The circular restricted three body problem (CR3BP) is a special case of the three body problem. In the CR3BP (much like in Keplerian 2-body dynamics), we neglect the mass of satellite S, while treating the larger celestial body m_1 and smaller celestial body m_2 as point masses. Crucially, these bodies must orbit one another in circular orbits. In other words, they both orbit about their inertially fixed barycenter c *at constant velocity and distance.*

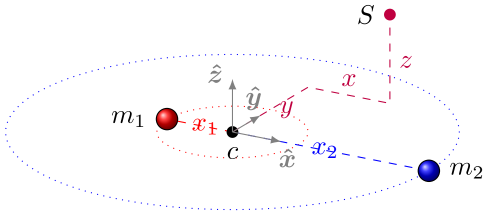

Figure 1: Geometry. The purple satellite is located by $\boldsymbol{r} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}$, and the spheres are the celestial bodies. Note that $\boldsymbol{r}$ is not constrained to the xy plane, but m_1 and m_2 definitionally are. The bodies are always along the x axis, as the c frame rotates with them

Because $\hat{\boldsymbol{x}}$ points from c to m_2, which is not inertially stationary, the xyz frame is rotating. Specifically, it is rotating positively about z. Because the celestial bodies are in a circular orbit, their rates of rotation about c are constant. This means that the xyz frame rotates at a constant rate of $\boldsymbol{\omega} = \Omega\hat{\boldsymbol{z}}$

We can also find a relationship between x_1, x_2, r_{12}, m_1, and m_2. Because c is the barycenter,

$$x_1 m_1 = x_2 m_2$$

or

$$\frac{x_1}{m_2} = \frac{x_2}{m_1}$$

We can use this to find that

$$\begin{aligned}
\frac{x_1}{x_1 + x_2} &= \frac{x_1/m_2}{x_1/m_2 + x_2/m_2} \\
&= \frac{x_2/m_1}{x_2/m_1 + x_2/m_2} \\
&= \frac{1/m_1}{1/m_1 + 1/m_2} \\
&= \frac{m_2}{m_2 + m_1}
\end{aligned}$$

Defining $r_{12} = x_1 + x_2$ as the distance between the two celestial bodies, and $M = m_1 + m_2$ as their total mass, we get that

$$\boxed{\frac{x_1}{r_{12}} = \frac{m_2}{M}}$$

and similarly

$$\boxed{\frac{x_2}{r_{12}} = \frac{m_1}{M}}$$

Which can be rewritten as

$$\boxed{\frac{x_1}{m_2} = \frac{x_2}{m_1} = \frac{r_{12}}{M}}$$

Kinematics

The transport theorem states that the inertial (fixed f frame) derivative of a vector u (expressed in the rotating c frame) is

$$\frac{\mathrm{d}^f \boldsymbol{u}}{\mathrm{d}t} = \frac{\mathrm{d}^c \boldsymbol{u}}{\mathrm{d}t} + \boldsymbol{\omega}^{cf} \times \boldsymbol{u}$$

Where $\frac{\mathrm{d}^f}{\mathrm{d}t}$ denotes the derivative in the coordinates of the fixed frame f, and $\frac{\mathrm{d}^c}{\mathrm{d}t}$ denotes derivative in the coordinates of the rotating frame c, and $\boldsymbol{\omega}^{cf}$ denotes the angular velocity of c in f. For this case, $\boldsymbol{\omega}^{cf} = \Omega\hat{\boldsymbol{z}}$. We can find Ω

For the satellite's position in the CR3BP frame $\boldsymbol{r} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}$, we will find the inertial acceleration to generate equations of motion.

$$\begin{aligned}
\dot{\boldsymbol{r}} &= \frac{\mathrm{d}^c \boldsymbol{r}}{\mathrm{d}t} + \boldsymbol{\omega}^{cf} \times \boldsymbol{r} \\
&= \frac{\mathrm{d}^c}{\mathrm{d}t}\left(x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}\right) + \left(\Omega\hat{\boldsymbol{z}} \times \left(x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}\right)\right) \\
&= \left(\dot{x}\hat{\boldsymbol{x}} + \dot{y}\hat{\boldsymbol{y}} + \dot{z}\hat{\boldsymbol{z}}\right) + \left(\Omega x\hat{\boldsymbol{y}} - \Omega y\hat{\boldsymbol{x}}\right) \\
&= \left(\dot{x} - \Omega y\right)\hat{\boldsymbol{x}} + \left(\dot{y} + \Omega x\right)\hat{\boldsymbol{y}} + \dot{z}\hat{\boldsymbol{z}}
\end{aligned}$$

$$\begin{aligned}
\ddot{\boldsymbol{r}} &= \dot{\boldsymbol{r}} \\
&= \frac{\mathrm{d}^c}{\mathrm{d}t}\left(\left(\dot{x} - \Omega y\right)\hat{\boldsymbol{x}} + \left(\dot{y} + \Omega x\right)\hat{\boldsymbol{y}} + \dot{z}\hat{\boldsymbol{z}}\right) \\
&\quad + \Omega\hat{\boldsymbol{z}} \times \left(\left(\dot{x} - \Omega y\right)\hat{\boldsymbol{x}} + \left(\dot{y} + \Omega x\right)\hat{\boldsymbol{y}} + \dot{z}\hat{\boldsymbol{z}}\right) \\
&= \left(\ddot{x} - \Omega\dot{y}\right)\hat{\boldsymbol{x}} + \left(\ddot{y} + \Omega\dot{x}\right)\hat{\boldsymbol{y}} + \ddot{z}\hat{\boldsymbol{z}} \\
&\quad + \left(\left(\Omega\dot{x} - \Omega^2 y\right)\hat{\boldsymbol{y}} - \left(\Omega\dot{y} + \Omega^2 x\right)\hat{\boldsymbol{x}}\right) \\
&= \left(\ddot{x} - 2\Omega\dot{y} - \Omega^2 x\right)\hat{\boldsymbol{x}} + \left(\ddot{y} + 2\Omega\dot{x} - \Omega^2 y\right)\hat{\boldsymbol{y}} + \ddot{z}\hat{\boldsymbol{z}}
\end{aligned}$$

$$\boxed{\ddot{\boldsymbol{r}} = \left(\ddot{x} - 2\Omega\dot{y} - \Omega^2 x\right)\hat{\boldsymbol{x}} + \left(\ddot{y} + 2\Omega\dot{x} - \Omega^2 y\right)\hat{\boldsymbol{y}} + \ddot{z}\hat{\boldsymbol{z}}}$$