Sec. 8.2

2. Determine which of these are linear homogeneous recurrence relations with constant coefficients. Also, find the degree of those that are.

a) $a_n = 3a_{n-2}$

linear, homogeneous, constant coefficients; degree 2

b) $a_n = 3$

linear with constant coefficients but not homogeneous

c) $a_n = a_{n-1}^2$

not linear

d) $a_n = a_{n-1} + 2a_{n-3}$

linear, homogeneous, constant coefficients; degree 3

e) $a_n = a_{n-1}/n$

linear and homogeneous, but not with constant coefficients

f) $a_n = a_{n-1} + a_{n-2} + n + 3$

linear with constant coefficients, but not homogeneous

g) $a_n = 4a_{n-2} + 5a_{n-4} + 9a_{n-7}$

linear, homogeneous, with constant coefficients; degree 7

4. Solve these recurrence relations together with the initial conditions given.

$a_{n+2} = -4a_{n+1} + 5a_n$ **for** $n \geq 0$**,** $a_0 = 2, a_1 = 8$

$r^2 = -4r + 5$ -> $r = 1, r = -5$

$a_n = \alpha_1 + \alpha_2(-5)^n$

$2 = \alpha_1 + \alpha_2$

$8 = \alpha_1 + -5\alpha_2$

so, $\alpha_1 = -1, \alpha_2 = 3$

$a_n = -(-5)^n + 3$

20. Find the general form of the solutions of the recurrence relation $a_n = 8a_{n-2} - 16a_{n-4}$.

$r^4 = 8r^2 - 16$ -> $(r^2 - 4)^2 = 0$ -> $r = 2, r = -2$

$a_n = \alpha_1 2^n + \alpha_2 n 2^n + \alpha_3(-2)^n + \alpha_4 n(-2)^n$

30.

a) Find all solutions of the recurrence relation $a_n = -5a_{n-1} - 6a_{n-2} + 42 \cdot 4^n$**.**

$r^2 = -5r - 6$ -> $r = -2, r = -3$

$a_n = \alpha(-2)^n + \beta(-3)^n + c4^n$

$c4^n = -5c * 4^{n-1} - 6c4^{n-2} + 42 * 4^n$ -> $16c = -20c - 6c + 42 * 16$ -> $c = 16$

$a_n = \alpha(-2)^n + \beta(-3)^n + 4^{n+2}$

b) Find the solution of this recurrence relation with $a_1 = 56$ **and** $a_2 = 278$**.**

$56 = a_1 = -2\alpha - 3\beta + 64$

$278 = a_2 = 4\alpha + 9\beta + 256$

So, α = 1 and β = 2

$a_n = (-2)^n + 2(-3)^n + 4^{n+2}$

35. Find the solution of the recurrence relation $a_n = 4a_{n-1} - 3a_{n-2} + 2^n + n + 3$ with $a_0 = 1$ and $a_1 = 4$.

$r^2 = 4r - 3$ -> $r = 1, r = 3$

- $a_n = \alpha + \beta 3^n + c2^n + dn + e$

  $c2^n + dn + e = 4c2^{n-1} + 4d(n-1) + 4e - 3c2^{n-2} - 3d(n-2) - 3e + 2^n + n + 3$

  $4c2^{n-2} = 8c2^{n-2} - 3c2^{n-2} + 4 * 2^{n-2}$ -> c = -4

  $dn = 4dn - 3dn + n$ -> this formula has no root.

- So, we need to reconstruct the a_n

  we assume $a_n = \alpha + \beta 3^n + c2^n + mn^2 + dn + e$

  then the formula is
  $c2^n + mn^2 + dn + e = 4c2^{n-1} + 4m(n-1)^2 + 4d(n-1) + 4e - 3c2^{n-2} - 3m(n-2)^2 - 3d(n-2) - 3e + 2^n + n + 3$

  $4c2^{n-2} = 8c2^{n-2} - 3c2^{n-2} + 4 * 2^{n-2}$ -> c = -4

  $mn^2 = 4mn^2 - 3mn^2$ -> m can be arbitrary value.

  $dn = -8mn + 4dn + 12mn - 3dn + n$ -> m = -1/4

  $e = 4m - 4d + 4e - 12m + 6d - 3e + 3$ ->$5 + 2d = 0$ -> d = -5/2, e isn't needed in the formula.

Now, $a_n = \alpha + \beta 3^n - 2^{n+2} - \frac{n^2}{4} - \frac{5}{2}n$

Since $a_0 = 1$ and $a_1 = 4$, we get

$1 = \alpha + \beta - 4$

$4 = \alpha + 3\beta - 8 - \frac{1}{4} - \frac{5}{2}$

so, $\alpha = \frac{1}{8}, \beta = \frac{39}{8}$

the final answer is $a_n = \frac{39}{8}3^n - 2^{n+2} - \frac{n^2}{4} - \frac{5}{2}n + \frac{1}{8}$.

Sec. 8.4

6. Find a closed form for the generating function for the sequence {a_n} ,where

d) $a_n = 1/(n+1)!$ **for n = 0,1,2,....**

We know that e^x is $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

$\sum_{n=0}^{\infty} \frac{x^n}{(n+1)!} = \frac{1}{x}\sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} = \frac{1}{x}(\sum_{n=0}^{\infty} \frac{x^n}{(n)!} - 1)$

So, the answer is $\frac{(e^x - 1)}{x}$

f) $a_n = C_{10}^{n+1}$ **for n = 0,1,2,....**

$\sum_{n=0}^{\infty} C_{10}^{n+1} x^n = \sum_{n=1}^{\infty} C_{10}^{n} x^{n-1} = \frac{1}{x}\sum_{n=1}^{\infty} C_{10}^{n} x^n = \frac{1}{x}((1+x)^{10} - 1)$

10. Find the coefficient of x^9 in the power series of each of these functions.

c) $(x^3 + x^5 + x^6)(x^3 + x^4)(x + x^2 + x^3 + x^4 + \cdots)$

$x^7(1 + x^2 + x^3)(1 + x)(1 + x + x^2 + x^3 + \cdots)$

We can easily find the the coefficient of x^2 by brute force.

So, the answer is 3.

d) $(x + x^4 + x^7 + x^{10} + \cdots)(x^2 + x^4 + x^6 + x^8 + \cdots)$

$x^3(1+x^3+x^6+x^9\dots)(1+x^2+x^4+x^6)$

We can only find 1 and x^6 or x^6 and 1, so the answer is 2.

e) $(1+x+x^2)^3$

The answer is 0., since the highest power of x is x^6.

16. Use generating functions to find the number of ways to choose a dozen bagels from three varieties—egg, salty, and plain—if at least two bagels of each kind but no more than three salty bagels are chosen.

$x^6(1+x)(1+x+x^2+\dots)^2$

the generating function is $\frac{(1+x)}{(1-x)^2}$

Since, $\frac{1}{(1-x)^n}=\sum_{k=0}^{\infty}C_{n+k-1}^{k}x^k$

the answer is $C_{2+6-1}^{6}+C_{2+5-1}^{5}=7+6=13$

24.

a) What is the generating function for {a_k}, where a_k is the number of solutions of $x_1+x_2+x_3+x_4=k$ when x_1,x_2,x_3 and x_4 are integers with $x_1\geq 3, 1\leq x_2\leq 5, 0\leq x_3\leq 4,$ and $x_4\geq 1$?

$(x^3+x^4+x^5+x^6+\dots)(x+x^2+x^3+x^4+x^5)(1+x+x^2+x^3+x^4)(x+x^2+x^3+\dots)$

=$x^5(1+x^2+x^3+\dots)^2(1+x+x^2+x^3+x^4)^2$

=$\frac{x^5(1+x+x^2+x^3+x^4)^2}{(1-x)^2}$

30. If G(x) is the generating function for the sequence{a_k}, what is the generating function for each of these sequences?

a) $2a_0, 2a_1, 2a_2, 2a_3, \dots$

2G(x)

b) $0, a_0, a_1, a_2, a_3, \dots$ **(assuming that terms follow the pattern of all but the first term)**

xG(x)

c) $0, 0, 0, 0, a_2, a_3, \dots$ **(assuming that terms follow the pattern of all but the first four terms)**

$x^2(G(x)-a_0-a_1x)$

d) $a_2, a_3, a_4, \dots$

$\frac{(G(x)-a_0-a_1x)}{x^2}$

e) $a_1, 2a_2, 3a_3, 4a_4, \dots$ **[Hint: Calculus required here.]**

$G'(x)$

f) $a_0^2, 2a_0a_1, a_1^2+2a_0a_2, 2a_0a_3+2a_1a_2, 2a_0a_4+2a_1a_3+a_2^2, \dots$

$(G(x))^2$

34. Use generating functions to solve the recurrence relation $a_k=3a_{k-1}+4^{k-1}$ with the initial condition $a_0=1$.

$G(x)=\sum_{k=0}^{\infty}a_kx^k$ $xG(x)=\sum_{k=0}^{\infty}a_kx^{k+1}=\sum_{k=1}^{\infty}a_{k-1}x^k$

$(1-3x)G(x)=G(x)-3xG(x)=\sum_{k=0}^{\infty}a_kx^k-3\sum_{k=1}^{\infty}a_{k-1}x^k$

=$a_0+\sum_{k=1}^{\infty}(a_k-3a_{k-1})x^k$

=$1+x\sum_{k=1}^{\infty}4^{k-1}x^{k-1}$

=$1+\frac{x}{1-4x}$

So, $G(x) = \frac{1}{1-4x}$, $a_k = 4^k$.

43. Use generating functions to prove Vandermonde's identity:
$C(m+n,r) = \sum_{k=0}^{r} C(m,r-k)C(n,k)$**, whenever m, n, and r are nonnegative integers with r not exceeding either m or n.[Hint: Look at the coefficient of x^r in both sides of $(1+x)^{m+n} = (1+x)^m(1+x)^n$.]**

$\sum_{r=0}^{m+n} C(m+n,r)x^r = \sum_{r=0}^{m} C(m,r)x^r * \sum_{r=0}^{n} C(n,r)x^r$

$=\sum_{r=0}^{m+n} [\sum_{k=0}^{r} C(m,r-k)C(n,k)]x^r$.