$W = \int_a^b F(r(t)) \cdot r'(t)\, dt$

¿Como se parametriza una curva?

Integral de linea

→ Evaluar W $\quad F(x,y,z) = xy\hat{i} + yz\hat{j} + zx\hat{k}$ y C es la cubica torcida $x = t$; $y = t^2$ $\quad z = t^3$ $\quad 0 \le t \le 1$

$r(t) = t + t^2 + t^3$

$r'(t) = \hat{i} + 2t\hat{j} + 3t^2\hat{k}$

$F(r(t)) = t^3\hat{i} + t^5\hat{j} + t^4\hat{k}$

// Steward

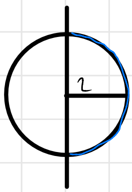
$t^3 + 2t^6 + 3t^6 = t^3 + 5t^6$

$$\int_0^1 (t^3 + 5t^6)\, dt = \frac{t^4}{4} + \frac{5}{7}t^7 \Big|_0^1 = \frac{27}{28}$$

33) Un alambre doblado en forma de semicircunferencia $x^2 + y^2 = 4$ calcule la masa y el centro de masa $x > 0$

$\int_a^b f \cdot ds$

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \cdot dt = 2\, dt$$



$$\bar{x} = \frac{1}{2k\pi} \int_{-\pi/2}^{\pi/2} 2\cos t \; k \; 2\, dt$$

$$\bar{x} = \frac{1}{m} \int_C x\, \rho(x,y)\, ds \qquad -\frac{\pi}{2} < t < \frac{\pi}{2}$$

$$\bar{y} = \frac{1}{m} \int_C y\, \rho(x,y)\, ds$$

$$\bar{x} = \frac{4k}{2k\pi} \int_{-\pi/2}^{\pi/2} \cos t\, dt$$

$x = 2\cos t$

$y = 2\sin t$

$m = \int_C \rho(x,y)\, ds$

$$\bar{x} = \frac{4}{2\pi} \sin t \Big|_{-\pi/2}^{\pi/2} = \frac{4}{\pi}$$

$m = k \int ds$

$m = 2kt \Big|_{-\pi/2}^{\pi/2}$

$-2\sin t$

$2\cos t$

$m = 2k\pi$

$$\bar{y} = \frac{1}{2\cancel{\mu}\pi} \int_{-\pi/2}^{\pi/2} 2\sin t \cdot \cancel{\mu}\, 2\,dt$$

$$\bar{y} = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} 2\sin t\ 2\,dt \rightarrow \quad 4 \int \sin t\,dt$$

$$4 \quad -\cos t \Big|_{-\pi/2}^{\pi/2}$$

$$= \emptyset$$

$\bar{y} = 0$

$$\bar{x} = \frac{1}{2\cancel{\mu}\pi} \int_{-\pi/2}^{\pi/2} 2\cos t \cdot \cancel{\mu}\, 2\,dt \quad = \int 2\cos t\, 2\,dt \quad = 4 \int \cos t\,dt$$

$$4 \quad \sin t \Big|_{-\pi/2}^{\pi/2}$$

$\left(\frac{4}{\pi}, 0\right)$

$$= 4 \cdot 2$$

$$= \cancel{8} \cdot \frac{1}{\cancel{2}\pi} = \frac{4}{\pi}$$

Teorema de Green

$$\int_C P\,dx + Q\,dy = \iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \partial A = \int_{\partial D} P\,dx + Q\,dy$$

$z = (x, y)$

Calcule $\oint (3y - e^{\sin x}) + (7x + \sqrt{y^4 + 1})\,dy$ donde C

es la circunferencia $x^2 + y^2 = 9$

$P = 3y - e^{\sin x}$

$r = 3$
$x = 3\cos\theta$
$y = 3\sin\theta$

$Q = 7x\sqrt{y^4 + 1}$

$\frac{\partial P}{\partial y} = 3$

$\frac{\partial Q}{\partial x} = 7$

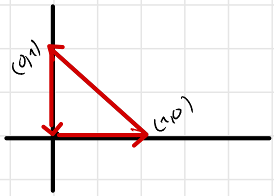
$$\int_0^{2\pi}\int_0^3 (7-3)\, r\, dr\, d\theta = 2r^2\Big|_0^3 \quad 18 \cdot \int_0^{2\pi} d\theta = 18\,\theta\Big|_0^{2\pi}$$

$$= 36\pi$$

$$4\int_0^{2\pi}\int_0^3 dA \qquad 4 \cdot \pi r^2$$

Utilize Green para hallar W que realiza $F(x,y) = x(x+y)\hat{i} + xy^2\hat{j}$ al desplazar una partícula desde el origen hasta (1,0) luego a lo largo del segmento rectilíneo (0,1) y después el regreso al origen por el eje y

$$W = \int F \cdot ds \qquad ds \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dt}{dt}\right)^2}$$



$$P = x(x+y) = 2x + xy$$

$$Q = xy^2$$

$$\frac{\partial P}{\partial y} = x$$

$$\frac{\partial Q}{\partial x} = y^2$$

$$\Rightarrow \int_0^1\int_0^{1-x} \frac{1}{2} x - y^2 \cdot dx\, dy = \frac{1}{2}\int_0^{1-x}\int_0^1 x - y^2\, dx\, dy =$$

$$\frac{1}{2}\int_0^{1-x} x - y^2\, dy = \frac{1}{2}xy - \frac{1}{2}\frac{y^3}{3}$$

$$= \frac{1}{2}xy - \frac{1}{6}y^3 \Big|_0^{1-x}$$

$$= \frac{1}{2}x(1-x) - \frac{1}{6}(1-x)^3$$

$$= \frac{1}{2}x - x^2 - \frac{1}{6}(1-x)^3$$

$$= \frac{1}{2}(x - x^2) - \frac{1}{6}(1 - x^3)$$

$$\int_0^1 \frac{1}{2}(x - x^2) - \frac{1}{6}(1 - x^3)\, dx$$

$$\frac{1}{3}\int_0^1 (x - x^2)\, dx - \int_0^1 (1 - x^3)\, dx$$

$$u = x - x^2$$

$$du = -2x\, dx = \frac{du}{-2x} = dx$$

Stokes $\longrightarrow$ $S \to \mathbb{R}^3$

$\operatorname{rot} \vec{F} = \nabla \times F$

$$\int_C F\, dr = \int_S \int \operatorname{rot} f \cdot ds = \iint_R \operatorname{rot} F \underbrace{\cdot N\, ds}_{dA}$$

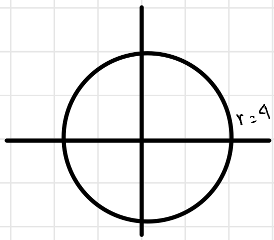
$r(t)$, $a \le t \le b \Rightarrow$ la integral de linea F a lo largo de C

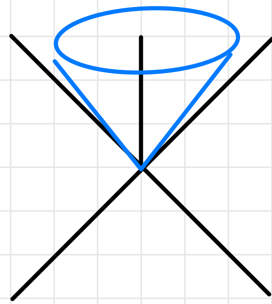
$$\int_C F \cdot dr = \int_a^b f(r(t)) \cdot r'(t) = \int_C F \cdot \overset{\to \text{ vector unitario tangente}}{T}\, ds$$

$\vec{f}(x,y,z) = -y\hat{\imath} + x\hat{\jmath} - z\hat{k}$ y $S \Rightarrow$ cono $z^2 = x^2 + y^2$ $\quad 0 \le z \le 4$ $\quad 4^2 = x^2 + y^2 = 4$

$F(r(t)) = 4\sin\theta\,\hat{\imath} + 4\cos\theta\,\hat{\jmath} - 2\hat{k}$

$r'(t) = -4\sin\theta - 4\cos t + 0\hat{k}$





$$\int_0^{2\pi} (-16\sin^2\theta - 16\cos^2)\, d\theta$$

$$-16\, \cancel{(\sin^2\theta - \cos^2\theta)}^{1}\, d\theta$$

$$-16\int_0^{2\pi} d\theta = 32\pi$$

$x = r\cos\theta$

$y = -r\sin\theta$

$z = -4$

$$\uparrow \iint_C F \cdot N\, ds = F\,[-g_x(x,y) - g_y(x,y) + k]\, dA \quad 0 \le \theta \le 2\pi$$

$r(t) = 4\cos\theta - 4\sin\theta + 4$

$r'(t) = -4\sin\theta - 4\cos t$

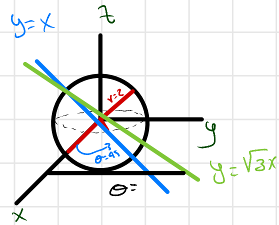
/la region que se esta study

$$\downarrow \iint_C F \cdot [g_x(x,y) + g_y(x,y) - k]\, dA$$

Estudio

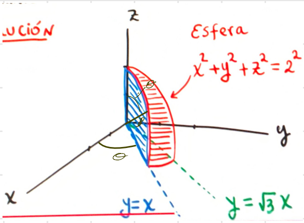
Triples en esfericas

1. Calcular el volumen del solido que esta acotado por $x^2+y^2+z^2=4$, $y=x$, $y=\sqrt{3}x$, $z=0$ en el primer octante



Si $y=x \rightarrow \frac{y}{x}=1$ $\quad \tan\theta=\frac{y}{x}=1$ $\quad \theta=\tan^{-1}(1)=45^\circ=\frac{\pi}{4}$

$\theta: \frac{y}{x}=\sqrt{3} = \tan^{-1}(\sqrt{3})=60=\frac{\pi}{3}$



$x=\rho\sin\phi\cos\theta \quad = \rho=r$

$y=\rho\sin\phi\sin\theta$

$z=\rho\cos\theta$

$x^2+y^2+z^2=2^2$

$r^2\sin^2\phi\cos^2\theta + r^2\sin^2\phi\sin\theta + r^2\cos\phi$

$r^2\sin^2\phi\,(\underbrace{\cancel{\cos^2\theta+\sin^2\theta}}_{=1}) + r^2\cos^2\phi$

$r^2\sin^2\phi + r^2\cos^2\phi = r^2(\underbrace{\cancel{\sin^2\phi+\cos^2\phi}}_{=1})$

$x^2+y^2+z^2=4 \rightsquigarrow r^2=4 = r=2$

$y=x$

$\cancel{r^2}\cancel{\sin\phi}\sin\theta = \cancel{r^2}\cancel{\sin\phi}\cos\theta$

$\sin\theta = \cos\theta \rightarrow \theta=\frac{\pi}{4}$

$r=2$ $\quad \pi/4 \le \theta \le \pi/3$ $\quad V=\iiint_D dv = \iiint r^2\sin\phi\,dr\,d\phi\,d\theta$

$0\le\phi\le\pi/2$

$0\le r\le 2$ $\quad \int_{\pi/4}^{\pi/3} d\theta \int_0^{\pi/2}\sin\phi\,d\phi \int_0^2 r^2\,dr$

$y=\sqrt{3}x$

$\theta=\frac{\pi}{3}$

$\theta\Big|_{\pi/4}^{\pi/3} \quad -\cos\phi\Big|_0^{\pi/2} \quad \frac{1}{3}r^3\Big|_0^2$

$\frac{\pi}{3}-\pi/4 \quad -0+1+\frac{8}{3}$

$\frac{\pi}{12}+1+\frac{8}{3} = \frac{\pi}{12}+\frac{11}{3}$

2 $\iiint_B e^{(x^2+y^2+z^2)^{3/2}}\, dv$, donde B es la Bola unitaria

$B = \{(\rho,\theta,\phi) \mid 0 \le \rho \le 1,\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi\}$

Si $x^2+y^2+z^2 = \rho^2$

$$\int_0^{\pi}\int_0^{2\pi}\int_0^{1} e^{(\rho^2)^{3/2}} \rho^2 \sin\phi \, d\rho\, d\theta\, d\phi$$

$$= \int_0^{\pi} \sin\phi \int_0^{2\pi} d\theta \int_0^1 \rho^2 e^{(\rho^2)^{3/2}}$$

$-\cos\phi \Big|_0^{\pi}$ $\quad$ $b\,\theta \Big|_0^{2\pi}$ $\quad$ $\int_0^1 \rho^2 e^{(\rho)} \to \frac{1}{3}\int_0^1 e^{(u)} = \frac{1}{3} e^{(\rho^3)} \Big|_0^1 = \frac{1}{3} e^{(1)} - \frac{1}{3} e^{(0)}$

$1+1 \quad + \quad 2\pi - 0$

si $\rho^3 = u$

$du = 3\rho^2$

$= \frac{1}{3} e - 1$

$$= 2 + 2\pi + \frac{1}{3} e - 1$$

$$= \frac{4}{3}\pi(e-1)$$

3. Cambiar de rectangulares a esferas $(\sqrt{3}, -1, 2\sqrt{3})$

$\rho^2 = (\sqrt{3})^2 + (-1)^2 + (\sqrt{3})^2$

$\rho^2 = 3 - 1 + 3 \quad = \sqrt{5}$

$\tan\theta = \frac{y}{x} = -\frac{1}{\sqrt{3}}$

$\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -30 = -\frac{1}{6}\pi$

$\cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) = \frac{\sqrt{3}}{\sqrt{5}} = 37{,}76$

$(\sqrt{5}, -30^\circ, 37{,}76^\circ)$

Escribir la ecuación en coordenadas esfericas

$x^2+z^2 = 9$

$x = \rho \sin\phi \cos\theta$

$y = \rho \sin\phi \sin\theta$

$z = \rho \cos\theta$

$$\rho^2 \sin^2\phi \cos^2\theta + \rho^2\cos^2\theta = 9$$

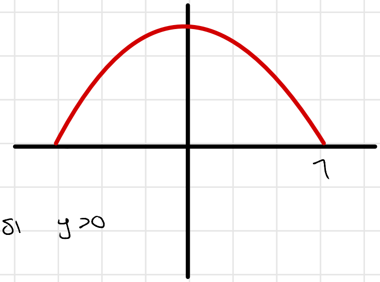
$$\rho^2(\cos^2\theta \sin^2\phi + \cos\theta)$$

$x + 2y + 3z = 1$

$\rho \sin\phi\cos\theta + 2(\rho\sin\phi\sin\theta) + 3(\rho\cos\theta) = 1$

Integral de línea

1 $\displaystyle\int_C (2+x^2y)\,ds$, donde C es la mitad superior de la circunferencia



Unitaria $x^2+y^2=1$ si $y>0$

parametrizar $x=\cos t$

$y=\sin t$

$$\int_C (2+x^2y)\,ds$$

$$\to \int_0^{\pi} (2+\cos^2 t\cdot\sin t)\cdot\sqrt{\frac{dx}{dt}(\cos t)^2+\frac{dy}{dt}(\sin t)^2}\;dt$$

$$\to \int_0^{\pi} (2+\cos^2 t\cdot\sin t)\cdot\sqrt{\sin^2 t+\cos^2 t}\;dt$$

$$\to 2\int_0^{\pi} 2(\cos^2 t\cdot\sin t)\cdot\sqrt{\cancel{\sin^2 t+\cos^2 t}}^{\,1}\;dt$$

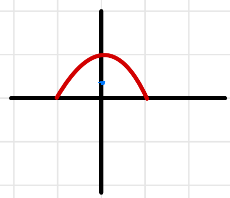
$$\int_0^{\pi} (\cos^2 t\cdot\sin t)\;dt \quad \text{sustitución}$$

$$\int_0^{\pi} 2\,dt \qquad \downarrow \qquad -\frac{\cos^3(t)}{3}\Big|_0^{\pi} \qquad = 2\pi-\underbrace{\frac{1}{3}\cos^3(\pi)}_{1/3}+\underbrace{\frac{1}{3}\cos^3(0)}_{1/3}$$

$$2\int_0^{\pi} dt \qquad\qquad = 2\pi+\frac{2}{3}$$

$$2t\Big|_0^{\pi}$$

2. **V EJEMPLO 3** Un alambre toma la forma de una semicircunferencia $x^2+y^2=1$, $y \geqslant 0$, y es más grueso cerca de su base que de la parte superior. Calcule el centro de masa del alambre si la densidad lineal en cualquier punto es proporcional a su distancia desde la recta $y=1$.



Parametrizamos $x=\cos t$ $y=\sin t$ $0\leq t\leq\pi$

Como la densidad es lineal $\rho(x,y)=k(1-y)$

$$m=\int_C k(1-y)\,ds=\int_0^{\pi} k(1-\sin t)\,dt = k\int(1-\sin t)\,dt$$

$$\int 1\,dt-\int \sin t\,dt$$

$$t\Big|_0^{\pi}+\cos t\Big|_0^{\pi}$$

$$k(\pi-2)$$

$$\bar{y} = \frac{1}{m}\int_c y\,\rho(x,y)\,ds = \frac{1}{\cancel{k}(\pi-2)}\int_C y\,\cancel{k}(1-y)\,ds$$

$$= \frac{1}{\pi-2}\int \sin t\,(1-\sin t)\,dt$$

$$\int \sin t - \int \sin^2 t\,dt$$

$$-\cos - \int \frac{1-\cos(2t)}{2}\,dt$$

$$\hookrightarrow \frac{1}{2}\int 1-\cos(2t)\,dt$$

$$\hookrightarrow \frac{1}{2}t - \frac{\sin(2t)}{2}$$

$$-\cos t\Big|_0^{\pi} \quad \frac{1}{2}t\Big|_0^{\pi} \qquad -\frac{\sin(2\pi)}{2}\Big|_0^{\pi}$$

$$= \frac{4-\pi}{2(\pi-2)}$$

Halle el trabajo realizado por el campo de fuerzas $\mathbf{F}(x, y) = x^2\,\mathbf{i} + xy\,\mathbf{j}$ sobre una partícula que se mueve una vez alrededor del círculo $x^2 + y^2 = 4$ orientado en sentido contrario a las manecillas del reloj.

$$W = \int_a^b F(r(t))\cdot r'(t)\,dt$$

$$W = \int F\cdot ds$$

$x = 2\cos t \qquad 0 \le t \le 2\pi$

$$F(r(t)) = (2\cos t)^2\,\hat{\imath} + (2\cos t)(2\sin t)\,\hat{\jmath} \qquad y = 2\sin t$$

$$4\cos^2 t\,\hat{\imath} + 2(\cos t\cdot\sin t) \qquad r(t) = 2\cos t + 2\sin t$$

$$r'(t) = -2\sin t + 2\cos t$$

$$4\cos^2 t\,\hat{\imath} + 4\cos t\cdot\sin t$$

$$\int_0^{2\pi} -8\cos^2 t\,\cancel{\sin t} + 8\cos^2 t\cdot\cancel{\sin t}\,dt = 0$$

Teorema fundamental de línea $(\vec{F} = \vec{\nabla} f)$

C es una curva a trozos sobre R región del objeto

$$F\hat{\imath} = \frac{\partial f}{\partial x} \; ; \; F\hat{\jmath} = \frac{\partial f}{\partial y} \; ; \; F\hat{k} = \frac{\partial f}{\partial z}$$

$$r(t) = x(t)\,\hat{\imath} + y(t)\,\hat{\jmath} + u(t)\,\hat{k} \qquad a \le t \le b$$

$F(x,y) = M\hat{\imath} + N\hat{\jmath} + (P\hat{k})$ $\quad F$ es conservativo si solo si : $\dfrac{\partial N}{\partial x} = \dfrac{\partial M}{\partial y}$

$$\int_C F \cdot dr = \int_C \nabla f \cdot dr = F(x(b), y(b)) - F(x(a), y(a))$$

Ejemplo: $F = \underbrace{e^x \sin y}_{M} + \underbrace{e^x \cos y}_{N}$ y C = cicloide $\quad x = \theta \sin\theta$ desde $(0,0)$ hasta $(\pi, 0)$

$\qquad y = 1 - \cos\theta$

$$\frac{\partial M}{\partial y} = e^x \cos y \qquad \frac{\partial N}{\partial x} = e^x \cos y$$

metodo 1

$$\left.\begin{aligned} \int e^x \sin y \, dx &= e^x \sin y + g(y) \\ \int e^x \cos y \, dy &= e^x \sin y + h(x) \end{aligned}\right\} \quad f(x,y) = e^x \sin y + g(y) + h(x) \implies g(y) \; ; \; h(y) = k$$

$$f(x,y) = e^x \sin y + k \Big|_{(0,0)}^{(\pi,0)}$$

$$e^{\pi} \sin(0) - e^{0} \sin(0) = 0$$

$$\text{rot } \vec{F} = \nabla \times F \qquad \begin{array}{ccc} + & - & + \end{array}$$
$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & -2 \end{vmatrix} = (0)\hat{\imath} - (0)\hat{\jmath} + (1-(-1))\hat{k} = 2\hat{k}$$

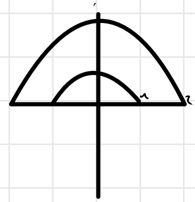
$$\iint_R -2 \, dA = -2 \int_0^{2\pi} \int_0^4 r \, dr \, d\theta = -2 \cdot \frac{r^2}{2} \Big|_0^4$$

$$2 \cdot \frac{16}{2}$$

$$2 \cdot 8 \cdot 2\pi = 32\pi$$

Teorema de Green

Evalue $\oint_C y^2\,dx + 3xy\,dy$, donde C es la frontera de la región semianular D entre las circunferencias $x^2+y^2=1$ y $x^2+y^2=4$ en el semiplano superior



r, θ

$1 \le r \le 2$

$0 \le \theta \le \pi$

$$\oint_C \underbrace{y^2\,dx}_{Q} + \underbrace{3xy\,dy}_{P} = \int_0^{\pi}\int_1^{2} r\sin\theta \; r\,dr\,d\theta$$

$dA = r\,dr\,d\theta$

$x = r\cos\theta$

$y = r\sin\theta$

$$Q_y = 2y \quad = \quad 2y - 3y = y$$

$$P_x = 3y$$

$$\int_0^{\pi}\int_1^{2} r^2 \sin\theta\,dr\,d\theta = \frac{r^3}{3}\sin\theta\Big|_1^2 = \frac{r^3}{3}\int_0^{\pi}\sin\theta\,d\theta$$

$$= -\cos\theta\Big|_0^{\pi} = \left(\frac{8}{3} - \frac{1}{3}\right) + \left(\cos(\pi) - \cos(0)\right)$$

$$\frac{7}{3} \quad + \quad 2 \quad = \frac{13}{3}$$