The circular restricted three body problem (CR3BP) is a special case of the three body problem. In the CR3BP (much like in Keplerian 2-body dynamics), we neglect the mass of satellite S, while treating the larger celestial body m_1 and smaller celestial body m_2 as point masses. Crucially, these bodies must orbit one another in circular orbits. In other words, they both orbit about their inertially fixed barycenter c *at constant velocity and distance.*

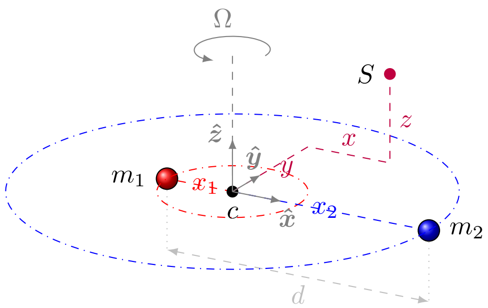


Figure 1: Geometry

The purple satellite is located by $\boldsymbol{r} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}$, and the spheres are the celestial bodies. The xyz frame is not inertially stationary. Instead, they rotate with the larger bodies at a rate Ω. The origin c is the center of mass of m_1 and m_2 (and therefore inertially fixed). The xy plane defines the plane in which m_1 and m_2 orbit, so $\hat{\boldsymbol{z}}$ is defined by the direction of their angular momenta- the constant rotation at a rate Ω is a consequence of this. Note that $\boldsymbol{r}$ is not constrained to the xy plane, but m_1 and m_2 definitionally are.

Useful Relations

We can also find a relationship between x_1, x_2, d, m_1, and m_2. Because c is the barycenter,

$$x_1 m_1 = x_2 m_2$$

or

$$\frac{x_1}{m_2} = \frac{x_2}{m_1}$$

We can use this to find that

$$\begin{aligned} \frac{x_1}{x_1 + x_2} &= \frac{x_1/m_2}{x_1/m_2 + x_2/m_2} \\ &= \frac{x_2/m_1}{x_2/m_1 + x_2/m_2} \\ &= \frac{1/m_1}{1/m_1 + 1/m_2} \\ &= \frac{m_2}{m_2 + m_1} \end{aligned}$$

Defining $d = x_1 + x_2$ as the distance between the two celestial bodies, and $m = m_1 + m_2$ as their total mass, we get that

$$\boxed{\frac{x_1}{d} = \frac{m_2}{m}}$$

and similarly

$$\boxed{\frac{x_2}{d} = \frac{m_1}{m}}$$

Which can be rewritten as

$$\boxed{\frac{x_1}{m_2} = \frac{x_2}{m_1} = \frac{d}{m}}$$

Lastly, we will define two additional vectors $\boldsymbol{r_1}$ and $\boldsymbol{r_2}$ which point from the first and second body respectively to the satellite.

$$\boldsymbol{r_1} = (x + x_1)\,\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}$$

and

$$\boldsymbol{r_2} = (x - x_2)\,\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}$$

Now we will solve for Ω.

$$\begin{aligned}
\boldsymbol{F}_{\text{on2}} &= -\frac{Gm_1m_2}{d^2}\hat{\boldsymbol{x}} \\
\cancel{m_2}\boldsymbol{a}_2^f &= -\frac{Gm_1\cancel{m_2}}{d^2}\hat{\boldsymbol{x}} \\
\frac{\mathrm{d}^f}{\mathrm{d}t}\left(\frac{\mathrm{d}^f}{\mathrm{d}t}x_2\hat{\boldsymbol{x}}\right) &= -\frac{Gm_1}{d^2}\hat{\boldsymbol{x}} \\
\frac{\mathrm{d}^f}{\mathrm{d}t}\left(\cancelto{0}{\frac{\mathrm{d}^c x_2\hat{\boldsymbol{x}}}{\mathrm{d}t}} + \Omega\hat{\boldsymbol{z}}\times x_2\hat{\boldsymbol{x}}\right) &= -\frac{Gm_1}{d^2}\hat{\boldsymbol{x}} \\
\frac{\mathrm{d}^f}{\mathrm{d}t}\Omega x_2\hat{\boldsymbol{y}} &= -\frac{Gm_1}{d^2}\hat{\boldsymbol{x}} \\
\cancelto{0}{\frac{\mathrm{d}^c}{\mathrm{d}t}\Omega x_2\hat{\boldsymbol{y}}} + \Omega\hat{\boldsymbol{z}}\times\Omega x_2\hat{\boldsymbol{y}} &= -\frac{Gm_1}{d^2}\hat{\boldsymbol{x}} \\
-\Omega^2 x_2\hat{\boldsymbol{x}} &= -\frac{Gm_1}{d^2}\hat{\boldsymbol{x}} \\
\Omega^2 x_2 &= \frac{Gm_1}{d^2} \\
\Omega &= \sqrt{\frac{G}{d^2}\frac{m_1}{x_2}} \\
\Omega &= \sqrt{\frac{Gm}{d^3}}
\end{aligned}$$

$$\boxed{\Omega = \sqrt{\frac{Gm}{d^3}}}$$

Kinematics

The transport theorem states that the inertial (fixed f frame) derivative of a vector u (expressed in the rotating c frame) is

$$\frac{\mathrm{d}^f\boldsymbol{u}}{\mathrm{d}t} = \frac{\mathrm{d}^c\boldsymbol{u}}{\mathrm{d}t} + \boldsymbol{\omega}^{cf}\times\boldsymbol{u}$$

Where $\frac{\mathrm{d}^f}{\mathrm{d}t}$ denotes the derivative in the coordinates of the fixed frame f, and $\frac{\mathrm{d}^c}{\mathrm{d}t}$ denotes derivative in the coordinates of the rotating frame c, and $\boldsymbol{\omega}^{cf}$ denotes the angular velocity of c in f. For this case, $\boldsymbol{\omega}^{cf} = \Omega\hat{\boldsymbol{z}}$. We can find Ω

For the satellite's position in the CR3BP frame $\boldsymbol{r} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}$, we will find the inertial acceleration to generate equations of motion.

$$\begin{aligned}
\dot{\boldsymbol{r}} &= \frac{\mathrm{d}^c\boldsymbol{r}}{\mathrm{d}t} + \boldsymbol{\omega}^{cf}\times\boldsymbol{r} \\
&= \frac{\mathrm{d}^c}{\mathrm{d}t}(x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}) + (\Omega\hat{\boldsymbol{z}}\times(x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}})) \\
&= (\dot{x}\hat{\boldsymbol{x}} + \dot{y}\hat{\boldsymbol{y}} + \dot{z}\hat{\boldsymbol{z}}) + (\Omega x\hat{\boldsymbol{y}} - \Omega y\hat{\boldsymbol{x}}) \\
&= (\dot{x} - \Omega y)\,\hat{\boldsymbol{x}} + (\dot{y} + \Omega x)\,\hat{\boldsymbol{y}} + \dot{z}\hat{\boldsymbol{z}}
\end{aligned}$$

$$
\begin{aligned}
\ddot{\boldsymbol{r}} &= \dot{\boldsymbol{r}} \\
&= \frac{\mathrm{d}^c}{\mathrm{d}t}\left((\dot{x}-\Omega y)\,\hat{\boldsymbol{x}} + (\dot{y}+\Omega x)\,\hat{\boldsymbol{y}} + \dot{z}\hat{\boldsymbol{z}}\right) \\
&\quad + \Omega\hat{\boldsymbol{z}} \times \left((\dot{x}-\Omega y)\,\hat{\boldsymbol{x}} + (\dot{y}+\Omega x)\,\hat{\boldsymbol{y}} + \dot{z}\hat{\boldsymbol{z}}\right) \\
&= (\ddot{x}-\Omega\dot{y})\,\hat{\boldsymbol{x}} + (\ddot{y}+\Omega\dot{x})\,\hat{\boldsymbol{y}} + \ddot{z}\hat{\boldsymbol{z}} \\
&\quad + \left(\left(\Omega\dot{x}-\Omega^2 y\right)\hat{\boldsymbol{y}} - \left(\Omega\dot{y}+\Omega^2 x\right)\hat{\boldsymbol{x}}\right) \\
&= \left(\ddot{x}-2\Omega\dot{y}-\Omega^2 x\right)\hat{\boldsymbol{x}} + \left(\ddot{y}+2\Omega\dot{x}-\Omega^2 y\right)\hat{\boldsymbol{y}} + \ddot{z}\hat{\boldsymbol{z}}
\end{aligned}
$$

$$
\boxed{\ddot{\boldsymbol{r}} = \left(\ddot{x}-2\Omega\dot{y}-\Omega^2 x\right)\hat{\boldsymbol{x}} + \left(\ddot{y}+2\Omega\dot{x}-\Omega^2 y\right)\hat{\boldsymbol{y}} + \ddot{z}\hat{\boldsymbol{z}}}
$$

Equations of Motion

We can now generate the equations of motion

$$
\begin{aligned}
\sum_i \boldsymbol{F_i} &= m\ddot{\boldsymbol{r}} \\
\boldsymbol{F_1} + \boldsymbol{F_1} &= m\ddot{\boldsymbol{r}} \\
-\frac{Gm_1 m}{r_1^3}\boldsymbol{r_1} - \frac{Gm_2 m}{r_2^3}\boldsymbol{r_2} &= m\ddot{\boldsymbol{r}} \\
-\frac{Gm_1}{r_1^3}\boldsymbol{r_1} - \frac{Gm_2}{r_2^3}\boldsymbol{r_2} &= \left(\ddot{x}-2\Omega\dot{y}-\Omega^2 x\right)\hat{\boldsymbol{x}} + \left(\ddot{y}+2\Omega\dot{x}-\Omega^2 y\right)\hat{\boldsymbol{y}} + \ddot{z}\hat{\boldsymbol{z}}
\end{aligned}
$$

We now write this as three equations, one each in x, y, and z

$$
\begin{aligned}
-\frac{Gm_1}{r_1^3}(x+x_1) - \frac{Gm_2}{r_2^3}(x-x_2) &= \ddot{x} - 2\Omega\dot{y} - \Omega^2 x \\
-\frac{Gm_1}{r_1^3}y - \frac{Gm_2}{r_2^3}y &= \ddot{y} + 2\Omega\dot{x} - \Omega^2 y \\
-\frac{Gm_1}{r_1^3}z - \frac{Gm_2}{r_2^3}z &= \ddot{z}
\end{aligned}
$$

Isolating the second derivatives,

$$
\begin{aligned}
\ddot{x} &= -\frac{Gm_1}{r_1^3}(x+x_1) - \frac{Gm_2}{r_2^3}(x-x_2) + 2\Omega\dot{y} + \Omega^2 x \\
\ddot{y} &= -\frac{Gm_1}{r_1^3}y - \frac{Gm_2}{r_2^3}y - 2\Omega\dot{x} + \Omega^2 y \\
\ddot{z} &= -\frac{Gm_1}{r_1^3}z - \frac{Gm_2}{r_2^3}z
\end{aligned}
$$

We can now substitute $\Omega = \sqrt{\frac{Gm}{d^3}} = \sqrt{\frac{Gm_1+Gm_2}{d^3}}$

$$
\boxed{
\begin{aligned}
\ddot{x} &= -\frac{Gm_1}{r_1^3}(x+x_1) - \frac{Gm_2}{r_2^3}(x-x_2) + 2\sqrt{\frac{Gm}{d^3}}\dot{y} + \frac{Gm}{d^3}x \\
\ddot{y} &= -\frac{Gm_1}{r_1^3}y - \frac{Gm_2}{r_2^3}y - 2\sqrt{\frac{Gm}{d^3}}\dot{x} + \frac{Gm}{d^3}y \\
\ddot{z} &= -\frac{Gm_1}{r_1^3}z - \frac{Gm_2}{r_2^3}z
\end{aligned}
}
$$

Nondimensional Equations of Motion

We will now begin to non-dimensionalize the EOMs.

First, we will change the time unit. We will pick $t^\star = t\sqrt{Gm/d^3}$ (recall that $\sqrt{Gm/d^3} = \Omega$, which has dimensions of time^{-1}). Everywhere that a derivative is present, it is implied to be with respect to the time unit of 1 second. We must therefore switch from implied $\frac{\mathrm{d}}{\mathrm{d}t}$ to implied $\frac{\mathrm{d}}{\mathrm{d}t^\star}$.

$$\begin{aligned}\frac{\mathrm{d}(\;)}{\mathrm{d}t} &= \frac{\mathrm{d}(\;)}{\mathrm{d}t^\star}\frac{\mathrm{d}t^\star}{\mathrm{d}t}\\ &= \frac{\mathrm{d}(\;)}{\mathrm{d}t^\star}\sqrt{\frac{Gm}{d^3}}\end{aligned}$$

From this, it can be seen that to make a derivative with time implied to be with the nondimensional time, $\sqrt{d^3/Gm}$ must be multiplied for each derivative taken. The EOMs can now be rewritten this way, with dots implied to be relative to the nondimensional time unit

$$\begin{aligned}\left(\frac{\mathrm{d}^2x}{\mathrm{d}t^{\star 2}}\frac{Gm}{d^3}\right) &= -\frac{Gm_1}{r_1^3}(x+x_1) - \frac{Gm_2}{r_2^3}(x-x_2) + 2\sqrt{\frac{Gm}{d^3}}\left(\frac{\mathrm{d}y}{\mathrm{d}t^\star}\sqrt{\frac{Gm}{d^3}}\right) + \frac{Gm}{d^3}x\\ \left(\frac{\mathrm{d}^2y}{\mathrm{d}t^{\star 2}}\frac{Gm}{d^3}\right) &= -\frac{Gm_1}{r_1^3}y - \frac{Gm_2}{r_2^3}y - 2\sqrt{\frac{Gm}{d^3}}\left(\frac{\mathrm{d}x}{\mathrm{d}t^\star}\sqrt{\frac{Gm}{d^3}}\right) + \frac{Gm}{d^3}y\\ \left(\ddot{z}\frac{Gm}{d^3}\right) &= -\frac{Gm_1}{r_1^3}z - \frac{Gm_2}{r_2^3}z\end{aligned}$$

Some algebraic simplifications can now be made

$$\begin{aligned}\frac{\mathrm{d}^2x}{\mathrm{d}t^{\star 2}} &= -\frac{m_1d^3}{mr_1^3}(x+x_1) - \frac{m_2d^3}{mr_2^3}(x-x_2) + 2\frac{\mathrm{d}y}{\mathrm{d}t^\star} + x\\ \frac{\mathrm{d}^2y}{\mathrm{d}t^{\star 2}} &= -\frac{m_1d^3}{mr_1^3}y - \frac{m_2d^3}{mr_2^3}y - 2\frac{\mathrm{d}x}{\mathrm{d}t^\star} + y\\ \frac{\mathrm{d}^2z}{\mathrm{d}t^{\star 2}} &= -\frac{m_1d^3}{mr_1^3}z - \frac{m_2d^3}{mr_2^3}z\end{aligned}$$

Next, we can define nondimensional distances and masses. The relationship between a nondimensional distance $L^\star$ and its dimensional counterpart is $L^\star = L/d$. Furthermore, we define nondimensional masses a similar relatonship $M^\star = M/m$. With this defined, we can make some substitutions in the EOMs.

$$\begin{aligned}\frac{\mathrm{d}^2x}{\mathrm{d}t^{\star 2}} &= -\frac{m_1^\star}{r_1^{\star 3}}(x+x_1) - \frac{m_2^\star}{r_2^{\star 3}}(x-x_2) + 2\frac{\mathrm{d}y}{\mathrm{d}t^\star} + x\\ \frac{\mathrm{d}^2y}{\mathrm{d}t^{\star 2}} &= -\frac{m_1^\star}{r_1^{\star 3}}y - \frac{m_2^\star}{r_2^{\star 3}}y - 2\frac{\mathrm{d}x}{\mathrm{d}t^\star} + y\\ \frac{\mathrm{d}^2z}{\mathrm{d}t^{\star 2}} &= -\frac{m_1^\star}{r_1^{\star 3}}z - \frac{m_2^\star}{r_2^{\star 3}}z\end{aligned}$$

By dividing both sides by d, the remaining distances can be made nondimensional.

$$\begin{aligned}\frac{\mathrm{d}^2x^\star}{\mathrm{d}t^{\star 2}} &= -\frac{m_1^\star}{r_1^{\star 3}}(x^\star+x_1^\star) - \frac{m_2^\star}{r_2^{\star 3}}(x^\star-x_2^\star) + 2\frac{\mathrm{d}y^\star}{\mathrm{d}t^\star} + x^\star\\ \frac{\mathrm{d}^2y^\star}{\mathrm{d}t^{\star 2}} &= -\frac{m_1^\star}{r_1^{\star 3}}y^\star - \frac{m_2^\star}{r_2^{\star 3}}y^\star - 2\frac{\mathrm{d}x^\star}{\mathrm{d}t^\star} + y^\star\\ \frac{\mathrm{d}^2z^\star}{\mathrm{d}t^{\star 2}} &= -\frac{m_1^\star}{r_1^{\star 3}}z^\star - \frac{m_2^\star}{r_2^{\star 3}}z^\star\end{aligned}$$

The final set of substitutions will now be made: $m_2 = m - m_1 \to m_2^\star = 1 - m_1^\star$. Recall from the useful relations that $\frac{x_1}{d} = \frac{m_2}{m}$, Therefore $x_1^\star = 1 - m_1^\star$ (and $x_2^\star = m_1^\star$). While $r_1^\star$ and $r_2^\star$ can be written in terms of $x_1^\star$, $x^\star$, $y^\star$, and $z^\star$, I will not do this as it makes the equations much less compact.

$$\begin{aligned}
\frac{\mathrm{d}^2 x^\star}{\mathrm{d}t^{\star 2}} &= -\frac{m_1^\star}{r_1^{\star 3}}(x^\star + 1 - m_1^\star) - \frac{1 - m_1^\star}{r_2^{\star 3}}(x^\star - m_1^\star) + 2\frac{\mathrm{d}y^\star}{\mathrm{d}t^\star} + x^\star \\
\frac{\mathrm{d}^2 y^\star}{\mathrm{d}t^{\star 2}} &= -\frac{m_1^\star}{r_1^{\star 3}} y^\star - \frac{1 - m_1^\star}{r_2^{\star 3}} y^\star - 2\frac{\mathrm{d}x^\star}{\mathrm{d}t^\star} + y^\star \\
\frac{\mathrm{d}^2 z^\star}{\mathrm{d}t^{\star 2}} &= -\frac{m_1^\star}{r_1^{\star 3}} z^\star - \frac{1 - m_1^\star}{r_2^{\star 3}} z^\star
\end{aligned}$$

I will write this with the nondimensionalization implicit instead of explicit. In this space, all parameters represent their nondimensional counterpart.

$$\begin{aligned}
\ddot{x} &= -\frac{m_1}{r_1^3}(x + 1 - m_1) - \frac{1 - m_1}{r_2^3}(x - m_1) + 2\dot{y} + x \\
\ddot{y} &= -\frac{m_1}{r_1^3} y - \frac{1 - m_1}{r_2^3} y - 2\dot{x} + y \\
\ddot{z} &= -\frac{m_1}{r_1^3} z - \frac{1 - m_1}{r_2^3} z
\end{aligned}$$

If instead m_2 is used as the characterizing parameter (rather than m_1), the EOMs change slightly. Conventionally, this parameter is written μ (not to be confused with the gravitational parameter $\mu = G(m_1 + m_2)$)

$$\boxed{\begin{aligned}
\ddot{x} &= -\frac{1 - \mu}{r_1^3}(x + \mu) - \frac{\mu}{r_2^3}(x - 1 + \mu) + 2\dot{y} + x \\
\ddot{y} &= -\frac{1 - \mu}{r_1^3} y - \frac{\mu}{r_2^3} y - 2\dot{x} + y \\
\ddot{z} &= -\frac{1 - \mu}{r_1^3} z - \frac{\mu}{r_2^3} z
\end{aligned}}$$

Noting that

$$\boxed{\begin{aligned}
r_1 &= \sqrt{(x + \mu)^2 + y^2 + z^2} \\
r_2 &= \sqrt{(x - 1 + \mu)^2 + y^2 + z^2}
\end{aligned}}$$

The solutions to this differential equation solve the non-dimensional CR3BP, in which the bodies' masses add to 1 and the distance between them is 1. To translate it into real solutions, the time scale, distance scale, and mass scale must be applied. Therefore the only thing in this equation that depends upon the speciifc two-body system of interest is m_1, or the fraction of the primary mass to the total masses of both planetary.

Energy Analysis

To find the energy of the system, we must find a potential function U for the fictitious centrifugal acceleration $-\Omega^2 (x\hat{x} + y\hat{y}) = -\frac{Gm}{d^3}(x\hat{x} + y\hat{y})$ The potential function that satisfies this is

$$U = -\frac{1}{2}\frac{Gm}{d^3}\left(x^2 + y^2\right)$$

We can analyse the energy of a CR3BP system to determine possible positions. The total energy of the system is.

$$\begin{aligned}
E &= \frac{1}{2} v^2_{\text{in CR3BP frame}} + U_{\text{grav}} + U_{\text{centrifugal}} \\
&= \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - \left(\frac{Gm_1}{r_1} + \frac{Gm_2}{r_2}\right) - \frac{1}{2}\frac{Gm}{d^3}\left(x^2 + y^2\right)
\end{aligned}$$

This can be nondimensionalized to

$$\frac{1}{2}\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right)-\left(\frac{1-\mu}{r_1}+\frac{\mu}{r_2}\right)-\frac{1}{2}\left(x^2+y^2\right)$$

This is a conserved quantity. Different sources treat it differently, with some doubling or negating it. In the interest of aligning with JPL's Three Body Periodic Orbits Database, I will double and negate this energy; this value is called the Jacobi constant (or Jacobi integral)

$$J=2\left(\frac{1-\mu}{r_1}+\frac{\mu}{r_2}\right)+\left(x^2+y^2\right)-\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right)$$

The zero-velocity contours are given by

$$0=2\left(\frac{1-\mu}{r_1}+\frac{\mu}{r_2}\right)+\left(x^2+y^2\right)-J$$

Keeping in mind that r_1 and r_2 are functions of x, y, z, μ. These curves can give some intuition for regions that a spacecraft cannot enter given a quantity of energy.

In the following figures, "colder" colors (towards blue) correspond to lower Jacobi constants, while "warmer" colors (yellows, oranges, and towards red) correspond to higher values. A spacecraft with a given amount of energy is bounded outside of any region encompassed by a colder color. For example, a spacecraft with some amount of energy corresponding to green cannot enter any area with colors cooler than green.

Figure 2: Level Curves of the Jacobi Constant for $\mu = 0.01$

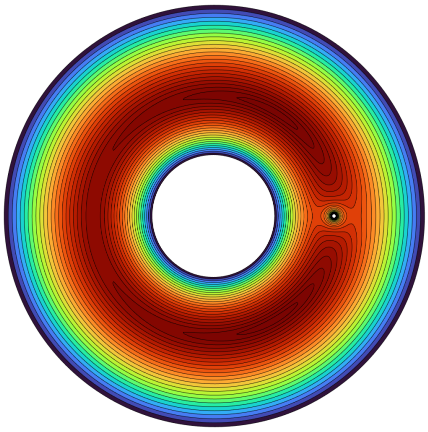


Figure 3: Level Curves of the Jacobi Constant for $\mu = 0.1$

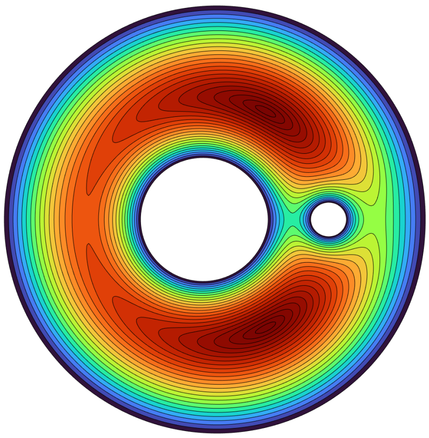


Figure 4: Level Curves of the Jacobi Constant for $\mu = 0.25$

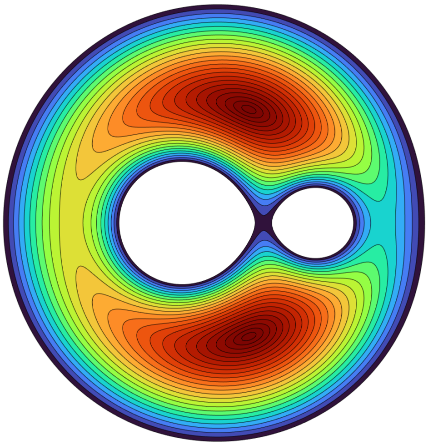
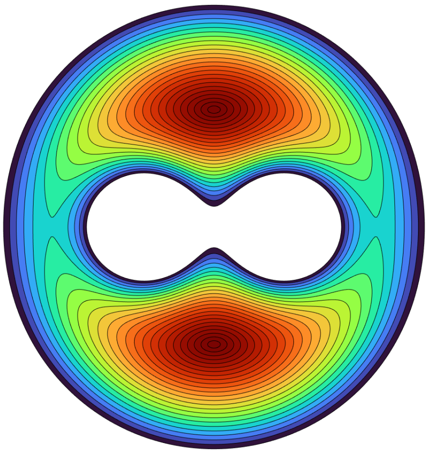


Figure 5: Level Curves of the Jacobi Constant for $\mu = 0.5$



For reference, the Jacobi value for the Earth-Moon system is $\mu \approx 0.012$, which is roughly that of the first diagram. Clearly, entering regions near the orbit of the second body is challenging, but exceeding those regions is quite feasible.

Zero-Velocity Solutions - Lagrange Points

There are some solutions that can exist with zero velocity (called Lagrange Points). To find these, we simply set the derivatives in the EOMs equal to zero.

$$\cancel{\ddot{x}} = -\frac{1-\mu}{r_1^3}(x+\mu) - \frac{\mu}{r_2^3}(x-1+\mu) + \cancel{2\dot{y}} + x$$
$$\cancel{\ddot{y}} = -\frac{1-\mu}{r_1^3}y - \frac{\mu}{r_2^3}y - \cancel{2\dot{x}} + y$$
$$\cancel{\ddot{z}} = -\frac{1-\mu}{r_1^3}z - \frac{\mu}{r_2^3}z$$

Looking at the z EOM, the right-hand-side must have the opposite sign to z. Therefore, for the velocity to remain zero, $z = 0 \forall t$. We will examine the case of $y = 0$ and $y \neq 0$ separately.

First, we look at $y = 0$, which eliminates the second equality above.

$$
\begin{aligned}
\cancel{\ddot{x}} &= -\frac{1-\mu}{r_1^3}(x+\mu) - \frac{\mu}{r_2^3}(x-1+\mu) + \cancel{2\dot{y}} + x \\
0 &= -\frac{1-\mu}{\sqrt{(x+\mu)^2}^3}(x+\mu) - \frac{\mu}{\sqrt{(x-1+\mu)^2}^3}(x-1+\mu) + x \\
0 &= -\frac{1-\mu}{|x+\mu|^3}(x+\mu) - \frac{\mu}{|x-1+\mu|^3}(x-1+\mu) + x
\end{aligned}
$$

This has three solutions, typically called L_1 (the solution between the two bodies), L_2 (the solution on the far side of the smaller body), and L_3 (the solution on the far side of the larger body).

Now, we examine $y \neq 0$

$$
\begin{aligned}
\cancel{\ddot{y}} &= -\frac{1-\mu}{r_1^3}y - \frac{\mu}{r_2^3}y - \cancel{2\dot{x}} + y \\
0 &= -\frac{1-\mu}{r_1^3}y - \frac{\mu}{r_2^3}y + y \\
0 &= -\frac{1-\mu}{r_1^3} - \frac{\mu}{r_2^3} + 1 \\
1 &= \frac{1-\mu}{r_1^3} + \frac{\mu}{r_2^3} \\
\frac{1-\mu}{r_1^3} &= 1 - \frac{\mu}{r_2^3}
\end{aligned}
$$

Bringing in the equation for x,

$$
\begin{aligned}
0 &= -\frac{1-\mu}{r_1^3}(x+\mu) - \frac{\mu}{r_2^3}(x-1+\mu) + x \\
0 &= -\left(1 - \frac{\mu}{r_2^3}\right)(x+\mu) - \frac{\mu}{r_2^3}(x-1+\mu) + x \\
0 &= -(x+\mu) + \frac{\mu}{r_2^3}(x+\mu) - \frac{\mu}{r_2^3}(x+\mu) + \frac{\mu}{r_2^3} + x \\
0 &= -x - \mu + \frac{\mu}{r_2^3} + x \\
0 &= -\mu + \frac{\mu}{r_2^3} \\
\frac{1}{r_2^3} &= 1 \\
r_2 &= 1
\end{aligned}
$$

Plugging this back into $\frac{1-\mu}{r_1^3} = 1 - \frac{\mu}{r_2^3}$,

$$
\begin{aligned}
\frac{1-\mu}{r_1^3} &= 1 - \frac{\mu}{r_2^3} \\
\frac{1-\mu}{r_1^3} &= 1 - \mu \\
\frac{1}{r_1^3} &= 1 \\
r_1 &= 1
\end{aligned}
$$

The conditions $r_1 = r_2 = 1$ locate the final two Lagrange points, forming equilateral triangles with the line connecting the two bodies.

$$\begin{aligned} r_1 &= r_2 \\ \sqrt{(x+\mu)^2 + y^2} &= \sqrt{(x-1+\mu)^2 + y^2} \\ (x+\mu)^2 + y^2 &= (x-1+\mu)^2 + y^2 \\ (x+\mu)^2 &= (x-1+\mu)^2 \\ x^2 + \mu^2 + 2x\mu &= x^2 - 2x + 2x\mu + 1 - 2\mu + \mu^2 \\ 0 &= -2x + 1 - 2\mu \\ 2x &= 1 - 2\mu \\ x &= \frac{1}{2} - \mu \end{aligned}$$

Plugging this once again into $r_1 = \sqrt{(x+\mu)^2 + y^2} = 1$,

$$\begin{aligned} \sqrt{(x+\mu)^2 + y^2} &= 1 \\ \left(\frac{1}{2} - \mu + \mu\right)^2 + y^2 &= 1 \\ \frac{1}{4} + y^2 &= 1 \\ y^2 &= \frac{3}{4} \\ y &= \pm\frac{\sqrt{3}}{2} \end{aligned}$$

The points $y = \pm\sqrt{3}/2$, $x = \frac{1}{2} - \mu$ locate L_4 and L_5.

The Lagrange points are shown below for the Earth-Moon system ($\mu = 0.01215058560962404$). L_4 and L_5 are known precisely, while L_1, L_2, L_3 are computed numerically from $y = 0$ and $-\frac{1-\mu}{|x+\mu|^3}(x+\mu) - \frac{\mu}{|x-1+\mu|^3}(x-1+\mu) + x = 0$.

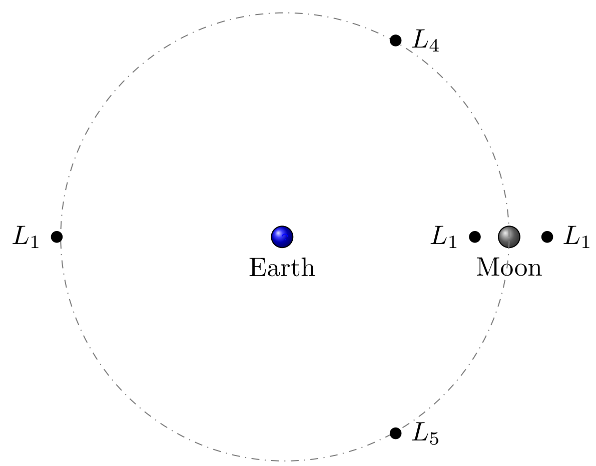


Figure 6: Earth-Moon Lagrange Points. Note that the barycenter lies within Earth's radius (in the diagram, and in dimensional units)