Sec.6.1

41. A palindrome is a string whose reversal is identical to the string. How many bit strings of length n are palindromes?

n=2, 2 bit string 00 and 11

n=3, 4 bit string 101,111,000,010

$2^{\lceil n/2 \rceil}$

56. The name of a variable in the C programming language is a string that can contain uppercase letters, lowercase letters, digits, or underscores. Further, the first character in the string must be a letter, either uppercase or lowercase, or an underscore. If the name of a variable is determined by its first eight characters, how many different variables can be named in C? (Note that the name of a variable may contain fewer than eight characters.)

1 character: 26*2 + 1 = 53

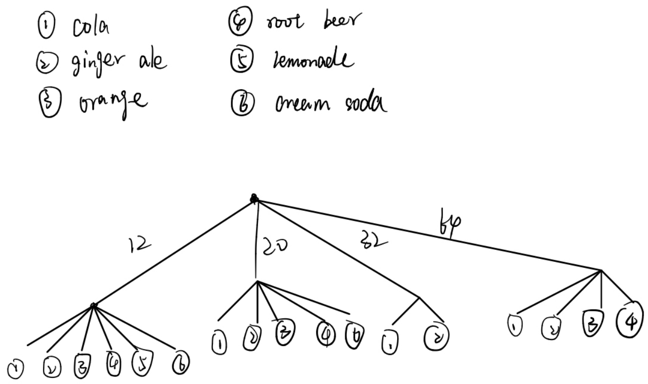
2 characters: 53 * (26*2+1+10) = 53* 63

...

$53 + 53 * 63 + 53 * 63^2 + \ldots + 53 * 63^7 = \frac{53*(63^8-1)}{62} \approx 2.1 * 10^{14}$

68.

a) Suppose that a store sells six varieties of soft drinks: cola, ginger ale, orange, root beer, lemonade, and cream soda. Use a tree diagram to determine the number of different types of bottles the store must stock to have all varieties available in all size bottles if all varieties are available in 12-ounce bottles, all but lemonade are available in 20-ounce bottles, only cola and ginger ale are available in 32-ounce bottles,and all but lemonade and cream soda are available in 64-ounce bottles?



b) Answer the question in part (a) using counting rules.

6+5+2+4 = 17

Sec. 6.2

10. Let (x_i, y_i), i = 1,2,3,4,5,be a set of five distinct points with integer coordinates in the xy plane. Show that the midpoint of the line joining at least one pair of these points has integer coordinates.

There are four possible pairs of parities: (odd,odd), (odd,even), (even,odd), and (even,even). Since we are given five points, the pigeonhole principle guarantees that at least two of them will have the same pair of parities. The midpoint of the segment joining these two points will therefore have integer coordinates.

38. Find the least number of cables required to connect eight computers to four printers to guarantee that for every choice of four of the eight computers, these four computers can directly access four different printers. Justify your answer.

The answer is 20. According to the pigeonhole principle, every printer needs to connect at least 5 computers, which means that for a printer, there is one of the four computers can be connected to it. Label the computers C_1 through C_8, and label the printers P_1 through P_4. If we connect C_k to P_k for k = 1,2,3,4 and connect each of the computers C5 through C8 to all the printers, then we have used a total of $4 + 4 \cdot 4 = 20$ cables.

40. Prove that at a party where there are at least two people, there are two people who know the same number of other people there.

Let K(x) be the number of other people at the party that person x knows. The possible values for K(x) are 1,...,n−1, since we assume that if you are at a party, there is at least one person you know. So, it's apparently that the possible values are smaller than the n. According to the pigeonhole principle, at a party where there are at least two people, there are two people who know the same number of other people there.

41. An arm wrestler is the champion for a period of 75 hours. (Here, by an hour, we mean a period starting from an exact hour, such as 1 p.m., until the next hour.) The arm wrestler had at least one match an hour, but no more than 125 total matches. Show that there is a period of consecutive hours during which the arm wrestler had exactly 24 matches.

x_i : the number of matches held on the i-th hour.

$a_i = \sum_{k=1}^{i} x_k : 1 \leq a_1 < a_2 <..< a_{75} \leq 125$

$c_i = a_i + 24 : 25 \leq c_1 < c_2 <..< c_{75} \leq 149$

A = $\{a_1, a_2, \ldots, a_{75}, c_1, c_2, \ldots, c_{75}\}$ B = {1,2,...,150}

$\exists i \neq j$ such that $a_i = c_j$

42. Is the statement in Exercise 41 true if 24 is replaced by

a) 2

$c_i = a_i + 2 : 3 \leq c_1 < c_2 <..< c_{75} \leq 127$

So, it's apparently that the statement is true.

b) 23

$c_i = a_i + 23 : 24 \leq c_1 < c_2 <..< c_{75} \leq 148$

So, it's apparently that the statement is true.

c) 25

$c_i = a_i + 25 : 26 \leq c_1 < c_2 <..< c_{75} \leq 150$

if $\exists i \neq j$ such that $a_i = c_j$, then the statement is true.

we assume that $\forall i \neq j, a_i \neq c_j$:

Since each of the numbers $a_i + 25$ is greater than or equal to 26, the numbers 1,2,...,25 must all appear among the a_i's. But since the a_i's are increasing, the only way this can happen is if $a_1 = 1, a_2 = 2, \ldots, a_{25} = 25$. Thus there were exactly 25 matches in the first 25 hours.

d) 30

- Let a_i, a_j be distinct numbers among $a_1, a_2, \ldots, a_{31}$ and a_i mod 30 == a_j mod 30

  if a_i, a_j are differ by 30, then we find the solution.

  Otherwise, they differ by 60 or more, so $a_{31} \geq 61$.
- Similarly, Let a_i, a_j be distinct numbers among $a_{31}, a_{32}, \ldots, a_{61}$ and a_i mod 30 == a_j mod 30

  if a_i, a_j are differ by 30, then we find the solution.

  Otherwise, they differ by 60 or more, so $a_{61} \geq 121$.

  But this means that a66 ≥126, a contradiction.

Sec. 6.3

20. How many bit strings of length 10 have

a) exactly three 0s?

$C_{10}^3 = 120$

b) more 0s than 1s?

$C_{10}^6 + C_{10}^7 + C_{10}^8 + C_{10}^9 + C_{10}^{10} = 386$

c) at least seven 1s?

$C_{10}^7 + C_{10}^8 + C_{10}^9 + C_{10}^{10} = 176$

d) at least three 1s?

$2^{10} - (C_{10}^0 + C_{10}^1 + C_{10}^2) = 968$

42. Find a formula for the number of ways to seat r of n people around a circular table, where seatings are considered the same if every person has the same two neighbors without regard to which side these neighbors are sitting on.

（离散数学）计数/排列与组合 n人的圆桌问题在考虑相对于固定位置的左右对称时有$(n-1)!$解，只考虑人不考虑对称时有$\frac{(n-1)!}{2}$解。

$C_n^r * \frac{(r-1)!}{2} = \frac{n!}{2*r!*(n-r)!} * (r-1)! = \frac{n!}{2*r*(n-r)!}$ for $n \geq 3$.

When r=1, the answer is n.

When r=2, the answer is C_n^2

44. How many ways are there for a horse race with four horses to finish if ties(并列) are possible? [Note: Any number of the four horses may tie.)

- There is no ties.

  $4! = 24.$
- There are two horses that tie, but the others have distinct finishes.

  $C_4^2 * A_3^3 = 36$
- There might be two groups of two horses that are tied.

  $C_4^2 = 6$
- There might be a group of three horses all tied.

  $C_4^3 * 2 = 8$
- There is only one way for all the horses to tie.

The answer is 24+36+6+8+1 = 75.

Sec. 6.4

14. Show that if n is a positive integer, then $1 = C_n^0 < C_n^1 < ... < C_n^{\lfloor n/2 \rfloor} = C_n^{\lceil n/2 \rceil} > ... > C_n^{n-1} > C_n^n = 1$

$C_n^{k-1} = \frac{k}{n-k+1} C_n^k.$

- if $k \le n/2$ then $\frac{k}{n-k+1} < 1$, so $1 = C_n^0 < C_n^1 < ... C_n^{\lfloor n/2 \rfloor}$
- if $k > n/2$ then $\frac{k}{n-k+1} > 1$, so $C_n^{\lceil n/2 \rceil} > ... > C_n^{n-1} > C_n^n = 1$
- As for $C_n^{\lfloor n/2 \rfloor} = C_n^{\lceil n/2 \rceil}$
  - if n is even, $\lfloor n/2 \rfloor = \lceil n/2 \rceil$
  - if n is odd, let $m = \lfloor n/2 \rfloor$, $m + 1 = \lceil n/2 \rceil$.

    since $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$. The equalities at the ends are clear.

22. Prove the identity $C_n^r * C_r^k = C_n^k * C_{n-k}^{r-k}$, whenever n, r, and k are nonnegative integers with r ≤ n and k ≤ r,

a) using a combinatorial argument.

We choose r from n elements and then from the chosen r elements select k elements as their leaders.

We first choose k leaders and then choose the ordinary elements.

b) using an argument based on the formula for the number of r-combinations of a set with n elements.

$C_n^r * C_r^k = \frac{n!}{r!(n-r)!} * \frac{r!}{k!(r-k)!} = \frac{n!}{k!(n-r)!(r-k)!}$

$C_n^k * C_{n-k}^{r-k} = \frac{n!}{k!(n-k)!} * \frac{(n-k)!}{(r-k)!(n-r)!} = \frac{n!}{k!(n-r)!(r-k)!}$

26. Let n and k be integers with 1 ≤ k ≤ n. Show that $\sum_{k=1}^{n} C_n^k * C_n^{k-1} = C_{2n+2}^{n+1}/2 - C_{2n}^{n}$

$\sum_{k=1}^{n} C_n^k * C_n^{k-1}$

=$\sum_{k=1}^{n} C_n^{n-k} * C_n^{k-1}$

=C_{2n}^{n-1}

$C_{2n+2}^{n+1}/2 - C_{2n}^{n}$

=$\frac{(2n+2)!}{2(n+1)!(n+1)!} - \frac{(2n)!}{n!n!}$

=$\frac{(2n)!*(n+1)*(2n+1)}{(n+1)!(n+1)!} - \frac{(2n)!*(n+1)^2}{(n+1)!(n+1)!}$

=$\frac{(2n)!}{(n+1)!(n-1)!}$

= C_{2n}^{n-1}

30. Give a combinatorial proof that $\sum_{k=1}^{n} k(C_n^k)^2 = nC_{2n-1}^{n-1}$.[Hint: Count in two ways the number of ways to select a committee, with n members from a group of n mathematics professors and n computer science professors, such that the chairperson of the committee is a mathematics professor.]

nC_{2n-1}^{n-1} means that we firstly choose a chairperson from the group and then choose n-1 people from the remaining.

$\sum_{k=1}^{n} k(C_n^k)^2$ means that we choose k people from n mathematics professor and n-k people from computer science professors. And then we choose a chairperson from the chosen k math professor.