Supplementary Materials:
Night-Rider: Nocturnal Vision-aided Localization in Streetlight Maps Using Invariant Extended Kalman Filtering

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This document provides supplementary materials for the ICRA 2024 submission "Night-Rider: Nocturnal Vision-aided Localization in Streetlight Maps Using Invariant Extended Kalman Filtering".

A. Derivation of Error Propagation Model

From the Eq. (8), we need to derive the linearized model of right invariant error $\boldsymbol{\eta}_t^r$ and bias error $\boldsymbol{\zeta}_t$.

For the states fitting into Lie group, the derivative of right invariant error $\boldsymbol{\eta}_t^r$ associates with the log of invariant error $\boldsymbol{\xi}_t$ based on $\boldsymbol{\eta}_t^r = \exp(\boldsymbol{\xi}_t^\wedge)$.

$$\frac{d}{dt}\boldsymbol{\eta}_t^r \approx \frac{d}{dt}(\mathbf{I}+\boldsymbol{\xi}_t^\wedge) = \begin{bmatrix} \frac{d}{dt}(\mathbf{I}+(\boldsymbol{\xi}_{R_t})_\times) & \frac{d}{dt}\boldsymbol{\xi}_{v_t} & \boldsymbol{\xi}_{p_t} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{27}$$

With Eq. (3), the derivative of $\boldsymbol{\xi}_t$ is deduced as below.

$$\begin{aligned}
&\frac{d}{dt}(\mathbf{I}+(\boldsymbol{\xi}_{R_t})_\times) = \frac{d}{dt}(\boldsymbol{\xi}_{R_t})_\times = \frac{d\hat{\mathbf{R}}_{b_t}^w}{dt}\mathbf{R}_{b_t}^{w\top} + \hat{\mathbf{R}}_{b_t}^w\frac{d\mathbf{R}_{b_t}^{w\top}}{dt} \\
&= \hat{\mathbf{R}}_{b_t}^w(\tilde{\boldsymbol{\omega}}_t - \hat{\mathbf{b}}_{\omega_t})_\times\mathbf{R}_{b_t}^{w\top} - \hat{\mathbf{R}}_{b_t}^w(\boldsymbol{\omega}_t)_\times\mathbf{R}_{b_t}^{w\top} \\
&= \hat{\mathbf{R}}_{b_t}^w(\tilde{\boldsymbol{\omega}}_t + \mathbf{b}_{\omega_t} + \mathbf{n}_{\omega_t} - \hat{\mathbf{b}}_{\omega_t} - \boldsymbol{\omega}_t)_\times\mathbf{R}_{b_t}^{w\top} \\
&= \hat{\mathbf{R}}_{b_t}^w(\mathbf{n}_{\omega_t} - \boldsymbol{\zeta}_{\omega_t})_\times\hat{\mathbf{R}}_{b_t}^{w\top}\hat{\mathbf{R}}_{b_t}^w\mathbf{R}_{b_t}^{w\top} \\
&= (\hat{\mathbf{R}}_{b_t}^w(\mathbf{n}_{\omega_t} - \boldsymbol{\zeta}_{\omega_t}))_\times\boldsymbol{\eta}_{R_t} \\
&\approx (\hat{\mathbf{R}}_{b_t}^w(\mathbf{n}_{\omega_t} - \boldsymbol{\zeta}_{\omega_t}))_\times(\mathbf{I}+\boldsymbol{\xi}_{R_t}) \approx (\hat{\mathbf{R}}_{b_t}^w(\mathbf{n}_{\omega_t} - \boldsymbol{\zeta}_{\omega_t}))_\times
\end{aligned} \tag{28}$$

$$\begin{aligned}
&\frac{d}{dt}\boldsymbol{\xi}_{v_t} = \frac{d}{dt}{}^w\hat{\mathbf{v}}_{b_t} - \frac{d}{dt}(\hat{\mathbf{R}}_{b_t}^w\mathbf{R}_{b_t}^{w\top}){}^w\mathbf{v}_{b_t} - \hat{\mathbf{R}}_{b_t}^w\mathbf{R}_{b_t}^{w\top}\frac{d}{dt}{}^w\mathbf{v}_{b_t} \\
&\approx \hat{\mathbf{R}}_{b_t}^w(\tilde{\boldsymbol{a}}_t - \hat{\mathbf{b}}_{a_t}) + \mathbf{g} - (\hat{\mathbf{R}}_{b_t}^w(\mathbf{n}_{\omega_t} - \boldsymbol{\zeta}_{\omega_t}))_\times{}^w\mathbf{v}_{b_t} \\
&\quad - \hat{\mathbf{R}}_{b_t}^w\mathbf{R}_{b_t}^{w\top}(\mathbf{R}_{b_t}^w\boldsymbol{a}_t + \mathbf{g}) \\
&= \hat{\mathbf{R}}_{b_t}^w(\boldsymbol{a}_t + \mathbf{b}_{a_t} + \mathbf{n}_{a_t} - \hat{\mathbf{b}}_{a_t} - \boldsymbol{a}_t) \\
&\quad - (\hat{\mathbf{R}}_{b_t}^w(\mathbf{n}_{\omega_t} - \boldsymbol{\zeta}_{\omega_t}))_\times{}^w\mathbf{v}_{b_t} + (\mathbf{I} - \hat{\mathbf{R}}_{b_t}^w\mathbf{R}_{b_t}^{w\top})\mathbf{g} \\
&\approx \hat{\mathbf{R}}_{b_t}^w(\mathbf{n}_{a_t} - \boldsymbol{\zeta}_{a_t}) - (\hat{\mathbf{R}}_{b_t}^w(\mathbf{n}_{\omega_t} - \boldsymbol{\zeta}_{\omega_t}))_\times{}^w\mathbf{v}_{b_t} - (\boldsymbol{\xi}_{R_t})_\times\mathbf{g} \\
&= (\mathbf{g})_\times\boldsymbol{\xi}_{R_t} + ({}^w\mathbf{v}_{b_t})_\times\hat{\mathbf{R}}_{b_t}^w(\mathbf{n}_{\omega_t} - \boldsymbol{\zeta}_{\omega_t}) + \hat{\mathbf{R}}_{b_t}^w(\mathbf{n}_{a_t} - \boldsymbol{\zeta}_{a_t})
\end{aligned} \tag{29}$$

$$\begin{aligned}
&\frac{d}{dt}\boldsymbol{\xi}_{p_t} = \frac{d}{dt}\hat{\mathbf{p}}_{b_t}^w - \frac{d}{dt}(\hat{\mathbf{R}}_{b_t}^w\mathbf{R}_{b_t}^{w\top})\mathbf{p}_{b_t}^w - \hat{\mathbf{R}}_{b_t}^w\mathbf{R}_{b_t}^{w\top}\frac{d}{dt}\mathbf{p}_{b_t}^w \\
&\approx {}^w\hat{\mathbf{v}}_{b_t} - (\hat{\mathbf{R}}_{b_t}^w(\mathbf{n}_{\omega_t} - \boldsymbol{\zeta}_{\omega_t}))_\times\mathbf{p}_{b_t}^w - \hat{\mathbf{R}}_{b_t}^w\mathbf{R}_{b_t}^{w\top}{}^w\mathbf{v}_{b_t} \\
&= ({}^w\hat{\mathbf{v}}_{b_t} - \hat{\mathbf{R}}_{b_t}^w\mathbf{R}_{b_t}^{w\top}{}^w\mathbf{v}_{b_t}) - (\hat{\mathbf{R}}_{b_t}^w(\mathbf{n}_{\omega_t} - \boldsymbol{\zeta}_{\omega_t}))_\times\mathbf{p}_{b_t}^w \\
&= \boldsymbol{\xi}_{v_t} + (\mathbf{p}_{b_t}^w)_\times\hat{\mathbf{R}}_{b_t}^w(\mathbf{n}_{\omega_t} - \boldsymbol{\zeta}_{\omega_t})
\end{aligned} \tag{30}$$

On the other hand, the differential of bias errors is calculated as

$$\frac{d}{dt}\boldsymbol{\zeta}_t = \begin{bmatrix} \frac{d}{dt}\hat{\mathbf{b}}_{\omega_t} - \frac{d}{dt}\mathbf{b}_{\omega_t} \\ \frac{d}{dt}\hat{\mathbf{b}}_{a_t} - \frac{d}{dt}\mathbf{b}_{a_t} \end{bmatrix} = \begin{bmatrix} \mathbf{n}_{b\omega_t} \\ \mathbf{n}_{ba_t} \end{bmatrix} \tag{31}$$

Therefore, the linearized model can be written as in Eq. (9).

B. Derivation of $\boldsymbol{\Sigma}_{t_{proj,ij}}$ and $\sigma^2_{t_{ang,ij}}$

The covariance matrix of reprojection error in Eq. (18) for the streetlight observation B_i and cluster L_j is derived as below:

$$\boldsymbol{\Sigma}_{t_{proj,ij}} = \boldsymbol{\Sigma}_{\tilde{\mathbf{p}}_{t_i}} + \frac{\partial\tilde{\mathbf{c}}_{t_j}^m}{\partial\begin{bmatrix}\hat{\mathbf{R}}_{b_t}^w & \hat{\mathbf{p}}_{b_t}^w\end{bmatrix}}\hat{\mathbf{P}}_t^-\left(\frac{\partial\tilde{\mathbf{c}}_{t_j}^m}{\partial\begin{bmatrix}\hat{\mathbf{R}}_{b_t}^w & \hat{\mathbf{p}}_{b_t}^w\end{bmatrix}}\right)^\top \tag{32}$$

$$\begin{aligned}
\frac{\partial\tilde{\mathbf{c}}_{t_j}^m}{\partial\hat{\mathbf{R}}_{b_t}^w} &= \frac{\partial\frac{1}{\tilde{Z}_j^c}\mathbf{K}(\mathbf{R}_b^c\hat{\mathbf{R}}_{b_t}^{w\top}(\tilde{\mathbf{C}}_j^w - \hat{\mathbf{p}}_{b_t}^w) + \mathbf{t}_b^c)}{\partial\hat{\mathbf{R}}^w{}_{b_t}} \\
&= \frac{\partial\frac{1}{\tilde{Z}_j^c}\mathbf{K}\tilde{\mathbf{C}}_j^c}{\partial\hat{\mathbf{R}}_{b_t}^w} = \frac{\partial\frac{1}{\tilde{Z}_j^c}\mathbf{K}\tilde{\mathbf{C}}_j^c}{\partial\tilde{\mathbf{C}}_j^c}\frac{\partial\tilde{\mathbf{C}}_j^c}{\partial\hat{\mathbf{R}}_{b_t}^w} \\
&= \frac{\partial\frac{1}{\tilde{Z}_j^c}\mathbf{K}\tilde{\mathbf{C}}_j^c}{\partial\tilde{\mathbf{C}}_j^c}\frac{\partial\mathbf{R}_b^c\hat{\mathbf{R}}_{b_t}^{w\top}(\tilde{\mathbf{C}}_j^w - \hat{\mathbf{p}}_{b_t}^w) + \mathbf{t}_b^c}{\partial\hat{\mathbf{R}}_{b_t}^w} \\
&= \begin{bmatrix} \frac{f_x}{\tilde{Z}_j^c} & 0 & -\frac{\tilde{X}_j^c f_x}{\tilde{Z}_j^{c2}} \\ 0 & \frac{f_y}{\tilde{Z}_j^c} & -\frac{\tilde{Y}_j^c f_y}{\tilde{Z}_j^{c2}} \end{bmatrix}\mathbf{R}_b^c(\hat{\mathbf{R}}_{b_t}^{w\top}(\tilde{\mathbf{C}}_j^w - \hat{\mathbf{p}}_{b_t}^w))_\times
\end{aligned} \tag{33}$$

$$\begin{aligned}
\frac{\partial\tilde{\mathbf{c}}_{t_j}^m}{\partial\hat{\mathbf{p}}_{b_t}^w} &= \frac{\partial\frac{1}{\tilde{Z}_j^c}\mathbf{K}(\mathbf{R}_b^c\hat{\mathbf{R}}_{b_t}^{w\top}(\tilde{\mathbf{C}}_j^w - \hat{\mathbf{p}}_{b_t}^w) + \mathbf{t}_b^c)}{\partial\hat{\mathbf{p}}_{b_t}^w} \\
&= \frac{\partial\frac{1}{\tilde{Z}_j^c}\mathbf{K}\tilde{\mathbf{C}}_j^c}{\partial\hat{\mathbf{p}}_{b_t}^w} = \frac{\partial\frac{1}{\tilde{Z}_j^c}\mathbf{K}\tilde{\mathbf{C}}_j^c}{\partial\tilde{\mathbf{C}}_j^c}\frac{\partial\tilde{\mathbf{C}}_j^c}{\partial\hat{\mathbf{p}}_{b_t}^w} \\
&= \frac{\partial\frac{1}{\tilde{Z}_j^c}\mathbf{K}\tilde{\mathbf{C}}_j^c}{\partial\tilde{\mathbf{C}}_j^c}\frac{\partial\mathbf{R}_b^c\hat{\mathbf{R}}_{b_t}^{w\top}(\tilde{\mathbf{C}}_j^w - \hat{\mathbf{p}}_{b_t}^w) + \mathbf{t}_b^c}{\partial\hat{\mathbf{p}}_{b_t}^w} \\
&= -\begin{bmatrix} \frac{f_x}{\tilde{Z}_j^c} & 0 & -\frac{\tilde{X}_j^c f_x}{\tilde{Z}_j^{c2}} \\ 0 & \frac{f_y}{\tilde{Z}_j^c} & -\frac{\tilde{Y}_j^c f_y}{\tilde{Z}_j^{c2}} \end{bmatrix}\mathbf{R}_b^c\hat{\mathbf{R}}_{b_t}^{w\top}
\end{aligned} \tag{34}$$

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where $\boldsymbol{\Sigma}_{\tilde{\mathbf{p}}_{t_i}} = \alpha \mathbf{I}_{2\times 2}$ and α is a preset parameter.

Similar to $\boldsymbol{\Sigma}_{t_{proj,ij}}$, the variance of angle error in Eq. (19) also originates from the covariance of detected streetlight observation and current pose.

$$\sigma^2_{t_{ang,ij}} = \frac{\partial \cos\theta_{ij}}{\partial \tilde{\mathbf{p}}_{t_i}} \boldsymbol{\Sigma}_{\bar{\tilde{\mathbf{p}}}_{t_i}} \left(\frac{\partial \cos\theta_{ij}}{\partial \tilde{\mathbf{p}}_{t_i}}\right)^\top + \frac{\partial \cos\theta_{ij}}{\partial \left[\mathbf{R}^w_{b_t} \quad \mathbf{p}^w_{b_t}\right]} \hat{\mathbf{P}}^-_t \left(\frac{\partial \cos\theta_{ij}}{\partial \left[\mathbf{R}^w_{b_t} \quad \mathbf{p}^w_{b_t}\right]}\right)^\top \tag{35}$$

$$\begin{aligned}
\frac{\partial \cos\theta_{ij}}{\partial \tilde{\mathbf{p}}_{t_i}} &= \frac{\partial \frac{(\mathbf{K}^{-1}\bar{\tilde{\mathbf{p}}}_{t_i})^\top (\mathbf{R}^c_b \hat{\mathbf{R}}^{w\top}_{b_t} (\tilde{\mathbf{C}}^w_j - \hat{\mathbf{p}}^w_{b_t}) + \mathbf{t}^c_b)}{\|\mathbf{K}^{-1}\bar{\tilde{\mathbf{p}}}_{t_i}\|_2 \cdot \|\mathbf{R}^c_b \hat{\mathbf{R}}^{w\top}_{b_t} (\tilde{\mathbf{C}}^w_j - \hat{\mathbf{p}}^w_{b_t}) + \mathbf{t}^c_b\|_2}}{\partial \tilde{\mathbf{p}}_{t_i}} \\
&= \frac{\tilde{\mathbf{C}}^{c\top}_j}{\|\tilde{\mathbf{C}}^c_j\|_2} \frac{\partial \frac{\tilde{\mathbf{P}}_{t_i}}{\|\tilde{\mathbf{P}}_{t_i}\|_2}}{\partial \tilde{\mathbf{P}}_{t_i}} \frac{\partial \tilde{\mathbf{P}}_{t_i}}{\partial \tilde{\mathbf{p}}_{t_i}} \\
&= \frac{\tilde{\mathbf{C}}^{c\top}_j}{\|\tilde{\mathbf{C}}^c_j\|_2} \left(\frac{\mathbf{I}}{\|\tilde{\mathbf{P}}_{t_i}\|_2} - \frac{\tilde{\mathbf{P}}_{t_i}\tilde{\mathbf{P}}^\top_{t_i}}{\|\tilde{\mathbf{P}}_{t_i}\|^3_2}\right) \mathbf{K}^{-1}
\end{aligned} \tag{36}$$

$$\begin{aligned}
\frac{\partial \cos\theta_{ij}}{\partial \hat{\mathbf{R}}^w_{b_t}} &= \frac{\partial \frac{(\mathbf{K}^{-1}\bar{\tilde{\mathbf{p}}}_{t_i})^\top (\mathbf{R}^c_b \hat{\mathbf{R}}^{w\top}_{b_t} (\tilde{\mathbf{C}}^w_j - \hat{\mathbf{p}}^w_{b_t}) + \mathbf{t}^c_b)}{\|\mathbf{K}^{-1}\bar{\tilde{\mathbf{p}}}_{t_i}\|_2 \cdot \|\mathbf{R}^c_b \hat{\mathbf{R}}^{w\top}_{b_t} (\tilde{\mathbf{C}}^w_j - \hat{\mathbf{p}}^w_{b_t}) + \mathbf{t}^c_b\|_2}}{\partial \hat{\mathbf{R}}^w_{b_t}} \\
&= \frac{\tilde{\mathbf{P}}^\top_{t_i}}{\|\tilde{\mathbf{P}}_{t_i}\|_2} \frac{\partial \frac{\tilde{\mathbf{C}}^c_j}{\|\tilde{\mathbf{C}}^c_j\|_2}}{\partial \tilde{\mathbf{C}}^c_j} \frac{\partial \tilde{\mathbf{C}}^c_j}{\partial \hat{\mathbf{R}}^w_{b_t}} \\
&= \frac{\tilde{\mathbf{P}}^\top_{t_i}}{\|\tilde{\mathbf{P}}_{t_i}\|_2} \left(\frac{\mathbf{I}}{\|\tilde{\mathbf{C}}^c_j\|_2} - \frac{\tilde{\mathbf{C}}^c_j \tilde{\mathbf{C}}^{c\top}_j}{\|\tilde{\mathbf{C}}^c_j\|^3_2}\right) \frac{\partial \tilde{\mathbf{C}}^c_j}{\partial \hat{\mathbf{R}}^w_{b_t}} \\
&= \frac{\tilde{\mathbf{P}}^\top_{t_i}}{\|\tilde{\mathbf{P}}_{t_i}\|_2} \left(\frac{\mathbf{I}}{\|\tilde{\mathbf{C}}^c_j\|_2} - \frac{\tilde{\mathbf{C}}^c_j \tilde{\mathbf{C}}^{c\top}_j}{\|\tilde{\mathbf{C}}^c_j\|^3_2}\right) \mathbf{R}^c_b (\hat{\mathbf{R}}^{w\top}_{b_t} (\tilde{\mathbf{C}}^w_j - \hat{\mathbf{p}}^w_{b_t}))_\times
\end{aligned} \tag{37}$$

$$\begin{aligned}
\frac{\partial \cos\theta_{ij}}{\partial \hat{\mathbf{p}}^w_{b_t}} &= \frac{\partial \frac{(\mathbf{K}^{-1}\bar{\tilde{\mathbf{p}}}_{t_i})^\top (\mathbf{R}^c_b \hat{\mathbf{R}}^{w\top}_{b_t} (\tilde{\mathbf{C}}^w_j - \hat{\mathbf{p}}^w_{b_t}) + \mathbf{t}^c_b)}{\|\mathbf{K}^{-1}\bar{\tilde{\mathbf{p}}}_{t_i}\|_2 \cdot \|\mathbf{R}^c_b \hat{\mathbf{R}}^{w\top}_{b_t} (\tilde{\mathbf{C}}^w_j - \hat{\mathbf{p}}^w_{b_t}) + \mathbf{t}^c_b\|_2}}{\partial \hat{\mathbf{p}}^w_{b_t}} \\
&= \frac{\tilde{\mathbf{P}}^\top_{t_i}}{\|\tilde{\mathbf{P}}_{t_i}\|_2} \frac{\partial \frac{\tilde{\mathbf{C}}^c_j}{\|\tilde{\mathbf{C}}^c_j\|_2}}{\partial \tilde{\mathbf{C}}^c_j} \frac{\partial \tilde{\mathbf{C}}^c_j}{\partial \hat{\mathbf{p}}^w_{b_t}} \\
&= \frac{\tilde{\mathbf{P}}^\top_{t_i}}{\|\tilde{\mathbf{P}}_{t_i}\|_2} \left(\frac{\mathbf{I}}{\|\tilde{\mathbf{C}}^c_j\|_2} - \frac{\tilde{\mathbf{C}}^c_j \tilde{\mathbf{C}}^{c\top}_j}{\|\tilde{\mathbf{C}}^c_j\|^3_2}\right) \frac{\partial \tilde{\mathbf{C}}^c_j}{\partial \hat{\mathbf{p}}^w_{b_t}} \\
&= -\frac{\tilde{\mathbf{P}}^\top_{t_i}}{\|\tilde{\mathbf{P}}_{t_i}\|_2} \left(\frac{\mathbf{I}}{\|\tilde{\mathbf{C}}^c_j\|_2} - \frac{\tilde{\mathbf{C}}^c_j \tilde{\mathbf{C}}^{c\top}_j}{\|\tilde{\mathbf{C}}^c_j\|^3_2}\right) \mathbf{R}^c_b \hat{\mathbf{R}}^{w\top}_{b_t}
\end{aligned} \tag{38}$$

where $\bar{(\cdot)}$ denotes the homogeneous coordinates. $\boldsymbol{\Sigma}_{\bar{\tilde{\mathbf{p}}}_{t_i}} = diag_block(\mathbf{I}_{2\times 2}, 0)$.

C. Derivation of the Jacobian Matrix of the Camera-based Observation Model

For the derivation of the Jacobian matrix $\mathbf{H}_{c_{t_i}}$ in Eq. (24), we transform the observation function in Eq. (23) as:

$$\begin{aligned}
\mathbf{z}_{c_{t_i}} &= \mathbf{y}_{t_i} - h_i(\hat{\boldsymbol{\Psi}}_{t_i}) = h_i(\boldsymbol{\Psi}_{t_i}) - h_i(\hat{\boldsymbol{\Psi}}_{t_i}) + \mathbf{n}_{c_{t_i}} \\
&\approx h_i(\hat{\boldsymbol{\Psi}}_{t_i}) + \mathbf{H}_i(\hat{\boldsymbol{\Psi}}_{\mathbf{t_i}})(\boldsymbol{\Psi}_{t_i} - \hat{\boldsymbol{\Psi}}_{t_i}) - h_i(\hat{\boldsymbol{\Psi}}_{t_i}) + \mathbf{n}_{c_{t_i}} \\
&= \mathbf{H}_i(\hat{\boldsymbol{\Psi}}_{\mathbf{t_i}})(\boldsymbol{\Psi}_{t_i} - \hat{\boldsymbol{\Psi}}_{t_i}) + \mathbf{n}_{c_{t_i}} \\
&= \mathbf{H}_i(\hat{\boldsymbol{\Psi}}_{\mathbf{t_i}}) \left[\mathbf{R}^{w\top}_{b_t}(\tilde{\mathbf{C}}^w_{t_j} - \mathbf{t}^w_{b_t}) - \hat{\mathbf{R}}^{w\top}_{b_t}(\tilde{\mathbf{C}}^w_{t_j} - \hat{\mathbf{t}}^w_{b_t})\right] + \mathbf{n}_{c_{t_i}} \\
&= \mathbf{H}_i(\hat{\boldsymbol{\Psi}}_{\mathbf{t_i}}) \Big[\hat{\mathbf{R}}^{w\top}_{b_t}(\hat{\mathbf{R}}^w_{b_t}\mathbf{R}^{w\top}_{b_t} - \mathbf{I})\tilde{\mathbf{C}}^w_{t_j} \\
&\qquad + \hat{\mathbf{R}}^{w\top}_{b_t}(\hat{\mathbf{t}}^w_{b_t} - \hat{\mathbf{R}}^w_{b_t}\mathbf{R}^{w\top}_{b_t}\mathbf{t}^w_{b_t})\Big] + \mathbf{n}_{c_{t_i}} \\
&\approx -\mathbf{H}_i(\hat{\boldsymbol{\Psi}}_{\mathbf{t_i}}) \left[\hat{\mathbf{R}}^{w\top}_{b_t}(\tilde{\mathbf{C}}^w_{t_j})_\times \boldsymbol{\xi}_{R_t} - \hat{\mathbf{R}}^{w\top}_{b_t}\boldsymbol{\xi}_{p_t}\right] + \mathbf{n}_{c_{t_i}}
\end{aligned} \tag{39}$$

The Jacobian matrix $\mathbf{H}_i(\hat{\boldsymbol{\Psi}}_{\mathbf{t_i}})$ is formulated as:

$$\begin{aligned}
\mathbf{H}_i(\hat{\boldsymbol{\Psi}}_{t_i}) &= \frac{\partial h(\boldsymbol{\Psi}_{t_i})}{\partial \boldsymbol{\Psi}_{t_i}}\Big|_{\boldsymbol{\Psi}_{t_i} = \hat{\boldsymbol{\Psi}}_{t_i}} \\
&= -\frac{1}{\hat{Z}^{c^2}_{t_j}}(\mathbf{R}^c_b \hat{\boldsymbol{\Psi}}_{t_i} + \mathbf{t}^c_b) \frac{\partial \hat{Z}^c_{t_j}}{\partial \boldsymbol{\Psi}_{t_i}}\Big|_{\boldsymbol{\Psi}_{t_i} = \hat{\boldsymbol{\Psi}}_{t_i}} + \frac{1}{\hat{Z}^c_{t_j}}\mathbf{R}^c_b \\
\frac{\partial \hat{Z}^c_{t_j}}{\partial \boldsymbol{\Psi}_{t_i}}\Big|_{\boldsymbol{\Psi}_{t_i} = \hat{\boldsymbol{\Psi}}_{t_i}} &= \frac{\partial \mathbf{R}^c_{b,3}\boldsymbol{\Psi}_{t_i} + t^c_{b,3}}{\partial \boldsymbol{\Psi}_{t_i}}\Big|_{\boldsymbol{\Psi}_{t_i} = \hat{\boldsymbol{\Psi}}_{t_i}} = \mathbf{R}^c_{b,3}
\end{aligned} \tag{40}$$

where $\mathbf{R}^c_{b,3}$ denotes the third row of $\mathbf{R}^c_b$ and $t^c_{b,3}$ is the third element of $\mathbf{t}^c_b$. By substituting Eq. (40) into Eq. (39), the Jacobian matrix $\mathbf{H}_{c_{t_i}}$ of the camera-based observation model can be derived.