Sec. 2.3

22. Determine whether each of these functions is a bijection from R to R.

c) f(x)= (x +1)/(x +2)

This is a bijection but not from R to R.

34. If f and f ◦g are one-to-one, does it follow that g is one-to-one? Justify your answer.

We assume that g is not one-to-one.

Then we know that there are two different elements a_1, a_2 and $g(a_1) = g(a_2)$.

So, It's apparently that $f(g(a_1)) = f(g(a_2)) \leftrightarrow (f \circ g)(a1) = (f \circ g)(a2)$

Since $a_1 \neq a_2$, this is contradict with f ◦g is one to one.

Hence , g is one-to-one.

40. Let f be a function from the set A to the set B. Let S and T be subsets of A. Show that

a) f(S∪T)= f(S)∪f(T) .

This really has two parts.

First suppose that b is in f(S∪T). Thus b = f(a) for some a∈S∪T . Either a∈S, in which case b∈f(S), or a∈T , in which case b∈f(T). Thus in either case b∈f(S)∪f(T). This shows that f(S∪T)⊆f(S)∪f(T).

Conversely, suppose b∈f(S)∪f(T). Then either b∈f(S) or b∈f(T). This means either that b = f(a) for some a ∈ S or that b = f(a) for some a ∈ T . In either case, b = f(a) for some a∈S∪T , so b∈f(S∪T). This shows that f(S)∪f(T)⊆f(S∪T).

72. Suppose that f is a function from A to B, where A and B are finite sets with|A|=|B|. Show that f is one-to-one if and only if it is onto.

If f is one-to-one, then every element of A gets sent to a different element of B. If in addition to the range of A there were another element in B, then |B| would be at least one greater than |A|. This cannot happen, so we conclude that f is onto.

Conversely, suppose that f is onto, so that every element of B is the image of some element of A. In particular, there is an element of A for each element of B. If two or more elements of A were sent to the same element of B, then |A| would be at least one greater than the |B|. This cannot happen, so we conclude that f is one-to-one.

74. Prove or disprove each of these statements about the floor and ceiling functions.

c) $\lceil \lceil x/2 \rceil /2 \rceil = \lceil x/4 \rceil$ **for all real numbers x.**

Let x = 4k + a, k donate to a integer and a is a real number $0 \leq a < 4$

if a = 0, both sides equal k

if $0 < a \leq 2$, $\lceil x/2 \rceil = 2k+1$, both sides equal k+1

if 2 < a < 4 , $\lceil x/2 \rceil = 2k+2$, both sides equals k+1

d) $\lceil \sqrt{\lceil x \rceil} \rceil = \lfloor \sqrt{x} \rfloor$ **for all positive real numbers x.**

For x = 8.5, the left-hand side is 3, whereas the right-hand side is 2.

Sec. 2.4

16. Find the solution to each of these recurrence relations with the given initial conditions. Use an iterative approach such as that used in Example 10.

f) $a_n = 2na_{n-1}, a_0 = 3$

$a_n = 3 * 2^n n!$

g) $a_n = -a_{n-1} + n - 1, a_0 = 7$

$an = n - 1 - a_{n-1}$

$= (n-1) - (n-2) + a_{n-2}$

=$(n-1) - (n-2) + (n-3) - a_{n-3}$

...

=$(n-1) - (n-2) + \cdots + (-1)^{n-1}(n-n) + (-1)^n a_{n-n}$

= $\frac{2n-1+(-1)^{n-1}}{4} + (-1)^n * 7$

26. For each of these lists of integers, provide a simple formula or rule that generates the terms of an integer sequence that begins with the given list. Assuming that your formula or rule is correct, determine the next three terms of the sequence.

g) 1,0,0,1,1,1,0,0,0,0,1,1,1,1,1,...

This sequence alternates between 0s and 1s and having one more item in each group than in the previous group. Thus six 0's will follow next, so the next three terms are 0, 0, 0.

28. Let a_n be the nth term of the sequence 1,2,2,3,3,3,4,4,4, 4,5,5,5,5,5,6,6,6,6,6,6,..., constructed by including the integer k exactly k times. Show that $a_n = \lfloor \sqrt{2n} + 1/2 \rfloor$.

It's apparently that $a_1 = \lfloor \sqrt{2*1} + 1/2 \rfloor = 1$

Let's assume $a_k = \lfloor \sqrt{2k} + 1/2 \rfloor$

We need to prove $a_{k+1} = \lfloor \sqrt{2k+2} + 1/2 \rfloor$

- case 1: $k = \frac{(m+1)m}{2}$, which means $a_k + 1 = a_{k+1}$ and m is an integer equals a_k.
  We know $a_k = \lfloor \sqrt{m^2+m} + 1/2 \rfloor$ = m
  We need to prove $a_{k+1} = \lfloor \sqrt{m^2+m+2} + 1/2 \rfloor$
  $m^2 + m + 2 > m^2 + m + \frac{1}{4} = (m + \frac{1}{2})^2 > m^2 + m$

hence, $a_{k+1} = \lfloor\sqrt{m^2+m+2}+1/2\rfloor \geq \lfloor m+1/2+1/2\rfloor = m+1$

- case 2: $\frac{(m-1)m+2}{2} \leq k \leq \frac{(m+1)m-2}{2}$, which means $a_k = a_{k+1}$ and m is an integer equals a_k.

  We know $a_k = m = \lfloor\sqrt{m^2+m}+1/2\rfloor = \lfloor\sqrt{2k}+1/2\rfloor$

  $a_{k+1} = \lfloor\sqrt{2k+2}+1/2\rfloor \leq \lfloor\sqrt{m^2+m}+1/2\rfloor = m$

  $a_{k+1} = \lfloor\sqrt{2k+2}+1/2\rfloor \geq \lfloor\sqrt{m^2-m+4}+1/2\rfloor = m$

  hence, $a_{k+1} = m$

Now, we can conclude that $a_n = \lfloor\sqrt{2n}+1/2\rfloor$

42. Find a formula for $\sum_{k=0}^{m}\lfloor\sqrt[3]{k}\rfloor$, when m is a positive integer.

There is also a special notation for products. The product of $a_m, a_{m+1}, \ldots, a_n$ is represented by $\prod_{j=m}^{n} a_j$, read as the product from $j=m$ to $j=n$ of a_j.

$m = (n+1)^3 - 1 + p$, n is a positive integer

p is a integer, $0 \leq p < (n+2)^3 - 1$

$\sum_{k=0}^{m}\lfloor\sqrt[3]{k}\rfloor = \sum_{i=1}^{n} i(3i^2+3i+1) = \frac{n(3n+4)(n+1)^2}{4} + p*(n+1)$

Sec. 2.5

4. Determine whether each of these sets is countable or uncountable. For those that are countably infinite, exhibit a one-to-one correspondence between the set of positive integers and that set.

c) the real numbers with decimal representations consisting of all 1s

We can list these numbers in the order 1,1.1,11,1.11,11.1,111...

The correspondence is given by 1↔1, 2↔1.1, 3↔11, 4↔1.11, 5↔11.1, and so on.

d) the real numbers with decimal representations of all 1s or 9s

uncountable

28. Show that the set $Z^+ \times Z^+$ is countable.

Z^+ is countable. The elements of the set $Z^+ \times Z^+$ can be listed by listing all terms (i,j) with i+j=2, and then i+j=3 and so on.

36. Show that there is a one-to-one correspondence from the set of subsets of the positive integers to the set real numbers between 0 and 1.Use this result and Exercises 34 and 35 to conclude that $\aleph_0 < |P(Z^+)| = |R|$.

[Hint: Assume that there is such a one-to-one correspondence. Represent a subset of the set of positive integers as an infinite bit string with i-th bit 1 if i belongs to the subset and 0 otherwise. Suppose that you can list these infinite strings in a sequence indexed by the positive integers. Construct a new bit string with its i-th bit equal to the complement of the i-th bit of the i-th string in the list. Show that this new bit string cannot appear in the list.]

According to Exercise 34, $|R| = (0, 1)$

We can encode subsets of the set of positive integers as strings of, say, 5's and 6's, where the i-th symbol is a 5 if i is in the subset and a 6 otherwise.

If we interpret this string as a real number by putting a 0 and a decimal point in front, then we have constructed a one-to-one function from $P(Z^+)$ to $(0, 1)$.

Also, we can construct a one-to-one function from $P(Z^+)$ to $(0, 1)$ by sending the number whose binary expansion is $0.d_1d_2d_3...$ to the set $\{i|d_i = 1\}$.

Thus, $|P(Z^+)| = |R|$

Since, we already know from Cantor's diagonal argument that $\aleph 0 < |R|$. We can conclude that $\aleph_0 < |P(Z^+)| = |R|$.

38. Show that the set of functions from the positive integers to the set {0,1,2,3,4,5,6,7,8,9} is uncountable. [Hint: First set up a one-to-one correspondence between the set of real numbers between 0 and 1 and a subset of these functions. Do this by associating to the real number $0.d_1d_2...d_n...$the function f with $f(n) = d_n$.]

只要证明不可数 所以可以只证明(0,1)区间到函数的单射

If x is a real number whose decimal representation is $0.d_1d_2d_3...$(with ambiguity resolved by forbidding the decimal to end with an infinite string of 9's), then we associate to x the function whose rule is given by $f(n) = d_n$. Clearly this is a one-to-one function from the set of real numbers between 0 and 1 and a subset of the set of all functions from the set of positive integers to the set {0,1,2,3,4,5,6,7,8,9}.

So, $|f| \geq |(0, 1)|$, which means it is uncountable.