Chapter 1

Force and moment equilibria

1.1 Statics, rigid bodies and force systems

1.2 Force couples and moments

Chapter 2

Bearings

2.1 Bearings and degrees of freedom

2.2 Free body diagrams

Chapter 3

Internal forces

3.1 Internal force variables

3.2 Distributed loads

Chapter 4

Elastostatic equilibrium

4.1 Mathematical notation

We will make extensive use of vector and tensor calculus in this and the following chapters, and it is important to clearly lay out the mathematical notation used here.

4.1.1 Derivatives

Given a function $f(x, y)$, we write the partial derivative of this function with respect to x as

$$\frac{\partial f}{\partial x} = \partial_x f = f_{,x}. \tag{4.1}$$

All variables following a comma in a subscript are derivatives. The second derivative with respect to x is then

$$\frac{\partial^2 f}{\partial x^2} = \partial_x^2 f = f_{,xx}. \tag{4.2}$$

Mixed derivatives are written as

$$\frac{\partial^2 f}{\partial x \partial y} = \partial_x \partial_y f = f_{,xy}. \tag{4.3}$$

The total derivative is indicated with the letter d, e.g.

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} = f_{,x}x_{,t} + f_{,y}y_{,t} \tag{4.4}$$

for $f = f(x, y)$, $x = x(t)$ and $y = y(t)$.

Sometimes the prime is used to a indicate derivative, e.g. $f'(x) = df/dx$ is the derivative of f. It is common to indicate the derivative with respect to

time by a dot, i.e. given $f(t)$ the derivative $\dot{f}(t) = df/dt$. We will use these notations occasionally for brevity but point out that writing the differential operator explicitly is less ambiguous. In particular, for functions of more than one variable the differential operator allows us to distinguish clearly between total and partial derivatives.

4.1.2 Einstein summation

Einstein summation is an implicit summation over repeated indices. As a simple example, consider the total derivative of a function $f(\vec{r})$ with $\vec{r} = (x, y)$ equivalent to Eq. (4.4). We can write this as

$$\frac{df}{dt} = f_{,x} r_{x,t} + f_{,y} r_{y,t} = \sum_{i=x,y} f_{,i} r_{i,t} = f_{,i} r_{i,t} \tag{4.5}$$

where in the right hand side the sum is implicit because the index i is repeated. We will come back to this notation when discussing explicit examples below.

4.1.3 Integrals

We write the integration variable next to the integral sign,

$$\int \mathrm{d}x\, x^2 = \frac{x^3}{3} + C, \tag{4.6}$$

and the integration is intended to be over the expression that follows integral and integration variable. We will use parenthesis to avoid ambiguities where appropriate.

4.1.4 Vectors & matrices

We use explicit arrows, $\vec{v}$ to indicate vectors and underline matrices, $\underline{M}$. A fourth order tensor is underlined twice, $\underline{\underline{M}}$. Note that we chose this notation over using, e.g., bold font to indicate vectors because it is blackboard friendly. It can be used on blackboards and typeset notes alike.

4.2 Stress and static equilibrium

We will treat elasticity exclusively in the limit of small or infinitesimal strains where all equations are linear. The generalization of this "small strain" theory is "finite strain" elasticity which we will not treat in this class. Note that

linear elasticity is a classical *field theory*, this means all quantities typically depend continuously on positions $\vec{r}$. Those quantities are called *fields*.

The central quantities of small strain elasticity are the (Cauchy) stress field $\underline{\sigma}(\vec{r})$ and the displacement field $\vec{u}(\vec{r})$. Given a material point has moved from position $\vec{r}$ to $\vec{r}'$, the displacement field is $\vec{u}(\vec{r}) = \vec{r}' - \vec{r}$. It therefore describes by how much a volume element in our deformed material has moved (or displaced) because of the deformation. The stress $\underline{\sigma}$ is a tensor that transforms an area (vector) $\vec{A}$ into a force vector $\vec{F}$,

$$\vec{F} = \underline{\sigma} \cdot \vec{A} \tag{4.7}$$

Note here the area is a vectorial quantity; the direction of this area vector points outwards on that area, i.e. $\vec{A} = A\hat{n}$ where $\hat{n}$ is the normal vector on the respective area.

4.2.1 Force equilibrium

We will now consider the equilibrium of forces inside a solid body. Specifically, we regard a small volume element inside this body. Figure 4.1a shows a sketch of some body with a volume element highlighted in red. If side lenghts of the element Δx, Δy and Δz are small enough, then the forces on opposite sites of the element must balance. We here denote the forces on the faces perpendicular to the x-direction by $\vec{X}$, the forces on the y-faces by $\vec{Y}$ and the forces on the z-faces by $\vec{Z}$ (see Fig. 4.1b, z-direction not shown). If we know the areas, we can get the forces from the stress tensor $\underline{\sigma}$ (that converts areas into forces, see Eq. (4.7)), specifically

$$\vec{X} = \begin{pmatrix} \sigma_{xx}\Delta y\Delta z \\ \sigma_{yx}\Delta y\Delta z \\ \sigma_{zx}\Delta y\Delta z \end{pmatrix} \tag{4.8}$$

$$\vec{Y} = \begin{pmatrix} \sigma_{xy}\Delta x\Delta z \\ \sigma_{yy}\Delta x\Delta z \\ \sigma_{zy}\Delta x\Delta z \end{pmatrix} \tag{4.9}$$

$$\vec{Z} = \begin{pmatrix} \sigma_{xz}\Delta x\Delta y \\ \sigma_{yz}\Delta x\Delta y \\ \sigma_{zz}\Delta x\Delta y \end{pmatrix}. \tag{4.10}$$

Note that $\vec{X}$, $\vec{Y}$, $\vec{Z}$ and $\underline{\sigma}$ are fields; they explicitly depend on position $\vec{r}$ within the body.

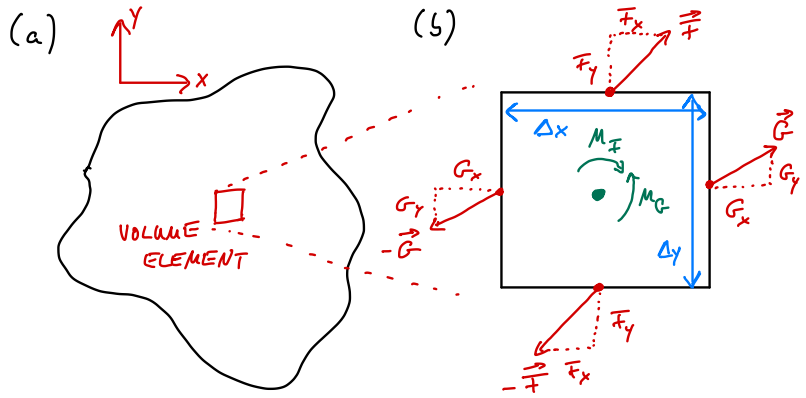

Figure 4.1: Force equilibrium in a small volume element inside a solid body.

For a volume element located at position $\vec{r} = (x, y, z)$, force equilibrium inside the element can be expressed as

$$\vec{X}(x + \Delta x, y, z) - \vec{X}(x, y, z) + \vec{Y}(x, y + \Delta y, z) - \vec{Y}(x, y, z) + \vec{Z}(x, y, z + \Delta z) - \vec{Z}(x, y, z) = \vec{F}(x, y, z) \tag{4.11}$$

where $\vec{F}(x, y, z)$ is an external force, often called the *body force*, acting on the volume element. We can insert Eqs. (4.8), (4.9) and (4.10) and divide by the volume of the element $\Delta x \Delta y \Delta z$ to obtain

$$\frac{\sigma_{xx}(x + \Delta x, y, z) - \sigma_{xx}(x, y, z)}{\Delta x} + \frac{\sigma_{xy}(x, y + \Delta y, z) - \sigma_{xy}(x, y, z)}{\Delta y} + \frac{\sigma_{xz}(x, y, z + \Delta z) - \sigma_{xz}(x, y, z)}{\Delta z} = f_x(x, y, z) \tag{4.12}$$

for the x-component of Eq. (4.11). Here $f_x = F_x / \Delta x \Delta y \Delta z$ is a volume force. In the limit $\Delta x \to 0$ and $\Delta y \to 0$ this becomes

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} = f_x. \tag{4.13}$$

From the y and z-component of Eq. (4.11) we get two more differential

equations,

$$\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} = f_y \tag{4.14}$$

$$\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = f_z. \tag{4.15}$$

These can be summarized to the compact notation

$$\nabla \cdot \underline{\sigma} = \vec{f}. \tag{4.16}$$

Equation (4.16) is the central expression of elastostatics that describes force balance within a solid body.

An alternative derivation of force balance invokes the divergence theorem. We can integrate Eq. (4.16) over a volume element inside this body of volume V and surface area $S(V)$. Using the divergence theorem (sometimes also called Gauss' theorem), we obtain

$$\int_V d^3r \ \nabla \cdot \underline{\sigma} = \int_{S(V)} d^2r \ \underline{\sigma} \cdot \vec{e}_S = \int_{S(V)} d^2r \ d\vec{F} = \int_V d^3r \ \vec{f} \tag{4.17}$$

where $\vec{e}_S$ is the normal vector pointing outwards on $S(V)$. The infinitesimal area vector $\vec{e}_S d^2r$ is hence transformed into an (infinitesimal) force vector $d\vec{F} = \underline{\sigma} \cdot \vec{e}_S \ d^2r$ and integrated over. Eq. (4.17) hence contains a sum over all forces acting on the surface of the volume element V, and these forces must sum to the body force. It is nothing else than a statement of force balance for any volume element within the solid body.

Note: The divergence theorem is an important result of vector analysis. It converts an integral over a volume V into an integral over the surface ∂V of this volume. For a vector field $\vec{f}(\vec{r})$, the divergence theorem states that

$$\int_V \mathrm{d}^3 r \, \nabla \cdot \vec{f}(\vec{r}) = \int_{\partial V} \mathrm{d}^2 r \, \vec{f}(\vec{r}) \cdot \hat{n}(\vec{r}) \tag{4.18}$$

Here, $\hat{n}(\vec{r})$ is the normal vector pointing outward on the edge ∂V of the volume V.

Note: Note that $\nabla \cdot \underline{\sigma} \equiv \mathrm{div} \ \underline{\sigma}$. Sometimes it is useful to make use of Einstein summation, i.e. implicit summation over repeated indices within the same quantity of in products. Examples are: $\nabla \cdot \underline{\sigma} = \partial_i \sigma_{ij} =$

$\sum_i \partial_i \sigma_{ij}, 3\,\sigma_h = \sigma_{kk} = \sum_k \sigma_{kk} = tr\,\underline{\sigma}$, where σ_h is the hydrostatic stress. In the solid mechanics literature, derivatives are often expressed as indices following a comma. For example, the derivative of the function $f(x, y, z)$ with respect to x would be written as $f_{,x}$. In this notation, Eq. (4.16) becomes $\nabla \cdot \underline{\sigma} = \partial_i \sigma_{ij} = \sigma_{ij,i} = 0$. By virtue of the Einstein summation convention we need to sum over the repeated index i in the right hand side expression. Vector/tensor and component notation with Einstein summation will be used intermixed throughout these notes.

Moment equilibrium

Besides equilibrium of forces, we also need to fulfills the equilibrium of moments acting on the volume element. The moment around the z-axis is given by

$$Y_x \Delta y + X_y \Delta x = 0 \tag{4.19}$$

which immediately implies $\sigma_{yx} = \sigma_{xy}$. The moment equilibrium around the x- and y-axes leads to conditions on the other off-diagonal components of $\underline{\sigma}$, $\sigma_{zx} = \sigma_{xz}$ and $\sigma_{zy} = \sigma_{yz}$. By virtue of moment balance, the stress tensor is a *symmetric* tensor, $\sigma_{ij} = \sigma_{ji}$ or $\underline{\sigma} = \underline{\sigma}^T$.

Chapter 5

Stress

5.1 Vectors

5.1.1 Euclidean space

A vector is an object that represents a direction and a magnitude. Geometrically, they are often represented as arrows. In a Cartesian coordinate system (or basis), a vector can be represented by a set of numbers. For example in two dimensions, we denote the components of the vector $\vec{v}$ as the x and y components and typically write the vector in the column-form

$$\vec{v} = v_x\hat{x} + v_y\hat{y} \equiv \begin{pmatrix} v_x \\ v_y \end{pmatrix} \tag{5.1}$$

where v_x and v_y are real numbers that are called the *components* of the vector $\vec{v}$. $\hat{x}$ and $\hat{y}$ are vectors of unit length that point along the x- and y-directions of the coordinates. The length of the vector is given by the 2-norm $|\vec{v}| \equiv \|\vec{v}\|_2 = (v_x^2 + v_y^2)^{1/2}$.

It is important to emphasize that this is a *representation* of the vector and the components $v_i \in \mathbb{R}$ depend on the specific basis $\hat{x}$ and $\hat{y}$. This representation is called a *tensor of order* 1. The order of a tensor is sometimes also called degree or rank. Any component-wise representation, such as the one on the right hand side of Eq. (5.1), implies a basis. Note that in component notation, the basis vectors are

$$\hat{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \hat{y} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{5.2}$$

More on this, including tensors of order higher than 1, will be discussed when we talk about linear transformations below. Note that we will restrict the

discussion here to orthogonal bases where $\hat{x} \cdot \hat{y} = 0$. The discussion below will use two-dimensional examples, but a generalization to more that two dimensions is straightforward

5.1.2 Vector spaces

Formally, a vector is an element of a *vector space*. A vector space V (often also called a *linear space*) is a set of objects (for example the set containing our basis vectors $\hat{x}$ and $\hat{y}$ and linear combinations thereof) along with two operations: Addition (of two vectors) and multiplication (of a vector) with a scalar. These operations again yield a vector, i.e. an element of V. This can be expressed as

- $\vec{u},\ \vec{v} \in V$, then $\vec{u} + \vec{v} \in V$
- $a \in \mathbb{R}$ and $\vec{u} \in V$, then $a \cdot \vec{u} \in V$

and hence Eq. (5.1) yields a vector. In general, we may multiply the vectors by an element of a *algebraic number field* $\mathbb{F}$ rather than $\mathbb{R}$. Then we say V is a vector space over the field $\mathbb{F}$. In this notes (and our lectures) we will always deal with either real ($\mathbb{F} = \mathbb{R}$) or complex ($\mathbb{F} = \mathbb{C}$) numbers.

Recall the concept of an algebraic number field in mathematics. A field is an algebraic structure that is a set along with two operations "+" and "$\cdot$" associating an element with two elements of the set. The operations are required to satisfy the field axioms:

- Associativity of addition and multiplication: $a + (b + c) = (a + b) + c$ and $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.
- Commutativity of addition and multiplication: $a + b = b + a$ and $a \cdot b = b \cdot a$.
- Additive and multiplicative identity: $0 \in \mathbb{F}$ with $a + 0 = a$ and $1 \in \mathbb{F}$ with $1 \cdot a = a$.
- Additive inverses: $\forall a \in \mathbb{F}$ we have an inverse element $i = -a$ with $a + i = a + (-a) = 0$
- Multiplicative inverses: $\forall a \neq 0$ we have an element a^{-1} with $a \cdot a^{-1} = 1$.
- Distributivity of multiplication over addition: $a \cdot (b + c) = a \cdot b + a \cdot c$

Note that in physics a *field* is typically a quantity that depends on position and not an algebraic number field. A scalar field $\phi(\vec{r})$ would be scalar quantity $\phi \in \mathbb{R}$ that depends on a vector (the position) $\vec{r} \in V$. While these two meanings of the term *field* exist, we will always refer to this latter physical meaning in the following.

5.2 Linear transformations

5.2.1 Formal definition

The simplest (and also most important) operation on a vector is a linear transformation. The simplest linear transformation is the multiplication with a scalar $a \in \mathbb{R}$. In terms of the geometric interpretation of a vector, this multiplication would change the length of the vector by a but not its direction.

A general linear transformation can in addition change the direction of the vector and therefore represent for example rotations. Given a vector $\vec{u}$ and a scalar α, a linear transformation $\mathcal{L}$ to a vector $\vec{u}' = \mathcal{L}\vec{u}$ has the properties:

$$\mathcal{L}(\alpha\vec{u}) = \alpha\mathcal{L}\vec{u} \tag{5.3}$$

$$\mathcal{L}(\vec{u} + \vec{v}) = \mathcal{L}\vec{u} + \mathcal{L}\vec{v} \tag{5.4}$$

5.2.2 Representation

We will only deal with linear operations that map between the same vector space V, $\mathcal{L} : V \mapsto V$. (It may map between quantities with different physical units.) Given a component-wise (Cartesian) representation of a vector, Eq. (5.1), a linear transformation can be expressed as a multiplication by a matrix. This is easily by applying Eqs. (5.3) and (5.4) to

$$\vec{w} = \mathcal{L}\vec{v} = \mathcal{L}(v_x\hat{x} + v_y\hat{y}) = v_x\mathcal{L}\hat{x} + v_y\mathcal{L}\hat{y}. \tag{5.5}$$

We can express any element of our vector space as a linear combination of the basis vectors, hence also

$$\mathcal{L}\hat{x} = L_{xx}\hat{x} + L_{yx}\hat{y} \tag{5.6}$$

$$\mathcal{L}\hat{y} = L_{xy}\hat{x} + L_{yy}\hat{y}. \tag{5.7}$$

Application of the linear operation to an arbitrary vector, Eq. (5.5), can therefore be expressed as

$$\vec{w} = \mathcal{L}\vec{v} = (L_{xx}v_x + L_{xy}v_y)\hat{x} + (L_{yx}v_x + L_{yy}v_y)\hat{y} \equiv \underline{L} \cdot \vec{v} \tag{5.8}$$

where $\underline{L} \cdot \vec{v}$ is the multiplication of the matrix $\underline{L}$ with the vector $\vec{v}$. A matrix is therefore a *representation* of a linear operation. Note that from Eq. (5.6) it is straightforward to see that formally we obtain the components of the matrix from

$$L_{ij} = \hat{i} \cdot \mathcal{L}\hat{j}. \tag{5.9}$$

Here the $\cdot$ indicates the two product or contraction of two vectors, as discussed in the next section.

5.2.3 Index and dyadic notation

In component-wise notation we can express Eq. (5.8) as

$$w_i = \sum_{j=x,y} L_{ij} v_j \equiv L_{ij} v_j \tag{5.10}$$

where the last term on the right-hand side uses the *Einstein summation* convention. In this convention, a summation over repeated indices within the same quantity or in products is implicit. This summation is also often called a *contraction* and in dyadic notation it is indicated by a centered dot, e.g. $\vec{w} = \underline{L} \cdot \vec{v}$.

It is straighforward to "convert" from dyadic notation to component-wise or *index notation*. Imagine the i, j component of the resultant matrix of the product

$$[\underline{A} \cdot \underline{B} \cdot \underline{C}]_{ij} = A_{ik} B_{kl} C_{lj}. \tag{5.11}$$

Converting from the dyadic notation to index notation involves identifying the indices of the resultant (first index of the first matrix in the product i and last index of the last matrix in the product j) and introducing repeated indices for the summation. These indices, k and l in the example, always sit next to each other and are the indices that are *contracted*. The advantage of the index notation is that it is unambiguous, but it may hide the physical structure of the underlying operations. In these notes, we will therefore intermix "dyadic" notation as in Eq. (5.8) and index notation as appropriate.

Note that there is also a double contraction, for example

$$\sigma_{ij} = C_{ijkl} \varepsilon_{kl}, \tag{5.12}$$

that in dyadic notation we would indicate with two dots: $\underline{\sigma} = \underline{\underline{C}} : \underline{\varepsilon}$.

5.3 Rotating the stress tensor

5.3.1 Vectors

A rotation $\mathcal{R}$ is a linear operation that does not affect the length (or norm) of the vector, $|\mathcal{R}\vec{x}| = |\vec{x}|$. Note that if $\mathcal{R}$ is represented by a matrix $\underline{R}$, then

$$|\underline{R} \cdot \vec{x}| = (R_{ij} x_j R_{ik} x_k)^{1/2} = (x_j R^T_{ji} R_{ik} x_k)^{1/2} = \left(\vec{x} \cdot \underline{R}^T \cdot \underline{R} \cdot \vec{x}\right)^{1/2} \tag{5.13}$$

which is is equal to $|\vec{x}|$ only if $\underline{R}^T \cdot \underline{R} = \underline{1}$, i.e. if $\underline{R}$ is an orthogonal matrix. Here the superscript ^{T} indicates the transpose operation, $R^T_{ij} = R_{ji}$, and $\underline{1}$ is the unit matrix. Note that this relationship implies $\underline{R}^{-1} = \underline{R}^T$, i.e. the inverse of an orthogonal matrix is simply its transpose.

Let us assume we want to rotate from a coordinate system (basis vectors) $\hat{x}$ and $\hat{y}$ to the primed coordinate system $\hat{x}'$ and $\hat{y}'$. Then the rotation should transform the basis vectors of the unprimed into the primed system,

$$\mathcal{R}\hat{x} = \hat{x}' \quad \text{and} \quad \mathcal{R}\hat{y} = \hat{y}'. \tag{5.14}$$

From Eq. (5.9) we obtain the components of the rotation matrix as

$$\underline{R} = \begin{pmatrix} R_{xx} & R_{xy} \\ R_{yx} & R_{yy} \end{pmatrix} = \begin{pmatrix} \hat{x} \cdot \mathcal{R}\hat{x} & \hat{x} \cdot \mathcal{R}\hat{y} \\ \hat{y} \cdot \mathcal{R}\hat{x} & \hat{y} \cdot \mathcal{R}\hat{y} \end{pmatrix} = \begin{pmatrix} \hat{x} \cdot \hat{x}' & \hat{x} \cdot \hat{y}' \\ \hat{y} \cdot \hat{x}' & \hat{y} \cdot \hat{y}' \end{pmatrix}. \tag{5.15}$$

If all vectors are expressed in the coordinate system $\hat{x}$ and $\hat{y}$, then those vectors have the simple representation given by Eq. (5.1) and the rotation matrix can be written as

$$\underline{R} = (\hat{x}', \hat{y}'), \tag{5.16}$$

i.e. we just stack the basis vectors as columns together to obtain $\underline{R}$. This is a simple prescription to construct the rotation matrix given the basis vectors of the rotated coordinate system (in terms of the original coordinate system). It is straightforward to see that $\underline{R}$ is an orthogonal matrix,

$$\underline{R}^T \cdot \underline{R} = \begin{pmatrix} \hat{x}' \cdot \hat{x}' & \hat{x}' \cdot \hat{y}' \\ \hat{y}' \cdot \hat{x}' & \hat{y}' \cdot \hat{y}' \end{pmatrix} = \underline{1}, \tag{5.17}$$

if $\hat{x}'$ and $\hat{y}'$ are orthogonal vectors of unit length.

The basis vectors of a coordinate system rotated counterclockwise by an angle θ are given by

$$\hat{x}' = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \quad \text{and} \quad \hat{y}' = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \tag{5.18}$$

and hence the rotation matrix is

$$\underline{R} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{5.19}$$

It is straightforward to verify that indeed $\underline{R}^T \cdot \underline{R} = \underline{1}$

5.3.2 Tensors

A tensor is a representation of a linear transformation: A tensor of order 2 (a matrix) transforms a tensor of order 1 (a vector) into another tensor of order 1. As an example, let us discuss the Cauchy stress tensor $\underline{\sigma}$ of solid mechanics. It transforms an area vector $\vec{A}$ into a force vector $\vec{F}$,

$$\vec{F} = \underline{\sigma} \cdot \vec{A} \quad \text{or in index notation} \quad F_i = \sigma_{ij} A_j. \tag{5.20}$$

We describe $\vec{F}$, $\vec{A}$ and $\underline{\sigma}$ in terms of their components, i.e. we describe a realization of the linear transformation (and the area and force vectors).

A rotation does not affect the effect of the linear transformation. Hence, if we change the coordinate system from $\hat{x}$, $\hat{y}$ to $\hat{x}'$, $\hat{y}'$, then the component of our vector $\vec{F}$ change to

$$F'_x = \vec{F} \cdot \hat{x}' = F_x \hat{x} \cdot \hat{x}' + F_y \hat{y} \cdot \hat{x}' \tag{5.21}$$

$$F'_y = \vec{F} \cdot \hat{y}' = F_y \hat{x} \cdot \hat{y}' + F_y \hat{y} \cdot \hat{y}'. \tag{5.22}$$

Comparing with Eq. (5.19) yields the compact notation

$$\vec{F}' = \underline{R}^T \cdot \vec{F} \quad \text{or in index notation} \quad F'_i = F_j R_{ji}. \tag{5.23}$$

Note that the transpose shows up because $\underline{R}$ describes the rotation of the basis and we are here rotating a vector expressed within this basis, which is the inverse operation.

Since we now understand how to rotate vectors, we can ask the question of how to rotate a tensor of order 2. Starting from Eq. (5.20) and using the inverse of Eq. (5.23), we write

$$\underline{R} \cdot \vec{F}' = \underline{\sigma} \cdot \left(\underline{R} \cdot \vec{A}' \right), \tag{5.24}$$

and multiply by $\underline{R}^T$ from the left to yield

$$\vec{F}' = \left(\underline{R}^T \cdot \underline{\sigma} \cdot \underline{R} \right) \cdot \vec{A}'. \tag{5.25}$$

In the rotated coordinate system, the tensor attains the representation

$$\underline{\sigma}' = \underline{R}^T \cdot \underline{\sigma} \cdot \underline{R} \quad \text{or in index notation} \quad \sigma'_{ij} = \sigma_{kl} R_{ki} R_{lj} \tag{5.26}$$

since this leave the expression for the linear transformation

$$\vec{F}' = \underline{\sigma}' \cdot \vec{A}'. \tag{5.27}$$

invariant. The trace and determinant of the rotated tensor are

$$\operatorname{tr} \underline{\sigma}' = \operatorname{tr} \left(\underline{R}^T \cdot \underline{\sigma} \cdot \underline{R} \right) = \operatorname{tr} \left(\underline{R} \cdot \underline{R}^T \cdot \underline{\sigma} \right) = \operatorname{tr} \underline{\sigma} \tag{5.28}$$

$$\det \underline{\sigma}' = \det \left(\underline{R}^T \cdot \underline{\sigma} \cdot \underline{R} \right) = \det \left(\underline{R} \cdot \underline{R}^T \cdot \underline{\sigma} \right) = \det \underline{\sigma} \tag{5.29}$$

and hence invariant under rotation. Note that in general for an $n \times n$ tensor, there are n invariants; more on this will be discussed below when talking about eigenvalues.

Since we now understand how to rotate tensors of order 2, we can ask the question how to rotate a tensor of order 4. As an example, we use the stiffness tensor $\underline{\underline{C}}$ that transforms a strain tensor $\underline{\varepsilon}$ into a stress tensor $\underline{\sigma}$,

$$\underline{\sigma} = \underline{\underline{C}} : \underline{\varepsilon} \quad \text{or in index notation} \quad \sigma_{ij} = C_{ijkl} \varepsilon_{kl}. \tag{5.30}$$

We can rewrite this using the inverse of Eq. (5.26) as

$$\underline{R} \cdot \underline{\sigma}' \cdot \underline{R}^T = \underline{\underline{C}} : \left(\underline{R} \cdot \underline{\varepsilon}' \cdot \underline{R}^T \right) \quad \text{or} \quad R_{ik} R_{jl} \sigma'_{kl} = C_{ijkl} R_{km} R_{ln} \varepsilon'_{mn}, \tag{5.31}$$

which we now multiply from the left with $\underline{R}^T$ and from the right with $\underline{R}$. This gives

$$\sigma'_{ij} = C_{mnop} R_{mi} R_{nj} R_{ok} R_{pl} \varepsilon'_{kl}, \tag{5.32}$$

and hence the transformation rule

$$C'_{ijkl} = C_{mnop} R_{mi} R_{nj} R_{ok} R_{pl}. \tag{5.33}$$

Quantities that transform as Eqs. (5.23), (5.26) and (5.33) are called *tensors*.

5.4 Principal stresses

5.5 Stress invariants

2×2 matrix

Let us discuss in more detail what happens if we rotate a symmetric tensor of order 2, i.e. a tensor that fulfills $\underline{\sigma}^T = \underline{\sigma}$. From Eq. (5.26) it is straightforward to see, that $\underline{\sigma}'^T = \underline{\sigma}'$, i.e. the transformed tensor is also symmetric.

We now explicitly write the rotation for the tensor

$$\underline{\sigma} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{pmatrix}, \tag{5.34}$$

using the rotation matrix Eq. (5.19). This gives symmetric $\underline{\sigma}' = \underline{R}^T \cdot \underline{\sigma} \cdot \underline{R}$ with the components

$$\sigma'_{xx} = \frac{1}{2}(\sigma_{xx} + \sigma_{yy}) + \frac{1}{2}(\sigma_{xx} - \sigma_{yy})\cos 2\theta + \sigma_{xy} \sin 2\theta \tag{5.35}$$

$$\sigma'_{yy} = \frac{1}{2}(\sigma_{xx} + \sigma_{yy}) - \frac{1}{2}(\sigma_{xx} - \sigma_{yy})\cos 2\theta - \sigma_{xy} \sin 2\theta \tag{5.36}$$

$$\sigma'_{xy} = -\frac{1}{2}(\sigma_{xx} - \sigma_{yy})\sin 2\theta + \sigma_{xy} \cos 2\theta. \tag{5.37}$$

There is a specific rotation angle θ_0 where the diagonal elements σ'_{xx} and σ'_{yy} become extremal. It is determined from $\sigma'_{xx,\theta} = \sigma'_{yy,\theta} = 0$,

$$\tan 2\theta_0 = \frac{2\sigma_{xy}}{\sigma_{xx} - \sigma_{yy}}. \tag{5.38}$$

At this rotation angle we find that the off-diagonal components vanish, $\sigma'_{xy}(\theta_0) = 0$, and the rotated matrix is diagonal,

$$\underline{\sigma} = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}, \tag{5.39}$$

with diagonal elements

$$\sigma_1 = \frac{\sigma_{xx} + \sigma_{yy}}{2} + \left[\left(\frac{\sigma_{xx} - \sigma_{yy}}{2} \right)^2 + \sigma_{xy}^2 \right]^{1/2} \tag{5.40}$$

$$\sigma_2 = \frac{\sigma_{xx} + \sigma_{yy}}{2} - \left[\left(\frac{\sigma_{xx} - \sigma_{yy}}{2} \right)^2 + \sigma_{xy}^2 \right]^{1/2}. \tag{5.41}$$

This is the simplest example of the diagonalization of a matrix.

The diagonalization of a symmetric matrix always corresponds to the rotation into a new coordinate system. We have explictly shown this for the two-dimensional case here and will now show it in more generality for the three-dimensional case.

5.5.1 3×3 matrix

Eq. (5.39) fulfills the eigenvalue equations $\underline{\sigma} \cdot \hat{x} = \sigma_1 \hat{x}$ and $\underline{\sigma} \cdot \hat{y} = \sigma_2 \hat{y}$. Rather than explicitly computing a rotation, we can ask the question whether we can find a scalar λ and a vector $\vec{v}$ that fulfills the eigenvalue equation

$$\underline{\sigma} \cdot \vec{v} = \lambda \vec{v}. \tag{5.42}$$

This equation of course has the trivial solution $\vec{v} = 0$. It can only have a nontrivial solution if

$$\det\left(\underline{\sigma} - \lambda \underline{1}\right) = 0. \tag{5.43}$$

For a $n \times n$ matrix, Eq. (5.43) leads to a polynomial of order n in λ with n (possibly complex valued) solutions.

For the case of a symmetric 3×3 matrix, we can write this down explicitly as

$$\det \begin{pmatrix} \sigma_{xx} - \lambda & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} - \lambda & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} - \lambda \end{pmatrix} = -\lambda^3 + I_1 \lambda^2 - I_2 \lambda + I_3 = 0 \tag{5.44}$$

with

$$I_1 = \operatorname{tr} \underline{\sigma} = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} \tag{5.45}$$

$$I_2 = \sigma_{yy}\sigma_{zz} + \sigma_{xx}\sigma_{zz} + \sigma_{xx}\sigma_{yy} - \sigma_{yz}^2 - \sigma_{xz}^2 - \sigma xy^2 \tag{5.46}$$

$$I_3 = \det \underline{\sigma} \tag{5.47}$$

The quantities I_1 to I_3 are called *invariants*. We have shown above explicitly that the trace and the determinat is invariant under rotation. The same holds true for all coefficients of the characteristic polynomial. This is because

$$\det\left(\underline{R}^T \cdot \underline{\sigma} \cdot \underline{R} - \lambda \underline{1}\right) = \det\left[\underline{R}^T \cdot \left(\underline{\sigma} - \lambda \underline{1}\right) \cdot \underline{R}\right] = \det\left(\underline{\sigma} - \lambda \underline{1}\right). \tag{5.48}$$

The 3-dimensional tensor therefore has three invariants. These invariants have important physical interpretation. For the stress tensor, I_1 is related to the hydrostatic stress and I_2 to the shear stress.

Note that for a diagonal matrix,

$$\underline{\sigma}' = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix}, \tag{5.49}$$

the invariants are

$$I_1 = \sigma_1 + \sigma_2 + \sigma_3 \tag{5.50}$$
$$I_2 = \sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_2\sigma_3 \tag{5.51}$$
$$I_3 = \sigma_1\sigma_2\sigma_3 \tag{5.52}$$

By equating Eqs. (5.45) to (5.47) with Eqs. (5.50) to (5.52) we can calculate the eigenvalues σ_1, σ_2 and σ_3.

Once we have computed the eigenvalues, we can obtain the corresponding eigenvectors by solving

$$\underline{\sigma}\vec{v}_1 = \sigma_1\vec{v}_1, \quad \underline{\sigma}\vec{v}_2 = \sigma_2\vec{v}_2, \quad \text{and} \quad \underline{\sigma}\vec{v}_3 = \sigma_3\vec{v}_3. \tag{5.53}$$

Note that the expressions only determine the direction of $\vec{v}_i$, not its length, and we are free to require $|\vec{v}_i| = 1$. Furthermore, let us regard scalar product $\vec{v}_1 \cdot \vec{v}_2$, then

$$\sigma_1\vec{v}_1 \cdot \vec{v}_2 = (\underline{\sigma} \cdot \vec{v}_1) \cdot \vec{v}_2 = \vec{v}_1 \cdot (\underline{\sigma}^T \cdot \vec{v}_2) = \vec{v}_1 \cdot (\underline{\sigma} \cdot \vec{v}_2) = \sigma_2\vec{v}_1 \cdot \vec{v}_2 \tag{5.54}$$

and if $\sigma_1 \neq \sigma_2$ we must have $\vec{v}_1 \cdot \vec{v}_2 = 0$. Hence the eigenvectors of a symmetric matrix are orthonormal, or in other words, they form the basis of a coordinate system.

We can write Eq. (5.53) in the more compact notation

$$\underline{\sigma} \cdot \underline{R} = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix} \cdot \underline{R} \tag{5.55}$$

with $\underline{R} = (\vec{v}_1, \vec{v}_2, \vec{v}_3)$. Multiplying from the left with $\underline{R}^{-1} = \underline{R}^T$ (this holds because the eigenvectors are orthonormal) we get

$$\underline{R}^T \cdot \underline{\sigma} \cdot \underline{R} = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix}. \tag{5.56}$$

This is nothing else than a coordinate transformation (rotation) of the tensor $\underline{\sigma}$. Since the diagonalization of a symmetric matrix leads to orthonormal eigenvectors, the diagonalization is a rotation of the coordinate system.

Chapter 6

Strain and displacement

6.1 Strain

6.2 Displacement

The displacement field $\vec{u}(\vec{r})$ describes how a point on our solid body moves during deformation. For a rigid body $\vec{u} \equiv 0$, but for a deformable body we obtain finite displacements during deformation. The point $\vec{r}$ then moves to the point $\vec{r}'(\vec{r}) = \vec{r} + \vec{u}(\vec{r})$. The displacement field $\vec{u}(\vec{r}) = \vec{r} - \vec{r}'(\vec{r})$ is hence the difference of the deformed ("displaced") point to its reference position $\vec{r}$. Note that the displacement field itself is defined as a function of this reference position $\vec{r}$.

6.3 Strain

The strain field in the small strain approximation is given by the symmetrized gradient of $\vec{u}(\vec{r})$,

$$\underline{\varepsilon}(\vec{r}) = \frac{1}{2}\left(\nabla\vec{u} + (\nabla\vec{u})^T\right). \tag{6.1}$$

The left hand side of Eq. (6.1) contains the gradient of a vector field, $\nabla\vec{u}$, which is a second rank tensor,

$$\nabla\vec{u} = \begin{pmatrix} \frac{\partial u_x}{\partial x} & \frac{\partial u_x}{\partial y} & \frac{\partial u_x}{\partial z} \\ \frac{\partial u_y}{\partial x} & \frac{\partial u_y}{\partial y} & \frac{\partial u_y}{\partial z} \\ \frac{\partial u_z}{\partial x} & \frac{\partial u_z}{\partial y} & \frac{\partial u_z}{\partial z} \end{pmatrix}, \tag{6.2}$$

whose components are given by $[\nabla\vec{u}]_{ij} = \partial u_i/\partial r_j = u_{i,j}$. It is *not* the divergence, $\nabla \cdot \vec{u}$ which would give a scalar. This potential source of confusion

can be avoided by writing the equation in the component-wise notation,

$$\varepsilon_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i) = \frac{1}{2}(u_{j,i} + u_{i,j}). \tag{6.3}$$

(Note that these expression do not contain a sum since there is no repeated index.)

The geometric interpretation of the strain is that is converts a vector $\vec{r}$ that has a direction and length into the change this vector undergoes under deformation: $\vec{u} = \underline{\varepsilon} \cdot \vec{r}$ with the new (transformed) vector $\vec{r}' = \vec{r} + \vec{u} = (1 + \underline{\varepsilon}) \cdot \vec{r}$. In terms of thinking about tensors as a representation of a linear transformation, the strain tensor transforms a position into a displacement. The strain tensor is dimensionless.

Chapter 7

Hooke's law

7.1 General form

Finally, we need a constitutive relation (material law) to close the equations of elastostatic equilibrium,

$$\nabla \cdot \underline{\sigma} = 0 \quad \text{and} \quad \underline{\varepsilon}(\vec{r}) = \frac{1}{2}\left(\nabla \vec{u} + (\nabla \vec{u})^T\right). \tag{7.1}$$

Since we will be working in linear elasticity, the constitutive equation is a linear relationship between $\underline{\sigma}$ and $\underline{\varepsilon}$. The most general form of this linear relationship is

$$\underline{\sigma} = \underline{\underline{C}} : \underline{\varepsilon}, \text{or using Einstein summation } \sigma_{ij} = C_{ijkl}\varepsilon_{kl}. \tag{7.2}$$

It is called *Hooke's law*. The quantity $\underline{\underline{C}}$ is a fourth order symmetric tensor of elastic constants that contains at most 21 independent elastic moduli. To see that there are only 21 independent coefficients, it is useful to remove the symmetric entries from $\underline{\sigma}$ and $\underline{\varepsilon}$ and express them as 6-vectors in what is called *Voigt notation*,

$$\vec{\sigma} = (\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{yz}, \sigma_{xz}, \sigma_{xy}) \tag{7.3}$$

and

$$\vec{\varepsilon} = (\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}, 2\varepsilon_{yz}, 2\varepsilon_{xz}, 2\varepsilon_{xy}). \tag{7.4}$$

Then $\sigma = \underline{C} \cdot \vec{\varepsilon}$ where $\underline{C}$ is a 6×6 symmetric matrix called the *stiffness matrix* containing the above-mentioned 21 independent elastic constants. (There are $6 \cdot 6 = 36$ components, but the matrix is symmetric.)

Note that the off-diagonal components of $\underline{\sigma}$ are often denoted by $\tau_{xy} \equiv \sigma_{xy}, \tau_{xz} \equiv \sigma_{xz}$ and $\tau_{yz} \equiv \sigma_{yz}$. The off-diagonal components of $\underline{\varepsilon}$ are often

denoted by $\gamma_{xy} \equiv 2\varepsilon_{xy}, \gamma_{xz} \equiv 2\varepsilon_{xz}$ and $\gamma_{yz} \equiv 2\varepsilon_{yz}$ and absorb the factor of 2 that occurs in Eq. (7.4). Voigt notation then becomes

$$\vec{\sigma} = (\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \tau_{yz}, \tau_{xz}, \tau_{xy}) \tag{7.5}$$

and

$$\vec{\varepsilon} = (\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}, \gamma_{yz}, \gamma_{xz}, \gamma_{xy}). \tag{7.6}$$

Note: It is important to keep in mind that the γ's contain a factor 2 but the τ's do not. The factor of 2 ensures that $\vec{\sigma} = \underline{C} \cdot \vec{\varepsilon}$ and $\underline{\sigma} = \underline{\underline{C}} : \underline{\varepsilon}$ are the same constitutive law.

7.2 Isotropic solids

For isotropic elasticity, the total 21 independent elastic constants reduce to two. The constitutive equation for isotropic elasticity is

$$\sigma_{ij} = \lambda \delta_{ij} \varepsilon_{kk} + 2\mu \varepsilon_{ij} \tag{7.7}$$

or its inverse

$$\varepsilon_{ij} = \frac{1}{2G}\sigma_{ij} - \frac{\nu}{E}\delta_{ij}\delta_{kk} = \frac{1}{E}[(1+\nu)\sigma_{ij} - \nu\delta_{ij}\sigma_{kk}], \tag{7.8}$$

where δ_{ij} is the Kronecker-Delta. These expressions have been conveniently written in their most simple form. The constants that show up in Eqs. (7.7) and (7.8) are the shear modulus μ, Lamé's first constant λ, Young's modulus E and Poisson number ν. Both λ and ν are often called Lamé's constants. Note that $\sigma_{kk} = 3\sigma_h$ (Einstein summation!) where $3\sigma_h$ is the hydrostatic stress.

The four moduli are not independent (only two are), and the following expressions relate the pairs λ, μ and E, ν to each other:

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \tag{7.9}$$

$$\mu = \frac{E}{2(1+\nu)} \tag{7.10}$$

$$\lambda + \mu = \frac{E}{2(1+\nu)(1-2\nu)} \tag{7.11}$$

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu} \tag{7.12}$$

$$\nu = \frac{\lambda}{2(\lambda + \mu)} \tag{7.13}$$

Note that the volumetric strain $\varepsilon_h = \frac{1}{3}\,\mathrm{tr}\,\underline{\varepsilon} = \frac{1}{E}[(1+\nu)\sigma_h - 3\,\nu\sigma_h] = \frac{1}{E}(1-2\nu)\sigma_h$ vanishes at $\nu = 1/2$. In this case, $\sigma_{ij} = 2\sigma_h\varepsilon_{ij}$ because the $\varepsilon_h = \varepsilon_{kk}$ must vanish. Note that another common symbol for the shear modulus μ is the Latin letter G.

We can also write down a free energy functional (often also called a hyperelastic energy density), which is quadratic in the strain $\underline{\varepsilon}$,

$$W = \frac{1}{2}\lambda\varepsilon_{ii}^2 + \mu\varepsilon_{ij}^2 \tag{7.14}$$

Using $\sigma_{ij} = \partial W/\partial\varepsilon_{ij}$ recovers the above constitutive expression Eq. (7.7). From the free energy functional we see that any isotropic material must have $\lambda > 0$ and $\mu > 0$, otherwise the energy could be made arbitrarily small by increasing the deformation of the solid. This limits the Poisson number to the range $-1 < \nu < 1/2$. Note that $\nu < 0$ is typically only achieved for architectured materials such as foams or metamaterials.

7.3 Example: Auxetic materials

Chapter 8

Plane problems

Plane problems are problems where the system has a symmetry in a certain direction. We will here use the y-direction as the direction in which the plane conditions hold. This symmetry implies that the relevant quantities do not vary in z-direction. Note that throughout this text, we will switch this direction.

8.1 Plane strain

For a plane strain situation, the system cannot elongate or shrink in that direction and hence $\varepsilon_{yy} = 0$. From Eq. (7.8) we see that

$$\varepsilon_{yy} = \frac{1+\nu}{E}\sigma_{yy} - \frac{\nu}{E}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}) = 0 \tag{8.1}$$

and hence

$$\sigma_{yy} = \nu(\sigma_{xx} + \sigma_{zz}) \tag{8.2}$$

With the two relations for ε_{yy} and σ_{yy} we can express Hooke's law as

$$\varepsilon_{xx} = \frac{1-\nu^2}{E}\sigma_{xx} - \frac{\nu(1+\nu)}{E}\sigma_{zz} \tag{8.3}$$

$$\varepsilon_{zz} = -\frac{\nu(1+\nu)}{E}\sigma_{xx} + \frac{1-\nu^2}{E}\sigma_{zz} \tag{8.4}$$

and its inverse

$$\sigma_{xx} = (\lambda + 2\nu)\varepsilon_{xx} + \lambda\varepsilon_{zz} \tag{8.5}$$

$$\sigma_{zz} = \lambda\varepsilon_{xx} + (\lambda + 2\mu)\varepsilon_{zz} \tag{8.6}$$

Note that the condition for elastic equilibrium Eq. (**??**) becomes (in Cartesian coordinates)

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xz}}{\partial z} = 0 \tag{8.7}$$

$$\frac{\partial \sigma_{zz}}{\partial z} + \frac{\partial \sigma_{xz}}{\partial x} = 0 \tag{8.8}$$

8.2 Plane stress conditions

For plane stress we consider a situation with $\sigma_{yy} = 0$, i.e. there is no stress in the y-direction. This is a good approximation for example for a thin plate. In this limit, Hooke's law becomes

$$\varepsilon_{xx} = \frac{1}{E}(\sigma_{xx} - \nu\sigma_{zz}) \tag{8.9}$$

$$\varepsilon_{zz} = \frac{}{E}(-\nu\sigma_{xx} + \sigma_{zz}) \tag{8.10}$$

and its inverse

$$\sigma_{xx} = \frac{E}{1-\nu^2}(\varepsilon_{xx} + \nu\varepsilon_{zz}) \tag{8.11}$$

$$\sigma_{zz} = \frac{E}{1-\nu^2}(\nu\varepsilon_{xx} + \varepsilon_{zz}). \tag{8.12}$$

Note that plane strain and plane stress are described by the same set of differential equation but with different elastic moduli. For example, we can convert the plane stress Eqs. (8.9)-(8.12) to the corresponding plane strain equation by substituting the elastic constants,

$$E \to \frac{E}{1-\nu^2} \quad \text{and} \quad \nu \to \frac{\nu}{1-\nu}. \tag{8.13}$$

Hence any plane stress solution can be converted into a plane strain solution using this substitution. In the following, we will continue to work with the plane stress expression (because they are simpler), but all results carry over to plane strain with this substitution.

8.3 Compatibility condition

Another condition to be fulfilled is the *compatibility condition*. For plane problems, the compatibility conditions is the single equation:

$$\frac{\partial^2 \varepsilon_{xx}}{\partial z^2} + \frac{\partial^2 \varepsilon_{zz}}{\partial x^2} = 2\,\frac{\partial^2 \varepsilon_{xz}}{\partial x \partial z} \tag{8.14}$$

That it must hold is easily seen by expressing the strain in Eq. (8.14) in terms of the displacements $\vec{u}$ using Eq. (6.1). It is therefore a consequence of the fact that the strain field is the gradient of the displacement field. It is similar to the well-known condition that the curl of a gradient has to disappear.

The compatibility condition has a simple geometric explanation. Imaging a jigsaw puzzle that you deform in its assembled state. Even in the deformed state, all pieces still have to fit together. They deformation of neighboring pieces can therefore not be independent of each other.

Chapter 9

Beams

9.1 Stresses

Assume a rectangular beam in plane stress or plane strain subject to bending. (The "plane" direction is the y-direction.) The surface normal of the beam is oriented in z-direction. Bending will give rise to internal stresses inside of the beam. We will require that these stresses comply with the external shear force $Q(x)$ and bending moment $M(x)$ in the *weak* or integral sense,

$$Q(x) = \int_A dydz\, \tau_{xz}(x,z) \tag{9.1}$$

$$M(x) = \int_A dydz\, z\sigma_{xx}(x,z). \tag{9.2}$$

Because of the plane state, integration in y-direction will only yield a constant factor, the width t of the beam.

We can derive a condition equivalent to static equilibrium, $\sigma_{ij,j} = 0$, for the weak quantities defined in Eqs. (9.1) and (9.2). Taking the derivative of $Q(x)$ yields

$$Q_{,x} = \int_A dydz\, \tau_{xz,x} = -\int_A dydz\, \sigma_{zz,z} = -t\left[\sigma_{zz}(x,h/2) - \sigma_{zz}(x,-h/2)\right]. \tag{9.3}$$

We call the quantity $q(x) \equiv t\left[\sigma_{zz}(x,h/2) - \sigma_{zz}(x,-h/2)\right]$ the line load of the beam. Next, we take the derivative of $M(x)$,

$$M_{,x} = \int_A dydz\, z\sigma_{xx,x} = -\int_A dydz\, z\tau xz, z = -t\left[z\tau_{xz}\right]_{-h/2}^{h/2} + \int_A dydz\, z\tau_{xz}. \tag{9.4}$$

The first term on the right hand side vanishes because the surfaces are traction free, $\tau_{xz} = 0$ at $z = h/2$ and $z = -h/2$. This yields

$$Q_{,x} = -q(x) \tag{9.5}$$
$$M_{,x} = Q(x) \tag{9.6}$$

for the weak form of the equilibrium conditions.

We now *assume* that the stress is a linear function of the position z perpendicular to the beam axis,

$$\sigma_{xx}(x, z) = C(x)z. \tag{9.7}$$

In what follow we show that this is a good assumption, i.e. we can fulfill force and moment equilibrium and the resulting strains fulfill the compatibility conditions.

Note: The theory derived in this chapter is commonly referred to as the *Euler-Bernoulli beam theory*. The starting point of this theory is typically not Eq. (9.7), but the assumption that each cross section will remain plane and undergo small rotations during deformation. These are sometimes called the *Bernoulli assumptions*. They implies that the strain $\varepsilon_{xx} \propto z$ rather than the stress. It is often argued that $\sigma_{xx} = E\varepsilon_{xx}$ but this of course ignores the other components of the strain tensor. As will be seen below, the Bernoulli assumptions are actually wrong but assuming a linear stress profile leads to the correct small strain expression for the deformation of a beam.

It is straightforward to compute the bending moment,

$$M(x) = \int_A dydz\, \sigma_{xx}(x, z)z = C(x) \int_A dydz\, z^2 = C(x)I_y, \tag{9.8}$$

where A is the cross-section of the beam and I_y is the *axial moment of inertia*. For a rectangular beam of height h and width t, $I_y = h^3t/12$. The moment of inertia is a geometric factor and depends on the shape of the cross-section of the beam.

We can rewrite Eq. (9.7) as

$$\sigma_{xx}(x, z) = \frac{M(x)}{I_y}z. \tag{9.9}$$

Note that an additional longitudinal force L will simple be an additive contribution to Eq. (9.9), $\sigma_{xx}(x, z) = M(x)/I_y z + L/A$.

We can now use the condition for static equilibrium to compute the full stress tensor $\underline{\sigma}$. From $\sigma_{xx,x} + \tau_{xz,z} = 0$ we obtain

$$\tau_{xz,z} = -\frac{z}{I_y} M_{,x} = -\frac{z}{I_y} Q(x). \tag{9.10}$$

We can integrate this across the height h of the beam, keeping in mind that $\tau_{xz} = 0$ at a traction-free surface, to yield

$$\tau_{xz}(x,z) = \frac{Q(x)}{2I_y}\left(\frac{h^2}{4} - z^2\right). \tag{9.11}$$

Next, we use $\sigma_{zz,z} + \tau_{xz,x} = 0$ to obtain

$$\sigma_{zz,z} = -\frac{1}{2I_y} Q_{,x}\left(\frac{h^2}{4} - z^2\right) = \frac{q(x)}{2I_y}\left(\frac{h^2}{4} - z^2\right). \tag{9.12}$$

We need to integrate this equation again, but now $\sigma_{zz} \neq 0$ at the surface since the beam is subject to a line load $q(x)$. Rather, we need the condition that the loads on top and bottom surface of the beam balance, $\sigma_{zz}(x, h/2) = -\sigma_{zz}(x, -h/2)$. This gives

$$\sigma_{zz}(x,z) = \frac{q(x)}{2I_y}\left(\frac{h^2}{4} - \frac{z^2}{3}\right) z \tag{9.13}$$

with

$$\sigma_{zz}(x, h/2) = \frac{q(x)h^3}{24I_y} = \frac{q(x)}{2t} \tag{9.14}$$

where t is the width of the beam.

Note that in this derivation, we have required that the stress gives rise to a certain bending moment through Eq. (9.8). Since we do not prescribe the specific stress state $\sigma_{xx}(x,z)$ but only its integral, this is a *weak* condition. Similarly, integration of Eq. (9.11) in accordance with Eq. (9.1) gives

$$\int_{-t/2}^{t/2} dy \int_{-h/2}^{h/2} dz\, \tau_{xz}(x,z) = Q(x), \tag{9.15}$$

i.e. the force acting on the cross-section at position x along the beam. Again, this condition is fulfilled in the integral, i.e. in the weak sense.

9.2 Displacements

Now that the know the stress inside the beam, we can compute the displacement and thereby the deformation of the beam from Hooke's law. Starting from Hooke's law,

$$\varepsilon_{xx} \equiv u_{x,x} = \sigma_{xx}/E - \nu\sigma_{zz}/E \tag{9.16}$$

$$2\varepsilon_{xz} \equiv u_{x,z} + u_{z,x} = 2(1+\nu)\tau_{xz}/E, \tag{9.17}$$

and taking the derivative of Eq. (9.17) with respect to x, we obtain

$$u_{x,xz} + u_{z,xx} = 2(1+\nu)\tau_{xz,x}/E \tag{9.18}$$

and

$$u_{x,xz} + u_{z,xx} = \partial_z(u_{x,x}) + u_{z,xx} = u_{z,xx} + (\sigma_{xx,z} - \nu\sigma_{zz,z})/E. \tag{9.19}$$

Combining these two equations yields

$$u_{z,xx} = \left[2(1+\nu)\tau_{xz,x} - \sigma_{xx,z} + \nu\sigma_{zz,z}\right]/E, \tag{9.20}$$

where we now insert Eqs. (9.9), (9.11) and (9.13). This gives

$$u_{z,xx} = \frac{1}{EI_y}\left[-\left(1+\frac{\nu}{2}\right)\left(\frac{h^2}{4} - z^2\right)q(x) - M(x)\right], \tag{9.21}$$

which at the surface of the beam $w(x) = u_z(x, h/2)$ becomes

$$EI_y w_{,xx} = -M(x). \tag{9.22}$$

This equation is called the *Euler-Bernoulli beam equation*. By using $M_{xx} = -q(x)$ we can write this as

$$(EI_y w_{,xx})_{,xx} = q(x), \tag{9.23}$$

in terms of the line load $q(x)$, or

$$EI_y w_{,xxxx} = q(x), \tag{9.24}$$

if EI_y does not depend on position x.

9.3 Buckling

Chapter 10

Plates

10.1 Stress

Kirchhoff plate theory is the straightforward generalization of Euler-Bernoulli beam theory to plates. We abandon the plane situation in which all derivatives in y-direction vanish. The weak boundary conditions then become

$$Q_x(x,y) = \int_h dz\, \tau_{xz}(x,y,z) \tag{10.1}$$

$$Q_y(x,y) = \int_h dz\, \tau_{yz}(x,y,z) \tag{10.2}$$

$$M_{xx}(x,y) = \int_h dz\, z\sigma_{xx}(x,y,z) \tag{10.3}$$

$$M_{yy}(x,y) = \int_h dz\, z\sigma_{yy}(x,y,z) \tag{10.4}$$

$$M_{xy}(x,y) = \int_h dz\, z\tau_{xy}(x,y,z), \tag{10.5}$$

where the integral is over the height h of the plate. Q_x and Q_y are called shear forces, M_{xx} and M_{yy} are bending moments and M_{xy} is the torsional moment.

Note that employing static equilibrium $\sigma_{ij,j} = 0$ we obtain

$$Q_{x,x} + Q_{y,y} = \int_{-h/2}^{h/2} dz\, \left(\tau_{zx,x} + \tau_{zy,y}\right) = -\int_{-h/2}^{h/2} dz\, \tau_{zz,z} \tag{10.6}$$

but

$$\int_{-h/2}^{h/2} dz\, \tau_{zz,z} = \tau_{zz}(x,y,h/2) - \tau_{zz}(x,y,-h/2) \equiv p(x,y) \tag{10.7}$$

where $p(x, y)$ is the pressure on the plate (cf. also the corresponding equation Eq. (9.14) for the beam). Similarly

$$M_{xx,x} + M_{xy,y} = \int_{-h/2}^{h/2} dz\, z \left(\tau_{xx,x} + \tau_{xy,y}\right) = -\int_{-h/2}^{h/2} dz\, z\tau_{xz,z} \tag{10.8}$$

and

$$\int_{-h/2}^{h/2} dz\, z\tau_{xz,z} = \left[z\tau_{xz}\right]_{-h/2}^{h/2} - \int_{-h/2}^{h/2} dz\, \tau_{xz} = -Q_x. \tag{10.9}$$

where the last equality holds because $\tau_{xz}(x, y, h/2) = -\tau_{xz}(x, y, -h/2)$. The condition for static equilibrium $\sigma_{ij,j} = 0$ therefore becomes

$$Q_{x,x} + Q_{y,y} = -p(x, y) \tag{10.10}$$
$$M_{xx,x} + M_{xy,y} = Q_x(x, y) \tag{10.11}$$
$$M_{xy,x} + M_{yy,y} = Q_y(x, y) \tag{10.12}$$

in the weak form. Note that this can be written in the compact form $Q_{i,i} = -p$ and $M_{ij,j} = Q_i$.

As in the Euler-Bernoulli case, we assume that the components σ_{xx}, σ_{yy} and τ_{xy} vary linearly with z. We can write

$$\sigma_{xx}(x, y, z) = \frac{M_{xx}(x, y)}{I} z \tag{10.13}$$
$$\sigma_{yy}(x, y, z) = \frac{M_{yy}(x, y)}{I} z \tag{10.14}$$
$$\tau_{xy}(x, y, z) = \frac{M_{xy}(x, y)}{I} z \tag{10.15}$$

with $I = \int dz\, z^2 = h^3/12$. The remaining components of the stress tensor are obtained from static equilibrium. Static equilibrium yields

$$\tau_{xz,z} = -\frac{z}{I} Q_x \quad \text{and} \quad \tau_{yz,z} = -\frac{z}{I} Q_y \tag{10.16}$$

which can be integrated under the condition $\tau_{xz}(x, y, h/2) = \tau_{xz}(x, y, -h/2) = 0$ to

$$\tau_{xz}(x, y, z) = \frac{Q_x}{2I}\left(\frac{h^2}{4} - z^2\right) \quad \text{and} \quad \tau_{yz}(x, y, z) = \frac{Q_y}{2I}\left(\frac{h^2}{4} - z^2\right). \tag{10.17}$$

This is analogous to Eq. (9.11) for the beam.

We are finally left with finding an expression for σ_{zz}. Again we use static equilibrium to obtain

$$\sigma_{zz,z} = -\tau_{xz,x} - \tau_{yz,y} = \frac{p(x,y)}{2I}\left(\frac{h^2}{4} - z^2\right). \tag{10.18}$$

Integration under the condition that the loads on top and bottom surface of the plate balance, $\sigma_{zz}(x, h/2) = -\sigma_{zz}(x, -h/2)$, gives

$$\sigma_{zz}(x,y,z) = \frac{p(x,y)}{2I}\left(\frac{h^2}{4} - \frac{z^2}{3}\right)z. \tag{10.19}$$

At the top and bottom of the plate we find $\sigma_{zz}(x, h/2) = -\sigma_{zz}(x, -h/2) = p(x,y)/2$.

10.2 Displacements

Now that we know the stress inside the plate, we can again compute the displacements from Hooke's law. In the full three-dimensional case, Hooke's law,

$$\varepsilon_{xx} \equiv u_{x,x} = (\sigma_{xx} - \nu\sigma_{yy} - \nu\sigma_{zz})/E \tag{10.20}$$
$$\varepsilon_{yy} \equiv u_{y,y} = (\sigma_{yy} - \nu\sigma_{xx} - \nu\sigma_{zz})/E \tag{10.21}$$
$$2\varepsilon_{xz} \equiv u_{x,z} + u_{z,x} = 2(1+\nu)\tau_{xz}/E \tag{10.22}$$
$$2\varepsilon_{yz} \equiv u_{y,z} + u_{z,y} = 2(1+\nu)\tau_{yz}/E \tag{10.23}$$
$$2\varepsilon_{xy} \equiv u_{x,y} + u_{y,x} = 2(1+\nu)\tau_{xy}/E, \tag{10.24}$$

and taking the derivative of Eq. (10.22) with respect to x and of Eq. (10.23) with respect to y, we obtain

$$u_{x,xz} + u_{z,xx} = 2(1+\nu)\tau_{xz,x}/E \tag{10.25}$$
$$u_{y,yz} + u_{z,yy} = 2(1+\nu)\tau_{yz,y}/E \tag{10.26}$$

and

$$u_{x,xz} + u_{z,xx} = \partial_z(u_{x,x}) + u_{z,xx} = u_{z,xx} + (\sigma_{xx,z} - \nu\sigma_{yy,z} - \nu\sigma_{zz,z})/E \tag{10.27}$$
$$u_{y,yz} + u_{z,yy} = \partial_z(u_{y,y}) + u_{z,yy} = u_{z,yy} + (\sigma_{yy,z} - \nu\sigma_{xx,z} - \nu\sigma_{zz,z})/E. \tag{10.28}$$

Combining Eqs. (10.25), (10.27) and Eqs. (10.26), (10.28) and noting that $\sigma_{zz,z} = -\tau_{xz,x} - \tau_{xz,y}$ yields

$$u_{z,xx} = \left[(2+\nu)\tau_{xz,x} - \nu\tau_{yz,y} - \sigma_{xx,z} + \nu\sigma_{yy,z}\right]/E \tag{10.29}$$

$$u_{z,yy} = \left[(2+\nu)\tau_{yz,y} - \nu\tau_{xz,x} - \sigma_{yy,z} + \nu\sigma_{xx,z}\right]/E. \tag{10.30}$$

We now create linear combination of these expressions such that $\sigma_{xx,z} = M_{xx}/I$ or $\sigma_{yy,z} = M_{yy}/I$ drop out,

$$u_{z,xx} + \nu u_{z,yy} = \left[(2+\nu-\nu^2)\tau_{xz,x} - \nu(1+\nu)\tau_{yz,y} - (1-\nu^2)M_{xx}/I\right]/E \tag{10.31}$$

$$u_{z,yy} + \nu u_{z,xx} = \left[(2+\nu-\nu^2)\tau_{yz,y} - \nu(1+\nu)\tau_{xz,x} - (1-\nu^2)M_{yy}/I\right]/E. \tag{10.32}$$

We now only consider the displacement at the surface, $w(x,y) \equiv u_z(x,y,h/2)$. Since the surfaces are traction free, all terms involving τ_{xz} and τ_{yz} vanish. Hence

$$M_{xx} = -K(w_{,xx} + \nu w_{,yy}) \tag{10.33}$$

$$M_{yy} = -K(w_{,yy} + \nu w_{,xx}) \tag{10.34}$$

with the *flexural rigidity* $K = EI/(1-\nu^2) = Eh^3/[12(1-\nu^2)]$.

Finally, we are looking for an expression for $M_{xy} = I\tau_{xy,z}$. We have from Eqs. (10.22)-(10.24)

$$\frac{2(1+\nu)}{EI}M_{xy} = u_{x,yz} + u_{y,xz} = \frac{2(1+\nu)}{E}(\tau_{xz,y} + \tau_{yz,x}) - 2u_{z,xy}, \tag{10.35}$$

which yields

$$M_{xy} = -K(1-\nu)w_{,xy}, \tag{10.36}$$

the desired expression.

We now plug Eqs. (10.33), (10.34) and (10.36) into the equilibrium conditions Eqs. (10.11) and (10.12). This yields

$$-K(w_{,xxx} + w_{,xyy}) = Q_x(x,y) \tag{10.37}$$

$$-K(w_{,yyy} + w_{,xxy}) = Q_y(x,y) \tag{10.38}$$

$$-K(w_{,xxxx} + 2w_{,xxyy} + w_{,yyyy}) = -p(x,y). \tag{10.39}$$

The last expression is Kirchhoff's equation,

$$w_{,xxxx} + 2w_{,xxyy} + w_{,yyyy} = \nabla^2\nabla^2 w = \nabla^4 w = \frac{p}{K}, \tag{10.40}$$

that governs the deformation of plates.

Chapter 11

Plastic failure

Chapter 12

Brittle failure

12.1 Stress near a crack tip

12.1.1 Airy stress function

First, we invoke Hooke's law, Eq. (7.8). Using Eq. (8.2) Hooke's law itself becomes

$$\varepsilon_{xx} = \frac{1}{E}[(1+\nu)\sigma_{xx} - \nu(\sigma_{xx} + \sigma_{yy} + \sigma_{zz})] = \frac{1+\nu}{E}[(1-\nu)\sigma_{xx} - \nu\sigma_{yy}] \tag{12.1}$$

$$\varepsilon_{yy} = \frac{1+\nu}{E}[(1-\nu)\sigma_{yy} - \nu\sigma_{xx}] \tag{12.2}$$

$$\varepsilon_{xy} = \frac{1}{2G}\sigma_{xy} = \frac{1+\nu}{E}\sigma_{xy} \tag{12.3}$$

under plane strain conditions. We have just eliminated reference to σ_{zz} and ε_{zz} from the equations. Inserting Hooke's law into the compatibility condition, Eq. (8.14), yields

$$(1-\nu)\left[\frac{\partial^2\sigma_{xx}}{\partial y^2} + \frac{\partial^2\sigma_{yy}}{\partial x^2}\right] - \nu\left[\frac{\partial^2\sigma_{xx}}{\partial x^2} + \frac{\partial^2\sigma_{yy}}{\partial y^2}\right] = 2\frac{\partial^2\sigma_{xy}}{\partial x\partial y}, \tag{12.4}$$

the compatibility condition for the stresses.

We now use a mathematical trick to solve the equations of elastic equilibrium. We define the *Airy stress function* $\phi(x, y)$, that gives the stresses (in

Cartesian coordinates) as

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2} \equiv [\nabla(\nabla\phi)]_{yy}, \tag{12.5}$$

$$\sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2} \equiv [\nabla(\nabla\phi)]_{xx}, \tag{12.6}$$

$$\sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} \equiv [\nabla(\nabla\phi)]_{xy} \tag{12.7}$$

(Note that the first ∇ is the gradient of the vector field $\nabla\phi$, not the divergence! Hence $\nabla(\nabla\phi)$ is a second order tensor with components $[\nabla(\nabla\phi)]_{ij} = \phi_{,ij}$!) Equation (8.7) becomes

$$\frac{\partial^3 \phi}{\partial x \partial y^2} - \frac{\partial^3 \phi}{\partial x \partial y^2} = 0 \tag{12.8}$$

and is automatically fulfilled! The same holds for Eq. (8.8) We can now insert Eqs. (12.5) to (12.7) into the compatibility condition Eq. (12.4) to give

$$(1-\nu)\left[\frac{\partial^2}{\partial y^2}\frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2}{\partial x^2}\frac{\partial^2 \phi}{\partial x^2}\right] - \nu\left[\frac{\partial^2}{\partial x^2}\frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2}{\partial y^2}\frac{\partial^2 \phi}{\partial x^2}\right] + 2\frac{\partial^2}{\partial x \partial y}\frac{\partial^2 \phi}{\partial x \partial y} = 0, \tag{12.9}$$

which can be rearranged to

$$\frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left[\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi\right] = \nabla^4 \phi = 0 \tag{12.10}$$

Conditions (8.7) to (8.14) are all simultaneously fulfilled if Eq. (12.10) is fulfilled! Equation (12.10) is the compatibility condition, but expressed for the Airy stress function.

The operator ∇^4 is called the *biharmonic* operator and any function ϕ satifying $\nabla^4\phi = 0$ is called a biharmonic function. (Functions ϕ that fulfill the Laplace equation $\nabla^2\phi = 0$ are called harmonic functions.) The derivation above tells us, that for isotropic elasticity a biharmonic Airy stress $\phi(x, y)$ automatically leads to a compatible strain field. This means for *any* biharmonic function $\phi(x, y)$, the stress field derived from Eqs. (12.5)-(12.7) will describe a system in static equilibrium. The only thing left to do is to find the function $\phi(x, y)$ that fulfills a specific boundary condition.

12.1.2 Westergaard stress function

The Westergaard stress function builds on top of the Airy function and eliminates the need to even satisfy the biharmonic equation $\nabla^4\phi = 0$. This

means that for any choice of the Westergaard stress function, the conditions for force and moment equilibrium and the compatibility condition are fulfilled automatically. The Westergaard stress function can then be chosen freely as to fulfill the boundary conditions of the problem.

Westergaard (1933) introduced a function $Z(z)$ that is now known as the *Westergaard stress function*. Note that here z is a complex variable that contains the x and y-position as its real and imaginary part, $z = x + iy$, and *not* the z-coordinate. Likewise, the function $Z(z)$ is complex valued. Following common notation, we denote its integrals by

$$\bar{\bar{Z}}_{,z} = \bar{Z} \quad \text{and} \quad \bar{Z}_{,z} = Z. \tag{12.11}$$

Westergaard defined the Airy stress function as

$$\phi(x, y) = \Re\left\{\bar{\bar{Z}}(x + iy)\right\} + y\Im\left\{\bar{Z}(x + iy)\right\} \tag{12.12}$$

where $\Re$ and $\Im$ denote the real and imaginary part of a complex number, respectively. This Airy stress function fulfills the biharmonic equation, Eq. (12.10), for any arbitrary function $Z(z)$, as we will show below.

We will now derive the expressions for the stresses. We first note that since $Z(z)$ is a function of a complex variable, it satisfies the Cauchy-Riemann conditions,

$$\Re Z_{,z} = (\Re Z)_{,x} = (\Im Z)_{,y} \tag{12.13}$$

$$\Im Z_{,z} = (\Im Z)_{,x} = -(\Re Z)_{,y}, \tag{12.14}$$

which implies that

$$\nabla^2(\Re Z) = (\Re Z)_{,xx} + (\Re Z)_{,yy} = 0 \tag{12.15}$$

$$\nabla^2(\Im Z) = (\Im Z)_{,xx} + (\Im Z)_{,yy} = 0, \tag{12.16}$$

$$\tag{12.17}$$

i.e. the real and imaginary parts of the complex function $Z(z)$ fulfill the Laplace equation.

Note: *Cauchy-Riemann equations* – The Cauchy-Riemann equations are a cornerstone of complex analysis. They hold for any differential function of a complex variable. Such functions are also called *holomorphic*. Given $f(z) = u(z) + iv(z)$ (hence $u = \Re f$ and $v = \Im f$) with $z = x + iy$, we can formally write the derivative as

$$f'(z_0) = \lim_{w \in \mathcal{C}, w \to 0} \frac{f(z_0 + w) - f(z_0)}{w} \tag{12.18}$$

as the limit of the difference quotient. For real numbers, we can approach z_0 from the left or the right and both limits must equal if the function is differentiable. In the complex plane, we can approach from any direction in two dimensions. For example, we can compute the derivative along the real (x-)axis, $w = x$:

$$f'(z_0) = \lim_{x \in \mathcal{R}, x \to 0} \frac{f(z_0 + x) - f(z_0)}{x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \tag{12.19}$$

We can equally well compute the derivative along the imaginary (y-)axis, $w = iy$, which gives

$$f'(z_0) = \lim_{y \in \mathcal{R}, y \to 0} \frac{f(z_0 + iy) - f(z_0)}{iy} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}. \tag{12.20}$$

Since both expressions have to be equal, we find

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}, \tag{12.21}$$

the Cauchy-Riemann equations.

Now we differentiate ϕ with respect to x and y. This yields

$$\phi_{,x} = \left(\Re\bar{\bar{Z}}\right)_{,x} + y\left(\Im\bar{Z}\right)_{,x} = \Re\bar{\bar{Z}}_{,z} + y\Im\bar{Z}_{,z} = \Re\bar{Z} + y\Im Z \tag{12.22}$$

$$\phi_{,y} = \left(\Re\bar{\bar{Z}}\right)_{,y} + \left(y\Im\bar{Z}\right)_{,y} = -\Im\bar{\bar{Z}}_{,z} + \Im\bar{Z} + y\Re\bar{Z}_{,z} = y\Re Z \tag{12.23}$$

for the first derivatives and

$$\phi_{,xx} = \left(\Re\bar{Z}\right)_{,x} + y\left(\Im Z\right)_{,x} = \Re\bar{Z}_{,z} + y\Im Z_{,z} = \Re Z + y\Im Z_{,z} \tag{12.24}$$

$$\phi_{,yy} = \left(y\Re Z\right)_{,y} = \Re Z + y\left(\Re Z\right)_{,y} = \Re Z - y\Im Z_{,z} \tag{12.25}$$

$$\phi_{,xy} = \left(\Re\bar{Z}\right)_{,y} + \left(y\Im Z\right)_{,y} = -\Im\bar{Z}_{,z} + \Im Z + y\Re Z_{,z} = y\Re Z_{,z} \tag{12.26}$$

for the second derivatives. In summary, we obtain

$$\sigma_{xx} = \phi_{,yy} = \Re Z - y\Im Z_{,z} \tag{12.27}$$

$$\sigma_{yy} = \phi_{,xx} = \Re Z + y\Im Z_{,z} \tag{12.28}$$

$$\sigma_{xy} = -\phi_{,xy} = -y\Re Z_{,z} \tag{12.29}$$

for the components of the stress tensor. Note that the Ansatz Eq. (12.12) fulfills the biharmonic equation. We know from summing Eqs. (12.27) and

(12.28) that

$$\psi \equiv \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \phi = 2\Re Z, \tag{12.30}$$

hence by virtue of Eq. (12.15), $\nabla^4 \phi = \nabla^2 \psi = 0$ is fulfilled for *any* function $Z(x + iy)$. The utility of the Westergaard function over the Airy stress function is hence that we have eliminated the need to explicitly fulfill the biharmonic equation.

12.1.3 Stress field

We here discuss only mode I fracture, i.e. crack opening displacement. Consider a plane (strain or stress) situation in which we can derive the stress field from the Westergaard stress function $Z(x+iy)$, Sec. 12.1.2. Westergaard (1933) made the Ansatz

$$Z(z) = \frac{\sigma_\infty}{\sqrt{1 - (a/z)^2}} \tag{12.31}$$

for the stress function of a central crack in a large sheet. Here $2a$ is the length of the crack and σ_∞ the stress at infinity. The derivative of the stress function is

$$Z_{,z} = \frac{\sigma_\infty}{\sqrt{z^2 - a^2}} \left[1 - \frac{1}{1 - (a/z)^2} \right] \tag{12.32}$$

and its integral is

$$\bar{Z} = \sigma_\infty \sqrt{z^2 - a^2}. \tag{12.33}$$

Let us first look in the plane of the crack where $y = 0$, $z = x$ and hence

$$Z(x) = \frac{\sigma_\infty}{\sqrt{1 - (a/x)^2}}. \tag{12.34}$$

For $|x| < a$ (inside the crack) the function is imaginary but for $|x| > a$ (outside the crack) the function is real. The stresses in the plane of the crack are given by $\sigma_{xx} = \Re Z$, $\sigma_{yy} = \Re Z$ and $\tau_{xy} = 0$. Hence they vanish inside the crack. This is the condition for the crack faces which are free surfaces and therefore tractionless. Note that σ_{xx} does not need to vanish from this condition but does here.

Outside the crack (but in the plane of the crack, $y = 0$), the stress is given by

$$\sigma_{xx} = \sigma_{yy} = \frac{\sigma_\infty}{\sqrt{1 - (a/x)^2}}. \tag{12.35}$$

It diverges as $x \to a$ from above and approaches the hydrostatic state $\sigma_{xx} = \sigma_{yy} = \sigma_\infty$ as $x \to \infty$.

We will now focus on the crack line and switch to the variable $z^* = z - a$. The stress function becomes

$$Z(z^*) = \frac{\sigma_\infty (z^* + a)}{\sqrt{(z^* + a)^2 - a^2}} = \frac{\sigma_\infty (z^* + a)}{\sqrt{(z^*)^2 + 2az^*}} \approx \sigma_\infty \sqrt{\frac{a}{2z^*}} \tag{12.36}$$

where the $\approx$ sign is valid for small z^*. We write this expression as

$$Z(z^*) = \frac{K_I}{\sqrt{2\pi z^*}} \quad \text{with} \quad K_I = \sigma_\infty \sqrt{\pi a}. \tag{12.37}$$

Note that we have absorbed both the stress at infinity σ_∞ and the crack length a into a single constant, the *stress intensity factor* (for mode I fracture), K_I. The stress field near the crack tip depends only on K_I, not on σ_∞ and a individually and hence the loading condition and geometry, independently.

To derive the component of the stress tensor, we switch to cylindrical coordinates and write $z^* = re^{i\theta}$, yielding

$$Z(r, \theta) = \frac{K_I}{\sqrt{2\pi r}} e^{-i\theta/2}. \tag{12.38}$$

In order to obtain the full stress field, we also need the expression for the derivative of the Westergaard stress function near the crack tip. From Eq. (12.32), we find

$$Z_{,z} \approx \frac{\sigma_\infty}{\sqrt{2az^*}} \left[1 - \frac{2az^* + a^2}{2az^*} \right] = -\frac{\sigma_\infty a^2}{(2az^*)^{3/2}} = -\frac{K_I \pi}{(2\pi z^*)^{3/2}} = -\frac{K_I}{\sqrt{8\pi r}} e^{-3i\theta/2}. \tag{12.39}$$

We now obtain the individual components of the stress tensor from Eq. (12.27)-(12.29):

$$\sigma_{xx} = \Re Z - y \Im Z_{,z} = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2} \left(1 - \sin\frac{\theta}{2} \sin\frac{3\theta}{2} \right) \tag{12.40}$$

$$\sigma_{yy} = \Re Z + y \Im Z_{,z} = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2} \left(1 + \sin\frac{\theta}{2} \sin\frac{3\theta}{2} \right) \tag{12.41}$$

$$\sigma_{xy} = -y \Re Z_{,z} = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2} \sin\frac{\theta}{2} \cos\frac{3\theta}{2} \tag{12.42}$$

Hence the stress field *near* the crack tip is entirely described by the stress intensity factor K_I. K_I is measured in (weird) units of $\mathrm{Pa}\sqrt{\mathrm{m}}$. The I indicates

that this is the stress intensity factor for mode I fracture. K_{II} and K_{III} are related quatities for the two other fracture modes.

Note that the stress intensity factor is the amplitude of the square-root singularity of the stress field at the crack tip. It is often defined from the stress field itself as the limit

$$K_I = \lim_{r\to 0} \sqrt{2\pi r}\sigma_{yy}(r,0) \tag{12.43}$$

$$K_{II} = \lim_{r\to 0} \sqrt{2\pi r}\sigma_{xy}(r,0) \tag{12.44}$$

$$K_{III} = \lim_{r\to 0} \sqrt{2\pi r}\sigma_{yz}(r,0), \tag{12.45}$$

$$\tag{12.46}$$

where the singularity has been removed by multiplying with $\sqrt{2\pi r}$. Note that the limit is taken at the angle $\theta = 0$, i.e. along $y = 0$ within the plane of the crack. From Eqs. (12.43) to (12.45) is also clear that only K_I is nonzero for the crack geometry that is discussed in this chapter.

12.2 Fracture toughness

12.2.1 Displacement field at the crack tip

In order to compute the displacement field near the crack, we first need the strain field. As usual, we obtain this from Hooke's law (in plain stress):

$$\varepsilon_{xx} \equiv u_{x,x} = \frac{1}{E}\left(\sigma_{xx} - \nu\sigma_{yy}\right) = \frac{1}{E}\left[(1-\nu)\Re Z - (1+\nu)y\Im Z_{,z}\right] \tag{12.47}$$

$$\varepsilon_{yy} \equiv u_{y,y} = \frac{1}{E}\left(\sigma_{yy} - \nu\sigma_{xx}\right) = \frac{1}{E}\left[(1-\nu)\Re Z + (1+\nu)y\Im Z_{,z}\right] \tag{12.48}$$

$$2\varepsilon_{xy} \equiv u_{x,y} + u_{y,x} = 2\frac{1+\nu}{E}\sigma_{xy} = -2\frac{1+\nu}{E}y\Re Z_{,z} \tag{12.49}$$

Now we use $Z = \bar{Z}_{,z}$ to express $\Re Z = \Re\bar{Z}_{,z} = (\Re\bar{Z})_{,x}$ and $\Im Z_{,z} = (\Im Z)_{,x}$ in Eq. (12.47) to identify

$$u_x = \frac{1}{E}\left[(1-\nu)\Re\bar{Z} - (1+\nu)y\Im Z\right] = \frac{1}{2\mu}\left[\frac{1-\nu}{1+\nu}\Re\bar{Z} - y\Im Z\right]. \tag{12.50}$$

The expression for u_y is more complicated because of the explicit y that shows up in the expressions for the strains. It is given by

$$u_y = \frac{1}{2\mu}\left[\frac{2}{1+\nu}\Im\bar{Z} - y\Re Z\right]. \tag{12.51}$$

It is straightforward to verify that from these two expressions we also recover the expression for ε_{xy}.

It is common to replace ν with $\kappa = (3-\nu)/(1+\nu)$. The displacements can then be written as

$$u_x = \frac{1}{4\mu}\left[(\kappa-1)\Re\bar{Z} - 2y\Im Z\right] \tag{12.52}$$

$$u_y = \frac{1}{4\mu}\left[(\kappa+1)\Im\bar{Z} - 2y\Re Z\right]. \tag{12.53}$$

Since the integral of the Westergaard function near the crack tip is

$$\bar{Z}(z^*) \approx \sigma_\infty\sqrt{2az^*} = K_I\sqrt{\frac{2z^*}{\pi}} = 2K_I\sqrt{\frac{r}{2\pi}}e^{i\theta/2}, \tag{12.54}$$

we can directly write the displacement field in cylindrical coordinates as

$$u_x = \frac{K_I}{2\mu}\sqrt{\frac{r}{2\pi}}\left[2\frac{1-\nu}{1+\nu}\cos\frac{\theta}{2} + \sin^2\frac{\theta}{2}\right] \tag{12.55}$$

$$u_y = \frac{K_I}{2\mu}\sqrt{\frac{r}{2\pi}}\left[2\frac{1-\nu}{1+\nu}\cos\frac{\theta}{2} + \sin^2\frac{\theta}{2}\right]. \tag{12.56}$$

Note that in the plane of the crack ($y = 0$, $z^* = x$), the displacement field is given by

$$u_x = \begin{cases} \frac{K_I}{2\mu}(\kappa-1)\sqrt{\frac{x}{2\pi}} & \text{if } x > 0 \\ 0 & \text{if } x \le 0 \end{cases} \tag{12.57}$$

$$u_y^+ = \begin{cases} 0 & \text{if } x \ge 0 \\ \frac{K_I}{2\mu}(\kappa+1)\sqrt{-\frac{x}{2\pi}} & \text{if } x < 0 \end{cases}. \tag{12.58}$$

The y-displacement here is denoted with a little $+$, u_y^+, to indicate that this is the displacement of the top crack face at a position $y = 0^+$, i.e. slightly above $y = 0$. The displacement u_y^- of the bottom crack face is the negative of this values, $u_y^- = -u_y^+$, for symmetry reasons. Mathematically, this property emerges because the square-root has a *branch cut* along the negative real axis.

12.2.2 Strain energy release rate

In order to formulate a fracture criterion we will require an expression for the elastic *energy* released during propagation of the crack. This will lead to the concept of the *strain energy release rate*.

We have not yet talked about the concept of energy in this notes. Since elasticity is fully reversible (an elastic object returns to its origin shape when unloaded), the work carried out when deforming an elastic body is *conservative*. This means we can define an elastic energy that is recoverable by unloading the body. In order to define the energy release rate, we will here focus on the work W done by a moving crack.

We first note that the crack faces do not contribute to the work because the normal stress is zero by definition, since we are dealing with a free surface. We therefore have to focus on the crack tip. The tip opens by the distance given by Eq. (12.58) to both sides. This displacement work against the stress σ_{yy} right at the crack tip. We now assume that the crack moves by a distance Δa which we will later take to zero. We assume that the stress before the crack has moved is taken to zero quasistatically during the crack opening process. The crack faces have opened a distance

$$u_y^+ - u_y^- = 2\frac{K_I}{2\mu}(\kappa+1)\sqrt{\frac{\Delta a - x^*}{2\pi}} \tag{12.59}$$

after the crack has moved by Δa. The stress was

$$\sigma_{yy} = \frac{K_I}{\sqrt{2\pi x^*}} \tag{12.60}$$

before the movement of the crack. The work on the crack faces is then $W = \int dx^* \sigma_{yy}(u_y^+ - u_y^-)/2$. (The factor $1/2$ enters because we are taking the stress to zero as the crack faces are displacing. Imagine a simple spring with force $f = kx$ taken from x_0 to $x = 0$. The work is $kx_0^2/2$, equal to the elastic energy of the string at extension x_0.) This gives

$$W(\Delta a) = \frac{K_I^2}{4\pi\mu}(\kappa+1)\int_0^{\Delta a} dx^* \sqrt{\frac{\Delta a - x^*}{x^*}} = \frac{K_I^2}{\mu}\frac{\kappa+1}{8}\Delta a = \frac{K_I^2}{E}\Delta a \tag{12.61}$$

where we have used $\int_0^1 dx\sqrt{(1-x)/x} = \pi/2$. The energy released per crack length is then

$$G = \frac{W}{\Delta a} = \frac{K_I^2}{E}. \tag{12.62}$$

G is called the *strain energy release rate* and has units of energy per area. Note that Eq. (12.62) is valid for a state of plain strain, which has entered through Hooke's law in Eqs. (12.47) to (12.49).

12.3 Griffith's fracture criterion

The value of K_I uniquely defines the stress field near the crack tip and therefore determines when the crack advances. We define a critical K_{Ic} beyond which the crack becomes unstable and a new crack opens when $K_I > K_{Ic}$. K_{Ic} is called the *fracture toughness*. Generally, K_I depends on crack geometry and loading condition. For the example worked above, a straight crack of length a in an infinite isotropic medium we get Eq. (12.37). We see that the stress intensity factor K_I growth with crack length a; hence there is a critical length a_c beyond which the crack becomes unstable.

As the crack advances it creates new surface area. An advance of the crack from length a to length $a + \Delta a$ requires the additional energy $\Delta E_{\text{surf}} = 2\gamma\Delta a$. Here, γ is the surface energy of the pristine crack surface and the factor of 2 enters because a crack has two faces. Since we are looking at a plane situation, ΔE_{surf} is an energy per unit length.

Griffith's fracture criterion now states that the crack advances when the energy released from the elastic field (described by the strain energy release rate G) is larger than the energy needed to create a new surface,

$$G > G_c \tag{12.63}$$

where G_c is Griffith's critical energy release rate. For an ideal brittle crack,

$$G_c = 2\gamma, \tag{12.64}$$

because we only need to "pay" for the new surface with elastic energy. This allows us to define a fracture toughness, $K_{Ic} = \sqrt{EG_c}$, from Griffith's theory.

Bibliography

H. M. Westergaard. Stresses at a crack, size of the crack, and the bending of reinforced concrete. *Journal Proceedings*, 30(11), 1933. doi: 10.14359/8300.