiation in logical form

- Can we represent derivative of an analytic mapn $f\colon U \subseteq \mathbb{C} \to \mathbb{C}$ by Stone duality?
- Equivalently: Given the predicate transformer representing the map, can we represent the derivative of an analytic map as a predicate transformer?



- $\mathrm{wp}(\mathbf{f}, O) = f^{-1}(O)$.
- $\mathrm{wp}(\mathbf{f}', O) = ?$.

Generalized Lipschitz constant(Edalat 2014)

- $f : U \subseteq \mathbb{C} \to \mathbb{C}$ a continuous function.
- **b** a nonempty convex and compact set in $\mathbb{C}$.
- **a** an open subset of U.
- f has set valued Lipschitz constant **b** in **a** if:

$$\forall x, y \in \mathbf{a}, x \neq y.\ \frac{f(x) - f(y)}{x - y} \in b$$

- Allowing **b** any compact set we have all the local differential properties of f.
- $\delta(b\chi_a)$, the tie of **a** with **b** collection of all function f which have Lipschitz constant **b** in **a**.

L-Derivative (Edalat 2014)

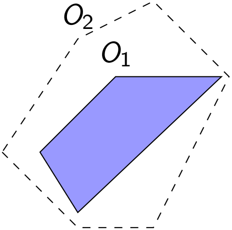
- Collecting all the local differential properties captured by ties we obtain the L-derivative:

$$\mathcal{L}f \colon U \subseteq \mathbb{C} \to \mathbf{C}(\mathbb{C})$$
$$\mathcal{L}f(x) = \bigcap \{b | f \in \delta(b\chi_a), x \in a\}$$

- $\mathcal{L}f$ is Scott continuous function.

Way-below relation in lattice of open sets

- In $(\Omega(\mathbb{C}), \subseteq)$ we have $O_1 \ll O_2$ iff $\overline{O_1}$ is compact and $\overline{O_1} \subseteq O_2$.



- $\Box O = \{C \in \mathbf{C}(\mathbb{C}) | C \subset O\}$: basic open set for Scott topology for $\mathbf{C}(\mathbb{C})$, where $O \subset \mathbb{C}$ is open set.
  way-below relation in $\Omega(\mathbf{C}(\mathbb{C}))$: $\Box O_1 \ll \Box O_2$ iff $O_1 \ll O_2$.

Localic approximable mapping

- We capture f by approximable mapping A_f which satisfies:

$$a\, A_f\, b \text{ iff } a \ll f^{-1}(b)$$

Stably locally compact spaces

- X is stably locally compact if:
  1. $\Omega(X)$ is a distributive continuous lattice.
  2. For $O, O_1, O_2 \in \Omega(X)$, $O \ll O_1 \wedge O_2$ if $O \ll O_1, O_2$.
- $\mathbb{C}$, $\mathbf{C}(\mathbb{C})$ are stably locally compact.
- **SLC** the category of stably locally compact spaces with continuous functions.
- A semi-strong proximity lattice is a lattice with an additional strong ordering satisfies some axioms.
- **PL** is the category of semi-strong proximity lattices with approximable mappings.

Equivalence of **PL** and **SLC** via functors A and G

- $A : \mathbf{SLC} \to \mathbf{PL}$.
- On objects: Let (X, B) be a stably locally compact spaces with basis B, then $A(X, B) = B$, the basis of X closed under finite intersection.
- On morphisms: $A(f) = A_f$.

- $G : \mathbf{PL} \to \mathbf{SLC}$.
- On objects: $G(B) = Spec(B)$, all prime filters of B.
- On morphisms: $G_R(F) = \{b_2 \in B_2 : \exists b_1 \in F.b_1 \, R \, b_2\}$, for $R : B_1 \to B_2$.

- $\mathbf{PL} \cong \mathbf{SLC}$ via A & G (Jung & Sünderhauf 1996).

Logical representation of $\mathbb{C}$ and $\mathbf{C}(\mathbb{C})$

Semi-strong proximity lattice of $\mathbb{C}$

- $B^0_{\mathbb{C}}$ basis of $\mathbb{C}$ consisting rational convex open polytopes.
- $B_{\mathbb{C}}$ consisting of finite join of elements of $B^0_{\mathbb{C}}$.
- $\prec$ is way-below relation in the lattice of open sets of $\mathbb{C}$ restricted to $B_{\mathbb{C}}$.
- $(B_{\mathbb{C}}, \prec)$ is a semi-strong proximity lattice.
- $\mathrm{Spec}(B_{\mathbb{C}}) \cong \mathbb{C}$.

Semi-strong proximity lattice of $\mathbf{C}(\mathbb{C})$

- Similarly, $B^0_{\mathbf{C}(\mathbb{C})}$ basis of Scott topology of $\mathbf{C}(\mathbb{C})$ consisting $\Box a$ for $a \in B^0_{\mathbb{C}}$.
- $(B_{\mathbf{C}(\mathbb{C})}, \prec)$ semi-strong proximity lattice generated by $B^0_{\mathbf{C}(\mathbb{C})}$.
- $\mathrm{Spec}(B_{\mathbf{C}(\mathbb{C})}) \cong \mathbf{C}(\mathbb{C})$.

Knot of approximable mappings

- Knot of approximable mapping dual to tie of a function.
- $R : B_U \to B_{\mathbb{C}}$ an approximable mapping.
- $O \in B^0_{\mathbb{C}}$, $a \in B^0_U$.
- $a, O \neq 0, 1$, the bottom and top elements of the lattice.
- R has Lipschitzian constant O in a, denoted $R \in \Delta(a, O)$ if,

$$\forall a_1 \forall a_2 \in B^0_U.\ a_1, a_2 \prec a \ \&\ \overline{a_1} \cap \overline{a_2} = \emptyset$$

$$\exists a'_1 \exists a'_2 \in B^0_{\mathbb{C}}.\ a_1 \, R \, a'_1 \ \&\ a_2 \, R \, a'_2 \ \&\ a'_1 - a'_2 \prec O \cdot (a_1 - a_2).$$

Stone duality of strong ties and strong knots

- Strong tie: $f \in \delta_s(b\chi_a)$, if $\exists a'.\ a \ll_{\Omega(U)} a'$ & $\exists b'.\ b \ll_{\mathbf{C}(\mathbb{C})} b'$ such that $f \in \delta(b'\chi_{a'})$.
- Strong knot: $R \in \Delta_s(a, O)$, if $\exists a'.\ a \prec a'$ & $\exists O'.\ O' \prec O$ such that $R \in \Delta(a', O')$.

Theorem:

- $R \in \Delta_s(a, O)$ iff $G_R \in \delta_s(\overline{O}\chi_a)$.
- $f \in \delta_s(b\chi_a)$ iff $A_f \in \Delta_s(a, b^\circ)$.

Example

- Consider conjugate function, i.e, $z \to \bar{z} : \mathbb{C} \to \mathbb{C}$.
- $f \in \delta(b\chi_a) \iff D(-1, 1) \subseteq b$.
- Thus, $\mathcal{L}f(z) = D(-1, 1)$.

Stone duality for analytic functions

- $R \in \Delta^1(U)$ if:

$$\forall a_0 \prec 1 \ \forall \epsilon > 0 \ \exists \delta > 0. \ a \prec a_0 \ \& \ \mathrm{diam}(a) < \delta$$
$$\exists O \in B^0_{\mathbb{C}}. \ \mathrm{diam}(O) < \epsilon \ \& \ R \in \Delta_s(a, O)$$

- $\Delta^1(U)$ class of approximable mapping representing analytic functions.

- f is analytic iff $A_f \in \Delta^1(U)$.

- $R \in \Delta^1(U)$ iff G_R is analytic.

Derivative of approximable mappings

- Single-step approximable mapping $\eta_{(a,O)} : B_U \to B_{\mathbf{C}(\mathbb{C})}$ defined by:

$$\eta_{(a,O)} = A_{\overline{O}\chi_a}$$

- Let $R : B_U \to B_{\mathbb{C}}$ be a Lipschitzian approximable mapping. The Lipschitzian derivative of R is defined as

$$\mathsf{L}(R) = \sup\{\eta_{(a,O)} : R \in \Delta_s(a, O)\}$$

- Stone duality, $\mathsf{L}(R) = A_{\mathcal{L}G_R}$.

- $\mathsf{L}(A_f) = A_{\mathcal{L}f}$.

Calculus of the Lipschitzian derivative

- $R_1, R_2 : B_U \to B_{\mathbb{C}}$ approximable mappings. Then:

$$\mathsf{L}(R_1) + \mathsf{L}(R_2) \subseteq \mathsf{L}(R_1 + R_2)$$

  equality holds if R_1 or R_2 be in $\Delta^1(U)$.

- $R_1, R_2 : B_U \to B_{\mathbb{C}}$ approximable mappings. Then:

$$R_1 \cdot \mathsf{L}(R_2) + R_2 \cdot \mathsf{L}(R_1) \subseteq \mathsf{L}(R_1 \cdot R_2)$$

  equality holds if R_1 or R_2 be in $\Delta^1(U)$.

- $R_1 : B_{U_1} \to B_{\mathbb{C}}$ and $R_2 : B_{U_2} \to B_{\mathbb{C}}$ approximable mappings, $U_1 \subseteq \mathbb{C}$, $U_2 \subseteq \mathbb{C}$ and $\mathrm{Im}(R_1) \subseteq B_{U_2}$. Then:

$$(\mathsf{L}(R_2) \circ R_1) \cdot \mathsf{L}(R_1) \subseteq \mathsf{L}(R_2 \circ R_1)$$

  equality holds if $R_1 \in \Delta^1(U_1)$ or $R_2 \in \Delta^1(U_2)$.

Conclusion and future work:

- Implementation in Haskell and Coq.
- Validation of Automatic Differentiation.