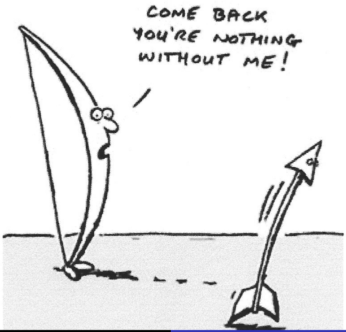
Topology and Domain Theory Interfaces

Jimmie Lawson

July, 2018

Introduction

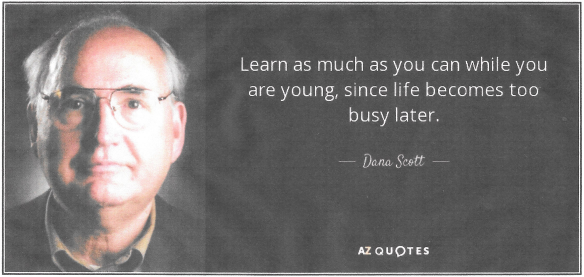
Domain theory began roughly 50 years ago as an outgrowth of theories of order. But progress in the theory rapidly required large doses of (non-Hausdorff) topology, much of which was developed in the context of domain theory. Indeed the relationship between domain theory and non-Hausdorff topology might well be viewed as a mathematical symbiosis.



The Scott topology

The Scott closed sets on a dcpo (directed complete partially ordered set) are those sets A that are lower sets ($\downarrow A = A$) and closed with respect to taking directed suprema.

Dana Scott introduced this non-Hausdorff topology in the early seventies when almost universally all spaces were assumed to be Hausdorff. The suitability of this topology for domain theory has been vindicated many times over, but in addition it has opened up new and fruitful directions of research in topology.



Interface 1: Scott Spaces

Dcpos equipped with the Scott topology (Scott spaces) have proved a rich source for a variety of non-Hausdorff examples with varied topological properties.

An early question was whether all Scott spaces were sober. Peter Johnstone provided the now classic counterexample, which John Isbell supercharged to a non-sober complete lattice.

On the positive side it was shown that continuous domains, the workhorses of domain theory, were sober in their Scott topologies, and this result extended to the more general quasicontinuous domains.

Well-filtered and Coherent Spaces

Compactness is a much weaker topological property in non-Hausdorff spaces. Some important, basic topological properties associated with compactness that are automatically satisfied in Hausdorff spaces assume a life of their own in the non-Hausdorff setting.

Definition

1. A topological space is *well-filtered* if any filtered family of compact saturated* sets with intersection in an open set must have some member of the family contained in the open set.
2. A topological space is *coherent* if the intersection of any two compact saturated sets is again compact.

*A subset is *saturated* if it is an intersection of open sets.

Dual and Patch Topologies

Defining the upper set of a point in a T_0-space (X, τ) to be the intersection of all open sets containing it defines a partial order, called the *order of specialization*. A topology τ^* is a *dual topology* if its order of specialization is the reverse order. Given a topology and a dual topology, one defines the *patch topology* to be the join of the two topologies. These various auxiliary topologies greatly enrich the study of T_0-topologies.

The *de Groot* dual takes as a subbasis of open sets all complements of compact saturated sets. The smallest dual topology on X has as subbasis all sets of the form $X \setminus \uparrow x$, where $\uparrow x$ is defined with respect to the order of specialization. For a Scott space, the patch topology of the Scott topology and the smallest dual topology is called the *Lawson topology*.

A sober space is well-filtered. R. Heckmann asked if the converse held for Scott spaces, and H. Kou (1999) gave a counterexample, but showed equivalence in the locally compact setting.
The next results show relationships among previously presented properties in the setting of Scott spaces.

Very Recent Results (X. Jia, A. Jung, Q. Li and X. Xi, Lawson)

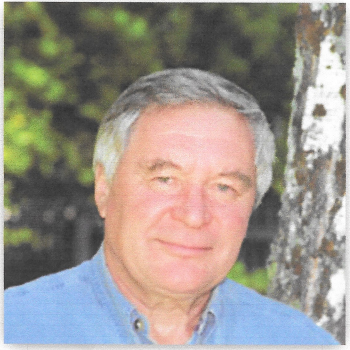
The following are equivalent in a Scott space X.

1. X is well-filtered and $\uparrow x \cap \uparrow y$ is compact for all $x, y \in X$;
2. X is well-filtered and coherent;
3. $\uparrow x$ is compact in the Lawson topology for all $x \in X$.

It follows immediately that the non-sober Isbell lattice is well-filtered.

Topological Domain Theory

Yuri Ershov has been the principal advocate of a topological approach to domain theory, in contrast to the more standard order-theoretic approach. At the foundational level this involves translating the two basic notions of directed completeness and a suitable approximation or "way-below" relation $\ll$ into the topological setting.



Monotone Convergence Spaces

To translate directed completeness we replace a dcpo equipped with the Scott topology by a *monotone convergence space*, a T_0-space in which every subset D that is directed in the order of specialization has a supremum to which it converges.

Such spaces were apparently first introduced by O. Wyler (1981) under the designation *d*-spaces. He gave a construction for the *d*-completion of a T_0-space, which is a categorical reflection of the category of T_0-spaces into the full subcategory of monotone convergence spaces.

Ershov (1999) later investigated d-spaces in their own right and showed that, for any subspace X of a monotone convergence space Y, there is a smallest directed complete subspace X^d containing X, and that the inclusion is a *d*-completion.

c-Spaces

The topological translation of $x \ll y$ is $y \in \mathrm{int}(\uparrow x)$, where $\uparrow x$ is taken in the order of specialization. A space X is a *c-space* if for $x \in U$ open, there exists $z \in U$ satisfying $x \in \mathrm{int}(\uparrow z)$. The terminology is that of M. Erné (1981); they were called α-spaces by Ershov (1997).

Theorem (Ershov)

The following are equivalent for a T_0-space X.

1. X is a c-space.
2. X is (homeomorphic to) a basis for a continuous domain E (with the relative Scott topology).

In the preceding case the inclusion of X into E is the d-completion of X, hence the continuous domain is uniquely determined.

A Rigidity Theorem. A monotone convergence c-space is a continuous domain with the Scott topology.

Given any topological space X, it is natural to consider its lattice of open sets $\mathcal{O}(X)$. This is part of a categorical duality, the other direction assigning to a (primally generated) complete distributive lattice its spectrum of prime elements. Klaus Keimel early on (1976?) noted a connection with domain theory: For a locally compact space X, the lattice $\mathcal{O}(X)$ is continuous.

Theorem (Hofmann, Lawson (1978))

For a space X the following are equivalent:

1. X is core compact (a slight generalization of locally compact).
2. The sobrification of X is locally compact.
3. $\mathcal{O}(X)$ is a continuous lattice.

A Surprising Characterization

Around 1979 R.-E. Hoffman and Lawson independently showed that the lattice of open sets of a continuous domain with the Scott topology is completely distributive. This was later mildly generalized to the following.

Theorem

The following are equivalent for a T_0-space X.

1. X is a c-space.
2. The sobrification of X is a continuous domain equipped with the Scott topology.
3. $\mathcal{O}(X)$ is a completely distributive lattice.

Furthermore, every completely distributive lattice is isomorphic to $\mathcal{O}(X)$ for a unique continuous domain X.

Quasicontinuity

By replacing the single element y in $y \ll x$ by a finite set $F \ll x$ in the definition of domain resp. c-space, one obtains a *quasicontinuous domain* resp. *qc-space*. Substantial portions of domain theory carry over to the quasicontinuous setting.

Theorem

The following are equivalent for a T_0-space X.

1. X is a qc-space.
2. The sobrification of X is a quasicontinuous domain equipped with the Scott topology.
3. $\mathcal{O}(X)$ is a hypercontinuous lattice (continuous + T_2 interval topology).

Rigidity Theorem II. A monotone convergence qc-space is a quasicontinuous domain equipped with the Scott topology.

Rudin's Lemma: The Backbone of Quasicontinuity

Rudin's Lemma: If $\mathcal{F}$ is a family of finite nonempty subsets of partially ordered set that is filtered in the sense that given $E, F \in \mathcal{F}$, there exists $G \in \mathcal{F}$ such that $G \subseteq \uparrow E \cap \uparrow F$, then there exists a directed set $D \subseteq \bigcup_{F \in \mathcal{F}} F$ such that $D \cap F \neq \emptyset$ for all $F \in \mathcal{F}$.

When the authors Gierz, Lawson, and Stralka were at a standstill in developing the theory of quasicontinuous domains, Stralka consulted a senior topologist colleague, F. Burton Jones, who contacted Mary Ellen Rudin, arguably the best-known female topologist of her day, who derived and supplied the crucial lemma needed to make things work. Another great example of topology and domain theory coming together.

Models

A *model* for a T_1-space X is a dcpo P such that X is homeomorphic to $\mathrm{MaX}(P)$ equipped with the relative Scott topology. Standard examples are the interval domain for the real numbers and the formal ball domain for a complete metric space.

A Brief Model Bibliography

1. K. Weihrauch, U. Schreiber, Embedding metric spaces into cpo's, 1981.
2. T. Kamimura, A. Tang, Total objects of domains, 1984.
3. J.D. Lawson, Spaces of maximal points, 1997.
4. A. Edalat, R. Heckmann, A computational model for metric spaces, 1998.
5. K. Martin, Domain theoretic models of topological spaces, 1998.
6. R. Kopperman, A. Künzi, P. Waszkiewicz, Bounded complete models of topological spaces, 2004.
7. P. Waszkiewicz, How do domains model topologies?, 2004.
8. H. Bennett, D. Lutzer, Domain representable spaces, 2006.
9. M. Erné, Algebraic models for T_1-spaces, 2011.
10. D. Zhao, X. Xi, Dcpo models of T_1 topological spaces, 2018.

Important Results

The results of the preceding papers tend to show how to build a domain model for certain classes of topological spaces, but another type of result states that spaces with domain models, sometimes ones of a certain type, must have certain topological properties.

Theorem (K. Martin)

A topological space with a domain model is Choquet complete, i.e., has a winning strategy for the (topological) Choquet game.

As a corollary, a metric space has a domain model if and only if is completely metrizable. In this setting directed completeness corresponds to metric completeness.

Open Problem: Characterize those spaces that have domain models.

In contrast to the domain setting D. Zhou and X. Xi (2018) have given a model construction for the following general result.

Theorem. Every T_1-space has a dcpo model.

Their model is one in which each principal ideal $\downarrow x$ is a quasicontinuous (actually quasialgebraic) dcpo. It is shown that sobriety and other topological properties hold for a space iff they hold for the model.
The authors also give an example of a T_1-space that has no bounded complete dcpo for a model.

Stably Compact Spaces

An important and basic class of T_0-spaces that has emerged primarily through the study of domain theory is the class of *stably compact spaces*: T_0-spaces that are locally compact, compact, coherent, and well-filtered.

These spaces play a role in the theory of T_0-spaces similar to that played by compact spaces in the Hausdorff setting.

Nachbin Pospaces

A *Nachbin pospace* is a compact Hausdorff space equipped with a closed partial order, i.e., a partial order such that $\{(x,y) : x \leq y\}$ is a closed subset of $X \times X$. Nachbin pospaces are a counterpart of stably compact spaces, as the following statements make clear.

(1). For any Nachbin pospace X, the collection $\mathcal{U}$ of open upper sets forms a topology and $(X,\mathcal{U})$ is a stably compact space.
(2). For any stably compact space $(X,\mathcal{U})$, the space $(X,\mathcal{U}^{\#})$ equipped with the order of specialization is a Nachbin pospace, where $\mathcal{U}^{\#}$ is the patch topology of $\mathcal{U}$ and $\mathcal{U}^d$, the cocompact topology of the de Groot dual.
(3). The operations of (1) and (2) are inverse operations.

Examples

(1). A compact Hausdorff space is a stably compact space and its associated Nachbin pospace is the same space equipped with the trivial identity order.

(2). The unit interval equipped with the Scott topology is stably compact, and its associated Nachbin space is the unit interval with its usual topology and order.

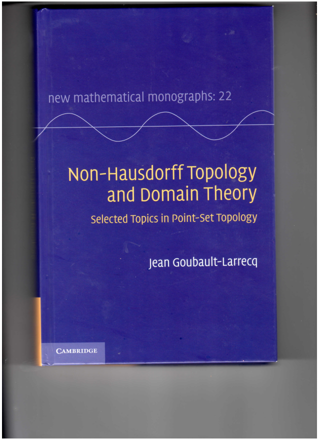
(3). For a compact Hausdorff space X, the space $\mathcal{K}(X)$ of nonempty compact subsets equipped with the upper Vietoris topology (= the Scott topology) with basic open sets $\Box U = \{K \in \mathcal{K}(X) : K \subseteq U\}$, U open in X, is a stably compact space. The cocompact topology is the lower Vietoris topology, and the associated Nachbin space is $\mathcal{K}(X)$ equipped with the Vietoris topology and the reverse order of inclusion.

Klaus Keimel

Much more about stably compact spaces can be found in Chapter VI of **Continuous Lattices and Domains**, and also other things I've talked about. Klaus Keimel assumed a lion's share of the work in bringing that book to publication, and in my opinion it would not have seen the light of day without his work. So I wish to salute him on this occasion and express my gratitude for a number of occasions that we were able to collaborate and for the privilege of knowing him over many years as a good friend.

And Also .,..

A more recent and much more detailed treatment of many themes of this talk is the text of Jean Goubault-Larrecq.



THE END