Exercise 0: Review of basic concepts

18.10.2021

This is just a short test of some basic concepts which are relevant for the course. It will not be graded.
Any problems in solving these exercises will be discussed in the first tutorial. Be prepared to ask your questions!

Question 1 .... 0 points

Consider the vectors $\vec{a}$ and $\vec{b}$, and the matrix C:

$$\vec{a} \equiv \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \quad \vec{b} \equiv \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \quad \text{and} \quad C \equiv \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}.$$

(a) Compute the Euclidean norm $|\vec{a}|$, the dot product $\vec{a} \cdot \vec{b}$, and the cross product $\vec{a} \times \vec{b}$!

Solution:

$$|a| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$
$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$
$$\vec{a} \times \vec{b} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ -a_1 b_3 + a_3 b_1 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}$$

(b) Use the *Einstein summation convention* to evaluate the following expressions!

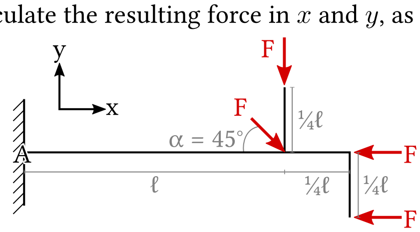
$$a_i b_i \tag{1}$$
$$C_{ij} b_j \tag{2}$$
$$C_{ji} \tag{3}$$

What is the meaning of these operations? *Hint:* For example, $\sqrt{a_i a_i}$ is the Euclidean norm of $\vec{a}$ and the matrix $C = C_{ij}$ where $C_{ij}^T := C_{ji}$.

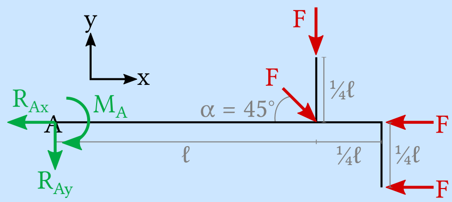
Solution:

$$a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3 \quad \text{(dot product)}$$
$$C_{ij} b_j = \begin{pmatrix} C_{11} b_1 + C_{12} b_2 + C_{13} b_3 \\ C_{21} b_1 + C_{22} b_2 + C_{23} b_3 \\ C_{31} b_1 + C_{32} b_2 + C_{33} b_3 \end{pmatrix} \quad \text{(matrix-vector product)}$$
$$C_{ji} = \begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix} \quad \text{(matrix transposition)}$$

(c) Consider the bar shown below. Calculate the resulting force in x and y, as well as the moment about point A!



Solution:



Note that $\cos(45^\circ) = \sin(45^\circ) = \sqrt{2}/2$.

$$-R_{Ax} + \frac{\sqrt{2}}{2}F - 2F = 0 \quad \text{(equilibrium in } x\text{-direction)}$$

$$\implies R_{Ax} = (\frac{\sqrt{2}}{2} - 2)F$$

$$-R_{Ay} - \frac{\sqrt{2}}{2}F - F = 0 \quad \text{(equilibrium in } y\text{-direction)}$$

$$\implies R_{Ay} = (-\frac{\sqrt{2}}{2} - 1)F$$

$$-M_A - \frac{\sqrt{2}}{2}F\ell - F\ell - F\frac{1}{4}\ell = 0 \quad \text{(equilibrium of rotation around } z \text{ about } A\text{)}$$

$$\implies M_A = F\ell(-\frac{\sqrt{2}}{2} - \frac{5}{4})$$

Question 2 . . . . 0 points

Consider the following functions

$$\begin{aligned}
f(x,y,z) &= xy + \exp\Big((x^2+y^2)z\Big)\\
g\left(x(t), y(t)\right) &= x^3 - y^2\\
u(x) &= x\exp(x)\\
\vec{h}(x,y,z) &= \begin{pmatrix} xy \\ z^2 \\ y^2 \end{pmatrix}\\
p(x) &= (3-x)^2 + (x-2)^4
\end{aligned}$$

(a) Compute the following derivatives!

$$\frac{\partial f}{\partial x}, \quad \frac{\partial f}{\partial y}, \quad \frac{\partial f}{\partial z}, \quad \frac{dg}{dt}, \quad \frac{\mathrm{d}u}{\mathrm{d}x}$$

Remember the difference between partial and total differentiation, as well as the chain rule and the product rule.

Solution:

$$\begin{aligned}
\frac{\partial f}{\partial x} &= y + 2xz \exp\Big((x^2+y^2)z\Big)\\
\frac{\partial f}{\partial y} &= x + 2yz \exp\Big((x^2+y^2)z\Big)\\
\frac{\partial f}{\partial z} &= (x^2+y^2) \exp\Big((x^2+y^2)z\Big)\\
\frac{\mathrm{d}g}{\mathrm{d}t} &= 3x^2\frac{\partial x}{\partial t} - 2y\frac{\partial y}{\partial t}\\
\frac{\mathrm{d}u}{\mathrm{d}x} &= \exp(x) + x\exp(x)
\end{aligned}$$

(b) Compute the following divergence, gradient and rotation!

$$\text{div}\,\vec{h} = \nabla\cdot\vec{h}\,,\quad \text{grad}\,f = \nabla f \quad \text{rot}\,\vec{h} = \nabla\times\vec{h}$$

Remember: $\nabla = (\partial/\partial x, \partial/\partial y, \partial/\partial z)$

Solution:

$$\begin{aligned}
\text{div}\,\vec{h} &= y\\
\text{grad}\,f &= \begin{pmatrix} y + 2xz\exp\big((x^2+y^2)z\big)\\ x + 2yz\exp\big((x^2+y^2)z\big)\\ (x^2+y^2)\exp\big((x^2+y^2)z\big)\end{pmatrix}\\
\text{rot}\,\vec{h} &= \begin{pmatrix} 2y-2z\\ 0\\ -x\end{pmatrix}
\end{aligned}$$

(c) Compute the Taylor expansion of $p(x)$ around the point $x_0 = 1$, up to the second order!

Solution: Let $p'(x_0)$ and $p''(x_0)$ be the first and second derivatives, respectively, of $p(x)$ at $x = x_0$. Then

$$p(x) \approx p(x_0) + \frac{p'(x_0)}{1!}(x-x_0) + \frac{p''(x_0)}{2!}(x-x_0)^2 + \mathcal{O}((x-x_0)^3).$$

We have

$$\begin{aligned}
p'(x) &= \frac{\partial p(x)}{\partial x} = -2(3-x) + 4(x-2)^3,\\
p'(1) &= -8,\\
p''(x) &= \frac{\partial^2 p(x)}{\partial x^2} = 2 + 12(x-2)^2,\\
p''(1) &= 14,
\end{aligned}$$

therefore

$$p(x) \approx 5 - 8(x-1) + 7(x-1)^2 + \mathcal{O}((x-1)^3).$$

Question 3 .... 0 points

(a) Compute the following integral using integration by parts!

$$\int x \sin(x)\,\mathrm{d}x$$

Solution: Recall

$$\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u.$$

Let $u \equiv x$ and $dv \equiv \sin(x)$, then $v = -\cos(x)$, $\mathrm{d}u = 1$, and the integral becomes

$$-x\cos(x) + \int \cos(x)\,\mathrm{d}x = -x\cos(x) + \sin(x). \tag{4}$$

(b) Compute the following integral using integration by substitution!

$$\int_0^1 x^2 \sin\left(x^3 - 4\right)\mathrm{d}x$$

Solution: Substitute $y \equiv (x^3 - 4)$. Then $\mathrm{d}x = \frac{1}{3x^2}\,\mathrm{d}y$ and $\int_{x=0}^{x=1}(\dots)\,\mathrm{d}x \to \int_{y=-4}^{y=-3}(\dots)\,\mathrm{d}y$. The integral becomes

$$\int_{-4}^{-3} \frac{1}{3}\sin(y)\,\mathrm{d}y = -\frac{1}{3}\cos(-3) + \frac{1}{3}\cos(-4) \approx 0.112.$$

(c) Compute the following multidimensional integrals!

$$O = \int_0^{2\pi} \mathrm{d}\varphi \int_0^{\pi} \sin(\theta) r^2\,\mathrm{d}\theta$$
$$V = \int_0^{2\pi} \mathrm{d}\varphi \int_0^{\pi} \mathrm{d}\theta \int_0^{R} \sin(\theta) r^2\,\mathrm{d}r$$

Note that O and V are the surface area and volume of a sphere with radius R, respectively.

Solution:

$$\begin{aligned} O &= \int_0^{2\pi} \mathrm{d}\varphi \int_0^{\pi} \sin(\theta) r^2\,\mathrm{d}\theta \\ &= 2\pi r^2 \int_0^{\pi} \sin(\theta)\,\mathrm{d}\theta \\ &= -2\pi r^2 \cos(\theta)|_0^{\pi} \\ &= 4\pi r^2 \end{aligned}$$

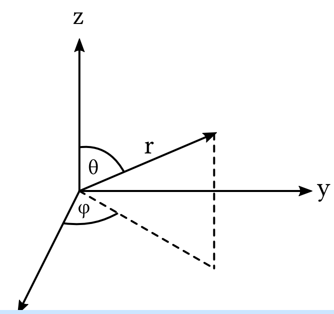
$$\begin{aligned} V &= \int_0^{2\pi} \mathrm{d}\varphi \int_0^{\pi} \mathrm{d}\theta \int_0^{R} \sin(\theta) r^2\,\mathrm{d}r \\ &= 2\pi \int_0^{\pi} \sin(\theta)\,\mathrm{d}\theta \int_0^{R} r^2 dr \\ &= \frac{2\pi}{3} R^3 \int_0^{\pi} \sin(\theta)\,\mathrm{d}\theta \\ &= \frac{4\pi}{3} R^3 \end{aligned}$$

(d) Use the divergence theorem (Gauss' theorem) to compute the surface integral of $\vec{I} = \begin{pmatrix} -x \\ 3y^2 \\ z^2 \end{pmatrix}$ over the surface ∂V of the unit sphere, i.e.

$$\int_{\partial V} \vec{I} \cdot \mathrm{d}\vec{S},$$

where $\partial V = \{(x, y, z) \in \mathbb{R} : x^2 + y^2 + z^2 = 1\}$. *Hint:* First use Gauss' theorem, $\int_{\partial V} \vec{F} \cdot \mathrm{d}\vec{S} = \int_V \operatorname{div} \vec{F}\ \mathrm{d}V$, to turn the surface integral into a volume integral. For the evaluation of the volume integral it is useful to switch to spherical coordinates, i.e.

$$\begin{aligned} x &= r \sin(\theta) \cos(\varphi) \\ y &= r \sin(\theta) \sin(\varphi) \\ z &= r \cos(\theta) \\ \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z &= r^2 \sin(\theta)\, \mathrm{d}r\, \mathrm{d}\theta\, \mathrm{d}\varphi \end{aligned}$$



Solution: We can use Gauss' theorem to transform the integral over the surface into an integral over the volume,

$$\int_{\partial V} \vec{I} \cdot \mathrm{d}\vec{S} = \int_V \operatorname{div}(\vec{I}) dV.$$

The divergence of $\vec{I}$ is

$$\begin{aligned} \operatorname{div}(\vec{I}) &= \frac{\partial(-x)}{\partial x} + \frac{\partial(3y^2)}{\partial y} + \frac{\partial(z^2)}{\partial z} \\ &= -1 + 6y + 2z. \end{aligned}$$

The integral becomes

$$\begin{aligned} \int_V \operatorname{div}(\vec{I}) dV &= \int_V (-1 + 6y + 2z)\, dV, \\ &= -\frac{4}{3}\pi + \int_V 6y dV + \int_V 2z dV. \end{aligned}$$

Keep in mind that we consider the unit sphere, therefore $\int_V 1 dV = \frac{4}{3}\pi$. In the following, we discuss the remaining integrals separately. Let $A \equiv \int_V 6y dV$ and $B = \int_V 2z dV$. We introduce spherical coordinates, as specified above, so $dV \to r^2 \sin(\theta) dr d\theta d\varphi$, with $r \in [0, 1]$, $\theta \in [0, \phi]$ and $\varphi \in [0, 2\pi]$. The first integral becomes

$$\begin{aligned}
A &= \int_0^{2\pi}\int_0^{\pi}\int_0^1 6\sin(\varphi)\sin^2(\theta)r^3 dr d\theta d\varphi \\
&= 6\int_0^{2\pi}\sin(\varphi)\int_0^{\pi}\sin^2(\theta)\int_0^1 r^3 dr d\theta d\varphi \\
&= \frac{3}{2}\int_0^{2\pi}\sin(\varphi)\int_0^{\pi}\sin^2(\theta) d\theta d\varphi \quad \text{(integrate out } r\text{)} \\
&= \frac{3}{2}\left[\frac{1}{2}\left(\theta - \sin(\theta)\cos(\theta)\right)\right]\Big|_0^{\pi}\int_0^{2\pi}\sin(\varphi)d\varphi \quad \text{(integrate out } \theta\text{)} \\
&= \frac{3}{4}\pi\int_0^{2\pi}\sin(\varphi)d\varphi \\
&= \frac{3}{4}\pi\left[-\cos(\varphi)\right]\big|_0^{2\pi} \\
&= 0.
\end{aligned}$$

The second integral becomes

$$\begin{aligned}
B &= \int_0^{2\pi}\int_0^{\pi}\int_0^1 2\cos(\theta)\sin(\theta)r^3 dr d\theta d\varphi \\
&= 2\int_0^{2\pi}\int_0^{\pi}\cos(\theta)\sin(\theta)\int_0^1 r^3 dr d\theta d\varphi \\
&= \pi\int_0^{\pi}\cos(\theta)\sin(\theta)d\theta \quad \text{(integrate out } \varphi \text{ and } r\text{)} \\
&= \pi\left[-\frac{1}{2}\cos^2(\theta)\right]\Big|_0^{\pi} \\
&= 0.
\end{aligned}$$

Thus, the solution is

$$\int_V \operatorname{div}(\vec{I})dV = -\frac{4}{3}\pi.$$

Question 4 .... 0 points

Compute eigenvalues and eigenvectors of the following matrices:

(a)

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \tag{5}$$

Solution: Since this matrix is diagonal, eigenvalues are 1 and 2 with eigenvectors $(1, 0)$ and $(0, 1)$.

(b)

$$\begin{pmatrix} 7 & 2 \\ 6 & 3 \end{pmatrix} \tag{6}$$

Solution: The eigenvalues are given by the characteristic polynomial:

$$\det\begin{pmatrix} 7-\lambda & 2 \\ 6 & 3-\lambda \end{pmatrix} = (7-\lambda)(3-\lambda) - 12 = 9 - 10\lambda + \lambda^2 = 0 \tag{7}$$

The eigenvectors given by $\lambda = 5 \pm \sqrt{25 - 9}$ or $\lambda_1 = 1$ and $\lambda_2 = 9$. The eigenvectors are now obtained from the null-space of

$$\begin{pmatrix} 6 & 2 \\ 6 & 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -2 & 2 \\ 6 & -6 \end{pmatrix} \tag{8}$$

which are $\vec{v}_1 = (1, -3)$ and $\vec{v}_2 = (1, 1)$.