Exercise 11: The Airy Stress Function
Jan. 24, 2022 - Jan. 28, 2022

The purpose of this exercise is to introduce the Airy stress function, which is part of a solution method for certain two-dimensional elastostatic problems. We will need this method especially for fracture mechanics, here the Airy stress function is a strong tool to derive analytic solutions to crack problems.
The principle here is to postulate a function whose derivatives are the components of the stress tensor. This postulated function allows to reduce the governing equations for the 2D case to one single equation. Then one tries to find a function which satisfies this equation and the boundary conditions.
Recall question 3 from exercise sheet 8, where you wrote the governing equations for plane strain in terms of only the displacement components. Similarly, one can write the governing equations for 2D problems in terms of only the stress components,

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + F_x = 0 \quad \text{(equilibrium)}, \tag{1}$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + F_y = 0, \quad \text{(equilibrium)}, \tag{2}$$

$$\nabla^2 \left(\sigma_{xx} + \sigma_{yy}\right) = -\left(\nu + 1\right)\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y}\right) \quad \text{(compatibility)}. \tag{3}$$

Note that the compatibility equation, expressed here in terms of stresses, is a consequence of Hooke's law.
We consider the case where the field of body forces is conservative, i.e. there is a scalar potential $V(x, y)$ such that

$$F_x = \frac{\partial V}{\partial x}, \tag{4}$$

$$F_y = \frac{\partial V}{\partial y}. \tag{5}$$

The Airy stress function $\phi(x, y)$ is defined by specifying its derivatives

$$\sigma_{xx} + V = \frac{\partial^2 \phi}{\partial y^2}, \tag{6}$$

$$\sigma_{yy} + V = \frac{\partial^2 \phi}{\partial x^2}, \tag{7}$$

$$\sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}. \tag{8}$$

If ϕ exists, then equilibrium is automatically satisfied. You can check this by entering the stress components from equations 6, 7, and 8 into equations 1 and 2. Remember your maths lectures and Schwarz's theorem (symmetry of second derivatives) which allows you to interchange the order of partial derivatives.
The remaining equation which needs to be satisfied — the only governing equation now — is the compatibility equation. It becomes

$$\frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = (1 - \nu)\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}\right) = 0 \tag{9}$$

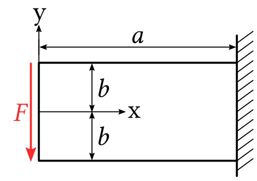
or in short-hand notation

$$\nabla^2\left(\nabla^2 \phi\right) \equiv \nabla^4 \phi = (1 - \nu)\ \nabla^2 V = 0. \tag{10}$$

If there are no body forces, the right hand side is zero.

Question 1 . . . .
Reference: Barber, Elasticity, Springer (2010), p. 59

The beam below is build into the wall at $x = a$ and a force F is applied at $x = 0$. The body force is zero, i.e. $V = 0$.



For the free surfaces of the beam we can derive boundary conditions as it was shown in question 1 of exercises sheet 8. We find the following boundary conditions

$$\sigma_{xy} = 0 \quad \text{at} \quad y = \pm b, \tag{11}$$
$$\sigma_{yy} = 0 \quad \text{at} \quad y = \pm b, \tag{12}$$
$$\sigma_{xx} = 0 \quad \text{at} \quad x = 0, \tag{13}$$
$$\int_{-b}^{b} \sigma_{xy} dy = F \quad \text{at} \quad x = 0. \tag{14}$$

The force boundary condition is applied in a *weak* form, that is we require merely that the *integral* of stress is equal to F. We do not impose it in the *strong* form, where the stress would have to be equal to the traction due to F *pointwise*. The importance of weak form versus strong form is that the Airy stress function is a polynomial with a finite number of terms when the weak form boundary condition is used. Under the strong form, it is an infinite series.

(a) Find the Airy stress function ϕ!
Hints:
- ϕ is a polynomial in x and y
- The bending moment should vary linearly in x. Remember the formula for the bending moment from the lecture and keep track of the coordinate system, which was slightly different orientated in the lecture, to find, $M(x) = \int_{-b}^{+b} y\sigma_{xx}(x, y)\mathrm{d}y$. What does that mean for σ_{xx}? How should σ_{xx} vary with y assuming the Bernoulli assumption?
In combination, what does that mean for $\phi(x, y)$?
- Based on the two hints above, you can find one term of ϕ – some proportionality of the form $\phi \propto Cx^n y^m$, where C is a, up to now unknown, constant and n and m are integers you can determine by the above hints. Use this term as a trial function and see whether it can fulfill the boundary conditions! If not, what other terms would have to be added so that both Eq. (9) and the boundary conditions are fulfilled? Determine the constants, e.g. C in $\phi \propto Cx^n y^m$!

(b) Determine the stress components using $\phi(x, y)$!

(c) Compare these results to those from Chapter 17: Beams - Stresses, slide 14!

Solution: (a) The bending moment is given by

$$M(x) = \int_{-b}^{b} y\sigma_{xx}(x, y)dy. \tag{15}$$

Since $\sigma_{xx} \propto M(x)$ and $M(x) \propto x$ we have $\sigma_{xx} \propto x$. Similarly, from the Bernoulli assumption we know that $\sigma_{xx} \propto y$. Now σ_{xx} is given by the second derivative of the Airy stress function with respect to y. In order to get $\sigma_{xx} \propto xy$, we use the Ansatz

$$\phi(x, y) = Cxy^3. \tag{16}$$

(Note that the beam has a different orientation than what you find in the lecture notes. In this exercise, z is the coordinate in which the plane conditions are fulfilled - this can be implicitly seen from Eqs. (1)-(3) - while we have used the y-direction in the notes.) Since we have no body forces, $V = 0$.

The stress components can be obtained from Eqs. (6)-(8), giving

$$\sigma_{xx} = 6Cxy \tag{17}$$
$$\sigma_{yy} = 0 \tag{18}$$
$$\sigma_{xy} = -3Cy^2. \tag{19}$$

This solution violates the boundary condition Eq. (11). To fix this, we simply add a constant to σ_{xy}. From Eq. (8), we see that this gives a contribution $\propto xy$ to the Airy function, hence

$$\phi(x,y) = C_1 xy^3 + C_2 xy, \tag{20}$$

leading to

$$\sigma_{xx} = 6C_1 xy \tag{21}$$
$$\sigma_{yy} = 0 \tag{22}$$
$$\sigma_{xy} = -3C_1 y^2 - C_2. \tag{23}$$

The constants can be obtained from the boundary conditions, Eq. (11) and (14). (Note that the other two boundary conditions are automatically fulfilled.) This gives

$$0 = \sigma_{xy}(x, y = \pm b) = -3C_1 b^2 - C_2 \tag{24}$$
$$F = \int_{-b}^{b} \sigma_{xy}(x = 0, y)dy = \left[-C_1 y^3 - C_2 y\right]_{-b}^{b} = -2C_1 b^3 - 2C_2 b \tag{25}$$

which can be solved for C_1 and C_2:

$$C_1 = F/4b^3 \tag{26}$$
$$C_2 = -3F/4b \tag{27}$$

The final Airy function is therefore

$$\phi(x,y) = \frac{F}{4b^3}(xy^3 - 3b^2 xy), \tag{28}$$

(b) The Airy function, eq. (28) is leading to the internal stress

$$\sigma_{xx}(x,y) = \frac{3F}{2b^3}xy \tag{29}$$
$$\sigma_{yy}(x,y) = 0 \tag{30}$$
$$\sigma_{xy}(x,y) = \frac{3F}{4b^3}(b^2 - y^2). \tag{31}$$

(c) This is identical to the expression derived in the lecture.
Note that in the lecture, we used $h = 2b$ and the force F was the actual force while here it is a force per length, F/w in the nomenclature of what you find on the slides. Further the beam in the lecture was in the x-z-plane while here it is in the x-y-plane. This leads us to the replacements $F \to F/w$, $b \to h/2$ and $y \to z$ to come from the expressions derived here to arrive at the formulas explained in the lecture.