The circular restricted three body problem (CR3BP) is a special case of the three body problem. In the CR3BP (much like in Keplerian 2-body dynamics), we neglect the mass of satellite S, while treating the larger celestial body m_1 and smaller celestial body m_2 as point masses. Crucially, these bodies must orbit one another in circular orbits. In other words, they both orbit about their inertially fixed barycenter c *at constant velocity and distance.*

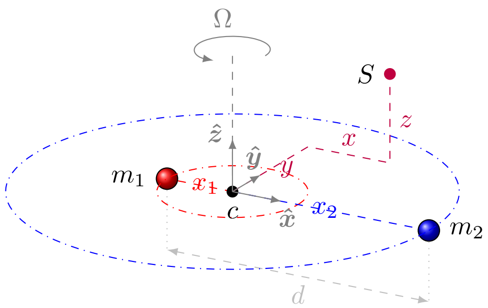

Figure 1: Geometry

The purple satellite is located by $\boldsymbol{r} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}$, and the spheres are the celestial bodies. The xyz frame is not inertially stationary. Instead, they rotate with the larger bodies at a rate Ω. The origin c is the center of mass of m_1 and m_2 (and therefore inertially fixed). The xy plane defines the plane in which m_1 and m_2 orbit, so $\hat{\boldsymbol{z}}$ is defined by the direction of their angular momenta- the constant rotation at a rate Ω is a consequence of this. Note that $\boldsymbol{r}$ is not constrained to the xy plane, but m_1 and m_2 definitionally are.

Useful Relations

We can also find a relationship between x_1, x_2, d, m_1, and m_2. Because c is the barycenter,

$$x_1 m_1 = x_2 m_2$$

or

$$\frac{x_1}{m_2} = \frac{x_2}{m_1}$$

We can use this to find that

$$\begin{aligned}\frac{x_1}{x_1 + x_2} &= \frac{x_1/m_2}{x_1/m_2 + x_2/m_2} \\ &= \frac{x_2/m_1}{x_2/m_1 + x_2/m_2} \\ &= \frac{1/m_1}{1/m_1 + 1/m_2} \\ &= \frac{m_2}{m_2 + m_1}\end{aligned}$$

Defining $d = x_1 + x_2$ as the distance between the two celestial bodies, and $M = m_1 + m_2$ as their total mass, we get that

$$\boxed{\frac{x_1}{d} = \frac{m_2}{M}}$$

and similarly

$$\boxed{\frac{x_2}{d} = \frac{m_1}{M}}$$

Which can be rewritten as

$$\boxed{\frac{x_1}{m_2} = \frac{x_2}{m_1} = \frac{d}{M}}$$

Lastly, we will define two additional vectors $\boldsymbol{r_1}$ and $\boldsymbol{r_2}$ which point from the first and second body respectively to the satellite.

$$\boldsymbol{r_1} = (x + x_1)\,\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}$$

and

$$\boldsymbol{r_2} = (x - x_2)\,\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}$$

Now we will solve for Ω.

$$\begin{aligned}
\boldsymbol{F}_{\text{on2}} &= -\frac{Gm_1m_2}{d^2}\hat{\boldsymbol{x}} \\
\cancel{m_2}\boldsymbol{a}_2^f &= -\frac{Gm_1\cancel{m_2}}{d^2}\hat{\boldsymbol{x}} \\
\frac{\mathrm{d}^f}{\mathrm{d}t}\left(\frac{\mathrm{d}^f}{\mathrm{d}t}x_2\hat{\boldsymbol{x}}\right) &= -\frac{Gm_1}{d^2}\hat{\boldsymbol{x}} \\
\frac{\mathrm{d}^f}{\mathrm{d}t}\left(\cancelto{0}{\frac{\mathrm{d}^c x_2\hat{\boldsymbol{x}}}{\mathrm{d}t}} + \Omega\hat{\boldsymbol{z}} \times x_2\hat{\boldsymbol{x}}\right) &= -\frac{Gm_1}{d^2}\hat{\boldsymbol{x}} \\
\frac{\mathrm{d}^f}{\mathrm{d}t}\Omega x_2\hat{\boldsymbol{y}} &= -\frac{Gm_1}{d^2}\hat{\boldsymbol{x}} \\
\cancelto{0}{\frac{\mathrm{d}^c}{\mathrm{d}t}\Omega x_2\hat{\boldsymbol{y}}} + \Omega\hat{\boldsymbol{z}} \times \Omega x_2\hat{\boldsymbol{y}} &= -\frac{Gm_1}{d^2}\hat{\boldsymbol{x}} \\
-\Omega^2 x_2\hat{\boldsymbol{x}} &= -\frac{Gm_1}{d^2}\hat{\boldsymbol{x}} \\
\Omega^2 x_2 &= \frac{Gm_1}{d^2} \\
\Omega &= \sqrt{\frac{G}{d^2}\frac{m_1}{x_2}} \\
\Omega &= \sqrt{\frac{GM}{d^3}}
\end{aligned}$$

Defining μ conventionally as $\mu = GM$,

$$\boxed{\Omega = \sqrt{\frac{\mu}{d^3}}}$$

Kinematics

The transport theorem states that the inertial (fixed f frame) derivative of a vector u (expressed in the rotating c frame) is

$$\frac{\mathrm{d}^f\boldsymbol{u}}{\mathrm{d}t} = \frac{\mathrm{d}^c\boldsymbol{u}}{\mathrm{d}t} + \boldsymbol{\omega}^{cf} \times \boldsymbol{u}$$

Where $\frac{\mathrm{d}^f}{\mathrm{d}t}$ denotes the derivative in the coordinates of the fixed frame f, and $\frac{\mathrm{d}^c}{\mathrm{d}t}$ denotes derivative in the coordinates of the rotating frame c, and $\boldsymbol{\omega}^{cf}$ denotes the angular velocity of c in f. For this case, $\boldsymbol{\omega}^{cf} = \Omega\hat{\boldsymbol{z}}$. We can find Ω

For the satellite's position in the CR3BP frame $\boldsymbol{r} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}$, we will find the inertial acceleration to generate equations of motion.

$$\begin{aligned}
\dot{\boldsymbol{r}} &= \frac{\mathrm{d}^c\boldsymbol{r}}{\mathrm{d}t} + \boldsymbol{\omega}^{cf} \times \boldsymbol{r} \\
&= \frac{\mathrm{d}^c}{\mathrm{d}t}\left(x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}\right) + \left(\Omega\hat{\boldsymbol{z}} \times \left(x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}\right)\right) \\
&= \left(\dot{x}\hat{\boldsymbol{x}} + \dot{y}\hat{\boldsymbol{y}} + \dot{z}\hat{\boldsymbol{z}}\right) + \left(\Omega x\hat{\boldsymbol{y}} - \Omega y\hat{\boldsymbol{x}}\right) \\
&= \left(\dot{x} - \Omega y\right)\hat{\boldsymbol{x}} + \left(\dot{y} + \Omega x\right)\hat{\boldsymbol{y}} + \dot{z}\hat{\boldsymbol{z}}
\end{aligned}$$

$$
\begin{aligned}
\ddot{\boldsymbol{r}} &= \dot{\boldsymbol{r}} \\
&= \frac{\mathrm{d}^c}{\mathrm{d}t}\left(\left(\dot{x}-\Omega y\right)\hat{\boldsymbol{x}}+\left(\dot{y}+\Omega x\right)\hat{\boldsymbol{y}}+\dot{z}\hat{\boldsymbol{z}}\right) \\
&\quad +\Omega\hat{\boldsymbol{z}}\times\left(\left(\dot{x}-\Omega y\right)\hat{\boldsymbol{x}}+\left(\dot{y}+\Omega x\right)\hat{\boldsymbol{y}}+\dot{z}\hat{\boldsymbol{z}}\right) \\
&= \left(\ddot{x}-\Omega\dot{y}\right)\hat{\boldsymbol{x}}+\left(\ddot{y}+\Omega\dot{x}\right)\hat{\boldsymbol{y}}+\ddot{z}\hat{\boldsymbol{z}} \\
&\quad +\left(\left(\Omega\dot{x}-\Omega^2 y\right)\hat{\boldsymbol{y}}-\left(\Omega\dot{y}+\Omega^2 x\right)\hat{\boldsymbol{x}}\right) \\
&= \left(\ddot{x}-2\Omega\dot{y}-\Omega^2 x\right)\hat{\boldsymbol{x}}+\left(\ddot{y}+2\Omega\dot{x}-\Omega^2 y\right)\hat{\boldsymbol{y}}+\ddot{z}\hat{\boldsymbol{z}}
\end{aligned}
$$

$$
\boxed{\ddot{\boldsymbol{r}} = \left(\ddot{x}-2\Omega\dot{y}-\Omega^2 x\right)\hat{\boldsymbol{x}}+\left(\ddot{y}+2\Omega\dot{x}-\Omega^2 y\right)\hat{\boldsymbol{y}}+\ddot{z}\hat{\boldsymbol{z}}}
$$

Equations of Motion

We can now generate the equations of motion

$$
\begin{aligned}
\sum_i \boldsymbol{F_i} &= m\ddot{\boldsymbol{r}} \\
\boldsymbol{F_1}+\boldsymbol{F_1} &= m\ddot{\boldsymbol{r}} \\
-\frac{\mu_1 m}{r_1^3}\boldsymbol{r_1}-\frac{\mu_2 m}{r_2^3}\boldsymbol{r_2} &= m\ddot{\boldsymbol{r}} \\
-\frac{\mu_1}{r_1^3}\boldsymbol{r_1}-\frac{\mu_2}{r_2^3}\boldsymbol{r_2} &= \left(\ddot{x}-2\Omega\dot{y}-\Omega^2 x\right)\hat{\boldsymbol{x}}+\left(\ddot{y}+2\Omega\dot{x}-\Omega^2 y\right)\hat{\boldsymbol{y}}+\ddot{z}\hat{\boldsymbol{z}}
\end{aligned}
$$

We now write this as three equations, one each in x, y, and z

$$
\begin{aligned}
-\frac{\mu_1}{r_1^3}(x+x_1)-\frac{\mu_2}{r_2^3}(x-x_2) &= \ddot{x}-2\Omega\dot{y}-\Omega^2 x \\
-\frac{\mu_1}{r_1^3}y-\frac{\mu_2}{r_2^3}y &= \ddot{y}+2\Omega\dot{x}-\Omega^2 y \\
-\frac{\mu_1}{r_1^3}z-\frac{\mu_2}{r_2^3}z &= \ddot{z}
\end{aligned}
$$

Isolating the second derivatives,

$$
\begin{aligned}
\ddot{x} &= -\frac{\mu_1}{r_1^3}(x+x_1)-\frac{\mu_2}{r_2^3}(x-x_2)+2\Omega\dot{y}+\Omega^2 x \\
\ddot{y} &= -\frac{\mu_1}{r_1^3}y-\frac{\mu_2}{r_2^3}y-2\Omega\dot{x}+\Omega^2 y \\
\ddot{z} &= -\frac{\mu_1}{r_1^3}z-\frac{\mu_2}{r_2^3}z
\end{aligned}
$$

We can now substitute $\Omega=\sqrt{\frac{\mu}{d^3}}=\sqrt{\frac{\mu_1+\mu_2}{d^3}}$

$$
\begin{aligned}
\ddot{x} &= -\frac{\mu_1}{r_1^3}(x+x_1)-\frac{\mu_2}{r_2^3}(x-x_2)+2\sqrt{\frac{\mu}{d^3}}\dot{y}+\frac{\mu}{d^3}x \\
\ddot{y} &= -\frac{\mu_1}{r_1^3}y-\frac{\mu_2}{r_2^3}y-2\sqrt{\frac{\mu}{d^3}}\dot{x}+\frac{\mu}{d^3}y \\
\ddot{z} &= -\frac{\mu_1}{r_1^3}z-\frac{\mu_2}{r_2^3}z
\end{aligned}
$$

We can also get rid of μ_2 and x_2

$$
\begin{aligned}
\ddot{x} &= -\frac{\mu_1}{r_1^3}(x+x_1)-\frac{\mu_2}{r_2^3}(x-d+x_1)+2\sqrt{\frac{\mu}{d^3}}\dot{y}+\frac{\mu}{d^3}x \\
\ddot{y} &= -\frac{\mu_1}{r_1^3}y-\frac{\mu-\mu_1}{r_2^3}y-2\sqrt{\frac{\mu}{d^3}}\dot{x}+\frac{\mu}{d^3}y \\
\ddot{z} &= -\frac{\mu_1}{r_1^3}z-\frac{\mu-\mu_1}{r_2^3}z
\end{aligned}
$$

Now, we will non-dimensionalize these to make the EOMs more general. The distance unit is $L = x_1 + x_2$. The time unit is $T = \sqrt{d/\mu}$ (the inverse of angular velocity). To non-dimensionalize, we must cancel units. The mass unit is $M = m_1 + m_2$

$$\ddot{x}\frac{T^2}{L} = -\frac{\mu_1\left(\frac{T^2}{L^3}\right)}{r_1^3\left(\frac{1}{L}\right)^3}(x + x_1) - \frac{\mu_2}{r_2^3}(x - d + x_1) + 2\sqrt{\frac{\mu}{d^3}}\dot{y} + \frac{\mu}{d^3}x$$

$$\ddot{y}\frac{T^2}{L} = -\frac{\mu_1\left(\frac{T^2}{L^3}\right)}{r_1^3\left(\frac{1}{L}\right)^3}y - \frac{\mu - \mu_1}{r_2^3}y - 2\sqrt{\frac{\mu}{d^3}}\dot{x} + \frac{\mu}{d^3}y$$

$$\ddot{z}\frac{T^2}{L} = -\frac{\mu_1\left(\frac{T^2}{L^3}\right)}{r_1^3\left(\frac{1}{L}\right)^3}z - \frac{\mu - \mu_1}{r_2^3}z$$