Sec. 6.5

10. A croissant shop has plain croissants, cherry croissants, chocolate croissants, almond croissants, apple croissants, and broccoli croissants. How many ways are there to choose

a) a dozen croissants?

$C(6+12-1, 12) = C(17, 12) = 6188$

b) three dozen croissants?

$C(6+36-1, 36) = C(41, 36) = 749398$

c) two dozen croissants with at least two of each kind?

We firstly picked 2·6 = 12 croissants.

$C(6+12-1, 12) = C(17, 12) = 6188$

d) two dozen croissants with no more than two broccoli croissants?

- 0 broccoli croissants

  $C(24+5-1, 24) = C(28, 24) = 20,475$
- 1 broccoli croissants

  $C(23+5-1, 23) = C(27, 23) = 17,550$
- 2 broccoli croissants

  $C(22+5-1, 23) = C(26, 22) = 14,950$

$14950 + 20475 + 17550 = 52,975$

e) two dozen croissants with at least five chocolate croissants and at least three almond croissants?

$C(16+6-1, 16) = C(21, 16) = 20,349$

f) two dozen croissants with at least one plain croissant, at least two cherry croissants, at least three chocolate croissants, at least one almond croissant, at least two apple croissants, and no more than three broccoli croissants?

- when at least four broccoli croissants:

  $C(24-13+6-1, 24-13) = C(16, 11) = 4368$
- when there is no limitation on broccoli croissants:

  $C(24-9+6-1, 24-9) = C(20, 15) = 15504$

$15504 - 4368 = 11136$

16. How many solutions are there to the equation $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 29$, where x_i, i = 1, 2, 3, 4, 5, 6, is a nonnegative integer such that

a) $x_i > 1 \ for \ i = 1, 2, 3, 4, 5, 6$**?**

$C(17+6-1,17) = C(22,17) = 26334$

b) $x_1 \geq 1, x_2 \geq 2, x_3 \geq 3, x_4 \geq 4, x_5 > 5, and\ x_6 \geq 6$**?**

$C(7+6-1,8) = C(12,7) = 792$

c) $x_1 \leq 5$**?**

$C(29+6-1,29) - C(23+6-1,23) = C(34,29) - C(28,23) = 179976$

d) $x_1 < 8$ **and** $x_2 > 8$**?**

$C(20+6-1,20) - C(12+6-1,12) = C(25,20) - C(17,12) = 46942$

26. How many positive integers less than 1,000,000 have exactly one digit equal to 9 and have a sum of digits equal to 13?

$x_1 + x_2 + x_3 + x_4 + x_5 = 4,\ x_6 = 9.\ x_1, x_2, x_3, x_4, x_5 \leq 8$

$C(4+5-1,4) = 70$

70 * 6 = 420, since x_6 has 6 positions to put.

32. How many different strings can be made from the letters in AARDVARK, using all the letters, if all three As must be consecutive?

$\frac{A_6^6}{A_2^2} = 360$

46. A shelf holds 12 books in a row. How many ways are there to choose five books so that no two adjacent books are chosen?

[Hint: Represent the books that are chosen by bars and the books not chosen by stars. Count the number of sequences of five bars and seven stars so that no two bars are adjacent.]

We firstly fixe the bars and then use four stars to avoid the bars being adjacent.

So,there are 3 books remaining.

$C(6+3-1,3) = C(8,3) = 56$

50. How many ways are there to distribute five distinguishable objects into three indistinguishable boxes?

$S(5,1)$ = 1

$S(5,2) = (32-2)/2 = 15$

$S(5,3) = (3^5 - 2^5 * 3 + 1 * 3)/6 = 25$

1+15+25 = 41

54. How many ways are there to distribute five indistinguishable objects into three indistinguishable boxes?

5 = 5, 5 = 4+1, 5 = 3+2, 5 = 3+1+1, and 5 = 2+2+1

5

61. Suppose that a weapons inspector must inspect each of five different sites twice, visiting one site per day. The inspector is free to select the order in which to visit these sites, but cannot visit site X, the most suspicious site, on two consecutive days. In how many different orders can the inspector visit these sites?

$\frac{10!}{2^5} - \frac{9!}{2^4} = 113400 - 22680 = 90720$

Sec. 6.6

6.Find the next larger permutation in lexicographic order after each of these permutations.

f) 23587416

23587461

7. Use Algorithm 1 to generate the 24 permutations of the first four positive integers in lexicographic order.

1234 1243 1324 1342 1423 1432

2134 2143 2314 2341 2413 2431

3124 3142 3214 3241 3412 3421

4123 4132 4213 4231 4312 4321

9. Use Algorithm 3 to list all the 3-combinations of {1,2,3,4,5}.

{123}、 {124}、 {125}、{134}、{135}、

{145}、{234}、{235}、{245}、{345}

Sec. 8.1

24. Find a recurrence relation for the number of bit sequences of length n with an even number of 0s.

Let e_n be the number of bit sequences of length n with an even number of 0's.

Therefore there are $2^n - e_n$ bit sequences with an odd number of 0's.

So, $e_n = e_{n-1} + 2^{n-1} - e_{n-1} = 2^{n-1}$

26.

a) Find a recurrence relation for the number of ways to completely cover a 2×n checkerboard with 1×2 dominoes. [Hint: Consider separately the coverings where the position in the top right corner of the checkerboard is covered by a domino positioned horizontally and where it is covered by a domino positioned vertically.]

Let a_n be the number of coverings.

If the right-most domino is positioned vertically, then we have a covering of the leftmost n−1 columns, and this can be done in a_{n-1} ways.

If the right-most domino is positioned horizontally, then there must be another domino directly beneath it, and these together cover the last two columns. The first n−2 columns therefore will need to contain a covering by dominoes, and this can be done in a_{n-2} ways. Thus we obtain the Fibonacci recurrence $a_n = a_{n-1} + a_{n-2}$.

b) What are the initial conditions for the recurrence relation in part (a)?

$a_1 = 1$ and $a_2 = 2$.

c) How many ways are there to completely cover a 2×17 checkerboard with 1×2 dominoes?

The sequence is thus 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, ..., so the answer to this part is 2584.

Exercises 38–45 involve the Reve's puzzle, the variation of the Tower of Hanoi puzzle with four pegs and n disks. Before presenting these exercises, we describe the Frame–Stewart algorithm for moving the disks from peg 1 to peg 4 so that no disk is ever on top of a smaller one. This algorithm, given the number of disks n as input, depends on a choice of an integer k with 1 ≤ k ≤ n. When there is only one disk, move it from peg 1 to peg 4 and stop. For n>1, the algorithm proceeds recursively, using these three steps. Recursively move the stack of the n−k smallest disks from peg 1 to peg 2, using all four pegs. Next move the stack of the k largest disks from peg 1 to peg 4, using the three-peg algorithm from the Tower of Hanoi puzzle without using the peg holding the n−k smallest disks. Finally, recursively move the smallest n−k disks to peg 4, using all four pegs. Frame and Stewart showed that to produce the fewest moves using their algorithm, k should be chosen to be the smallest integer such that n does not exceed $t_k = k(k+1)/2$, the kth triangular number, that is, $t_{k-1} < n \leq t_k$. The unsettled conjecture, known as Frame's conjecture, is that this algorithm uses the fewest number of moves required to solve the puzzle, no matter how the disks are moved.

41. Show that if R(n) is the number of moves used by the Frame–Stewart algorithm to solve the Reve's puzzle with n disks, where k is chosen to be the smallest integer with n ≤ k(k+1)/2, then R(n) satisfies the recurrence relation $R(n) = 2R(n-k) + 2^k - 1$, with R(0) = 0 and R(1) = 1.

42. Show that if k is as chosen in Exercise 41, then $R(n) - R(n-1) = 2^{k-1}$.

- $R(1) - R(0) = 2^0 = 1$.
- we assume $R(m) - R(m-1) = 2^{k-1}$ with $m \le k(k+1)/2$, and we need to prove $R(m+1) - R(m) = 2^{k-1}$.
  - when $1 + 2 + ... + k \ge m + 1$

    $R(m) = 2R(m-k) + 2^k - 1$

    $R(m+1) = 2R(m+1-k) + 2^k - 1$

    Subtracting yields $R(m+1) - R(m) = 2R(m+1-k) - 2R(m-k)$

    Since k −1 is the value selected for n−k, the inductive hypothesis tells us that this difference is $2 \cdot 2^{k-2} = 2^{k-1}$, as desired.
  - when $1 + 2 + ... + k + k + 1 \ge m + 1$

    $R(m) = 2R(m-k) + 2^k - 1$

    $R(m+1) = 2R(m-k) + 2^{k+1} - 1$

    Subtracting yields $R(m+1) - R(m) = 2R(m-k) - 2R(m-k) + 2^k = 2^k$

    Since the smallest integer with m+1 ≤ (k+2)(k+1)/2, the inductive hypothesis tells us that this difference is $2^{k+1-1} = 2^k$, as desired.

45. Show that R(n) is $O(\sqrt{n}2^{\sqrt{2n}})$.

Let $\{a_n\}$ be a sequence of real numbers. The backward differences of this sequence are defined recursively as shown next. The first difference ∇a_n is $\nabla a_n = a_n - a_{n-1}$. The $(k+1)st$ difference $\nabla^{k+1} a_n$ is obtained from $\nabla^k a_n$ by $\nabla^{k+1} a_n = \nabla^k a_n - \nabla^k a_{n-1}$.

By Exercise 43, R(n) is no larger than $\sum_{i=1}^{k} i2^{i-1}$. It can be shown that this sum equals $(k+1)2^k - 2^{k+1} + 1$, so it is no greater than $(k+1)2^k$. Because $n > k(k-1)/2$, the quadratuc formula can be used to show that $k < 1 + \sqrt{2n}$ for all n>1.

Therefore, R(n) is bounded above by $(1 + \sqrt{2n} + 1)2^{1+\sqrt{2n}} < 8\sqrt{n}2^{\sqrt{2n}}$,for all n > 2.

Hence, R(n) is $O(\sqrt{n}2^{\sqrt{2n}})$.

48. Show that $a_{n-1} = a_n - \nabla a_n$

This follows immediately (by algebra) from the definition.