Exercise 6: Stress invariants

29.11.2021 - 03.12.2021

Question 1 ....

Analyse the plane stress

$$\underline{\sigma} = \begin{pmatrix} 3 & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & 2 \end{pmatrix}$$

(a) Find the rotation angle $\phi_{\sigma,\text{max}}$ at which the diagonal stress entries are maximal.

(b) Which values take the principal or main stresses σ_1 and σ_2? You can use the formulas derived in the lecture.

(c) As well find the rotation angle $\phi_{\tau,\text{max}}$ at which the shear stress is maximal and compute the value for the maximal shear stress τ_{max}.

(d) In the lecture it was shown that not only the principal stresses can characterize a stress state but also the stress invariants. Compute the stress invariants I_1 and I_2.

(e) The dimension of a stress as well as the dimension of the principal stresses is force per area. What are the dimensions of the two stress invariants I_1 and I_2.

Solution:

(a) In the lecture we have derived the formula

$$\tan(2\phi_{\sigma,\text{max}}) = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

By plugging in the numbers and equating out we find

$$\phi_{\sigma,\text{max}} = \frac{\arctan(\sqrt{3})}{2} = \frac{\pi}{6} = 30°$$

(b) Method 1: use the formulas

$$\sigma_{1/2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_y - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \frac{5}{2} \pm 1 = \begin{cases} \sigma_1 = \frac{7}{2} \\ \sigma_2 = \frac{3}{2} \end{cases}$$

Method 2: rotate the stress by the angle $\phi_{\sigma,\text{max}} = 30°$

$$\underline{R} = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix} \overset{\alpha=\phi_{\sigma,\text{max}}}{=} \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$$

$$\underline{\sigma}' = \underline{R}^T \cdot \underline{\sigma} \cdot \underline{R} = \begin{pmatrix} \frac{7}{2} & 0 \\ 0 & \frac{3}{2} \end{pmatrix}$$

(c) We use again the formula from the lecture

$$-\tan(2\phi_{\tau,\text{max}}) = \frac{\sigma_x - \sigma_y}{2\tau_{xy}} \qquad \Rightarrow \qquad \phi_{\tau,\text{max}} = -\frac{\arctan(\frac{1}{\sqrt{3}})}{2} = -\frac{\pi}{12} = -15° = \begin{cases} -15° \\ -15° + 90° \end{cases}$$

As in part (b) we can either use the formula

$$\tau_{\text{max}} = \pm\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \pm\frac{\sigma_1 - \sigma_2}{2} = \pm 1$$

or rotate the matrix by the angle $\phi_{\tau,\max} = \begin{cases} 15° \\ 75° \end{cases}$

$$\underline{R}_{-15°} = \begin{pmatrix} \frac{\sqrt{3}+1}{2\sqrt{2}} & \frac{\sqrt{3}-1}{2\sqrt{2}} \\ -\frac{\sqrt{3}-1}{2\sqrt{2}} & \frac{\sqrt{3}+1}{2\sqrt{2}} \end{pmatrix} \qquad \underline{R}_{75°} = \begin{pmatrix} \frac{\sqrt{3}-1}{2\sqrt{2}} & -\frac{\sqrt{3}+1}{2\sqrt{2}} \\ \frac{\sqrt{3}+1}{2\sqrt{2}} & \frac{\sqrt{3}-1}{2\sqrt{2}} \end{pmatrix}$$
$$\sigma' = \underline{R}^T_{-15°} \cdot \underline{\sigma} \cdot \underline{R}_{-15°} = \begin{pmatrix} \frac{5}{2} & 1 \\ 1 & \frac{5}{2} \end{pmatrix} \qquad \sigma' = \underline{R}^T_{75°} \cdot \underline{\sigma} \cdot \underline{R}_{75°} = \begin{pmatrix} \frac{5}{2} & -1 \\ -1 & \frac{5}{2} \end{pmatrix}$$

(d) Method 1: The stress invariants I_i are the *negative* coefficients of the characteristic polynomial

$$\det(\sigma - \lambda \mathbb{1}) = \lambda^2 - 5\lambda + \frac{21}{4} \qquad \Rightarrow \qquad I_1 = 5\,, \quad I_2 = -\frac{21}{4}$$

Method 2: By the formulas derived in the lecture

$$I_1 = \mathrm{tr}(\sigma) = 5$$
$$I_2 = -\sigma_x\sigma_y + \tau_{xy}^2 = -\sigma_1\sigma_2 = -\det(\sigma) = -\frac{21}{4}$$

(e) I_1 is the trace of the stress and thus has the same dimension as the stress $(force/area)$. I_2 is the product of the two main stresses and thus hast the dimension $(force/area)^2$.

Question 2 . . . .

The following stress tensor characterises a special stress state

$$\underline{\sigma} = \begin{pmatrix} \sqrt{2} & \sqrt{2} \\ \sqrt{2} & -\sqrt{2} \end{pmatrix}$$

(a) Compute the angle $\phi_{\sigma,\max}$ at which the normal stresses takes its maximal value.

(b) Use the general rotation matrix and the computed angle $\phi_{\sigma,\max}$ to rotate the stress state in the coordinate system of maximal normal stress. What are the values for the principal stresses?

(c) What is the special name for the stress state found in (b)?

(d) In the lecture we have derived two rotation angles to rotate the stress from maximal diagonal elements into the coordinate system where the shear stress is maximal. Take one of the two angles and rotate the stress computed in (b) by the rotation matrix to find the stress state with maximal shear stress.

Solution:

(a) As in Q1 (a) we use the formula derived in the lecture and find

$$\phi_{\sigma,\max} = \frac{\arctan(1)}{2} = \frac{\pi}{8} = 22.5°$$

(b) Now we are explicitly asked to rotate the stress into the coordinate system of maximal normal stresses.

$$\underline{R}_{22.5°} = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix} \overset{\alpha=\phi_{\sigma,\max}}{=} \begin{pmatrix} \frac{\sqrt{2+\sqrt{2}}}{2} & -\frac{\sqrt{2-\sqrt{2}}}{2} \\ \frac{\sqrt{2-\sqrt{2}}}{2} & \frac{\sqrt{2+\sqrt{2}}}{2} \end{pmatrix}$$
$$\sigma_{\sigma,\max} = \underline{R}^T_{22.5°} \cdot \sigma \cdot \underline{R}_{22.5°} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$$

We find the two principal stresses $\sigma_1 = 2$, $\sigma_2 = -2$.

(c) The two principal stresses are the negative of each other. This stress state is called *pure shear* stress. By transforming into the coordinate system of maximal shear stress in part (d) we will see more clearly why this state is called pure shear.

(d) In the lecture we have shown that a rotation by $45°$ or $135°$ rotates a stress state from maximal normal stresses to maximal shear stresses. The two rotation matrices are given by

$$\underline{R}_{45°} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \qquad \underline{R}_{135°} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

You can use one of them to find the stress state in the coordinate system where the shear stress is maximal

$$\sigma_{\tau,\max} = \underline{R}^T_{45°} \cdot \underline{\sigma}_{\sigma,\max} \cdot R_{45°} = \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}$$

$$\sigma_{\tau,\max} = R^T_{135°} \cdot \sigma_{\sigma,\max} \cdot R_{135°} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$$

Question 3 ....

Now we have a more general three dimensional stress state given by

$$\underline{\sigma} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 2 \\ 3 & 2 & 1 \end{pmatrix}$$

(a) Compute the three principal stresses which are the eigenvalues of the stress tensor.

(b) What are the values of the three invariants I_1, I_2 and I_3 of the given stress state?

(c) Compute the hydrostatic stress σ_h.

(d) Compute the deviatoric stress s_{ij} which is also called stress deviator.

(e) Which values take the invariants J_1, J_2 and J_3 of the stress deviator.

(f) What is the value of the von Mises stress?

(g) What is special about J_2 and why is the von Mises stress derived from J_2?

Solution:

(a) Compute the eigenvalues from the characteristic polynomial

$$\det(\sigma - \lambda \mathbb{1}) = -\lambda^3 + 6\lambda^2 + 8\lambda - 16 \stackrel{!}{=} 0$$

Either you use a computer or you can solve this equation also by hand. You can guess the solution $\lambda_1 = -2$ and use polynomial long division to find the other two roots.

$$(-\lambda^3 + 6\lambda^2 + 8\lambda - 16) : (\lambda + 2) = -\lambda^2 + 8\lambda - 8$$

$$-\lambda^2 + 8\lambda - 8 \stackrel{!}{=} 0 \quad \Rightarrow \quad \lambda_{2/3} = 2(2 \mp \sqrt{2})$$

So we find:

$$\sigma_1 = -2 \quad , \qquad \sigma_2 = 2(2 - \sqrt{2}) \quad , \qquad \sigma_3 = 2(2 + \sqrt{2})$$

(b) The three invariants I_1, I_2 and I_3 can be computed by the formulas given in the lecture or more easy can be directly read from the coefficients of the characteristic polynomial.

$$\begin{aligned} 0 &= -\lambda^3 \underbrace{+6}_{=I_1} \lambda^2 \underbrace{+8}_{=I_2} \lambda \underbrace{-16}_{=I_3} \\ I_1 &= \mathrm{tr}(\sigma) = 1 + 4 + 1 = +6 \\ I_2 &= \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_1\sigma_3 = -8 \\ I_3 &= \sigma_1\sigma_2\sigma_3 = -16 \end{aligned}$$

(c) The hydrostatic stress in three dimensions is given by

$$\sigma_h = \frac{1}{3}\mathrm{tr}(\sigma) = \frac{1}{3}I_1 = +2$$

(d) The deviatoric stress s_{ij} is a pure shear stress and given by

$$s = \sigma - \sigma_h \mathbb{1} = \begin{pmatrix} -1 & 2 & 3 \\ 2 & 2 & 2 \\ 3 & 2 & -1 \end{pmatrix}$$

(e) The invariants J_1, J_2 and J_3 of the stress deviator can be either be found by constructing the characteristic polynomial or by using the formulas from the lecture. As the trace of the deviatoric stress is zero by construction J_1 is always equal to zero.

$$\begin{aligned} 0 &= -\lambda^3 \underbrace{+0}_{=J_1} \lambda^2 \underbrace{+20}_{=J_2} \lambda \underbrace{+16}_{J_3} \\ J_1 &= 0 \\ J_2 &= \frac{1}{6}\left((\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2\right) = +20 \\ J_3 &= \det(s) = +16 \end{aligned}$$

(f) The von Mises stress can be computed directly from J_2

$$\sigma_{1/2}\mathrm{vM} = \sqrt{3J_2} = \sqrt{60} = 2\sqrt{15}$$

(g) The von Mises stress is a measure for a material if it undergoes plastic deformation or other failure. Typically materials only deform plastically under shear stress. In the lecture we have derived in two dimensions that $\sqrt{J_2}$ is equal to the maximal shear stress $\sigma_{\tau,\mathrm{max}}$. Thus the von Mises stress is the maximal shear stress and therefore a direct measure to propose if a material undergoes plastic deformation under a certain stress state.