The circular restricted three body problem (CR3BP) is a special case of the three body problem. In the CR3BP (much like in Keplerian 2-body dynamics), we neglect the mass of satellite S, while treating the larger celestial body m_1 and smaller celestial body m_2 as point masses. Crucially, these bodies must orbit one another in circular orbits. In other words, they both orbit about their inertially fixed barycenter c *at constant velocity and distance.*

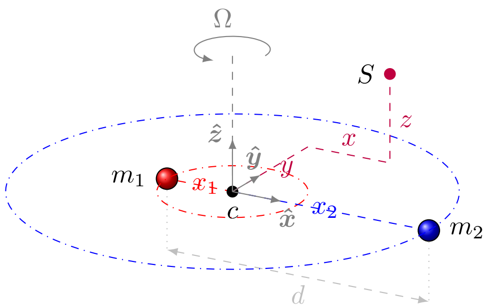

Figure 1: Geometry

The purple satellite is located by $\boldsymbol{r} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}$, and the spheres are the celestial bodies. The xyz frame is not inertially stationary. Instead, they rotate with the larger bodies at a rate Ω. The origin c is the center of mass of m_1 and m_2 (and therefore inertially fixed). The xy plane defines the plane in which m_1 and m_2 orbit, so $\hat{\boldsymbol{z}}$ is defined by the direction of their angular momenta- the constant rotation at a rate Ω is a consequence of this. Note that $\boldsymbol{r}$ is not constrained to the xy plane, but m_1 and m_2 definitionally are.

Useful Relations

We can also find a relationship between x_1, x_2, d, m_1, and m_2. Because c is the barycenter,

$$x_1 m_1 = x_2 m_2$$

or

$$\frac{x_1}{m_2} = \frac{x_2}{m_1}$$

We can use this to find that

$$\begin{aligned}
\frac{x_1}{x_1 + x_2} &= \frac{x_1/m_2}{x_1/m_2 + x_2/m_2} \\
&= \frac{x_2/m_1}{x_2/m_1 + x_2/m_2} \\
&= \frac{1/m_1}{1/m_1 + 1/m_2} \\
&= \frac{m_2}{m_2 + m_1}
\end{aligned}$$

Defining $d = x_1 + x_2$ as the distance between the two celestial bodies, and $m = m_1 + m_2$ as their total mass, we get that

$$\boxed{\frac{x_1}{d} = \frac{m_2}{m}}$$

and similarly

$$\boxed{\frac{x_2}{d} = \frac{m_1}{m}}$$

Which can be rewritten as

$$\boxed{\frac{x_1}{m_2} = \frac{x_2}{m_1} = \frac{d}{m}}$$

Lastly, we will define two additional vectors $\boldsymbol{r_1}$ and $\boldsymbol{r_2}$ which point from the first and second body respectively to the satellite.

$$\boldsymbol{r_1} = (x + x_1)\,\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}$$

and

$$\boldsymbol{r_2} = (x - x_2)\,\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}$$

Now we will solve for Ω.

$$\begin{aligned}
\boldsymbol{F}_{\text{on2}} &= -\frac{Gm_1m_2}{d^2}\hat{\boldsymbol{x}} \\
\cancel{m_2}\boldsymbol{a}_2^f &= -\frac{Gm_1\cancel{m_2}}{d^2}\hat{\boldsymbol{x}} \\
\frac{\mathrm{d}^f}{\mathrm{d}t}\left(\frac{\mathrm{d}^f}{\mathrm{d}t}x_2\hat{\boldsymbol{x}}\right) &= -\frac{Gm_1}{d^2}\hat{\boldsymbol{x}} \\
\frac{\mathrm{d}^f}{\mathrm{d}t}\left(\cancelto{0}{\frac{\mathrm{d}^c x_2\hat{\boldsymbol{x}}}{\mathrm{d}t}} + \Omega\hat{\boldsymbol{z}}\times x_2\hat{\boldsymbol{x}}\right) &= -\frac{Gm_1}{d^2}\hat{\boldsymbol{x}} \\
\frac{\mathrm{d}^f}{\mathrm{d}t}\Omega x_2\hat{\boldsymbol{y}} &= -\frac{Gm_1}{d^2}\hat{\boldsymbol{x}} \\
\cancelto{0}{\frac{\mathrm{d}^c}{\mathrm{d}t}\Omega x_2\hat{\boldsymbol{y}}} + \Omega\hat{\boldsymbol{z}}\times\Omega x_2\hat{\boldsymbol{y}} &= -\frac{Gm_1}{d^2}\hat{\boldsymbol{x}} \\
-\Omega^2 x_2\hat{\boldsymbol{x}} &= -\frac{Gm_1}{d^2}\hat{\boldsymbol{x}} \\
\Omega^2 x_2 &= \frac{Gm_1}{d^2} \\
\Omega &= \sqrt{\frac{G}{d^2}\frac{m_1}{x_2}} \\
\Omega &= \sqrt{\frac{Gm}{d^3}}
\end{aligned}$$

Defining μ conventionally as $\mu = Gm$,

$$\boxed{\Omega = \sqrt{\frac{\mu}{d^3}}}$$

Kinematics

The transport theorem states that the inertial (fixed f frame) derivative of a vector u (expressed in the rotating c frame) is

$$\frac{\mathrm{d}^f\boldsymbol{u}}{\mathrm{d}t} = \frac{\mathrm{d}^c\boldsymbol{u}}{\mathrm{d}t} + \boldsymbol{\omega}^{cf}\times\boldsymbol{u}$$

Where $\frac{\mathrm{d}^f}{\mathrm{d}t}$ denotes the derivative in the coordinates of the fixed frame f, and $\frac{\mathrm{d}^c}{\mathrm{d}t}$ denotes derivative in the coordinates of the rotating frame c, and $\boldsymbol{\omega}^{cf}$ denotes the angular velocity of c in f. For this case, $\boldsymbol{\omega}^{cf} = \Omega\hat{\boldsymbol{z}}$. We can find Ω

For the satellite's position in the CR3BP frame $\boldsymbol{r} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}$, we will find the inertial acceleration to generate equations of motion.

$$\begin{aligned}
\dot{\boldsymbol{r}} &= \frac{\mathrm{d}^c\boldsymbol{r}}{\mathrm{d}t} + \boldsymbol{\omega}^{cf}\times\boldsymbol{r} \\
&= \frac{\mathrm{d}^c}{\mathrm{d}t}(x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}) + (\Omega\hat{\boldsymbol{z}}\times(x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}})) \\
&= (\dot{x}\hat{\boldsymbol{x}} + \dot{y}\hat{\boldsymbol{y}} + \dot{z}\hat{\boldsymbol{z}}) + (\Omega x\hat{\boldsymbol{y}} - \Omega y\hat{\boldsymbol{x}}) \\
&= (\dot{x} - \Omega y)\,\hat{\boldsymbol{x}} + (\dot{y} + \Omega x)\,\hat{\boldsymbol{y}} + \dot{z}\hat{\boldsymbol{z}}
\end{aligned}$$

$$
\begin{aligned}
\ddot{\boldsymbol{r}} &= \dot{\boldsymbol{r}} \\
&= \frac{\mathrm{d}^c}{\mathrm{d}t}\left((\dot{x}-\Omega y)\,\hat{\boldsymbol{x}} + (\dot{y}+\Omega x)\,\hat{\boldsymbol{y}} + \dot{z}\hat{\boldsymbol{z}}\right) \\
&\quad + \Omega\hat{\boldsymbol{z}} \times \left((\dot{x}-\Omega y)\,\hat{\boldsymbol{x}} + (\dot{y}+\Omega x)\,\hat{\boldsymbol{y}} + \dot{z}\hat{\boldsymbol{z}}\right) \\
&= (\ddot{x}-\Omega\dot{y})\,\hat{\boldsymbol{x}} + (\ddot{y}+\Omega\dot{x})\,\hat{\boldsymbol{y}} + \ddot{z}\hat{\boldsymbol{z}} \\
&\quad + \left(\left(\Omega\dot{x}-\Omega^2 y\right)\hat{\boldsymbol{y}} - \left(\Omega\dot{y}+\Omega^2 x\right)\hat{\boldsymbol{x}}\right) \\
&= \left(\ddot{x}-2\Omega\dot{y}-\Omega^2 x\right)\hat{\boldsymbol{x}} + \left(\ddot{y}+2\Omega\dot{x}-\Omega^2 y\right)\hat{\boldsymbol{y}} + \ddot{z}\hat{\boldsymbol{z}}
\end{aligned}
$$

$$
\boxed{\ddot{\boldsymbol{r}} = \left(\ddot{x}-2\Omega\dot{y}-\Omega^2 x\right)\hat{\boldsymbol{x}} + \left(\ddot{y}+2\Omega\dot{x}-\Omega^2 y\right)\hat{\boldsymbol{y}} + \ddot{z}\hat{\boldsymbol{z}}}
$$

Equations of Motion

We can now generate the equations of motion

$$
\begin{aligned}
\sum_i \boldsymbol{F_i} &= m\ddot{\boldsymbol{r}} \\
\boldsymbol{F_1} + \boldsymbol{F_1} &= m\ddot{\boldsymbol{r}} \\
-\frac{\mu_1 m}{r_1^3}\boldsymbol{r_1} - \frac{\mu_2 m}{r_2^3}\boldsymbol{r_2} &= m\ddot{\boldsymbol{r}} \\
-\frac{\mu_1}{r_1^3}\boldsymbol{r_1} - \frac{\mu_2}{r_2^3}\boldsymbol{r_2} &= \left(\ddot{x}-2\Omega\dot{y}-\Omega^2 x\right)\hat{\boldsymbol{x}} + \left(\ddot{y}+2\Omega\dot{x}-\Omega^2 y\right)\hat{\boldsymbol{y}} + \ddot{z}\hat{\boldsymbol{z}}
\end{aligned}
$$

We now write this as three equations, one each in x, y, and z

$$
\begin{aligned}
-\frac{\mu_1}{r_1^3}(x+x_1) - \frac{\mu_2}{r_2^3}(x-x_2) &= \ddot{x}-2\Omega\dot{y}-\Omega^2 x \\
-\frac{\mu_1}{r_1^3}y - \frac{\mu_2}{r_2^3}y &= \ddot{y}+2\Omega\dot{x}-\Omega^2 y \\
-\frac{\mu_1}{r_1^3}z - \frac{\mu_2}{r_2^3}z &= \ddot{z}
\end{aligned}
$$

Isolating the second derivatives,

$$
\begin{aligned}
\ddot{x} &= -\frac{\mu_1}{r_1^3}(x+x_1) - \frac{\mu_2}{r_2^3}(x-x_2) + 2\Omega\dot{y} + \Omega^2 x \\
\ddot{y} &= -\frac{\mu_1}{r_1^3}y - \frac{\mu_2}{r_2^3}y - 2\Omega\dot{x} + \Omega^2 y \\
\ddot{z} &= -\frac{\mu_1}{r_1^3}z - \frac{\mu_2}{r_2^3}z
\end{aligned}
$$

We can now substitute $\Omega = \sqrt{\frac{\mu}{d^3}} = \sqrt{\frac{\mu_1+\mu_2}{d^3}}$

$$
\boxed{
\begin{aligned}
\ddot{x} &= -\frac{\mu_1}{r_1^3}(x+x_1) - \frac{\mu_2}{r_2^3}(x-x_2) + 2\sqrt{\frac{\mu}{d^3}}\dot{y} + \frac{\mu}{d^3}x \\
\ddot{y} &= -\frac{\mu_1}{r_1^3}y - \frac{\mu_2}{r_2^3}y - 2\sqrt{\frac{\mu}{d^3}}\dot{x} + \frac{\mu}{d^3}y \\
\ddot{z} &= -\frac{\mu_1}{r_1^3}z - \frac{\mu_2}{r_2^3}z
\end{aligned}
}
$$

Nondimensional Equations of Motion

We will now begin to non-dimensionalize the EOMs.

First, we will change the time unit. We will pick $t^\star = t\sqrt{\mu/d^3}$ (recall that $\sqrt{\mu/d^3} = \Omega$, which has dimensions of time^{-1}). Everywhere that a derivative is present, it is implied to be with respect to the time unit of 1 second. We must therefore switch from implied $\frac{\mathrm{d}}{\mathrm{d}t}$ to implied $\frac{\mathrm{d}}{\mathrm{d}t^\star}$.

$$\begin{aligned}\frac{\mathrm{d}(\ )}{\mathrm{d}t} &= \frac{\mathrm{d}(\ )}{\mathrm{d}t^\star}\frac{\mathrm{d}t^\star}{\mathrm{d}t} \\ &= \frac{\mathrm{d}(\ )}{\mathrm{d}t^\star}\sqrt{\frac{\mu}{d^3}}\end{aligned}$$

From this, it can be seen that to make a derivative with time implied to be with the nondimensional time, $\sqrt{d^3/\mu}$ must be multiplied for each derivative taken. The EOMs can now be rewritten this way, with dots implied to be relative to the nondimensional time unit

$$\begin{aligned}\left(\frac{\mathrm{d}^2x}{\mathrm{d}t^{\star 2}}\frac{\mu}{d^3}\right) &= -\frac{\mu_1}{r_1^3}(x+x_1) - \frac{\mu_2}{r_2^3}(x-x_2) + 2\sqrt{\frac{\mu}{d^3}}\left(\frac{\mathrm{d}y}{\mathrm{d}t^\star}\sqrt{\frac{\mu}{d^3}}\right) + \frac{\mu}{d^3}x \\ \left(\frac{\mathrm{d}^2y}{\mathrm{d}t^{\star 2}}\frac{\mu}{d^3}\right) &= -\frac{\mu_1}{r_1^3}y - \frac{\mu_2}{r_2^3}y - 2\sqrt{\frac{\mu}{d^3}}\left(\frac{\mathrm{d}x}{\mathrm{d}t^\star}\sqrt{\frac{\mu}{d^3}}\right) + \frac{\mu}{d^3}y \\ \left(\ddot{z}\frac{\mu}{d^3}\right) &= -\frac{\mu_1}{r_1^3}z - \frac{\mu_2}{r_2^3}z\end{aligned}$$

Some algebraic simplifications can now be made

$$\begin{aligned}\frac{\mathrm{d}^2x}{\mathrm{d}t^{\star 2}} &= -\frac{m_1d^3}{mr_1^3}(x+x_1) - \frac{m_2d^3}{mr_2^3}(x-x_2) + 2\frac{\mathrm{d}y}{\mathrm{d}t^\star} + x \\ \frac{\mathrm{d}^2y}{\mathrm{d}t^{\star 2}} &= -\frac{m_1d^3}{mr_1^3}y - \frac{m_2d^3}{mr_2^3}y - 2\frac{\mathrm{d}x}{\mathrm{d}t^\star} + y \\ \frac{\mathrm{d}^2z}{\mathrm{d}t^{\star 2}} &= -\frac{m_1d^3}{mr_1^3}z - \frac{m_2d^3}{mr_2^3}z\end{aligned}$$

Next, we can define nondimensional distances and masses. The relationship between a nondimensional distance $L^\star$ and its dimensional counterpart is $L^\star = L/d$. Furthermore, we define nondimensional masses a similar relatonship $M^\star = M/m$. With this defined, we can make some substitutions in the EOMs.

$$\begin{aligned}\frac{\mathrm{d}^2x}{\mathrm{d}t^{\star 2}} &= -\frac{m_1^\star}{r_1^{\star 3}}(x+x_1) - \frac{m_2^\star}{r_2^{\star 3}}(x-x_2) + 2\frac{\mathrm{d}y}{\mathrm{d}t^\star} + x \\ \frac{\mathrm{d}^2y}{\mathrm{d}t^{\star 2}} &= -\frac{m_1^\star}{r_1^{\star 3}}y - \frac{m_2^\star}{r_2^{\star 3}}y - 2\frac{\mathrm{d}x}{\mathrm{d}t^\star} + y \\ \frac{\mathrm{d}^2z}{\mathrm{d}t^{\star 2}} &= -\frac{m_1^\star}{r_1^{\star 3}}z - \frac{m_2^\star}{r_2^{\star 3}}z\end{aligned}$$

By dividing both sides by d, the remaining distances can be made nondimensional.

$$\begin{aligned}\frac{\mathrm{d}^2x^\star}{\mathrm{d}t^{\star 2}} &= -\frac{m_1^\star}{r_1^{\star 3}}(x^\star+x_1^\star) - \frac{m_2^\star}{r_2^{\star 3}}(x^\star-x_2^\star) + 2\frac{\mathrm{d}y^\star}{\mathrm{d}t^\star} + x^\star \\ \frac{\mathrm{d}^2y^\star}{\mathrm{d}t^{\star 2}} &= -\frac{m_1^\star}{r_1^{\star 3}}y^\star - \frac{m_2^\star}{r_2^{\star 3}}y^\star - 2\frac{\mathrm{d}x^\star}{\mathrm{d}t^\star} + y^\star \\ \frac{\mathrm{d}^2z^\star}{\mathrm{d}t^{\star 2}} &= -\frac{m_1^\star}{r_1^{\star 3}}z^\star - \frac{m_2^\star}{r_2^{\star 3}}z^\star\end{aligned}$$

$m_2 = m - m_1$, $m_2^\star = 1 - m_1^\star$. Similarly, $x_2^\star = 1 - x_1^\star$. While $r_1^\star$ and $r_2^\star$ can be written in terms of $x_1^\star$, $x^\star$, $y^\star$, and $z^\star$, I will not do this as it makes the equations much less compact and offers no benifit in computational implimentation.

$$
\begin{aligned}
\frac{\mathrm{d}^2 x^\star}{\mathrm{d}t^{\star 2}} &= -\frac{m_1^\star}{r_1^{\star 3}}(x^\star + x_1^\star) - \frac{1 - m_1^\star}{r_2^{\star 3}}(x^\star - 1 + x_1^\star) + 2\frac{\mathrm{d}y^\star}{\mathrm{d}t^\star} + x^\star \\
\frac{\mathrm{d}^2 y^\star}{\mathrm{d}t^{\star 2}} &= -\frac{m_1^\star}{r_1^{\star 3}}y^\star - \frac{1 - m_1^\star}{r_2^{\star 3}}y^\star - 2\frac{\mathrm{d}x^\star}{\mathrm{d}t^\star} + y^\star \\
\frac{\mathrm{d}^2 z^\star}{\mathrm{d}t^{\star 2}} &= -\frac{m_1^\star}{r_1^{\star 3}}z^\star - \frac{1 - m_1^\star}{r_2^{\star 3}}z^\star
\end{aligned}
$$

I will write this with the nondimensionalization implicit instead of explicit. In this space, all parameters represent their nondimensional counterpart.

$$
\boxed{
\begin{aligned}
\ddot{x} &= -\frac{m_1}{r_1^3}(x + x_1) - \frac{1 - m_1}{r_2^3}(x - 1 + x_1) + 2\dot{y} + x \\
\ddot{y} &= -\frac{m_1}{r_1^3}y - \frac{1 - m_1}{r_2^3}y - 2\dot{x} + y \\
\ddot{z} &= -\frac{m_1}{r_1^3}z - \frac{1 - m_1}{r_2^3}z
\end{aligned}
}
$$

The solutions to this differential equation solve the non-dimensional CR3BP, in which the bodies' masses add to 1 and the distance between them is 1. To translate it into real solutions, the time scale, distance scale, and mass scale must be applied.