Exercise 0: Review of basic concepts

18.10.2021

This is just a short test of some basic concepts which are relevant for the course. It will not be graded.
Any problems in solving these exercises will be discussed in the first tutorial. Be prepared to ask your questions!

Question 1 . . . . 0 points

Consider the vectors $\vec{a}$ and $\vec{b}$, and the matrix C:

$$\vec{a} \equiv \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \quad \vec{b} \equiv \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \quad \text{and} \quad C \equiv \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}.$$

(a) Compute the Euclidean norm $|\vec{a}|$, the dot product $\vec{a} \cdot \vec{b}$, and the cross product $\vec{a} \times \vec{b}$!

(b) Use the *Einstein summation convention* to evaluate the following expressions!

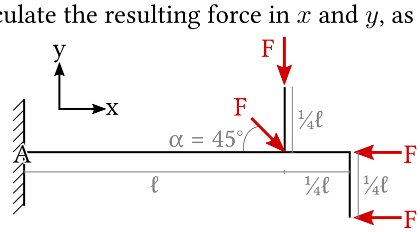
$$a_i b_i \tag{1}$$

$$C_{ij} b_j \tag{2}$$

$$C_{ji} \tag{3}$$

What is the meaning of these operations? *Hint:* For example, $\sqrt{a_i a_i}$ is the Euclidean norm of $\vec{a}$ and the matrix $C = C_{ij}$ where $C^T_{ij} := C_{ji}$.

(c) Consider the bar shown below. Calculate the resulting force in x and y, as well as the moment about point A!



Question 2 . . . . 0 points

Consider the following functions

$$f(x, y, z) = xy + \exp\left((x^2 + y^2)z\right)$$
$$g\left(x(t), y(t)\right) = x^3 - y^2$$
$$u(x) = x \exp(x)$$
$$\vec{h}(x, y, z) = \begin{pmatrix} xy \\ z^2 \\ y^2 \end{pmatrix}$$
$$p(x) = (3 - x)^2 + (x - 2)^4$$

(a) Compute the following derivatives!

$$\frac{\partial f}{\partial x}, \quad \frac{\partial f}{\partial y}, \quad \frac{\partial f}{\partial z}, \quad \frac{dg}{dt}, \quad \frac{\mathrm{d}u}{\mathrm{d}x}$$

Remember the difference between partial and total differentiation, as well as the chain rule and the product rule.

(b) Compute the following divergence, gradient and rotation!

$$\operatorname{div} \vec{h} = \nabla \cdot \vec{h}\,, \quad \operatorname{grad} f = \nabla f \quad \operatorname{rot} \vec{h} = \nabla \times \vec{h}$$

Remember: $\nabla = (\partial/\partial x, \partial/\partial y, \partial/\partial z)$

(c) Compute the Taylor expansion of $p(x)$ around the point $x_0 = 1$, up to the second order!

Question 3 .... 0 points

(a) Compute the following integral using integration by parts!

$$\int x \sin(x)\,\mathrm{d}x$$

(b) Compute the following integral using integration by substitution!

$$\int_0^1 x^2 \sin\left(x^3 - 4\right)\mathrm{d}x$$

(c) Compute the following multidimensional integrals!

$$O = \int_0^{2\pi} \mathrm{d}\varphi \int_0^{\pi} \sin(\theta) r^2\,\mathrm{d}\theta$$
$$V = \int_0^{2\pi} \mathrm{d}\varphi \int_0^{\pi} \mathrm{d}\theta \int_0^{R} \sin(\theta) r^2\,\mathrm{d}r$$

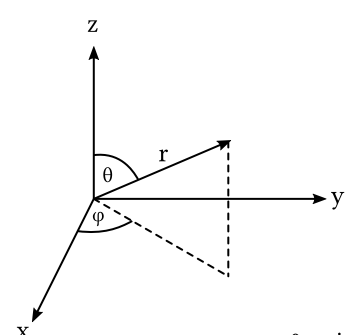
Note that O and V are the surface area and volume of a sphere with radius R, respectively.

(d) Use the divergence theorem (Gauss' theorem) to compute the surface integral of $\vec{I} = \begin{pmatrix} -x \\ 3y^2 \\ z^2 \end{pmatrix}$ over the surface ∂V of the unit sphere, i.e.

$$\int_{\partial V} \vec{I} \cdot \mathrm{d}\vec{S},$$

where $\partial V = \{(x, y, z) \in \mathbb{R} : x^2 + y^2 + z^2 = 1\}$. *Hint:* First use Gauss' theorem, $\int_{\partial V} \vec{F} \cdot \mathrm{d}\vec{S} = \int_V \operatorname{div} \vec{F}\ \mathrm{d}V$, to turn the surface integral into a volume integral. For the evaluation of the volume integral it is useful to switch to spherical coordinates, i.e.

$$x = r\sin(\theta)\cos(\varphi)$$
$$y = r\sin(\theta)\sin(\varphi)$$
$$z = r\cos(\theta)$$
$$\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = r^2 \sin(\theta)\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\varphi$$



Question 4 .... 0 points

Compute eigenvalues and eigenvectors of the following matrices:

(a)

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \tag{5}$$

(b)

$$\begin{pmatrix} 7 & 2 \\ 6 & 3 \end{pmatrix} \tag{6}$$