Fundamentals of Orbital Mechanics

Sir Fig Newton and Johannes Momma
Purdon't University, School of Aeronautics and Astronautics

There are many publications that purportedly cover the basics of astrodynamics and orbital mechanics, and many even claim to be catered to an uninitiated audience. However, many of them take for granted some prior knowledge of orbital mechanics, and neglect the ever-important step of ensuring that the reader can follow *every* step of the process. This document intends to rectify this by covering the basics of orbital dynamics, from derivations of orbital motion to complex orbital maneuvers, while ensuring the reader can follow every step. This means algebraic steps are shown, motivations are discussed (where possible) instead of completing procedures seemingly out of the blue, and conclusions are summarized. It is reccomended that the reader view the appendix first, to get a sense for some terminology that is used throughout this document.

Contents

1 Definitions of Motion

Before any orbital parameters can be defined, basic equations describing orbits must be found. The radial nature at which gravity acts lends itself quite well to polar expression, so this process will begin with expression of motion in polar coordinates

$$\vec{r}(t) = \langle r\cos(\theta), r\sin(\theta)\rangle$$

This position vector can be differentiated to obtain a velocity vector.

$$\begin{aligned}
\vec{v}(t) &= \dot{\vec{r}}(t) \\
&= \langle \frac{d}{dt} r\cos(\theta), \frac{d}{dt} r\sin(\theta)\rangle \\
&= \langle \dot{r}\cos(\theta) - r\dot{\theta}\sin(\theta), \dot{r}\sin(\theta) + r\dot{\theta}\cos(\theta)\rangle \\
&= \dot{r}\hat{u}_r + r\dot{\theta}\hat{u}_\theta
\end{aligned}$$

This can, in turn, be differentiated again to yield acceleration.

$$\begin{aligned}
\vec{a}(t) &= \dot{\vec{v}}(t) \\
&= \langle \frac{d}{dt}\dot{r}\cos(\theta) - \frac{d}{dt} r\dot{\theta}\sin(\theta), \frac{d}{dt}\dot{r}\sin(\theta) + \frac{d}{dt} r\dot{\theta}\cos(\theta)\rangle \\
&= \langle \ddot{r}\cos(\theta) - 2\dot{r}\dot{\theta}\sin(\theta) - r\ddot{\theta}\sin(\theta) - r\dot{\theta}^2\cos(\theta), \ddot{r}\sin(\theta) + 2\dot{r}\dot{\theta}\cos(\theta) + r\ddot{\theta}\cos(\theta) - r\dot{\theta}^2\sin(\theta)\rangle
\end{aligned}$$

This expression can now reorganized. The motivation of this is to allow the setup of a differential equation.

$$\begin{aligned}
\vec{a}(t) &= \langle \ddot{r}\cos(\theta) - 2\dot{r}\dot{\theta}\sin(\theta) - r\ddot{\theta}\sin(\theta) - r\dot{\theta}^2\cos(\theta), \ddot{r}\sin(\theta) + 2\dot{r}\dot{\theta}\cos(\theta) + r\ddot{\theta}\cos(\theta) - r\dot{\theta}^2\sin(\theta)\rangle \\
&= \langle (\ddot{r} - r\dot{\theta}^2)\cos(\theta) + (2\dot{r}\dot{\theta} + r\ddot{\theta})(-\sin(\theta)), (\ddot{r} - r\dot{\theta}^2)\sin(\theta) + (2\dot{r}\dot{\theta} + r\ddot{\theta})\cos(\theta)\rangle \\
&= (\ddot{r} - r\dot{\theta}^2)\langle \cos(\theta), \sin(\theta)\rangle + (2\dot{r}\dot{\theta} + r\ddot{\theta})\langle -\sin(\theta), \cos(\theta)\rangle \\
&= (\ddot{r} - r\dot{\theta}^2)\hat{u}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{u}_\theta
\end{aligned}$$

This can also all be calculated using the Transport Theorem (or Basic Kinematic Equation) so that it is more generalized to allow for inclined orbits. The Transport Theorem allows differentiation in the basis frame of a vector $\vec{u}$ expressed in the polar frame. The polar frame will be defined with basis vectors $\hat{u}_r$ (pointing to the satellite), $\hat{u}_\theta$ (perpendicular to $\hat{r}$ in the general direction of orbit, and in plane with the orbit), and $\hat{u}_n$ (normal to the orbit and in line with the right-hand-rule such that $\hat{u}_r \times \hat{u}_\theta = \hat{u}_n$). This basis frame has basis vectors $\hat{x}$, $\hat{y}$, and $\hat{z}$. The polar frame is rotating counterclockwise (as viewed from the $+\hat{u}_n$ direction looking down) at an angular velocity of ${}^b\omega^p = \dot{\theta}\hat{u}_n$. Note that $\hat{u}_n$ is not necessarily in the direction of $\hat{z}$ See Figures 26 and 27.

The transport theorem states that

$$\frac{d^{xyz}}{dt}\vec{V}^{r\theta n} = \frac{d^{xyz}}{dt}\vec{V}^{r\theta n} + {}^{xyz}\omega^{r\theta n} \times \vec{V}^{r\theta n}$$

where $\vec{V}^{r\theta n}$ denotes the vector $\vec{V}$ expressed in the polar frame, $\frac{d^{xyz}}{dt}$ and $\frac{d^{r\theta n}}{dt}$ denote the time derivatives in the basis and polar frames respectively, and ${}^{xyz}\omega^{r\theta n}$ denotes the angular velocity of the polar frame in basis frame.

$$\vec{r}(t) = r\hat{u}_r$$

$$\vec{v}(t) = \frac{d^{xyz}}{dt}\vec{r}(t)$$

$$
\begin{aligned}
&= \dot{r}\hat{u}_r + (\dot{\theta}\hat{u}_n \times r\hat{u}_r) \\
&= \dot{r}\hat{u}_r + r\dot{\theta}\hat{u}_\theta
\end{aligned}
$$

$$
\begin{aligned}
\vec{a}(t) &= \frac{d^{xyz}}{dt}\vec{v}(t) \\
&= \ddot{r}\hat{u}_r + (\dot{r}\dot{\theta} + r\ddot{\theta})\hat{u}_\theta + (\dot{\theta}\hat{u}_n \times (\dot{r}\hat{u}_r + r\dot{\theta}\hat{u}_\theta)) \\
&= \ddot{r}\hat{u}_r + (\dot{r}\dot{\theta} + r\ddot{\theta})\hat{u}_\theta + (\dot{r}\dot{\theta}\hat{u}_\theta - r\dot{\theta}^2\hat{u}_r) \\
&= (\ddot{r} - r\dot{\theta}^2)\hat{u}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{u}_\theta + 0\hat{u}_n
\end{aligned}
$$

To summarize the findings from this section:

$$\vec{r} = r\hat{u}_r \tag{1}$$

$$\vec{v} = \dot{r}\hat{u}_r + r\dot{\theta}\hat{u}_\theta \tag{2}$$

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{u}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{u}_\theta + 0\hat{u}_n \tag{3}$$

1.1 Gravity

Before any orbital parameters can be defined, basic equations describing orbits must be found. The radial nature at which gravity acts lends itself quite well to polar expression, so this process will begin with expression of motion in polar coordinates

Sir Isaac Newton's equation of gravity (replacing the product GM with μ by convention) is

$$\vec{F} = \frac{-\mu m}{r^2}\hat{r}$$

This can be expressed easily in the polar frame as

$$\vec{F} = \frac{-\mu m}{r^2}\hat{u}_r + 0\hat{u}_\theta + 0\hat{u}_n \tag{4}$$

1.2 Differential Equation Setup

With both force (Equation (4)) and acceleration (Equation (3)) known, Newton's second law ($\vec{F} = m\vec{a}$) can be applied.

$$
\begin{aligned}
\vec{F} &= m\vec{a} \\
\frac{-\mu m}{r^2}\hat{u}_r + 0\hat{u}_\theta + 0\hat{u}_n &= m((\ddot{r} - r\dot{\theta}^2)\hat{u}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{u}_\theta + 0\hat{u}) \\
\frac{-\mu}{r^2}\hat{u}_r + 0\hat{u}_\theta + 0\hat{u}_n &= (\ddot{r} - r\dot{\theta}^2)\hat{u}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{u}_\theta + 0\hat{u}
\end{aligned}
$$

This can be broken down into a set of two differential equations (one for equality in $\hat{u}_r$ and another for equality in $\hat{u}_\theta$. Equality in $\hat{u}_n$ is trivial).

$$\frac{-\mu}{r^2} = \ddot{r} - r\dot{\theta}^2 \tag{5a}$$

$$0 = 2\dot{r}\dot{\theta} + r\ddot{\theta} \tag{5b}$$

The formation of these differential equations allows for a solution to be found.

1.3 Conservation of Angular Momentum

This brief interruption from pure calculus and differential equations, while sudden, will prove quite useful soon. For reasons that will become apparent in future sections, conservation of angular momentum must be proven. Torque causes change in angular momentum. A satellites angular momentum can therefore only be changed by forces acting along some vector that is not parallel to the displacement vector of the satellite from the orbited body. The only force applied to the satellite is gravity, which acts along that vector. From this physical reasoning, angular momentum is constant. However, this can also be determined mathematically. Instead of looking at total angular momentum which is defined as

$$L = |\vec{r} \times \vec{p}|$$

This section will analyze *specific* angular momentum, which is defined as

$$\begin{aligned} h &= \frac{L}{m} \\ &= |\frac{\vec{r} \times \vec{p}}{m}| \\ &= |\vec{r} \times \vec{v}| \\ &= rv_{\perp} \\ &= r(r\dot{\theta}) \\ &= r^2\dot{\theta} \end{aligned}$$

This equation will be rewritten here for future use

$$h = r^2\dot{\theta} \tag{6}$$

Note for future derivations that (6) can be rewritten as

$$\dot{\theta} = \frac{h}{r^2}$$

Angular momentum can be differentiated to prove conservation

$$\begin{aligned} \dot{h} &= (r^2\dot{\theta})' \\ &= (2r\dot{r}\dot{\theta} + r^2\ddot{\theta}) \\ &= r(2\dot{r}\dot{\theta} + r\ddot{\theta}) \\ &= ra_{\theta} \\ &= 0 \end{aligned}$$

With a derivative of zero, angular momentum must be conserved

1.4 Differential Equations Solution

We now return to the set of differential equations (Equations (5a) and (5b))

$$\ddot{r} - r\dot{\theta}^2 = \frac{-\mu}{r^2} \qquad\qquad 2\dot{r}\dot{\theta} + r\ddot{\theta} = 0$$

These differential equations, however, are not independently solvable. This differential equation will instead be solved for r in terms of θ. While the current differential equation describes r and θ both in terms of t, the switch must now be made to a time-invariant approach to express independent variable r in terms of dependant variable θ.

$$\frac{dr}{d\theta} = \frac{dr/dt}{d\theta/dt} = \frac{\dot{r}}{\dot{\theta}}$$

So far, there is no formula for $\dot{r}$ or $\dot{\theta}$, so a substitution must be made.

$$u = r^{-1} = \frac{1}{r}$$
$$\dot{u} = -r^{-2}\dot{r} = \frac{-\dot{r}}{r^2}$$

The first and second derivatives of u with respect to θ will now be found. Recall from Equation (6) that $\dot{\theta} = \frac{h}{r^2}$

$$\begin{aligned}
\frac{du}{d\theta} &= \frac{\dot{u}}{\dot{\theta}} \\
&= \frac{-\dot{r}/r^2}{h/r^2} \\
&= \frac{-\dot{r}}{h} \\
\frac{d^2u}{d\theta^2} &= \frac{d}{d\theta}(\frac{du}{d\theta}) \\
&= \frac{d}{dt}\frac{dt}{d\theta}\frac{du}{d\theta} \\
&= \frac{\frac{d}{dt}\frac{du}{d\theta}}{\frac{d\theta}{dt}} \\
&= \frac{\frac{d}{dt}(-\dot{r}/h)}{\dot{\theta}} \\
&= \frac{-\ddot{r}/h}{h/r^2} \\
&= \frac{-r^2\ddot{r}}{h^2}
\end{aligned}$$

Returning now to the differential equations, there are now enough equations to apply the requisite substitutions.

$$\begin{aligned}
\ddot{r} - r\dot{\theta}^2 &= \frac{-\mu}{r^2} \\
\ddot{r} - r(\frac{h}{r^2})^2 &= \frac{-\mu}{r^2} \\
\ddot{r}(\frac{-r^2}{h^2}) - r(\frac{h}{r^2})^2(\frac{-r^2}{h^2}) &= \frac{-\mu}{r^2}(\frac{-r^2}{h^2}) \\
\frac{-\ddot{r}r^2}{h^2} + \frac{1}{r} &= \frac{\mu}{h^2} \\
\frac{d^2u}{d\theta^2} + u &= \frac{\mu}{h^2}
\end{aligned}$$

This is a non homogeneous second-order differential equation. The homogeneous solution is

$$u_h(\theta) = \cos(\theta - \omega)$$

While the specific solution is

$$u_s(\theta) = \frac{\mu}{h^2}$$

The solution to the differential equation is therefore

$$\begin{aligned} u(\theta) &= u_h(\theta) + C_1 r_s(\theta) \\ &= \frac{\mu}{h^2} + C_1 \cos(\theta - \omega) \end{aligned}$$

This can now be expressed in terms of r, substituting $u = \frac{1}{r}$

$$\frac{1}{r} = \frac{\mu}{h^2} + C_1 \cos(\theta - \omega) \tag{7}$$

2 Orbit Equation

This section focuses on deriving numerous expressions describing orbits, with the goal of ultimately finding geometric relationships between parameters of the orbit.

2.1 Explicit Orbit Equation

The first equation found will describe r as a function of θ

$$\frac{1}{r} = \frac{\mu}{h^2} + C_1 \cos(\theta - \omega)$$
$$r = \frac{1}{\frac{\mu}{h^2} + C_1 \cos(\theta - \omega)}$$
$$r = \frac{\frac{h^2}{\mu}}{1 + C_1 \frac{h^2}{\mu} \cos(\theta - \omega)} \tag{8}$$

Equation (8) describes a conic section centered at a focus (as will be be proven in Section 3.1), with the major axis being slanted at an angle of ω (which is called the Argument of Periapsis; see Figure 23). In order for this to describe a *closed* orbit, $-1 < \frac{C_1 h^2}{\mu} < 1$ so that the denominator is never zero or negative.

2.2 Semi-Major Axis

The Semi-Major Axis, usually expressed SMA or simply a, is half of the length of the major axis. The major axis of the ellipse spans from the closest point to the focus to the furthest point (which can be found by maximizing and minimizing the denominator of Equation (8) repsectively). The major axis $2a$ is therefore the sum of the two extreme points on on the conic section (see Figure 22).

$$\begin{aligned}
2a &= r_{\text{min}} + r_{\text{max}} \\
&= \frac{\frac{h^2}{\mu}}{1 + C_1 \frac{h^2}{\mu}} + \frac{\frac{h^2}{\mu}}{1 - C_1 \frac{h^2}{\mu}} \\
&= \frac{\frac{h^2}{\mu}(1 - C_1 \frac{h^2}{\mu}) + \frac{h^2}{\mu}(1 + C_1 \frac{h^2}{\mu})}{(1 + C_1 \frac{h^2}{\mu})(1 - C_1 \frac{h^2}{\mu})} \\
&= \frac{2\frac{h^2}{\mu}}{(1 + C_1 \frac{h^2}{\mu})(1 - C_1 \frac{h^2}{\mu})} \\
a &= \frac{\frac{h^2}{\mu}}{(1 + C_1 \frac{h^2}{\mu})(1 - C_1 \frac{h^2}{\mu})}
\end{aligned}$$

2.3 Eccentricity

The eccentricity e of an ellipse is defined as the ratio of the distance from the foci to the center and the semimajor axis (again, see Figure 22).

$$\begin{aligned}
e &= \frac{\text{Focus Distance from Center}}{a} \\
&= \frac{a - r_{\text{min}}}{a}
\end{aligned}$$

$$= \frac{\frac{\frac{h^2}{\mu}}{(1+C_1\frac{h^2}{\mu})(1-C_1\frac{h^2}{\mu})} - \frac{\frac{h^2}{\mu}}{1+C_1\frac{h^2}{\mu}}}{\frac{\frac{h^2}{\mu}}{(1+C_1\frac{h^2}{\mu})(1-C_1\frac{h^2}{\mu})}}$$

$$= \frac{\frac{h^2}{\mu} - \frac{h^2}{\mu}(1 - C_1\frac{h^2}{\mu})}{\frac{h^2}{\mu}}$$

$$= C_1\frac{h^2}{\mu}$$

This allows the orbit equation Equation (8) to be redefined to

$$r = \frac{\frac{h^2}{\mu}}{1 + e\cos(\theta - \omega)} \tag{9}$$

2.4 Semi-Latus Rectum

In an ellipse, the semi-latus rectum p is defined as the perpendicular distance between the focus and the ellipse (see Figure 22). In other words, it is half of the width of the ellipse at the focus. Because Equation (9) has the focus at the origin, the semi-latus rectum occurs when θ is 90 degrees (or $\frac{\pi}{2}$ radians) offset from the apses. The semi-latus rectum can therefore be found by simply evaluating Equation (9) at that point.

$$p = \frac{\frac{h^2}{\mu}}{1 + e\cos(90°)}$$

$$= \frac{\frac{h^2}{\mu}}{1 + e(0)}$$

$$= \frac{h^2}{\mu}$$

Note that this expression for the semi-latus rectum follows only from physical inputs (the gravitational parameter and the angular momentum).

$$p = \frac{h^2}{\mu} \tag{10}$$

Which allows Equation (9) to transform to

$$r = \frac{p}{1 + e\cos(\theta - \omega)} \tag{11}$$

However, the semi-latus rectum term is still unideal; the objective is to express everything in terms of a and e. Recall the logic behind apoapsis and periapsis radii: the apoapsis occurs when the denominator is minimized, while the periapsis occurs when it is maximized. Applying this to Equation (11):

$$r_{\text{ap}} = \frac{p}{1-e} \qquad \frac{p}{1+e} = r_{\text{pe}}$$

$$r_{\text{ap}}(1-e) = p \qquad p = r_{\text{pe}}(1+e)$$

Setting these two equations equal to eachother, we get

$$r_{\text{ap}}(1-e) = r_{\text{pe}}(1+e)$$

From Figure 22, $r_{\text{ap}} + r_{\text{pe}} = 2a$, meaning that $r_{\text{pe}} = 2a - r_{\text{ap}}$

$$\begin{aligned}
r_{\text{ap}}(1-e) &= r_{\text{pe}}(1+e) \\
r_{\text{ap}}(1-e) &= (2a - r_{\text{ap}})(1+e) \\
r_{\text{ap}} - er_{\text{ap}} &= 2a - r_{\text{ap}} + 2ae - er_{\text{ap}} \\
2r_{\text{ap}} &= 2a + 2ae \\
r_{\text{ap}} &= a(1+e) \\
\frac{p}{1-e} &= a(1+e) \\
p &= a(1+e)(1-e)
\end{aligned}$$

$$p = a(1-e^2) \tag{12}$$

This allows Equation (11) to be written as

$$r(\theta) = \frac{a(1-e^2)}{1 + e\cos(\theta - \omega)}$$

The ω term will be dropped, and this equation describes r with respect the the angle θ between a satellite and its periapsis.

$$r(\theta) = \frac{a(1-e^2)}{1 + e\cos(\theta)} \tag{13}$$

Note that for hyperbolic and parabolic orbits, this equation (as well as the similar equations for r in terms of θ) is only valid for $\frac{-\pi}{2} < \theta < \frac{\pi}{2}$.

3 Proofs

3.1 Kepler's First Law

Kepler's first law states that all orbits are conic sections. A conic section, when graphed in cartesian coordinates, follows

$$\left(\frac{x \pm c}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

Where a is the semi-major axis, b is the semi-minor axis, and c is the distance from the origin to the center of the ellipse. Note the $\pm$ is to accommodate an ellipse with either focus on the origin.

The orbit equation with the semi-latus rectum (Equation (11)) will be used, as it is slightly more concise than Equation (13). ω will be set to zero, as it does not dictate the shape of the orbit. For the sake of conversion between polar and cartesian coordinates, the inclination of the orbit will be assumed to be zero ($\hat{u}_n = \hat{z}$). Note that this algebra is somewhat tedious. If there is any section in which the "throw it into Desmos and see if it lines up" approach to math should be taken, this is the section.

$$\begin{aligned}
r &= \frac{p}{1 + e\cos(\theta)} \\
r(1 + e\cos(\theta)) &= p \\
r &= p - er\cos(\theta) \\
r^2 &= (p - er\cos(\theta))^2 \\
x^2 + y^2 &= (p - ex)^2 \\
x^2 + y^2 &= e^2x^2 - 2epx + p^2 \\
x^2 - e^2x^2 + 2epx - p^2 &= -y^2 \\
(1 - e^2)x^2 + 2epx - p^2 &= -y^2 \\
\left(\sqrt{1 - e^2}x + \frac{ep}{\sqrt{1 - e^2}}\right)^2 - \left(\frac{p}{\sqrt{1 - e^2}}\right)^2 &= -y^2 \\
\left(x + \frac{ep}{1 - e^2}\right)^2 - \left(\frac{p}{1 - e^2}\right)^2 &= -\left(\frac{y}{\sqrt{1 - e^2}}\right)^2 \\
\left(\frac{x + \frac{ep}{1-e^2}}{\frac{p}{1-e^2}}\right)^2 - 1 &= -\left(\frac{\frac{y}{\sqrt{1-e^2}}}{\frac{p}{1-e^2}}\right)^2 \\
\left(\frac{x(1 - e^2)}{p} + e\right)^2 - 1 &= -\left(\frac{y\sqrt{1 - e^2}}{p}\right)^2 \\
\left(\frac{x(1 - e^2)}{p} + e\right)^2 + \left(\frac{y\sqrt{1 - e^2}}{p}\right)^2 &= 1 \\
\left(\frac{x(1 - e^2)}{a(1 - e^2)} + e\right)^2 + \left(\frac{y\sqrt{1 - e^2}}{a(1 - e^2)}\right)^2 &= 1 \\
\left(\frac{x}{a} + e\right)^2 + \left(\frac{y}{a\sqrt{1 - e^2}}\right)^2 &= 1
\end{aligned}$$

$$\left(\frac{x+ea}{a}\right)^2+\left(\frac{y}{a\sqrt{1-e^2}}\right)^2=1 \tag{14}$$

This is in the form of

$$\left(\frac{x+c}{a}\right)^2+\left(\frac{y}{b}\right)^2=1$$

The + denotes that the right focus (as opposed to the left one) is at the origin. It was already known that a was the semi-major axis, however this equation also yields some other useful information. This formula implies that the semi-minor axis $b=a\sqrt{1-e^2}$, and that the distance from the origin (which is at the focus of the ellipse) to its center is $c=ea$.

This equation is also valid for hyperbolic orbits, as $a\sqrt{1-e^2}$ will be purely complex, but then will be squared to become negative. For parabolic orbits, however, this equation is not valid. A similar approach is valid for hyperbolic orbits in which $e=1$.

$$\begin{aligned}
r &= \frac{p}{1+e\cos(\theta)} \\
r &= \frac{p}{1+\cos(\theta)} \\
r(1+\cos(\theta)) &= p \\
r+r\cos(\theta) &= p \\
r &= p-r\cos(\theta) \\
r^2 &= p^2-2pr\cos(\theta)+(r\cos(\theta))^2 \\
x^2+y^2 &= p^2-2px+x^2 \\
y^2 &= p^2-2px
\end{aligned}$$

Notice that the multiplicity of y is 2, and the multiplicity of x is 1. This is a parabola that opens up in the $-x$ direction.

3.2 Kepler's Second Law

Kepler's second law states that the line connecting an orbiting body and the body it orbits sweeps out equal area in equal quantities of time.

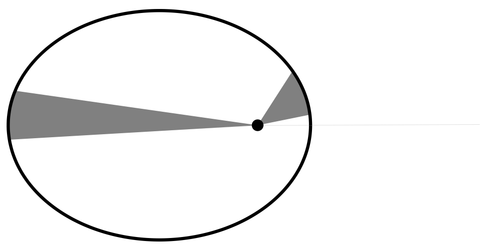

Figure 1: An orbit with multiple areas shaded, each of which is swept out in equal time

To prove this law, the area will be found of a segment of the ellipse tranversed from t_1 to t_2.

$$\text{Area}=\iint dA$$

$$
\begin{aligned}
&= \int_{\theta(t_1)}^{\theta(t_2)} \int_0^r r dr d\theta \\
&= \int_{\theta(t_1)}^{\theta(t_2)} \frac{1}{2} r^2 d\theta \\
&= \int_{t_1}^{t_2} \frac{1}{2} r^2 \frac{d\theta}{dt} dt \\
&= \int_{t_1}^{t_2} \frac{1}{2} r^2 \dot{\theta} dt
\end{aligned}
$$

Recall from Equation (6) that $h = r^2\dot{\theta}$.

$$
\begin{aligned}
\text{Area} &= \int_{t_1}^{t_2} \frac{1}{2} r^2 \dot{\theta} dt \\
&= \int_{t_1}^{t_2} \frac{1}{2} h dt \\
&= \frac{h}{2} \int_{t_1}^{t_2} dt \\
&= \frac{h}{2} \Delta t
\end{aligned}
$$

This expression is expression is independent of where in the orbit a satellite is.

$$
\text{Area Traversed} = \frac{h}{2} \Delta t \tag{15}
$$

3.3 Conservation of Specific Energy

Specific energy ε is, like angular momentum, conserved. It is defined as the sum of specific kinetic and specific potential energy.

$$
\begin{aligned}
\varepsilon &= \frac{\text{KE} + \text{PE}}{m} \\
&= \frac{1}{m} (\text{KE} + \text{PE}) \\
&= \frac{1}{m} \left(\frac{1}{2} m v^2 + \frac{-\mu m}{r} \right)
\end{aligned}
$$

$$
\varepsilon = \frac{v^2}{2} - \frac{\mu}{r} \tag{16}
$$

Conservation of energy can be proven quite simply with physical reasoning, as there is no external force (external to the system, with said system comprising of the orbited body and the orbiting body) to do work. With no external work done, there is no source of energy change. However, it can now be shown that specific energy is conserved through differentiation with respect to distance travelled. Recall that $a_r = \frac{-\mu}{r^2}$ (Equation (5a)) and $a_\theta = 0$ (Equation (5b)) from Section 1.2.

$$\begin{aligned}
\frac{d\varepsilon}{ds} &= \frac{d}{ds}\left(\frac{|v|^2}{2}\right) - \frac{d}{ds}\left(\frac{\mu}{|r|}\right) \\
&= |v|\frac{d|v|}{ds} + \frac{\mu}{|r|^2}\frac{d|r|}{ds} \\
&= |v|\frac{d|v|}{dt}\frac{dt}{ds} + \frac{\mu}{|r|^2}\frac{d|r|}{dt}\frac{dt}{ds} \\
&= \frac{dt}{ds}\left(|v|\frac{d|v|}{dt} + \frac{\mu}{|r|^2}\frac{d|r|}{dt}\right) \\
&= \frac{dt}{ds}\left(|v|a_{\parallel} - a_r\frac{d|r|}{dt}\right) \\
&= \frac{dt}{ds}(|v|(\vec{a}\cdot\hat{v}) - a_r v_r) \\
&= \frac{dt}{ds}(\vec{a}\cdot\vec{v} - a_r v_r) \\
&= \frac{dt}{ds}((a_r v_r + a_\theta v_\theta) - a_r v_r) \\
&= \frac{dt}{ds}(a_\theta v_\theta) \\
&= \frac{dt}{ds}(0 v_\theta) \\
&= 0
\end{aligned}$$

Because the derivative of ε with respect to distance travelled is zero, it does not change across an orbit and will be conserved.

4 Orbit Geometry

Various elements of an orbit can now be expressed simply. Note that there is a lot of variation in standards when it comes to hyperbolas and parabolas. In this document, the convention will be used that $a, c < 0$ in a hyperbola. Note that p is always positive. In a parabola, a is infinite.

Throughout this section, it will be shown that an orbit can be defined uniquely by a and e (except for parabolic orbits).

4.1 Periapsis

From Figure 22, a geometric observation can be made.

$$\begin{aligned} r_{\text{pe}} + c + a &= 2a \\ r_{\text{pe}} &= a - c \\ &= a - ea \\ &= a(1-e) \end{aligned}$$

TThis equation is valid for all elliptic and hyperbolic orbits, but not for parabolic ones (the semi-major axis of a parabola is infinite, while $1 - e$ equals zero in a parabola).

$$r_{\text{pe}} = a(1-e) \tag{17}$$

4.2 Apoapsis

Using the same figure and similar observations and reasoning, the apoapsis radius can be found.

$$\begin{aligned} r_{\text{ap}} &= a + c \\ &= a + ea \\ &= a(1+e) \end{aligned}$$

The apoapsis is not defined for hyperbolic or parabolic orbits.

$$r_{\text{ap}} = a(1+e) \tag{18}$$

4.3 Semi-Minor Axis

The Semi-Minor Axis b is slightly harder to find than the apoapsis and periapsis. Of course, various definitions of eccentricity can be applied, such as $e = \sqrt{1 - \frac{a^2}{b^2}}$, to solve for b, however that's not quite intellectually fulfilling; after all, it's not immediately obvious where that equation came from.

Instead, the definition of an ellipse wil be brought in. In a circle, the distance to the center is always constant. In an ellipse, the sum of distances to the foci is always constant. This distance will be denoted $2d$.

$$\begin{aligned} 2d(\text{At Pe}) &= r_{\text{pe}} + (r_{\text{ap}}) \\ &= 2a \end{aligned}$$

$$d = a$$

A right triangle can then be drawn with which further reasoning can be applied

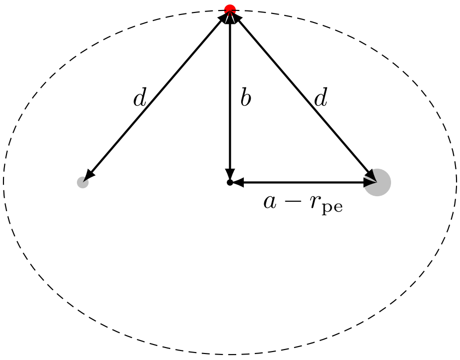

Figure 2: An orbit with relevant parameters labeled, and with the point on the semi-major axis drawn in red . The vacant focus is the smaller circle, with the orbited body being the large gray circle.

The right triangle that will be examined is the one formed by the center of the orbit, the left focus, and the point in the orbit on the semi-minor axis. The Pythagorean Theorem can be applied to find the unknown b, keeping in mind that $d = a$.

$$\begin{aligned}(a - r_{\text{pe}})^2 + b^2 &= d^2 \\ (a - r_{\text{pe}})^2 + b^2 &= a^2 \\ b^2 &= a^2 - (a - r_{\text{pe}})^2\end{aligned}$$

Note from Figure 22 that $r_{\text{pe}} + c = a$, meaning that $a - r_{\text{pe}} = c$, which in turn is defined as $c = ae$

$$\begin{aligned}b^2 &= a^2 - (c)^2 \\ b^2 &= a^2 - (ae)^2 \\ b^2 &= a^2 - a^2e^2 \\ b^2 &= a^2(1 - e^2) \\ b &= \sqrt{a^2(1 - e^2)} \\ b &= a\sqrt{1 - e^2}\end{aligned}$$

The semi-minor axis is not well defined for hyperbolic orbits. The argument of the square root can be put in an absolute value to ensure that the semi-minor axis is real. Note that for some hyperbolic orbits (namely where $e > \sqrt{2}$), $|b| > |a|$.

$$b = a\sqrt{|1 - e^2|} \tag{19}$$

4.4 Geometry in Terms of Measurable Parameters

While e and a are conventionally used as the defining parameters of an orbit, the apses can instead be used to describe elliptic orbits. Once e and a are known, other geometric parameters can be found as before. Hyperbolic and parabolic orbits do not have a defined apoapsis, so this does not apply to them.

4.4.1 Semi-Major Axis in terms of Apses

From Figure 22, a can be found in terms of r_{pe} and r_{ap}

$$2a = r_{\text{pe}} + r_{\text{ap}}$$

$$a = \frac{r_{\text{pe}} + r_{\text{ap}}}{2} \tag{20}$$

4.4.2 Eccentricity in Terms of Apses

The eccentricity can also be put in terms of r_{ap} and r_{pe} using Equations (17) and (18).

$$\begin{aligned} r_{\text{ap}} &= a(1+e) \\ \frac{r_{\text{ap}}}{1+e} &= a \end{aligned} \qquad \begin{aligned} r_{\text{pe}} &= a(1-e) \\ \frac{r_{\text{pe}}}{1-e} &= a \end{aligned}$$

Setting both sides equal to eachother,

$$\begin{aligned}
\frac{r_{\text{ap}}}{1+e} &= \frac{r_{\text{pe}}}{1-e} \\
\frac{1+e}{r_{\text{ap}}} &= \frac{1-e}{r_{\text{pe}}} \\
\frac{1}{r_{\text{ap}}} + \frac{e}{r_{\text{ap}}} &= \frac{1}{r_{\text{pe}}} - \frac{e}{r_{\text{pe}}} \\
\frac{e}{r_{\text{ap}}} + \frac{e}{r_{\text{pe}}} &= \frac{1}{r_{\text{pe}}} - \frac{1}{r_{\text{ap}}} \\
e\left(\frac{1}{r_{\text{ap}}} + \frac{1}{r_{\text{pe}}}\right) &= \frac{1}{r_{\text{pe}}} - \frac{1}{r_{\text{ap}}} \\
e &= \frac{\frac{1}{r_{\text{pe}}} - \frac{1}{r_{\text{ap}}}}{\frac{1}{r_{\text{pe}}} + \frac{1}{r_{\text{ap}}}} \\
&= \frac{\frac{r_{\text{pe}} r_{\text{ap}}}{r_{\text{pe}}} - \frac{r_{\text{pe}} r_{\text{ap}}}{r_{\text{ap}}}}{\frac{r_{\text{pe}} r_{\text{ap}}}{r_{\text{pe}}} + \frac{r_{\text{pe}} r_{\text{ap}}}{r_{\text{ap}}}} \\
&= \frac{r_{\text{ap}} - r_{\text{pe}}}{r_{\text{ap}} + r_{\text{pe}}}
\end{aligned}$$

$$e = \frac{r_{\text{ap}} - r_{\text{pe}}}{r_{\text{ap}} + r_{\text{pe}}} \tag{21}$$

4.5 Conclusion

Throughout this section, many geometric relationships have been found describing orbits. Most importantly, it has been shown that an orbital trajectory can be described in its plane using only two parameters: the semi-major axis a and the eccentricity e. These relationships are summarized here.

The periapsis and apoapsis radii are

$$r_{\text{pe}} = a(1-e) \qquad \text{and} \qquad r_{\text{ap}} = a(1+e)$$

The semi-minor axis of the orbit is

$$b = a\sqrt{|1-e^2|}$$

The semi-major axis and eccentricity of an elliptic orbit can be stated in terms of known periapsis and apoapsis radii

$$a = \frac{r_{\text{pe}} + r_{\text{ap}}}{2} \qquad\qquad e = \frac{r_{\text{ap}} - r_{\text{pe}}}{r_{\text{ap}} + r_{\text{pe}}}$$

Because hyperbolas don't have physically meaningful apoapses, the semi-major axis and eccentricity can be defined in terms of the entry/exit angle θ_{hyp} and the periapsis radius.

$$e = \sec(\theta_{\text{hyp}}) \qquad\qquad a = \frac{r_{\text{pe}}}{1 - \sec(\theta_{\text{hyp}})}$$

5 Physical Orbital Parameters from Geometry

With the geometry of an entire orbit known, physical quantities can be determined.

5.1 Angular Momentum

The angular momentum h of a satellite is, as was proven earlier, conserved. From the physical definition of angular momentum ($\vec{h} = \vec{r} \times \vec{v}$), the following can be observed:

$$h = rv_{\perp} \tag{22}$$

Equation (10), which describes the semi-latus rectum from physical parameters, can be set equal to (12), which describes the semi-latus rectum from geometric definitions. This will allow for the angular momentum to be isolated.

$$\begin{aligned} \frac{h^2}{\mu} &= a(1-e^2) \\ h^2 &= a(1-e^2)\mu \\ h &= \sqrt{a\mu(1-e^2)} \end{aligned}$$

This equation is not defined for parabolic orbits. Note that in a hyperbolic orbit, the signs of a and $1-e^2$ are both negative, so the square root is still real.

$$h = \sqrt{a\mu(1-e^2)} \tag{23}$$

5.2 Period

The area of a closed ellipse is known from geometry.

$$\text{Area} = \pi ab$$

From Kepler's first law (which was proven in Section 3.1), an equation exists that tells us the area of an orbit as a function of time (Equation (15)). If the entire period T is used as the time step, then the area can be found. Because the period is not defined for a hyperbolic orbit, the semi-minor axis will not use an absolute value. However, it turns out that this omission is inconsequential anywho, as the $\sqrt{1-e^2}$ term will cancel out.

$$\begin{aligned} \text{Area} &= \frac{h}{2}P \\ \pi ab &= \frac{h}{2}P \\ \pi a^2\sqrt{1-e^2} &= \frac{h}{2}P \\ \pi a^2\sqrt{1-e^2} &= \frac{\sqrt{a\mu(1-e^2)}}{2}P \\ T &= \frac{2\pi a^2\sqrt{1-e^2}}{\sqrt{a\mu(1-e^2)}} \end{aligned}$$

$$= 2\pi a^2 \sqrt{\frac{1-e^2}{a\mu(1-e^2)}}$$

$$= \sqrt{\frac{4\pi^2 a^4}{a\mu}}$$

$$T = \sqrt{\frac{4\pi^2 a^3}{\mu}} \tag{24}$$

Hyperbolic and parabolic orbits do not have periods, as they are not closed.

5.3 Specific Energy

Specific energy ε is the total energy in an orbit, normalized with respect to the mass of the satellite. Total energy is the sum of the kinetic energy and gravitational potential energy (with the potential constant set so that, as distance approaches infinity, the potential function vanishes).

$$\begin{aligned} \varepsilon &= \frac{\mathrm{KE} + \mathrm{PE}}{m} \\ &= \frac{1}{m}(\mathrm{KE} + \mathrm{PE}) \\ &= \frac{1}{m}\left(\frac{1}{2}mv^2 + \frac{-\mu m}{r}\right) \\ &= \frac{v^2}{2} - \frac{\mu}{r} \end{aligned}$$

Conservation of specific energy was proven earlier, but a more useful expression for it can be found.

Specific energy is constant, so it will be found at periapsis. First, velocity must be found by setting Equation (22) (the physical definition of angular momentum, $rv_\perp$) equal to its derived counterpart ($\sqrt{a\mu(1-e^2)}$) from Equation (23). Note that at periapsis, velocity is entirely normal to the radial vector, so $v_\perp = v$, and recall that Equation (17) states $r_{\mathrm{pe}} = a(1-e)$.

$$\begin{aligned} r_{\mathrm{pe}} v_{\mathrm{pe}} &= \sqrt{\mu a(1-e^2)} \\ a(1-e) v_{\mathrm{pe}} &= \sqrt{\mu a(1-e)(1+e)} \\ v_{\mathrm{pe}} &= \sqrt{\frac{\mu a(1-e)(1+e)}{a^2(1-e)^2}} \\ v_{\mathrm{pe}} &= \sqrt{\frac{\mu(1+e)}{a(1-e)}} \end{aligned}$$

Now, specific energy can be found.

$$\begin{aligned} \varepsilon &= \frac{1}{2}v^2 - \frac{\mu}{r} \\ &= \frac{1}{2}\frac{\mu(1+e)}{a(1-e)} - \frac{\mu}{a(1-e)} \end{aligned}$$

$$
\begin{aligned}
&= \frac{\mu(1+e)}{2a(1-e)} - \frac{2\mu}{2a(1-e)} \\
&= \frac{\mu(1+e) - 2\mu}{2a(1-e)} \\
&= \frac{\mu(e-1)}{2a(1-e)} \\
&= \frac{-\mu}{2a}
\end{aligned}
$$

Note that because a is infinite for a parabola (see 7.2), $\varepsilon = 0$ for parabolic orbits, Since $a < 0$ for hyperbolas, $\varepsilon > 0$ in hyperbolic orbits.

$$\varepsilon = \frac{-\mu}{2a} \tag{25}$$

This is valid for *all* orbits. Of note is the fact that eccentricity plays no role in specific energy of an orbit, aside from to determine its sign in extreme cases. This has an interesting ramification for orbital dynamics; since a burn normal to the velocity vector and in plane with the orbit performs no work, it must not change the specific energy of the orbit. While other parameters, such as inclination, can change without the specific energy changing, that requires out-of-plane maneuvers. This implies (correctly) that burning normal to the velocity has the effect of changing its eccentricity or inclination but not its semi-major axis.

5.4 Velocity

Specific energy found in Equation (25) can be set equal to the physical definition of specific energy, so that v can be solved for.

$$
\begin{aligned}
\frac{1}{2}v^2 - \frac{\mu}{r} &= \frac{-\mu}{2a} \\
\frac{1}{2}v^2 &= \frac{\mu}{r} - \frac{\mu}{2a} \\
v^2 &= \frac{2\mu}{r} - \frac{\mu}{a} \\
v &= \sqrt{\frac{2\mu}{r} - \frac{\mu}{a}}
\end{aligned}
$$

The $\frac{-\mu}{a}$ term vanishes in parabolic orbits, showing that in a parabolic orbit velocity is determined solely by radius (and is independent of periapsis). In a hyperbolic orbit, both terms in the square root are positive, showing that velocity at a given radius is greater in a hyperbolic orbit than in a parabolic or elliptic orbit.

$$v = \sqrt{\frac{2\mu}{r} - \frac{\mu}{a}} \tag{26}$$

This equation describes velocity of a satellite as a function of its distance from the orbited body, the semi-major axis of its orbit, and the standard gravitational parameter. Note that it is defined for elliptic, parabolic, and hyperbolic orbits.

For circular orbits where $a = r$, this simplifies to

$$v_{\text{circ}} = \sqrt{\frac{\mu}{r}} \tag{27}$$

Escape is achieved when a satellite is able to approach infinite distance with a real-valued velocity. The minimum such velocity that the satellite can approach is zero. From Equation (26), this can only happen if a is infinite. This gives escape velocity as a function of radius to be

$$v_{\text{escape}} = \sqrt{\frac{2\mu}{r}} \tag{28}$$

5.5 Flight Path Angle

The flight path angle ϕ is the angle between a satellite's velocity vector and the line (in plane with the orbit) tangent to the orbited body (in other words, the angle between $\vec{v}$ and $\hat{u}_\theta$).

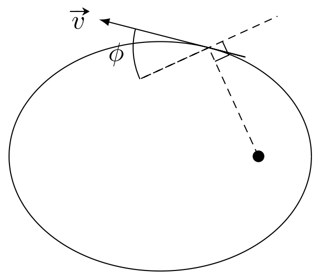

Figure 3: An orbit with the flight path angle ϕ indicated

Angular momentum (Equations (23) and (22)) can be used to find ϕ.

$$\begin{aligned}
rv_\perp &= \sqrt{a\mu(1-e^2)} \\
rv\cos(\phi) &= \sqrt{a\mu(1-e^2)} \\
r\cos(\phi)\sqrt{\frac{2\mu}{r}-\frac{\mu}{a}} &= \sqrt{a\mu(1-e^2)} \\
\cos(\phi)\sqrt{2r\mu-\frac{r^2\mu}{a}} &= \sqrt{a\mu(1-e^2)} \\
\cos(\phi) &= \frac{\sqrt{a\mu(1-e^2)}}{\sqrt{2r\mu-\frac{r^2\mu}{a}}} \\
\cos(\phi) &= \sqrt{\frac{a\mu(1-e^2)}{2r\mu-\frac{r^2\mu}{a}}} \\
\cos(\phi) &= \sqrt{\frac{a^2(1-e^2)}{2ra-r^2}} \\
\cos(\phi) &= \sqrt{\frac{a^2(1-e^2)}{r(2a-r)}}
\end{aligned}$$

Note that even in hyperbolic orbits the argument of the square root is positive, so $\cos(\phi)$ is always real. ϕ can be found with an arccosine, keeping in mind that arccosine loses a solution. ϕ is positive when the satellite is getting further from the orbited body, and negative when it's getting closer. Therefore, the sign of ϕ is the same as the sign as $\vec{r}\cdot\vec{v}$

$$\phi = \arccos\sqrt{\frac{a^2(1-e^2)}{r(2a-r)}} \tag{29}$$

An equation has already been found relating r and θ, meaning this formula *can* be put in terms of θ. Solving for this expression, however, is left as an exercise to the reader.

5.6 True Anomaly

Using the orbit equation (Equation (13)), the angle θ to the periapsis can be found.

$$\begin{aligned} r &= \frac{a(1-e^2)}{1+e\cos(\theta)} \\ 1+e\cos(\theta) &= \frac{a(1-e^2)}{r} \\ \cos(\theta) &= \frac{a(1-e^2)}{er} - \frac{1}{e} \end{aligned}$$

When taking the inverse cosine, a solution will be lost. On the side of the orbit where r is increasing (moving from periapsis to apoapsis) we know that $0 < \theta < \pi$, while on the other side of the orbit $\pi < \theta < 2\pi$. Therefore, the sign of θ (assuming $-\pi < \theta < \pi$ is taken to the the domain) is the same as the sign of v_r, which in turn is the same as the sign of $\vec{v} \cdot \vec{r}$.

$$\theta = \cos^{-1}\left(\frac{a(1-e^2)-r}{er}\right) \tag{30}$$

5.7 Time Since Periapsis

Because of Kepler's second law (Section 3.2), there is a direct analogue between area swept out and time spent on the orbit. This allows the following relationship to be made between the time since periapsis T_p and θ.

$$\begin{aligned} T_p(\theta) &= \frac{T}{\text{Area}} \times \text{Area swept from periapsis to } \theta \\ &= \frac{\sqrt{\frac{4\pi^2 a^3}{\mu}}}{\pi a^2\sqrt{1-e^2}} \int_0^\theta \int_0^{r(\theta)} r dr d\theta \\ &= \frac{2\pi\sqrt{a^3}}{\pi a^2\sqrt{\mu(1-e^2)}} \int_0^\theta \frac{1}{2} r^2(\theta) d\theta \\ &= \frac{2}{\sqrt{\mu a(1-e^2)}} \int_0^\theta \frac{1}{2}\left(\frac{a(1-e^2)}{1+e\cos(\theta)}\right)^2 d\theta \\ &= \frac{\sqrt{a^3(1-e^2)^3}}{\sqrt{\mu}} \int_0^\theta \frac{1}{(1+e\cos(\theta))^2} d\theta \end{aligned}$$

At this point, the transformation will be made to integrate with respect to r instead of θ. To do this, $\frac{d\theta}{dr}$ is needed.

$$\frac{d\theta}{dr} = \frac{1}{dr/d\theta}$$

$$\begin{aligned}
\frac{1}{d\theta/dr} &= \frac{dr}{d\theta} \\
&= \frac{d}{d\theta}\frac{a(1-e^2)}{1+e\cos(\theta)} \\
&= a(1-e^2)\frac{d}{d\theta}(1+e\cos(\theta))^{-1} \\
&= -a(1-e^2)(1+e\cos(\theta))^{-2}(-e\sin(\theta)) \\
&= \frac{ea\sin(\theta)(1-e^2)}{(1+e\cos(\theta))^2} \\
\frac{d\theta}{dr} &= \frac{(1+e\cos(\theta))^2}{ea\sin(\theta)(1-e^2)}
\end{aligned}$$

Now, the integral can be transformed to be with respect to r.

$$\begin{aligned}
T_p &= \frac{\sqrt{a^3(1-e^2)^3}}{\sqrt{\mu}}\int_{r_{\mathrm{pe}}}^{r(\theta)} \frac{1}{\left(1+e\cos\left(\cos^{-1}\left(\frac{a(1-e^2)-r}{er}\right)\right)\right)^2}\frac{\left(1+e\cos\left(\cos^{-1}\left(\frac{a(1-e^2)-r}{er}\right)\right)\right)^2}{ea\sin\left(\cos^{-1}\left(\frac{a(1-e^2)-r}{er}\right)\right)(1-e^2)}dr \\
&= \frac{\sqrt{a^3(1-e^2)^3}}{\sqrt{\mu}}\int_{r_{\mathrm{pe}}}^{r(\theta)} \frac{1}{\left(1+e\left(\frac{a(1-e^2)-r}{er}\right)\right)^2}\frac{\left(1+e\left(\frac{a(1-e^2)-r}{er}\right)\right)^2}{ea\sin\left(\cos^{-1}\left(\frac{a(1-e^2)-r}{er}\right)\right)(1-e^2)}dr \\
&= \frac{\sqrt{a^3(1-e^2)^3}}{\sqrt{\mu}}\int_{r_{\mathrm{pe}}}^{r(\theta)} \frac{1}{ea\sin\left(\cos^{-1}\left(\frac{a(1-e^2)-r}{er}\right)\right)(1-e^2)}dr
\end{aligned}$$

The property can now be applied that $\sin(\arccos(\theta)) = \sqrt{1-\theta^2}$

$$\begin{aligned}
T_p &= \frac{\sqrt{a^3(1-e^2)^3}}{\sqrt{\mu}}\int_{r_{\mathrm{pe}}}^{r(\theta)} \frac{1}{ea\sin\left(\cos^{-1}\left(\frac{a(1-e^2)-r}{er}\right)\right)(1-e^2)}dr \\
&= \frac{\sqrt{a^3(1-e^2)^3}}{\sqrt{\mu}}\int_{r_{\mathrm{pe}}}^{r(\theta)} \frac{1}{ea\sqrt{1-\left(\frac{a(1-e^2)-r}{er}\right)^2}(1-e^2)}dr \\
&= \frac{\sqrt{a^3(1-e^2)^3}}{\sqrt{\mu}}\int_{r_{\mathrm{pe}}}^{r(\theta)} \frac{1}{ea\sqrt{1-\left(\frac{a^2(1-e^2)^2-2ar(1-e^2)+r^2}{e^2r^2}\right)}(1-e^2)}dr \\
&= \frac{\sqrt{a^3(1-e^2)^3}}{\sqrt{\mu}}\int_{r_{\mathrm{pe}}}^{r(\theta)} \frac{1}{ea\sqrt{\frac{e^2r^2-a^2(1-e^2)^2+2ar(1-e^2)-r^2}{e^2r^2}}(1-e^2)}dr \\
&= \frac{\sqrt{a^3(1-e^2)^3}}{\sqrt{\mu}}\int_{r_{\mathrm{pe}}}^{r(\theta)} \frac{er}{ea\sqrt{-r^2(1-e^2)-a^2(1-e^2)^2+2ar(1-e^2)}(1-e^2)}dr \\
&= \frac{\sqrt{a^3(1-e^2)^3}}{\sqrt{\mu}}\int_{r_{\mathrm{pe}}}^{r(\theta)} \frac{r}{a(1-e^2)^{3/2}\sqrt{-r^2-a^2(1-e^2)+2ar}}dr
\end{aligned}$$

$$
\begin{aligned}
&= \sqrt{\frac{a}{\mu}} \int_{r_{\rm pe}}^{r(\theta)} \frac{r}{\sqrt{-r^2 - a^2(1-e^2) + 2ar}} dr \\
&= \sqrt{\frac{a}{\mu}} \int_{r_{\rm pe}}^{r(\theta)} \frac{r}{\sqrt{-(r-a)^2 + a^2(1-(1-e^2))}} dr \\
&= \sqrt{\frac{a}{\mu}} \int_{r_{\rm pe}}^{r(\theta)} \frac{r}{\sqrt{-(a-r)^2 + a^2e^2}} dr \\
&\qquad \text{Substitution: } (a-r) = u_1 \qquad du_1 = -dr \\
&= \sqrt{\frac{a}{\mu}} \int_{a-r_{\rm pe}}^{a-r(\theta)} \frac{u_1 - a}{\sqrt{-u_1^2 + a^2e^2}} du_1 \\
&= \sqrt{\frac{a}{\mu}} \left(\int_{a-r_{\rm pe}}^{a-r(\theta)} \frac{u_1}{\sqrt{a^2e^2 - u_1^2}} du_1 - \int_{a-r_{\rm pe}}^{a-r(\theta)} \frac{a}{\sqrt{a^2e^2 - u_1^2}} du_1 \right) \\
&\qquad \text{Substitution: } a^2e^2 - u_1^2 = u_2 \qquad du_2 = -2u_1 du_1 \\
&= \sqrt{\frac{a}{\mu}} \left(\int_{(ae)^2-(a-r_{\rm pe})^2}^{(ae)^2-(a-r(\theta))^2} \frac{-0.5}{\sqrt{u_2}} du_2 - \int_{a-r_{\rm pe}}^{a-r(\theta)} \frac{a}{ae\sqrt{1-\left(\frac{u_1}{ae}\right)^2}} du_1 \right) \\
&\qquad \text{Substitution: } \frac{u_1}{ae} = u_3 \qquad du_3 = \frac{1}{ae} du_1 \\
&= \sqrt{\frac{a}{\mu}} \left(\frac{-1}{2} \int_{(ae)^2-(a-r_{\rm pe})^2}^{(ae)^2-(a-r(\theta))^2} u_2^{-1/2} du_2 + a \int_{(a-r_{\rm pe})/ae}^{(a-r(\theta))/ae} \frac{-1}{\sqrt{1-u_3^2}} du_3 \right) \\
&= \sqrt{\frac{a}{\mu}} \left(-\Big(\sqrt{u_2}\Big)_{(ae)^2-(a-r_{\rm pe})^2}^{(ae)^2-(a-r(\theta))^2} + a \Big(\cos^{-1}(u_3) \Big)_{(a-r_{\rm pe})/ae}^{(a-r(\theta))/ae} \right) \\
&= \sqrt{\frac{a}{\mu}} \left(a \Big(\cos^{-1}(u_3) \Big)_{(a-r_{\rm pe})/ae}^{(a-r(\theta))/ae} - \Big(\sqrt{u_2}\Big)_{(ae)^2-(a-r_{\rm pe})^2}^{(ae)^2-(a-r(\theta))^2} \right) \\
&= \sqrt{\frac{a}{\mu}} \left(a \cos^{-1}\left(\frac{a-r}{ae} \right) \Big|_{r_{\rm pe}}^{r(\theta)} - \sqrt{(ae)^2 - (a-r)^2} \Big|_{r_{\rm pe}}^{r(\theta)} \right) \\
&= \sqrt{\frac{a}{\mu}} \left(a \cos^{-1}\left(\frac{a-r}{ae} \right) - \sqrt{(ae)^2 - (a-r)^2} \right)_{r_{\rm pe}}^{r(\theta)} \\
&= \sqrt{\frac{a}{\mu}} \left(a \cos^{-1}\left(\frac{a-r}{ae} \right) - a\sqrt{e^2 - \left(1 - \frac{r}{a}\right)^2} \right)_{r_{\rm pe}}^{r(\theta)} \\
&= \sqrt{\frac{a^3}{\mu}} \left(\cos^{-1}\left(\frac{a-r}{ae} \right) - \sqrt{e^2 - \left(1 - \frac{r}{a}\right)^2} \right)_{r_{\rm pe}}^{r(\theta)}
\end{aligned}
$$

When the function is evaluated at $r_{\rm pe}$, the result is 0. This means that the lower bound of the function can be removed, and the upper bound can be set as the singular argument of the function.

The time since periapsis for an orbit can be evaluated at any radius r with

$$
t_p(r) = \sqrt{\frac{a^3}{\mu}} \left(\cos^{-1}\left(\frac{a-r}{ae} \right) - \sqrt{e^2 - \left(1 - \frac{r}{a}\right)^2} \right) \tag{31}
$$

This function is only defined from the periapsis radius to the apoapsis radius, and will return a complex value for any radius outside these bounds. For hyperbolic and parabolic orbits, the function does not apply either. In a circular orbit, the argument of arccosine is $\frac{0}{0}$, so the function cannot be evaluated. This function only covers the first half of an orbit. After apoapsis, an orbit will return to every radius between the apoapsis and periapsis before once more arriving at periapsis. To find the time it takes to reach a point on this side of the orbit, simply subtract the result of this function from the period of the orbit.

5.8 Conclusion

Throughout this section, many geometric relationships have been found describing orbits. Most importantly, it has been shown that an orbital trajectory can be described in its plane using only two parameters: the semi-major axis a and the eccentricity e. These relationships are summarized here.

The angular momentum is

$$h = \sqrt{a\mu(1-e^2)}$$

The specific energy of an orbit is

$$\varepsilon = \frac{v^2}{2} - \frac{\mu}{r} = \frac{-\mu}{2a}$$

The period of an elliptic orbit is

$$P = \sqrt{\frac{4\pi^2 a^3}{\mu}}$$

The velocity at any point in an orbit can be found with

$$v = \sqrt{\frac{2\mu}{r} - \frac{\mu}{a}}$$

Velocity for circular orbit is

$$v_{\text{circ}} = \sqrt{\frac{\mu}{r}}$$

Escape velocity at any altitude is

$$v_{\text{escape}} = \sqrt{\frac{2\mu}{r}} = v_{\text{circ}}\sqrt{2}$$

While the flight path angle can be found as

$$\phi = \arccos\left(\sqrt{\frac{a^2(1-e^2)}{r(2a-r)}}\right)$$

The true anomaly can be found in an orbit with

$$\theta = \arccos\left(\frac{a(1-e^2)-r}{er}\right)$$

The time since periapsis in any noncircular elliptic orbit can be found with

$$t_p(r) = \sqrt{\frac{a^3}{\mu}}\left(\cos^{-1}\left(\frac{a-r}{ae}\right) - \sqrt{e^2 - \left(1-\frac{r}{a}\right)^2}\right)$$

6 Geometry from Physical Parameters

A plausible scenario for a spacecraft is that telemetry systems provide a distance from the orbited planet and current velocity (as a vector in $\vec{u}_r$ and $\vec{u}_\theta$, noting that $v_n = 0$ at all points), while measured or estimated values of μ would be known. In this case, equations so far derived in terms of geometric values such as a and e will prove widely useless until the precise orbit of the spacecraft is known. Therefore, equations will be derived in terms of these physical values to determine geometric parameters of the orbit. Throughout this chapter, v will be used to refer to the magnitude of $\vec{v}$, while v_r and v_θ, and ϕ may also be used. Note that ϕ can be found using the fact that $v_\theta = v\cos(\phi)$ so $\phi = \arccos(v_\theta/v)$, with the sign of ϕ being the same as the sign of v_r.

6.1 Orbit Shape

Many parameters will depend on the type of orbit that a satellite is in. As such, an easy way of determining this must be found. Section 5.3, specifically Equation (16) defined specific energy to be

$$\varepsilon = \frac{1}{2}v^2 - \frac{\mu}{r}$$

For an orbit to be hyperbolic, the satellite must be able to approach infinite distance from the orbited body, with the velocity always remaining positive. This means the kinetic energy term must be greater in magnitude than the potential energy term, thus making the specific energy positive. For the orbit to be parabolic, the satellite's velocity will approach zero as its distance approaches infinity. This requires the kinetic and potential energy terms to have the same magnitude, making specific energy zero. For a closed orbit, the satellite cannot approach infinite distance without kinetic energy being negative. This means the kinetic term is less than the potential term, making elliptic orbit specific energy negative. This can be summarized with the following statement

$$\text{Orbit Shape}: \begin{cases} \text{Elliptic} & v < \sqrt{\frac{2\mu}{r}} \\ \text{Parabolic} & v = \sqrt{\frac{2\mu}{r}} \\ \text{Hyperbolic} & v > \sqrt{\frac{2\mu}{r}} \end{cases} \tag{32}$$

Note that it is astronomically (I will not apologize for that) improbable for an orbit to be parabolic, given that there is only one exact state for a parabolic orbit and any deviation from that state will result in an elliptic or hyperbolic orbit.

This can be used to prove an interesting relationship

$$\begin{aligned} \varepsilon &= \frac{1}{2}v^2 - \frac{\mu}{r} \\ &= \frac{1}{2}\left(v^2 - \frac{\mu}{2r}\right) \\ &= \frac{1}{2}\left(v^2 - \left(\sqrt{\frac{\mu}{2r}}\right)^2\right) \\ &= \frac{1}{2}\left(v^2 - v_{\text{escape}}^2\right) \end{aligned}$$

While this may not be immediately useful, it is nonetheless intriguing.

6.2 Semi-Major Axis

If v and r are known, Equation (26) can be solved for a.

$$
\begin{aligned}
v &= \sqrt{\frac{2\mu}{r} - \frac{\mu}{a}} \\
v^2 &= \frac{2\mu}{r} - \frac{\mu}{a} \\
v^2 - \frac{2\mu}{r} &= -\frac{\mu}{a} \\
\frac{\mu}{a} &= \frac{2\mu}{r} - v^2 \\
\frac{a}{\mu} &= \frac{1}{\frac{2\mu}{r} - v^2} \\
a &= \frac{\mu}{\frac{2\mu}{r} - v^2} \\
a &= \frac{\mu r}{2\mu - rv^2}
\end{aligned}
$$

Note that $\frac{2\mu}{r} - v^2 = v_{\text{escape}}^2 - v^2$.

$$
a = \frac{\mu r}{2\mu - rv^2} = \frac{\mu}{v_{\text{escape}}^2 - v^2} \tag{33}
$$

This shows that the semi-major axis of an orbit is inversely proportional to the difference between the squares of escape velocity and current velocity. This also implies that this difference is conserved throughout an orbit.

6.3 Eccentricity

To determine the eccentricity at any point given v, r, and ϕ, angular momentum must be used.

$$
\begin{aligned}
\sqrt{a\mu(1-e^2)} &= h \\
\sqrt{a\mu(1-e^2)} &= rv\cos(\phi) \\
a\mu(1-e^2) &= r^2v^2\cos^2(\phi) \\
\frac{\mu r}{2\mu - rv^2}\mu(1-e^2) &= r^2v^2\cos^2(\phi) \\
\mu(1-e^2) &= r^2v^2\cos^2(\phi)\frac{2\mu - rv^2}{\mu r} \\
1 - e^2 &= r^2v^2\cos^2(\phi)\frac{2\mu - rv^2}{\mu^2 r} \\
e^2 &= 1 - r^2v^2\cos^2(\phi)\frac{2\mu - rv^2}{\mu^2 r}
\end{aligned}
$$

$$e = \sqrt{1 - r^2 v^2 \cos^2(\phi) \frac{2\mu - rv^2}{\mu^2 r}} \tag{34}$$

6.4 Conclusion

An orbit can take one of three shapes. If the velocity is $v < \sqrt{\frac{2\mu}{r}}$, the orbit is elliptic. If $v = \sqrt{\frac{2\mu}{r}}$ the orbit is parabolic. If the velocity is above escape velocity $v > \sqrt{\frac{2\mu}{r}}$, then the trajectory is be hyperbolic.

The semi-major axis a can be found from measured values

$$a = \frac{\mu r}{2\mu - rv^2} = \frac{\mu}{v_{\text{escape}}^2 - v^2}$$

The eccentricity is

$$e = \sqrt{1 - r^2 v^2 \cos^2(\phi) \frac{2\mu - rv^2}{\mu^2 r}}$$

With the two defining geometric parameters able to be determined from instrumentation readings, equations from Sections 4 and 5 can be used.

7 Analysis of Unbound Trajectories

Hyperbolic and parabolic trajectories present some interesting differences to their elliptic counterparts. These will be examined in this section.

7.1 Hyperbolic Trajectory

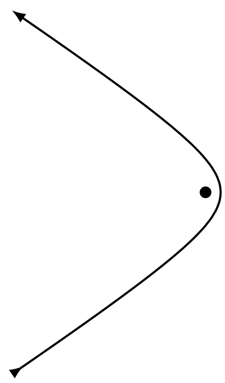

Figure 4: A hyperbolic trajectory

An orbit is described by Equation (13). However, for the sake of analysis, this will be transformed into cartesian coordinates. Equation (14) will be rewritten so that there are no complex arguments. Recall that for hyperbolic orbits, $e > 1$ and $a < 0$.

$$\left(\frac{x+ea}{a}\right)^2 + \left(\frac{y}{a\sqrt{1-e^2}}\right)^2 = 1$$
$$\left(\frac{x+ea}{a}\right)^2 + \left(\frac{y}{ai\sqrt{e^2-1}}\right)^2 = 1$$
$$\left(\frac{x+ea}{a}\right)^2 + \frac{1}{i^2}\left(\frac{y}{a\sqrt{e^2-1}}\right)^2 = 1$$
$$\left(\frac{x+ea}{a}\right)^2 - \left(\frac{y}{a\sqrt{e^2-1}}\right)^2 = 1$$

This trajectory can be parameterized. It can be seen that the radius in the x direction is a, while the radius in the y direction is $a\sqrt{e^2-1}$. The hyperbola is also translated $-ea$. This can be parameterized using hyperbolic sine and hyperbolic cosine. The semi-major axis will also be factored out to simplify the expression.

$$\vec{r}(s) = a\left\langle \cosh(s) - e, \sqrt{e^2-1}\,\sinh(s)\right\rangle \tag{35}$$

The parameter s is used instead of the typical t to show that this parameterization does *not* describe the location of a satellite with respect to time, but instead describes the geometry of the trajectory.

Hyperbolas approach a tangent line in their end behavior. Finding the angle θ_{hyp} of this tangent line to the horizontal will prove useful to analysis of the hyperbola. This will be done by finding the end behavior of the derivative of the parameterization.

$$\vec{r}'(s) = a\left\langle \sinh(s) - e, \sqrt{e^2-1}\,\cosh(s)\right\rangle$$

$$
\begin{aligned}
\theta_{\text{hyp}} &= \lim_{s\to\infty} \arctan\left(\frac{y}{x}\right) \\
&= \lim_{s\to\infty} \arctan\left(\frac{\sqrt{e^2-1}\cosh(s)}{\sinh(s)-e}\right) \\
&= \lim_{s\to\infty} \arctan\left(\sqrt{e^2-1}\frac{\cosh(s)}{\sinh(s)-e}\right) \\
&= \arctan\left(\lim_{s\to\infty}\left(\sqrt{e^2-1}\frac{\cosh(s)}{\sinh(s)-e}\right)\right) \\
&= \arctan\left(\sqrt{e^2-1}\lim_{s\to\infty}\left(\frac{\cosh(s)}{\sinh(s)-e}\right)\right) \\
&= \arctan\left(\sqrt{e^2-1}\lim_{s\to\infty}\left(\frac{\frac{d}{ds}\cosh(s)}{\frac{d}{ds}(\sinh(s)-e)}\right)\right) \\
&= \arctan\left(\sqrt{e^2-1}\lim_{s\to\infty}\left(\frac{\sinh(s)}{\cosh(s)}\right)\right) \\
&= \arctan\left(\sqrt{e^2-1}\lim_{s\to\infty}(\tanh(s))\right) \\
&= \arctan\left(\sqrt{e^2-1}\right)
\end{aligned}
$$

The angle from the horizontal (that is to say, a line passing through the periapsis and the planet) to the beginning/end tangent line is

$$\theta_{\text{hyp}} = \arctan\left(\sqrt{e^2-1}\right) \tag{36}$$

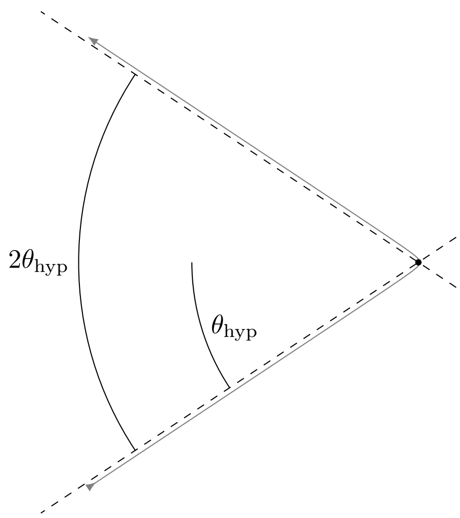

Figure 5: A hyperbolic trajectory with the tangent lines indicated

7.1.1 Hyperbola Eccentricity in Terms of θ_{hyp}

From equation (36), the eccentricity can be isolated.

$$\theta_{\text{hyp}} = \arctan\left(\sqrt{e^2-1}\right)$$

$$
\begin{aligned}
\tan(\theta_{\text{hyp}}) &= \sqrt{e^2 - 1} \\
e^2 - 1 &= \tan^2(\theta_{\text{hyp}}) \\
e^2 &= 1 + \tan^2(\theta_{\text{hyp}}) \\
e^2 &= \sec^2(\theta_{\text{hyp}})
\end{aligned}
$$

$$
e = \sec(\theta_{\text{hyp}}) \tag{37}
$$

7.1.2 Hyperbola Semi-Major Axis in Terms of Periapsis

Equations (17) and (37) can be combined to isolate the semi-major axis in terms of the periapsis radius.

$$
\begin{aligned}
r_{\text{pe}} &= a(1 - e) \\
r_{\text{pe}} &= a(1 - \sec(\theta_{\text{hyp}}))
\end{aligned}
$$

$$
a = \frac{r_{\text{pe}}}{1 - \sec(\theta_{\text{hyp}})} \tag{38}
$$

7.2 Parabolic Trajectories

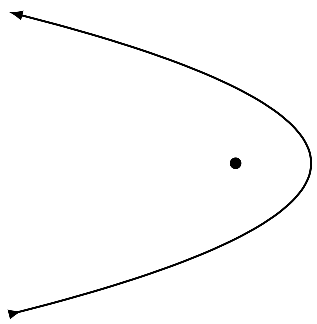

Figure 6: A parabolic trajectory

The eccentricity of a parabola is, by definition, $e = 1$. It is also proven in Section 3.1 that when $e = 1$ the orbit equation creates a parabola. For the numerator of the orbit equation $r = \frac{a(1-e^2)}{1+e\cos(\theta)}$ to be nonzero, a must be infinite. The velocity in a parabolic orbit can be investigated with Equation (26).

$$
\begin{aligned}
v &= \sqrt{\frac{2\mu}{r} - \frac{\mu}{a}} \\
&= \sqrt{\frac{2\mu}{r} - \lim_{a\to\infty}\left(\frac{\mu}{a}\right)} \\
&= \sqrt{\frac{2\mu}{r} - \cancelto{0}{\lim_{a\to\infty}\left(\frac{\mu}{a}\right)}}
\end{aligned}
$$

$$v_{\text{parabola}} = v_{\text{escape}} = \sqrt{\frac{2\mu}{r}} \tag{39}$$

Since a parabola has a determined unique shape, there is only one parameter for it; its size. The defining parameter for a parabolic trajectory is either the periapsis r_{pe} or the semi-latus rectum p, which can be related using Equation (11) keeping in mind that at periapsis the cosine will evaluate to 1

$$\begin{aligned} r &= \frac{p}{1 + e\cos(\theta - \omega)} \\ r_{\text{pe}} &= \frac{p}{1 + \cos(0)} \\ r_{\text{pe}} &= \frac{p}{2} \end{aligned}$$

$$p_{\text{parabola}} = 2r_{\text{pe,parabola}} \tag{40}$$

To determine the periapsis of a parabolic trajectory from a measured velocity, angular momentum must be used

$$\begin{aligned} h_{\text{pe}} &= h \\ r_{\text{pe}} v_{\text{pe}} &= rv\cos(\phi) \\ r_{\text{pe}} &= \frac{rv\cos(\phi)}{v_{\text{pe}}} \\ &= \frac{rv\cos(\phi)}{\sqrt{\frac{2\mu}{r_{\text{pe}}}}} \\ r_{\text{pe}}^2 &= \frac{r^2 v^2 \cos^2(\phi)}{\frac{2\mu}{r_{\text{pe}}}} \\ r_{\text{pe}}^2 &= \frac{r_{\text{pe}} r^2 v^2 \cos^2(\phi)}{2\mu} \end{aligned}$$

$$r_{\text{pe,parabola}} = \frac{r^2 v^2 \cos^2(\phi)}{2\mu} \tag{41}$$

7.3 Conclusion

Hyperbolic orbits approach a linear trajectory with an angle of

$$\theta_{\text{hyp}} = \arctan\left(\sqrt{e^2 - 1}\right)$$

Because hyperbolas don't have physically meaningful apoapses, the semi-major axis and eccentricity can be defined in terms of the entry/exit angle θ_{hyp} and the periapsis radius r_{pe}.

$$e = \sec(\theta_{\text{hyp}}) \qquad a = \frac{r_{\text{pe}}}{1 - \sec(\theta_{\text{hyp}})}$$

Parabolic trajectories occur at exactly escape velocity, meaning that at all points

$$v = \sqrt{\frac{2\mu}{r}}$$

In a parabolic trajectory, the semi-latus rectum and periapsis are related by

$$p_{\text{parabola}} = 2r_{\text{pe,parabola}}$$

From telemetry data, the periapsis radius can be found with

$$r_{\text{pe,parabola}} = \frac{r^2 v^2 \cos^2(\phi)}{2\mu}$$

8 Orbital Manuevers Basics

Throughout this section, burns are assumed to be pseudo-impulsive. That is to say, they occur over an infinitesimal amount of distance and time. The n vectors can change throughout the burn (and in fact, they do) with change in direction of velocity. It is assumed for ease of analysis that the spacecraft's thrust vector shifts to keep up with the m vectors. It is important to remember that the m vectors are based on velocity, not on position. That is to say, $\hat{m}_r$ does not necessarily point outward from the planet, but rather it points perpendicular to the velocity vector and in plane with the orbit. Only at the apses does $\hat{m}_v$ point perpendicular to the displacement from the planet. Similarly, only at the apses does $\hat{m}_r$ point radially outward from the planet.

8.1 Rocket Equation

The rocket equation describes the relationship between the obtained change in velocity Δv is related to the change in mass of a spacecraft.

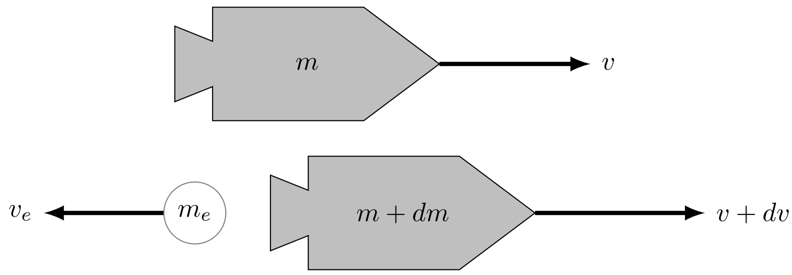

Figure 7: A spacecraft with some exhaust exiting it

First, conservation of energy will be applied. For mass to be conserved, $m_e = -dm$.

$$
\begin{aligned}
p_1 &= p_2 \\
p_{\text{rocket},1} &= p_{\text{rocket},2} + p_{\text{exhaust}} \\
mv &= (m+dm)(v+dv) + m_e(v - v_e) \\
mv &= (m+dm)(v+dv) - dm(v - v_e) \\
mv &= mv + mdv + vdm + dmdv - vdm + v_e dm \\
\cancel{mv} &= \cancel{mv} + mdv + \cancel{vdm} + dmdv - \cancel{vdm} + v_e dm \\
0 &= mdv + dmdv + v_e dm \\
&\text{The second-order differential } dmdv \text{ vanishes} \\
0 &= mdv + v_e dm \\
-mdv &= v_e dm \\
dv &= \frac{-v_e dm}{m} \\
\int_{v_1}^{v_2} dv &= -v_e \int_{m_1}^{m_2} \frac{1}{m} dm \\
\Delta v &= -v_e(\ln(m_2) - \ln(m_1)) \\
\Delta v &= -v_e \ln(\frac{m_2}{m_1})
\end{aligned}
$$

$$\Delta v = v_e \ln\left(\frac{m_1}{m_2}\right) \tag{42}$$

This equation has another (equivalent) form. Because exhaust velocity can come in many different units, making it hard to compare between spacecraft. Instead of the exhaust velocity, a specific impulse I_sp is defined. The specific impulse is the exhaust velocity divided by Earth's standard gravity at sea level. This allows the units to cancel out to seconds. $I_{sp} = \frac{v_e}{g_0} \rightarrow v_e = I_{sp} g_0$

$$\Delta v = I_{sp} g_0 \ln\left(\frac{m_1}{m_2}\right) \tag{43}$$

Note that Δv is not actually the change in velocity; a spacecraft can burn for a change in velocity of some arbitrary amount δ in one direction, then perform a burn of equal magnitude in a perpendicular direction. The magnitude of the change in velocity of the spacecraft is $\sqrt{2}\delta$, while the change in magnitude of velocity would depend on the initial velocity vector of the spacecraft. Instead, Δv can be treated as a quantity that a spacecraft expends as it performs manuevers. For the rest of this chapter, the uppercase ΔV will be used to make this distinction, with the differential form dV being used as well. Keep in mind that, while dV and ΔV have units of velocity, they are *not* necessarily equal to differential or absolute changes in the actual magnitude of velocity. When dV is integrated, the integration bounds will be V- think of this as the currency which is being used, or a means of measuring fuel requirements for maneuvers.

8.2 Burns in the $\hat{m}_r$ Direction

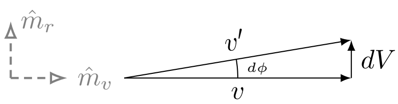

Figure 8: Velocity change due to thrust in the $\hat{m}_r$ direction

It can be shown that $v' = v$.

$$\begin{aligned} v' &= \sqrt{v^2 + dV^2} \\ (v')^2 &= v^2 + dV^2 \\ (v')^2 &= v^2 + \cancel{dV^2} \quad \text{smaller order differential} \\ (v')^2 &= v^2 \\ v' &= v \end{aligned}$$

This proves that, when dV is always perpendicular to the velocity vector, there is no change in magnitude of velocity. Because the magnitude of velocity is conserved, Equation (16) implies that specific energy is conserved. Equation (25) shows that the semi-major axis is also unchanged due to the conservation of specific energy.

However, the flight path angle ϕ does change. Remember throughout this process that v is constant.

$$dV = v d\phi$$

$$
\begin{aligned}
d\phi &= \frac{dV}{v} \\
\int_{\phi_1}^{\phi_2} &= \frac{1}{v}\int_{V_1}^{V_2} dV \\
\Delta\phi &= \frac{\Delta V}{v}
\end{aligned}
$$

While the magnitude of velocity is not changed, its direction relative to the u vectors (and any inertial base frame) is changed. Equation (29) shows the relationship between ϕ and geometric parameters of the orbit.

$$
\cos(\phi) = \sqrt{\frac{a^2(1-e^2)}{r(2a-r)}}
$$

Because a and r are unchanged, e must change. The precise change in e can be investigated by solving for e_2, the eccentricity of the orbit post-burn.

$$
\begin{aligned}
\sqrt{\frac{a^2(1-e_2^2)}{r(2a-r)}} &= \cos(\phi+\Delta\phi) \\
&= \cos(\phi)\cos(\Delta\phi) - \sin(\phi)\sin(\Delta\phi) \\
&= \cos(\phi)\cos(\Delta\phi) - \sqrt{1-\cos^2(\phi)}\sin(\Delta\phi) \\
&= \sqrt{\frac{a^2(1-e^2)}{r(2a-r)}}\cos(\Delta\phi) - \sqrt{1-\frac{a^2(1-e^2)}{r(2a-r)}}\sin(\Delta\phi) \\
\sqrt{\frac{a^2(1-e_2^2)}{r(2a-r)}}\Big/\sqrt{\frac{a^2(1-e^2)}{r(2a-r)}} &= \cos(\Delta\phi) - \frac{\sqrt{1-\frac{a^2(1-e^2)}{r(2a-r)}}}{\sqrt{\frac{a^2(1-e^2)}{r(2a-r)}}}\sin(\Delta\phi) \\
\sqrt{\frac{a^2(1-e_2^2)}{a^2(1-e^2)}} &= \cos(\Delta\phi) - \frac{\sqrt{\frac{r(2a-r)-a^2(1-e^2)}{r(2a-r)}}}{\sqrt{\frac{a^2(1-e^2)}{r(2a-r)}}}\sin(\Delta\phi) \\
\sqrt{\frac{1-e_2^2}{1-e^2}} &= \cos(\Delta\phi) - \sqrt{\frac{\frac{r(2a-r)-a^2(1-e^2)}{r(2a-r)}}{\frac{a^2(1-e^2)}{r(2a-r)}}}\sin(\Delta\phi) \\
\sqrt{\frac{1-e_2^2}{1-e^2}} &= \cos(\Delta\phi) - \sqrt{\frac{r(2a-r)-a^2(1-e^2)}{a^2(1-e^2)}}\sin(\Delta\phi) \\
\sqrt{\frac{1-e_2^2}{1-e^2}} &= \cos(\Delta\phi) - \sqrt{\frac{r(2a-r)}{a^2(1-e^2)}-1}\sin(\Delta\phi) \\
\sqrt{\frac{1-e_2^2}{1-e^2}} &= \cos(\Delta\phi) - \sqrt{1/\left(\sqrt{\frac{a^2(1-e^2)}{r(2a-r)}}\right)^2-1}\sin(\Delta\phi) \\
\sqrt{\frac{1-e_2^2}{1-e^2}} &= \cos(\Delta\phi) - \sqrt{\frac{1}{\cos^2(\phi)}-1}\sin(\Delta\phi) \\
\sqrt{\frac{1-e_2^2}{1-e^2}} &= \cos(\Delta\phi) - \sqrt{\sec^2(\phi)-1}\sin(\Delta\phi)
\end{aligned}
$$

$$\sqrt{\frac{1-e_2^2}{1-e^2}} = \cos(\Delta\phi) - \sqrt{\tan^2(\phi)}\sin(\Delta\phi)$$
$$\sqrt{\frac{1-e_2^2}{1-e^2}} = \cos(\Delta\phi) - \tan(\phi)\sin(\Delta\phi)$$
$$\frac{1-e_2^2}{1-e^2} = (\cos(\Delta\phi) - \tan(\phi)\sin(\Delta\phi))^2$$
$$1-e_2^2 = (\cos(\Delta\phi) - \tan(\phi)\sin(\Delta\phi))^2(1-e^2)$$
$$e_2^2 = 1-(\cos(\Delta\phi) - \tan(\phi)\sin(\Delta\phi))^2(1-e^2)$$
$$e_2^2 = 1-\Big(\cos^2(\Delta\phi) + \tan^2(\phi)\sin^2(\Delta\phi) - \tan(\phi)\sin(\Delta\phi)\cos(\Delta\phi)\Big)(1-e^2)$$
$$e_2^2 = 1-\cos^2(\Delta\phi)\Big(1 + \tan^2(\phi)\tan^2(\Delta\phi) - \tan(\phi)\tan(\Delta\phi)\Big)(1-e^2)$$
$$e_2^2 = 1-\cos^2(\Delta\phi)\Big(1 - \tan(\phi)\tan(\Delta\phi)\Big)^2(1-e^2)$$
$$e_2 = \sqrt{1-(1-e^2)\cos^2(\Delta\phi)\Big(1 - \tan(\phi)\tan(\Delta\phi)\Big)^2}$$

$$e_2 = \sqrt{1-(1-e^2)\cos^2\left(\frac{\Delta V}{v}\right)\left(1 - \tan(\phi)\tan\left(\frac{\Delta V}{v}\right)\right)^2}$$

This formula is here for the sake of completeness, however in it is utterly useless for analysis.

Recall that ϕ is constrained $\frac{-\pi}{2} < \phi < \frac{\pi}{2}$ (at $\phi = \pm\frac{\pi}{2}$, $\hat{m}_r$ is undefined as there are infinitely many vectors normal to velocity and in plane with the displacement vector). Any thrust in the $\hat{m}_r$ direction will increase ϕ.

The following equation will be analyzed to determine how manuevers in $\hat{m}_r$ impact e.

$$\cos(\phi) = \sqrt{\frac{a^2(1-e^2)}{r(2a-r)}}$$

As ϕ increases within the domain $\frac{-\pi}{2} < \phi < 0$, $\cos(\phi)$ increases, meaning that $1-e^2$ must increase as well, so e must decrease. On the domain $0 < \phi < \frac{\pi}{2}$, the opposite holds true; increase in ϕ means an increase in e. When ϕ is negative, the spacecraft's distance to the planet is decreasing. This means that the spacecraft is approaching periapsis. When ϕ is positive, the spacecraft is approaching apoapsis.

From Equation (30), an increase in eccentricity will increase the angle to the periapsis, while a decrease in eccentricity will decrease the angle to periapsis.

This means that if the spacecraft is heading towards periapsis, a radial out burn (by decreasing eccentricity) will bring the periapsis up and closer to the spacecraft's true anomaly. When the spacecraft is heading towards apoapsis, a radial out burn will (by increasing efficiency) further raise the apoapsis while increasing the angle to periapsis and therefore decreasing the angle to apoapsis.

Conclusion: any burn in the $\hat{m}_r$ direction raises apsis that the spacecraft is approaching while lowering the apsis that the spacecraft is coming from. The orbit "pivots" about the spacecraft, with the eccentricity changing and the semi-major axis remaining the same. The increase in one apsis is equal to the decrease in the other apsis (see Figure 22). The true anomaly of whichever apsis the spacecraft is heading towards will be brought closer to the spacecraft. Because the plane of the orbit is unchanged, Ω and i are kept the same with radial manuevers. To keep the spacecraft where it is, $\omega + \theta$ is also not changed.

Radial in burns are the reverse of radial out burns.

8.3 Burns in the $\hat{m}_n$ Direction

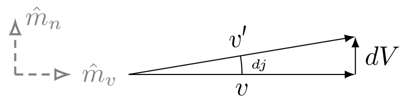

Figure 9: Velocity change due to thrust in the $\hat{m}_n$ direction

Much like in Section 8.2, the magnitude of velocity does not change. Instead of ϕ changing, the inclination of the orbit changes. Note that instead of using i, j is used here. This is because ΔV causes a linear change in the angle between the planes of the pre-burn and post-burn orbits. However, the inclination measures the angle between the post-burn orbit and some arbitrary horizon. Δj is very easy to find, with Δi requiring some geometric reasoning.

Using the same mathematics as in Section 8.2, it is apparent that $\Delta j = \frac{\Delta V}{v}$.

This means that normal burns have the effect of "tilting" the plane of the orbit about the line connecting the spacecraft and the orbited body. The two points at which the orbit intersects the reference plane are the ascending and descending nodes. At the ascending node, the spacecraft goes from negative latitude to positive latitude. At the descending node, it goes from positive latitude to negative latitude.

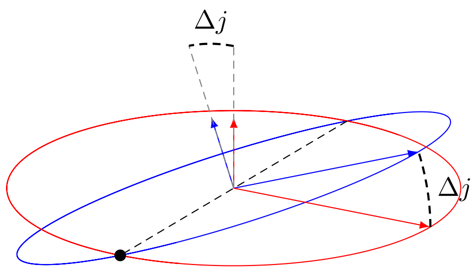

Figure 10: A manuever in the $\hat{m}_n$ direction performed at position indicated by the black dot

If the original orbit (red) is inclined by some angle i in such a way that i and Δj are not in the same direction, finding the change in inclination is not simple addition. If Δj is not in the same direction as i, $\Delta i \neq \Delta j$, and the longitude of ascending node Ω will be changed.

It will now become useful to express inclination as a vector. A normal burn has the effect of adding another inclination vector to the original inclination vector. The inclination is a rotation of the orbit about a vector that points from the orbited body to the ascending node.

The inclination can be expressed as a rotation of $i\hat{b}_1$ (see Appendix Figure 24). A burn in the $\hat{m}_n$ direction will cause a rotation of $\Delta j \hat{u}_r$ vector. Therefore,

$$i_2 = |i\hat{b}_1 + \Delta j \hat{u}_r|$$

Conclusion: normal burns rotate the orbit about the polar $\hat{u}_r$ vector.

8.4 Burns in the $\hat{m}_v$ Direction

Returning once more to the simple realm of in-plane orbits, a burn in the $\hat{m}_v$ direction carries the unique property that ΔV actually *is* in the direction of velocity, meaning ΔV is equal to the change in magnitude of velocity and the magnitude of change in velocity. This means that

$$v_2 = v + \Delta v$$

Using Equation (16), the change in specific energy can be found.

$$\begin{aligned}
\Delta\varepsilon &= \varepsilon_2 - \varepsilon \\
&= \left(\frac{(v+\Delta V)^2}{2} - \frac{\mu}{r}\right) - \left(\frac{v^2}{2} - \frac{\mu}{r}\right) \\
&= \frac{1}{2}\left((v+\Delta V)^2 - v^2\right) \\
&= \frac{1}{2}\left(\left(v^2 + (\Delta V)^2 + v\Delta V\right) - v^2\right) \\
&= \frac{1}{2}\left((\Delta V)^2 + v\Delta V\right) \\
&= \frac{1}{2}\Delta V\left(\Delta V + v\right)
\end{aligned}$$

This means that, to maximize specific energy gained for any given ΔV, a burn should be done when the spacecraft is at its maximum speed, which occurs at Periapsis.

Because ε changes, Equation (25) implies that the semi-major axis changes as well.

$$\begin{aligned}
\varepsilon &= \frac{-\mu}{2a} \\
a &= \frac{-\mu}{2\varepsilon}
\end{aligned}$$

The proportional change in a will be found. That is to say, the ratio of a_2 and a.

$$\begin{aligned}
\frac{a_2}{a} &= (\frac{-\mu}{2\varepsilon})/(\frac{-\mu}{2(\varepsilon + \frac{1}{2}\Delta V\left(\Delta V + v\right))}) \\
&= \left(\frac{-\mu}{2\varepsilon}\right) / \left(\frac{-\mu}{\varepsilon + \Delta V\left(\Delta V + v\right)}\right) \\
&= \frac{\varepsilon + \Delta V\left(\Delta V + v\right)}{2\varepsilon} \\
&= 1 + \frac{\Delta V\left(\Delta V + v\right)}{2\varepsilon} \\
&= 1 + \frac{\Delta V\left(\Delta V + v\right)}{2\left(\frac{v^2}{2} - \frac{\mu}{r}\right)} \\
&= 1 + \frac{\Delta V\left(\Delta V + v\right)}{v^2 - \frac{2\mu}{r}}
\end{aligned}$$

However, a more simple means of finding Δa can be found using Equation (26).

$$\begin{aligned}
v + \Delta V &= \sqrt{\frac{2\mu}{r} - \frac{\mu}{a_2}} \\
(v + \Delta V)^2 &= \left(\sqrt{\frac{2\mu}{r} - \frac{\mu}{a_2}}\right)^2
\end{aligned}$$

$$
\begin{aligned}
v^2 + (\Delta V)^2 + 2v\Delta V &= \frac{2\mu}{r} - \frac{\mu}{a_2} \\
\frac{v^2 + (\Delta V)^2 + 2v\Delta V}{\mu} &= \frac{2}{r} - \frac{1}{a_2} \\
\frac{1}{a_2} &= \frac{2}{r} - \frac{v^2 + (\Delta V)^2 + 2v\Delta V}{\mu} \\
\frac{1}{a_2} &= \frac{2\mu}{r\mu} - \frac{rv^2 + r(\Delta V)^2 + 2rv\Delta V}{r\mu} \\
\frac{1}{a_2} &= \frac{2\mu - rv^2 - r(\Delta V)^2 - 2rv\Delta V}{r\mu} \\
a_2 &= \frac{r\mu}{2\mu - rv^2 - r(\Delta V)^2 - 2rv\Delta V}
\end{aligned}
$$

Conclusion: burns in the $\hat{m}_v$ direction increase the semi-major axis of the orbit. Because the point that the spacecraft occupies must remain fixed, the opposite side of the orbit exhibits the most change in radius.

8.5 Conclusion

The rocket equation describes the change in velocity (or, more accurately, the magnitude of the vector change in velocity) due to the expenditure of propellant in a spacecraft. Because all maneuvers require changing velocity, this ΔV is treated as a means of measuring mission requirements, almost like a fuel requirement.

$$\Delta V = I_{sp} g_0 \ln\left(\frac{m_1}{m_2}\right)$$

If a spacecraft burns radially outward (that is to say, perpendicular from and in plane with its velocity and generally away from the planet), this will have the effect of raising the next apsis and lowering the opposite one, while bringing the next apsis closer to the spacecraft's position. This will change the eccentricity of the orbit.

Out-of-plane burns change the inclination of an orbit about the point at which the maneuver is performed. Because the plane of the orbit is defined by the velocity vector and the radial vector, knowledge of the direction of the velocity vector and radial vector allow the plane of the orbit to be known.

Burns along the velocity vector increase the semi-major axis by raising the opposite side of the orbit, while burns opposite the velocity vector have the opposite effect.

9 Single-Body Maneuvers

9.1 Inclination Change

In Section 8.3, maneuvers were done at a constant angle to the velocity vector. However, it is more efficient to transfer to the desired orbit directly by finding the vector difference between the old velocity and the desired velocity. To maximize the effect of such a manuever, it should be done at either the ascending or descending node (such that $\Delta j = \Delta i$).

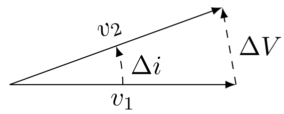

Figure 11: Inclination change velocity

Because the only effect of this maneuver is to change the direction of velocity, ΔV is the magnitude of the vector difference in the old velocity and the new velocity. Note that, throughout this maneuver, the magnitude of velocity will first decrease and then increase, so there is no net effect on the shape of the orbit in its own plane.

$$
\begin{aligned}
\Delta V &= |\vec{v}_2 - \vec{v}_1| \\
&= |(v\cos(\Delta i)\hat{m}_v + v\sin(\Delta i)\hat{m}_n) - (v\hat{m}_v)| \\
&= |v(\cos(\Delta i) - 1)\hat{m}_v + v\sin(\Delta i)\hat{m}_n| \\
&= \sqrt{(v(\cos(\Delta i) - 1))^2 + (v\sin(\Delta i))^2} \\
&= \sqrt{v^2(\cos(\Delta i) - 1)^2 + v^2\sin^2(\Delta i)} \\
&= v\sqrt{(\cos(\Delta i) - 1)^2 + \sin^2(\Delta i)} \\
&= v\sqrt{\cos^2(\Delta i) + 1 - 2\cos(\Delta i) + \sin^2(\Delta i)} \\
&= v\sqrt{(\cos^2(\Delta i) + \cos^2(\Delta i)) + 1 - 2\cos(\Delta i)} \\
&= v\sqrt{2 - 2\cos(\Delta i)} \\
&= v\sqrt{2(1 - \cos(\Delta i))} \\
&= v\sqrt{2\left(1 - \cos\left(2\frac{\Delta i}{2}\right)\right)} \\
&= v\sqrt{4\left(\frac{1 - \cos\left(2\frac{\Delta i}{2}\right)}{2}\right)} \\
&= v\sqrt{4\sin^2\left(\frac{\Delta i}{2}\right)}
\end{aligned}
$$

This gives us the ΔV required for an inclination change maneuver

$$\Delta V = 2\sin\left(\frac{\Delta i}{2}\right) \tag{44}$$

This shows the minimum required ΔV for an orbital inclination change. If the inclination change occurs at any point other than the ascending or descending node, it will require a greater ΔV and will also have the effect of chaning the longitude of ascending node Ω (and therefore moving the ascending and descending nodes). In-plane maneuvers will change neither the inclination nor the longitude of ascending node.

9.2 Hohmann Transfer

A Hohmann Transfer is a transfer between two circular orbits from radius r_1 to radius r_2. The transfer is done via an elliptical orbit, and takes two burns. The first burn puts the orbit into an elliptical orbit in which the periapsis is the lower orbit, and the apoapsis is the higher orbit. The second burn transfers from the elliptic orbit into the final circular orbit.

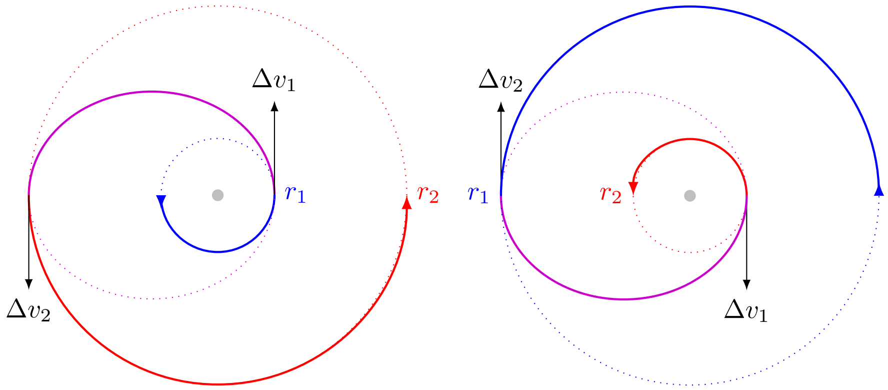

Figure 12: A Hohmann Transfer from a low orbit to a high orbit (left) and from a high orbit to a lower one (right)

Because both burns are along the velocity vector, $\Delta V = \Delta v_1 + \Delta v_2$. $a_1 = r_1 = r_{\mathrm{pe,tr}}$ will refer to the semi-major axis of the smaller orbit, with $a_2 = r_2 = r_{\mathrm{ap,tr}}$ being the semi-major axis of the larger orbit. The semi-major axis of the transfer orbit is $a_{\mathrm{tr}} = \frac{1}{2}(r_1 + r_2)$. v_1 and v_2 refer to the circular orbit velocities in the original and new orbit. The below calculations assume a transfer from a low orbit into a higher one.

$$\begin{aligned}
\Delta V &= \Delta v_1 + \Delta v_2 \\
&= |v_{\mathrm{pe,tr}} - v_1| + |v_2 - v_{\mathrm{ap,tr}}| \\
&= \left| \sqrt{\frac{2\mu}{r_{\mathrm{pe,tr}}} - \frac{\mu}{a_{\mathrm{tr}}}} - \sqrt{\frac{2\mu}{r_1} - \frac{\mu}{a_1}} \right| + \left| \sqrt{\frac{2\mu}{r_{\mathrm{ap,tr}}} - \frac{\mu}{a_2}} - \sqrt{\frac{2\mu}{r_2} - \frac{\mu}{a_{\mathrm{tr}}}} \right| \\
&= \left| \sqrt{\frac{2\mu}{r_1} - \frac{\mu}{\frac{1}{2}(r_1 + r_2)}} - \sqrt{\frac{2\mu}{r_1} - \frac{\mu}{r_1}} \right| + \left| \sqrt{\frac{2\mu}{r_2} - \frac{\mu}{r_2}} - \sqrt{\frac{2\mu}{r_2} - \frac{\mu}{\frac{1}{2}(r_1 + r_2)}} \right| \\
&= \left| \sqrt{\frac{2\mu}{r_1} - \frac{\mu}{\frac{1}{2}(r_1 + r_2)}} - \sqrt{\frac{\mu}{r_1}} \right| + \left| \sqrt{\frac{\mu}{r_2}} - \sqrt{\frac{2\mu}{r_2} - \frac{\mu}{\frac{1}{2}(r_1 + r_2)}} \right| \\
&= \left| \sqrt{\frac{2\mu}{r_1} - \frac{\mu}{\frac{1}{2}(r_1 + r_2)}} - \sqrt{\frac{\mu}{r_1}} \right| + \left| \sqrt{\frac{\mu}{r_2}} - \sqrt{\frac{2\mu}{r_2} - \frac{\mu}{\frac{1}{2}(r_1 + r_2)}} \right| \\
&= \left| \sqrt{\frac{2\mu}{r_1} - \frac{2\mu}{r_1 + r_2}} - \sqrt{\frac{\mu}{r_1}} \right| + \left| \sqrt{\frac{\mu}{r_2}} - \sqrt{\frac{2\mu}{r_2} - \frac{2\mu}{r_1 + r_2}} \right|
\end{aligned}$$

$$= \left| \sqrt{\frac{2\mu(r_1+r_2)}{r_1(r_1+r_2)} - \frac{2\mu r_1}{r_1(r_1+r_2)}} - \sqrt{\frac{\mu}{r_1}} \right| + \left| \sqrt{\frac{\mu}{r_2}} - \sqrt{\frac{2\mu(r_1+r_2)}{r_2(r_1+r_2)} - \frac{2\mu r_2}{r_2(r_1+r_2)}} \right|$$
$$= \left| \sqrt{\frac{2\mu(r_1+r_2) - 2\mu r_1}{r_1(r_1+r_2)}} - \sqrt{\frac{\mu}{r_1}} \right| + \left| \sqrt{\frac{\mu}{r_2}} - \sqrt{\frac{2\mu(r_1+r_2) - 2\mu r_1}{r_2(r_1+r_2)}} \right|$$
$$= \left| \sqrt{\frac{2\mu r_2}{r_1(r_1+r_2)}} - \sqrt{\frac{\mu}{r_1}} \right| + \left| \sqrt{\frac{\mu}{r_2}} - \sqrt{\frac{2\mu r_1}{r_2(r_1+r_2)}} \right|$$

Note that both absolute value terms will have the same sign; they will either both be positive, or both be negative. Therefore, they can be absorbed under the same absolute value.

$$\Delta V = \left| \sqrt{\frac{2\mu r_2}{r_1(r_1+r_2)}} + \sqrt{\frac{\mu}{r_2}} - \sqrt{\frac{2\mu r_1}{r_2(r_1+r_2)}} - \sqrt{\frac{\mu}{r_1}} \right| \tag{45}$$

Note that this equation is the same if r_1 and r_2 are swapped. Therefore, ΔV does not depend on the direction of the transfer; a transfer between the same two circular orbits will have the same ΔV regardless of whether the transfer is from a high orbit to a low one, or from the low orbit to a the higher one.

The time for this transfer is simply half the period of the transfer orbit, which can be calculated using Equation (24).

$$\begin{aligned} t &= \frac{T_{\text{tr}}}{2} \\ &= \frac{1}{2}\sqrt{\frac{4\pi^2 a_{\text{tr}}^3}{\mu}} \\ &= \pi\sqrt{\frac{a_{\text{tr}}^3}{\mu}} \\ &= \pi\sqrt{\frac{\left(\frac{1}{2}(r_1+r_2)\right)^3}{\mu}} \end{aligned}$$

Giving us a total transfer time of

$$T_{\text{tr}} = \pi\sqrt{\frac{(r_1+r_2)^3}{8\mu}} \tag{46}$$

9.2.1 ΔV Analysis for Hohmann Transfers

Equation (45) will be written with $r_{\text{high}} = r_2$ and $r_{\text{low}} = r_1$. A parameter α is defined as $\alpha = r_{\text{high}}/r_{\text{low}}$, allowing ΔV to be made a function of α.

$$\begin{aligned} \Delta V &= \left| \sqrt{\frac{2\mu r_{\text{high}}}{r_{\text{low}}(r_{\text{low}}+r_{\text{high}})}} + \sqrt{\frac{\mu}{r_{\text{high}}}} - \sqrt{\frac{2\mu r_{\text{low}}}{r_{\text{high}}(r_{\text{low}}+r_{\text{high}})}} - \sqrt{\frac{\mu}{r_{\text{low}}}} \right| \\ &= \left| \sqrt{\frac{2\mu (r_{\text{low}}\alpha)}{r_{\text{low}}(r_{\text{low}}+(r_{\text{low}}\alpha))}} + \sqrt{\frac{\mu}{r_{\text{low}}\alpha}} - \sqrt{\frac{2\mu r_{\text{low}}}{(r_{\text{low}}\alpha)(r_{\text{low}}+(r_{\text{low}}\alpha))}} - \sqrt{\frac{\mu}{r_{\text{low}}}} \right| \end{aligned}$$

$$
\begin{aligned}
&= \left| \sqrt{\frac{2\mu\alpha}{r_{\text{low}} + r_{\text{low}}\alpha}} + \sqrt{\frac{\mu}{r_{\text{low}}\alpha}} - \sqrt{\frac{2\mu}{\alpha(r_{\text{low}} + r_{\text{low}}\alpha)}} - \sqrt{\frac{\mu}{r_{\text{low}}}} \right| \\
&= \left| \sqrt{\frac{2\mu\alpha}{r_{\text{low}}(1+\alpha)}} + \sqrt{\frac{\mu}{r_{\text{low}}\alpha}} - \sqrt{\frac{2\mu}{r_{\text{low}}\alpha(1+\alpha)}} - \sqrt{\frac{\mu}{r_{\text{low}}}} \right| \\
&= \left| \sqrt{\frac{2\mu}{r_{\text{low}}}}\sqrt{\frac{\alpha}{1+\alpha}} + \sqrt{\frac{\mu}{r_{\text{low}}}}\sqrt{\frac{1}{\alpha}} - \sqrt{\frac{2\mu}{r_{\text{low}}}}\sqrt{\frac{1}{\alpha(1+\alpha)}} - \sqrt{\frac{\mu}{r_{\text{low}}}} \right| \\
&= \left| \sqrt{\frac{\mu}{r_{\text{low}}}}\sqrt{\frac{2\alpha}{1+\alpha}} + \sqrt{\frac{\mu}{r_{\text{low}}}}\sqrt{\frac{1}{\alpha}} - \sqrt{\frac{\mu}{r_{\text{low}}}}\sqrt{\frac{2}{\alpha(1+\alpha)}} - \sqrt{\frac{\mu}{r_{\text{low}}}} \right| \\
&= \sqrt{\frac{\mu}{r_{\text{low}}}} \left| \sqrt{\frac{2\alpha}{1+\alpha}} + \sqrt{\frac{1}{\alpha}} - \sqrt{\frac{2}{\alpha(1+\alpha)}} - 1 \right|
\end{aligned}
$$

Because $\alpha > 1$, the absolute value can vanish as its argument will always be positive.

$$
\Delta V(\alpha, r_{\text{low}}) = \sqrt{\frac{\mu}{r_{\text{low}}}} \left(\sqrt{\frac{2\alpha}{1+\alpha}} + \sqrt{\frac{1}{\alpha}} - \sqrt{\frac{2}{\alpha(1+\alpha)}} - 1 \right) \tag{47}
$$

Notice that the coefficient $\sqrt{\mu/r_{\text{low}}}$ is the circular velocity at the low orbit.

By plotting ΔV against α, a trend can be found. From Equation (47), the exact value of r_{low} only scales the plot of $\Delta V(\alpha)$ but does not effect its shape.

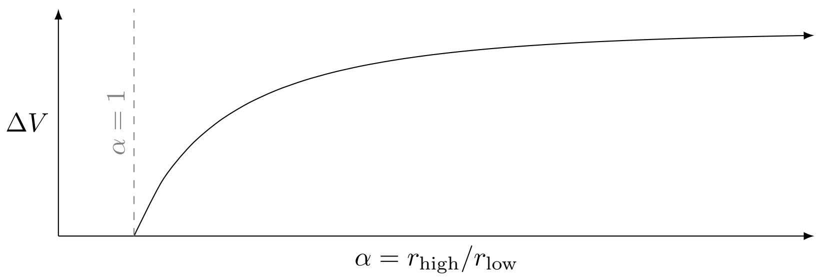

Figure 13: Plot of ΔV vs α with r_{low} held constant

It is important to note that the apparent behavior of constant increase is not the true behavior of this function. The function exhibits a maximum at $\alpha \approx 15.58172$ (the work for this is not shown, as differentiation of (47) is rather tedious, and the roots of the derivative cannot be found analytically), and then exhibits the following limiting behavior

$$
\lim_{\alpha \to \infty} \Delta V(\alpha, r_{\text{low}}) = (\sqrt{2} - 1)\sqrt{\frac{\mu}{r_{\text{low}}}}
$$

If instead α is held constant, then the relationship between r_{low} and ΔV can be explored. As can be seen in Equation (47), the value of α does not impact the shape of the plot of $\Delta V(r_{\text{low}})$, but instead scales it. The general behavior of $\Delta V(r_{\text{low}})$ follows $r_{\text{low}}^{-0.5}$.

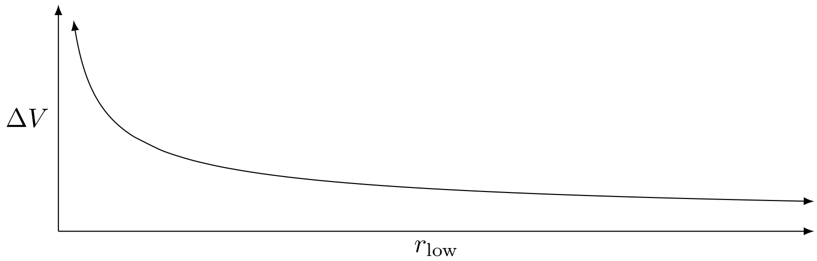

Figure 14: Plot of ΔV vs r_{low} with α held constant

The ΔV for a Hohmann Transfer can also be plotted as a heatmap, with one axis being origin orbit radius and the other being destination orbit radius.

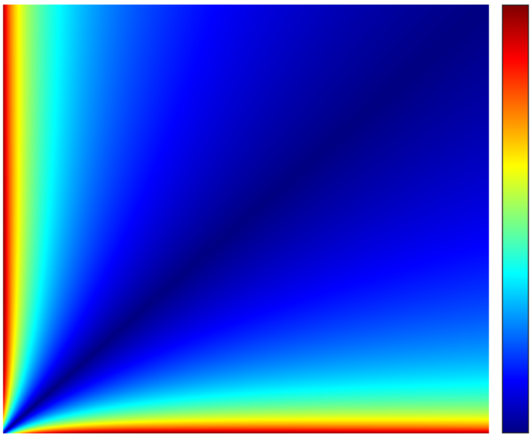

Figure 15: ΔV required for a Hohmann transfer between two orbits (created with MATLAB)

In the above figure, ΔV is represented by the colors, while the axes correspond to the destination and origin orbit radii. It can be seen that the highest ΔV expenditure occurs when one orbit is very low, while the other is very high.

9.3 Expedited Transfer

It may at times be the case that the Hohmann Transfer is simply too slow for some applications. If the transfer is from a low orbit to a higher one, then the apoapsis of the transfer orbit can simply be raised higher than the destination orbit. Calculating the ΔV of such a transfer is elementary, as it requires only application of Equation (26). Using Equation (31), the transfer time can be evaluated.

9.4 Bi-Elliptic Transfer

A bi-elliptic transfer is another means of transferring from one circular orbit to another. While a Hohmann transfer takes a spacecraft from one circular orbit to another via a singular transfer orbit, the bi-elliptic transfer utilizes a series of two transfer orbits. First, the spacecraft is put into an orbit much higher than either the source or destination orbit. Then, at the apoapsis of the high transfer orbit, the periapsis is brought to the new orbit. Bi-elliptic transfers take advantage of the efficiencies of apsidal changes at high altitudes, and as such the transfer orbit will *always* be higher than the origin or destination orbits.

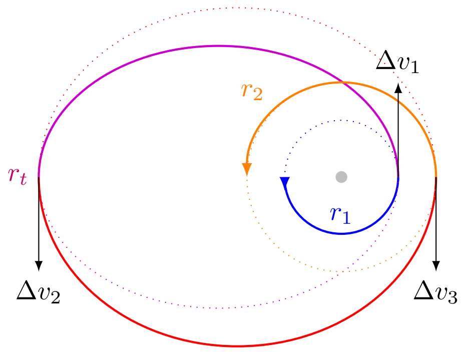

Figure 16: A bi-elliptic transfer from a low orbit to a high orbit (left)

Figure 16 shows a satellite undergoing a bi-elliptic transfer. It begins in one orbit (blue) at some radius r_1, then performs a burn to enter a high transfer orbit with apoapsis $r_t > r_1, r_2$. It then performs a small burn to either raise or lower its periapsis to the destination orbit altitude. Finally, at periapsis, it slows down to lower its apoapsis into a circular orbit. Equation (26) allows this orbit to be analyzed easily much like was done in Section 9.2.

$a_1 = r_1 = r_{\text{pe,tr}}$ is the origin orbit radius, and $a_2 = r_2 = r_{\text{ap,tr}}$ is the destination orbit radius. The semi-major axis of the transfer orbits are $a_{\text{tr1}} = \frac{1}{2}(r_1 + r_t)$ and $a_{\text{tr2}} = \frac{1}{2}(r_2 + r_t)$. v_1 and v_2 refer to the circular orbit velocities in the original and new orbit. Transfer 1 refers to the first transfer orbit (purple in Figure 16), while transfer 2 refers to the second (red in the same figure).

$$\begin{aligned}\Delta V &= \Delta v_1 + \Delta v_2 + \Delta v_3 \\ &= |v_{\text{pe,tr1}} - v_1 + |v_{\text{ap,tr2}} - v_{\text{ap,tr1}}| + |v_2 - v_{\text{pe,tr2}}|\end{aligned}$$

The first burn will always be an accelerative one, while the last will always be declarative. However, the direction of the second burn will depend on whether the origin orbit is higher or lower than the destination orbit. Therefore, the middle burn must have absolute value bars to ensure that its ΔV contribution is always positive.

$$\begin{aligned}\Delta V &= |(v_{\text{pe,tr1}} - v_1)| + |v_{\text{ap,tr2}} - v_{\text{ap,tr1}}| + |v_2 - v_{\text{pe,tr2}}| \\ &= (v_{\text{pe,tr1}} - v_1) + |v_{\text{ap,tr2}} - v_{\text{ap,tr1}}| + (v_{\text{pe,tr2}} - v_2) \\ &= \left(\sqrt{\frac{2\mu}{r_{\text{pe,tr1}}} - \frac{\mu}{a_{\text{tr1}}}} - \sqrt{\frac{2\mu}{r_1} - \frac{\mu}{a_1}}\right) \\ &\quad + \left|\sqrt{\frac{2\mu}{r_{\text{ap,tr2}}} - \frac{\mu}{a_{\text{tr2}}}} - \sqrt{\frac{2\mu}{r_{\text{ap,tr1}}} - \frac{\mu}{a_{\text{tr1}}}}\right|\end{aligned}$$

$$+\left(\sqrt{\frac{2\mu}{r_{\text{pe,tr2}}}-\frac{\mu}{a_{\text{tr2}}}}-\sqrt{\frac{2\mu}{r_2}-\frac{\mu}{a_2}}\right)$$

$$\begin{aligned}
&=\left(\sqrt{\frac{2\mu}{r_1}-\frac{2\mu}{r_1+r_t}}-\sqrt{\frac{2\mu}{r_1}-\frac{\mu}{r_1}}\right)\\
&\quad+\left|\sqrt{\frac{2\mu}{r_t}-\frac{2\mu}{r_2+r_t}}-\sqrt{\frac{2\mu}{r_t}-\frac{2\mu}{r_1+r_t}}\right|\\
&\quad+\left(\sqrt{\frac{2\mu}{r_2}-\frac{2\mu}{r_2+r_t}}-\sqrt{\frac{2\mu}{r_2}-\frac{\mu}{r_2}}\right)
\end{aligned}$$

$$\begin{aligned}
&=\left(\sqrt{\frac{2\mu(r_1+r_t)-2\mu r_1}{r_1(r_1+r_t)}}-\sqrt{\frac{\mu}{r_1}}\right)\\
&\quad+\left|\sqrt{\frac{2\mu(r_2+r_t)-2\mu r_t}{r_t(r_2+r_t)}}-\sqrt{\frac{2\mu(r_1+r_t)-2\mu r_t}{r_t(r_1+r_t)}}\right|\\
&\quad+\left(\sqrt{\frac{2\mu(r_2+r_t)-2\mu r_2}{r_2(r_2+r_t)}}-\sqrt{\frac{\mu}{r_2}}\right)
\end{aligned}$$

$$\begin{aligned}
&=\left(\sqrt{\frac{2\mu r_t}{r_1(r_1+r_t)}}-\sqrt{\frac{\mu}{r_1}}\right)\\
&\quad+\left|\sqrt{\frac{2\mu r_2}{r_t(r_2+r_t)}}-\sqrt{\frac{2\mu r_1}{r_t(r_1+r_t)}}\right|\\
&\quad+\left(\sqrt{\frac{2\mu r_t}{r_2(r_2+r_t)}}-\sqrt{\frac{\mu}{r_2}}\right)\\
&=\sqrt{\frac{2\mu r_t}{r_1(r_1+r_t)}}+\sqrt{\frac{2\mu r_t}{r_2(r_2+r_t)}}-\sqrt{\frac{\mu}{r_1}}-\sqrt{\frac{\mu}{r_2}}+\left|\sqrt{\frac{2\mu r_2}{r_t(r_2+r_t)}}-\sqrt{\frac{2\mu r_1}{r_t(r_1+r_t)}}\right|
\end{aligned}$$

This equation is valid for all r_1, r_2, r_t that obey $r_t \geq r_1, r_2$

$$\Delta V=\sqrt{\frac{2\mu r_t}{r_1(r_1+r_t)}}+\sqrt{\frac{2\mu r_t}{r_2(r_2+r_t)}}-\sqrt{\frac{\mu}{r_1}}-\sqrt{\frac{\mu}{r_2}}+\left|\sqrt{\frac{2\mu r_2}{r_t(r_2+r_t)}}-\sqrt{\frac{2\mu r_1}{r_t(r_1+r_t)}}\right| \tag{48}$$

Much like (45), this equation is the same if r_1 and r_2 are swapped. This shows that the ΔV for a bi-elliptic transfer does not depend on which orbit is the origin orbit and which is the destination.

The time it takes to complete this transfer can be calculated with Equation (24) as the sum of the half-periods of both transfer orbits.

$$\begin{aligned}
t&=\frac{T_{\text{tr1}}}{2}+\frac{T_{\text{tr2}}}{2}\\
&=\frac{1}{2}\left(\sqrt{\frac{4\pi^2 a_{\text{tr1}}^3}{\mu}}+\sqrt{\frac{4\pi^2 a_{\text{tr2}}^3}{\mu}}\right)\\
&=\pi\left(\sqrt{\frac{a_{\text{tr1}}^3}{\mu}}+\sqrt{\frac{a_{\text{tr2}}^3}{\mu}}\right)
\end{aligned}$$

$$= \pi \left(\sqrt{\frac{(\frac{1}{2}(r_1 + r_t))^3}{\mu}} + \sqrt{\frac{(\frac{1}{2}(r_2 + r_t))^3}{\mu}} \right)$$

$$T_{\text{tr}} = \pi \left(\sqrt{\frac{(r_1 + r_t)^3}{8\mu}} + \sqrt{\frac{(r_2 + r_t)^3}{8\mu}} \right) \tag{49}$$

9.4.1 ΔV Analysis for Bi-Elliptic Transfers

Just as was done in 9.2.1 before, Equation (48) will be rewritten. r_{low} and r_{high} will be introduced with $\alpha = r_{\text{high}}/r_{\text{low}}$. When r_1 of (48) is taken to be r_{low} and r_2 to be r_{high} (recalling that the ΔV does not depend on which orbit is the origin and which is the destination), the absolute value's argument will be positive and as such the absolute value can be removed.

$$\begin{aligned}
\Delta V &= \sqrt{\frac{2\mu r_t}{r_{\text{low}}(r_{\text{low}} + r_t)}} + \sqrt{\frac{2\mu r_t}{r_{\text{high}}(r_{\text{high}} + r_t)}} - \sqrt{\frac{\mu}{r_{\text{low}}}} - \sqrt{\frac{\mu}{r_{\text{high}}}} + \sqrt{\frac{2\mu r_{\text{high}}}{r_t(r_{\text{high}} + r_t)}} - \sqrt{\frac{2\mu r_{\text{low}}}{r_t(r_{\text{low}} + r_t)}} \\
&= \sqrt{\frac{2\mu r_t}{r_{\text{low}}(r_{\text{low}} + r_t)}} + \sqrt{\frac{2\mu r_t}{\alpha r_{\text{low}}(\alpha r_{\text{low}} + r_t)}} - \sqrt{\frac{\mu}{r_{\text{low}}}} - \sqrt{\frac{\mu}{\alpha r_{\text{low}}}} + \sqrt{\frac{2\mu \alpha r_{\text{low}}}{r_t(\alpha r_{\text{low}} + r_t)}} - \sqrt{\frac{2\mu r_{\text{low}}}{r_t(r_{\text{low}} + r_t)}} \\
&= \sqrt{\frac{\mu}{r_{\text{low}}}} \left(\sqrt{\frac{2r_t}{r_{\text{low}} + r_t}} + \sqrt{\frac{2r_t}{\alpha(\alpha r_{\text{low}} + r_t)}} - \sqrt{\frac{1}{\alpha}} - 1 \right) + \sqrt{\frac{\mu r_{\text{low}}}{r_t}} \left(\sqrt{\frac{2\alpha}{\alpha r_{\text{low}} + r_t}} - \sqrt{\frac{2}{r_{\text{low}} + r_t}} \right)
\end{aligned}$$

Now, a new parameter β will be defined such that $\beta = \frac{r_t}{r_{\text{low}}}$, or $r_t = \beta r_{\text{low}}$. For the transfer to be bi-elliptic, β must be greater than α

$$\begin{aligned}
\Delta V &= \sqrt{\frac{\mu}{r_{\text{low}}}} \left(\sqrt{\frac{2r_t}{r_{\text{low}} + r_t}} + \sqrt{\frac{2r_t}{\alpha(\alpha r_{\text{low}} + r_t)}} - \sqrt{\frac{1}{\alpha}} - 1 \right) + \sqrt{\frac{\mu r_{\text{low}}}{r_t}} \left(\sqrt{\frac{2\alpha}{\alpha r_{\text{low}} + r_t}} - \sqrt{2\frac{1}{r_{\text{low}} + r_t}} \right) \\
&= \sqrt{\frac{\mu}{r_{\text{low}}}} \left(\sqrt{\frac{2\beta r_{\text{low}}}{r_{\text{low}} + \beta r_{\text{low}}}} + \sqrt{\frac{2\beta r_{\text{low}}}{\alpha(\alpha r_{\text{low}} + \beta r_{\text{low}})}} - \sqrt{\frac{1}{\alpha}} - 1 \right) \\
&\quad + \sqrt{\frac{\mu r_{\text{low}}}{\beta r_{\text{low}}}} \left(\sqrt{\frac{2\alpha}{\alpha r_{\text{low}} + \beta r_{\text{low}}}} - \sqrt{\frac{2}{r_{\text{low}} + \beta r_{\text{low}}}} \right) \\
&= \sqrt{\frac{\mu}{r_{\text{low}}}} \left(\sqrt{\frac{2\beta}{1 + \beta}} + \sqrt{\frac{2\beta}{\alpha(\alpha + \beta)}} - \sqrt{\frac{1}{\alpha}} - 1 \right) + \sqrt{\frac{\mu}{r_{\text{low}}}} \left(\sqrt{\frac{2\alpha}{\beta(\alpha + \beta)}} - \sqrt{\frac{2}{\beta(1 + \beta)}} \right) \\
&= \sqrt{\frac{\mu}{r_{\text{low}}}} \left(\sqrt{\frac{2\beta}{1 + \beta}} + \sqrt{\frac{2\beta}{\alpha(\alpha + \beta)}} - \sqrt{\frac{1}{\alpha}} - 1 + \sqrt{\frac{2\alpha}{\beta(\alpha + \beta)}} - \sqrt{\frac{2}{\beta(1 + \beta)}} \right)
\end{aligned}$$

$$\Delta V_{\text{BE}}(\alpha, \beta, r_{\text{low}}) = \sqrt{\frac{\mu}{r_{\text{low}}}} \left(\sqrt{\frac{2\beta}{\alpha(\alpha + \beta)}} + \sqrt{\frac{2\alpha}{\beta(\alpha + \beta)}} + \sqrt{\frac{2\beta}{1 + \beta}} - \sqrt{\frac{2}{\beta(1 + \beta)}} - \sqrt{\frac{1}{\alpha}} - 1 \right) \tag{50}$$

This equation is valid for all $\beta \geq \alpha > 1$. Because r_{low} shows up only in the denominator of the coefficient square root, it is clear that, much like a Hohmann transfer, a bi-elliptic transfer's ΔV scales with the inverse of the square root of the lower orbit altitude. Recall that, again like a Hohmann transfer, the ΔV for a bi-elliptic transfer does not depend on whether the origin orbit or destination orbit are higher (assuming the same r_t).

Finally, the ΔV for the limiting bi-elliptic case of a bi-parabolic transfer will be found.

$$\begin{aligned}
\Delta V_{\text{bi-parabolic}} &= \lim_{\beta\to\infty} \Delta V_{\text{BE}}(\alpha,\beta,r_{\text{low}}) \\
&= \lim_{\beta\to\infty} \sqrt{\frac{\mu}{r_{\text{low}}}}\left(\sqrt{\frac{2\beta}{\alpha(\alpha+\beta)}}+\sqrt{\frac{2\alpha}{\beta(\alpha+\beta)}}+\sqrt{\frac{2\beta}{1+\beta}}-\sqrt{\frac{2}{\beta(1+\beta)}}-\sqrt{\frac{1}{\alpha}}-1\right) \\
&= \sqrt{\frac{\mu}{r_{\text{low}}}}\left(\sqrt{\frac{2}{\alpha}}+\sqrt{2}-\sqrt{\frac{1}{\alpha}}-1\right) \\
&= \sqrt{\frac{\mu}{r_{\text{low}}}}\left(\sqrt{\frac{1}{\alpha}}+1\right)(\sqrt{2}-1)
\end{aligned}$$

A bi-parabolic transfer has ΔV

$$\Delta V_{\text{BP}}(\alpha,r_{\text{low}}) = \sqrt{\frac{\mu}{r_{\text{low}}}}\left(\sqrt{\frac{1}{\alpha}}+1\right)(\sqrt{2}-1) \tag{51}$$

9.5 Comparison of Hohmann and Bi-Elliptic Transfers

With mathematic analyses completed of both Hohmann and bi-elliptic transfers, the two can be compared to determine when each is more efficient in terms of time and in terms of ΔV expenditure.

First, transfer times will be analyzed. Equations (46) and (49) that the transfers have a transfer time of

$$t_{\text{Hohmann}} = \frac{\pi}{\sqrt{8\mu}}(r_1+r_2)^{3/2} \qquad \text{and} \qquad t_{\text{Bi-Elliptic}} = \frac{\pi}{\sqrt{8\mu}}\left((r_1+r_t)^{3/2}+(r_2+r_t)^{3/2}\right)$$

It is clear that the bi-elliptic transfer takes substantially longer than the Hohmann transfer, taking at best almost twice as long (as for a Hohmann transfer, only 180 degrees of the orbit must be traversed, whereas the bi-elliptic transfer requires 360 degrees to be traversed). If the bi-elliptic transfer is allowed to approach a bi-parabolic transfer, then the transfer time approaches infinity.

Comparing the ΔV efficiencies is substantially harder.

We will subtract the Hohmann transfer ΔV from the bi-elliptic transfer one. When this is negative, it means that $\Delta V_{\text{Hohmann}} > \Delta V_{\text{Bi-Elliptic}}$, and a bi-elliptic transfer is more efficient. If this difference is set to zero (and the condition is applied that $\beta > \alpha$), then the α, β pairs can be found at which the two transfers are equally efficient. Bi-elliptic will be abbreviated in subscripts as BE, while Hohmann will be abbreviated H. For any α where there is no solution for $\Delta V_{\text{Hohmann}} = \Delta V_{\text{Bi-Elliptic}}$, then either the Hohmann transfer or bi-elliptic transfer will be more efficient regardless of β.

$$\begin{aligned}
0 &= \Delta V_{\text{BE}} - \Delta V_{\text{H}} \\
0 &= \sqrt{\frac{\mu}{r_{\text{low}}}}\left(\sqrt{\frac{2\beta}{\alpha(\alpha+\beta)}}+\sqrt{\frac{2\alpha}{\beta(\alpha+\beta)}}+\sqrt{\frac{2\beta}{1+\beta}}-\sqrt{\frac{2}{\beta(1+\beta)}}-\sqrt{\frac{1}{\alpha}}-1\right) \\
&\quad -\sqrt{\frac{\mu}{r_{\text{low}}}}\left(\sqrt{\frac{2\alpha}{1+\alpha}}+\sqrt{\frac{1}{\alpha}}-\sqrt{\frac{2}{\alpha(1+\alpha)}}-1\right)
\end{aligned}$$

$$0 = \left(\sqrt{\frac{2\beta}{\alpha(\alpha+\beta)}}+\sqrt{\frac{2\alpha}{\beta(\alpha+\beta)}}+\sqrt{\frac{2\beta}{1+\beta}}-\sqrt{\frac{2}{\beta(1+\beta)}}-\sqrt{\frac{1}{\alpha}}-1\right)$$

$$-\left(\sqrt{\frac{2\alpha}{1+\alpha}}+\sqrt{\frac{1}{\alpha}}-\sqrt{\frac{2}{\alpha(1+\alpha)}}-1\right)$$

$$\begin{aligned}
\sqrt{\frac{2\alpha}{1+\alpha}}+\sqrt{\frac{1}{\alpha}}-\sqrt{\frac{2}{\alpha(1+\alpha)}} &= \sqrt{\frac{2\beta}{\alpha(\alpha+\beta)}}+\sqrt{\frac{2\alpha}{\beta(\alpha+\beta)}}+\sqrt{\frac{2\beta}{1+\beta}}-\sqrt{\frac{2}{\beta(1+\beta)}}-\sqrt{\frac{1}{\alpha}} \\
\sqrt{\frac{2\alpha}{1+\alpha}}+\sqrt{\frac{2}{\beta(1+\beta)}}+2\sqrt{\frac{1}{\alpha}} &= \sqrt{\frac{2\beta}{\alpha(\alpha+\beta)}}+\sqrt{\frac{2\alpha}{\beta(\alpha+\beta)}}+\sqrt{\frac{2\beta}{1+\beta}}+\sqrt{\frac{2}{\alpha(1+\alpha)}}
\end{aligned}$$

$$\sqrt{\frac{1}{\beta(1+\beta)}}+\sqrt{\frac{\alpha}{1+\alpha}}+\sqrt{\frac{2}{\alpha}}=\sqrt{\frac{\beta}{\alpha(\alpha+\beta)}}+\sqrt{\frac{\alpha}{\beta(\alpha+\beta)}}+\sqrt{\frac{\beta}{1+\beta}}+\sqrt{\frac{1}{\alpha(1+\alpha)}} \tag{52}$$

Analyzing Equation (50), it can be seen that at any α, ΔV increases with β up to some point, and then decreases with increase in β. This means that if a bi-elliptic transfer is as efficient as a Hohmann transfer at some $\beta > \alpha$, then it will be more efficient for *all* β above that β. By allowing β to approach infinity for the limiting case of a bi-parabolic transfer, the minimum α can be found such that it satisfies Equation (52).

$$\begin{aligned}
\lim_{\beta\to\infty}\sqrt{\frac{1}{\beta(1+\beta)}}+\sqrt{\frac{\alpha}{1+\alpha}}+\sqrt{\frac{2}{\alpha}} &= \lim_{\beta\to\infty}\sqrt{\frac{\beta}{1+\beta}}+\lim_{\beta\to\infty}\sqrt{\frac{\beta}{\alpha(\alpha+\beta)}}+\lim_{\beta\to\infty}\sqrt{\frac{\alpha}{\beta(\alpha+\beta)}}+\sqrt{\frac{1}{\alpha(1+\alpha)}} \\
\sqrt{\frac{\alpha}{1+\alpha}}+\sqrt{\frac{2}{\alpha}} &= 1+\sqrt{\frac{1}{\alpha}}+\sqrt{\frac{1}{\alpha(1+\alpha)}} \\
\sqrt{\frac{\alpha^2}{\alpha(1+\alpha)}}+\sqrt{\frac{2(1+\alpha)}{\alpha(1+\alpha)}} &= \sqrt{\frac{\alpha(1+\alpha)}{\alpha(1+\alpha)}}+\sqrt{\frac{1+\alpha}{\alpha(1+\alpha)}}+\sqrt{\frac{1}{\alpha(1+\alpha)}} \\
0 &= \alpha+\sqrt{2(\alpha+1)}-\sqrt{\alpha(\alpha+1)}-\sqrt{\alpha+1}-1
\end{aligned}$$

This is unfortunately not analytically solvable, but it can be solved with numerical methods. It yields

$$\alpha \approx 11.93876$$

For $\alpha \approx 11.93876$, a bi-elliptic transfer requires $\beta = \infty$ to be as efficient as a Hohmann transfer. Therefore, for all $\alpha < 11.93876$, a Hohmann transfer is more efficient than a bi-elliptic transfer.

Recall that a bi-elliptic transfer becomes less efficient with increase in β up to a point, and then becomes more efficient with increasing β. We will search for the case in which a bi-elliptic transfer is more efficient than a Hohmann transfer for all $\beta > \alpha$. Unfortunately setting β equal to α and evaluating Equation (52) does not yield any helpful information, as equality is satisfied for all $\beta = \alpha$ in that equation (as when $\beta = \alpha$ the bi-elliptic transfer is a Hohmann transfer, so their ΔVs must be the same). Instead, the bi-elliptic ΔV (Equation (50)) will be differentiated with respect to β, and then the derivative will be set to zero with $\alpha = \beta$. Any α which the derivative of ΔV with respect to β at $\beta = \alpha$ is negative or zero will be indicative of the case where a bi-elliptic transfer is as efficient as a Hohmann transfer and becomes only more efficient with increase in β. Because of how unwieldy this equation is, fewer steps are shown than would be elsewhere.

$$\frac{\partial \Delta V}{\partial \beta}=\frac{\partial}{\partial \beta}\sqrt{\frac{\mu}{r_{\text{low}}}}\left(\sqrt{\frac{2\beta}{\alpha(\alpha+\beta)}}+\sqrt{\frac{2\alpha}{\beta(\alpha+\beta)}}+\sqrt{\frac{2\beta}{1+\beta}}-\sqrt{\frac{2}{\beta(1+\beta)}}-\sqrt{\frac{1}{\alpha}}-1\right)$$

$$= \sqrt{\frac{\mu}{r_1}} \left(\frac{3\beta + 1}{\sqrt{2\beta^3(1+\beta)^3}} - \frac{\alpha}{\sqrt{2\alpha\beta^3(\alpha+\beta)}} \right)$$

We now apply the condition that $\alpha = \beta$

$$\begin{aligned}
0 &= \left(\frac{\partial \Delta V}{\partial \beta}\right)_{\alpha=\beta} \\
0 &= \sqrt{\frac{\mu}{r_1}} \left(\frac{3\beta + 1}{\sqrt{2\beta^3(1+\beta)^3}} - \frac{\alpha}{\sqrt{2\alpha\beta^3(\alpha+\beta)}} \right)_{\alpha=\beta} \\
0 &= \sqrt{\frac{\mu}{r_1}} \left(\frac{3\alpha + 1}{\sqrt{2\alpha^3(1+\alpha)^3}} - \frac{\alpha}{\sqrt{2\alpha\alpha^3(\alpha+\alpha)}} \right) \\
0 &= \frac{3\alpha + 1}{\sqrt{2\alpha^3(1+\alpha)^3}} - \frac{\alpha}{\sqrt{2\alpha\alpha^3(\alpha+\alpha)}} \\
0 &= \frac{3\alpha + 1}{\sqrt{2\alpha^3(1+\alpha)^3}} - \frac{1}{\sqrt{4\alpha^3}} \\
\frac{3\alpha + 1}{\sqrt{2\alpha^3(1+\alpha)^3}} &= \frac{1}{\sqrt{4\alpha^3}} \\
\frac{3\alpha + 1}{\sqrt{2\alpha^3(1+\alpha)^3}} &= \frac{2^{-1/2}(1+\alpha)^{3/2}}{\sqrt{2\alpha^3(1+\alpha)^3}} \\
3\alpha + 1 &= 2^{-1/2}(1+\alpha)^{3/2} \\
9\alpha^2 + 6\alpha + 1 &= \frac{1}{2}(1+\alpha)^3 \\
18\alpha^2 + 12\alpha + 2 &= 1 + 3\alpha + 3\alpha^2 + 3\alpha^3 \\
0 &= -\alpha^3 + 15\alpha^2 + 9\alpha + 1
\end{aligned}$$

This is not factorable. While there is a formula much like the quadratic formula that can solve cubics, it is a very long and unwieldy formula. Instead, this will be solved numerically. The single positive root of this cubic is

$$\alpha = 15.58172\ldots$$

At this α, a bi-elliptic transfer's ΔV is always decreasing for $\beta \geq \alpha$ (which it always is; the transfer orbit apoapsis must be above both the origin and destination orbits). Beyond this α, a bi-elliptic transfer is *always* more efficient than a Hohmann transfer, regardless of the transfer orbit apoapsis.

For all $\alpha < 11.93876$, a Hohmann transfer is more efficient than a bi-elliptic transfer. For all $\alpha > 15.58172$, a bi-elliptic transfer is always more efficient. Between these two α values, the choice of β determines which is more efficient. Of course, the transfer time increases with β, and is infinite in the limiting case of a bi-parabolic transfer.

Because the ΔV is the same between any two orbits, α, defined as $r_{\text{high}}/r_{\text{low}}$, can be replaced with either r_2/r_1 or r_1/r_2. The most efficient maneuver selection is summarized below, with α' defined as the ratio of final orbit to initial orbit

$$\text{Most } \Delta V \text{ efficient transfer: } \begin{cases} 0 < \alpha' < \frac{1}{15.58172} & \text{Bi-elliptic} \\ \frac{1}{15.58172} < \alpha' < \frac{1}{11.93876} & \text{Depends on } \beta \\ \frac{1}{11.93876} < \alpha' < 1 & \text{Hohmann} \\ 1 < \alpha' < 11.93876 & \text{Hohmann} \\ 11.93876 < \alpha' < 15.58172 & \text{Depends on } \beta \\ 15.58172 < \alpha' < \infty & \text{Bi-elliptic} \end{cases}$$

9.6 Combined Manuever

Any single-burn maneuver can be done most efficiently by burning such that ΔV takes the velocity from some original velocity $\vec{v}_1$ to some desired velocity $\vec{v}_2$. In all probability, the velocity vector of a satellite will be known by measurement. If instead the geometric parameters of the orbit are known, Equations (26) and (29) can be used to determine the velocity vector in the plane of the orbit. The desired velocity vector can be found with the same equations, and using basic trigonometry changes in inclination can be accounted for as well. For maneuvers in which the location of the ascending and descending nodes changes, some geometry must be applied. For any maneuver where the only change in inclination occurs at the ascending or descending node, the calculation is relatively simple. We will assume that the original velocity v_1 and the new velocity v_2 are both known, the change in flight path angle $\Delta\phi$ is known, and the angle between the old and new orbital planes Δj is known. Using this, the total ΔV required can be calculated. The law of cosines can be used to simplify this calculation, however for ease of following it will not be used.

$$
\begin{aligned}
\Delta V &= |\vec{v}_2 - \vec{v}_1| \\
&= |v_2(\cos(\Delta\phi)\cos(\Delta j)\hat{m}_v + \sin(\Delta\phi)\cos(\Delta j)\hat{m}_r + \sin(\Delta j)\hat{m}_n) - v_1\hat{m}_v| \\
&= |(v_2\cos(\Delta\phi)\cos(\Delta j) - v_1)\hat{m}_v + v_2\sin(\Delta\phi)\cos(\Delta j)\hat{m}_r + v_2\sin(\Delta j)\hat{m}_n| \\
&= \sqrt{(v_2\cos(\Delta\phi)\cos(\Delta j) - v_1)^2 + (v_2\sin(\Delta\phi)\cos(\Delta j))^2 + (v_2\sin(\Delta j))^2} \\
&= \sqrt{v_2^2\cos^2(\Delta\phi)\cos^2(\Delta j) + v_1^2 - 2v_1v_2\cos(\Delta\phi)\cos(\Delta j) + v_2^2\sin^2(\Delta\phi)\cos^2(\Delta j)^2 + v_2^2\sin^2(\Delta j)} \\
&= \sqrt{v_1^2 + v_2^2\cos^2(\Delta\phi)\cos^2(\Delta j) + v_2^2\sin^2(\Delta\phi)\cos^2(\Delta j) + v_2^2\sin^2(\Delta j) - 2v_1v_2\cos(\Delta\phi)\cos(\Delta j)} \\
&= \sqrt{v_1^2 + v_2^2\cos^2(\Delta j)(\cos^2(\Delta\phi) + \sin^2(\Delta\phi)) + v_2^2\sin^2(\Delta j) - 2v_1v_2\cos(\Delta\phi)\cos(\Delta j)} \\
&= \sqrt{v_1^2 + v_2^2\cos^2(\Delta j) + v_2^2\sin^2(\Delta j) - 2v_1v_2\cos(\Delta\phi)\cos(\Delta j)} \\
&= \sqrt{v_1^2 + v_2^2(\cos^2(\Delta j) + \sin^2(\Delta j)) - 2v_1v_2\cos(\Delta\phi)\cos(\Delta j)} \\
&= \sqrt{v_1^2 + v_2^2 - 2v_1v_2\cos(\Delta\phi)\cos(\Delta j)}
\end{aligned}
$$

For any manuever where the beginning and end velocity vectors are known, the required ΔV is

$$\Delta V = \sqrt{v_1^2 + v_2^2 - 2v_1v_2\cos(\Delta\phi)\cos(\Delta j)} \tag{53}$$

If instead the angle Φ between the vectors is known, this simplifies to

$$\Delta V = \sqrt{v_1^2 + v_2^2 - 2v_1v_2\cos(\Phi)} \tag{54}$$

9.7 Conclusion

To change the inclination of an orbit, a normal burn should be done at either the ascending or descending node. The ΔV required to change the inclination by a given amount is

$$\Delta V = 2\sin\left(\frac{\Delta i}{2}\right)$$

To transfer between two circular orbits, the most simple trajectory is a Hohmann transfer, in which a satellite first enters a transfer orbit that is tangent to both the destination and origin orbits, and then circularizes at the destination orbit. If α describes the ratio of lower orbit to higher orbit and r_{low} is the lower orbit's radius, the ΔV required for a Hohmann transfer is

$$\Delta V_{\text{Hohmann}} = \sqrt{\frac{\mu}{r_{\text{low}}}} \left(\sqrt{\frac{2\alpha}{1+\alpha}} + \sqrt{\frac{1}{\alpha}} - \sqrt{\frac{2}{\alpha(1+\alpha)}} - 1 \right)$$

Time time it takes for a Hohmann transfer is

$$t_{\text{Hohmann}} = \pi \sqrt{\frac{(r_1 + r_2)^3}{8\mu}}$$

A bi-elliptic transfer, much like a Hohmann transfer, transfers between two orbits with a ratio $\alpha = r_{\text{low}}/r_{\text{high}}$. A bi-elliptic transfer takes three burns, instead of the two that are needed for a Hohmann transfer. In a bi-elliptic transfer, the transfer orbit has an apoapsis above both the origin and destination orbits. If β describes the ratio of this apoapsis to the lower orbit radius, then the ΔV required for a bi-elliptic transfer is

$$\Delta V_{\text{Bi-Elliptic}} = \sqrt{\frac{\mu}{r_{\text{low}}}} \left(\sqrt{\frac{2\beta}{\alpha(\alpha+\beta)}} + \sqrt{\frac{2\alpha}{\beta(\alpha+\beta)}} + \sqrt{\frac{2\beta}{1+\beta}} - \sqrt{\frac{2}{\beta(1+\beta)}} - \sqrt{\frac{1}{\alpha}} - 1 \right)$$

with a longer transfer time of

$$t_{\text{Bi-Elliptic}} = \pi \left(\sqrt{\frac{(r_1 + r_t)^3}{8\mu}} + \sqrt{\frac{(r_2 + r_t)^3}{8\mu}} \right)$$

For all $\alpha < 11.93876$, a Hohmann transfer is always more efficient than a bi-elliptic transfer. For all $\alpha > 15.58172$, a bi-elliptic transfer is more efficient. For all α between these values, there exists a β above which the bi-elliptic transfer is more efficient and below which the Hohmann transfer is more efficient.

For complex single-burn maneuvers in which both the beginning velocity vector and desired velocity vector are known, the ΔV required to complete the maneuver is

$$\Delta V = \sqrt{v_1^2 + v_2^2 - 2v_1 v_2 \cos(\Delta\phi)\cos(\Delta j)}$$

10 Multi-Body Maneuvers

10.1 Gravity Assist

One maneuver that is used frequently in movies is the gravity assist (sometimes in pop culture referred to as a slingshot maneuver). Intuitively, this may seem to violate what we have learned so far. As a satellite approaches a gravitational source, its velocity increases, trading kinetic energy for potential energy. However, as the satellite departs from the gravitational source, the tradeoff will occur in reverse, with no kinetic energy being gained. This would seem to imply that a spacecraft cannot gain speed just by flinging around a planet.

A flyby will take the shape of a hyperbolic trajectory, entering at some angle relative to reference direction and leaving at some other angle. From Equation (26), the speed is only distance-dependant. A spacecraft's velocity at some distance r from the gravity source will not change whether the spacecraft is approaching the gravity source or leaving it. Of course, the existence of a gravity assist maneuver means that there must be some error in this logic.

For the sake of this exploration, a gravity assist from Jupiter in a solar orbit will be considered.

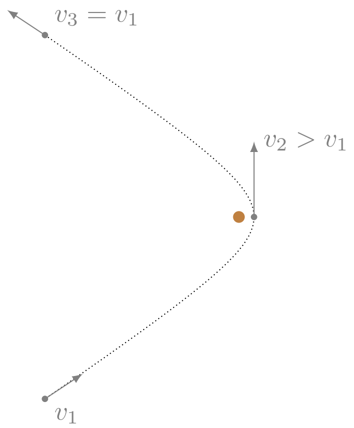

Figure 17: A hyperbolic trajectory of a satellite (gray) past Jupiter (brown)

The critical piece that is missing from this understanding so far is the relative nature of velocity. Equations so far have all been in terms of the body being orbited. It is indeed true that, in the above image, $|v_3| = |v_1|$ relative to Jupiter. However, this is not the complete picture. When in a hyperbolic trajectory past Jupiter, the spacecraft's speed *relative to Jupiter* is dependant only on its position. However, relative to the sun this may not be true.

Figure 17 can be redrawn to include velocities relative to the sun. It will be decided that Jupiter is moving to the left sun. The sun will not be shown. Velocities relative to the sun will be drawn in red, while velocities relative to Jupiter will be drawn in brown. Subscripts j and s mean relative to Jupiter and to the sun, respectively.

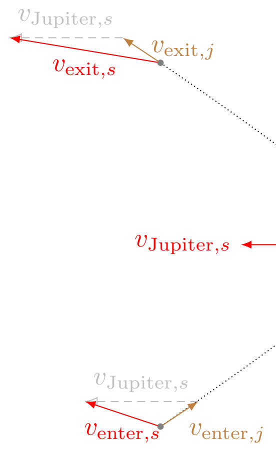

Figure 18: Jupiter Flyby With Relative Velocities

Note that Jupiter's velocity relative to the sun is also shown at the spacecraft's location in order to make the vector addition visually intuitive. As can be seen in Figure 18, the spacecraft's velocity relative to the sun is changed by Jupiter.

While this may seem to violate conservation of energy and momentum, it does not. Any momentum or energy gained by the spacecraft will be lost by the planet that's providing the gravity assist. While this does have an effect on the orbit of the planet that's providing a gravity assist (Jupiter, in this case), this effect is negligible. Jupiter has a mass on the order of 10^{27} kilograms, while most satellites will have masses between 10^2 and 10^4 kilograms. This means that an energy transfer that tremendously impacts the velocity of a satellite will have a negligible effect on Jupiter.

To illustrate the varying effects that gravity assists can have, a few possible assists will be drawn.

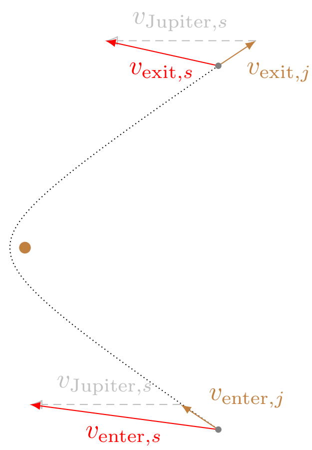

Figure 19: Gravity Assist In Front of Planet

From Figures 18 and 19, it is clear that passing behind the planet acts to accelerate the spacecraft, while passing behind the planet decelerates the spacecraft by the same amount. Note that, while these orbits both start "below" Jupiter and end "above" it, the effect is the exact same if the direction of the flyby is reversed. Visualization of this effect is left as an exercise to the reader. Next, the effect of passing "above" or "below" the planet will be investigated.

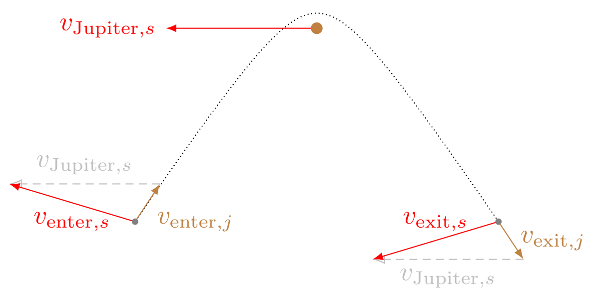

Figure 20: Radial Gravity Assists

In Figure 20, the magnitude of velocity of the satellite did not change; instead, its direction changed. The planet has the net effect of redirecting the spacecraft's velocity with respect to the planet that is giving the gravity assist.

In Figures 17, 18, 19, and 20, the hyperbolic flyby had the same eccentricity (precisely 1.2). However, the direction and eccentricity of the flyby can be fine-tuned to ensure the desired effect on the spacecraft's orbit.

10.1.1 Mathematical Analysis of Gravity Assists

Because a hyperbolic trajectory occurs above escape velocity (See Equation (32)), the speed that the spacecraft approaches (relative to the planet) in the limit can be determined.

$$\begin{aligned} v_\infty &= \lim_{r\to\infty} v \\ &= \lim_{r\to\infty}\left(\sqrt{\frac{2\mu}{r}-\frac{\mu}{a}}\right) \\ &= \sqrt{\frac{\mu}{-a}} \end{aligned}$$

Using this, the semi-major axis can be found

$$\begin{aligned} v_\infty &= \sqrt{\frac{\mu}{-a}} \\ v_\infty^2 &= \frac{\mu}{-a} \\ a &= \frac{-\mu}{v_\infty^2} \end{aligned}$$

The semi-major axis is uniquely determined by the limiting velocity of the spacecraft, and is unaffected by the entry/exit angle θ_{hyp}. A more useful way of phrasing this is that the entry/exit angle is unaffected by the entry velocity.

Because a is not determined by the chosen hyperbolic trajectory, from Equation (17) the eccentricity can be determined by the periapsis radius. Solving Equation (17) for the eccentricity,

$$\begin{aligned} r_{\text{pe}} &= a(1-e) \\ r_{\text{pe}} &= \frac{-\mu}{v_\infty^2}(1-e) \end{aligned}$$

$$r_{\text{pe}} = \frac{\mu}{v_\infty^2}(e-1)$$
$$\frac{r_{\text{pe}}v_\infty^2}{\mu} = e - 1$$

$$e = \frac{r_{\text{pe}}v_\infty^2}{\mu} + 1 \tag{55}$$

Recall from Equation (36) that θ_{hyp} is determined uniquely by e.

$$\theta_{\text{hyp}} = \arctan\left(\sqrt{e^2-1}\right)$$
$$= \arctan\left(\sqrt{\left(\frac{r_{\text{pe}}v_\infty^2}{\mu}+1\right)^2-1}\right)$$

$$\theta_{\text{hyp}} = \arctan\left(\sqrt{\left(\frac{r_{\text{pe}}v_\infty^2}{\mu}+1\right)^2-1}\right) \tag{56}$$

With Equation (56), we now have the tools to determine the effect of a gravity assist on a spacecraft's velocity relative to the sun. For the sake of simplicity, the argument of arctan in Equation (56) will be referred to as simply m_{hyp}

The net change in a spacecraft's velocity vector relative to the sun can be expressed as follows. Subscripts s and p will be used to denote velocity relative to a planet versus that relative to the sun. The maneuvering basis vectors are relative to $\vec{v}_{\text{enter},p}$. The exit velocity vector will be rotated by $2\theta_{\text{hyp}}$, and then reversed in direction (See figure 4). This has the net effect of a rotation of $\pi - 2\theta_{\text{hyp}}$ Recall also that $\hat{m}_r$ points *out* of the orbit.

$$\begin{aligned}
\Delta\vec{v}_s &= \Delta\vec{v}_p \\
&= \vec{v}_{\text{exit},p} - \vec{v}_{\text{enter},p} \\
&= v_\infty\hat{v}_{\text{exit},p} - v_\infty\hat{v}_{\text{enter},p} \\
&= v_\infty(\cos(\pi-2\theta_{\text{hyp}})\hat{m}_v - \sin(\pi-2\theta_{\text{hyp}})(-\hat{m}_r)) - v_\infty\hat{m}_v \\
&= v_\infty(\cos(\pi-2\theta_{\text{hyp}})\hat{m}_v + \sin(\pi-2\theta_{\text{hyp}})\hat{m}_r) - v_\infty\hat{m}_v \\
&= v_\infty(-\cos(2\theta_{\text{hyp}})\hat{m}_v + \sin(2\theta_{\text{hyp}})\hat{m}_r) - v_\infty\hat{m}_v \\
&= v_\infty((-1-\cos(2\theta_{\text{hyp}}))\hat{m}_v + \sin(2\theta_{\text{hyp}})\hat{m}_r) \\
&= v_\infty(-(1+\cos(2\theta_{\text{hyp}}))\hat{m}_v + \sin(2\theta_{\text{hyp}})\hat{m}_r) \\
&= v_\infty(-(1+\cos(2\arctan(m_{\text{hyp}})))\hat{m}_v + \sin(2\arctan(m_{\text{hyp}}))\hat{m}_r)
\end{aligned}$$

It can be shown from trigonometry that

$$\cos(2\arctan(\theta)) = \frac{1-\theta^2}{1+\theta^2} \qquad \text{and} \qquad \sin(2\arctan(\theta)) = \frac{2\theta}{1+\theta^2}$$

$$\Delta\vec{v}_s = v_\infty(-(1+\cos(2\arctan(m_{\text{hyp}})))\hat{m}_v + \sin(2\arctan(m_{\text{hyp}}))\hat{m}_r)$$

$$
\begin{aligned}
&= v_\infty \left(-\left(1 + \frac{1 - m_{\text{hyp}}^2}{1 + m_{\text{hyp}}^2}\right)\hat{m}_v + \frac{2m_{\text{hyp}}}{1 + m_{\text{hyp}}^2}\hat{m}_r \right) \\
&= v_\infty \left(-\left(1 + \frac{1 - \left(\frac{r_{\text{pe}}v_\infty^2}{\mu} + 1\right)^2 + 1}{1 + \left(\frac{r_{\text{pe}}v_\infty^2}{\mu} + 1\right)^2 - 1}\right)\hat{m}_v + \frac{2\sqrt{\left(\frac{r_{\text{pe}}v_\infty^2}{\mu} + 1\right)^2 - 1}}{1 + \left(\frac{r_{\text{pe}}v_\infty^2}{\mu} + 1\right)^2 - 1}\hat{m}_r \right) \\
&= v_\infty \left(-\left(\frac{2}{\left(\frac{r_{\text{pe}}v_\infty^2}{\mu} + 1\right)^2}\right)\hat{m}_v + \frac{2\sqrt{\left(\frac{r_{\text{pe}}v_\infty^2}{\mu} + 1\right)^2 - 1}}{\left(\frac{r_{\text{pe}}v_\infty^2}{\mu} + 1\right)^2}\hat{m}_r \right) \\
&= v_\infty \left(\frac{-2}{\left(\frac{r_{\text{pe}}v_\infty^2}{\mu} + 1\right)^2}\hat{m}_v + \frac{2\sqrt{\left(\frac{r_{\text{pe}}v_\infty^2}{\mu} + 1\right)^2 - 1}}{\left(\frac{r_{\text{pe}}v_\infty^2}{\mu} + 1\right)^2}\hat{m}_r \right)
\end{aligned}
$$

While this may not intuitively seem that useful (and, frankly, it isn't), the magnitude proves to be much simpler.

$$
\begin{aligned}
|\Delta \vec{v}_s| &= v_\infty \left| \frac{-2}{\left(\frac{r_{\text{pe}}v_\infty^2}{\mu} + 1\right)^2}\hat{m}_v + \frac{2\sqrt{\left(\frac{r_{\text{pe}}v_\infty^2}{\mu} + 1\right)^2 - 1}}{\left(\frac{r_{\text{pe}}v_\infty^2}{\mu} + 1\right)^2}\hat{m}_r \right| \\
&= v_\infty \sqrt{\left(\frac{-2}{\left(\frac{r_{\text{pe}}v_\infty^2}{\mu} + 1\right)^2}\right)^2 + \left(\frac{2\sqrt{\left(\frac{r_{\text{pe}}v_\infty^2}{\mu} + 1\right)^2 - 1}}{\left(\frac{r_{\text{pe}}v_\infty^2}{\mu} + 1\right)^2}\right)^2} \\
&= v_\infty \sqrt{\frac{4}{\left(\frac{r_{\text{pe}}v_\infty^2}{\mu} + 1\right)^4} + \frac{4\left(\left(\frac{r_{\text{pe}}v_\infty^2}{\mu} + 1\right)^2 - 1\right)}{\left(\frac{r_{\text{pe}}v_\infty^2}{\mu} + 1\right)^4}} \\
&= 2v_\infty \sqrt{\frac{1 + \left(\frac{r_{\text{pe}}v_\infty^2}{\mu} + 1\right)^2 - 1}{\left(\frac{r_{\text{pe}}v_\infty^2}{\mu} + 1\right)^4}} \\
&= 2v_\infty \sqrt{\frac{\left(\frac{r_{\text{pe}}v_\infty^2}{\mu} + 1\right)^2}{\left(\frac{r_{\text{pe}}v_\infty^2}{\mu} + 1\right)^4}} \\
&= 2v_\infty \sqrt{\frac{1}{\left(\frac{r_{\text{pe}}v_\infty^2}{\mu} + 1\right)^2}} \\
&= \frac{2v_\infty}{\frac{r_{\text{pe}}v_\infty^2}{\mu} + 1}
\end{aligned}
$$

The magnitude of ΔV from a gravity assist is

$$\Delta V_{\text{assist}} = \frac{2\mu_p v_\infty}{r_{\text{pe}} v_\infty^2 + \mu_p} \tag{57}$$

Where μ_p refers to the standard gravitational parameter of the planet that is giving a gravity assist.

The direction of this gravity assist is the direction of the change relative velocity of the planet. From Figures 18, 19, and 20, the direction of change in velocity is the vector facing from the periapsis of the hyperbola to the planet.

Equation (57) can be optimized to find the peak ΔV. It is clear that any decrease in r_{pe} will increase the ΔV, so the optimization will be for peak v_∞. This will be done by differentiating (57) with respect to v_∞, and setting the derivative to zero.

$$\begin{aligned}
\frac{d}{dv_\infty}\Delta V_{\text{assist}} &= \frac{d}{dv_\infty}\Delta\left(\frac{2\mu_p v_\infty}{r_{\text{pe}} v_\infty^2 + \mu_p}\right) \\
&= \frac{d}{dv_\infty}\left(\frac{2\mu_p v_\infty}{r_{\text{pe}} v_\infty^2 + \mu_p}\right) \\
&= 2\mu_p \frac{d}{dv_\infty}\left(v_\infty (r_{\text{pe}} v_\infty^2 + \mu_p)^{-1}\right) \\
&= 2\mu_p \left(v_\infty \frac{d}{dv_\infty}(r_{\text{pe}} v_\infty^2 + \mu_p)^{-1} + (r_{\text{pe}} v_\infty^2 + \mu_p)^{-1}\frac{d}{dv_\infty} v_\infty\right) \\
&= 2\mu_p \left(-v_\infty (r_{\text{pe}} v_\infty^2 + \mu_p)^{-2}\frac{d}{dv_\infty}(r_{\text{pe}} v_\infty^2 + \mu_p) + (r_{\text{pe}} v_\infty^2 + \mu_p)^{-1}\right) \\
&= 2\mu_p \left(-v_\infty (r_{\text{pe}} v_\infty^2 + \mu_p)^{-2}(2 r_{\text{pe}} v_\infty) + (r_{\text{pe}} v_\infty^2 + \mu_p)^{-1}\right) \\
&= 2\mu_p \left(\frac{-2 r_{\text{pe}} v_\infty^2}{(r_{\text{pe}} v_\infty^2 + \mu_p)^2} + \frac{1}{r_{\text{pe}} v_\infty^2 + \mu_p}\right) \\
&= 2\mu_p \left(\frac{-2 r_{\text{pe}} v_\infty^2 + r_{\text{pe}} v_\infty^2 + \mu_p}{(r_{\text{pe}} v_\infty^2 + \mu_p)^2}\right) \\
&= 2\mu_p \left(\frac{-r_{\text{pe}} v_\infty^2 + \mu_p}{(r_{\text{pe}} v_\infty^2 + \mu_p)^2}\right)
\end{aligned}$$

Now, this will be set to zero to find v_∞ for maximum ΔV

$$\begin{aligned}
2\mu_p \left(\frac{-r_{\text{pe}} v_\infty^2 + \mu_p}{(r_{\text{pe}} v_\infty^2 + \mu_p)^2}\right) &= 0 \\
\frac{-r_{\text{pe}} v_\infty^2 + \mu_p}{(r_{\text{pe}} v_\infty^2 + \mu_p)^2} &= 0 \\
-r_{\text{pe}} v_\infty^2 + \mu_p &= 0 \\
r_{\text{pe}} v_\infty^2 &= \mu_p \\
v_\infty^2 &= \frac{\mu_p}{r_{\text{pe}}} \\
v_\infty &= \sqrt{\frac{\mu_p}{r_{\text{pe}}}}
\end{aligned}$$

Notice that this is equal to the velocity for a circular orbit at periapsis.

$$v_{\infty,\text{best}} = \sqrt{\frac{\mu_p}{r_{\text{pe}}}} = v_{\text{circ@pe}} \tag{58}$$

This can be plugged into Equation (57) to find the optimal ΔV for a gravity assist.

$$\begin{aligned}
\Delta V_{\text{assist}} &= \frac{2\mu_p v_\infty}{r_{\text{pe}} v_\infty^2 + \mu_p} \\
&= \frac{2\mu_p \sqrt{\frac{\mu_p}{r_{\text{pe}}}}}{r_{\text{pe}} \left(\sqrt{\frac{\mu_p}{r_{\text{pe}}}}\right)^2 + \mu_p} \\
&= \frac{2\mu_p \sqrt{\frac{\mu_p}{r_{\text{pe}}}}}{2\mu_p} \\
&= \sqrt{\frac{\mu_p}{r_{\text{pe}}}}
\end{aligned}$$

The maximum ΔV from a gravity assist occurs when the limiting velocity of the satellite is $v_\infty = \sqrt{\frac{\mu_p}{r_{\text{pe}}}}$, and it results in an assist with $\Delta V = \sqrt{\frac{\mu_p}{r_{\text{pe}}}}$ along the vector pointing from periapsis to the planet.

For the sake of curiosity, we shall now find (using Equation (55)) the eccentricity that this hyperbolic trajectory will have.

$$\begin{aligned}
e &= \frac{r_{\text{pe}} v_\infty^2}{\mu_p} + 1 \\
&= \frac{r_{\text{pe}} (\sqrt{\mu_p / r_{\text{pe}}})^2}{\mu_p} + 1 \\
&= \frac{r_{\text{pe}} (\mu_p / r_{\text{pe}})}{\mu_p} + 1 \\
&= \frac{r_{\text{pe}} \mu_p}{\mu_p r_{\text{pe}}} + 1 \\
&= 1 + 1 \\
&= 2
\end{aligned}$$

The eccentricity to maximize ΔV of a gravity assist is $e = 2$. Equation (36) shows that this gives $\theta_{\text{hyp}} = 60° = \frac{\pi}{3}$.

10.1.2 Gravity Assist Conclusion

The term "gravity assist" is possibly a bit of a misnomer, as the gravity serves only to redirect the velocity of the spacecraft, but does not change its magnitude. A spacecraft can be accelerated in its orbit by a planet if it passes behind the planet. Really what's happening is the spacecraft is using the planet's mass to redirect its speed, resulting in a change in the spacecraft's orbit. A well-timed flyby can act to change the velocity of the spacecraft in many different directions. From equation (57), it can be seen that ΔV is maximized by minimizing r_{pe} and bringing v_∞ as close as possible to $\sqrt{\frac{\mu_p}{r_{\text{pe}}}}$. At this value for v_∞, $\Delta V = \sqrt{\frac{\mu_p}{r_{\text{pe}}}}$ and the hyperbola will have an eccentricity of $e = 2$. It can also be seen that the greater the mass of the planet (and therefore μ), the larger the effect will be. Another way of stating this is that to maximize the effect of a gravity assist, the hyperbolic trajectory should maximize its change in direction, which means minimizing its eccentricity (provided the eccentricity is greater than one). From Equation (55), it can be seen that this entails minimizing r_{pe} for any given v_∞.

10.2 Patched Conic Approximation

So far, the assumption has been made that a satellite has, at any moment, only one gravitational attractor. This leads to convenient results, such as conic section trajectories and simple analytical expressions for position, velocity, and the like. This approximation falls apart when dealing with orbits in which multiple bodies are referenced. The patched conic approximation is one in which you view each body as having its own "sphere of influence", and neglect the gravitational effect of all bodies aside from the one whose sphere of influence a spacecraft is in. Its name eludes to the patching together Keplarian trajectories about different bodies into a continuous set of conic sections.

To motivate this discussion of patched concis, we will examine a mission to Saturn. A spacecraft begins in a low parking orbit about Earth, and then performs a burn to exceed escape velocity. It then flies off from Earth, and enters an interplanetary trajectory about the sun. It orbits the sun for some fraction of its orbital period, after which it flies past Jupiter, getting a gravity assist. At this point, its orbit about the sun has changed, and it is on a trajectory Saturn. Once near Saturn, the spacecraft performs a capture burn to enter a circular parking orbit. This entire trajectory consisted of only two burns; an escape burn and a capture burn. This creates, theoretically, three distinct trajectories (initial parking orbit, transfer orbit, and final parking orbit).

While the actual mission consisted of only three trajectories, a total of 7 conic sections are needed to analyze the trajectory with Keplarian conic sections. The spacecraft's initial orbit about Earth is a circle (conic 1). After the escape burn it becomes hyperbolic (conic 2). Its trajectory around the sun is some elliptic trajectory (conic 3). As it encounters Jupiter, its trajectory from the Jupiterian frame of reference is hyperbolic (conic 4). It then enters another ellipse around the sun (conic 5). As it approaches Saturn, its trajectory about Saturn will be hyperbolic (conic 6). Finally, it circularizes into a parking orbit around Saturn (conic 7). Throughout this process, the reference frame was implicitly switched from Earth, to the sun, to Jupiter, back to the sun, and finally to Saturn.

The conic section shape of a trajectory comes from single-gravitational-attractor systems (note that Equation (5) assumed a single gravitational attractor). To approximate the trajectory as a series of conic sections, all other gravitational influences aside from that of the primary reference must be neglected. Therefore, to approximate a trajectory as a series of patched conic sections, an approximation must be made that at any point there is only one gravitational influence on a spacecraft. Because the vast majority of the time in this interplanetary trajectory is spent under primarily the influence of the sun, with only brief interludes of Terran, Jupiterian, and Saturnian gravity, the approximation proves largely valid. The exact construction of spheres of influence varies by application and publication. The radius of a sphere of influence generally depends on the mass of a body, the mass of its parent body, and the separation of the two bodies.

While the patched conics approximation is only an approximation, and has its flaws, it it a necessary approximation in order to generate trajectories analytically. Much research has been done on the n-body problem (that is to say, the problem of finding trajectories under multiple gravitational influences), with the unfortunate conclusion being that it is not possible to calculate with analytical methods the trajectory of a spacecraft under more than one gravitational influence. The only alternative, therefore, to the patched conics approximation is to numerically integrate gravitational acceleration.

10.3 Conclusion

A gravity assist is a method of using a hyperbolic flyby around a planet to change a satellite's velocity vector around the sun. If the limiting velocity of the hyperbolic orbit and the closest approach altitude are known, then the semi-major axis and eccentricity of the flyby can be calculated as

$$a = \frac{-\mu_p}{v_\infty^2} \qquad \text{and} \qquad e = \frac{r_{\text{pe}} v_\infty^2}{\mu_p} + 1$$

The magnitude of the change in velocity vector about the sun from a flyby is given by

$$\Delta V_{\text{assist}} = \frac{2\mu_p v_\infty}{r_{\text{pe}} v_\infty^2 + \mu_p}$$

To optimize this ΔV, the closest approach distance should be minimized. For any given closest approach distance,

$$v_{\infty,\text{best}} = \sqrt{\frac{\mu_p}{r_{\text{pe}}}} = v_{\text{circ@pe}}$$

Yielding a ΔV of

$$v_{\text{assist,best}} = \sqrt{\frac{\mu_p}{r_{\text{pe}}}}$$

The patched conic approximation allows trajectories to be calculated between multiple bodies easily, without requiring the use of numerical integration. While the approximation does not hold for all instances, it is generally a helpful tool for ΔV and trajectory approximation.

11 Special Orbits

11.1 Geosynchronous Orbits (WIP)

11.2 Lagrange Points (WIP)

Up to this point, trajectories have all assumed a single gravitational influence. Section 10.2 described how to use this approximation to generate trajectories between multiple bodies. However, by taking into account multiple gravitational attractors, otherwise impossible trajectories can be generated.

To motivate this subject, we will consider a lunar signal relay spacecraft. An orbital engineer wants to find an orbit for a satellite that will keep it at the same spot between Earth and the moon at all times, with the purpose of relaying signals from Earth-based communications networks to moon-orbiting satellites. For the sake of this example, the moon's orbit about Earth can be approximated (within reason) to be perfectly circular. Because the satellite is between Earth and the moon, $0 < a_{\text{sat}} < a_{\text{moon}}$.In order for this spacecraft to achieve the desired orbit, its orbital period must match that of the moon. Recall from Eqaution (24) that the period of an orbit is

$$T = \sqrt{\frac{4\pi^2 a}{\mu}}$$

Using this equation, it is clearly impossible for the satellite and the moon to have the same period, as their semi-major axes are different. Neglecting any other gravitational influences, a satellite between Earth and the moon would have to move faster than the moon in order to remain in a circular orbit, and would also have to a further distance to go (a larger orbit means a larger circumference). However, if some force can pull the satellite away from Earth, it will be able to orbit slower at the same altitude. The moon can provide such a force on the satellite. If instead the satellite wants to keep the moon between it and Earth at all times, it would require a downward force, which, if the moon is between it and Earth, the moon can provide. Because of the moon's gravitational influence on the satellite, most orbits at lunar orbital altitude would be unstable. However, the gravitational influence of the moon makes three of them stable. These points at which the satellite can orbit are called Lagrange points, and they exist in any two-body system. Lagrange points will be generalized to be between any two masses M_1 and M_2, some distance R apart.

The most interesting Lagrange points are the ones that lie on either side of the smaller body along the line extending between both. These are known as L1 (in the Earth-moon system, between Earth and the moon), and L2 (on the opposite side of the moon as Earth). The Lagrange points can be found analytically. To make calculations easier, it will be assumed that $M_2 << M_1$, so that M_2 orbits about M_1's center of mass, with M_1's inertial acceleration being negligible.

Figure 21: Locations of L1 and L2 in a system with two masses

11.2.1 L1 Lagrange Point

We will begin with Newton's second law

$$\vec{F}_{\text{net}} = m\vec{a}$$

To use the Transport Theorem (see Section 1) to find the required inertial acceleration $\vec{a}$ in the above equation, the angular velocity ω of the smaller body (in the Earth-moon system, this would be the moon) must be found. ω can be expressed as the velocity divided by the radius. V denotes the velocity of M_2 about M_1 at their distance R apart. Recall circular velocity from Equation (27).

$$\omega = \frac{V}{R}$$
$$\omega = \frac{\sqrt{\mu/R}}{R}$$
$$\omega = \sqrt{\frac{\mu}{R^3}}$$

With this calculated, the required polar acceleration can be calculated. The satellite's orbital radius about M_1 is r

$$\begin{aligned}
\vec{v}^{r\theta n} &= \frac{d^{xyz}}{dt}\vec{r}^{r\theta n} + {}^{xyz}\omega^{r\theta n} \times \vec{r}^{r\theta n} \\
&= \frac{d^{xyz}}{dt} r\hat{u}_r + \sqrt{\frac{\mu}{R^3}}\hat{u}_n \times r\hat{u}_r \\
&= \sqrt{\frac{\mu r^2}{R^3}}\hat{r}_\theta
\end{aligned}$$

This can be differentiated to find acceleration

$$\begin{aligned}
\vec{a}^{r\theta n} &= \frac{d^{xyz}}{dt}\vec{v}^{r\theta n} + {}^{xyz}\omega^{r\theta n} \times \vec{v}^{r\theta n} \\
&= \frac{d^{xyz}}{dt}\sqrt{\frac{\mu r^2}{R^3}}\hat{r}_\theta + \sqrt{\frac{\mu}{R^3}}\hat{u}_n \times \sqrt{\frac{\mu r^2}{R^3}}\hat{r}_\theta \\
&= -\sqrt{\frac{\mu}{R^3}}\sqrt{\frac{\mu r^2}{R^3}}\hat{r}_n \\
&= -\sqrt{\frac{\mu^2 r^2}{R^6}}\hat{r}_n \\
&= -\frac{\mu r}{R^3}\hat{r}_n
\end{aligned}$$

The polar acceleration on the satellite for it to maintain its orbit is

$$\vec{a}_{\text{polar}} = -\frac{\mu r}{R^3}\hat{r}_n$$

From Equation (4) (and Newton's second law), the polar acceleration of gravity in the radial direction is

$$a_g = \frac{GM}{d^2}$$

Where μ has been replaced with GM, and d is the distance to the body. The direction of the acceleration is toward the attracting body. At $L1$, M_2 will provide radially outward acceleration while M_1 will provide radially inward acceleration. An equation can now be set up to find α_{r1}, the ratio of distance between $L1$ and M_1 and that of M_1 and M_2 (in other words, $\alpha_{L1} = r/R$).

$$\begin{aligned}
a_1 + a_2 &= a_{\text{net}} \\
-\frac{GM_1}{(R\alpha_{L1})^2} + \frac{GM_2}{(R(1-\alpha_{L1}))^2} &= -\frac{\mu_1 r}{R^3} \\
-\frac{GM_1}{(R\alpha_{L1})^2} + \frac{GM_2}{(R(1-\alpha_{L1}))^2} &= -\frac{GM_1 R\alpha_{L1}}{R^3} \\
-\frac{M_1}{\alpha_{L1}^2} + \frac{M_2}{(1-\alpha_{L1})^2} &= -M_1\alpha_{L1} \\
-M_1(1-\alpha_{L1})^2 + M_2\alpha_{L1}^2 &= -M_1\alpha_{L1}^3(1-\alpha_{L1})^2 \\
M_1\alpha_{L1}^3(1-\alpha_{L1})^2 - M_1(1-\alpha_{L1})^2 + M_2\alpha_{L1}^2 &= 0 \\
M_1(\alpha_{L1}^3 - 2\alpha_{L1}^4 + \alpha_{L1}^5) + M_1(-1 + 2\alpha_{L1} - \alpha_{L1}^2) + M_2\alpha_{L1}^2 &= 0 \\
M_1\alpha_{L1}^5 - 2M_1\alpha_{L1}^4 + M_1\alpha_{L1}^3 + (M_2 - M_1)\alpha_{L1}^2 + 2M_1\alpha_{L1} - M_1 &= 0
\end{aligned}$$

This is not solveable analytically, but it can be solved numerically. The entire expression will be divided by M_1 to make it simpler.

$$\left(\frac{r_{L1}}{R}\right)^5 - 2\left(\frac{r_{L1}}{R}\right)^4 + M_1\left(\frac{r_{L1}}{R}\right)^3 + \left(\frac{M_2}{M_1} - 1\right)\left(\frac{r_{L1}}{R}\right)^2 + 2\left(\frac{r_{L1}}{R}\right) - 1 = 0 \tag{59}$$

11.3 Time over the Horizon (WIP)

11.4 Conclusion

12 Appendix

12.1 Orbit Planar Geometry

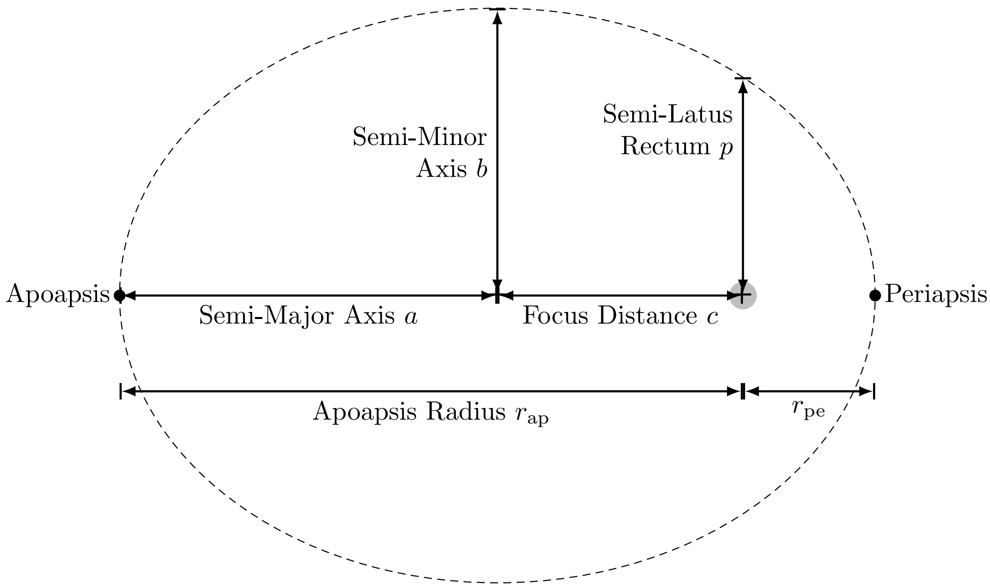

Figure 22: An orbit with important geometric values labeled

In the above figure, the semi-major axis a, semi-minor axis b, semi-latus rectum p, focus distance c, apoapsis radius r_{ap}, and periapsis radius r_{pe} are all labeled. The apoapsis is the highest point in the orbit, and the periapsis is the lowest. The major axis is the largest distance across the ellipse, while the minor axis is the smallest. The latus rectum is the width of the ellipse perpendicular to the foci. The focus distance is the distance between the foci and the center of the ellipse.

12.2 Positional Diagram

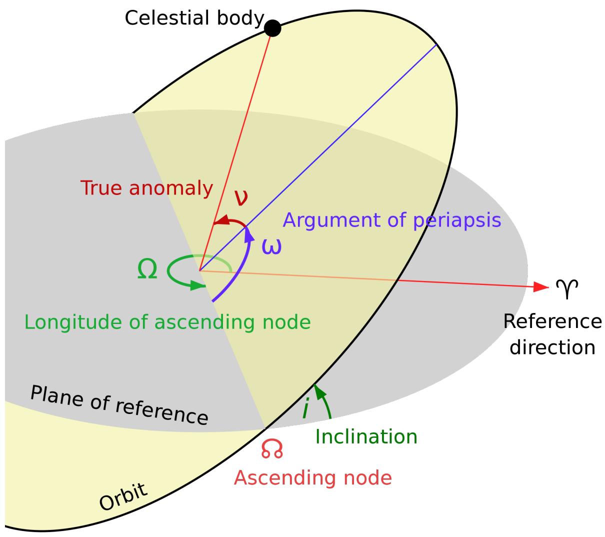

Figure 23: Figure from wikipedia.org showing orbital elements

There are a variety of parameters that are used to position the orbit in space, and to position the satellite in the orbit. Note that ν (nu) and θ (theta) are used interchangeably, with θ being used predominantly in this document. The blue line points to the periapsis. Note also that the blue line and the red line are not necessarily aligned with eachother.

The base frame a can be defined such that $\hat{a}_1$ points in the reference direction, $\hat{a}_2$ points 90 degrees clockwise of the reference direction in place with the reference plane, and $\hat{a}_3$ pointing normal to the base frame (generally "up" in the image above).

Another frame b can be defined such that b is the same as the a frame, but rotated Ω about $\hat{a}_3$. This means that $\hat{b}_1$ points toward the ascending node, $\hat{b}_2$ is normal to $\hat{b}_1$ and in the reference plane, and $\hat{b}_3 = \hat{a}_3$.

Frame c can be defined as a rotation of frame b about $\hat{b}_1$ by i. This frame is in the plane of the orbit, with $\hat{c}_1 = \hat{b}_1$ and $\hat{c}_3$ being normal to the orbit.

Next, d can be defined as a rotation of c by ω about $\hat{c}_3$. This frame is also in the plane of the orbit, with $\hat{c}_1$ pointing toward the periapsis.

Finally, another frame can be defined as a rotation of d by θ about $\hat{d}_3$. This creates the polar frame (see Figures 26 and 27).

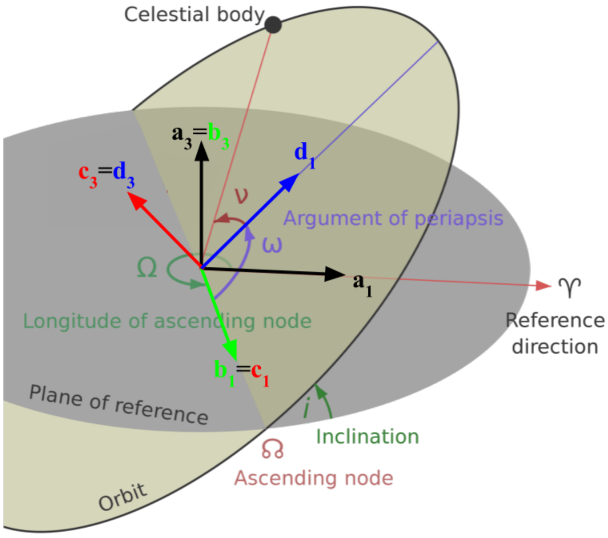

Figure 24: Reference frames (aside from polar and maneuvering frames)

To get a better sense for how orbital parameters work, it is highly reccomended that the reader experiment with https://orbitalmechanics.info/

12.3 Orbit elements

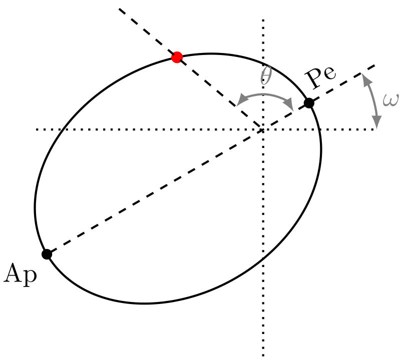

Figure 25: True Anomaly and Argument of Periapsis

Above, an orbit is shown with the major axis indicated by a dashed line, and the apoapsis (Ap) and periapsis (Pe) labeled. The origin of this coordinate system is the focus about which the body orbits. The argument of periapsis ω is the angle between the periapsis of the orbit and the ascending node (the +x axis in this case), and the true anomaly θ (sometimes designated ν) is the angle between the satellite's current location and its periapsis

12.4 Coordinate Systems

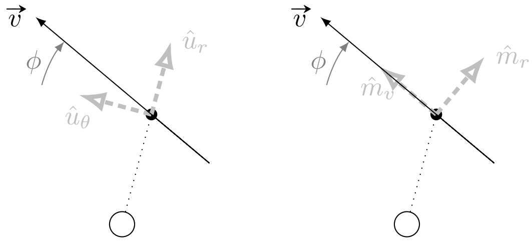

Figure 26: Polar and maneuvering vector frames

A satellite (black dot) is shown orbiting a source (hollow circle) with the flight path angle ϕ and velocity vector $\vec{v}$ indicated. On the left image, the polar basis vectors u are shown. On the right, the manuevering vectors m are shown.

The polar basis vectors are defined such that $\hat{u}_r$ points radially outward from the planet, with $\hat{u}_\theta$ being perpendicular to it and generally in the direction of the velocity vector (and in plane with both $\vec{v}$ and $\hat{u}_r$). $\hat{u}_n$ is defined $\hat{u}_n = \hat{u}_r \times \hat{u}_\theta$. In this figure, $\hat{u}_n$ points out-of-page.

The manuevering basis vectors m are defined such that $\hat{m}_v$ points in the direction of the velocity, with $\hat{m}_r$ being normal to it and generally facing away from the planet (and in the plane defined by the velocity vector and the planet). $\hat{m}_n$ will always be equal to $\hat{u}_n$, and is defined as $\hat{m}_n = \hat{m}_r \times \hat{m}_v$.

The flight path angle is the angle between the two basis vectors

12.5 Coordinate System 3D

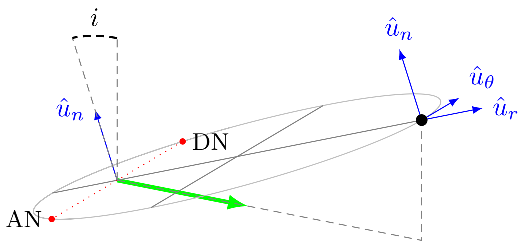

Figure 27: Inclined orbit showing the polar basis vectors

A satellite (black dot) is shown above orbiting a body with the polar coordinate system shown. The semi-major and semi-minor axes are shown with a thin line. Note that in this position (at the apoapsis in this case, however this also holds true at periapsis), the polar u and maneuvering m basis vectors are equal.

Note that the inclination i of the orbit is also shown. In this figure the satellite is shown with $\theta = \pi$ and $\omega = \frac{3}{2}\pi$ (the satellite is 180 degrees offset from its periapsis, and the periapsis is 270 degrees offset from the ascending node). The red dots are the ascending and descending nodes. These are the points at which the orbit intersects the horizon. The ascending node (AN) is the one at which the latitude changes from negative to positive, and the desending node (DN) is the point at which latitude goes from positive to negative. In this figure, $\Omega = \frac{3}{2}\pi$ (with the reference direction being the green arrow). Note that Ω does not always equal ω; Ω positions the ascending node relative to a reference direction, while ω positions the periapsis relative to the ascending node.