The circular restricted three body problem (CR3BP) is a special case of the three body problem. In the CR3BP (much like in Keplerian 2-body dynamics), we neglect the mass of satellite S, while treating the larger celestial body B_1 and smaller celestial body B_2 as point masses. Crucially, these bodies must orbit one another in circular orbits. In other words, they both orbit about their inertially fixed barycenter c *at constant velocity and distance*.

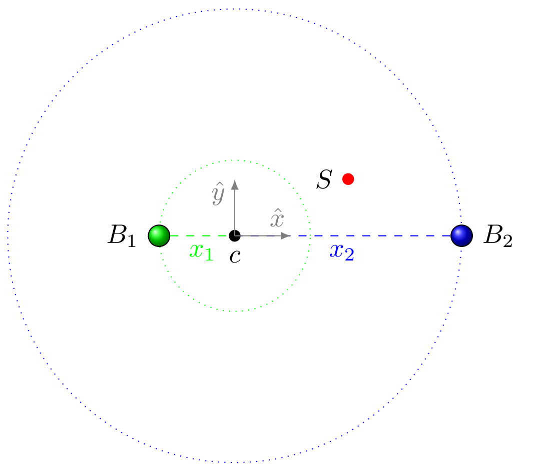

Figure 1: Geometry. S is the satellite, B are the celestial bodies. $\hat{z}$ implied out of page. The bodies are always along the x axis, as the c frame rotates with them

Because $\hat{x}$ points from c to B_2, which is not inertially stationary, the xyz frame is rotating. Specifically, it is rotating positively about z. Because the celestial bodies are in a circular orbit, their rates of rotation about c are constant. This means that the xyz frame rotates at a constant rate of $\overline{\omega} = \Omega \hat{z}$

We can also find a relationship between x_1, x_2, r_{12}, m_1, and m_2. Because c is the barycenter,

$$\boxed{x_1 m_1 = x_2 m_2}$$

Kinematics

The transport theorem states that the inertial (fixed f frame) derivative of a vector u (expressed in the rotating c frame) is

$$\frac{\mathrm{d}^f \overline{u}}{\mathrm{d}t} = \frac{\mathrm{d}^c \overline{u}}{\mathrm{d}t} + \overline{\omega}^{cf} \times \overline{u}$$

Where $\frac{\mathrm{d}^f}{\mathrm{d}t}$ denotes the derivative in the coordinates of the fixed frame f, and $\frac{\mathrm{d}^c}{\mathrm{d}t}$ denotes derivative in the coordinates of the rotating frame c, and $\overline{\omega}^{cf}$ denotes the angular velocity of c in f. For this case, $\overline{\omega}^{cf} = \Omega \hat{z}$.

For a position vector $\overline{r}$, we will find the velocity $\overline{v} = \dot{\overline{r}}$ and acceleration $\overline{a} = \ddot{\overline{r}}$. Note that dot notation implies inertial.

$$\begin{aligned}
\overline{v} &= \dot{\overline{r}} \\
&= \frac{\mathrm{d}^c \overline{r}}{\mathrm{d}t} + \overline{\omega}^{cf} \times \overline{r} \\
&= \overline{v}^c + (\Omega \hat{z} \times \overline{r})
\end{aligned}$$

$$\begin{aligned}
\overline{a} &= \dot{\overline{v}} \\
&= \frac{\mathrm{d}^c}{\mathrm{d}t} \left(\overline{v}^c + (\Omega \hat{z} \times \overline{r})\right) + \overline{\omega}^{cf} \times \left(\overline{v}^c + (\Omega \hat{z} \times \overline{r})\right) \\
&= \frac{\mathrm{d}^c}{\mathrm{d}t} \left(\overline{v}^c + (\Omega \hat{z} \times \overline{r})\right) + \Omega \hat{z} \times \left(\overline{v}^c + (\Omega \hat{z} \times \overline{r})\right) \\
&= \overline{a}^c + (\Omega \hat{z} \times \overline{v}^c) + \Omega \hat{z} \times \left(\overline{v}^c + (\Omega \hat{z} \times \overline{r})\right) \\
&= \overline{a}^c + (\Omega \hat{z} \times \overline{v}^c) + (\Omega \hat{z} \times \overline{v}^c) + \Omega \hat{z} \times (\Omega \hat{z} \times \overline{r}) \\
&= \overline{a}^c + 2\left(\Omega \hat{z} \times \overline{v}^c\right) - \Omega^2 \overline{r}
\end{aligned}$$

$$\boxed{\overline{v} = \overline{v}^c + (\Omega \hat{z} \times \overline{r})}$$

$$\boxed{\overline{a} = \overline{a}^c + 2\left(\Omega \hat{z} \times \overline{v}^c\right)}$$

Now we will solve for Ω. Note that, because c is the center of mass of B_1 and B_2,

$$\begin{aligned}
F_{\text{on2}} &= -\frac{G m_1 m_2}{r_{12}^2} \hat{x} \\
\cancel{m_2} \overline{a}_2^f &= -\frac{G m_1 \cancel{m_2}}{r_{12}^2} \hat{x} \\
\left(\cancelto{0}{\overline{a}^c} + 2\left(\Omega \hat{z} \times \cancelto{0}{\overline{v}^c}\right) - \Omega^2 \overline{r}\right) &= \frac{G m_1}{r_{12}^2} \\
-\Omega^2 x_2 \hat{x} &= -\frac{G m_1}{r_{12}^2} \hat{x} \\
\Omega^2 x_2 &= \frac{G m_1}{r_{12}^2} \\
\Omega &= \sqrt{\frac{G m_1}{x_2 r_{12}^2}}
\end{aligned}$$