Higher-dimensional categories: induction on extensivity

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- Enrich this to define weak *n*-dimensional V-categories.
- Build dimensions through iterated enrichment.

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Examples: **Set**, ω -**Cpo**, **Cat**, \mathcal{V} -**Cat** and \mathcal{V} -**Gph** (for extensive \mathcal{V}).

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V-Cat⁽ⁿ⁾ and V-Gph⁽ⁿ⁾

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When $V = \mathbf{Set}$, \mathbf{Set} - $\mathbf{Cat}^{(n)} = n$ - \mathbf{Cat} , the category of strict n-categories.

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Corollary

If V is extensive and finitely complete, then V- $\mathbf{Gph}^{(n)}$ and V- $\mathbf{Cat}^{(n)}$ are also extensive and finitely complete.

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and the induced monad $T^{(n)}$ is cartesian.

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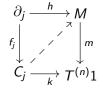
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- Classical operads: $C = \mathbf{Set}$, T = free monoid.
- For weak *n*-dimensional \mathcal{V} -categories, use $\mathcal{C} = \mathcal{V}$ -**Gph**⁽ⁿ⁾, $T = T^{(n)}$.

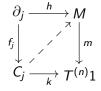
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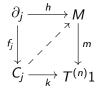


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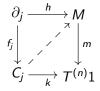
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A weak n-dimensional V-category is an algebra for the initial $T^{(n)}$ -operad with contraction.

For $V = \mathbf{Set}$, this agrees with Leinster's definition of weak *n*-category.