INSTRUCTOR'S SOLUTIONS MANUAL

to accompany

ADVANCED ENGINEERING MATHEMATICS

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Contents

Chapter One - First Order Differential Equations

Section 1.1 Preliminary Concepts

1. For $x > 1, 2\varphi\varphi' = 2\sqrt{x-1}\dfrac{1}{2\sqrt{x-1}} = 1$, so φ is a solution.

2. With $\varphi(x) = ce^{-x}$ we have $\varphi' + \varphi = -ce^{-x} + ce^{-x} = 0$, so φ is a solution.

3. For $x > 0$, we rewrite the equation as $2xy' + 2y = -e^x$. Then with $\varphi = \dfrac{1}{2}x^{-1}(c - e^x)$, we have $2x\varphi' + 2\varphi = -x^{-1}(c - e^x) + (-e^x) + x^{-1}(c - e^x) = -e^x$, so φ is a solution.

4. For $x \neq \pm\sqrt{2}$, we have $\varphi' = \dfrac{-2xc}{(x^2-2)^2} = \left(\dfrac{2x}{2-x^2}\right)\left(\dfrac{c}{x^2-2}\right) = \dfrac{2x\varphi}{2-x^2}$, so φ is a solution.

5. On any interval not containing $x = 0$ we have, $x\varphi' = x\left(\dfrac{1}{2} + \dfrac{3}{2x^2}\right) = x + \left(\dfrac{3}{2x} - \dfrac{x}{2}\right) = x - \left(\dfrac{x^2-3}{2x}\right) = x - \varphi$, so φ is a solution.

6. For all $x, \varphi' + \varphi = -ce^{-x} + (1 + ce^{-x}) = 1$, thus $\varphi(x) = 1 + ce^{-x}$ is a solution for all x.

In 7 - 11, recall that for y defined implicitly by $F(x,y) = C$ we have $\dfrac{\partial F}{\partial x} + \dfrac{\partial F}{\partial y} \cdot y' = 0$ for all (x,y) for which the partials F_x and F_y exist.

7. With $F(x,y) = y^2 + xy - 2x^2 - 3x - 2y = C$, we have $y - 4x - 3 + (2y + x - 2)y' = 0$.

8. With $F(x,y) = xy^3 - y = C$, we have $y^3 + (3xy^2 - 1)y' = 0$.

9. With $F(x,y) = y^2 - 4x^2 + e^{xy} = C$, we have $8x - ye^{xy} - (2y + xe^{xy})y' = 0$.

10. With $F(x,y) = 8\ln|x-2y+4| - 2x + 6y = C$, we have $\dfrac{8}{x-2y+4} - 2 + \left(\dfrac{-16}{x-2y+4} + 6\right)y' = 0$. Solving for y' gives $y' = \dfrac{x-2y}{3x-6y+4}$.

11. With $F(x,y) = \tan^{-1}\left(\dfrac{y}{x}\right) + x^2 = C$, we have $\dfrac{-y}{x^2+y^2} + 2x + y'\left(\dfrac{x}{x^2+y^2}\right) = 0$.

12. Direct integration gives $y = x^2 + C$. The initial condition $y(2) = 1$ gives $1 = 4 + C$ so $C = -3$. The unique solution is $y = x^2 - 3$.

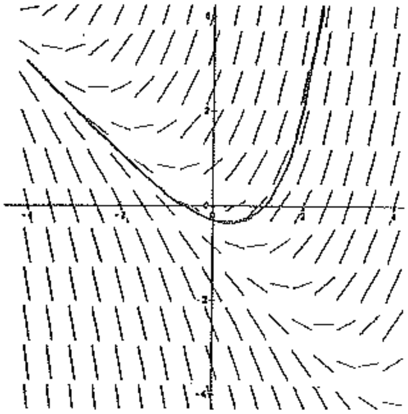
13. Direct integration gives $y = -e^{-x} + C$. The initial condition gives $2 = -1 + C$ so $C = 3$. The unique solution is $y = 3 - e^{-x}$.

14. $y = x^2 + 2x + 2$

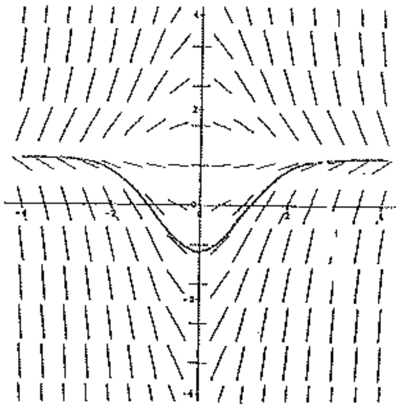
15. $y = 2\sin^2(x) - 2$

16. $y = 4x^2 + \dfrac{1}{2}\sin(2x) - 3$

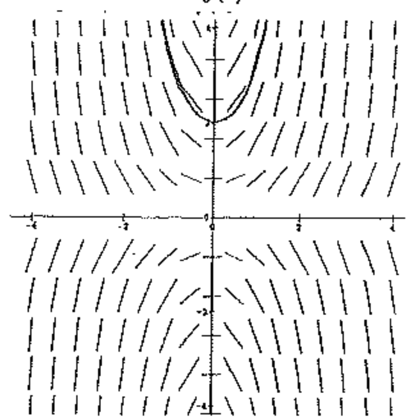
17. Direction field for $y' = x + y$;
Solution for $y(2) = 2$.



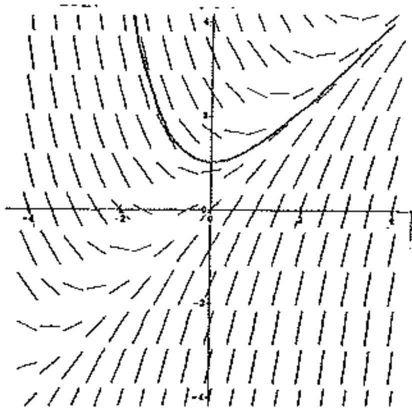
18. Direction field for $y' = x - xy$;
Solution for $y(0) = -1$.



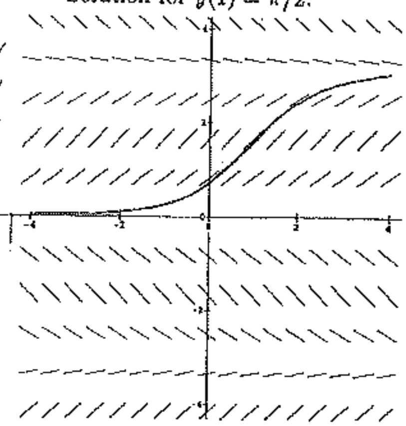
19. Direction field for $y' = xy$;
Solution for $y(0) = 2$.



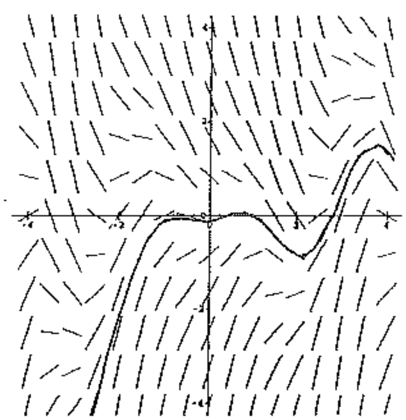
20. Direction field for $y' = x - y + 1$;
Solution for $y(0) = 1$.



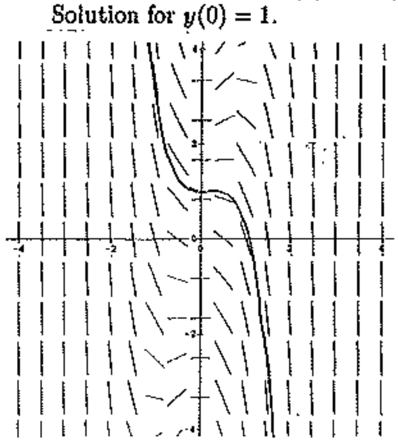
21. Direction field for $y' = \sin(y)$;
Solution for $y(1) = \pi/2$.



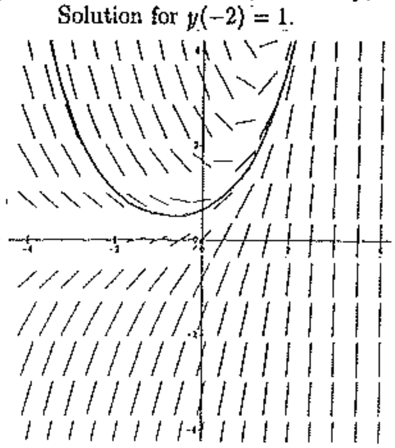
22. Dir. field for $y' = x\cos(2x) - y$;
Solution for $y(1) = 0$.



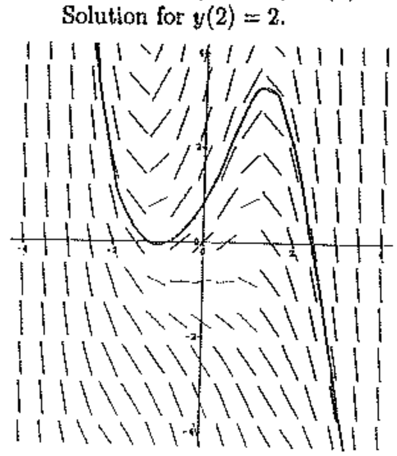
23. Dir. field for $y' = y\sin(x) - 3x^2$;
Solution for $y(0) = 1$.



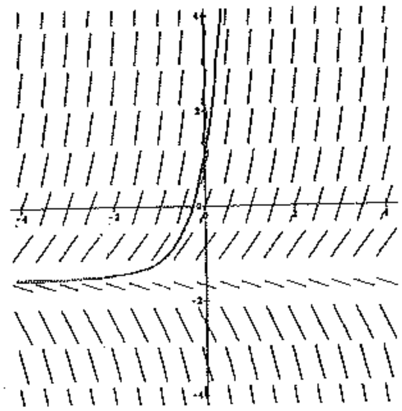
24. Direction field for $y' = e^x - y$;
Solution for $y(-2) = 1$.



25. Dir. field for $y' = 1 + y\cos(x) - x^2$;
Solution for $y(2) = 2$.



26. Direction field for $y' = 2y + 3$;
Solution for $y(0 = 1$.



27. Choose a point (x_0, z) on the line $x = x_0$. Then the solution curve through this point has slope $y'(x_0, z) = q(x_0) - p(x_0)z$, and the lineal element through this point has equation $y = [q(x_0) - p(x_0)z](x - x_0) + z$. The claim is that all these lines pass through the common point $\left(x_0 + \frac{1}{p(x_0)}, \frac{q(x_0)}{p(x_0)}\right)$. To see this, put $x = x_0 + \frac{1}{p(x_0)}$ in the tangent line equation and get

$$y = [q(x_0) - p(x_0)z](x_0 + \frac{1}{p(x_0)} - x_0) + z = \frac{q(x_0)}{p(x_0)}.$$

Section 1.2 Separable Equations

1. Separation of variables gives $3y^2dy = 4xdx$, and integration gives $y^3 = 2x^2 + K$.

2. The equation is separable as $\frac{dy}{y} = -\frac{dx}{x}$, where we have assumed $y \neq 0$ and $x \neq 0$. Integration gives $\ln|y| = -\ln|x| + c$, or more compactly, $\ln|xy| = c$. Exponentiation gives $xy = e^c = K$ where we have renamed the constant e^c as K. Removing any restrictions on K, we get all solutions in the form $xy = K$.

3. This equation is not separable, since $\sin(x + y)$ cannot be expressed as $A(x)B(y)$ for any A, B.

4. Since $e^{x+y} = e^x e^y$, the equation is separable as $e^y dy = 3xe^{-x}dx$. Integration gives $e^y = -3e^{-x}(x + 1) + C$.

5. The equation separates as $\frac{dx}{x} = \frac{dy}{y^2 - 1} + \left(\frac{1}{y} - \frac{1}{y-1}\right) dy$. Integration gives $\ln|x| + C = \ln\left|\frac{y}{y-1}\right|$, which can be rewritten as $y = Ax(y - 1)$. This can easily be solved for y to give $y = \frac{1}{1 - Ax}$. $y(x) \equiv 0$ is also a singular solution.

6. The equation is not separable.

7. The equation separates as $\dfrac{\sin(y)dy}{\cos(y)} = \dfrac{dx}{x}$, and integration gives $\sec(y) = Ax$.

8. Separation and use of partial fractions gives $\left(\dfrac{1}{y} - \dfrac{2y}{y^2+1}\right) dy = \left(\dfrac{1}{x} - \dfrac{1}{x+1}\right) dx$. Integration and exponentiation gives the implicitly defined solution $\dfrac{y^2}{y^2+1} = A\left(\dfrac{x}{x+1}\right)^2$.

9. Not separable

10. Expand the coefficient of dy by using addition formulas for $\cos(x+y)$ and $\sin(x-y)$ to get $[\cos(x+y) + \sin(x-y)] = [\cos(x)\cos(y) - \sin(x)\sin(y) + \sin(x)\cos(y) - \cos(x)\sin(y)] = [\cos(x)+\sin(x)][\cos(y)-\sin(y)]$. Also by a trig identity $\cos(2x) = \cos^2(x) - \sin^2(x) = [\cos(x)+\sin(x)][\cos(x)-\sin(x)]$. Thus the equation separates as $[\cos(y)-\sin(y)]dy = [\cos(x)-\sin(x)]dx$. The general solution is given implicitly by $\cos(y) + \sin(y) = \cos(x) + \sin(x) + C$.

11. Separation of variables and a bit of algebra gives $\left(y - 1 + \dfrac{1}{y+1}\right) dy = \dfrac{dx}{x}$. Integration yields $\dfrac{y^2}{2} - y + \ln|y+1| = \ln|x| + C$. The initial condition gives $2 - 2 + \ln(3) = \ln(3) + 2 + C$ so $C = -2$, and $\dfrac{y^2}{2} - y + \ln(y+1) = \ln(x) - 2$. Note: the absolute values on $\ln|y+1|$ and $\ln|x|$ can both be removed since the initial conditions ensure we are in a region of the xy-plane where $y > -1$ and $x > 0$.

12. $\dfrac{dy}{y+2} = 3x^2 dx$ integrates to give $\ln|y+2| = x^3 + C$. The initial condition gives $C = \ln(10) - 8$, so $\ln\left(\dfrac{y+2}{10}\right) = x^3 - 8$.

13. Writing $\ln(y^x) = x\ln(y)$, separation of variables gives $\dfrac{\ln(y)}{y} dy = 3x dx$, with solution $(\ln(y))^2 = 3x^2 + C$. Initial conditions give $9 = 12 + C$ so $C = -3$ and $(\ln(y))^2 = 3x^2 - 3$.

14. Since $e^{x-y^2} = e^x \cdot e^{-y^2}$, separation gives $2ye^{y^2} dy = e^x dx$, with solution $e^{y^2} = e^x + C$. The initial condition requires that $C = 0$ so $e^{y^2} = e^x$ or $y^2 = x$. Since $y < 0$, we have $y = -\sqrt{x}$.

15. Separation gives $y\cos(3y)dy = 2x dx$, with solution $\dfrac{y\sin(3y)}{3} + \dfrac{\cos(3y)}{9} = x^2 + C$. The initial condition requires $\dfrac{\pi}{9}\sin(\pi) + \dfrac{1}{9}\cos(\pi) = \dfrac{4}{9} + C$, so $C = -\dfrac{5}{9}$. The solution can be written as $3y\sin(3y) + \cos(3y) = 9x^2 - 5$.

16. By Newton's law of cooling the temperature is given by the solution of $\dfrac{dT}{dt} = k(T - 60)$; $T(0) = 90$ and $T(10) = 88$. Separation of variables and the initial condition $T(0) = 90$ easily give $T(t) = 60 + 30e^{kt}$. The condition $T(10) = 88$ gives $88 = 60 + 30e^{10k}$ or $e^{10k} = \dfrac{14}{15}$. At this point we could either solve for $k = \dfrac{1}{10}\ln\left(\dfrac{14}{15}\right) \approx -6.899287 \times 10^{-3}$, or noting that $e^{10k} = \dfrac{14}{15}$, write the solution as $T(t) = 60 + 30(e^{10k})^{t/10} = 60 + 30\left(\dfrac{14}{15}\right)^{t/10}$. In 20 minutes,

$T(20) = 60 + 30\left(\frac{14}{15}\right)^2 = 86.13°F$. To reach 65°, solve $65 = 60 + 30\left(\frac{14}{15}\right)^{t/10}$ to get $t = \frac{10\ln(1/6)}{\ln(14/15)} \approx 259.7$ minutes.

17. Let $t \geq 0$ denote the time in minutes since the thermometer was removed from the house which was at $70°F$. Let A denote the unknown outside ambient temperature (assumed constant). The temperature of the thermometer is modeled by the problem $\frac{dT}{dt} = k(T - A)$, with $T(0) = 70, T(5) = 60$ and $T(15) = 50.4$. We wish to find A. Separation of variables and $T(0) = 70$ readily yields $T(t) = A + (70 - A)e^{kt}$. The other two temperature readings at $t = 5$ and $t = 15$ give the equations. $T(5) = 60 = A + (70 - A)e^{5k}$ and $T(15) = 50.4 = A+(70-A)e^{15k}$. To solve these equations for A, solve the first for $e^{5k} = \left(\frac{60 - A}{70 - A}\right)$, substitute into the second to get $(70 - A)\left(\frac{60 - A}{70 - A}\right)^3 = (50.4 - A)$ and simplify to get the quadratic equation $10.4A^2 - 1156A + 30960 = 0$ which has solutions $A = 45$ and $A = 66.15$. Clearly we must have $A < 50.4$, so $A = 45°$.

18. The solution of the equation which models this bacterial growth is $P(t) = P_0e^{kt}$ where P_0 is a constant, $k > 0$ and t is measured in hours. For the particular problem we are given $P(0) = 100,000$ with time $= 0$ at 10:00 a.m. on Tuesday, so $P(t) = 10^5e^{kt}$. From 10:00 a.m. Tuesday until noon Thursday is 50 hours, and $P(50) = 3 \cdot 10^5 = 10^5e^{50k}$. Solve for $e^k = (3)^{1/50}$ to get $P(t) = 10^5(3)^{t/50}$. Time 3:00 p.m. the following Sunday is $t = 125$ hours so $P(125) = 10^5(3)^{125/50} = 9 \cdot 10^5\sqrt{3} \approx 1.56 \times 10^6$ bacteria. Monday at 4:00 p.m. corresponds to $t = 150$, so $P(150) = 10^5(3)^3 = 2 \cdot 7 \times 10^6$ bacteria. Assuming the Earth is a perfect sphere of radius 3960 miles, we find the Earth area to be $A = 4\pi[3960 \cdot 5280 \cdot 12]^2 \approx 7.91 \times 10^{17}$ square inches. At 10^5 bacteria per square inch, it would take $\hat{t}$ hours to overrun the Earth where $10^5(3)^{\hat{t}/50} = 7.91 \times 10^{22}$. Solve for $\hat{t} = \frac{50\ln(7.91 \times 10^{17})}{\ln(3)} \approx 1876$ hrs $= 78$ days.

19. Since $A = 4\pi r^2$ and $V = \frac{4}{3}\pi r^3$, we have $A = \sqrt[3]{36\pi}V^{2/3}$, so $\frac{dV}{dt} = kA = k\sqrt[3]{36\pi}V^{2/3}$. Assuming $V = V_0$ at time $t = 0$, this separable equation has solution $V(t) = \left[V_0^{1/3} + k\left(\frac{4\pi}{3}\right)^{1/3} t\right]^3$.

20. The amount of radioactive material is given by the solution of $\frac{dA}{dt} = kA, A(0) = e^3, A(\ln(2)) = \frac{e^3}{2}$, with t in weeks. Solving and fitting initial conditions gives $A(t) = e^3\left(\frac{1}{2}\right)^{t/\ln 2}$ tons, so $A(3) = e^3\left(\frac{1}{2}\right)^{3/\ln 2} = 1$ ton.

21. We easily find $U(t) = 10\left(\frac{1}{2}\right)^{t/4.5\times 10^9}$, t in years, so $U(10^9) = 10\left(\frac{1}{2}\right)^{1/4.5} \approx 8.57$ kg.

22. At any time t there will be $A(t) = 12e^{kt}$ gms. present, and $A(4) = 9.1$ requires $e^{4k} = \frac{9.1}{12}$, so $k = \frac{1}{4}\ln\left(\frac{9.1}{12}\right) \approx -.06915805$. Half life is the time t^* so $A(t^*) = 6$, or $e^{kt^*} = \frac{1}{2}$. This gives $t^* = -\frac{\ln 2}{k} \approx 10.02$ minutes.

23. With $I(x) = \int_0^\infty e^{-t^2-(\frac{x}{t})^2}dt$, we have $I'(x) = \int_0^\infty -\frac{2x}{t^2}e^{-[t^2+(\frac{x}{t})^2]}dt$. Now make the substitution $u = x/t$ in the integral above to get $I'(x) = 2\int_\infty^0 e^{-[(\frac{x}{u})^2+u^2]}du = -2\int_0^\infty e^{-[u^2+(\frac{x}{u})^2]}du = -2I(x)$. Solving the separable equation $I'(x) = -2I(x)$ gives $I(x) = Ce^{-2x}$. Now $I(0) = \int_0^\infty e^{-t^2}dt = \frac{\sqrt{\pi}}{2}$, so $I(x) = \frac{\sqrt{\pi}}{2}e^{-2x}$. Putting $x = 3$ gives $\int_0^\infty e^{-t^2-\frac{9}{t^2}}dt = \frac{\sqrt{\pi}}{2}e^{-6}$.

24. We use a conservation of energy argument. Let state-1 describe the particle at rest (velocity $= 0$) h feet above some reference point and state -2 describe the particle located at the reference point traveling with velocity v. Then since total energy is conserved we have $mgh + \frac{1}{2}m(0)^2 = mg(0) + \frac{1}{2}mv^2$. Solve for v to get the result $v(t) = \sqrt{2gh(t)}$, independent of m.

25. This problem is modeled using Torricelli's law and the geometry of the hemispherical tank. Let $h(t)$ denote the depth of the liquid in the hemispherical dome, $r(t)$ the radius of the exposed surface of draining liquid, and $V(t)$ the volume of water in the container. Then $dV/dt = -kA\sqrt{2gh}$ and $dV/dt = \pi r^2(dh/dt)$, where $r^2 + h^2 = 18^2$ from the geometry of the hemisphere. We are given $k = .8$ and $A = \pi\left(\frac{1}{4}\right)^2 = \frac{\pi}{16}$, the area of the drain hole. Using $g = 32$ ft/sec^2 we obtain after substitution and simplification $\pi(324 - h^2)\frac{dh}{dt} = .4\pi\sqrt{h}, h(0) = 18$. This separable equation has solution $1620\sqrt{h} - h^{5/2} = -t + K$, and $h(0) = 18$ gives $K = 3888\sqrt{2}$. Thus $1620\sqrt{h} - h^{5/2} = 3888\sqrt{2} - t$. The hemisphere runs empty when $h = 0$, so $t = 3888\sqrt{2}$ sec ≈ 91 min, 39 sec.

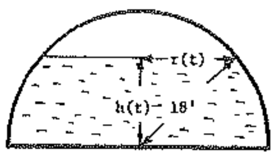


Figure for Problem 25

26. (a) For water h feet deep in the cylindrical hot tub we get $V = 25\pi h$, so
$25\pi\frac{dh}{dt} = -.6\pi\left(\frac{5}{16}\right)^2\sqrt{64h}, h(0) = 4$, thus $\frac{dh}{dt} = -\frac{3\sqrt{h}}{160}$.

(b) To drain the tank will require $T = \int_H^0 \left(\frac{dt}{dh}\right) dh = \int_4^0 -\frac{160}{3\sqrt{h}}dh = \frac{640}{3}$ sec.

(c) To drain the upper half requires $T_1 = \int_4^2 -\frac{160}{3\sqrt{h}}dh = \frac{320}{3}[2-\sqrt{2}]$ seconds ≈ 62.5 seconds.

To drain the lower half requires $T_2 = \int_2^0 -\frac{160}{3\sqrt{h}}dh = \frac{320}{3}\sqrt{2}$ seconds ≈ 150.8 sec.

27. (a) Let $r(t)$ be the radius of the exposed water surface, and $h(t)$ the depth of the draining water. Since cross sections of the cone are similar, we have $\pi r^2(dh/dt) = -kA\sqrt{2gh}$, with $h(0) = 9$. From similar triangles we have $r/h = 4/9$, so $r = (4/9)h$. Substituting $k = 0.6$, $g = 32$ and $A = \pi(1/12)^2$, and simplifying, we get $h^{3/2}(dh/dt) = -27/160$, with $h(0) = 9$. This separable equation has solution $h^{5/2} = -\dfrac{27}{64}t + K$, and $h(0) = 9$ gives $K = 243$. The conical tank runs empty when $h = 0$, so $t = 243\left(\dfrac{64}{27}\right) = 576\,\text{sec} \approx 9$ min. 36 sec.

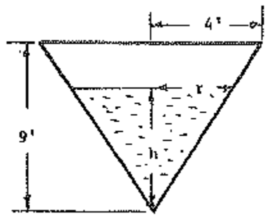


Figure for Problem 27(a)

(b) This problem is modeled like part (a), except the cone is now inverted. This changes the similar triangle proportionality to

$$\frac{r}{9-h} = \frac{4}{9},$$

or $r = (4/9)(9-h)$. The separable differential equation now becomes

$$\frac{(9-h)^2}{\sqrt{h}}dh = -\frac{27}{160}, h(0) = 9,$$

with solution $162\sqrt{h} - 12h^{3/2} + \frac{2}{5}h^{5/2} = -(27/160)t + (1296/5)$. The tank runs dry at $h = 0$, so

$$t = \frac{160}{27}\left(\frac{1296}{5}\right) = 1536 \text{ seconds},$$

about 25 minutes, 36 seconds.

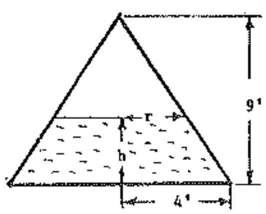


Figure for Problem 27(b)

28. From the geometry of the cone and Torricelli's Law $\dfrac{dV}{dt} = \pi\left(\dfrac{16}{81}\right)h^2\dfrac{dh}{dt} = -\dfrac{.6(8\pi)}{144}\sqrt{(h-2)}$ when the drain hole is two feet above the vertex. With the drain hole at the bottom of the tank we get $\dfrac{dV}{dt} = \pi\left(\dfrac{16}{81}\right)^2 h^2\dfrac{dh}{dt} = -\dfrac{.6(8\pi)}{144}\sqrt{h}$. If one knows the rates of change of depth of water in these two instances, then one could locate the drain hole height above the bottom of the tank without knowing hole size, since $\pi\left(\dfrac{16}{81}\right)h^2\left(\dfrac{dh}{dt}\right)_1 = -kA\sqrt{2g(h-h_0)}$ divided by $\pi\left(\dfrac{16}{81}\right)^2 h^2\left(\dfrac{dh}{dt}\right)_2 = -kA\sqrt{2gh}$ gives $\dfrac{\sqrt{h-h_0}}{\sqrt{h}} = \dfrac{(dh/dt)_1}{(dh/dt)_2} = r$, a known constant, and we can solve for h_0, the location of the hole above tank bottom.

29. From Problem 27, we have $\dfrac{dV}{dt} = \pi\left(\dfrac{16}{81}\right)h^2\dfrac{dh}{dt} = -.\dfrac{6(8\pi)}{144}\sqrt{h}$ for the draining conical tank. Thus if we pour water in at $\dfrac{\pi}{10}$ ft^3/sec, the differential equation would be $\pi\left(\dfrac{16}{81}\right)h^2\dfrac{dh}{dt} = -\dfrac{\pi}{30}\sqrt{h} + \dfrac{\pi}{10} = \dfrac{\pi}{10}\left(1 - \dfrac{\sqrt{h}}{3}\right)$, $h(0) = 0$, since the tank starts empty. From $h(0) = 0$, we see $\dfrac{dh}{dt} > 0$ so the water level rises, but as $h \to 9$, the right side of the equation $\to 0$, hence $\dfrac{dh}{dt} \to 0$, and the tank never overflows.

30. From the geometry of the sphere, the equation $\frac{dV}{dt} = -kA\sqrt{2gh}$ becomes $\pi[32A - (h-8)^2]\frac{dh}{dt} = -0.8\pi\left(\frac{1}{4}\right)^2\sqrt{64h}$, $h(0) = 36$, where h is the height above the bottom of the sphere. This simplifies to $(36\sqrt{h} - h^{3/2})dh = -0.4dt$ with solution $h\sqrt{h}(60-h) = -t + K$. Then $t = 0$ when $h = 36$ gives $K = 5184$, The tank runs empty when $h = 0$, so $t = 5184$ seconds, about 86.4 minutes. This is the time it takes to drain the spherical tank.

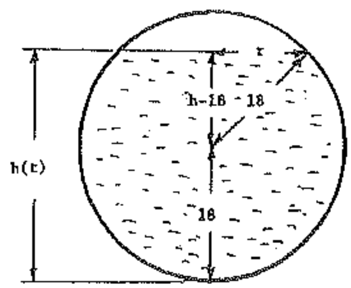


Figure for Problem 30.

Section 1.3 Linear Differential Equations

Solution of the equations in this section is routine once an integrating factor is found. Details are given for Problem 1. For most of the remaining problems only answers are given. A few additional details are included where the solution involves unusual features.

1. Identifying $p(x) = -3/x$, an integrating factor is $e^{\int p(x)dx} = e^{-3\ln(x)} = x^{-3}$. The equation (after multiplying by x^{-3}) can be written $\frac{d}{dx}(yx^{-3}) = \frac{2}{x}$ which gives by a routine integration $yx^{-3} = 2\ln(x) + c$ or $y = cx^3 + 2x^3\ln(x)$. We have assumed here that $x > 0$. A solution for $x < 0$ can be found by the same technique except we need $\int \frac{dx}{x} = \ln(-x)$ if $x < 0$. This would give the general form $y = cx^3 + 2x^3\ln(-x)$, or $y = cx^3 + 2x^3\ln|x|$ in one formula.

2. $y = ce^x + \frac{1}{4}e^{-x} + \frac{1}{2}xe^x$.

3. $y = ce^{-2x} + \frac{1}{2}x - \frac{1}{4}$

4. To find an integrating factor write $\sin(2x) = 2\sin(x)\cos(x)$ and put in standard form $y' + \frac{\sin(x)}{\cos(x)}y = \frac{1}{\cos(x)}$. The solution is $y = \sin(x) + c\cos(x)$.

5. $y = 4x^2 + 4x + 2 + ce^{2x}$

6. Standard form gives $p(x) = \frac{3x}{x^2 - x - 2} = \left[\frac{2}{x-2} + \frac{1}{x+1}\right]$, and an integrating factor is $(x+1)(x-2)^2$. Thus $\frac{d}{dx}(y(x+1)(x-2)^2) = (x-2)^3$, and we find $y = \frac{(x-2)^2}{4(x+1)} + \frac{c}{(x+1)(x-2)^4}$.

7. An integrating factor is easily found to be e^x. Then $\frac{d}{dx}(ye^x) = \left(\frac{x-1}{x^2}\right)e^x$ gives $y = e^{-x}\int\left(\frac{x-1}{x^2}\right)e^x dx + ce^{-x}$. The integral here cannot be evaluated in closed form.

8. An integrating factor is $\sec(x) + \tan(x)$ giving solution $y = \frac{x\cos(x) - \cos^2(x)}{1 + \sin(x)} + \frac{c\cos(x)}{1 + \sin(x)}$

9. $y = x^2 - x - 2$

10. $y = e^{2x} - 2 + 3e^{-3x}$

11. $y = x + 1 + 4(x+1)^{-2}$

12. The equation can be written $y' + \left(1 - \frac{2}{x} - \frac{1}{x-2}\right)y = x^2e^{-x}$, which lets us identify an integrating factor of $\frac{e^x}{x^2(x-2)}$. The solution is $y = x^2(x-2)\ln(x-2)e^{-x} + 2x^2(x-2)e^{-x}$.

13. $y = \frac{2}{3}e^{4x} - \frac{11}{3}e^x$

14. $y = \frac{27}{41}x^4 + \frac{9}{23}x^2 - \frac{2782}{943}x^{-5/9}$

15. Let (x,y) be a point on the curve. We want the tangent line at (x,y) to pass through $(0,2x^2)$, so the tangent line has slope $\dfrac{y-2x^2}{x}$. But this slope is also given by y', hence the curve satisfies $y' = \dfrac{y-2x^2}{x}$. The solution of this first order linear equation is $y = -2x^2 + cx$.

16. Let $A(t)$ be the amount of salt in the tank at any time $t \geq 0$. Then $\dfrac{dA}{dt}$ = (rate salt added) − (rate salt removed) $= 6 - 2\left(\dfrac{A}{50+t}\right)$; $A(0) = 28$. An integrating factor is $(50+t)^2$ and the solution is $A(t) = 2(50+t) + \dfrac{C}{(50+t)^2}$. The initial condition gives $C = -180,000$, so $A(t) = 2(50+t) - \dfrac{180000}{(50+t)^2}$. The tank contains 100 gal. when $t = 50$ and $A(50) = 176$ pounds of salt.

17. If A_1 and A_2 denote the amounts of salt respectively in tank 1 and tank 2 at time $t \geq 0$ we have $A_1' = \dfrac{5}{2} - \dfrac{5A_1}{100}$, $A_1(0) = 20$ and $A_2' = \dfrac{5A_1}{100} - \dfrac{5A_2}{150}$, $A_2(0) = 90$. Solving the linear equation for $A_1(t)$ we get $A_1(t) = 50 - 30e^{-t/20}$. Then A_2 is given by the solution of $A_2' + \dfrac{1}{30}A_2 = \dfrac{5}{2} - \dfrac{3}{2}e^{-t/20}$, $A_2(0) = 90$. Solving gives $A_2(t) = 75 + 90e^{-t/20} - 75e^{-t/30}$. Tank 2 will have a minimum when $A_2'(t) = 0$, thus $2.5e^{-t/30} - 4.5e^{-t/20} = 0$. This gives $e^{t/60} = \dfrac{9}{5}$ or $t = 60\ln\left(\dfrac{9}{5}\right)$, with $A_2(t)_{\min} = A_2\left(60\ln\left(\dfrac{9}{5}\right)\right) = \dfrac{5450}{81}$ pounds.

Section 1.4 Exact Differential Equations

1. Since $\dfrac{\partial M}{\partial y} = 4y + e^{xy} + xye^{xy} = \dfrac{\partial N}{\partial x}$, the equation is exact in the entire plane. A potential function is $\phi(x,y) = 2xy^2 + e^{xy} + y^2$; solutions are defined implicitly by $2xy^2 + e^{xy} + y^2 = c$.

2. Since $\dfrac{\partial M}{\partial y} = 4x = \dfrac{\partial N}{\partial x}$, the equation is exact everywhere. A potential function is $\phi(x,y) = 2x^2y + x^2 + y^3$; solutions are $2xy^2 + x^2 + y^3 = c$.

3. $\dfrac{\partial M}{\partial y} = 4x + 2x^2$ and $\dfrac{\partial N}{\partial x} = 4x$, so the equation is not exact.

4. $\dfrac{\partial M}{\partial y} = -2\sin(x+y) - 2x\cos(x+y) = \dfrac{\partial N}{\partial x}$, so the equation is exact everywhere. Potential function is $\phi(x,y) = 2x\cos(x+y)$; solutions are $2x\cos(x+y) = c$.

5. $\dfrac{\partial M}{\partial y} = 1 = \dfrac{\partial N}{\partial x}$, so the equation is exact for all $x \neq 0$. $\phi(x,y) = \ln|x| + xy + y^3$; with solutions $\ln|x| + xy + y^3 = c$.

6. $\dfrac{\partial M}{\partial y} = 2ye^x(1+x)\cos(y^2)$ and $\dfrac{\partial N}{\partial x} = 2ye^x(1+x)\sin(y^2)$, so the equation is not exact.

7. $\dfrac{\partial M}{\partial y} = \sinh(x)\cosh(y) = \dfrac{\partial N}{\partial x}$ so the equation is exact; $\phi(x,y) = \cosh(x)\sinh(y)$ is a potential function; solutions are $\cosh(x)\sinh(y) = c$

8. $\frac{\partial M}{\partial y} = 16y^3 = \frac{\partial N}{\partial x}$, so the equation is exact everywhere; a potential function is $\phi(x,y) = 4xy^3 + 3\sin(x) - 3\sin(y)$; solutions are $4xy^3 + 3\sin(x) - 3\sin(y) = c$.

9. $\frac{\partial M}{\partial y} = 12y^3 = \frac{\partial N}{\partial x}$ so the equation is exact everywhere; $3xy^4 - x = 47$

10. $2xy - \tan(xy^2) = 4 - \tan(4)$

11. $x\sin(2y - x) = \frac{\pi}{24}$

12. $x + xe^{y/x} = 1 + e^{-5}$

13. Not exact

14. $xe^y - y = 5$

15. To be exact we need $\frac{\partial M}{\partial y} = 6xy^2 - 3 = \frac{\partial N}{\partial x} = -3 - 2\alpha xy^2$, hence choose $\alpha = -3$. A potential function will be $\phi(x,y) = x^2y^3 - 3xy - 3y^2$ and solutions are $x^2y^3 - 3xy - 3y^2 = c$.

16. To be exact we need $\frac{\partial M}{\partial y} = x\alpha y^{\alpha-1} = \frac{\partial N}{\partial x} = -2xy^{\alpha-1}$, so choose $\alpha = -2$. A potential function will be $\phi(x,y) = x^3 + \frac{x^2}{2y^2}$ and solutions are $x^3 + \frac{x^2}{2y^2} = c$ for $y \neq 0$.

17. If ϕ is a potential function, then $\frac{\partial \phi}{\partial x} = M$ and $\frac{\partial \phi}{\partial y} = N$. But then $\phi + c$ is also a potential function, since $\frac{\partial}{\partial x}(\phi + c) = \frac{\partial \phi}{\partial x} = M$ and $\frac{\partial(\phi + c)}{\partial y} = \frac{\partial \phi}{\partial y} = N$. The solution obtained by using ϕ and $\phi + c$ are the same curves.

Section 1.5 Integrating Factors

1. A function of y only, $\nu(y)$, is an integrating factor for $M + Ny' = 0$ if and only if $\frac{\partial}{\partial y}(\nu M) = \frac{\partial}{\partial x}(\nu N)$. Since $\nu = \nu(y)$ we get the condition $\nu' M + \nu M_y = \nu N_x$. Solving for $\frac{\nu'}{\nu} = \frac{1}{M}(N_x - M_y)$, a sufficient condition for such a ν is that $\frac{1}{M}(N_x - M_y)$ is a function of y only (since ν'/ν is). If $\frac{1}{M}(N_x - M_y) = g(y)$, then $\nu(y) = e^{\int g(y)dy}$ will produce an integrating factor.

2. $x^a y^b$ will be an integrating factor for $M + Ny' = 0$ if $\frac{\partial}{\partial y}(x^a y^b M) = \frac{\partial}{\partial x}(x^a y^b N)$. Simplifying this equation gives a sufficient condition as $M_y - N_x = a\frac{N}{x} - b\frac{M}{y}$, for some constants a and b.

3. (a) $M_y = 1$ and $N_x = -1$ so never exact.
(b) Since $\frac{1}{N}(M_y - N_x) = -\frac{2}{x}$, $\mu(x) = \frac{1}{x^2}$
(c) Since $\frac{1}{M}(N_x - M_y) = -\frac{2}{y}$, $\nu(y) = \frac{1}{y^2}$
(d) By Problem 2, $M_y - N_x = 2 = a\frac{(-x)}{x} - b\frac{(y)}{y} = -(a+b)$ for any a, b satisfying $a + b = -2$.

4. (a) $M_y = -3, N_x = 1$ so not exact.
(b) Since $\frac{1}{N}(M_y - N_x) = -\frac{4}{x}, \mu(x) = e^{-\int \frac{4}{x}dx} = \frac{1}{x^4}$.
(c) Now $\left(-\frac{3y}{x^4} - \frac{2}{x}\right) + \frac{1}{x^3}y' = 0$ is exact with solutions defined implicitly by $\frac{y}{x^3} - 2\ln|x| = c$, or $y = cx^3 + 2x^3 \ln|x|$.

5. (a) $M_y = 0, N_x = 3$; (b) $\nu(y) = e^{3y}$; (c) $xe^{3y} - e^y = c$

6. (a) $M_y = 6x^2 + 12x + 2y, N_x = 12x$; (b) $\mu(x) = e^x$; (c) $6x^2ye^x + y^2e^x = c$.

7. (a) $M_y = 4x + 12y, N_x = 4x + 6y$
(b) By problem 2, $M_y - N_x = 6y = a(2x + 6y) - b(4x + 6y)$ holds for $a = 2, b = 1$, so $\mu = x^2y$
(c) $x^4y^2 + 2x^3y^3 = c$.

8. (a) $M_y = 2y + 1, N_x = -1$; (b) $\nu(y) = \frac{1}{y^2}$; (c) $xy + x = cy$; (d) $y = 0$ is a singular solution.

9. (a) $M_y = 4xy + 2x, N_x = 2xy + 2x$; (b) $\nu(y) = \frac{1}{y+1}$; (c) $x^2y = c$ (d) $y = -1$

10. (a) $M_y = 4y - 9x, N_x = 3y - 12x$;
(b) By Problem 2, $M_y - N_x = y + 3x = a(3y - 6x) - b(2y - 9x)$ holds for $a = b = 1$, so $\mu = xy$;
(c) $x^2y^3 - 3x^3y^2 = c$

11. (a) $M_y = 1 - 4y^3, N_x = 0$;
(b) The hint produces $\mu(x, y) = e^{-3x}y^{-4}$
(c) $y^3 - 1 = ky^3e^{3x}$

12. (a) $M_y = x - \frac{3}{2}y^{-5/2}, N_x = 2x$;
(b) By Problem 2, we get $\mu = x^{1/2}y^{3/2}$;
(c) $\frac{1}{5}x^{5/2}y^{5/2} + \frac{1}{3}x^{3/2} = c$.

13. $\mu(x) = \frac{1}{x}; \ln|x| + y = c; y = 4 - \ln(x)$

14. $\mu(x) = \frac{1}{x^{1/4}}; x^{3/4}y = c; x^{3/4}y = 6, x > 0$.

15. $\mu(x) = x; x^2(y^3 - 2) = c; x^2(y^3 - 2) = -9$

16. $\mu(x) = \frac{e^{x/2}}{\sqrt{x}}; \sqrt{x}e^{x/2}y = c; \sqrt{x}e^{x/2}y = 12e^2$

17. $\nu(y) = \frac{1}{y}; x^2 + 3\ln|y| = c; y = 4e^{-x^2/3}$

18. $\mu(x) = xe^{x^2}; yx^2e^{x^2} = c; yx^2e^{x^2} = 12e^4$

19. $\mu(x) = e^x; e^x \sin(x - y) = c; e^x \sin(x - y) = \frac{1}{2}$

20. $\nu(y) = \frac{1}{y}; x^3 + xy^2 = c; x^3 + xy^2 = 10$

21. $\frac{\partial}{\partial y}(c\mu M) = c\frac{\partial}{\partial y}(\mu M) = c\frac{\partial}{\partial x}(\mu N) = \frac{\partial}{\partial x}(c\mu N)$

22. Let $\mu(x,y)$ be an integrating factor for $M + Ny' = 0$ and suppose the general solution is defined by $\phi(x,y) = 0$. Consider

$$\frac{\partial}{\partial y}(\mu M G(\phi(x,y)) - \frac{\partial}{\partial x}(\mu N G(\phi(x,y))$$

$$= G(\phi(x,y)\left[\frac{\partial}{\partial y}(\mu M) - \frac{\partial}{\partial x}(\mu N)\right]$$
$$+G'(\phi(x,y))\left[\mu M\frac{\partial \phi}{\partial y} - \mu N\frac{\partial \phi}{\partial x}\right]$$

$$= G(\phi(x,y))0 + G'(\phi(x,y))\mu[M\phi_y - N\phi_x] = 0 + 0 = 0.$$

The first bracket term is zero because μ is an integrating factor, the second because $\phi(x,y) = 0$ are solution curves and $y' = -\dfrac{\phi_x}{\phi_y}$, hence $M + Ny' = 0$ is equivalent to $M\phi_y - N\phi_x = 0$.

Section 1.6 Homogeneous, Bernoulli, and Riccati Equations

1. This is a Riccati equation with solution $S(x) = x$. Thus put $y = x + \dfrac{1}{z}$ and substitute to get $1 - \dfrac{z'}{z^2} = \dfrac{1}{x^2}\left(x + \dfrac{1}{z}\right)^2 - \dfrac{1}{x}\left(x + \dfrac{1}{z}\right) + 1$. Simplification gives $z' + \dfrac{1}{x}z = -\dfrac{1}{x^2}$, with solution $z = -\dfrac{\ln(x)}{x} + \dfrac{c}{x}$. Then $y = x + \dfrac{x}{c - \ln(x)}$

2. This is a Bernoulli equation with $\alpha = -4/3$, so we put $v = y^{7/3}$, or $y = v^{3/7}$. Substitution gives $\dfrac{3}{7}v^{-4/7}v' + \dfrac{1}{x}v^{3/7} = \dfrac{2}{x^3}v^{-4/7}$ or after simplifying $v' + \dfrac{7}{3x}v = \dfrac{14}{3x^2}$. An integrating factor for this first order equation is $x^{7/3}$ so $(vx^{7/3})' = \dfrac{14}{3}x^{1/3}$ and by integrating $vx^{7/3} = \dfrac{7}{2}x^{4/3} + c$. But $v = y^{7/3}$, so $2y^{7/3}x^{7/3} - 7x^{4/3} = k$.

3. Bernoulli equation with $\alpha = 2; y = \dfrac{1}{1 + ce^{x^2/2}}$

4. This is a homogeneous equation so we put $u = y/x$ or $y = xu$. This gives $u + xu' = \dfrac{1}{u} + u$ which is separable as $udu = \dfrac{dx}{x}$, so $u^2 = 2\ln|x| + c$, and $\dfrac{y^2}{x^2} = 2\ln|x| + c$.

5. Homogeneous; $y\ln|y| - x = cy$

6. Riccati equation with solution $S(x) = 4; y = 4 + \dfrac{6x^3}{c - x^3}$

7. Exact; $xy - x^2 - y^2 = c$

8. Homogeneous; $\sec\left(\dfrac{y}{x}\right) + \tan\left(\dfrac{y}{x}\right) = cx$

9. Bernoulli with $\alpha = -3/4$; $5x^{7/4}y^{7/4} + 7x^{-5/4} = c$

10. Homogeneous; $\frac{2\sqrt{3}}{3}\tan^{-1}\left(\frac{2y-x}{\sqrt{3}x}\right) = \ln|x| + c$

11. Bernoulli with $\alpha = 2$; $y = 2 + \frac{2}{cx^2-1}$

12. Homogeneous; $\frac{1}{2}\frac{x^2}{y^2} = \ln|x| + c$

13. Riccati with solution $S(x) = e^x$; $y = \frac{2e^x}{ce^{2x}-1}$

14. Bernoulli equation; $y = \frac{2}{3+cx^2}$

15. (a) We have $F\left(\frac{ax+by+c}{dx+ey+r}\right) = F\left(\frac{a+b(y/x)+c/y}{d+e(y/x)+r/y}\right) = f\left(\frac{y}{x}\right)$ if and only if $c = 0$ and $r = 0$.

(b) With $Y = y+k$ and $X = x+h$, we have by the chain rule $\frac{dY}{dX} = \frac{dY}{dx}\frac{dx}{dX} = \frac{d}{dx}(y+k) = \frac{dy}{dx}$ so the equation $\frac{dy}{dx} = F\left(\frac{ax+by+c}{dx+ey+r}\right)$ becomes $\frac{dY}{dX} = F\left(\frac{a(X-h)+b(Y-k)+c}{d(X-h)+e(Y-k)+r}\right) = F\left(\frac{aX+bY+c-ah-bk}{dX+eY+r-dh-ek}\right)$. By (a), this will be homogeneous iff $ah+bk = c$ and $dh+ek = r$. The given condition $ae - bd \neq 0$ ensures that for any c, r there will be unique values of h and k so that $ah+bk = c$ and $dh+ek = r$ and the new equation in X and Y will be homogeneous. We find these values of h and k to be $h = \frac{ce-br}{ae-bd}, k = \frac{ar-cd}{ae-bd}$.

16. From 15(b), choose h, k so that $k = 3$, and $h + k = 1$, or $h = -2$. The new equation in X and Y becomes $\frac{dY}{dX} = \frac{Y}{X+Y} = \frac{Y/X}{1+Y/X}$. Now let $W = Y/X$ or $Y = WX$ to get $W + W'X = \frac{W}{1+W}$ which is separable and has solution $W\ln|W| - 1 = -W\ln|X| + CW$. Replace $W = \frac{Y}{X} = \frac{y-3}{x+2}$ and simplify to get $(y-3)\ln|y-3| - (x+2) = c(y-3)$.

17. To make homogeneous, choose h and k so $3h - k = 9$ and $h + k = 1$, thus $h = 2, k = -3$. Then solve $\frac{dY}{dX} = \frac{3X-Y}{X+Y}$ as a homogeneous equation and substitute to get $3(x-2)^2 - 2(x-2)(y+3) - (y+3)^2 = c$

18. $(x+5)^2 + 4(x+5)(y+1) - (y+1)^2 = c$.

19. $(2x+y-3)^2 = c(y-x+3)$

20. Putting $u = \frac{ax+by}{a}$, or $y = \frac{a}{b}(u-x)$ gives $\frac{dy}{dx} = \frac{a}{b}\left[\frac{du}{dx} - 1\right] = F\left(\frac{au+c}{d(au-by)/a+ey+r}\right) - F\left(\frac{au+c}{du+(ae-bd)y/a+r}\right) = F\left(\frac{au+c}{du+r}\right)$, since $ae - bd = 0$.

Hence we get $\frac{du}{dx} = 1 + \frac{b}{a}F\left(\frac{au+c}{du+r}\right)$, which is separable.

21. From Problem 20, let $u = x - y$, to get $\dfrac{du}{dx} = 1 - \dfrac{u+2}{u+3} = \dfrac{1}{u+3}$ so $(u+3)du = dx$ or $\dfrac{u^2}{2} + 3u = x + c$. Hence $\dfrac{(x-y)^2}{2} + 3(x-y) = x + c$

22. Let $u = 3x + y$ to get the separable equation $\dfrac{du}{dx} = 3 + \dfrac{u-1}{2u-3} = \dfrac{7u-10}{2u-3}$. Separate to get $\left(\dfrac{2u-3}{7u-10}\right) du = dx$ and integrate to find $2u - \dfrac{1}{7}\ln|7u-10| = 7x + c$. Since $u = 3x + y$, we get $14(y+3x) - \ln|21x + 7y - 10| = 49x + k$.

23. With $u = x - 2y$ we find a solution $x - 3y - 4\ln|x - 2y + 4| = c$

24. With $u = x - y$ we find a solution $x - 3y + 7\ln|x - y - 1| = c$.

25. At time $t = 0$, assume the dog is at the origin of an xy system and the man is located at $(A, 0)$ on the x axis. The man moves directly upward into the first quadrant and at time t is located at (A, vt). The position of the dog at time $t > 0$ is at (x, y) and, as stated, the dog runs with speed $2v$, always directly toward his master. At time $t > 0$, the man is at (A, vt), the dog is at (x, y) and the tangent to the dog's path joins these two points. Thus $\dfrac{dy}{dx} = \dfrac{vt - y}{A - x}$, for $x < A$. To eliminate t from this equation use the fact that during the time the man has moved vt units upward into the first quadrant, the dog has run $2vt$ units along his path. Thus $2vt = \displaystyle\int_0^x \left[1 + \left(\frac{dy}{d\xi}\right)^2\right]^{1/2} d\xi$. Use this integral to eliminate the vt term in the original differential equation to get $2(A-x)y'(x) = \displaystyle\int_0^x \left[1 + \left(\frac{dy}{d\xi}\right)^2\right]^{1/2} d\xi - 2y$. Now differentiate this equation and obtain $2(A-x)y'' - 2y' = [1 + (y')^2]^{1/2} - 2y'$; or $2(A-x)y'' = [1 + (y')^2]^{1/2}$, subject to $y(0) = 0, y'(0) = 0$. Let $u = y'$ to get the separable equation $\dfrac{du}{\sqrt{1+u^2}} = \dfrac{dx}{2(A-x)}$, which has solution $\ln[u + \sqrt{1+u^2}] = -\dfrac{1}{2}\ln(A - x) + c$. Using $y'(0) = u(0) = 0$ gives $u + \sqrt{1+u^2} = \dfrac{\sqrt{A}}{\sqrt{A-x}}$, or equivalently $y' + \sqrt{1+(y')^2} = \dfrac{\sqrt{A}}{\sqrt{A-x}}, y(0) = 0$. From the equation for y'' (before the substitution) get $\sqrt{1+(y')^2} = 2(A-x)y''$ so $y' + 2(A-x)y'' = \dfrac{\sqrt{A}}{\sqrt{A-x}}, y(0) = 0, y'(0) = 0, x < A$. Let $w = y'$ to write this as a linear first order equation $w' + \dfrac{1}{2(A-x)}w = \dfrac{\sqrt{A}}{2(A-x)^{3/2}}$. An integrating factor is $\dfrac{1}{\sqrt{A-x}}$ and we get $\dfrac{d}{dx}\left[\dfrac{w'}{\sqrt{A-x}}\right] = \dfrac{\sqrt{A}}{2(A-x)^2}$. The solution of this, subject to $w(0) = 0$ is $w = \dfrac{A}{\sqrt{2}}\dfrac{1}{\sqrt{A-x}} - \dfrac{1}{2\sqrt{A}}\sqrt{A-x} = \dfrac{dy}{dx}$. Integrate once more to get $y(x) = -\sqrt{A}\sqrt{A-x} + \dfrac{1}{3\sqrt{A}}(A-x)^{3/2} + \dfrac{2}{3}A$, using $y(0) = 0$ to determine the constant of integration. The dog catches the man at $x = A$, so they meet at $(A, 2A/3)$. Since this is also (A, vt) when they meet, we find $vt = \dfrac{2}{3}A$, so they meet at time $t = \dfrac{2A}{3v}$.

26. (a) Clearly each bug follows the same curve of pursuit relative to the corner from which it starts. Place a polar coordinate system as suggested and determine the pursuit curve for the bug starting at $\theta = 0, r = a/\sqrt{2}$. At any time $t > 0$, the bug will be at $(f(\theta), \theta)$, its target at $(f(\theta), \theta + \pi/2)$, and $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f'(\theta)\sin(\theta) + f(\theta)\cos(\theta)}{f'(\theta)\cos(\theta) - f(\theta)\sin(\theta)}$ by the chain rule. On the other hand, this tangent direction must be from location $(f(\theta), \theta)$ to target $(f(\theta), \theta + \pi/2)$. Hence $\frac{dy}{dx} = \frac{f(\theta)\sin(\theta + \pi/2) - f(\theta)\sin(\theta)}{f(\theta)\cos(\theta + \pi/2) - f(\theta)\cos(\theta)} = \frac{\cos(\theta) - \sin(\theta)}{-\sin(\theta) - \cos(\theta)} = \frac{\sin(\theta) - \cos(\theta)}{\sin(\theta) + \cos(\theta)}$. Equate these two expressions for $\frac{dy}{dx}$ and simplify to get $f'(\theta) + f(\theta) = 0$, with $f(0) = a/\sqrt{2}$. Thus $r = f(\theta) = \frac{a}{\sqrt{2}}e^{-\theta}$ gives polar equation of the pursuit curve.

(b) Distance traveled is $D = \int_0^\infty \sqrt{(r')^2 + r^2}d\theta = \int_0^\infty \left[\left(\frac{a}{\sqrt{2}}e^{-\theta}\right)^2 + \left(-\frac{a}{\sqrt{2}}e^{-\theta}\right)^2\right]^{1/2} d\theta = a\int_0^\infty e^{-\theta}d\theta = a$.

(c) Since $r = f(\theta) = \frac{a}{\sqrt{2}}e^{-\theta} > 0$ for all θ, no bug ever catches it quarry. The actual distance between pursuer and quarry will be $ae^{-\theta}$.

27. Assume the disk rotates counterclockwise with angular velocity ω radians/second, and the bug steps on the rotating disk at point $(a, 0)$. Then by the chain rule $\frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt}$ so $\frac{dr}{d\theta} = -\frac{v}{\omega}$. Thus $r = c - \frac{\theta v}{\omega}$ and $r(0) = a$ gives $r(\theta) = a - \frac{\theta v}{\omega}$. This is a spiral.

(b) To reach the center, solve $r = 0 = a - \frac{\theta v}{\omega}$ to get $\theta = \frac{a\omega}{v}$ radians or $\theta = \frac{a\omega}{2\pi v}$ revolutions.

(c) Distance traveled $s = \int_0^{a\omega/v} \sqrt{(r)^2 + (r')^2}d\theta = \int_0^{a\omega/v} \sqrt{\left(a - \frac{v\theta}{\omega}\right)^2 + \left(\frac{v}{\omega}\right)^2} d\theta$. To evaluate this, let $\theta = \frac{a\omega}{v} - z$, so $s = \frac{v}{\omega}\int_0^{a\omega/v} \sqrt{z^2 + 1}dz = \frac{1}{2}\left[\frac{a\omega}{v^2}\sqrt{a\omega^2 + v^2} + \ln\left(\frac{a\omega + \sqrt{\omega^2 + v^2}}{v}\right)\right]$.

Section 1.7 Applications

1. Take $x = 0$ as the point at which one end of the chain just touches the floor and let $x > 0$ denote the upward movement of the shorter end of chain. Then the mass of moving chain is $m = (16 - x)\frac{\rho}{g}$ slugs and the net force that acts on this is $F = (2 + x)\rho$ pounds. The motion is modeled by $\frac{d}{dt}[v(16 - x)\frac{\rho}{g}] = (2 + x)\rho, v = 0$ if $x = 0$. By the chain rule $\frac{dy}{dt}(x) = \frac{dy}{dx}\frac{dx}{dt} = v\frac{dy}{dx}$, so $v\frac{d}{dx}[v(16 - x)] = 32(2 + x)$. Rewrite as the Barnoulli equation $(\alpha = -1)$, $\frac{dv}{dx} - \frac{1}{(16 - x)}v = \frac{32(2 + x)}{(16 - x)}v^{-1}$. Let $z = v^2$ to get the linear equation $z' - \left(\frac{2}{16 - x}\right)z = \frac{64(2 + x)}{16 - x}$, which

has integrating factor $\mu = (16-x)^2$ and solution $z(16-x)^2 = 64\left(32x + 7x^2 - \frac{x^3}{3}\right)$, since $z(0) = 0$ or $v^2(x) = \frac{64(32x + 7x^2 - x^3/3)}{(16-x)^2}$. Put $x = 7$ to get $v(7) = \frac{8}{9}\sqrt{\frac{1358}{3}} \approx 18.91$ ft/sec.

2. The time require for the chain to leave the pulley is $t_f = \int_0^{t_f} dt = \int_0^{v(7)} \left(\frac{dt}{dv}\right) dv = \int_0^7 \left(\frac{dt}{dx}\right) dx = \int_0^7 \frac{1}{v(x)} dx = \int_0^7 \frac{(16-x)}{8\sqrt{32 + 7x^2 - x^3/3}} dx \approx 1.397$ seconds.

3. Since 10 feet of chain hang down initially, the chain touches the floor at $t = 0$. The force pulling chain from the platform is $F = 10\rho$ pounds, a constant, and the amount of chain moving is 10 feet with mass $= \frac{10\rho}{g}$. Let v be the velocity of the center of mass of the hanging chain, so $10\rho = \frac{10\rho}{g}\frac{dv}{dt}$ or $\frac{dv}{dt} = g, v(0) = 0$. We find $v(t) = 32t$, and $s(t) = 16t^2$. The chain leaves the support when $s = 30$, at time $t = \frac{1}{4}\sqrt{30}$ at a velocity of $v\left(\frac{1}{4}\sqrt{30}\right) = 8\sqrt{30} \approx 43.82$ ft/sec.

4. Take $x = 0$ to be at equilibrium with 24 feet of chain on one side of the pulley and 16 feet of chain and the 8ρ weight on the other. The system will be released from $x = 1$ at $v = 0$. Total mass in motion is $\frac{48\rho}{g}$ slugs, and for $x \geq 1$, the net force acting is $F = 2x\rho$ pounds. Thus $2x\rho = \frac{48\rho}{g}\frac{dv}{dt} = \frac{3}{2}\rho v \frac{dv}{dx}, v(1) = 0$. Solve by separation of variables to get $3v^2 = 4x^2 + c$. $v(1) = 0$ gives $c = -4$, so $v^2 = \frac{4}{3}(x^2 - 1)$. The chain leaves the pulley at $x = 23$, with $v = 8\sqrt{11} \approx 26.5$ ft/sec.

5. Let $x(t)$ denote the amount of chain hanging down from the table, and note that once the chain starts moving, all 24 feet move with velocity v. The motion is modeled by $\rho x = \frac{24\rho}{g}\frac{dv}{dt} = \frac{3\rho}{4} v \frac{dv}{dx}, v(6) = 0$. Thus $x^2 = \frac{3}{4}v^2 + c$, and $v(6) = 0$ gives $c = 36$ so $v^2 = \frac{4}{3}(x^2 - 36)$. When the end leaves the table, $x = 24$, so $v = 12\sqrt{5} \approx 26.84$ ft/sec. The time is $t_f = \int_6^{24} \frac{1}{v(x)} dx = \int_6^{24} \frac{\sqrt{3}}{2\sqrt{x^2 - 36}} dx = \frac{\sqrt{3}}{2}\ln[6 + \sqrt{35}] \approx 2.15$ seconds.

6. The force which pulls chain off the table is due to the four feet of chain hanging between the table and floor. Let x denote the distance the free end of chain on the table has moved. The motion is modeled by $4\rho = \frac{d}{dt}\left[(22-x)\frac{\rho}{g}v\right], v = 0$ when $x = 0$. Rewrite as $128 + v^2 = (22-x)v\frac{dv}{dx}$, and solve by separation of variables to get $c - \ln(22-x) = \frac{1}{2}\ln(128 + v^2)$, and $v = 0, x = 0$ gives $c = \ln(176\sqrt{2})$. The end of the chain leaves the table when $x = 18$ so $v = \sqrt{3744} \approx 61.19$ ft/sec.

7. The time for the chain to leave it's support is $t_f = \int_0^{t_f} dt = \int_{x_0}^{x_f} \frac{1}{v(x)} dx = \frac{\sqrt{3}}{8} \int_{10}^{40} \frac{x}{\sqrt{x^3 - 1000}} dx$. To evaluate this improper integral numerically, let $u = \sqrt{x^3 - 1000}$ to get $t_f = \frac{\sqrt{3}}{12} \int_0^{30\sqrt{70}} (u^2 + 1000)^{-1/3} du \approx 1.7117$ seconds.

8. Compute potential energy by $PE = mgh$ where h is the height of the center of mass of the object above the floor. Initially, $PE + KE = 7\rho(16.5) + 9\rho(15.5) + \frac{1}{2}\frac{\rho}{g}(0)^2 = 255\rho$ foot pounds. At any later time, the chain moves with velocity v, and the ends have moved x feet up and down from the equilibrium position, so for $x \geq 1, PE + KE = (8-x)\rho\left(16 + \frac{x}{2}\right) + (8+x)\rho\left(16 - \frac{x}{2}\right) + \frac{1}{2}\frac{16\rho}{g}v^2(x)$. Equate these two expressions to get $v^2(x) = 4(x^2 - 1), x \geq 1$.

9. Since the mass in motion is not constant, we need to derive a conservation law by finding a constant of the motion. Consider

$$\begin{aligned} &\frac{d}{dt}\left[\frac{\rho x^2 v^2}{64} - \frac{x^3\rho}{3}\right] \\ &= \frac{d}{dx}\left[\frac{\rho x^2 v^2}{64} - \frac{x^3\rho}{3}\right]\frac{dx}{dt} \\ &= xv\left[\frac{\rho}{32}v^2 + \frac{\rho x}{32}v\frac{dv}{dx} - x\rho\right] = 0, \end{aligned}$$

since the equation of motion is $\frac{x\rho}{32}v\frac{dv}{dx} + \frac{\rho v^2}{32} = x\rho$. Thus $\frac{\rho x^2 v^2}{64} - \frac{x^3\rho}{3} =$ constant. With $t = 0$ when $x = 10$ and $v = 0$ we get $-\frac{1000}{3}\rho = \frac{\rho x^2 v^2}{64} - \frac{x^3\rho}{3}$, or $v^2 = \frac{64}{3}\left[x - \frac{1000}{x^2}\right]$, for $x \geq 10$. The chain leaves the pulley when $x = 40$, so $v = 2\sqrt{210} \approx 29$ ft/sec.

10. Let $y = y(x)$ be the shape of the curve, then the normal line to the curve has slope $m = -\frac{1}{\left(\frac{dy}{dx}\right)} = -\frac{dx}{dy}$, which must be the equilibrium direction for the acting forces. Thus $-\frac{dx}{dy} = -\frac{mg}{m\omega^2 x}$ or $\frac{dx}{dy} = \frac{g}{\omega^2 x}$, a separable equation with solution $y = \frac{\omega^2}{2g}x^2 + y_0$, a parabola.

11. Once released, the only force acting on the ballast bag is due to gravity, so if y represent distance above the ground, $\frac{d^2y}{dt^2} = -g = -32, y(0) = 342, y'(0) = 4$. By two integrations obtain $y'(t) = 4 - 32t, y(t) = 342 + 4t - 16t^2$. Maximum height is reached when $y'(t) = 0$ or $t = 1/8$ second. Max height $= y(1/8) = 342.25$ ft. The ballast bag remains aloft until $y(t) = 0$ or $-16t^2 + 4t + 342 = 0$, hence $t = \frac{19}{4}$ seconds at which time it hits the ground with speed $= |y'(19/4)| = 148$ft/sec.

12. With a gradient of $7/24$ the plane is inclined at an angle θ for which $\sin(\theta) = 7/25$ and $\cos(\theta) = 24/25$. The velocity of the box satisfies $\left(\frac{48}{32}\right)\frac{dv}{dt} = -48\left(\frac{24}{25}\right)\left(\frac{1}{3}\right) + 48\left(\frac{7}{25}\right) - \frac{3}{2}v, v(0) = 16$. Solve this linear equation for $v(t) = \frac{432}{25}e^{-t} - \frac{32}{25}$ ft/sec, which reaches zero

when $t_s = \ln\left(\frac{27}{2}\right)$ sec. The box will travel a distance of $s(t_s) = \int_0^{t_s} v(\xi)d\xi = \frac{432}{25}(1-e^{-t_s}) - \frac{32}{25}t_s = \frac{432}{25}\left(1-\frac{2}{27}\right) - \frac{32}{25}\ln\left(\frac{27}{2}\right) \approx 12.7$ feet.

13. Until the chute is opened (at $t = 4$ sec) the velocity is given by $\left(\frac{192}{32}\right)\frac{dv}{dt} = 192 - 6v, v(0) = 0$. The solution for $0 \le t \le 4$ is easily found to be $v(t) = 32(1-e^{-t})$. When her chute opens ($t = 4$) she has a velocity of $v(4) = 32(1-e^{-4})$ ft/sec. Velocity with an open chute satisfies $\left(\frac{192}{32}\right)\frac{dv}{dt} = 192 - 3v^2, v(4) = 32(1-e^{-4})$ for $t \ge 4$. This equation is separable and can be integrated by partial fractions as $\int\left[\frac{1}{v+8} - \frac{1}{v-8}\right]dv = -\int 8dt$ to get $\ln\left[\frac{v+8}{v-8}\right] = -8t + \ln\left[\frac{5-4e^{-4}}{3-4e^{-4}}\right] + 32$. Solve for v to get $v(t) = \frac{8(1+ke^{-8(t-4)})}{1-ke^{-8(t-4)}}, t \ge 4$ where $k = \frac{3-4e^{-4}}{5-4e^{-4}}$. Terminal velocity is $\lim_{t\to\infty} v(t) = 8$ ft/sec. Distance fallen is

$$s(t) = \int_0^t v(\xi)d\xi = 32[t - 1 + e^{-t}] \text{ for } 0 \le t \le 4,$$

and for $t \ge 4$,

$$s(t) = 32(3+e^{-4}) + 8(t-4) + 2\ln(1-ke^{-8(t-4)}) - 2\ln\left(\frac{2}{5-4e^{-4}}\right).$$

14. When fully submerged the buoyant force will be $F_B = 1 \times 2 \times 3 \times 62.5 = 375$ pounds upward. The mass, $m = \frac{384}{32} = 12$ slugs. Velocity of the sinking box satisfies $12\frac{dv}{dt} = 384 - 375 - \frac{1}{2}v, v(0) = 0$. This linear equation has solution $v(t) = 18(1-e^{-t/24})$. In t seconds the box will have sunk $s(t) = 18(t + 24e^{-t/24} - 24)$ feet. From $v(t)$ we find terminal velocity $= \lim_{t\to\infty} v(t) = 18$ ft/sec.
To answer the question about velocity when the box reaches the bottom ($s = 100$) we would normally solve $s(t) = 100$ and substitute the t value into velocity. This equation can be solved numerically to use this approach. We can answer the question by an alternative approach. Find t_s so $v(t_s) = 10$ ft/sec, and calculate $s(t_s)$ to see how far the box has fallen. With this approach we solve $18(1-e^{-t/24}) = 10$ to get $t_s = 24\ln\left(\frac{9}{4}\right)$ seconds, and compute $s(t_s) = 432\ln\left(\frac{9}{4}\right) - 240 \approx 110.3$ feet. We conclude that at the bottom ($s = 100$) the box has not yet reached a velocity of 10 ft/sec.

15. If the box loses 32 pounds of material on impact with the bottom, $m = 11$ slugs. So $11\frac{dv}{dt} = -352 + 375 - \frac{1}{2}v, v(0) = 0$, where we have taken up as positive. Solving gives $v(t) = 46(1-e^{-t/22})$, so distance traveled up from the bottom is $s(t) = 46(t + 22e^{-t/22} - 22)$

feet. Solving $s(t) = 100$ gives $t \approx 10.56$ seconds and a surfacing velocity of approximately $v(10.56) \approx 17.5$ ft/sec.

16. The statement of gravitational attraction inside the Earth gives $\frac{dv}{dt} = -kr$, where r is distance to the Earth center. When $r = R$ we know the acceleration is g, so $k = -\frac{g}{R}$ and $\frac{dv}{dt} = -\frac{gr}{R}$. By the chain rule $\frac{dv}{dt} = \frac{dv}{dr}\frac{dr}{dt} = v\frac{dv}{dr}$, which gives the separable equation $v\frac{dv}{dr} = -\frac{gr}{R}$, with $v(R) = 0$. Integrate to get $v^2 = gR - \frac{gr^2}{R}$. Put $r = 0$ to get the speed at Earth's center of $v = \sqrt{gR} = \sqrt{24} \approx 4.9$ miles/second.

17. With θ the angle that the chord makes with the vertical we have $m\frac{dv}{dt} = mg\cos(\theta), v(0) = 0$. Then $s(t) = \frac{1}{2}gt^2\cos(\theta)$; so the time of descent is given by $t = \left(\frac{2s}{g\cos\theta}\right)^{1/2}$ where s is the length of the chord. By the law of cosines, the length of this chord satisfies $s^2 = 2R^2 - 2R^2\cos(\pi - 2\theta) = 2R^2(1+\cos 2\theta) = 4R^2\cos^2\theta$. Thus $t = 2\sqrt{\frac{R}{g}}$, independent of θ.

18. The loop currents in Figure 1.22satisfy the equations $10i_1 + 15(i_1 - i_2) = 10, 15(i_2 - i_1) + 30i_2 = 0$ so $i_1 = \frac{1}{2}$ amp. and $i_2 = \frac{1}{6}$ amp.

19. The capacitor charge is modeled by $250 \times 10^3 i + \frac{1}{2\times 10^{-6}}q = 80, q(0) = 0$. Simplify and put $i = q'$ to get $q' + 2q = 32 \times 10^{-5}$, with solution $q(t) = 16 \times 10^{-5}(1 - e^{-2t})$. The capacitor voltage is $E_c = \frac{1}{C}q = 80(1 - e^{-2t})$. This voltage reaches 76 volts when $t = \frac{1}{2}\ln(20) \approx 1.498$ seconds after the switch is closed. Calculate current at this time by $i(\ln(20)/2) = q'(\ln(20)/2) = 32 \times 10^{-5}e^{-\ln(20)} = 16$ micro amps.

20. From Problem 19, the general solution is $q(t) = 16 \times 10^{-5} + ce^{-2t}$ for charge or $E_c = \frac{q}{C} = 80 + Ae^{-2t}$. If $E_c(0) = 50$, then $A = -30$ and $E_c = 80 - 30e^{-2t}$. Now this reaches 76 volts when $t = \frac{1}{2}\ln\left(\frac{15}{2}\right) \approx 1.00$ seconds.

21. The loop currents satisfy the equations $5(i_1' - i_2') + 10i_1 = 6, -5i_1' + 5i_2' + 30i_2 + 10(q_2 - q_3) = 0, -10q_2 + 10q_3 + 15i_3 + \frac{10}{4}q_3 = 0$. Since $q_1(0+) = q_2(0+) = q_3(0+) = 0$, from equation 3, $i_3(0+) = 0$. Add the three equations to get $10i_1(0+) + 30i_2(0+) = 6$. From the upper node between loops 1 and 2 we reason that $i_1(0^+) = i_2(0^+)$. So $i_1(0+) = i_2(0+) = \frac{3}{20}$ amps.

22. (a) Calculate $i'(t) = \frac{E}{R}e^{-Rt/L} > 0$, so $i(t)$ increases.
(b) Note that $(1 - e^{-1}) = .63^+$, so the inductive time constant is $t_0 = L/R$.
(c) For $i(0) \neq 0$, the time to reach 63% of E/R is $t_0 = \frac{L}{R}\ln\left[\frac{e(E - Ri(0))}{E}\right]$, which decreases with $i(0)$.

23. (a) For $q' + \frac{1}{RC}q = \frac{E}{R}, q(0) = q_0$ an integrating factor is $e^{t/RC}$ so $(qe^{t/RC})' = \frac{E}{R}e^{t/RC}$ and $q(t) = EC + Ae^{-t/RC}$. $q(0) = q_0$ gives $A = q_0 - EC$, so $q(t) = EC + (q_0 - EC)e^{-t/RC}$.
(b) $\lim_{t\to\infty} q(t) = EC$, independent of q_0.

(c) If $q_0 > EC, q_{\max} = q(0) = q_0$, there is no minimum in this case but $q(t)$ decreases toward EC. If $q_0 = EC, q(t) = EC$ for all t. If $q_0 < EC, q_{\min} = q(0) = q_0$, there is no maximum in this case but $q(t)$ increases toward EC.
(d) To reach 99% of the steady state value solve $EC + (q_0 - EC)e^{-t/RC} = EC(1 \pm .01)$, so $t = RC \ln\left|\dfrac{q_0 - EC}{.01EC}\right|$.

24. Differentiating $x+2y = K$ implicitly we find that the given family satisfies the differential equation $y' = -\dfrac{1}{2}$. Thus the orthogonal trajectories satisfy $y' = 2$, and are given by $y = 2x+c$.

25. The differential equation of the given family is $\dfrac{dy}{dx} = \dfrac{4x}{3}$; orthogonal trajectories satisfy $\dfrac{dy}{dx} = -\dfrac{3}{4x}$ and are given by $y = -\dfrac{3}{4}\ln|x| + c$.

26. The differential equation of the given family is $\dfrac{dy}{dx} = -\dfrac{x}{2y}$; orthogonal trajectories satisfy $\dfrac{dy}{dx} = \dfrac{2y}{x}$ and are given by $y = Ax^2$, a family of parabolas.

27. The differential equation of the given family is $y' = 2Kx = \dfrac{2x(y-1)}{x^2} = \dfrac{2(y-1)}{x}$; orthogonal trajectories satisfy $\dfrac{dy}{dx} = -\dfrac{x}{2(y-1)}$ and are given by $(y-1)^2 = -\dfrac{x^2}{2} + c$, a family of ellipses.

28. The differential equation of the given family is $x - yy'K = 0$ or $x - yy'\left(\dfrac{x^2-1}{y^2}\right) = 0$ so $y' = \dfrac{xy}{x^2-1}$; orthogonal trajectories satisfy $\dfrac{dy}{dx} = \dfrac{1-x^2}{xy}$, which is separable with solutions $\dfrac{y^2}{2} = \ln|x| - \dfrac{x^2}{2} + c$.

29. The differential equation of the given family is found by solving for K and differentiating to get $K = \dfrac{\ln(y)}{x}$ so $-\dfrac{1}{x^2}\ln(y) + \dfrac{1}{xy}y' = 0$ or $y' = \dfrac{y\ln(y)}{x}$; orthogonal trajectories satisfy $\dfrac{dy}{dx} = -\dfrac{x}{y\ln(y)}$ which is separable with solutions $y^2[\ln(y^2) - 1] = c - 2x^2$.

Section 1.8 Existence and Uniqueness

1. Both $f(x,y) = 2y^2+3xe^y\sin(xy)$ and $\dfrac{\partial f}{\partial y} = 4y+3xe^y\sin(xy)+3x^2e^y\cos(xy)$ are continuous everywhere.

2. Both $f(x,y) = 4xy + \cosh(x)$ and $\dfrac{\partial f}{\partial y} = 4x$ are continuous everywhere so Theorem 1 applies. Since the equation is linear we could apply Theorem 2.

3. Both $f(x,y) = (xy)^3 - \sin(y)$ and $\dfrac{\partial f}{\partial y} = 3x^3y^2 - \cos(y)$ are continuous everywhere.

4. Both $f(x,y) = x^5 - y^5 + 2xe^y$ and $\dfrac{\partial f}{\partial y} = -5y^4 + 2xe^y$ are continuous everywhere.

5. Both $f(x,y) = x^2ye^{-2x} + y^2$ and $\dfrac{\partial f}{\partial y} = x^2e^{-2x} + 2y$ are continuous everywhere.

6. (a) Taking $|y'| = y'$ we get $y' = 2y$ with solution $y(x) = y_0e^{2(x-x_0)}$. Taking $|y'| = -y'$ we get $y' = -2y$ with solution $y = y_0e^{-2(x-x_0)}$.
(b) Solving $|y'| = 2y$ we get $y' = \pm 2y = f(x,y)$, but $f(x,y) = \pm 2y$ is not even a function (unless $y = 0$), let alone continuous, so Theorem 1 does not apply.

7. (a) Since both $f(x,y) = 2 - y$ and $\dfrac{\partial f}{\partial y} = -1$ are continuous everywhere, the initial value problem has a unique solution.
(b) $y = 2 - e^{-x}$
(c) $y_0 = 1$ and $y_1 = 1 + \int_0^x 1dt = 1 + x$;

$$y_2 = 1 + \int_0^x (1-t)dt = 1 + x - \frac{x^2}{2};$$
$$y_3 = 1 + \int_0^x (1 - t + \frac{t^2}{2})dt = 1 + x - \frac{x^2}{2} + \frac{x^3}{3!};$$
$$y_4 = 1 + \int_0^x (1 - t + \frac{t^2}{2!} - \frac{t^3}{3!})dt = 1 + x - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!};$$
$$y_5 = 1 + \int_0^x (1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!})dt = 1 + x - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + \frac{x^5}{5!};$$
$$y_6 = 1 + \int_0^x y_5(t)dt = 1 + x - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + \frac{x^5}{5!} - \frac{x^6}{6!};$$
$$y_n = 1 + x - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + \cdots + (-1)^{n+1}\frac{x^n}{n!};$$

(d) $2 - e^{-x} = 2 - \left[1 - x + \dfrac{x^2}{2!} - \dfrac{x^3}{3!} + \cdots + (-1)^n\dfrac{x^n}{n!} + \cdots\right] = 1 + x - \dfrac{x^2}{2!} + \dfrac{x^3}{3!} - \cdots + (-1)^{n+1}\dfrac{x^n}{n!} + \cdots$

Since $2 - e^{-x} = 2 - \lim_{n\to\infty}\sum_{k=0}^{n}(-1)^n\dfrac{x^n}{n!} = \lim_{n\to\infty} y_n(x)$, the Picard iterates converge to the unique solution.

8. (a) Since both $f(x,y) = 4 + y$ and $\dfrac{\partial f}{\partial y} = 1$ are continuous everywhere, the initial value problem has a unique solution.
(b) $y = -4 + 7e^x$
(c) $y_0 = 3$ and

$$y_1 = 3 + \int_0^x 7dt = 3 + 7x;$$
$$y_2 = 3 + \int_0^x (7 + 7t)dt = 3 + 7x + 7\frac{x^2}{2};$$
$$y_3 = 3 + \int_0^x (7 + 7t + 7\frac{t^2}{2!})dt = 3 + 7x + 7\frac{x^2}{2} + 7\frac{x^3}{3!};$$
$$y_4 = 3 + \int_0^x (7 + 7t + 7\frac{t^2}{2!} + 7\frac{t^3}{3!})dt = 3 + 7x + 7\frac{x^2}{2} + 7\frac{x^3}{3!} + 7\frac{x^4}{4!};$$
$$y_5 = 3 + \int_0^x y_4(t)dt = 3 + 7x + 7\frac{x^2}{2} + 7\frac{x^3}{3!} + 7\frac{x^4}{4!} + 7\frac{x^5}{5!};$$

$y_6 = 3 + \int_0^x y_5(t)dt = 3 + 7x + 7\frac{x^2}{2} + 7\frac{x^3}{3!} + 7\frac{x^4}{4!} + 7\frac{x^5}{5!} + 7\frac{x^6}{6!}$;

(d) $y_n = 3 + 7x + 7\frac{x^2}{2!} + \cdots + 7\frac{x^n}{n!}$;

Note $y_n(x) = -4 + 7\left(\sum_{k=0}^{n} \frac{x^k}{k!}\right)$ and $\lim_{n\to\infty} y_n(x) = -4 + 7\sum_{k=0}^{\infty} \frac{x^k}{k!} = -4 + 7e^x$ since the series converges. Thus the Picard iterates converge to the unique solution.

9. (a) Both $f(x,y) = 2x^2$ and $\frac{\partial f}{\partial y} = 0$ are continuous everywhere.

(b) $y = \frac{2}{3}x^3 + \frac{7}{3}$

(c) $y_0 = 3$,

$y_1 = 3 + \int_1^x 2t^2 dt = 3 + \frac{2}{3}x^3 - \frac{2}{3} = \frac{2}{3}x^3 + \frac{7}{3}$,

(d) Since $f(x,y) = 2x^2$ is independent of y, we have $y_n(x) = y_1(x) = \frac{2}{3}x^3 + \frac{7}{3}$, for all $n \geq 1$, and the sequence of Picard iterates is a constant sequence.

$y = \frac{2}{3}x^3 + \frac{7}{3} = 3 + 2(x-1) + 2(x-1)^2 + \frac{2}{3}(x-1)^3$ is the Taylor series of the solution. For $n \geq 3$, the n^{th} partial sum of the series is the solution. Certainly $\{y_n\} \to y$ so the Picard iterates converge to the solution.

10. (a) $f(x,y) = \cos(x)$ and $\frac{\partial f}{\partial y} = 0$ are continuous everywhere.

(b) $y = 1 + \sin(x)$

(c) $y_0 = 1$, and

$y_1 = 1 + \int_\pi^x \cos(t)dt = 1 + \sin(x)$,

(d) Since $f(x,y) = \cos(x)$ is independent of $y, y_n(x) = y_1(x) = 1 + \sin(x)$, for all Picard iterates, $n \geq 1$.

$y = 1 + \sin(x) = 1 + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = 1 + x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$. The n^{th} partial sum of the Taylor series $T_n(x)$ does not agree with the n^{th} Picard iterate $y_n(x)$. However $\lim_{n\to\infty} T_n(x) = \lim_{n\to\infty} y_n(x) = 1 + \sin(x)$ so both sequences converge to the unique solution.

Chapter Two - Second Order Differential Equations

Section 2.2 Theory of Solutions

In Problems 1 - 6 direct substitution of y_1 and y_2 verifies that each is a solution of the given differential equation.

1. (b) $W = \begin{vmatrix} \cosh(2x) & \sinh(2x) \\ 2\sinh(2x) & 2\cosh(2x) \end{vmatrix} = 2[\cosh^2(2x) - \sinh^2(2x)] = 2;$
(c) $y = c_1 \cosh(2x) + c_2 \sinh(2x);$
(d) $y = \cosh(2x)$

2. (b) $W = \begin{vmatrix} \cos(3x) & \sin(3x) \\ -3\sin(3x) & 3\cos(3x) \end{vmatrix} = 3;$
(c) $y = c_1 \cos(3x) + c_2 \sin(3x);$
(d) $y = -\frac{1}{3}\sin(3x)$

3. (b) $W = \begin{vmatrix} e^{-3x} & e^{-8x} \\ -3e^{-3x} & -8e^{-8x} \end{vmatrix} = -5e^{-11x};$
(c) $y = c_1 e^{-3x} + c_2 e^{-8x};$
(d) $y = \frac{12}{5}e^{-3x} - \frac{7}{5}e^{-8x}$

4. (b) $W = \begin{vmatrix} e^{-x}\cos(\sqrt{7}x) & e^{-x}\sin(\sqrt{7}x) \\ e^{-x}(\cos(\sqrt{7}x) + \sqrt{7}\sin(\sqrt{7}x)) & e^{-x}(-\sin(\sqrt{7}x) + \sqrt{7}\cos(\sqrt{7}x)) \end{vmatrix} = \sqrt{7}e^{-x};$
(c) $y = e^{-x}[c_1\cos(\sqrt{7}x) + c_2\sin(\sqrt{7}x)];$
(d) $y = e^{-x}[2\cos(\sqrt{7}x) + \frac{2}{\sqrt{7}}\sin(\sqrt{7}x)]$

5. (b) $W = \begin{vmatrix} x^4 & x^4\ln x \\ 4x^3 & 4x^3\ln x + x^3 \end{vmatrix} = x^7;$
(c) $y = x^4(c_1 + c_2\ln(x))$;
(d) $y = 2x^4 - 4x^4\ln(x)$

6. (b) $W = \frac{2}{\pi}\begin{vmatrix} \frac{\cos(x)}{\sqrt{x}} & \frac{\sin(x)}{\sqrt{x}} \\ -(\frac{\sin(x)}{\sqrt{x}} + \frac{\cos(x)}{2x\sqrt{x}}) & (\frac{\cos(x)}{\sqrt{x}} - \frac{\sin(x)}{2x\sqrt{x}}) \end{vmatrix} = \frac{2}{\pi x};$
(c) $y = c_1\sqrt{\frac{2}{\pi}}\frac{\cos(x)}{\sqrt{x}} + c_2\sqrt{\frac{2}{\pi}}\frac{\sin(x)}{\sqrt{x}};$
(d) $y = -5\sqrt{\pi}\frac{\cos(x)}{\sqrt{x}} - \frac{(16\pi + 5)}{2\sqrt{\pi}}\frac{\sin(x)}{\sqrt{x}}$

7. $W = \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix} = x^4; y_1$ and y_2 are linearly independent solutions of the differential

equation $x^2y'' - 4xy' + 6y = 0$ or $y'' - \frac{4}{x}y' + \frac{6}{x^2}y = 0$. Theorem 3 applies for this equations only on intervals <u>not</u> containing $x = 0$, and on any such interval $W = x^4 \neq 0$.

8. Clearly y_1 and y_2 are linearly independent on $[-1, 1]$ since $y_1(x) \neq ky_2(x)$. The differential equation can be written $y'' - \frac{2}{x}y' + \frac{2}{x^2}y = 0$, so Theorem 2.3 applies only on intervals <u>not</u> containing $x = 0$.

9. $y'' - y' - 2y = 0$ has solution $y_1 = e^{-x}, y_2 = e^{2x}$, but $y_1y_2 = e^x$ is not a solution.

10. Theorem 2.2 applies only to linear equations and $yy'' + 2y' - (y')^2 = 0$ is non-linear.

11. At a relative extremum of a differentiable function y, we have $y'(x_0) = 0$. Thus $W(x_0) = \begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} = \begin{vmatrix} y_1(x_0) & y_2(x_0) \\ 0 & 0 \end{vmatrix} = 0$, and by Theorem 2.3, y_1 and y_2 are linearly dependent.

12. By Theorem 1, $y'' + p(x)y' + q(x)y = 0, y(x_0) = 0, y'(x_0) = 0$ has a unique solution, which is clearly $y(x) \equiv 0$. If $\phi'(x_0) = 0$, then $\phi(x) \equiv 0$ which contradicts the fact that ϕ is non-zero. Hence $\phi'(x_0) \neq 0$.

13. We have $W(x_0) = \begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} = 0$, and by Theorem 2.3, y_1 and y_2 are linearly dependent.

<u>Section 2.3</u> Reduction of Order

In problems 1 - 10 we put $y_2(x) = u(x)y_1(x)$, derive the equation satisfied by u, give its solution for $u(x)$, and give the general solution of the second order equation.

1. $u''\cos(2x) - 4\sin(2x)u' = 0; u(x) = \tan(2x); y = c_1\cos(2x) + c_2\sin(2x)$
2. $u'' + 6u' = 0; u(x) = e^{-6x}; y = c_1e^{3x} + c_2e^{-3x}$
3. $u'' = 0; u(x) = x; y = c_1e^{5x} + c_2xe^{5x}$
4. $xu'' + u' = 0; u = \ln(x); y = c_1x^4 + c_2x^4\ln(x)$.
5. $xu'' + u' = 0; u = \ln(x); y = c_1x^2 + c_2x^2\ln(x)$
6. $(2x^3 + x)u'' + 2u' = 0;\ u = 2x - \frac{1}{x};\ y = c_1x + c_2(2x^2 - 1)$.
7. $xu'' + 7u' = 0; u = x^{-6}; y = c_1x^4 + c_2x^{-2}$
8. $xu'' + \frac{2}{1+x^2}u' = 0$ which can be written as $\left[u'(\frac{x^2}{1+x^2})\right]' = 0$. Thus $u' = 1 + \frac{1}{x^2}$ and $u = \frac{x^2-1}{x}$ and $y = c_1(x^2 - 1) + c_2x$
9. $x^{-1/2}\cos(x)u'' - 2x^{-1/2}\sin(x)u' = 0; u(x) = \tan(x); y = c_1\left(\frac{\cos(x)}{\sqrt{x}}\right) + c_2\left(\frac{\sin(x)}{\sqrt{x}}\right)$
10. $(2x^3 + 3x^2 + x)u'' + (6x^2 + 6x + 2)u' = 0$ which gives $u' = \frac{2x+1}{x^2(x+1)^2}$ and $u = \frac{1}{x(x+1)}$ and the general solution is $y = c_1x + c_2\left(\frac{1}{x+1}\right)$

11. $y = c_1 e^{-ax} + c_2 x e^{-ax}$

12. (a) With $u = y'$ we get $xu' - u = 2$, first order linear with solution $u = -\frac{2}{x} + c_1 x$. Integrate to get $y = \int \left(-\frac{2}{x} + c_1 x\right) dx = -2\ln|x| + \hat{c}_1 x^2 + c_2$

(b) $xu' + 2u = x$ gives $u = \frac{x^2}{4} + \frac{c_1}{x^2}$ and then $y = \frac{x^2}{6} + \frac{\hat{c}_1}{x} + c_2$.

(c) $1 - u = 4u'$ gives $u = c_1 e^{-x/4} + 1$ and then $y = x + \hat{c}_1 e^{-x/4} + c_2$

(d) $u' + u^2 = 0$ gives $u = \frac{1}{(x + c_1)}$ and then $y = \ln|x + c_1| + c_2$

(e) $u' = 1 + u^2$ gives $u = \tan(x + c_1)$ and then $y = \ln|\sec(x + c_1)| + c_2$

13. (a) $yu\frac{du}{dy} + 3u^2 = 0$ is separable as $\frac{du}{u} = -\frac{3dy}{y}$. Integration gives $\ln|u| = -3\ln|y| + c$ or $uy^3 = A$. Thus $y^3 dy = Adx$ and $\frac{y^4}{4} = Ax + B$ or $y^4 = c_1 x + c_2$

(b) $(y-1)e^y = c_1 x + c_2$ or $y = c_3$

(c) $y = \frac{c_1 e^{c_1 x}}{c_2 - e^{c_1 x}}$ or $y = \frac{1}{c_3 - x}$

(d) $y = \ln|\sec(x + c_1)| + c_2$

(e) $y = \ln|c_1 x + c_2|$

14. With $y = uy_1$ we get $y'' + Ay' + By = \left[u'' - Au' + \frac{A^2}{4}u + A\left(u' - \frac{A}{2}u\right) + Bu\right]e^{-Ax/2} = u''e^{-Ax/2} = 0$ iff $u'' = 0$. Thus $u = c_1 + c_2 x$and $y = c_1 e^{-Ax/2} + c_2 x e^{-Ax/2}$.

15. With $y = uy_1$ we get $y'' + \frac{A}{x}y' + \frac{B}{x^2}y =$ $\left[u''x^2 + (1-A)xu' - \left(\frac{1-A}{2}\right)\left(\frac{1+A}{2}\right)u + A\left(xu' + \frac{(1-A)}{2}u\right) + Bu\right]x^{-(3+A)/2} = [xu'' + u']x^{(1+A)/2} = 0$ iff $xu'' + u' = 0$. Thus $u = c_1 + c_2\ln(x)$and $y = c_1 x^{(1-A)/2} + c_2 x^{(1-A)/2}\ln(x)$.

Section 2.4 The Constant Coefficient Homogeneous Linear Equation

1. The characteristic equation is $\lambda^2 - \lambda - 6 = 0$ which has roots $\lambda = -2$ and $\lambda = 3$; thus the general solution is $y = c_1 e^{-2x} + c_2 e^{3x}$

2. The characteristic equation is $\lambda^2 - 2\lambda + 10 = 0$ which has roots $\lambda = 1 + 3i$ and $\lambda = 1 - 3i$; thus the general solution is $y = e^x[c_1\cos(3x) + c_2\sin(3x)]$

3. The characteristic equation is $\lambda^2 + 6\lambda + 9 = 0$ which has repeated roots $\lambda = -3$ and $\lambda = -3$; thus the general solution is $y = e^{-3x}[c_1 + c_2 x]$

4. The characteristic equation is $\lambda^2 - 3\lambda = 0$ which has roots $\lambda = 0$ and $\lambda = 3$; thus the general solution is $y = c_1 + c_2 e^{3x}$

5. The characteristic equation is $\lambda^2 + 10\lambda + 26 = 0$ which has roots $\lambda = -5 + i$ and $\lambda = -5 - i$; thus the general solution is $y = e^{-5x}[c_1\cos(x) + c_2\sin(x)]$

6. The characteristic equation is $\lambda^2 + 6\lambda - 40 = 0$ which has roots $\lambda = -10$ and $\lambda = 4$; thus the general solution is $y = c_1 e^{-10x} + c_2 e^{4x}$

7. The characteristic equation is $\lambda^2+3\lambda+18=0$ which has roots $\lambda=-\frac{3}{2}+i\frac{3\sqrt{7}}{2}$ and $\lambda=-\frac{3}{2}-i\frac{3\sqrt{7}}{2}$; thus the general solution is $y=e^{-\frac{3}{2}x}\left[c_1\cos\left(\frac{3\sqrt{7}}{2}x\right)+c_2\sin\left(\frac{3\sqrt{7}}{2}x\right)\right]$

8. The characteristic equation is $\lambda^2+16\lambda+64=0$ which has repeated roots $\lambda=-8$ and $\lambda=-8$; thus the general solution is $y=e^{-8x}[c_1+c_2x]$

9. The characteristic equation is $\lambda^2-14\lambda+49=0$ which has repeated roots $\lambda=7$ and $\lambda=7$; thus the general solution is $y=e^{7x}[c_1+c_2x]$

10. The characteristic equation is $\lambda^2-6\lambda+7=0$ which has roots $\lambda=3+i\sqrt{2}$ and $\lambda=3-i\sqrt{2}$; thus the general solution is $y=e^{3x}[c_1e^{\sqrt{2}x}+c_2e^{-\sqrt{2}x}]$

11. The characteristic equation is $\lambda^2+4\lambda+9=0$ which has roots $\lambda=-2+i\sqrt{5}$ and $\lambda=-2-i\sqrt{5}$; thus the general solution is $y=e^{-2x}[c_1\cos(\sqrt{5}x)+c_2\sin(\sqrt{5}x)]$

12. The characteristic equation is $\lambda^2+5\lambda=0$ which has roots $\lambda=0$ and $\lambda=-5$; thus the general solution is $y=c_1+c_2e^{-5x}$

13. $y=5-2e^{-3x}$

14. $y=4e^{x}+2e^{-3x}$

15. $y=0$ for all x

16. $y=e^{2x}[3-x]$

17. $y=\frac{9}{7}e^{3(x-2)}+\frac{5}{7}e^{-4(x-2)}$

18. $y=\frac{\sqrt{6}}{4}e^{x}[e^{\sqrt{6}x}-e^{-\sqrt{6}x}]$

19. $y=e^{(x-1)}[29-17x]$

20. $y=-4(5-\sqrt{23})e^{5(x-2)/2}\sin\left(\frac{\sqrt{23}}{2}(x-2)\right)$

21. $y=e^{(x+2)/2}\left[\cos\left(\frac{\sqrt{15}}{2}(x+2)\right)+\frac{5}{\sqrt{15}}\sin\left(\frac{\sqrt{15}}{2}(x+2)\right)\right]$

22. (a) $\phi=e^{ax}[c_1+c_2x]$; (b) $\phi_\epsilon=e^{ax}[c_1e^{\epsilon x}+c_2e^{-\epsilon x}]$; (c) $\lim\limits_{\epsilon\to 0}\phi_\epsilon(x)=e^{ax}[c_1+c_2]\neq\phi(x)$ in general.

23. (a) $\psi=e^{ax}[c+(d-ac)x]$
(b) $\psi_\epsilon=e^{ax}\left[\frac{(d-ac+\epsilon c)e^{\epsilon x}+(ac-d+\epsilon c)e^{-\epsilon x}}{2\epsilon}\right]$
(c) $\lim\limits_{\epsilon\to 0}\psi_\epsilon(x)=\frac{e^{ax}}{2}\lim\limits_{\epsilon\to 0}[x(d-ac+\epsilon c)e^{\epsilon x}-x(ac-d+\epsilon c)e^{-\epsilon x}+c(e^{\epsilon x}+e^{-\epsilon x})]=e^{ax}[c+(d-ac)x]=\psi(x)$, by L'Hopital's rule.

24. The characteristic equation has roots $\lambda_1=-\frac{A}{2}+\frac{1}{2}\sqrt{A^2-4B}$ and $\lambda_2=-\frac{A}{2}-\frac{1}{2}\sqrt{A^2-4B}$. With $B>0$ we have $A^2-4B<A^2$, so λ_1 and λ_2 are either both
negative real numbers or are complex conjugates with negative real part. In the first case $\phi(x)=c_1e^{\lambda_1 x}+c_2e^{\lambda_2 x}$ and $\lim\limits_{x\to+\infty}\phi(x)=0$.

In the second case

$$\phi(x) = e^{-\frac{A}{2}x}\left[c_1\cos\left(\frac{1}{2}\sqrt{4B-A^2}x\right) + c_2\sin\left(\frac{1}{2}\sqrt{4B-A^2}x\right)\right]$$

so

$$|\phi(x)| \leq e^{-\frac{A}{2}}\sqrt{c_1^2+c_2^2} \text{ and } \lim_{x\to\infty}|\phi(x)| = 0, \text{hence } \lim_{x\to\infty}\phi(x) = 0.$$

Section 2.5 Euler's Equation

1. $y = c_1x^2 + c_2x^{-3}$
2. $y = x^{-1}[c_1 + c_2\ln(x)]$
3. $y = c_1\cos[2\ln(x)] + c_2\sin[2\ln(x)]$
4. $y = c_1x^2 + c_2x^{-2}$
5. $y = c_1x^4 + c_2x^{-4}$
6. $y = x^{-2}\{c_1\cos[3\ln(x)] + c_2\sin[3\ln(x)]\}$
7. $y = c_1x^{-2} + c_2x^{-3}$
8. $y = x^2\{c_1\cos[7\ln(x)] + c_2\sin[7\ln(x)]\}$
9. $y = x^{-12}[c_1 + c_2\ln(x)]$
10. $y = c_1x^7 + c_2x^5$
11. $y = x^{3/2}\left\{c_1\cos\left[\frac{\sqrt{39}}{2}\ln(x)\right] + c_2\sin\left[\frac{\sqrt{39}}{2}\ln(x)\right]\right\}$
12. $y = x^{1/2}\left\{c_1\cos\left[\frac{\sqrt{15}}{2}\ln(x)\right] + c_2\sin\left[\frac{\sqrt{15}}{2}\ln(x)\right]\right\}$
13. $y = x^{-2}\{3\cos[4\ln(-x)] - 2\sin[4\ln(-x)]\}$
14. $y = \frac{7}{10}\left(\frac{x}{2}\right)^3 + \frac{3}{10}\left(\frac{x}{2}\right)^{-7}$
15. $y = -3 + 2x^2$
16. $y = x^2[4 - 3\ln(x)]$
17. $y = -x^{-3}\cos[2\ln(-x)]$
18. $y = \frac{7}{2}\left(\frac{x}{2}\right)^{-1} - \frac{5}{2}\left(\frac{x}{2}\right)$
19. $y = -4x^{-12}[1 + 12\ln(x)]$
20. $y = 3x^6 - 2x^4$
21. $y = \frac{11}{4}x^2 + \frac{17}{4}x^{-2}$
22. The transformation $x = e^t$ transforms the Euler equation $x^2y'' + Axy' + By = 0$ into $Y'' + (A-1)Y' + BY = 0$. Let $\lambda_1$and λ_2 designate the characteristic roots of this constant coefficient equation. Suppose on the other hand we substitute x^r directly into $x^2y''+Axy'+By = 0$ to get $r(r-1)x^r + Arx^r + Bx^r = [r^2 + (A-1)r + B]x^r = 0$. Then r must satisfy $r^2 + (A-1)r + B = 0$ and the values of r are exactly $r_1 = \lambda_1$ and $r_2 = \lambda_2$. Thus both the transformation method, $x = e^t$, and direct substitution of x^r lead to the same solutions.

Section 2.6 The Nonhomogeneous Equation

1. By variation of parameters with $y_1 = \cos(x), y_2 = \sin(x)$, and $f(x) = \tan(x)$ we get $u' = \dfrac{-\sin^2(x)}{\cos(x)} = \cos(x) - \sec(x), v' = \sin(x)$. Thus $u(x) = \sin(x) - \ln|\sec(x) + \tan(x)| + c_1$ and $v(x) = -\cos(x) + c_2$ and the general solution is $y = c_1\cos(x) + c_2\sin(x) - \cos(x)\ln|\sec(x) + \tan(x)|$.

2. $y = c_1e^x + c_2e^{3x} + \frac{1}{5}\cos(x+3) - \frac{2}{5}\sin(x+3)$

3. $y = c_1\cos(3x) + c_2\sin(3x) + 4x\sin(3x) + \frac{4}{3}\ln[\cos(3x)]\cos(3x)$ by variation of parameters.

4. First write $2\sin^2(x) = 1 - \cos(2x)$ and get $y = c_1e^{3x} + c_2e^{-x} - \frac{1}{3} + \frac{7}{65}\cos(2x) + \frac{4}{65}\sin(2x)$

5. $y = c_1e^x + c_2e^{2x} - e^{2x}\cos(e^{-x})$

6. First write $8\sin^2(4x) = 4\cos(8x) - 4$ to get $y = c_1e^{3x} + c_2e^{2x} + \frac{2}{3} + \frac{58}{1241}\cos(8x) + \frac{40}{1241}\sin(8x)$

7. $y = c_1e^{2x} + c_2e^{-x} - x^2 + x - 4$

8. $y = c_1e^{3x} + c_2e^{-2x} - 2e^{2x}$

9. $y = e^x[c_1\cos(3x) + c_2\sin(3x)] + 2x^2 + x - 1$

10. $y = e^{2x}[c_1\cos(x) + c_2\sin(x)] + 21e^{2x}$

11. $y = c_1e^{2x} + c_2e^{4x} + e^x$

12. $y = e^{-3x}[c_1 + c_2x] + \frac{1}{2}\sin(3x)$

13. $y = c_1e^x + c_2e^{2x} + 3\cos(x) + \sin(x)$

14. $y = c_1 + c_2e^{-4x} - \frac{2}{3}x^3 - \frac{1}{2}x^2 - \frac{1}{4}x - \frac{2}{3}e^{3x}$

15. $y = e^{2x}[c_1\cos(3x) + c_2\sin(3x)] + \frac{1}{3}e^{2x} - \frac{1}{2}e^{3x}$

16. $y = e^x[c_1 + c_2x] + 3x + 6 + \frac{3}{2}\cos(3x) - 2\sin(3x)$

17. By undetermined coefficients $y = c_1e^{2x} + c_2e^{-x} + \frac{1}{3}xe^{2x}$

18. By variation of parameters $y = c_1x^2 + c_2x^{-6} - \frac{1}{12}\ln(x) - \frac{1}{36}$

19. By undetermined coefficients $y = c_1e^{2x} + c_2e^{-3x} - \frac{1}{6}x - \frac{1}{36}$

20. First write $2\sinh^2(x) = \cosh(2x) - 1$ and use undetermined coefficients to get $y = c_1e^{4x} + c_2e^{-3x} - \frac{2}{15}\cosh(2x) + \frac{1}{30}\sinh(2x) - \frac{1}{12}$

21. By variation of parameters $y = c_1x^2 + c_2x^4 + x$

22. By variation of parameters $y = x^{-1}[c_1 + c_2\ln(x)] + 2x^{-1}\ln^2(x)$

23. By variation of parameters $y = c_1\cos[2\ln(x)] + c_2\sin[2\ln(x)] - \frac{1}{4}\cos[2\ln(x)]\ln(x)$

24. By variation of parameters $y = c_1x^2 + c_2x^{-3} + \frac{1}{5}x^2\ln(x) + \frac{1}{3}$

25. By undetermined coefficients $y = c_1e^{2x} + c_2xe^{2x} + e^{3x} - \frac{1}{4}$

25. By undetermined coefficients $y = c_1e^{2x} + c_2e^{-x} - \frac{1}{2}x - \frac{1}{4}$

27. $y = \frac{7}{4}e^{2x} - \frac{3}{4}e^{-2x} - \frac{7}{4}xe^{2x} - \frac{1}{4}x$

28. $y = 3 + 2e^{-4x} - 2\cos(x) + 8\sin(x) + 2x$

29. $y = \frac{3}{8}e^{-2x} - \frac{19}{120}e^{-6x} + \frac{1}{5}e^{-x} + \frac{7}{12}$

30. $y = \frac{1}{5} + e^{3x} - \frac{1}{5}e^{2x}[\cos(x) + 3\sin(x)]$

31. $y = 2e^{4x} + 2e^{-2x} - 2e^{-x} - e^{2x}$

32. $y = e^{3x} - xe^{3x} + 2x^2e^{3x}$

33. $y = -\frac{17}{4}e^{2x} + \frac{55}{13}e^{3x} + \frac{1}{52}\cos(2x) - \frac{5}{52}\sin(2x)$

34. The general solution is given by $y = e^{x/2}\left[c_1\cos\left(\frac{\sqrt{3}}{2}x\right) + c_2\sin\left(\frac{\sqrt{3}}{2}x\right)\right] + 1$, but this can also be written in the form $y = e^{x/2}\left[d_1\cos\left(\frac{\sqrt{3}}{2}(x-1)\right) + d_2\sin\left(\frac{\sqrt{3}}{2}(x-1)\right)\right] + 1$ to facilitate fitting the initial conditions specified at $x = 1$. We get $y(1) = e^{1/2}d_1 + 1 = 4$, and $y'(1) = \frac{1}{2}e^{1/2}d_1 + \frac{\sqrt{3}}{2}e^{1/2}d_2 = -2$. We find $d_1 = 3e^{-1/2}$and $d_2 = -\frac{7}{\sqrt{3}}e^{-1/2}$.
The general solution can be written in the form

$$y = e^{(x-1)/2}\left[3\cos\left(\frac{\sqrt{3}}{2}(x-1)\right) - \frac{7}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}(x-1)\right)\right] + 1.$$

35. The general solution is given by $y = e^{4x}[c_1e^{\sqrt{14}x} + c_2e^{-\sqrt{14}x}] + \frac{e^{-x}}{11}$. By the results of Problem 27, Section 2.4 and properties of the hyperbolic functions this solution can be written $y = e^{4x}[A\cosh(\sqrt{14}(x+1) + \frac{B}{\sqrt{14}}\sinh(\sqrt{14}(x+1))] + \frac{e^{-x}}{11}$. This form will greatly facilitate fitting the initial conditions specified at $x = -1$. We get $y(-1) = Ae^{-4} + \frac{e}{11} = 5$ and $y'(-1) = 4Ae^{-4} + Be^{-4} - \frac{e}{11} = 2$. Solving for A and B gives the general solution

$$y = \frac{e^{4(x+1)}}{11}[(55-e)\cosh(\sqrt{14}(x+1))$$

$$+\frac{(5e-198)}{\sqrt{14}}\sinh(\sqrt{14}(x+1))] + \frac{e^{-x}}{11}.$$

36. $y = \frac{1}{100}[e^{-3x}(108 - 270x) - 8\cos(x) - 6\sin(x)]$.

37. $y = 4e^{-x} - \sin^2(x) - 2$

38. $y = 4\cos(x) + 4\sin(x) - \cos(x)\ln|\sec(x) + \tan(x)|$

39. $y = 2x^3 + x^{-2} - 2x^2$

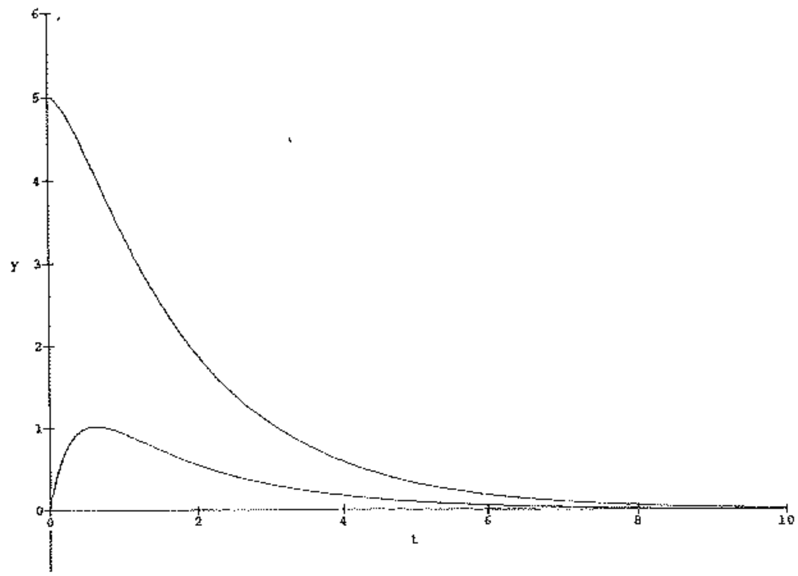
40. $y = 3x^{-3} + 2x^{-3}\ln(x) + 3\ln(x) - 2$

41. $y = x - x^2 + 3\cos[\ln(x)] + \sin[\ln(x)]$

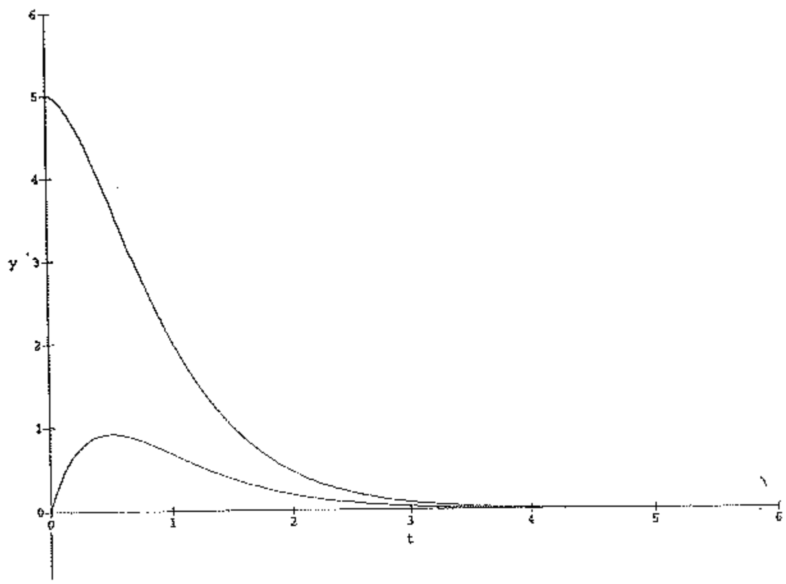
42. $y = (e^2 - 2)x^2 + \left(\frac{5}{4} - e^2\right)x^3 + x^2e^x$

Section 2.7 Applications

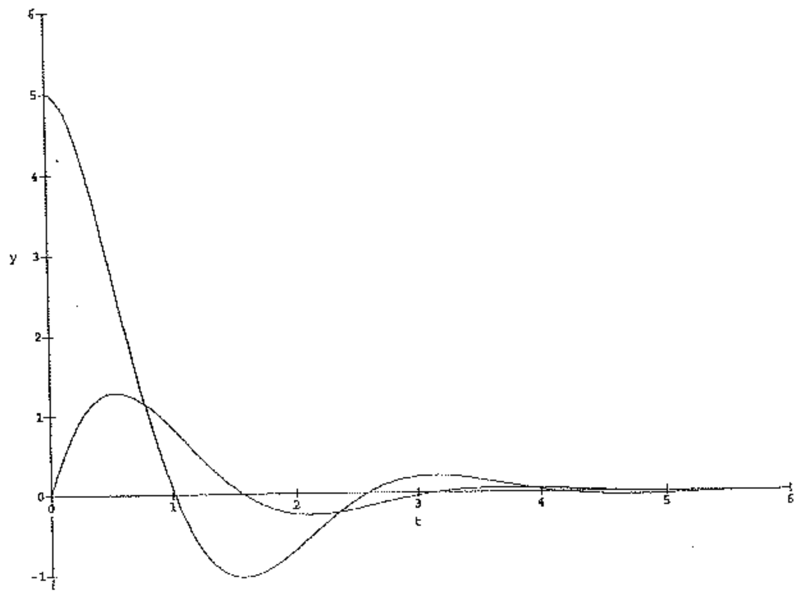
1. The solution with initial conditions $y(0) = 5, y'(0) = 0$ is $y_1(t) = 5e^{-2t}[\cosh(\sqrt{2}t) + \sqrt{2}\sinh(\sqrt{2}t)]$; with initial conditions $y(0) = 0, y'(0) = 5$ is $y_2(t) = \frac{5}{\sqrt{2}}e^{-2t}\sinh(\sqrt{2}t)$.



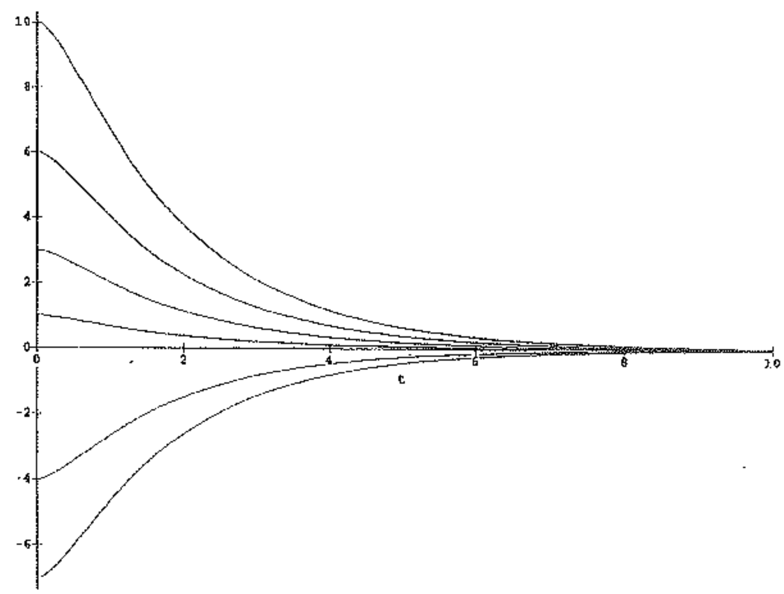
2. The solution with initial conditions $y(0) = 5, y'(0) = 0$ is $y_1(t) = 5e^{-2t}(1 + 2t)$; with initial conditions $y(0) = 0, y'(0) = 5$ is $y_2(t) = 5te^{-2t}$.



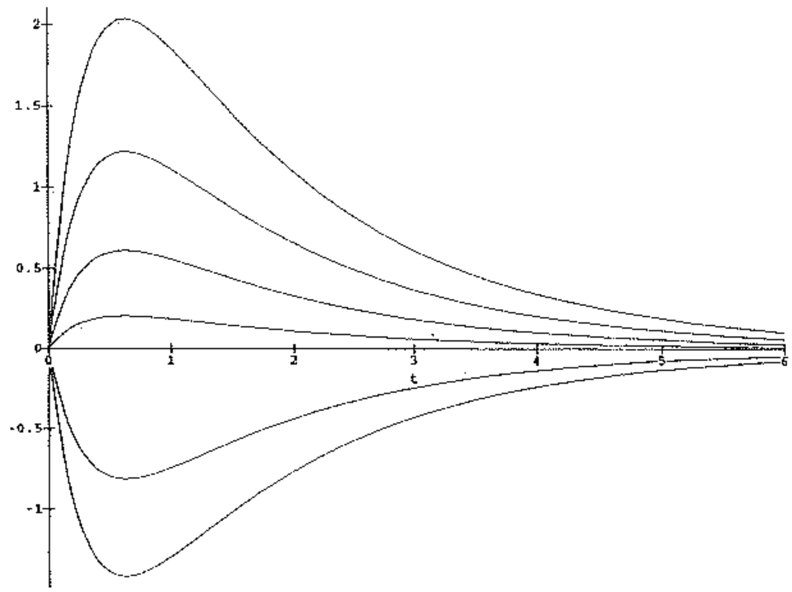
3. The solution with initial conditions $y(0)=5, y'(0)=0$ is $y_1(t)=\frac{5}{2}e^{-t}[2\cos(2t)+\sin(2t)]$; with initial conditions $y(0)=0, y'(0)=5$ is $y_2(t)=\frac{5}{2}e^{-t}\sin(2t)$.



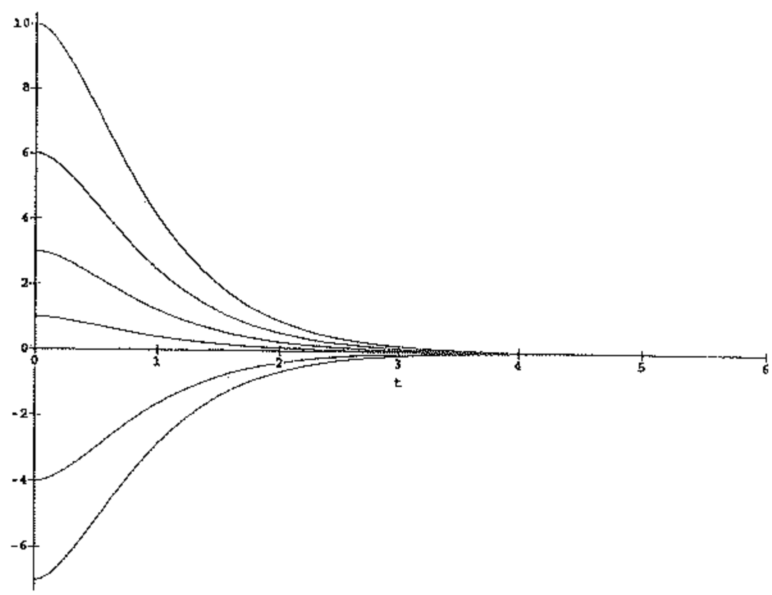
4. The solution is $y(t)=Ae^{-2t}[\cosh(\sqrt{2}t)+\sqrt{2}\sinh(\sqrt{2}t)]$ and is graphed for $A=1,3,6,10,-4$ and -7.



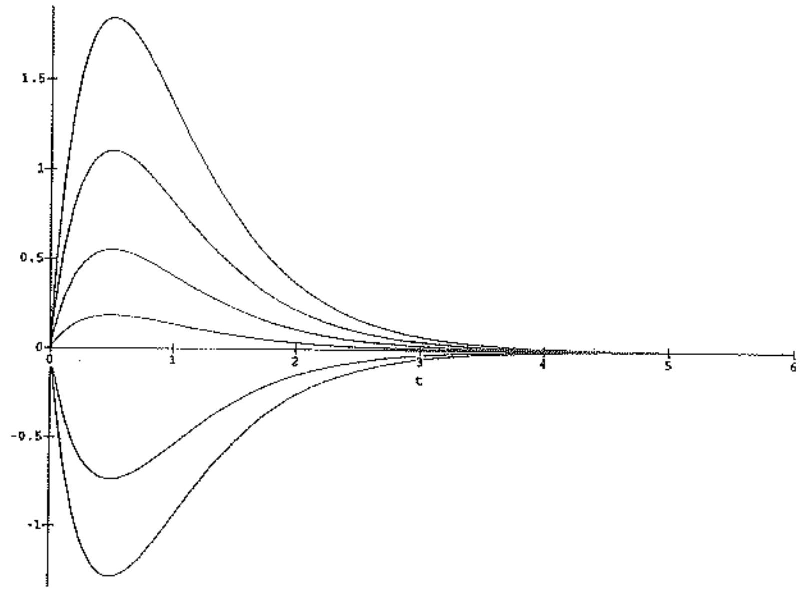
5. The solution is $y(t) = \dfrac{A}{\sqrt{2}} e^{-2t} \sinh(\sqrt{2}t)$ and is graphed for $A = 1, 3, 6, 10, -4$ and -7.



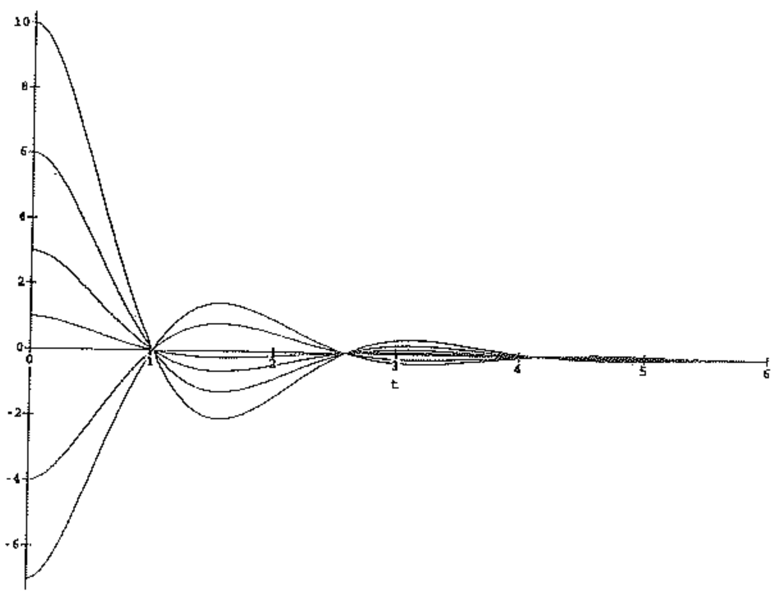
6. The solution is $y(t) = Ae^{-2t}(1 + 2t)$ and is graphed for $A = 1, 3, 6, 10, -4$ and -7.



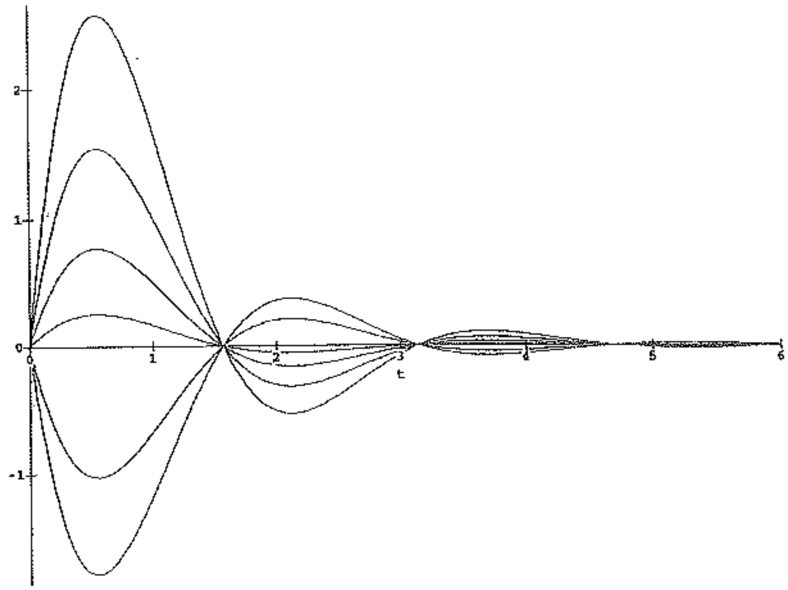
7. The solution is $y(t) = Ate^{-2t}$ and is graphed for $A = 1, 3, 6, 10, -4$ and -7.



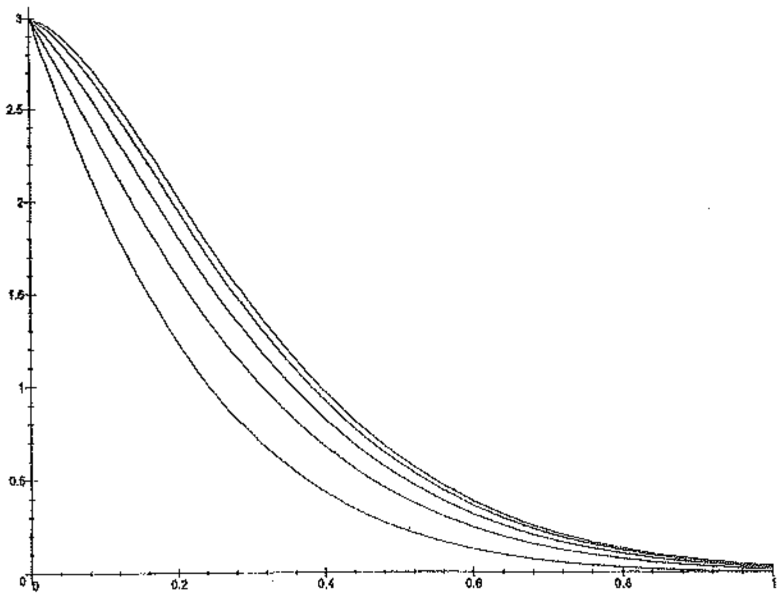
8. The solution is $y(t) = \dfrac{A}{2}e^{-t}[2\cos(2t)+\sin(2t)]$ and is graphed for $A = 1, 3, 6, 10, -4$ and -7.



9. The solution is $y(t) = \dfrac{A}{2}e^{-t}\sin(2t)$ and is graphed for $A = 1, 3, 6, 10, -4$ and -7.

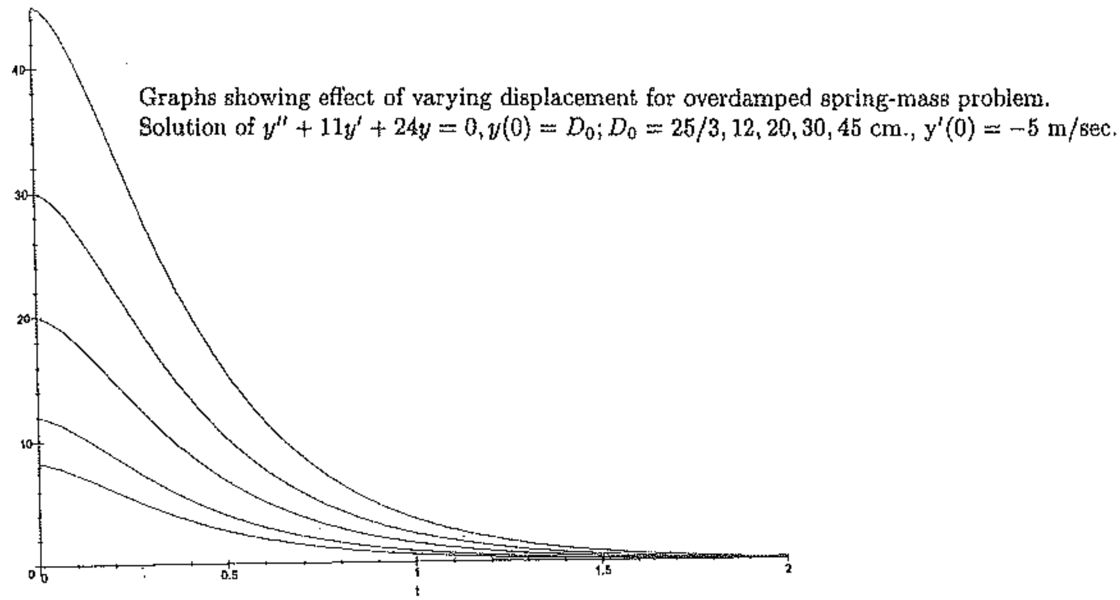


10. From Newton's second law we have $y'' = \sum \text{forces} = -29y - 10y'$, so the motion is described by the solution of $y''+10y'+29y = 0, y(0) = 3, y'(0) = -1$. The solution of this underdamped case is $y(t) = e^{-5t}[3\cos(2t)+7\sin(2t)] = \sqrt{58}e^{-5t}\cos(2t-\phi)$, where $\phi = \tan^{-1}(7/3)$. Comparative graphs are shown below for $y(0) = 3$ cm. $y'(0) = -1, -2, -4, -7, -12$ cm./sec. (recall down is the positive direction).



Graphs showing effect of varying initial velocity for underdamped spring-mass problem. Solution of $y'' + 10y' + 29y = 0, y(0) = 3, y'(0) = V_0; V_0 = -1, -2, -4, -7, -12$ cm/sec.

11. The motion is described by the solution of $y'' + 11y' + 24y = 0; y(0) = \dfrac{1}{12}$ m, $y'(0) = -5$ m/sec. The displacement is $y(t) = \dfrac{1}{60}[57e^{-8t} - 52e^{-3t}]$. Comparative graphs are shown below for $y(0) = \frac{25}{3}, 12, 20, 30, 45$ cm; $y'(0) = -5$ m/sec.



12. Since one pound $\doteq 4.45$ Newton and one inch $= 2.54$ cm we calculate the spring modulus $k = 4$ lbs/in $= \dfrac{4(4.45)}{.0254}$ Newton/m ≈ 700 Newton/m. The equation in the mks system will be $7y'' + 700y = 0; y(0) = 0, y'(0) = -4$ with solution $y = -\dfrac{2}{5}\sin(10t)$ meters.

13. For overdamped motion, the displacement is given by $y(t) = e^{-\alpha t}(A + Be^{\beta t})$ where $\alpha > 0$ ($-\alpha$ is the smaller characteristic root) and $\beta > 0$ is the difference $\beta =$ (larger root - smaller root). The factor $A + Be^{\beta t}$ could be zero at most once and only for some $t > 0$ if $-A/B > 1$. The values of A and B are determined by given initial conditions, in fact if $y_0 = y(0)$ and $v_0 = y'(0)$ we have $A + B = y_0$ and $-\alpha(A + B) + \beta B = v_0$. With a bit of algebra we find $-\dfrac{A}{B} = 1 - \dfrac{\beta y_0}{v_0 + \alpha y_0}$. To ensure that $-\dfrac{A}{B} \le 1$ we see that no condition on only y_0 will be sufficient. If we also specify that $v_0 > -\alpha y_0$, this will ensure that the overdamped bob never passes through the equilibrium point.

14. For critically damped motion, the displacement is given by $y(t) = e^{-\alpha t}(A + Bt)$ with $\alpha > 0$ and A and B determined by the initial conditions. From the linear factor we see that the bob could pass through equilibrium at most once, and will for some $t > 0$ if and only if $B \neq 0$ and $AB < 0$. Now note that $y_0 = y(0) = A$ and $v_0 = y'(0) = -\alpha A + B$. Thus to ensure that the bob never passes through equilibrium we need $AB > 0$, which becomes $(v_0 + \alpha y_0)y_0 > 0$. No condition on $y_0 = y(0)$ alone can ensure this. We would also need to specify $v_0 > -\alpha y_0$, and this will ensure that the critically damped bob never passes through the equilibrium point.

15. For underdamped motion we have $y(t) = e^{-ct/2m}[c_1\cos(\sqrt{4km-c^2}t/2m)+c_2\sin(\sqrt{4km-c_2}t/2m)]$ which has frequency $\omega = \dfrac{\sqrt{4km-c^2}}{2m}$. Thus increasing c decreases the frequency and decreasing c increases the frequency.

16. For critical damping $y(t) = e^{-ct/2m}(A+Bt)$. For maximum displacement we need $y'(t^*) = 0$ from which we find $t^* = \dfrac{2mB-cA}{Bc}$. Now $y(0) = A, y'(0) = -\dfrac{Ac}{2m}+B$ and since we are given $y(0) = y'(0) \neq 0$ we find $t^* = \dfrac{4m^2}{2mc+c^2}$, independent of $y(0)$. The maximum displacement is $y(t^*) = \dfrac{y(0)}{c}(2m+c)e^{-(\frac{2m}{2m+c})}$.

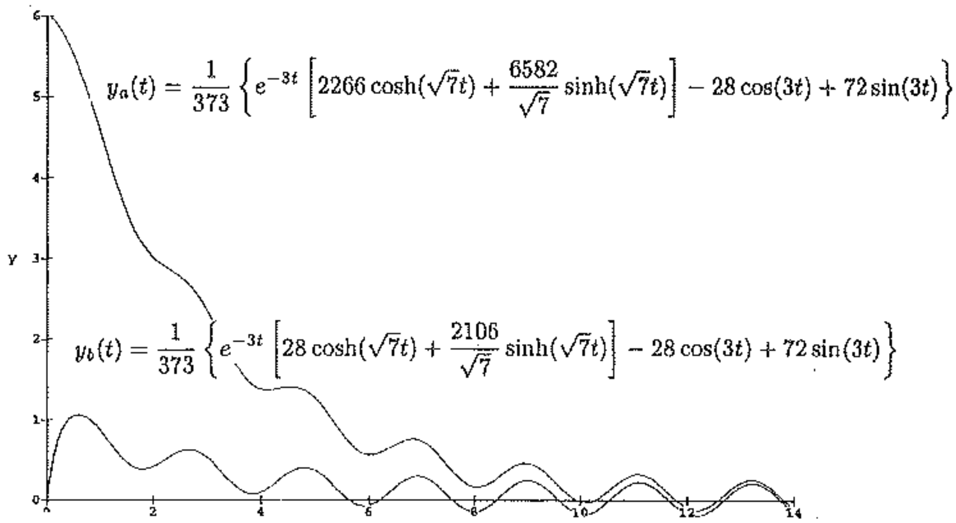
17. In the case of undamped motion, $y = c_1\cos(\omega t) + c_2\sin(\omega t)$, therefore $y'' = -\omega^2 y$. When $y = d, y'' = a$, so $\omega^2 = -\dfrac{a}{d}$, or $\omega = \sqrt{-\dfrac{a}{d}}$. The period of the motion is $T = \dfrac{2\pi}{\omega} = 2\pi\sqrt{-\dfrac{d}{a}}$. Note that d and a (as scalars) will have opposite signs.

18. The period of the original system with mass m_1 is $p = 2\pi\sqrt{\frac{m_1}{k}}$. The new system with mass (m_1+m_2) will have period $p_2 = 2\pi\sqrt{\frac{m_1+m_2}{k}} = 2\pi\sqrt{\frac{m_1}{k}}\sqrt{1+\frac{m_2}{m_1}} = p\sqrt{1+\frac{m_2}{m_1}}$.

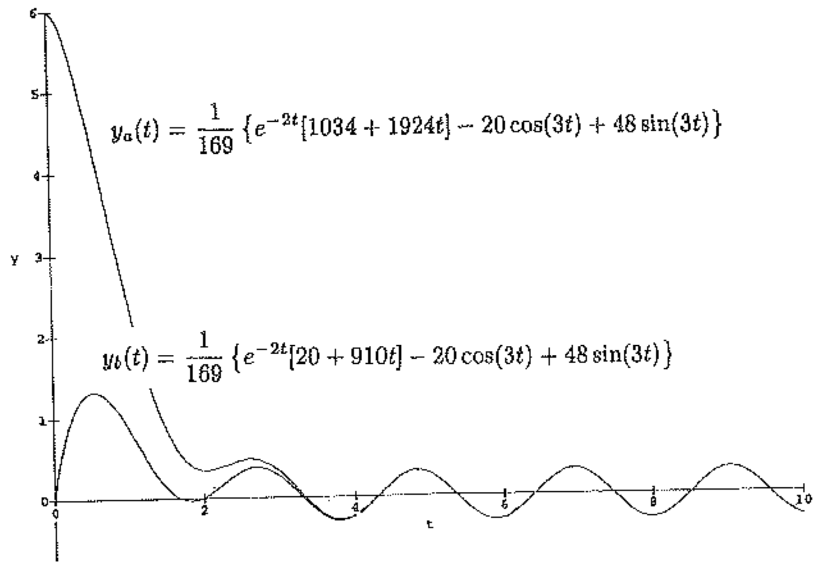
19. With $\omega \neq \omega_0$ the solution of $y'' + \omega_0^2 y = \dfrac{A}{m}\cos(\omega t); y(0) = y'(0) = 0$ is $y(t) = \dfrac{A}{m}\left[\dfrac{\cos(\omega_0 t)-\cos(\omega t)}{(\omega^2-\omega_0^2)}\right]$. Letting $\omega \to \omega_0$ and using L'Hopitals' rule we find $\lim\limits_{\omega\to\omega_0} y(t) = \dfrac{A}{2m\omega_0}t\sin(\omega_0 t)$ which is the solution of $y''+\omega_0^2 y = \dfrac{A}{m}\cos(\omega_0 t); y(0) = y'(0) = 0$.

20. The spring constant is $k = 16\cdot\dfrac{11}{8} = 22$ pounds/ft, mass m$= \dfrac{16}{32} = \dfrac{1}{2}$ slug so the equation of motion is $\dfrac{1}{2}y'' + 2y' + 22y = 4\cos(\omega t)$. The general solution of this equation is $y(t) = e^{-2t}[A\cos(2\sqrt{10}t)+B\sin(2\sqrt{10}t)] + \dfrac{8}{(44-\omega^2)^2+16\omega^2}\{(44-\omega^2)\cos(\omega t)+4\omega\sin(\omega t)\}$. As $t \to +\infty$, the exponential term dies out and the steady state solution can be written $y_{ss} = \dfrac{8}{\sqrt{(44-\omega^2)^2+16\omega^2}}\cos(\omega t+\delta)$. The amplitude is maximized when ω is chosen to minimize the radicand $(44-\omega^2)^2+16\omega^2 = (\omega^2-36)^2+640$. From this form we see we should choose $\omega = 6$ to get maximum amplitude of $\dfrac{8}{\sqrt{640}} = \dfrac{1}{\sqrt{10}}$ feet.

21. The general solution of the overdamped problem $y'' + 6y' + 2y = 4\cos(3t)$ can be written as $y(t) = e^{-3t}[c_1 \cosh(\sqrt{7}t) + c_2 \sinh(\sqrt{7}t)] - \frac{28}{373}\cos(3t) + \frac{72}{373}\sin(3t)$.
(a) The initial conditions $y(0) = 6, y'(0) = 0$ give $c_1 = \frac{2266}{373}$ and $c_2 = \frac{6582}{373\sqrt{7}}$ and unique solution $y_a(t) = \frac{1}{373}\{e^{-3t}[2266\cosh(\sqrt{7}t) + \frac{6582}{\sqrt{7}}\sinh(\sqrt{7}t)] - 28\cos(3t) + 72\sin(3t)\}$
(b) The initial conditions $y(0) = 0, y'(0) = 6$ give $c_1 = \frac{28}{373}$ and $c_2 = \frac{2106}{373}$ and unique solution $y_b(t) = \frac{1}{373}\{e^{-3t}[28\cosh(\sqrt{7}t) + \frac{2106}{\sqrt{7}}\sinh(\sqrt{7}t)] - 28\cos(3t) + 72\sin(3t)\}$.
(c) These two solutions are graphed on the axes below.



22. The general solution of the critically damped problem $y'' + 4y' + 4y = 4\cos(3t)$ is given by $y(t) = e^{-2t}[c_1 + c_2 t] - \dfrac{20}{169}\cos(3t) + \dfrac{48}{169}\sin(3t)$.
(a) The initial conditions $y(0) = 6, y'(0) = 0$ give $c_1 = \frac{1034}{169}$ and $c_2 = \frac{1924}{169}$ and unique solution $y_a(t) = \dfrac{1}{169}\{e^{-2t}[1034 + 1924t] - 20\cos(3t) + 48\sin(3t)\}$
(b) The initial conditions $y(0) = 0$ and $y'(0) = 6$ give $c_1 = \frac{20}{169}$ and $c_2 = \frac{910}{169}$ and unique solution $y_b(t) = \dfrac{1}{169}\{e^{-2t}[20 + 910t] - 20\cos(3t) + 48\sin(3t)\}$.
(c) These two solutions are graphed on the axes below.

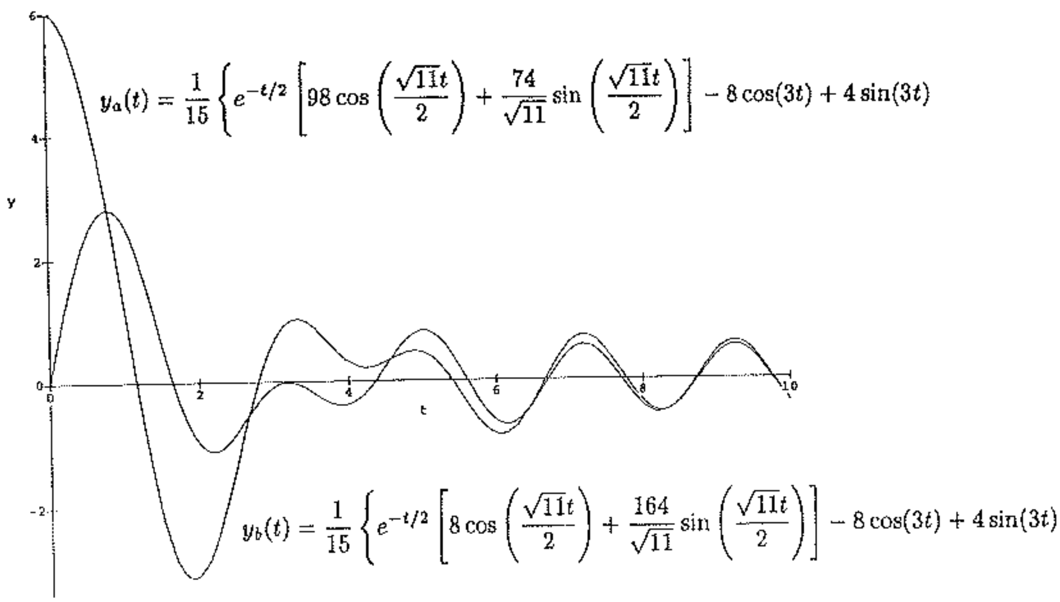


23. The general solution of the underdamped problem $y'' + y' + 3y = 4\cos(3t)$ is given by $y(t) = e^{-t/2}\left[c_1\cos\left(\frac{\sqrt{11}t}{2}\right) + c_2\sin\left(\frac{\sqrt{11}t}{2}\right)\right] - \frac{24}{45}\cos(3t) + \frac{12}{45}\sin(3t)$.

(a) The initial conditions $y(0) = 6, y'(0) = 0$ give $c_1 = \frac{98}{15}$ and $c_2 = \frac{74}{15\sqrt{11}}$ and unique solution
$y_a(t) = \frac{1}{15}\left\{e^{-t/2}\left[98\cos\left(\frac{\sqrt{11}t}{2}\right) + \frac{74}{\sqrt{11}}\sin\left(\frac{\sqrt{11}t}{2}\right)\right] - 8\cos(3t) + 4\sin(3t)\right\}$

(b) The initial conditions $y(0) = 0, y'(0) = 6$ give $c_1 = \frac{8}{15}$ and $c_2 = \frac{164}{15\sqrt{11}}$ and unique solution
$y_b(t) = \frac{1}{15}\left\{e^{-t/2}\left[8\cos\left(\frac{\sqrt{11}t}{2}\right) + \frac{164}{\sqrt{11}}\sin\left(\frac{\sqrt{11}t}{2}\right)\right] - 8\cos(3t) + 4\sin(3t)\right\}$

(c) These two solutions are graphed on the axes below.



In Problems 24 through 27 the RLC circuit driven by the potential $E(t)$ is modeled by the differential equation $Lq'' + Rq' + \frac{1}{C}q = E(t)$ for charge q. Since current $i = q'$ this can be written $Li' + Ri + \frac{1}{C}q = E(t)$ and by differentiation we get the second order equation $Li'' + Ri' + \frac{1}{C}i = E'(t)$. With $q(0) = i(0) = 0$ we get $i'(0+) = \frac{E(0+)}{L}$ as the second initial condition for the current problem. In the answers below, some terms with exceedingly small coefficients ($\approx 10^{-6}$) have been dropped so initial conditions may not be satisfied exactly.

24. $i(t) = -0.005027e^{-0.8337t} + (0.001003t + 0.005027)e^{-t}$

25. $i(t) = -0.000938e^{-0.0625t} + 0.018000e^{-3333.27t} - 0.000862\cos(20t) + 0.299998\sin(20t)$

26. $i(t) = 0.0007695e^{-0.1334t} - 0.0007704e^{-t}$

27. $i(t) = -0.000511e^{-0.3176t} + 0.001633e^{-t} - 0.001121e^{-t}\cos(6t) = 0.000044e^{-t}\sin(6t)$

Chapter Three - The Laplace Transform

Section 3.1 Definition and Basic Properties

1. $\mathcal{L}[2\sinh(t) - 4] = \mathcal{L}[e^t - e^{-t} - 4] = \dfrac{1}{s-1} - \dfrac{1}{s+1} - \dfrac{4}{s}$

2. $\dfrac{s}{s^2+1} - \dfrac{1}{s^2+1}$

3. $\dfrac{16s}{(s^2+4)^2}$

4. $\dfrac{2}{s^3} - \dfrac{3}{s^2} + \dfrac{5}{s}$

5. $\dfrac{1}{s^2} - \dfrac{s}{s^2+25}$

6. $\dfrac{4}{(s+3)^3} - \dfrac{4}{s^2} + \dfrac{1}{s}$

7. $\dfrac{2}{s^3} + \dfrac{8}{s^2} + \dfrac{16}{s}$

8. $\dfrac{3}{s+1} + \dfrac{6}{s^2+36}$

9. $\dfrac{6}{s^4} - \dfrac{3}{s^2} + \dfrac{s}{s^2+16}$

10. $-\dfrac{3s}{s^2+4} + \dfrac{20}{s^2+16}$

11. $\mathcal{L}^{-1}\left[-\dfrac{2}{s+16}\right] = -2\mathcal{L}^{-1}\left[\dfrac{1}{s+16}\right] = -2e^{-16t}$

12. $4\cosh(\sqrt{14}t)$

13. $\mathcal{L}^{-1}\left[\dfrac{2s-5}{s^2+16}\right] = 2\mathcal{L}^{-1}\left[\dfrac{s}{s^2+16}\right] - \dfrac{5}{4}\mathcal{L}^{-1}\left[\dfrac{4}{s^2+16}\right] = 2\cos(4t) - \dfrac{5}{4}\sin(4t)$

14. $3\cosh(\sqrt{7}t) + \dfrac{17}{\sqrt{7}}\sinh(\sqrt{7}t)$

15. $3e^{7t} + t$

16. $5te^{-7t}$

17. $e^{4t} - 6te^{4t}$

18. $\dfrac{1}{12}t^4 - \dfrac{1}{20}t^5 + \dfrac{8}{9!}t^9$

19. $\mathcal{L}[f(t)] = F(s) = \lim_{R\to\infty}\int_0^R e^{-st}f(t)dt$. For each R, let k be the largest integer so that $(k+1)T \le R$, and use additivity of the integral to write

$$\int_0^R e^{-st}f(t)dt = \sum_{n=0}^{k}\int_{nT}^{(n+1)T} e^{-st}f(t)dt$$
$$+\int_{(k+1)T}^{R} e^{-st}f(t)dt.$$

Since $F(s)$ exists, for R sufficiently large, $\int_{(k+1)T}^{R} e^{-st} f(t)dt$ can be made arbitrarily small. Also as $R \to \infty, k \to +\infty$. Thus we have $\int_0^\infty e^{-st} f(t)dt = \sum_{n=0}^{\infty} \int_{nT}^{(n+1)T} e^{-st} f(t)dt$.

20. This result is established by a change of variable $u = t - nT$,and the periodic property of f as follows: $\int_{nT}^{(n+1)T} e^{-st} f(t)dt = \int_0^T e^{-s(u+nT)} f(u+nT)du = e^{-snT} \int_0^T e^{-su} f(u)du$, since $f(u+nT) = f(u)$.

21. $\mathcal{L}[f(t)] = \sum_{n=0}^{\infty} \int_{nT}^{(n+1)T} e^{-st} f(t)dt = \sum_{n=0}^{\infty} e^{-snT} \int_0^T e^{-st} f(t)dt$, by Problems 21, 22 and the fact that $\int_0^T e^{-st} f(t)dt$ is independent of n.

22. For all $s > 0, |e^{-st}| < 1$, hence $\sum_{n=0}^{\infty} e^{-nsT} = \sum_{n=0}^{\infty} (e^{-sT})^n = \frac{1}{1-e^{-sT}}$. By Problem 23, $L[f](s) = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)dt$.

23. f has period $T = 6$, and $\int_0^6 e^{-st} f(t)dt = \int_0^3 5e^{-st}dt + \int_3^6 e^{-st} \cdot 0dt = \frac{5}{s}(1-e^{-3s})$. By Problem 24, $L[f](s) = \frac{5}{s}\frac{(1-e^{-3s})}{(1-e^{-6s})} = \frac{5}{s}\frac{(1-e^{-3s})}{(1-e^{-3s})(1+e^{-3s})} = \frac{5}{s(1+e^{-3s})}$.

24. Since $\int_0^{\pi/\omega} e^{-st} E\sin(\omega t)dt = \frac{E\omega}{s^2+\omega^2}[1+e^{-s\pi/\omega}]$ and f has period $T = \pi/\omega$, we have
$\mathcal{L}[f](s) = \frac{E\omega}{s^2+\omega^2}\left[\frac{1+e^{-s\pi/\omega}}{1-e^{-s\pi/\omega}}\right] = \frac{E\omega}{s^2+\omega^2}\left[\frac{e^{s\pi/2\omega}+e^{-s\pi/2\omega}}{e^{s\pi/2\omega}-e^{-s\pi/2\omega}}\right] = \frac{E\omega}{s^2+\omega^2}\coth\left(\frac{s\pi}{2\omega}\right)$

25. From the graph we have $f(t) = \begin{cases} 0 & \text{if } 0 < t \le 5, \\ 5 & \text{if } 5 < t \le 10, \\ 0 & \text{if } 10 < t \le 25 \end{cases}$ and $f(t+25) = f(t)$ so $T = 25$, and $\int_0^{25} e^{-st} f(t)dt = \int_5^{10} 5e^{-st}dt = \frac{5}{s}e^{-5s}(1-e^{-5s})$. Then $\mathcal{L}[f](s) = \frac{5e^{-5s}(1-e^{-5s})}{s(1-e^{-25s})}$.

26. $f(t) = \frac{1}{3}t$ for $0 \le t < 6$, and $f(t+6) = f(t)$, and $\int_0^6 \frac{1}{3} te^{-st}dt$
$= \frac{1}{3s^2}(1-6se^{-6s}-e^{-6s})$, we have

$$\mathcal{L}[f](s) = \frac{1}{3s^2}\frac{(1-6se^{-6s}-e^{-6s})}{(1-e^{-6s})}.$$

27. $\frac{E\omega}{s^2+\omega^2}\frac{1}{1-e^{-\pi s/\omega}}$

28. $f(t) = \frac{3}{2}t$ for $0 < t < 2, f(t) = 0$ for $2 \le t \le 8$, and $f(t+8) = f(t)$ so $T = 8$.

$$\mathcal{L}[f](s) = \frac{3}{2s^2}\frac{(1-2se^{-2s}-e^{-2s})}{(1-e^{-8s})}.$$

29. $f(t) = \begin{cases} h & \text{if } 0 < t \leq a, \\ 0 & \text{if } a < t \leq 2a \end{cases}$ and $f(t+2a) = f(t)$ so $T = 2a$, and $\mathcal{L}[f](s) = \dfrac{h}{s(1+e^{-as})}$.

30. $f(t) = \begin{cases} \frac{ht}{a} & \text{if } 0 < t \leq a, \\ -\frac{h}{a}(t-2a) & \text{if } a < t \leq 2a \end{cases}$ and $f(t+2a) = f(t)$ so $T = 2a$.
Now $\displaystyle\int_0^{2a} f(t)e^{-st}dt = \frac{h}{a}\int_0^a te^{-st}dt - \frac{h}{a}\int_a^{2a}(t-2a)e^{-st}dt = \frac{h}{as^2}(1-e^{-as})^2$, hence

$$\mathcal{L}[f](s) = \frac{h}{as^2}\frac{(1-e^{-as})^2}{(1-e^{-2as})}$$

$$= \frac{h}{as^2}\frac{(1-e^{-as})}{(1+e^{-as})} = \frac{h}{as^2}\tanh\left(\frac{as}{2}\right).$$

Section 3.2 Solution of Initial Value Problems Using the Laplace Transform

1. Transforming the initial value problem yields $\mathcal{L}[y'] + 4\mathcal{L}[y] = \mathcal{L}[1]$ which becomes $sY - y(0) + 4Y = \frac{1}{s}$. Putting $y(0) = -3$ and solving for $Y(s)$ we get $Y(s) = \dfrac{1}{s+4}\left[\dfrac{1}{s} - 3\right] = -\dfrac{13}{4}\left(\dfrac{1}{s+4}\right) + \dfrac{1}{4}\left(\dfrac{1}{s}\right)$. By inverting the transform we get

$$y(t) = -\frac{13}{4}\mathcal{L}^{-1}\left(\frac{1}{s+4}\right)$$
$$+\frac{1}{4}\mathcal{L}^{-1}\left(\frac{1}{s}\right)$$
$$= -\frac{13}{4}e^{-4t} + \frac{1}{4}.$$

2. $Y(s) = \dfrac{1}{s-9}\left(\dfrac{1}{s^2} + 5\right)$
$= \dfrac{406}{81}\left(\dfrac{1}{s-9}\right) - \dfrac{1}{9}\left(\dfrac{1}{s^2}\right) - \dfrac{1}{81}\left(\dfrac{1}{s}\right)$;
$y(t) = \dfrac{406}{81}e^{9t} - \dfrac{1}{9}t - \dfrac{1}{81}$

3. $Y(s) = \dfrac{1}{s+4}\left(\dfrac{s}{s^2+1}\right)$
$= -\dfrac{4}{17}\left(\dfrac{1}{s+4}\right) + \dfrac{1}{17}\dfrac{(4s+1)}{s^2+1}$;
$y(t) = -\dfrac{4}{17}e^{-4t} + \dfrac{4}{17}\cos(t) + \dfrac{1}{17}\sin(t)$

4. $Y(s) = \dfrac{1}{s+2}\left(\dfrac{1}{s+1} + 1\right) = \dfrac{1}{s+1}$; $y(t) = e^{-t}$

5. $Y(s) = \dfrac{1}{s-2}\left(\dfrac{1}{s} - \dfrac{1}{s^2} + 4\right)$

$$= \frac{1}{2s^2} - \frac{1}{4s} + \frac{17}{4}\left(\frac{1}{s-2}\right);\ y(t) = \frac{1}{2}t - \frac{1}{4} + \frac{17}{4}e^{2t}$$

6. $Y(s) = \frac{1}{s^2+1}\left(\frac{1}{s} - 6s\right) = \frac{1}{s} - 7\left(\frac{s}{s^2+1}\right);\ y(t) = 1 - 7\cos(t)$

7. $Y(s) = \frac{1}{(s-2)^2}\left[\frac{s}{s^2+1} + s - 1 + 4\right] =$

$$-\frac{13}{5}\left(\frac{1}{(s-2)^2}\right) + \frac{22}{25}\left(\frac{1}{s-2}\right) + \frac{3}{25}\left(\frac{s}{s^2+1}\right) - \frac{4}{25}\left(\frac{1}{s^2+1}\right);$$

$$y(t) = -\frac{13}{5}te^{2t} + \frac{22}{25}e^{2t}$$

$$+\frac{3}{25}\cos(t) - \frac{4}{25}\sin(t)$$

8. $Y(s) = \frac{1}{(s^2+9)}\left(\frac{2}{s^3}\right) = \frac{2}{9}\left(\frac{1}{s^3}\right) - \frac{2}{81}\left(\frac{1}{s}\right) + \frac{2}{81}\left(\frac{s}{s^2+9}\right);\ y(t) = \frac{1}{9}t^2 - \frac{2}{81} + \frac{2}{81}\cos(3t)$

9. $Y(s) = \frac{1}{(s^2+16)}\left[\frac{1}{s} + \frac{1}{s^2} - 2s + 1\right]$

$$= \frac{1}{16}\left(\frac{1}{s^2}\right) + \frac{1}{16}\left(\frac{1}{s}\right) - \frac{33}{16}\left(\frac{s}{s^2+16}\right) + \frac{15}{16}\left(\frac{1}{s^2+16}\right);$$

$$y(t) = \frac{1}{16}t + \frac{1}{16} - \frac{33}{16}\cos(4t) + \frac{15}{64}\sin(4t)$$

10. $Y(s) = \frac{1}{(s-2)(s-3)}\left[\frac{1}{s+1} - 10\right]$

$$= \frac{29}{3}\left(\frac{1}{s-2}\right) - \frac{39}{4}\left(\frac{1}{s-3}\right) + \frac{1}{12}\left(\frac{1}{s+1}\right);$$

$$y(t) = \frac{29}{3}e^{2t} - \frac{39}{4}e^{3t} + \frac{1}{12}e^{-t}$$

11. $\mathcal{L}[f'] = \lim_{\epsilon\to 0^+}\int_\epsilon^\infty e^{-st}f'(t)dt$

$$= \lim_{\epsilon\to 0^+}[f(t)e^{-st}|_\epsilon^\infty + s\int_\epsilon^\infty e^{-st}f(t)dt] = sF(s) - f(0^+)$$

12. $\mathcal{L}[f'] = \int_0^c e^{-st}f'(t)dt + \int_c^\infty e^{-st}f'(t)dt$

$$= \lim_{a\to c^-}[f(t)e^{-st}|_0^a + s\int_0^a e^{-st}f(t)dt] + \lim_{b\to c^+}[f(t)e^{-st}|_b^\infty$$

$$+s\int_b^\infty e^{-st}f(t)dt] = e^{-sc}f(c-) - f(0) - e^{-sc}f(c+) + s\int_0^\infty e^{-st}f(t)dt$$

$$= sF(s) - f(0) - e^{-sc}[f(c+) - f(c-)]$$

13. Letting $f(t) = \int_a^t g(\tau)d\tau$, we easily have $f'(t) = g(t)$ and $f(0) = -\int_0^a g(\tau)d\tau$. Transforming gives $\mathcal{L}[g(t)] = \mathcal{L}[f'(t)] = s\mathcal{L}[f(t)] - f(0) = s\mathcal{L}[\int_a^t g(\tau)d\tau] + \int_0^a g(\tau)d\tau$. Solving for $\mathcal{L}[\int_a^t g(\tau)d\tau] = \frac{1}{s}G(s) - \frac{1}{s}\int_0^a g(\tau)d\tau$ yields the result.

Section 3.3 Shifting Theorems and the Heaviside Function

1. By the first shifting theorem,

$$\begin{aligned}\mathcal{L}[(t^3 - 3t + 2)e^{-2t}] &= \mathcal{L}[t^3 - 3t + 2]|_{s\to s+2}\\ &= \frac{6}{(s+2)^4} - \frac{3}{(s+2)^2} + \frac{2}{(s+2)}\end{aligned}$$

2.

$$\frac{1}{(s+3)^2} - \frac{2}{s+3}$$

3. Write

$$\begin{aligned}f(t) &= [1 - H(t-7)] + \cos(t)H(t-7)\\ &= [1 - H(t-7)] + \cos((t-7)+7)H(t-7)\\ &= [1 - H(t-7)] + \cos(7)\cos(t-7)H(t-7) - \sin(7)\sin(t-7)H(t-7).\end{aligned}$$

Then

$$\mathfrak{L}[f(t)] = \frac{1}{s}(1 - e^{-7s}) + \frac{s}{s^1+1}\cos(7)e^{-7s} - \frac{1}{s^2+1}\sin(7)e^{-7s}.$$

4.

$$\frac{1}{(s-4)^2} - \frac{s-4}{(s-4)^2+1}$$

5. Write

$$\begin{aligned}f(t) &= t + (1-4t)H(t-3) = t + (1 - 4(t-3) + 3)H(t-3)\\ &= t - 11H(t-3) - 4(t-3)H(t-3),\end{aligned}$$

so

$$\mathfrak{L}[f(t)] = \frac{1}{s^2} - \frac{11}{s}e^{-3s} - \frac{4}{s^2}e^{-3s}.$$

6. $f(t) = [2(t-\pi) + 2\pi + \sin(t-\pi)][1 - H(t-\pi)]$, so

$$\mathfrak{L}[f(t)] = \frac{2}{s^2} - \frac{1}{s^2+1} - \frac{2}{s^2}e^{-\pi s} - \frac{2\pi}{s}e^{-\pi s} - \frac{1}{s^2+1}e^{-\pi s}.$$

7.

$$\frac{1}{s+1} - \frac{2}{(s+1)^3} + \frac{1}{(s+1)^2+1}$$

8.

$$f(t) = t^2 + (1 - t - 4t^2)H(t-2)$$

$$= t^2 + [1 - (t-2) - 2 - 4((t-2)+2)]H(t-2)$$
$$= t^2 - [17 - 17(t-2) + 4(t-2)^2]H(t-2)$$

so

$$\mathfrak{L}[f(t)] = \frac{2}{s^3} - \left(\frac{17}{s} + \frac{17}{s^2} + \frac{8}{s^3}\right)e^{-2s}.$$

9.

$$\frac{s}{s^2+1} + \left(\frac{2}{s} - \frac{s}{s^2+1} - \frac{1}{s^2+1}\right)e^{-2\pi s}$$

10. $f(t) = -4 + 4H(t-1) + e^{-3}e^{-(t-3)}H(t-3)$, so

$$\mathfrak{L}[f(t)] = -\frac{4}{s} + \frac{4}{s}e^{-s} + \frac{e^{-3}}{s+1}e^{-3s}.$$

11.

$$\frac{s^2+4s-5}{(s^2+4s+13)^2}$$

12.

$$\frac{1}{s-1} - \frac{1}{2(s-2)} - \frac{1}{2s}$$

13.

$$\frac{1}{s^2} - \frac{2}{s} - \left(\frac{1}{s^2} + \frac{15}{s}\right)e^{-16s}$$

14. $f(t) = 1 - \cos(2t) + [\cos(2(t-3\pi)) - 1]H(t-3\pi)$, so

$$\mathfrak{L}[f(t)] = \left(\frac{1}{s} - \frac{s}{s^2+4}\right)(1 - e^{-3\pi s}).$$

15.

$$\frac{24}{(s+5)^5} + \frac{4}{(s+5)^3} + \frac{1}{(s+5)^2}$$

16.

$$F(s) = \frac{1}{(s+2)^2+8}, \text{ so } f(t) = \frac{1}{2\sqrt{2}}e^{-2t}\sin(2\sqrt{2}t).$$

17.

$$F(s) = \frac{1}{(s-2)^2+1}, \text{ so } f(t) = e^{2t}\sin(t).$$

18.

$$\mathfrak{L}^{-1}\left[\frac{e^{-5s}}{s^3}\right] = \mathfrak{L}^{-1}\left[\frac{1}{s^3}\right]_{t\to t-5} H(t-5) = \frac{1}{2}(t-5)^2H(t-5).$$

19. $\cos(3(t-2))H(t-2)$

20. $3e^{-(t-4)}H(t-2)$

21.

$$F(s) = \frac{1}{(s+3)^2 - 2}, \text{ so } f(t) = \frac{e^{-3t}}{\sqrt{2}} \sinh(\sqrt{2}t).$$

22.

$$F(s) = \frac{s-4}{(s-4)^2 - 6}, \text{ so } f(t) = e^{4t} \cosh(\sqrt{6}t).$$

23.

$$F(s) = \frac{(s+3)-1}{(s+3)^2 - 8}, \text{ so } f(t) = e^{-3t} \cosh(2\sqrt{2}t) - \frac{1}{2\sqrt{2}} e^{-3t} \sinh(2\sqrt{2}t).$$

24.

$$\frac{1}{2} e^{5(t-1)}(t-1)^2 H(t-1)$$

25.

$$\frac{1}{16}[1 - \cos(4(t-2))]H(t-2)$$

26. By the first shifting theorem, $\mathfrak{L}\left[e^{-2t} \int_0^t e^{2w} \cos(3w)dw\right] = F(s+2)$, where $F(s) = \mathfrak{L}\left[\int_0^t e^{2w} \cos(3w)dw\right]$. This transform can be found by applying the property in equation (3.4) of Section 3.2 to get

$$F(s) = \frac{1}{s} \mathfrak{L}\left[e^{2t} \cos(3t)\right] = \frac{1}{s} \frac{s-2}{(s-2)^2 + 9}.$$

Then

$$\mathfrak{L}\left[e^{-2t} \int_0^t e^{2w} \cos(3w)dw\right] = \frac{s}{(s+2)(s^2+9)}.$$

27. The initial value problem $y'' + 4y = 3H(t-4) = 1; y(0) = 1, y'(0) = 0$ transforms to

$$(s^2 + 4)Y = \frac{3}{s} e^{-4s} + s.$$

Then

$$Y(s) = \frac{3}{4}\left[\frac{1}{s} - \frac{s}{s^2+4}\right] e^{-4s} + \frac{s}{s^2+4},$$

and inverting gives us

$$y(t) = \cos(2t) + \frac{3}{4}[1 - \cos(2(t-4))]H(t-4).$$

28. The problem is $y'' - 2y' - 3y = 12H(t-4); y(0) = 1, y'(0) = 0$. Transforming and solving for $Y(s)$ yields

$$Y(s) = \frac{1}{4}\left[\frac{1}{s-3} + \frac{3}{s+1}\right] + \left[\frac{1}{s-3} + \frac{3}{s+1} - \frac{4}{s}\right] e^{-4s}$$

and inversion gives

$$y(t) = \frac{1}{4}\left[e^{3t} + 3e^{-t}\right] + \left[e^{3(t-4)} + 3e^{-(t-4)} - 4\right] H(t-4).$$

29. The problem is $y^{(3)} - 8y = 2H(t-6); y(0) = y'(0) = y''(0) = 0$. Transforming gives us

$$Y(s) = \left[-\frac{1}{4s} + \frac{1}{12}\frac{1}{s-2} + \frac{1}{6}\frac{s}{s^2+2s+4} \right] e^{-6s}.$$

Inverting gives us

$$y(t) = \left[-\frac{1}{4} + \frac{1}{12}e^{-2(t-6)} + \frac{1}{6}e^{-(t-6)}\cos(\sqrt{3}(t-6)) \right] H(t-6).$$

30. We have $y'' + 5y' + 6y = -2[1 - H(t-3)]; y(0) = y'(0) = 0$. Transforming gives us

$$Y(s) = \left[\frac{1}{s+2} - \frac{2}{3}\frac{1}{s+3} - \frac{1}{3s} \right] (1 - e^{-3s}),$$

so

$$y(t) = e^{-2t} - \frac{2}{3}e^{-3t} - \frac{1}{3} - \left[e^{-2(t-3)} - \frac{2}{3}e^{-3(t-3)} \right] H(t-3).$$

31. $y^{(3)} - y'' + 4y' - 4y = 1 + H(t-5); y(0) = y'(0) = 0, y''(0) = 1$, so

$$Y(s) = \left[-\frac{1}{4s} + \frac{2}{5}\frac{1}{s-1} - \frac{3}{20}\frac{s}{s^2+4} - \frac{2}{5}\frac{1}{s^2+4} \right] (1 - e^{-5s}),$$

so

$$y(t) = -\frac{1}{4} + \frac{2}{5}e^t - \frac{3}{20}\cos(2t) - \frac{1}{5}\sin(2t)$$
$$- \left[-\frac{1}{4} + \frac{2}{5}e^{t-5} - \frac{3}{20}\cos(2(t-5)) - \frac{1}{5}\sin(2(t-5)) \right] H(t-5).$$

32. $y'' - 4y' + 4y = t + 2H(t-3); y(0) = -2, y'(0) = 1$. Transforming gives us

$$Y(s) = \frac{1}{4s} + \frac{1}{4s^2} - \frac{9}{4}\frac{1}{s-2} - \frac{43}{4}\frac{1}{(s-2)^2}$$
$$+ \left[\frac{1}{2s} - \frac{1}{2}\frac{1}{s-2} + \frac{1}{s-2)^2} \right] e^{-3s}.$$

Then

$$y(t) = \frac{1}{4} + \frac{1}{4}t - \frac{9}{4}e^{2t} - \frac{43}{4}te^{2t}$$
$$+ \left[\frac{1}{2} - \frac{1}{2}e^{2(t-3)} + (t-3)e^{2(t-3)} \right] H(t-3).$$

33. Assume that the switch is held in position B for five seconds, then switched to position A and left there. The charge q on the capacitor is modeled by the initial value problem

$$250{,}000q' + 10^6 q = 10[H(t-5)], q(0) = C, E(0) = 5(10^{-6}).$$

Transform and solve for Q to get

$$Q(s) = \frac{5(10^{-6})}{s+4} + \frac{4(10^{-6})e^{-5s}}{s} = \frac{5(10^{-6})}{s+4} + 10^{-5}\left[\frac{1}{s} - \frac{1}{s+4} \right] e^{-5s}.$$

Inverting the transform, we get

$$E_{out} = \frac{q(t)}{C} = 10^6 q(t) = 5e^{-4t} + 10[1 - e^{-4(t-5)}]H(t-5).$$

34. The current is modeled by the initial value problem $Li' + Ri = 2H(t-5), i(0) = 0$. Transforming, we get

$$I(s) = \frac{2}{s(Ls+4)}e^{-5s} = \frac{2}{R}\left[\frac{1}{s} - \frac{1}{s+R/L}\right]e^{-5s}.$$

Inverting gives us

$$i(t) = \frac{2}{R}[1 - e^{-R(t-5)/L}]H(t-5).$$

35. The current is modeled by $Li' + Ri = k[1 - H(t-5)], i(0) = 0$. Transforming, we get

$$I(s) = \frac{k}{s(Ls+R)}[1 - e^{-5s}] = \frac{k}{R}\left[\frac{1}{s} - \frac{1}{s+R/L}\right](1 - e^{-5s}).$$

Inverting gives the current

$$i(t) = \frac{k}{R}[1 - e^{-Rt/L}] - \frac{k}{R}[1 - e^{-R(t-5)/L}]H(t-5) \text{ amps.}$$

36. The current is modeled by $Li' + Ri = Ae^{-t}H(t-4), i(0) = 0$. Before transforming write the impressed voltage as $Ae^{-4}e^{-(t-4)}H(t-4)$. Transforming and solving for I gives us

$$I(s) = A\frac{e^{-4}e^{-4s}}{(s+1)(Ls+R)}.$$

If $R \neq L$,

$$I(s) = \frac{Ae^{-4}}{R-L}\left[\frac{1}{s+1} - \frac{1}{s+R/L}\right]e^{-4s}$$

and inverting gives us

$$i(t) = \frac{Ae^{-4}}{R-L}\left[e^{-(t-4)} - e^{-R(t-4)/L}\right]H(t-4).$$

If $R = L$ we have

$$I(s) = \frac{Ae^{-4}}{R}\frac{e^{-4s}}{(s+1)^2},$$

so

$$i(t) = \frac{Ae^{-4}}{R}(t-4)e^{-(t-4)}H(t-4).$$

37. $f(t) = K[H(t-a) - H(t-b)]$ and

$$F(s) = \frac{K}{s}\left[e^{-as} - e^{-bs}\right].$$

38.

$$f(t) = \frac{M(t-a)}{b-a}[H(t-a) - H(t-b)] + \frac{N(t-c)}{c-b}[H(t-b) - H(t-c)].$$

To transform this function, rewrite

$$f(t) = \frac{M}{b-a}(t-a)H(t-a)$$

$$+\left[\frac{N}{c-b}\left\{(t-b)-(c-b)\right\} - \frac{M}{b-a}\left\{(t-b)-(a-b)\right\}\right]H(t-b)$$

$$-\frac{N}{c-b}(t-c)H(t-c),$$

or

$$f(t) = \frac{M}{b-a}(t-a)H(t-a) + \left[\frac{N}{c-b} - \frac{M}{b-a}\right](t-a)H(t-b)$$

$$+(M-N)H(t-b) - \frac{N}{c-b}(t-c)H(t-c).$$

Then

$$F(s) = \frac{M}{b-a}\frac{e^{-as}}{s^2} + \left[\frac{N}{c-b} - \frac{M}{b-a}\right]\frac{e^{-bs}}{s^2}$$

$$+(M-N)\frac{e^{-bs}}{s} - \frac{N}{c-b}\frac{e^{-cs}}{s^2}.$$

39. Since f is piecewise linear we can write

$$f(t) = \frac{h}{b-a}(t-a)[H9t-a) - H(t-b)]$$

$$+\frac{h}{b-c}(t-c)[H(t-b)H(t-c)],$$

or

$$f(t) = \frac{h}{b-a}(t-a)H(t-a) + \frac{h(c-a)}{(b-c)(b-a)}\left[\frac{t-c}{b-c} - \frac{t-a}{b-a}\right]H(t-b)$$

$$= \frac{h}{b-c}(t-c)H(t-c)$$

$$= \frac{h}{b-a}(t-a)H(t-a) + \frac{h(c-a)}{(b-c)(b-a)}(t-b)H(t-b)$$

$$-\frac{h}{b-c}(t-c)H(t-c).$$

Then

$$F(s) = \frac{h}{s^2}\left[\frac{e^{-as}}{b-a} + \frac{c-a}{(b-c)(b-a)}e^{-bs} - \frac{e^{-cs}}{b-c}\right].$$

40. The current is modeled by the initial value problem $Li' + Ri = E(t)$, where $E(t) = 10[1 - H(t-2)]$ for $0 \le t \le 4$ and $E(t+4) = E(t)$. The transform of this differential equation gives us $(Ls+R)I(s) = \mathfrak{L}[E(t)]$. Since $E(t)$ is a periodic function with period $T = 4$ we get

$$\mathfrak{L}[E(t)] = \frac{1}{1-e^{-4s}}\int_0^2 10e^{-st}dt = \frac{10}{1-e^{-4s}}\frac{1-e^{-2s}}{s} = \frac{10}{s(1+e^{-2s})}.$$

Then

$$I(s) = \frac{10}{s(Ls+R)}\left(\frac{1}{1+e^{-2s}}\right).$$

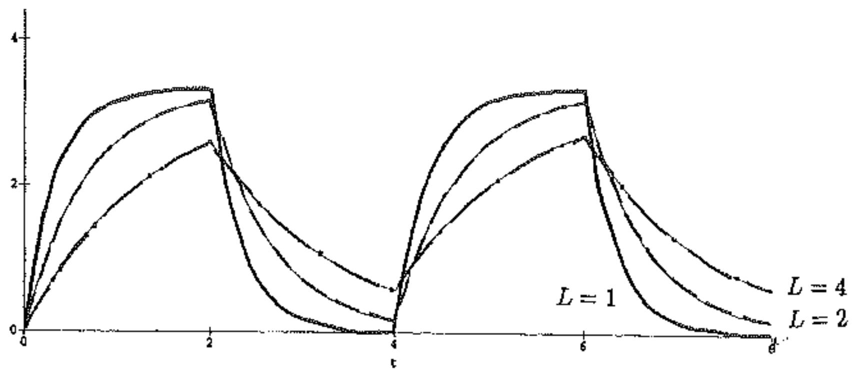
Since $e^{-2s} > 1$ for $s > 0$, expand $1/(1+e^{-2s})$ in the geometric series $\sum_{n=0}^{\infty}(-1)^n e^{-2ns}$ to get

$$I(s) = \frac{10}{R}\sum_{n=0}^{\infty}(-1)^n\left[\frac{1}{s} - \frac{1}{s+R/L}\right]e^{-2ns}.$$

Inverting this transform gives us

$$i(t) = \frac{10}{R}\sum_{n=0}^{\infty}(-1)^n\left[1 - e^{-R(t-2n)/L}\right]H(t-2n).$$

The graph of this current function is shown for $0 \le t \le 8$, with $R = 3$ and $L = 1$, 2 and 4.



Current for Problem 50 with $R = 3, L = 1,\ 2,\ 4$.

Section 3.4 Convolution

1. Let $F(s) = \dfrac{1}{s^2+4}$ and $G(s) = \dfrac{1}{s^2-4}$, so $\mathcal{L}^{-1}[F(s)] = \dfrac{\sin(2t)}{2}$ and $\mathcal{L}^{-1}[G(s)] = \dfrac{\sinh(2t)}{2}$.
By the convolution theorem, $\mathcal{L}^{-1}\left[\left(\dfrac{1}{s^2+4}\right)\left(\dfrac{1}{s^2-4}\right)\right] = \dfrac{1}{4}\sin(2t) * \sinh(2t) =$
$\dfrac{1}{4}\displaystyle\int_0^t \sin(2(t-\tau))\sinh(2\tau)d\tau = \dfrac{1}{16}\left[\sin(2(t-\tau))\cosh(2\tau) + \cos(2(t-\tau))\sinh(2\tau)\right]\Big|_0^t =$
$\dfrac{1}{16}[\sinh(2t) - \sin(2t)]$.

2. Take $F(s) = \dfrac{s}{s^2+16}$, $G(s) = \dfrac{e^{-2s}}{s}$, and $\mathcal{L}^{-1}[F(s)G(s)] = \cos(4t) * H(t-2) =$
$\displaystyle\int_0^t \cos(4(t-\tau))H(\tau-2)d\tau$. Since $H(\tau-2) = \begin{cases} 0, & \tau < 2 \\ 1, & \tau \ge 2 \end{cases}$ we have
$\mathcal{L}^{-1}[F(s)G(s)] = \displaystyle\int_2^t \cos(4(t-\tau))d\tau = -\frac{1}{4}\sin[4(t-\tau)]\Big|_2^t = \frac{1}{4}\sin[4(t-2)]$ for $t \ge 2$,
and zero otherwise. We can write this as

$$\mathcal{L}^{-1}\left[\frac{e^{-2s}}{s^2+16}\right] = \frac{1}{4}\sin[4(t-2)]H(t-2).$$

1. 3. If $a^2 \neq b^2, \mathcal{L}^{-1}\left[\frac{s}{(s^2+a^2)}\frac{1}{(s^2+b^2)}\right] = \cos(at) * \frac{\sin(bt)}{b} = \frac{1}{b}\int_0^t \cos[a(t-\tau)]\sin(b\tau)d\tau =$

$$\frac{1}{2b}\int_0^t \sin[(b-a)\tau + at] + \sin[(b+a)\tau - at]d\tau =$$

$$\frac{1}{2b}\left\{-\frac{\cos[(b-a)\tau+at]}{(b-a)} - \frac{\cos[(b+a)\tau-at]}{(b+a)}\right\}\Bigg|_0^t =$$

$$\frac{1}{2b}\left\{-\frac{\cos(bt)}{(b-a)} - \frac{\cos(bt)}{(b+a)} + \frac{\cos(at)}{(b-a)} + \frac{\cos(at)}{(b+a)}\right\} = \frac{\cos(at)-\cos(bt)}{(b-a)(b+a)}.$$

If $b^2 = a^2, \mathcal{L}^{-1}\left[\left(\frac{s}{s^2+a^2}\right)\cdot\left(\frac{1}{s^2+a^2}\right)\right] = \cos(at) * \frac{\sin(at)}{a} = \frac{1}{a}\int_0^t \cos[a(t-\tau)]\sin(a\tau)d\tau =$

$$\frac{1}{2a}\int_0^t \{\sin(at) + \sin[2a\tau - at]\}\, d\tau = \frac{1}{2a}\left\{\tau\sin(at) - \frac{\cos[a(2\tau-t)]}{2a}\right\}\Bigg|_0^t = \frac{t\sin(at)}{2a}.$$

4. Express $\frac{s}{(s-3)(s^2+5)} = \frac{1}{(s-3)}\cdot\left(\frac{s}{s^2+5}\right)$ so $\mathcal{L}^{-1}\left[\frac{s}{(s-3)(s^2+5)}\right] = e^{3t} * \cos(\sqrt{5}t)$

$= \int_0^t \cos(\sqrt{5}\tau)e^{3(t-\tau)}d\tau = e^{3t}\int_0^t \cos(\sqrt{5}\tau)e^{-3\tau}d\tau$

$= e^{3t}\left[\frac{e^{-3\tau}}{14} - 3\cos(\sqrt{5}\tau) + \sqrt{5}\sin(\sqrt{5}\tau)\right]\Bigg|_0^t$, yielding

$$\mathcal{L}^{-1}\left[\frac{s}{(s-3)(s^2+5)}\right] = \frac{3}{14}e^{3t} - \frac{3}{14}\cos(\sqrt{5}t) + \frac{\sqrt{5}}{14}\sin(\sqrt{5}t).$$

5. Since $\mathcal{L}^{-1}\left[\frac{1}{s(s^2+a^2)}\right] = \frac{1-\cos(at)}{a^2}$ and $\mathcal{L}^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{\sin(at)}{a}$,

we have $\mathcal{L}^{-1}\left[\frac{1}{s(s^2+a^2)^2}\right] = \frac{1}{a^3}[1-\cos(at)] * \sin(at) = \frac{1}{a^3}\int_0^t \{1-\cos[a(t-\tau)]\}\sin(a\tau)d\tau =$

$$\frac{1}{a^3}\left\{-\frac{\cos(a\tau)}{a} - \frac{\tau\sin(at)}{2} + \frac{\cos[2a\tau - at]}{4a}\right\}\Bigg|_0^t = \frac{1}{a^4}[1-\cos(at)] - \frac{t}{2a^3}\sin(at).$$

6. $\mathcal{L}^{-1}\left[\frac{1}{s^4}\cdot\frac{1}{s-5}\right] = e^{5t} * \frac{t^3}{6} = \frac{1}{6}\int_0^t e^{5(t-\tau)}\tau^3 d\tau = \frac{1}{6}e^{5t}\int_0^t \tau^3 e^{-5\tau}d\tau$

$$= \frac{1}{625}e^{5t} - \frac{1}{30}t^3 - \frac{1}{50}t^2 - \frac{1}{125}t - \frac{1}{625}.$$

7. $\mathcal{L}^{-1}\left[\frac{1}{s+2}\cdot\frac{e^{-4s}}{s}\right] = e^{-2t} * H(t-4) = \int_4^t e^{-2(t-\tau)}d\tau = \frac{1}{2}[1-e^{-2(t-4)}]$ if $t > 4$ and zero otherwise. So $\mathcal{L}^{-1}\left[\frac{1}{s+2}\cdot\frac{e^{-4s}}{s}\right] = \frac{1}{2}[1-e^{-2(t-4)}]H(t-4)$.

8. $\mathcal{L}^{-1}\left[\frac{2}{s^3}\cdot\frac{1}{s^2+5}\right] = t^2 * \frac{\sin(\sqrt{5}t)}{\sqrt{5}} = \frac{1}{\sqrt{5}}\int_0^t \tau^2 \sin[\sqrt{5}(t-\tau)]d\tau = \frac{1}{5}t - \frac{2}{25} + \frac{2}{25}\cos(\sqrt{5}t)$.

9. Taking transforms we get $(s^2-5s+6)Y(s) = F(s)$ or

$$Y(s) = \frac{F(s)}{s^2-5s+6} = \left[\frac{1}{s-3} - \frac{1}{s-2}\right]F(s).$$

By the convolution theorem, $y(t) = e^{3t} * f(t) - e^{2t} * f(t)$.

10. Taking transforms we get $(s^2+10s+24)Y(s) = F(s) + s$, since $y(0) = 1$. Solving for

$$Y(s) = \frac{F(s)}{(s+6)(s+4)} + \frac{s}{(s+6)(s+4)} = \frac{1}{2}F(s)\left[\frac{1}{s+4} - \frac{1}{s+6}\right] + \left[\frac{3}{s+6} - \frac{2}{s+4}\right]$$

and inverting by the convolution theorem gives $y(t) = \frac{1}{2}e^{-4t}*f(t) - \frac{1}{2}e^{-6t}*f(t) + 3e^{-6t} - 2e^{-4t}$.

11. $y(t) = \frac{1}{4}e^{6t} * f(t) - \frac{1}{4}e^{2t} * f(t) + 2e^{6t} - 5e^{2t}$

12. $y(t) = \frac{1}{6}e^{5t} * f(t) - \frac{1}{6}e^{-t} * f(t) + \frac{1}{2}e^{5t} + \frac{3}{2}e^{-t}$

13. $y(t) = \frac{1}{3}\sin(3t) * f(t) - \cos(3t) + \frac{1}{3}\sin(3t)$

14. $y(t) = \frac{1}{k}\sinh(kt) * f(t) - 2\cosh(kt) - \frac{4}{k}\sinh(kt)$

15. $y(t) = \frac{1}{4}e^{2t} * f(t) + \frac{1}{12}e^{-2t} * f(t) - \frac{1}{3}e^{t} * f(t) - \frac{1}{4}e^{2t} - \frac{1}{12}e^{-2t} + \frac{4}{3}e^{t}$

16. $y(t) = \frac{1}{42}e^{3t} * f(t) - \frac{1}{42}e^{-3t} * f(t) - \frac{\sqrt{2}}{28}e^{\sqrt{2}t} * f(t) - \frac{\sqrt{2}}{28}e^{-\sqrt{2}t} * f(t)$

17. The integral equation can be expressed as $f(t) = -1 + f(t) * e^{-3t}$, and transforming gives $F(s) = \frac{-1}{s} + \frac{F(s)}{s+3}$. When solved for $F(s)$ we get

$$F(s) = -\frac{s+3}{s(s+2)} = \frac{1}{2(s+2)} - \frac{3}{2s}.$$

Inverting the transform gives $f(t) = \frac{1}{2}e^{-2t} - \frac{3}{2}$.

18. Rewrite as $f(t) = -t + f(t)*\sin(t)$, transform and solve for $F(s) = -\frac{(s^2+1)}{s^4} = -\frac{1}{s^2} - \frac{1}{s^4}$. Invert to get $f(t) = -t - \frac{1}{6}t^3$.

19. Rewrite as $f(t) = e^{-t} + f(t) * 1$, transform and solve for

$$F(s) = \frac{s}{(s+1)(s-1)} = \frac{1}{2}\left[\frac{1}{s+1} + \frac{1}{s-1}\right].$$

Invert to get $f(t) = \frac{1}{2}e^{-t} + \frac{1}{2}e^{t} = \cosh(t)$.

20. Rewrite as $f(t) = -1 + t - 2f(t) * \sin(t)$, transform and solve for

$$F(s) = \frac{(1-s)(s^2+1)}{s^2(s^2+3)} = \frac{1}{3s^2} - \frac{1}{3s} - \frac{2}{3}\left(\frac{s}{s^2+3}\right) + \frac{2}{3}\left(\frac{1}{s^2+3}\right).$$

Invert to get $f(t) = \frac{1}{3}t - \frac{1}{3} - \frac{2}{3}\cos(\sqrt{3}t) + \frac{2\sqrt{3}}{9}\sin(\sqrt{3}t)$.

21. Rewrite as $f(t) = 3 + f(t) * \cos(2t)$, transform and solve for

$$F(s) = \frac{3(s^2+4)}{s(s^2-s+4)} = \frac{3}{s} + \frac{3}{s^2-s+4}.$$

Invert to get $f(t) = 3 + \frac{2\sqrt{15}}{5}e^{t/2}\sin\left(\frac{\sqrt{15}}{2}t\right)$.

22. Rewrite as $f(t) = \cos(t) + \int_0^t f(\alpha)e^{-2(t-\alpha)}d\alpha = \cos(t) + f(t) * e^{-2t}$, transform and solve for $F(s) = \frac{s(s+2)}{(s+1)(s^2+1)} = -\frac{1}{2(s+1)} + \frac{3}{2}\left(\frac{s}{s^2+1}\right) + \frac{1}{2}\left(\frac{1}{s^2+1}\right)$. Invert to get $f(t) = -\frac{1}{2}e^{-t} + \frac{3}{2}\cos(t) + \frac{1}{2}\sin(t)$.

23. Rewrite as $f(t) = e^{-2t} - 3\int_0^t f(\alpha)e^{-3(t-\alpha)}d\alpha = e^{-2t} - 3f(t) * e^{-3t}$, transform and solve for $F(s) = \frac{s+3}{(s+2)(s+6)} = \frac{1}{4(s+2)} + \frac{3}{4(s+6)}$. Invert to get $f(t) = \frac{1}{4}e^{-2t} + \frac{3}{4}e^{-6t}$.

24. Let $f(t)$ satisfy conditions which ensure f has a Laplace transform; e.g. f piecewise continuous on $[0,T]$, all $T > 0$, and $|f(t)| \le Me^{bt}$. Then $\mathcal{L}\left[\int_0^t f(w)dw\right] = \mathcal{L}[1 * f] = \frac{1}{s}F(s)$.

25. With $f(t) = \cos(t)$ we have $(f * 1)(t) = \int_0^t \cos(t-\tau)d\tau = -\sin(t-\tau)|_0^t = \sin(t) \neq \cos(t)$.

26. $f(t) = e^{-2t}\int_0^t e^{2w}\cos(3w)dw = \int_0^t e^{-2(t-w)}\cos(3w)dw = e^{-2t} * \cos(3t)$, so $\mathfrak{L}[f(t)] = \frac{1}{s+2}\frac{s}{s^2+9}$.

27. Changing the order of integration in the double integral we have $\int_0^t \int_0^w f(\alpha)d\alpha dw =$ $\int_0^t \int_\alpha^t f(\alpha)dwd\alpha = \int_0^t f(\alpha)\int_\alpha^t dwd\alpha = \int_0^t f(\alpha)(t-\alpha)d\alpha = f(t) * t = \mathcal{L}^{-1}\left[F(s)\cdot\frac{1}{s^2}\right]$

Section 3.5 Unit Impulses and the Dirac Delta Function

1. Transforming the initial value problem gives $(s^2+5s+6)Y(s) = 3e^{-2s} - 4e^{-5s}$. Solving for $Y(s)$ gives

$$Y(s) = 3\left[\frac{1}{s+2} - \frac{1}{s+3}\right]e^{-2s} - 4\left[\frac{1}{s+2} - \frac{1}{s+3}\right]e^{-5s}$$

and inverting we obtain

$$y(t) = 3[e^{-2(t-2)} - e^{-3(t-2)}]H(t-2) - 4[e^{-2(t-5)} - e^{-3(t-5)}]H(t-5).$$

2. $y(t) = \frac{4}{3}e^{2(t-3)}\sin[3(t-3)]H(t-3)$

3. $y(t) = 6(e^{-2t} - e^{-t} + te^{-t})$

4. $y(t) = 3\cos(4t) + 3\sin[4(t-5\pi/8)]H(t-5\pi/8)$

5. $\phi(t) = (B+9)e^{-2t} - (B+6)e^{-3t}$; $\phi(0) = 3, \phi'(0) = B$. The Dirac delta function applied at time $t_0, \delta(t-t_0)$, imparts a unit velocity to the unit mass.

6. The proof of Theorem 3.12 can be modified as follows. $\int_0^\infty f(t)\delta(t-a)dt = \lim_{\epsilon\to 0^+}\int_0^\infty f(t)\delta_\epsilon(t-a)dt = \lim_{\epsilon\to 0^+}\frac{1}{\epsilon}\int_a^{a+\epsilon} f(t)dt = \lim_{\epsilon\to 0^+}\frac{1}{\epsilon}\cdot\epsilon f(a+\theta\epsilon)$, where $0<\theta<1$ by the Mean Value Theorem for Integrals. But $\lim_{\epsilon\to 0^+} f(a+\theta\epsilon) = f(a+)$, which establishes the result.

7. By the Filtering Property, Theorem 3.12, we have $\int_0^\infty \frac{\sin(t)}{t}\delta(t-\pi/6)dt = \frac{6}{\pi}\sin\left(\frac{\pi}{6}\right) = \frac{3}{\pi}$

8. Write $\int_0^2 t^2\delta(t-3)dt = \int_0^\infty t^2[1-H(t-2)]\delta(t-3)dt = 0$ by the Filtering Property.

9. Since $f(t) = \begin{cases} t & \text{if } 0\le t<2 \\ 5 & \text{if } t=2 \\ t^2 & \text{if } t>2 \end{cases}$ and by Problem 6 $\int_0^\infty f(t)\delta(t-a)dt = f(a^+)$, we have $\int_0^\infty f(t)\delta(t-2)dt = 4.$

10. Formally $H'(t) = \lim_{h\to 0}\frac{H(t+h)-H(t)}{h}$. Now note for $h>0, H(t+h)-H(t) = 0$ for all $t>0$. So consider $h<0$, and let $\epsilon = -h$. Then $H'(t) = \lim_{\epsilon\to 0^+}\frac{H(t-\epsilon)-H(t)}{-\epsilon} = \lim_{\epsilon\to 0^+}\frac{H(t)-H(t-\epsilon)}{\epsilon} = \delta(t)$ for $t>0$.

11. From Example 3.17 of Section 3.3, the initial value problem is given by $25\cdot 10^4 i + 10^6 q = 10[H(t-2) - H(t-3)]; i(0) = q(0) = 0$. Differentiating the equation and using $H'(t) = \delta(t)$ gives $25\cdot 10^4 i' + 10^6 i = 10[\delta(t-2) - \delta(t-3)]$. Transforming and solving for $I(s)$ gives $I(s) = \frac{10^{-3}}{(25s+100)}[e^{-2s} - e^{-3s}]$. Invert the transform to get $i(t)$ and then $E_{out} = Ri(t) = 10[e^{-4(t-2)}H(t-2) - e^{-4(t-3)}H(t-3)]$

12. $\mathcal{L}^{-1}[H'(t)] = s\mathcal{L}^{-1}[H(t)] - H(0^+) = s\left(\dfrac{e^{-0s}}{s}\right) - 1 = 0$. But if $H'(t) = \delta(t)$, then $\mathcal{L}[H'(t)] = \mathcal{L}[\delta(t)] = 1$. These results are not compatible.

13. $\delta(t-a) * f(t) = \displaystyle\int_0^t f(\tau)\delta(t-\tau-a)d\tau = \int_0^t f(\tau)\delta((t-a)-\tau)d\tau = \begin{cases} 0 & \text{if } t < a \\ f(t-a) & \text{if } t \geq a \end{cases}$

by the filtering property of the Dirac delta function.

14. The motion is modeled by the initial value problem $my'' + ky = 0; y(0) = 0, y'(0) = v_0$. Laplace transforms give $Y(s) = \dfrac{mv_0}{ms^2+k}$, and inverting gives $y(t) = v_0\sqrt{\frac{m}{k}}\sin\left[\sqrt{\frac{k}{m}}t\right]$. The initial momentum is mv_0.

15. The motion is modeled by the problem $my'' + ky = mv_0\delta(t); y(0) = y'(0) = 0$. Tranforms give $Y(s) = \dfrac{mv_0}{ms^2+k}$, and inverting gives $y(t) = v_0\sqrt{\frac{m}{k}}\sin\left[\sqrt{\frac{k}{m}}t\right]$.

16. $F = kx$ gives $k = 2(\frac{8}{3})12 = 9$ pounds/ft. and $m = \dfrac{2}{32} = \dfrac{1}{16}$ slugs. The motion is modeled by $\dfrac{1}{16}y'' + 9y = \dfrac{1}{4}\delta(t); y(0) = y'(0) = 0$. Transforming gives $Y(s) = \dfrac{4}{s^2+144}$, and hence $y(t) = \dfrac{1}{3}\sin(12t)$. Initial velocity $= y'(0) = 4$ ft/second; frequency $= \dfrac{6}{\pi}$ hertz; amplitude $= \dfrac{1}{3}$ ft $= 4$ inches.

Section 3.6 Laplace Transform Solution of Systems

1. Taking the Laplace transform of the system gives $sX - 2sY = 1/s$; $sX - X + Y = 0$. Solving these equations for $X(s)$ and $Y(s)$ gives

$$X(s) = \frac{1}{s^2(2s-1)} = -\frac{1}{s^2} - \frac{2}{s} + \frac{4}{2s-1};$$

$$Y(s) = \frac{-s+1}{s^2(2s-1)} = -\frac{1}{s^2} - \frac{1}{s} + \frac{2}{2s-1}.$$

Inverting the transform gives the solution

$$x(t) = -t - 2 + 2e^{t/2};\ y(t) = -t - 1 + e^{t/2}.$$

2. Taking the Laplace transform of the system gives $2sX + (2s-3)Y = 0$; $sX + sY = 1/s^2$. Solving these equations for $X(s)$ and $Y(s)$ gives

$$X(s) = \frac{-2s+3}{3s^2} = -\frac{2}{3}\left(\frac{1}{s^2}\right) + \frac{1}{s^3};\ [Y(s) = \frac{2}{3}(\frac{1}{s^2}).$$

Inverting the transform gives the solution

$$x(t) = -\frac{2}{3}t + \frac{1}{2}t^2;\ y(t) = \frac{2}{3}t.$$

3. Taking the Laplace transform of the system gives $sX + (2s-1)Y = 1/s$; $sX + Y = 0$. Solving these equations for $X(s)$ and $Y(s)$ gives

$$X(s) = \frac{-1}{s^2(4s-3)} = \frac{1}{3}\left(\frac{1}{s^2}\right) + \frac{4}{9}\left(\frac{1}{s}\right) - \frac{16}{9}\left(\frac{1}{4s-3}\right);$$

$$Y(s) = \frac{2}{s(4s-3)} = -\frac{2}{3}\left(\frac{1}{s}\right) + \frac{8}{3}\left(\frac{1}{4s-3}\right).$$

Inverting the transform gives the solution

$$x(t) = \frac{1}{3}t + \frac{4}{9} - \frac{4}{9}e^{3t/4};\ y(t) = -\frac{2}{3} + \frac{2}{3}e^{3t/4}.$$

4. Taking the Laplace transform of the system gives $(s-1)X + sY = \frac{s}{s^2+4}$; $sX + 2sY = 0$. Solving these equations for $X(s)$ and $Y(s)$ gives

$$X(s) = \frac{2s^2}{(s^2+4)(s^2-3s)} = \frac{1}{2}\left(\frac{1}{s-1}\right) - \frac{1}{2}\left(\frac{s}{s^2+4}\right) + \frac{1}{s^2+4};$$

$$Y(s) = \frac{-s^2}{(s^2+4)(s^2-3s)} = -\frac{1}{4}\left(\frac{1}{s-1}\right) + \frac{1}{4}\left(\frac{s}{s^2+4}\right) - \frac{1}{2}\left(\frac{1}{s^2+4}\right).$$

Inverting the transform gives the solution

$$x(t) = \frac{1}{2}e^t - \frac{1}{2}\cos(2t) + \frac{1}{2}\sin(2t);$$

$$y(t) = -\frac{1}{4}e^t + \frac{1}{4}\cos(2t) - \frac{1}{4}\sin(2t).$$

5. Taking the Laplace transform of the system gives $3sX - Y = 2/s^2$; $sX + (s-1)Y = 0$. Solving these equations for $X(s)$ and $Y(s)$ gives

$$X(s) = \frac{2(s-1)}{s^3(3s-2)} = \frac{1}{s^3} + \frac{1}{2}\left(\frac{1}{s^2}\right) + \frac{3}{4}\left(\frac{1}{s}\right) - \frac{9}{4}\left(\frac{1}{3s-2}\right);$$

$$Y(s) = \frac{-2}{s^2(3s-2)} = \frac{1}{s^2} + \frac{3}{2s} - \frac{9}{2}\left(\frac{1}{3s-2}\right).$$

Inverting the transform gives the solution

$$x(t) = \frac{t^2}{2} + \frac{1}{2}t + \frac{3}{4} - \frac{3}{4}e^{2t/3};$$

$$y(t) = t + \frac{3}{2} - \frac{3}{2}e^{2t/3}.$$

6. Taking the Laplace transform of the system gives $sX + (4s-1)Y = 0;\ sX + 2Y = \dfrac{1}{s+1}$. Solving these equations for $X(s)$ and $Y(s)$ gives

$$X(s) = \frac{-4s+1}{(s+1)(4s^2-3s)} = \frac{5}{7}\left(\frac{1}{s+1}\right) - \frac{1}{3s} - \frac{32}{21}\left(\frac{1}{4s-3}\right);$$

$$Y(s) = \frac{s}{(s+1)(4s^2-3s)} = -\frac{1}{7}\left(\frac{1}{s+1}\right) + \frac{4}{7}\left(\frac{1}{4s-3}\right).$$

Inverting the transform gives the solution

$$x(t) = \frac{5}{7}e^{-t} - \frac{1}{3} - \frac{8}{21}e^{3t/4};\ y(t) = -\frac{1}{7}e^{-t} + \frac{1}{7}e^{3t/4}.$$

7. Taking the Laplace transform of the system gives $(s+2)X - sY = 0;\ (s+1)X + Y = \dfrac{2}{s^3}$. Solving these equations for $X(s)$ and $Y(s)$ gives

$$X(s) = \frac{2}{s^2(s^2+2s+2)} = \frac{1}{s^2} - \frac{1}{s} + \frac{s+1}{(s+1)^2+1};$$

$$Y(s) = \frac{2(s+2)}{s^3(s^2+2s+2)} = \frac{2}{s^3} - \frac{1}{s^2} + \frac{1}{(s+1)^2+1}.$$

Inverting the transform gives the solution

$$x(t) = t - 1 + e^{-t}\cos(t);\ y(t) = t^2 - t + e^{-t}\sin(t).$$

8. Taking the Laplace transform of the system gives $(s+4)X - Y = 0;\ sX + sY = 1/s^2$. Solving these equations for $X(s)$ and $Y(s)$ gives

$$X(s) = \frac{1}{s^2(s^2+5s)} = \frac{1}{125s} - \frac{1}{25s^2} + \frac{1}{5s^3} - \frac{1}{125}\left(\frac{1}{s+5}\right);$$

$$Y(s) = \frac{s+4}{s^2(s^2+5s)} = -\frac{1}{125s} + \frac{1}{25s^2} + \frac{4}{5s^3} + \frac{1}{125}\left(\frac{1}{s+5}\right).$$

Inverting the transform gives the solution

$$x(t) = \frac{1}{125} - \frac{1}{25}t + \frac{1}{10}t^2 - \frac{1}{125}e^{-5t};\ y(t) = -\frac{1}{125} + \frac{1}{25}t + \frac{2}{5}t^2 + \frac{1}{125}e^{-5t}.$$

9. Taking the Laplace transform of the system gives $(s+1)X + (s-1)Y = 0;\ (s+1)X + 2sY = 1/s$. Solving these equations for $X(s)$ and $Y(s)$ gives

$$X(s) = \frac{1-s}{s(s+1)^2} = \frac{1}{s} - \frac{1}{s+1} - \frac{2}{(s+1)^2};$$

$$Y(s) = \frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1}.$$

Inverting the transform gives the solution

$$x(t) = 1 - e^{-t} - 2te^{-t};\ y(t) = 1 - e^{-t}.$$

10. Taking the Laplace transform of the system gives $(s-1)X + 2sY = 0;\ 4sX + (3s+1)Y = -6/s$. Solving these equations for $X(s)$ and $Y(s)$ gives

$$X(s) = \frac{-12}{5s^2+2s} = -\frac{6}{s} + \frac{30}{5s+2};$$

$$Y(s) = \frac{6(s-1)}{s(5s^2+2s)} = -\frac{21}{2s} + \frac{3}{s^2} + \frac{105}{2}\left(\frac{1}{5s+2}\right).$$

Inverting the transform gives the solution

$$x(t) = -6 + 6e^{-2t/5};\ y(t) = -\frac{21}{2} + 3t + \frac{21}{2}e^{-2t/5}.$$

11. Taking the Laplace transform of the system gives $sY_1 - 2sY_2 + 3Y_3 = 0;\ Y_1 - 4sY_2 + 3Y_3 = 1/s^2$;
$Y_1 - 2sY_2 + 3sY_3 = -1/s$. Solving these equations for $Y_1(s), Y_2(s)$ and $Y_3(s)$ gives

$$Y_1(s) = \frac{1+s-s^2}{s^2(s^2-1)} = -\frac{1}{s^2} - \frac{1}{s} + \frac{1}{2(s-1)} + \frac{1}{2(s+1)};$$

$$Y_2(s) = -\frac{(s+1)}{2s^3} = -\frac{1}{2}\left(\frac{1}{s^2}\right) - \frac{1}{2}\left(\frac{1}{s^3}\right);$$

$$Y_3(s) = \frac{-2s^2+1}{3s^2(s^2-1)} = -\frac{1}{3s^2} - \frac{1}{6(s-1)} + \frac{1}{6(s+1)}.$$

Inverting the transform gives the solution

$$y_1(t) = -t - 1 + \frac{1}{2}e^t + \frac{1}{2}e^{-t};\ y_2(t) = -\frac{1}{2}t - \frac{1}{4}t^2;\ y_3(t) = -\frac{1}{3}t - \frac{1}{6}e^t + \frac{1}{6}e^{-t}.$$

12. The loop currents, i_1, and i_2, satisfy the system of differential equation
$2i_1 + 5\frac{d}{dt}(i_1 - i_2) + 3i_1 = E(t) = 2H(t-4) - H(t-5);\ i_2 + 4i_2 + 5\frac{d}{dt}(i_2 - i_1) = 0.$
Simplifying these equations and transforming gives

$$5(s+1)I_1 - 5sI_2 = \frac{2}{s}e^{-4s} - \frac{1}{s}e^{-5s};\quad -5sI_1 + 5(s+1)I_2 = 0.$$

Solving for $I_1(s)$ and $I_2(s)$ gives

$$I_1(s) = \frac{2}{5}\left[\frac{1}{s} - \frac{2}{2s+1}\right]e^{-4s} - \frac{1}{5}\left[\frac{1}{s} - \frac{2}{2s+1}\right]e^{-5s};$$

$$I_2(s) = -\frac{2}{5(2s+1)}e^{-4s} + \frac{1}{5(2s+1)}e^{-5s}$$

and inverting the transform gives

$$i_1(t) = \frac{2}{5}[1 - e^{-(t-4)}]H(t-4) - \frac{1}{5}[1 - e^{-(t-5)}]H(t-5);$$

$$i_2(t) = -\frac{2}{5}e^{-(t-4)}H(t-4) + \frac{1}{5}e^{-(t-5)}H(t-5).$$

13. The loop currents, i_1, and i_2, satisfy the system of differential equations $5i_1' + 5i_1 - 5i_2' = 1 - H(t-4)\sin[2(t-4)]$; $-5i_1' + 5i_2' + 5i_2 = 0$. Simplifying these equations and transforming gives

$$(s+1)I_1 - sI_2 = \frac{1}{5}\left[\frac{1}{s} - \frac{2e^{-4s}}{s^2+4}\right]; \quad -sI_1 + (s+1)I_2 = 0.$$

Solving for $I_1(s)$ and $I_2(s)$ gives

$$I_1(s) = \frac{s+1}{5(2s+1)}\left[\frac{1}{s} - \frac{2e^{-4s}}{s^2+4}\right] = \frac{1}{5}\left[\frac{1}{s} - \frac{1}{2s+1}\right] - \frac{2}{85}\left\{\frac{2}{2s+1} - \frac{s}{s^2+4} + \frac{9}{s^2+4}\right\}e^{-4s};$$

$$I_2(s) = \frac{1}{5(2s+1)}\left[1 - \frac{2se^{-4s}}{s^2+4}\right] = \frac{1}{5(2s+1)} + \frac{2}{85}\left\{\frac{2}{2s+1} - \frac{s}{s^2+4} - \frac{8}{s^2+4}\right\}e^{-4s}.$$

and inverting the transform gives

$$i_1(t) = \frac{1}{5}\left[1 - \frac{1}{2}e^{-t/2}\right] - \frac{2}{85}\{e^{-(t-4)/2} - \cos[2(t-4)] + \frac{9}{2}\sin[2(t-4)]\}H(t-4);$$

$$i_2(t) = \frac{1}{10}e^{-t/2} + \frac{2}{85}\{e^{-(t-4)/2} - \cos[2(t-4)] - 4\sin[2(t-4)]\}H(t-4).$$

14. Letting x_1 and x_2 denote, respectively, the downward displacements of masses m_1 and m_2 from equilibrium and applying Newton's second law we obtain the equations of motion $m_1x_1'' = \sum \text{forces} = -k_1x_1 + k_2(x_2 - x_1) + f_1(t)$; $m_2x_2'' = \sum \text{forces} = -k_3x_2 + k_2(x_1 - x_2) + f_2(t)$. Substituting values $k_1 = 6, k_2 = 2, k_3 = 3, m_1 = 1, m_2 = 1, f_1(t) = 2, f_2(t) = 0$, simplifying and transforming gives $(s^2+8)X_1 - 2X_2 = \frac{2}{s}$; $-2X_1 + (s^2+5)X_2 = 0$. Solving for $X_1(s)$ and $X_2(s)$ gives

$$X_1(s) = \frac{2(s^2+5)}{s(s^4+13s^2+36)} = \frac{5}{18s} - \frac{1}{10}\left(\frac{s}{s^2+4}\right) - \frac{8}{45}\left(\frac{s}{s^2+9}\right);$$

$$X_2(s) = \frac{-4}{s(s^4+13s^2+36)} = -\frac{1}{9s} + \frac{1}{5}\left(\frac{s}{s^2+4}\right) - \frac{4}{45}\left(\frac{s}{s^2+9}\right)$$

and inverting gives the displacement functions

$$x_1(t) = \frac{5}{18} - \frac{1}{10}\cos(2t) - \frac{8}{45}\cos(3t); \; x_2(t) = \frac{1}{9} - \frac{1}{5}\cos(2t) + \frac{4}{45}\cos(3t).$$

15. Following Problem 14 above we have the equations of motion $x_1'' + 8x_1 - 2x_2 = 1 - H(t-2)$; $-2x_1 + x_2'' + 5x_2 = 0$ which upon transforming gives $(s^2+8)X_1 - 2X_2 = \frac{1}{s}[1 - e^{-2s}]$; $-2X_1 + (s^2+5)X_2 = 0$. Solving these gives

$$X_1(s) = \frac{s^2+5}{s(s^4+13s^2+36)}[1-e^{-2s}];\ X_2(s) = \frac{2}{s(s^4+13s^2+36)}[1-e^{-2s}]$$

and inverting gives the solution

$$x_1(t) = \left[\frac{5}{36} - \frac{1}{20}\cos(2t) - \frac{4}{45}\cos(3t)\right] - \left[\frac{5}{36} - \frac{1}{20}\cos(2(t-2)) - \frac{4}{45}\cos(3(t-2))\right] H(t-2);$$

$$x_2(t) = \left[\frac{1}{18} - \frac{1}{10}\cos(2t) + \frac{2}{45}\cos(3t)\right] - \left[\frac{1}{18} - \frac{1}{10}\cos(2(t-2)) + \frac{2}{45}\cos(3(t-2))\right] H(t-2).$$

16. (a) The equations of motion are $my_1'' + k_1y_1 - k_2(y_2 - y_1) + c_1y_1' = A\sin(\omega t)$; $my_2'' - k_2(y_1 - y_2) = 0$, with initial conditions $y_1(0) = y_1'(0) = y_2(0) = y_2'(0) = 0$. Transforming and solving for $Y_1(s)$ and $Y_2(s)$ gives $Y_1(s) = \dfrac{ms^2+k_2}{k_2}Y_2(s)$ and

$$Y_2(s) = \frac{A\omega k_2}{(s^2+\omega^2)(Mms^4 + mc_1s^3 + (mk_1 + mk_2 + Mk_2)s^2 + k_2c_1s + k_1k_2)}.$$

(b) If $\omega = \sqrt{\frac{k_2}{m}}$ then

$$Y_1(s) = \frac{(s^2+\omega^2)}{\omega^2}Y_2(s) = \frac{A\omega}{Ms^4 + c_1s^3 + (k_1 + k_2 + M\omega^2)s^2 + \omega^2c_1s + k_1\omega^2}.$$

The absence of the factor $(s^2+\omega^2)$ in the denominator indicates that the forced vibrations of frequency ω have been absorbed.

17. The equations of motion and initial conditions are $m_1y_1'' = k(y_2 - y_1)$; $m_2y_2'' = k(y_1 - y_2)$; $y_1(0) = y_1'(0) = y_2'(0) = 0, y_2(0) = d$. Transforming and solving for $Y_1(s)$ and $Y_2(s)$ gives $Y_1(s) = \dfrac{kd}{m_1s[s^2 + (\frac{m_1+m_2}{m_1m_2})k]}$, and $Y_2(s) = \dfrac{d(m_1s^2+k)}{m_1s[s^2 + (\frac{m_1+m_2}{m_1m_2})k]}$. The quadratic factor in the denominator shows that the motion has frequency $\omega = \sqrt{(\frac{m_1+m_2}{m_1m_2})k}$, and hence period $2\pi\sqrt{\frac{m_1m_2}{k(m_1+m_2)}}$.

18. The equations for the loop currents can be written as $20i_1' + 10(i_1 - i_2) = E(t) = 5H(t-5), 30i_2' + 10i_2 + 10(i_2 - i_1) = 0$; with $i_1(0) = i_2(0) = 0$. Transforming and solving for $I_1(s)$ and $I_2(s)$ gives

$$I_1(s) = \frac{5(30s+20)e^{-5s}}{s(600s^2+700s+100)} = \left[\frac{1}{s} - \frac{1}{10(s+1)} - \frac{27}{5}\left(\frac{1}{6s+1}\right)\right]e^{-5s};$$

$$I_2(s) = \frac{50e^{-5s}}{s(600s^2+700s+100)} = \left[\frac{1}{2s} + \frac{1}{10(s+1)} - \frac{18}{5}\left(\frac{1}{6s+1}\right)\right]e^{-5s}$$

and the currents are

$$i_1(t) = \left[1 - \frac{1}{10}e^{-(t-5)} - \frac{9}{10}e^{-(t-5)/6}\right] H(t-5);$$

$$i_2(t) = \left[\frac{1}{2} + \frac{1}{10}e^{-(t-5)} - \frac{3}{10}e^{-(t-5)/6}\right] H(t-5).$$

19. Following Problem 18 above with $E(t) = 5\delta(t-1)$, the transformed equations have solution

$$I_1(s) = \frac{5(30s+20)e^{-s}}{(600s^2+700s+100)} = \left[\frac{1}{10(s+1)} + \frac{9}{10(6s+1)}\right] e^{-s};$$

$$I_2(s) = \frac{50e^{-s}}{(600s^2+700s+100)} = \left[-\frac{1}{10(s+1)} + \frac{3}{5(6s+1)}\right] e^{-s};$$

and the currents are

$$i_1(t) = \left[\frac{1}{10}e^{-(t-1)} + \frac{3}{20}e^{-(t-1)/6}\right] H(t-1);\ i_2(t) = \left[-\frac{1}{10}e^{-(t-1)} + \frac{1}{10}e^{-(t-1)/6}\right] H(t-1).$$

20. Let $x_1(t)$ and $x_2(t)$ denote, respectively, the amounts of salt (in pounds) in tanks 1 and 2 at any time $t \geq 0$. Then $\dfrac{dx_1}{dt}$ = rate of change of salt in tank 1 = (rate salt added) − (rate salt removed). With this notation, $x_1(t)$ and $x_2(t)$ satisfy the system of differential equations

$$x_1' = \frac{1}{3} + \frac{3}{18}x_2 - \frac{5}{60}x_1;\ x_1(0) = 11;\ x_2' = \frac{5}{60}x_1 - \frac{5}{18}x_2 + 11[H(t-4) - H(t-6)];\ x_2(0) = 7.$$

Simplifying and transforming these equations gives

$$(12s+1)X_1 - 2X_2 = \frac{4}{s} + 132;\ -3X_1 + (36s+10)X_2 = \frac{396}{s}[e^{-4s} - e^{-6s}] + 252$$

and solving for $X_1(s)$ and $X_2(s)$ gives

$$X_1(s) = \frac{[4752s^2 + 1968s + 40 + 792[e^{-4s} - e^{-6s}]}{s(432s^2 + 156s + 4)}$$

$$= \frac{10}{s} - \frac{6}{3s+1} + \frac{108}{36s+1} + 2\left[\frac{99}{s} + \frac{27}{3s+1} - \frac{3888}{36s+1}\right][e^{-4s} - e^{-6s}];$$

$$X_2(s) = \frac{3024s^2 + 648s + 12 + 396(12s+1)[e^{-4s} - e^{-6s}]}{s(432s^2 + 156s + 4)} =$$

$$\frac{3}{s} + \frac{9}{3s+1} + \frac{36}{36s+1} + \left[\frac{99}{s} - \frac{81}{3s+1} - \frac{2592}{36s+1}\right][e^{-4s} - e^{-6s}].$$

Inverting the transform gives

$$x_1(t) = 10 - 2e^{-t/3} + 3e^{-t/36} + 2[99 + 9e^{-(t-4)/3} - 108e^{-(t-4)/36}]H(t-4)$$

$$-2[99 + 9e^{-(t-6)/3} - 108e^{-(t-6)/36}]H(t-6);$$

$$x_2(t) = 3 + 3e^{-t/3} + e^{-t/36} + [99 - 27e^{-(t-4)/3} - 72e^{-(t-4)/36}]H(t-4)$$

$$-[99 - 27e^{-(t-6)/3} - 72e^{-(t-6)/36}]H(t-6).$$

21. Using the notation of Problem 20 above we have

$$x_1' = -\frac{6}{200}x_1 + \frac{3}{100}x_2;\ x_1(0) = 10;\ x_2' = \frac{4}{200}x_1 - \frac{4}{100}x_2 + 5\delta(t-3);\ x_2(0) = 5.$$

Simplifying and transforming gives

$$(100s+3)X_1 - 3X_2 = 1000;\ -2X_1 + (100s+4)X_2 = 500 + 500e^{-3s}$$

and solving for $X_1(s)$ and $X_2(s)$ gives

$$X_1(s) = \frac{100000s + 5500 + 1500e^{-3s}}{10000s^2 + 700s + 6} = \frac{50}{(50s+3)} + \frac{900}{(100s+1)} + e^{-3s}\left[\frac{300}{(100s+1)} - \frac{150}{(50s+3)}\right];$$

$$X_2(s) = \frac{50000s + 3500 + (50000s + 1500)e^{-3s}}{10000s^2 + 700s + 6}$$

$$= -\frac{50}{(50s+3)} + \frac{600}{(100s+1)} + e^{-3s}\left[\frac{150}{(50s+3)} + \frac{200}{(100s+1)}\right].$$

Inverting the transform gives

$$x_1(t) = e^{-3t/50} + 9e^{-t/100} + 3[e^{-(t-3)/100} - e^{-3(t-3)/50}]H(t-3);$$

$$x_2(t) = -e^{-3t/50} + 6e^{-t/100} + [3e^{-3(t-3)/50} + 2e^{-(t-3)/100}]H(t-3).$$

Section 3.7 Differential Equations With Polynomial Coefficients

1. Before transforming the equation, make the substitution $u = 1/t$. Letting $z(u) = y(t(u)) = y(1/u)$ we find $\frac{dy}{dt} = \frac{dz}{du}\cdot\frac{du}{dt} = -\frac{1}{t^2}\frac{dz}{du}$ by the chain rule and $t^2\frac{dy}{dt} - 2y = 2$ becomes $-\frac{dz}{du} - 2z = 2$. Now transform to get $-sZ + z(0) - 2Z = \frac{2}{s}$. Then $Z(s) = -\frac{2}{s(s+2)} + \frac{z(0)}{s+2} = \frac{1+z(0)}{s+2} - \frac{1}{s}$. Invert to get $z(u) = ce^{-2u} - 1$, or $y(t) = -1 + ce^{-2/t}$. This is also easily solved by first order linear techniques.

2. Transforming the initial value problem gives $[s^2Y - sy(0) - y'(0)] - 4\frac{d}{ds}[sY - y(0)] - 4Y = 0$. Compute the indicated derivative, substitute initial values and rearrange to get the first order linear equation $4sY' + (8 - s^2)Y = 7$. This equation has solution $Y(s) = -\frac{7}{s^2} + \frac{c}{s^2}e^{s^2/8}$ where c is the arbitrary constant of integration. Since $\lim_{s\to+\infty} Y(s) = 0$, we must choose $c = 0$. Thus $Y(s) = -\frac{7}{s^2}$, and inverting gives $y(t) = -7t$.

3. $y = 7t^2$

4. $y = -4t$

5. $y = ct^2e^{-t}$

6. $y = 3t^2$

7. $y = 4$

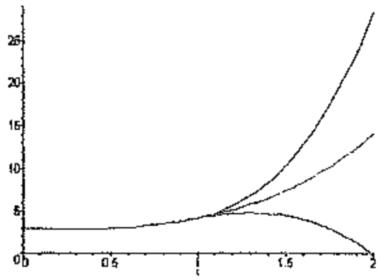
8. $y = 10t$

9. $y = \frac{3}{2}t^2$

10. Transforming gives $[s^2Y - sy(0) - y'(0)] + \frac{d}{ds}[s^2Y - sy(0) - y'(0)] - \frac{d}{ds}[sY - y(0)] - Y = 0$. Computing the indicated derivatives and substituting values for $y(0) = 3, y'(0) = -1$, we have $(s^2 - s)Y' + (s^2 + 2s - 2)Y = 3s + 2$, first order linear in $Y'(s)$. This equation has an integrating factor of $\mu = se^s$, and multiplying by this factor gives $\frac{d}{ds}[e^s(s^3 - s^2)Y] = 3s^2e^s + 2se^s$. Integrate and solve for

$$Y(s) = \frac{3s^2 - 4s + 4}{s^2(s-1)} + K\frac{e^{-s}}{s^2(s-1)} = \frac{3}{s-1} - \frac{4}{s^2} + K\left(\frac{1}{s-1} - \frac{1}{s} - \frac{1}{s^2}\right)e^{-s}.$$

Inverting the transform gives the solution $y(t) = 3e^t - 4t + K[e^{t-1} - t]H(t-1)$, valid for all values of K. Note the appearance of the term with coefficient K, which arises in solving by Laplace transform. This term gives a bifurcation of solutions at $t = 1$, which are shown in the graph below for $K = -20, 0, 20$. Notice that the hypotheses of the existence, uniqueness theorem are not satisfied at $t = 1$, which is a singular point of the solution. If this original initial value problem is solved by series methods (see Chapter Four), or by using a software package, this bifurcation may be missed in the solution.



Chapter Four - Series Solutions

Section 4.1 Power Series Solutions of Initial Value Problems

1. $y = -2 - \frac{1}{3}x^3 + \frac{1}{12}x^4 - \frac{1}{60}x^5 - \frac{1}{120}x^6 + \cdots$

2. $y = 1 + 2x + \frac{1}{2}x^2 - \frac{1}{2}x^3 - \frac{7}{24}x^4 + \cdots$

3. $y = 3 + \frac{5}{2}(x-1)^2 + \frac{5}{6}(x-1)^3 + \frac{5}{24}(x-1)^4 + \frac{1}{6}(x-1)^5 + \cdots$

4. $y = 1 - 4(x-2) - \frac{1}{2}(x-2)^2 + \frac{4}{3}(x-2)^3 + \frac{5}{12}(x-2)^4 + \cdots$

5. $y = 7 + 3(x-1) - 2(x-1)^2 - (x-1)^3 + (x-1)^4 + \cdots$

6. $y = -2 + 7x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{5}{24}x^4 + \cdots$

7. $y = -3 + x + 4x^2 + \frac{7}{6}x^3 + \frac{1}{3}x^4 + \cdots$

8. $y = -2x + x^2 - \frac{1}{15}x^5 + \frac{1}{90}x^6 - \frac{3}{70}x^7 + \cdots$

9. $y = 1 + x - \frac{1}{4}x^2 + \frac{1}{4}x^3 + \frac{1}{32}x^4 + \cdots$

10. $y = 2(x-4) + \frac{3}{2}(x-4)^2 + \frac{5}{12}(x-4)^3 + \frac{1}{12}(x-4)^4 + \frac{7}{480}(x-4)^5 + \cdots$

11. $y = a_0\left[1 - \frac{1}{2}x^2 + \frac{1}{6}x^4 - \frac{31}{720}x^6 + \frac{379}{40,320}x^8 + \cdots\right]$
$- \left[\frac{1}{2}x^2 - \frac{1}{8}x^4 + \frac{5}{144}x^6 - \frac{43}{5760}x^8 + \frac{1741}{1,209,600}x^{10} + \cdots\right]$

12. $y = a_0\left[1 - \frac{x^3}{3} + \frac{x^6}{3^2\cdot 2!} - \frac{x^9}{3^3\cdot 3!} + \frac{x^{12}}{3^4\cdot 4!} - \cdots\right]$
$+ \left[x - \frac{x^4}{3} + \frac{x^7}{3^2\cdot 2!} - \frac{x^{10}}{3^3\cdot 3!} + \frac{x^{13}}{3^4\cdot 4!} - \cdots\right]$. The series here can be recognized which allows us to find the solution $y = a_0e^{-x^3/3} + xe^{-x^3/3}$ in closed form.

13. $y = x - 1 + A\left[1 - \frac{x^2}{2} + \frac{x^4}{4\cdot 2!} - \frac{x^6}{8\cdot 3!} + \frac{x^8}{16\cdot 4!} - \cdots\right] = x - 1 + A\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{n!2^n} = x - 1 + Ae^{-\frac{x^2}{2}}$. Here again the solution can be expressed in closed form since the series can be recognized.

14. $y = a_0\left[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right] + \left[\frac{1}{2}x^2 + \frac{1}{12}x^4 - \frac{1}{30}x^5 + \frac{1}{36}x^6 - \frac{25}{126}x^7 + \cdots\right]$

15. $y = a_0\left[1 - \frac{1}{6}x^3 + \frac{1}{180}x^6 - \frac{1}{12,960}x^9 + \frac{1}{1,710,720}x^{12} + \cdots\right]$
$+ a_1\left[x - \frac{1}{12}x^4 + \frac{1}{504}x^7 - \frac{1}{45,360}x^{10} + \frac{1}{7,076,160}x^{13} + \cdots\right]$

16. $y = a_0\left[1 - \frac{1}{6}x^3 - \frac{1}{12}x^4 - \frac{1}{40}x^5 - \frac{1}{180}x^6 - \cdots\right] + a_1\left[x + x^2 + \frac{2}{3}x^3 + \frac{1}{4}x^4 + \frac{1}{20}x^5 + \cdots\right]$

17. $y = a_0\left[1 + \frac{1}{4\cdot 5}x^5 + \frac{1}{4\cdot 5\cdot 9\cdot 10}x^{10} + \frac{1}{4\cdot 5\cdot 9\cdot 10\cdot 14\cdot 15}x^{15}\right.$
$\left.+ \frac{1}{4\cdot 5\cdot 9\cdot 10\cdot 14\cdot 15\cdot 19\cdot 20}x^{20} + \cdots\right] + a_1\left[x + \frac{1}{5\cdot 6}x^6 + \frac{1}{5\cdot 6\cdot 10\cdot 11}x^{11}\right.$
$\left.+ \frac{1}{5\cdot 6\cdot 10\cdot 11\cdot 15\cdot 16}x^{16} + \frac{1}{5\cdot 6\cdot 10\cdot 11\cdot 15\cdot 16\cdot 20\cdot 21}x^{21} + \cdots\right]$

$$+\left[\frac{1}{2}x^2+\frac{1}{2\cdot 6\cdot 7}x^7+\frac{1}{2\cdot 6\cdot 7\cdot 11\cdot 12}x^{12}+\frac{1}{2\cdot 6\cdot 7\cdot 11\cdot 12\cdot 16\cdot 17}x^{17}\right.$$
$$\left.+\frac{1}{2\cdot 6\cdot 7\cdot 11\cdot 12\cdot 16\cdot 17\cdot 21\cdot 22}x^{22}+\cdots\right]$$

18. $y=a_0\left[1-\frac{1}{3}x^3+\frac{1}{12}x^4-\frac{1}{15}x^5+\frac{1}{45}x^6-\cdots\right]+a_1\left[x-\frac{1}{2}x^2+\frac{1}{3}x^4+\frac{1}{6}x^5-\cdots\right]$

19. $y=a_0\left[1+\frac{1}{12}x^4-\frac{1}{60}x^5+\frac{1}{360}x^6-\frac{1}{2520}x^7+\cdots\right]+a_1\left[x-\frac{1}{2}x^2+\frac{1}{6}x^3-\frac{1}{24}x^4+\frac{7}{120}x^5-\cdots\right]$

20. $y=a_0\left[1+\frac{4}{3}x^3+\frac{16}{45}x^6+\frac{16}{405}x^9+\frac{32}{13,365}x^{12}+\cdots\right]$

$$+a_1\left[x+\frac{2}{3}x^4+\frac{8}{63}x^7+\frac{32}{2835}x^{10}+\frac{64}{110,565}x^{13}+\cdots\right]$$
$$+\left[\frac{1}{2}x^2+\frac{1}{5}x^5+\frac{1}{35}x^8+\frac{39}{1925}x^{11}+\frac{156}{175,175}x^{14}+\cdots\right]$$

21. $y=a\left[1-\frac{1}{2\cdot 3}x^3+\frac{1}{2\cdot 3\cdot 5\cdot 6}x^6-\frac{1}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9}x^9+\frac{1}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9\cdot 11\cdot 12}x^{12}-\cdots\right]+$
$b\left[x-\frac{1}{3\cdot 4}x^4+\frac{1}{3\cdot 4\cdot 6\cdot 7}x^7-\frac{1}{3\cdot 4\cdot 6\cdot 7\cdot 9\cdot 10}x^{10}+\frac{1}{3\cdot 4\cdot 6\cdot 7\cdot 9\cdot 10\cdot 12\cdot 13}x^{13}-\cdots\right]$

22. $y=1-\cos(x)=1-\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}=\frac{1}{2}x^2-\frac{1}{24}x^4+\frac{1}{720}x^6-\frac{1}{8!}x^8+\frac{1}{10!}x^{10}+\cdots$

23. $y=2-3e^{-x}=-1+3x-\frac{3}{2}x^2+\frac{1}{2}x^3-\frac{1}{8}x^4+\cdots$

24. $y=2e^{-x}-\frac{5}{4}e^{-2x}+\frac{x}{2}-\frac{3}{4}=x-\frac{3}{2}x^2+\frac{4}{3}x^3-\frac{11}{12}x^4+\frac{19}{60}x^5+\cdots$

25. $y=\frac{1}{5}+\frac{e^{2x}}{5}(32\sin(x)-6\cos(x))=-1+4x+11x^2+\frac{34}{3}x^3+\frac{27}{4}x^4+\cdots$

Section 4.2 Power Series Solutions Using Recurrence Relations

1. $y'-xy=\sum_{n=1}^{\infty}na_nx^{n-1}-\sum_{n=0}^{\infty}a_nx^{n-1}=a_1+[2a_2-a_0]x+\sum_{n=3}^{\infty}[na_n-a_{n-2}]x^{n-1}=1-x$, which gives the recurrence relation a_0 arbitrary, $a_1=1, 2a_2-a_0=-1, a_n=\frac{a_{n-2}}{n}n\geq 3$. Letting $a_0=c_0+1$ we get coefficients $a_2=\frac{c_0}{2}, a_4=\frac{c_0}{2\cdot 4}, a_6=\frac{c_0}{2\cdot 4\cdot 6}, a_8=\frac{c_0}{2\cdot 4\cdot 6\cdot 8}, a_1=1, a_3=\frac{1}{3}, a_5=\frac{1}{3\cdot 5}, a_7=\frac{1}{3\cdot 5\cdot 7}, a_9=\frac{1}{3\cdot 5\cdot 7\cdot 9}$, and solution
$$y=1+\sum_{n=0}^{\infty}\frac{1}{1\cdot 3\cdot 5\cdots(2n+1)}x^{2n+1}+c_0\left(1+\sum_{n=1}^{\infty}\frac{1}{2\cdot 4\cdot 6\cdots 2n}x^{2n}\right)$$

2. $y'-x^3y=\sum_{n=1}^{\infty}na_nx^{n-1}-\sum_{n=0}^{\infty}a_nx^{n+3}=a_1+2a_2+3a_3x^2+\sum_{n=4}^{\infty}[na_n-a_{n-4}]x^{n-1}=4$; the recurrence relation is a_0 arbitrary, $a_1=4, a_2=a_3=0, a_n=\frac{a_{n-4}}{n}, n\geq 4$. This gives the solution $y=4\sum_{n=0}^{\infty}\frac{1}{1\cdot 5\cdot 9\cdots(4n+1)}x^{4n+1}+a_0\left(1+\sum_{n=1}^{\infty}\frac{1}{4\cdot 8\cdot 12\cdots 4n}x^{4n}\right)$

3. $y'+(1-x^2)y=\sum_{n=1}^{\infty}na_nx^{n-1}+\sum_{n=0}^{\infty}a_nx^n-\sum_{n=0}^{\infty}a_nx^{n+2}=(a_1+a_0)+(2a_2+a_1)x+\sum_{n=3}^{\infty}[na_n+a_{n-1}-a_{n-3}]x^{n-1}=x$; the recurrence relation is a_0 arbitrary, $a_1+a_0=0, 2a_2+a_1=1, na_n+a_{n-1}-a_{n-3}=0$ for $n\geq 3$. This gives the solution
$y=a_0\left[1-x+\frac{1}{2!}x^2+\frac{1}{3!}x^3-\frac{7}{4!}x^4+\frac{19}{5!}x^5+\cdots\right]+\left[\frac{1}{2!}x^2-\frac{1}{3!}x^3+\frac{1}{4!}x^4+\frac{11}{5!}x^5-\frac{31}{6!}x^6+\cdots\right]$

4. $y''+2y'+xy=\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}+\sum_{n=1}^{\infty}2na_nx^{n-1}+\sum_{n=0}^{\infty}a_nx^{n+1}=(2a_2+2a_1)+(3\cdot 2a_3+2\cdot 2a_2+a_0)x+\sum_{n=4}^{\infty}[n(n-1)a_n+2(n-1)a_{n-1}+a_{n-3}]x^{n-2}=0$; the recurrence relation is a_0, a_1 arbitrary, $a_2=-a_1, 6a_3+4a_2+a_0=0, n(n-1)a_n+2(n-1)a_{n-1}+a_{n-3}=0$ for $n\geq 4$. Taking $a_0=1$ and $a_1=0$ gives one solution $y_1=1-\frac{1}{6}x^3+\frac{1}{12}x^4-\frac{1}{30}x^5+\frac{1}{60}x^6+\cdots$; taking $a_0=0$ and $a_1=1$ gives a second solution $y_2=x-x^2+\frac{2}{3}x^3-\frac{5}{12}x^4+\frac{7}{60}x^5+\cdots$. The general solution is of the form $y=a_0y_1(x)+a_1y_2(x)$ where $a_0=y(0)$ and $a_1=y'(0)$ are arbitrary constants.

5. $y''-xy'+y=\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}-\sum_{n=1}^{\infty}na_nx^n+\sum_{n=0}^{\infty}a_nx^n=(2a_2+a_0)+\sum_{n=3}^{\infty}[n(n-1)a_n-(n-3)a_{n-2}]x^{n-2}=3$. Taking $a_0=c_0+3$ with c_0 arbitrary,a_1 arbitrary we get the recurrence relation $a_2=-\frac{c_0}{2}$, and $a_n=\frac{(n-3)}{n(n-1)}a_{n-2}, n\geq 3$. This gives the general solution $y=3+a_1x+c_0\left[1+\sum_{n=1}^{\infty}\frac{(-1)\cdot 1\cdot 3\cdots(2n-3)}{(2n)!}x^{2n}\right]$ in which $a_1=y'(0)$ and $c_0=y(0)-3$ are arbitrary.

6. $y''+xy'+xy=\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}+\sum_{n=1}^{\infty}na_nx^n+\sum_{n=0}^{\infty}a_nx^{n+1}=2a_2+\sum_{n=3}^{\infty}[n(n-1)a_n+(n-2)a_{n-2}+a_{n-3}]x^{n-2}=0$. The recurrence relation is a_0 and a_1 arbitrary, $a_2=0, a_n=\frac{-(n-2)a_{n-2}-a_{n-3}}{n(n-1)}$ for $n\geq 3$. Taking $a_0=1$ and $a_1=0$ gives one solution $y_1=1-\frac{1}{2\cdot 3}x^3+\frac{3}{2\cdot 3\cdot 4\cdot 5}x^5+\frac{1}{2\cdot 3\cdot 5\cdot 6}x^6-\frac{3\cdot 5}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}x^7+\cdots$; taking $a_0=0$ and $a_1=1$ gives a second solution
$y_2=x-\frac{1}{2\cdot 3}x^3-\frac{1}{3\cdot 4}x^4+\frac{3}{2\cdot 3\cdot 4\cdot 5}x^5+\frac{3}{2\cdot 3\cdot 5\cdot 6}x^6+\cdots$ Since the differential equation is homogeneous, the general solution is $y=a_0y_1(x)+a_1y_2(x)$ where $a_0=y(0)$ and $a_1=y'(0)$ are arbitrary constants.

7. $y''-x^2y'+2y=\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}-\sum_{n=1}^{\infty}na_nx^{n+1}+\sum_{n=0}^{\infty}2a_nx^n=(2a_2+2a_0)+(3\cdot 2a_3+2a_1)x+\sum_{n=4}^{\infty}[n(n-1)a_n-(n-3)a_{n-3}+2a_{n-2}]x^{n-2}=x$. The recurrence relation is a_0, a_1 arbitrary, $a_2=-a_0, 6a_3+2a_1=1, a_n=\frac{(n-3)a_{n-3}-2a_{n-2}}{n(n-1)}$ for $n\geq 4$. The general solution of the nonhomogeneous differential equation can be obtained in the form

$y = a_0\left[1 - x^2 + \frac{1}{6}x^4 - \frac{1}{10}x^5 - \frac{1}{90}x^6 + \cdots\right] + a_1\left[x - \frac{1}{3}x^3 + \frac{1}{12}x^4 + \frac{1}{30}x^5 - \frac{7}{180}x^6 + \cdots\right]$
$+ \left[\frac{1}{6}x^3 - \frac{1}{60}x^5 + \frac{1}{60}x^6 + \frac{1}{1260}x^7 - \frac{1}{480}x^8 + \cdots\right]$ where $a_0 = y(0)$ and $a_1 = y'(0)$ are arbitrary constants. The third bracket above is a series representation of a particular solution obtained from the recurrence by putting $a_0 = a_1 = 0$.

8. $y'' + x^2y' + 2y = \sum_{n=2}^{\infty} n(n-1)a_nx^{n-2} + \sum_{n=1}^{\infty} na_nx^{n+1} + \sum_{n=0}^{\infty} 2a_nx^n = (2a_2 + 2a_0) + (3\cdot 2a_3 + 2a_1)x + \sum_{n=4}^{\infty}[n(n-1)a_n + (n-3)a_{n-3} + 2a_{n-2}]x^{n-2} = 0$. The recurrence relation is a_0, a_1 arbitrary, $a_2 = -a_0, a_3 = -\frac{a_1}{3}, a_n = \frac{-(n-3)a_{n-3} - 2a_{n-2}}{n(n-1)}$ for $n \geq 4$. Taking $a_0 = 1$ and $a_1 = 0$ gives one solution $y_1 = 1 - x^2 + \frac{1}{6}x^4 + \frac{1}{10}x^5 - \frac{1}{90}x^6 + \cdots$; taking $a_0 = 0$ and $a_1 = 1$ gives a second solution $y_2 = x - \frac{1}{3}x^3 - \frac{1}{12}x^4 + \frac{1}{30}x^5 + \frac{7}{180}x^6 + \cdots$. The general solution of this homogeneous equation is $y = a_0y_1(*) + a_1y_2(x)$ where $a_0 = y(0)$ and $a_1 = y'(0)$ are arbitrary constants.

9. $y'' + (1-x)y' + 2y = \sum_{n=2}^{\infty} n(n-1)a_nx^{n-2} + \sum_{n=1}^{\infty} na_nx^{n-1} - \sum_{n=1}^{\infty} na_nx^n + \sum_{n=0}^{\infty} 2a_nx^n = (2a_2 + a_1 + 2a_0) + \sum_{n=3}^{\infty}[n(n-1)a_n + (n-1)a_{n-1} - (n-4)a_{n-2}]x^{n-2} = 1 - x^2$. The recurrence relation is a_0, a_1 arbitrary, $2a_2 + a_1 + 2a_0 = 1, 6a_3 + 2a_2 + a_1 = 0, 12a_4 + 3a_3 = -1, a_n = \frac{-(n-1)a_{n-1} + (n-4)a_{n-2}}{n(n-1)}$ for $n \geq 5$. The general solution of this nonhomogeneous differential equation can be obtained in the form $y = a_0\left[1 - x^2 + \frac{1}{3}x^3 - \frac{1}{12}x^4 + \frac{1}{30}x^5 - \cdots\right] + a_1\left[x - \frac{x^2}{2}\right] + \left[\frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{24}x^4 - \frac{1}{360}x^6 + \frac{1}{2520}x^7 + \cdots\right]$ where $a_0 = y(0)$ and $a_1 = y'(0)$ are arbitrary constants. The third bracketed term above is a particular solution of the nonhomogeneous equation.

10. $y'' + y' - (1 - x + x^2)y = \sum_{n=2}^{\infty} n(n-1)a_nx^{n-2} + \sum_{n=1}^{\infty} na_nx^{n-1} - \sum_{n=0}^{\infty} a_nx^n + \sum_{n=0}^{\infty} a_nx^{n+1} - \sum_{n=0}^{\infty} a_nx^{n+2} = (2a_2 + a_1 - a_0) + (6a_3 + 2a_2 - a_1 + a_0)x + \sum_{n=4}^{\infty}[n(n-1)a_n + (n-1)a_{n-1} - a_{n-2} + a_{n-3} - a_{n-4}]x^{n-2} = -5$. The recurrence relation is a_0, a_1 arbitrary, $2a_2 + a_1 - a_0 = -5, 6a_3 + 2a_2 - a_1 + a_0 = 0$,
$a_n = \frac{-(n-1)a_{n-1} + a_{n-2} - a_{n-3} + a_{n-4}}{n(n-1)}$ for $n \geq 4$.
The general solution of this nonhomogeneous differential equation has the form
$y = a_0\left[1 + \frac{1}{2}x^2 - \frac{1}{3}x^3 + \frac{5}{24}x^4 - \frac{1}{12}x^5 + \cdots\right] + a_1\left[x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{5}{24}x^4 + \frac{2}{15}x^5 - \cdots\right]$
$- 5\left[\frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{12}x^4 - \frac{1}{20}x^5 + \frac{1}{30}x^6 - \cdots\right]$ where $a_0 = y(0)$ and $a_1 = y'(0)$ are arbitrary constants. The third term above is a particular solution of the nonhomogeneous equation.

11. $y' + xy = \sum_{n=1}^{\infty} na_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^{n+1} = a_1 + \sum_{k=0}^{\infty}[2ka_{2k} + a_{2k-2}]x^{2k-1} + \sum_{k=1}^{\infty}[(2k+1)a_{2k+1} + a_{2k-1}]x^{2k} = \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{(2k)!}$. The recurrence relation is a_0 arbitrary, $a_1 = 1, a_{2k} = -\frac{a_{2k-2}}{2k}, a_{2k+1} = \frac{-a_{2k-1} + \frac{(-1)^k}{(2k)!}}{(2k+1)}$, for $k \geq 1$. The solution is $y = a_0\left[1 - \frac{1}{2}x^2 + \frac{1}{2\cdot 4}x^4 - \frac{1}{2\cdot 4\cdot 6}x^+ \frac{1}{2\cdot 4\cdot 6\cdot 8}x^8 - \cdots\right]$ $+ \left[x - \frac{3}{3!}x^3 + \frac{13}{5!}x^5 - \frac{79}{7!}x^7 + \frac{633}{9!}x^9 - \cdots\right]$ where $a_0 = y(0)$ is an arbitrary constant.

12. $y'' + xy' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=1}^{\infty} na_n x^n = 2a_2 + \sum_{n=3}^{\infty}[n(n-1)a_n + (n-2)a_{n-2}]x^{n-2} = -\sum_{n=3}^{\infty}\frac{1}{(n-2)!}x^{n-2}$. The recurrence relation is a_0, a_1 arbitrary, $a_2 = 0, a_n = \frac{-(n-2)a_{n-2} - \frac{1}{(n-2)!}}{n(n-1)}$ for $n \geq 3$. The solution is $y = a_0 + a_1\left[x - \frac{1}{3!}x^3 + \frac{3}{5!}x^5 - \frac{15}{7!}x^7 + \frac{105}{9!}x^9 + \cdots\right]$ $+\left[-\frac{1}{3!}x^3 - \frac{1}{4!}x^4 + \frac{2}{5!}x^5 + \frac{3}{6!}x^6 - \frac{11}{7!}x^7 - \frac{19}{8!}x^8 + \frac{76}{9!}x^9 + \frac{151}{10!}x^{10} + \cdots\right]$ where $a_0 = y(0)$ and $a_1 = y'(0)$ are arbitrary constants. The third bracketed term above is a particular solution of the nonhomogeneous equation.

Section 4.3 Singular Points and the Method of Frobenius

1. From $P(x) = x^2(x-3)^3 = 0$, singular points are $x = 0$ and $x = 3$. For $x = 0$, $\frac{xQ(x)}{P(x)} = \frac{4(x+2)}{x-3}$ and $x^2\frac{R(x)}{P(x)} = \frac{x^2 - x - 2}{(x-3)^2}$ are both analytic at $x = 0$, so $x = 0$ is a regular singular point. For $x = 3$, $\frac{(x-3)Q(x)}{P(x)} = \frac{4(x+2)}{x}$ and $(x-3)^2\frac{R(x)}{P(x)} = \frac{x^2-x-2}{x^2}$ are both analytic at $x = 3$, so $x = 3$ is a regular singular point.

2. Since $P(x) = (x+1)^2(x-4)$, $x = -1$and $x = 4$ are singular points. $x = 4$ is regular, however $\frac{(x+1)Q(x)}{P(x)} = \frac{-2(x^2+1)}{(x+1)(x-4)}$ is not analytic at $x = -1$, so $x = -1$ is an irregular singular point.

3. $x = 0$ and $x = 2$ are singular points and both are regular.

4. $x = 3$ and $x = -3$ are singular points and both are regular.

5. $x = 2$ and $x = 0$ are singular points. $x = 2$ is regular, however $\frac{x^2Q(x)}{P(x)} = \frac{x-2}{\sqrt{x}}$ is not analytic at $x = 0$, so $x = 0$ is an irregular singular point.

6. $x = 0$ and $x = k\pi, k$ any non-zero integer are all singular points. $\frac{xQ(x)}{P(x)} = \frac{1}{x\sin(x-\pi)\cos(x-\pi)}$ is not analytic at $x = 0$, so $x = 0$ is an irregular singular point.
For $x = k\pi (k \neq 0)$, $\frac{(x-k\pi)Q(x)}{P(x)} = \frac{(x-k\pi)\tan(x)}{x^2\sin(x-\pi)\cos(x-\pi)}$ has a removable singularity at

$x = k\pi$, as does $\dfrac{(x-k\pi)^2R(x)}{P(x)} = \left[\dfrac{(x-k\pi)}{\sin(x-\pi)}\right]^2 \dfrac{\cos(x)}{x^2}$;
thus $x = k\pi(k \neq 0)$ is a regular singular point. Since $Q(x) = \tan(x-\pi)\tan(x)$ is not analytic at any $x = (2k+1)\dfrac{\pi}{2}$ (k any integer), each point $x = (2k+1)\dfrac{\pi}{2}$ is an irregular singular point.

7. (a) $\dfrac{xQ(x)}{P(x)} = \dfrac{1}{2}$ and $x^2\dfrac{R(x)}{P(x)} = -\dfrac{x}{4}$ are both analytic at $x = 0$, so $x = 0$ is a regular singular point. With $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ we get $4x^2y'' + 2xy' - xy = \sum_{n=0}^{\infty} 4(n+r)(n+r-1)c_n x^{n+r} + \sum_{n=0}^{\infty} 2(n+r)x^{n+r} - \sum_{n=0}^{\infty} c_n x^{n+r+1} = 2r(2r-1)c_0x^r + \sum_{n=1}^{\infty}[2(n+r)(2(n+r)-1)c_n - c_{n-1}]x^{n+r} = 0.$

(b) Since $c_0 \neq 0, r$ must satisfy the indicial equation $2r(2r-1) = 0$ with roots $r_1 = \dfrac{1}{2}$ and $r_2 = 0$.

(c) The recurrence relation is $c_n = \dfrac{1}{2(n+r)(2(n+r)-1)}c_{n-1}, n \geq 1$

(d) Using $r = \dfrac{1}{2}$ gives $c_n = \dfrac{1}{2n(2n+1)}c_{n-1}$ and one solution $y_1 = x^{1/2}\left[1 + \sum_{n=1}^{\infty}\dfrac{x^n}{(2n+1)!}\right]$;

whereas using $r = 0$ gives $c_n = \dfrac{1}{(2n-1)(2n)}c_{n-1}$ and a second solution $y_2 = 1 + \sum_{n=1}^{\infty}\dfrac{x^n}{(2n)!}$

8. (a) $\dfrac{xQ(x)}{P(x)} = -\dfrac{x}{4}, \dfrac{x^2R(x)}{P(x)} = \dfrac{3}{16}$ both analytic at $x = 0$.

(b) The indicial equation is $16r^2 - 16r + 3 = 0$ with roots $r_1 = \dfrac{3}{4}$ and $r_2 = \dfrac{1}{4}$.

(c) $c_n = \dfrac{4(n+r-1)}{16(n+r)^2 - 16(n+r) + 3}c_{n-1}, n \geq 1$

(d) Using $r = \dfrac{3}{4}$ gives $c_n = \dfrac{4n-1}{8n(2n+1)}c_{n-1}$ and $y_1 = x^{3/4}\left[1 + \sum_{n=1}^{\infty}\dfrac{3\cdot 7\cdot 11\cdots(4n-1)}{8^n n![3\cdot 5\cdot 7\cdots(2n+1)]}x^n\right] = x^{3/4}\left[1 + \dfrac{1}{8}x + \dfrac{7}{640}x^2 + \dfrac{11}{15360}x^3 + \dfrac{11}{294912}x^4 + \cdots\right]$;

whereas $r = \dfrac{1}{4}$ gives $c_n = \dfrac{4n-3}{8n(2n-1)}c_{n-1}$ and $y_2 = x^{1/4}\left[1 + \sum_{n=1}^{\infty}\dfrac{1\cdot 5\cdot 9\cdots(4n-3)}{8^n n![1\cdot 3\cdot 5\cdots(2n-1)]}x^n\right] = x^{1/4}\left[1 + \dfrac{1}{8}x + \dfrac{5}{384}x^2 + \dfrac{1}{1024}x^3 + \dfrac{13}{229376}x^4 + \cdots\right]$

9. (a) $\dfrac{xQ(x)}{P(x)} = 0$ and $\dfrac{x^2R(x)}{P(x)} = \dfrac{2(2x+1)}{9}$, both analytic at $x = 0$.

(b) $9r^2 - 9r + 2 = 0$ with roots $r_1 = 2/3$ and $r_2 = 1/3$

(c) $c_n = \dfrac{-4}{(3(n+r)-2)(3(n+r)-1)}c_{n-1}, n \geq 1$

(d) Using $r = 2/3$ gives $c_n = \dfrac{-4}{3n(3n+1)}c_{n-1}$ and $y_1 = x^{2/3}\left[1 + \sum_{n=1}^{\infty}\dfrac{(-1)^n 4^n}{3^n n![4\cdot 7\cdot 10\cdots(3n+1)]}x^n\right]$;

whereas $r = 1/3$ gives $c_n = \dfrac{-4}{(3n-1)3n}c_{n-1}$ and $y_2 = x^{1/3}\left[1 + \sum_{n=1}^{\infty}\dfrac{(-1)^n 4^n}{3^n n![2\cdot 5\cdot 8\cdots(3n-1)]}x^n\right]$

10. (a) $\dfrac{xQ(x)}{P(x)} = \dfrac{5}{12}$ and $\dfrac{x^2R(x)}{P(x)} = \dfrac{1-2x^2}{12}$, both analytic at $x=0$.
(b) $12r^2 - 7r + 1 = 0$ which has roots $r_1 = 1/3$ and $r_2 = 1/4$
(c) $c_n = \dfrac{2}{(4(n+r)-1)(3(n+r)-1)}c_{n-2}, n \geq 2$
(d) Using $r = 1/3$ gives $c_n = \dfrac{2}{n(12n+1)}c_{n-2}$, or with $n = 2k$, $c_{2k} = \dfrac{1}{k(24k+1)}c_{2k-2}, k \geq 1$ and $y_1 = x^{1/3}\left[1 + \displaystyle\sum_{k=1}^{\infty}\frac{1}{k![25\cdot 49\cdot 79\cdots(24k+1)]}x^{2k}\right]$;
whereas $r = 1/4$ gives $c_n = \dfrac{2}{n(12n-1)}c_{n-2}$, or $c_{2k} = \dfrac{1}{k(24k-1)}c_{2k-2}, k \geq 1$ and $y_2 = x^{1/4}\left[1 + \displaystyle\sum_{k=1}^{\infty}\frac{1}{k![23\cdot 47\cdot 71\cdots(24k-1)]}x^{2k}\right]$

11. (a) $\dfrac{xQ(x)}{P(x)} = \dfrac{2x+1}{2}$ and $\dfrac{x^2R(x)}{P(x)} = x$, both analytic at $x=0$.
(b) $2r^2 - r = 0$ with roots $r_1 = 1/2$ and $r_2 = 0$
(c) $c_n = -\dfrac{2(n+r)+2}{(n+r)(2(n+r)-1)}c_{n-1}, n \geq 1$
(d) Using $r_1 = 1/2$ gives $c_n = -\dfrac{(2n+3)}{n(2n+1)}c_{n-1}$, and $y_1 = x^{1/2}\left[1 + \displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n(2n+3)}{3(n!)}x^n\right]$
$= x^{1/2}\left[1 - \dfrac{5}{3}x + \dfrac{7}{6}x^2 - \dfrac{1}{2}x^3 + \dfrac{11}{72}x^4 + \cdots\right]$;
whereas $r = 0$ gives $c_n = -\dfrac{2(n+1)}{n(2n-1)}c_{n-1}$, and $y_2 = \left[1 + \displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{2^n(n+1)}{[1\cdot 3\cdot 5\cdots(2n-1)]}x^n\right] = \left[1 - 4x + 4x^2 - \dfrac{32}{15}x^3 + \dfrac{16}{21}x^4 + \cdots\right]$

12. (a) $\dfrac{xQ(x)}{P(x)} = -\dfrac{1}{2}$ and $\dfrac{x^2R(x)}{P(x)} = \dfrac{1-x^2}{2}$, both analytic at $x=0$.
(b) $2r^2 - 3r + 1 = 0$ with roots $r_1 = 1$ and $r_2 = 1/2$
(c) $c_n = \dfrac{1}{(2(n+r)-1)(n+r-1)}c_{n-2}, n \geq 2$
(d) Using $r = 1$ gives $c_n = \dfrac{1}{n(2n+1)}c_{n-2}$, and $y_1 = x\left[1 + \displaystyle\sum_{n=1}^{\infty}\frac{1}{2^n n![5\cdot 9\cdot 13\cdots(4n+1)]}x^{2n}\right]$;
whereas $r = 1/2$ gives $c_n = \dfrac{1}{n(2n-1)}c_{n-2}$, and $y_2 = x^{1/2}\left[1 + \displaystyle\sum_{n=1}^{\infty}\frac{1}{2^n n![3\cdot 7\cdot 11\cdots(4n-1)]}x^{2n}\right]$

13. (a) $\dfrac{xQ(x)}{P(x)} = \dfrac{2x+1}{2}$ and $\dfrac{x^2R(x)}{P(x)} = -\dfrac{(2x^2+1)}{2}$, both analytic at $x=0$.
(b) $2r^2 - r - 1 = 0$ with roots $r_1 = 1$ and $r_2 = -1/2$
(c) $(2r+3)c_1 + 2c_0 = 0, c_n = \dfrac{2c_{n-2} - 2(n+r-1)c_{n-1}}{(n+r-1)(2(n+r)+1)}, n \geq 2$
(d) Using $r = 1$ gives $c_1 = -\dfrac{2}{5}c_0, c_n = \dfrac{2c_{n-2} - 2nc_{n-1}}{n(2n+3)}, n \geq 2$ and
$y_1 =$

$$x\left[1-\frac{2}{5}x+\frac{18}{2!5\cdot 7}x^2-\frac{164}{3!5\cdot 7\cdot 9}x^3+\frac{2284}{4!5\cdot 7\cdot 9\cdot 11}x^4-\cdots\right];$$

whereas $r=-1/2$ gives $c_1=-c_0$, $c_n=\dfrac{2c_{n-2}-(2n-3)c_{n-1}}{n(2n-3)}$, $n\geq 2$, and

$$y_2=x^{-1/2}\left[1-x+\frac{3}{2!}x^2-\frac{13}{3!3}x^3+\frac{119}{4!3\cdot 5}x^4-\cdots\right]$$

14. (a) $\dfrac{xQ(x)}{P(x)}=\dfrac{4}{3}$ and $\dfrac{x^2R(x)}{P(x)}=-\dfrac{(3x+2)}{3}$, both analytic at $x=0$.

(b) $3r^2+r-2=0$ with roots $r_1=2/3$ and $r_2=-1$

(c) $c_n=\dfrac{3}{(3(n+r)-2)(n+r+1)}, c_{n-1}, n\geq 1$

(d) Using $r=2/3$ gives $c_n=\dfrac{3}{n(3n+5)}c_{n-1}$ and

$$y_1=x^{2/3}\left[1+\sum_{n=1}^{\infty}\frac{3^n}{n![8\cdot 11\cdot 14\cdots(3n+5)]}x^n\right];$$

whereas $r=-1$ gives $c_n=\dfrac{3}{n(3n-5)}c_{n-1}$, and

$$y_2=x^{-1}\left[1+\sum_{n=1}^{\infty}\frac{3^n}{n![-2\cdot 1\cdot 4\cdots(3n-5)]}x^n\right]$$

15. (a) $\dfrac{xQ(x)}{P(x)}=1$ and $\dfrac{x^2R(x)}{P(x)}=\dfrac{9x^2-4}{9}$, both analytic at $x=0$.

(b) $9r^2-4=0$ with roots $r_1=2/3$ and $r_2=-2/3$

(c) $c_1=0, c_n=\dfrac{-9}{(3(n+r)-2)(3(n+r)+2)}, c_{n-2}, n\geq 2$

(d) Using $r=2/3$ gives $c_1=0, c_n=\dfrac{-3}{(n(3n+4)}c_{n-2}, n\geq 2$ and

$$y_1=x^{2/3}\left[1+\sum_{n=1}^{\infty}\frac{(-1)^n 3^n}{4^n n![5\cdot 8\cdot 11\cdots(3n+2)]}x^{2n}\right];$$

whereas $r=-2/3$ gives $c_1=0$,

$c_n=\dfrac{-3}{n(3n-4)}c_{n-2},\ n\geq 2$ and

$$y_2=x^{-2/3}\left[1+\sum_{n=1}^{\infty}\frac{(-1)^n 3^n}{4^n n![1\cdot 4\cdot 7\cdots(3n-2)]}x^{2n}\right]$$

Section 4.4 Second Solutions and Logarithm Factors

1. (a)

$xy''+(1-x)y'+y=\displaystyle\sum_{n=0}^{\infty}(n+r)(n+r-1)c_nx^{n+r-1}+\sum_{n=0}^{\infty}(n+r)c_nx^{n+r-1}-\sum_{n=0}^{\infty}(n+r)c_nx^{n+r}+$ $\displaystyle\sum_{n=0}^{\infty}c_nx^{n+r}=r^2c_0x^{r-1}+\sum_{n=1}^{\infty}[(n+r)^2c_n-(n+r-2)c_{n-1}]x^{n+r-1}=0$. Since $c_0\neq 0, r$ must satisfy the indicial equation $r^2=0$, which has equal roots $r_1=r_2=0$.

(b) By Theorem 4.4 - (2) there will be one solution of the form $y_1 = \sum_{n=0}^{\infty} c_n x^n$ and a second solution of the form $y_2 = y_1(x)\ln(x) + \sum_{n=1}^{\infty} c_n^* x^n$.

(c) For a first solution, y_1, choose the coefficients to satisfy $c_0 = 1, c_n = \frac{(n-2)}{n^2} c_{n-1}, n \geq 1$. This choice of coefficients yields the solution $y_1 = 1 - x$. It follows that $y_2 = (1-x)\ln(x) + \sum_{n=1}^{\infty} c_n^* x^n$ and by substitution into the differential equation we obtain $x\left[-\frac{2}{x} - \frac{(1-x)}{x^2}\right] + (1-x)\left[-\ln(x) + \frac{(1-x)}{x}\right] + (1-x)\ln(x) + \sum_{n=2}^{\infty} n(n-1)c_n^* x^{n-1} + (1-x)\sum_{n=1}^{\infty} nc_n^* x^{n-1} + \sum_{n=1}^{\infty} c_n^* x^n = (-3 + c_1^*) + x(1 + 4c_2^*) + \sum_{n=3}^{\infty}[n^2 c_n^* - (n-2)c_{n-1}^*]x^{n-1} = 0$. The coefficients c_n^* are given by $c_1^* = 3, c_2^* = -1/4, c_n^* = \frac{n-2}{n^2} c_{n-1}^*, n \geq 3$, and a second solution is $y_2 = (1-x)\ln(x) + 3x - \sum_{n=2}^{\infty} \frac{1}{n(n-1)n!} x^n$

2. (a) $r(r-1) = 0$ with roots $r_1 = 1$ and $r_2 = 0$.

(b) Since $r_1 - r_2 = 1$, a positive integer, by Theorem 4.4-(3), $y_1 = \sum_{n=0}^{\infty} c_n x^{n+1}$ with $c_0 \neq 0$, and $y_2 = ky_1(x)\ln(x) + \sum_{n=0}^{\infty} c_n^* x^n$.

(c) For y_1 the recurrence relation for the coefficients is $c_0 \neq 0, c_n = \frac{2(n+r-2)}{(n+r)(n+r-1)} c_{n-1}, n \geq 1$. With $r = 1$ and $c_0 = 1$ we obtain $y_1 = x$ as a first solution. Substitution of $y_2 = kx\ln(x) + \sum_{n=0}^{\infty} c_n^* x^n$ into the differential equation gives (after simplification) $(2c_0^* + k) + 2(c_2^* - k)x + \sum_{n=3}^{\infty}[n(n-1)c_n^* - 2(n-2)c_{n-1}^*]x^{n-1} = 0$. For simplification take $c_0^* = 1$, thus $k = -2$, c_1^* is arbitrary (which we choose to be zero), $c_2^* = -2$, and $c_n^* = \frac{2(n-2)}{n(n-1)} c_{n-1}^*, n \geq 3$. This gives a second solution $y_2 = -2x\ln(x) + 1 - \sum_{n=2}^{\infty} \frac{2^n}{n!(n-1)} x^n$.

3. (a) $-r^2 + 4r = 0$ with roots $r_1 = 4$ and $r_2 = 0$.

(b) Since $r_1 - r_2 = 4$, by Theorem 4.4-(3), $y_1 = \sum_{n=0}^{\infty} c_n x^{n+4}$ with $c_0 \neq 0$, and $y_2 = ky_1(x)\ln(x) + \sum_{n=0}^{\infty} c_n^* x^n$ where k may be zero.

(c) Using $r = 4$ gives the recurrence $c_n = \left(\frac{n+1}{n}\right) c_{n-1}$ and solution $y_1 = x^4[1 + 2x + 3x^2 +$

$4x^3 + \cdots] = x^4 \dfrac{d}{dx}(1 + x + x^2 + x^3 + x^4 + \cdots) = \dfrac{x^4}{(x-1)^2}$. A second solution is $y_2 = \dfrac{3-4x}{(x-1)^2}$

4. (a) $4r^2 - 9 = 0$ with roots $r_1 = 3/2$ and $r_2 = -3/2$.

(b) By Theorem 4.4 - (3), $y_1 = \sum_{n=0}^{\infty} c_n x^{n+3/2}, c_0 \neq 0$;

and $y_2 = ky_1(x)\ln(x) + \sum_{n=0}^{\infty} c_n^* x^{n-3/2}$ where k may be zero.

(c) $y_1 = x^{3/2}\left[1 + \sum_{n=1}^{\infty} \dfrac{(-1)^n}{2^n n![5\cdot 7\cdot 9\cdots(2n+3)]} x^{2n}\right] =$

$x^{3/2}\left[1 - \dfrac{1}{10}x^2 + \dfrac{1}{280}x^4 - \dfrac{1}{15120}x^6 + \dfrac{1}{887040}x^8 + \cdots\right]$;

$y_2 = x^{-3/2}\left[1 + \sum_{n=1}^{\infty} \dfrac{(-1)^n}{2^n n![(-1)(1)(3)\cdots(2n-3)]} x^{3n}\right]$

$= x^{-3/2}\left[1 + \dfrac{1}{2}x^2 - \dfrac{1}{8}x^4 + \dfrac{1}{144}x^6 - \dfrac{1}{5760}x^8 + \cdots\right]$.

5. (a) $4r^2 - 2r = 0$ with roots $r_1 = \dfrac{1}{2}$ and $r_2 = 0$.

(b) $y_1 = x^{1/2}\sum_{n=0}^{\infty} c_n x^n, c_0 \neq 0$; $y_2 = \sum_{n=0}^{\infty} c_n^* x^n$.

(c) $y_1 = x^{1/2}\left[1 + \sum_{n=1}^{\infty} \dfrac{(-1)^n}{2^n n![3\cdot 5\cdot 7\cdots(2n+1)]} x^n\right]$

$= x^{1/2}\left[1 - \dfrac{1}{6}x + \dfrac{1}{120}x^2 - \dfrac{1}{5040}x^3 + \dfrac{1}{362880}x^4 + \cdots\right]$;

$y_2 = \left[1 + \sum_{n=1}^{\infty} \dfrac{(-1)^n}{2^n n![1\cdot 3\cdot 5\cdots(2n-1)]} x^n\right]$

$= \left[1 - \dfrac{1}{2}x + \dfrac{1}{24}x^2 - \dfrac{1}{720}x^3 + \dfrac{1}{40320}x^4 - \cdots\right]$.

6. (a) $4r^2 - 1 = 0$ with roots $r_1 = \dfrac{1}{2}$ and $r_2 = -\dfrac{1}{2}$.

(b) $y_1 = \sum_{n=0}^{\infty} c_n x^{n+1/2}, c_0 \neq 0$ and $y_2 = ky_1(x)\ln(x) + \sum_{n=0}^{\infty} c_n^* x^{n-1/2}$.

(c) Note this is an Euler equation with solutions $y_1 = x^{1/2}$ and $y_2 = x^{-1/2}$.

7. (a) $r^2 - 3r + 2 = 0$ with roots $r_1 = 2$ and $r_2 = 1$.

(b) $y_1 = \sum_{n=0}^{\infty} c_n x^{n+2}, c_0 \neq 0$ and $y_2 = ky_1(x)\ln(x) + \sum_{n=0}^{\infty} c_n^* x^{n+1}$.

(c) $y_1 = x^2 + \dfrac{1}{3!}x^4 + \dfrac{1}{5!}x^6 + \dfrac{1}{7!}x^8 + \cdots = x\sinh(x)$;

$y_2 = x - x^2 + \dfrac{1}{2!}x^3 - \dfrac{1}{3!}x^4 + \dfrac{1}{4!}x^5 - \cdots = xe^{-x}$.

8. (a) $r^2 - 2r = 0$ with roots $r_1 = 2$ and $r_2 = 0$

(b) $y_1 = \sum_{n=0}^{\infty} c_n x^{n+2}, c_0 \neq 0$ and $y_2 = ky_1(x)\ln(x) + \sum_{n=0}^{\infty} c_n^* x^n$

(c) For y_1, the recurrence relation is c_0 arbitrary, $c_n = \dfrac{-2c_{n-1}}{n(n-2)}, n \geq 1$ and a solution is (with $c_0 = 1$) $y_1 = \sum_{n=0}^{\infty} \dfrac{(-1)^n 2^{n+1}}{n!(n+2)!} x^{n+2} = x^2 - \dfrac{2}{3}x^3 + \dfrac{1}{6}x^4 - \dfrac{1}{45}x^5 + \dfrac{1}{540}x^6 - \cdots$. Substitution of the form of y_2, the series solution for y_1, and simplification gives $(2c_0^* - c_1^*) + \sum_{n=2}^{\infty}[n(n-2)c_n^* + 2c_{n-1}^* + \dfrac{(-1)^n 2^n k}{[(n-2)!]^2 n}]x^{n-1} = 0$. With $c_0^* = 1$ (for simplicity) we get $c_1^* = 2, k = -2, c_2^*$ arbitrary (which we take to be 0) $c_n^* = -\dfrac{1}{n(n-2)}\left[2c_{n-1}^* + \dfrac{(-1)^n 2^{n+1}}{n[(n-2)!]^2}\right], n \geq 3$, and get a second solution $y_2 = -2y_1(x)\ln(x) + 1 + 2x + \dfrac{16}{9}x^3 - \dfrac{25}{36}x^4 + \dfrac{157}{1350}x^5 - \cdots$.

9. (a) $2r^2 = 0$ with roots $r_1 = r_2 = 0$

(b) $y_1 = \sum_{n=0}^{\infty} c_n x^n, c_0 \neq 0$ and $y_2 = y_1(x)\ln(x) + \sum_{n=1}^{\infty} c_n^* x^n$

(c) $y_1 = 1 - x$; $y_2 = (1-x)\ln\left(\dfrac{x}{x-2}\right) - 2$

10. (a) $r^2 - 1 = 0$ with roots $r_1 = 1$ and $r_2 = -1$

(b) $y_1 = \sum_{n=0}^{\infty} c_n x^{n+1}$; $y_2 = ky_1(x)\ln(x) + \sum_{n=0}^{\infty} c_n^* x^{n-1}$

(c) $y_1 = x\left[1 + \sum_{n=1}^{\infty} \dfrac{(-1)^n[1 \cdot 4 \cdot 7 \cdots (3n-2)]}{3^n n![5 \cdot 8 \cdot 11 \cdots (3n+2)]} x^{3n}\right]$;

$y_2 = x^{-1}\left[1 + \sum_{n=1}^{\infty} \dfrac{(-1)^n[-1 \cdot 2 \cdot 5 \cdots (3n-4)]}{3^n n![1 \cdot 4 \cdot 7 \cdots (3n-2)]} x^{3n}\right]$

11. (a) $25r^2 + 15r - 4 = 0$ with roots $r_1 = 1/5$ and $r_2 = -4/5$

(b) $y_1 = \sum_{n=0}^{\infty} c_n x^{n+1/5}, c_0 \neq 0$ and $y_2 = ky_1(x)\ln(x) + \sum_{n=0}^{\infty} c_n^* x^{n-4/5}$

12. (a) $-192r^2 - 144r - 27 = 0$ with roots $r_1 = r_2 = -3/8$

(b) $y_1 = \sum_{n=0}^{\infty} c_n x^{n-3/8}, c_0 \neq 0$ and $y_2 = y_1(x)\ln(x) + \sum_{n=1}^{\infty} c_n^* x^{n-3/8}$

13. (a) $48r^2 - 20r - 8 = 0$ with roots $r_1 = 2/3$ and $r_2 = -1/4$

(b) $y_1 = \sum_{n=0}^{\infty} c_n x^{n+2/3}$ with $c_0 \neq 0$; $y_2 = \sum_{n=0}^{\infty} c_n^* x^{n-1/4}$ with $c_0^* \neq 0$

14. (a) $9r^2 + 3r - 56 = 0$ with roots $r_1 = 7/3$ and $r_2 = -8/3$

(b) $y_1 = c_n x^{n+7/3}$; $c_0 \neq 0$ and $y_2 = ky_1(x)\ln(x) + \sum_{n=0}^{\infty} c_n^* x^{n-8/3}$ where k may or may not be zero.

15. (a) $4r^2 - 10r = 0$ with roots $r_1 = 5/2$ and $r_2 = 0$

(b) $y_1 = \sum_{n=0}^{\infty} c_n x^{n+5/2}$ with $c_0 \neq 0$; $y_2 = \sum_{n=0}^{\infty} c_n^* x^n$ with $c_0^* \neq 0$

16. (a) $r^2 - 2r + 2 = 0$ with roots $r_1 = 1 + i$ and $r_2 = 1 - i$

(b) $y_1 = x^{1+i}\sum_{n=0}^{\infty} c_n x^n$ with $c_0 \neq 0$, $y_2 = x^{1-i}\sum_{n=0}^{\infty} c_n^* x^n$ with $c_0^* \neq 0$

Chapter Five - Numerical Methods

Section 5.1 Euler's Method

In Problems 1 through 6, approximate solutions were computed by Euler's method with $h = 0.2$, then $h = 0.1$ and $h = 0.05$.

1. Approximate solution of $y' = y\sin(x); y(0) = 1$ on $[0, 4]$

x_k	$h = 0.2$	$h = 0.1$	$h = 0.05$	Exact
0.0	1	1	1	1
0.2	1	1.009098344	1.011503107	1.02013342
0.4	1.03973387	1.06048863	1.07118496	1.08213832
0.6	1.12071215	1.15460844	1.17239843	1.19084648
0.8	1.24727249	1.29838437	1.32567276	1.35431161
1	1.42622019	1.50052665	1.54082744	1.58359518
1.2	1.66624478	1.77177248	1.82981081	1.89201471
1.4	1.97684583	2.12354101	2.205173	2.29339409
1.6	2.36646226	2.56550146	2.67726796	2.79882454
1.8	2.83955291	3.10178429	3.24991245	3.41167064
2.0	3.39261128	3.72595725	3.91473574	4.12121011
2.2	4.00958982	4.41563127	4.64536355	4.89640429
2.4	4.65793762	5.12853123	5.39367649	5.68251387
2.6	5.28719068	5.90260482	6.09104775	6.403782
2.8	5.83230149	6.36250556	6.65682772	6.97423289
3	6.22305187	6.73296375	7.01389293	7.31547887
3.2	6.39869129	6.85637053	7.10744867	7.37646684
3.4	6.32398766	6.70882212	6.92043335	7.14775408
3.6	6.0007799	6.30806368	6.47904791	6.66425664
3.8	5.46968635	5.70948505	5.8457521	5.99525134
4	4.80035219	4.99149302	5.10259695	5.22598669

2. Solution of $y' = x + y; y(1) = -3$ on $[1, 5]$. The exact solution is $y(x) = e^{x-1} - x - 1$.

x_k	$h = 0.2$	$h = 0.1$	$h = 0.05$	Exact
1.0	-3	-3	-3	-3
1.2	-3.4	-3.41	-3.41550625	-3.42140276
1.4	-3.84	-3.8641	-3.87745544	-3.8918247
1.6	-4.328	-4.371561	-4.39585633	-4.4221188
1.8	-4.8736001	-4.94358881	-4.98287549	-5.02554093
2.0	-5.48832001	-5.69374246	-5.65329771	-5.71828183
2.2	-6.185984	-6.3342838	-7.32012915	-7.45519997
2.4	-6.98318081	-7.19749834	-7.32012915	-7.4551997
2.6	-7.89981697	-8.19497299	-8.36494148	-8.55303243
2.8	-8.95978036	-9.35992731	-9.59181615	-9.84964746

3.0	−10.1917364	−10.7275	−11.0399887	−11.3890561
3.2	−11.6300837	−12.3402749	−12.7571503	−13.2250135
3.4	−13.3161005	−14.2497327	−14.8012697	−15.4231764
3.6	−15.2993205	−16.5181765	−17.2428083	−18.063738
3.8	−17.6391846	−19.2209936	−20.1674125	−21.2446468
4.0	−20.4070216	−22.4494023	−23.6791859	−25.0855369
4.2	−23.6884259	−26.3137768	−27.9046673	−29.7325302
4.4	−27.5861111	−30.9476699	−32.9976649	−35.3641001
4.6	−32.2233333	−36.5126806	−39.1451342	−42.1982344
4.8	−37.7479999	−43.2043435	−46.5743203	−50.5011845
5.0	−44.3375999	−51.2592556	−55.5614412	−60.5981501

3. Approximate solution of $y' = 3xy; y(0) = 5$ on $[0, 4]$

x_k	$h = 0.2$	$h = 0.1$	$h = 0.05$	Exact
0.0	5	5	5	5
0.2	5	5.15	5.22810641	5.30918274
0.4	5.6	5.90531	6.14480554	6.35624575
0.6	6.944	7.66399928	8.09129921	8.58003432
0.8	9/44384	10.9426582	11.8990307	13.0584824
1	13.9768832	17.2324981	19.4849022	22.4084454
1.2	22.3630131	29.7949892	35.4287514	43.3556883
1.4	38.4643826	56.3244477	71.3385803	94.5792316
1.6	70.774464	115.972038	158.672557	232.627372
1.8	138.717949	259.17431	388.901099	645.121012
2.0	288.533335	626.631648	1047.95246	2017.14397
2.2	634.773337	1634.25534	3097.73286	7111.2827
2.4	1472.67414	4584.73992	10024.1822	28266.6493
2.6	3593.32491	13800.0672	35439.1284	126682.333
2.8	9198.91176	44461.0564	136620.046	640137.266
3	24653.0835	152981.603	573253.363	3647081.85
3.2	69028.6339	560983.536	2613461.49	23427893.9
3.4	201563.611	2188060.18	12923807.1	169682214
3.6	612753.378	9060757.21	69209615.8	$1.38565379E+9$
3.8	1936300.68	39765851.2	400745113	$1,27581728E+10$
4	6351066, 22	184664659	$2,50522053E+9$	$1,32445611E+11$

4. Approximate solution of $y' = 2-x; y(0) = 1$ on $[0, 4]$. The exact solution is $y = (2+4x-x^2)/2$.

x_k	$h = 0.2$	$h = 0.1$	$h = 0.05$	Exact
0.0	1	1	1	1
0.2	1.4	1.39	1.385	1.38
0.4	1.76	1.74	1.73	1.72
0.6	2.08	2.05	2.035	2.28
0.8	2.36	2.32	2.3	2.5
1.0	2.6	2.55	2.525	2.68
1.2	2.8	2.74	2.71	2.68
1.4	2.96	2.89	2.855	2.82
1.6	3.08	3	2.96	2.92
1.8	3.16	3.07	3.025	2.98

2.0	3.2	3.1	3.05	3
2.2	3.2	3.09	3.035	2.98
2.4	3.16	3.04	2.98	2.92
2.6	3.08	2.95	2.885	2.82
2.8	2.96	2.82	2.75	2.68
3.0	2.8	2.65	2.575	2.5
3.2	2.6	2.44	2.36	2.28
3.4	2.36	2.19	2.105	2.02
3.6	2.08	1.9	1.81	1.72
3.8	1.76	1.57	1.475	1.38
4.0	1.4	1.2	1.1	1

5. Approximate solution of $y' = y - \cos(x); y(1) = -2$ on $[1, 5]$. The exact solution is

$$y = \left[\frac{\sin(1) - \cos(1)}{2} - 2\right] e^{x-1} + \frac{1}{2}[\cos(x) - \sin(x)].$$

x_k	$h = 0.2$	$h = 0.1$	$h = 0.05$	Exact
1.0	−2	−2	−2	−2
1.2	−2.50806046	−2.52479287	−2.53395265	−2.54372206
1.4	−3.0821441	−3.1216086	−3.14338517	−3.16674525
1.6	−3.73256636	−3.80291652	−3.84303839	−3.88424161
1.8	−4.47323972	−4.58543259	−4.6482934	−4.71647511
2	−5.32244724	−5.49105224	−5.58620527	−5.68995512
2.2	−6.30370733	−6.54791245	−6.68669139	−6.838775
2.4	−7.44674857	−7.79161134	−7.98890878	−8.20617878
2.6	−8.78861954	−9.26662206	−9.54186557	−9.84641081
2.8	−10.3749657	−11.0279477	−11.4063233	−11.826918
3.0	−12.2615144	−13.1430765	−13.6570618	−14.2309923
3.2	−14.5158187	−15.6943098	−16.3855953	−17.1609616
3.4	−17.2193235	−18.7815545	−19.7034487	−20.7420636
3.6	−20.4698286	−22.5256874	−23.7461298	−25.127167
3.8	−24.2844426	−27.0726284	−28.6779576	−30.5025422
4	−29.1031376	−32.5982807	−34.6979486	−37.0949271
4.2	−34.7930364	−39.3145364	−42.046999	−45.1801881
4.4	−41.6536915	−47.4765804	−51.0166552	−55.0939413
4.6	−49.9228433	−57.3917761	−61.9598251	−67.2445787
4.8	−59.8849815	−69.4304735	−75.3038559	−82.1292389
5	−71.8794775	−84.03149	−91.5664948	−100.35338

6. Approximate solution of $y' = x - y^2; y(0) = 4$ on $[0, 4]$.

x_k	$h = 0.2$	$h = 0.1$	$h = 0.05$
0.0	4	4	4
0.2	0.8	1.834	2.06878526
0.4	0.712	1.31731995	1.46706786
0.6	0.6906112	1.09365165	1.19624308
0.8	0.715222434	0.997119506	1.07054336
1.0	0.772913808	0.972106068	1.02550076
1.2	0.853434657	0.992035493	1.03065387
1.4	0.947764514	1.04087909	1.06818001
1.6	1.048113	1.10750278	1.12608679
1.8	1.14840483	1.18377918	1.19578843
2.0	1.2446381	1.26391494	1.27116397
2.2	1.3348133	1.34408173	1.34806934
2.4	1.41846799	1.42203937	1.42393537
2.6	1.4960577	1.49671239	1.49736677
2.8	1.56841997	1.5677814	1.56776551
3.0	1.63643173	1.63535114	1.63501891
3.2	1.70084997	1.69972016	1.6992724
3.4	1.76227185	1.76124314	1.76078252
3.6	1.82115144	1.82026104	1.8198329
3.8	1.87783293	1.87707327	1.87669254
4.0	1.93248163	1.93193214	1.9315994

7. The exact solution of $y' = y; y(0) = 1$ is $y = e^x$, so $e = y(1)$. Using Euleer's method with $h = 0.01$ to approximate $y(1)$ gives us $e \approx 2.70481383$. This approximation is too small because $y''(x) = y(x) = e^x > 0$, so the solution curve is concave up. Since Euler's method follows the tangent line at each point, the approximate values will always lie below the actual solution values.

8. The exact solution is $y(x) = \ln(x)$, so $\ln(2) = y(2)$. Using Euler's method with $h = 0.01$ to approximate $y(2)$ gives us $\ln(2) \approx 0.6956534306$. This approximation is too large because $y''(x) = -1/x^2 < 0$ for $x > 0$, so the solution curve is concave down and the tangent line at each step (on which approximate solution values are taken) lies above the actual solution.

9. The table and the graph show the effect on $y(40)$, the depth required to reach a velocity of 40 feet per second, due to varying the drag coefficient D. Approximate values of $y(40)$ were computed using Euler's method.

D	$y(40)$
0.2	225.3
0.3	238.8
0.4	254.2
0.5	272.3
0.6	293.6
0.7	319.4
0.8	351.4
0.9	392.8
1.0	448.9

10. The table and the graph show the effect of $y(40)$ due to varying the coefficient p. Approximate values of $y(40)$ were computed using Euler's method.

p	$y(40)$
1.0	260
1.05	275
1.1	294
1.15	323
1.2	369
1.25	455
1.3	719
1.319	1557

11. With $h = 0.1$ we get $y(40) \approx 342.6$ feet, so drums will likely rupture on impact.

Section 5.2 One-Step Methods

In Problems 1 through 9, the approximate solutions were computed for the indicated value of h, using the modified Euler, Taylor and RK4 methods.

1. Approximate solution of $y' = \sin(x+y); y(0) = 2$ on $[0,4]$.

$h = 0.2$

x_k	Mod Euler	Taylor	RK4
0.0	2	2	2
0.2	2.16276019	2.16596852	2.162573
0.4	2.27710203	2.28172312	2.27782433
0.6	2.33995722	2.34461272	2.34197299
0.8	2.35626076	2.36030271	2.35937518
1.0	2.33366881	2.33693853	2.33748836
1.2	2.29794712	2.28230603	2.28390814
1.4	2.20094708	2.20292281	2.20518645
1.6	2.10243707	2.10395777	2.10658823
1.8	1.98824842	1.98942164	1.99221666
2.0	1.86149654	1,86240627	1.86523474
2.2	1.72458851	1.72529835	1.72807915
2.4	1.57939294	1.57995044	1.58263585
2.6	1.42737086	1.4278115	1.43037559
2.8	1.26967441	1.27002472	1.2724554
3.0	1.10722027	1.10750021	1.10979405
3.2	0.94074422	0.940969099	0.943127937
3.4	0.77084298	0.771023935	0.773052969
3.6	0.598002896	0.598149009	0.600054304
3.8	0.422625177	0.422743336	0.424532579
4.0	0.245042591	0.245138181	0.246819036

$h = 0.1$

x_k	Mod Euler	Taylor	RK4
0.0	2	2	2
0.2	2.16260835	2.16331964	2.16257799
0.4	2.27764497	2.27864781	2.27783452
0.6	2.34149618	2.34249201	2.34198641
0.8	2.35864818	2.35950334	2.35938954
1.0	2.33660171	2.33728697	2.33750216
1.2	2.28294468	2.28347612	2.28392071
1.4	2.20420589	2.20461236	2.2.0519759
1.6	2.10562833	2.2.10593799	2.106598
1.8	1.99129888	2.99153503	2.99222519
2.0	1.86436976	1.86455048	1.8652422
2.2	1.72727096	1.72740984	1.72808569
2.4	1.58188451	1.58199169	1.58264163
2.6	1.42967893	1.42976193	1.4303807
2.8	1.27181015	1.27187458	1.27245995
3.0	1.10919644	1.10924652	1.10979812
3.2	0.942574127	0.942613018	0.94313596
3.4	0.772538925	0.772569046	0.773056001
3.6	0.599577036	0.599600237	0.60057301
3.8	0.424088544	0.424106255	0.424535308
4.0	0.246405248	0.24641858	0.24682153

$h = 0.05$

x_k	Mod Euler	Taylor	RK4
0.0	2	2	2
0.2	2.16258437	2.16275169	2.16257832
0.4	2.2777872	2.27802032	2.27783517
0.6	2.34186673	2.34209633	2.34198724
0.8	2.3592098	2.5940579	2.35939041
1.0	2.33728455	2.33744082	2.337503
1.2	2.28368518	2.2838058	2.28392147
1.4	2.20495844	2.20505023	2.20519827
1.6	2.10636421	2.10643374	2.1065986
1.8	2.99200183	2.99205451	2.99222571
2.0	1.86503179	1.86507179	1.86524266
2.2	1.72788915	1.7279196	1.72808609
2.4	1.58245892	1.58248215	1.58264198
2.6	1.43021129	1.43022904	1.43038101
2.8	1.27230303	1.27231659	1.27246022
3.0	1.10965277	1.1096631	1.10979836
3.2	0.942996879	0.943004703	0.943131815
3.4	0.7772931004	0.772936875	0.773056196
3.6	0.599941166	0.599945507	0.600057476
3.8	0.424427241	0.424430376	0.424535465
4.0	0.246720806	0.246722988	0.24682167

2. Approximate solution of $y' = y - x^2; y(0) = 2$ on $[1,5]$. The exact solution is $y = -9e^{x-1} +$

$x^2 + 2x + 2$.

$h = 0.2$

x_k	mod Euler	Taylor	RK4	Exact
1.0	−4	−4	−4	−4
1.2	−5.14200001	−5.14	−5.15260667	−5.15262483
1.4	−6.64004	−6.6356	−6.66637645	−6.66642228
1.6	−8.5900488	−8.582632	−8.63898286	−8.3906921
1.8	−11.1090595	−11.0980111	−11.1897243	−11.1898684
2.0	−14.3398526	−14.3243735	−14.464312	−14.4645365
2.2	−18.4566202	−18.4357356	−18.6407173	−18.6410523
2.4	−23.6718767	−23.6443975	−23.9363148	−23.9367997
2.6	−30.2448895	−30.2093649	−30.6166056	−30.6172918
2.8	−38.4919652	−38.4466252	−39.0058727	−39.0068272
3.0	−48.7989976	−48.7416828	−49.5001956	−49.5015049
3.2	−61.636777	−61.564853	−62.5833456	−62.5851215
3.4	−77.579668	−77.4899206	−78.846201	−78.8485875
3.6	−97.328395	−97.2169032	−99.0104605	−99.0136424
3.8	−121.737842	−121.599822	−123.957607	−123.961821
4.0	−151.850967	−151.680583	−154.764284	−154.769832
4.2	−188.94018	−188.730311	−192.745503	−192.752772
4.4	−234.557819	−234.299779	−239.50742	−239.5169
4.6	−290.59774	−290.280931	−297.011794	−297.02411
4.8	−359.370442	−358.981935	−367.654716	−367.670661
5.0	−443.69474	−443.218761	−454.362772	−454.38335

$h = 0.1$

x_k	Mod Euler	Taylor	RK4
1.0	−4	−4	−4
1.2	−5.14975125	−5.149225	−5.15262358
1.4	−6.65928727	−6.65811846	−6.66641914
1.6	−8.62581149	−8.62385809	−8.63906329
1.8	−11.1680117	−11.1651003	−11.1898585
2.0	−14.4308088	−14.426776	−14.4645211
2.2	−18.5911545	−18.5856451	−18.6410294
2.4	−23.8651117	−23.8578583	−23.9367666
2.6	−30.5164982	−30.5071155	30.617245
2.8	−38.8674425	−38.8554597	−38.006762
3.0	−49.3112712	−49.2961136	−49.5014155
3.2	−62.3282462	−62.3092121	−62.5850004
3.4	−78.5047791	−78.4810118	−78.8484247
3.6	−98.5568931	−98.5273463	−99.0134254
3.8	−123.358871	−123.322267	−123.961534
4.0	−153.978232	−153.933012	−154.7694543
4.2	−191.718447	−191.662706	−192.752277
4.4	−238.171084	−238.102497	−239.516255
4.6	−295.279488	−295.195215	−297.023271
4.8	−365.416533	−365.313107	−367.669575
5.0	−451.479554	−451.352742	−454.381949

3. Approximate solution of $y' = \cos(y) + e^{-x}; y(0) = 1$ on $[0, 4]$

$h = 0.2$

x_k	Mod Euler	Taylor	RK4
0.0	1	1	1
0.2	1.26192858	1.26213807	1.26465161
0.4	1.44915876	1.44887644	1.45389723
0.6	1.57895669	1.57813647	1.58483705
0.8	1.66583668	1.66462541	1.67216598
1.0	1.7211279	1.71969487	1.72743772
1.2	1.75344332	1.75189313	1.75944359
1.4	1.76927425	1.76770426	1.77479969
1.6	1.77349631	1.77196464	1.77846513
1.8	1.76975866	1.75941776	1.77414403
2.0	1.76077645	1.75941776	1.76458702
2.2	1.74855122	1.747302	1.75181791
2.4	1.73453923	1.73340407	1.73730594
2.6	1.71978118	1.71875959	1.7220984
2.8	1.70500301	1.70409106	1.70692335
3.0	1.69069427	1.68988581	1.69226943
3.2	1.67716896	1.67645649	1.6784478
3.4	1.66461241	1.66398777	1.66563982
3.6	1.65311732	1.65257212	1/.65393361
3.8	1.64271103	1.64223704	1.64335178
4.0	1.63337612	1.63296547	1.63387228

$h = 0.1$

x_k	Mod Euler	Taylor	RK4
0.0	1	1	1
0.2	1.26402944	1.26403233	1.26466198
0.4	1.45281544	1.45269679	1.45391187
0.6	1.58349547	1.1.58325838	1.58485267
0.8	1.67072292	1.67040421	1.67218108
1.0	1.72600007	1.7256358	1.7274517
1.2	1.75807746	1.75769492	1.75945615
1.4	1.77354268	1.77316065	1.77481079
1.6	1.77733578	1.77696658	1.7784748
1.8	1.7731483	1.77279968	1.77415235
2.0	1.7637228	1.76339926	1.76459409
2.2	1.75107799	1.75078179	1.75182385
2.4	1.73668018	1.73641198	1.73731088
2.6	1.72157515	1.72133454	1.72210246
2.8	1.7064905	1.7062763	1.70692664
3.0	1.69191512	1.69172569	1.69227208
3.2	1.67816081	1.67799425	1.6784499
3.4	1.6650987	1.66526414	1.66564147
3.6	1.65375148	1.65362453	1.65393488
3.8	1.64320934	1.64309917	1.64335274
4.0	1.63384681	1.63382381	1.63387303

$h = 0.05$

x_k	Mod Euler	Taylor	RK4
0.0	1	1	1
0.2	1.26451	1.26450553	1.26466259
0.4	1.45364883	1.45361432	1.45391273
0.6	1.58452735	1.58446508	1.58485357
0.8	1.67183176	1.67175087	1.67218196
1.0	1.72710406	1.7270131	1.72745248
1.2	1.75912611	1.75903146	1.75945687
1.4	1.77450732	1.77441334	1.77481143
1.6	1.77820233	1.77811186	1.77847536
1.8	1.77391225	1.77382707	1.77415282
2.0	1.76438583	1.76430695	1.76459449
2.2	1.75164565	1.75157357	1.75182419
2.4	1.73716027	1.7370951	1.73731116
2.6	1.72197661	1.72191823	1.72210269
2.8	1.70682263	1.70677071	1.70692683
3.0	1.69218701	1.69214114	1.69227223
3.2	1.67838106	1.67834077	1.67845002
3.4	1.66558637	1.66555114	1.66564156
3.6	1.6538913	1.65386063	1.65393495
3.8	1.64331871	1.64329212	1.64335279
4.0	1.63384681	1.63382382	1.63387303

4. Approximate solution of $y' = y^3 - 2xy; y(3) = 2$ on $[3, 7]$.

$h = 0.2$

x_k	Mod Euler	Taylor	RK4
3.0	2	2	2
3.2	0.8352	0.64	0.927007472
3.4	0.370657996	0.244141523	0.281610758
3.6	0.197852483	0.1231716	0.0797232508
3.8	0.114751596	0.0678810294	0.0218981447
4.0	0.071557123	0,0402752591	$5.92711033E-03$
4.2	0.0480268873	0.0257474268	$1.60552109E-03$
4.4	0.0347909252	0.0177885656	$4.42481887E-04$
4.6	0.0272685199	0.0133172411	$1.26188752E-04$
4.8	0.0231635246	0.018229877	$3.78589406E-05$

5.0	0.0213450684	$9.55814261E-03$	$1.21355052E-05$
5.2	0.0213430845	$9.17529297E-03$	$4.21329018E-06$
5.4	0.0231509167	$9.57117744E-03$	$1.60328477E-06$
5.6	0.0272227469	0.0108416476	$6.75106871E-07$
5.8	0.0346662836	0.0133213296	$3.16918711E-07$
6.0	0.0477464063	0.017731633	$1.66774625E-07$
6.2	0.0710199197	0.0255293145	$9.87576172E-08$
6.4	0.113881513	0.0396900127	$6.59626297E-08$
6.6	0.196396857	0.0665006996	$4.97544678E-08$
6.8	0.36274873	0.1197688	$4.23925682E-08$
7.0	0.70960326	0.230731523	$4.07825998E-08$

$h = 0.1$

x_k	Mod Euler	Taylor	RK4
3	2	2	2
3.2	0.904305462	0.845628172	0.914284078
3.4	0.279892401	0.248496901	0.260933358
3.6	0.0819935333	0.0715668658	0.0649662457
3.8	0.0230404273	0.0.0198269382	0.014894893
4.0	$6.23828219E-03$	$5.28890271E-03$	$3.15911106E-03$
4.2	$1.6362622E-03$	$1.36538795E-03$	$6.20382675E-04$
4.4	$4.18193702E-04$	$3.43121118E-04$	$1.12894506E-04$
4.6	$1.04778096E-04$	$8.4446909E-05$	$1.90555447E-05$
4.8	$2.58967132E-05$	$2.04830842E-05$	$2.98678483E-06$
5.0	$6.35431692E-06$	$4.9280285E-06$	$4.35324023E-07$
5.2	$1.55791012E-06$	$1.18370259E-06$	$5.90943245E-08$
5.4	$3.84120812E-07$	$2.85716734E-07$	$7.48550889E-09$
5.6	$9.58570797E-08$	$6.9753169E-08$	$8.86729294E--10$
5.8	$2.43635531E-08$	$1.73337015E-08$	$9.84815667E-11$
6.0	$6.34558634E-09$	$4.41172795E-09$	$1.02843249E-11$
6.2	$1.70363003E-09$	$1.15694036E-09$	$1.01319557E-12$
6.4	$4.74124499E-10$	$3.1439558E-10$	$9.4522947E-14$
6.6	$1.37509096E-10$	$8.90123053E-11$	$8.38569922E-15$
6.8	$4.17690897E-11$	$2.63894211E-11$	$7.10791162E-16$
7.0	$1.33496752E-11$	$8.23108632E-12$	$5.78635946E-17$

$h = 0.05$

x_k	Mod Euler	Taylor	RK4
3	2	2	2
3.2	0.911797601	0.896832109	0.913321781
3.4	0.264080584	0.256884168	0.259941336
3.6	0.0676674935	0.0656024585	0.0644117662
3.8	0.0160650959	0.0155328476	0.0146711175
4.0	$3.54890589E-03$	$3.42162659E-03$	$3.08425981E-03$
4.2	$7.30400741E-04$	$7.0207723E-04$	$5.98611489E-04$
4.4	$1.40211725E-04$	$1.34339863E-04$	$1.07266492E-04$
4.6	$2.51360145E-05$	$2.40006424E-05$	$1.77469601E-05$
4.8	$4.21373054E-06$	$4.00872133E-06$	$2.71108538E-06$

5.0	$6.61458233E-07$	$6.26843076E-07$	$3.82422355E-07$
5.2	$9.73753501E-08$	$9.10920738E-08$	$4.98137453E-08$
5.4	$1.34645363E-08$	$1.26528438E-08$	$5.99224883E-09$
5.6	$1.75167733E-09$	$1.63858906E-09$	$6.65730408E-10$
5.8	$2.14784002E-10$	$1.99955812E-10$	$6.83144938E-11$
6.0	$2.48679565E-11$	$2.30347607E-11$	$6.47554821E-12$
6.2	$2.72404622E-12$	$2.50994066E-12$	$5.67072592E-13$
6.4	$2.82886027E-13$	$2.59213972E-13$	$4.58833263E-14$
6.6	$2.79098825E-14$	$2.54269499E-14$	$2.43074215E-15$
6.8	$2.62192862E-15$	$2.37430736E-15$	$2.37087613E-16$
7.0	$2.35074951E-16$	$2.11539743E-16$	$1,51459485E-17$

5. Approximate solution of $y' = -y + e^{-x}; y(0) = 4$ on $[0, 4]$.

$h = 0.2$

x_k	Mod Euler	Taylor	RK4	Exact
0.0	4	4	4	4
0.2	3.44096748	3.44	3.34867474	3.43866916
0.4	2.95338237	2.95179692	2.94941776	2.9494082
0.6	2.52967327	2.52772468	2.52454578	2.52453353
0.8	2.16267291	2.1605441	2.15679297	2.15677903
1.0	1.84571914	1.8435388	1.83941205	1.83939721
1.2	1.57270632	1.57056252	1.56622506	1.5662099
1.4	1.33810166	1.33605234	1.33163863	1.33162361
1.6	1.13693745	1.13501844	1.13063507	1.1306205
1.8	0.964787486	0.96301856	0.958747437	0.958733552
2.0	0.817733484	0.816123042	0.812024757	0.812011699
2.2	0.692326037	0.690874539	0.686991726	0.686979582
2.4	0.585543056	0.584245628	0.580606091	0.580594901
2.6	0.494747947	0.493596287	0.490215845	0.490205616
2.8	0.417648947	0.416632728	0.413517713	0.413508426
3.0	0.352260579	0.351368447	0.348517861	0.348509479
3.2	0.296867774	0.296088057	0.293495395	0.293487869
3.4	0.249992964	0.24931416	0.246968924	0.246962198
3.6	0.210366242	0.209777334	0.207666276	0.27660291
3.8	0.176898549	0.17638921	0.174497327	0.174492021
4.0	0.148657777	0.148218475	0.1465298	0.146525111

$h = 0.1$

x_k	Mod Euler	Taylor	RK4
0.0	4	4	4
0.2	3.43920787	3.43898537	3.43866949
0.4	2.95033866	2.94997425	2.94940876
0.6	2.5257356	2.52528799	2.52453424
0.8	2.1581561	2.15766738	2.15677984
1.0	1.84087289	1.84037264	1.83939807
1.2	1.56772495	1.56723338	1.56621078
1.4	1.33313308	1.33266345	1.33162448
1.6	1.13209122	1.13165172	1.13062135
1.8	0.960142926	0.959737036	0.958734358

2.0	0.813341791	0.812973396	0.812012458
2.2	0.688221511	0.687889673	0.686980287
2.4	0.581743491	0.581447053	0.58059555
2.6	0.491259275	0.4909963	0.49020621
2.8	0.414468233	0.414236234	0.413508964
3.0	0.349378445	0.349174974	0.348509965
3.2	0.294270343	0.294092618	0.293488305
3.4	0.247663405	0.247508773	0.246962588
3.6	0.208285963	0.20815189	0.207660638
3.8	0.175048125	0.174932237	0.174492328
4.0	0.14701764	0.146917746	0.146525383

$h = 0.05$

x_k	Mod Euler	Taylor	RK4
0.0	4	4	4
0.2	3.43879955	3.4387462	3.43866918
0.4	2.9496331	2.94954594	2.94940824
0.6	2.52482424	2.52471694	2.52453357
0.8	2.15711194	2.15699481	2.15677908
1.0	1.83975385	1.83963397	1.83939726
1.2	1.56657595	1.56645816	1.56620996
1.4	1.3319882	1.33187568	1.33162366
1.6	1.13097564	1.13087036	1.13062055
1.8	0.959073547	0.95897657	0.958733602
2.0	0.812332718	0.812244494	0.812011746
2.2	0.687279252	0.687199794	0.686979625
2.4	0.580871987	0.580801015	0.580594941
2.6	0.490459746	0.490396795	0.490205653
2.8	0.413739872	0.413684365	0.413509508
3.0	0.348718977	0.348670284	0.348509508
3.2	0.293676479	0.293633952	0.293487895
3.4	0.247131187	0.247094192	0.246962221
3.6	0.207811048	0.207778977	0.207660312
3.8	0.174625992	0.174598274	0.174492309
4.0	0.146643746	0.146619857	0.146525127

6. Approximate solution of $y' = \sec(1/y) - xy^2; y(\pi/4) = 1$ on $[\pi/4, 4 + \pi/4]$.

$h = 0.2$

x_k	Mod Euler	Taylor	RK4
0.78	1	1	1
0.98	1.10643889	1.09819165	1.12861758
1.18	1.13133284	1.12796467	1.15326609
1.38	1.11242798	1.11108997	1.12873771
1.58	1.07670951	1.07567835	1.08670113
1.78	1.03881192	1.03741125	1.04361713
1.98	1.00426366	1.00261973	1.00583831
2.18	0.974614076	0.972818282	0.974432628
2.38	0.949574406	0.947647407	0.948494123
2.58	0.928416471	0.926287198	0.926795228

2.78	0.910474273	0.907963997	0.908332291
2.98	0.895314578	0.8920663	0.892384814
3.18	0.882864974	0.878139331	0.878450533
3.38	0.873740995	0.865863694	0.86186968
3.58	0.870256999	0.855063851	0.855394723
3.78	0.87498838	0.845815738	0.846088608
3.98	0.921710205	0.838900598	0.838882738
4.18	1.04485648	0.837554717	0.837128055
4.38	1.33250821	0.8532474	0.86979209
4.58	2.53839586	0.911278323	0.652966304
4.78	2.24842891	1.00809716	−22.0813377

$h = 0.1$

x_k	Mod Euler	Taylor	RK4
0.78	1	1	1
0.98	1.12460026	1.12370845	1.12896304
1.18	1.14972786	1.14946367	1.15378701
1.38	1.1264371	1.1262473	1.1292014
1.58	1.08555129	1.08527232	1.08697272
1.78	1.04320034	1.04286594	1.04366394
1.98	1.00574444	1.00541846	1.0057016
2.18	0.974409415	0.974127074	0974182134
2.38	0.948437533	0.948204534	0.948187488
2.58	0.926679173	0.926487132	0.92646048
2.78	0.90815266	0.90799054	0.9079104
2.98	0.892132825	0.891991339	0.89198005
3.18	0.878099907	0.877972054	0.877966797
3.38	0.865676389	0.865556786	0.865555607
3.58	0.854580957	0.854465038	0.854466656
3.78	0.844598715	0.84448198	0.844495602
3.98	0.83556224	0.835439409	0.835444523
4.18	0.827339492	0.827203217	0.0.827209445
4.38	0.819826502	0.819664889	0.819671853
4.58	0.81294455	0.812735609	0.812742736
4.78	0.806644616	0.806342237	0.806348304

$h = 0.05$

x_k	Mod Euler	Taylor	RK4
0.78	1	1	1
0.98	1.12802967	1.12791221	1.12897598
1.18	1.15293644	1.1529082	1.1538066
1.38	1.12863964	1.12860199	1.12922007
1.58	1.08669708	1.08663327	1.08698233
1.78	1.04358141	1.04350758	1.04366376
1.98	1.0057125	1.00564467	1.00569426
2.18	0.974220801	0.974165659	0.974171102
2.38	0.948224581	0.948181835	0.948175352
2.58	0.926488396	0.926455222	0.926448449

2.78	0.907990438	0.907964067	0.907959373
2.98	0.891993128	0.891971558	0.891968617
3.18	0.877975302	0.877957205	0.877955391
3.38	0.865560663	0.865545156	0.86554404
3.58	0.854468957	0.85445543	0.85445476
3.78	0.0.844485567	0.844473584	0.844473215
3.98	0.835442383	0.83543162	0.835431462
4.18	0.827205283	0.82719549	0.827195488
4.38	0.81966561	0.819656593	0.819656709
4.58	0.812734187	0.81275787	0.812725995
4.78	0.806336992	0.806329075	0.806329358

9. Let $f(x_k, y_k) = f_k$. By Euler's method with step size h,

$$y_{k+1} = y_k + hy'(x_k)$$

$$= y_k + hf(x_k, y_k) = y_k + hf_k.$$

Now use $y_{k+1} = y_k + hf_k$ from Euler's method to get

$$f_k = \frac{1}{2}\left(f_k + f_{k+1}\right) = \frac{1}{2}\left(f_k + f(x_{k+1}, y_{k+1})\right) = \frac{1}{2}\left(f_k + f(x_{k+1}, y_k + hf_k)\right).$$

Then

$$y_{k+1} = y_k + \frac{1}{2}h\left(f_k + f(x_{k+1}, y_k + hf_k)\right).$$

10. Approximate solution of $y' = 1 - y; y(0) = 2$ on $[0, 4]$. The exact solution is $y = 1 + e^{-x}$.
$h = 0.2$

x_k	Euler	Mod Euler	Imp Euler	Exact
0.0	2	2	2	2
0.2	1.8	1.82	1.82	1.81873075
0.4	1.64	1.6724	1.6724	1.67032005
0.6	1.512	1.551368	1.551368	1.54881164
0.8	1.4096	1.45212176	1.45212176	1.44932896
1.0	1.32768	1.37073984	1.37073984	1.36787944
1.2	1.262144	1.30400667	1.30400667	1.30119421
1.4	1.2097152	1.24928547	1.24928547	1.24659696
1.6	1.16777216	1.20441409	1.20441409	1.20189652
1.8	1.13421773	1.16761955	1.16761955	1.16529889
2.0	1.10737418	1.13744803	1.13744803	1.13533528
2.2	1.08589935	1.11270739	1.11270739	1.11080316
2.4	1.06871948	1.09242006	1.09242006	1.09071795
2.6	1.05497558	1.07578445	1.07578445	1.07427358
2.8	1.04398047	1.06214325	1.06214325	1.06081006
3.0	1.03518437	1.05095746	1.05095746	1.04978707
3.2	1.0281475	1.04178512	1.04178512	1.0407622
3.4	1.022518	1.0342638	1.0342638	1.03337327
3.6	1.0180144	1.02809631	1.02809631	1.02732372
3.8	1.01441152	1.02303898	1.02303898	1.02237077
4.0	1.01152922	1.01889196	1.01889196	1.01831564

$h = 0.1$

x_k	Euler	Mod Euler	Imp Euler
0.0	2	2	2
0.2	1.81	1.819025	1.819025
0.4	1.6561	1.67080195	1.67080195
0.6	1.531441	1.54940357	1.54940357
0.8	1.43046721	1.44997526	1.44997526
1.0	1.34867844	1.36854099	1.36854099
1.2	1.28242954	1.30184428	1.30184428
1.4	1.22876792	1.24721801	1.24721801
1.6	1.18530202	1.20247773	1.20247773
1.8	1.15009464	1.16583432	1.16583432
2.0	1.12157665	1.13582246	1.13582246
2.2	1.09847709	1.11124199	1.11124199
2.4	1.07976644	1.09110997	1.09110997
2.6	1.06461082	1.07462134	1.07462134
2.8	1.05233476	1.06111675	1.06111675
3.0	1.04239116	1.05005614	1.05005614
3.2	1.03433684	1.04099723	1.04099723
3.4	1.02781284	1.03357776	1.03357776
3.6	1.0225284	1.02750102	1.02750102
3.8	1.018248	1.02252403	1.02252403
4.0	1.01478088	1.01844774	1.01844774

$h = 0.05$

x_k	Euler	Mod Euler	Imp Euler
0.0	2	2	2
0.2	1.81450625	1.81880159	1.81880159
0.4	1.66342043	1.67043605	1.67043605
0.6	1.54036009	1.54895411	1.54895411
0.8	1.44012667	1.4494845	1.4494845
1.0	1.35848592	1.30135061	1.30135061
1.2	1.29198902	1.30135061	1.30135061
1.4	1.23782689	1.24674636	1.24674636
1.6	1.19371148	1.20203631	1.20203631
1.8	1.15777921	1.16542766	1.16542766
2.0	1.12851216	1.13545243	1.13545243
2.2	1.10467396	1.11090866	1.11090866
2.4	1.08525759	1.09081219	1.09081219
2.6	1.06944284	1.07435717	1.07435717
2.8	1.05656163	1.06088377	1.06088377
3.0	1.0460698	1.04985172	1.04985172
3.2	1.03752414	1.04081867	1.04081867
3.4	1.03056365	1.03342239	1.03342239
3.6	1.02489428	1.02736631	1.02736631
3.8	1.02027655	1.02240758	1.02240758
4.0	1.01651537	1.01834736	1.01834736

11. Approximate solution of $y' = -\frac{y}{x} + x; y(1) = 1$ on $[1, 5]$

$h = 0.2$

x_k	Euler	Mod Euler	Imp Euler	Exact
1.0	1	1	1	1
1.2	1	1.03818182	1.03666667	1.03555556
1.4	1.07333333	1.13331002	1.13142857	1.12952381
1.6	1.2	1.27432878	1.2725	1.27
1.8	1.37	1.45495428	1.45333333	1.45037037
2.0	1.57777778	1.67136234	1.67	1.66666667
2.2	1.82	1.92109622	1.92	1.91636364
2.4	2.09454546	2.20250721	2.20166667	1.19777778
2.6	2.4	2.51444832	2.51384616	2.50974359
2.8	2.73538462	2.8560973	2.85571429	2.85142857
3.0	3.1	3.22684945	3.22666667	3.22222222
3.2	3.49333334	3.62625024	3.62625	3.62166667
3.4	3.915	4.05395145	4.05411765	4.04941177
3.6	4.36470588	4.50968188	4.51000001	4.50518519
3.8	4.84222223	4.99322713	4.99368422	4.98877193
4.0	5.34736843	5.50441551	5.50500001	5.5
4.2	5.88000001	6.04310794	6.04380953	6.03873016
4.4	6.44000001	6.60919053	6.61000001	6.60484849
4.6	7.02727274	7.20256912	7.20347827	7.19826087
4.8	7.64173914	7.82316522	7.82416667	7.81888889
5.0	8.28333334	8.47091287	8.472	8.46666667

$h = 0.1$

x_k	Euler	Mod Euler	Imp Euler
1.0	1	1	1
1.2	1.01909091	1.03616507	1.03583333
1.4	1.10307692	1.13040682	1.13
1.6	1.23666667	1.27101358	1.270625
1.8	1.41176471	1.45144731	1.45111111
2.0	1.62368421	1.66777308	1.6675
2.2	1.86952381	1.9174816	1.91727273
2.4	2.1473913	2.19889755	2.19875
2.6	2.456	2.51085981	2.51076923
2.8	2.79444445	2.85253832	2.8525
3.0	3.16206897	3.223324	3.22333334
3.2	3.5583871	3.62275978	3.6228125
3.4	3.9830303	4.050496	4.05058824
3.6	4.43571428	4.50626061	4.50638889
3.8	4.91621621	4.98983879	4.99
4.0	5.42435897	5.50105862	5.50125
4.2	5.95999999	6.03978091	6.03999999
4.4	6.52302235	6.6.60589176	6.60613636
4.6	7.11333332	7.19929706	7.19956521
3.8	7.73085105	7.81991837	7.82020832
5.0	8.37551019	8.46768981	8.46799999

$h = 0.05$

x_k	Euler	Mod Euler	Imp Euler
1.0	1	1	1
1.2	1.0276087	1.0357021	1.035625
1.4	1.1166667	1.12973676	1.12964286
1.6	1.25370968	1.27024501	1.27015625
1.8	1.43142857	1.4506312	1.45055555
2.0	1.64551282	1.66693507	1.666875
2.2	1.89325581	1.91663523	1.91659091
2.4	2.17287234	1.19805015	2.19802083
2.6	2.48313725	2.5100154	2.50999999
2.8	2.82318181	2.85169908	2.85169642
3.0	3.19237288	3.22249103	2.22249999
3.2	3.59023809	3.62193359	3.62195321
3.4	4.0164179	4.04967673	4.04970587
3.6	4.47063379	4.50544819	4.5054861
3.8	4.95266665	4.98903301	4.98907893
4.0	5.46234176	5.50025924	5.50031248
4.2	5.99951805	6.03898763	6.0390476
4.4	6.56408044	6.60510426	6.60517044
4.6	7.15593404	7.19851504	7.19858694
4.8	7.77499998	7.81914155	7.81921873
5.0	8.42121211	8.46691789	8.46699998

12. Approximate solution of $y' = y - e^x; y(-1) = 4$ on $[-1, 3]$. The exact solution is $y = (4e - 1 - x)e^x$.

$h = 0.2$

x_k	Euler	Mod Euler	Imp Euler	Exact
−1.0	4	4	4	4
−0.8	4.72642411	4.79132848	4.79092157	4.79574524
−0.6	5.58184314	5.73711711	5.73612368	5.74777414
−0.4	6.58844944	6.8670005	6.86518148	6.88628318
−0.2	7.77207532	8.21617057	8.21320993	8.24717912
0.0	9.16274423	9.826386	9.82186843	9.87312732
0.2	10.7952931	11.7471567	11.7405392	11.8147844
0.4	12.7100711	14.0371314	14.027707	14.1322453
0.6	14.9537204	16.7657196	16.7525717	16.8967396
0.8	17.5800408	20.0149849	19.9969292	20.1926607

1.0	20.5800408	23.8818502	23.8573605	24.1196607
1.2	24.2374725	28.2806584	28.4477743	28.7957968
1.4	28.4209436	33.9461415	33.9023506	34.3602256
1.6	33.2940923	40.4368508	40.3789405	40.9770679
1.8	38.9623043	48.1391079	48.0629788	48.8395742
2.0	45.5448357	57.2715398	57.1719708	58.1749794
2.2	53.1759917	68.0902635	67.9606163	69.2500776
2.4	62.0061873	80.8947847	80.7266326	82.3776006
2.6	72.2027894	96.034675	95.8173368	97.923481
2.8	83.9505977	113.917082	113.637038	116.31508
3.0	97.4517903	135.015119	134.655275	138.050452

$h = 0.1$

x_k	Euler	Mod Euler	Imp Euler
−1.0	4	4	4
−0.8	4.7588763	4.79455824	4.79445137
−0.6	5.6591556	5.74490893	5.74464791
−0.4	6.72655593	6.88109682	6.88061868
−0.2	7.99131565	8.23883551	8.23805697
0.0	9.48894781	9.86054529	9.85935683
0.2	11.2611098	11.7965726	11.7948309
0.4	13.3566026	14.1066209	14.1041395
0.6	15.8325163	16.8614271	16.8579639
0.8	18.7555364	20.1447208	20.1399628
1.0	22.2034293	24.0555116	24.0490555
1.2	26.2667218	28.710752	28.7020792
1.4	31.0505909	34.2484304	34,.2368762
1.6	36.676974	40.8311551	40.8158692
1.8	43.2869103	48.6502943	48.630191
2.0	51.0431108	57.930743	57.904439
2.2	60.1327509	68.9363912	68.9021268
2.4	70.7704589	81.9763712	81.9319118
2.6	83.2014564	97.4121605	97.3546722
2.8	97.704778	115.665613	115.591507
3.0	114.596456	137.227982	137.13272

$h = 0.05$

x_k	Euler	Mod Euler	Imp Euler
−1	4	4	4
−0.8	4.77669693	4.79543751	4.79541012
−0.6	5.70188503	5.74703124	5.74696434
−0.4	6.80338236	6.88493831	6.88481574
−0.2	8.11407589	8.24501533	8.24481573
0.0	9.67280884	9.86986402	9.86955928
0.2	11.5254138	11.8100604	11.8096138
0.4	13.7259134	14.1255979	14.1249615
0.6	16.3379111	16.887578	16.8866896
0.8	19.4362003	20.1801886	20.178968

1.0	23.1086182	24.103014	24.1013575
1.2	27.4581759	28.7737251	28.7714998
1.4	32.6054974	34.3312082	34.3282431
1.6	3.6915995	40.9391907	40.9352675
1.8	45.8810461	48.790434	48.7852738
2.0	54.3655083	58.1115643	58.1048118
2.2	64.367755	69.1686207	69.1598237
2.4	76.1460949	82.2733988	82.2619829
2.6	89.9992753	97.7906722	97.7759092
2.8	106.271825	116.14637	116.127337
3.0	125.359801	137.836776	137.812307

Section 5.3 Multistep Methods

In Problems 1 through 5, approximate solutions wθere computed using $h = 0.1$ and the Taylor, modified Euler, RK4 and Adams-Bashforth-Moulton methods.

1. Approximate solution of $y' = 4y^2 - x; y(3) = 0$ on $[3, 7]$.

$h = 0.02$

x_k	Taylor	Mod Euler	RK4	Adams-Moulton
3	0	0	0	0
3.2	−0.62	−0.548	−0.534175479	−0.534175749
3.4	−0.80756992	−0.760542319	−0.800211467	−0.800211467
3.6	−0.883586757	−0.856173691	−0.901774383	−0.901774383
3.8	−0.931557148	−0.914689066	−0.94958559	−0.887947739
4.0	−0.96831005	−0.95736816	−0.981533021	−0.908078807
4.2	−0.999555229	−0.992105032	−1.00860355	−1.0359039
4.4	−1.02771202	−1.02240058	−1.03388159	−1.12481081
4.6	−1.05394433	−1.04998501	−1.05824251	−1.05377481
4.8	−1.07886314	−1.07578031	−1.08195365	−0.901642354
5.0	−1.10281209	−1.10030768	−1.1051143	−0.97490577
5.2	−1.12599657	−1.12387702	−1.12777544	−1.26839477
5.4	−1.14854662	−1.14668128	−1.14997251	−1.34044722
5.6	−1.17054911	−1.16884579	−1.17173458	−0.965939027
5.8	−1.19206515	−1.19045507	−1.19308708	−0.26900074
6.0	−1.2131397	−1.21156783	−1.21405284	−0.895645065
6.2	−1.23380713	−1.23222554	−1.23465257	−1.57930763
6.4	−1.2540944	−1.25245726	−1.25490513	−1.39055949
6.6	−1.274023	−1.27228215	−1.27482779	0.996892302
6.8	−1.29360996	−1.29171044	−1.29443644	0.125659833
7.0	−1.31286829	−1.3107431	−1.31374567	−0.631007544

$h = 0.1$

x_k	Taylor	Mod Euler	RK4	Ad-Bash-Moult
3.0	0	0	0	0
3.2	−0.54950962	−0.535839146	−0.53549231	−0.535492931
3.4	−0.806871572	−0.798461491	−0.806778016	−0.80455223
3.6	−0903569977	−0.900545864	−0.907274543	−0.905024282
3, 8	−0.950052097	−0.949091031	−0.952253424	−0.952144216
4.0	−0.981689238	−0.981381742	−0.982583	−0.982752774
4.2	−1.00868179	−1.00856977	−1.00898001	−1.00908338
4.4	−1.03393561	−1.03388121	−1.03403163	−1.03408442
4.6	−1.05828751	−1.05825061	−1.05831574	−1.05832243
4.8	−1.08199512	−1.0819642	−1.08200303	−1.0820113
5.0	−1.10515444	−1.10512619	−1.10515651	−1.10515675
5.2	−1.1278152	−1.1277886	−1.12781558	−1.12781814
5.4	−1.1500123	−1.14998697	−1.15001214	−1.15001326
5.6	−1.17177457	−1.17175035	−1.17177421	−1.17177585
5.8	−1.19312738	−1.19310414	−1.1931269	−1.19312833
6.0	−1.21409353	−1.21407118	−1.21409298	−1.21409441
6.2	−1.23469371	−1.23467218	−1.23469309	−1.23469455
6.4	−1.25494679	−1.25492601	−1.25494612	−1.25494751
6.6	−1.27487005	−1.27487995	−1.27486934	−1.26487076
6.8	−1.29447936	−1.29445989	−1.29447861	−1.29447997
7.0	−1.31378934	−1.31377045	−1.31378855	−1.31378993

$h = 0.05$

x_k	Taylor	Mod Euler	RK4	Ad-Moult
3	0	0	0	0
3.2	−0.538575012	−0.535559414	−0.535607221	−0.535632364
3.4	−0.806970999	−0.80527647	−0.807053647	−0.806944105
3.6	−0.906708736	−0.906180366	−0.907465216	−0.907432876
3.8	−0.951937066	−0.951795773	−0.952334311	−0.95234005
4.0	−0.982468487	−0.982429582	−0.982610954	−0.982616899
4.2	−1.00895536	−1.00894161	−1.00899824	−1.00900073
4.4	−1.03402352	−1.0340159	−1.03403522	−1.03403605
4.6	−1.0583147	−1.05830873	−1.03401777	−1.05831806
4.8	−1.08200379	−1.08199844	−1.08200466	−1.08200479
5.0	−1.10515769	−1.10515271	−1.10515802	−1.10515811
5.2	−1.12781685	−1.12781217	−1.12781704	−1.12781712
5.4	−1.15001344	−1.15000902	−1.15001358	−1.15001365
5.6	−1.17177551	−1.17177132	−1.17177562	−1.1717757
5.8	−1.19131281	−1.19312423	−1.1931283	−1.19312837
6.0	−1.21409428	−1.21409049	−1.21409435	−1.21409442
6.2	−1.23469439	−1.23469078	−1.23469446	−1.23469452
6.4	−1.2549742	−1.25494397	−1.25494747	−1.25494754
6.6	−1.27487063	−1.27487333	−1.27487067	−1.27487074
6.8	−1.29447989	−1.29447673	−1.29447993	−1.29447999
7.0	−1.31378983	−1.31378679	−1.31378985	−1.31378992

2. Approximate solution of $y' = x\sin(y0 - x^2; y(1) = -3$ on $[1, 5]$.

$h = 0.2$

x_k	Taylor	Mod Euler	RK4	Ad-Moult
1	-3	-3	-3	-3
1.2	-3.2484524	-3.24804498	-3.24651073	-3.24651073
1.4	-3.52541303	-3.52482278	-3.52248135	-3.52248135
1.6	-3.82379739	-3.82397069	-3.82158019	-3.82158019
1.8	-4.14689708	-4.14936572	-4.14743315	-4.14769638
2.0	-4.51287072	-4.52007961	-4.51936538	-4.52004327
2.2	-4.96503229	-4.9821068	-4.98602187	-4.98770222
2.4	-5.60543892	-5.6459567	-5.67070095	-5.67345478
2.6	-6.69902382	-6.7946522	-6.8918032	-6.88206844
2.8	-8.744684	-8.77504425	-8.80582348	-8.66276891
3.0	-10.3713991	-10.2836162	-10.3721013	-10.4143115
3.2	-11.5580734	-11.5860896	-11.7174168	-11.6365956
3.4	-13.4328755	-13.6187121	-13.962184	-13.7785296
3.6	-17.0373894	-16.4855869	-16.5802825	-16.622165
3.8	-18.8919556	-18.5066385	-18.7295178	-18.7048922
4.0	-23.0737015	-22.1935066	-22.1915762	-21.5444148
4.2	-25.2416611	-24.7448582	-24.9313906	-25.0920224
4.4	-30.54225	-29.249671	-29.0071697	-28.805366
4.6	-34.8034179	-32.7955652	-32.6932086	-32.8285012
4.8	-37.2023547	-36.5355499	-36.9504256	-38.2605886
5.0	-43.2867464	-41.9686528	-42.0675946	-43.0848163

$h = 0.1$

x_k	Taylor	Mod Euler	RK4	Ad-Moult
1	-3	-3	-3	-3
1.2	-3.24695865	-3.24685213	-3.24650426	-3.24650426
1.4	-3.52311817	-3.52299655	-3.52246816	-3.52247584
1.6	-3.82198866	-3.82210566	-3.82156093	-3.82157796
1.8	-4.14710675	-4.1478628	-4.14741057	-4.14743901
2.0	-4.5174237	-4.51951221	-4.51934832	-4.51941407
2.2	-4.9798868	-4.98492708	-4.98601764	-4.9861958
2.4	-5.65082604	-5.6634307	-5.67069388	-5.67108192
2.6	-6.836927	-6.86457152	-6.89318249	-6.89188775
2.8	-8.80968697	-8.79427022	-8.81009295	-8.80884613
3.0	-10.3662565	-10.3510751	-10.3738256	-10.3832545
3.2	-11.6682433	-11.6846368	-11.721474	-11.7321468
3.4	-13.8413497	-13.8896238	-13.9812302	-13.9782572
3.6	-16.5682156	-16.5240231	-16.5849961	-16.6316291
3.8	-18.5854051	-18.627576	-18.7457093	-18.8009969
4.0	-22.2363038	-22.1117617	-22.2076657	-11.2530543
4.2	-24.7749149	-24.7958629	-24.97225	-24.07476
4.4	-29.0538292	-28.8336633	-28.996905	-28.1084926
4.6	-32.3838368	-32.3442329	-32.6923004	-32.6440587
3.8	-36.8616443	-36.6820651	-36.9104519	-36.8863821
5.0	-42.1817796	-41.6452747	-41.8812336	-41.9460734

$h = 0.05$

x_k	Taylor	Mod Euler	RK4	Ad-Moult
1	−3	−3	−3	−3
1.2	−3.24661362	−3.24658684	−3.24650389	−3.24650411
1.4	−3.52262005	−3.52259319	−3.52246739	−3.52246823
1.6	−3.8216521	−3.82169028	−3.82155975	−3.82156092
1.8	−4.14731389	−4.14751947	−4.14740913	−4.14741102
2.0	−4.51883051	−4.51938849	−4.51934704	−4.51935173
2.2	−4.9843705	−4.98573519	−4.98601648	−4.98603074
2.4	−5.66523559	−5.66873777	−5.67068973	−5.67072651
2.6	−6.87847108	−6.88554907	−6.8932823	−6.89315282
2.8	−8.80995039	−8.80539974	−8.810358	−8.81104726
3.0	−10.3708626	−10.3680212	−10.3739986	−10.3741634
3.2	−11.7064388	−11.7119768	−11.7217732	−11.72199
3.4	−13.9501887	−13.9587611	−13.9826687	−13.9815364
3.6	−16.5768573	−16.5690402	−16.5857689	−16.5877236
3.8	−18.6986359	−18.7140276	−18.7473156	−18.7490278
4.0	−22.1997616	−22.1795489	−22.2103734	−22.2295699
4.2	−24.9049566	−24.9233028	−24.9760805	−24.9831371
4.4	−28.9761177	−28.9531288	−29.0023129	−29.0207174
4.6	−32.5817023	−32.6070501	−32.7071482	−32.7218566
4.8	−36.8277714	−36.8367133	−36.9211694	−36.951908
5.0	−41.8338787	−41.807109	−41.8937799	−41,9423929

3. Approximate solution of $y' = x^2 + 4y; y(0) = -2$ on $[0, 4]$

$h = 0.2$

x_k	Taylor	Mod Euler	RK4	Ad-Moult	Exact
0.0	−2	−2	−2	−2	−2
0.2	−4.24	−4.238	−4.44149333	−4.44414933	−4.4477837
0.4	−8.96960001	−8.96336	−9.84468412	−9.84468412	−9.87253259
0.6	−18.954752	−18.9395232	−21.8055033	−21.8055033	−21.8981285
0.8	−40.0592743	−40.0249892	−48.3155479	−48.1843246	−48.5896688
1.0	−84.7144614	−84.6397771	−107.135343	−106.51773	−107.896358
1.2	−179.274658	−179.114327	−237.73588	−235.652355	−239.764885
1.4	−379.611075	−379.269174	−527.837489	−521.638006	−533.09824
1.6	−804.17068	−803.443849	−1172.38983	−1155.13001	−1185.75367
1.8	−1704.06104	−1702.51816	−2604.64904	−2558.57459	−1185.75367
2.0	−3611.63021	−3608.3573	−5787.48354	−5667.99096	−5870.04229
2.2	−7655.45604	−7648.51548	−12860.7721	−12557.3303	−13062.6841
2.4	−16228.1236	−16213.4076	−28580.1814	−27821.8776	−29069.935
2.6	−34401.9133	−34370.7133	−63514.7048	−61643.4748	−64694.4344
2.8	−72930.0593	−72863.9135	−141152.874	−136582.183	−143977.899
3.0	−154609.419	−154469.187	−313695.599	302624.668	−320426.152
3.2	−327769.328	−327472.035	−697154.188	−670527.141	−713118.596
3.4	−694867.979	−694237.718	−1549352.17	−1485693.8	−1587071.31
3.6	−1473116.74	−1471780.59	−3443276.54	−3291870.79	−3532088.44
3.8	−3123003.72	−3120171.07	−7652333.62	−7293842.79	−7860803.19
4.0	−6620763.7	−6614757.47	−17006541.6	−16161072.7	−17494534.6

$h = 0.1$

x_k	Taylor	Mod Euler	RK4	Ad-Bash-Moult
0.0	−2	−2	−2	−2
0.2	−4.3786	−4.37798	−4.44723874	−4.44723874
0.4	−9.56702144	−9.56504339	−9.87011722	−9.86993267
0.6	−20.8862678	−20.8813151	−21.8900872	−21.8887964
0.8	−45.6106649	−45.5991965	−48.5658508	−48.561028
1.0	−99.6738964	−99.648156	−107.830182	−107.815098
1.2	−217.977103	−217.920101	−239.588317	−239.545057
1.4	−476.967742	−476.842265	−532.640112	−532.522265
1.6	−1044.09633	−1043.82086	−1184.58908	−1184.2788
1.8	−2286.14646	−2285.54246	−2635.15604	−264.35871
2.0	−5006.52093	−5005.19732	−5862.83785	−5860.82572
2.2	−10964.9933	−10962.0934	−13045.0526	−13040.0457
2.4	−24016.1713	−24009.8188	−29027.1396	−29014.8203
2.6	−52603.1881	−52589.273	−64591.2795	−64561.245
2.8	−115219.882	−115189.402	−143730.717	−143658.051
3.0	−252375.158	−252308.393	−319836.86	−319662.181
3.2	−552799.719	−552653.478	−711719.918	−711302.336
3.4	−1210489.3	−1210528.97	−1583764.49	−1582771.01
3.6	−2652240.7	−2651539.05	−3524297.25	−3521943.51
3.8	−5809463.99	−5807927.1	−7842502.92	−7836947.07
4.0	−12725045.4	−12721679	−17451668.8	−17438597.6

$h = 0.05$

x_k	Taylor	Mod Euler	RK4	Ad-Moult
0.0	−2	−2	−2	−2
0.2	−4.42768992	−4.42751728	−4.44774355	−4.44775754
0.4	−9.78329859	−9.7827435	−9.87235457	−9.87252343
0.6	−21.6009202	−21.5995178	−21.8975357	−21.8981722
0.8	−47.7097555	−47.7064763	−48.5879127	−48.5899582
1.0	−105.454104	−105.456667	−107.891478	−107.897429
1.2	−233.257392	−233.240743	−239.751863	−239.768221
1.4	−516.240361	−516.203306	−533.064451	−533.107782
1.6	−1142.97394	−1142.89167	−1185.66777	−1185.77962
1.8	−2531.2058	−2531.02339	−2637.85558	−2638.13881
2.0	−5606.38685	−5605.98257	−5869.51081	−5870.2175
2.2	−12418.7004	−12417.8046	−13061.3833	−13063.126
2.4	27509.9884	−27508.0038	−29066.7775	−29071.034
2.6	−60941.9504	−60937.5537	−64686.8229	−64697.1377
2.8	−135004.617	−134994.877	−143959.659	−143984.488
3.0	−299077.863	−299056.285	−320382.665	−320442.091
3.2	−662554.634	−662506.83	−713015.372	−713156.905
3.4	−1467776.9	−1467671	−1586827.25	−1587162.88
3.6	−3251613.2	−3251378.59	−3531513.36	−3532306.28
3.8	−7203406.97	−7202887.23	−7859452.38	−7861319.3
4.0	−15957951.8	−15956800.4	−17491370.3	−17495652.8

4. Approximate solution of $y' = 1 - \cos(x - y) + x^2; y(3) = 0$ on $3, 7]$

$h = 0.2$

x_k	Taylor	Mod Euler	RK4	Ad-Moult
3.0	6	6	6	6
3.2	8.34619425	8.22588014	8.23052065	8.23052065
3.4	10.4599268	10.4365983	10.4451202	10.4451202
3.6	13.1173498	13.1391115	13.1512808	13.1512808
3.8	16.2266922	16.0893271	16.0987876	16.2044495
4.0	19.2311894	19.2354373	19.2842974	19.2561985
4.2	23.123893	22.8668203	22.8829435	22.9719003
4.4	26.8451661	26.7472694	26.7814503	26.7193294
4.6	31.0952036	30.8517175	30.9532362	31.22957
4.8	36.2798394	35.6694703	35.658142	35.7645574
5.0	40.8218613	40.6916977	40.6704424	40.6628742
5.2	45.7997401	45.9216311	46.0038564	46.2362496
5.4	51.9436528	51.560188	51.7553377	52.2494167
5.6	58.6878124	57.7252626	57.9652387	58.686008
5.8	65.776231	64.3618911	64.6276263	65.5774007
6.0	72.9329889	71.3938815	71.7326167	72.9080671
6.2	80.0887389	78.836022	79.3045364	80.5735335
6.4	87.446352	86.8968556	87.448371	88.5427238
6.6	95.4569271	95.7397258	96.1982746	97.1417675
6.8	105.175898	104.777041	105.268807	106.41776
7.0	114.225625	114.609065	115.120266	115.987763

$h = 0.1$

x_k	Taylor	Mod Euler	RK4	Ad-Moult
3	6	6	6	6
3.2	8.2507291	8.23369295	8.22967569	8.22967569
3.4	10.4394927	10.4418832	10.4457615	10.4446675
3.6	13.1465841	13.148787	13.1528387	13.1464095
3.8	16.1094296	16.0956396	16.0992489	16.1046833
4.0	19.2594085	19.2745034	19.2872731	19.2799514
4.2	22.890562	22.8738044	22.8818386	22.8918358
4.4	26.765283	26.7715934	26.7818089	26.7612537
4.6	30.9317221	30.9418549	30.9643702	30.9550772
4.8	35.6745326	35.6366817	35.6479526	35.6457561
5.0	40.6633235	40.6487982	40.6571216	40.5886006
5.2	45.9602607	45.9912774	46.0185911	45.9445744
5.4	51.7482391	51.7729066	51.8119687	51.7383767
5.6	57.99959	58.0162529	58.055699	57.9703634
5.8	64.703494	64.7227709	64.757674	64.6495991
6.0	71.9102868	71.9134198	71.9376171	71.7884816
6.2	79.6770173	79.5971662	79.6047034	79.4434934
6.4	87.8862486	87.7148415	87.7257471	87.6500051
6.6	96.3404078	96.2771643	96.3344672	96.3672427
6.8	105.686747	105.580538	105.568534	105.441081
7.0	115.355925	115.164595	115.2962	115.307915

$h = 0.05$

x_k	Taylor	Mod Euler	RK4	Ad-Moult
3	6	6	6	6
3.2	8.23419098	8.23056606	8.22964436	8.22966882
3.4	10.4431915	10.4447587	10.4458133	10.445672
3.6	13.1518268	13.1517996	13.1529273	13.1525766
3.8	16.1002677	16.0982334	16.0993195	16.0996007
4.0	19.2809396	19.2841627	19.2875476	19.2870428
4.2	22.8817309	22.8796025	22.8820311	22.8828946
4.4	26.7793173	26.7789918	26.7821396	26.7821471
4.6	30.9535325	30.9587591	30.9651235	309658788
4.8	35.6499651	35.644397	35.6483098	35.6516244
5.0	40.6586698	40.6535337	40.6573965	40.6596025
5.2	46.0073894	46.0120691	46.0198728	46.0198916
5.4	51.7935503	51.8029304	51.8137989	51.8141422
5.6	58.0344809	58.045062	58.0576058	58.0581417
5.8	64.7345646	64.7457112	64.7591988	64.7582984
6.0	71.9173141	71.9252788	71.9376167	71.9331926
6.2	79.6008015	79.5937715	79.6024046	79.5981186
6.4	87.724633	87.7117002	87.7237714	87.7284662
6.6	96.3083406	96.3203455	96.3404767	96.3297379
6.8	105, 561564	105.550152	105.56088	105.53758
7.0	115.209011	115.216756	115.2395	115.220034

5. Approximate solution of $y' = 4x^3 - xy + \cos(y); y(0) = 4$ on $[0, 4]$
$h = 0.2$

x_k	Taylor	Mod Euler	RK4	Ad-Moult
0.0	4	4	4	4
0.2	3.77937769	3.7817712	3.78003807	3.78003807
0.4	3.39585336	3.40892579	3.41113608	3.41113608
0.6	2.95844535	2.98911514	2.99815198	2.99815198
0.8	2.63644019	2.68362434	2.6939363	2.68963491
1.0	2.57410268	2.63198688	2.63406773	2.62865502
1.2	2.86123301	2.92765631	2.9194424	2.91676056
1.4	3.57876742	3.66118348	3.64788255	3.64974904
1.6	4.89071754	5.00831485	4.99580399	4.99442617
1.8	7.04694486	7.12490142	7.05772847	6.99742903
2.0	9.39036697	9.40456226	9.38041487	9.41845601
2.2	12.1092386	12.2787828	12.2467327	12.2562991
2.4	15.8058673	15.6440336	156093858	15.5628496
2.6	19.3918856	19.4924985	19.4082914	19.4603801
2.8	23.4820972	23.4305212	23.5006254	23.6030269
3.0	28.4971066	28.2469139	28.1243054	28.2019881
3.2	33.1969888	33.3039945	33.0682128	32.9768377
3.4	37.8589178	38.3976867	38.2778064	38.1869018
3.6	43.6740568	43.7081616	43.8121637	44.0155361
3.8	49.9799643	49.464134	49.7032917	50.0324907
4.0	56.4028404	55.6739098	55.9382614	56.2119126

$h = 0.1$

x_k	Taylor	Mod Euler	RK4	Ad-Bash-Moult
0.0	4	4	4	4
0.2	3.7794333	3.78043259	3.77999805	3.77999805
0.4	3.4063912	3.41050545	3.41104815	3.4110294
0.6	2.98742511	2.99596552	2.99802794	2.99786651
0.8	2.67888513	2.69113585	2.69319067	2.69290979
1.0	2.61904369	2.63355977	2.63372362	2.63352302
1.2	2.90467175	2.9211945	2.91895169	2.91894526
1.4	3.62976051	3.65076771	3.64732119	3.64756467
1.6	4.97054479	4.99929765	4.99555968	4.99474741
1.8	7.06207082	7.07183105	7.05743353	7.05537836
2.0	9.37312564	9.38527061	9.37941313	9.3863398
2.2	12.2268713	12.2600988	12.2487484	12.2388627
2.4	15.6095086	15.6111703	15.605291	15.6243401
2.6	19.4141045	19.4310567	19.408775	19.377544
2.8	23.4681005	23.5044464	23.5079285	23.5176327
3.0	28.1280137	28.1104867	28.1077469	28.1137305
3.2	33.10814	33.0802212	33.052868	32.9831436
3.4	38.2813467	38.3144963	38.2882438	38.2409524
3.6	43.7987621	43.8554581	43.8484638	43.8459653
3.8	49.6967946	49.746549	49.7487578	49.7574115
4.0	55.936434	55.9835947	55.9845424	55.9910457

$h = 0.05$

x_k	Taylor	Mod Euler	RK4	Ad-Moult
0.0	4	4	4	4
0.2	3.77980105	3.78010467	3.77999556	3.77999594
0.4	3.40977125	3.41090778	3.41104294	3.41104123
0.6	2.99528031	2.99752291	2.99802027	2.99800967
0.8	2.6895847	2.69270074	2.69317759	2.69316373
1.0	2.63006493	2.63369368	2.63370176	2.63369387
1.2	2.91538281	2.91949818	2.91892039	2.9189233
1.4	3.64287275	3.64816427	3.6472847	3.64729887
1.6	4.98954176	4.99653975	4.994555048	4.99545614
1.8	7.05886786	7.06073696	7.05740858	7.05754168
2.0	9.37696907	9.38079089	9.37936789	9.37955898
2.2	12.2453842	12.2517694	12.248905	12.2481896
2.4	15.6043401	15.6065622	15.6052289	15.6061482
2.6	19.4126103	19.4136268	19.4088447	19.4084339
2.8	23.4978433	23.5078993	23.5083899	23.5079779
3.0	28.1085153	28.108165	28.1075924	28.1110261
3.2	33.0629406	33.0567381	33.0522514	33.0552094
3.4	38.2929232	38.2933477	38.2883211	38.2859525
3.6	43.843662	43.8518368	43.8491852	43.8447666
3.8	49.7387829	49.750584	49.7497394	49.7451907
4.0	55.9732413	55.9860986	55.9855852	55.9804132

In problems 6, 7 and 8, approximations are generated using the Adams-Bashforth-Moulton method, first with $h = 0.2$, then with $h = 0.1$.

6. Approximate solution of $y' = y - x^3; y(-2) = -4$ on $[-2, 2]$. The exact solution is $y = -2e^{x+2} + x^3 + 3x^2 + 6x + 6$.

x_k	$h = 0.2$	$h = 0.1$	Exact
-2	-4	-4	-4
-1.8	-3.354774	-3.35480347	-3.35480552
-1.6	-2.99958216	-2.99964626	-2.99964939
-1.4	-2.90812926	-2.9082347	-2.9082376
-1.2	-3.05895311	-3.05907945	-3.05908186
-1.0	-3.43641228	-3.4365621	-3.43656366
-0.8	-4.03205546	-4.03223363	-4.03223385
-0.6	-4.8461901	-4.84640173	-4.84639993
-0.4	-5.88981841	-5.89006957	-5.89006485
-0.2	-7.18700596	-7.18730377	-7.18729493
0.0	-8.77777395	-8.77812676	-8.7781122
0.2	-10.7216318	-10.7220494	-10.722027
0.4	-13.101892	-13.1023857	-13.1023528
0.6	-16.0309401	-16.0315232	-16.0314761
0.8	-19.6566716	-19.6573594	-19.6572936
1.0	-24.1703541	-24.1711645	-14.1710739
1.2	-29.8162301	-29.8171836	-29.817004
1.4	-36.9032455	-36.9043659	-36.9042001
1.6	-45.8193758	-45.82069	-45.8204689
1.8	-57.0491227	-57.0506618	-57.050369
2.0	-71.1948864	-71.1966855	-71.1963001

7. Approximate solution of $y' = 2xy - y^3; y(0) = 2$ on $[0, 4]$

x_k	$h = 0.2$	$h = 0.1$
0.0	2	2
0.2	1.22921715	1.27928486
0.4	1.07091936	1.08638842
0.6	1.04836859	1.05688972
0.8	1.01157152	1.10836346
1.0	1.14419902	1.21475378
1.2	1.31901324	1.35628443
1.4	1.49026217	1.51174278
1.6	1.65343971	1.66272411
1.8	1.7992031	1.79992887

2.0	1.92265942	1.92272594
2.2	2.02954279	2.03410264
2.4	2.1312142	2.13707792
2.6	2.23463857	2.23374303
2.8	2.23692569	2.32542008
3.0	2.42896574	2.41296935
3.2	2.50245762	2.49699181
3.4	2.55106451	2.57793593
3.6	2.55394614	2.65615241
3.8	2.38180227	2.7319243
4.0	2.10320803	2.80548524

8. Approximate solution of $y' = \ln(x) + x^2 y; y(2) = 1$ on $[2, 6]$.

x_k	$h = 0.2$	$h = 0.1$
2.0	1	1
2.2	2.65026902	2.65599503
2.4	7.89892485	7.94440496
2.6	27.6949689	278.0075572
2.8	111.600028	119.953799
3.0	535.770648	636.471761
3.2	3084.87485	4242.03453
3.4	21286.8706	35888.2231
3.6	175576.11	388524.253
3.8	1724629.75	5414269.61
4.0	20087193.5	97497139.9
4.2	276121968	$2.27312799E+09$
4.4	$4.45804515E+09$	$6.86371441E+10$
4.6	$8.41299341E+10$	$2.68108863E+12$
4.8	$1.84695066E+12$	$1.35177074E+14$
5.0	$4.69519264E+13$	$8.7690622E+15$
5.2	$1.37597864E+15$	$7.29302779E+17$
5.4	$4.62887885E+16$	$7.74201731E+19$
5.6	$1.78021704E+18$	$1.04437124E+22$
5.8	$7.79671448E+19$	$1.78182913E+24$
6.0	$3.87418078E+21$	$3.82651095E+25$

Chapter Six - Vectors and Vector Spaces

Section 6.1 The Algebra and Geometry of Vectors

1. $\mathbf{F}+\mathbf{G}=(2+\sqrt{2})\mathbf{i}+3\mathbf{j}$; $\mathbf{F}-\mathbf{G}=(2-\sqrt{2})\mathbf{i}-9\mathbf{j}+10\mathbf{k}$; $\|\mathbf{F}\|=\sqrt{38}$; $\|\mathbf{G}\|=\sqrt{63}$; $2\mathbf{F}=4\mathbf{i}-6\mathbf{j}+10\mathbf{k}$; $3\mathbf{G}=3\sqrt{2}\mathbf{i}+18\mathbf{j}-15\mathbf{k}$

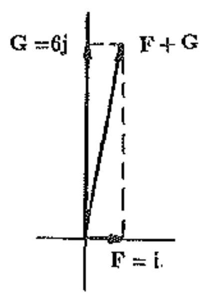
2. $\mathbf{F}+\mathbf{G}=\mathbf{i}+4\mathbf{j}-3\mathbf{k}$; $\mathbf{F}-\mathbf{G}=\mathbf{i}-4\mathbf{j}-3\mathbf{k}$; $\|\mathbf{F}\|=\sqrt{10}$, $\|\mathbf{G}\|=4$; $2\mathbf{F}=2\mathbf{i}-6\mathbf{k}$; $3\mathbf{G}=12\mathbf{j}$

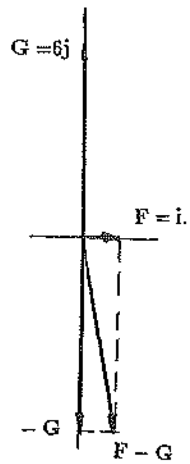
3. $\mathbf{F}+\mathbf{G}=3\mathbf{i}-\mathbf{k}$; $\mathbf{F}-\mathbf{G}=\mathbf{i}-10\mathbf{j}+\mathbf{k}$; $\|\mathbf{F}\|=\sqrt{29}$; $\|\mathbf{G}\|=3\sqrt{3}$; $2\mathbf{F}=4\mathbf{i}-10\mathbf{j}$; $3\mathbf{G}=3\mathbf{i}+15\mathbf{j}-3\mathbf{k}$

4. $\mathbf{F}+\mathbf{G}=(\sqrt{2}+8)\mathbf{i}+\mathbf{j}-4\mathbf{k}$; $\mathbf{F}-\mathbf{G}=(\sqrt{2}-8)\mathbf{i}+\mathbf{j}-8\mathbf{k}$; $\|\mathbf{F}\|=\sqrt{41}$; $\|\mathbf{G}\|=\sqrt{68}$; $2\mathbf{F}=2\sqrt{2}\mathbf{i}+2\mathbf{j}-12\mathbf{k}$; $3\mathbf{G}=24\mathbf{i}+6\mathbf{k}$

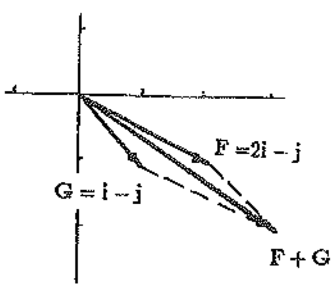
5. $\mathbf{F}+\mathbf{G}=3\mathbf{i}-\mathbf{j}+3\mathbf{k}$; $\mathbf{F}-\mathbf{G}=-\mathbf{i}+3\mathbf{j}-\mathbf{k}$; $\|\mathbf{F}\|=\sqrt{3}$; $\|\mathbf{G}\|=2\sqrt{3}$; $2\mathbf{F}=2\mathbf{i}+2\mathbf{j}+2\mathbf{k}$; $3\mathbf{G}=6\mathbf{i}-6\mathbf{j}+6\mathbf{k}$

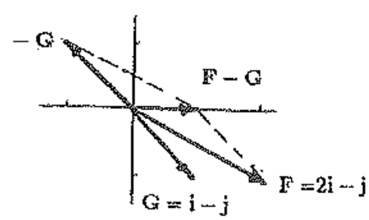
6. $\mathbf{F}+\mathbf{G}=\mathbf{i}+6\mathbf{j}$; $\mathbf{F}-\mathbf{G}=\mathbf{i}-6\mathbf{j}$



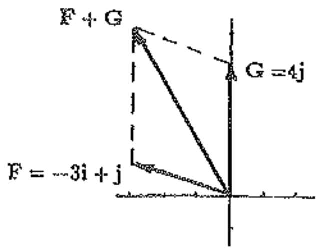


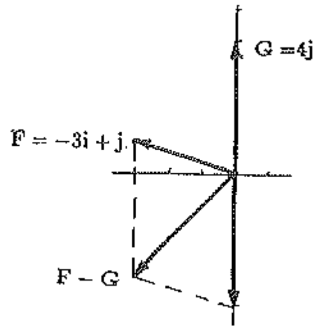
7. $\mathbf{F}+\mathbf{G}=3\mathbf{i}-2\mathbf{j}$; $\mathbf{F}-\mathbf{G}=\mathbf{i}$



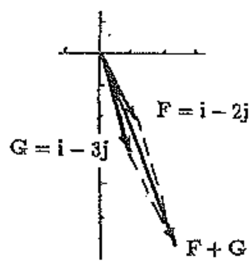


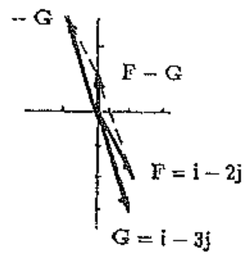
8. $\mathbf{F} + \mathbf{G} = -3\mathbf{i} + 5\mathbf{j}; \mathbf{F} - \mathbf{G} = -3\mathbf{i} - 3\mathbf{j}$



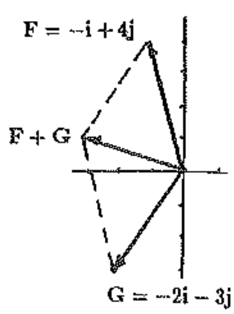


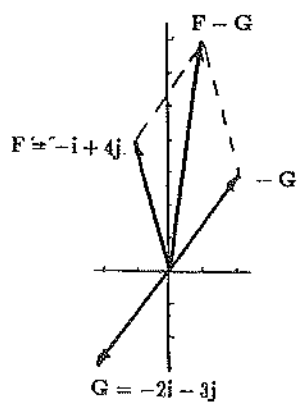
9. $\mathbf{F} + \mathbf{G} = 2\mathbf{i} - 5\mathbf{j}; \mathbf{F} - \mathbf{G} = \mathbf{j}$



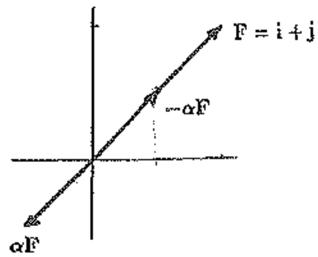


10. $\mathbf{F} + \mathbf{G} = -3\mathbf{i} + \mathbf{j}; \mathbf{F} - \mathbf{G} = \mathbf{i} + 7\mathbf{j}$

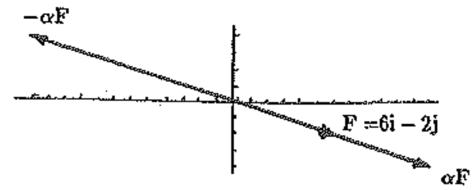




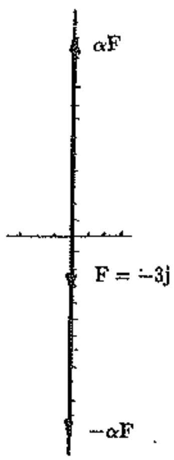
11. $\alpha\mathbf{F} = -\frac{1}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}$



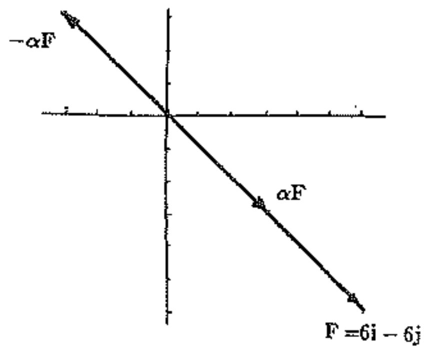
12. $\alpha\mathbf{F} = 12\mathbf{i} - 4\mathbf{j}$



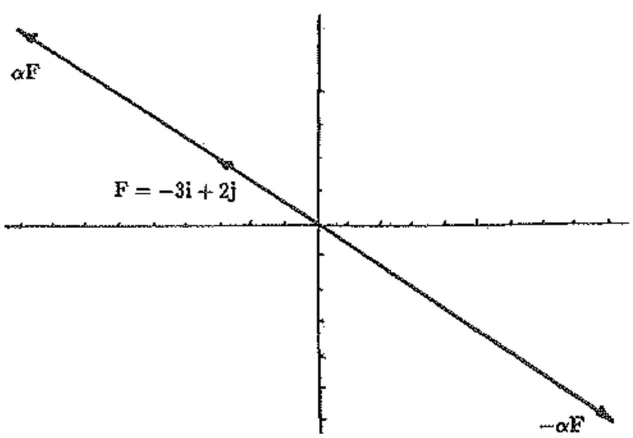
13. $\alpha\mathbf{F} = 12\mathbf{j}$



14. $\alpha\mathbf{F} = 3\mathbf{i} - 3\mathbf{j}$



15. $\alpha\mathbf{F} = -9\mathbf{i} + 6\mathbf{j}$



16. A vector from $(1,0,4)$ to $(2,1,1)$ is $\mathbf{M} = \mathbf{i}+\mathbf{j}-3\mathbf{k}$. A vector from $(1,0,4)$ on L to (x,y,z) on L is $(x-1)\mathbf{i}+(y-0)\mathbf{j}+(z-4)\mathbf{k}$, which must be parallel to $\mathbf{M}$. Thus $x-1=t, y-0=t, z-4=-3t$, for some scalar t. Parametric equations of the line are $x=1+t, y=t, z=4-3t$, for $-\infty < t < \infty$. Eliminating the parameter t gives normal equations $\dfrac{x-1}{1}=\dfrac{y}{1}=\dfrac{z-4}{-3}$.

17. Parametric equations are $x=3+6t, y=-t, z=0; -\infty<t<\infty$; Normal equations are $\dfrac{x-3}{6}=\dfrac{y}{-1}, z=0$.

18. Parametric equations are $x=2, y=1, z=1-3t; -\infty<t<\infty$; Normal equations are $x=2, y=1, z=1-3t$. This line is parallel to the z axis through $(2,1,0)$.

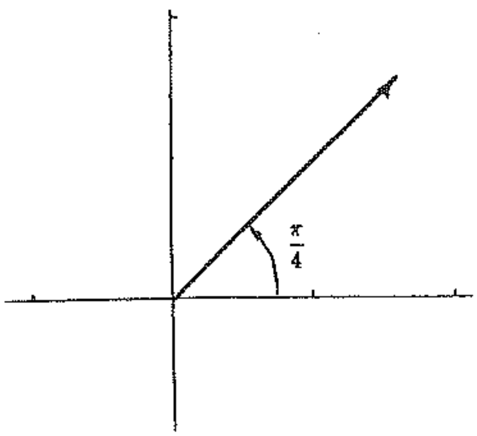
19. Parametric equations are $x=0, y=1+t, z=3+2t; -\infty<t<\infty$; Normal equations are $x=0, y-1=\dfrac{z-3}{2}$.

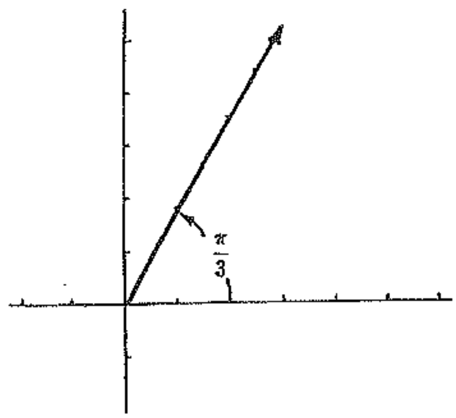
20. Parametric equations are $x=1-3t, y=0-2t, z=-4+9t; -\infty<t<\infty$; Normal equations are $\dfrac{x-1}{-3}=\dfrac{y}{-2}=\dfrac{z+4}{9}$.

21. Parametric equations are $x=2-3t, y=-3+9t, z=6-2t; -\infty<t<\infty$; Normal equations are $\dfrac{x-2}{-3}=\dfrac{y+3}{9}=\dfrac{z-6}{-2}$.

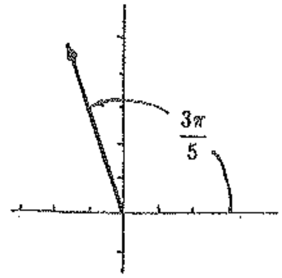
22. $\mathbf{F}=\sqrt{5}\left(\cos\left(\frac{\pi}{4}\right)\mathbf{i}+\sin\left(\frac{\pi}{4}\right)\mathbf{j}\right)=\frac{\sqrt{10}}{2}(\mathbf{i}+\mathbf{j})$

23. $\mathbf{F}=6\left(\cos\left(\frac{\pi}{3}\right)\mathbf{i}+\sin\left(\frac{\pi}{3}\right)\mathbf{j}\right)=3\mathbf{i}+$

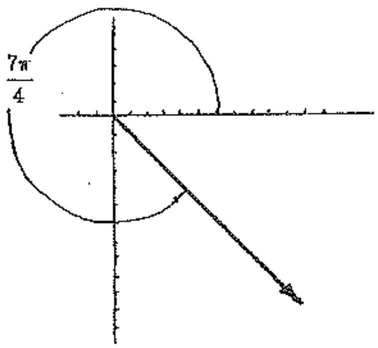




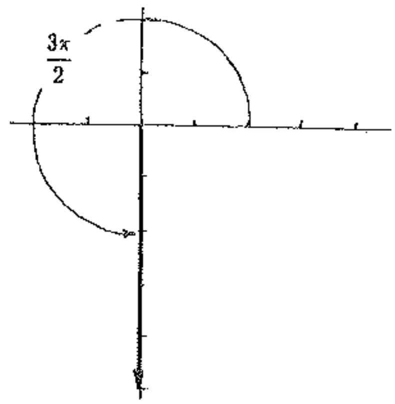
24. $\mathbf{F} = 5\left(\cos\left(\frac{3\pi}{5}\right)\mathbf{i} + \sin\left(\frac{3\pi}{5}\right)\mathbf{j}\right) \approx -1.545\mathbf{i} + 4.755\mathbf{j}$



25. $\mathbf{F} = 15\left(\cos\left(\frac{7\pi}{4}\right)\mathbf{i} + \sin\left(\frac{7\pi}{4}\right)\mathbf{j}\right) = \frac{15}{\sqrt{2}}(\mathbf{i} - \mathbf{j})$



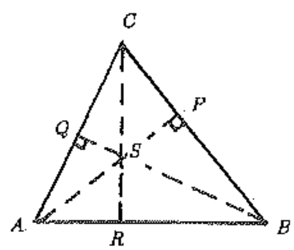
26. $\mathbf{F} = 25\left(\cos\left(\frac{3\pi}{2}\right)\mathbf{i} + \sin\left(\frac{3\pi}{2}\right)\mathbf{j}\right) = -25$



27. Let the coordinates of $P_i = (x_i, y_i, z_i)$. Then $\mathbf{F}_i = (x_{i+1} - x_i)\mathbf{i} + (y_{i+1} - y_i)\mathbf{j} + (z_{i+1} - z_i)\mathbf{k}$, and $\mathbf{F}_1 + \mathbf{F}_2 + \cdots + \mathbf{F}_{n-1} = \sum_{r=1}^{n-1}\{(x_{r+1} - x_r)\mathbf{i} + (x_{r+1} - y_r)\mathbf{j} + (z_{r+1} - z_r)\mathbf{k} = (x_n - x_1)\mathbf{i} + (y_n - y_1)\mathbf{j} + (z_n - z_1)\mathbf{k} = -\mathbf{F}_n$, so $\mathbf{F}_1 + \mathbf{F}_2 + \cdots + \mathbf{F}_n = 0$.

28. Since $\|t\mathbf{F}\| = |t|\|\mathbf{F}\|$, $\|t\mathbf{F}\| = 1$ implies $|t| = \frac{1}{\|\mathbf{F}\|}$. Two choices for the scalar t are $t = \pm\frac{1}{\|\mathbf{F}\|}$.

29. Let A, B, C be the vertices of the triangle, P and Q the feet of the altitudes from A and B to sides BC and AC respectively, and S the point where these two altitudes intersect. Finally, let R be the point where the vector $\mathbf{CS}$ extended intersects side $\mathbf{AB}$. It will be sufficient to show $\mathbf{CR}$ is perpendicular to $\mathbf{AB}$ (so the altitude from C also passes through S). We have $\mathbf{AS} \cdot \mathbf{BC} = \mathbf{BS} \cdot \mathbf{AC} = 0$, and then $\mathbf{CS} \cdot \mathbf{AB} = \mathbf{CS} \cdot (\mathbf{AC} + \mathbf{CB}) = \mathbf{CS} \cdot \mathbf{AC} + \mathbf{CS} \cdot \mathbf{CB} = (\mathbf{CB} + \mathbf{BS}) \cdot \mathbf{AC} + (\mathbf{CA} + \mathbf{AS}) \cdot \mathbf{CB} = \mathbf{CB} \cdot \mathbf{AC} + \mathbf{BS} \cdot \mathbf{AC} + \mathbf{CA} \cdot \mathbf{CB} + \mathbf{AS} \cdot \mathbf{CB} = \mathbf{CB} \cdot \mathbf{AC} + 0 + \mathbf{CB} \cdot \mathbf{CA} + 0 = \mathbf{CB} \cdot (\mathbf{AC} + \mathbf{CA}) = \mathbf{CB} \cdot \mathbf{0} = 0$, which completes the proof.



Section 6.2 The Dot Product

In 1 - 6, we use the first vector as $\mathbf{F}$, the second as $\mathbf{G}$.

1. $\mathbf{F}\cdot\mathbf{G}=2; \cos(\theta)=\dfrac{\mathbf{F}\cdot\mathbf{G}}{\|\mathbf{F}\|\|\mathbf{G}\|}=\dfrac{2}{\sqrt{14}}$; not orthogonal; $|\mathbf{F}\cdot\mathbf{G}|=2<\sqrt{14}=\|\mathbf{F}\|\|\mathbf{G}\|$.

2. $\mathbf{F}\cdot\mathbf{G}=8; \cos(\theta)=\dfrac{8}{\sqrt{41}\sqrt{2}}$; not orthogonal; $|\mathbf{F}\cdot\mathbf{G}|=8<\sqrt{82}=\|\mathbf{F}\|\|\mathbf{G}\|$.

3. $\mathbf{F}\cdot\mathbf{G}=-23; \cos(\theta)=\dfrac{-23}{\sqrt{29}\sqrt{41}}$; not orthogonal; $|\mathbf{F}\cdot\mathbf{G}|=23=\sqrt{529}<\sqrt{1189}=\|\mathbf{F}\|\|\mathbf{G}\|$

4. $\mathbf{F}\cdot\mathbf{G}=-63; \cos(\theta)=\dfrac{-63}{\sqrt{75}\sqrt{74}}$; not orthogonal; $|\mathbf{F}\cdot\mathbf{G}|=63=\sqrt{3969}<\sqrt{5550}=\|\mathbf{F}\|\|\mathbf{G}\|$

5. $\mathbf{F}\cdot\mathbf{G}=-18; \cos(\theta)=\dfrac{-18}{\sqrt{10}\sqrt{40}}$; not orthogonal; $|\mathbf{F}\cdot\mathbf{G}|=18=\sqrt{324}<\sqrt{400}=\|\mathbf{F}\|\|\mathbf{G}\|$

6. $\mathbf{F}\cdot\mathbf{G}=4; \cos(\theta)=\dfrac{4}{\sqrt{6}\sqrt{6}}$; not orthogonal; $|\mathbf{F}\cdot\mathbf{G}|=4<6=\|\mathbf{F}\|\|\mathbf{G}\|$

7. $3x-y+4z=4$

8. $x-2y=-1$

9. $4x-3y+2z=25$

10. $-3x+2y=1$

11. $7x+6y-5z=-26$

12. $4x+3y+z=-6$

13. Let M = midpoint of $\mathbf{BC}$, and $\mathbf{V}=\mathbf{AM}$. Then $\cos(\theta)=\dfrac{\mathbf{V}\cdot\mathbf{AB}}{\|\mathbf{V}\|\|\mathbf{AB}\|}$. Here we have $M=(7/2,1,-3), \mathbf{V}=5/2\mathbf{i}+3\mathbf{j}-9\mathbf{k}, \mathbf{AB}=2\mathbf{i}+2\mathbf{j}-5\mathbf{k}, \cos(\theta)=\dfrac{112}{\sqrt{385}\sqrt{33}}=\dfrac{112}{11\sqrt{105}}$

14. $M=(-1/2,1/2,4), \mathbf{V}=-7/2\mathbf{i}+5/2\mathbf{j}+7\mathbf{k}, \mathbf{AB}=-5\mathbf{i}+2\mathbf{j}+4\mathbf{k}, \cos(\theta)=\dfrac{101}{\sqrt{270}\sqrt{45}}=\dfrac{101}{45\sqrt{6}}$

15. $M=(-3/2,1,2), \mathbf{V}=-5/2\mathbf{i}+3\mathbf{j}-4\mathbf{k}, \mathbf{AB}=-\mathbf{i}+6\mathbf{j}-9\mathbf{k}, \cos(\theta)=\dfrac{113}{\sqrt{125}\sqrt{118}}=\dfrac{113}{5\sqrt{590}}$

16. $M=(4,-1,2), \mathbf{V}=4\mathbf{i}-6\mathbf{j}+3\mathbf{k}, \mathbf{AB}=\mathbf{i}-7\mathbf{j}+6\mathbf{k}, \cos(\theta)=\dfrac{64}{\sqrt{61}\sqrt{86}}=\dfrac{64}{\sqrt{5246}}$

17. That $\mathbf{F}=\mathbf{0}$, since $\mathbf{F}\cdot\mathbf{F}=\|\mathbf{F}\|^2=0$.

18. That $\mathbf{F}=\mathbf{0}$, since *every* vector $\mathbf{X}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ for some x,y,z. Then $\mathbf{F}\cdot\mathbf{X}=0$ for *every* vector $\mathbf{X}$ and $\mathbf{F}=\mathbf{0}$ by Problem 17.

19. For any unit vector $\mathbf{U}$, $\mathbf{F}\cdot\mathbf{U}=\|\mathbf{F}\|\cos\theta$, so $|\mathbf{F}\cdot\mathbf{U}|=\|\mathbf{F}\||\cos\theta|\leq\|\mathbf{F}\|$. The maximum value of $|\mathbf{F}\cdot\mathbf{U}|$ is achieved when $|\cos\theta|=1$ so $\theta=0$ or π and hence $\mathbf{U}$ is parallel to $\mathbf{F}$. Observe that $(\mathbf{F}\cdot\mathbf{U})_{\max}=\|\mathbf{F}\|$ and $(\mathbf{F}\cdot\mathbf{U})_{\min}=-\|\mathbf{F}\|$.

20. Let $\mathbf{F}=a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$. Then $\mathbf{F}\cdot\mathbf{i}=a$ since $(\mathbf{i}\cdot\mathbf{i})=1, (\mathbf{i}\cdot\mathbf{j})=(\mathbf{i}\cdot\mathbf{k})=0$. Similarly $(\mathbf{F}\cdot\mathbf{j})=b$ and $(\mathbf{F}\cdot\mathbf{k})=c$ and $\mathbf{F}=(\mathbf{F}\cdot\mathbf{i})\mathbf{i}+(\mathbf{F}\cdot\mathbf{j})\mathbf{j}+(\mathbf{F}\cdot\mathbf{k})\mathbf{k}$.

Section 6.3 The Cross Product

1. $\mathbf{F}\times\mathbf{G} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 6 & 1 \\ -1 & -2 & 1 \end{vmatrix} = 8\mathbf{i}+2\mathbf{j}+12\mathbf{k} = -\mathbf{G}\times\mathbf{F}$; $\cos(\theta) = \dfrac{\mathbf{F}\cdot\mathbf{G}}{\|\mathbf{F}\|\|\mathbf{G}\|} = -\dfrac{4}{\sqrt{69}}$; $\sin(\theta) = \sqrt{1-\cos^2\theta} = \sqrt{1-\dfrac{16}{69}} = \dfrac{\sqrt{53}}{\sqrt{69}}$; $\|\mathbf{F}\|\|\mathbf{G}\|\sin(\theta) = \sqrt{46}\sqrt{6}\dfrac{\sqrt{53}}{\sqrt{69}} = \sqrt{212} = \sqrt{64+4+144} = \|\mathbf{F}\times\mathbf{G}\|$.

In Problems 2 - 10, the values are given in order for $\mathbf{F}\times\mathbf{G}; \cos(\theta); \sin(\theta)$; and the common value of $\|\mathbf{F}\|\|\mathbf{G}\|\sin(\theta) = \|\mathbf{F}\times\mathbf{G}\|$.

2. $\mathbf{i}+12\mathbf{j}+6\mathbf{k}; -\dfrac{2}{\sqrt{185}}; \dfrac{\sqrt{181}}{\sqrt{185}}; \sqrt{181}$

3. $-8\mathbf{i}-12\mathbf{j}-5\mathbf{k}; \dfrac{-12}{\sqrt{29}\sqrt{13}}; \dfrac{\sqrt{233}}{\sqrt{29}\sqrt{13}}; \sqrt{233}$

4. $112\mathbf{k}; \dfrac{3}{5}; \dfrac{4}{5}; 112$

5. $18\mathbf{i}+50\mathbf{j}-60\mathbf{k}; \dfrac{62}{5\sqrt{2}\sqrt{109}}; \dfrac{\sqrt{1606}}{5\sqrt{2}\sqrt{109}}; 2\sqrt{1606}$

6. $2\mathbf{i}+16\mathbf{j}; 0; 1, 2\sqrt{65}$

In Problems 7- 11, denote the points in order by P, Q, R and form vectors $\mathbf{F} = \mathbf{PQ}, \mathbf{G} = \mathbf{PR}$. The points are colinear if and only $\mathbf{F} = \lambda\mathbf{G}$. Otherwise $\mathbf{F}\times\mathbf{G}$ will be a normal to the plane containing P, Q, R. If $\mathbf{F}\times\mathbf{G} = n_1\mathbf{i}+n_2\mathbf{j}+n_3\mathbf{k}$, and $P = (x_0, y_0, z_0)$, an equation of the plane will be $n_1(x-x_0)+n_2(y-y_0)+n_3(z-z_0) = 0$.

7. $\mathbf{F} = 3\mathbf{i}-\mathbf{j}-5\mathbf{k}, \mathbf{G} = 4\mathbf{i}-\mathbf{j}-6\mathbf{k}$, so P, Q, R are not collinear. $\mathbf{N} = \mathbf{F}\times\mathbf{G} = \mathbf{i}-2\mathbf{j}+\mathbf{k}$, and the plane containing P, Q, R is given by $(x+1)-2(y-1)+(z-6) = 0$ or $x-2y+z = 3$.

8. not colinear; $x+2y+6z = 12$

9. not colinear; $2x-11y+z = 0$

10. not colinear; $5x+16y-2z = -4$

11. not colinear; $29x+37y-12z = 30$

In Problems 12 through 16, with $\mathbf{F}$ and $\mathbf{G}$ being vectors along a pair of adjacent sides of the parallelogram, area $= \|\mathbf{F}\times\mathbf{G}\|$.

12. Form $\mathbf{F} = \mathbf{PQ} = \mathbf{i}+4\mathbf{j}-6\mathbf{k}, \mathbf{G} = \mathbf{PR} = 5\mathbf{i}+2\mathbf{j}-5\mathbf{k}$, and compute $\mathbf{F}\times\mathbf{G} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 4 & -6 \\ 5 & 2 & -5 \end{vmatrix} = -8\mathbf{i}-25\mathbf{j}-18\mathbf{k}$. Then area $= \|\mathbf{F}\times\mathbf{G}\| = \sqrt{1013}$

13. $7\sqrt{2}$

14. 98

15. $2\sqrt{209}$

16. $20\sqrt{41}$

In Problems 17 - 21, we take the points given as P, Q, R, S and form the vectors $\mathbf{PQ}, \mathbf{PR}, \mathbf{PS}$. Then the required volume is given by $|\mathbf{PQ} \cdot \mathbf{PR} \times \mathbf{PS}|$

17. $\mathbf{PQ} = -5\mathbf{i} + \mathbf{j} + 6\mathbf{k}, \mathbf{PR} = 2\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}, \mathbf{PS} = -\mathbf{i} + 0\mathbf{j} + 5\mathbf{k}$ and $\mathbf{PQ} \cdot \mathbf{PR} \times \mathbf{PS} = \begin{vmatrix} -5 & 1 & 6 \\ 2 & 4 & 6 \\ -1 & 0 & 5 \end{vmatrix} = -92$, so volume $= |-92| = 92$.

18. Volume $= -\begin{vmatrix} -3 & 0 & 10 \\ 1 & 6 & 8 \\ -3 & -1 & 10 \end{vmatrix} = 34$

19. Volume $= -\begin{vmatrix} -3 & -2 & 1 \\ 2 & -6 & -1 \\ 1 & -4 & -5 \end{vmatrix} = 98$

20. Volume $= -\begin{vmatrix} 9 & 0 & -4 \\ -2 & 3 & -6 \\ 3 & -1 & -10 \end{vmatrix} = 296$

21. Volume $= -\begin{vmatrix} 1 & 1 & 1 \\ 5 & 0 & 2 \\ -3 & 3 & 5 \end{vmatrix} = 22$

22. $\mathbf{N} = 8\mathbf{i} - \mathbf{j} + \mathbf{k}$

23. $\mathbf{N} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$

24. $\mathbf{N} = \mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$

25. $\mathbf{N} = 7\mathbf{i} + \mathbf{j} - 7\mathbf{k}$

26. $\mathbf{N} = 4\mathbf{i} + 6\mathbf{j} + 4\mathbf{k}$

27. This proof uses a property of determinants.

$$\mathbf{F} \times (\mathbf{G} + \mathbf{H}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ f_1 & f_2 & f_3 \\ g_1 + h_1 & g_2 + h_2 & g_3 + h_3 \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ f_1 & f_2 & f_3 \\ g_1 & g_2 & g_3 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ f_1 & f_2 & f_3 \\ h_1 & h_2 & h_3 \end{vmatrix}$$
$= (\mathbf{F} \times \mathbf{G}) + (\mathbf{F} \times \mathbf{H})$

28. By a property of determinants we have

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \alpha f_1 & \alpha f_2 & \alpha f_3 \\ g_1 & g_2 & g_3 \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ f_1 & f_2 & f_3 \\ \alpha g_1 & \alpha g_2 & \alpha g_3 \end{vmatrix} = \alpha \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ f_1 & f_2 & f_3 \\ g_1 & g_2 & g_3 \end{vmatrix}$$
As vector calculations these represent, respectively $(\alpha\mathbf{F}) \times \mathbf{G} = \mathbf{F} \times (\alpha\mathbf{G}) = \alpha(\mathbf{F} \times \mathbf{G})$

29. Since $\mathbf{F} \times (\mathbf{G} \times \mathbf{H})$ is parallel to the plane of $\mathbf{G}$ and $\mathbf{H}$ and orthogonal to $\mathbf{F}$ we have $\mathbf{F} \times (\mathbf{G} \times \mathbf{H}) = \alpha\mathbf{G} + \beta\mathbf{H}$ for some scalars α, β and $\mathbf{F} \times (\mathbf{G} \times \mathbf{H}) \cdot \mathbf{F} = 0 = \alpha(\mathbf{F} \cdot \mathbf{G}) + \beta(\mathbf{F} \cdot \mathbf{H})$. Thus $\alpha = \lambda(\mathbf{F} \cdot \mathbf{H})$, and $\beta = -\lambda(\mathbf{F} \cdot \mathbf{G})$ for some scalar λ. This gives

$$\mathbf{F} \times (\mathbf{G} \times \mathbf{H}) = \lambda[(\mathbf{F} \cdot \mathbf{H})\mathbf{G} - (\mathbf{F} \cdot \mathbf{G})\mathbf{H}] \qquad (1)$$

To determine λ we construct a special orthogonal triple of unit vectors, $\mathbf{e}_3$ along $\mathbf{H}$, so $\mathbf{H} = h_3\mathbf{e}_3$, $\mathbf{e}_2$ in the plane of $\mathbf{G}$ and $\mathbf{H}$, so $\mathbf{G} = g_2\mathbf{e}_2 + g_3\mathbf{e}_3$, and $\mathbf{e}_1 = \mathbf{e}_2 \times \mathbf{e}_3$ so $\mathbf{F} = f_1\mathbf{e}_1 + f_2\mathbf{e}_2 + f_3\mathbf{e}_3$. Substitute these expressions for $\mathbf{F}, \mathbf{G}, \mathbf{H}$ in (1) and calculate $\mathbf{F} \times (\mathbf{G} \times \mathbf{H}) =$

$\mathbf{F}\times(g_2h_3\mathbf{e}_1) = -f_2g_2h_3\mathbf{e}_3 + f_3g_2h_3\mathbf{e}_2$. Also $\lambda[(\mathbf{F}\cdot\mathbf{H})\mathbf{G} - (\mathbf{F}\cdot\mathbf{G})\mathbf{H}] = \lambda[(f_3h_3)(g_2\mathbf{e}_2 + g_3\mathbf{e}_3) - (f_2g_2+f_3g_3)(h_3\mathbf{e}_3)] = \lambda[f_3h_3g_2\mathbf{e}_2 - f_2g_2h_3\mathbf{e}_3]$, so $\lambda = 1$, and $\mathbf{F}\times(\mathbf{G}\times\mathbf{H}) = (\mathbf{F}\cdot\mathbf{H})\mathbf{G} - (\mathbf{F}\cdot\mathbf{G})\mathbf{H}$.

30. Let $P = (a_1, b_1, c_1), Q = (a_2, b_2, c_2)$ and $R = (a_3, b_3, c_3)$ be the three given points. Form the vectors $\mathbf{V} = \mathbf{PQ} = (a_2 - a_1)\mathbf{i} + (b_2 - b_1)\mathbf{j} + (c_2 - c_1)\mathbf{k}$ and $\mathbf{W} = \mathbf{PR} = (a_3 - a_1)\mathbf{i} + (b_3 - b_1)\mathbf{j} + (c_3 - c_1)\mathbf{k}$. The points PQR are colinear if and only if $\mathbf{V}$ and $\mathbf{W}$ are parallel (or equivalently $\mathbf{V} = \lambda\mathbf{W}$). The condition that at least two of the ratios $\frac{a_2 - a_1}{a_3 - a_1}, \frac{b_2 - b_1}{b_3 - b_1}, \frac{c_2 - c_1}{c_3 - c_1}$ are different will ensure P, Q, R are not colinear. The area of triangle PQR is given by

$$A = \frac{1}{2}\|\mathbf{V}\|\|\mathbf{W}\|\sin(\theta) = \frac{1}{2}\|\mathbf{V}\times\mathbf{W}\| = \left\| \begin{array}{ccc} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_2 - a_1 & b_2 - b_1 & c_2 - c_1 \\ a_3 - a_1 & b_3 - b_1 & c_3 - c_1 \end{array} \right\|,$$

so

$$\text{area } = \frac{1}{2}\left\{ \left| \begin{array}{cc} b_2 - b_1 & c_2 - c_1 \\ b_3 - b_1 & c_3 - c_1 \end{array} \right|^2 + \left| \begin{array}{cc} a_2 - a_1 & c_2 - c_1 \\ a_3 - a_1 & c_3 - c_1 \end{array} \right|^2 + \left| \begin{array}{cc} a_2 - a_1 & b_2 - _1 \\ a_3 - a_1 & b_3 - b_1 \end{array} \right|^2 \right\}^{1/2}.$$

Using properties of determinants we get

$$\left| \begin{array}{cc} b_2 - b_1 & c_2 - c_1 \\ b_3 - b_1 & c_3 - c_1 \end{array} \right| = \left| \begin{array}{ccc} 1 & 0 & 0 \\ 0 & b_2 - b_1 & c_2 - c_1 \\ 0 & b_3 - b_1 & c_3 - c_1 \end{array} \right| = \left| \begin{array}{ccc} 1 & b_1 & c_1 \\ 1 & b_2 & c_2 \\ 1 & b_3 & c_3 \end{array} \right|$$

and similar results for the other two terms. Thus, in terms of coordinates of P, Q, R,

$$\text{area of } \triangle PQP = \frac{1}{2}\left\{ \left| \begin{array}{ccc} 1 & b_1 & c_1 \\ 1 & b_2 & c_2 \\ 1 & b_3 & c_3 \end{array} \right|^2 + \left| \begin{array}{ccc} a_1 & 1 & c_1 \\ a_2 & 1 & c_2 \\ a_3 & 1 & c_3 \end{array} \right|^2 + \left| \begin{array}{ccc} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{array} \right|^2 \right\}^{1/2}.$$

31. $\mathbf{F}\cdot\mathbf{G}\times\mathbf{H} = (a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}) \cdot \left| \begin{array}{ccc} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{array} \right| = a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$

$= \left| \begin{array}{ccc} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{array} \right|$

Section 6.4 The Vector Space R^n

In 1-6 we give in order, the sum of the two vectors, the dot product, and the cosine of the angle between them.

1. $\mathbf{F} + \mathbf{G} = 5\mathbf{e}_1 + 5\mathbf{e}_2 + 6\mathbf{e}_3 + 5\mathbf{e}_4 + \mathbf{e}_5; \mathbf{F}\cdot\mathbf{G} = 0; \cos(\theta) = 0$

2. $\mathbf{F} + \mathbf{G} = 3\mathbf{e}_1 + 9\mathbf{e}_2 + 10\mathbf{e}_3 - 7\mathbf{e}_4; \mathbf{F}\cdot\mathbf{G} = 22; \cos(\theta) = \dfrac{\sqrt{22}}{\sqrt{26}\sqrt{120}}$

3. $\mathbf{F}+\mathbf{G} = 17\mathbf{e}_1 - 4\mathbf{e}_2 + 3\mathbf{e}_3 + 6\mathbf{e}_4; \mathbf{F}\cdot\mathbf{G} = 24; \cos(\theta) = \dfrac{6}{\sqrt{30}\sqrt{17}}$

4. $\mathbf{F}+\mathbf{G} = 10\mathbf{e}_1 + 4\mathbf{e}_2 + 6\mathbf{e}_3 + 0\mathbf{e}_4 + 0\mathbf{e} + 0\mathbf{e}_5; \mathbf{F}\cdot\mathbf{G} = 27; \cos(\theta) = \dfrac{27}{\sqrt{43}\sqrt{55}}$

5. $\mathbf{F}+\mathbf{G} = 6\mathbf{e}_1 + 7\mathbf{e}_2 - 6\mathbf{e}_3 - 2\mathbf{e}_5 - \mathbf{e}_6 + 11\mathbf{e}_7 + 4\mathbf{e}_8; \mathbf{F}\cdot\mathbf{G} = -94; \cos(\theta) = \dfrac{-47}{\sqrt{37}\sqrt{303}}$

6. $\mathbf{F}+\mathbf{G} = -4\mathbf{e}_1 + 3\mathbf{e}_2 + 3\mathbf{e}_3 - 15\mathbf{e}_4 - \mathbf{e}_5; \mathbf{F}\cdot\mathbf{G} = -1; \cos(\theta) = \dfrac{-1}{\sqrt{146}\sqrt{116}}$

7. Clearly $\mathbf{0}$ is in S by taking $x = y = z = 0$. For $\mathbf{F} = (x, y, z, x, x)$ and $\mathbf{G} = (a, b, c, a, a)$ we have $\mathbf{F}+\mathbf{G} = (x+a, y+b, z+c, x+a, x+a)$ is in S and $\alpha\mathbf{F} = (\alpha x, \alpha y, \alpha z, \alpha x, \alpha x)$ is in S, so S is a subspace of R^5.

8. S is a subspace of R^4 since by taking $x = y = 0$ we get $\mathbf{0}$ in S, and for $\mathbf{F} = (x, 2x, 3x, y)$ and $\mathbf{G} = (a, 2a, 3a, b)$ we have $\mathbf{F}+\mathbf{G} = (x+a, 2(x+a), 3(x+a), y+b)$ is in S, and $\alpha\mathbf{F} = (\alpha x, 2\alpha x, 3\alpha x, \alpha y)$ is in S.

9. S is not a subspace of R^6 since every vector in S has 4^{th} component equal to 1. Hence $\mathbf{0} = (0, 0, 0, 0, 0, 0)$ is not in S.

10. S is a subspace of R^3. With $x = y = 0$ we have $\mathbf{0}$ in S; $\alpha(0, x, y) = (0, \alpha x, \alpha y)$ is in S, and $(0, x, y) + (0, a, b) = (0, x+a, y+b)$ is in S.

11. S is a subspace of R^4. With $x = y = 0$ we have $\mathbf{0}$ in S; $\alpha(x, y, x+y, x-y) = (\alpha x, \alpha y, \alpha(x+y), \alpha(x-y))$ is in S and $(x, y, x+y, x-y) + (a, b, a+b, a-b) = (x+a, y+b, (x+a)+(y+b), (x+a)-(y+b))$ is in S.

12. S is a subspace of R^7. Clearly $\mathbf{0}$ is in S. Also for $\mathbf{F}, \mathbf{G}$ in $S, \mathbf{F}+\mathbf{G}$ and $\alpha\mathbf{F}$ will have third and fifth components equal to zero.

13. S is a subspace of R^4 since $\mathbf{0}, \mathbf{F}+\mathbf{G}$, and $\alpha\mathbf{F}$ will all have equal first and second components whenever $\mathbf{F}$ and $\mathbf{G}$ have.

14. S is not a subspace of R^3 because $\mathbf{0}$ is not in S. Note $\mathbf{0} = (0, 0, 0)$ is not on the plane $ax + by + cz = k$ for $k \neq 0$.

15. For $\mathbf{F}, \mathbf{G}$ in R^n we have $\|\mathbf{F}+\mathbf{G}\|^2 = (\mathbf{F}+\mathbf{G})\cdot(\mathbf{F}+\mathbf{G}) = (\mathbf{F}\cdot\mathbf{F}) + (\mathbf{F}\cdot\mathbf{G}) + (\mathbf{G}\cdot\mathbf{F}) + (\mathbf{G}\cdot\mathbf{G}) = \|\mathbf{F}\|^2 + 2(\mathbf{F}\cdot\mathbf{G}) + \|\mathbf{G}\|^2$, and $\|\mathbf{F}-\mathbf{G}\|^2 = \|\mathbf{F}\|^2 - 2(\mathbf{F}\cdot\mathbf{G}) + \|\mathbf{G}\|^2$. Thus $\|\mathbf{F}+\mathbf{G}\|^2 + \|\mathbf{F}-\mathbf{G}\|^2 = 2(\|\mathbf{F}\|^2 + \|\mathbf{G}\|^2)$.

16. For $\mathbf{F}, \mathbf{G}$ in R^n we have $\|\mathbf{F}+\mathbf{G}\|^2 = \mathbf{F}\cdot\mathbf{F} + \mathbf{F}\cdot\mathbf{G} + \mathbf{G}\cdot\mathbf{F} + \mathbf{G}\cdot\mathbf{G} = \|\mathbf{F}\|^2 + \|\mathbf{G}\|^2$, since $\mathbf{F}\cdot\mathbf{G} = \mathbf{G}\cdot\mathbf{F} = 0$ because $\mathbf{F}$ and $\mathbf{G}$ are orthogonal.

17. Yes.
For any $\mathbf{F}, \mathbf{G}$ in R^n we have
$\|\mathbf{F}+\mathbf{G}\|^2 = \|\mathbf{F}\|^2 + 2(\mathbf{F}+\mathbf{G}) + \|\mathbf{G}\|^2$. If in addition, $\mathbf{F}$ and $\mathbf{G}$ satisfy Pythagoras's Theorem, then $\|\mathbf{F}+\mathbf{G}\|^2 = \|\mathbf{F}\|^2 + \|G\|^2$. It follows that $(\mathbf{F}\cdot\mathbf{G}) = 0$, and $\mathbf{F}$ and $\mathbf{G}$ are orthogonal.

Section 6.5 Linear Independence, Spanning Sets and Dimension

1. Suppose $\alpha(3\mathbf{i}+2\mathbf{j}) + \beta(\mathbf{i}-\mathbf{j}) = \mathbf{0}$. Then $3\alpha+\beta = 0$ and $2\alpha - \beta = 0$. But the only solution to these two equations is $\alpha = \beta = 0$. Thus $3\mathbf{i}+2\mathbf{j}$ and $\mathbf{i}-\mathbf{j}$ are linearly independent

2. Suppose $\alpha_1(2\mathbf{i}) + \alpha_2(3\mathbf{j}) + \alpha_3(5\mathbf{i} - 12\mathbf{k}) + \alpha_4(\mathbf{i} + \mathbf{j} + \mathbf{k}) = \mathbf{0}$. Then $2\alpha_1 + 5\alpha_3 + \alpha_4 = 0, 3\alpha_2 + \alpha_4 = 0, -12\alpha_3 + \alpha_4 = 0$. By inspection we see $\alpha_1 = -17/2, \alpha_2 = -4, \alpha_3 = 1, \alpha_4 = 12$ is a non-trivial solution of these equations. It follows that $2\mathbf{i}, 3\mathbf{j}, 5\mathbf{i} - 12\mathbf{k}, \mathbf{i} + \mathbf{j} + \mathbf{k}$ are linearly dependent.

3. These vectors are linearly independent by Lemma 5.3.

4. Since $4(1, 0, 0, 0) - 6(0, 1, 1, 0) + (-4, 6, 6, 0) = \mathbf{0}$ these vectors are linearly dependent.

5. Since $2(1, 2, -3, 1) + (4, 0, 0, 2) - (6, 4, -6, 4) = \mathbf{0}$ these vectors are linearly dependent.

6. Suppose $\alpha_1(0, 1, 1, 1) + \alpha_2(-3, 2, 4, 4) + \alpha_3(-2, 2, 34, 2) + \alpha_4(1, 1, -6, 2) = \mathbf{0}$. This gives the matrix system of equations

$$\begin{pmatrix} 0 & -3 & -2 & 1 \\ 1 & 2 & 2 & 1 \\ 1 & 4 & 34 & -6 \\ 1 & 4 & 2 & 2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Since the determinant of the coefficient matrix equals 129, the system has only the trivial solution $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4$. Thus the vectors are linearly independent.

7. Suppose $\alpha_1(1, -2) + \alpha_2(4, 1) + \alpha_3(6, 6) = \mathbf{0}$. Then $\alpha_1 + 4\alpha_2 + 6\alpha_3 = 0$ and $-2\alpha_1 + \alpha_2 + 6\alpha_3 = 0$. By subtracting those two equations we see $3\alpha_1 + 3\alpha_2 = 0$. Thus choose $\alpha_1 = 2, \alpha_2 = -2$ and $\alpha_3 = 1$ and we see the vectors are linearly dependent.

8. Suppose $\alpha_1(-1, 1, 0, 0, 0) + \alpha_2(0, -1, 1, 0, 0) + \alpha_3(0, 1, 1, 1, 0) = \mathbf{0}$. Then $-\alpha_1 = 0, \alpha_1 - \alpha_2 + \alpha_3 = 0, \alpha_2 + \alpha_3 = 0, \alpha_3 = 0$ from which we see the only solution is $\alpha_1 = \alpha_2 = \alpha_3 = 0$. Thus the vectors are linearly independent.

9. Suppose $\alpha_1(-2, 0, 0, 1, 1) + \alpha_2(1, 0, 0, 0, 0) + \alpha_3(0, 0, 0, 0, 2) + \alpha_4(1, -1, 3, 3, 1) = \mathbf{0}$. Then certainly $\alpha_4 = 0$ and $-2\alpha_1 + \alpha_2 + \alpha_4 = 0, \alpha_1 + 3\alpha_4 = 0$ and $\alpha_1 + 2\alpha_3 + \alpha_4 = 0$. The only solution of these equations is $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$. Thus the vectors are linearly independent.

10. Suppose $\alpha_1(3, 0, 0, 4) + \alpha_2(2, 0, 0, 8) = \mathbf{0}$. Then $3\alpha_1 + 2\alpha_2 = 0$ and $4\alpha_1 + 8\alpha_2 = 0$. The only solution of these equations is $\alpha_1 = \alpha_2 = 0$. Thus the vectors are linearly independent.

11. Suppose $\mathbf{F}, \mathbf{G}, \mathbf{H}$ are linearly dependent so that $\alpha\mathbf{F} + \beta\mathbf{G} + \gamma\mathbf{H} = 0$ with at least one of α, β, γ nonzero. Specifically, suppose $\gamma \neq 0$ so that $\mathbf{H} = -\frac{\alpha}{\gamma}\mathbf{F} - \frac{\beta}{\gamma}\mathbf{G}$. Then $[\mathbf{F}, \mathbf{G}, \mathbf{H}] = \mathbf{F} \cdot \mathbf{G} \times \mathbf{H} = 0$ by Problem 42, Section 5.3. Conversely, let $\mathbf{F} = (a_1, b_1, c_1), \mathbf{G} = (a_2, b_2, c_2), \mathbf{H} = (a_3, b_3, c_3)$ and suppose $[\mathbf{F}, \mathbf{G}, \mathbf{H}] = 0$. Then By Problem 39, Section 5.3, the determinant $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$. Thus the system of equations $\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ has a nontrivial solution, i.e. at least one of α, β, γ nonzero. But then $\alpha\mathbf{F} + \beta\mathbf{G} + \gamma\mathbf{H} = 0$ and $\mathbf{F}, \mathbf{G}, \mathbf{H}$ are linearly dependent.

12. $[3\mathbf{i} + 6\mathbf{j} - \mathbf{k}, 8\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}, \mathbf{i} - \mathbf{j} + \mathbf{k}] = \begin{vmatrix} 3 & 6 & -1 \\ 8 & 2 & -4 \\ 1 & -1 & 1 \end{vmatrix} = -68 \neq 0$, so the vectors are linearly independent.

13. $[\mathbf{i}+6\mathbf{j}-2\mathbf{k}, -\mathbf{i}+4\mathbf{j}-3\mathbf{k}, \mathbf{i}+16\mathbf{j}-7\mathbf{k}] = \begin{vmatrix} 1 & 6 & -2 \\ -1 & 4 & -3 \\ 1 & 16 & -7 \end{vmatrix} = 0$, so the vectors are linearly dependent.

14. $[4\mathbf{i}-3\mathbf{j}+\mathbf{k}, 10\mathbf{i}-3\mathbf{j}, 2\mathbf{i}-6\mathbf{j}+3\mathbf{k}] = \begin{vmatrix} 4 & -3 & 1 \\ 10 & -3 & 0 \\ 2 & -6 & 3 \end{vmatrix} = 0$, so the vectors are linearly dependent.

15. $[8\mathbf{i}+6\mathbf{j}, 2\mathbf{i}-4\mathbf{j}, \mathbf{i}+\mathbf{k}] = \begin{vmatrix} 8 & 6 & 0 \\ 2 & -4 & 0 \\ 1 & 0 & 1 \end{vmatrix} = -44 \neq 0$, so the vectors are linearly independent.

16. $[12\mathbf{i}-3\mathbf{k}, \mathbf{i}+2\mathbf{j}-\mathbf{k}, -3\mathbf{i}+4\mathbf{j}] = \begin{vmatrix} 12 & 0 & -3 \\ 1 & 2 & -1 \\ -3 & 4 & 0 \end{vmatrix} = 18 \neq 0$, so the vectors are linearly independent.

17. A basis for S is the set of vectors $(1,0,0,-1)$ and $(0,1,-1,0)$, so S has dimension two.

18. A basis for S is the set of vectors $(1,0,2,0)$ and $(0,1,0,3)$, so S has dimension two.

19. A point (x,y,z) is on the plane $2x-y+z=0$ if and only if $y=2x+z$. Thus all vectors in the plane have the form $(x, 2x+z, z)$ and every such vector can be expressed as a linear combination $x(1,2,0)+z(0,1,1)$ for some x,z. Since $(1,2,0)$ and $(0,1,1)$ are linearly independent, they form a basis for S, so S has dimension two as we would expect for a plane.

20. A basis for S is the set of vectors $(1,0,0,1,0), (0,1,-1,-1,0), (0,0,0,0,1)$, so S has dimension three.

21. A basis for S is the set of vectors $(1,0,0,0), (0,0,1,0), (0,0,0,1)$, so S has dimension three.

22. A basis for S is the set of vectors $(-1,1,0,0)$ and $(0,0,1,2)$, so S has dimension two.

23. Every vector in R^2 parallel to the line $y=4x$ has the form $(x,4x)$ for some x. A basis for S is the vector $(1,4)$, so S has dimension one, as we would expect for a line.

24. Every vector in S has the form $(x,y,4x+2y) = x(1,0,4)+y(0,1,2)$ for some x,y. A basis for S is the vectors $(1,0,4)$ and $(0,1,2)$, so S has dimension two.

Chapter Seven - Matrices and Systems of Linear Equations

Section 7.1 Matrices

1. $2A - 3B = \begin{pmatrix} 14 & -2 & 6 \\ 10 & -5 & -6 \\ -26 & -43 & -8 \end{pmatrix}$

2. $-5A + 3B = \begin{pmatrix} 19 & 2 \\ 6 & -2 \\ -28 & 38 \\ -27 & 35 \end{pmatrix}$

3. $A^2 + 2AB = \begin{pmatrix} 2 + 2x - x^2 & -12x + (1-x)(x + e^x + 2\cos(x)) \\ 4 + 2x + 2e^x + 2xe^x & -22 - 2x + e^{2x} + 2e^x \cos x \end{pmatrix}$

4. $-3A - 5B = (18)$

5. $4A + 8B = \begin{pmatrix} -36 & 0 & 68 & 196 & 20 \\ 128 & -40 & -36 & -8 & 72 \end{pmatrix}$

6. $A^2 - B^2 = \begin{pmatrix} 47 & -11 \\ 4 & 43 \end{pmatrix} - \begin{pmatrix} -40 & 8 \\ -5 & -39 \end{pmatrix} = \begin{pmatrix} 27 & 10 \\ 11 & 40 \end{pmatrix}$

7. $AB = \begin{pmatrix} -10 & -34 & -16 & -30 & -14 \\ 10 & -2 & -11 & -8 & -45 \\ -5 & 1 & 15 & 61 & -63 \end{pmatrix}$; BA is not defined

8. $AB = \begin{pmatrix} -16 & 0 \\ 17 & 28 \end{pmatrix}$; $BA = \begin{pmatrix} 12 & -32 \\ -14 & 0 \end{pmatrix}$

9. $AB = (115)$; $BA = \begin{pmatrix} 3 & -18 & -6 & -42 & 66 \\ -2 & 12 & 4 & 28 & -44 \\ -6 & 36 & 12 & 84 & -132 \\ 0 & 0 & 0 & 0 & 0 \\ 4 & -24 & -8 & -56 & 88 \end{pmatrix}$

10. $AB = \begin{pmatrix} 48 & 1 & 1 & -58 \\ -96 & 2 & 2 & 220 \\ -288 & -22 & -22 & -68 \\ -16 & 6 & 6 & 184 \end{pmatrix}$; $BA = \begin{pmatrix} 76 & 152 \\ 50 & 136 \end{pmatrix}$

11. AB not defined; $BA = \begin{pmatrix} 410 & 36 & -56 & 227 \\ 17 & 253 & 40 & -1 \end{pmatrix}$

12. $AB = \begin{pmatrix} -22 & 30 & -10 & -4 \\ -42 & 45 & 30 & 6 \end{pmatrix}$; BA not defined.

13. AB not defined; $BA = (-16 \quad -13 \quad -5)$

14. Neither AB nor BA is defined.

15. $AB = \begin{pmatrix} 39 & -84 & 21 \\ -23 & 38 & 3 \end{pmatrix}$; BA not defined.

16. AB not defined; $BA = (28\ \ 30)$

17. AB is 14×14; BA is 21×21

18. Neither AB nor BA is defined.

19. AB is not defined; BA is 4×2

20. AB is 1×3; BA is not defined.

21. AB is not defined; BA is 7×6

22. Take $A = \begin{pmatrix} 2 & 1 \\ 8 & 4 \end{pmatrix}, B = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}, C = \begin{pmatrix} 6 & 0 \\ -1 & 1 \end{pmatrix}$. Then $BA = CA = \begin{pmatrix} 12 & 6 \\ 6 & 3 \end{pmatrix}$, but $B \neq C$. Many other solutions are possible.

23. For the graph G the adjacency matrix is $A = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix}$

We find

$$A^3 = \begin{pmatrix} 2 & 7 & 7 & 4 & 4 \\ 7 & 8 & 9 & 9 & 9 \\ 7 & 9 & 8 & 9 & 9 \\ 4 & 9 & 9 & 6 & 7 \\ 4 & 9 & 9 & 7 & 6 \end{pmatrix} \text{ and } A^4 = \begin{pmatrix} 14 & 17 & 17 & 18 & 18 \\ 17 & 34 & 33 & 26 & 26 \\ 17 & 33 & 34 & 26 & 26 \\ 18 & 26 & 26 & 25 & 24 \\ 18 & 26 & 26 & 24 & 25 \end{pmatrix}$$

The number of distinct $v_1 - v_4$ walks of length 3 is $(A^3)_{14} = 4$, and of length 4 is $(A^4)_{14} = 18$; the number of distinct $v_2 - v_3$ walks of length 3 is $(A^3)_{23} = 9$; the number of distinct $v_2 - v_4$ walks of length 4 is $(A^4)_{24} = 26$.

24. For the graph G the adjacency matrix is $A = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix}$

We find

$$A^2 = \begin{pmatrix} 3 & 2 & 1 & 2 & 1 \\ 2 & 3 & 1 & 2 & 1 \\ 1 & 1 & 3 & 0 & 3 \\ 2 & 2 & 0 & 2 & 0 \\ 1 & 1 & 3 & 0 & 3 \end{pmatrix} \text{ and} A^4 = \begin{pmatrix} 19 & 18 & 11 & 14 & 11 \\ 18 & 19 & 11 & 14 & 11 \\ 11 & 11 & 20 & 4 & 20 \\ 14 & 14 & 4 & 12 & 4 \\ 11 & 11 & 20 & 4 & 20 \end{pmatrix}$$

The number of distinct $v_1 - v_4$ walks of length 4 is $(A^4)_{14} = 14, v_2 - v_3$ walks of length 2 is $(A^2)_{23} = 1$

25. For the graph G the adjacency matrix is $A = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{pmatrix}$

We find

$$A^2 = \begin{pmatrix} 4 & 2 & 3 & 3 & 2 \\ 2 & 3 & 2 & 2 & 3 \\ 3 & 2 & 4 & 3 & 2 \\ 3 & 2 & 3 & 4 & 2 \\ 2 & 3 & 2 & 2 & 3 \end{pmatrix}, \; A^3 = \begin{pmatrix} 10 & 10 & 11 & 11 & 10 \\ 10 & 6 & 10 & 10 & 6 \\ 11 & 10 & 10 & 11 & 10 \\ 11 & 10 & 11 & 10 & 10 \\ 10 & 6 & 10 & 10 & 6 \end{pmatrix}$$

and

$$A^4 = \begin{pmatrix} 42 & 32 & 41 & 41 & 32 \\ 32 & 30 & 32 & 32 & 30 \\ 41 & 32 & 42 & 41 & 32 \\ 41 & 32 & 41 & 42 & 32 \\ 32 & 30 & 32 & 32 & 30 \end{pmatrix}$$

The number of distinct $v_4 - v_5$ walks of length 2 is $(A^2)_{45} = 2; v_2 - v_3$ walks of length 3 is $(A^3)_{23} = 10; v_1 - v_2$ walks of length 4 is $(A^4)_{12} = 32$; and $v_4 - v_5$ walks of length 4 is $(A^4)_{45} = 32$.

26. (a) The ii element of A^2 is the number of distinct $v_i - v_i$ walks of length 2 in G. Such a walk must have the form $v_i \to v_j \to v_i$ for same $j \neq i$. Thus $(A^2)_{ii}$ enumerates the number of vertices in G adjacent to v_i.
(b) The ii element of A^3 is the number of distinct walks of length 3 in G which start and end at v_i. Such a walk must have the form $v_i \to v_j \to v_k \to v_i$ for same $j \neq k$, and neither j nor k equal i. These three vertices constitute a triangle in G, but each triangle gets counted twice since such a walk could be $v_i \to v_j \to v_k \to v_i$ or $v_i \to v_k \to v_j \to v_i$. Thus $(A^3)_{ii} =$ twice the number of triangles in G containing v_i as a point.

Section 7.2 Elementary Row Operations and Elementary Matrices

In Problems 1 through 8, the first matrix given below is the result of performing the listed row operations directly on A. The second matrix is a matrix Ω which will produce the same modified matrix as the matrix product ΩA. Where several row operations are listed, the matrix Ω can be computed as a product of elementary matrices, $\Omega = E_k E_{k-1} \ldots E_1 I_n$, where E_j is the matrix which corresponds to operation E_j.

1. $\begin{pmatrix} -2 & 1 & 4 & 2 \\ 0 & \sqrt{3} & 16\sqrt{3} & 3\sqrt{3} \\ 1 & -2 & 4 & 8 \end{pmatrix}; \; \Omega = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & 1 \end{pmatrix}$

2. $\begin{pmatrix} 3 & -6 \\ 1 & 1 \\ 14 & 4 \\ 0 & 5 \end{pmatrix}; \; \Omega = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 6 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

3. $\begin{pmatrix} 40 & 5 & -15 \\ -2+2\sqrt{13} & 14+9\sqrt{13} & 6+5\sqrt{13} \\ 2 & 9 & 5 \end{pmatrix}$;

$\Omega = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & \sqrt{13} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 5 & 0 \\ 1 & 0 & \sqrt{13} \\ 0 & 0 & 1 \end{pmatrix}$

4. $\begin{pmatrix} -4 & 6 & -3 \\ 5 & -3 & 3 \\ 12 & 4 & -4 \end{pmatrix}$; $\Omega = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

5. $\begin{pmatrix} 30 & 120 \\ -3+2\sqrt{3} & 15+8\sqrt{3} \end{pmatrix}$; $\Omega = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 15 \end{pmatrix} \begin{pmatrix} 1 & \sqrt{3} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 15 \\ 1 & \sqrt{3} \end{pmatrix}$

6. $\begin{pmatrix} 3 & -4 & 5 & 9 \\ 2+3\sqrt{3} & 1-4\sqrt{3} & 3+5\sqrt{3} & -6+9\sqrt{3} \\ 18+3\sqrt{3} & 37-4\sqrt{3} & 31+5\sqrt{3} & 54+9\sqrt{3} \end{pmatrix}$;

$\Omega = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \sqrt{3} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \sqrt{3} & 1 & 0 \\ 4+\sqrt{3} & 1 & 4 \end{pmatrix}$

7. $\begin{pmatrix} -1 & 0 & 3 & 0 \\ -36 & 28 & -20 & 28 \\ -13 & 3 & 44 & 9 \end{pmatrix}$; $\Omega = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 14 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 4 \\ 14 & 1 & 0 \end{pmatrix}$

8. $\begin{pmatrix} 28 & 50 & 2 \\ 9 & 15 & 0 \\ 0 & -45 & 70 \end{pmatrix}$; $\Omega = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 3 \\ 0 & 0 & 1 \\ 5 & 0 & 0 \end{pmatrix}$

9. Let $A = (a_{ij})$ be any $m \times n$ matrix. Since B and E are obtained, respectively, by interchanging rows s and t of A and I_m, we have, for $i \neq s$ and $i \neq t, b_{ij} = a_{ij}, e_{ij} = \delta_{ij}$; for $i = s, b_{sj} = a_{tj}, e_{sj} = \delta_{tj}$; and for $i = t, b_{tj} = a_{sj}, e_{tj} = \delta_{sj}$ for $j = 1, 2, \ldots, m$. Now consider the ij^{th} element of EA. For $i \neq s$ and $i \neq t, (EA)_{ij} = \sum_{k=1}^{m} e_{ik}a_{kj} = a_{ij} = b_{ij}$; for $i = s, (EA)_{sj} = \sum_{k=1}^{m} e_{sk}a_{kj} = \sum_{k=1}^{m} \delta_{tk}a_{kj} = a_{tj} = b_{sj}$; and for $i = t, (EA)_{tj} = \sum_{k=1}^{m} e_{tk}a_{kj} = \sum_{k=1}^{m} \delta_{sk}a_{kj} = a_{sj} = b_{tj}$ for $j = 1, 2, \ldots, n$. Thus $EA = B$.

10. Let $A = (a_{ij})$ be any $m \times n$ matrix. Since B and E are obtained, respectively, by multiplying row s of A and I_m by α, we have for $i \neq s, b_{ij} = a_{ij}, e_{ij} = \delta_{ij}$, whereas for $i = s, b_{sj} = \alpha a_{sj}, e_{sj} = \alpha\delta_{sj}$. Now consider the ij^{th} element of EA. For $i \neq s, (EA)_{ij} = \sum_{k=1}^{m} e_{ik}a_{kj} = \sum_{k=1}^{m} \alpha\delta_{ik}a_{kj} = a_{ij} = b_{ij}$, whereas $(EA)_{sj} = \sum_{k=1}^{m} e_{sk}a_{kj} = \sum_{k=1}^{m} \alpha\delta_{sk}a_{kj} = \alpha a_{sj} = b_{sj}$, for $j = 1, 2, \ldots, n$. Thus $EA = B$.

11. Let $A = (a_{ij})$ be any $m \times n$ matrix. Since B and E are obtained, respectively, from A and I_m by adding α times row s to row t, we have for $i \neq t, b_{ij} = a_{ij}, e_{ij} = \delta_{ij}$, whereas

for $i = t, b_{tj} = a_{tj} + \alpha a_{sj}, e_{tj} = \delta_{tj} + \alpha\delta_{sj}$. Now consider the ij^{th} element of EA. For $i \neq t, (EA)_{ij} = \sum_{k=1}^{m} e_{ik}a_{kj} = \sum_{k=1}^{m} \delta_{ik}a_{kj} = a_{ij}$; whereas, for $i = t, (EA)_{tj} = \sum_{k=1}^{m} e_{tk}a_{kj} = \sum_{k=1}^{m}(\delta_{tk} + \alpha\delta_{sj})a_{kj} = a_{tj} + \alpha a_{sj} = b_{sj}$. Thus $EA = B$.

Section 7.3 The Row Echelon Form of a Matrix

In Problems 1 through 12, the matrix A_R is the reduced matrix; the matrix Ω is a matrix which will reduce A to A_R by $A_R = \omega A$.

1. $A_R = \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$; $\Omega = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

2. $A_R = \begin{pmatrix} 1 & 0 & 1/3 & 4/3 \\ 0 & 1 & 0 & 0 \end{pmatrix}$; $\Omega = \begin{pmatrix} 1/3 & -1/3 \\ 0 & 1 \end{pmatrix}$

3. $A_R = \begin{pmatrix} 1 & -4 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$; $\Omega = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$

4. $A_R = \begin{pmatrix} 1 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & 0 & 2 \end{pmatrix}$; $\Omega = I_2$

5. $A_R = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$; $\Omega = \begin{pmatrix} 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -6 & 17 \\ 0 & 1 & 0 & 0 \end{pmatrix}$

6. $A_R = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$; $\Omega = \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}$

7. $A_R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$; $\Omega = \frac{1}{270}\begin{pmatrix} -8 & -2 & 38 \\ 37 & 43 & -7 \\ 19 & -29 & 11 \end{pmatrix}$

8. $A_R = \begin{pmatrix} 1 & -4/3 & -4/3 \\ 0 & 0 & 0 \end{pmatrix}$; $\Omega = \begin{pmatrix} -1/3 & 0 \\ 0 & 1 \end{pmatrix}$

9. $A_R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 3/2 & 1/2 \end{pmatrix}$; $\Omega = \begin{pmatrix} 0 & 1 \\ 1/2 & 1/2 \end{pmatrix}$

10. $A_R = \begin{pmatrix} 1 & 0 & 0 & -3/4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{pmatrix}$; $\Omega = \frac{1}{4}\begin{pmatrix} 0 & 0 & 1 \\ 4 & -4 & -8 \\ -4 & 8 & 8 \end{pmatrix}$

11. $A_R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$; $\Omega = \begin{pmatrix} 0 & 1/2 & -1 \\ 0 & 0 & 1 \\ -1/7 & 2/7 & -3/7 \end{pmatrix}$

12. $A_R = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$; $\Omega = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 1 & 0 & -6 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}$

Section 7.4 The Row and Column Spaces and Rank of a Matrix

1. (a) $A = \begin{pmatrix} -4 & 1 & 3 \\ 2 & 2 & 0 \end{pmatrix}$ has reduced form $A_R = \begin{pmatrix} 1 & 0 & -3/5 \\ 0 & 1 & 3/5 \end{pmatrix}$, so rank $(A) = 2$.

(b) From the leading one's in rows 1 and 2 of A_R we see $\mathbf{R}_1 = (-4, 1, 3)$ and $\mathbf{R}_2 = (2, 2, 0)$ are linearly independent vectors, hence form a basis for the row space of A which is a two dimensional subspace of R^3.

(c) By similar reasoning, the leading one's in columns 1 and 2 of A_R indicate that columns $\mathbf{C}_1 = \begin{pmatrix} -4 \\ 2 \end{pmatrix}$ and $\mathbf{C}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ are linearly independent columns of A, hence form a basis for the column space of A which is a two dimensional subspace of R^2 and hence is all of R^2.

2. (a) $A = \begin{pmatrix} 1 & -1 & 4 \\ 0 & 1 & 3 \\ 2 & -1 & 11 \end{pmatrix}$ has reduced form $A_R = \begin{pmatrix} 1 & 0 & 7 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$, so rank $(A) = 2$.

(b) From A_R we see $\mathbf{R}_1 = (1, -1, 4)$ and $\mathbf{R}_2 = (0, 1, 3)$ constitute a basis for the row space of A, which is a dimension 2 subspace of R^3.

(c) From A_R we see $\mathbf{C}_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ and $\mathbf{C}_2 = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$ constitute a basis for the column space of A, which is a dimension 2 subspace of R^3.

3. (a) $A = \begin{pmatrix} -3 & 1 \\ 2 & 2 \\ 4 & -3 \end{pmatrix}$ has reduced form $A_R = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$, so rank $(A) = 2$.

(b) $\mathbf{R}_1 = (-3, 1)$ and $\mathbf{R}_2 = (2, 2)$ constitute a basis for the row space of A, which has dimension 2, hence is all of R^2.

(c) $\mathbf{C}_1 = \begin{pmatrix} -3 \\ 2 \\ 4 \end{pmatrix}$, $\mathbf{C}_2 = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$ constitute a basis for the column space of A, which is a dimension 2 subspace of R^3.

4. (a) $A = \begin{pmatrix} 6 & 0 & 0 & 1 & 1 \\ 12 & 0 & 0 & 2 & 2 \\ 1 & -1 & 0 & 0 & 0 \end{pmatrix}$ has reduced form $A_R = \begin{pmatrix} 1 & 0 & 0 & 1/6 & 1/6 \\ 0 & 1 & 0 & 1/6 & 1/6 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$, which involved interchanging rows 2 and 3. From A_R we see rank $(A) = 2$.

(b) Since rows 2 and 3 were interchanged to get A_R we see $\mathbf{R}_1 = (6, 0, 0, 1, 1)$ and $\mathbf{R}_3 = (1, -1, 0, 0, 0)$ constitute a basis for the row space of A, which is a dimension 2 subspace of R^5.

(c) $\mathbf{C}_1 = \begin{pmatrix} 6 \\ 12 \\ 1 \end{pmatrix}$, $\mathbf{C}_2 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$ constitute a basis for the column space of A, which is a dimension 2 subspace of R^3.

5. (a) $A = \begin{pmatrix} 8 & -4 & 3 & 2 \\ 1 & -1 & 1 & 0 \end{pmatrix}$ has reduced form $A_R = \begin{pmatrix} 1 & 0 & -1/4 & 1/2 \\ 0 & 1 & -5/4 & 1/2 \end{pmatrix}$, so rank $(A) = 2$.

(b) $\mathbf{R}_1 = (8, -4, 3, 2)$ and $\mathbf{R}_2 = (1, -1, 1, 0)$ constitute a basis for the row space of A, which is a dimension 2 subspace of R^4.

(c) $\mathbf{C}_1 = \begin{pmatrix} 8 \\ 1 \end{pmatrix}, \mathbf{C}_2 = \begin{pmatrix} -4 \\ -1 \end{pmatrix}$ constitute a basis for the column space of A, which is all of R^2.

6. (a) $A = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 0 & 1. \end{pmatrix}$ is already reduced, so rank $(A) = 2$.

(b) $\mathbf{R}_1 = (1, 3, 0), \mathbf{R}_2 = (0, 0, 1)$ constitute a basis for the row space of A, which is a dimension 2 subspace of R^3.

(c) $\mathbf{C}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{C}_3 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ constitute a basis for the column space of A, which is all of R^2.

7. (a) $A = \begin{pmatrix} 2 & 2 & 1 \\ 1 & -1 & 3 \\ 0 & 0 & 1 \\ 4 & 0 & 7 \end{pmatrix}$ reduces to $A_R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, so rank $(A) = 3$.

(b) Since the reduction $A \to A_R$ required no row interchanges, $\mathbf{R}_1 = (2, 2, 1), \mathbf{R}_2 = (1, -1, 3), \mathbf{R}_3 = (0, 0, 1)$ constitute a basis for the row space of A, which has dimension 3 and is all of R^3.

(c) $\mathbf{C}_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 4 \end{pmatrix}, \mathbf{C}_2 = \begin{pmatrix} 2 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{C}_3 = \begin{pmatrix} 1 \\ 3 \\ 1 \\ 7 \end{pmatrix}$ constitute a basis for the column space of A, which is a dimension 3 subspace of R^4.

8. (a) $A = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 2 \end{pmatrix}$ reduces to $A_R = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, so rank $(A) = 2$.

(b) $\mathbf{R}_1 = (0, -1, 0), \mathbf{R}_2 = (0, 0, -1)$ constitute a basis for the row space of A, which is a dimension 2 subspace of R^3.

(c) $\mathbf{C}_2 = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{C}_3 = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}$ constitute a basis for the column space of A, which is a dimension 2 subspace of R^3.

9. (a) $A = \begin{pmatrix} 0 & 4 & 3 \\ 6 & 1 & 0 \\ 2 & 2 & 2 \end{pmatrix}$ reduces to $A_R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, so rank $(A) = 3$.

(b) $\mathbf{R}_1 = (0, 4, 3), \mathbf{R}_2 = (6, 1, 0), \mathbf{R}_3 = (2, 2, 2)$ constitute a basis for the row space of A, which has dimension 3 and is all of R^3.

(c) $\mathbf{C}_1 = \begin{pmatrix} 0 \\ 6 \\ 2 \end{pmatrix}, \mathbf{C}_2 = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}, \mathbf{C}_3 = \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}$ constitute a basis for the column space of A, which has dimension 3 and is all of R^3.

10. (a) $A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 1 & 0 & -1 \\ 3 & 0 & 0 \end{pmatrix}$ reduces to $A_R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, so rank $(A) = 2$.

(b) The reduction required interchanging rows 2 and 3, so $\mathbf{R}_1 = (1,0,0)$ and $\mathbf{R}_3 = (1,0,-1)$ constitute a basis for the row space of A, which is a dimension 2 subspace of R^3.

(c) $\mathbf{C}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 3 \end{pmatrix}, \mathbf{C}_2 = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix}$ constitute a basis for the column space of A, which is a dimension 2 subspace of R^4.

11. (a) $A = \begin{pmatrix} -3 & 2 & 2 \\ 1 & 0 & 5 \\ 0 & 0 & 2 \end{pmatrix}$ reduces to $A_R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ so rank $(A) = 3$.

(b) $\mathbf{R}_1 = (-3,2,2), \mathbf{R}_2 = (1,0,5), \mathbf{R}_3 = (0,0,2)$ constitute a basis for the row space of A, which has dimension 3 and is all of R^3.

(c) $\mathbf{C}_1 = \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}, \mathbf{C}_2 = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \mathbf{C}_3 = \begin{pmatrix} 2 \\ 5 \\ 2 \end{pmatrix}$ constitute a basis for the column space of A, which has dimension 3 and is all of R^3.

12. (a) $A = \begin{pmatrix} -4 & -2 & 1 & 6 \\ 0 & 4 & -4 & 2 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ reduces to $A_R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -13/2 \\ 0 & 0 & 1 & -7 \end{pmatrix}$ so rank $(A) = 3$.

(b) $\mathbf{R}_1 = (-4,-2,1,6), \mathbf{R}_2 = (0,4,-4,2), \mathbf{R}_3 = (1,0,0,0)$ constitute a basis for the row space of A, which is a dimension 3 subspace of R^4.

(c) $\mathbf{C}_1 = \begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix}, \mathbf{C}_2 = \begin{pmatrix} -2 \\ 4 \\ 0 \end{pmatrix}, \mathbf{C}_3 = \begin{pmatrix} 1 \\ -4 \\ 2 \end{pmatrix}$ constitute a basis for the column space of A, which has dimension 3 and is all of R^3.

13. (a) $A = \begin{pmatrix} -2 & 5 & 7 \\ 0 & 1 & -3 \\ -4 & 11 & 11 \end{pmatrix}$ reduces to $A_R = \begin{pmatrix} 1 & 0 & -11 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{pmatrix}$ so rank $(A) = 2$.

(b) $\mathbf{R}_1 = (-2,5,7), \mathbf{R}_2 = (0,1,-3)$ constitute a basis for the row space of A, which is a dimension 2 subspace of R^3.

(c) $\mathbf{C}_1 = \begin{pmatrix} -2 \\ 0 \\ -4 \end{pmatrix}, \mathbf{C}_2 = \begin{pmatrix} 5 \\ 1 \\ 11 \end{pmatrix}$ constitutes a basis for the column space of A, which is a dimension 2 subspace of R^3.

14. (a) $A = \begin{pmatrix} -3 & 2 & 1 & 1 & 0 \\ 6 & -4 & -2 & -2 & 0 \end{pmatrix}$ reduces to $A_R = \begin{pmatrix} 1 & -2/3 & -1/3 & -1/3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$, so rank $(A) = 1$.

(b) $\mathbf{R}_1 = (-3,2,1,1,0)$ constitutes a basis for the row space of A, which is a dimension 1 subspace of R^5.

(c) $\mathbf{C}_1 = \begin{pmatrix} -3 \\ 6 \end{pmatrix}$ constitutes a basis for the column space of A, which is a dimension 1 subspace of R^2.

15. By Theorem 2, rank (A) = dim (row space of A), and by Theorem 3, dim (row space of A) = dim (column space of A). But the row space of A and the column space of A^t are the same. So rank (A) = dim (column space of A) = dim (row space of A^t) = rank (A^t).

Section 7.5 Solution of Homogeneous Systems of Linear Equations

1. The coefficient matrix $A = \begin{pmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & -1 & 1 \end{pmatrix}$ has reduced form $A_R = \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 1 \end{pmatrix}$. Since rank $(A) = 2$, the general solution will have $m-$rank $(A) = 4-2 = 2$ arbitrary constants. Letting $x_3 = \alpha, x_4 = \beta$ we find all solutions can be written as $X = \alpha \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix} =$ $\alpha V_1 + \beta V_2$. Since V_1 and V_2 are linearly independent and are a spanning set for the solution space of $AX = 0, V_1$ and V_2 form a basis for the solution space of $AX = 0$ which has dimension $2 = \#$ of unknowns $-$ rank $(A) = 4 - 2$.

2. The coefficient matrix $A = \begin{pmatrix} -3 & 1 & -1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 4 \\ 0 & 0 & -3 & 2 & 1 \end{pmatrix}$ has reduced form $A_R = \begin{pmatrix} 1 & 0 & 0 & 1/9 & 11/9 \\ 0 & 1 & 0 & 2/3 & 13/3 \\ 0 & 0 & 1 & -2/3 & -1/3 \end{pmatrix}$

With $x_4 = \alpha$ and $x_5 = \beta$ the general solution can be written $X = \alpha \begin{pmatrix} -1/9 \\ -2/3 \\ 2/3 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -11/9 \\ -13/3 \\ 1/3 \\ 0 \\ 1 \end{pmatrix}$,

so the solution space of $AX = 0$ has dimension $2 = m-$ rank $(A) = 5 - 3$.

3. The coefficient matrix $A = \begin{pmatrix} -2 & 1 & 2 \\ 1 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$ has reduced form $A_R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. The only solution is $X = 0$, so the solution space of $AX = 0$ has dimension $0 = m-$ rank $(A) = 3 - 3$.

4. The coefficient matrix $A = \begin{pmatrix} 4 & 1 & -3 & 1 \\ 2 & 0 & -1 & 0 \end{pmatrix}$ has reduced form $A_R = \begin{pmatrix} 1 & 0 & -1/2 & 0 \\ 0 & 1 & -1 & 1 \end{pmatrix}$.

With $x_3 = \alpha$ and $x_4 = \beta$ the general solution can be written $X = \alpha \begin{pmatrix} 1/2 \\ 1 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$,

and these two vectors are a basis for the solution space of $AX = 0$ which has dimension $2 = m-$ rank $(A) = 4 - 2$.

5. The coefficient matrix $A = \begin{pmatrix} 1 & -1 & 3 & -1 & 4 \\ 2 & -2 & 1 & 1 & 0 \\ 1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix}$ has reduced form $A_R = \begin{pmatrix} 1 & 0 & 0 & 0 & 9/4 \\ 0 & 1 & 0 & 0 & 7/4 \\ 0 & 0 & 1 & 0 & 5/8 \\ 0 & 0 & 0 & 1 & -13/8 \end{pmatrix}$

With $x_5 = \alpha$ the general solution can be written as $X = \alpha \begin{pmatrix} -9/4 \\ -7/4 \\ -5/8 \\ 13/8 \\ 1 \end{pmatrix}$ so the solution space of $AX = 0$ has dimension $1 = m-$ rank $(A) = 5 - 4$.

6. The coefficient matrix $A = \begin{pmatrix} 6 & -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 2 \\ 1 & 0 & 0 & 0 & -2 \end{pmatrix}$ has reduced form $A_R = \begin{pmatrix} 1 & 0 & 0 & 0 & -2 \\ 0 & 1 & -1 & 0 & -12 \\ 0 & 0 & 0 & 1 & -4 \end{pmatrix}$

With $x_3 = \alpha, x_5 = \beta$ the general solution can be written as $X = \alpha \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ 12 \\ 0 \\ 4 \\ 1 \end{pmatrix}$ so the solution space of $AX = 0$ has dimension $2 = m-$ rank $(A) = 5 - 3$.

7. The coefficient matrix $A = \begin{pmatrix} -10 & -1 & 4 & -1 & 1 & -1 \\ 0 & 1 & -1 & 3 & 0 & 0 \\ 2 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{pmatrix}$ has reduced form

$A_R = \begin{pmatrix} 1 & 0 & 0 & 0 & 5/6 & 5/9 \\ 0 & 1 & 0 & 0 & 2/3 & 10/9 \\ 0 & 0 & 1 & 0 & 8/3 & 13/9 \\ 0 & 0 & 0 & 1 & 2/3 & 1/9 \end{pmatrix}$. With $x_5 = \alpha, x_6 = \beta$ the general solution can be written as $X = \alpha \begin{pmatrix} -5/6 \\ -2/3 \\ -8/3 \\ -2/3 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -5/9 \\ -10/9 \\ -13/9 \\ -1/9 \\ 0 \\ 1 \end{pmatrix}$, so the solution space of $AX = 0$ has dimension $2 = m-$ rank $(A) = 6 - 4$.

8. The coefficient matrix $A = \begin{pmatrix} 8 & 0 & -2 & 0 & 0 & 1 \\ 2 & -1 & 0 & 3 & 0 & -1 \\ 0 & 1 & 1 & 0 & -2 & -1 \\ 0 & 0 & 0 & 1 & -3 & 2 \end{pmatrix}$ has reduced form

$A_R = \begin{pmatrix} 1 & 0 & 0 & 0 & 7/6 & -5/4 \\ 0 & 1 & 0 & 0 & -20/3 & 9/2 \\ 0 & 0 & 1 & 0 & 14/3 & -11/2 \\ 0 & 0 & 0 & 1 & -3 & 2 \end{pmatrix}$. With $x_5 = \alpha, x_6 = \beta$ the general solution can be written as $X = \alpha \begin{pmatrix} -7/6 \\ 20/3 \\ -14/3 \\ 3 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 5/4 \\ -9/2 \\ 11/2 \\ -2 \\ 0 \\ 1 \end{pmatrix}$, so the solution space of $AX = 0$ has dimension $2 = m-$ rank $(A) = 6 - 4$.

9. The coefficient matrix $A = \begin{pmatrix} 0 & 1 & -3 & 1 \\ 2 & -1 & 1 & 0 \\ 2 & -3 & 0 & 4 \end{pmatrix}$ has reduced form $A_R = \begin{pmatrix} 1 & 0 & 0 & -5/14 \\ 0 & 1 & 0 & -11/7 \\ 0 & 0 & 1 & -6/7 \end{pmatrix}$.

With $x_4 = \alpha$ the general solution can be written $X = \alpha \begin{pmatrix} 5/14 \\ 11/7 \\ 6/7 \\ 1 \end{pmatrix}$, so the solution space of $AX = 0$ has dimension $1 = m-$ rank $(A) = 4 - 3$.

10. The coefficient matrix $A = \begin{pmatrix} 4 & -3 & 0 & 1 & 1 & -3 \\ 0 & 2 & 0 & 4 & -1 & -6 \\ 3 & -2 & 0 & 0 & 4 & -1 \\ 2 & 1 & -3 & 4 & 0 & 0 \end{pmatrix}$ has reduced form

$A_R = \begin{pmatrix} 1 & 0 & 0 & 0 & -41/6 & -2/3 \\ 0 & 1 & 0 & 0 & -49/6 & -1/3 \\ 0 & 0 & 1 & 0 & -13/6 & -7/3 \\ 0 & 0 & 0 & 1 & 23/6 & -4/3 \end{pmatrix}$. With $x_5 = \alpha, x_6 = \beta$ the general solution can be written as $X = \alpha \begin{pmatrix} 41/6 \\ 49/6 \\ 13/6 \\ -23/6 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 2/3 \\ 1/3 \\ 7/3 \\ 4/3 \\ 0 \\ 1 \end{pmatrix}$, so the solution space of $AX = 0$ has dimension $2 = m-$ rank $(A) = 6 - 4$.

11. The coefficient matrix $A = \begin{pmatrix} 1 & -2 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 & 1 & -2 & 3 \\ 1 & 0 & 0 & 0 & -1 & 2 & 0 \\ 2 & 0 & 0 & -3 & 1 & 0 & 0 \end{pmatrix}$ has reduced form

$A_R = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 1 & 0 & 0 & -1 & 3/2 & -1/2 \\ 0 & 0 & 1 & 0 & 0 & -2/3 & 3 \\ 0 & 0 & 0 & 1 & -1 & 4/3 & 0 \end{pmatrix}$. With $x_5 = \alpha, x_6 = \beta, x_7 = \gamma$ the general solution can be written as $X = \alpha \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -2 \\ -3/2 \\ 2/3 \\ -4/3 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} 0 \\ 1/2 \\ -3 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$, so the solution space of

$AX = 0$ has dimension $3 = m-$ rank $(A) = 7 - 4$.

12. The coefficient matrix $A = \begin{pmatrix} 2 & 0 & 0 & 0 & -4 & 0 & 1 & 1 \\ 0 & 2 & 0 & 0 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & -4 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & -1 & 0 \end{pmatrix}$ has reduced form

$A_R = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -3 & 7/2 & -1/2 \\ 0 & 1 & 0 & 0 & 0 & -1/2 & 1/2 & -1/2 \\ 0 & 0 & 1 & 0 & 0 & -2/3 & 2/3 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1/6 & 1/6 & -1/2 \\ 0 & 0 & 0 & 0 & 1 & -3/2 & 3/2 & -1/2 \end{pmatrix}$. With $x_6 = \alpha, x_7 = \beta, x_8 = \gamma$ the general solution can be written as $X = \alpha \begin{pmatrix} 3 \\ 1/2 \\ 2/3 \\ 1/6 \\ 3/2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -7/2 \\ -1/2 \\ -2/3 \\ -1/6 \\ -3/2 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} 1/2 \\ 1/2 \\ 1 \\ 1/2 \\ 1/2 \\ 0 \\ 0 \\ 1 \end{pmatrix}$, so the solution space of $AX = 0$ has dimension $3 = m-$ rank $(A) = 8 - 5$.

Section 7.6 The Solution Space of $AX = O$

1. The solution space of $AX = 0$ has dimension 2, as shown by the two arbitrary parameters in the general solution.

2. The solution space of $AX = 0$ has dimension 2, as shown by the two arbitrary parameters in the general solution.

3. The solution space of $AX = 0$ has dimension 0, and $AX = 0$ has only the trivial solution.

4. The solution space of $AX = 0$ has dimension 2, as shown by the two arbitrary parameters in the general solution.

5. The solution space of $AX = 0$ has dimension 1, as shown by the single arbitrary parameter in the general solution.

6. The solution space of $AX = 0$ has dimension 2, as shown by the two arbitrary parameters in the general solution.

7. The solution space of $AX = 0$ has dimension 2, as shown by the two arbitrary parameters in the general solution.

8. The solution space of $AX = 0$ has dimension 2, as shown by the two arbitrary parameters in the general solution.

9. The solution space of $AX = 0$ has dimension 1, as shown by the single arbitrary parameter in the general solution.

10. The solution space of $AX = 0$ has dimension 2, as shown by the two arbitrary parameters in the general solution.

11. The solution space of $AX = 0$ has dimension 3, as shown by the three arbitrary parameters in the general solution.

12. The solution space of $AX = 0$ has dimension 3, as shown by the three arbitrary parameters in the general solution.

13. Yes, provided rank$(A) <$ number of unknowns.

14. Let A be $m \times n$ with $m < n$. Then rank$(A) = r \leq m < n$ and there will be $(n - r) > 0$ variables which can be assigned arbitrary (non-zero) values. Thus $AX = 0$ will have non-trivial solutions.

15. If A is $n \times n$ and $A_R = I_N$, then the unique solution of $AX = 0$ will be $x_1 = x_2 = \cdots = x_n = 0$ Conversely, if $AX = 0$ has only the trivial solution, then rank$(A) = n$, and $A_R = I_n$.

Section 7.7 Nonhomogeneous Systems of Linear Equations

1. The augmented matrix is $\begin{pmatrix} 3 & -2 & 1 & \vdots & 6 \\ 1 & 10 & -1 & \vdots & 2 \\ -3 & -2 & 1 & \vdots & 0 \end{pmatrix}$. The reduced form of this matrix is $\begin{pmatrix} 1 & 0 & 0 & \vdots & 1 \\ 0 & 1 & 0 & \vdots & 1/2 \\ 0 & 0 & 1 & \vdots & 4 \end{pmatrix}$. Since rank $(A) =$ rank $(A \vdots B) =$ number of unknowns $= 3$, the system has a unique solution $X = \begin{pmatrix} 1 \\ 1/2 \\ 4 \end{pmatrix}$.

2. The augmented matrix is $\begin{pmatrix} 4 & -2 & 3 & 10 & \vdots & 1 \\ 1 & 0 & 0 & -3 & \vdots & 8 \\ 2 & -3 & 0 & 1 & \vdots & 16 \end{pmatrix}$. The reduced form of this matrix is $\begin{pmatrix} 1 & 0 & 0 & -3 & \vdots & 8 \\ 0 & 1 & 0 & -7/3 & \vdots & 0 \\ 0 & 0 & 1 & 52/9 & \vdots & -31/3 \end{pmatrix}$. Since rank $(A) =$ rank $(A \vdots B)$ the system has solutions which can be expressed as $X = \begin{pmatrix} 8 \\ 0 \\ -31/3 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 3 \\ 7/3 \\ -52/9 \\ 1 \end{pmatrix}$ where α is arbitrary.

3. The augmented matrix is $\begin{pmatrix} 2 & -3 & 0 & 1 & 0 & -1 & \vdots & 0 \\ 3 & 0 & -2 & 0 & 1 & 0 & \vdots & 1 \\ 0 & 1 & 0 & -1 & 0 & 6 & \vdots & 3 \end{pmatrix}$. The reduced form of this

matrix is $\begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 17/2 & \vdots & 9/2 \\ 0 & 1 & 0 & -1 & 0 & 6 & \vdots & 3 \\ 0 & 0 & 1 & -3/2 & -1/2 & 51/4 & \vdots & 25/4 \end{pmatrix}$. Since rank (A) = rank $(A\vdots B)$ the system has solutions which can be expressed as $X = \begin{pmatrix} 9/2 \\ 3 \\ 25/4 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ 1 \\ 3/2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 0 \\ 1/2 \\ 0 \\ 1 \\ 0 \end{pmatrix} +$ $\gamma \begin{pmatrix} -17/2 \\ -6 \\ -51/4 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ where α, β, γ are arbitrary.

4. The augmented matrix is $\begin{pmatrix} 2 & -3 & \vdots & 1 \\ -1 & 3 & \vdots & 0 \\ 1 & -4 & \vdots & 3 \end{pmatrix}$. The reduced form of this matrix is $\begin{pmatrix} 1 & 0 & \vdots & 0 \\ 0 & 1 & \vdots & 0 \\ 0 & 0 & \vdots & 1 \end{pmatrix}$. Since rank $(A) = 2$ and rank $(A\vdots B) = 3$, this system has no solutions.

5. The augmented matrix is $\begin{pmatrix} 0 & 3 & 0 & -4 & 0 & 0 & \vdots & 10 \\ 1 & -3 & 0 & 0 & 4 & -1 & \vdots & 8 \\ 0 & 1 & 1 & -6 & 0 & 1 & \vdots & -9 \\ 1 & -1 & 0 & 0 & 0 & 1 & \vdots & 0 \end{pmatrix}$. The reduced form of this matrix is $\begin{pmatrix} 1 & 0 & 0 & 0 & -2 & 2 & \vdots & -4 \\ 0 & 1 & 0 & 0 & -2 & 1 & \vdots & -4 \\ 0 & 0 & 1 & 0 & -7 & 9/2 & \vdots & -38 \\ 0 & 0 & 0 & 1 & -3/2 & 3/4 & \vdots & -11/2 \end{pmatrix}$. Since rank (A) = rank $(A\vdots B)$ the system has solutions which can be expressed as $X = \begin{pmatrix} -4 \\ -4 \\ -38 \\ -11/2 \\ 0 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 2 \\ 2 \\ 7 \\ 3/2 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -2 \\ -1 \\ -9/2 \\ -3/4 \\ 0 \\ 1 \end{pmatrix}$, where α and β are arbitrary.

6. The augmented matrix is $\begin{pmatrix} 2 & -3 & 0 & 1 & \vdots & 1 \\ 0 & 3 & 1 & -1 & \vdots & 0 \\ 2 & -3 & 10 & 0 & \vdots & 0 \end{pmatrix}$. The reduced form of this matrix is $\begin{pmatrix} 1 & 0 & 0 & 1/20 & \vdots & 11/20 \\ 0 & 1 & 0 & -9/30 & \vdots & 1/30 \\ 0 & 0 & 1 & -1/10 & \vdots & -1/10 \end{pmatrix}$. Since rank (A) = rank $(A\vdots B)$ the system has solutions which can be expressed as $X = \begin{pmatrix} 11/20 \\ 1/30 \\ -1/10 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} -1/20 \\ 9/30 \\ 1/10 \\ 1 \end{pmatrix}$, where α is arbitrary.

7. The augmented matrix is $\begin{pmatrix} 8 & -4 & 0 & 0 & 10 & \vdots & 1 \\ 0 & 1 & 0 & 1 & -1 & \vdots & 2 \\ 0 & 0 & 1 & -3 & 2 & \vdots & 0 \end{pmatrix}$. (Note the x_1 column has been omitted since x_1 does not appear in the equations). The reduced form of this matrix is $\begin{pmatrix} 1 & 0 & 0 & 1/2 & 3/4 & \vdots & 9/8 \\ 0 & 1 & 0 & 1 & -1 & \vdots & 2 \\ 0 & 0 & 1 & -3 & 2 & \vdots & 0 \end{pmatrix}$. Since rank (A) = rank $(A\vdots B)$ the system has solutions which can be expressed as

$$X = \begin{pmatrix} 9/8 \\ 2 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} -1/2 \\ -1 \\ 3 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -3/4 \\ 1 \\ -2 \\ 0 \\ 1 \end{pmatrix},$$ where α and β are arbitrary.

8. The augmented matrix is $\begin{pmatrix} 2 & 0 & -3 & \vdots & 1 \\ 1 & -1 & 1 & \vdots & 1 \\ 2 & -4 & 1 & \vdots & 2 \end{pmatrix}$. The reduced form of this matrix is $\begin{pmatrix} 1 & 0 & 0 & \vdots & 3/4 \\ 0 & 1 & 0 & \vdots & -1/12 \\ 0 & 0 & 1 & \vdots & 1/6 \end{pmatrix}$. Since rank (A) = rank $(A\vdots B)$ = number of unknowns = 3, the system has a unique solution $X = \begin{pmatrix} 3/4 \\ -1/12 \\ 1/6 \end{pmatrix}$.

9. The augmented matrix is $\begin{pmatrix} 0 & 0 & 14 & 0 & -3 & 0 & 1 & \vdots & 2 \\ 1 & 1 & 1 & -1 & 0 & 1 & 0 & \vdots & -4 \end{pmatrix}$. The reduced form of this matrix is $\begin{pmatrix} 1 & 1 & 0 & -1 & 3/14 & 1 & -1/14 & \vdots & -29/7 \\ 0 & 0 & 1 & 0 & -3/14 & 0 & 1/14 & \vdots & 1/7 \end{pmatrix}$.

Since rank (A) = rank $(A\vdots B)$ the system has solutions, which can be expressed as

$$X = \begin{pmatrix} -29/7 \\ 0 \\ 1/7 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \alpha \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} -3/14 \\ 0 \\ 3/14 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \delta \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \epsilon \begin{pmatrix} 1/14 \\ 0 \\ -1/14 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

where $\alpha, \beta, \gamma, \delta, \epsilon$ are arbitrary.

10. The augmented matrix is $\begin{pmatrix} 3 & -2 & \vdots & -1 \\ 4 & 3 & \vdots & 4 \end{pmatrix}$. The reduced form of this matrix is $\begin{pmatrix} 1 & 0 & \vdots & 5/17 \\ 0 & 1 & \vdots & 16/17 \end{pmatrix}$. Since rank (A) = rank $(A\vdots B)$ = number of unknowns = 2, the system has a unique solution $X = \begin{pmatrix} 5/17 \\ 16/17 \end{pmatrix}$.

11. The augmented matrix is $\begin{pmatrix} 7 & -3 & 4 & 0 & \vdots & -7 \\ 2 & 1 & -1 & 4 & \vdots & 6 \\ 0 & 1 & 0 & -3 & \vdots & -5 \end{pmatrix}$. The reduced form of this matrix is
$\begin{pmatrix} 1 & 0 & 0 & 19/15 & \vdots & 22/15 \\ 0 & 1 & 0 & -3 & \vdots & -5 \\ 0 & 0 & 1 & -67/15 & \vdots & -121/15 \end{pmatrix}$. Since rank (A) = rank $(A\vdots B)$ the system has solutions which can be expressed $X = \begin{pmatrix} 22/15 \\ -5 \\ -121/15 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} -19/15 \\ 3 \\ 67/15 \\ 1 \end{pmatrix}$ where α is arbitrary.

12. The augmented matrix is $\begin{pmatrix} -4 & 5 & -6 & \vdots & 2 \\ 2 & -6 & 1 & \vdots & -5 \\ -6 & 16 & -11 & \vdots & 1 \end{pmatrix}$. The reduced form of this matrix is
$\begin{pmatrix} 1 & 0 & 0 & \vdots & -137/48 \\ 0 & 1 & 0 & \vdots & 1/6 \\ 0 & 0 & 1 & \vdots & 41/24 \end{pmatrix}$. Since rank (A) = rank $(A\vdots B)$ = number of unknowns = 3, the system has a unique solution $X = \begin{pmatrix} -137/48 \\ 1/6 \\ 41/24 \end{pmatrix}$.

13. The augmented matrix is $\begin{pmatrix} 4 & -1 & 4 & \vdots & 1 \\ 1 & 1 & -5 & \vdots & 0 \\ -2 & 1 & 7 & \vdots & 4 \end{pmatrix}$. The reduced form of this matrix is $\begin{pmatrix} 1 & 0 & 0 & \vdots & 16/57 \\ 0 & 1 & 0 & \vdots & 99/57 \\ 0 & 0 & 1 & \vdots & 23/57 \end{pmatrix}$. Since rank (A) = rank $(A\vdots B)$ = number of unknowns = 3, the system has a unique solution $X = \begin{pmatrix} 16/57 \\ 99/57 \\ 23/57 \end{pmatrix}$.

14. The augmented matrix is $\begin{pmatrix} -6 & 2 & -1 & 1 & \vdots & 0 \\ 1 & 4 & 0 & -1 & \vdots & -5 \\ 1 & 1 & 1 & -7 & \vdots & 0 \end{pmatrix}$. The reduced form of this matrix is
$\begin{pmatrix} 1 & 0 & 0 & 21/23 & \vdots & -15/23 \\ 0 & 1 & 0 & -11/23 & \vdots & -25/23 \\ 0 & 0 & 1 & -171/23 & \vdots & 40/23 \end{pmatrix}$. Since rank (A) = rank $(A\vdots B)$ the system has solutions which can be expressed $X = \begin{pmatrix} -15/23 \\ -25/23 \\ 40/23 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} -21/23 \\ 11/23 \\ 171/23 \\ 1 \end{pmatrix}$ where α is arbitrary.

15. $A_R X = B$ has a solution if and only if rank (A_R) = rank $(A_R \vdots B)$. Since A_R is reduced this occurs only if the last $(n-r)$ rows of B have only zero entries; hence $b_{r+1} = b_{r+2} = \ldots = b_n = 0$.

<u>Section 7.8</u> Matrix Inverses

1. $A^{-1} = -\frac{1}{5}\begin{pmatrix} 1 & -2 \\ -2 & -1 \end{pmatrix}$

2. A^{-1} does not exist since the reduced form of the matrix is $A_R = \begin{pmatrix} 1 & 1/4 \\ 0 & 0 \end{pmatrix} \neq I_2$.

3. $A^{-1} = -\frac{1}{12}\begin{pmatrix} 2 & -2 \\ -1 & -5 \end{pmatrix}$

4. $A^{-1} = -\frac{1}{4}\begin{pmatrix} 4 & 0 \\ -4 & -1 \end{pmatrix}$

5. $A^{-1} = \frac{1}{12}\begin{pmatrix} 3 & -2 \\ -3 & 6 \end{pmatrix}$

6. $A^{-1} = \frac{1}{56}\begin{pmatrix} 64 & -4 & 49 \\ -8 & 4 & -7 \\ 0 & 0 & 14 \end{pmatrix}$

7. $A^{-1} = \frac{1}{31}\begin{pmatrix} -6 & 11 & 2 \\ 3 & 10 & -1 \\ 1 & -7 & 10 \end{pmatrix}$

8. A^{-1} does not exist since the reduced form of the matrix is $A_R = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \neq I_3$.

9. $A^{-1} = -\frac{1}{12}\begin{pmatrix} 6 & -6 & 0 \\ -3 & -9 & 2 \\ 3 & -3 & -2 \end{pmatrix}$

10. A^{-1} does not exist since the reduced form of the matrix is $A_R = \begin{pmatrix} 1 & 0 & 28/27 \\ 0 & 1 & 14/9 \\ 0 & 0 & 0 \end{pmatrix} \neq I_3$.

11. $X = A^{-1}B = \frac{1}{11}\begin{pmatrix} -1 & -1 & 8 & 4 \\ -9 & 2 & -5 & 14 \\ 2 & 2 & -5 & 3 \\ 3 & 3 & -2 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 0 \\ -5 \end{pmatrix} = \frac{1}{11}\begin{pmatrix} -23 \\ -75 \\ -9 \\ 14 \end{pmatrix}$

12. $X = A^{-1}B = \frac{1}{55}\begin{pmatrix} 5 & 5 & 5 \\ -10 & 34 & 23 \\ 5 & 6 & 17 \end{pmatrix}\begin{pmatrix} 4 \\ 0 \\ 5 \end{pmatrix} = \frac{1}{11}\begin{pmatrix} 9 \\ 15 \\ 13 \end{pmatrix}$

13. $X = A^{-1}B = -\frac{1}{28}\begin{pmatrix} -11 & -12 & -9 \\ -3 & -16 & -5 \\ -8 & -24 & -4 \end{pmatrix}\begin{pmatrix} -4 \\ 5 \\ 8 \end{pmatrix} = -\frac{1}{28}\begin{pmatrix} -88 \\ -108 \\ -120 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 22 \\ 27 \\ 30 \end{pmatrix}$

14. $X = A^{-1}B = \frac{1}{52}\begin{pmatrix} 4 & 4 & 0 \\ 7 & -6 & 39 \\ 1 & 14 & 13 \end{pmatrix}\begin{pmatrix} 4 \\ -5 \\ 0 \end{pmatrix} = \frac{1}{52}\begin{pmatrix} -4 \\ 58 \\ -66 \end{pmatrix}$

15. $X = A^{-1}B = -\frac{1}{25}\begin{pmatrix} 5 & -15 & -15 \\ -10 & 15 & 10 \\ -5 & 10 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ -7 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -21 \\ 14 \\ 0 \end{pmatrix}$

16. By conclusion (2) of Theorem 6.26, $AA = A^2$ is nonsingular. To show $(A^2)^{-1} = (A^{-1})^2$, consider $A^2(A^{-1})^2 = AAA^{-1}A^{-1} = I$, and $(A^{-1})^2A^2 = A^{-1}A^{-1}AA = I$, so $(A^2)^{-1} = (A^{-1})^2$. The general result follows by a simple induction argument.

17. Suppose $BA = AC = I_n$, then $B = BI_n = B(AC) = (BA)C = I_nC = C$.

Chapter Eight - Determinants

Section 8.1 Permutations

1. Even permutations of $1, 2, 3$ are: $p_1 : 1, 2, 3$; $p_4 : 2, 3, 1$; $p_5 : 3, 1, 2$;
Odd permutations of $1, 2, 3$ are: $p_2 : 1, 3, 2$; $p_3 : 2, 1, 3$; $p_6 : 3, 2, 1$;

2. Even permutations of $1, 2, 3, 4$ are:
$1,2,3,4$; $1,3,4,2$; $1,4,2,3$;
$2,1,4,3$; $2,3,1,4$; $2,4,1,3$;
$3,1,2,4$; $3,2,4,1$; $3,4,1,2$;
$4,1,3,2$; $4,2,1,3$; $4,3,2,1$;

Odd permutations of $1, 2, 3, 4$ are:
$1,2,4,3$; $1,3,2,4$; $1,4,3,2$;
$2,1,3,4$; $2,3,4,1$; $2,4,3,1$;
$3,1,4,2$; $3,2,1,4$; $3,4,2,1$;
$4,1,2,3$; $4,2,3,1$; $4,3,1,2$;

3. First observe that interchanging two adjacent entries in any permutation of $1, 2, \ldots, n$ changes the permutation from even to odd, or from odd to even. Now consider the set of all even permutations of $1, 2, \ldots, n$, which we claim is half of all the permutations of $1, 2, \ldots, n$. To see this, in each even permutations interchange the first two elements. This creates a set of distinct odd permutations of $1, 2, \ldots, n$ which we claim is all the odd permutations. If this were not the case, take any one of the odd permutations not obtained from the even ones by interchanging elements and interchange the first two elements in this odd permutation. This would produce an even permutation different from those with which we started. This is a contradiction. It follows that the number of even and odd permutations of $1, 2, \ldots, n$ are the same, each equal to one half the total number, $n!$.

Section 8.2 Definition of the Determinant

1. $|A| = (1)(2)(1) - (1)(-1)(1) - (6)(1)(1) + (6)(-1)(0) + (0)(1)(1) - (0)(2)(0) = -3$

2. $|A| = (-1)(2)(4) - (-1)(0)(1) - (3)(2)(4) + (3)(0)(1) + (1)(2)(1) - (1)(2)(1) = -32$

3. $|A| = (6)(1)(-4) - (6)(4)(1) - (-3)(2)(-4) + (-3)(4)(0) + (5)(2)(1) - (5)(1)(0) = -62$

4. $|A| = (-4)(1)(0) - (-4)(1)(0) - (0)(0)(0) + (0)(1)(0) + (1)(0)(0) - (1)(1)(0) = 0$

5. Referring to the list of even and odd permutations of $1, 2, 3, 4$ in Problem 2, Section 7.1 we find the formula to be:
$|A| = a_{11}a_{22}a_{33}a_{44} + a_{11}a_{23}a_{34}a_{42} + a_{11}a_{24}a_{32}a_{43} + a_{12}a_{21}a_{34}a_{43} + a_{12}a_{23}a_{31}a_{44} + a_{12}a_{24}a_{31}a_{43} + a_{13}a_{21}a_{32}a_{44} + a_{13}a_{22}a_{34}a_{41} + a_{13}a_{24}a_{31}a_{42} + a_{14}a_{21}a_{33}a_{42} + a_{14}a_{22}a_{31}a_{43} + a_{14}a_{23}a_{32}a_{41} - (a_{11}a_{22}a_{34}a_{43} + a_{11}a_{23}a_{32}a_{44} + a_{11}a_{24}a_{33}a_{42} + a_{12}a_{21}a_{33}a_{44} + a_{12}a_{23}a_{34}a_{41} + a_{12}a_{24}a_{33}a_{41} + a_{13}a_{21}a_{34}a_{42} + a_{13}a_{22}a_{31}a_{44} + a_{13}a_{24}a_{32}a_{41} + a_{14}a_{21}a_{32}a_{43} + a_{14}a_{22}a_{33}a_{41} + a_{14}a_{23}a_{31}a_{42})$

Section 8.3 Properties of Determinants

1. We have $B = (\alpha I_n)A$ so $|B| = |\alpha I_n||A|$, by Theorem 7.4. By definition and direct calculation, the only nonzero product in the definition of $|\alpha I_n|$ is $\underbrace{\alpha \cdot \alpha \cdots \alpha}_{n \text{ factors}}$, so $|\alpha I_n| = \alpha^n$, and $|B| = \alpha^n|A|$.

2. In the 2×2 case we have $B = \begin{pmatrix} a_{11} & \frac{1}{\alpha}a_{12} \\ \alpha a_{21} & a_{22} \end{pmatrix}$, and $|B| = |A|$ by direct evaluation. In the 3×3 case we have $B = \begin{pmatrix} a_{11} & \frac{1}{\alpha}a_{12} & \frac{1}{\alpha^2}a_{13} \\ \alpha a_{21} & a_{22} & \frac{1}{\alpha}a_{23} \\ \alpha^2 a_{31} & \alpha a_{32} & a_{33} \end{pmatrix}$ and $|B| = |A|$. In the $n \times n$ case we also have $|B| = |A|$.

3. Let A be skew symmetric and of odd order. Then $A^t = -A$, hence $|A| = |A^t| = |-A| = (-1)^n|A| = -|A|$, by Problem 1. But $|A| = -|A|$ implies that $|A| = 0$.

Section 8.4 Evaluation of Determinants by Elementary Row and Column Operations

There are many correct solutions to each of these problems. One such solution is shown in Problems 1-6, with only final determinant values given in Problems 7-10.

1. $\begin{vmatrix} -2 & 4 & 1 \\ 1 & 6 & 3 \\ 7 & 0 & 4 \end{vmatrix} = \begin{vmatrix} 0 & 16 & 7 \\ 1 & 6 & 3 \\ 0 & -42 & -17 \end{vmatrix} = (-1)^{2+1}(1)\begin{vmatrix} 16 & 7 \\ -42 & -17 \end{vmatrix} = -22$

2. $\begin{vmatrix} 2 & -3 & 7 \\ 14 & 1 & 1 \\ -13 & -1 & 5 \end{vmatrix} = \begin{vmatrix} 44 & 0 & 10 \\ 14 & 1 & 1 \\ 1 & 0 & 6 \end{vmatrix} = (-1)^{2+2}(1)\begin{vmatrix} 44 & 10 \\ 1 & 6 \end{vmatrix} = 254$

3. $\begin{vmatrix} -4 & 5 & 6 \\ -2 & 3 & 5 \\ 2 & -2 & 6 \end{vmatrix} = \begin{vmatrix} 1 & 5 & 21 \\ 1 & 3 & 14 \\ 0 & -2 & 0 \end{vmatrix} = (-1)^{3+2}(-2)\begin{vmatrix} 1 & 21 \\ 1 & 14 \end{vmatrix} = -14$

4. $\begin{vmatrix} 2 & -5 & 8 \\ 4 & 3 & 8 \\ 13 & 0 & -4 \end{vmatrix} = \begin{vmatrix} 28 & -5 & 0 \\ 30 & 3 & 0 \\ 13 & 0 & -4 \end{vmatrix} = (-1)^{3+3}(-4)\begin{vmatrix} 28 & -5 \\ 30 & 3 \end{vmatrix} = -936$

5. $\begin{vmatrix} 17 & -2 & 5 \\ 1 & 12 & 0 \\ 14 & 7 & -7 \end{vmatrix} = \begin{vmatrix} 27 & 3 & 5 \\ 1 & 12 & 0 \\ 0 & 0 & -7 \end{vmatrix} = (-1)^{3+3}(-7)\begin{vmatrix} 27 & 3 \\ 1 & 12 \end{vmatrix} = -2,247$

6. $\begin{vmatrix} -3 & 3 & 9 & 6 \\ 1 & -2 & 15 & 6 \\ 7 & 1 & 1 & 5 \\ 2 & 1 & -1 & 3 \end{vmatrix} = \begin{vmatrix} -3 & 0 & 0 & 0 \\ 1 & -1 & 18 & 8 \\ 7 & 8 & 22 & 19 \\ 2 & 3 & 5 & 7 \end{vmatrix} = (-1)^{1+1}(-3)\begin{vmatrix} -1 & 18 & 8 \\ 8 & 22 & 19 \\ 3 & 5 & 7 \end{vmatrix} = -3\begin{vmatrix} -1 & 0 & 0 \\ 8 & 166 & 83 \\ 3 & 59 & 31 \end{vmatrix} =$
$(-3)(-1)\begin{vmatrix} 166 & 83 \\ 59 & 31 \end{vmatrix} = 747$

7. -122

8. 293

9. -72

10. $-2,667$

Section 8.5 Cofactor Expansions

A solution is shown only for Problems 1 and 2; answers only are given for Problems 3-10.

1. Expand by the 3^{rd} column to get
$$\begin{vmatrix} -4 & 2 & -8 \\ 1 & 1 & 0 \\ 1 & -3 & 0 \end{vmatrix} = (-1)^{1+3}(-8)\begin{vmatrix} 1 & 1 \\ 1 & -3 \end{vmatrix} = (-8)(-4) = 32$$

2. Use row operations to reduce column one, then expand by the 1^{st} column to get
$$\begin{vmatrix} 1 & 1 & 6 \\ 2 & -2 & 1 \\ 3 & -1 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 16 \\ 0 & -4 & -11 \\ 0 & -4 & -14 \end{vmatrix} = \begin{vmatrix} -4 & -11 \\ -4 & -14 \end{vmatrix} = \begin{vmatrix} -4 & -11 \\ 0 & -3 \end{vmatrix} = 12$$

3. 3

4. 124

5. -773

6. 3,775

7. -152

8. 4,882

9. 1,693

10/ 3,372

11. $$\begin{vmatrix} 1 & \alpha & \alpha^2 \\ 1 & \beta & \beta^2 \\ 1 & \gamma & \gamma^2 \end{vmatrix} = \begin{vmatrix} 1 & \alpha & \alpha^2 \\ 0 & \beta-\alpha & \beta^2-\alpha^2 \\ 0 & \gamma-\alpha & \gamma^2-\alpha^2 \end{vmatrix} = (\beta-\alpha)(\gamma-\alpha)\begin{vmatrix} 1 & \alpha & \alpha^2 \\ 0 & 1 & \beta+\alpha \\ 0 & 1 & \gamma+\alpha \end{vmatrix} =$$
$$(\beta-\alpha)(\gamma-\alpha)\begin{vmatrix} 1 & \beta+\alpha \\ 1 & \gamma+\alpha \end{vmatrix} = (\beta-\alpha)(\gamma-\alpha)(\gamma-\beta).$$

12. First add columns two, three and four to column one and factor $(\alpha+\beta+\gamma+\delta)$ out of column one to get

$$\begin{vmatrix} \alpha & \beta & \gamma & \delta \\ \beta & \gamma & \delta & \alpha \\ \gamma & \delta & \alpha & \beta \\ \delta & \alpha & \beta & \gamma \end{vmatrix} = (\alpha+\beta+\gamma+\delta)\begin{vmatrix} 1 & \beta & \gamma & \delta \\ 1 & \gamma & \delta & \alpha \\ 1 & \delta & \alpha & \beta \\ 1 & \alpha & \beta & \gamma \end{vmatrix}.$$

Next add $\{(-1)$ row 2+ row 3− row 4$\}$ to row 1 and factor out $(\beta-\alpha+\delta-\gamma)$ from the new row 1 to get

$$(\alpha+\beta+\gamma+\delta)(\beta-\alpha+\delta-\gamma)\begin{vmatrix} 0 & 1 & -1 & 1 \\ 1 & \gamma & \delta & \alpha \\ 1 & \delta & \alpha & \beta \\ 1 & \alpha & \beta & \gamma \end{vmatrix}.$$

13. Let A be a square matrix such that $A^t = A^{-1}$. Then $I = AA^{-1} = AA^t$, and $|I| = |AA^t| = |A||A^t| = |A|^2$, since $|A^t| = |A|$. From $|A|^2 = 1$ we deduce $|A| = \pm 1$.

14. We define the expression $L(x,y) = \begin{vmatrix} 1 & x & y \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$. By expanding L along the first row, we see that $L(x,y)$ is a linear expression in x and y, hence the equation $L(x,y) = 0$ is a line in the xy-plane. But $L(x_2,y_2) = L(x_3,y_3) = 0$, so the line contains the points (x_2,y_2) and (x_3,y_3). Finally (x_1,y_1) lies on this line if and only if $L(x_1,y_1) = 0 = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$.

Section 8.6 Determinants of Triangular Matrices

1. $|A| = (-4)(7)(2)(-2)(5)(4) = 2,240$
2. $|A| = (6)(-4)(-5)(14)(13)(3) = 65,520$
3. $|A| = (3)(-6)(2)(8)(5) = -1,440$

Section 8.7 A Determinant Formula for a Matrix Inverse

1. $A^{-1} = \frac{1}{13}\begin{pmatrix} 6 & 1 \\ -1 & 2 \end{pmatrix}$
2. $A^{-1} = \frac{1}{12}\begin{pmatrix} 4 & 0 \\ -1 & 3 \end{pmatrix}$
3. $A^{-1} = -\frac{1}{5}\begin{pmatrix} 4 & -1 \\ -1 & -1 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -4 & 1 \\ 1 & 1 \end{pmatrix}$
4. $A^{-1} = \frac{1}{29}\begin{pmatrix} -3 & -5 \\ 7 & 2 \end{pmatrix}$
5. $A^{-1} = \frac{1}{32}\begin{pmatrix} 5 & 3 & 1 \\ -8 & -24 & 24 \\ -2 & -14 & 6 \end{pmatrix}$
6. $A^{-1} = \frac{1}{120}\begin{pmatrix} -10 & -10 & 0 \\ -11 & -95 & 36 \\ 3 & 15 & 12 \end{pmatrix}$
7. $A^{-1} = \frac{1}{29}\begin{pmatrix} -1 & 25 & -21 \\ -8 & -3 & 6 \\ -1 & -4 & 8 \end{pmatrix}$
8. $A^{-1} = \frac{1}{119}\begin{pmatrix} 9 & 35 & 5 \\ 0 & 119 & 0 \\ -4 & 77 & 11 \end{pmatrix}$

9. $A^{-1}=\frac{1}{378}\begin{pmatrix} 210 & -42 & 42 & 0 \\ 899 & -124 & 223 & -135 \\ 275 & -64 & 109 & -27 \\ -601 & 122 & -131 & 81 \end{pmatrix}$

10. $A^{-1}=\frac{1}{784}\begin{pmatrix} -52 & 131 & -62 & 54 \\ 208 & -132 & 248 & -216 \\ -496 & 360 & -320 & 304 \\ -212 & 127 & -102 & 190 \end{pmatrix}$

Section 8.8 Cramer's Rule

1. $|A|=47\neq 0$ so Cramer's rule applies.
$x_1=\frac{1}{47}\begin{vmatrix} 5 & -4 \\ -4 & 1 \end{vmatrix}=-\frac{11}{47},\ x_2=\frac{1}{47}\begin{vmatrix} 15 & 5 \\ 8 & -4 \end{vmatrix}=-\frac{100}{47}$

2. $|A|=-3\neq 0$ so Cramer's rule applies.
$x_1=-\frac{1}{3}\begin{vmatrix} 3 & 4 \\ 0 & 1 \end{vmatrix}=-\frac{3}{3}=-1,\ x_2=-\frac{1}{3}\begin{vmatrix} 1 & 3 \\ 1 & 0 \end{vmatrix}=\frac{3}{3}=1$

3. $|A|=132\neq 0$ so Cramer's rule applies.
$x_1=\frac{1}{132}\begin{vmatrix} 0 & -4 & 3 \\ -5 & 5 & -1 \\ -4 & 6 & 1 \end{vmatrix}=-\frac{66}{132}=-\frac{1}{2},$

$x_2=\frac{1}{132}\begin{vmatrix} 8 & 0 & 3 \\ 1 & -5 & -1 \\ -2 & -4 & 1 \end{vmatrix}=-\frac{114}{132}=-\frac{19}{22},$

$x_3=\frac{1}{132}\begin{vmatrix} 8 & -4 & 0 \\ 1 & 5 & -5 \\ -2 & 6 & -4 \end{vmatrix}=\frac{24}{132}=\frac{2}{11}$

4. $|A|=108\neq 0$ so Cramer's rule applies.
$x_1=-\frac{63}{108}=-\frac{7}{12},x_2=-\frac{165}{108}=-\frac{55}{36},x_3=-\frac{243}{108}=-\frac{9}{4}.$

5. $|A|=-6\neq 0$ so Cramer's rule applies.
$x_1=\frac{5}{6},x_2=-\frac{10}{3},x_3=-\frac{5}{6}$

6. $|A|=-130\neq 0$ so Cramer's rule applies.
$x_1=\frac{197}{130},x_2=\frac{255}{130},x_3=\frac{1260}{130},x_4=\frac{42}{130},x_5=\frac{173}{130}$

7. $|A|=4\neq 0$ so Cramer's rule applies.
$x_1=-\frac{172}{2},x_2=-\frac{109}{2},x_3=-\frac{43}{2},x_4=\frac{37}{2}$

8. $|A|=12\neq 0$ so Cramer's rule applies.
$x_1=\frac{117}{12},x_2=\frac{63}{12},x_3=\frac{3}{2},x_4=-\frac{21}{12}$

9. $|A|=93\neq 0$ so Cramer's rule applies.
$x_1=\frac{33}{93},x_2=-\frac{409}{33},x_3=-\frac{1}{93},x_4=\frac{116}{93}$

10. $|A| = 42 \neq 0$ so Cramer's rule applies.
$x_1 = \frac{69}{21}, x_2 = \frac{162}{21}, x_3 = \frac{24}{21}, x_4 = -\frac{54}{21}$

Section 8.9 The Matrix Tree Theorem

1. The tree matrix is $T = \begin{pmatrix} 2 & 0 & -1 & 0 & -1 \\ 0 & 2 & -1 & -1 & 0 \\ -1 & -1 & 4 & -1 & -1 \\ 0 & -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & -1 & 3 \end{pmatrix}$. Evaluate any cofactor to get 21 for the number of spanning trees in G.

2. The tree matrix is $T = \begin{pmatrix} 4 & -1 & -1 & -1 & 0 & -1 \\ -1 & 3 & -1 & -1 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 & 0 \\ -1 & -1 & 0 & 4 & -1 & -1 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & -1 & 3 \end{pmatrix}$. Evaluate any cofactor to get 55 for the number of spanning trees in G.

3. The tree matrix is $T = \begin{pmatrix} 4 & -1 & 0 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 3 & -1 & -1 & 0 \\ -1 & 0 & -1 & 4 & -1 & -1 \\ -1 & 0 & -1 & -1 & 3 & 0 \\ -1 & 0 & 0 & -1 & 0 & 2 \end{pmatrix}$. Evaluate any cofactor to get 61 for the number of spanning trees in G.

4. The tree matrix is $T = \begin{pmatrix} 4 & -1 & -1 & 0 & -1 & -1 \\ -1 & 3 & -1 & -1 & 0 & 0 \\ -1 & -1 & 3 & -1 & 0 & 0 \\ 0 & -1 & -1 & 3 & -1 & 0 \\ -1 & 0 & 0 & -1 & 3 & -1 \\ -1 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$. Evaluate any cofactor to get 64 for the number of spanning trees in G.

5. The tree matrix is $T = \begin{pmatrix} 3 & -1 & 0 & 0 & -1 & -1 \\ -1 & 3 & -1 & 0 & -1 & 0 \\ 0 & -1 & 4 & -1 & -1 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ -1 & -1 & -1 & -1 & 4 & 0 \\ -1 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}$. Evaluate any cofactor to get 61 for the number of spanning trees in G.

6. The tree matrix for a complete graph on n points will be the $n \times n$ matrix

$$T_n = \begin{pmatrix} n-1 & -1 & -1 & \cdots & -1 \\ -1 & n-1 & -1 & \cdots & -1 \\ -1 & -1 & n-1 & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & n-1 \end{pmatrix}.$$

To compute the number of spanning trees, we compute the cofactor $(-1)^{1+1}M_{11}$, which is the determinant of the $(n-1) \times (n-1)$ tree matrix T_{n-1}. To evaluate $|T_{n-1}|$, add the last $(n-2)$ rows of $|T_{n-1}|$ to row 1 to get

$$|T_{n-1}| = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ -1 & n-1 & -1 & \cdots & -1 \\ -1 & -1 & n-1 & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & n-1 \end{vmatrix}.$$

Now subtract column 1 from each of columns 2 through $(n-2)$ to get $\begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1 & n & 0 & \cdots & 0 \\ -1 & 0 & n & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & n \end{vmatrix} =$

n^{n-2}

Chapter Nine - Eigenvalues, Diagonalization and Special Matrices

Section 9.1 Eigenvalues and Eigenvectors

For Problems 1 through 16, we give the characteristic polynomial of A as $p_A(\lambda)$, the eigenvalues of A along with associated eigenvectors. Following Problem 20 are some sketches of Gerschgorin circles for a few selected (interesting) Problems. The circles are drawn solid and eigenvalues are plotted with an $\times$ mark.

1. $p_A(\lambda) = |\lambda I - A| = \lambda^2 - 2\lambda - 5 = 0$ has roots $\lambda_1 = 1 + \sqrt{6}, \lambda_2 = 1 - \sqrt{6}$.
Eigenvectors are $\mathbf{V}_1 = \begin{pmatrix} \sqrt{6} \\ 2 \end{pmatrix}$, $\mathbf{V}_2 = \begin{pmatrix} -\sqrt{6} \\ 2 \end{pmatrix}$.

2. $p_A(\lambda) = \lambda^2 - 2\lambda - 8 = 0; \lambda_1 = 4, \mathbf{V}_1 = \begin{pmatrix} 0 \\ 4 \end{pmatrix}; \lambda_2 = -2, \mathbf{V}_2 = \begin{pmatrix} 6 \\ -1 \end{pmatrix}$.

3. $p_A(\lambda) = \lambda^2 + 3\lambda - 10 = 0; \lambda_1 = -5, \mathbf{V}_1 = \begin{pmatrix} 7 \\ -1 \end{pmatrix}; \lambda_2 = 2, \mathbf{V}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

4. $p_A(\lambda) = \lambda^2 - 10\lambda + 18 = 0; \lambda_1 = 5 + \sqrt{7}, \mathbf{V}_1 = \begin{pmatrix} 2 \\ 1 - \sqrt{7} \end{pmatrix}; \lambda_2 = 5 - \sqrt{7}, \mathbf{V}_2 = \begin{pmatrix} 2 \\ 1 + \sqrt{7} \end{pmatrix}$.

5. $p_A(\lambda) = \lambda^2 - 3\lambda + 14 = 0; \lambda_1 = \dfrac{3 + \sqrt{14}i}{2}, \mathbf{V}_1 = \begin{pmatrix} -1 + \sqrt{47}i \\ 4 \end{pmatrix}; \lambda_2 = \dfrac{3 - \sqrt{14}i}{2}, \mathbf{V}_2 \begin{pmatrix} -1 - \sqrt{47}i \\ 4 \end{pmatrix}$.

6. $p_A(\lambda) = \lambda^2 = 0; \lambda_1 = \lambda_2 = 0$. There is only one associated eigenvector, $\mathbf{V}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

7. $p_A(\lambda) = \lambda^3 - 5\lambda^2 + 6\lambda = 0, \lambda_1 = 0, \mathbf{V}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}; \lambda_2 = 2, \mathbf{V}_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}; \lambda_3 = 3, \mathbf{V}_3 = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}$.

8. $p_A(\lambda) = (\lambda + 1)(\lambda^2 - \lambda - 7) = 0$;

$$\lambda_1 = 1, \mathbf{V}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}; \lambda_2 = \frac{1 + \sqrt{29}}{2}, \mathbf{V}_2 = \begin{pmatrix} 2 \\ 5 + \sqrt{29} \\ 0 \end{pmatrix}; \lambda_3 = \frac{1 - \sqrt{29}}{2}, \mathbf{V}_3 = \begin{pmatrix} 2 \\ 5 - \sqrt{29} \\ 0 \end{pmatrix}.$$

9. $p_A(\lambda) = \lambda^2(\lambda + 3) = 0; \lambda_1 = -3, \mathbf{V}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}; \lambda_2 = \lambda_3 = 0, \mathbf{V}_2 = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$ is only one eigenvector for $\lambda = 0$.

10. $p_A(\lambda) = \lambda^3 + 2\lambda = 0$;

$$\lambda_1 = 0, \mathbf{V}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}; \lambda_2 = \sqrt{2}i, \mathbf{V}_2 = \begin{pmatrix} 1 \\ -1 \\ -\sqrt{2}i \end{pmatrix}; \lambda_3 = -\sqrt{2}i, \mathbf{V}_3 = \begin{pmatrix} 1 \\ -1 \\ \sqrt{2}i \end{pmatrix}.$$

11. $p_A(\lambda) = (\lambda+14)(\lambda-2)^2 = 0; \lambda_1 = -14, \mathbf{V}_1 = \begin{pmatrix} -16 \\ 0 \\ 1 \end{pmatrix}; \ \lambda_2 = \lambda_3 = 2, \mathbf{V}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$,
only one eigenvector for $\lambda = 2$.

12. $p_A(\lambda) = (\lambda-3)(\lambda^2+\lambda-42) = 0;$

$$\lambda_1 = 6, \mathbf{V}_1 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}; \lambda_2 = 3, \mathbf{V}_2 = \begin{pmatrix} 30 \\ -2 \\ 5 \end{pmatrix}; \lambda_3 = -7, \mathbf{V}_3 = \begin{pmatrix} 0 \\ 8 \\ 5 \end{pmatrix}.$$

13. $p_A(\lambda) = \lambda(\lambda^2 - 8\lambda + 7) = 0, \lambda_1 = 0, \mathbf{V}_1 = \begin{pmatrix} 14 \\ 7 \\ 10 \end{pmatrix}; \lambda_2 = 1, \mathbf{V}_2 = \begin{pmatrix} 6 \\ 0 \\ 5 \end{pmatrix}; \lambda_3 = 7, \mathbf{V}_3 =$
$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

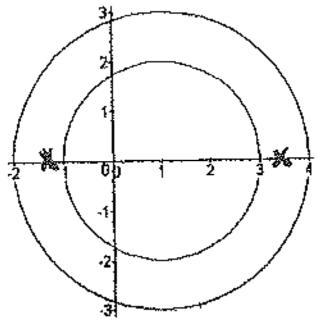
14. $p_A(\lambda) = \lambda^2(\lambda^2 + 2\lambda - 1) = 0; \lambda_1 = \lambda_2 = 0$ with two associated eigenvectors, $\mathbf{V}_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \\ -1 \end{pmatrix}$,
$\mathbf{V}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}; \ \lambda_3 = -1 + \sqrt{2}, \mathbf{V}_3 = \begin{pmatrix} 1 \\ 1+\sqrt{2} \\ 0 \\ 0 \end{pmatrix}; \ \lambda_4 = -1 - \sqrt{2}, \mathbf{V}_4 = \begin{pmatrix} 1 \\ 1-\sqrt{2} \\ 0 \\ 0 \end{pmatrix}$.

15. $p_A(\lambda) = (\lambda-1)(\lambda-2)(\lambda^2+\lambda-13) = 0; \lambda_1 = 1, \mathbf{V}_1 = \begin{pmatrix} -2 \\ -11 \\ 0 \\ 1 \end{pmatrix}; \lambda_2 = 2, \mathbf{V}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix};$
$\lambda_3 = \dfrac{-1+\sqrt{53}}{2}, \mathbf{V}_3 = \begin{pmatrix} \sqrt{53}-7 \\ 0 \\ 0 \\ 2 \end{pmatrix}; \lambda_4 = \dfrac{-1-\sqrt{53}}{2}, \mathbf{V}_4 = \begin{pmatrix} -\sqrt{53}-7 \\ 0 \\ 0 \\ 2 \end{pmatrix}$.

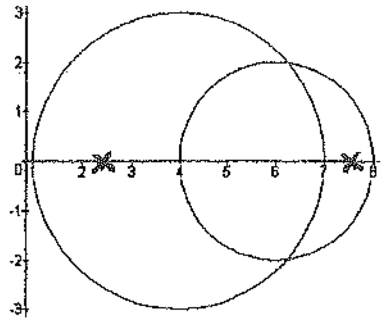
16. $p_A(\lambda) = (\lambda-1)(\lambda-5)\lambda^2 = 0; \lambda_1 = 1, \mathbf{V}_1 = \begin{pmatrix} 1 \\ -4 \\ 0 \\ 0 \end{pmatrix}; \lambda_2 = 5, \mathbf{V}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix};$
$\lambda_3 = \lambda_4 = 0, \mathbf{V}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$, only one eigenvector for $\lambda = 0$.

Sketches of Gerschgorin circles for some selected Problems 1 through 20

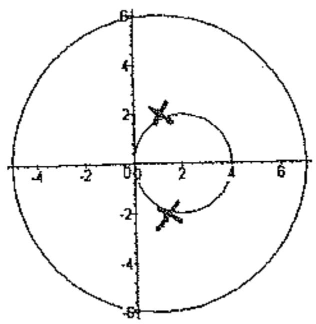
Problem 1



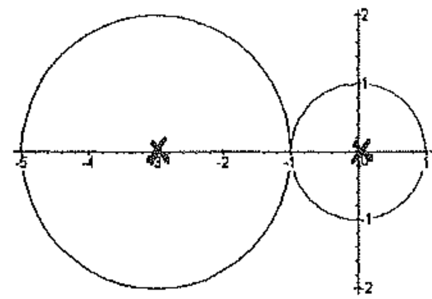
Problem 4



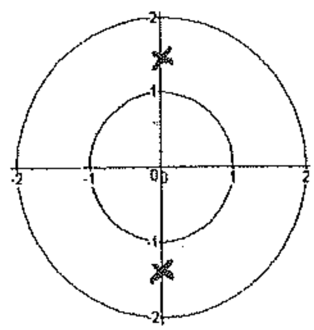
Problem 5



Problem 9



Problem 10



17. We have $p_A(\lambda) = \lambda^2 - (\alpha+\gamma)\lambda + (\alpha\gamma - \beta^2)$. The discriminant of this quadratic is $B^2 - 4AC = (\alpha+\gamma)^2 - 4(\alpha\gamma - \beta^2) = (\alpha-\gamma)^2 + 4\beta^2 \geq 0$. Thus the quadratic characteristic equation has real roots.

18. To establish this result we define the inner product for complex entry $n \times 1$ matrices x and y by $(x, y) = x^t\overline{y}$. With this definition it is easy to establish the results $(\lambda x, y) = \lambda(x, y)$ and $(x, \lambda y) = \overline{\lambda}(x, y)$ for any complex scalar λ, and the fact that $(x, x) = \|x\|^2 = \sum_{i=1}^{n}|x_i|^2$. Let S be the given 3×3 symmetric matrix having real entries and note that $S^t = S$, and $\overline{S} = S$. Finally let λ be any eigenvalue of S with associated eigenvector x. Then $\lambda\|x\|^2 = \lambda(x, x) = (\lambda x, x) = (Sx, x) = (Sx)^t\overline{x} = x^tS^t\overline{x} = x^tS\overline{x} = x^t\overline{Sx} = (x, Sx) = (x, \lambda x) = \overline{\lambda}(x, x) = \overline{\lambda}\|x\|^2$. From this it follows that $\lambda = \overline{\lambda}$, and hence λ is real. Note that this proof establishes that the eigenvalues of any $n \times n$ real symmetric matrix are real.

19. We are given that $AE = \lambda E$, with $E \neq 0$. Now $A^2E = A(AE) = A(\lambda E) = \lambda(AE) = \lambda^2 E$, so λ^2 is an eigenvalue of A^2 with eigenvector E. The general result follows by an induction proof to show $A^kE = \lambda^kE$.

20. Since E and L are eigenvectors it follows that neither is the zero vector. Assume now that E and L are linearly dependent. Then $E = kL$ for some nonzero scalar k. Now $AE = A(kL) = kAL = k\mu L = \mu E$, and $AE = \lambda E$, where we have used the fact that E and L are eigenvectors associated with λ and μ respectively. Subtract these two equations to get $(\mu - \lambda)E = \theta$. Since $E \neq \theta$ we must have $\mu - \lambda = 0$ or $\mu = \lambda$ which is a contradiction. Thus E and L are linearly independent.

21. We have $p_A(\lambda) = |\lambda I - A|$, and the constant term of this polynomial is found by putting $\lambda = 0$, which yields $|-A| = (-1)^n|A|$. Recall that the constant term of a polynomial is zero if and only if $\lambda = 0$, is a root of the polynomial. From this we have $\lambda = 0$ is an eigenvalue of A if and only if $|A| = 0$ and A is singular.

Section 9.2 Diagonalization of Matrices

1. $p_A(\lambda) = \lambda^2 - 3\lambda + 4 = 0$ gives distinct roots $\lambda_1 = \dfrac{3+\sqrt{7}i}{2}$, with $\mathbf{V}_1 = \begin{pmatrix} -3+\sqrt{7}i \\ 8 \end{pmatrix}$ and $\lambda_2 = \dfrac{3-\sqrt{7}i}{2}$, with $\mathbf{V}_2 = \begin{pmatrix} -3-\sqrt{7}i \\ 8 \end{pmatrix}$. Form $P = \begin{pmatrix} -3+\sqrt{7}i & -3-\sqrt{7}i \\ 8 & 8 \end{pmatrix}$, and

$$P^{-1}AP = \begin{pmatrix} \frac{3+\sqrt{7}i}{2} & 0 \\ 0 & \frac{3-\sqrt{7}i}{2} \end{pmatrix}$$

2. $p_A(\lambda) = \lambda^2 - 8\lambda + 12 = 0$ gives distinct roots $\lambda_1 = 2$, with $\mathbf{V}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, $\lambda_2 = 6, \mathbf{V}_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$.

$P = \begin{pmatrix} -1 & 3 \\ 1 & 1 \end{pmatrix}$ diagonalizes $A, P^{-1}AP = \begin{pmatrix} 2 & 0 \\ 0 & 6 \end{pmatrix}$.

3. $p_A(\lambda) = \lambda^2 - 2\lambda + 1 = 0$ gives $\lambda_1 = \lambda_2 = 1$, with only one eigenvector $\mathbf{V}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, so A is not diagonalizable.

4. $p_A(\lambda) = \lambda^2 - 4\lambda - 45 = 0$ gives $\lambda_1 = -5, \mathbf{V}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$; $\lambda_2 = 9, \mathbf{V}_2 = \begin{pmatrix} 3 \\ 14 \end{pmatrix}$, $P = \begin{pmatrix} 1 & 3 \\ 0 & 14 \end{pmatrix}$ diagonalizes A, $P^{-1}AP = \begin{pmatrix} -5 & 0 \\ 0 & 9 \end{pmatrix}$.

5. $p_A(\lambda) = \lambda(\lambda - 5)(\lambda + 2) = 0$ gives roots $\lambda_1 = 0, \mathbf{V}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$; $\lambda_2 = 5, \mathbf{V}_2 = \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix}$;

$\lambda_3 = -2, \mathbf{V}_3 = \begin{pmatrix} 0 \\ -3 \\ 2 \end{pmatrix}$. So $P = \begin{pmatrix} 0 & 5 & 0 \\ 1 & 1 & -3 \\ 0 & 0 & 2 \end{pmatrix}$ diagonalizes A, $P^{-1}AP = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -2 \end{pmatrix}$.

6. $p_A(\lambda) = \lambda(\lambda^2 - 3\lambda - 2) = 0$ gives roots

$$\lambda_1 = 0, \mathbf{V}_1 = \begin{pmatrix} -2 \\ -3 \\ 1 \end{pmatrix}; \lambda_2 = \frac{3 + \sqrt{17}}{2}, \mathbf{V}_2 = \begin{pmatrix} 0 \\ 4 \\ 3 + \sqrt{17} \end{pmatrix}; \lambda_3 = \frac{3 - \sqrt{17}}{2}, \mathbf{V}_3 = \begin{pmatrix} 0 \\ 4 \\ 3 - \sqrt{17} \end{pmatrix};$$

$$P = \begin{pmatrix} -2 & 0 & 0 \\ -3 & 4 & 4 \\ 1 & 3 + \sqrt{17} & 3 - \sqrt{17} \end{pmatrix} \text{ diagonalizes } A, P^{-1}AP = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{3+\sqrt{17}}{2} & 0 \\ 0 & 0 & \frac{3-\sqrt{17}}{2} \end{pmatrix}.$$

7. $p_A(\lambda) = (\lambda-1)(\lambda+2)^2 = 0$ gives roots $\lambda_1 = 1, \mathbf{V}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$; $\lambda_2 = \lambda_3 = -2, \mathbf{V}_2 = \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}$.
Since A has only two eigenvectors, A is not diagonalizable.

8. $p_A(\lambda) = (\lambda - 2)(\lambda^2 - 4\lambda + 5) = 0$ gives roots

$$\lambda_1 = 2, \mathbf{V}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}; \lambda_2 = 2 + i, \mathbf{V}_2 = \begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix}; \lambda_3 = 2 - i, \mathbf{V}_3 = \begin{pmatrix} 0 \\ 1 \\ -i \end{pmatrix}.$$

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & i & -i \end{pmatrix} \text{ diagonalizes } A, P^{-1}AP = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 + i & 0 \\ 0 & 0 & 2 - i \end{pmatrix}.$$

9. $p_A(\lambda) = (\lambda - 1)(\lambda - 4)(\lambda^2 + 5\lambda + 5) = 0$ gives roots $\lambda_1 = 1, \mathbf{V}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$;

$\lambda_2 = 4, \mathbf{V}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}; \lambda_3 = \dfrac{-5+\sqrt{5}}{2}, \mathbf{V}_3 = \begin{pmatrix} 0 \\ \frac{2-3\sqrt{5}}{41} \\ \frac{-1+\sqrt{5}}{2} \\ 1 \end{pmatrix}, \lambda_4 = \dfrac{-5-\sqrt{5}}{2}, \mathbf{V}_4 = \begin{pmatrix} 0 \\ \frac{2+3\sqrt{5}}{41} \\ \frac{-1-\sqrt{5}}{2} \\ 1 \end{pmatrix};$

$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{2-3\sqrt{5}}{41} & \frac{2+3\sqrt{5}}{41} \\ 0 & 0 & \frac{-1+\sqrt{5}}{2} & \frac{-1-\sqrt{5}}{2} \\ 0 & 0 & 1 & 1 \end{pmatrix}$ diagonalizes $A, P^{-1}AP = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & \frac{-5+\sqrt{5}}{2} & 0 \\ 0 & 0 & 0 & \frac{-5-\sqrt{5}}{2} \end{pmatrix}$.

10. $p_A(\lambda) = (\lambda+2)^4 = 0$ gives roots $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = -2$, with $\mathbf{V}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$,

$\mathbf{V}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{V}_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$. Since A has only three eigenvectors, A is not diagonalizable.

11. Since A^2 is diagonalizable, A^2 has n linearly independent eigenvectors $X_1, X_2, \ldots, X_n$ with associated eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and $(A^2 - \lambda_j I_n)X_j = 0$. Now $(A^2 - \lambda_j I_n) = (A - \sqrt{\lambda_j} I_n)(A + \sqrt{\lambda_j} I)$ for $j = 1, 2, \ldots, n$. We now have $p_A(\sqrt{\lambda_j})p_A(-\sqrt{\lambda_j}) = p_{A^2}(\lambda_j) = 0$, so either $\sqrt{\lambda_j}$ or $-\sqrt{\lambda_j}$ is an eigenvalue of A and X_j will be an associated eigenvector. Thus A will have n linearly independent eigenvectors and be diagonalizable by Theorem 2.

12. Since P diagonalizes $A, P^{-1}AP = D$, or $PDP^{-1} = A$.
Then $A^k = (PDP^{-1})^k = \underbrace{(PDP^{-1})(PDP^{-1})\cdots(PDP^{-1})}_{k \text{ times}} = PD^kP^{-1}$, as was to be shown.

13. $A = \begin{pmatrix} -1 & 0 \\ 1 & -5 \end{pmatrix}$ has $p_A(\lambda) = \lambda^2 + 6\lambda + 5$, so eigenvalues are $\lambda_1 = -1, \lambda_2 = -5$.

$P = \begin{pmatrix} 4 & 0 \\ 1 & 1 \end{pmatrix}$ diagonalizes A. Then $A^{18} = P\begin{pmatrix} -1 & 0 \\ 0 & -5 \end{pmatrix}^{18} P^{-1} =$

$$\begin{pmatrix} 4 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 5^{18} \end{pmatrix} \begin{pmatrix} 1/4 & 0 \\ -1/4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{1-5^{18}}{4} & 5^{18} \end{pmatrix}.$$

14. $A = \begin{pmatrix} -3 & -3 \\ -2 & 4 \end{pmatrix}$ has $p_A(\lambda) = \lambda^2 - \lambda - 18$, so $\lambda_1 = \dfrac{1+\sqrt{73}}{2}, \lambda_2 = \dfrac{1-\sqrt{73}}{2}$.

$P = \begin{pmatrix} -6 & -6 \\ 7+\sqrt{73} & 7-\sqrt{73} \end{pmatrix}, P^{-1} = \dfrac{1}{12\sqrt{73}}\begin{pmatrix} 7-\sqrt{73} & 6 \\ -7-\sqrt{73} & -6 \end{pmatrix}$.

Then $A^{16} = P\begin{pmatrix} (\frac{1+\sqrt{73}}{2})^{16} & 0 \\ 0 & (\frac{1-\sqrt{73}}{2})^{16} \end{pmatrix} P^{-1} = \begin{pmatrix} 6(2^{16}) - 3^{16} & 3(2^{16}) - 3^{17} \\ -2^{17} + 2(3^{16}) & -2^{16} + 6(3^{16}) \end{pmatrix}$.

15. $A = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$ has eigenvalues $\lambda_1 = \sqrt{2}, \lambda_2 = -\sqrt{2}$ with associated eigenvectors $\mathbf{V}_1 =$

$\begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix}$, $\mathbf{V}_2 = \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix}$. Thus $P = \begin{pmatrix} \sqrt{2} & -\sqrt{2} \\ 1 & 1 \end{pmatrix}$ and $P^{-1} = \begin{pmatrix} \sqrt{2}/4 & 1/2 \\ -\sqrt{2}/4 & 1/2 \end{pmatrix}$. Then $A^{43} = \begin{pmatrix} \sqrt{2} & -\sqrt{2} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} (\sqrt{2})^{43} & 0 \\ 0 & (-\sqrt{2})^{43} \end{pmatrix} \begin{pmatrix} \sqrt{2}/4 & 1/2 \\ -\sqrt{2}/4 & 1/2 \end{pmatrix} = \begin{pmatrix} 0 & 2^{22} \\ 2^{21} & 0 \end{pmatrix}$.

16. $A = \begin{pmatrix} -2 & 3 \\ 3 & -4 \end{pmatrix}$ has eigenvalues $\lambda_1 = -3 + \sqrt{10}, \lambda_2 = -3 - \sqrt{10}$, with associated eigenvectors $\mathbf{V}_1 = \begin{pmatrix} 3 \\ 1 - \sqrt{10} \end{pmatrix}$, $\mathbf{V}_2 = \begin{pmatrix} 3 \\ 1 + \sqrt{10} \end{pmatrix}$. Thus

$$P = \begin{pmatrix} 3 & 3 \\ 1 - \sqrt{10} & 1 + \sqrt{10} \end{pmatrix}, P^{-1} = \begin{pmatrix} (10 + \sqrt{10})/60 & -\sqrt{10}/20 \\ (10 - \sqrt{10})/60 & \sqrt{10}/20 \end{pmatrix}.$$

Then

$$A^{31} = P \begin{pmatrix} (-3 + \sqrt{10})^{31} & 0 \\ 0 & (-3 - \sqrt{10})^{31} \end{pmatrix} P^{-1} = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix} \text{ where}$$

$$k_{11} = \frac{1}{2\sqrt{10}}[(1 + \sqrt{10})(-3 + \sqrt{10})^{31} + (\sqrt{10} - 1)(-3 - \sqrt{10})^{31}];$$
$$k_{12} = \frac{3}{2\sqrt{10}}[(-3 + \sqrt{10})^{31} + (3 + \sqrt{10})^{31}];$$
$$k_{21} = \frac{1}{2\sqrt{10}}[(-1 + \sqrt{10})(-3 + \sqrt{10})^{31} + (\sqrt{10} + 1)(3 + \sqrt{10})^{31}];$$
$$k_{22} = \frac{3}{2\sqrt{10}}[(-3 + \sqrt{10})^{31} + (3 + \sqrt{10})^{31}].$$

Section 9.3 Orthogonal and Symmetric Matrices

1. $p_A(\lambda) = \lambda^2 - 5\lambda$, so $\lambda_1 = 0, \lambda_2 = 5$. Eigenvectors are $\mathbf{V}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\mathbf{V}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$.
To form Q, take columns of Q to be $\frac{1}{\|\mathbf{V}_1\|}\mathbf{V}_1$, and $\frac{1}{\|\mathbf{V}_2\|}\mathbf{V}_2$, so $Q = \begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}$.
Then $Q^{-1} = Q^t$ and $Q^tAQ = \begin{pmatrix} 0 & 0 \\ 0 & 5 \end{pmatrix}$.

2. $p_A(\lambda) = \lambda^2 - \lambda - 37$, so $\lambda_1 = \dfrac{1 + \sqrt{149}}{2}, \lambda_2 = \dfrac{1 - \sqrt{149}}{2}$.
We can take $\mathbf{V}_1 = \begin{pmatrix} 10 \\ 7 + \sqrt{149} \end{pmatrix}$, $\mathbf{V}_2 = \begin{pmatrix} 10 \\ 7 - \sqrt{149} \end{pmatrix}$.

Then $(\mathbf{V}_1 \cdot \mathbf{V}_2) = V_1^t V_2 = 100 + 49 - 149 = 0$.

$$Q = \begin{pmatrix} \dfrac{10}{\sqrt{298 + 14\sqrt{149}}} & \dfrac{10}{\sqrt{298 - 14\sqrt{149}}} \\ \dfrac{7 + \sqrt{149}}{\sqrt{298 + 14\sqrt{149}}} & \dfrac{7 - \sqrt{149}}{\sqrt{298 - 14\sqrt{149}}} \end{pmatrix}.$$

3. $p_A(\lambda) = \lambda^2 - 10\lambda - 23$, so $\lambda_1 = 5 + \sqrt{2}, \lambda_2 = 5 - \sqrt{2}$.
Take $\mathbf{V}_1 = \begin{pmatrix} 1 + \sqrt{2} \\ 1 \end{pmatrix}; \mathbf{V}_2 = \begin{pmatrix} 1 - \sqrt{2} \\ 1 \end{pmatrix}$.
Then $(\mathbf{V}_1 \cdot \mathbf{V}_2) = V_1^t V_2 = 1 - 2 + 1 = 0$.

$$Q = \begin{pmatrix} \frac{1+\sqrt{2}}{\sqrt{4+2\sqrt{2}}} & \frac{1-\sqrt{2}}{\sqrt{4-2\sqrt{2}}} \\ \frac{1}{\sqrt{4+2\sqrt{2}}} & \frac{1}{\sqrt{4-2\sqrt{2}}} \end{pmatrix}.$$

4. $p_A(\lambda) = \lambda^2 + 9\lambda - 53$, so $\lambda_1 = \dfrac{-9 + \sqrt{293}}{2}, \lambda_2 = \dfrac{-9 - \sqrt{293}}{2}$.
Take $\mathbf{V}_1 = \begin{pmatrix} 2 \\ 17 + \sqrt{293} \end{pmatrix}, \mathbf{V}_2 = \begin{pmatrix} 2 \\ 17 - \sqrt{293} \end{pmatrix}$.
Then $(\mathbf{V}_1 \cdot \mathbf{V}_2) = V_1^t V_2 = 4 + 289 - 293 = 0$.

$$Q = \begin{pmatrix} \frac{2}{\sqrt{586-34\sqrt{298}}} & \frac{2}{\sqrt{586+34\sqrt{298}}} \\ \frac{17-\sqrt{293}}{\sqrt{586-34\sqrt{298}}} & \frac{17+\sqrt{293}}{\sqrt{586+34\sqrt{298}}} \end{pmatrix}.$$

5. $p_A(\lambda) = (\lambda - 3)(\lambda^2 + 2\lambda - 1)$, so $\lambda_1 = 3, \lambda_2 = -1 + \sqrt{2}, \lambda_3 = -1 - \sqrt{2}$.
Take $\mathbf{V}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \mathbf{V}_2 = \begin{pmatrix} 1 + \sqrt{2} \\ 1 \\ 0 \end{pmatrix}, \mathbf{V}_3 = \begin{pmatrix} 1 - \sqrt{2} \\ 1 \\ 0 \end{pmatrix}$.
Clearly $(\mathbf{V}_1 \cdot \mathbf{V}_2) = (\mathbf{V}_1 \cdot \mathbf{V}_3) = 0, (\mathbf{V}_2 \cdot \mathbf{V}_3) = 1 - 2 + 1 = 0$.

$$Q = \begin{pmatrix} 0 & \frac{1+\sqrt{2}}{\sqrt{4+2\sqrt{2}}} & \frac{1-\sqrt{2}}{\sqrt{4-2\sqrt{2}}} \\ 0 & \frac{1}{\sqrt{4+2\sqrt{2}}} & \frac{1}{\sqrt{4-2\sqrt{2}}} \\ 1 & 0 & 0 \end{pmatrix}.$$

6. $p_A(\lambda) = (\lambda - 2)(\lambda^2 - 2\lambda - 2)$, so $\lambda_1 = 2, \lambda_2 = 1 + \sqrt{3}, \lambda_3 = 1 - \sqrt{3}$.
Take $\mathbf{V}_1 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \mathbf{V}_2 = \begin{pmatrix} -1 + \sqrt{3} \\ 1 \\ 1 \end{pmatrix}, \mathbf{V}_3 = \begin{pmatrix} -1 - \sqrt{3} \\ 1 \\ 1 \end{pmatrix}$.
Then $(\mathbf{V}_1 \cdot \mathbf{V}_2) = (\mathbf{V}_1 \cdot \mathbf{V}_3) = 0$, and $(\mathbf{V}_2 \cdot \mathbf{V}_3) = 1 - 3 + 1 + 1 = 0$.

$$Q = \begin{pmatrix} 0 & (-1 + \sqrt{3})/\sqrt{6} & (-1 - \sqrt{3})/\sqrt{6} \\ -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{6} \end{pmatrix}.$$

7. $p_A(\lambda) = \lambda(\lambda^2 - 5\lambda - 4)$, so $\lambda_1 = 0, \lambda_2 = \dfrac{5+\sqrt{41}}{2}, \lambda_3 = \dfrac{5-\sqrt{41}}{2}$.

Take $\mathbf{V}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{V}_2 = \begin{pmatrix} 5+\sqrt{41} \\ 0 \\ 4 \end{pmatrix}, \mathbf{V}_3 = \begin{pmatrix} 5-\sqrt{41} \\ 0 \\ 4 \end{pmatrix}$.

Then $(\mathbf{V}_1 \cdot \mathbf{V}_2) = (\mathbf{V}_1 \cdot \mathbf{V}_3) = 0$, and $(\mathbf{V}_2 \cdot \mathbf{V}_3) = 25 - 41 + 16 = 0$.

$$Q = \begin{pmatrix} 0 & (5+\sqrt{41})/\sqrt{82+10\sqrt{41}} & (5-\sqrt{41})/\sqrt{82-10\sqrt{41}} \\ 1 & 0 & 0 \\ 0 & 4/\sqrt{82+10\sqrt{41}} & 4/\sqrt{82-10\sqrt{41}} \end{pmatrix}.$$

8. $p_A(\lambda) = \lambda(\lambda^2 - 2\lambda - 16)$, so $\lambda_1 = 0, \lambda_2 = 1 + \sqrt{17}, \lambda_3 = 1 - \sqrt{17}$.

Take $\mathbf{V}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \mathbf{V}_2 = \begin{pmatrix} 1+\sqrt{17} \\ -4 \\ 0 \end{pmatrix}, \mathbf{V}_3 = \begin{pmatrix} 1-\sqrt{17} \\ -4 \\ 0 \end{pmatrix}$.

Then $(\mathbf{V}_1 \cdot \mathbf{V}_2) = (\mathbf{V}_1 \cdot \mathbf{V}_3) = 0$, and $(\mathbf{V}_2 \cdot \mathbf{V}_3) = 1 - 17 + 16 = 0$.

$$Q = \begin{pmatrix} 0 & (1+\sqrt{17})/\sqrt{34+2\sqrt{17}} & (1-\sqrt{17})/\sqrt{34-2\sqrt{17}} \\ 0 & -4/\sqrt{34+2\sqrt{17}} & -4/\sqrt{34-2\sqrt{17}} \\ 1 & 0 & 0 \end{pmatrix}.$$

9. $p_A(\lambda) = \lambda(\lambda^2 - \lambda - 4)$, so $\lambda_1 = 0, \lambda_2 = \dfrac{1+\sqrt{17}}{2}, \lambda_3 = \dfrac{1-\sqrt{17}}{2}$.

Take $\mathbf{V}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{V}_2 = \begin{pmatrix} 0 \\ -1-\sqrt{17} \\ 4 \end{pmatrix}, \mathbf{V}_3 = \begin{pmatrix} 0 \\ -1-\sqrt{17} \\ 4 \end{pmatrix}$.

Then $(\mathbf{V}_1 \cdot \mathbf{V}_2) = (\mathbf{V}_2 \cdot \mathbf{V}_3) = 0$, and $(\mathbf{V}_2 \cdot \mathbf{V}_3) = 1 - 17 + 16 = 0$.

$$Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & (-1-\sqrt{17})/\sqrt{34+2\sqrt{17}} & (-1+\sqrt{17})/\sqrt{34+2\sqrt{17}} \\ 0 & 4/\sqrt{34+2\sqrt{17}} & 4/\sqrt{34+2\sqrt{17}} \end{pmatrix}.$$

10. $p_A(\lambda) = (\lambda - 1)(\lambda^2 - \lambda - 10)$, so $\lambda_1 = 1, \lambda_2 = \dfrac{1+\sqrt{41}}{2}, \lambda_3 = \dfrac{1-\sqrt{41}}{2}$.

Take $\mathbf{V}_1 = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}, \mathbf{V}_2 = \begin{pmatrix} 6 \\ -1+\sqrt{41} \\ 2 \end{pmatrix}, \mathbf{V}_3 = \begin{pmatrix} 6 \\ -1-\sqrt{41} \\ 2 \end{pmatrix}$.

Then $(\mathbf{V}_1 \cdot \mathbf{V}_2) = (\mathbf{V}_1 \cdot \mathbf{V}_3) = 6 - 6 = 0$, and $(\mathbf{V}_2 \cdot \mathbf{V}_3) = 36 + 1 - 41 + 4 = 0$.

$$Q = \begin{pmatrix} 1/\sqrt{10} & 6/\sqrt{82-2\sqrt{41}} & 6/\sqrt{82+2\sqrt{41}} \\ 0 & (-1+\sqrt{41})/\sqrt{82-2\sqrt{41}} & (-1-\sqrt{41})/\sqrt{82-2\sqrt{41}} \\ -3/\sqrt{10} & 2/\sqrt{82-2\sqrt{41}} & 2/\sqrt{82-2\sqrt{41}} \end{pmatrix}.$$

11. $p_A(\lambda) = \lambda^2(\lambda^2 - 2\lambda - 3)$, so $\lambda_1 = \lambda_2 = 0, \lambda_3 = -1, \lambda_4 = 3$.
Take $\mathbf{V}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \mathbf{V}_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \mathbf{V}_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \mathbf{V}_4 = \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}$.
We easily see $(\mathbf{V}_i \cdot \mathbf{V}_j) = 0, i \neq j$.

$$Q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

12. $p_A(\lambda) = \lambda(\lambda - 5)(\lambda^2 - 1)$, so $\lambda_1 = 0, \lambda_2 = 5, \lambda_3 = 1, \lambda_4 = -1$.
Take $\mathbf{V}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \mathbf{V}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \mathbf{V}_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \mathbf{V}_4 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$.
We easily check $(\mathbf{V}_i \cdot \mathbf{V}_j) = 0, i \neq j$.

$$Q = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Section 9.4 Quadratic Forms

1. $A = \begin{pmatrix} 1 & 1 \\ 1 & 6 \end{pmatrix}$

2. $A = \begin{pmatrix} 3 & -2 & -3/2 \\ -2 & 3 & 1 \\ -3/2 & 1 & 1 \end{pmatrix}$

3. $A = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}$

4. $A = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}$

5. $A = \begin{pmatrix} -1 & 0 & -1/2 & -1 \\ 0 & 0 & 2 & 3/2 \\ -1/2 & 2 & 0 & 0 \\ -1 & 3/2 & 0 & 1 \end{pmatrix}$

6. $A = \begin{pmatrix} 1 & 0 & -1/2 \\ 0 & -1 & 2 \\ -1/2 & 2 & 0 \end{pmatrix}$

7. This form is X^tAX with $A = \begin{pmatrix} -5 & 2 \\ 2 & 3 \end{pmatrix}$.

The characteristic polynomial of A is $p_A(\lambda) = \lambda^2 + 2\lambda - 19$ with roots $\lambda_1 = -1 + 2\sqrt{5}$, $\lambda_2 = -1 - 2\sqrt{5}$.

By the Principal Axis Theorem, the standard form is

$$(-1 + 2\sqrt{5})y_1^2 + (-1 - 2\sqrt{5})y_2^2.$$

8. $p_A(\lambda) = \lambda^2 - 5\lambda - 32$ with roots $\lambda_1 = \dfrac{5 + \sqrt{153}}{2}, \lambda_2 = \dfrac{5 - \sqrt{153}}{2}$.
Standard form is

$$\left(\frac{5 + \sqrt{153}}{2}\right) y_1^2 + \left(\frac{5 - \sqrt{153}}{2}\right) y_2^2.$$

9. $p_A(\lambda) = \lambda^2 - 4\lambda - 25$ with roots $\lambda_1 = 2 + \sqrt{29}$, $\lambda_2 = 2 - \sqrt{29}$.
Standard form is

$$(2 + \sqrt{29})y_1^2 + (2 - \sqrt{29})y_2^2.$$

10. $p_A(\lambda) = \lambda^2 - 3\lambda - 2$ with roots $\lambda_1 = \dfrac{3 + \sqrt{17}}{2}$, $\lambda_2 = \dfrac{3 - \sqrt{17}}{2}$.
Standard form is

$$\left(\frac{3 + \sqrt{17}}{2}\right) y_1^2 + \left(\frac{3 - \sqrt{17}}{2}\right) y_2^2.$$

11. $p_A(\lambda) = \lambda^2 - 4\lambda - 9$ with roots $\lambda_1 = 2 + \sqrt{13}$, $\lambda_2 = 2 - \sqrt{13}$.
Standard form is

$$(2 + \sqrt{13})y_1^2 + (2 - \sqrt{13})y_2^2.$$

12. $p_A(\lambda) = \lambda^2 - 7\lambda - 6$ with roots $\lambda_1 = 1$, $\lambda_2 = 6$.
Standard form is

$$y_1^2 + 6y_2^2.$$

13. This form is X^tAX with $A = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

The eigenvalues of A are $\lambda = -1, 1, 2$. By the Principal Axis Theorem the standard form of this quadratic form is

$$-y_1^2 + y_2^2 + 2y_3^2.$$

14. This equation can be written $X^tAX = 6$ with $A = \begin{pmatrix} 1 & -1 \\ -1 & 4 \end{pmatrix}$.

Eigenvalues of A are $\lambda = \dfrac{5 \pm \sqrt{13}}{2}$. The standard form of the conic is

$$\left(\frac{5+\sqrt{13}}{2}\right) y_1^2 + \left(\frac{5-\sqrt{13}}{2}\right) y_2^2 = 6.$$

Since both eigenvalues are positive, this conic is an ellipse.

15. This equation can be written $X^tAX = 5$ with $A = \begin{pmatrix} 3 & 5/2 \\ 5/2 & -3 \end{pmatrix}$.

Eigenvalues of A are $\lambda = \pm\dfrac{\sqrt{61}}{2}$. The standard form of the conic is

$$\frac{\sqrt{61}}{2} y_1^2 - \frac{\sqrt{61}}{2} y_2^2 = 5,$$

which is an hyperbola.

16. This equation can be written $X^tAX = 5$ with $A = \begin{pmatrix} -2 & 1/2 \\ 1/2 & 3 \end{pmatrix}$.

Eigenvalues of A are $\lambda = \dfrac{2 \pm \sqrt{29}}{4}$. The standard form of the conic is

$$\left(\frac{2+\sqrt{29}}{4}\right) y_1^2 + \left(\frac{2-\sqrt{29}}{4}\right) y_2^2 = 5.$$

17. This equation can be written $X^tAX = 8$ with $A = \begin{pmatrix} 4 & 3 \\ 3 & -4 \end{pmatrix}$.

The eigenvalues of A are $\lambda = 5, -5$. The standard form of the conic is

$$5y_1^2 - 5y_2^2 = 8,$$

which is an hyperbola.

18. This equation can be written $X^tAX = 14$ with $A = \begin{pmatrix} 6 & 1 \\ 1 & 5 \end{pmatrix}$. Eigenvalues of A are $\lambda = \dfrac{11 \pm \sqrt{5}}{2}$. The standard form of the conic is

$$\left(\frac{11+\sqrt{5}}{2}\right) y_1^2 + \left(\frac{11-\sqrt{5}}{2}\right) y_2^2 = 14,$$

which is an ellipse.

19. $-2x_1^2 + 2x_1x_2 + 6x_2^2$

20. $14x_1^2 - 6x_1x_2 + 2x_2^2 + 2x_2x_3 + 7x_3^2$

21. $6x_1^2 + 2x_1x_2 - 14x_1x_3 + 2x_1^2 + x_3^2$

22. $7x_1^2 + 2x_1x_2 - 4x_1x_3 - 2x_2x_3 + 3x_3^2$

23. Take $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$. Then $P_A(\lambda) = (\lambda - 2)(\lambda^2 + 1)$ so A has eigenvalues $2, i, -i$ and is diagonalizable. But the standard form associated with X^tAX would have complex coefficients.

Section 9.5 Unitary, Hermitian and Skew-Hermitian Matrices

1. The matrix is not hermitian, not skew-hermitian and not unitary. $p_A(\lambda) = \lambda(\lambda - 4) + 4 = (\lambda - 2)^2$, so $\lambda_1 = \lambda_2 = 2$. There is only one eigenvector, $\mathbf{V}_1 = \begin{pmatrix} i \\ 1 \end{pmatrix}$, so the matrix is not diagonalizable.

2. The matrix is not hermitian, not skew-hermitian, not unitary. $p_A(\lambda) = \lambda^2 + 2\lambda + 1 = (\lambda + 1)^2$, so $\lambda_1 = \lambda_2 = -1$. There is only one associated eigenvector, $\mathbf{V}_1 = \begin{pmatrix} i \\ -1 \end{pmatrix}$, so the matrix is not diagonalizable.

3. The matrix is skew-hermitian since $S^t = -\overline{S}$. $p_S(\lambda) = \lambda(\lambda^2 + 3)$, so $\lambda_1 = 0, \mathbf{V}_1 = \begin{pmatrix} 2 \\ 0 \\ 1+i \end{pmatrix}$; $\lambda_2 = \sqrt{3}i, \mathbf{V}_2 = \begin{pmatrix} 1 \\ \sqrt{3}i \\ -1-i \end{pmatrix}$; $\lambda_3 = -\sqrt{3}i, \mathbf{V}_3 = \begin{pmatrix} 1 \\ -\sqrt{3}i \\ -1-i \end{pmatrix}$.
Take $P = (\mathbf{V}_1\mathbf{V}_2\mathbf{V}_3)$ and

$$P^{-1}SP = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}.$$

4. Compute $\overline{U}U^t = I$, so $U^t = \overline{U}^{-1}$ and U is unitary. $p_U(\lambda) = (\lambda - 1)\left(\lambda^2 - \frac{(1+i)}{\sqrt{2}}\lambda + i\right)$, so $\lambda_1 = 1, \mathbf{V}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$; $\lambda_2 = \frac{1+\sqrt{3}}{2\sqrt{2}} + i\left(\frac{1-\sqrt{3}}{2\sqrt{2}}\right), \mathbf{V}_2 = \begin{pmatrix} 1+i \\ i(1-\sqrt{3}) \\ 0 \end{pmatrix}$; $\lambda_3 = \frac{1-\sqrt{3}}{2\sqrt{2}} + i\left(\frac{1+\sqrt{3}}{2\sqrt{2}}\right), \mathbf{V}_3 = \begin{pmatrix} 1+i \\ i(1+\sqrt{3}) \\ 0 \end{pmatrix}$. Take $P = (\mathbf{V}_1\mathbf{V}_2\mathbf{V}_3)$ and $P^{-1}UP = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$.

5. The matrix is hermitian since $H^t = \overline{H}$. $p_H(\lambda) = \lambda^3 - 3\lambda^2 - 5\lambda + 3$, so eigenvalues are (approximately) $\lambda_1 = 4.051374, \mathbf{V}_1 = \begin{pmatrix} 1 \\ .525687 \\ -.129755i \end{pmatrix}$; $\lambda_2 = .482696, \mathbf{V}_2 = \begin{pmatrix} 1 \\ -1.258652 \\ 2.607546i \end{pmatrix}$;

$\lambda_3 = -1.53407, \mathbf{V}_3 = \begin{pmatrix} 1 \\ -2.267035 \\ -1.477791i \end{pmatrix}$. Take $P = (\mathbf{V}_1\mathbf{V}_2\mathbf{V}_3)$ and

$$P^{-1}HP = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}.$$

6. The matrix is hermitian since $H^t = \overline{H}$. $p_H(\lambda) = (\lambda-1)(\lambda^2+\lambda-10)$, so $\lambda_1 = 1$, $\mathbf{V}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$; $\lambda_2 = \dfrac{-1+\sqrt{41}}{2}, \mathbf{V}_2 = \begin{pmatrix} 6-2i \\ 0 \\ 1+\sqrt{41} \end{pmatrix}$; $\lambda_3 = \dfrac{-1-\sqrt{41}}{2}, \mathbf{V}_3 = \begin{pmatrix} 6-2i \\ 0 \\ 1-\sqrt{41} \end{pmatrix}$.
Take $P = (\mathbf{V}_1\mathbf{V}_2\mathbf{V}_3)$ and $P^{-1}HP = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$.

7. The matrix is skew-hermitian since $S^t = -\overline{S}$. $p_S(\lambda) = \lambda^3 - i\lambda^2 + 5\lambda - 4i$, so (approximately) $\lambda_1 = -2.164248i, \mathbf{V}_1 = \begin{pmatrix} -i \\ -3.164248 \\ 2.924109 \end{pmatrix}$; $\lambda_2 = .772866i, \mathbf{V}_2 = \begin{pmatrix} i \\ .227134 \\ .587771 \end{pmatrix}$; $\lambda_3 = 2.391382i, \mathbf{V}_3 = \begin{pmatrix} i \\ -1.391382 \\ -1.163664 \end{pmatrix}$. Take $P = (\mathbf{V}_1\mathbf{V}_2\mathbf{V}_3)$ and $P^{-1}SP = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$.

8. The matrix is not hermitian, not skew-hermitian, not unitary. $p_A(\lambda) = (\lambda^2-1)(\lambda-3i)$, so $\lambda_1 = 1, \mathbf{V}_1 = \begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix}$; $\lambda_2 = -1, \mathbf{V}_2 = \begin{pmatrix} 0 \\ i \\ -1 \end{pmatrix}$; $\lambda_3 = 3i, \mathbf{V}_3 = \begin{pmatrix} -10i \\ 3 \\ -1 \end{pmatrix}$. Take $P = (\mathbf{V}_1\mathbf{V}_2\mathbf{V}_3)$ and

$$P^{-1}AP = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}.$$

9. The matrix is hermitian since $H^t = \overline{H}$. $p_H(\lambda) = \lambda(\lambda^2 - 8\lambda - 2)$, so $\lambda_1 = 0, \mathbf{V}_1 = \begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix}$; $\lambda_2 = 4+3\sqrt{2}, \mathbf{V}_2 = \begin{pmatrix} 4+3\sqrt{2} \\ -1 \\ -i \end{pmatrix}$; $\lambda_3 = 4-3\sqrt{2}, \mathbf{V}_3 = \begin{pmatrix} 4-3\sqrt{2} \\ -1 \\ -i \end{pmatrix}$. Take $P = (\mathbf{V}_1\mathbf{V}_2\mathbf{V}_3)$ and

$$P^{-1}HP = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}.$$

10. If A is hermitian, then $A^t = \overline{A}$. Then $A = \overline{\overline{A}} = (\overline{A})^t$, so $A(\overline{A})^t = A^2 = (\overline{A})^tA$. If A is skew-hermitian, then $A^t = -\overline{A}$. Then $-A = -\overline{\overline{A}} = (\overline{A})^t$, so $A(\overline{A})^t = -A^2 = -AA = (\overline{A})^tA$. If$A$ is unitary, then $A^{-1} = (\overline{A})^t$, so $A(\overline{A})^t = AA^{-1} = A^{-1}A = (\overline{A})^tA$.

11. If S is skew-hermitian then $S^t = -\overline{S}$, so $s_{rr} = -\overline{s}_{rr}$ for $r = 1, 2, \ldots, n$. But if $s_{rr} = a_{rr} + ib_{rr}$ and $s_{rr} = -\overline{s}_{rr}$ then $a_{rr} = -a_{rr}$ for $r = 1, 2, \ldots, n$, hence $a_{rr} = 0$. Hence $s_{rr} = ib_{rr}$ is either pure imaginary or zero.

12. If H is hermitian, then $H^t = \overline{H}$ so the diagonal elements satisfy $h_{rr} = \overline{h}_{rr}$, but then h_{rr} is real for $r = 1, 2, \ldots, n$.

13. Let U and V be $n \times n$ unitary matrices. Then $U^{-1} = \overline{U}^t$ and $V^{-1} = \overline{V}^t$. Compute $(UV)^{-1} = V^{-1}U^{-1} = \overline{V}^t\overline{U}^t = (\overline{U} : \overline{V})^t = (\overline{UV})^t$, so UV is unitary.

Chapter Ten - Systems of Linear Differential Equations

Section 10.1 Theory of Systems of Linear First-Order Differential Equations

1. The system can be written $X' = AX$ where

$$A = \begin{pmatrix} 5 & 3 \\ 1 & 3 \end{pmatrix} \text{ and } X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -e^{2t} & 3e^{6t} \\ e^{2t} & e^{6t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

(a) By direct calculation

$$X' = \begin{pmatrix} -2e^{2t} & 18e^{6t} \\ 2e^{2t} & 6e^{6t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix},$$

and

$$AX = \begin{pmatrix} 5 & 3 \\ 1 & 3 \end{pmatrix} X = \begin{pmatrix} -2e^{2t} & 18e^{6t} \\ 2e^{2t} & 6e^{6t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

(b) Let $\Omega(t) = \begin{pmatrix} -e^{2t} & 3e^{6t} \\ e^{2t} & e^{6t} \end{pmatrix}$.

Since the columns of Ω are solutions of $X' = AX$
and $|\Omega(0)| = \begin{vmatrix} -1 & 3 \\ 1 & 1 \end{vmatrix} = -4 \neq 0, \Omega(t)$ is a fundamental matrix
for the system.

(c) Clearly $\Omega(t)C = X(t)$ gives the general solution.

(d) To satisfy $x_1(0) = 0, x_2(0) = 4$, find $C = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ so $\Omega(0)C = X(0) = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$.
Since $\Omega^{-1}(0)$ exists,

$$C = \Omega^{-1}(0) \begin{pmatrix} 0 \\ 4 \end{pmatrix} = -\frac{1}{4} \begin{pmatrix} 1 & -3 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix},$$

and

$$X(t) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \Omega(t) \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} -3e^{2t} + 3e^{6t} \\ 3e^{2t} + e^{6t} \end{pmatrix}.$$

2. The coefficient matrix is $A = \begin{pmatrix} 2 & 1 \\ -3 & 6 \end{pmatrix}$.

(b) A fundamental matrix is

$$\Omega(t) = \begin{pmatrix} e^{4t}\cos(t) & e^{4t}\sin(t) \\ 2e^{4t}(\cos(t) - \sin(t)) & 2e^{4t}(\cos(t) - \sin(t)) \end{pmatrix}$$

since the columns of Ω are solutions of $X' = AX$ and $|\Omega(0)| = \begin{vmatrix} 1 & 0 \\ 2 & 2 \end{vmatrix} = 2 \neq 0$.

(c) $X(t) = \Omega(t)C$ gives the general solution.

(d) $C = \Omega^{-1}(0)X(0) = \frac{1}{2}\begin{pmatrix} 2 & 0 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 5/2 \end{pmatrix}$.

Thus

$$X(t) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \Omega(t)\begin{pmatrix} -2 \\ 5/2 \end{pmatrix} = \begin{pmatrix} e^{4t}(-2\cos(t) + 5/2\sin(t)) \\ e^{4t}(\cos(t) + \sin(t)) \end{pmatrix}$$

is the unique solution of the initial value problem.

3. The coefficient matrix is $A = \begin{pmatrix} 3 & 8 \\ 1 & -1 \end{pmatrix}$.

(b) A fundamental matrix is

$$\Omega(t) = \begin{pmatrix} 4e^{(1+2\sqrt{3})t} & 4e^{(1-2\sqrt{3})t} \\ (-1+\sqrt{3})e^{(1+2\sqrt{3})t} & (-1-\sqrt{3})e^{(1-2\sqrt{3})t} \end{pmatrix}$$

since $|\Omega(0)| = \begin{vmatrix} 4 & 4 \\ -1+\sqrt{3} & -1-\sqrt{3} \end{vmatrix} = -8\sqrt{3} \neq 0$.

(c) $X(t) = \Omega(t)C$ gives the general solution.

(d) $C = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \Omega^{-1}(0)X(0) = -\frac{1}{8\sqrt{3}}\begin{pmatrix} -1-\sqrt{3} & -4 \\ 1-\sqrt{3} & 4 \end{pmatrix}\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} (3+5\sqrt{3})/12 \\ (3-5\sqrt{3})/12 \end{pmatrix}$

Thus

$$X(t) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \Omega(t)\begin{pmatrix} (3+5\sqrt{3})/12 \\ (3-5\sqrt{3})/12 \end{pmatrix}$$

gives

$$x_1(t) = 2e^t\cosh(2\sqrt{3}t) + \frac{10\sqrt{3}}{3}e^t\sinh(2\sqrt{3}t); x_2(t) = 2e^t\cosh(2\sqrt{3}t) - \frac{\sqrt{3}}{3}e^t\sinh(2\sqrt{3})t)$$

as the unique solution of the initial value problem.

4. The coefficient matrix is $A = \begin{pmatrix} 1 & -1 \\ 4 & 2 \end{pmatrix}$.

(b) A fundamental matrix is

$$\Omega(t) = e^{3t/2}\begin{pmatrix} 2\cos\left(\frac{\sqrt{15}}{2}t\right) & 2\sin\left(\frac{\sqrt{15}}{2}t\right) \\ -\cos\left(\frac{\sqrt{15}}{2}t\right) + \sqrt{15}\sin\left(\frac{\sqrt{15}}{2}t\right) & -\sin\left(\frac{\sqrt{15}}{2}t\right) + \sqrt{15}\cos\left(\frac{\sqrt{15}}{2}t\right) \end{pmatrix}$$

since the columns are solution of $X' = AX$ and $|\Omega(0)| = \begin{vmatrix} 2 & 0 \\ -1 & \sqrt{15} \end{vmatrix} = 2\sqrt{15} \neq 0$.

(c) $X(t) = \Omega(t)C$ is the general solution.

(d) $C = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \Omega^{-1}(0)X(0) = \frac{1}{2\sqrt{15}}\begin{pmatrix} \sqrt{15} & 0 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} -2 \\ 7 \end{pmatrix} = \begin{pmatrix} -1 \\ \frac{2\sqrt{15}}{5} \end{pmatrix}$.

Thus

$$X(t) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \Omega(t)\begin{pmatrix} -1 \\ \frac{2\sqrt{15}}{5} \end{pmatrix}$$

gives

$$x_1(t) = e^{3t/2}\left[\frac{4\sqrt{15}}{5}\sin\left(\frac{\sqrt{15}}{2}t\right) - 2\cos\left(\frac{\sqrt{15}}{2}t\right)\right];$$

$$x_2(t) = e^{3t/2}\left[7\cos\left(\frac{\sqrt{15}}{2}t\right) - \frac{7}{5}\sqrt{15}\sin\left(\frac{\sqrt{15}}{2}t\right)\right];$$

as the unique solution of the initial value problem.

5. The coefficient matrix is $A = \begin{pmatrix} 5 & -4 & 4 \\ 12 & -11 & 12 \\ 4 & -4 & 5 \end{pmatrix}$.

(b) A fundamental matrix is

$$\Omega(t) = \begin{pmatrix} e^t & 0 & e^{-3t} \\ 0 & e^t & 3e^{-3t} \\ -e^t & e^t & e^{-3t} \end{pmatrix}$$

since the columns are solution of $X' = AX$ and $|\Omega(0)| = -1 \neq 0$.

(c) $X(t) = \Omega(t)C$ is the general solution.

(d) $C = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \Omega^{-1}(0)X(0) = \begin{pmatrix} 2 & -1 & 1 \\ 3 & -2 & 3 \\ -1 & 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix} = \begin{pmatrix} 10 \\ 24 \\ -9 \end{pmatrix}$.

Thus

$$X(t) = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \Omega(t)\begin{pmatrix} 10 \\ 24 \\ -9 \end{pmatrix}$$

gives

$$x_1(t) = 10e^t - 9e^{-3t};\ x_2(t) = 24e^t - 27e^{-3t};\ x_3(t) = 14e^t - 9e^{-3t}$$

as the unique solution of the initial value problem.

Section 10.2 Solution of $X' = AX$ When A Is Constant

1. The coefficient matrix is $A = \begin{pmatrix} 3 & 0 \\ 5 & -4 \end{pmatrix}$, and we need eigenvalues and eigenvectors of A. Solve $|\lambda I - A| = (\lambda - 3)(\lambda + 4) = 0$ to get eigenvalues $\lambda_1 = 3$ and $\lambda_2 = -4$. Corresponding to $\lambda_1 = 3$ we find an eigenvector $\begin{pmatrix} 7 \\ 5 \end{pmatrix}$, and for $\lambda_2 = -4$ we find eigenvector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Construct a fundamental matrix

$$\Omega = \begin{pmatrix} 7e^{3t} & 0 \\ 5e^{3t} & e^{-4t} \end{pmatrix}$$

and the general solution can be written

$$X(t) = \Omega(t)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 7c_1e^{3t} \\ 5c_1e^{3t} + c_2e^{-4t} \end{pmatrix}$$

In Problem 2 through 7, only the answer is given in the form $X(t) = \Omega(t)C$ where $\Omega(t)$ is a fundamental matrix. Note that we can read the eigenvalues and eigenvectors of A directly from Ω.

2. $X(t) = \Omega(t)C = \begin{pmatrix} 2e^t & e^{6t} \\ -3e^t & e^{6t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 2c_1e^t + c_2e^{6t} \\ -3c_1e^t + c_2e^{6t} \end{pmatrix}$

3. $X(t) = \Omega(t)C = \begin{pmatrix} 1 & e^{2t} \\ -1 & e^{2t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} c_1 + c_2e^{2t} \\ -c_1 + c_2e^{2t} \end{pmatrix}$

4. $X(t) = \Omega(t)C = \begin{pmatrix} e^t & e^{-t} & e^{2t} \\ e^t & e^{-t} & 2e^{2t} \\ e^t & 2e^{-t} & e^{2t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} c_1e^t + c_2e^{-t} + c_3e^{2t} \\ c_1e^t + c_2e^{-t} + 2c_3e^{2t} \\ c_1e^t + 2c_2e^{-t} + c_3e^{2t} \end{pmatrix}$

5. $X(t) = \Omega(t)C = \begin{pmatrix} 1 & 2e^{3t} & -e^{-4t} \\ 6 & 3e^{3t} & 2e^{-4t} \\ -13 & -2e^{3t} & e^{-4t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} c_1 + 2c_2e^{3t} - c_3e^{-4t} \\ 6c_1 + 3c_2e^{3t} + 2c_3e^{-4t} \\ -13c_1 - 2c_2e^{3t} + c_3e^{-4t} \end{pmatrix}$

In Problems 6 through 11, the general solution can be written as $X(t) = \Omega(t)C$, and then C is chosen to satisfy initial conditions, $X(0) = \Omega(0)C$ or $C = \Omega^{-1}(0)X(0)$. We can then write the general solution as $X(t) = \Omega(t)\Omega^{-1}(0)X(0)$. This representation is used to answer 8 through 13.

6. $X(t) = \begin{pmatrix} 2e^t & e^{-t} \\ e^t & e^{-t} \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 7 \\ 5 \end{pmatrix} = \begin{pmatrix} 4e^t & +3e^{-t} \\ 2e^t & +3e^{-t} \end{pmatrix}$

7. $X(t) = \begin{pmatrix} 2e^{4t} & e^{-3t} \\ -3e^{4t} & 2e^{-3t} \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -3 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ -19 \end{pmatrix} = \begin{pmatrix} 6e^{4t} - 5e^{-3t} \\ -9e^{4t} - 10e^{-3t} \end{pmatrix}$

8. $X(t) = \begin{pmatrix} 2e^{-3t} & -5e^{4t} \\ e^{-3t} & e^{4t} \end{pmatrix} \begin{pmatrix} 2 & -5 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} -3 \\ 6 \end{pmatrix} = \frac{1}{7} \begin{pmatrix} 54e^{-3t} - 75e^{4t} \\ 27e^{-3t} + 15e^{4t} \end{pmatrix}$

9. $X(t) = \begin{pmatrix} 0 & e^{2t} & 3e^{3t} \\ 1 & e^{2t} & e^{3t} \\ 1 & 0 & e^{3t} \end{pmatrix} \begin{pmatrix} 0 & 1 & 3 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 4e^{2t} - 3e^{3t} \\ 2 + 4e^{2t} - e^{3t} \\ 2 - e^{3t} \end{pmatrix}$

10. $X(t) = \begin{pmatrix} e^t & e^{-t} & -e^{-3t} \\ e^t & 3e^{-t} & 3e^{-3t} \\ e^t & 3e^{-t} & -e^{-3t} \end{pmatrix} \begin{pmatrix} 1 & 1 & -1 \\ 1 & 3 & 3 \\ 1 & 3 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 7 \\ 3 \end{pmatrix} = \begin{pmatrix} e^t + e^{-t} - e^{3t} \\ e^t + 3e^{-t} + 3e^{-3t} \\ e^t + 3e^{-t} - e^{-3t} \end{pmatrix}$

11. $X(t) = \begin{pmatrix} e^{4t} & 3e^{-t} & 0 \\ 0 & -5e^{-t} & -e^{2t} \\ 0 & 0 & e^{2t} \end{pmatrix} \begin{pmatrix} 1 & 3 & 0 \\ 0 & -5 & -1 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 9 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{51}{5}e^{4t} & -\frac{6}{5}e^{-t} & \\ & 2e^{-t} & -3e^{2t} \\ & & 3e^{2t} \end{pmatrix}$

12. If $z = \ln(t), t > 0$, let $y(z) = x(t(z))$, then $t\frac{dy}{dt} = t\frac{dy}{dz}\frac{dz}{dt} = t\frac{dy}{dz}\frac{1}{t} = \frac{dy}{dz}$.
It follows that

$$t\frac{dx_1}{dt} = \frac{dy_1}{dz} = ay_1 + by_2, t\frac{dx_2}{dt} = \frac{dy_2}{dz} = cy_1 + dy_2.$$

13. With $z = \ln(t)$, the transformed system is $\begin{array}{l} \frac{dy_1}{dz} = 6y_1 + 2y_2 \\ \frac{dy_2}{dz} = 4y_1 + 4y_2 \end{array} = \begin{pmatrix} 6 & 2 \\ 4 & 4 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$.
Eigenvalues of A are solutions of $|\lambda I - A| = \lambda^2 - 10\lambda + 16 = (\lambda - 8)(\lambda - 2) = 0$.

For $\lambda = 8$ we find an eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, and for $\lambda = 2$ we find an eigenvector $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$. Thus a fundamental matrix is

$$\Omega(z) = \begin{pmatrix} e^{8z} & e^{2z} \\ e^{8z} & -2e^{2z} \end{pmatrix} = \Omega(z(t)) = \begin{pmatrix} t^8 & t^2 \\ t^8 & -2t^2 \end{pmatrix}.$$

General solutions for $X(t)$ are given by

$$X(t) = \Omega(z(t))C = \begin{pmatrix} t^8 & t^2 \\ t^8 & -2t^2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

or

$$x_1(t) = c_1 t^8 + c_2 t^2; x_2(t) = c_1 t^8 - 2c_2 t^2.$$

14. Let $z = \ln(t)$ to get the transformed system $Y' = \begin{pmatrix} -1 & -3 \\ 1 & -5 \end{pmatrix} Y$ with general solution

$$Y(z) = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \Omega(z)C = \begin{pmatrix} e^{-4z} & 3e^{-2z} \\ e^{-4z} & e^{-2z} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

Then

$$X = Y(z(t)) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \Omega(z(t))C = \begin{pmatrix} t^{-4} & 3t^{-2} \\ t^{-4} & t^{-2} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

or

$$x_1(t) = c_1 t^{-4} + 3c_2 t^{-2}; x_2(t) = c_1 t^{-4} + c_2 t^{-2}.$$

15. Eigenvalues of A satisfy $\lambda^2 - 4\lambda + 8 = 0$ and are $\lambda_1 = 2 + 2i, \lambda_2 = 2 - 2i$ with a corresponding set of eigenvectors $\begin{pmatrix} 2 \\ -i \end{pmatrix}$ and $\begin{pmatrix} 2 \\ i \end{pmatrix}$. A real fundamental matrix is

$$\Omega(t) = \begin{pmatrix} 2e^{2t}\cos(2t) & 2e^{2t}\sin(2t) \\ e^{2t}\sin(2t) & -e^{2t}\cos(2t) \end{pmatrix}.$$

16. Eigenvalues of A satisfy $\lambda^2 + 2\lambda + 5 = 0$ and are $\lambda_1 = -1 + 2i, \lambda_2 = -1 - 2i$ with a corresponding set of eigenvectors $\begin{pmatrix} 5 \\ -1+2i \end{pmatrix}$ and $\begin{pmatrix} 5 \\ -1-2i \end{pmatrix}$.
A real fundamental matrix is

$$\Omega(t) = \begin{pmatrix} 5e^{-t}\cos(2t) & 5e^{-t}\sin(2t) \\ e^{-t}[-\cos(2t) - 2\sin(2t)] & e^{-t}[2\cos(2t) - \sin(2t)] \end{pmatrix}.$$

17. Eigenvalues of A satisfy $\lambda^2 - 2\lambda + 2 = 0$ and are $\lambda_1 = 1+i, \lambda_2 = 1-i$ with a corresponding set of eigenvectors $\begin{pmatrix} 5 \\ 2-i \end{pmatrix}$ and $\begin{pmatrix} 5 \\ 2+i \end{pmatrix}$.
A real fundamental matrix is

$$\Omega(t) = \begin{pmatrix} 5e^t\cos(t) & 5e^t\sin(t) \\ e^t[(2\cos(t) + \sin(t)] & e^t[2\sin(t) - \cos(t)] \end{pmatrix}.$$

18. Eigenvalues of A satisfy $(\lambda+1)(\lambda^2+1)=0$ and are $\lambda_1=-1, \lambda_2=i, \lambda_3=-i$ with a corresponding set of eigenvectors $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1+i \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1-i \\ 1 \\ 1 \end{pmatrix}$.

A real fundamental matrix is

$$\Omega(t)=\begin{pmatrix} 0 & \cos(t)-\sin(t) & \sin(t)+\cos(t) \\ e^{-t} & \cos(t) & \sin(t) \\ -e^{-t} & \cos(t) & \sin(t) \end{pmatrix}.$$

19. Eigenvalues of A satisfy $(\lambda+2)(\lambda^2+2\lambda+5)=0$ and are $\lambda_1=-2, \lambda_2=-1+2i, \lambda_3=-1-2i$ with a corresponding set of eigenvectors $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1+2i \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1-2i \\ 3 \end{pmatrix}$.

A real fundamental matrix is

$$\Omega(t)=\begin{pmatrix} 0 & e^{-t}\cos(2t) & e^{-t}\sin(2t) \\ 0 & e^{-t}[\cos(2t)-2\sin(2t)] & e^{-t}[\sin(2t)+2\cos(2t)] \\ e^{-2t} & 3e^{-t}\cos(2t) & 3e^{-t}\sin(2t) \end{pmatrix}.$$

In Problems 20 through 23, the solution is expressed in the form $X(t)=\Omega(t)\Omega^{-1}(0)X(0)$ where $\Omega(t)$ is a real fundamental matrix.

20. Eigenvalues and a set of eigenvectors of A are
$\lambda_1=2+3i, \begin{pmatrix} 2 \\ -1+3i \end{pmatrix}; \lambda_2=2-3i, \begin{pmatrix} 2 \\ -1-3i \end{pmatrix}$.
Then

$$\Omega(t)=\begin{pmatrix} 2e^{2t}\cos(3t) & 2e^{2t}\sin(3t) \\ e^{2t}[-\cos(3t)-3\sin(3t)] & e^{2t}[-\sin(3t)+3\cos(3t)] \end{pmatrix}$$

and

$$X(t)=\Omega(t)\begin{pmatrix} 2 & 0 \\ -1 & 3 \end{pmatrix}^{-1}\begin{pmatrix} 2 \\ 8 \end{pmatrix}=\Omega(t)\begin{pmatrix} 1 \\ 3 \end{pmatrix}=\begin{pmatrix} e^{2t}[2\cos(3t)+6\sin(3t)] \\ e^{2t}[8\cos(3t)-6\sin(3t)] \end{pmatrix}.$$

21. Eigenvalues and a set of eigenvectors of A are $\lambda_1=i, \begin{pmatrix} 2 \\ 3-i \end{pmatrix}; \lambda_2=-i, \begin{pmatrix} 2 \\ 3+i \end{pmatrix}$.
Then

$$\Omega(t)=\begin{pmatrix} 2\cos(t) & 2\sin(t) \\ 3\cos(t)+\sin(t) & 3\sin(t)-\cos(t) \end{pmatrix}$$

and

$$X(t)=\Omega(t)\Omega^{-1}(0)X(0)=\Omega(t)\begin{pmatrix} 2 & 0 \\ 3 & -1 \end{pmatrix}^{-1}\begin{pmatrix} 2 \\ 10 \end{pmatrix}$$

$$=\Omega(t)\begin{pmatrix} 1 \\ -7 \end{pmatrix}=\begin{pmatrix} 2\cos(t) & -14\sin(t) \\ 10\cos(t) & -20\sin(t) \end{pmatrix}.$$

22. Eigenvalues and a set of eigenvectors of A are $\lambda_1 = i, \begin{pmatrix} 5 \\ 2-i \end{pmatrix}; \lambda_2 = -i, \begin{pmatrix} 5 \\ 2+i \end{pmatrix}$.
Then

$$\Omega(t) = \begin{pmatrix} 5\cos(t) & 5\sin(t) \\ 2\cos(t)+\sin(t) & 2\sin(t)-\cos(t) \end{pmatrix}$$

and

$$X(t) = \Omega(t)\Omega^{-1}(0)X(0) = \Omega(t)\begin{pmatrix} 5 & 0 \\ 2 & -1 \end{pmatrix}^{-1}\begin{pmatrix} 5 \\ 0 \end{pmatrix} = \Omega(t)\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 5\cos(t)+10\sin(t) \\ 5\sin(t) \end{pmatrix}.$$

23. Eigenvalues and a set of eigenvectors of A are
$\lambda_1 = 1, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}; \lambda_2 = 1+i, \begin{pmatrix} 2+i \\ 2 \\ 1 \end{pmatrix}, \lambda_3 = 1-i, \begin{pmatrix} 2-i \\ 2 \\ 1 \end{pmatrix}$.
Then

$$\Omega(t) = \begin{pmatrix} e^t & e^t[2\cos(t)-\sin(t)] & e^t[2\sin(t)+\cos(t)] \\ e^t & 2e^t\cos(t) & 2e^t\sin(t) \\ e^t & e^t\cos(t) & e^t\sin(t) \end{pmatrix}$$

and

$$X(t) = \Omega(t)\Omega^{-1}(0)X(0) = \Omega(t)\begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 0 \\ 1 & 1 & 0 \end{pmatrix}^{-1}\begin{pmatrix} 7 \\ 4 \\ 3 \end{pmatrix}$$

$$= \Omega(t)\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = e^t\begin{pmatrix} 2+5\cos(t)+5\sin(t) \\ 2+2\cos(t)+6\sin(t) \\ 2+\cos(t)+3\sin(t) \end{pmatrix}.$$

24. Yes, $A = \begin{pmatrix} 1 & 0 \\ i & -1 \end{pmatrix}$ has eigenvalues $\lambda = \pm 1$.

In Problems 25 through 30 in this section, this matrix given is not necessarily the matrix exponential e^{At} produced by many standard computer algebra systems, e.g. Maple, but rather a matrix produced by the method described in the text. The matrix so produced is not unique (since eigenvectors are not unique) but will be a fundamental matrix. If the matrix is denoted $\Omega(t)$, then $\Omega(t)\Omega^{-1}(0)$ will be the unique matrix e^{At} produced by computer software.

25. A has $\lambda_1 = \lambda_2 = 3$, with $\mathbf{V}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$;

$\Omega(t) = e^{At} = \begin{pmatrix} e^{3t} & 2te^{3t} \\ 0 & e^{3t} \end{pmatrix}; \; X(t) = \Omega(t)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} c_1e^{3t}+2c_2te^{3t} \\ c_2e^{3t} \end{pmatrix}$

26. A has $\lambda_1 = \lambda_2 = 2$, with $\mathbf{V}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$;

$\Omega(t) = \begin{pmatrix} 0 & e^{2t} \\ e^{2t} & 5te^{2t} \end{pmatrix}; \; X(t) = \Omega(t)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} c_2e^{2t} \\ c_1e^{2t}+5c_2te^{2t} \end{pmatrix}$

27. A has $\lambda_1 = 2$, with $\mathbf{V}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$); $\lambda_2 = \lambda_3 = 5$, with $\mathbf{V}_2 = \begin{pmatrix} -3 \\ -3 \\ 1 \end{pmatrix}$;

$$\Omega(t) = \begin{pmatrix} e^{2t} & 3e^{5t} & 27te^{5t} \\ 0 & 3e^{5t} & (3+27t)e^{5t} \\ 0 & -e^{5t} & (2-9t)e^{5t} \end{pmatrix}; \; X(t) = \Omega(t)\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} c_1e^{2t} + & [3c_2 + 27c_3t]e^{5t} \\ & [3c_2 + 3c_3(1+9t)]e^{5t} \\ & [-c_2 + c_3(2-9t)]e^{5t} \end{pmatrix}$$

28. A has $\lambda_1 = \lambda_2 = \lambda_3 = 1$, with $\mathbf{V}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$;

$$\Omega(t) = e^{At} = \begin{pmatrix} e^t & 5te^t & 0 \\ 0 & e^t & 0 \\ 4te^t & (10t^2+8t)e^t & e^t \end{pmatrix}; \; X(t) = \Omega(t)\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} c_1e^t & +5c_2te^t & \\ & c_2e^t & \\ 4c_1te^t & +c_2(10t^2+8t)e^t & +c_3e^t \end{pmatrix}$$

29. A has $\lambda_1 = 0$, with $\mathbf{V}_1 = \begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \end{pmatrix}$; $\lambda_2 = 3$, with $\mathbf{V}_2 = \begin{pmatrix} 3 \\ 2 \\ 2 \\ 0 \end{pmatrix}$; $\lambda_3 = \lambda_4 = 1$, with

$$\mathbf{V}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}; \; \mathbf{V}_4 = \begin{pmatrix} 0 \\ -2 \\ -2 \\ 1 \end{pmatrix};$$

$$\Omega(t) = \begin{pmatrix} 2 & 3e^{3t} & e^t & 0 \\ 0 & 2e^{3t} & 0 & -2e^t \\ 1 & 2e^{3t} & 0 & -2e^t \\ 0 & 0 & 0 & e^t \end{pmatrix}; \; X(t) = \Omega(t)\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 2c_1 & +3c_2e^{3t} & +c_3e^t & \\ & 2c_2e^{3t} & & -2c_4e^t \\ c_1 & 2c_2e^{3t} & & -2c_4e^t \\ & & & c_4e^t \end{pmatrix}$$

30. A has $\lambda_1 = \lambda_3 = i$, with $\mathbf{V}_1 = \begin{pmatrix} i \\ -1 \\ -i \\ 1 \end{pmatrix}$; $\lambda_2 = \lambda_4 = -i$, with $\mathbf{V}_2 = \begin{pmatrix} -i \\ -1 \\ i \\ 1 \end{pmatrix}$;

$$\Omega(t) = \begin{pmatrix} \cos(t) & \cos(t) + t\sin(t) & \sin(t) & \sin(t) - t\cos(t) \\ -\sin(t) & t\cos(t) & \cos(t) & t\sin(t) \\ -\cos(t) & \cos(t) - t\sin(t) & -\sin(t) & \sin(t) + t\cos(t) \\ \sin(t) & -t\cos(t) - 2\sin(t) & -\cos(t) & -t\sin(t) + 2\cos(t) \end{pmatrix}; X(t) = \Omega(t)\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix}$$

In Problem 31 through 35 the unique solution of the initial value problem is expressed in the matrix form $X(t) = \Omega(t)\Omega^{-1}(0)X^0$ where $\Omega(t)$ is a fundamental matrix. In the case that $\Omega(t) = e^{At}$, then $\Omega(0) = I_n$ and this factor is omitted. The general solution can be recovered by replacing X^0 by an $n \times 1$ matrix having arbitrary entries $c_1, c_2, \ldots, c_n$.

31. A has $\lambda_1 = \lambda_2 = 6$, with $\mathbf{V}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$;

$$X(t) = \begin{pmatrix} e^{6t} & (1+t)e^{6t} \\ e^{6t} & te^{6t} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{-1}\begin{pmatrix} 5 \\ 3 \end{pmatrix}, \text{ so } x_1 = (5+2t)e^{6t}; \; x_2 = (3+2t)e^{6t}.$$

32. See Problem 26. $X(t) = \begin{pmatrix} e^{2t} & 0 \\ 5te^{2t} & e^{2t} \end{pmatrix} \begin{pmatrix} 4 \\ 3 \end{pmatrix}$, so $x_1(t) = 4e^{2t}$; $x_2(t) = (20t+3)e^{2t}$.

33. A has $\lambda_1 = 2$, with $\mathbf{V}_1 = \begin{pmatrix} 1 \\ 6 \\ 0 \end{pmatrix}$; $\lambda_2 = \lambda_3 = -4$, with $\mathbf{V}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$;

$X(t) = \begin{pmatrix} e^{2t} & e^{-4t} & 11te^{-4t} \\ 6e^{2t} & 0 & 5e^{-4t} \\ 0 & 0 & 6e^{-4t} \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 6 & 0 & 5 \\ 0 & 0 & 6 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 4 \\ 12 \end{pmatrix}$, so

$$x_1 = -e^{2t} + e^{-4t}(1+22t);\ x_2 = -6e^{2t} + 10e^{-4t};\ x_3 = 12e^{-4t}.$$

34. A has $\lambda_1 = \lambda_2 = \lambda_3 = -5$, with $\mathbf{V}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$;

$X(t) = \begin{pmatrix} e^{-5t} & 2te^{-5t} & (t+3t^2)e^{-5t} \\ 0 & e^{-5t} & 3te^{-5t} \\ 0 & 0 & e^{-5t} \end{pmatrix} \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}$, so

$$x_1 = (2 - 2t + 12t^2)e^{-5t};\ x_2 = (-3+12t)e^{-5t};\ x_3 = 4e^{-5t}.$$

35. A has $\lambda_1 = i$, with $\mathbf{V}_1 = \begin{pmatrix} 2 \\ 1-i \\ 0 \\ 0 \end{pmatrix}$, $\lambda_2 = \overline{\lambda_1}, \mathbf{V}_2 = \overline{\mathbf{V}_2}; \lambda_3 = \lambda_4 = 2$, with $\mathbf{V}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$;

$X(t) = \begin{pmatrix} 2\cos(t) & 2\sin(t) & 0 & 0 \\ \cos(t)+\sin(t) & \sin(t)-\cos(t) & 0 & 0 \\ 0 & 0 & e^{2t} & 3te^{2t} \\ 0 & 0 & e^{2t} & (-1+3t)e^{2t} \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 2 \\ -2 \\ 1 \\ 4 \end{pmatrix}$,
so

$$x_1 = 2\cos(t) + 6\sin(t);\ x_2 = -2\cos(t) + 4\sin(t);\ x_3 = (1-9t)e^{2t};\ x_4 = (4-9t)e^{2t}.$$

36. $A = \begin{pmatrix} -2 & 1 \\ -4 & 3 \end{pmatrix}$ has eigenvalues $\lambda_1 = -1, \lambda_2 = 2$ and $P = \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}$.
We find $P^{-1} = \frac{1}{3}\begin{pmatrix} 4 & -1 \\ -1 & 1 \end{pmatrix}$, so Z satisfies $Z' = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} Z$.
Solving for Z gives

$$z_1 = c_1 e^{-t};\ z_2 = c_2 e^{2t}$$

Finally

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = PZ = \begin{pmatrix} c_1 e^t + c_2 e^{-2t} \\ c_1 e^t + 4c_2 e^{-2t} \end{pmatrix}.$$

37. For $A = \begin{pmatrix} 3 & 3 \\ 1 & 5 \end{pmatrix}, \lambda_1 = 2, \lambda_2 = 6, P = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}$, $Z' = \begin{pmatrix} 2 & 0 \\ 0 & 6 \end{pmatrix} Z$. Solving for Z gives

$$z_1 = c_1 e^{2t};\ z_2 = c_2 e^{6t}.$$

So

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = PZ = \begin{pmatrix} 3c_1e^{2t} + c_2e^{6t} \\ -c_1e^{2t} + c_2e^{6t} \end{pmatrix}$$

38. For $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \lambda_1 = 0, \lambda_2 = 2, P = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix},\ Z' = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} Z$. Solving for Z gives

$$z_1 = c_1;\ z_2 = c_2e^{2t}.$$

So

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = PZ = \begin{pmatrix} c_1 + c_2e^{2t} \\ -c_1 + c_2e^{2t} \end{pmatrix}.$$

39. For $A = \begin{pmatrix} 6 & 5 \\ 1 & 2 \end{pmatrix}, \lambda_1 = 1, \lambda_2 = 7, P = \begin{pmatrix} 1 & 5 \\ -1 & 1 \end{pmatrix},\ Z' = \begin{pmatrix} 1 & 0 \\ 0 & 7 \end{pmatrix} Z +$. Solving for Z gives $z_1 = c_1e^t;\ z_2 = c_2e^{7t}$. So

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = PZ = \begin{pmatrix} c_1e^t + 5c_2e^{7t} \\ -c_1e^t + c_2e^{7t} \end{pmatrix}.$$

40. For $A = \begin{pmatrix} 3 & -2 \\ 9 & -3 \end{pmatrix}, \lambda_1 = 3i, \lambda_2 = -3i, P = \begin{pmatrix} 1+i & 1-i \\ 3 & 3 \end{pmatrix},\ Z' = \begin{pmatrix} 3i & 0 \\ 0 & -3i \end{pmatrix} Z$. Solving for Z gives $z_1 = d_1e^{3it};\ z_2 = d_2e^{-3it}$, with d_1 and d_2 complex constants. Letting $d_1 = \frac{1}{2}(c_1 + c_2i), d_2 = \frac{1}{2}(c_1 - c_2i)$ with c_1, c_2 real we get

$$z_1 = \frac{1}{2}[c_1\cos(3t) - c_2\sin(3t)] + \frac{1}{2}[c_2\cos(3t) + c_1\sin(3t)]i + \left(\frac{2-i}{6}\right)e^{2t};$$

$$z_2 = \frac{1}{2}[c_1\cos(3t) - c_2\sin(3t)] - \frac{1}{2}[c_2\cos(3t) + c_1\sin(3t)]i + \left(\frac{2+i}{6}\right)e^{2t}.$$

So

$$X = PZ = \begin{pmatrix} c_1[\cos(3t) - \sin(3t)] & -c_2[\sin(3t) + \cos(3t)] \\ 3c_1\cos(3t) & -3c_2\sin(3t) \end{pmatrix}.$$

In Problems 41 through 45 in this section, this matrix given is not necessarily the matrix exponential e^{At} produced by many standard computer algebra systems, e.g. Maple, but rather a matrix produced by the method described in the text. The matrix so produced is not unique (since eigenvectors are not unique) but will be a fundamental matrix. If the matrix is denoted $\Omega(t)$, then $\Omega(t)\Omega^{-1}(0)$ will be the unique matrix e^{At} produced by computer software.

41. $\Omega(t) = e^{At} = \begin{pmatrix} e^{3t} & 2te^{3t} \\ 0 & e^{3t} \end{pmatrix};\ X(t) = \Omega(t)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

42. $\Omega(t) = \begin{pmatrix} 0 & e^{2t} \\ e^{2t} & 5te^{2t} \end{pmatrix};\ X(t) = \Omega(t)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

43. $\Omega(t) = \begin{pmatrix} 2e^{4t} & (1-2t)e^{4t} \\ -e^{4t} & te^{4t} \end{pmatrix}$; $X(t) = \Omega(t)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

44. $\Omega(t) = \begin{pmatrix} e^{2t} & (1+3t)e^{2t} \\ e^{2t} & 3te^{2t} \end{pmatrix}$; $X(t) = \Omega(t)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

45. $\Omega(t) = \begin{pmatrix} e^{2t} & 3e^{5t} & 27te^{5t} \\ 0 & 3e^{5t} & (3+27t)e^{5t} \\ 0 & -e^{5t} & (2-9t)e^{5t} \end{pmatrix}$; $X(t) = \Omega(t)\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$

In Problem 46 through 50 the unique solution of the initial value problem is expressed in the matrix form $X(t) = \Omega(t)\Omega^{-1}(0)X^0$ where $\Omega(t)$ is a fundamental matrix. In the case that $\Omega(t) = e^{At}$, then $\Omega(0) = I_n$ and this factor is omitted. The general solution can be recovered by replacing X^0 by an $n \times 1$ matrix having arbitrary entries $c_1, c_2, \ldots, c_n$.

46. $X(t) = \begin{pmatrix} e^{6t} & (1+t)e^{6t} \\ e^{6t} & te^{6t} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{-1}\begin{pmatrix} 5 \\ 3 \end{pmatrix}$, so $x_1 = (5+2t)e^{6t}$; $x_2 = (3+2t)e^{6t}$.

47. $X(t) = \begin{pmatrix} e^{2t} & 0 \\ 5te^{2t} & e^{2t} \end{pmatrix}\begin{pmatrix} 4 \\ 3 \end{pmatrix}$, so $x_1(t) = 4e^{2t}$; $x_2(t) = (20t+3)e^{2t}$.

48. $X(t) = \begin{pmatrix} e^{2t} & e^{-4t} & 11te^{-4t} \\ 6e^{2t} & 0 & 5e^{-4t} \\ 0 & 0 & 6e^{-4t} \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 6 & 0 & 5 \\ 0 & 0 & 6 \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ 4 \\ 12 \end{pmatrix}$, so

$$x_1 = -e^{2t} + e^{-4t}(1+22t);\; x_2 = -6e^{2t} + 10e^{-4t};\; x_3 = 12e^{-4t}.$$

49. $X(t) = \begin{pmatrix} e^{-5t} & 2te^{-5t} & (t+3t^2)e^{-5t} \\ 0 & e^{-5t} & 3te^{-5t} \\ 0 & 0 & e^{-5t} \end{pmatrix}\begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}$, so

$$x_1 = (2-2t+12t^2)e^{-5t};\; x_2 = (-3+12t)e^{-5t};\; x_3 = 4e^{-5t}.$$

50. $X(t) = \begin{pmatrix} 2\cos(t) & 2\sin(t) & 0 & 0 \\ \cos(t)+\sin(t) & \sin(t)-\cos(t) & 0 & 0 \\ 0 & 0 & e^{2t} & 3te^{2t} \\ 0 & 0 & e^{2t} & (-1+3t)e^{2t} \end{pmatrix}\begin{pmatrix} 2 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}^{-1}\begin{pmatrix} 2 \\ -2 \\ 1 \\ 4 \end{pmatrix}$,

so

$$x_1 = 2\cos(t) + 6\sin(t);\; x_2 = -2\cos(t) + 4\sin(t);\; x_3 = (1-9t)e^{2t};\; x_4 = (4-9t)e^{2t}.$$

Section 10.3 Solution of $X' = AX + G$

1. $A = \begin{pmatrix} 5 & 2 \\ -2 & 1 \end{pmatrix}$ has repeated eigenvalues $\lambda_1 = \lambda_2 = 3$, with one eigenvector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Compute $(A-3I)^2 = 0$ and choose $K = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ so $(A-3I)^2K = 0$ but $(A-3I)K \neq 0$. A fundamental matrix is found to be $W(t) = e^{3t}\begin{pmatrix} 1 & 1+2t \\ -1 & -2t \end{pmatrix}$. If we wish we can easily compute $e^{At} = \Omega(t) = W(t)W^{-1}(0) = e^{3t}\begin{pmatrix} 1+2t & 2t \\ -2t & 1-2t \end{pmatrix}$, and we will use $\Omega(t) = e^{At}$ in the remainder of this solution since this is matrix produced by most computer software. Compute $\Omega^{-1}(t) = e^{-3t}\begin{pmatrix} 1-2t & -2t \\ 2t & 1+2t \end{pmatrix}$ and take $u(t)$ to be the particular solution given by

$$u(t) = \int \Omega^{-1}(t)G(t)dt = \int e^{-3t}\begin{pmatrix} 1-2t & -2t \\ 2t & 1+2t \end{pmatrix}\begin{pmatrix} -3e^{t} \\ e^{3t} \end{pmatrix}dt =$$

$$\int \begin{pmatrix} 6te^{-2t} & -3e^{-2t} - 2t \\ -6te^{-2t} & +1+2t \end{pmatrix}dt = \begin{pmatrix} -3te^{-2t} - t^2 \\ \frac{3}{2}(1+2t)e^{-2t} + t + t^2 \end{pmatrix}.$$

Then $X(t) = \Omega(t)C + \Omega(t)u(t)$

$$= e^{3t}\begin{pmatrix} 1+2t & 2t \\ -2t & 1-2t \end{pmatrix}\left[\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} -3te^{-2t} - t^2 \\ \frac{3}{2}(1+2t)e^{-2t} + t + t^2 \end{pmatrix}\right]$$

so

$$x_1 = e^{3t}[c_1(1+2t) + 2tc_2] + t^2e^{3t};$$

$$x_2 = e^{3t}[-2tc_1 + (1-2t)c_2] + (t-t^2)e^{3t} + \frac{3}{2}e^{t}$$

In Problems 2 through 5, there may be some variations in the appearance of the solution obtained by a correct solution method and the answers given below. This variation is due to the nonuniqueness of a fundamental matrix, $\Omega(t)$, and to variations as to whether terms in the particular solution obtained which duplicate terms in the solution of $X' = AX$ have been absorbed in the complementary solution or not.

2. A has repeated eigenvalues $\lambda_1 = \lambda_2 = 0$, with eigenvector $\mathbf{V} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

A fundamental matrix is

$$\Omega(t) = \begin{pmatrix} 2 & 1+2t \\ 1 & t \end{pmatrix}.$$

We find a solution

$$x_1 = 2c_1 + c_2(1+2t) + t + t^2 - 2t^3; \; x_2 = c_1 + c_2t + 2t^2 - t^3.$$

3. A has repeated eigenvalues $\lambda_1 = \lambda_2 = 6$, with eigenvector $\mathbf{V} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

A fundamental matrix is

$$\Omega(t) = e^{6t}\begin{pmatrix} 1 & 1+t \\ 1 & t \end{pmatrix}.$$

We find a solution

$$x_1 = [c_1 + c_2(1+t) + 2t + t^2 - t^3]e^{6t};\ x_2 = [c_1 + c_2 t + 4t^2 - t^3]e^{6t}.$$

4. A has repeated eigenvalues $\lambda_1 = \lambda_2 = \lambda_3 = 2$, with eigenvectors

$$\mathbf{V}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{V}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

A fundamental matrix is

$$\Omega(t) = e^{2t}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -4t \\ 0 & 1 & 1-4t \end{pmatrix}.$$

We find a solution

$$x_1 = c_1 e^{2t};\ x_2 = [c_2 - 4tc_3]e^{2t} + 1;\ x_3 = [c_2 + c_3(1-4t)]e^{2t} + 1.$$

5. A has repeated eigenvalues $\lambda_1 = \lambda_2 = 1$, with a single eigenvector

$$\mathbf{V}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

and $\lambda_3 = \lambda_4 = 3$ with a pair of eigenvectors

$$\mathbf{V}_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \text{ and } \mathbf{V}_4 \begin{pmatrix} 0 \\ -9 \\ 2 \\ 0 \end{pmatrix}.$$

A fundamental matrix is

$$\Omega(t) = \begin{pmatrix} 0 & e^t & 0 & 0 \\ 0 & -2e^t & e^{3t} & -9e^{3t} \\ 0 & 0 & 0 & 2e^{3t} \\ e^t & -5te^t & e^{3t} & 0 \end{pmatrix}.$$

We find a solution

$$x_1 = c_2 e^t;\ x_2 = -2c_2 e^t + (c_3 - 9c_4)e^{3t} + e^t;\ x_3 = 2c_4 e^{3t};\ x_4 = (c_1 - 5c_2 t)e^t + c_3 e^{3t} + e^t + 3te^t.$$

6. $x_1 = -1 + e^{2t}$;
$x_2 = -5t + (3+5t)e^{2t}$.

7. $x_1 = (-1 - 14t)e^t$;
$x_2 = (3 - 14t)e^t$.

8. $x_1 = 13e^t - (8 + 12t + 3t^2)e^{2t}$;
$x_2 = 4e^t + (7 + 2t)e^{2t}$;
$x_3 = -e^t - e^{2t}$.

9. $x_1 = \left(6 + 12t + \frac{1}{2}t^2\right) e^{-2t}$;
$x_2 = \left(2 + 12t + \frac{1}{2}t^2\right) e^{-2t}$;
$x_3 = \left(3 + 38t + 66t^2 + \frac{13}{6}t^3\right) e^{-2t}$.

10. (a) If $\Omega(t)$ is a transition matrix, then $\Omega(t) = e^{At}$. Then $\Omega(-t) = e^{-At}$ and $\Omega(t)\Omega(-t) = e^{At} \cdot e^{-At} = I$ by properties of the matrix exponential. Similarly $\Omega(-t)\Omega(t) = I$, so $\Omega(-t) = \Omega^{-1}(t)$. Also $\Omega(t+s) = e^{A(t+s)} = e^{At} \cdot e^{As} = \Omega(t)\Omega(s)$.

(b) Since $\Omega(t)$ is a fundamental matrix, the columns of $\Omega(t)$ are linearly independent solutions of $X' = AX$. Then $\Omega^{-1}(0)$ is a nonsingular matrix of constants and the columns of $\Phi(t) = \Omega(t)\Omega^{-1}(0)$ are linear combinations of the columns of $\Omega(t)$, and are hence solutions. The columns of $\Phi(t)$ are linearly independent because rank $(\Phi(t))$ = rank $(\Omega(t)\Omega^{-1}(0))$ = rank $(\Omega(t)) = n$. Finally $\Phi(0) = \Omega(0)\Omega^{-1}(0) = I_n$.

11. Compute $\frac{d\Omega(t)}{dt} = \begin{pmatrix} 2e^t & 6e^{6t} \\ -3e^t & 6e^{6t} \end{pmatrix}$, and

$$A\Omega(t) = \begin{pmatrix} 4 & 2 \\ 3 & 3 \end{pmatrix} \Omega(t) = \begin{pmatrix} 2e^t & 6e^{6t} \\ -3e^t & 6e^{6t} \end{pmatrix}.$$

Since $\frac{d\Omega(t)}{dt} = A\Omega(t)$, the columns of $\Omega(t)$ are solutions of $X' = AX$.
Calculate $|\Omega(0)| = \begin{vmatrix} 2 & 1 \\ -3 & 1 \end{vmatrix} = 5 \neq 0$, so the columns of $\Omega(t)$ are linearly independent.
By Problem 10(b),

$$\Phi(t) = \Omega(t)\Omega^{-1}(0) = \frac{1}{5}\begin{pmatrix} 2e^t & e^{6t} \\ -3e^t & e^{6t} \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 3 & 2 \end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix} 2e^t + 3e^{6t} & -2e^t + 2e^{6t} \\ -3e^t + 3e^{6t} & 3e^t + 2e^{6t} \end{pmatrix}$$

is a transition matrix.

12. $\frac{d\Omega(t)}{dt} = \begin{pmatrix} -10e^{-5t} & -25e^{-5t} \\ -5e^{-5t} & (-\frac{25}{2}t + \frac{5}{2})e^{-5t} \end{pmatrix}$, and

$$A\Omega(t) = \begin{pmatrix} 0 & -10 \\ \frac{5}{2} & -10 \end{pmatrix} \Omega(t) = \begin{pmatrix} -10e^{-5t} & -25e^{-5t} \\ -5e^{-5t} & (-\frac{25}{2}t + \frac{5}{2})e^{-5t} \end{pmatrix}.$$

Since $\dfrac{d\Omega(t)}{dt} = A\Omega(t)$, the columns of $\Omega(t)$ are solutions of $X' = AX$.
Calculate $|\Omega(0)| = \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} = -1 \neq 0$, so the columns of $\Omega(t)$ are linearly independent.
By Problem 10(b),

$$\Phi(t) = \Omega(t)\Omega^{-1}(0) = \begin{pmatrix} 2e^{-5t} & (1+5t)e^{-5t} \\ e^{-5t} & \frac{5}{2}te^{-5t} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} (1+5t)e^{-5t} & -10te^{-5t} \\ \frac{5}{2}te^{-5t} & (1+t)e^{-5t} \end{pmatrix}$$

is a transition matrix.

13. $\dfrac{d\Omega(t)}{dt} = \begin{pmatrix} -3e^{-3t} & e^t & 0 \\ -9e^{-3t} & 0 & e^t \\ -3e^{-3t} & -e^t & e^t \end{pmatrix}$, and

$$A\Omega(t) = \begin{pmatrix} 5 & -4 & 4 \\ 12 & -11 & 12 \\ 4 & -4 & 5 \end{pmatrix} \Omega(t) = \begin{pmatrix} -3e^{-3t} & e^t & 0 \\ -9e^{-3t} & 0 & e^t \\ -3e^{-3t} & -e^t & e^t \end{pmatrix}.$$

Since $\dfrac{d\Omega(t)}{dt} = A\Omega(t)$, the columns of $\Omega(t)$ are solutions of $X' = AX$.
Calculate $|\Omega(0)| = -1 \neq 0$, so the columns of $\Omega(t)$ are linearly independent.
By Problem 10(b),

$$\Phi(t) = \Omega(t)\Omega^{-1}(0) = \begin{pmatrix} e^{-3t} & e^t & 0 \\ 3e^{-3t} & 0 & e^t \\ e^{-3t} & -e^t & e^t \end{pmatrix} \begin{pmatrix} -1 & 1 & -1 \\ 2 & -1 & 1 \\ 3 & -2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 2e^t - e^{-3t} & e^{-3t} - e^t & e^t - e^{-3t} \\ 3e^t - 3e^{-3t} & 3e^{-3t} - 2e^t & 3e^t - 3e^{-3t} \\ e^t - e^{-3t} & e^{-3t} - e^t & 2e^t - e^{-3t} \end{pmatrix}$$

is a transition matrix.

In Problems 14 through 18 we express the solution in the form $X(t) = PZ(t)$ where P is a matrix having eigenvectors of A as columns, $Z(t)$ is the solution of the uncoupled system $Z' = DZ + P^{-1}G$, D is a diagonal matrix with $d_{ii} = i^{th}$ eigenvalue of A.

14. $A = \begin{pmatrix} -2 & 1 \\ -4 & 3 \end{pmatrix}$ has eigenvalues $\lambda_1 = -1, \lambda_2 = 2$ and $P = \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}$.
We find $P^{-1} = \dfrac{1}{3}\begin{pmatrix} 4 & -1 \\ -1 & 1 \end{pmatrix}$, so Z satisfies

$$Z' = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} Z + \frac{1}{3}\begin{pmatrix} 4 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 10\cos(t) \end{pmatrix}.$$

Solving for Z gives

$$z_1 = c_1e^{-t} - \frac{5}{3}\cos(t) - \frac{5}{3}\sin(t);\; z_2 = c_2e^{2t} - \frac{4}{3}\cos(t) + \frac{2}{3}\sin(t).$$

Finally

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = PZ = \begin{pmatrix} c_1e^t + c_2e^{-2t} - 3\cos(t) - \sin(t) \\ c_1e^t + 4c_2e^{-2t} - 7\cos(t) + \sin(t) \end{pmatrix}.$$

15. For $A = \begin{pmatrix} 3 & 3 \\ 1 & 5 \end{pmatrix}, \lambda_1 = 2, \lambda_2 = 6, P = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}$,

$$Z' = \begin{pmatrix} 2 & 0 \\ 0 & 6 \end{pmatrix} Z + \frac{1}{4}\begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 8 \\ 4e^{3t} \end{pmatrix}.$$

Solving for Z gives

$$z_1 = c_1e^{2t} - 1 - e^{3t};\; z_2 = c_2e^{6t} - 1/3 - e^{3t}.$$

So

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = PZ = \begin{pmatrix} 3c_1e^{2t} + c_2e^{6t} - 4e^{3t} - 10/3 \\ -c_1e^{2t} + c_2e^{6t} + 2/3 \end{pmatrix}$$

16. For $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \lambda_1 = 0, \lambda_2 = 2, P = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$,

$$Z' = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} Z + \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 6e^{3t} \\ 4 \end{pmatrix}.$$

Solving for Z gives

$$z_1 = c_1 - 2t + e^{3t};\; z_2 = c_2e^{2t} - 1 + 3e^{3t}.$$

So

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = PZ = \begin{pmatrix} c_1 + c_2e^{2t} - 1 - 2t + 4e^{3t} \\ -c_1 + c_2e^{2t} - 1 + 2t + 2e^{3t} \end{pmatrix}.$$

17. For $A = \begin{pmatrix} 6 & 5 \\ 1 & 2 \end{pmatrix}, \lambda_1 = 1, \lambda_2 = 7, P = \begin{pmatrix} 1 & 5 \\ -1 & 1 \end{pmatrix}$,

$$Z' = \begin{pmatrix} 1 & 0 \\ 0 & 7 \end{pmatrix} Z + \frac{1}{6}\begin{pmatrix} 1 & -5 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -4\cos(3t) \\ 8 \end{pmatrix}.$$

Solving for Z gives

$$z_1 = c_1e^t + \frac{1}{15}\cos(3t) - \frac{3}{15}\sin(3t) + \frac{20}{3};\; z_2 = c_2e^{7t} + \frac{7}{87}\cos(3t) - \frac{2}{58}\sin(3t) - \frac{4}{21}.$$

So

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = PZ = \begin{pmatrix} c_1e^t + 5c_2e^{7t} + \frac{68}{145}\cos(3t) - \frac{54}{145}\sin(3t) + \frac{40}{7} \\ -c_1e^t + c_2e^{7t} + \frac{2}{145}\cos(3t) + \frac{24}{145}\sin(3t) - \frac{48}{7} \end{pmatrix}.$$

18. For $A=\begin{pmatrix}3 & -2\\ 9 & -3\end{pmatrix}$, $\lambda_1=3, \lambda_2=-3, P=\begin{pmatrix}1+i & 1-i\\ 3 & 3\end{pmatrix}$,

$$Z'=\begin{pmatrix}3i & 0\\ 0 & -3i\end{pmatrix}Z+\frac{1}{6}\begin{pmatrix}-3i & 1+i\\ 3i & 1-i\end{pmatrix}\begin{pmatrix}3e^{2t}\\ e^{2t}\end{pmatrix}.$$

Solving for Z gives

$$z_1=d_1e^{3it}+\left(\frac{2-i}{6}\right)e^{2t};\ z_2=d_2e^{-3it}+\left(\frac{2+i}{6}\right)e^{2t}.$$

Letting $d_1=\dfrac{1}{2}(c_1+c_2i), d_2=\dfrac{1}{2}(c_1-c_2i)$ with c_1, c_2 real we get

$$z_1=\frac{1}{2}[c_1\cos(3t)-c_2\sin(3t)]+\frac{1}{2}[c_2\cos(3t)+c_1\sin(3t)]i+\left(\frac{2-i}{6}\right)e^{2t};$$

$$z_2=\frac{1}{2}[c_1\cos(3t)-c_2\sin(3t)]-\frac{1}{2}[c_2\cos(3t)+c_1\sin(3t)]i+\left(\frac{2+i}{6}\right)e^{2t}.$$

So

$$X=PZ=\begin{pmatrix}c_1[\cos(3t)-\sin(3t)]-c_2[\sin(3t)+\cos(3t)]+e^{2t}\\ 3c_1\cos(3t)-3c_2\sin(3t)+2e^{2t}\end{pmatrix}.$$

19. For $A=\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$, $\lambda_1=0, \lambda_2=2; P=\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}$. With $X=PZ$ the uncoupled system is $Z'=\begin{pmatrix}0 & 0\\ 0 & 2\end{pmatrix}Z+\begin{pmatrix}1/2 & -1/2\\ 1/2 & 1/2\end{pmatrix}\begin{pmatrix}6e^{2t}\\ 2e^{2t}\end{pmatrix}$, with initial condition $Z(0)=P^{-1}X(0)=\begin{pmatrix}3\\ 3\end{pmatrix}$. Solving for Z gives $z_1=2+e^{2t}, z_2=3e^{2t}+4te^{2t}$, and then $X=PZ=\begin{pmatrix}x_1\\ x_2\end{pmatrix}=\begin{pmatrix}2+4(1+t)e^{2t}\\ -2+2(1+2t)e^{2t}\end{pmatrix}$.

20. For $A=\begin{pmatrix}1 & -2\\ -1 & 2\end{pmatrix}$, $\lambda_1=0, \lambda_2=3; P=\begin{pmatrix}2 & -1\\ 1 & 1\end{pmatrix}$. With $X=PZ$ the uncoupled system is $Z'=\begin{pmatrix}0 & 0\\ 0 & 3\end{pmatrix}Z+\begin{pmatrix}1/3 & 1/3\\ -1/3 & 2/3\end{pmatrix}\begin{pmatrix}2t\\ 5\end{pmatrix}$, with initial condition $Z(0)=P^{-1}X(0)=\begin{pmatrix}25/3\\ 11/3\end{pmatrix}$. Solving for Z gives $z_1=\frac{1}{3}t^2+\frac{5}{3}t+\frac{25}{3}, z_2=\frac{127}{27}e^{3t}+\frac{2}{9}t-\frac{28}{27}$, and then $X=PZ=\begin{pmatrix}x_1\\ x_2\end{pmatrix}=\begin{pmatrix}-\frac{127}{27}e^{3t}+\frac{2}{3}t^2+\frac{28}{9}t+\frac{478}{27}\\ \frac{127}{27}e^{3t}+\frac{1}{3}t^2+\frac{17}{9}t+\frac{197}{27}\end{pmatrix}$.

21. For $A=\begin{pmatrix}2 & -5\\ 1 & -2\end{pmatrix}$, $\lambda_1=i, \lambda_2=-i; P=\begin{pmatrix}5 & 5\\ 2-i & 2+i\end{pmatrix}$. With $X=PZ$ the uncoupled system is $Z'=\begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix}Z+\begin{pmatrix}\frac{1-2i}{10} & \frac{i}{2}\\ \frac{1+2i}{10} & -\frac{i}{2}\end{pmatrix}\begin{pmatrix}5\sin(t)\\ 0\end{pmatrix}$, with initial condition

$Z(0) = P^{-1}X(0) = \begin{pmatrix} 1+i/2 \\ 1-i/2 \end{pmatrix}$. Solving for Z gives $z_1 =, z_2 =$, and then
$X = PZ = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 10\cos(t) + \frac{5}{2}t\sin(t) - 5t\cos(t) \\ 5\cos(t) + \frac{5}{2}\sin(t) - \frac{5}{2}t\cos(t) \end{pmatrix}$.

22. For $A = \begin{pmatrix} 5 & -4 & 4 \\ 12 & -11 & 12 \\ 4 & -4 & 5 \end{pmatrix}, \lambda_1 = 1, \lambda_2 = 1, \lambda_3 = -3; P = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 3 \\ 0 & 1 & 1 \end{pmatrix}$. With $X = PZ$ the uncoupled system is $Z' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{pmatrix} Z + \begin{pmatrix} 3 & -2 & 3 \\ 1 & -1 & 2 \\ -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} -3e^{-3t} \\ t \\ 0 \end{pmatrix}$, with initial condition $Z(0) = P^{-1}X(0) = \begin{pmatrix} 11 \\ 6 \\ -4 \end{pmatrix}$. Solving for Z gives $z_1 =, z_2 =, z_3 =$, and then
$X = PZ = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{5}{2}e^t - \frac{48}{18}e^{-3t} + 3te^{-3t} + \frac{8}{9} + \frac{4}{3}t \\ \frac{27}{4}e^t - \frac{113}{12}e^{-3t} + 9te^{-3t} + \frac{5}{3} + 3t \\ \frac{17}{4}e^t - \frac{113}{36}e^{-3t} + 3te^{-3t} + \frac{8}{9} + \frac{4}{3}t \end{pmatrix}$.

23. For $A = \begin{pmatrix} 3 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}, \lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 2; P = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$. With $X = PZ$ the uncoupled system is $Z' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} Z + \begin{pmatrix} -1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ t \\ 2e^t \end{pmatrix}$, with initial condition $Z(0) = P^{-1}X(0) = \begin{pmatrix} -1 \\ 3 \\ -1 \end{pmatrix}$. Solving for Z gives $z_1 =, z_2 =, z_3 =$, and then
$X = PZ = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -\frac{1}{4}e^{2t} + (2+2t)e^t - \frac{3}{4} - \frac{1}{2}t \\ e^{2t} + (2+2t)e^t - 1 - t \\ -\frac{5}{4}e^{2t} + 2te^t - \frac{3}{4} - \frac{1}{2}t \end{pmatrix}$.

Chapter Eleven - Qualitative Methods and Systems of Nonlinear Differential Equations

Section 11.1 Nonlinear Systems and Existence of Solutions

1. For the nonlinear damped pendulum we have $x' = F_1(x,y) = y; y' = F_2(x,y) = -\omega^2\sin(x) - \gamma y$. Calculate $\frac{\partial F_1}{\partial x} = 0, \frac{\partial F_1}{\partial y} = 1, \frac{\partial F_2}{\partial x} = -\omega^2\cos(x), \frac{\partial F_2}{\partial y} = -\gamma$. Since F_1, F_2 and all four first partials are continuous at all points of R^3, there exists a unique solution for any given set of initial conditions $x(0) = a, y(0) = b$. the value of a denotes initial angular displacement of the pendulum from the vertical, while the value of b denotes the time rate of change of this angular displacement. Mathematically, there are no restrictions on the values of a and b, even though there may be physical restrictions.

2. For the nonlinear spring we have $F_1(x,y) = y, F_2(x,y) = -\frac{k}{m}x - \frac{c}{m}y + \frac{\alpha}{m}x^3$. $F_1(x,y), F_2(x,y)$ and all four first partials $\frac{\partial F_1}{\partial x} = 0, \frac{\partial F_1}{\partial y} = 1, \frac{\partial F_2}{\partial x} = -\frac{k}{m} + \frac{3\alpha}{m}x^2, \frac{\partial F_2}{\partial y} = -\frac{c}{m}$ are continuous on all of R^3. Thus there exists a unique solution for any given set of initial conditions $x(0) = a, y(0) = b$. The values of a and b represent initial displacement and initial velocity respectively of the mass on the end of the nonlinear spring. There are no restrictions on these values for mathematical reasons, although there may be restrictions for physical reasons.

3. For the given driving force calculate $F_1(x,y) = y, F_2(x,y) = -\frac{k}{m}x - \frac{c}{m}y + \frac{\alpha}{m}x^3 + \frac{\beta}{m}x^5$. $F_1(x,y), F_2(x,y)$ and all first partials are continuous everywhere in R^3, hence there is a unique solution for any given initial conditions $x(0) = a, y(0) = b$, and such a solution will be defined on some interval $-h < t < h$.

Section 11.2 The Phase Plane, Phase Portraits, and Direction Fields

1. $x = -c_1\cos(t) - c_2\sin(t);\ y = (4c_1 - c_2)\cos(t) + (c_1 + 4c_2)\sin(t)$.
2. $x = c_1e^{2t};\ y = 8c_1te^{2t} + c_2e^{2t}$
3. $x = c_1e^{-3t} + 7c_2e^{2t};\ y = c_1e^{-3t} + 2c_2e^{2t}$
4. $x = e^{-t}[c_1\cos(2t) + c_2\sin(2t)];\ y = e^{-t}[(2c_1 - c_2)\cos(2t) + (c_1 + 2c_2)\sin(2t)]$
5. $x = c_1e^{5t} + 2c_2e^{4t};\ y = c_2e^{4t}$
6. $x = 2c_1e^{-7t} + 3c_2e^{-8t};\ y = c_1e^{-7t} + 2c_2e^{-8t}$
7. $\frac{dy}{dx} = -\frac{4x}{9y}$; $9y^2 + 4x^2 = c$; integral curves are ellipses centered at $(0,0)$.
8. $\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$, a homogeneous equation with general solution $x^2+y^2 = kx$ or $(x-c)^2+y^2 = c^2$; integral curves are circles tangent to y axis.
9. $\frac{dy}{dx} = \frac{(x-1)}{(y+2)}$; $(x-1)^2 - (y+2)^2 = c$; integral curves are hyperbolas centered at $(1,-2)$.

10. $\dfrac{dy}{dx} = \dfrac{y}{\csc(x)} = y\sin(x)$; separable with solution $y = ke^{-\cos(x)}$

11. $\dfrac{dy}{dx} = \dfrac{x+y}{x}$, a homogeneous equation with solution $y = x\ln|cx| = x\ln|x| + kx$

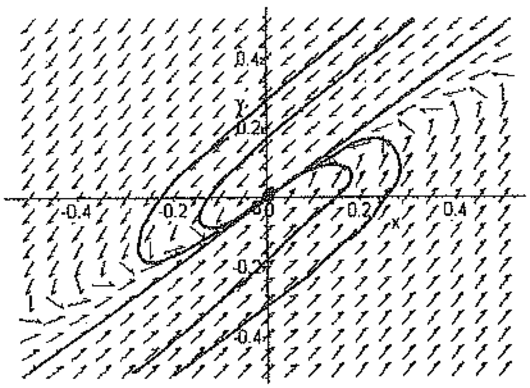
12. $\dfrac{dy}{dx} = \dfrac{y}{x^2}$, separable with solution $y = ke^{-1/x}$

Section 11.3

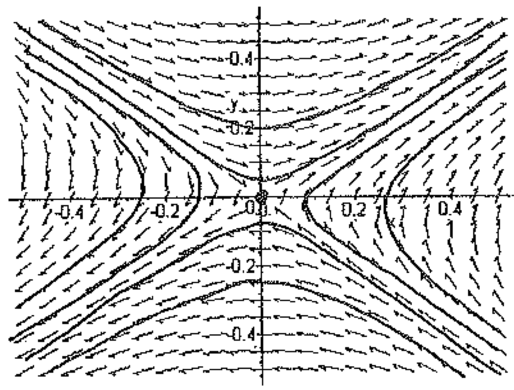
Problems 1 through 6 have phase portraits which illustrate all possibilities. Because of the repetition of appearance in the phase portraits of the remaining exercises, only these six are shown here.

1. The characteristic equation $\lambda^2 + 4\lambda + 4 = 0$ has equal roots $\lambda_1 = \lambda_2 = -2$ with a single eigenvector. Thus the origin is an improper node. Solution is $x = c_1e^{-2t} + 5(c_1 - c_2)te^{-2t}$; $y = c_2e^{-2t} + 5(c_1 - c_2)te^{-2t}$

2. $\lambda_1 = 4, \lambda_2 = -3$; $(0,0)$ is a saddle point. Solution is $x = \dfrac{4}{3}c_1e^{4t} - c_2e^{3t}; y = c_1e^{4t} + c_2e^{3t}$



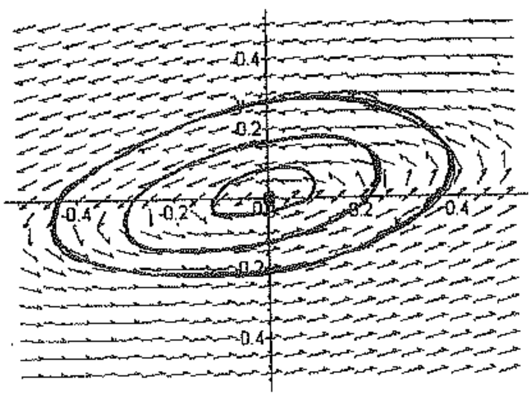
Problem 1 - Improper Node



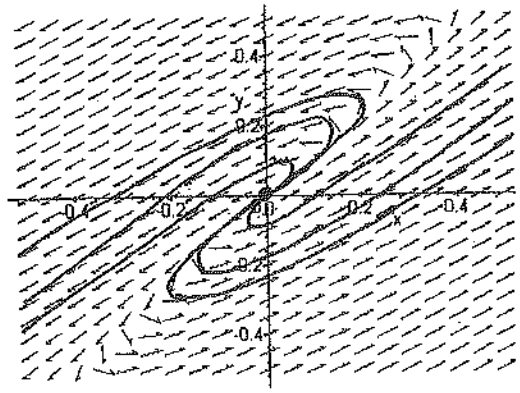
Problem 2 - Saddle Point

3. $\lambda_1 = 2i, \lambda_2 = -2i$; $(0,0)$ is a center. Solution is $x = (c_1 - 2c_2)\sin(2t) + (2c_1 + c_2)\cos(2t); y = c_1\sin(2t) + c_2\cos(2t)$

4. $\lambda_1 = 2, \lambda_2 = 3$; $(0,0)$ is a nodal source. Solution is $x = 7c_1e^{3t} + c_2e^{2t}; y = 6c_1e^{3t} + c_2e^{2t}$



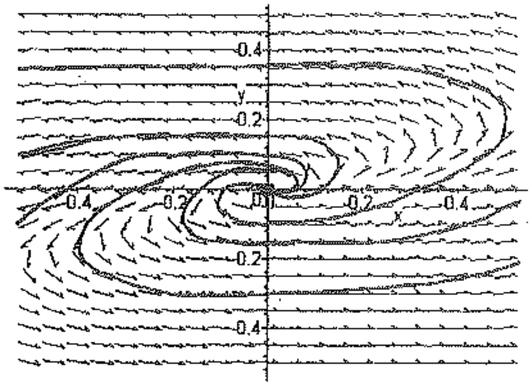
Problem 3 - Center



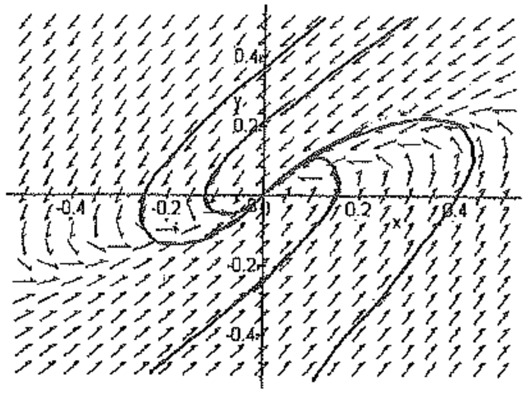
Problem 4 - Nodal Source

5. $\lambda_1 = 4+5i, \lambda_2 = 4-5i$; $(0,0)$ is a spiral point. Solution is $x = (3c_1 - 5c_2)e^{4t}\sin(5t) + (5c_1+3c_2)e^{4t}\cos(5t); y = 2c_1e^{4t}\sin(5t) + 2c_2e^{4t}\cos(5t)$

6. $\lambda_1 = -5, \lambda_2 = -3$; $(0,0)$ is a nodal sink. Solution is $x = 7c_1e^{-3t} + c_2e^{-5t}; y = 5c_1e^{-3t} + c_2e^{-5t}$



Problem 5 - Spiral Point



Problem 6 - Nodal Sink

7. $\lambda_1 = \lambda_2 = 3$ with a single eigenvector; $(0,0)$ is an improper node. Solution is $x = c_1e^{3t} + c_2te^{3t}; y = (c_1+c_2)e^{3t} + c_2te^{3t}$

8. $\lambda_1 = \sqrt{31}i, \lambda_2 = -\sqrt{31}i$; $(0,0)$ is a center. Solution is $x = (3c_1 - \sqrt{31}c_2)\sin(\sqrt{31}t) + (\sqrt{31}c_1 + 3c_2)\cos(\sqrt{31}t); y = 8c_1\sin(\sqrt{31}t) + 8c_2\cos(\sqrt{31}t)$

9. $\lambda_1 = -2+\sqrt{3}i, \lambda_2 = -2-\sqrt{3}i$; $(0,0)$ is a spiral point. Solution is $x = c_1e^{-2t}\cos(\sqrt{3}t) - c_2e^{-2t}\sin(\sqrt{3}t); y = c_1e^{-2t}\sin(\sqrt{3}t) + 3c_2e^{-2t}\cos(\sqrt{3}t)$

10. $\lambda_1 = \lambda_2 = -13$ with a single eigenvector; $(0,0)$ is an improper node. Solution is $x = c_1e^{-13t} + 7(c_1-c_2)te^{-13t}; y = c_2e^{-13t} + 7(c_1-c_2)te^{-13t}$

Section 11.4 Critical Points and Stability

1. The characteristic equation $\lambda^2 + 4\lambda + 4 = 0$ has equal roots $\lambda_1 = \lambda_2 = -2$ with a single eigenvector. Since both eigenvalues are negative, the origin is an improper node and is both stable and asymptotically stable.

2. $\lambda_1 = 4, \lambda_2 = -3$; $(0,0)$ is a saddle point which is unstable.

3. $\lambda_1 = 2i, \lambda_2 = -2i$; $(0,0)$ is a center which is stable but not asymptotically stable.

4. $\lambda_1 = 2, \lambda_2 = 3$; $(0,0)$ is a nodal source and is unstable.

5. $\lambda_1 = 4+5i, \lambda_2 = 4-5i$; $(0,0)$ is a spiral point. Since eigenvalues have positive real part, the spiral point is unstable.

6. $\lambda_1 = -5, \lambda_2 = -3$; $(0,0)$ is a nodal sink and is both stable and asymptotically stable.

7. $\lambda_1 = \lambda_2 = 3$ with a single eigenvector; $(0,0)$ is an improper node. Since the eigenvalue is positive this node is unstable.

8. $\lambda_1 = \sqrt{31}i, \lambda_2 = -\sqrt{31}i$; $(0,0)$ is a center which is stable but not asymptotically stable.

9. $\lambda_1 = -2 + \sqrt{3}i, \lambda_2 = -2 - \sqrt{3}i$; $(0,0)$ is a spiral point. Since eigenvalues have negative real part, the spiral point is both stable and asymptotically stable.

10. $\lambda_1 = \lambda_2 = -13$ with a single eigenvector; $(0,0)$ is an improper node. The eigenvalue is negative so the node is both stable and asymptotically stable.

11. (a) For $\epsilon = 0$ the eigenvalues are $\lambda_1 = \sqrt{5}i$ and $\lambda_2 = -\sqrt{5}i$ so $(0,0)$ is a center which is stable but not asymptotically stable.

(b) For $\epsilon > 0$ the eigenvalues are $\lambda_1 = \frac{\epsilon}{2} + \frac{1}{2}\sqrt{(\epsilon-2)^2 - 24}$ and $\lambda_2 = \frac{\epsilon}{2} - \frac{1}{2}\sqrt{(\epsilon-2)^2 - 24}$ which have positive real part. Thus $(0,0)$ is an unstable spiral point whenever $0 < \epsilon < 2\sqrt{6} + 2$. For $\epsilon > 2\sqrt{6} + 2$ the origin becomes an unstable saddle point. For $\epsilon = 2\sqrt{6} + 2$ the origin is an unstable improper node.

12. (a) For $\epsilon = 0$ the characteristic equation is $\lambda^2 + 6\lambda + 9 = 0$ which has repeated roots $\lambda_1 = \lambda_2 = -3$. This repeated eigenvalue has only one eigenvector, namely $\mathbf{V}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Since the eigenvalue is negative, $(0,0)$ is stable and asymptotically stable improper node.

(b) For $\epsilon > 0$ the eigenvalues are $\lambda_1 = \frac{\epsilon - 6}{2} + \frac{1}{2}\sqrt{(\epsilon + 10)^2 - 100}$ and $\lambda_2 = \frac{\epsilon - 6}{2} - \frac{1}{2}\sqrt{(\epsilon + 10)^2 - 100}$. Thus for $\epsilon > 0$ these eigenvalues are real and distinct. Furthermore for $0 < \epsilon < 9/8$ these eigenvalues are both negative, so $(0,0)$ is a stable and asymptotically stable nodal sink. For $\epsilon > 9/8$ the eigenvalues have opposite sign, so $(0,0)$ is an unstable saddle point. For $\epsilon = 9/8$ the origin is not an isolated singularity.

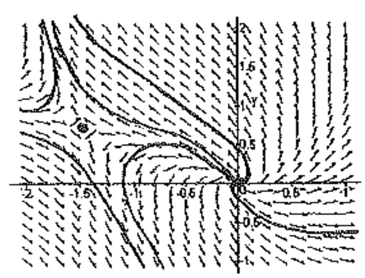
Section 11.5 Almost Linear Systems

1.

(a) We have $f(x,y) = x - y + x^2$ and $g(x,y) = x + 2y$, with first partial derivatives $f_x = 1 + 2x, f_y = -1, g_x = 1, g_y = 2$. Since the first partials are continuous everywhere, the system is almost linear.

(b) Critical points are the simultaneous solutions of $f(x,y) = x - y + x^2 = 0$ and $g(x,y) = x + 2y = 0$ which are easily found to be $(0,0)$ and $(-3/2, 3/4)$.

(c) We easily identify $\mathbf{A}_{(0,0)} = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}$, which has eigenvalues $\lambda_1 = \dfrac{3}{2} + \dfrac{\sqrt{3}}{2}i$ and $\lambda_2 = \dfrac{3}{2} - \dfrac{\sqrt{3}}{2}i$. Since these are complex conjugate eigenvalues with positive real part, the origin of the nonlinear system is an unstable spiral point. The matrix $\mathbf{A}_{(-3/2,3/4)} = \begin{pmatrix} -2 & -1 \\ 1 & 2 \end{pmatrix}$, which has eigenvalues $\lambda_1 = \sqrt{3}$ and $\lambda_2 = -\sqrt{3}$. Since these are of opposite sign, the point $(-3/2, 3/4)$ is an unstable saddle point of the nonlinear system.

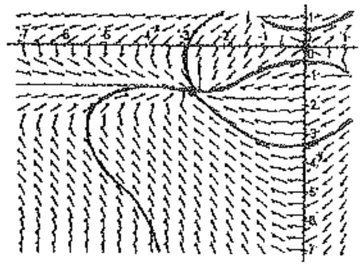


2.

(a) We have $f(x,y) = x + 3y - x^2\sin(y)$ and $g(x,y) = 2x + y - xy^2$, with first partial derivatives $f_x = 1 - 2x\sin(y), f_y = 3 - x^2\cos(y), g_x = 2 - y^2, g_y = 1 - 2xy$. Since the first partials are continuous everywhere, the system is almost linear.

(b) Critical points are the simultaneous solutions of $f(x,y) = x + 3y - x^2\sin(y) = 0$ and $g(x,y) = 2x + y - xy^2 = 0$ one of which is easily found to be $(0,0)$. The other is determined numerically to be $\approx (-2.7533, -1.6074)$

(c) We identify $\mathbf{A}_{(0,0)} = \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}$, which has eigenvalues $\lambda_1 = 1 + \sqrt{6}$ and $\lambda_2 = 1 - \sqrt{6}$. Since these are of opposite sign, the origin of the nonlinear system is an unstable saddle point. We also find $\mathbf{A}_{(-2.7533,-1.6074)} \approx \begin{pmatrix} -4.5029 & 3.2774 \\ .5837 & -7.8513 \end{pmatrix}$, which has eigenvalues approximately -7.7386 and -4.8156, so $(-2.7533, -1.6074)$ is an asymptotically stable nodal sink.

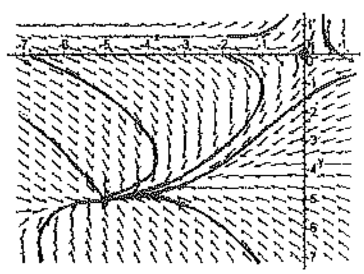


3.

(a) We have $f(x,y) = -2x + 2y$ and $g(x,y) = x + 4y + y^2$, with first partial derivatives $f_x = -2, f_y = 2, g_x = 1, g_y = 4 + 2y$. Since the first partials are continuous everywhere, the system is almost linear.

(b) Critical points are the simultaneous solutions of $f(x,y) = -2x + 2y = 0$ and $g(x,y) = x + 4y + y^2 = 0$ which are easily found to be $(0,0)$ and $(-5,-5)$.

(c) We easily identify $\mathbf{A}_{(0,0)} = \begin{pmatrix} -2 & 2 \\ 1 & 4 \end{pmatrix}$, which has eigenvalues $\lambda_1 = 1 + \sqrt{11}$ and $\lambda_2 = 1 - \sqrt{11}$. Since these are of opposite sign, the origin of the nonlinear system is an unstable saddle point. The matrix $\mathbf{A}_{(-5,-5)} = \begin{pmatrix} -2 & 2 \\ 1 & -6 \end{pmatrix}$, which has eigenvalues $\lambda_1 = -4 + \sqrt{6}$ and $\lambda_2 = -4 - \sqrt{6}$. Since these are unequal and both negative, the point $(-5,-5)$ is an asymptotically stable nodal sink of the nonlinear system.

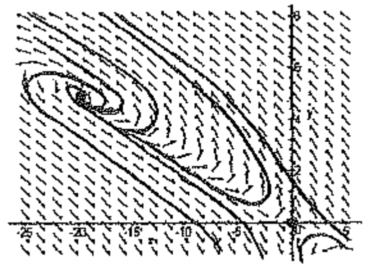


4.

(a) We have $f(x, y) = -2x - 3y - y^2$ and $g(x, y) = x + 4y$, with first partial derivatives $f_x = -2, f_y = -3 - 2y, g_x = 1, g_y = 4$. Since the first partials are continuous everywhere, the system is almost linear.

(b) Critical points are the simultaneous solutions of $f(x, y) = -2x - 3y - y^2 = 0$ and $g(x, y) = x + 4y = 0$ which are easily found to be $(0, 0)$ and $(-20, 5)$.

(c) We easily identify $\mathbf{A}_{(0,0)} = \begin{pmatrix} -2 & -3 \\ 1 & 4 \end{pmatrix}$, which has eigenvalues $\lambda_1 = 1 + \sqrt{6}$ and $\lambda_2 = 1 - \sqrt{6}$. Since these are of opposite sign, the origin of the nonlinear system is an unstable saddle point. The matrix $\mathbf{A}_{(-20,5)} = \begin{pmatrix} -2 & -13 \\ 1 & 4 \end{pmatrix}$, which has eigenvalues $\lambda_1 = 1 + 2i$ and $\lambda_2 = 1 - 2i$. Since these are complex conjugates with positive real part, the point $(-20, 5)$ is an unstable spiral point of the nonlinear system.

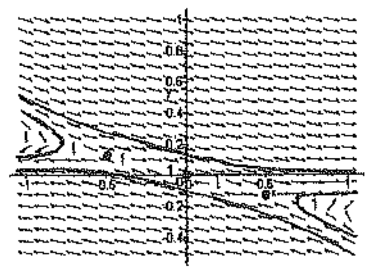


5.

(a) We have $f(x, y) = 3x + 12y$ and $g(x, y) = -x - 3y + x^3$, with first partial derivatives $f_x = 3, f_y = 12, g_x = -1 + 3x^2, g_y = -3$. Since the first partials are continuous everywhere, the system is almost linear.

(b) Critical points are the simultaneous solutions of $f(x, y) = 3x + 12y = 0$ and $g(x, y) = -x - 3y + x^3 = 0$ which are easily found to be $(0, 0), (1/2, -1/8)$ and $(-1/2, 1/8)$.

(c) We easily identify $\mathbf{A}_{(0,0)} = \begin{pmatrix} 3 & 12 \\ -1 & -3 \end{pmatrix}$, which has eigenvalues $\lambda_1 = 3i$ and $\lambda_2 = -3i$. Since these are pure imaginary, the origin of the linear system is a center, whereas the origin of the nonlinear system could be either a center or a spiral point, and could be unstable or asymptotically stable. The matrices $\mathbf{A}_{(1/2,-1/8)} = \mathbf{A}_{(-1/2,1/8)} \begin{pmatrix} 3 & 12 \\ -1/4 & -3 \end{pmatrix}$, which has eigenvalues $\lambda_1 = \sqrt{6}$ and $\lambda_2 = -\sqrt{6}$. Since these are of opposite sign, the points $(1/2, -1/8)$ and $(-1/2, 1/8)$ are both unstable saddle points of the nonlinear system.

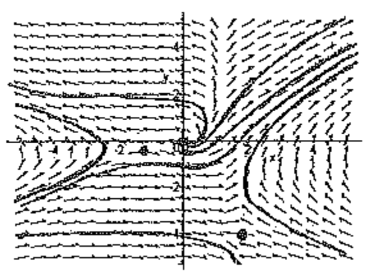


6.

(a) We have $f(x, y) = 2x - 4y + 3xy$ and $g(x, y) = x + y + x^2$, with first partial derivatives $f_x = 2+3y, f_y = -4+3x, g_x = 1+2x, g_y = 1$. Since the first partials are continuous everywhere, the system is almost linear.

(b) Critical points are the simultaneous solutions of $f(x, y) = 2x - 4y + 3xy = 0$ and $g(x, y) = x + y + x^2 = 0$ which are found to be $(0, 0)$ and $((1 \pm \sqrt{73})/6, -(20 \pm 2\sqrt{73})/9)$.

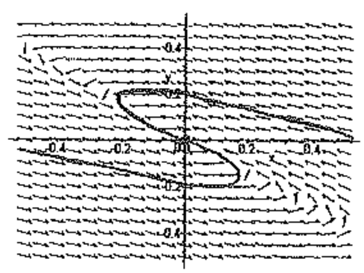
(c) We easily identify $\mathbf{A}_{(0,0)} = \begin{pmatrix} 2 & -4 \\ 1 & 1 \end{pmatrix}$, which has eigenvalues $\lambda_1 = \frac{3}{2} + \frac{1}{2}\sqrt{13}i$ and $\lambda_2 = \frac{3}{2} - \frac{1}{2}\sqrt{13}i$. Since these are complex with positive real part, the origin of the nonlinear system is an unstable spiral point. The other two critical points are both unstable saddle points of the nonlinear system.



7.

(a) We have $f(x, y) = -3x - 4y + x^2 - y^2$ and $g(x, y) = x + y$, with first partial derivatives $f_x = -3+2x, f_y = -4-2y, g_x = 1, g_y = 1$. Since the first partials are continuous everywhere, the system is almost linear.

(b) Critical points are the simultaneous solutions of $f(x, y) = -3x - 4y + x^2 - y^2 = 0$ and $g(x, y) = x + y = 0$ which is easily found to be only $(0, 0)$.



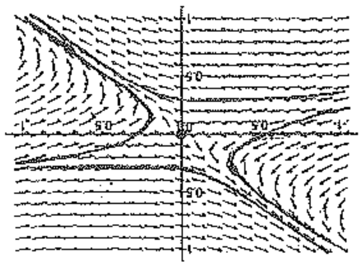
(c) We easily identify $\mathbf{A}_{(0,0)} = \begin{pmatrix} -3 & -4 \\ 1 & 1 \end{pmatrix}$, which has eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -1$. Since these are equal and negative, the origin of the linear approximation is an asymptotically stable improper node, whereas the nonlinear system could have an asymptotically stable node or spiral point.

8.

(a) We have $f(x,y) = -3x - 4y$ and $g(x,y) = -x + y - x^2y$, with first partial derivatives $f_x = -3, f_y = -4, g_x = -1 - 2xy, g_y = 1 - x^2$. Since the first partials are continuous everywhere, the system is almost linear.

(b) Critical points are the simultaneous solutions of $f(x,y) = -3x - 4y = 0$ and $g(x,y) = -x + y - x^2y = 0$ which is found to be only $(0,0)$.

(c) We easily identify $\mathbf{A}_{(0,0)} = \begin{pmatrix} -3 & -4 \\ -1 & 1 \end{pmatrix}$, which has eigenvalues $\lambda_1 = -1 + 2\sqrt{2}$ and $\lambda_2 = -1 - 2\sqrt{2}$. Since these are of opposite sign, the origin of the nonlinear system is an unstable saddle point.

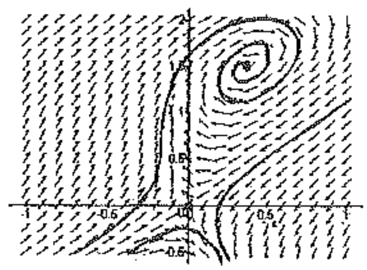


9.

(a) We have $f(x,y) = -2x - y + y^2$ and $g(x,y) = -4x + y$, with first partial derivatives $f_x = -2, f_y = -1+2y, g_x = -4, g_y = 1$. Since the first partials are continuous everywhere, the system is almost linear.

(b) Critical points are the simultaneous solutions of $f(x,y) = -2x - y + y^2 = 0$ and $g(x,y) = -4x + y = 0$ which are easily found to be $(0,0)$ and $(3/8, 3/2)$.

(c) We easily identify $\mathbf{A}_{(0,0)} = \begin{pmatrix} -2 & -1 \\ -4 & 1 \end{pmatrix}$, which has eigenvalues $\lambda_1 = 3$ and $\lambda_2 = -2$. Since these are of opposite sign , the origin of the nonlinear system is an unstable saddle point. The matrix $\mathbf{A}_{(3/8,3/2)} = \begin{pmatrix} -2 & 2 \\ -4 & 1 \end{pmatrix}$, which has eigenvalues $\lambda_1 = -\frac{1}{2} + \frac{\sqrt{23}}{2}i$ and $\lambda_2 = -\frac{1}{2} - \frac{\sqrt{23}}{2}i$. Since these are complex with negative real part, the point $(3/8, 3/2)$ is an asymptotically stable spiral point of the nonlinear system.

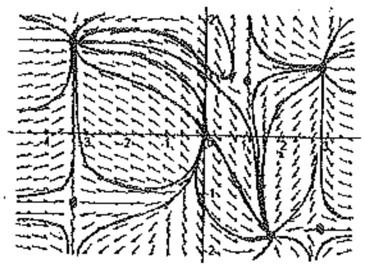


10.

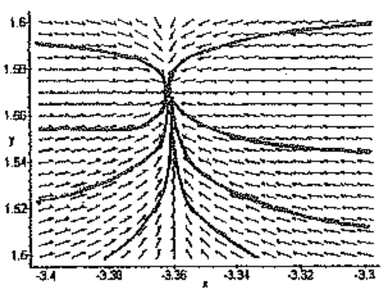
(a) We have $f(x,y) = 2x - y - x^3\sin(x)$ and $g(x,y) = -2x + y + xy^2$, with first partial derivatives $f_x = 2 - 3x^2\sin(x) - x^3\cos(x)$, $f_y = -1$, $g_x = -2+y^2$, $g_y = 1+2xy$. Since the first partials are continuous everywhere, the system is almost linear.

(b) Critical points are the simultaneous solutions of $f(x,y) = 2x - y - x^3\sin(x) = 0$ and $g(x,y) = -2x + y + xy^2 = 0$,of which there are seven. $(0,0)$ is easily found by inspection, the others are determined numerically to be approximately $(-3.3617, 1.5707), (-3.2912, -1.2704), (1.0918, 1.0285),$ $(1.7428, -1.7299), (2.8100, -1.6033), (2.9608, 1.2554)$.

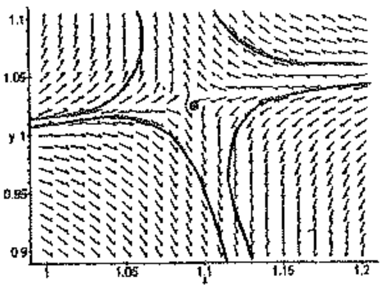
(c) We easily identify $\mathbf{A}_{(0,0)} = \begin{pmatrix} 2 & -1 \\ -2 & 1 \end{pmatrix}$, which has eigenvalues $\lambda_1 = 3$ and $\lambda_2 = 0$. Since one of these is zero, the matrix $\mathbf{A}_{(0,0)}$ is singular and the critical points of the linear approximation are not isolated; in fact every point on the line $y = 2x$ is a critical point of the linear approximation. The current theory does not provide a classification of the stability of the origin of the almost linear system. The remaining six critical points are found to be respectively asymptotically stable nodal sink, unstable saddle point, unstable saddle point, asymptotically stable nodal sink, unstable saddle point, asymptotically unstable nodal source. Enlarge phase portraits of these points are shown below.



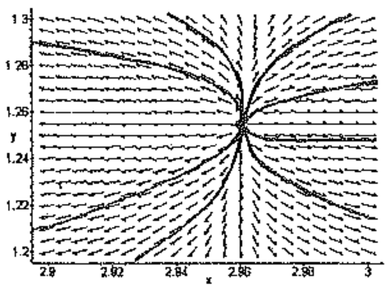
Local Enlargements of Phase Portrait Near Each Critical Point



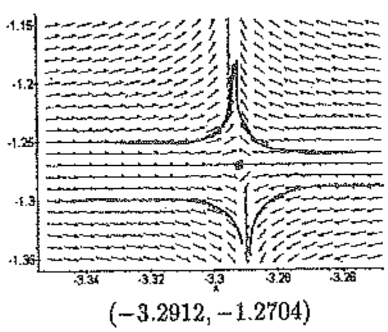
(−3.3617.1.5707)
Nodal Sink



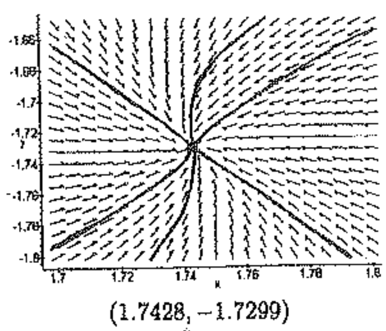
(1.0918, 1.0285)
Saddle Point



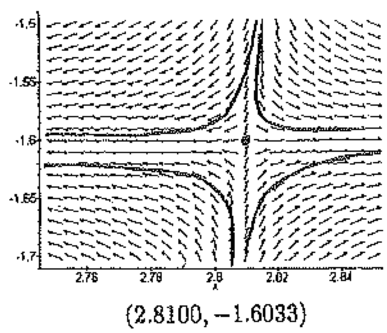
(2.9608, 1.2554)
Nodal Source



(−3.2912, −1.2704)
Saddle Point



(1.7428, −1.7299)
Nodal Sink



(2.8100, −1.6033)
Saddle Point

11. (a) By using Taylor series we find that both systems have the linearization $X' = Y, Y' = -X$, for which the eigenvalues are $\lambda_1 = i, \lambda_2 = -i$, so $(0,0)$ is a center for the associated linear system of both systems.

(b) Using polar coordinates we get $\left|\dfrac{x\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}\right| = |r\cos(\theta)| < r \to 0$ as $r \to 0$. Similarly $\left|\dfrac{y\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}\right| < r \to 0$ as $r \to 0$ so both systems are almost linear.

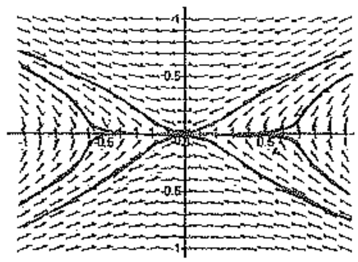
(c) Since $r^2 = x^2 + y^2$, we easily get by differentiating with respect to t, that $rr' = xx' + yy'$.

(d) For the first system, compute $r\dfrac{dr}{dt} = xx' + yy' = -(x^2+y^2)\sqrt{x^2+y^2} = -r^3$, or $\dfrac{dr}{dt} = -r^2$. Thus $r'(t) < 0$ for all t. Solve for $r(t)$ subject to $r(t_0) = r_0$ to get $r(t) = \dfrac{1}{t - t_0 + \frac{1}{r_0}}$, and note as $t \to \infty, r(t) \to 0$, so the origin is asymptotically stable for the first system.

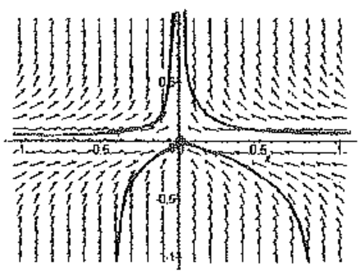
(e) For the second system, compute $r\dfrac{dr}{dt} = xx' + yy' = r^3$ or $\dfrac{dr}{dt} = r^2$. Solve for $r(t)$ subject to $r(t_0) = r_0$ to get $r(t) = \dfrac{1}{(\frac{1}{r_0} + t_0) - t}$. As $t \to \left(t_0 + \dfrac{1}{r_0}\right)^-$, note that $r(t) \to \infty$, so the origin is unstable for the second system.

Section 11.6 Lyapunov's Stability Criteria

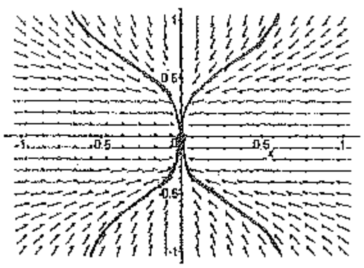
1. Compute the orbital derivative $\dot{V} = -4ax^2y^2 + b(-2xy^3 - x^3y) - 2cx^2y^2$. Choosing $b = 0, a = c = 1$ gives $V = x^2 + y^2$ which is positive definite and $\dot{V} = -6x^2y^2$ which is negative semidefinite. By Theorem 10.4, $(0,0)$ is stable.



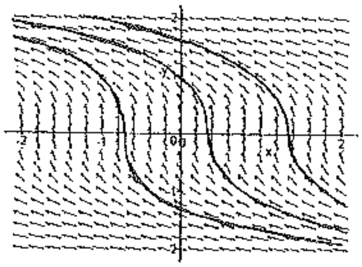
2. The orbital derivative is $\dot{V} = -2ax^2\cos^2(y) + b(-xy\cos^2 y + (6-x)xy^2) + 2c(6-x)y^3$. This expression is neither positive definite nor negative definite for any choice of constants a, b, c. Thus it seems that Lyapunov's method is not suitable for this problem. The phase diagram certainly seems to show that the origin $(0,0)$ is unstable, acting like a nodal sink for trajectories approaching from the lower half plane, while appearing to have saddle point structure in the upper half plane.



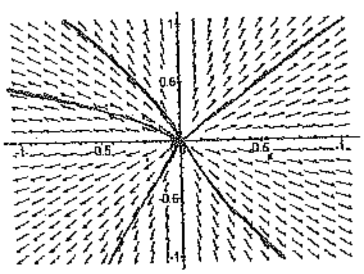
3. The orbital derivative is $\dot{V} = -4ax^2 + b(-2xy - 3xy^3) - 6cy^4$. Choosing $b = 0, a = c = 1$ gives $V = x^2 + y^2$ which is positive definite and $\dot{V} = -4x^2 - 6y^4$ which is negative definite. By Theorem 10.4, $(0,0)$ is asymptotically stable.



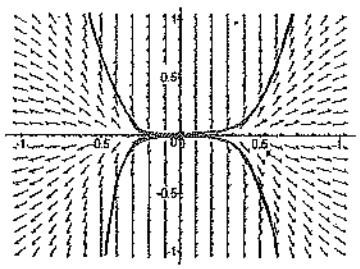
4. The orbital derivative is $\dot{V} = -2ax^3y + b(x^3 - x^2y^2) + 2cx^2y$. This expression is neither positive semidefinite nor negative semidefinite, so Lyapunov's method does not seem to be suitable for classifying $(0,0)$. The phase portrait indicates that the origin is a point with saddle point characteristics, and as such is unstable.



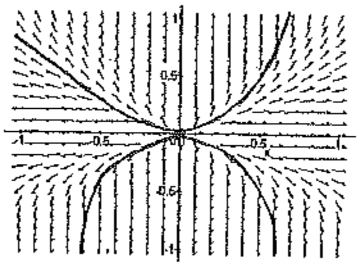
5. The orbital derivative is $\dot{V} = 2ax^2y^2 + 2bxy^3 + 2cy^4$. Choosing $b = 0, a = c = -1$ gives $V = -x^2 - y^2$ which is negative definite and $\dot{V} = -2x^2y^2 - 2y^4$ which is negative definite. By Theorem 10.5, $(0,0)$ is unstable.



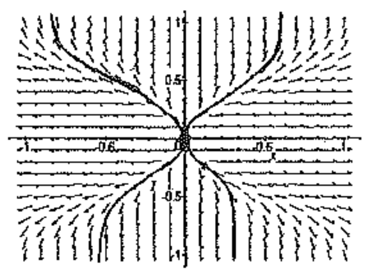
6. The orbital derivative is $\dot{V} = 2ax^4(1+y^2)+b[x^3y(2+y^2)+xy^3]+2c(x^2y^2+y^4)$. Choosing $b = 0, a = c = -1$ gives $V = -x^2-y^2$ which is negative definite and $\dot{V} = -2x^4(1+y^2)-2(x^2y^2+y^4)$ which is negative definite. By Theorem 10.5, $(0,0)$ is unstable.



7. The orbital derivative is $\dot{V} = 2ax^4(1+y) + b[yx^3(1+y) + xy^3(4+x^2)] + 2cy^4(4+x^2)$. Choosing $b = 0, a =, c = -1$ gives $V = -x^2 - y^2$ which is negative definite and $\dot{V} = -2x^4(1+y) - 2y^4(4+x^2)$ which is negative definite. By Theorem 10.5, $(0,0)$ is unstable.

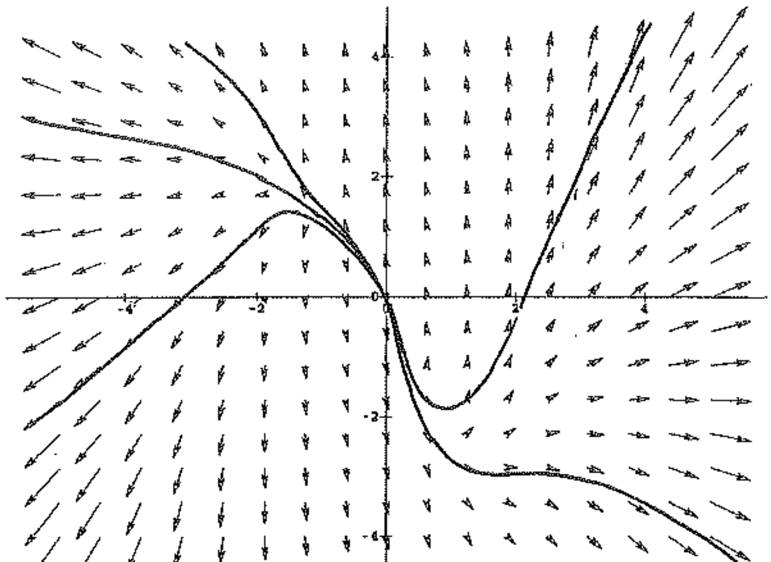


8. The orbital derivative is $\dot{V} = 2ax^4\cot^2(y) + b[x^3y\cot^2(y) + xy^3(2+x^4)] + 2cy^4(2+x^4)$. Choosing $b = 0, a = c = -1$ given $V = -x^2 - y^2$ which is negative definite and $\dot{V} = -2x^4\cot^2(y) - 2y^4(2+x^4)$ which is negative definite. By Theorem 10.5, $(0,0)$ is unstable.



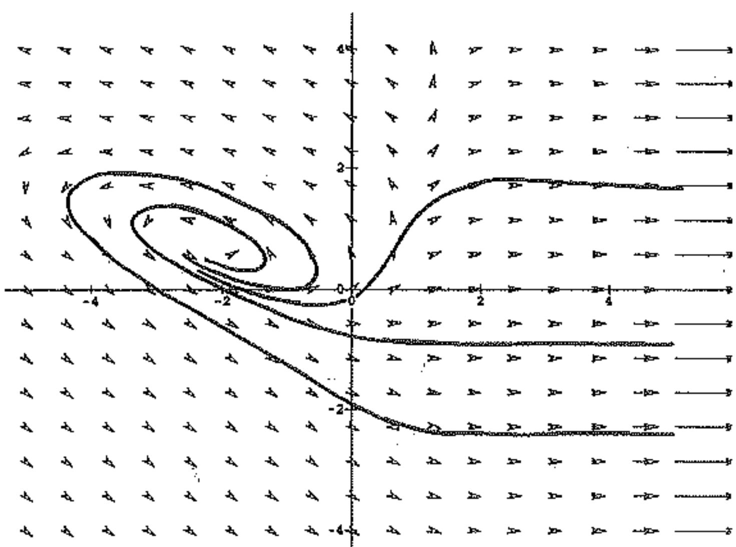
Section 11.7 Limit Cycles and Periodic Solutions

1. Calculate $\dfrac{\partial f}{\partial x}+\dfrac{\partial g}{\partial y}=-2+3x^2+5+x^2-2\sin(x)=3+4x^2-2\sin(x)\geq 1+4x^2>0$ for all (x,y). By Bendixson's Theorem the system has no closed trajectories.



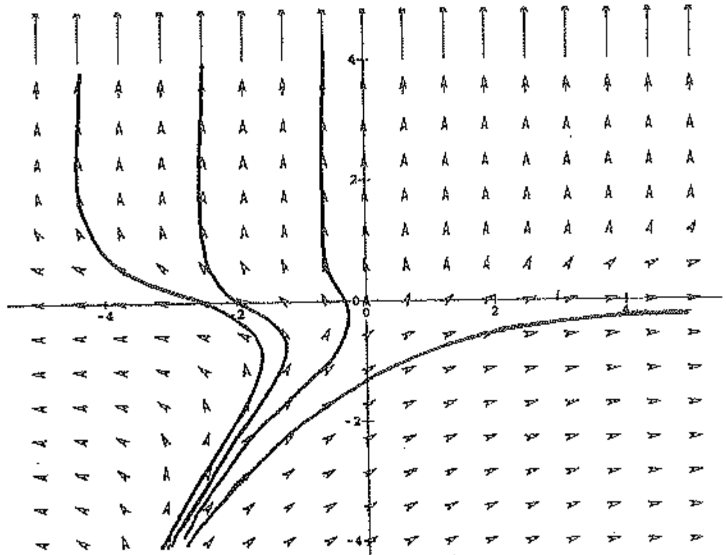
1. Phase Portrait

2. Calculate $\dfrac{\partial f}{\partial x}+\dfrac{\partial g}{\partial y}=-1+2e^{2x}+2-\sin(y)\geq 2e^{2x}>0$ for all (x,y). Thus no closed trajectories.



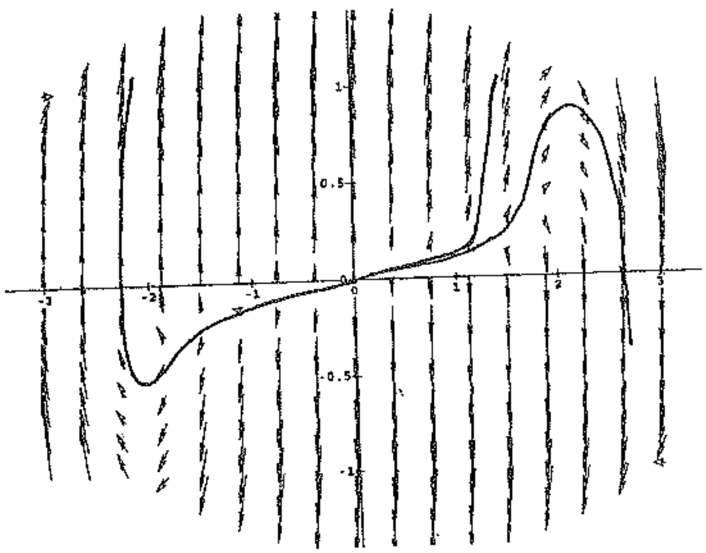
2. Phase Portrait

3. Calculate $\dfrac{\partial f}{\partial x}+\dfrac{\partial g}{\partial y}=3+\cosh(x)-4+15e^{3y}\geq 0$ for all (x,y). Thus no closed trajectories.



3. Phase Portrait

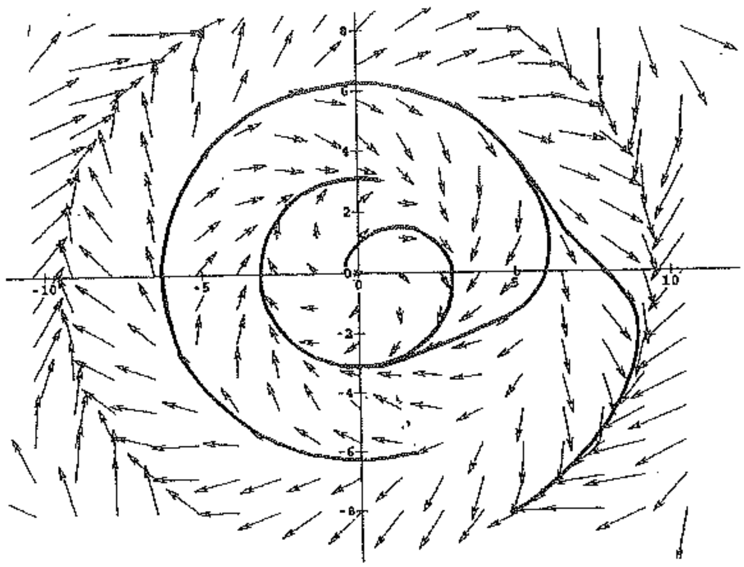
4. Calculate $\dfrac{\partial f}{\partial x}+\dfrac{\partial g}{\partial y}=(9-x^2-3y^2)-6y^2=9-x^2-9y^2>0$ for all (x,y) inside $\dfrac{x^2}{9}+y^2<1$. Thus no closed trajectories.



4. Phase Portrait

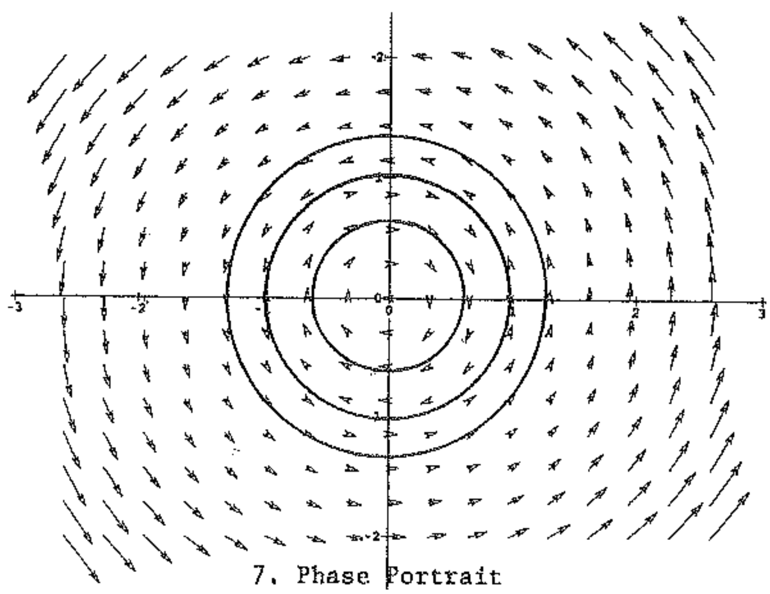
5. Calculate $\dfrac{dr}{dt}=\dfrac{1}{r}\left(x\dfrac{dx}{dt}+y\dfrac{dy}{dt}\right)=\dfrac{1}{r}rf(r)=f(r)$. Hence any circle $r=r_0$ for which $f(r_0)=0$ is a closed trajectory. To determine orientation calculate $\dfrac{d\theta}{dt}=\dfrac{1}{r^2}\left[x\dfrac{dy}{dt}-x^2\dfrac{dx}{dt}\right]=-1$, so the closed circular trajectories have clockwise orientation.

6. First observe that the origin $(0,0)$ is a critical point and hence a closed trajectory. For other trajectories calculate $\dfrac{dr}{dt} = \dfrac{xx' + yy'}{r} = r\sin(r)$, and $\dfrac{d\theta}{dt} = \dfrac{1}{r^2}[xy' - yx'] = -1$. It follows that each circle $r = n\pi, n = 1, 2, 3\ldots$ is a closed trajectory. For $(2n-1)\pi < r < 2n\pi$ we have $\dfrac{dr}{dt} < 0$, so trajectories spiral in; whereas for $2n\pi < r < (2n-1)\pi$ we have $\dfrac{dr}{dt} > 0$, so trajectories spiral out. Thus $r = (2n-1)\pi, n \geq 1$ are all stable limit cycles; $r = 2n\pi, n \geq 1$ are all unstable limit cycles.



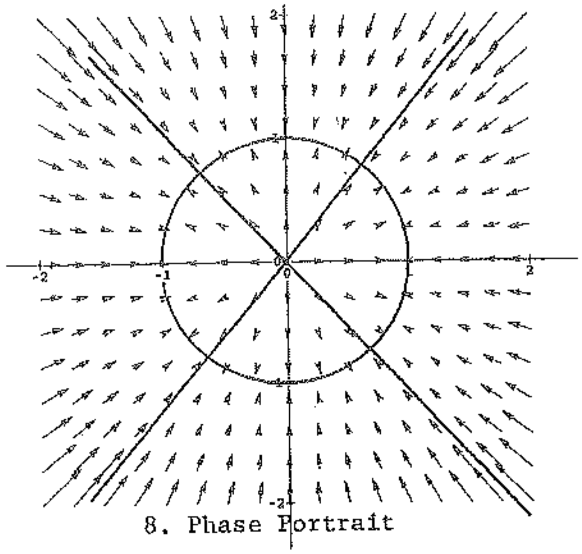
6. Phase Portrait

7. Calculate $\dfrac{dr}{dt} = 0, \dfrac{d\theta}{dt} = r^2 - 1$. Trajectories are the origin (a critical point), all points on $r = 1$ (each is a critical point), and the closed circles $r = a (a \neq 1)$. Each closed trajectory $r = a, a \neq 1$ is a neutrally stable trajectory.



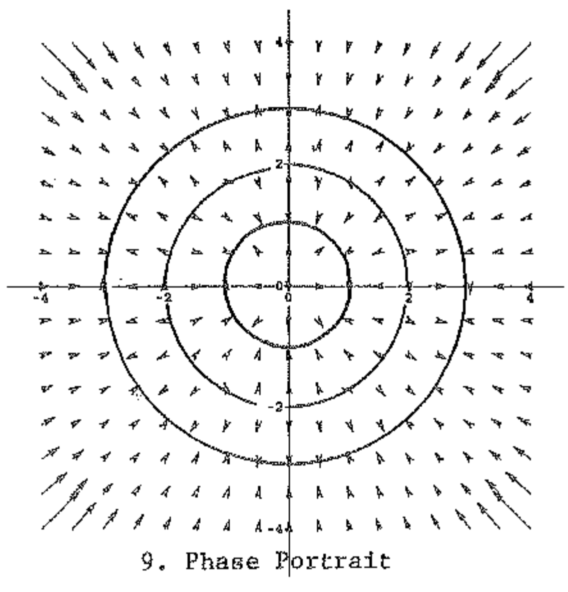
7. Phase Portrait

8. Calculate $\frac{dr}{dt} = r(1-r^2), \frac{d\theta}{dt} = 0$. The origin is a trajectory (critical point) as is every point on the unit circle $r = 1$. All other trajectories are radial lines, $\theta =$ constant, which move outward toward the unit circle from inside the unit circle, and move in toward the unit circle from outside the unit circle.



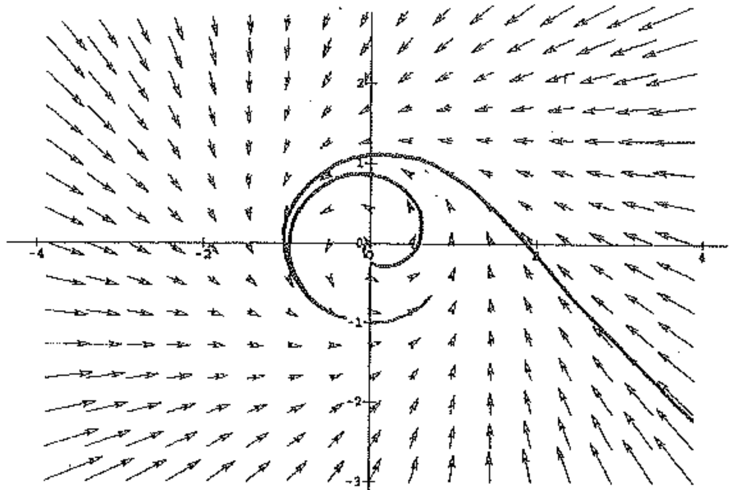
8. Phase Portrait

9. Calculate $\frac{dr}{dt} = r(1-r^2)(4-r^2)(9-r^2), \frac{d\theta}{dt} = -1$. The origin is a critical point. Closed trajectories are the circles $r = 1, r = 2, r = 3$. For $0 < r < 1$, $\frac{dr}{dt} > 0$, so other trajectories in this region spiral outward toward $r = 1$ with clockwise orientation; for $1 < r < 2$, $\frac{dr}{dt} < 0$, so trajectories in this region spiral inward toward $r = 1$. Thus $r = 1$ is an asymptotically stable limit cycle. By similar reasoning we deduce that $r = 2$ is an unstable limit cycle, and $r = 3$ is an asymptotically stable stable limit cycle.



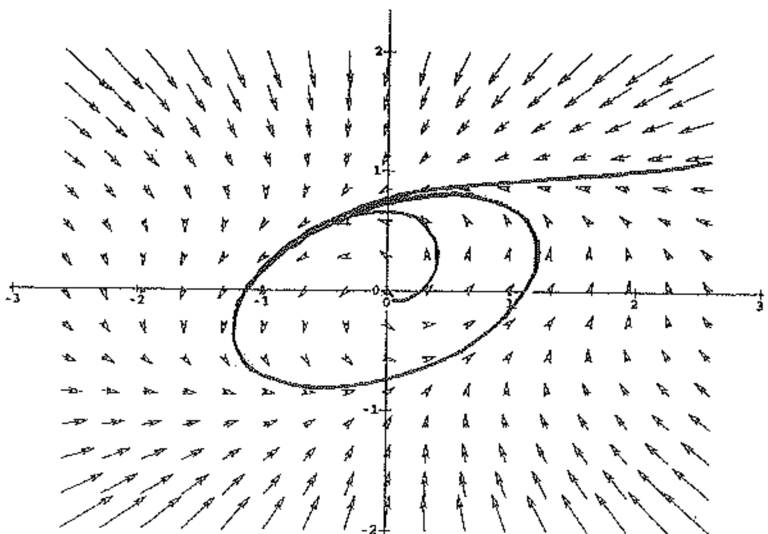
9. Phase Portrait

10. Calculate $xx' + yy' = r^2 - r^3 = r^2(1-r)$. Let R be the annular region $1/2 \le r \le 2$. The origin is the only critical point of the system, hence R contains no critical points. Now consider any trajectory C starting in R at time t_0. On the outer circle of $R, r = 2$, we have $\frac{dr}{dt} = 2(1-2) = -2 < 0$, so no trajectory can leave R across $r = 2$. Similarly on $r = 1/2$, we have $\frac{dr}{dt} = \frac{1}{2}\left(1 - \frac{1}{2}\right) = \frac{1}{4} > 0$, so no trajectory can leave R across $r = 1/2$. By the Poincare-Bendixson Theorem we conclude R contains a closed trajectory. Observe that $\frac{dr}{dt} = r(1-r)$ shows that $r = 1$ is such a closed trajectory.



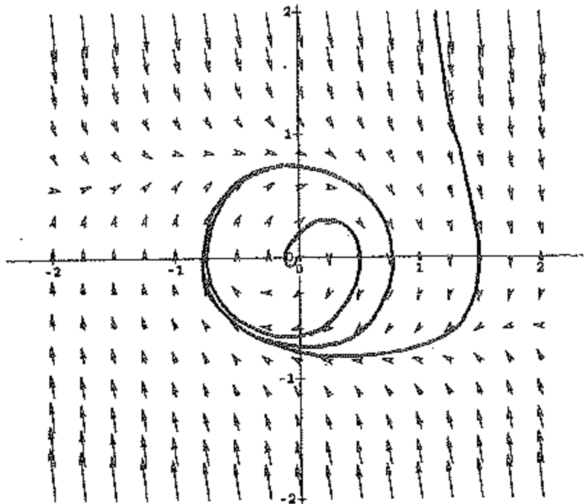
10. Phase Portrait

11. Calculate $xx' + yy' = 4(x^2+y^2)[4-(x^2+9y^2)] = 4r^2[4-r^2-8y^2] = 4r^2[4-9r^2+8x^2]$. Let R be the annular region $\frac{1}{3} \le r \le 3$, which contains no critical points. From the above calculation of $xx'+yy'$ we see that on $r = 1/3, \frac{dr}{dt} = 4\left(\frac{1}{3}\right)(3+8x^2) > 0$, and on $r = 3, \frac{dr}{dt} = 4(3)(-5-8y^2) < 0$. Thus trajectories starting in R at t_0 remain in R for $t > t_0$. Thus R contains a closed trajectory.



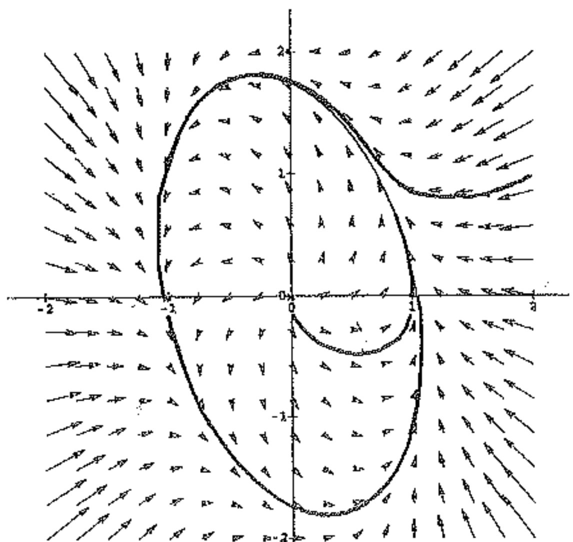
11. Phase Portrait

12. Calculate $xx' + yy' = y^2(1 - x^2 - 2y^2) = y^2[1 - (x^2 + y^2) - y^2] = y^2[1 - 2(x^2 + y^2) + x^2]$. Let R be the annular region $\frac{1}{2} \le r \le 1$. On $r = \frac{1}{2}$ we have $\frac{dr}{dt} = 2y^2\left[\frac{1}{2} + x^2\right] > 0$, and on $r = 1$, $\frac{dr}{dt} = y^2[-y^2] \le 0$. R contains no critical points, and these calculations show that trajectories starting in R at t_0 remain in R for $t > t_0$. Thus R contains a closed trajectory.



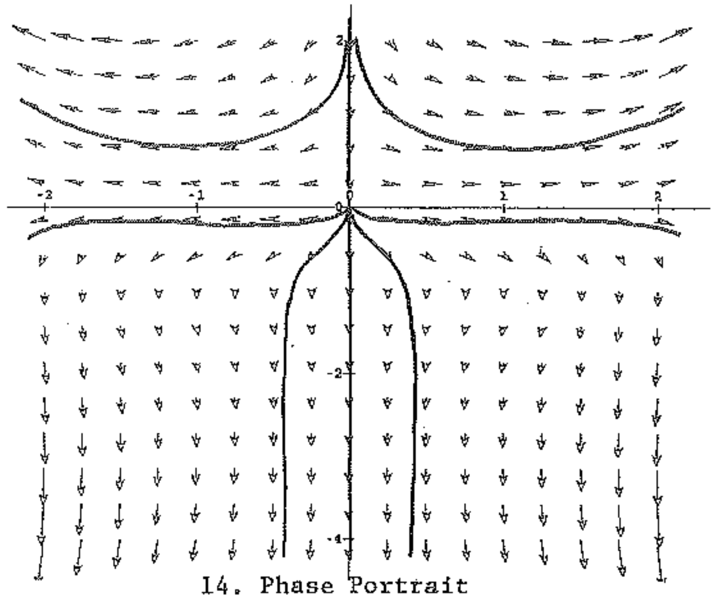
12. Phase Portrait

13. Calculate $xx' + yy' = 4(x^2 + y^2) - (x^2 + y^2)(4x^2 + y^2) = r^2[4 - 4x^2 - y^2] = r^2[4 - r^2 - 3x^2] = r^2[4 - 4r^2 + 3y^2]$. Let R be the annular region $1 \le r \le 2$, which contains no critical points. On $r = 1$, $\frac{dr}{dt} = [3y^2] \ge 0$ and on $r = 2$, $\frac{dr}{dt} = 2[-3x^2] \le 0$. Trajectories starting in R remain in R, so R contains a closed trajectory. The ellipse $4x^2 + y^2 = 4$ is a limit cycle.



13. Phase Portrait

14. Calculate $\dfrac{\partial f}{\partial x}+\dfrac{\partial g}{\partial y}=3+4y+y^2-4y+x^4=3+y^2+x^4>0$ for all (x,y). By Bendixson's Theorem there are no closed trajectories.



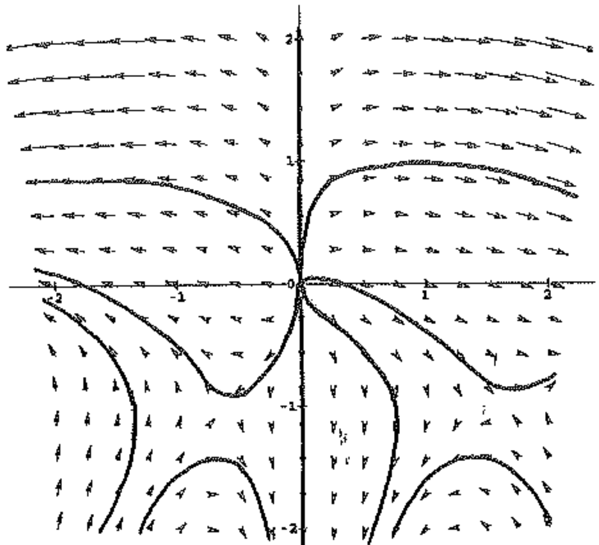
14. Phase Portrait

15. Calculate $\dfrac{dr}{dt}=\dfrac{1}{r}[xx'+yy']=r(1+r^2)>0$ for all $r>0$, so there are no closed trajectories.

16. This equation is solvable by writing $\dfrac{dy}{dx}=\dfrac{3x+2x^3}{-y^2}$ and separating variables to get $9x^2+3x^4+2y^3=K$. Since $x'=-y^2\le 0$ at all points, there can be no closed trajectories.

17. As in Problem 16, we have $\dfrac{dy}{dt}>0$ at all points, so there can be no closed trajectories.

18. We can prove the existence of a closed trajectory using Lienard's Theorem. Take $p(x)=x^2-1, q(x)=x$. Then q is odd, p is even, $q(x)>0$ for $x>0$. We find $F(x)=\int_0^x p(\xi)d\xi=\dfrac{x^3}{3}-x$ has exactly one positive root at $x=\sqrt{3}$. Also $F(x)<0$ for $0<x<\sqrt{3}, F(x)>0$ for $x>\sqrt{3}$, and is nondecreasing for $x>1$. Finally $F(x)\to+\infty$ as $x\to+\infty$. By Lienard's Theorem, the system has a unique closed trajectory.



18. Phase Portrait

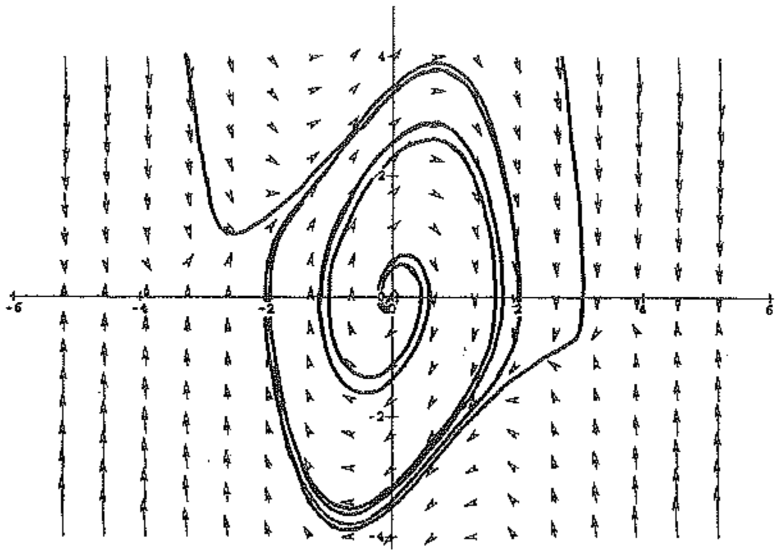
19. Calculate $\dfrac{\partial f}{\partial x} + \dfrac{\partial g}{\partial y} = 1 + 3y^2 + 35y^4 \geq 1 > 0$, so there are no closed trajectories.

20. Calculate $\dfrac{dr}{dt} = y^2 e^{-y} > 0$ for all $y \neq 0$. The origin $x = y = 0$ is the only trajectory having $y = 0$, so there are no closed trajectories.

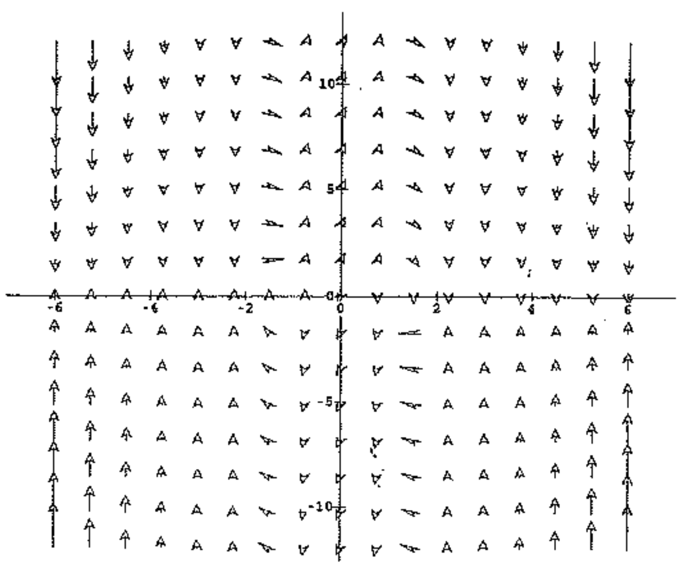
21. Separate variables and solve to get $2y^2 + 4x^4 = C$, which is a closed trajectory for all $C > 0$.

22. Calculate $\dfrac{dr}{dt} = r[9 - x^2 - 9y^2]$, and apply the Poincare-Bendixson Theorem on the annular region $R, \dfrac{1}{2} \leq r \leq 4$, which contains no critical points. On the inner circle $r = 1/2, \dfrac{dr}{dt} = \dfrac{1}{2}\left[\dfrac{27}{4} + 8x^2\right] > 0$, and on $r = 4, \dfrac{dr}{dt} = 4[-7 - 8y^2] < 0$. So trajectories starting in R stay in R. By the Poincare-Bendixson Theorem there is a closed trajectory in R.

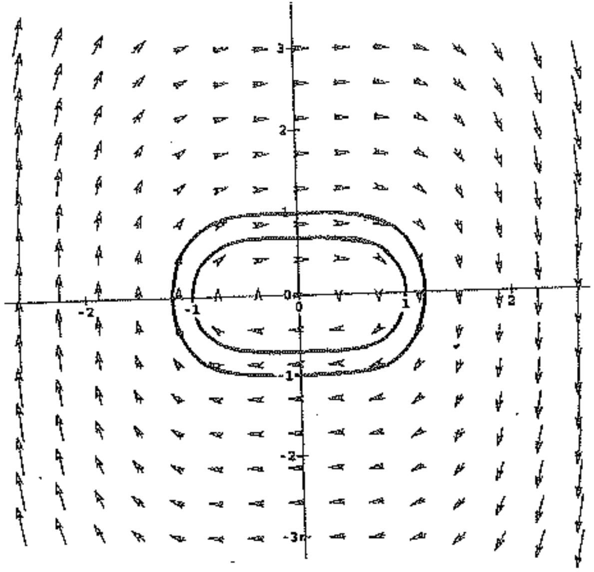
23. Apply Lienard's Theorem with $p(x) = x^2 - 1, q(x) = 2x + \sin(x)$, which satisfy all necessary conditions. $F(x) = \int_0^x p(t)dt = \frac{x^3}{3} - x$ has unique positive solution $x = \sqrt{3}$ and satisfies all other conditions. Thus there exists a unique closed trajectory and hence a periodic solution.



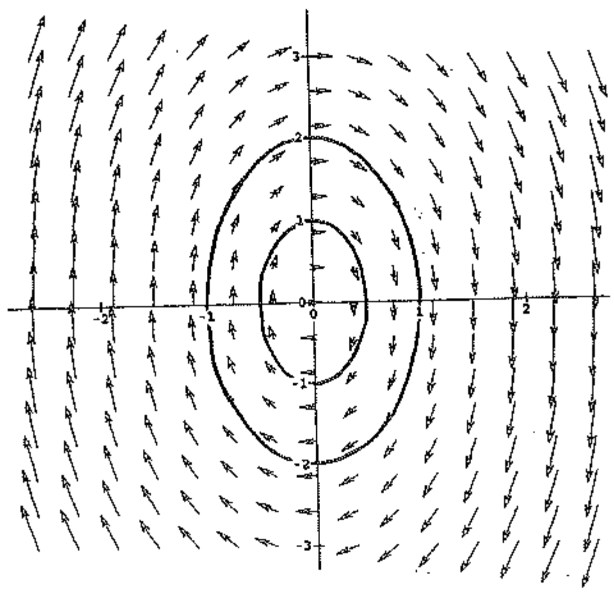
24. Apply Lienard's Theorem with $p(x) = 5x^4 + 9x^2 - 4, q(x) = \sinh(x)$, which satisfy all conditions. $F(x) = \int_0^x p(t)dt = x^5 + 3x^3 - 4x$ has unique positive solution $x = 1$ and satisfies all other conditions.



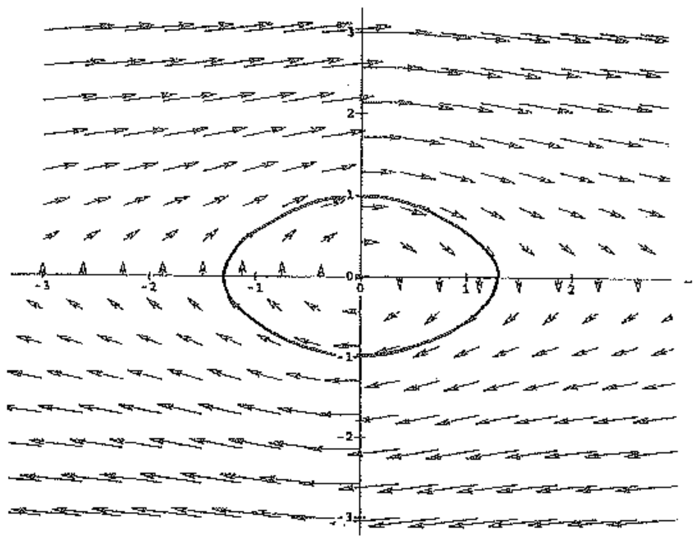
25. The equivalent system $x' = y, y' = -x^3$ can be solved by separation of variables to give $x^4 + 2y^2 = C$, a closed trajectory for all $C > 0$.



26. The general solution of this equation is $x(t) = c_1 \cos(2t) + c_2 \sin(2t)$, a periodic function (of period π) for any choice of c_1, c_2.



27. The equivalent system $x' = y$; $y' = \dfrac{x}{1+x^2}$ can be explicitly solved to give $\ln(1+x^2)+y^2 = C$, a closed trajectory for any $C > 0$.



28. Calculate $f_x + g_y = -\alpha(x^2 - 1)$. On each of the regions given in (a), (b), (c), this expression has one sign.
(a) $-1 < x < 1$ gives $f_x + g_y > 0$,
(b) $x > 1$ gives $f_x + g_y < 0$,
(c) $x < -1$ gives $f_x + g_y < 0$.
By Bendixson Theorem, none of these regions could completely contain a closed trajectory.

Chapter Twelve - Vector DifferentialCalculus

Section 12.1 Vector Functions of One Variable

1. (a) $\frac{d}{dt}[f(t)\mathbf{F}(t)] = \frac{d}{dt}[4\cos(3t)\mathbf{i} + 12t^2\cos(3t)\mathbf{j} + 8t\cos(3t)\mathbf{k}] =$
$-12\sin(3t)\mathbf{i} + [24t\cos(3t) - 36t^2\sin(3t)]\mathbf{j} + 8[\cos(3t) - 3t\sin(3t)]\mathbf{k}$
(b) $\frac{d}{dt}[f(t)\mathbf{F}(t)] = -12\sin(3t)\mathbf{F}(t) + f(t)[6t\mathbf{j} + 2\mathbf{k}] =$
$-12\sin(3t)\mathbf{i} + [-36t^2\sin(3t) + 24t\cos(3t)]\mathbf{j} + [8\cos(3t) - 24t\sin(3t)]\mathbf{k}$

2. (a) $\frac{d}{dt}[\mathbf{F}(t)\cdot\mathbf{G}(t)] = \frac{d}{dt}[t - 3t^2\cos(t)] = 1 - 6t\cos(t) + 3t^2\sin(t)$
(b) $\frac{d}{dt}[\mathbf{F}(t)\cdot\mathbf{G}(t)] = (\mathbf{i} - 6t\mathbf{k})\cdot(\mathbf{i} + \cos(t)\mathbf{k}) + (t\mathbf{i} - 3t^2\mathbf{k})\cdot(-\sin(t))\mathbf{k} = 1 - 6t\cos(t) + 3t^2\sin(t)$

3. (a) Since $\mathbf{F}(t)\times\mathbf{G}(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t & 1 & 4 \\ 1 & -\cos(t) & t \end{vmatrix} = (t + 4\cos(t))\mathbf{i} + (4 - t^2)\mathbf{j} - (t\cos(t) + 1)\mathbf{k}$,
$\frac{d}{dt}(\mathbf{F}(t)\times\mathbf{G}(t)) = (1 - 4\sin(t))\mathbf{i} - 2t\mathbf{j} - [\cos(t) - t\sin(t)]\mathbf{k}$
(b) $\mathbf{F}'(t)\times\mathbf{G}(t) + \mathbf{F}(t)\times\mathbf{G}'(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 1 & -\cos(t) & t \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t & 1 & 4 \\ 0 & \sin(t) & 1 \end{vmatrix} =$
$-t\mathbf{j} - \cos(t)\mathbf{k} + (1 - 4\sin(t))\mathbf{i} - t\mathbf{j} + t\sin(t)\mathbf{k}$

4. (a) Since $\mathbf{F}(t)\times\mathbf{G}(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & \sinh(t) & -t \\ t & t^2 & -t^2 \end{vmatrix} = [t^3 - t^2\sinh(t)]\mathbf{i} - t^2\mathbf{j} - t\sinh(t)\mathbf{k}$,
$\frac{d}{dt}(\mathbf{F}(t)\times\mathbf{G}(t)) = [3t^2 - 2t\sinh(t) - t^2\cosh(t)]\mathbf{i} - 2t\mathbf{j} - [\sinh(t) + t\cosh(t)]\mathbf{k}$
(b) $\mathbf{F}'(t)\times\mathbf{G}(t) + \mathbf{F}(t)\times\mathbf{G}'(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & \cosh(t) & -1 \\ t & t^2 & -t^2 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & \sinh(t) & -t \\ 1 & 2t & -2t \end{vmatrix} =$
$[-t^2\cosh(t) + t^2]\mathbf{i} - t\mathbf{j} - t\cosh(t)\mathbf{k} - 2t\sinh(t)\mathbf{i} - t\mathbf{j} - \sinh(t)\mathbf{k}$

5. (a) $\frac{d}{dt}[f(t)\mathbf{F}(t)] = \frac{d}{dt}[(t - 2t^4)\mathbf{i} - (1 - 2t^3)\cosh(t)\mathbf{j} + (e^t - 2t^3e^t)\mathbf{k}] =$
$(1 - 8t^3)\mathbf{i} + [6t^2\cosh(t) - (1 - 2t^3)\sinh(t)]\mathbf{j} + [e^t - 6t^2e^t - 2t^3e^t]\mathbf{k}$
(b) $f'(t)\mathbf{F}(t) + f(t)\mathbf{F}'(t) = -6t^2(t\mathbf{i} - \cosh(t)\mathbf{j} + e^t\mathbf{k}) + (1 - 2t^3)(\mathbf{i} - \sinh(t)\mathbf{j} + e^t\mathbf{k})$

6. (a) $\frac{d}{dt}[\mathbf{F}(t)\cdot\mathbf{G}(t)] = \frac{d}{dt}[t\sin(t) + 4t + t^5] = \sin(t) + t\cos(t) + 4 + 5t^4$
(b) $\mathbf{F}'(t)\cdot\mathbf{G}(t) + \mathbf{F}(t)\cdot\mathbf{G}'(t) = (\mathbf{i} - \mathbf{j} + 2t\mathbf{k})\cdot(\sin(t)\mathbf{i} - 4\mathbf{j} + t^3\mathbf{k}) + (t\mathbf{i} - t\mathbf{j} + t^2\mathbf{k})\cdot(\cos(t)\mathbf{i} + 3t^2\mathbf{k}) =$
$\sin(t) + 4 + 2t^4 + t\cos(t) + 3t^4$

7. (a) $\mathbf{F}(t)\times\mathbf{G}(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -9 & t^2 & t^2 \\ e^t & 0 & 0 \end{vmatrix} = t^2e^t\mathbf{j} - t^2e^t\mathbf{k}$, so $\frac{d}{dt}[\mathbf{F}(t)\times\mathbf{G}(t)] =$
$(2te^t + t^2e^t)\mathbf{j} - (2te^t + t^2e^t)\mathbf{k} = te^t(2 + t)[\mathbf{j} - \mathbf{k}]$
(b) $\mathbf{F}'(t)\times\mathbf{G}(t) + \mathbf{F}(t)\times\mathbf{G}'(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 2t & 2t \\ e^t & 0 & 0 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -9 & t^2 & t^2 \\ e^t & 0 & 0 \end{vmatrix} = 2te^t\mathbf{j} - 2te^t\mathbf{k} + t^2e^t\mathbf{j} - t^2e^t\mathbf{k}$

8. (a) $\frac{d}{dt}[\mathbf{F}(t)\cdot\mathbf{G}(t)] = \frac{d}{dt}[-16\sin(t)\cos(t)] = -16\cos^2(t) + 16\sin^2(t)$

(b) $\mathbf{F}'(t)\cdot\mathbf{G}(t) + \mathbf{F}(t)\cdot\mathbf{G}'(t) = 4\sin(t)\mathbf{k}\cdot(-t^2\mathbf{i} + 4\sin(t)\mathbf{k} + (-4\cos(t)\mathbf{k})\cdot(-2t\mathbf{i} + 4\cos(t)\mathbf{k})$

9. (a) A position vector is $\mathbf{F}(t) = \sin(t)\mathbf{i} + \cos(t)\mathbf{j} + 45t\mathbf{k}$, for $0 \le t \le 2\pi$;
a tangent vector is $\mathbf{F}'(t) = \cos(t)\mathbf{i} - \sin(t)\mathbf{j} + 45\mathbf{k}$.
(b) $s(t) = \int_0^t \|\mathbf{F}'(\xi)\|d\xi = \int_0^t \sqrt{2026}d\xi = \sqrt{2026}t$.
(c) Since $t = \frac{s}{\sqrt{2026}}, 0 \le s \le 2\pi\sqrt{2026}$, a position vector is

$$\mathbf{G}(s) = \mathbf{F}(t(s)) = \sin\left(\frac{s}{\sqrt{2026}}\right)\mathbf{i} + \cos\left(\frac{s}{\sqrt{2026}}\right)\mathbf{j} + \frac{45s}{\sqrt{2026}}\mathbf{k},$$

so

$$\mathbf{G}'(s) = \frac{1}{\sqrt{2026}}\left[\cos\left(\frac{s}{\sqrt{2026}}\right)\mathbf{i} - \sin\left(\frac{s}{\sqrt{2026}}\right)\mathbf{j} + 45\mathbf{k}\right] \text{ and } \|\mathbf{G}'(s)\| = 1.$$

10. (a) A position vector is $\mathbf{F}(t) = t^3(\mathbf{i}+\mathbf{j}+\mathbf{k})$, for $-1 \le t \le 1$,
a tangent vector is
$\mathbf{F}'(t) = 3t^2(\mathbf{i}+\mathbf{j}+\mathbf{k})$.
(b) $s(t) = \int_0^t \|\mathbf{F}'(\xi)\|d\xi = 3\sqrt{3}\int_0^t \xi^2 d\xi = \sqrt{3}t^3$.
(c) Since $t = \left(\frac{s}{\sqrt{3}}\right)^{1/3}$, a position vector is

$$\mathbf{G}(s) = \mathbf{F}(t(s)) = \frac{s}{\sqrt{3}}(\mathbf{i}+\mathbf{j}+\mathbf{k}),$$

so

$$\mathbf{G}'(s) = \frac{1}{\sqrt{3}}(\mathbf{i}+\mathbf{j}+\mathbf{k}) \text{ and } \|\mathbf{G}(s)\| = 1.$$

11. (a) A position vector is $\mathbf{F}(t) = t^2(2\mathbf{i}+3\mathbf{j}+4\mathbf{k})$, for $1 \le t \le 3$;
a tangent vector is $\mathbf{F}'(t) = 2t(2\mathbf{i}+3\mathbf{j}+4\mathbf{k})$.
(b) $s(t) = \int_1^t \|\mathbf{F}'(\xi)\|d\xi = 2\sqrt{29}\int_1^t \xi d\xi = \sqrt{29}(t^2-1)$.
(c) Since $t = \sqrt{\frac{s}{\sqrt{29}} + 1}, 0 \le s \le 8\sqrt{29}$, a position vector is

$$\mathbf{G}(s) = \mathbf{F}(t(s)) = \left(\frac{s}{\sqrt{29}} + 1\right)(2\mathbf{i}+3\mathbf{j}+4\mathbf{k}),$$

so

$$\mathbf{G}'(s) = \frac{1}{\sqrt{29}}(2\mathbf{i}+3\mathbf{j}+4\mathbf{k}) \text{ and } \|\mathbf{G}'(s)\| = 1.$$

12. If $\mathbf{F}\times\mathbf{F}' = 0$, then either $\mathbf{F}(t) = \mathbf{0}, \mathbf{F}'(t) = 0$ or $\mathbf{F}(t)$ and $\mathbf{F}'(t)$ are parallel for all t. We can assume $\mathbf{F}(t) \ne 0$. If $\mathbf{F}'(t) = \mathbf{0}$, then there is no motion and the particle is at rest. If $\mathbf{F}(t)$ and $\mathbf{F}'(t)$ are parallel, then the velocity vector is always directed along the path of motion and we would have straight line motion.

Section 12.2 Velocity, Acceleration, Curvature, and Torsion

Before giving solutions of Problems 1 through 10, we make same observations about the various quantities to be computed which will reduce the plethora of calculations to two differentiations followed by some routine vector calculations. We assume that $\mathbf{F}(t)$ is position (given) and that t is time. Clearly then, velocity is $\mathbf{V}(t) = \mathbf{F}'(t)$ and acceleration is $\mathbf{a}(t) = \mathbf{F}''(t)$, both easily found. Then speed is $v(t) = \|\mathbf{F}'(t)\|$ and the unit tangent is $\mathbf{T} = \dfrac{1}{v(t)}\mathbf{F}'(t)$. Now recall $\mathbf{F}'' = a_T\mathbf{T} + \kappa\left(\dfrac{ds}{dt}\right)^2\mathbf{N}$, and observe $\mathbf{F}' \times \mathbf{F}'' = \kappa\left(\dfrac{ds}{dt}\right)^3\mathbf{B}$ where this step uses the fact that $\mathbf{T}\times\mathbf{N} = \mathbf{B}$. Since $\left(\dfrac{ds}{dt}\right) > 0$ and $\kappa > 0$ we see that $\mathbf{B}$ is a unit vector with the same direction as $\mathbf{F}'\times\mathbf{F}''$ and this is easily calculated. It further follows that $\mathbf{N}$ is a unit vector having the direction of $(\mathbf{F}'\times\mathbf{F}'')\times\mathbf{F}'$. Finally, having now found $\mathbf{T}$ and $\mathbf{N}$, we easily find $a_T = (\mathbf{a}\cdot\mathbf{T})$ and $a_N = (\mathbf{a}\cdot\mathbf{N})$; also since $a_N = \kappa v^2$ we have $\kappa = \dfrac{(\mathbf{a}\cdot\mathbf{N})}{v^2}$ thus completing the calculations. In summary; find $\mathbf{V} = \mathbf{F}', \mathbf{a} = \mathbf{F}''$; then $\mathbf{T}, \mathbf{B}, \mathbf{N}$ are unit vectors in the respective directions of $\mathbf{F}', \mathbf{F}'\times\mathbf{F}'', (\mathbf{F}'\times\mathbf{F}'')\times\mathbf{F}'$; then $a_T = (\mathbf{a}\cdot\mathbf{T}), a_N = (\mathbf{a}\cdot\mathbf{N}), \kappa = \dfrac{a_N}{v^2}$.
We now apply these results to Problem 1 and give answers only for Problems 2 through 10.

1. We have $\mathbf{V} = \mathbf{F}' = 3\mathbf{i} + 2t\mathbf{k}, \mathbf{a} = \mathbf{F}'' = 2\mathbf{k}$, speed $= \sqrt{9+4t^2}$; Now $\mathbf{T} = \dfrac{1}{\sqrt{9+4t^2}}[3\mathbf{i} + 2t\mathbf{k}]$; we compute $\mathbf{F}'\times\mathbf{F}'' = -6\mathbf{j}$, so $\mathbf{B} = -\mathbf{j}$; and $(\mathbf{F}\times\mathbf{F}'')\times\mathbf{F}' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -6 & 0 \\ 3 & 0 & 2t \end{vmatrix} = -12t\mathbf{i} + 18\mathbf{j}$, so $\mathbf{N} = \dfrac{1}{\sqrt{4t^2+9}}[-2t\mathbf{i} + 3\mathbf{j}]$. Finally $a_T = (\mathbf{a}\cdot\mathbf{T}) = \dfrac{4t}{\sqrt{9+4t^2}}; a_N = (\mathbf{a}\cdot\mathbf{N}) = \dfrac{6}{\sqrt{9+4t^2}}$ and $\kappa = \dfrac{6}{(9+4t^2)^{3/2}}$.

2. $\mathbf{V} = [\sin(t) + t\cos(t)]\mathbf{i} + [\cos(t) - t\sin(t)]\mathbf{j}; \mathbf{a} = [2\cos(t) - t\sin(t)]\mathbf{i} - [2\sin(t) + t\cos(t)]\mathbf{j}$;
$v = [1+t^2]^{1/2}; \mathbf{T} = \dfrac{1}{v}\mathbf{V} = \dfrac{1}{\sqrt{1+t^2}}\left[[\sin(t) + t\cos(t)]\mathbf{i} + [\cos(t) - t\sin(t)]\mathbf{j}\right]; \mathbf{B} = -\mathbf{k}$;
$\mathbf{N} = \dfrac{1}{\sqrt{1+t^2}}\left[[\cos(t) - t\sin(t)]\mathbf{i} - [\sin(t) + t\cos(t)]\mathbf{j}\right]; a_T = \dfrac{t}{\sqrt{1+t^2}}; a_N = \dfrac{t^2+2}{\sqrt{1+t^2}}$;
$\kappa = \dfrac{t^2+2}{(1+t^2)^{3/2}}$

3. $\mathbf{V} = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}; \mathbf{a} = 0; v = 3; T = \dfrac{1}{3}(2\mathbf{i} - 2\mathbf{j} + \mathbf{k}); \mathbf{N} = \dfrac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j})$ or any unit vector perpendicular to $\mathbf{T}$; $\mathbf{B} = \dfrac{\sqrt{2}}{6}(-\mathbf{i} - \mathbf{j} + 4\mathbf{k}); a_T = 0; a_N = 0; \kappa = 0$.

4. $\mathbf{V} = e^t[(\sin(t) + \cos(t))\mathbf{i} + (\cos(t) - \sin(t))\mathbf{k}]; \mathbf{a} = 2e^t[\cos(t)\mathbf{i} - \sin(t)\mathbf{k}]; v(t) = \sqrt{2}e^t$;
$\mathbf{T} = \dfrac{1}{\sqrt{2}}\{[\sin(t) + \cos(t)]\mathbf{i} + [\cos(t) - \sin(t)]\mathbf{k}\}; \mathbf{B} = \mathbf{j}$;
$\mathbf{N} = \dfrac{1}{\sqrt{2}}\{[\cos(t) - \sin(t)]\mathbf{i} - [\sin(t) + \cos(t)]\mathbf{k}\}; a_T = \sqrt{2}e^t; a_N = \sqrt{2}e^t; \kappa = \dfrac{1}{\sqrt{2}}e^{-t}$

5. $\mathbf{V} = -3e^{-t}(\mathbf{i}+\mathbf{j}-2\mathbf{k}); \mathbf{a} = 3e^{-t}(\mathbf{i}+\mathbf{j}-2\mathbf{k}); v = 3\sqrt{6}e^{-t}; \mathbf{T} = \dfrac{1}{\sqrt{6}}(-\mathbf{i}-\mathbf{j}+2\mathbf{k}); \mathbf{N} = \dfrac{1}{\sqrt{2}}(\mathbf{i}-\mathbf{j})$

or any unit vector perpendicular to $\mathbf{T}$; $\mathbf{B} = \frac{1}{\sqrt{3}}(\mathbf{i}+\mathbf{j}+\mathbf{k})$; $a_T = -3\sqrt{6}e^{-t}$; $a_N = 0$; $\kappa = 0$.

6. $\mathbf{V} = -\alpha\sin(t)\mathbf{i} + \beta\mathbf{j} + \alpha\cos(t)\mathbf{k}$; $\mathbf{a} = -\alpha\cos(t)\mathbf{i} - \alpha\sin(t)\mathbf{k}$; $v = \sqrt{\alpha^2+\beta^2}$;
$\mathbf{T} = \frac{1}{\sqrt{a^2+\beta^2}}[-\alpha\sin(t)\mathbf{i} + \beta\mathbf{j} + \alpha\cos(t)\mathbf{k}]$; $\mathbf{B} = \frac{\alpha}{\beta\sqrt{\alpha^2+\beta^2}}[-\beta\sin(t)\mathbf{i} - \alpha\mathbf{j} + \beta\cos(t)\mathbf{k}]$;
$\mathbf{N} = -\cos(t)\mathbf{i} - \sin(t)\mathbf{k}$; $a_T = 0$; $a_N = \alpha$; $\kappa = \frac{\alpha}{\alpha^2+\beta^2}$

7. $\mathbf{V} = 2\cosh(t)\mathbf{j} - 2\sinh(t)\mathbf{k}$; $\mathbf{a} = 2\sinh(t)\mathbf{j} - 2\cosh(t)\mathbf{k}$; $v = 2\sqrt{\cosh(2t)}$;
$\mathbf{T} = \frac{1}{\sqrt{\cosh(2t)}}[\cosh(t)\mathbf{j} - \sinh(t)\mathbf{k}]$;
$\mathbf{N} = \frac{1}{\sqrt{\cosh(2t)}}[-\sinh(t)\mathbf{j} - \cosh(t)\mathbf{k}]$;
$\mathbf{B} = -\mathbf{i}$; $a_T = \frac{2\sinh(2t)}{\sqrt{\cosh(2t)}}$; $a_N = \frac{2}{\sqrt{\cosh(2t)}}$; $\kappa = \frac{1}{2[\cosh(2t)]^{3/2}}$.

8. $\mathbf{V} = \frac{1}{t}(\mathbf{i}-\mathbf{j}+2\mathbf{k})$; $\mathbf{a} = -\frac{1}{t^2}(\mathbf{i}-\mathbf{j}+2\mathbf{k})$; $v = \frac{\sqrt{6}}{t}$; $\mathbf{T} = \frac{i-j+2k}{\sqrt{6}}$; $\mathbf{N} = \frac{(i+j)}{\sqrt{2}}$ or any unit vector perpendicular to $\mathbf{T}$; $\mathbf{B} = \frac{1}{\sqrt{3}}(-\mathbf{i}+\mathbf{j}+\mathbf{k})$; $a_T = -\frac{\sqrt{6}}{t^2}$; $a_N = 0$; $\kappa = 0$.

9. $\mathbf{V} = 2t(\alpha\mathbf{i} + \beta\mathbf{j} + \gamma\mathbf{k})$; $\mathbf{a} = 2(\alpha\mathbf{i} + \beta\mathbf{j} + \gamma\mathbf{k})$; $v = 2|t|\sqrt{\alpha^2+\beta^2+\gamma^2}$;
$\mathbf{T} = \frac{1}{\sqrt{\alpha^2+\beta^2+\gamma^2}}(\alpha\mathbf{i}+\beta\mathbf{j}+\gamma\mathbf{k})$; $\mathbf{N}$ = any unit vector perpendicular to $\mathbf{T}$, and $\mathbf{B} = \mathbf{T}\times\mathbf{N}$;
$a_T = 2(\mathrm{sgn}(t))\sqrt{\alpha^2+\beta^2+\gamma^2}$; $a_N = 0$; $\kappa = 0$.

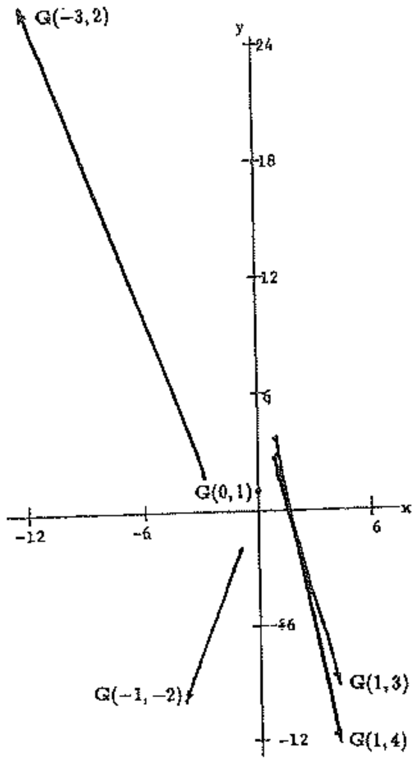
10. $\mathbf{V} = [3\cos(t) - 3t\sin(t)]\mathbf{j} - [3\sin(t) + 3t\cos(t)]\mathbf{k}$;
$\mathbf{a} = [-6\sin(t) - 3t\cos(t)]\mathbf{j} - [6\cos(t) - 3t\sin(t)]\mathbf{k}$; $v = 3\sqrt{1+t^2}$;
$\mathbf{T} = \frac{1}{\sqrt{1+t^2}}\{[\cos(t) - t\sin(t)]\mathbf{j} - [\sin(t) + t\cos(t)]\mathbf{k}\}$;
$\mathbf{N} = \frac{-1}{\sqrt{1+t^2}}\{[\sin(t) + t\cos(t)]\mathbf{j} + [\cos(t) - t\sin(t)]\mathbf{k}\}$; $\mathbf{B} = -\mathbf{i}$;
$a_T = \frac{3t}{\sqrt{1+t^2}}$; $a_N = \frac{(3t^2+6)^2}{\sqrt{1+t^2}}$; $\kappa = \frac{(3t^2+6)^2}{9(1+t^2)^{3/2}}$.

11. Suppose that $\mathbf{T}(s)$ = constant. Then $\mathbf{F}'(s) = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$ for some scalars c_1, c_2, c_3, where $\mathbf{F}(s)$ is the position vector expressed as a function of arc length s. But then $x'(s) = c_1, y'(s) = c_2, z'(s) = c_3$, so $x(s) = c_1 s + a_1, y(s) = c_2 s + a_2, z(s) = c_3 s + a_3$ and these are parametric equations of a line.

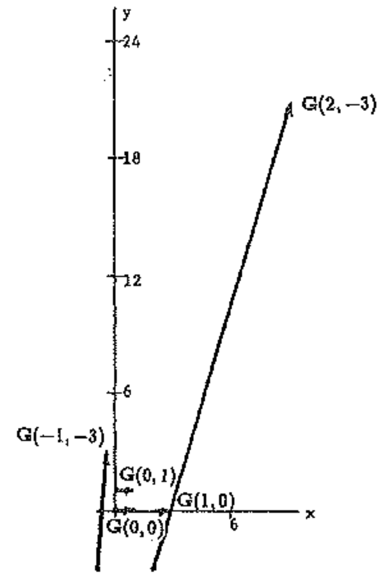
12. Yes, since from the first Frenet formula, if $\kappa = 0$, then $\frac{d\mathbf{T}}{ds} = 0$ and $\mathbf{T}(s)$ is a constant vector. By Problem 11, the curve is a straight line.

Section 12.3 Vector Fields and Streamlines

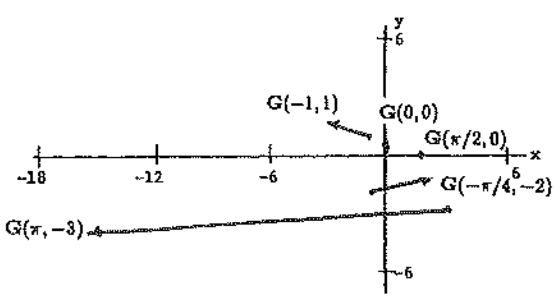
1. $\dfrac{\partial \mathbf{G}}{\partial x} = 3\mathbf{i} - 4y\mathbf{j};\ \dfrac{\partial \mathbf{G}}{\partial y} = -4x\mathbf{j}$



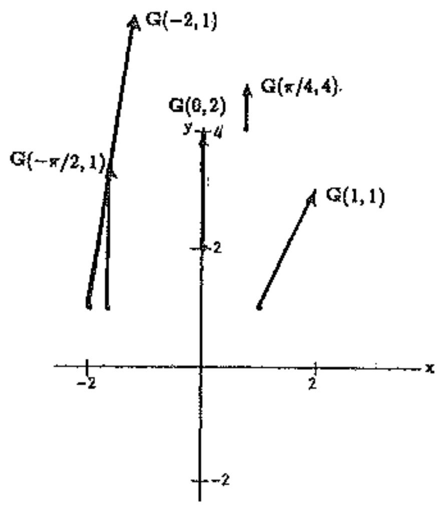
2. $\dfrac{\partial \mathbf{G}}{\partial x} = e^x\mathbf{i} - 4xy\mathbf{j};\ \dfrac{\partial \mathbf{G}}{\partial y} = -2x^2\mathbf{j}$



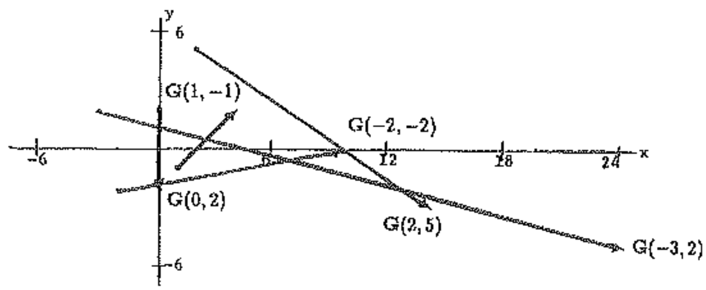
3. $\dfrac{\partial \mathbf{G}}{\partial x} = 2y\mathbf{i} - \sin(x)\mathbf{j};\ \dfrac{\partial \mathbf{G}}{\partial y} = 2x\mathbf{i}$



4. $\dfrac{\partial \mathbf{G}}{\partial x} = 2y\cos(2xy)\mathbf{i} + 2x\mathbf{j};\ \dfrac{\partial \mathbf{G}}{\partial y} = 2x\cos(2xy)\mathbf{i} + \mathbf{j}$



5. $\dfrac{\partial \mathbf{G}}{\partial x} = 6x\mathbf{i} + \mathbf{j}; \dfrac{\partial \mathbf{G}}{\partial y} = -2\mathbf{j}$



6. $\mathbf{F}_x = ye^{xy}\mathbf{i} - 4x\mathbf{j}; \mathbf{F}_y = xe^{xy}\mathbf{i} + \sinh(z+y)\mathbf{k}; \mathbf{F}_z = \sinh(z+y)\mathbf{k}$

7. $\mathbf{F}_x = -4z^2\sin(x)\mathbf{i} - 3x^2yz\mathbf{j} + 3x^2y\mathbf{k}; \mathbf{F}_y = -x^3z\mathbf{j} + x^3\mathbf{k}; \mathbf{F}_z = 8z\cos(x)\mathbf{i} - x^3y\mathbf{j}$

8. $\mathbf{F}_x = 3y^3\mathbf{i} + \dfrac{1}{x+y+z}\mathbf{j} + yz\sinh(xyz)\mathbf{k}; \mathbf{F}_y = 9xy^2\mathbf{i} + \dfrac{1}{x+y+z}\mathbf{j} + xz\sinh(xyz)\mathbf{k};$
$\mathbf{F}_z = \dfrac{1}{x+y+z}\mathbf{j} + xy\sinh(xyz)\mathbf{k}$

9. $\mathbf{F}_x = -yz^4\cos(xy)\mathbf{i} + 3y^4z\mathbf{j} - \sinh(z-x)\mathbf{k}; \mathbf{F}_y = -xz^4\cos(xy)\mathbf{i} + 12xy^3z\mathbf{j};$
$\mathbf{F}_z = -4z^3\sin(xy)\mathbf{i} + 3xy^4\mathbf{j} + \sinh(z-x)\mathbf{k}$

10. $\mathbf{F}_x = 14\mathbf{i} + 2x\mathbf{j} + 5y\mathbf{k}; \mathbf{F}_y = -2\mathbf{i} - 2y\mathbf{j} + 5x\mathbf{k}; \mathbf{F}_z = -2z\mathbf{j}$

11. Since $\mathbf{F} = f_1\mathbf{i} + f_2\mathbf{j} + f_3\mathbf{k} = \mathbf{i} - y^2\mathbf{j} + z\mathbf{k}$, the streamlines satisfy $\dfrac{dx}{1} = -\dfrac{dy}{y^2} = \dfrac{dz}{z}$. Integrate $dx = -\dfrac{dy}{y^2}$ to get $x = \dfrac{1}{y} + c_1$; integrate $dx = \dfrac{dz}{z}$ to get $x = \ln|z| + c_2$. In terms of x we can write the streamlines as $x = x, y = \dfrac{1}{x - c_1}, z = e^{(x-c_2)}$. The particular streamline through $(2,1,1)$ requires $1 = \dfrac{1}{2-c_1}$ and $1 = e^{(2-c_2)}$ so $c_1 = 1, c_2 = 2$ and we get $x = x, y = \dfrac{1}{x-1}, z = e^{(x-2)}$.

12. Streamlines satisfy $\dfrac{dx}{1} = \dfrac{dy}{-2} = \dfrac{dz}{1}$. Integration gives $y = -2x + c_1, z = x + c_2$. To pass through $(0,1,1)$ requires $c_1 = 1, c_2 = 1$ to give $x = x, y = -2x + 1, z = x + 1$.

13. Streamlines satisfy $xdx = \dfrac{dy}{e^x} = \dfrac{dz}{-1}$. Integrate $xdx = \dfrac{dy}{e^x}$ as $xe^x dx = dy$ to get $y = xe^x - e^x + c_1$, from $xdx = -dz$ we get $x^2 = -2z + c_2$. To pass through (2, 0, 4) we need $e^2 + c_1 = 0$ and $4 = -8 + c_2$ so $c_1 = -e^2$ and $c_2 = 12$. Using x as parameter we have $x = x, y = xe^x - e^x - e^2, z = \dfrac{1}{2}(12 - x^2)$.

14. Streamlines satisfy $\dfrac{dx}{\cos(y)} = \dfrac{dy}{\sin(x)}$ and $dz = 0$. Integrate $\sin(x)dx = \cos(y)dy$ to get $-\cos(x) + c_1 = \sin(y), z = c_2$. To pass through $(\pi/2, 0, -4)$ we need $c_1 = 0$ and $c_2 = -4$. Using x as parameter we have $x = x, y = \sin^{-1}[-\cos(x)], z = -4$.

15. Streamlines satisfy $dx = 0$ and $\dfrac{dy}{2e^z} = -\dfrac{dz}{\cos(y)}$. Integration gives $x = c_1$ and from $\cos(y)dy = -2e^z dz$ we get $\sin(y) = c_2 - 2e^z$. To pass through $(3, \pi/4, 0)$ we need $c_1 = 3$ and

$c_2 = \frac{\sqrt{2}}{2} + 2$. With y as parameter we have $x = 3, y = y, z = \ln\left[\frac{\sqrt{2}}{4} + 1 - \frac{1}{2}\sin(y)\right]$.

16. Streamlines satisfy $\frac{dx}{3x^2} = -\frac{dy}{y} = \frac{dz}{z^3}$. Integration of $\frac{dx}{3x^2} = -\frac{dy}{y}$ gives $\frac{1}{x} = 3\ln|y| + c_1$, and of $-\frac{dy}{y} = \frac{dz}{z^3}$ gives $-\ln|y| = -\frac{1}{2z^2} + c_2$. To pass through (2, 1, 6) we need $c_1 = \frac{1}{2}$ and $c_2 = \frac{1}{72}$. With y as parameter we have $x = \frac{2}{1 + 6\ln(y)}, y = y, z = \frac{6}{\sqrt{1 + 72\ln(y)}}$.

17. Any non-zero constant vector field will have streamlines which are straight lines. If $\mathbf{F} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ it is easy to produce the streamlines $x = at + x_0, y = bt + y_0, z = ct + z_0$, which are lines.

18. Circular streamlines about the origin in the xy-plane could be written $x^2 + y^2 = r^2$ so $xdx + ydy = 0$, or $\frac{dx}{y} = -\frac{dx}{x}, dz = 0$. A vector field having these streamlines is $\mathbf{F} = y\mathbf{i} - x\mathbf{j} + 0\mathbf{k}$.

19. This is impossible. If $\mathbf{F} = f_1(x,y,z)\mathbf{i} + f_2(x,y,z)\mathbf{j} + f_3(x,y,z)\mathbf{k}$ has streamlines $(x(t), y(t), z(t))$ lying only in the x,y-plane then $x'(t) = \alpha f_1, y'(t) = \alpha f_2, z'(t) = \alpha f_3$. But by assumption, $z'(t) = 0$, so $f_3(x,y,z) = 0$ and $z =$ constant α. Thus $\mathbf{F} = f_1(x,y,\alpha)\mathbf{i} + f_2(x,y,\alpha)\mathbf{j}$ is a function of only two variables.

Section 12.4 The Gradient Field and Directional Derivatives

1. $\nabla\varphi(x,y,z) = \frac{\partial}{\partial x}(xyz)\mathbf{i} + \frac{\partial}{\partial y}(xyz)\mathbf{j} + \frac{\partial}{\partial z}(xyz)\mathbf{k} = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}; \nabla\varphi(1,1,1) = \mathbf{i} + \mathbf{j} + \mathbf{k}$;
The maximum value of $D_{\mathbf{u}}\varphi(1,1,1) = \|\nabla\varphi(1,1,1)\| = \sqrt{3}$, and the minimum value of $D_{\mathbf{u}}\varphi(1,1,1) = -\|\nabla\varphi(x,1,1)\| = -\sqrt{3}$.

2. $\nabla\varphi(x,y,z) = (2x - z\cos(zx))\mathbf{i} + x^2\mathbf{j} - x\cos(xz)\mathbf{k}; \nabla\varphi(1,-1,\pi/4) = (2 - \frac{\sqrt{2}\pi}{8})\mathbf{i} + \mathbf{j} - \frac{\sqrt{2}}{2}\mathbf{k}$;
$D_{\mathbf{u}}\varphi(1,-1,\pi/4)_{\max} = \|\nabla\varphi(1,-1,\pi/4)\| = (176 + \pi - 16\sqrt{2}\pi)/32$,
$D_{\mathbf{u}}\varphi(1,-1,\pi/4)_{\min} = -\|\nabla\varphi(1,-1,\pi/4)\| = -(176 + \pi - 16\sqrt{2}\pi)/32$.

3. $\nabla\varphi(x,y,z) = (2y + e^z)\mathbf{i} + 2x\mathbf{j} + xe^z\mathbf{k}; \nabla\varphi(-2,1,6) = (2 + e^6)\mathbf{i} - 4\mathbf{j} - 2e^6\mathbf{k}$;
$D_{\mathbf{u}}\varphi(-2,1,6)_{\max} = \|\nabla\varphi(-2,1,6)\| = \sqrt{20 + 4e^6 + 5e^{12}}; D_{\mathbf{u}}\varphi(-2,1,6)_{\min} = -\|\nabla\varphi(-2,1,6)\| = -\sqrt{20 + 4e^6 + 5e^{12}}$

4. $\nabla\varphi(x,y,z) = -yz\sin(xyz)\mathbf{i} - xz\sin(xyz)\mathbf{j} - xy\sin(xyz)\mathbf{k}; \nabla\varphi(-1,1,\pi/2) = \frac{\pi}{2}\mathbf{i} - \frac{\pi}{2}\mathbf{j} - \mathbf{k}$,

$D_{\mathbf{u}}\varphi(-1,1,\pi/2)_{\max} = \|\nabla\varphi(-1,1,\pi/2)\| = \sqrt{\frac{\pi^2}{2} + 1}$

$D_{\mathbf{u}}\varphi(-1,1,\pi/2)_{\min} = -\|\nabla\varphi(-1,1,\pi/2)\| = -\sqrt{\frac{\pi^2}{2} + 1}$

5. $\nabla\varphi(x,y,z) = 2y\sinh(2xy)\mathbf{i} + 2x\sinh(2xy)\mathbf{j} - \cosh(z)\mathbf{k}; \nabla\varphi(0,1,1) = -\cosh(1)\mathbf{k}$;
$D_{\mathbf{u}}\varphi(0,1,1)_{\max} = \|\nabla\varphi(0,1,1)\| = \cosh(1)$
$D_{\mathbf{u}}\varphi(0,1,1)_{\min} = -\|\nabla\varphi(0,1,1)\| = -\cosh(1)$

6. $\nabla\varphi(x,y,z) = \dfrac{x}{\sqrt{x^2+y^2+z^2}}\mathbf{i} + \dfrac{y}{\sqrt{x^2+y^2+z^2}}\mathbf{j} + \dfrac{z}{\sqrt{x^2+y^2+z^2}}\mathbf{k}$; $\nabla\varphi(2,2,2) = \dfrac{1}{\sqrt{3}}\mathbf{i} + \dfrac{1}{\sqrt{3}}\mathbf{j} + \dfrac{1}{\sqrt{3}}\mathbf{k}$,
$D_{\mathbf{u}}\varphi(2,2,2)_{\max} = \|\nabla\varphi(2,2,2)\| = 1$
$D_{\mathbf{u}}\varphi(2,2,2)_{\min} = -\|\nabla\varphi(2,2,2)\| = -1$

7. $D_{\mathbf{u}}\varphi(x,y,z) = \nabla\varphi(x,y,z)\cdot\mathbf{u} = [(8y^2-z)\mathbf{i}+16xy\mathbf{j}-x\mathbf{k}]\cdot\left[\dfrac{\mathbf{i}+\mathbf{j}+\mathbf{k}}{\sqrt{3}}\right] = \dfrac{1}{\sqrt{3}}(8y^2-z+16xy-x)$

8. $D_{\mathbf{u}}\varphi(x,y,z) = \nabla\varphi(x,y,z)\cdot\mathbf{u} = [-\sin(x-y)\mathbf{i}+\sin(x-y)\mathbf{j}+e^z\mathbf{k}]\cdot\left[\dfrac{\mathbf{i}-\mathbf{j}+2\mathbf{k}}{\sqrt{6}}\right] = \dfrac{1}{\sqrt{6}}(-2\sin(x-y)+2e^z)$

9. $D_{\mathbf{u}}\varphi(x,y,z) = \nabla\varphi(x,y,z)\cdot\mathbf{u} = [2xyz^3\mathbf{i}+x^2z^3\mathbf{j}+3x^2yz^2\mathbf{k}]\cdot\left[\dfrac{2\mathbf{j}+\mathbf{k}}{\sqrt{5}}\right] = \dfrac{1}{\sqrt{5}}(2x^2z^3+3x^2yz^2)$

10. $D_{\mathbf{u}}\varphi(x,y,z) = \nabla\varphi(x,y,z)\cdot\mathbf{u} = [(z+y)\mathbf{i}+(z+x)\mathbf{j}+(y+x)\mathbf{k}]\cdot\left[\dfrac{\mathbf{i}-4\mathbf{k}}{\sqrt{17}}\right] = \dfrac{1}{\sqrt{17}}(z-3y-4x)$

11. With $\varphi(x,y,z) = x^2+y^2+z^2$, the surface is the level surface $\varphi(x,y,z) = 4$ so a normal vector to the surface at $(1,1,\sqrt{2})$ is $\mathbf{N} = \nabla\varphi(1,1,\sqrt{2}) = 2\mathbf{i}+2\mathbf{j}+2\sqrt{2}\mathbf{k}$. The tangent plane to the surface at $(1,1,\sqrt{2})$ is given by $2(x-1)+2(y-1)+2\sqrt{2}(z-\sqrt{2}) = 0$ or $x+y+\sqrt{2}z = 4$. A normal line to the surface is given by the parametric equations $x = 1+2t, y = 1+2t, z = \sqrt{2}+2\sqrt{2}t, -\infty < t < \infty$.

12. With the surface written $x^2+y-z = 0$, a normal vector at $(-1,1,2)$ is $\mathbf{N} = \nabla(x^2+y-z)_{(-1,1,2)} = -2\mathbf{i}+\mathbf{j}-\mathbf{k}$. The tangent plane is $-2x+y-z = 1$, and normal line is $x = -1-2t, y = 1+t, z = 2-t, -\infty < t < \infty$.

13. $\mathbf{N} = \nabla(x^2-y^2-z^2)_{(1,1,0)} = 2\mathbf{i}-2\mathbf{j}$; tangent plane is $2x-2y = 0$ or $y = x$; normal line is $x = 1+2t, y = 1-2t, z = 0, -\infty < t < \infty$.

14. $\mathbf{N} = \nabla(x^2-y^2+z^2)_{(1,1,0)} = 2\mathbf{i}-2\mathbf{j}$; tangent plane is $2x-2y = 0$ or $y = x$; normal line is $x = 1+2t, y = 1-2t, z = 0, -\infty < t < \infty$

15. $\mathbf{N} = \nabla(2x-\cos(xyz))_{(1,\pi,1)} = 2\mathbf{i}$; tangent plane is $x = 1$; normal line is $x = 1+2t, y = \pi, z = 1, -\infty < t < \infty$

16. $\mathbf{N} = \nabla(3x^4+3y^4+6z^4)_{(1,1,1)} = 12\mathbf{i}+12\mathbf{j}+24\mathbf{k}$; tangent plane is $12x+12y+24z = 48$ or $x+y+2z = 4$; normal line is $x = 1+12t, y = 1+12t, z = 1+24t, -\infty < t < \infty$

17. Write the surfaces as level surfaces in the form $3x^2+2y^2-z = 0$ and $-2x+7y^2-z = 0$ respectively. We check that the point given, $(1,1,5)$, lies on both surfaces since $3(1)^2+2(1)^2-5 = 0$ and $-2(1)+7(1)^2-5 = 0$. At the point $(1,1,5)$ we can find normals to the respective surfaces as $\mathbf{N}_1 = \nabla(3x^2+2y^2-z)_{(1,1,5)} = 6\mathbf{i}+4\mathbf{j}-\mathbf{k}$ and $\mathbf{N}_2 = \nabla(-2x+7y^2-z)_{(1,1,5)} = -2\mathbf{i}+14\mathbf{j}-\mathbf{k}$. The angle between the surfaces at $(1,1,5)$ will be the angle between $\mathbf{N}_1$ and $\mathbf{N}_2$. We get

$$\cos(\theta) = \frac{\mathbf{N}_1\cdot\mathbf{N}_2}{\|\mathbf{N}_1\|\|\mathbf{N}_2\|} = \frac{-12+56+1}{\sqrt{56}\sqrt{201}} = \frac{45}{\sqrt{10,653}} \text{ so } \cos^{-1}\left(\frac{45}{\sqrt{10,653}}\right) \approx 1.11966 \text{ radians.}$$

18. Normals to the two surfaces at $(1,\sqrt{2},1)$ are given by $\mathbf{N}_1 = \nabla(x^2+y^2+z^2)_{(1,\sqrt{2},1)} = 2\mathbf{i}+2\sqrt{2}\mathbf{j}+2\mathbf{k}$, and $\mathbf{N}_2 = \nabla(z^2+x^2)_{(1,\sqrt{2},1)} = 2\mathbf{i}+2\mathbf{k}$. The angle between the surfaces is

the angle between $\mathbf{N}_1$ and $\mathbf{N}_2$ for which we have $\cos(\theta) = \dfrac{\mathbf{N}_1 \cdot \mathbf{N}_2}{\|\mathbf{N}_1\|\|\mathbf{N}_2\|} = \dfrac{4+4}{(4)\sqrt{8}} = \dfrac{1}{\sqrt{2}}$, so $\theta = \cos^{-1}\left(\dfrac{1}{\sqrt{2}}\right) = \dfrac{\pi}{4}$ radians.

19. Normals to the two surfaces at $(2, 2, \sqrt{8})$ are given by $\mathbf{N}_1 = \nabla(\sqrt{x^2+y^2} - z)_{(2,2,\sqrt{8})} = \dfrac{1}{\sqrt{2}}\mathbf{i} + \dfrac{1}{\sqrt{2}}\mathbf{j} - \mathbf{k}$, and $\mathbf{N}_2 = \nabla(x^2+y^2)_{(2,2,\sqrt{8})} = 4\mathbf{i} + 4\mathbf{j}$. The angle between the surfaces is the angle between $\mathbf{N}_1$ and $\mathbf{N}_2$ for which we have $\cos(\theta) = \dfrac{\mathbf{N}_1 \cdot \mathbf{N}_2}{\|\mathbf{N}_1\|\|\mathbf{N}_2\|} = \dfrac{4\sqrt{2}}{\sqrt{2}(4\sqrt{2})} = \dfrac{1}{\sqrt{2}}$, so $\theta = \cos^{-1}\left(\dfrac{1}{\sqrt{2}}\right) = \dfrac{\pi}{4}$ radians.

20. Normals to the two surfaces at $(2, 2, 1)$ are given by $\mathbf{N}_1 = \nabla\left(\dfrac{1}{2}x^2 + \dfrac{1}{2}y^2 + z^2\right)_{(2,2,1)} = 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$, and $\mathbf{N}_2 = \nabla(x+y+z)_{(2,2,1)} = \mathbf{i} + \mathbf{j} + \mathbf{k}$. The angle between the surfaces is the angle between $\mathbf{N}_1$ and $\mathbf{N}_2$ for which we have $\cos(\theta) = \dfrac{\mathbf{N}_1 \cdot \mathbf{N}_2}{\|\mathbf{N}_1\|\|\mathbf{N}_2\|} = \dfrac{6}{2\sqrt{3}\sqrt{3}} = 1$, so $\theta = \cos^{-1}(1) = 0$. Note that the two surfaces are tangent at the point $(2, 2, 1)$ since their respective normals are parallel.

21. Since $\nabla\varphi = \mathbf{i} + \mathbf{k}$ for all (x, y, z) the normal to the surface $\varphi(x, y, z) = K$ is the constant vector $\mathbf{N} = \mathbf{i} + \mathbf{k}$, thus the surface must be $x + z = K$, a plane. The streamlines of the vector field $\nabla\varphi = \mathbf{i} + \mathbf{k}$ are given by solutions of $\dfrac{dx}{1} = \dfrac{dz}{1}$ and $dy = 0$. We easily find these to be $x = z + c_1, y = c_2$, or using t as parameter $x = t + c_1, y = c_2, z = t, -\infty < t < \infty$. These streamlines are lines in three space which are orthogonal to the surface $x + z = K$.

Section 12.5 Divergence and Curl

1. $\nabla \cdot \mathbf{F} = \dfrac{\partial}{\partial x}(x) + \dfrac{\partial}{\partial y}(y) + \dfrac{\partial}{\partial z}(2z) = 4;$

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & 2z \end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0};$$

$\nabla \cdot (\nabla \times \mathbf{F}) = 0.$

2. $\nabla \cdot \mathbf{F} = \dfrac{\partial}{\partial x}(0) + \dfrac{\partial}{\partial y}(\sinh(xyz)) + \dfrac{\partial}{\partial z}(0) = xz\cosh(xyz);$

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & \sinh(xyz) & 0 \end{vmatrix} = -xy\cosh(xyz)\mathbf{i} + yz\cosh(xyz)\mathbf{k};$$

$\nabla \cdot (\nabla \times \mathbf{F}) = \dfrac{\partial}{\partial x}(-xy\cosh(xyz)) + \dfrac{\partial}{\partial z}(yz\cosh(xyz)) = \cosh(xyz)(-y+y) + \sinh(xyz)(-xy^2z + xy^2z) = 0.$

3. $\nabla \cdot \mathbf{F} = \dfrac{\partial}{\partial x}(2xy) + \dfrac{\partial}{\partial y}(xe^y) + \dfrac{\partial}{\partial z}(2z) = 2y + xe^y + 2;$

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy & xe^y & 2z \end{vmatrix} = (e^y - 2x)\mathbf{k};$$

$$\nabla \cdot (\nabla \times \mathbf{F}) = \frac{\partial}{\partial z}(e^y - 2x) = 0.$$

4. $\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(\sinh(x)) + \frac{\partial}{\partial y}(\cosh(xyz)) - \frac{\partial}{\partial z}(x+y+z) = \cosh(x) + xz\sinh(xyz) - 1;$

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sinh(x) & \cosh(xyz) & -(x+y+z) \end{vmatrix} = [-1 - xy\sinh(xyz)]\mathbf{i} - \mathbf{j} + yz\sinh(xyz)\mathbf{k};$$

$$\nabla \cdot (\nabla \times \mathbf{F}) = \frac{\partial}{\partial x}(-1 - xy\sinh(xyz)) + \frac{\partial}{\partial z}(yz\sinh(xyz)) =$$
$$(-y+y)\sinh(xyz) + \cosh(xyz)(-xy^2z + xy^2z) = 0.$$

5. $\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(z^2) = 2(x+y+z);$

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & y^2 & z^2 \end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0};$$
$$\nabla \cdot (\nabla \times \mathbf{F}) = 0.$$

6. $\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(\sinh(x-z)) + \frac{\partial}{\partial y}(2y) + \frac{\partial}{\partial z}(z-y^2) = \cosh(x-z) + 2 + 1 = \cosh(x-z) + 3;$

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sinh(x-z) & 2y & z-y^2 \end{vmatrix} = -2y\mathbf{i} - \cosh(x-z)\mathbf{j};$$
$$\nabla \cdot (\nabla \times \mathbf{F}) = \frac{\partial}{\partial x}(-2y) + \frac{\partial}{\partial y}(-\cosh(x-z)) + \frac{\partial}{\partial z}(0) = 0.$$

7. $\nabla\phi = \nabla(x - y + 2z^2) = \mathbf{i} - \mathbf{j} + 4z\mathbf{k}; \nabla \times (\nabla\phi) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 1 & -1 & 4z \end{vmatrix} = \mathbf{0}.$

8. $\nabla\phi = \nabla(18xyz + e^x) = (18yz + e^x)\mathbf{i} + 18xz\mathbf{j} + 18xy\mathbf{k};$

$$\nabla \times (\nabla\phi) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 18yz + e^x & 18xz & 18xy \end{vmatrix} = \mathbf{i}(18x - 18x) + \mathbf{j}(18y - 18y) + \mathbf{k}(18z - 18z) = \mathbf{0}.$$

9. $\nabla\phi = \nabla(-2x^3yz^2) = -6x^2yz^2\mathbf{i} - 2x^3z^2\mathbf{j} - 4x^3yz\mathbf{k};$

$$\nabla \times (\nabla\phi) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -6x^2yz^2 & -2x^3z^2 & -4x^3yz \end{vmatrix} =$$
$$(-4x^3z + 4x^3z)\mathbf{i} + (-12x^2yz + 12x^2yz)\mathbf{j} + (-6x^2z^2 + 6x^2z^2)\mathbf{k} = \mathbf{0}.$$

10. $\nabla\phi = \nabla(\sin(xz)) = z\cos(xz)\mathbf{i} + x\cos(xz)\mathbf{k};$

$$\nabla \times (\nabla\phi) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z\cos(xz) & 0 & x\cos(xz) \end{vmatrix} =$$
$$0\mathbf{i} + [\cos(xz) - xz\sin(xz) - \cos(xz) + xz\sin(xz)]\mathbf{j} + 0\mathbf{k} = \mathbf{0}.$$

11. $\nabla\phi = \nabla(x\cos(x+y+z)) = [\cos(x+y+z) - x\sin(x+y+z)]\mathbf{i} - x\sin(x+y+z)\mathbf{j} - x\sin(x+y+z)\mathbf{k};$

$$\nabla \times (\nabla\phi) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \cos(x+y+z) - x\sin(x+y+z) & -x\sin(x+y+z) & -x\sin(x+y+z) \end{vmatrix}$$

$= [-x\cos(x+y+z) + x\cos(x+y+z)]\mathbf{i}$
$+ [-\sin(x+y+z) - x\cos(x+y+z) + \sin(x+y+z) + x\cos(x+y+z)]\mathbf{j}$
$+ [-\sin(x+y+z) - x\cos(x+y+z) + \sin(x+y+z) + x\cos(x+y+z)]\mathbf{k} = 0.$

12. $\nabla\phi = \nabla(e^{x+y+z}) = e^{x+y+z}(\mathbf{i}+\mathbf{j}+\mathbf{k})$;

$$\nabla \times (\nabla\phi) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{x+y+z} & e^{x+y+z} & e^{x+y+z} \end{vmatrix} =$$

$[e^{x+y+z} - e^{x+y+z}]\mathbf{i} + [e^{x+y+z} - e^{x+y+z}]\mathbf{j} + [e^{x+y+z} - e^{x+y+z}]\mathbf{k} = 0.$

13. Let $\mathbf{F} = f_1(x,y,z)\mathbf{i} + f_2(x,y,z)\mathbf{j} + f_3(x,y,z)\mathbf{k}$. Then

$$\nabla \cdot (\phi\mathbf{F}) = \nabla \cdot (\phi f_1\mathbf{i} + \phi f_2\mathbf{j} + \phi f_3\mathbf{k}) = \frac{\partial}{\partial x}(\phi f_1) + \frac{\partial}{\partial y}(\phi f_2)$$

$$+\frac{\partial}{\partial z}(\phi f_3) = \left[\frac{\partial\phi}{\partial x}f_1 + \frac{\partial\phi}{\partial y}f_2 + \frac{\partial\phi}{\partial z}f_3\right] + \phi\left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\right)$$

We now recognize this first term as $\nabla\phi \cdot \mathbf{F}$, and the second as $\phi(\nabla \cdot \mathbf{F})$, thus $\nabla \cdot (\phi\mathbf{F}) = \nabla\phi \cdot \mathbf{F} + \phi(\nabla \cdot \mathbf{F})$.

Also $\nabla \times (\phi\mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \phi f_1 & \phi f_2 & \phi f_3 \end{vmatrix} =$

$$\left[\frac{\partial}{\partial y}(\phi f_3) - \frac{\partial}{\partial z}(\phi f_2)\right]\mathbf{i} + \left[\frac{\partial}{\partial z}(\phi f_1) - \frac{\partial}{\partial x}(\phi f_3)\right]\mathbf{j} + \left[\frac{\partial}{\partial x}(\phi f_2) - \frac{\partial}{\partial y}(\phi f_1)\right]\mathbf{k}.$$

To identify the above expression we apply the product rule to each term above and regroup as $\nabla \times (\phi\mathbf{F}) =$

$$[\phi_y f_3 - \phi_z f_2]\mathbf{i} + [\phi_z f_1 - \phi_x f_3]\mathbf{j} + [\phi_x f_2 - \phi_y f_1]\mathbf{k}$$

$$+\phi\left\{\left[\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right]\mathbf{i} + \left[\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right]\mathbf{j} + \left[\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right]\mathbf{k}\right\} = \nabla\phi \times \mathbf{F} + \phi(\nabla \times \mathbf{F}).$$

14. Since the expressions on each side of this identity are vectors, we establish the equality of the $\mathbf{i}, \mathbf{j}, \mathbf{k}$ components of the vectors on the left hand and right hand sides of the identity. Let $\mathbf{F} = f_1\mathbf{i} + f_2\mathbf{j} + f_3\mathbf{k}$ and $\mathbf{G} = g_1\mathbf{i} + g_2\mathbf{j} + g_3\mathbf{k}$, then $\mathbf{F} \cdot \mathbf{G} = f_1g_1 + f_2g_2 + f_3g_3$ and the $\mathbf{i}$ component on the left hand side is $\frac{\partial}{\partial x}[f_1g_1 + f_2g_2 + f_3g_3]$. For the right side, we first compute $\mathbf{F} \times (\nabla \times \mathbf{G})$ and $\mathbf{G} \times (\nabla \times \mathbf{F})$ so we can identify the $\mathbf{i}$ component of each. We have

$$\mathbf{F} \times (\nabla \times \mathbf{G}) = \mathbf{F} \times \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ g_1 & g_2 & g_3 \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ f_1 & f_2 & f_3 \\ \left(\frac{\partial g_3}{\partial y} - \frac{\partial g_2}{\partial z}\right) & \left(\frac{\partial g_1}{\partial z} - \frac{\partial g_3}{\partial x}\right) & \left(\frac{\partial g_2}{\partial x} - \frac{\partial g_1}{\partial y}\right) \end{vmatrix}.$$

From this we easily get the $\mathbf{i}$ component of

$$\mathbf{F} \times (\nabla \times \mathbf{G}) = \left[f_2\left(\frac{\partial g_2}{\partial x} - \frac{\partial g_1}{\partial y}\right) - f_3\left(\frac{\partial g_1}{\partial z} - \frac{\partial g_3}{\partial x}\right)\right],$$

and the $\mathbf{i}$ component of $\mathbf{G} \times (\nabla \times \mathbf{F})$ is found by interchanging the letters f and g above. Thus the $\mathbf{i}$ component on the right side is

$$\left[f_1\frac{\partial g_1}{\partial x} + f_2\frac{\partial g_1}{\partial y} + f_3\frac{\partial g_1}{\partial z}\right] + \left[g_1\frac{\partial f_1}{\partial x} + g_2\frac{\partial f_1}{\partial y} + g_3\frac{\partial f_1}{\partial z}\right] +$$

$$\left[f_2\left(\frac{\partial g_2}{\partial x} - \frac{\partial g_1}{\partial y}\right) - f_3\left(\frac{\partial g_1}{\partial z} - \frac{\partial g_3}{\partial x}\right)\right] + \left[g_2\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) - g_3\left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)\right]$$

$$= f_1\frac{\partial g_1}{\partial x} + g_1\frac{\partial f_1}{\partial x} + f_2\frac{\partial g_2}{\partial x} + g_2\frac{\partial f_2}{\partial x} + f_3\frac{\partial g_3}{\partial x} + g_3\frac{\partial f_3}{\partial x}$$

where the other eight terms cancel. But this expression is exactly $\dfrac{\partial}{\partial x}[f_1g_1 + f_2g_2 + f_3g_3]$. Thus the $\mathbf{i}$ components on the left and right are equal. Similarly the $\mathbf{j}, \mathbf{k}$ components are equal.

15. Let $\mathbf{F} = f_1\mathbf{i} + f_2\mathbf{j} + f_3\mathbf{k}$, and $\mathbf{G} = g_1\mathbf{i} + g_2\mathbf{j} + g_3\mathbf{k}$. Then $\mathbf{F} \times \mathbf{G} = [f_2g_3 - f_3g_2]\mathbf{i} + [f_3g_1 - f_1g_3]\mathbf{j} + [f_1g_2 - f_2g_1]\mathbf{k}$, and $\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \dfrac{\partial}{\partial x}[f_2g_3 - f_3g_2] + \dfrac{\partial}{\partial y}[f_3g_1 - f_1g_3] + \dfrac{\partial}{\partial z}[f_1g_2 - f_2g_1] = g_1\left[\dfrac{\partial f_3}{\partial y} - \dfrac{\partial f_2}{\partial z}\right] + g_2\left[\dfrac{\partial f_1}{\partial z} - \dfrac{\partial f_3}{\partial x}\right] + g_3\left[\dfrac{\partial f_2}{\partial x} - \dfrac{\partial f_1}{\partial y}\right] + f_1\left[\dfrac{\partial g_2}{\partial z} - \dfrac{\partial g_3}{\partial y}\right] + f_2\left[\dfrac{\partial g_3}{\partial x} - \dfrac{\partial g_1}{\partial z}\right] + f_3\left[\dfrac{\partial g_1}{\partial y} - \dfrac{\partial g_2}{\partial x}\right] = \mathbf{G} \cdot (\nabla \times \mathbf{F}) - \mathbf{F} \cdot (\nabla \times \mathbf{G})$.

16. To prove the assertion here we apply Problem 15 to $\nabla \cdot (\nabla\phi \times \nabla\psi)$ where $\nabla\phi = \mathbf{F}$ and $\nabla\psi = \mathbf{G}$ to get $\nabla \cdot (\nabla\phi \times \nabla\psi) = \nabla\psi \cdot (\nabla \times \nabla\phi) - \nabla\phi(\nabla \times \nabla\psi)$. By Theorem 1, $\nabla \times \nabla\phi = 0$ and $\nabla \times \nabla\psi = 0$ which gives the conclusion $\nabla \cdot (\nabla\phi \times \nabla\psi) = \nabla\psi \cdot 0 + \nabla\psi \cdot 0 = 0$. The application of Theorem 1 here assumes the continuity (and hence equality) of mixed second order partials.

Chapter Thirteen - Vector Integral Calculus

Section 13.1 Line Integrals

1. On the curve $C, x = t, y = t, z = t^3$, so $\mathbf{F} = t\mathbf{i} - \mathbf{j} + t^3\mathbf{k}, d\mathbf{R} = (\mathbf{i} + \mathbf{j} + 3t^2\mathbf{k})dt$, so $\mathbf{F} \cdot d\mathbf{R} = (t - 1 + 3t^5)dt$ and $\int_C \mathbf{F} \cdot d\mathbf{R} = \int_0^1 (t - 1 + 3t^5)dt = 0$.

2. On the curve $C, \mathbf{F} = 4t^2\mathbf{i}, d\mathbf{R} = -2tdt\mathbf{i}$, so $\int_C \mathbf{F} \cdot d\mathbf{R} = \int_0^1 -8t^3 dt = -2$.

3. For the curve $C, x = t, y = t, z = t^2$, so $ds = \sqrt{1 + 1 + (2t)^2}dt = \sqrt{2 + 4t^2}dt$, and $\int_C (x + y)ds = \int_0^2 (2t)\sqrt{2 + 4t^2}dt = \frac{1}{6}(2 + 4t^2)^{3/2}\Big|_0^2 = \frac{26\sqrt{2}}{3}$

4. Parametric equations for C are $x = t, y = 1 + t, z = 1 - 2t, 0 \le t \le 1$. Then $\int_c x^2 z dz = \int_0^1 t^2(1 - 2t)(-2)dt = \frac{1}{3}$.

5. On the curve $C, x = t, y = -t^2, z = 1$, so $\mathbf{F} = \cos(t)\mathbf{i} + t^2\mathbf{j} + t\mathbf{k}, d\mathbf{R} = (\mathbf{i} - 2t\mathbf{j})dt$ and $\int_C \mathbf{F} \cdot d\mathbf{R} = \int_0^3 [\cos(t) - 2t^3]dt = \sin(3) - \frac{81}{2}$

6. For the curve $C, x = t, y = t, z = 2t; 1 \le t \le 2$, so $ds = \sqrt{6}dt$, and $\int_C 4xyds = \int_1^2 4t^2\sqrt{6}dt = \frac{28\sqrt{6}}{3}$.

7. The circle C can be parametrized as $x = 2\cos(t), y = 2\sin(t), z = 0; 0 \le t \le 2\pi$, so $\int_C \mathbf{F} \cdot d\mathbf{R} = \int_0^{2\pi} 2\cos(t)(-2\sin(t)) + 2\sin(t)(2\cos(t))dt = 0$

8. Parametric equations for C are $x = 1, y = t, z = t^2; 0 \le t \le 2$. Then $ds = \sqrt{1 + 4t^2}dt$, and $\int_C yzds = \int_0^2 t^3\sqrt{1 + 4t^2}dt$. This integral can be integrated by parts to give

$$\int_0^2 t\sqrt{1 + 4t^2}dt = \frac{1}{120}\left[10t^2(1 + 4t^2)^{3/2} - (1 + 4t^2)^{5/2}\right]\Big|_0^2 = \frac{1}{120}[391\sqrt{17} + 1].$$

9. With z as parameter we get, $\int_C -xyzdz = \int_4^9 -(1)\sqrt{z}zdz = \frac{-2}{5}\left(z^{5/2}\right)\Big|_4^9 = -\frac{422}{5}$

10. $\int_C xzdy = \int_1^3 (t)(-4t^2)dt = -t^2\Big|_1^4 = -80$.

11. For the curve $C, ds = \sqrt{(4t)^2 + (4t)^2}dt = 4\sqrt{2}t, dt$, so $\int_C 8z^2 ds = \int_1^2 8 \cdot 4\sqrt{2}tdt = 16\sqrt{2}t^2\Big|_1^2 = 48\sqrt{2}$.

12. On the curve $C, \mathbf{F} \cdot d\mathbf{R} = (\mathbf{i} - \cos(t)\mathbf{j} + \mathbf{k}) \cdot (-\sin(t)\mathbf{i} - \cos(t)\mathbf{j} + \mathbf{k})dt = (-\sin(t) + \cos^2(t) + 1)dt$, so $\int_C \mathbf{F} \cdot d\mathbf{R} = \int_0^\pi (-\sin(t) + \cos^2(t) + 1)dt = -2 + \frac{\pi}{2} + \pi = \frac{3\pi}{2} - 2$.

13. On the curve $C, \mathbf{F} \cdot d\mathbf{R} = (8e^{2t}\mathbf{j}) \cdot (e^t\mathbf{i} - 2t\mathbf{j} + \mathbf{k})dt = -16te^{2t}dt$, so $\int_C \mathbf{F} \cdot d\mathbf{R} = \int_1^2 -16te^{2t}dt = \left[-8te^{2t} + 4e^{2t}\right]\Big|_1^2 = -12e^4 + 4e^2$.

14. $\int_C xdy - ydx = \int_0^3 [(2t)2 - (2t)(2)]dt = 0.$

15. For the curve $C, ds = \sqrt{1+4+9}dt = \sqrt{14}dt$, so $\int_C \sin(x)ds = \int_1^3 \sqrt{14}\sin(t)dt = \sqrt{14}[\cos(1) - \cos(3)]$

16. We parametrize the wire as $x = y = z = t, 0 \le t \le 3$. Then the mass of the wire is given by $M = \int_C \delta(x,y,z)ds = \int_0^3 (3t)\sqrt{3}dt = \frac{27\sqrt{3}}{2}$ grams. Because the density function and the position of the wire are symmetric in the first octant, the three Moments, M_x, M_y, M_z will be equal and hence $\overline{x} = \overline{y} = \overline{z}$. Computing $M_x = \int_C x\delta(x,y,z)ds = \int_0^3 t(3t)\sqrt{3}dt = 27\sqrt{3}$ gram cm, we find $\overline{x} = \overline{y} = \overline{z} = \frac{M_x}{M} = 2$, so centroid $= (2,2,2)$

17. First parametrize the line segment as $x = y = z = 1 + 3t; 0 \le t \le 1$. Then the work done by $\mathbf{F}$ moving along C is given by, work $= \int_C \mathbf{F}\cdot d\mathbf{R} = \int_0^1 [(1+3t)^2 - 2(1+3t)^2 + (1+3t)]3dt = \left(\frac{(1+3t)^2}{2} - \frac{(1+3t)^3}{3}\right)\Big|_0^1 = -\frac{27}{2}$

18. Take $\mathbf{F}(x) = f(x)\mathbf{i}$, and let $\mathbf{R}$ be the vector $\mathbf{R}(t) = t\mathbf{i}, a \le t \le b$. Thus C (traced by $\mathbf{R}$) is an interval of the x axis, and $\int_C \mathbf{F}\cdot d\mathbf{R} = \int_a^b f(x)dx$. This show line integrals are a generalization of Riemann integrals.

Section 13.2 Green's Theorem

1. Work done by $\mathbf{F} = \oint_C \mathbf{F}\cdot d\mathbf{R} = \oint_C xydx + xdy = \iint_D \left[\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(xy)\right] dA = \int_0^1\int_0^{6x} (1-x)dydx + \int_1^4\int_0^{8-2x} (1-x)dydx = \int_0^1 6x(1-x)dx + \int_1^4 (8-2x)(1-x)dx = -8.$

2. Work done by $\mathbf{F} = \oint_C \mathbf{F}\cdot d\mathbf{R} = \oint_C (e^x - y + x\cosh(x))dx + (y^{3/2} + x)dy = \iint_D \left[\frac{\partial}{\partial x}(y^{3/2} + x) - \frac{\partial}{\partial y}(e^x - y + x\cosh(x))\right] dA = \iint_D (1+1)dA = 2(\text{area of}D) = 2(\pi(6)^2) = 72\pi.$

3. Work done by $\mathbf{F} = \oint_C \mathbf{F}\cdot d\mathbf{R} = \oint_C (-\cosh(4x^4) + xy)dx + (e^{-y} + x)dy = \iint_D \left[\frac{\partial}{\partial x}(e^{-y} + x) - \frac{\partial}{\partial y}(-\cosh(4x^4) + xy)\right] dA = \iint_D (1-x)dA = \int_1^3\int_1^7 (1-x)dydx = \int_1^3 6(1-x)dx = -12.$

4. $\oint_C \mathbf{F}\cdot d\mathbf{R} = \iint_D \left[\frac{\partial}{\partial x}(-x) - \frac{\partial}{\partial y}(2y)\right] dA = \iint_D (-3)dA = -3(\text{area of}D) = -3(16\pi) = -48\pi.$

5. $\oint_C \mathbf{F}\cdot d\mathbf{R} = \iint_D \left[\frac{\partial}{\partial x}(-2xy) - \frac{\partial}{\partial y}(x^2)\right] dA = \iint_D (-2y)dA = \int_1^6 \int_{\frac{y+4}{5}}^{\frac{22-2y}{5}} -2ydxdy = \int_1^6 \frac{2y}{5}(3y - 18)dy = -40.$

6. $\oint_C \mathbf{F}\cdot d\mathbf{R} = \iint_D \left[\frac{\partial}{\partial x}(x-y) - \frac{\partial}{\partial y}(x+y)\right] dA = \iint_D 0\ dA = 0.$

7. $\oint_C \mathbf{F}\cdot d\mathbf{R} = \iint_D \frac{\partial}{\partial x}(8y^2)dA = \iint_D 8y^2 dA$. To evaluate this integral change to polar coordinates for D which are $0 \le r \le 4, 0 \le \theta \le 2\pi$ to get $\oint_C \mathbf{F}\cdot d\mathbf{R} = \int_0^{2\pi}\int_0^4 8(r\sin\theta)^2 r dr d\theta = \int_0^{2\pi}\sin^2\theta d\theta \int_0^4 8r^3 dr = (\pi)(512) = 512\pi$

8. $\oint_C \mathbf{F}\cdot d\mathbf{R} = \iint_D \left[\frac{\partial}{\partial x}(\cos(2y) - e^{3y} + 4x) - \frac{\partial}{\partial y}(x^2 - y)\right] dA = \iint_D (4{+}1)\ dA = 5(\text{area of}D) = 125.$

9. $\oint_C \mathbf{F}\cdot d\mathbf{R} = \iint_D \left[\frac{\partial}{\partial x}(e^x\sin(y)) - \frac{\partial}{\partial y}(e^x\cos(y)\right] dA = \iint_D [-e^x\sin(y) + e^x\sin(y)]dA = 0$

10. $\oint_C \mathbf{F}\cdot d\mathbf{R} = \iint_D \left[\frac{\partial}{\partial x}(-x^2y) - \frac{\partial}{\partial y}(x^2y)\right] dA = \iint_D (-y^2 - x^2)dA = \int_0^{\pi/2}\int_0^2 (-r^2) r dr d\theta = \frac{\pi}{2}\int_0^2 -r^3 dr = -2\pi$

11. $\oint_C \mathbf{F}\cdot d\mathbf{R} = \iint_D \left[\frac{\partial}{\partial x}(xy^2 - e^{\cos(y)}) - \frac{\partial}{\partial y}(xy)\right] dA = \iint_D (y^2 - x)dA = \int_0^3\int_0^{5-5x/3}(y^2-x)dydx = \int_0^3 \frac{1}{3}\left(5 - \frac{5x}{3}\right)^3 dx - \int_0^3 x\left(5 - \frac{5x}{3}\right)dx = \frac{95}{4}$

12. (a) By Green's Theorem, with $\mathbf{F} = -y\mathbf{i} + 0\mathbf{j}$, $\oint_C -ydx = \iint_D \left[\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial y}(-y)\right] dA = \iint_D dA = \text{area of } D$;

(b) Apply Green's Theorem, with $\mathbf{F} = 0\mathbf{i} + x\mathbf{j}$, $\oint_C \mathbf{F}\cdot d\mathbf{R} = \iint_D \left[\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(0)\right] dA = \iint_D dA = \text{area of } D$;

(c) Add the results of (a) and (b) to get 2 (area of D) $= \oint_C -ydx + xdy$, so that area of $D = \frac{1}{2}\oint_C -ydx + xdy$.

13. By Green's Theorem,

$$\oint_C -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy = \iint_D \left[\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right)\right] dA = \iint_D \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] dA.$$

An Extension of Green's Theorem

1. If the path C does not enclose the origin, by Green's Theorem we have

$$\oint_C \mathbf{F}\cdot d\mathbf{R} = \iint_D \left[\frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right)\right] dA = 0.$$

If C does enclose the origin, we use the extension of Green's Theorem where K is a circle of radius r lying entirely within C but enclosing the origin and

$$\oint_C \mathbf{F}\cdot d\mathbf{R} = \oint_K \mathbf{F}\cdot d\mathbf{R} = \int_0^{2\pi}\left[\frac{r\cos(\theta)}{r^2}(-r\sin(\theta)) + \frac{r\sin(\theta)}{r^2}r\cos(\theta)\right] d\theta = 0.$$

2. If C does not enclose the origin, by Green's Theorem we have

$$\oint_C \mathbf{F}\cdot d\mathbf{R} = \iint_D \left[\frac{\partial}{\partial x}\left(\frac{y}{(x^2+y^2)^{3/2}}\right) - \frac{\partial}{\partial y}\left(\frac{x}{(x^2+y^2)^{3/2}}\right)\right] dA = 0,$$

since

$$\frac{\partial}{\partial x}\left(\frac{y}{(x^2+y^2)^{3/2}}\right) = \frac{\partial}{\partial y}\left(\frac{x}{(x^2+y^2)^{3/2}}\right) = \frac{-3xy}{(x^2+y^2)^{5/2}}.$$

If C does enclose the origin, choose a smaller circle K inside C, disjoint from C which also encloses the origin, and

$$\oint_C \mathbf{F}\cdot d\mathbf{R} = \oint_K \mathbf{F}\cdot d\mathbf{R} = \int_0^{2\pi}\left[\frac{r\cos(\theta)}{r^3}(-r\sin(\theta)) + \frac{r\sin(\theta)}{r^3}(r\cos(\theta))\right] d\theta = 0.$$

3. If C does not enclose the origin, then by Green's Theorem,

$$\oint_C \mathbf{F}\cdot d\mathbf{R} = \iint_D \left[\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2} - 2y\right) - \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2} + x^2\right)\right] dA = 0$$

since

$$\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2} - 2y\right) = \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2} + x^2\right) = \frac{y^2-x^2}{(x^2+y^2)^2}.$$

If C does enclose the origin, choose a smaller circle K inside C, disjoint from C which also encloses the origin, and $\oint_C \mathbf{F}\cdot d\mathbf{R} = \oint_K \mathbf{F}\cdot d\mathbf{R} =$

$$\int_0^{2\pi}\left[\left(\frac{-r\sin(\theta)}{r^2} + r^2\cos^2(\theta)\right)(-r\sin(\theta)) + \left(\frac{r\cos(\theta)}{r^2} - 2r\sin(\theta)\right)(r\cos(\theta))\right] d\theta =$$

$$\int_0^{2\pi} [1 - r^3\cos^2(\theta)\sin(\theta) - 2r^2\sin(\theta)\cos(\theta)] d\theta =$$

$$\theta + \frac{r^3}{3}\cos^3(\theta) - r^2\sin^2(\theta)\Big|_0^{2\pi} = 2\pi$$

4. If C does not enclose the origin, then by Green's Theorem,

$$\oint_C \mathbf{F}\cdot d\mathbf{R} = \iint_D \left[\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2} - y\right) - \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2} + 3x\right)\right] dA = 0,$$

since

$$\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2} - y\right) = \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2} + 3x\right) = \frac{y^2-x^2}{(x^2+y^2)^2}.$$

If C does enclose the origin, then by the extension of Green's Theorem $\oint_C \mathbf{F}\cdot d\mathbf{R} = \oint_K \mathbf{F}\cdot d\mathbf{R} =$

$$\int_0^{2\pi}\left(\frac{-r\sin(\theta)}{r^2} + 3r\cos(\theta)\right)(-r\sin(\theta)) + \left(\frac{r\cos(\theta)}{r^2} - r\sin(\theta)\right)(r\cos(\theta)) d\theta$$

$$= \int_0^{2\pi} [1 - 4r^2 \sin(\theta)\cos(\theta)]d\theta = 2\pi$$

5. If C does not enclose the origin, then by Green's Theorem,

$$\oint_C \mathbf{F} \cdot d\mathbf{R} = \iint_D \left[\frac{\partial}{\partial x}\left(\frac{y}{\sqrt{x^2+y^2}} - 3y^2\right) - \frac{\partial}{\partial y}\left(\frac{x}{\sqrt{x^2+y^2}} + 2x\right)\right] dA = 0,$$

since

$$\frac{\partial}{\partial x}\left(\frac{y}{\sqrt{x^2+y^2}}\right) = \frac{\partial}{\partial y}\left(\frac{x}{\sqrt{x^2+y^2}}\right) = -\frac{xy}{(x^2+y^2)^{3/2}}.$$

If C does enclose the origin, then by the extension of Green's Theorem $\oint_C \mathbf{F}\cdot d\mathbf{R} = \oint_K \mathbf{F}\cdot d\mathbf{R} =$

$$\int_0^{2\pi}\left[\left(\frac{r\cos(\theta)}{r} + 2r\cos(\theta)\right)(-r\sin(\theta)) + \left(\frac{r\sin(\theta)}{r} - 3r^2\sin^2(\theta)\right)(r\cos(\theta))\right] d\theta =$$

$$-r^2\int_0^{2\pi}[2\cos(\theta)\sin(\theta) + 3\sin^2(\theta)\cos(\theta)]d\theta = -r^2(\sin^2(\theta) + \sin^3(\theta))\Big|_0^{2\pi} = 0.$$

Section 13.3 Independence of Path and Potential Theory in the Plane

1. $\frac{\partial}{\partial y}[y^3] = 3y^2 = \frac{\partial}{\partial x}[3x^2y - 4]$, so $\mathbf{F}$ is conservative. To find a potential function, ϕ, start with $\frac{\partial\phi}{\partial x} = y^3$ and integrate to get $\phi(x,y) = xy^3 + k(y)$. Then calculate $\frac{\partial\phi}{\partial y} = 3x^2y + k'(y) = 3x^2y - 4$, so $k'(y) = -4$, and we choose $k(y) = -4y$. A potential function is $\phi(x,y) = xy^3 - 4y$.

2. $\frac{\partial}{\partial y}[6y + ye^{xy}] = 6 + e^{xy} + xye^{xy} = \frac{\partial}{\partial x}[6x + xe^{xy}]$, so $\mathbf{F}$ is conservative. To find a potential function, ϕ, we write $\frac{\partial\phi}{\partial x} = 6y + ye^{xy}$ and $\frac{\partial\phi}{\partial y} = 6x + xe^{xy}$. Now integrate the first equation with respect to x, the second with respect to y, and get $\phi(x,y) = 6xy + e^{xy} + c(y)$ or $\phi(x,y) = 6xy + e^{xy} + k(x)$. Since these expressions must agree we choose $c(y)$ and $k(x)$ accordingly. Thus take $c(y) = k(x) = 0$ and a potential function is $\phi(x,y) = 6xy + e^{xy}$.

3. $\frac{\partial}{\partial y}[16x] = 0 = \frac{\partial}{\partial x}[2 - y^2]$, so $\mathbf{F}$ is conservative. Integrate $\frac{\partial\phi}{\partial x} = 16x$ and $\frac{\partial\phi}{\partial y} = 2 - y^2$ to get $\phi(x,y) = 8x^2 + c(y) = 2y - \frac{y^3}{3} + k(x)$. Thus choose $c(y) = 2y - \frac{y^3}{3}$, and $k(x) = 8x^2$ to get a potential function $\phi(x,y) = 8x^2 + 2y - \frac{y^3}{3}$.

4. $\frac{\partial}{\partial y}[2xy\cos(x^2)] = 2x\cos(x^2) = \frac{\partial}{\partial x}[\sin(x^2)]$, so $\mathbf{F}$ is conservative. Integrate $\frac{\partial\phi}{\partial x} = 2xy\cos(x^2)$ and $\frac{\partial\phi}{\partial y} = \sin(x^2)$ to get $\phi(x,y) = y\sin(x^2) + c(y) = y\sin(x^2) + k(y)$. Choose $c(y) = k(x) = 0$ to get a potential function $\phi(x,y) = y\sin(x^2)$.

5. $\dfrac{\partial}{\partial y}\left[\dfrac{2xy}{x^2+y^2}\right] = \dfrac{-4xy}{(x^2+y^2)^2} = \dfrac{\partial}{\partial x}\left[\dfrac{2y}{x^2+y^2}\right]$, so $\mathbf{F}$ is conservative on any region not containing $(0,0)$. Integrate $\dfrac{\partial\phi}{\partial x} = \dfrac{2xy}{x^2+y^2}$ and $\dfrac{\partial\phi}{\partial y} = \dfrac{2y}{x^2+y^2}$ to get $\phi(x,y) = \ln(x^2+y^2) + c(y) = \ln(x^2+y^2) + k(x)$. Choose $c(y) = k(x) = 0$ to get a potential function $\phi(x,y) = \ln(x^2+y^2)$.

6. $\dfrac{\partial}{\partial y}[\sinh(x+y)] = \cosh(x+y) = \dfrac{\partial}{\partial x}[\sinh(x+y)]$, so $\mathbf{F}$ is conservative. Integrate $\dfrac{\partial\phi}{\partial x} = \sinh(x+y)$ and $\dfrac{\partial\phi}{\partial y} = \sinh(x+y)$ to get $\phi(x,y) = \cosh(x+y) + c(y) = \cosh(x+y) + k(x)$. Choose $c(y) = k(x)$ to get a potential function $\phi(x,y) = \cosh(x+y)$.

7. $\dfrac{\partial}{\partial y}[2\cos(2x)e^y] = 2\cos(2x)e^y = \dfrac{\partial}{\partial x}[e^y\sin(2x) - y]$, so $\mathbf{F}$ is conservative. Integrate $\dfrac{\partial\phi}{\partial x} = 2\cos(2x)e^y$ and $\dfrac{\partial\phi}{\partial y} = e^y\sin(2x) - y$ to get $\phi(x,y) = \sin(2x)e^y + c(y) = \sin(2x)e^y - \dfrac{y^2}{2} + k(x)$. Choose $c(y) = -\dfrac{y^2}{2}$ and $k(x) = 0$ to get a potential function $\phi(x,y) = \sin(2x)e^y - \dfrac{y^2}{2}$.

8. $\dfrac{\partial}{\partial y}[3x^2y - \sin(x) + 1] = 3x^2 = \dfrac{\partial}{\partial x}[x^3 + e^y]$, so $\mathbf{F}$ is conservative. Integrate $\dfrac{\partial\phi}{\partial x} = 3x^2y - \sin(x) + 1$ and $\dfrac{\partial\phi}{\partial y} = x^3 + e^y$ to get $\phi(x,y) = x^3y + \cos(x) + x + c(y) = x^3y + e^y + k(x)$. Choose $c(y) = e^y$ and $k(x) = \cos(x) + x$ to get a potential function $\phi(x,y) = x^3y + \cos(x) + x + e^y$.

9. First, $\dfrac{\partial}{\partial y}[3x^2(y^2 - 4y)] = 3x^2(2y-1) = \dfrac{\partial}{\partial x}[2x^3y - 4x^3]$, so $\mathbf{F}$ is conservative. To find a potential function for $\mathbf{F}$, integrate $\dfrac{\partial\phi}{\partial x} = 3x^2(y^2-4y)$ and $\dfrac{\partial\phi}{\partial y} = 2x^3y - 4x^3$ to get $\phi(x,y) = x^3(y^2-4y) + c(y) = x^3y^2 - 4y^3 + k(x)$. Choose $c(y) = 0$ and $k(x) = 0$ to get $\phi(x,y) = x^3(y^2-4y)$. Finally, $\int_C \mathbf{F}\cdot d\mathbf{R} = \phi(2,3) - \phi(-1,1) = -24 - 3 = -27$.

10. First, $\dfrac{\partial}{\partial y}[e^x\cos(y)] = -e^x\sin(y) = \dfrac{\partial}{\partial x}[-e^x\sin(y)]$, so $\mathbf{F}$ is conservative. A potential function for $\mathbf{F}$, is easily found to be $\phi(x,y) = e^x\cos(y)$. Finally, $\int_C \mathbf{F}\cdot d\mathbf{R} = \phi(2,\pi/4) - \phi(0,0) = \dfrac{e^2}{\sqrt{2}} - 1$.

11. First, $\dfrac{\partial}{\partial y}[2xy] = 2x = \dfrac{\partial}{\partial x}\left[x^2 - \dfrac{1}{y}\right]$, so $\mathbf{F}$ is conservative in any region not containing $y = 0$. A potential function for $\mathbf{F}$, is easily found to be $\phi(x,y) = x^2y - \ln|y|$. Finally, for any path not crossing the x axis, $\int_C \mathbf{F}\cdot d\mathbf{R} = \phi(2,2) - \phi(1,3) = 8 - \ln(2) - 3 + \ln(3) = 5 + \ln\left(\dfrac{3}{2}\right)$

12. First, $\dfrac{\partial}{\partial y}[1] = 0 = \dfrac{\partial}{\partial x}[6y + \sin(y)]$, so $\mathbf{F}$ is conservative. A potential function for $\mathbf{F}$ is $\phi(x,y) = x + 3y^2 - \cos(y)$. Finally, $\int_C \mathbf{F}\cdot d\mathbf{R} = \phi(1,3) - \phi(0,0) = 28 - \cos(3) + 1 = 29 - \cos(3)$.

13. $\dfrac{\partial}{\partial y}[3x^2y^2 - 6y^3] = 6x^2y - 18y^2 = \dfrac{\partial}{\partial x}[2x^3y - 18xy^2]$, so $\mathbf{F}$ is conservative. A potential

function for $\mathbf{F}$ is $\phi(x,y) = x^3y^2 - 6xy^3$, so $\int_C \mathbf{F}\cdot d\mathbf{R} = \phi(1,1) - \phi(0,0) = -5$.

14. In any region for which $x \neq 0$, $\frac{\partial}{\partial y}\left[\frac{y}{x}\right] = \frac{1}{x} = \frac{\partial}{\partial x}[\ln(x)]$, so $\mathbf{F}$ is conservative in any such region. A potential function for $\mathbf{F}$ is $\phi(x,y) = y\ln(x)$, so $\int_C \mathbf{F}\cdot d\mathbf{R} = \phi(2,2) - \phi(1,1) = 2\ln(2)$.

15. $\frac{\partial}{\partial y}[-8e^y + e^x] = -8e^y = \frac{\partial}{\partial x}[-8xe^y]$, so $\mathbf{F}$ is conservative. A potential function for $\mathbf{F}$ is $\phi(x,y) = -8xe^y + e^x$, so $\int_C \mathbf{F}\cdot d\mathbf{R} = \phi(3,1) - \phi(-1,-1) = e^3 - 24e - 9e^{-1}$

16. In any region for which $x \neq 0$, $\frac{\partial}{\partial y}\left[4xy + \frac{3}{x^2}\right] = 4x = \frac{\partial}{\partial x}[2x^2]$, so $\mathbf{F}$ is conservative in any such region. A potential function for $\mathbf{F}$ is $\phi(x,y) = 2x^2y - \frac{3}{x}$, so $\int_C \mathbf{F}\cdot d\mathbf{R} = \phi(3,3) - \phi(1,2) = 53 - 1 = 52$.

17. Let C be any smooth path of motion described by $\mathbf{R}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, and let L = kinetic energy + potential energy, so $L(t) = \frac{m}{2}\|\mathbf{R}'(t)\|^2 - \phi = \frac{m}{2}\mathbf{R}'(t)\cdot\mathbf{R}'(t) - \phi(\mathbf{R}(t))$. Then $\frac{dL}{dt} = \frac{m}{2}[\mathbf{R}''\cdot\mathbf{R}' + \mathbf{R}'\cdot\mathbf{R}''] - \frac{\partial\phi}{\partial x}x'(t) - \frac{\partial\phi}{\partial y}y'(t) - \frac{\partial\phi}{\partial z}z'(t) = (m\mathbf{R}''\cdot\mathbf{R}') - \nabla\phi\cdot\mathbf{R}' = [m\mathbf{R}'' - \nabla\phi]\cdot\mathbf{R}'$. But $\nabla\phi$ is the force acting upon the particle, so by Newton's second law, $m\mathbf{R}'' = \nabla\phi$, and hence $\frac{dL}{dt} = 0$. It follows that $L(t)$ = constant.

Section 13.4 Surfaces in 3 - Space and Surface Integrals

1. On the surface $\sum$, $z = 10 - x - 4y$, so $d\sigma = \sqrt{1 + (\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2}dA = \sqrt{18}dA = 3\sqrt{2}dA$, and $\iint_\Sigma x d\sigma = \iint_D x3\sqrt{2}dA = 3\sqrt{2}\int_0^{5/2}\int_0^{10-4y} x dx dy = \frac{3\sqrt{2}}{2}\int_0^{5/2}(10-4y)^2dy = -\frac{\sqrt{2}}{8}(10-4y)^3\Big|_0^{5/2} = 125\sqrt{2}$.

2. On the surface $\sum$, $z = x$, so $d\sigma = \sqrt{1 + (1)^2 + (0)^2}dA = \sqrt{2}dA$, and

$$\iint_\Sigma y^2 d\sigma = \iint_D y^2\sqrt{2}dA = \sqrt{2}\int_0^2\int_0^4 y^2 dx dy = \frac{128\sqrt{2}}{3}.$$

3. We have $d\sigma = \sqrt{1 + (2x)^2 + (2y)^2}dA = \sqrt{1 + 4(x^2+y^2)}dA$, and D is the annular region $2 \leq x^2 + y^2 \leq 7$. Thus $\iint_\Sigma 1 d\sigma = \iint_D \sqrt{1 + 4(x^2+y^2)}dA$. To evaluate the integral over D, use polar coordinates, i.e. $D = \{(r,\theta) | \sqrt{2} \leq r \leq \sqrt{7}, 0 \leq \theta \leq 2\pi\}$ to get

$$\iint_\Sigma 1 d\sigma = \int_0^{2\pi}\int_{\sqrt{2}}^{\sqrt{7}} r\sqrt{1+4r^2}dr d\theta = 2\pi\left[\frac{1}{12}(1+4r^2)^{3/2}\Big|_{\sqrt{2}}^{\sqrt{7}}\right] = \frac{\pi}{6}[(29)^{3/2} - 27].$$

4. We have on $\sum$, $z = \frac{1}{10}(25 - 4x - 8y)$, so $d\sigma = \sqrt{1 + (\frac{2}{5})^2 + (\frac{4}{5})^2}dA = \frac{3}{\sqrt{5}}dA$.
Thus $\iint_\Sigma (x+y)d\sigma = \iint_D (x+y)\frac{3}{\sqrt{5}}dA = \frac{3}{\sqrt{5}}\int_0^1\int_0^x (x+y)dy dx = \frac{3}{\sqrt{5}}\int_0^1 \frac{3}{2}x^2 dx = \frac{3}{2\sqrt{5}}$.

5. On $\sum$, we have $\dfrac{\partial z}{\partial x} = \dfrac{x}{\sqrt{x^2+y^2}}, \dfrac{\partial z}{\partial y} = \dfrac{y}{\sqrt{x^2+y^2}}$, so $d\sigma = \sqrt{1+\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}}dA = \sqrt{2}dA$. Thus $\int\int_\Sigma z d\sigma = \int\int_D \sqrt{x^2+y^2}\sqrt{2}dA = \sqrt{2}\int_0^{\pi/2}\int_2^4 r^2 dr d\theta = \dfrac{28\pi}{3}\sqrt{2}$.

6. We have $d\sigma = \sqrt{1+1+1}dA = \sqrt{3}dA$, and $z = x+y$, so $\int\int_\Sigma xyz d\sigma = \sqrt{3}\int\int_D xy(x+y)dA = \sqrt{3}\int_0^1\int_0^1(x^2y+xy^2)dydx = \dfrac{\sqrt{3}}{3}$.

7. We have $d\sigma = \sqrt{1+(2x)^2}dA = \sqrt{1+4x^2}dA$, so $\int\int_\Sigma y d\sigma = \int\int_D y\sqrt{1+4x^2}dA = \int_0^2\int_0^3 y\sqrt{1+4x^2}dydx = \dfrac{9}{2}\int_0^2\sqrt{1+4x^2}dx = \dfrac{9}{8}[\ln(4+\sqrt{17})+4\sqrt{17}]$

8. We have $d\sigma = \sqrt{1+(2x)^2+(2y)^2}dA = \sqrt{1+4(x^2+y^2)}dA$, so $\int\int_\Sigma x^2 d\sigma = \int\int_D x^2\sqrt{1+4(x^2+y^2)}dA = \int_0^{2\pi}\int_0^2 (r\cos(\theta))^2\sqrt{1+4r^2}r dr d\theta = \int_0^{2\pi}\cos^2(\theta)d\theta\int_0^2 r^3(1+4r^2)^{1/2}dr$
To evaluate the first integral, use the identity $\cos^2(\theta) = \dfrac{1+\cos 2(\theta)}{2}$; and in the second, use the substitution $u = 1+4r^2$, so $r^2 = \dfrac{u-1}{4}, rdr = \dfrac{du}{8}$ to get

$$\int\int_\Sigma x^2 d\sigma = \frac{1}{2}\int_0^{2\pi}(1+\cos(2\theta))d\theta\frac{1}{32}\int_1^{17}(u^{3/2}-u^{1/2})du = \frac{\pi}{240}[782\sqrt{17}+2].$$

9. We have $d\sigma = \sqrt{1+1+1}dA = \sqrt{3}dA, z = x-y$, so $\int\int_\Sigma z d\sigma = \int\int_D (x-y)\sqrt{3}dA = \sqrt{3}\int_0^1\int_0^5(x-y)dydx = -10\sqrt{3}$.

10. We have $d\sigma = \sqrt{1+(2y)^2}dA = \sqrt{1+4y^2}dA$, and $z = 1+y^2$, so $\int\int_\Sigma xyz d\sigma = \int\int_D xy(1+y^2)\sqrt{1+4y^2}dA = \int_0^1\int_0^1 xy(1+y^2)\sqrt{1+4y^2}dydx = \dfrac{1}{2}\int_0^1 y(1+y^2)\sqrt{1+4y^2}dy$, here let $u = 1+4y^2, \dfrac{du}{8} = ydy, 1+y^2 = \dfrac{u+3}{4}$ to get

$$\int\int_\Sigma xyz d\sigma = \frac{1}{2}\cdot\frac{1}{32}\int_1^5(u^{3/2}+3u^{1/2})du = \frac{1}{16}\left[5\sqrt{5}-\frac{3}{5}\right].$$

Section 13.5 Applications of Surface Integrals

1. An equation for the triangular shell having the given vertices is $6x+2y+3z=6$. To find the mass and center of mass, we need $M = \int\int_\Sigma(xz+1)d\sigma$ first, and then $\overline{x} = \dfrac{1}{M}\int\int_\Sigma x(xz+1)d\sigma, \overline{y} = \dfrac{1}{M}\int\int_\Sigma y(xz+1)d\sigma, \overline{z} = \dfrac{1}{M}\int\int_\Sigma z(xz+1)d\sigma$. Solving for $z = (2-\dfrac{2}{3}y-2x)$ we get $\dfrac{\partial z}{\partial x} = -2, \dfrac{\partial z}{\partial y} = -\dfrac{2}{3}$ and $d\sigma = \sqrt{1+4+\frac{4}{9}}dA = \dfrac{7}{3}dA$. Projecting $\sum$ onto the xy plane we get $D = \{(x,y)|0 \le y \le 3-3x, 0 \le x \le 1\}$. Then Mass $= M = \int\int_\Sigma(xz+$

$1)d\sigma = \frac{7}{3}\int_0^1\int_0^{3-3x}\left[x\left(2-\frac{2}{3}y-2x\right)+1\right]dydx$. The evaluation of this integral is routine but tedious and we find $M = \frac{49}{12}$. Similarly we evaluate the integrals for $\overline{x}, \overline{y}, \overline{z}$, and find $(\overline{x}, \overline{y}, \overline{z}) = \left(\frac{12}{35}, \frac{33}{35}, \frac{24}{35}\right)$. Since the surface $\sum$ is a portion of a plane, the center of mass must lie on this plane. This gives a method of checking the coordinates of the center of mass. We have $6\left(\frac{12}{35}\right)+2\left(\frac{33}{35}\right)+3\left(\frac{24}{35}\right) = \frac{210}{35} = 6$.

2. We first observe that $\overline{x} = \overline{y} = 0$ by symmetry of the surface and the fact that $\delta = K$, a constant. We can easily find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ implicitly from the equation $x^2+y^2+z^2 = 9$. We have $2xdx+2ydy+2zdz = 0$, so $dz = -\left(\frac{x}{z}\right)dx - \left(\frac{y}{z}\right)dy$, hence $\frac{\partial z}{\partial x} = -\frac{x}{z}, \frac{\partial z}{\partial y} = -\frac{y}{z}$. Thus $d\sigma = \sqrt{1+\frac{x^2}{z^2}+\frac{y^2}{z^2}}dA = \frac{3}{z}dA$. The portion of the hemisphere lying above $z = 1$ projects onto the xy plane in the region $D = \{(x,y)|x^2+y^2 \leq 8\}$. We now compute mass $M = \int\int_\Sigma K\left(\frac{3}{z}\right)dA = 3K\int\int_D \frac{1}{\sqrt{9-(x^2+y^2)}}dA$, which we evaluate by using polar coordinates.

$$M = 3K\int_0^{2\pi}\int_0^{\sqrt{8}} \frac{r}{\sqrt{9-r^2}}drd\theta = 6\pi K[-(9-r^2)^{1/2}]\Big|_0^{\sqrt{8}} = 12\pi K.$$

Finally

$$\overline{z} = \frac{1}{M}\int\int_\Sigma Kzd\sigma = \frac{1}{M}\int\int_D Kz\left(\frac{3}{z}\right)dA = \frac{1}{M}3K(\text{area of} D) = \frac{1}{M}(24\pi K), \text{ so } \overline{z} = 2.$$

Thus $(\overline{x}, \overline{y}, \overline{z}) = (0,0,2)$.

3. By geometric symmetry of $\sum$ and the constant density function, $\overline{x} = \overline{y} = 0$. Now $d\sigma = \sqrt{1+\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}}dA = \sqrt{2}dA$, so Mass $= M = \int\int_\Sigma Kd\sigma = K\sqrt{2}\int_0^{2\pi}\int_0^3 rdrd\theta = 9\pi K\sqrt{2}$. Then $\overline{z} = \frac{1}{M}\int\int_\Sigma zd\sigma = \frac{1}{M}\sqrt{2}K\int_0^{2\pi}\int_0^3 r^2drd\theta = \frac{1}{M}18K\pi\sqrt{2} = 2$. So $(\overline{x}, \overline{y}, \overline{z}) = (0,0,2)$.

4. On the surface $\Sigma, \frac{\partial z}{\partial x} = -2x, \frac{\partial z}{\partial y} = -2y$, so $d\sigma = \sqrt{1+4(x^2+y^2)}dA$. Σ projects onto the xy plane to give the quarter annulus $D = \{(x,y)|x \geq 0, y \geq 0, 1 \leq x^2+y^2 \leq 9\}$ or $D = \{(r,\theta)|0 \leq \theta \leq \pi/2, 1 \leq r \leq 3\}$. We have Mass $= \int\int_\Sigma \frac{xy}{\sqrt{1+4(x^2+y^2)}}d\sigma = \int\int_D xydA = \int_0^{\pi/2}\int_1^3 r^3\cos(\theta)\sin(\theta)drd\theta = \left(\frac{1}{2}\sin^2(\theta)\Big|_0^{\pi/2}\right)\left(\frac{r^4}{4}\Big|_1^3\right) = 10$. Since the region is symmetric about the line $y = x$ and $\delta(x,y,z) = \delta(r,\theta) = \frac{1}{\sqrt{1+4r^2}}$ is independent of $\theta, \overline{x} = \overline{y}$, which saves some work. $\overline{x} = \frac{1}{M}\int\int_\Sigma x\delta(x,y,z)d\sigma = \frac{1}{10}\int\int x^2ydA = \frac{1}{10}\int_0^{\pi/2}\int_1^3 r^4\cos^2(\theta)\sin(\theta)drd\theta = \frac{121}{75}$. Finally $\overline{z} = \frac{1}{M}\int\int_\Sigma z\delta(x,y,z)d\sigma = \frac{1}{10}\int\int_D(16-x^2-y^2)xydA = \frac{1}{10}\int_0^{\pi/2}\int_1^3(16-r^2)r^3\sin(\theta)\cos(\theta)drd\theta = \frac{331}{48}$. Thus $(\overline{x}, \overline{y}, \overline{z}) = \left(\frac{121}{75}, \frac{121}{75}, \frac{331}{48}\right)$.

5. By symmetry of Σ and $\delta(x,y,z)$ we have $\overline{x}=\overline{y}=0$. On Σ we have $d\sigma=\sqrt{1+(2x)^2+(2y)^2}dA=\sqrt{1+4(x^2+y^2)}dA$.
For Mass,

$$M=\iint_{\Sigma}\sqrt{1+4(x^2+y^2)}d\sigma=\iint_{D}(1+4(x^2+y^2))dA=\int_0^{2\pi}\int_0^{\sqrt{6}}(1+4r^2)rdrd\theta=78\pi.$$

For $\overline{z}$ calculate

$$\overline{z}=\frac{1}{M}\iint_{\Sigma}z\delta(x,y,z)d\sigma=\frac{1}{M}int\int_{D}(6-(x^2+y^2))(1+4(x^2+y^2))dA=$$

$$\frac{1}{M}\int_0^{2\pi}\int_0^{\sqrt{6}}(6-r^2)(1+4r^2)rdrd\theta=\frac{162}{M}\pi=\frac{27}{13}.$$

Thus $(\overline{x},\overline{y},\overline{z})=\left(0,0,\frac{27}{13}\right)$.

6. By symmetry, $\overline{x}=\overline{y}=\overline{z}$. On Σ, we have $\dfrac{\partial z}{\partial x}=-\dfrac{x}{z},\dfrac{\partial z}{\partial y}=-\dfrac{y}{z}$, so $d\sigma=\sqrt{1+\frac{x^2}{z^2}+\frac{y^2}{z^2}}dA=\dfrac{1}{z}dA$. We compute Mass,

$$M=\iint_{\Sigma}Kd\sigma=K(\text{area of}\Sigma)=\frac{K(4\pi)}{4}=\pi K,$$

and

$$\overline{z}=\frac{1}{M}\iint_{\Sigma}Kzd\sigma=\frac{K}{\pi K}\int\int_{D}z\left(\frac{1}{z}\right)dA=\frac{1}{\pi}(\text{area of }D)=\frac{1}{4}.$$

7. A unit normal to the plane $x+2y+z=8$ is $\mathbf{N}=\dfrac{\mathbf{i}+2\mathbf{j}+\mathbf{k}}{\sqrt{6}}$ and $\mathbf{F}\cdot\mathbf{N}=\dfrac{x+2y-z}{\sqrt{6}}$. But on Σ, $z=8-x-2y$; so $\mathbf{F}\cdot\mathbf{N}=\dfrac{2x+4y-8}{\sqrt{6}}$. Now $\dfrac{\partial z}{\partial x}=1,\dfrac{\partial z}{\partial y}=2$ so $d\sigma=\sqrt{1+4+1}dA=\sqrt{6}dA$. Hence, Flux of $\mathbf{F}$ across Σ is Flux $=\iint_{\Sigma}\mathbf{F}\cdot\mathbf{N}d\sigma=\iint_{D}(2x+4y-8)dA=\int_0^4\int_0^{8-2y}(2x+4y-8)dxdy=\dfrac{128}{3}$.

8. A unit normal to the sphere $x^2+y^2+z^2=4$ is given by $\mathbf{N}=\dfrac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{2}$ so $\mathbf{F}\cdot\mathbf{N}=\dfrac{1}{2}(x^2z-yz)$. Now $d\sigma\doteq\sqrt{1+(\frac{-x}{z})^2+(-\frac{y}{z})^2}dA=\dfrac{2}{z}dA$. Thus we have, Flux $=\iint_{\Sigma}\mathbf{F}\cdot\mathbf{N}d\sigma=\iint_{D}(x^2-y)dA$. Change to polar coordinates for $D=\{(r,\theta)|0\le r\le\sqrt{3},0\le\theta\le\pi\}$, to get Flux $=\displaystyle\int_0^{2\pi}\int_0^{\sqrt{3}}(r^2\cos^2(\theta)-r\sin(\theta))rdrd\theta=\frac{9}{4}\int_0^{2\pi}\cos^2(\theta)d\theta=\frac{9\pi}{4}$.

Section 13.6 Preparation for the Theorems of Gauss and Stokes

1. We apply Green's Theorem in the plane to the line integral

$$\oint_C -\phi\frac{\partial\psi}{\partial y}dx + \phi\frac{\partial\psi}{\partial y}dy = \iint_D \left[\frac{\partial}{\partial x}\left(\phi\frac{\partial\psi}{\partial x}\right) - \frac{\partial}{\partial y}\left(-\phi\frac{\partial\psi}{\partial y}\right)\right] dA =$$

$$\iint_D \left[\frac{\partial\phi}{\partial x}\cdot\frac{\partial\psi}{\partial x} + \frac{\partial\phi}{\partial y}\cdot\frac{\partial\psi}{\partial y} + \phi\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right)\right] dA = \iint_D \nabla\phi\cdot\nabla\psi\; dA + \iint_D \phi\nabla^2\psi\; dA.$$

Rearranging this gives the result.

2. Apply the result of Problem 1 to both the integrals

$$\iint_D \phi\nabla^2\psi\; dA = \int_C -\phi\frac{\partial\psi}{\partial y}dx + \phi\frac{\partial\psi}{\partial x}dy - \iint_D \nabla\phi\cdot\nabla\psi\; dA$$

and

$$\iint_D \psi\nabla^2\phi\; dA = \int_C -\psi\frac{\partial\phi}{\partial y}dx + \psi\frac{\partial\phi}{\partial x}dy - \iint_D \nabla\psi\cdot\nabla\phi\; dA$$

and subtract to get

$$\iint_D (\phi\nabla^2\psi - \psi\nabla^2\phi)dA = \oint_C \left(\psi\frac{\partial\phi}{\partial y} - \phi\frac{\partial\psi}{\partial y}\right)dx + \oint_C \left(\phi\frac{\partial\psi}{\partial x} - \psi\frac{\partial\phi}{\partial x}\right)dy.$$

3. Under the condition stated on ϕ, C, and D we have $\phi_{\mathbf{N}}(x,y) = \nabla\phi\cdot\mathbf{N}$, and $\mathbf{N} = \frac{dy}{ds}\mathbf{i} - \frac{dx}{ds}\mathbf{j}$, so

$$\oint_C \phi_{\mathbf{N}}(x,y)ds = \oint_C \left[\frac{\partial\phi}{\partial x}\left(\frac{dy}{ds}\right) - \frac{\partial\phi}{\partial y}\left(\frac{dx}{ds}\right)\right] ds = \oint_C -\frac{\partial\phi}{\partial y}dx + \frac{\partial\phi}{\partial x}dy.$$

Now apply Green's theorem to this line integral to get $\oint_C \phi_{\mathbf{N}}(x,y)ds =$

$$\iint_D \left[\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}\right) - \frac{\partial}{\partial y}\left(-\frac{\partial\phi}{\partial y}\right)\right] dA = \iint_D (\phi_{xx} + \phi_{yy})dA = \iint_D \nabla^2\phi\; dA.$$

Section 13.7 The Divergence Theorem of Gauss

1. Since $div(\mathbf{F}) = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} - \frac{\partial z}{\partial z} = 1$, we compute $\iiint_M div(\mathbf{F})dV = \iiint_M dV = vol(M) = \frac{4}{3}\pi(4)^3 = \frac{256}{3}\pi$.

2. Since $div(\mathbf{F}) = 4 - 6 = -2$, we compute $\iiint_M div(\mathbf{F})dV = \iiint_M (-2)dV = -2vol(M) = -2(\pi(2)^2 2) = -16\pi$.

3. Since $div(\mathbf{F}) = 0$, $\iiint_M div(\mathbf{F})dV = 0$

4. Since $div(\mathbf{F}) = 3x^2+3y^2+3z^2$, we calculate $\iiint_M div(\mathbf{F})dV = \iiint_M 3(x^2+y^2+z^2)dV$ by changing to spherical coordinates. This gives $\iiint_M div(\mathbf{F})dV = \int_0^{2\pi}\int_0^{\pi}\int_0^1 3\rho^4\sin(\phi)d\rho d\phi d\theta = \int_0^{2\pi} d\theta \int_0^{\pi}\sin(\phi)d\phi\int_0^1 3\rho^4 d\rho = (2\pi)(2)\left(\frac{3}{5}\right) = \frac{12\pi}{5}$.

5. Since $div(\mathbf{F}) = 4$ and Σ is a closed surface, we calculate $\iiint_M div(\mathbf{F})dV = \iiint_M (4)dV = 4vol(M) = 4\left(\frac{2\pi}{3}(1)^3\right) = \frac{8\pi}{3}$.

6. Since $div(\mathbf{F}) = 1+1+x = 2+x$, we calculate $\iiint_M div(\mathbf{F})dV = \int_0^3\int_0^2\int_0^4 (2+x)dxdydz = 6\frac{(2+x)^2}{2}\bigg|_0^4 = 96$.

7. With $div(\mathbf{F}) = 2(x+y+z)$, we compute $\iiint_M div(\mathbf{F})dV$ which we will evaluate in cylindrical coordinates. We have $\iiint_M div(\mathbf{F})dV = 2\int_0^{2\pi}\int_0^{\sqrt{2}}\int_r^{\sqrt{2}}(r\cos(\theta)+r\sin(\theta)+z)rdzdrd\theta = 4\pi\int_0^{\sqrt{2}} \frac{rz^2}{2}\bigg|_r^{\sqrt{2}} dr = 2\pi\int_0^{\sqrt{2}}(2r-r^3)dr = 2\pi$.

8. With $div(\mathbf{F}) = (2x+1)$, we compute the volume integral $\iiint_M (1+2x)dV = \int_0^2 dz \iint_{D_{xy}} (1+2x)dA$. To evaluate the double integral we switch to plane polar coordinates for
$D_{xy} : 0 \le \theta \le 2\pi, 0 \le r \le \sqrt{2}$, to get

$$\iiint_M (1+2x)dV = \int_0^2 dz \int_0^{2\pi}\int_0^{\sqrt{2}}(1+2r\cos\theta)rdrd\theta = 2\left[2\pi + \frac{4\sqrt{2}}{3}\int_0^{2\pi}\cos\theta d\theta\right] = 4\pi.$$

9. With the stated conditions on $\mathbf{F}, \Sigma, M$ we have $\iint_\Sigma (\nabla\times\mathbf{F})\cdot\mathbf{N}d\sigma = \iiint_M (\nabla\cdot\nabla\times\mathbf{F})dV$. By Theorem 11.7, Section 11.5, $\nabla\cdot\nabla\times\mathbf{F} = 0$, so $\iint_\Sigma (\nabla\times\mathbf{F})\cdot\mathbf{N}d\sigma = 0$

10. Let $\mathbf{F} = \phi\nabla\psi$ and apply Gauss's Theorem to $\iint_\Sigma (\phi\nabla\psi)\cdot\mathbf{N}d\sigma = \iiint_M \nabla(\phi\nabla\psi)dV$. By Problem 13, Section 12.5, we have $\nabla\cdot(\phi\nabla\psi) = \nabla\phi\cdot\nabla\psi + \phi\nabla^2\psi$, thus $\iint_\Sigma (\phi\nabla\psi)\cdot\mathbf{N}d\sigma = \iiint_M (\phi\nabla^2\psi + \nabla\phi\cdot\nabla\psi)dV$.

11. Use the result of Problem 12 to write $\iint_\Sigma (\phi\nabla\psi)\cdot\mathbf{N}d\sigma = \iiint_M (\phi\nabla^2\psi + \nabla\phi\cdot\nabla\psi)dV$; now interchange the roles of ϕ and ψ to get $\iint_\Sigma (\psi\nabla\phi)\cdot\mathbf{N}d\sigma = \iiint_M (\psi\nabla^2\phi + \nabla\psi\cdot\nabla\phi)dV$. Subtract these two integrals to get $\iint_\Sigma (\phi\nabla\psi - \psi\nabla\phi)\cdot\mathbf{N}d\sigma = \iiint_M (\phi\nabla^2\psi - \psi\nabla^2\phi)dV$ where the other terms have canceled.

12. Apply Gauss's Theorem to $\frac{1}{3}\iint_{\Sigma}\mathbf{R}\cdot\mathbf{N}d\sigma = \frac{1}{3}\iiint_{M}(\nabla\cdot\mathbf{R})dV = \frac{1}{3}\iiint_{M}(3)dV =$ volume of M.

13. Let f, g be two solutions of $\nabla^2 u = 0$ on M, and $\frac{\partial u}{\partial\eta} = 0$ on Σ. Define $w = f - g$. Then $\nabla^2 w = \nabla^2 f - \nabla^2 g = 0$ on M, and $\frac{\partial w}{\partial\eta} = \frac{\partial f}{\partial\eta} - \frac{\partial g}{\partial\eta} = 0$ on Σ. Now $\nabla\cdot(w\nabla w) = \nabla w\cdot\nabla w + w\nabla^2 w = \|\nabla w\|^2$ throughout M, since $\nabla^2 w = 0$ on M. Therefore we have $\iiint_M \|\nabla w\|^2 dV = \iiint_M \nabla\cdot(w\nabla w)dV = \iint_\Sigma w\nabla w\cdot\mathbf{N}d\sigma = \iint_\Sigma w\frac{\partial w}{\partial\eta}d\sigma = 0$, since $\frac{\partial w}{\partial\eta} = 0$ on Σ. From $\iiint_M \|\nabla w\|^2 dV = 0$, we conclude $\|\nabla w\|^2 = 0$, hence $\frac{\partial w}{\partial x} = \frac{\partial w}{\partial y} = \frac{\partial w}{\partial z} = 0$ and $w =$ constant. Thus $f(x,y,z) = g(x,y,z) + k$ throughout M.

Section 13.8 The Integral Theorem of Stokes

1. To decide whether the line integral $\oint_C \mathbf{F}\cdot d\mathbf{R}$ or the surface integral $\iint_\Sigma(\nabla\times\mathbf{F})\cdot\mathbf{N}d\sigma$ is easier to evaluate we need some preliminary calculations. The boundary curve C can be described by $x = 2\cos(t), y = 2\sin(t), z = 0, 0 \le t \le 2\pi$, so on C,

$$\mathbf{F}\cdot d\mathbf{R} = [-16\cos^2(t)\sin^2(t) - 16\cos^2(t)\sin^2(t)]d\sigma = -8\sin^2(2t)d\sigma = -4(1-\cos(4t))d\sigma.$$

We easily find

$$\oint_C \mathbf{F}\cdot d\mathbf{R} = \int_0^{2\pi} -4(1-\cos(4))d\sigma = -8\pi.$$

For comparison, we investigate the surface integral. $\nabla\times\mathbf{F} = 0\mathbf{i} + 0\mathbf{j} - (x^2+y^2)\mathbf{k}$. A normal to Σ is $\nabla(x^2+y^2+z^2) = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$, so a unit normal is $\mathbf{N} = \frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{2}$. Now $\frac{\partial z}{\partial x} = -\frac{x}{z}, \frac{\partial z}{\partial y} = -\frac{y}{z}$, so $d\sigma = \sqrt{1 + \frac{x^2}{z^2} + \frac{y^2}{z^2}}dA = \frac{2}{z}\,dA$. Putting all this together we find

$$\iint_\Sigma \nabla\times\mathbf{F}\cdot\mathbf{N}d\sigma = -\iint_D (x^2+y^2)dA = -\int_0^{2\pi}\int_0^2 r^3 dr d\theta = -2\pi(4) = -8\pi.$$

It appears the line integral was less work to calculate here.

2. In this problem, the curve C is a circle of radius 3 in the plane $z = 9$, so it appears $\oint_C \mathbf{F}\cdot d\mathbf{R}$ will be easy to evaluate. Parametrize C by $x = 3\cos(t), y = 3\sin(t), z = 9, 0 \le t \le 2\pi$, evaluate $\mathbf{F}$ along C to get $\oint_C \mathbf{F}\cdot d\mathbf{R} = \int_0^{2\pi} 9\cos(t)\sin(t)(-3\sin(t) + 27\sin(t)(3\cos(t))d\sigma = 27\int_0^{2\pi} -\sin^2(t)\cos(t) + 3\sin(t)\cos(t)d\sigma = 0$. The surface integral $\iint_\Sigma(\nabla\times\mathbf{F}\cdot\mathbf{N})d\sigma$ is much harder to compute here.

3. If we compute $\nabla\times\mathbf{F}$ we find $\nabla\times\mathbf{F} = \mathbf{i}+\mathbf{j}+\mathbf{k}$, and a unit normal to Σ is given by $\mathbf{N} = \frac{x\mathbf{i}+y\mathbf{j}-2\mathbf{k}}{\sqrt{x^2+y^2+z^2}} = \frac{x\mathbf{i}+y\mathbf{j}-2\mathbf{k}}{\sqrt{2}\sqrt{x^2+y^2}}$. Finally $d\sigma = \sqrt{1 + \frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2}}dA = \sqrt{2}dA$. So $\iint_\Sigma \nabla\times\mathbf{F}\cdot\mathbf{N}d\sigma = \iint_D \frac{(x+y-z)}{\sqrt{x^2+y^2}}dA = \iint_D(\frac{x+y}{\sqrt{x^2+y^2}} - 1)dA$, since $z = \sqrt{x^2+y^2}$.

Changing to polar coordinates we get $= \int_0^{2\pi}\int_0^4 [\cos(\theta) + \sin(\theta) - 1]r\,dr\,d\theta = -16\pi$. For the line integral, parametrize C by $x = 4\sin(t), y = 4\cos(t), z = 4, 0 \le t \le 2\pi$ to get $\oint_C \mathbf{F}\cdot d\mathbf{R} = \int_0^{2\pi} [4(4\cos(t)) + 4\sin(t)(-4\sin(t))]d\sigma = -16\pi$. Note, the parametrization of C was chosen so the orientation of C is consistent with the choice of normal $\mathbf{N}$ on Σ.

4. The boundary curve of Σ is the circle $x^2 + y^2 = 6$ in the xy plane, which we parametrize as $x = \sqrt{6}\cos(t), y = \sqrt{6}\sin(t), z = 0, 0 \le t \le 2\pi$. Then we get $\oint_C \mathbf{F}\cdot d\mathbf{R} = \int_0^{2\pi} 6\cos^2(t)\sqrt{6}\cos(t)d\sigma = 6\sqrt{6}\int_0^{2\pi}(1 - \sin^2(t))\cos(t)d\sigma = 0$.

5. Since the boundary curve would require piecewise parametrization, we select $\iint_\Sigma \nabla\times\mathbf{F}\cdot\mathbf{N}d\sigma$ to evaluate. We have $\nabla\times\mathbf{F} = (x-y)\mathbf{i} - y\mathbf{j} - x\mathbf{k}$, and $\mathbf{N} = \dfrac{2\mathbf{i}+4\mathbf{j}+\mathbf{k}}{\sqrt{21}}$, also $d\sigma = \sqrt{21}dA$. So $\iint_\Sigma \nabla\times\mathbf{F}\cdot\mathbf{N}d\sigma = \iint_D (x - 6y)dA = \int_0^2\int_0^{4-2y}(x - 6y)dx\,dy = -\dfrac{32}{3}$.

6. The circulation is $\oint_C \mathbf{F}\cdot d\mathbf{R}$. Take Σ to be the disk $0 \le x^2 + y^2 \le 1$, with boundary C given by $x = \cos(t), y = \sin(t), z = 0, 0 \le t \le 2\pi$. Then the proper normal to Σ is $\mathbf{N} = \mathbf{k}$. We have $\nabla\times\mathbf{F} = -za\mathbf{j} + (2xy + 1)\mathbf{k}$, so $\nabla\times\mathbf{F}\cdot\mathbf{N} = (2xy + 1)$, and $d\sigma = dA$. Then $\oint_C \mathbf{F}\cdot d\mathbf{R} = \iint_\Sigma \nabla\times\mathbf{F}\cdot\mathbf{N}d\sigma = \iint_D (2xy + 1)dA = \text{area of}A = \pi$, since $\iint_D 2xy\,dA = 0$.

7. The integral $\oint_C \mathbf{F}\cdot\mathbf{T}ds = \oint_C \mathbf{F}\cdot d\mathbf{R} = \iint_\Sigma \nabla\times\mathbf{F}\cdot\mathbf{N}d\sigma$. So calculate $\nabla\times\mathbf{F} = -(\mathbf{i}+\mathbf{j}+\mathbf{k}), \mathbf{N} = \dfrac{\mathbf{i}+4\mathbf{j}+\mathbf{k}}{3\sqrt{2}}$ and $d\sigma = 3\sqrt{2}dA$. Then $\oint_C \mathbf{F}\cdot\mathbf{T}ds = \iint_\Sigma \nabla\times\mathbf{F}\cdot\mathbf{N}d\sigma = \iint_D(-6)dA = -6(\text{area of}D) = -108$.

8. $\nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x & -2y & 2z \end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}$, so $\mathbf{F}$ is conservative. To find a potential function we write $\dfrac{\partial\phi}{\partial x} = f_1 = 2x, \dfrac{\partial\phi}{\partial y} = f_2 = -2y, \dfrac{\partial\phi}{\partial z} = f_3 = 2z$. Integrate all three partials and get

$\phi(x,y,z) = x^2 + c(y,z)$,
$\phi(x,y,z) = -y^2 + k(x,z)$,
$\phi(x,y,z) = z^2 + h(x,y)$.

We see the choice $c(y,z) = z^2 - y^2, k(x,z) = x^2 + z^2$ and $h(x,y) = x^2 - y^2$ satisfies all three equations and gives a potential function $\phi(x,y,z) = x^2 - y^2 + z^2$.

9. $\nabla\times\mathbf{F} = \mathbf{0}$ so $\mathbf{F}$ is conservative. Clearly $\phi(x,y,z) = x - 2y + z$ is a potential function.

10. $\nabla\times\mathbf{F} = [\sin(x) - \sin(x)]\mathbf{i} + [y\cos(x) - y\cos(x)]\mathbf{j} + [z\cos(x) - z\cos(x)]\mathbf{k} = \mathbf{0}$ so $\mathbf{F}$ is conservative. Then $\dfrac{\partial\phi}{\partial x} = yz\cos(x), \dfrac{\partial\phi}{\partial y} = z\sin(x) + 1, \dfrac{\partial\phi}{\partial z} = y\sin(x)$. Integrate these three with respect to x, y, z respectively and get

$\phi(x,y,z) = yz\sin(x) + c(y,z)$,
$\phi(x,y,z) = yz\sin(x) + y + k(x,z)$,
$\phi(x,y,z) = yz\sin(x) + h(x,y)$.

Choosing $c(y,z) = h(x,y) = y$ and $k(x,z) = 0$ gives a potential function $\phi(x,y,z) = yz\sin(x) + y$.

11. $\nabla \times \mathbf{F} = (-z^2 - xy)\mathbf{i} + 0\mathbf{j} + yz\mathbf{k} \neq 0$, so $\mathbf{F}$ is not conservative.

12. $\nabla \times \mathbf{F} = (x^2 - x^2)\mathbf{i} + e^{xyz}[2xy + x^2y^2z]\mathbf{j} + [2xz - e^{xyz}(2xz + x^2yz^2)]\mathbf{k} \neq 0$, so $\mathbf{F}$ is not conservative.

13. $\nabla \times \mathbf{F} = 0\mathbf{i} + 0\mathbf{j} + xy\cos(xy)\mathbf{k} \neq 0$, so $\mathbf{F}$ is not conservative.

14. $\nabla \times \mathbf{F} = (6xy - 6xy)\mathbf{i} + (3y^2 - 3y^2)\mathbf{j} + (6yz - 6yz)\mathbf{k} = 0$, so F is conservative. Then $\dfrac{\partial\phi}{\partial x} = 2x^2 + 3y^2z$, $\dfrac{\partial\phi}{\partial y} = 6xyz$, $\dfrac{\partial\phi}{\partial z} = 3xy^2$. Integrate with respect to x, y, and z respectively and get
$\phi(x,y,z) = \dfrac{2}{3}x^3 + 3xy^2z + c(y,z)$,
$\phi(x,y,z) = 3xy^2z + k(x,z)$,
$\phi(x,y,z) = 3xy^2z + h(x,y)$.
Choose $k(x,z) = h(x,y) = \dfrac{2}{3}x^3$ and $c(x,y) = 0$ to get a potential function $\phi(x,y,z) = \dfrac{2}{3}x^3 + 3xy^2z$.

15. A potential function for $\mathbf{F}$ is easily found to be $\phi(x,y,z) = x - 3y^3z$, so $\int_C \mathbf{F}\cdot d\mathbf{R} = \phi(0,3,5) - \phi(1,1,1) = -403$.

16. The easiest way to find a potential function for $\mathbf{F}$ here is to start with $\dfrac{\partial\phi}{\partial y} = x\cos(xz)$ and integrate with respect to y to get $\phi(x,y,z) = xy\cos(xz) + k(x,z)$. This function can easily be checked to be a potential for $\mathbf{F}$ when $k(x,z) = 0$. Thus $\int_C \mathbf{F}\cdot d\mathbf{R} = \phi(1,1,7) - \phi(1,0,\pi) = \cos(7)$.

17. A potential function is easily found to be $\phi(x,y,z) = 2x^3e^{yz}$, so $\int_C \mathbf{F}\cdot d\mathbf{R} = \phi(1,2,-1) - \phi(0,0,0) = 2e^{-2}$.

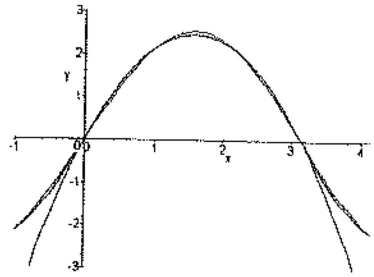
18. A potential function is easily found to be $\phi(x,y,z) = -8xy^2 - 4zy$, so $\int_C \mathbf{F}\cdot d\mathbf{R} = \phi(1,3,2) - \phi(-2,1,1) = -108$.

19. A potential function for $\mathbf{F}$ is easily found to be $\phi(x,y,z) = -x + 2yz^2$, so $\int_C \mathbf{F}\cdot d\mathbf{R} = \phi(1,1,6) - \phi(0,0,-4) = 71$.

20. A potential function for $\mathbf{F}$ is easily found to be $\phi(x,y,z) = xy - 2x^2z + z^3$, so $\int_C \mathbf{F}\cdot d\mathbf{R} = \phi(3,1,4) - \phi(1,1,1) = -5$.

Chapter Fourteen - Fourier Series

Section 14.1 Why Fourier Series?



1. The graph at the right shows the function $f(x) = x(\pi - x)$ along with the first three different partial sums of the Fourier series $S_2(x), S_4(x), S_6(x)$. On the interval $[0,\pi]$ all four graphs very nearly coincide at each point $x \in [0,\pi]$. Outside $[0,\pi]$ the partial sums stay close together, but the graph of f diverges from these partial sums.

2. Suppose $P(x)$ is a polynomial of degree k and that $P(x) = \alpha\sin(nx)$ for some n and all $x \in (0,\pi)$. Differentiate this equation $k+1$ times to get either $0 = \sin(nx)$ or $0 = \cos(nx)$ for all $x \in (0,\pi)$. Both of these statements are clearly false, thus $P(x)$ could not be any constant multiple of $\sin(nx)$ for all $x \in (0,\pi)$.

3. The same kind of reasoning used in Problem 2 would give (after $k+1$ differentiations) that $\sum_{j=1}^{n} \hat{c_j}\sin(jx) = 0$. But the functions $\{\sin jx\}_{j=1}^{n}$ are linearly independent on $[0,\pi]$, which would imply that $\hat{c_j} = 0$ and hence $c_j = 0$, for $j = 1, \ldots n$. But this would imply that $P(x) = 0$ for all $x \in [0,\pi]$, a clear contradiction.

Section 14.2 The Fourier Series of a Function

1. The Fourier series of $f(x) = 4$ on $[-3,3]$ is $\dfrac{a_0}{2} + \sum\limits_{n=1}^{\infty}\left[a_n\cos\left(\dfrac{n\pi x}{3}\right) + b_n\sin\left(\dfrac{n\pi x}{3}\right)\right]$. Since $f(x) = 4$ is even, we have $a_0 = \dfrac{2}{3}\int_0^3 4dx = 8, a_n = \dfrac{2}{3}\int_0^3 4\cos\left(\dfrac{n\pi x}{3}\right)dx = 0, n \geq 1$ and $b_n = 0$. The Fourier series is then 4.

2. The Fourier series of $f(x) = -x$ on $[-1,1]$ is $\dfrac{a_0}{2} + \sum\limits_{n=1}^{\infty}[a_n\cos(n\pi x) + b_n\sin(n\pi x)]$. Since $f(x) = -x$ is odd, we have $a_n = 0, n \geq 0$, and $b_n = 2\int_0^1 (-x)\sin(n\pi x)\,dx = \dfrac{2}{n\pi}(-1)^n, n \geq 1$. The Fourier series is $\dfrac{2}{\pi}\sum\limits_{n=1}^{\infty}\dfrac{(-1)^n}{n}\sin(n\pi x)$.

3. Since $f(x) = \cosh(\pi x)$ is even, we have $b_n = 0, n \geq 1, a_0 = \int_0^1 \cosh(\pi x)\,dx = \dfrac{\sinh(\pi)}{\pi}$, $a_n = 2\int_0^1 \cosh(\pi x)\cos(n\pi x)\,dx = \dfrac{2\sinh(\pi)}{\pi}\dfrac{(-1)^n}{n^2+1}, n \geq 1$.
The Fourier series is

$$\frac{\sinh(\pi)}{\pi} + \frac{2\sinh(\pi)}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}\cos(n\pi x)$$

4. $\dfrac{8}{\pi^2}\sum\limits_{n=1}^{\infty}\dfrac{1}{(2n-1)^2}\cos\left[\dfrac{(2n-1)\pi x}{2}\right]$.

5. $\dfrac{16}{\pi}\sum\limits_{n=1}^{\infty}\left(\dfrac{1}{2n-1}\right)\sin[(2n-1)x]$

6. Note that since $f(x)$ is odd and periodic of period π, $f(x) = \sin(2x)$ is its own Fourier series.

7. Write $f(x) = (3+x^2) - x$ which has even first term and odd second term to simplify calculation of coefficients. Compute

$a_0 = \int_0^2 \left(3+x^2\right) dx = \frac{26}{3}$,

$a_n = \int_0^2 \left(3+x^2\right) \cos\left(\frac{n\pi x}{2}\right) dx = \frac{16}{n^2\pi^2}(-1)^n, n \geq 1$,

$b_n = \int_0^2 (-x) \sin\left(\frac{n\pi x}{2}\right) dx = \frac{4}{n\pi}(-1)^n, n \geq 1.$

The Fourier series of $x^2 - x + 3$ is

$$\frac{13}{3} + \sum_{n=1}^{\infty} (-1)^n \left[\frac{16}{n^2\pi^2}\cos\left(\frac{n\pi x}{2}\right) + \frac{4}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\right].$$

8. Compute

$$a_0 = \frac{1}{5}\left[\int_{-5}^{0}(-x)\,dx + \int_0^5 \left(1+x^2\right) dx\right] = \frac{71}{3},$$

$$a_n = \frac{1}{5}\left[\int_{-5}^{0}(-x)\cos\left(\frac{n\pi x}{5}\right) dx + \int_0^5 \left(1+x^2\right)\cos\left(\frac{n\pi x}{5}\right) dx\right]$$
$$= \frac{25}{n^2\pi^2}\left[11(-1)^n - 1\right], n \geq 1,$$

$$b_n = \frac{1}{5}\left[\int_{-5}^{0}(-x)\sin\left(\frac{n\pi x}{5}\right) dx + \int_0^5 \left(1+x^2\right)\sin\left(\frac{n\pi x}{5}\right) dx\right]$$

$$= \frac{5}{n\pi}[1 - 21(-1)^n] + \frac{25}{n^3\pi^3}[(-1)^n - 10], n \geq 1.$$

The Fourier series is

$$\frac{71}{6} + \sum_{n=1}^{\infty}\left[\frac{25}{n^2\pi^2}[11(-1)^n - 1]\cos\left(\frac{n\pi x}{5}\right)\right.$$
$$\left. + \left\{\frac{5}{n\pi}[1 - 21(-1)^n] + \frac{25}{n^3\pi^3}[(-1)^n - 10]\right\}\sin\left(\frac{n\pi x}{5}\right)\right].$$

9. $\frac{3}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\left(\frac{1}{2n-1}\right)\sin[(2n-1)x]$

10. $\frac{2}{\pi} - \sin x - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{(4n^2-1)}\cos(nx).$

11. $\frac{\sin(3)}{3} + 6\sin(3)\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2\pi^2 - 9}\cos\left(\frac{n\pi x}{3}\right)$

12. $\frac{3}{4}-\sum_{n=1}^{\infty}\left\{\frac{1-(-1)^n}{n^2\pi^2}\cos(n\pi x)+\left[\frac{1-2(-1)^n}{n\pi}\right]\sin(n\pi x)\right\}$.

13. The Fourier series of f and g will be the same. Since changing the value of an integrand, f, at a finite set of x values does not change the value of the integral of f, f and g clearly have the same Fourier coefficients, hence same Fourier series. This shows that the value of the Fourier series of a function at x_0 need not be the value $f(x_0)$.

14. Let f be even on $[-L, L]$. Then $\int_{-L}^{L} f(x)\,dx = \int_{-L}^{0} f(x)\,dx + \int_{0}^{L} f(x)\,dx$. In the first integral on the right, let $t = -x$ to get

$$\int_{-L}^{L} f(x)\,dx = -\int_{L}^{0} f(-t)\,dt + \int_{0}^{L} f(x)\,dx = \int_{0}^{L} [f(-t)\,dt + \int_{0}^{L} f(x)\,dx.$$

But $f(-t) = f(t)$ since f is even. Hence $\int_{-L}^{L} f(x)\,dx = 2\int_{0}^{L} f(x)\,dx$.

15. Let f be odd on $[-L, L]$. Then $\int_{-L}^{L} f(x)\,dx = \int_{-L}^{0} f(x)\,dx + \int_{0}^{L} f(x)\,dx$.
Now let $t = -x$ in the first of these two integrals, and recall $f(-t) = -f(t)$ to get

$$\int_{-L}^{L} f(x)\,dx = -\int_{L}^{0} f(-t)\,dt + \int_{0}^{L} f(x)\,dx = -\int_{0}^{L} f(t)\,dt + \int_{0}^{L} f(x)\,dx = 0.$$

Section 14.3 Convergence of Fourier Series

In Problems 1 through 10, the function f given in the problem is piecewise continuous on $[-L, L]$, and for each $x \in (-L.L)$, f has both a left hand derivative and a right hand derivative. At each endpoint, one-sided derivatives from the interior of the interval exist. It follows by Theorems 14.1 and 14.2 that at each $x \in (-L, L)$ the Fourier series converges to $\frac{1}{2}(f(x^-) + f(x^+))$, while at $x = -L$ and $x = L$ the series converges to $\frac{1}{2}(f(-L^+) + f(L^-))$. Note that at interior points of continuity of f, the series converges to $f(x)$. The Fourier series and the sum of the series at each point in $[-L, L]$ is given in each problem below. Selected exercises show a plot of the twenty-fifth partial sum of the Fourier series along with $f(x)$. Observe the Gibbs phenomenon of overshooting near each jump discontinuity of f.

1. The Fourier series of f is

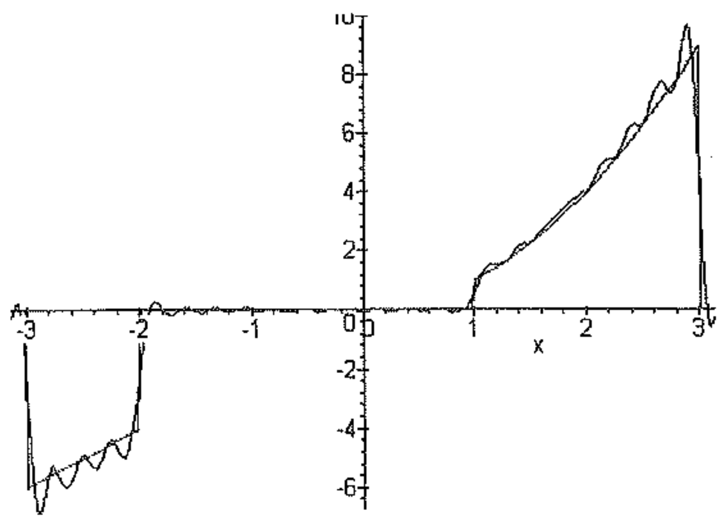
$$\frac{11}{18}+\sum_{n=1}^{\infty}\left\{\frac{1}{n\pi}\left[4\sin\left(\frac{2n\pi}{3}\right)-\sin\left(\frac{n\pi}{3}\right)\right]+\right.$$

$$\left.\frac{6}{n^2\pi^2}\left[\cos\left(\frac{2n\pi}{3}\right)-\cos\left(\frac{n\pi}{3}\right)+2(-1)^n\right]+\frac{18}{n^3\pi^3}\sin\left(\frac{n\pi}{3}\right)\right\}\cos\left(\frac{n\pi x}{3}\right)$$

$$+\sum_{n=1}^{\infty}\left\{\frac{1}{n\pi}\left[4\cos\left(\frac{2n\pi}{3}\right)+\cos\left(\frac{n\pi}{3}\right)-15(-1)^n\right]\right.$$

$$-\frac{6}{n^2\pi^2}\left[\sin\left(\frac{2n\pi}{3}\right)+\sin\left(\frac{n\pi}{3}\right)\right]-\frac{18}{n^3\pi^3}\left[\cos\left(\frac{n\pi}{3}\right)-(-1)^n\right]\right\}\sin\left(\frac{n\pi x}{3}\right)$$

$$=\begin{cases} 3/2 & \text{if} \quad x=-3 \text{ or } x=3 \\ 2x & \text{if} \quad -3<x<-2 \\ -2 & \text{if} \quad x=-2 \\ 0 & \text{if} \quad -2<x<1 \\ 1/2 & \text{if} \quad x=1 \\ x^2 & \text{if} \quad 1<x<3 \end{cases}$$



2. Since $f(x)=x^2$ is even, $b_n=0, n\geq 1$. Calculate $a_0=\int_0^2 x^2dx=\frac{8}{3}, a_n=\int_0^2 x^2\cos\left(\frac{n\pi x}{2}\right)dx=\frac{16}{n^2\pi^2}(-1)^n, n\geq 1$. The Fourier series of f is

$$\frac{4}{3}+\frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\left(\frac{n\pi x}{2}\right)$$

and converges to $f(x)=x^2$ for all $-2\leq x\leq 2$.

3. The Fourier series of f is

$$\left[\frac{11}{3}\sinh(3)-2\cosh(3)\right]$$

$$+\sum_{n=1}^{\infty}(-1)^n\left\{\sinh(3)\left[\frac{1}{(\alpha^2+1)}+\frac{4(1-3\alpha^2)}{3(\alpha^2+1)^3}\right]+\frac{4(\alpha^2-1)\cosh(3)}{(\alpha^2+1)^2}\right\}\cos(\alpha x)$$

$$+\sum_{n=1}^{\infty}(-1)^n\left\{\sinh(3)\left[\frac{6\alpha}{(\alpha^2+1)}+\frac{4\alpha(\alpha^2-3)}{3(\alpha^2+1)^3}\right]-\frac{8\alpha\cosh(3)}{(\alpha^2+1)^2}\right\}\sin(\alpha x)$$

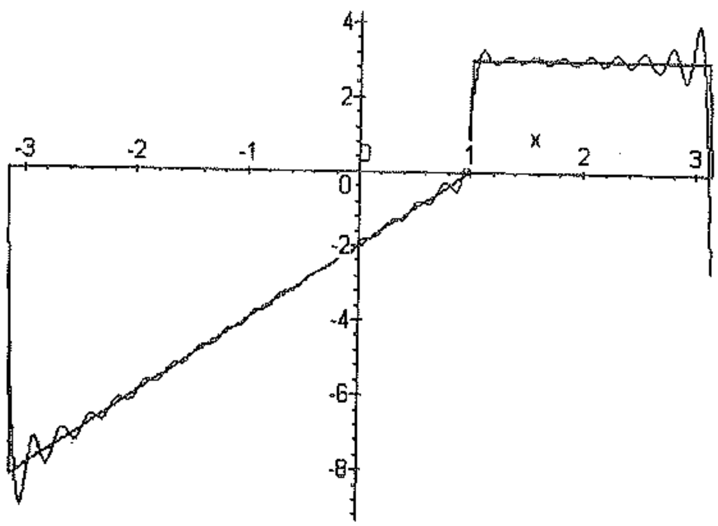
$$= \begin{cases} 18\cosh(3) & \text{if} \quad x = -3 \text{ or } x = 3 \\ x^2 e^{-x} & \text{if} \quad -3 < x < 3 \end{cases} \; ; \text{ where } \alpha = \frac{n\pi}{3}$$

4. The Fourier series of f is

$$-\frac{\pi}{2}\left(\pi^2 - \pi + 4\right) + \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{2\cos(n) + 2(-1)^{(n+1)} - 3n\sin(n)}{n^2}\cos(nx)$$

$$+\frac{1}{\pi}\sum_{n=1}^{\infty} \frac{2\sin(n) + (2\pi+5)\,n\,(-1)^{n+1} + 3n\cos(n)}{n^2}\sin(nx)$$

$$= \begin{cases} (1-2\pi)/2 & \text{if} \quad x = -\pi \text{ or } x = \pi \\ 3/2 & \text{if} \quad x = 1 \\ f(x) & \text{elsewhere} \end{cases}$$

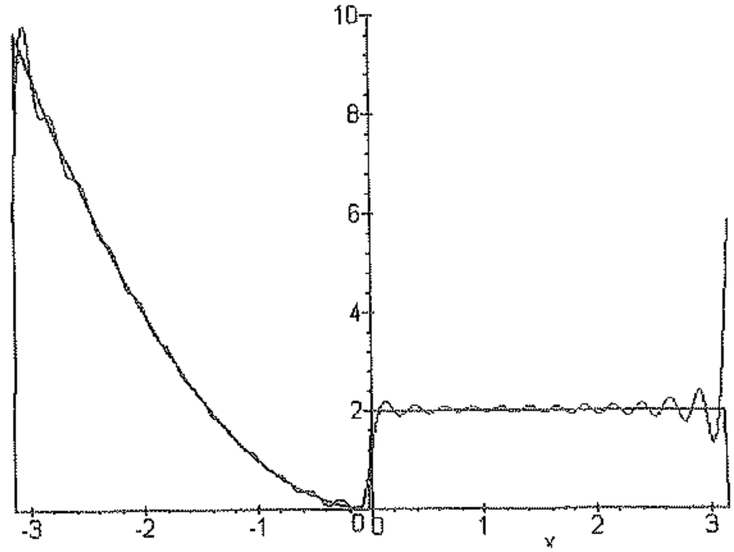


5. The Fourier series of f is

$$\frac{6+\pi^2}{6} + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos(nx)$$

$$+\frac{1}{\pi}\sum_{n=1}^{\infty} \left[\left(\frac{2}{n^3} + \frac{2}{n}\right)(1-(-1)^n) + \frac{\pi^2}{n}(-1)^n\right]\sin(nx)$$

$$= \begin{cases} (\pi^2+2)/2 & \text{if} \quad x = -\pi \text{ or } x = \pi \\ x^2 & \text{if} \quad -\pi < x < 0 \\ 1 & \text{if} \quad x = 0 \\ 2 & \text{if} \quad 0 < x < \pi \end{cases}$$

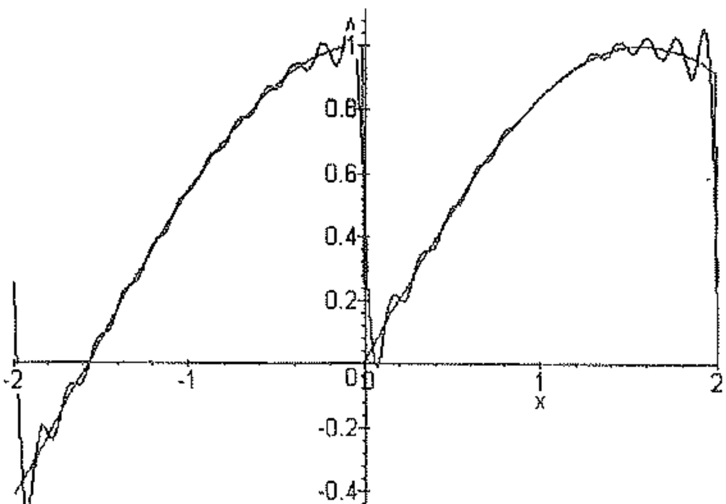


6. The Fourier series of f is

$$\frac{1}{4}\left[1+\sin(2)-\cos(2)\right]+\sum_{n=1}^{\infty}\frac{(-1)^n\,2\sin(2)}{\pi^2n^2-4}\cos\left(\frac{n\pi x}{2}\right)$$

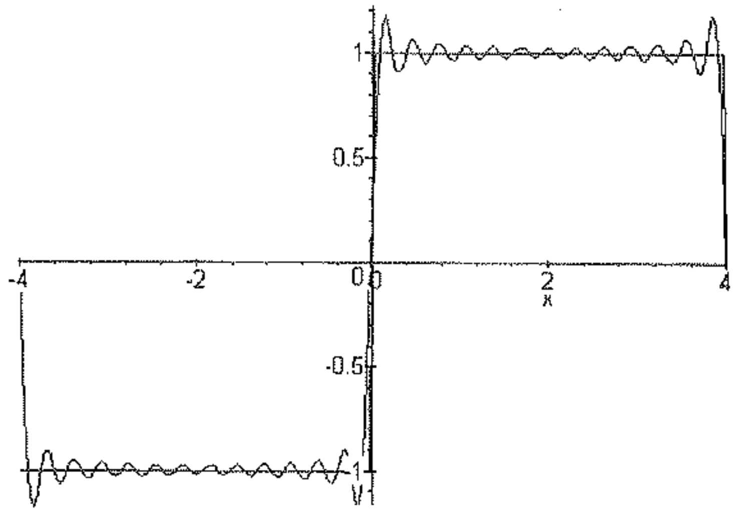
$$+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\,n\pi}{n^2\pi^2-4}\left[1+(-1)^n\left(\sin(2)-\cos(2)\right)\right]\sin\left(\frac{n\pi x}{2}\right)$$

$$=\begin{cases}(\cos(2)+\sin(2))/2 & \text{if} \quad x=-2 \text{ or } x=2\\ \cos(x) & \text{if} \quad -2<x<0\\ 1/2 & \text{if} \quad x=0\\ \sin(x) & \text{if} \quad 0<x<2\end{cases}$$



7. The Fourier series of f is $\dfrac{4}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{1}{(2n-1)}\sin\left[\frac{(2n-1)\,\pi x}{4}\right]$

$$= \begin{cases} -1 & \text{if} \quad -4 < x < 0 \\ 0 & \text{if} \quad x = -4, 0 \text{ or } 4 \\ 1 & \text{if} \quad 0 < x < 4 \end{cases}$$

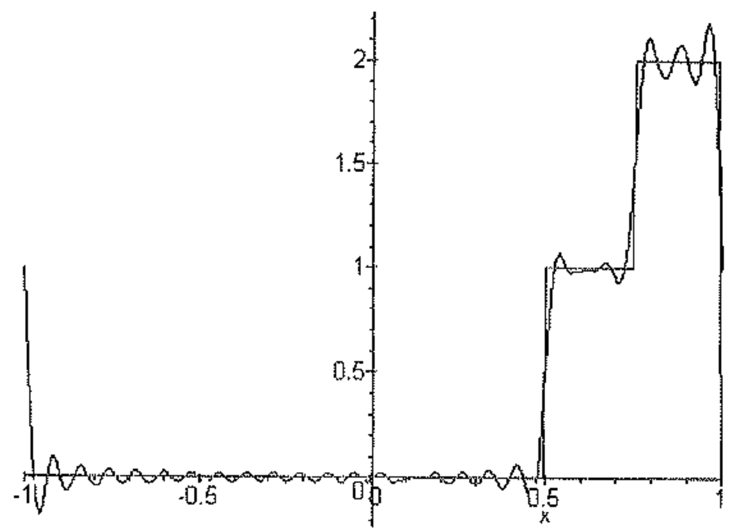


8. The Fourier series of f is

$$\frac{3}{8} - \sum_{n=1}^{\infty} \frac{1}{n\pi} \left[\sin\left(\frac{n\pi}{2}\right) + \sin\left(\frac{3n\pi}{4}\right) \right] \cos(n\pi x)$$

$$+ \sum_{n=1}^{\infty} \frac{1}{n\pi} \left[\cos\left(\frac{n\pi}{2}\right) + \cos\left(\frac{3n\pi}{4}\right) - 2(-1)^n \right] \sin(n\pi x)$$

$$= \begin{cases} 1 & \text{if} \quad x = -1, x = 1, \text{ or } 1/2 < x < 3/4 \\ 0 & \text{if} \quad -1 < x < 1/2 \\ 2 & \text{if} \quad 3/4 < x < 1 \\ 1/2 & \text{if} \quad x = 1/2 \\ 3/2 & \text{if} \quad x = 3/4 \end{cases}$$



9. The Fourier series of f is

$$\frac{1-e^{-\pi}}{\pi}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n e^{-\pi}}{1+n^2}\cos(nx),$$

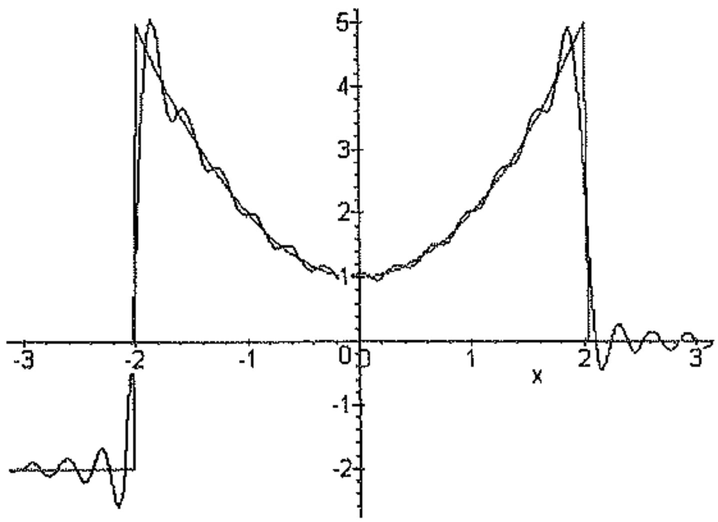
and this series converges to $f(x)$ for $-\pi \le x \le \pi$.

10. The Fourier series of f is

$$\frac{2}{3}+\frac{2}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^3}\left[(4n^2\pi^2-32)\sin\left(\frac{n\pi}{2}\right)+16n\pi\cos\left(\frac{n\pi}{2}\right)\right]\cos\left(\frac{n\pi x}{2}\right)$$

$$+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[\cos\left(\frac{n\pi}{2}\right)-(-1)^n\right]\sin\left(\frac{n\pi x}{2}\right)$$

$$=\begin{cases} -1 \text{ if } x=-4 \text{ or } x=4 \\ 3/2 \text{ if } x=-2 \\ 5/2 \text{ if } x=2 \\ f(x) \text{ elsewhere} \end{cases}$$



11. The Fourier series of $f(x) = x^2/2$ on $[-\pi, \pi]$ is easily found to be

$$\frac{\pi^2}{6} + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nx).$$

Since $f(x)$ is differentiable on $(-\pi, \pi)$, and $f_R'(-\pi)$ and $f_L'(\pi)$ both exist, we have by Theorem 13.2 that the Fourier seies converges to $x^2/2$ for $-\pi \leq x \leq \pi$. Thus put $x = \pi$ to get

$$\frac{\pi^2}{2} = \frac{\pi^2}{6} + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} (-1)^n.$$

Then

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{2}\left(\frac{\pi^2}{2} - \frac{\pi^2}{6}\right) = \frac{\pi^2}{6}.$$

12. Put $x = 0$ into the Fourier series of $f(x) = x^2/2$ on $[-\pi, \pi]$, found in Problem 11. This gives us

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}.$$

Section 14.4 Fourier Cosine and Sine Series

In Problem 1 through 10 the cosine series of f on $[0, L]$ is denoted by $C(x)$, the sine series is denoted by $S(x)$. The sums of the respective series are determined from Theorem 14.3 and 14.4 of this section.

1. $C(x) = 4$ for $0 \leq x \leq 3$;

$$S(x) = \frac{16}{\pi}\sum_{n=1}^{\infty} \frac{1}{2n-1} \sin\left[\frac{(2n-1)\pi x}{3}\right]$$

$$= \begin{cases} 0 & \text{if} \quad x = 0 \text{ or } x = 3 \\ 4 & \text{if} \quad 0 < x < 3 \end{cases}$$

2. $C(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)}\cos\left[\frac{(2n-1)\pi x}{2}\right]$

$$= \begin{cases} 1 & \text{if} \quad 0 \le x < 1 \\ 0 & \text{if} \quad x = 1 \\ -1 & \text{if} \quad 1 < x \le 2 \end{cases}$$

$S(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)}\sin\left[\frac{(2n-1)\pi x}{2}\right] =$

$$\begin{cases} 1 & \text{if} \quad 0 < x < 1 \\ 0 & \text{if} \quad x = 0 \text{ or } 1 \text{ or } 2 \\ -1 & \text{if} \quad 1 < x < 2 \end{cases}$$

3. $C(x) = \frac{1}{2}\cos(x) - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n(2n-1)}{(2n-3)(2n+1)}\cos\left[\frac{(2n-1)x}{2}\right]$

$$= \begin{cases} 0 & \text{if} \quad 0 \le x < \pi \\ -1/2 & \text{if} \quad x = \pi \\ \cos(x) & \text{if} \quad \pi < x < 2\pi \\ 0 & \text{if} \quad x = 2\pi \end{cases}$$

$$S(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n(2n-1)}{(2n-3)(2n+1)}\sin\left[\frac{(2n-1)x}{2}\right] - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{n}{n^2-1}\sin(nx)$$

$$= \begin{cases} 0 & \text{if} \quad 0 \le x < \pi \\ -1/2 & \text{if} \quad x = \pi \\ \cos(x) & \text{if} \quad \pi < x < 2\pi \\ 0 & \text{if} \quad x = 2\pi \end{cases}$$

4. $C(x) = 1 - \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos[(2n-1)\pi x] = 2x$ for $0 \le x \le 1$;

$S(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin(n\pi x) = \begin{cases} 2x & \text{if} \quad 0 \le x < 1 \\ 0 & \text{if} \quad x = 1 \end{cases}$

5. $C(x) = \frac{4}{3} + \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\left(\frac{n\pi x}{2}\right) = x^2$ for $0 \le x \le 2$;

$S(x) = -\frac{8}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n}{n} + \frac{2[1-(-1)^n]}{n^3\pi^2}\right]\sin\left(\frac{n\pi x}{2}\right)$

$$= \begin{cases} x^2 & \text{if} \quad 0 \le x < 2 \\ 0 & \text{if} \quad x = 2 \end{cases}$$

6. $C(x) = 1 - e^{-1} + 2\sum_{n=1}^{\infty} \frac{1}{1+n^2\pi^2}\left[1 - (-1)^n e^{-1}\right]\cos(n\pi x) = e^{-x}$ for $0 \le x \le 1$;

$$S(x) = 2\pi \sum_{n=1}^{\infty}\left[\frac{n}{1+n^2\pi^2}\left[1-(-1)^n e^{-1}\right]\right]\sin(n\pi x) = \begin{cases} e^{-x} & \text{if } 0 < x < 1 \\ 0 & \text{if } x = 0 \text{ or } 1 \end{cases}$$

7.

$$C(x) = \frac{1}{2}$$

$$+\sum_{n=1}^{\infty}\left[\frac{4}{n\pi}\sin\left(\frac{2n\pi}{3}\right) + \frac{12}{n^2\pi^2}\cos\left(\frac{2n\pi}{3}\right) - \frac{6}{n^2\pi^2}\left[1+(-1)^n\right]\right]\cos\left(\frac{n\pi x}{3}\right)$$

$$= \begin{cases} x & \text{if } 0 \le x < 2 \\ 1 & \text{if } x = 2 \\ 2 - x & \text{if } 2 < x \le 3 \end{cases}$$

$$S(x) = \sum_{n=1}^{\infty}\left[\frac{12}{n^2\pi^2}\sin\left(\frac{2n\pi}{3}\right) - \frac{4}{n\pi}\cos\left(\frac{2n\pi}{3}\right) + \frac{2}{n\pi}(-1)^n\right]\sin\left(\frac{n\pi x}{3}\right)$$

$$= \begin{cases} x & \text{if } 0 \le x < 2 \\ 1 & \text{if } x = 2 \\ 2 - x & \text{if } 2 < x < 3 \\ 0 & \text{if } x = 3 \end{cases}$$

8. $C(x) = -\frac{1}{5} + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\cos\left(\frac{n\pi}{5}\right)\sin\left(\frac{2n\pi}{5}\right)\cos\left(\frac{n\pi x}{5}\right)$

$$= \begin{cases} 1 & \text{if } 0 \le x < 1 \\ 1/2 & \text{if } x = 1 \\ 0 & \text{if } 1 < x < 3 \\ -1/2 & \text{if } x = 3 \\ -1 & \text{if } 3 < x \le 5 \end{cases}$$

$$S(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n}\left[1 + (-1)^n - 2\cos\left(\frac{n\pi}{5}\right)\cos\left(\frac{2n\pi}{5}\right)\right]\sin\left(\frac{n\pi x}{5}\right)$$

$$= \begin{cases} 1 & \text{if } 0 < x < 1 \\ 1/2 & \text{if } x = 1 \\ 0 & \text{if } 1 < x < 3 \text{ or } x = 0 \text{ or } x = 5 \\ -1/2 & \text{if } x = 3 \\ -1 & \text{if } 3 < x < 5 \end{cases}$$

9. $C(x) = \frac{5}{6} + \frac{16}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{1}{n^2}\cos\left(\frac{n\pi}{4}\right) - \frac{4}{n^3\pi}\sin\left(\frac{n\pi}{4}\right)\right]\cos\left(\frac{n\pi x}{4}\right)$

$$= \begin{cases} x^2 & \text{if } 0 \le x \le 1 \\ 1 & \text{if } 1 < x \le 4 \end{cases}$$

$$S(x) = \sum_{n=1}^{\infty}\left[\frac{16}{n^2\pi^2}\sin\left(\frac{n\pi}{4}\right) + \frac{64}{n^3\pi^3}\left[\cos\left(\frac{n\pi}{4}\right) - 1\right] - \frac{2(-1)^n}{n\pi}\right]\sin\left(\frac{n\pi x}{4}\right)$$

$$= \begin{cases} x^2 & \text{if } 0 \le x \le 1 \\ 1 & \text{if } 1 < x < 4 \\ 0 & \text{if } x = 4 \end{cases}$$

10. $C(x) = -1 - \frac{24}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[2(-1)^n + \frac{4}{n^2\pi^2}[1-(-1)^n]\right]\cos\left(\frac{n\pi x}{2}\right) = 1 - x^3$ if $0 \le x \le 2$;

$$S(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[1 + 7(-1)^n - \frac{48}{n^2\pi^2}(-1)^n\right]\sin\left(\frac{n\pi x}{2}\right)$$

$$= \begin{cases} 1 - x^3 & \text{if } \quad 0 < x < 2 \\ 0 & \text{if } \quad x = 0 \text{ or } 2 \end{cases}$$

11. Define $e(x) = \frac{1}{2}(f(x) + f(-x))$ and $o(x) = \frac{1}{2}(f(x) - f(-x))$. Then $e(x) + o(x) = f(x)$ for $-L \le x \le L$ and $e(-x) = e(x)$, so e is an even function, while $o(-x) = -o(x)$, so o is an odd function.

12. The only such function is the identically zero function. If f is both even and odd, then in Problem 11, $e(x) = o(x) = 0$ for all x, so $f(x) = 0$ for all x.

13. The Fourier cosine series of $\sin(x)$ on $[0, \pi]$ is $\frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\cos(2nx)$. By Theorem 14.3, this converges to $\sin(x)$ for $0 \le x \le \pi$. Now put $x = \pi/2$ to get

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2-1} = \frac{\pi}{4}\left(\frac{2}{\pi} - 1\right) = \frac{1}{2} - \frac{\pi}{4}.$$

Section 14.5 Integration and Differentiation of Fourier Series

1. Let f satisfy the hypotheses of Theorem 14.6, and suppose f and f' have the respective Fourier series

$$f : \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right),$$

$$f' : \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + B_n \sin\left(\frac{n\pi x}{L}\right)$$

Then

$$A_0 = \frac{1}{L}\int_{-L}^{L} f'(x)\,dx = \frac{1}{L}[f(L) - f(-L)] = 0$$

$$A_n = \frac{1}{L}\int_{-L}^{L} f'(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{1}{L}\left\{f(x)\cos\left(\frac{n\pi x}{L}\right)\Big|_{-L}^{L} + \frac{n\pi}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx\right\} = \frac{n\pi}{L}b_n, n \ge 1;$$

$$B_n = \frac{1}{L}\int_{-L}^{L} f'(x)\sin\left(\frac{n\pi x}{L}\right)dx =$$

$$\frac{1}{L}\left\{f(x)\sin\left(\frac{n\pi x}{L}\right)\Big|_{-L}^{L} - \frac{n\pi}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx\right\} = -\frac{n\pi}{L}a_n, n \ge 1.$$

Thus the Fourier coefficients of the series for f' are precisely what one would get by termwise differentiation of the series for f.

2. For $f(x) = |x|$ or $[-1,1]$, certainly f is continuous. Also $f'(x) = -1, -1 < x < 0$ and $f'(x) = 1, 0 < x < 1$; $\lim_{x\to 0^-} f'(x) = -1$, $\lim_{x\to 0^+} f'(x) = 1$, $\lim_{x\to -1^+} f'(x) = -1$, $\lim_{x\to 1^-} f'(x) = 1$, so f is piecewise smooth on $[-1,1]$. Clearly $f(-1) = f(1) = 1$. Also $f''(x) = 0$ for all $x \neq 0$ in $(-1,1)$.
The Fourier series of f is easily found to be

$$\frac{1}{2} - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos[(2n-1)\pi x].$$

Termwise differentiation of this series gives

$$\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)}\sin[(2n-1)\pi x].$$

This is easily seen to be the Fourier series of

$$g(x) = \begin{cases} -1, & -1 < x < 0 \\ 1, & 0 < x < 1 \end{cases}$$

by noting g is odd so $a_n = 0, n \geq 0$, and computing

$$b_n = 2\int_0^1 \sin(n\pi x)\,dx = -\frac{2}{n\pi}\cos(n\pi x)\Big|_0^1 = \frac{2}{n\pi}[1-\cos(n\pi)] = \frac{2}{n\pi}[1-(-1)^n].$$

So $b_{2n} = 0, b_{2n-1} = \dfrac{4}{\pi(2n-1)}, n \geq 1$.

3. For the Fourier series of f calculate
$a_0 = \dfrac{1}{\pi}\displaystyle\int_0^\pi x\,dx = \dfrac{\pi}{2}$,
$a_n = \dfrac{1}{\pi}\displaystyle\int_0^\pi x\cos(nx)\,dx = \dfrac{1}{n^2\pi}[(-1)^n - 1], n \geq 1$,
$b_n = \dfrac{1}{\pi}\displaystyle\int_0^\pi x\sin(nx)\,dx = -\dfrac{1}{n}(-1)^n, n \geq 1$. Since $f'(x) = 0, -\pi < x < 0, f'(x) = 1, 0 < x < \pi$, $\lim_{x\to 0^-} f'(x) = 0$, $\lim_{x\to 0^+} f'(x) = 1$, $\lim_{x\to \pi^-} f'(x) = 1$, $\lim_{x\to -\pi^+} f'(x) = 0$, f is piecewise smooth on $[-\pi,\pi]$. By Theorem 14.2,

$$f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left\{\frac{1}{n^2\pi}[(-1)^n - 1]\cos(nx) - \frac{1}{n}(-1)^n\sin(nx)\right\}$$

for all x in $(-\pi,\pi)$.
Now integrate this Fourier series termwise over $[-\pi, x]$ to get

$$\frac{\pi}{4}(x+\pi) + \sum_{n=1}^{\infty}\left\{\frac{1}{n^3\pi}[(-1)^n - 1]\sin(nx) + \frac{1}{n^2}(-1)^n\cos(nx) - \frac{1}{n^2}\right\}.$$

From Problem 11 of Section 14.3, $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$, and the Fourier series for $g(x) = x$ on $[-\pi,\pi]$ is easily found to be $\displaystyle\sum_{n=1}^{\infty}\frac{2(-1)^n}{n}\sin(nx)$. Substitution of these results above gives the Fourier series for $\int_{-\pi}^{x} f(t)\,dt$ to be

$$\frac{\pi^2}{12} + \sum_{n=1}^{\infty} \left[\left\{ \frac{\pi(-1)^n}{2n} + \frac{1}{n^3\pi}\left[(-1)^n - 1\right] \right\} \sin(nx) + \frac{(-1)^n}{n^2}\cos(nx) \right].$$

4. (a) Calculate $a_0 = \frac{1}{3}\int_{-3}^{3} x^2 dx = 6, a_n = \frac{1}{3}\int_{-3}^{3} x^2 \cos\left(\frac{n\pi x}{3}\right) dx = \frac{36(-1)^n}{n^2\pi^2}, n \geq 1; b_n = 0, n \geq 1$ since $f(x) = x^2$ is even. The Fourier series of $f(x) = x^2$ on $[-3, 3]$ is

$$3 + \frac{36}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos\left(\frac{n\pi x}{3}\right).$$

(b) Certainly $f(x) = x^2$ is continuous on $[-3, 3]$, and $f(3) = f(-3) = 9, f'(x) = 2x$ is continuous on $[-3, 3]$, and $f''(x) = 2$ exists on $(-3, 3)$. By Theorem 14.6, $f'(x) = 2x$ is given on $(-3, 3)$ by term-by-term differentiation of the series for f. Thus

$$2x = -\frac{12}{\pi}\sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi x}{3}\right); -3 < x < 3.$$

(c) By direct calculation of the Fourier coefficients for $g(x) = 2x$ we get $a_n = 0, n \geq 0$ since $g(x) = 2x$ is odd; $b_n = \frac{1}{3}\int_{-3}^{3} 2x \sin\left(\frac{n\pi x}{3}\right) dx = -\frac{12(-1)^n}{npi}, n \geq 1$. So the Fourier series of $g(x) = 2x = -\frac{12}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin\left(\frac{n\pi x}{3}\right); -3 < x < 3$. This result agrees with term-by-term differentiation in (b).

5. (a) Calculate $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(x)\cos(nx) dx = 2; a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(x)\cos(x) dx = \frac{1}{2}; a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(x)\cos(nx) dx = -\frac{2(-1)^n}{(n^2-1)}, n \geq 2$. Note all $b_n = 0$ since $x\sin(x)$ is even on $[-\pi, \pi]$. Thus the Fourier series of $x\sin(x)$ is

$$1 - \frac{1}{2}\sin(x) - 2\sum_{n=2}^{\infty} \frac{(-1)^n}{(n^2-1)} \cos(nx).$$

(b) The hypotheses of Theorem 14.6 are satisfied, so term by term differentiation yields the Fourier series for $f'(x)$ on the interval $(-\pi, \pi)$ to be

$$f'(x) = x\cos(x) + \sin(x) = \frac{1}{2}\sin(x) + 2\sum_{n=2}^{\infty} \frac{n(-1)^n}{(n^2-1)} \sin(nx).$$

(c) Since $x\cos(x) + \sin(x)$ is odd, $a_n = 0, n \geq 0$. Calculate $b_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos(x)\sin^2(x) dx = \frac{1}{2}; b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos(x)\sin(x)\sin(nx) dx = \frac{2n(-1)^n}{(n^2-1)}, n \geq 2$. This gives the same Fourier series as term by term differentiation in part (b).

Section 14.6 The Phase Angle Form of a Fourier Series

1. $(\alpha f + \beta g)(t + P) = \alpha f(t + P) + \beta g(t + P) = \alpha f(t) + \beta g(t) = (\alpha f + \beta g)(t)$.

2. Since $g(t + P/a) = f(a(t + P/a)) = f(at + P) = f(at) = g(t)$, g has period P/a, and since $h(t + bP) = f((t + bP)/b) = f(t/b + P) = f(t/b) = h(t)$, h has period bP.

3. Since $f'(t + P) = \lim\limits_{h\to 0} \dfrac{f(t + P + h) - f(t + P)}{h} = \lim\limits_{h\to 0} \dfrac{f(t + h) - f(t)}{h} = f'(t)$, f' has period P.

4. For any real number, α, we have $\int_{\alpha}^{\alpha+P} f(x)\,dx = \int_{\alpha}^{0} f(x)\,dx + \int_{0}^{P} f(x)\,dx + \int_{P}^{\alpha+P} f(x)\,dx$. In the third integral, let $u = x + P$ and use the periodicity of f to get $\int_{P}^{\alpha+P} f(x)\,dx = \int_{0}^{\alpha} f(u - P)\,du = -\int_{\alpha}^{0} f(u)\,du$, which cancels the first integral. Finally let $\alpha = -P/2$ to get the final assertion.

5. Expanding f in a Fourier series on $0 \le x \le 2$ gives $1 - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin(n\pi x)$. The trigonometric identity $\sin(n\pi x) = \cos\left(n\pi x - \frac{\pi}{2}\right)$ gives the phase angle form

$$1 - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\cos\left[n\pi x - \frac{\pi}{2}\right].$$

6. The Fourier series of f is

$$\frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)}\sin[(2n-1)\pi x],$$

with phase angle form

$$1 + \frac{2}{\pi}\sum_{n=1}^{\infty}\cos\left[(2n-1)\pi x - \frac{\pi}{2}\right].$$

7. The Fourier series of f is

$$16 + \frac{48}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\left(\frac{n\pi x}{2}\right) - \frac{\pi}{n}\sin\left(\frac{n\pi x}{2}\right),$$

with phase angle form

$$16 + \frac{48}{\pi^2}\sum_{n=1}^{\infty}\frac{\sqrt{1+n^2\pi^2}}{n^2}\cos\left[\frac{n\pi x}{2} + \arctan(n\pi)\right].$$

8. The Fourier series of f is

$$\frac{19}{8} + \sum_{n=1}^{\infty}\frac{2}{n^2\pi^2}\left[n\pi\sin\left(\frac{3n\pi}{2}\right) + \cos\left(\frac{3n\pi}{2}\right) - 1\right]\cos\left(\frac{n\pi x}{2}\right)$$

$$+\left[\sin\left(\frac{3n\pi}{2}\right)-\frac{n\pi}{2}-n\pi\cos\left(\frac{3n\pi}{2}\right)\right]\sin\left(\frac{n\pi x}{2}\right),$$

with phase angle form

$$\frac{19}{8}+\frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sqrt{8+5n^2\pi^2-12n\pi\sin\left(\frac{3n\pi}{2}\right)+4\left(n^2\pi^2-2\right)\cos\left(\frac{3n\pi}{2}\right)}\cos\left(\frac{n\pi x}{2}+\delta_n\right)$$

where $\delta_n=\tan^{-1}\left[\dfrac{\frac{n\pi}{2}1+2\cos\left(\frac{3n\pi}{2}\right)-\sin\left(\frac{3n\pi}{2}\right)}{n\pi\sin\left(\frac{3n\pi}{2}\right)+\cos\left(\frac{3n\pi}{2}\right)-1}\right]$.

9. The Fourier series of f is

$$\frac{8}{\pi}\sum_{n=1}^{\infty}\frac{n}{(4n^2-1)}\sin(2n\pi x),$$

with phase angle form

$$\frac{8}{\pi}\sum_{n=1}^{\infty}\frac{n}{(4n^2-1)}\cos\left[2n\pi x-\frac{\pi}{2}\right].$$

10. We can write $f(x)=kx$ for $0\le x\le 1, f(x)=k(2-x)$ for $1\le x\le 2$, and $f(x+2)=f(x)$ which has Fourier series

$$k+\frac{2k}{\pi^2}\sum_{n=1}^{\infty}\frac{-1}{(2n-1)^2}\cos\left[(2n-1)\pi x\right],$$

with phase angle form

$$k+\frac{2k}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\left[(2n-1)\pi x-\pi\right].$$

11. We can write $f(x)=x$ for $0\le x<1, f(x)=x-2$ for $1<x\le 2$ and $f(x+2)=f(x)$ which has Fourier series

$$\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(n\pi x),$$

with phase angle form

$$\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\cos\left[n\pi x+(-1)^{n+1}\frac{\pi}{2}\right].$$

12. We can write $f(x)=k$ for $0<x<1, f(x)=0$ for $1<x<2$ and $f(x+2)=f(x)$ which has Fourier series

$$\frac{k}{2}+\frac{2k}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)}\sin\left[(2n-1)\pi x\right],$$

with phase angle form

$$\frac{k}{2}+\frac{2k}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)}\cos\left[(2n-1)\pi x-\frac{\pi}{2}\right].$$

13. We can write $f(x) = 1$ for $0 \le x < 1, f(x) = 2$ for $1 < x < 3, f(x) = 1$ for $3 < x < 4$ and $f(x+4) = f(x)$ which has Fourier series

$$\frac{3}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)}\cos\left[(2n-1)\frac{\pi x}{2}\right],$$

with phase angle form

$$\frac{3}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)}\cos\left[(2n-1)\frac{\pi x}{2} + \frac{\pi}{2}\left[1-(-1)^n\right]\right].$$

14. We have $f(x) = x$ for $0 < x < 1, f(x+1) = f(x)$ with Fourier series

$$\frac{1}{2} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{-1}{n}\sin(2n\pi x),$$

with phase angle form

$$\frac{1}{2} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\cos\left[2n\pi x + \frac{\pi}{2}\right].$$

15. The current is modeled by the equation $10i' + 100i + 100q = 100t(\pi - t^2)$ for $-\pi < t < \pi$, with $E(t+2\pi) = E(t)$. Differentiate and divide by 10 to get $i'' + 10i' + 10i = 10(\pi - 3t^2) = E'(t)$. Since we desire only the steady state current, we will seek only a particular solution of this equation for $i(t)$. To this end, expand $E'(t) = 10(\pi - 3t^2)$ in a Fourier series, then substitute $i(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n\cos(nt) + B_n\sin(nt)$ and solve for the coefficients, $A_n, n \ge 0, B_n, n \ge 1$. We have $10(\pi - 3t^2) = 120\pi\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\sin(nt)$ and upon substituting for $i(t)$ we get

$$5A_0 = 0, \left(10 - n^2\right)A_n + 10B_n = 0, -10nA_n + \left(10-n^2\right)B_n = \frac{120\pi(-1)^{n+1}}{n^2}.$$

Thus

$$A_0 = 0, A_n = \frac{1200\pi(-1)^{n+1}(10-n^2)}{n^2\left[(10-n^2)^2 + (10n)^2\right]}, n \ge 1, B_n = \frac{1200\pi(-1)^{n+1}}{n\left[(10-n^2)^2+(10n)^2\right]}, n \ge 1,$$

and steady state current is

$$i(t) = 120\pi\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\left[(10-n^2)^2+(10n)^2\right]}\left\{\frac{(10-n^2)}{n^2}\cos(nt) + \frac{10}{n}\sin(nt)\right\}.$$

16. The circuit is modeled by $5i' + 500i + 5\times 10^6 q = |10\sin(800\pi t)| = E(t)$. Replace $E(t)$ by its Fourier series which is $\frac{20}{\pi}\left[1 - 2\sum_{n=1}^{\infty}\frac{\cos(1600n\pi t)}{(4n^2-1)}\right]$, divide by 5 and differentiate the equation to get

$$i'' + 100i' + 10^6 i = \sum_{n=1}^{\infty}\frac{12800n}{4n^2-1}\sin(1600n\pi t).$$

Substitute

$$i(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(1600n\pi t) + B_n \sin(1600n\pi t)$$

into the differential equation to get $A_0 = 0$, and for $n \geq 1$, (after simplifying),

$$\left(100 - 256n^2\pi^2\right) A_n + 16n\pi B_n = 0, -16n\pi A_n + \left(100 - 256n^2\pi^2\right) B_n = \frac{1.28n}{4n^2 - 1}.$$

Solve these by Cramer's Rule to get steady state current

$$i(t) = 1.28 \sum_{n=1}^{\infty} \frac{n(-16n\pi \cos(1600n\pi t) + (100 - 256n^2\pi^2)\sin(1600n\pi t))}{(4n^2 - 1)\left[(100 - 256n^2\pi^2)^2 + (256n^2\pi^2)\right]}.$$

Section 14.7 Complex Fourier Series and the Frequency Spectrum

1. Compute $d_0 = \frac{1}{3}\int_0^3 2x\,dx = 3$, and for $n \geq 1, d_n = \frac{1}{3}\int_0^3 2xe^{-i2n\pi x/3}dx = \frac{3}{n\pi}i$. The complex Fourier series of f is

$$3 + \frac{3i}{\pi} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1}{n} e^{2n\pi i x/3} = \begin{cases} 3 & \text{if} \quad x = 0 \text{ or } x = 3 \\ 2x & \text{if} \quad 0 < x < 3 \end{cases}$$

2. The complex Fourier series of f is $\frac{4}{3} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left[\frac{2}{n^2\pi^2} - \frac{2i}{n\pi}\right] e^{n\pi i x}$,

and converges to $g(x) = \begin{cases} 2 & \text{if} \quad x = 0 \text{ or } x = 2 \\ x^2 & \text{if} \quad 0 < x < 2 \end{cases}$

3. The complex Fourier series of f is $\frac{3}{4} - \frac{1}{2\pi} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1}{n}\left\{\sin\left(\frac{n\pi}{2}\right) + i\left[\cos\left(\frac{n\pi}{2}\right) - 1\right]\right\} e^{n\pi i x/2}$,

and converges to $g(x) = \begin{cases} 1/2 & \text{if} \quad x = 0 \text{ or } x = 1 \text{ or } x = 4 \\ 0 & \text{if} \quad 0 < x < 1 \\ 1 & \text{if} \quad 1 < x < 4 \end{cases}$

4. The complex Fourier series of f is $-\frac{1}{2} - \frac{3i}{\pi} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1}{n} e^{n\pi i x/3}$,

and converges to $g(x) = \begin{cases} -2 & \text{if} \quad x = 0 \text{ or } x = 6 \\ 1 - x & \text{if} \quad 0 < x < 6 \end{cases}$

5. The complex Fourier series of f is $\frac{1}{2} + \frac{3i}{\pi} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1}{2n-1} e^{(2n-1)\pi i x/2}$,

and converges to $g(x) = \begin{cases} 1/2 & \text{if} \quad x = 0, 2, 4 \\ -1 & \text{if} \quad 0 < x < 2 \\ 2 & \text{if} \quad 2 < x < 4 \end{cases}$

6. The complex Fourier series of f is $\sum_{n=-\infty}^{\infty} \frac{1-e^{-5}}{5+2n\pi i} e^{2n\pi i x/5}$,

and converges to $g(x) = \begin{cases} \left(1-e^{-5}\right)/2 & \text{if} \quad x=0 \text{ or } x=5 \\ e^{-x} & \text{if} \quad 0<x<5 \end{cases}$

7. The complex Fourier series of f is $\frac{1}{2} - \frac{2}{\pi^2} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1}{(2n-1)^2} e^{(2n-1)\pi i x}$,

and converges to $f(x)$ for all $0 \leq x \leq 2$.

8. We have $f(x) = 2x, 0 \leq x < 4, f(x+4) = f(x)$, so the complex Fourier series of f is $4 + \frac{16i}{\pi} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1}{n} e^{n\pi i x/2}$, and converges to $g(x) = \begin{cases} 4 & \text{if} \quad x=0 \text{ or } x=4, \\ 2x & \text{if} \quad 0<x<4 \end{cases}$

9. The complex Fourier series of f and g are

$$f(x) = \frac{5}{3} + \frac{5}{\pi} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi}{3}\right) e^{n\pi i x/6}$$

and

$$g(x) = \frac{5}{3} + \frac{5i}{2\pi} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1}{n} \left\{ \left[\cos\left(\frac{2n\pi}{3}\right) - 1\right] - i \sin\left(\frac{2n\pi}{3}\right) \right\} e^{n\pi i x/6}.$$

To compare frequency spectra and phase spectra, we write the coefficients of each series in polar complex form, $d_n = a_n + ib_n = r_n e^{-i\phi_n}$ where $r_n = \sqrt{a_n^2 + b_n^2}, \phi_n = \tan^{-1}(-b_n/a_n)$. The series for f already has this form with

$$r_0 = \frac{5}{3}, r_n = \frac{5}{n\pi} \sin\left(\frac{n\pi}{3}\right), \phi_n = 0, n \geq 0.$$

For g, compute

$$r_n = \frac{5}{2n\pi} \left[\left(\cos\left(\frac{2n\pi}{3}\right) - 1\right)^2 + \left(\sin\left(\frac{2n\pi}{3}\right)\right)^2 \right]^{1/2} =$$

$$\frac{5}{2n\pi} \sqrt{2\left(1 - \cos\left(\frac{2n\pi}{3}\right)\right)} = \frac{5}{n\pi} \sin\left(\frac{n\pi}{3}\right).$$

Thus f and g give the same frequency spectra.
For the phase spectrum of g,

$$\phi_n = \tan^{-1}\left(\frac{\sin\left(\frac{2n\pi}{3}\right)}{\cos\left(\frac{2n\pi}{3}\right) - 1}\right) = -\tan^{-1}\left(\cot\left(\frac{n\pi}{3}\right)\right).$$

Thus

$$\phi_n = -\frac{\pi}{2} \text{ if } n = 0 \pmod 3, \phi_n = -\frac{\pi}{6} \text{ if } n = 1 \pmod 3, \phi_n = \frac{\pi}{6} \text{ if } n = 2 \pmod 3,$$

so f and g have different frequency spectra.

Chapter Fifteen - The Fourier Integral and Fourier Transforms

Section 15.1 The Fourier Integral

1. First note $\int_{-\infty}^{\infty} |f(x)|dx = \int_{-\pi}^{\pi} |x|dx = 2\int_0^{\pi} xdx = \pi^2$. For the Fourier integral representation of f compute $A_\omega = \int_{-\infty}^{\infty} t\cos(\omega t)\,dt = 0$, since $t\cos(\omega t)$ is odd; and

$$B_\omega = \int_{-\infty}^{\infty} t\sin(\omega t)\,dt = \int_{-\pi}^{\pi} t\sin(\omega t)\,dt = 2\int_0^{\pi} t\sin(\omega t)\,dt =$$

$$2\left[-\frac{t}{\omega}\cos(\omega t) + \frac{\sin(\omega t)}{\omega^2}\bigg|_0^{\pi}\right] = 2\left[\frac{\sin(\omega\pi)}{\omega^2} - \frac{\pi}{\omega}\cos(\omega\pi)\right].$$

Then the Fourier integral representation of f is

$$\int_0^{\infty}\left[\frac{2\sin(\omega\pi)}{\pi\omega^2} - \frac{2\cos(\omega\pi)}{\omega}\right]\sin(\omega x)\,d\omega = \begin{cases} -\pi/2 & \text{if} \quad x = -\pi \\ x & \text{if} \quad -\pi < x < \pi \\ \pi/2 & \text{if} \quad x = \pi \\ 0 & \text{if} \quad |x| > \pi \end{cases}$$

2. Certainly $\int_{-\infty}^{\infty} |f(x)|dx = \int_{-10}^{10} kdx = 20k$ converges.
Compute

$$A_\omega = \int_{-\infty}^{\infty} f(t)\cos(\omega t)\,dt = \int_{-10}^{10} k\cos(\omega t)\,dt = \frac{2k}{\omega}\sin(10\omega) \text{ if } |\omega| > 0,$$

and $A(0) = 20k = \lim_{\omega\to 0} A_\omega$.
Also

$$B_\omega = \int_{-\infty}^{\infty} f(t)\sin(\omega t)\,dt = 0, \text{ since} f(t)\sin(\omega t) \text{ is odd.}$$

The Fourier integral representation of f is

$$\int_0^{\infty} \frac{2k}{\pi\omega}\sin(10\omega)\cos(\omega x)\,d\omega = \begin{cases} k & \text{if} \quad -10 < x < 10 \\ 0 & \text{if} \quad |x| > 10 \\ k/2 & \text{if} \quad x = -10 \text{ or } x = 10 \end{cases}$$

3. Certainly $\int_{-\infty}^{\infty} |f(x)|dx$ converges. Also f is odd, so $A_\omega = 0$.
Now $B_\omega = \int_{-\pi}^{\pi} f(t)\sin(\omega t)\,dt = 2\int_0^{\pi} \sin(\omega t)\,dt = \frac{2}{\omega}[1 - \cos(\omega\pi)]$.
The Fourier integral of f is

$$\int_0^{\infty} \frac{2}{\pi\omega}[1 - \cos(\pi\omega)]\sin(\omega x)\,d\omega = \begin{cases} -1/2 & \text{if} \quad x = -\pi \\ -1 & \text{if} \quad -\pi < x < 0 \\ 0 & \text{if} \quad x = 0, \text{ or } |x| > \pi \\ 1 & \text{if} \quad 0 < x < \pi \\ 1/2 & \text{if} \quad x = \pi \end{cases}$$

4. Certainly $\int_{-\infty}^{\infty} |f(x)|dx$ converges, so calculate

$$A_\omega = \int_{-4}^{0} \sin(t)\cos(\omega t)\,dt + \int_0^4 \cos(t)\cos(\omega t)\,dt =$$

$$\frac{1}{2}\left[\frac{1-\cos[4(\omega-1)]}{\omega-1} - \frac{1-\cos[4(\omega+1)]}{\omega+1} + \frac{\sin[4(\omega-1)]}{\omega-1} + \frac{\sin[4(\omega+1)]}{\omega+1}\right], |\omega| \neq 1,$$

and

$$B_\omega = \frac{1}{2}\left[\frac{1-\cos[4(\omega-1)]}{\omega-1} + \frac{1-\cos[4(\omega+1)]}{\omega+1} + \frac{\sin[4(\omega-1)]}{\omega-1} - \frac{\sin[4(\omega+1)]}{\omega+1}\right], |\omega| \neq 1.$$

If $|\omega| = 1$, then A_ω and B_ω should be replaced by appropriate limits, i.e. $A(1) = \lim_{\omega\to 1} A_\omega$. The Fourier integral of f is

$$\frac{1}{\pi}\int_0^\infty [A_\omega \cos(\omega x) + B_\omega \sin(\omega x)]\,d\omega = \begin{cases} 1/2 & \text{if} \quad x = 0 \\ \cos(4)/2 & \text{if} \quad x = 4 \\ -\sin(4)/2 & \text{if} \quad x = -4 \\ f(x) & \text{otherwise} \end{cases}$$

5. Since x^2 is even, we have $B_\omega = \frac{1}{\pi}\int_{-100}^{100} \xi^2 \sin(\omega\xi)d\xi = 0$;

and

$$A_\omega = \frac{1}{\pi}\int_{-100}^{100} \xi^2 \cos(\omega\xi)d\xi$$

$$= \frac{2}{\pi}\int_0^{100} \xi^2 \cos(\omega\xi)d\xi = \frac{2}{\pi}\left\{\frac{\xi^2\sin(\omega\xi)}{\omega} + \frac{2\xi\cos(\omega\xi)}{\omega^2} - \frac{2\sin(\omega\xi)}{\omega^3}\Bigg|_0^{100}\right\}$$

$$= \frac{20000\sin(100\omega)}{\pi\omega} - \frac{4\sin(100\omega)}{\pi\omega^3} + \frac{400\cos(100\omega)}{pi\omega^2}.$$

The Fourier integral representation is

$$\int_0^\infty \left[\frac{400\cos(100\omega)}{\pi\omega^2} + \frac{20000\omega^2 - 4}{\pi\omega^3}\sin(100\omega)\right]\cos(\omega x)\,d\omega = \begin{cases} x^2 & \text{if} \quad -100 < x < 100 \\ 0 & \text{if} \quad |x| > 100 \\ 5000 & \text{if} \quad x = 100 \text{ or } x = -100 \end{cases}$$

6. Certainly $\int_{-\infty}^{\infty} |f(x)|dx$ converges, so compute

$$A_\omega = \int_{-\pi}^{2\pi} |x|\cos(\omega x)\,dx = 2\int_0^\pi x\cos(\omega x)\,dx + \int_\pi^{2\pi} x\cos(\omega x)\,dx =$$

$$\frac{\pi}{\omega}\sin(\pi\omega) + \frac{2\pi}{\omega}\sin(2\pi\omega) + \frac{1}{\omega^2}\cos(\omega\pi) + \frac{\cos(2\pi\omega)}{\omega^2} - \frac{1}{\omega^2},$$

and

$$B_\omega = \int_{-\pi}^{2\pi} |x|\sin(\omega x)\,dx = \int_\pi^{2\pi} x\sin(\omega x)\,dx =$$

$$\frac{\sin(2\pi\omega)}{\omega^2} - \frac{\sin(\omega\pi)}{\omega^2} - \frac{2\pi\cos(2\pi\omega)}{\omega} + \frac{\pi}{\omega}\cos(\omega\pi).$$

The Fourier integral representation of f is

$$\frac{1}{\pi}\int_0^\infty A_\omega \cos(\omega x) + B_\omega \sin(\omega x)\, d\omega$$

$$= \begin{cases} |x| & \text{if} \quad -\pi < x < 2\pi \\ 0 & \text{if} \quad x < -\pi \text{ or } x > 2\pi \\ \pi/2 & \text{if} \quad x = -\pi \\ \pi & \text{if} \quad x = 2\pi \end{cases}$$

7. Certainly $\int_{-\infty}^{\infty} |f(x)|\,dx$ converges, so compute

$$A_\omega = \int_{-3\pi}^{\pi} \sin(x)\cos(\omega x)\,dx = \frac{2\sin(\omega\pi)\sin(2\omega\pi)}{\omega^2 - 1},$$

and

$$B_\omega = \int_{-3\pi}^{\pi} \sin(x)\sin(\omega x)\,dx = -\frac{2\cos(\pi\omega)\sin(2\pi\omega)}{\omega^2 - 1}.$$

The Fourier integral representation of f is

$$\int_0^\infty \frac{2}{\pi(\omega^2 - 1)}\left[\sin(\omega\pi)\sin(2\omega\pi)\cos(\omega x) - \cos(\omega\pi)\sin(2\omega\pi)\sin(\omega x)\right] d\omega$$

$$= \begin{cases} \sin(x) & \text{if} \quad -3\pi \le x \le \pi \\ 0 & \text{for} \quad x < -3\pi \text{ or } x > \pi \end{cases} = f(x) \text{ for all } x.$$

8. $$\int_0^\infty \frac{1}{2\pi\omega}\left\{[3\sin(5\omega) - \sin(\omega)]\cos(\omega x) - [\cos(5\omega) - \cos(\omega)]\sin(\omega x)\right\} d\omega$$

$$= \begin{cases} 1/4 & \text{if} \quad x = -5 \\ 3/4 & \text{if} \quad x = 1 \\ 1/2 & \text{if} \quad x = 5 \\ f(x) & \text{otherwise} \end{cases}$$

9. $\displaystyle\int_0^\infty \frac{2}{\pi(\omega^2+1)}\cos(\omega x)\,d\omega = e^{-|x|}$ for all x.

10. $\displaystyle\int_0^\infty \frac{2(\omega^2-1)}{\pi(\omega^2+1)^2}\sin(\omega x)\,d\omega = xe^{-|x|}$ for all x.

11. From equation 15.3, we have the Fourier integral representation of $f(t)$ given by $\displaystyle\frac{1}{\pi}\int_0^\infty \int_{-\infty}^{\infty} f(t)\cos(\omega(t-x))\,dt\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\cos(\omega(t-x))\,d\omega\,dt$, by switching order of

integration and because $f(t)\cos(\omega(t-x))$ is an even function of ω. Now $f(t)\sin(\omega(t-x))$ is an odd function of ω so, $\frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\sin(\omega(t-x))d\omega = 0$. It follows that we can write the Fourier integral representation of $f(t)$ as

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)[\cos(\omega(t-x)) + i\sin(\omega(t-x))]d\omega dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\left[\lim_{r\to\infty}\int_{-r}^{r} e^{i\omega(t-x)}d\omega\right]dt$$
$$= \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\left[\lim_{r\to\infty}\frac{e^{ir(t-x)} - e^{-ir(t-x)}}{2i(t-x)}\right]dt = \frac{1}{\pi}\lim_{\omega\to\infty}\int_{-\infty}^{\infty} f(t)\frac{\sin(\omega(t-x))}{t-x}dt.$$

Section 15.2 Fourier Cosine and Sine Integrals

In Problems 1 through 10, $S(x)$ is the Fourier sine integral of f and $C(x)$ is the Fourier cosine integral of f.

1. $S(x) = \int_0^\infty \frac{4}{\pi\omega^3}\left[10\omega\sin(10\omega) - \left(50\omega^2 - 1\right)\cos(10\omega) - 1\right]\sin(\omega x)\,d\omega;$
$C(x) = \int_0^\infty \frac{4}{\pi\omega^3}\left[10\omega\cos(10\omega) - \left(50\omega^2 - 1\right)\sin(10\omega)\right]\cos(\omega x)\,d\omega;$
both converge to x^2 for $0 \le x < 10$, to 0 for $x > 10$, and to 50 at $x = 10$.

2. $S(x) = \int_0^\infty \frac{2\sin(2\pi\omega)}{\pi(\omega^2-1)}\sin(\omega x)\,d\omega;$
$C(x) = \int_0^\infty \frac{2[\cos(2\pi\omega) - 1]}{\pi(\omega^2-1)}\cos(\omega x)\,d\omega;$
both converge to $f(x)$ for all x.

3. $S(x) = \int_0^\infty \frac{2}{\pi\omega}[1 + \cos(\omega) - 2\cos(4\omega)]\sin(\omega x)\,d\omega;$
$C(x) = \int_0^\infty \frac{2}{\pi\omega}[2\sin(4\omega) - \sin(\omega)]\cos(\omega x)\,d\omega;$
both converge to 1 for $0 < x < 1$, to $\frac{3}{2}$ for $x = 1$, to 2 for $1 < x < 4$, to 1 for $x = 4$, to 0 for $x > 4$, while $S(0) = 0$ and $C(0) = 1$.

4. $S(x) = \int_0^\infty \frac{2}{\pi(\omega^2+1)}[\sinh(5)(5\omega) - \omega\cosh(5)\cos(5\omega) + \omega]\sin(\omega x)\,d\omega;$
$C(x) = \int_0^\infty \frac{2}{\pi(\omega^2+1)}[\sinh(5)\cos(5\omega) + \omega\cosh(5)\sin(5\omega)]\cos(\omega x)\,d\omega;$
both converge to $\cosh(x)$ for $0 < x < 5$, to $\frac{\cosh(5)}{2}$ for $x = 5$, to 0 for $x > 5$, while $S(0) = 0$ and $C(0) = 1$.

5. $S(x) = \int_0^\infty \left\{\frac{2}{\pi\omega}[1 + (1-2\pi)\cos(\omega\pi) - 2\cos(3\omega\pi)] + \frac{4}{\pi\omega^2}\sin(\omega\pi)\right\}\sin(\omega x)\,d\omega;$
$C(x) = \int_0^\infty \left\{\frac{2}{\pi\omega}[(2\pi-1)\sin(\omega\pi) + 2\sin(3\omega\pi)] + \frac{4}{\pi\omega^2}[\cos(\omega\pi) - 1]\right\}\cos(\omega x)\,d\omega;$
both converge to $1 + 2x$ for $0 < x < \pi$, to $\frac{3+2\pi}{2}$ for $x = \pi$, to 2 for $\pi < x < 3\pi$, to 1 for $x = 3\pi$, to 0 for $x > 3\pi$, while $S(0) = 0$ and $C(0) = 1$.

6. $S(x) = \int_0^\infty \left\{\frac{2}{\pi\omega^2}[\sin(2\omega) - 3\omega\cos(2\omega)\} + \omega\cos(\omega)\right]\sin(\omega x)\,d\omega;$

$C(x) = \int_0^\infty \left\{ \frac{2}{\pi\omega^2} \left[\cos(2\omega) - 1 + 3\omega \sin(2\omega) - \omega \sin(\omega) \right] \right\} \cos(\omega x)\, d\omega;$
both converge to $f(x)$ for $0 \le x < 1, 1 < x < 2$ and $x > 2$, to $\frac{3}{2}$ for $x = 1$ and $x = 2$.

7. $S(x) = \int_0^\infty \frac{2}{\pi} \left(\frac{\omega^3}{4 + \omega^4} \right) \sin(\omega x)\, d\omega;$
$C(x) = \int_0^\infty \frac{2}{\pi} \left(\frac{2 + \omega^2}{4 + \omega^4} \right) \cos(\omega x)\, d\omega;$
both converge to $e^{-x} \cos(x)$ for $x > 0$, while $S(0) = 0$ and $C(0) = 1$.

8. $S(x) = \int_0^\infty \frac{36\omega}{\pi(\omega^2 + 9)^2} \sin(\omega x)\, d\omega;$
$C(x) = \int_0^\infty \frac{2}{\pi} \frac{(9 - \omega^2)}{(\omega^2 + 9)^2} \cos(\omega x)\, d\omega;$
both converge to $f(x)$ for all $x \ge 0$.

9. $S(x) = \int_0^\infty \frac{2k}{\pi\omega} [1 - \cos(c\omega)] \sin(\omega x)\, d\omega;$
$C(x) = \int_0^\infty \frac{2k}{\pi\omega} \sin(c\omega) \cos(\omega x)\, d\omega;$
both converge to k for $0 < x < c$, to $\frac{k}{2}$ for $x = c$, to 0 for $x > c$, while $S(0) = 0$ and $C(0) = k$.

10. $S(x) = \int_0^\infty \frac{2}{\pi} \left(\frac{\omega^3 + 1}{(\omega^2 + 5)^2 - 4} \right) \sin(\omega x)\, d\omega;$
$C(x) = \int_0^\infty \frac{2}{\pi} \left(\frac{\omega^2 + 5}{(\omega^2 + 5)^2 - 4} \right) \cos(\omega x)\, d\omega;$
both converging to $e^{-2x} \cos(x)$ for $x > 0$, while $S(0) = 0$ and $C(0) = 1$.

11. From the Laplace integrals given in the text and the convergence theorem we have

$$e^{-kx} = \frac{2k}{\pi} \int_0^\infty \frac{1}{k^2 + \omega^2} \cos(\omega x)\, d\omega \text{ for all } x \ge 0,$$

and

$$e^{-kx} = \frac{2}{\pi} \int_0^\infty \frac{\omega}{k^2 + \omega^2} \sin(\omega x) d\omega \text{ for all } x > 0.$$

Put $k = 1$ in these results and interchange letters x and ω to get

$$A_\omega = \frac{\pi e^{-\omega}}{2k} = \int_0^\infty \frac{1}{1 + x^2} \cos(x\omega) dx,$$

and

$$B_\omega = \frac{\pi e^{-\omega}}{2} = \int_0^\infty \frac{x}{1 + x^2} \sin(\omega x) dx.$$

From these results it follows that the cosine integral formula of $\frac{1}{1 + x^2}$ is

$$C(x) = \int_0^\infty e^{-\omega} \cos(\omega x)\, d\omega = \frac{1}{1 + x^2} \text{ for all } x \ge 0,$$

and the sine integral formula of $\frac{x}{1+x^2}$ is

$$S(x) = \int_0^\infty e^{-\omega} \sin(\omega x)\, d\omega = \frac{x}{1+x^2} \text{ for all } x > 0,$$

and by direct evalution this last result holds also for $x = 0$.

Section 15.3 The Complex Fourier Integral and the Fourier Transform

1. $C_\omega = \int_{-\infty}^{0} x e^{x} e^{-i\omega x} dx + \int_0^\infty x e^{-x} e^{-i\omega x} dx = \frac{-4i\omega}{(\omega^2+1)^2}$;
$f(x) = -\frac{2i}{\pi}\int_{-\infty}^{\infty} \frac{\omega}{(\omega^2+1)^2} e^{i\omega x} d\omega$, for all x.

2. $C_\omega = \int_{-1}^{1} (1-x) e^{-i\omega x} dx = \frac{2}{\omega^2}[\sin(\omega) - i(\omega\cos(\omega) - \sin(\omega))]$;
Complex Fourier integral of $f(x)$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{\omega^2}[\sin(\omega) - i(\omega\cos(\omega) - \sin(\omega))] e^{i\omega x} d\omega = \begin{cases} 1 & \text{for } x = -1 \\ 1-x & \text{for } -1 < x \le 1 \\ 0 & \text{for } |x| > 1 \end{cases}$$

3. $C_\omega = \int_{-5}^{5} \sin(\pi x) e^{-i\omega x} dx = i\left[\frac{\sin(5(\omega+\pi)}{\omega+\pi} - \frac{\sin(5(\omega-\pi)}{\omega-\pi}\right]$;
$f(x) = \frac{i}{2\pi}\int_{-\infty}^{\infty}\left[\frac{\sin(5(\omega+\pi)}{\omega+\pi} - \frac{\sin(5(\omega-\pi)}{\omega-\pi}\right] e^{i\omega x} d\omega$; for all x.

4. $C_\omega = \int_{-2}^{0} -x e^{-i\omega x} dx + \int_0^2 x e^{-i\omega x} dx = -\int_2^o x e^{i\omega x} dx + \int_0^2 x e^{-1\omega x} dx$
$= 2\int_0^2 x \frac{e^{i\omega x} + e^{-i\omega x}}{2} dx = 2\int_0^2 x\cos(\omega x) dx = \frac{2}{\omega^2}[2\omega\sin(2\omega) + \cos(2\omega) - 1]$;
Complex Fourier integral of $f(x)$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{[2\omega\sin(2\omega) + \cos(2\omega) - 1]}{\omega^2} e^{i\omega x} d\omega$$

$$= \begin{cases} 1 & \text{for } x = -2, 2 \\ |x| & \text{for } -2 < x < 2 \\ 0 & \text{for } |x| > 2 \end{cases}$$

5. $C_\omega = \int_{-\infty}^{-1} e^{x} e^{-i\omega x} dx + \int_{-1}^{1} x e^{-i\omega x} dx + \int_1^\infty e^{-x} e^{-i\omega x} dx$

$$= \frac{2e^{-1}}{1+\omega^2}(\cos(\omega) - \omega\sin(\omega)) + \frac{2i}{\omega^2}(\omega\cos(\omega) - \sin(\omega);$$

Complex Fourier integral of $f(x)$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty}\left[\frac{e^{-1}}{1+\omega^2}(\cos(\omega) - \omega\sin(\omega)) + \frac{i}{\omega^2}(\omega\cos(\omega) - \sin(\omega))\right] e^{i\omega x} d\omega$$

$$= \begin{cases} e^{-|x|} & \text{for } |x| > 1 \\ x & \text{for } |x| < 1 \\ \frac{e+1}{2e} & \text{for } x = 1 \\ \frac{1-e}{2e} & \text{for } x = -1 \end{cases}$$

6. $C_\omega = \int_{-k}^{0} - e^{-i\omega x}dx + \int_0^k e^{-i\omega x}dx = \int_k^0 e^{i\omega x}dx + \int_0^k e^{-i\omega x}dx = -2i\int_0^k \frac{e^{i\omega x} - e^{i\omega x}}{2i}dx =$
$-2i\int_0^k \sin(\omega x)dx = \frac{2i}{\omega}[\cos(\omega k) - 1]$;

Complex Fourier integral of $f(x) = \frac{i}{\pi}\int_{-\infty}^{\infty} \frac{[\cos(\omega k) - 1]}{\omega} e^{i\omega x} d\omega =$

$$\begin{cases} 0 & \text{for } |x| > k \\ 1 & \text{for } 0 < x < k \\ -1 & \text{for } -k < x < 0 \\ -1/2 & \text{for } x = -k \\ 1/2 & \text{for } x = k \\ 0 & \text{for } x = 0 \end{cases}$$

7. $C_\omega = \int_{-\pi/2}^{0} \sin(x)e^{-i\omega x}dx + \int_0^{\pi/2} \cos(x)e^{-i\omega x}dx$

$$= \frac{1}{\omega^2 - 1}\left[\left\{1 - \omega\sin\left(\frac{\pi\omega}{2}\right) - \cos\left(\frac{\pi\omega}{2}\right)\right\} - i\left\{\omega - \omega\cos\left(\frac{\pi\omega}{2}\right) - \sin\left(\frac{\pi\omega}{2}\right)\right\}\right];$$

Complex Fourier integral of $f(x)$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{\omega^2 - 1}\left[\left\{1 - \omega\sin\left(\frac{\pi\omega}{2}\right) - \cos\left(\frac{\pi\omega}{2}\right)\right\} - i\left\{\omega - \omega\cos\left(\frac{\pi\omega}{2}\right) - \sin\left(\frac{\pi\omega}{2}\right)\right\}\right] e^{i\omega x} d\omega$$

$$= \begin{cases} 0 & \text{for } |x| > \pi/2 \\ \cos(x) & \text{for } 0 < x \le \pi/2 \\ \sin(x) & \text{for } -\pi/2 < x < 0 \\ -1/2 & \text{for } x = -\pi/2 \\ 1/2 & \text{for } x = 0 \end{cases}$$

8. $C_\omega = \int_{-\infty}^{0} x^2 e^{x(3-i\omega)}dx + \int_0^{\infty} x^2 e^{-x(3+i\omega)}dx = \frac{-36(\omega^2 - 3)}{(\omega^2 + 9)^2}$;
$f(x) = -\frac{18}{\pi}\int_{-\infty}^{\infty} \frac{(\omega^2 - 3)}{(\omega^2 + 9)^2} e^{i\omega x} d\omega$, for all x.

9. $\hat{f}(\omega) = \int_{-1}^{0} - e^{-i\omega t}dt + \int_0^1 e^{-i\omega t}dt = \int_0^1 -(e^{i\omega t} - e^{-i\omega t})dt = \frac{2i}{\omega}[\cos(\omega) - 1]$.

10. Write $f(t) = \sin(t)[H(t+K) - H(t-K)]$, and use the Modulation Theorem 15.6, so

$$\hat{f}(\omega) = \frac{i}{2}\left[\frac{2\sin(K(\omega+1))}{\omega+1} - \frac{2\sin(K(\omega-1))}{\omega-1}\right] = i\left[\frac{\sin(K(\omega+1))}{\omega+1} - \frac{\sin(K(\omega-1))}{\omega-1}\right].$$

11. $f(t) = 5[H(t-3) - H(t-11)] = 5[H(t+4-7) + H(t+4+7)]$, so

$$\hat{f}(\omega) = 5e^{-7i\omega}\left(\frac{2\sin(4\omega)}{\omega}\right) = \frac{10}{\omega}e^{-7i\omega}\sin(4\omega).$$

12. $\hat{f}(\omega) = 5\sqrt{\frac{\pi}{3}}e^{-\omega^2/12}e^{-5i\omega}$ by the Time Shifting Theorem 15.1.

13. $\hat{f}(\omega) = \int_K^\infty e^{-t/4}e^{-i\omega t}dt = \left.\frac{e^{-(1/4+i\omega)t}}{-(1/4+i\omega)}\right|_K^\infty = \frac{4e^{-(1/4+i\omega)K}}{1+4i\omega}$

14. $\hat{f}(\omega) = \frac{2}{\omega^3}[K^2\omega^2\sin(\omega K) + 2K\omega\cos(\omega K) - 2\sin(\omega K)]$

15. $\hat{f}(\omega) = \pi e^{-|\omega|}$

16. $f(t) = 3H(t-2)e^{-3(t-2)}e^{-6} = 3e^{-6}H(t-2)e^{-3(t-2)}$, so

$$\hat{f}(\omega) = 3e^{-6}\mathcal{F}[H(t-2)e^{-3(t-2)}] = \frac{3e^{-6}e^{-2i\omega}}{3+i\omega} = \frac{3e^{-2(3+i\omega)}}{3+i\omega}$$

17. $\hat{f}(\omega) = \frac{24}{16+\omega^2}e^{2i\omega}$

18. $f(t) = H(t-3)e^{-2(t-3)}e^{-6}$, so $\hat{f}(\omega) = \frac{e^{-6}e^{-3i\omega}}{2+i\omega} = \frac{e^{-3(2+i\omega)}}{2+i\omega}$

19. $f(t) = 18\sqrt{2/\pi}e^{-4it}e^{-8t^2}$

20. $\hat{f}(\omega) = \frac{e^{-4(\omega-5)i}}{3+(\omega-5)i}$, so $f(t) = e^{5it}\ \hat{f}^{-1}\left[\frac{e^{-4i\omega}}{3+\omega i}\right] = e^{5it}H(t-4)e^{-3(t-4)}$

21. $\hat{f}(\omega) = \frac{e^{2(\omega-3)i}}{5+(\omega-3)i}$ so $f(t) = e^{3it}\ \hat{f}^{-1}\left[\frac{e^{2i\omega}}{5+\omega i}\right] = e^{3it}H(t+2)e^{-5(t+2)} = H(t+2)e^{-(10+(5-3i)t)}$

22. $\hat{f}(\omega) = \frac{10\sin(3\omega)}{\omega+\pi} = \frac{-10\sin(3(\omega+\pi))}{\omega+\pi}$, so

$$f(t) = -5e^{-\pi it}\ \hat{f}^{-1}\left[\frac{2\sin(3\omega)}{\omega}\right] = -5e^{-\pi it}[H(t+3) - H(t-3)].$$

23. $\hat{f}(\omega) = \frac{1+\omega i}{(3+\omega i)(2+\omega i)} = \frac{2}{3+\omega i} - \frac{1}{2+\omega i}$, so $f(t) = H(t)[2e^{-3t} - e^{-2t}]$.

24. $\hat{f}(\omega) = \frac{10(4+\omega i)}{9-\omega^2+8i\omega} = \frac{-10i(\omega-4i)}{(\omega-4i)^2+7} = -5i\left[\frac{1}{(\omega-4i)-\sqrt{7}i} + \frac{1}{(\omega-4i)+\sqrt{7}i}\right] =$

$$5e^{-4t}\ \hat{f}^{-1}\left[\frac{1}{\sqrt{7}+i\omega} + \frac{1}{-\sqrt{7}+i\omega}\right] = 5e^{-4t}[e^{\sqrt{7}t} + e^{-\sqrt{7}t}] = 10e^{-4t}\cosh(\sqrt{7}t)$$

Section 15.4 Additional Properties and Applications of the Fourier Transform

1. By Theorem 15.9, $\hat{f}\left[\frac{t}{9+t^2}\right] = i\left(\frac{\pi}{3}e^{-3|\omega|}\right)'$
$= \frac{\pi i}{3}[H(\omega)e^{-3\omega} + H(-\omega)e^{3\omega}]' = \pi i[H(-\omega)e^{3\omega} - H(\omega)e^{-3\omega}]$

2. $\hat{f}[3te^{-9t^2}] = 3i\sqrt{\frac{\pi}{9}}(e^{-\omega^2/36})' = \frac{-i\omega\sqrt{\pi}}{18}e^{-\omega^2/36}$

3. $\hat{f}[26H(t)te^{-2t}] = 26i\left(\frac{1}{2+i\omega}\right)' = \frac{26}{(2+i\omega)^2}$

4. $\hat{f}[H(t-3)(t-3)e^{-4t}] = e^{-12}\hat{f}[H(t-3)(t-3)e^{-4(t-3)}] = ie^{-12}e^{-3i\omega}\left(\dfrac{1}{4+i\omega}\right)' = \dfrac{e^{-3(4+i\omega)}}{(4+i\omega)^2}$

5. By Theorem 15.8, since $H(t)e^{-3t}$ has a jump discontinuity at $t = 0$, $\hat{f}\left[\dfrac{d}{dt}H(t)e^{-3t}\right] = \dfrac{1}{3+i\omega} - 1$

6. $\hat{f}[t\{H(t+1) - H(t-1)\}] = i\left(\dfrac{2\sin(\omega)}{\omega}\right)' = \dfrac{2i}{\omega^2}[\omega\cos(\omega) - \sin(\omega)]$.

7. $\hat{f}\left[\dfrac{5e^{3it}}{(t-2)^2+9}\right] = 5\hat{f}(\omega-3)$, where $\hat{f}(\omega) = \hat{f}\left[\dfrac{1}{(t-2)^2+9}\right] = e^{-2i\omega}\left(\dfrac{\pi}{3}e^{-3|\omega|}\right)$, so

$$\hat{f}\left[\frac{5e^{3it}}{(t-2)^2+9}\right] = \frac{5\pi}{3}e^{-2i(\omega-3)}e^{-3|\omega-3|}.$$

8. Write $f(t) = H(t-3)e^{-2(t-3)}e^{-6}$, so $\hat{f}[f(t)] = e^{-6}\dfrac{e^{-3i\omega}}{2+i\omega} = \dfrac{e^{-3(2+i\omega)}}{2+i\omega}$

9. $\hat{f}^{-1}\left(\dfrac{1}{(1+i\omega)^2}\right) = H(t)e^{-t} * H(t)e^{-t} = \displaystyle\int_{-\infty}^{\infty} H(\tau)e^{-\tau}H(t-\tau)e^{-(t-\tau)}d\tau = H(t)e^{-t}\int_0^t d\tau = te^{-t}$, if $t > 0$, or $f(t) = H(t)te^{-t}$

10. $\hat{f}^{-1}\left[\dfrac{1}{(1+i\omega)(2+i\omega)}\right] = H(t)e^{-t} * H(t)e^{-2t} = \displaystyle\int_{-\infty}^{\infty} H(\tau)e^{-\tau}H(t-\tau)e^{-2(t-\tau)}d\tau$
$= H(t)e^{-2t}\displaystyle\int_0^t e^{\tau}d\tau = H(t)e^{-2t}[e^t - 1]$, if $t > 0 = H(t)[e^{-t} - e^{-2t}]$

11. $\hat{f}^{-1}\left[\dfrac{\sin(3\omega)}{\omega(2+i\omega)}\right] = \dfrac{1}{2}[H(t+3) - H(t-3)] * H(t)e^{-2t}$
$= \dfrac{1}{2}\displaystyle\int_{-\infty}^{\infty}[H(\tau+3) - H(\tau-3)]\cdot H(t-\tau)e^{-2(t-\tau)}d\tau = \dfrac{e^{-2t}}{2}[H(t+3)\int_{-3}^{t}e^{2\tau}d\tau - H(t-3)\int_3^t e^{2\tau}dt]$
$= \dfrac{1}{4}H(t+3)[1 - e^{-2(t+3)}] - \dfrac{1}{4}H(t-3)[1 - e^{-2(t-3)}]$

12. $\hat{f}^{-1}\left[\dfrac{6e^{4i\omega}\sin(2\omega)}{9+\omega^2}\right] = \hat{f}^{-1}\left[\dfrac{6e^{4i\omega}}{(9+\omega^2)}\left(\dfrac{e^{2i\omega} - e^{-2i\omega}}{2i}\right)\right] = -3i\hat{f}^{-1}\left[\dfrac{e^{6i\omega}}{9+\omega^2} - \dfrac{e^{2i\omega}}{9+\omega^2}\right]$
$= -3if(t+6) + 3if(t+2)$, where $f(t) = \hat{f}^{-1}\left[\dfrac{1}{9+\omega^2}\right] = \dfrac{1}{6}e^{-3|t|}$.
So

$$\hat{f}^{-1}\left[\frac{6e^{4i\omega}\sin(2\omega)}{9+\omega^2}\right] = -\frac{i}{2}e^{-3|t+6|} + \frac{i}{2}e^{-3|t+2|}$$

13. $\hat{f}(\omega) = e^{-3|\omega+4|}\cos[2(\omega+4)] = \hat{g}(\omega+4)$, where

$$\hat{g}(\omega) = e^{-3|\omega|}\cos(2\omega) = \frac{1}{2}e^{-3|\omega|}(e^{2i\omega} + e^{-2i\omega}) = \frac{1}{2}[e^{-3|\omega|}e^{2i\omega} + e^{-3|\omega|}e^{-2i\omega}].$$

Then

$$\hat{f}^{-1}[\hat{f}(\omega)] = e^{4it}\hat{f}^{-1}[\hat{g}(\omega)] = e^{4it}\frac{3\pi}{2}\left[\frac{1}{9+(t+2)^2} + \frac{1}{9+(t-2)^2}\right]$$

14. $\hat{f}(\omega) = e^{-\omega^2/9}\sin(8\omega) = e^{-\omega^2/9}\left(\frac{e^{8i\omega}-e^{-8i\omega}}{2i}\right) = \frac{-i}{2}\left[e^{-\omega^2/9}e^{8i\omega} - e^{-\omega^2/9}e^{-8i\omega}\right]$, so

$$f(t) = \frac{3i}{4\sqrt{\pi}}[e^{-9(t-8)^2/4} - e^{-9(t+8)^2/4}]$$

15. $\int_{-\infty}^{\infty}[f(t)]^2dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\omega)\overline{\hat{f}(\omega)}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}|\hat{f}(\omega)|^2d\omega.$

16. We find $\hat{f}[H(t)e^{-2t}] = \frac{1}{2+i\omega}$, so by Parseval's identity in Problem 15

$$\int_{-\infty}^{\infty}[f(t)]^2dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\frac{1}{2+i\omega}\right|^2d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{4+\omega^2}d\omega = \frac{1}{4\pi}\tan^{-1}\left(\frac{\omega}{2}\right)\Big|_{-\infty}^{\infty} = \frac{1}{4}$$

or by direct evaluation $\int_{-\infty}^{\infty}[H(t)e^{-2t}]^2dt = \int_0^{\infty}e^{-4t}dt = \frac{1}{4}$.

17. By direct calculation, $\hat{f}\left[\frac{H(t+3)-H(t-3)}{2}\right] = \frac{1}{2}\int_{-3}^{3}e^{-1\omega t}dt = \frac{e^{i3\omega}-e^{-i3\omega}}{2i\omega} = \frac{\sin(3\omega)}{\omega}$, so by the Symmetry Theorem 15.5 we have $\hat{f}\left[\frac{\sin(3t)}{t}\right] = \pi[H(-\omega+3) - H(-\omega-3)] = \pi[H(\omega+3) - H(\omega-3)]$. Then by Parseval's identity in Problem 17 it follows that

$$\int_{-\infty}^{\infty}\left[\frac{\sin(3t)}{t}\right]^2dt = \frac{1}{2\pi}\int_{-3}^{3}(\pi)^2d\omega = 3\pi.$$

18. Let $\hat{y}(\omega) = \hat{f}[y(t)]$ and transform to get $\hat{y}[-\omega^2 + 6i\omega + 5] = \hat{f}[\delta(t-3)] = e^{-3i\omega}$, so

$$\hat{y}(\omega) = \frac{e^{-3i\omega}}{-\omega+6i\omega+5} = \frac{e^{-3i\omega}}{(1+i\omega)(5+i\omega)} = \frac{1}{4}\left[\frac{e^{-3i\omega}}{1+i\omega} - \frac{e^{-3i\omega}}{5+i\omega}\right].$$

Now invert to get $y(t) = \frac{1}{4}[H(t-3)e^{-(t-3)} - H(t-3)e^{-5(t-3)}]$.

19. Calculate $\widehat{f_{win}}(\omega) = \int_{-5}^{5}t^2e^{-i\omega t}dt$

$$= \frac{2}{\omega^3}\left[25\omega^2\sin(5\omega) + 10\omega\cos(5\omega) - 2\sin(5\omega)\right]$$

Since $g(t) = 1$ and the support of g is $[-5,5]$, $t_C = 0$. For the RMS bandwidth of the window function, $g_{RMS} = 2\left(\int_{-5}^{5}t^2dt/\int_{-5}^{5}dt\right)^{1/2} = \frac{10}{\sqrt{3}}$.

20. Calculate $\widehat{f_{win}}(\omega) = \int_{-4\pi}^{4\pi}\cos(at)e^{-i\omega t}dt$

$$= \frac{2}{\omega^2-a^2}\left[\omega\sin(4\omega\pi)\cos(4a\pi) - a\cos(4\omega\pi)\sin(4a\pi)\right]$$

Since $g(t)$ is constant 0n $[-4\pi, 4\pi]$, $t_C = 0$. For the RMS bandwidth of the window function, $g_{RMS} = 2\left(\int_{-4\pi}^{4\pi} t^2 dt / \int_{-4\pi}^{4\pi} dt\right)^{1/2} = \dfrac{8\pi}{\sqrt{3}}$

21. Calculate $\widehat{f_{win}}(\omega) = \displaystyle\int_0^4 e^{-t} e^{-i\omega t} dt = \frac{1}{1+i\omega}\left[1 - e^{-4(1+i\omega)}\right]$

$$= \frac{1}{1+\omega^2}\left[1 - e^{-4}(\cos(4\omega) - i\sin(4\omega))\right](1 - i\omega) = \frac{1 - e^{-4}\cos(4\omega) + e^{-4}\sin(4\omega)}{1+\omega^2}$$

$$+i\left[\frac{e^{-4}\sin(4\omega) + (e^{-4}\cos(4\omega) - 1)\omega}{1+\omega^2}\right].$$

We find $t_C = \dfrac{\int_0^4 t\,dt}{\int_0^4 dt} = \dfrac{t^2/2|_0^4}{4} = 2$. For the RMS bandwidth of the window function, $g_{RMS} = 2\left(\int_0^4 |t-2|^2 dt / \int_0^4 dt\right)^{1/2} = \dfrac{4}{\sqrt{3}}$

22. Calculate $\widehat{f_{win}}(\omega) = \displaystyle\int_{-1}^1 e^t \sin(\pi t) e^{-i\omega t} = \int_{-1}^1 \sin(\pi t) e^{(1-i\omega)t} dt$

$$= \frac{\pi}{(1+(\pi+\omega)^2)(1+(\pi-\omega))}\Big\{\Big(\sinh(1)(2(1+\pi^2)\cos(\omega) - 2\omega^2\cos(\omega)) + \cosh(1)(\omega\sin(\omega))\Big)$$

$$+i(\sinh(1)(2\omega^2\sin(\omega) - 2(1+\pi^2)\sin(\omega)) + \cosh(1)\omega\cos(\omega))\Big\}$$

Since $g(t) = 1$ and the support of $g = [-1, 1]$, $t_C = 0$. For the RMS bandwidth of the window function, $g_{RMS} = 2\left(\int_{-1}^1 t^2 dt / \int_{-1}^1 dt\right)^{1/2} = \dfrac{2}{\sqrt{3}}$

23. Calculate $\widehat{f_{win}}(\omega) = \displaystyle\int_{-2}^2 (t+2)^2 e^{-i\omega t} dt$

$$= \frac{2}{\omega^3}\left[\sin(2\omega)(8\omega^2 - 1) + 4\omega\cos(2\omega) - 2\sin(2\omega)\right]$$

$$+i\frac{2}{\omega^3}\left[\cos(2\omega)(8\omega^2 - 1) - 4\omega\sin(2\omega) + \cos(2\omega)\right]$$

24. Calculate $\widehat{f_{win}}(\omega) = \displaystyle\int_{3\pi}^{5\pi} e^{-i\omega t} dt = -\frac{1}{i\omega}[e^{5\omega\pi i} - e^{3\omega\pi i}]$

$$= -\frac{2e^{i\pi\omega}}{\omega}\left[\frac{e^{4\pi\omega i} - e^{-4\pi\omega i}}{2i}\right]$$

$$= -2e^{i\pi\omega}\sin(4\pi\omega) = -2\cos(\pi\omega)\sin(4\pi\omega) - 2i\sin(\pi\omega)\sin(4\pi\omega).$$

We find $t_C = \dfrac{\int_{3\pi}^{5\pi} t\,dt}{2\pi} = 4\pi$. For the RMS bandwidth of the window function, $g_{RMS} = 2\left(\int_{3\pi}^{5\pi} |t - 4\pi|^2 dt / \int_{3\pi}^{5\pi} dt\right)^{1/2} = \dfrac{2\pi}{\sqrt{3}}$

Section 15.5 The Fourier Cosine and Sine Transforms

1. $\widehat{f}_c(\omega) = \int_0^\infty e^{-t}\cos(\omega t)dt = \dfrac{1}{1+\omega^2}$; $\widehat{f}_s(\omega) = \int_0^\infty e^{-t}\sin(\omega t)dt = \dfrac{\omega}{1+\omega^2}$

2. $\widehat{f}_c(\omega) = \dfrac{a^2-\omega^2}{(a^2+\omega^2)^2}$; $\widehat{f}_s(\omega) = \dfrac{2a\omega}{(a^2+\omega^2)^2}$

3. $\widehat{f}_c(\omega) = \dfrac{1}{2}\left[\dfrac{\sin[(\omega+1)K]}{\omega+1} + \dfrac{\sin[(\omega-1)K]}{\omega-1}\right]$, $\omega \neq \pm 1$;

$\widehat{f}_c(1) = \widehat{f}_c(-1) = \dfrac{K}{2} + \dfrac{\sin(2K)}{2}$;

$\widehat{f}_s(\omega) = \dfrac{\omega}{w^2-1} - \dfrac{1}{2}\left[\dfrac{\cos[(\omega+1)K]}{\omega+1} + \dfrac{\cos[(\omega-1)K]}{\omega-1}\right]$, $\omega \neq \pm 1$; $\widehat{f}_s(1) = \dfrac{1}{4}[1-\cos(2K)]$,

$\widehat{f}_s(-1) = -\dfrac{1}{4}[1-\cos(2K)]$

4. $\widehat{f}_c(\omega) = \dfrac{1}{\omega}[2\sin(K\omega) - \sin(2K\omega)]$; $\widehat{f}_s(\omega) = \dfrac{1}{\omega}[1 - 2\cos(K\omega) + \cos(2K\omega)]$.

5. $\widehat{f}_c(\omega) = \dfrac{1}{2}\left[\dfrac{1}{1+(\omega+1)^2} + \dfrac{1}{1+(\omega-1)^2}\right]$;

$\widehat{f}_s(\omega) = \dfrac{1}{2}\left[\dfrac{\omega+1}{1+(\omega+1)^2} + \dfrac{\omega-1}{1+(\omega-1)^2}\right]$

6. $\widehat{f}_c(\omega) = \dfrac{1}{1+\omega^2}[\cosh(2K)\cos(2K\omega) - \cosh(K)\cos(K\omega)$

$+\omega\sinh(2K)\sin(2K\omega) - \omega\sinh(K)\sin(K\omega)]$;

$\widehat{f}_s(\omega) = \dfrac{1}{1+\omega^2}[\cosh(2\kappa)\sin(2K\omega) - \cosh(K)\sin(K\omega) - \omega\sinh(2K)\cos(2K\omega) + \omega\sinh(K)\cos(K\omega)]$

7. Suppose that for each $L > 0$, we have on $[0, L]$ that $f^{(4)}(t)$ is piecewise continuous, $f^{(3)}(t)$ is continuous and as $t \to +\infty$ that $f^{(3)}(t) \to 0, f''(t) \to 0, f(t) \to 0$. Then we can integrate by parts four times to get

$$\mathcal{F}_s\{f^{(4)}(t)\} = \int_0^\infty f^{(4)}(t)\sin(\omega t)dt$$

$$= \left[f^{(3)}(t)\sin(\omega t) - \omega f''(t)\cos(\omega t) - \omega^2 f'(t)\sin(\omega t) + \omega^3\cos(\omega t)f(t)\right]\Big|_0^\infty$$

$$+\omega^4\int_0^\infty \sin(\omega t)f(t)dt = \omega^4\widehat{f}_s(\omega) - \omega^3 f(0) + \omega f''(0).$$

8. Assume the same conditions on f as in Problem 7 and integrate by parts four times to get

$$\mathcal{F}_c\{f^{(4)}(t)\} = \int_0^\infty f^{(4)}(t)\cos(\omega t)dt$$

$$= \left[f^{(3)}(t)\cos(\omega t) + \omega f''(t)\sin(\omega t) - \omega^2 f'(t)\cos(\omega t) - \omega^3 f(t)\sin(\omega t)\right]\Big|_0^\infty$$

$$+\omega^4 \int_0^\infty \cos(\omega t) f(t) dt = \omega^4 \widehat{f}_c(\omega) + \omega^2 f'(0) - f^{(3)}(0).$$

Section 15.6 The Finite Fourier Cosine and Sine Transforms

1. $\widetilde{f_S}[K](n) = \int_0^\pi K \sin(nx) dx = -\frac{K}{n} \cos(nx)\Big|_0^\pi = \frac{K}{n}[1 - (-1)^n], n \geq 1.$

2. $\widetilde{f_S}[x](n) = \int_0^\pi x \sin(nx) dx = \frac{\pi}{n}(-1)^{n+1}, n \geq 1.$

3. $\widetilde{f_S}[x^2](n) = \int_0^\pi x^2 \sin(nx) dx = \frac{2}{n^3}[(-1)^n - 1] + \frac{\pi^2}{n}(-1)^{n+1}, n \geq 1$

4. $\widetilde{f_S}[x^5](n) = \int_0^\pi x^5 \sin(nx) dx = \left[\frac{\pi^5}{n} - \frac{20\pi^3}{n^3} + \frac{120\pi}{n^5}\right](-1)^{n+1}, n \geq 1$

5. $\widetilde{f_S}[\sin(ax)](n) = \int_0^\pi \sin(ax)\sin(nx) dx = \frac{(-1)^n n \sin(a\pi)}{a^2 - n^2}, n \geq 1$ if a is not an integer; If $a = m, \widetilde{f_S}[\sin(mx)](n) = 0$ for $n \neq m, \widetilde{f_S}[\sin(nx)](n) = \frac{\pi}{2}.$

6. $\widetilde{f_S}[\cos(ax)](n) = \int_0^\pi \cos(ax)\sin(nx) dx = \frac{n}{n^2 - a^2}[1 - (-1)^n \cos(a\pi)], n \geq 1$ if a is not an integer;
If $a = m, \widetilde{f_S}[\cos(mx)](n) = \frac{n}{n^2 - m^2}[1 - (-1)^n \cos(m\pi)]$, for $n \neq m, \widetilde{f_S}[\cos(nx)](n) = 0.$

7. $\widetilde{f_S}[e^{-x}](n) = \int_0^\pi e^{-x} \sin(nx) dx = \frac{n}{n^2 + 1}[1 - (-1)^n e^{-\pi}], n \geq 1.$

8. $\widetilde{f_C}[f(x)](n) = \int_0^{1/2} \cos(nx) dx - \int_{1/2}^\pi \cos(nx) dx$, so

$$\widetilde{f_C}(0) = 1 - \pi; \widetilde{f_C}(n) = \frac{1}{n}[\sin(nx)\Big|_0^{1/2} - \sin(nx)|_{1/2}^\pi] = \frac{2}{n}\sin\left(\frac{n}{2}\right), n \geq 1.$$

9. $\widetilde{f_C}[x](0) = \int_0^\pi x dx = \frac{\pi^2}{2}, \widetilde{f_C}(n) = \int_0^\pi x \cos(nx) dx = \frac{(-1)^n - 1}{n^2}, n \geq 1.$

10. $\widetilde{f_C}[x^2](0) = \int_0^\pi x^2 dx = \frac{\pi^3}{3}, \widetilde{f_C}(n) = \int_0^\pi x^2 \cos(nx) dx = \frac{2\pi}{n^2}(-1)^n, n \geq 1.$

11. $\widetilde{f_C}[x^3](0) = \int_0^\pi x^3 dx = \frac{\pi^4}{4}, \widetilde{f_C}(n) = \int_0^\pi x^3 \cos(nx) dx = \frac{6}{n^4} + (-1)^n \left(\frac{3\pi^2}{n^2} - \frac{6}{n^4}\right), n \geq 1.$

12. $\widetilde{f_C}(n) = \int_0^\pi \cosh(ax)\cos(nx) dx = \frac{a(-1)^n}{n^2 + a^2}\sinh(a\pi), n \geq 0.$

13. Assuming $a \neq 0$ we have $\widetilde{f_C}[\sin(ax)](0) = \int_0^\pi \sin(ax) dx = \frac{1 - \cos(a\pi)}{a}, \widetilde{f_C}[\sin(ax)](n) = \int_0^\pi \sin(ax)\cos(nx) dx = \frac{a}{a^2 - n^2}[1 - (-1)^n \cos(a\pi)], n \geq 1$ for a not an integer.

If $a = m, \widetilde{f_C}[\sin(mx)](n) = \int_0^\pi \sin(mx)\cos(nx)dx = \frac{m}{m^2-n^2}[1-(-1)^n\cos(m\pi)], n \neq m, \widetilde{f_C}[\sin(mx)](m) = 0.$

14. $\widetilde{f_C}[e^{-x}](n) = \int_0^\pi e^{-x}\cos(nx)dx = \frac{1}{n^2+1}[1-(-1)^n e^{-\pi}], n \geq 0.$

15. $\widetilde{f_S}[f'(x)](n) = \int_0^\pi f'(x)\sin(nx)dx = f(x)\sin(nx)\Big|_0^\pi - n\int_0^\pi f(x)\cos(nx)dx = -n\widetilde{f_C}[f(x)](n).$

16. $\widetilde{f_C}[f'(x)](n) = \int_0^\pi f'(x)\cos(nx)dx = f(x)\cos(nx)\Big|_0^\pi + n\int_0^\pi f(x)\sin(nx)dx = (-1)^n f(\pi) - f(0) + n\widetilde{f_S}[f(x)](n).$

Section 15.7 The Discrete Fourier Transform

The six point Discrete Fourier Transform of the sequence $u(j)$ is calculated by the formula

$$\mathcal{D}[u](k) = \sum_{j=0}^{5} u(j)e^{-\pi kji/3}, k = -4,-3,-2,-1,0,1,2,3,4$$

These values were computed with Maple and rounded to the five decimal places in the tables below for Problems One through Six.

1.

$$\begin{array}{llrcl}
\mathcal{D}[u](-4) & \approx & .13292 & - & .01658i \\
\mathcal{D}[u](-3) & \approx & .09624 & + & .72830\times 10^{-9}i \\
\mathcal{D}[u](-2) & \approx & .13292 & + & .01658i \\
\mathcal{D}[u](-1) & \approx & 2.93687 & + & .42794i \\
\mathcal{D}[u](0) & \approx & 1.82396 & + & 0i \\
\mathcal{D}[u](1) & \approx & 2.93687 & - & .42794i \\
\mathcal{D}[u](2) & \approx & .13292 & - & .01658i \\
\mathcal{D}[u](3) & \approx & .09624 & - & .72830\times 10^{-9}i \\
\mathcal{D}[u](4) & \approx & .13292 & + & .01658i
\end{array}$$

2.

$$\begin{array}{llrcl}
\mathcal{D}[u](-4) & \approx & .24922 & + & .10702i \\
\mathcal{D}[u](-3) & \approx & .09624 & + & .12883i \\
\mathcal{D}[u](-2) & \approx & .01662 & + & .14018i \\
\mathcal{D}[u](-1) & \approx & -.06520 & + & .15184i \\
\mathcal{D}[u](0) & \approx & 1.82396 & + & 8.17616i \\
\mathcal{D}[u](1) & \approx & 5.93894 & - & .70403i \\
\mathcal{D}[u](2) & \approx & .24922 & + & .10702i \\
\mathcal{D}[u](3) & \approx & .09624 & + & .12883i \\
\mathcal{D}[u](4) & \approx & .01662 & + & .14018i
\end{array}$$

3.

$$\begin{array}{lcrcl}
\mathcal{D}[u](-4) & \approx & .65000 & - & .17321i \\
\mathcal{D}[u](-3) & \approx & .61667 & - & .25346 \times 10^{-9}i \\
\mathcal{D}[u](-2) & \approx & .65000 & + & .17321i \\
\mathcal{D}[u](-1) & \approx & .81667 & + & .40415i \\
\mathcal{D}[u](0) & \approx & 2.45000 & + & 0i \\
\mathcal{D}[u](1) & \approx & .81667 & - & .40415i \\
\mathcal{D}[u](2) & \approx & .65000 & - & .17321i \\
\mathcal{D}[u](3) & \approx & .61667 & + & .25346 \times 10^{-9}i \\
\mathcal{D}[u](4) & \approx & .65000 & + & .17321i
\end{array}$$

4.

$$\begin{array}{lcrcl}
\mathcal{D}[u](-4) & \approx & .84806 & - & .13087i \\
\mathcal{D}[u](-3) & \approx & .81083 & - & .14161 \times 10^{-9}i \\
\mathcal{D}[u](-2) & \approx & .84806 & + & .13087i \\
\mathcal{D}[u](-1) & \approx & 1.0008 & + & .25403i \\
\mathcal{D}[u](0) & \approx & 1.49139 & + & 0i \\
\mathcal{D}[u](1) & \approx & 1.0008 & - & .25403i \\
\mathcal{D}[u](2) & \approx & .84806 & - & .13087i \\
\mathcal{D}[u](3) & \approx & .81083 & + & .14161 \times 10^{-9}i \\
\mathcal{D}[u](4) & \approx & .84806 & + & .13087i
\end{array}$$

5.

$$\begin{array}{lcrcl}
\mathcal{D}[u](-4) & \approx & -14.00000 & + & 10.39230i \\
\mathcal{D}[u](-3) & \approx & -15.00000 & + & .22023 \times 10^{-7}i \\
\mathcal{D}[u](-2) & \approx & -14.00000 & - & 10.10.39230i \\
\mathcal{D}[u](-1) & \approx & -6.00000 & - & 31.17691i \\
\mathcal{D}[u](0) & \approx & 55.00000 & + & 0i \\
\mathcal{D}[u](1) & \approx & -6.00000 & + & 31.17691i \\
\mathcal{D}[u](2) & \approx & -14.00000 & + & 10.39230i \\
\mathcal{D}[u](3) & \approx & -15.00000 & - & .22023 \times 10^{-7}i \\
\mathcal{D}[u](4) & \approx & -14.00000 & - & 10.39230i
\end{array}$$

6.

$$\begin{array}{lcrcl}
\mathcal{D}[u](-4) & \approx & .00932 & + & .09972i \\
\mathcal{D}[u](-3) & \approx & -.03259 & + & .21350 \times 10^{-8}i \\
\mathcal{D}[u](-2) & \approx & .00932 & - & .09972i \\
\mathcal{D}[u](-1) & \approx & 3.21296 & - & 2.57414i \\
\mathcal{D}[u](0) & \approx & -.41198 & + & 0i \\
\mathcal{D}[u](1) & \approx & 3.21296 & + & 2.57414i \\
\mathcal{D}[u](2) & \approx & .00932 & + & .09972i \\
\mathcal{D}[u](3) & \approx & -.03259 & - & .21350 \times 10^{-8}i \\
\mathcal{D}[u](4) & \approx & .00932 & - & .09972i
\end{array}$$

In problems 7 through 12, the N-point inverse discrete Fourier transform of the given sequence, $\{U_j\}_{j=0}^{N-1}$, is the sequence $\{u_j\}_{j=0}^{N-1}$ computed by the formula $u_j = \dfrac{1}{N}\displaystyle\sum_{k=0}^{N-1} U_k e^{2\pi ijk/N}$. The values of this sequence were computed using Maple to nine decimal places, with the results recorded below rounded to six decimal places.

7. For the given sequence and $N = 6$, We calculate $u_j = \frac{1}{6}\sum_{k=0}^{5}(1+i)^k e^{2\pi ijk/6}$. Approximate values are

$$\begin{aligned}
u_0 &= -1.333333 + .166667i \\
u_1 &= -.427030 + .549038i \\
u_2 &= -.016346 + .561004i \\
u_3 &= .333333 + .500000i \\
u_4 &= .849679 + .272329i \\
u_5 &= 1.593696 - 2.049038i
\end{aligned}$$

8. For the given sequence and $N = 5$, We calculate $u_j = \frac{1}{5}\sum_{k=0}^{4}(i^{-k})e^{2\pi ijk/5}$. Approximate values are

$$\begin{aligned}
u_0 &= .200000 \\
u_1 &= .731375 - .531375i \\
u_2 &= -.096261 + .296261i \\
u_3 &= .049047 + .150953i \\
u_4 &= .115838 + .084162
\end{aligned}$$

9. For the given sequence and $N = 7$, We calculate $u_j = \frac{1}{7}\sum_{k=0}^{6}(e^{-ik})e^{2\pi ijk/7}$. Approximate values are

$$\begin{aligned}
u_0 &= .103479 + .014751i \\
u_1 &= .933313 - .296094 \\
u_2 &= -.094163 + .088785i \\
u_3 &= -.023947 + .062482i \\
u_4 &= .004307 + .051899i \\
u_5 &= .025788 + .043852i \\
u_6 &= .051222 + .034325i
\end{aligned}$$

10. For the given sequence and $N = 5$, We calculate $u_j = \frac{1}{5}\sum_{k=0}^{4}(k^2)e^{2\pi ijk/5}$. Approximate values are

$$\begin{aligned}
u_0 &= 6.000000 \\
u_1 &= -1.052786 - 3.440955i \\
u_2 &= -1.947214 - .812299i \\
u_3 &= -1.947214 + .812299i \\
u_4 &= -1.052786 + 3.440955i
\end{aligned}$$

11. For the given sequence and $N = 5$, We calculate $u_j = \frac{1}{5}\sum_{k=0}^{4}(\cos(k))e^{2\pi ijk/5}$. Approximate values are

$$\begin{aligned} u_0 &= -.103896 \\ u_1 &= .420513 + .294562i \\ u_2 &= .131434 + .031205i \\ u_3 &= .131434 - .031205i \\ u_4 &= .420513 - .294562i \end{aligned}$$

12. For the given sequence and $N = 6$, We calculate $u_j = \frac{1}{6}\sum_{k=0}^{5} \ln(k+1)e^{2\pi ijk/6}$. Approximate values are

$$\begin{aligned} u_0 &= 1.096542 \\ u_1 &= -.249644 - .232302i \\ u_2 &= -.201697 - .084840i \\ u_3 &= -.193858 \\ u_4 &= -.201697 + .084840i \\ u_5 &= -.249644 + .232302i \end{aligned}$$

For Problems 13 through 16, the complex Fourier coefficients of the function $F(t)$ having period P are calculated by $d_k = \frac{1}{P}\int_0^P f(\xi)e^{-2\pi ik\xi}d\xi$ then evaluated for $k = -3, -2, \ldots, 1, 2, 3$. The discrete Fourier transform (DFT) with $N = 128$ is used to approximate these coefficients using $f_k = \frac{1}{128}\sum_{j=0}^{127} f\left(\frac{jP}{128}\right)e^{-2\pi ijk/128}$, for $k = -3, -2, \ldots, 1, 2, 3$. The values were computed using Maple to nine decimal places and are given below rounded to six decimals.

13. For $f(t) = \cos(t), P = 2$ we have

$$d_k = \frac{1}{2}\int_0^2 \cos(\xi)e^{-i\pi k\xi}d\xi = -\frac{\sin(2)}{2(\pi^2k^2 - 1)} + \frac{ik\pi(\cos(2) - 1)}{2(\pi^2k^2 - 1)}.$$

k	Complex Coefficients; d_k	DFT Approximations; f_k
-3	$-.005177 + .075984i$	$.000346 + .075849i$
-2	$-.011816 + .115622i$	$-.006293 + .115532i$
-1	$-.051259 + .250780i$	$-.045737 + .250753i$
0	$.454649$	$.460171$
1	$-.051259 - .250798i$	$-.045737 - .250753i$
2	$-.011816 - .115622i$	$-.006293 - .115532i$
3	$-.005177 - .075984i$	$.000346 - .075849i$

14. For $f(t) = e^{-t}, P = 3$ we have

$$d_k = \frac{1}{3}\int_0^3 e^{-\xi}e^{-i2\pi k\xi/3}d\xi = \frac{3(1 - e^{-3})}{9 + 4\pi^2k^2} - \frac{2\pi k(1 - e^{-3})}{9 + 4\pi^2k^2}.$$

k	Complex Coefficients			DFT Approximations		
-3	.007825	+	.049165i	.011551	+	.049074i
-2	.017079	+	.071538i	.020805	+	.071478i
-1	.058802	+	.123155i	.062528	+	.123125i
0	.316738			.320464		
1	.058802	$-$	.123155i	.062528	$-$	.123125i
2	.017079	$-$	.071538i	.020804	$-$	.071478i
3	.007825	$-$	.049165i	.011551	$-$	.049074i

15. For $f(t) = t^2, P = 1$ we have

$$d_k = \int_0^1 \xi^2 e^{-i2\pi k\xi} d\xi = \frac{1}{2\pi^2 k^2} + i\frac{1}{2\pi k}.$$

k	Complex Coefficients			DFT Approximations		
-3	.005629	$-$	.053051i	.001733	$-$	.052956i
-2	.012665	$-$	.079577i	.008769	$-$	.079514i
-1	.050661	$-$	.159155i	.046765	$-$	.159123i
0	.333333			.329437		
1	.050661	+	.159155i	.046765	+	.159123i
2	.012665	+	.079577i	.008769	+	.079514i
3	.005629	+	.053052i	.001733	+	.052956i

16. For $f(t) = te^{2t}, P = 4$ we have $d_k = \frac{1}{4}\int_0^4 \xi e^{2\xi} e^{-i\pi k\xi/2} d\xi$

$$= \frac{7e^8 + (16 - \pi^2 k^2) + 16e^8\pi^2 k^2}{(16 - \pi^2 k^2)^2 + 64\pi^2 k^2} + i\frac{56e^8 - 2e^8\pi k(16 - \pi^2 k^2) + 8k\pi}{(16 - \pi^2 k^2)^2 + 64\pi^2 k^2}.$$

k	Complex Coefficients			DFT Approximations		
-3	247.246215	$-$	515.579355i	201.215105	$-$	514.436038i
-2	452.586443	$-$	626.547636i	406.555000	$-$	625.785580i
-1	894.543813	$-$	612.101891i	848.512176	$-$	611,720909i
0	1304.231619			1258.199915		
1	894.543813	+	612.101891i	848.512177	+	611.720909i
2	452.586443	+	626.547636i	406.555000	+	625.785580i
3	247.246215	+	515.579355i			

Section 15.8 Sampled Fourier Series

For Problems 1 through 6, the complex Fourier coefficients of $f(t)$ having period P are computed by $d_n = \frac{1}{P}\int_0^P f(t)e^{-2k\pi i t/P}dt$. The 10^{th} partial sum of the series is formed and evaluated at t_0 to give $S_{10}(t_0)$. Next using $N = 128$, the DFT approximation to $S_{10}(t_0)$ requires the values $\{U_n\}_{n=0}^{10}$ and $\{U_n\}_{n=118}^{127}$ computed by $U_n = \sum_{j=0}^{127} f\left(\frac{jP}{128}\right) e^{-2\pi i j n/128}$. Then with $V_n =$

$U_n, n = 0, 1, \ldots, 10$ and $118, \ldots 127, V_n = 0, 11 \le n \le 117$, we get the DFT approximation $w = \dfrac{1}{128}\sum_{k=0}^{127} V_k e^{2\pi i k t_0/P}$. The non-zero values of $\{U_n\}$ (rounded to six decimal places) are recorded for each example; followed by the DFT approximation w and the difference, denoted by var $= |S_{10}(t_0) - w|$, between the actual value and the DFT approximation. All values were computed with *Maple* to nine decimal places, with tables entries rounded to 6 decimal places.

1. Calculate $d_n = \dfrac{1}{2}\displaystyle\int_0^2 (1+t)e^{-n\pi i t}dt$, for $n = 0, \pm 1, \pm 2, \ldots$ to get
$d_0 = 2, d_n = \dfrac{1 - 2n\pi i}{n^2\pi^2}, n \ne 0$. The complex Fourier series of f is given by $2 + \displaystyle\sum_{\substack{n=-\infty \\ n\ne 0}}^{\infty} \dfrac{1-2n\pi i}{n^2\pi^2} e^{n\pi i t}$.

Using the 10^{th} partial sum we find $S_{10}(1/8) \approx 1.020712$
For the DFT approximation we have

Calculated values of U_n

U_0	255		
U_1	$-1 + 40.735484i$	U_{118}	$-1 - 3.992224i$
U_2	$-1 + 20.355468i$	U_{119}	$-1 - 4.453202i$
U_3	$-1 + 13.556669i$	U_{120}	$-1 - 5.027339i$
U_4	$-1 + 10.153170i$	U_{121}	$-1 - 5.763142i$
U_5	$-1 + 8.107786i$	U_{122}	$-1 - 6.741452i$
U_6	$-1 + 6.741452i$	U_{123}	$-1 - 8.107786i$
U_7	$-1 + 5.763142i$	U_{124}	$-1 - 10.153170i$
U_8	$-1 + 5.027339i$	U_{125}	$-1 - 13.556670i$
U_9	$-1 + 4.453202i$	U_{126}	$-1 - 20.355468i$
U_{10}	$-1 + 3.992224i$	U_{127}	$-1 - 40.735484i$

Using these values of U_n we obtain the DFT approximation $w = 1.055233 + .278759 \times 10^{-9} i$ with var $= |S_{10}(1/8) - w| \approx .034520$.

2. Calculate $d_n = \int_0^1 t^2 e^{-2n\pi i t}dt$, for $n = 0, \pm 1, \pm 2, \ldots$ to get $d_0 = \frac{1}{3}, d_n = \frac{2n\pi + i(2n^2\pi^2 - 1)}{n^3\pi^3}, n \neq 0$. The complex Fourier series of f is given by $\frac{1}{3} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{2n\pi + i(2n^2\pi^2 - 1)}{n^3\pi^3} e^{2n\pi i t}$. Using the 10^{th} partial sum we find $S_{10}(1/2) \approx .2504564$
For the DFT approximation we have

Calculated values of U_n

U_0	42.167969		
U_1	$5.985858 + 20.367742i$	U_{118}	$-.433837 - 1.996112i$
U_2	$1.122442 + 10.177734i$	U_{119}	$-.418629 - 2.226601i$
U_3	$.221810 + 6.778335i$	U_{120}	$-.397367 - 2.513670i$
U_4	$-.093411 + 5.076585i$	U_{121}	$-.366352 - 2.881571i$
U_5	$-.239312 + 4.053893i$	U_{122}	$-.318566 - 3.370726i$
U_6	$-.318566 + 3.370726i$	U_{123}	$-.239312 - 4.053893i$
U_7	$-.366352 + 2.881571i$	U_{124}	$-.093411 - 5.076585i$
U_8	$-.397367 + 2.513670i$	U_{125}	$.221810 - 6.678335i$
U_9	$-.418629 + 2.226601i$	U_{126}	$1.122442 - 10.177734i$
U_{10}	$-.433837 + 1.996112i$	U_{127}	$5.985857 - 20.367742i$

Using these values of U_n we obtain the DFT approximation $w = .246560 + .156250 \times 10^{-9}i$ with var $= |S_{10}(1/2) - w| \approx .003896$.

3. Calculate $d_n = \frac{1}{2}\int_0^2 \cos(t)e^{-n\pi i t}dt$, for $n = 0, \pm 1, \pm 2, \ldots$ to get $d_0 = \frac{\sin(2)}{2}, d_n = \frac{-\sin(2) + in\pi(\cos(2) - 1)}{2(n^2\pi^2 - 1)}, n \neq 0$. The complex Fourier series of f is given by $\frac{\sin(2)}{2} + \frac{1}{2}\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{-\sin(2) + in\pi(\cos(2) - 1)}{(n^2\pi^2 - 1)} e^{n\pi i t}$. Using the 10^{th} partial sum we find $S_{10}(1/8) \approx 1.067161$ For the DFT approximation we have

Calculated values of U_n

U_0	58.901925		
U_1	$-5.854287 - 32.096339i$	U_{118}	$.647851 + 2.829713i$
U_2	$-.805518 - 14.788044i$	U_{119}	$.633992 + 3.157208i$
U_3	$.044274 - 9.708611i$	U_{120}	$.614603 + 3.565443i$
U_4	$.336014 - 7.235154i$	U_{121}	$.586989 + 4.089267i$
U_5	$.470070 - 5.764387i$	U_{122}	$.542633 + 4.787014i$
U_6	$.542633 - 4.787014i$	U_{123}	$.470070 + 5.764387i$
U_7	$.586299 - 4.089267i$	U_{124}	$.336014 + 7.235154i$
U_8	$.614603 - 3.565443i$	U_{125}	$.044274 + 9.7 - 8611i$
U_9	$.633991 - 3.157208i$	U_{126}	$-.805518 + 14.788044i$
U_{10}	$.647851 - 2.829712i$	U_{127}	$-5.854287 + 32.096339i$

Using these values of U_n we obtain the DFT approximation $w = 1.042757 - .267410 \times 10^{-9}i$ with var $= |S_{10}(1/8) - w| \approx .024403$.

4. Calculate $d_n = \int_0^1 e^{-t}e^{-2n\pi i t}dt$, for $n = 0, \pm 1, \pm 2, \ldots$ to get $d_0 = (e-1), d_n = \dfrac{(e-1)(1+2n\pi i)}{1+4n^2\pi^2}, n \neq 0$. The complex Fourier series of f is given by $(e-1) + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \dfrac{(e-1)(1+2n\pi i)}{1+4n^2\pi^2} e^{2n\pi i t}$. Using the 10^{th} partial sum we find $S_{10}(1/4) \approx .827534 - .9 \times 10^{-10} i$ For the DFT approximation we have

Calculated values of U_n

U_0	31.907298		
U_1	$9.553181 - 14.227057i$	U_{118}	$.620232 + 1.951481i$
U_2	$3.383468 - 9.071385i$	U_{119}	$.649821 + 2.174758i$
U_3	$1.847063 - 6.366930i$	U_{120}	$.691097 + 2.451901i$
U_4	$1.269469 - 4.860090i$	U_{121}	$.751110 + 2.805358i$
U_5	$.994545 - 3.915837i$	U_{122}	$.843127 + 3.271884i$
U_6	$.843127 - 3.271884i$	U_{123}	$.994545 + 3.915837i$
U_7	$.741110 - 2.805358i$	U_{124}	$1.269469 + 4.860090i$
U_8	$.691097 - 2.451901i$	U_{125}	$1.847063 + 6.366930i$
U_9	$.649821 - 2.174658i$	U_{126}	$3.383468 + 9.071385i$
U_{10}	$.620232 - 1.951481i$	U_{127}	$9.553181 + 14.227057i$

Using these values of U_n we obtain the DFT approximation $w = .810504 - .954242 \times 10^{-11} i$ with var $= |S_{10}(1/4) - w| \approx .017031$.

5. Calculate $d_n = \int_0^1 t^3 e^{-2n\pi i t}dt$, for $n = 0, \pm 1, \pm 2, \ldots$ to get $d_0 = 1/4, d_n = \dfrac{3n\pi + i(2n^2\pi^2 - 3)}{4n^3\pi^3}, n \neq 0$. The complex Fourier series of f is given by $\dfrac{1}{4} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \dfrac{3n\pi + i(2n^2\pi^2 - 3)}{4n^3\pi^3} e^{2n\pi i t}$. Using the 10^{th} partial sum we find $S_{10}(1/4) \approx -.000729$

For the DFT approximation we have

Calculated values of U_n

U_0	31.501953		
U_1	$9.228787 + 17.271595i$	U_{118}	$-.400755 - 1.993017i$
U_2	$1.933662 + 9.790716i$	U_{119}	$-.377943 - 2.222355i$
U_3	$.582715 + 6.663663i$	U_{120}	$-.346050 - 2.507623i$
U_4	$.109884 + 5.028208i$	U_{121}	$-.299528 - 2.872545i$
U_5	$-.108968 + 4.029124i$	U_{122}	$-.227849 - 3.356393i$
U_6	$-.227849 + 3.356393i$	U_{123}	$-.108968 - 4.029124i$
U_7	$-.299528 + 2.872544i$	U_{124}	$.109884 - 5.028208i$
U_8	$-.346050 + 2.507623i$	U_{125}	$.582715 - 6.663663i$
U_9	$-.377943 + 2.222355i$	U_{126}	$1.933662 - 9.790715i$
U_{10}	$-.400755 + 1.993017i$	U_{127}	$9.228787 - 17.271595i$

Using these values of U_n we obtain the DFT approximation $w = .003483 - .78125o \times 10^{-10} i$ with var $= |S_{10}(1/4) - w| \approx .004212$.

6. Calculate $d_n = \int_0^1 t\sin(t)e^{-2n\pi it}dt$, for $n = 0, \pm1, \pm2, \ldots$ to get $d_0 = \sin(1) - \cos(1)$,

$$d_n = \frac{\cos(1)(4n^2\pi^2 - 1) + \sin(1)(4n^2\pi^2 + 1)}{(4n^2\pi^2 - 1)^2} + i\frac{4n\pi(1 - \cos(1)) - 2n\pi\sin(1) + 8n^3\pi^3}{(4n^2\pi^2 - 1)^2}, n \neq 0.$$

The complex Fourier series of f is given by

$$\sin(1) - \cos(1) + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left\{ \frac{\cos(1)(4n^2\pi^2 - 1) + \sin(1)(4n^2\pi^2 + 1)}{(4n^2\pi^2 - 1)^2} \right.$$

$$\left. +i\,\frac{4n\pi(1 - \cos(1)) - 2n\pi\sin(1) + 8n^3\pi^3}{(4n^2\pi^2 - 1)^2} \right\} e^{2n\pi it}.$$

Using the 10^{th} partial sum we find $S_{10}(1/8) \approx ..053390 - .6 \times 10^{-10}i$
For the DFT approximation we have

Calculated values of U_n

U_0	60.953531		
U_1	$-94.509581 - 26.479226i$	U_{118}	$1.064899 + 6.040217i$
U_2	$-11.274203 - 30.060278i$	U_{119}	$.961000 + 6.737012i$
U_3	$-3.690170 - 20.379819i$	U_{120}	$.815252 + 7.604519i$
U_4	$-1.331661 - 15.319442i$	U_{121}	$.601640 + 8.715653i$
U_5	$-.286124 - 12.249522i$	U_{122}	$.270130 = 10.191613i$
U_6	$.270130 - 10.191613i$	U_{123}	$-.286124 + 12.249522i$
U_7	$.601640 - 8.715653i$	U_{124}	$-1.331661 + 15.319442i$
U_8	$.815252 - 7.604519i$	U_{125}	$-3.690170 + 20.379819i$
U_9	$.961000 - 6.737012i$	U_{126}	$-11.274203 + 30.060278i$
U_{10}	$1.064899 - 6.040217i$	U_{127}	$-.94509581 + 26.479226i$

Using these values of U_n we obtain the DFT approximation $w = .149844 + .607562 \times 10^{-9}i$ with var $= |S_{10}(1/8) - w| \approx .096453$.

7. The Fourier transform of $f(t) = e^{-4t}$ is $\hat{f}(\omega) = \int_0^\infty e^{-4t}e^{i\omega t}dt = \frac{4 - 4i\omega}{\omega^2 + 16}$ so $\hat{f}(4) = \frac{1}{8} - \frac{1}{8}i$. The DFT approximation of $\hat{f}(4)$ for $L = 3, N = 512$ is given by approx $= \frac{3\pi}{256}\sum_{j=0}^{511} f\left(\frac{3\pi j}{256}\right)e^{-3\pi ij/64} = .143860 - .124549i$, with $|\hat{f}(4) - approx| \approx .018887$

8. The Fourier transform of $f(t) = \cos(2t)$ does not exist since the defining integral diverges. We can, however approximate $\int_0^{12\pi}\cos(2t)e^{-i\omega t}dt$ using the DFT. With $L = 6, N = 512, \omega = 2$ we get

$$\hat{f}(\omega) \approx \frac{3\pi}{128}\sum_{j=0}^{511} f\left(\frac{3\pi j}{128}\right)e^{-3\pi j/64} = 18.84956.$$

The actual value computed using *Maple* or a simple integration is found to be 18.84956.

9. The Fourier transform of $f(t) = te^{-2t}$ is $\widehat{f}(\omega) = \int_0^\infty te^{-2t}e^{i\omega t}dt = \frac{4-\omega^2}{(\omega^2+4)^2} - \frac{4\omega i}{(\omega^2+4)^2}$ so $\widehat{f}(12) = -.006392 - .002191i$. The DFT approximation of $\widehat{f}(12)$ for $L = 3, N = 512$ is given by approx $= \frac{3\pi}{256}\sum_{j=0}^{511} f\left(\frac{3\pi j}{256}\right) e^{-9\pi ij/64} = -.006506 - .002191i$, with $|\widehat{f}(12) - approx| \approx .000114$

10. Note that $\int_0^\infty |t^2\cos(t)|dt$ diverges, so the basic assumption that the Fourier transform of this function can be well aproximated by the finite integral $\int_0^{2\pi L} t^2\cos(t)e^{-i\omega t}dt$ is not valid here. We can however attempt to approximate the formal symbol $\widehat{f}(4) = \int_0^{8\pi} t^2\cos(t)e^{-4it}dt = 3.797836 + 168.441249i$ by the DFT. With $L = 4, \omega = 4, N = 512$ we have approx $= \frac{\pi}{64}\sum_{j=0}^{511} f\left(\frac{\pi j}{64}\right) e^{-\pi ij/16} = -11.695188 + 167.933522i$ with $|\widehat{f}(4) - approx| = 15.501342$

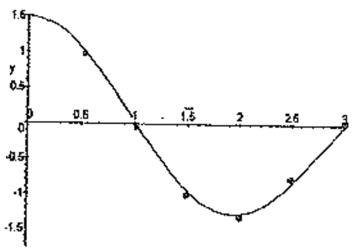
For Problems 11 through 14, the Fourier transform of the function $f(t)$ is calculated, then the DFT using $N = 256,\ L = 4$ is used to approximate values of $Re(\widehat{f}(\omega)), Im(\widehat{f}(\omega)), |\widehat{f}(\omega)|$ at values $\omega = k/2, k = 0, 1, 2, \ldots, 6$. Since most of these functions are zero outside an interval of compact support, we must make sure that the sum in the DFT is properly adjusted so we stay in the interval where $f(t)$ is non-zero. In fact, if the support of $f(t)$ is $[a, b]$ then the sum should start at the smallest integer $j_s \geq \frac{aN}{2\pi L}$ and finish at the largest integer $j_f \leq \frac{bN}{2\pi L}$. These values will need to be adjusted for each different choice of N and L. The values obtained are displayed in tables below and then plotted as point plots on the continuous graphs of each of these respective functions shown below. Actual values of $\widehat{f}(\omega)$ are given in the tables for comparison.

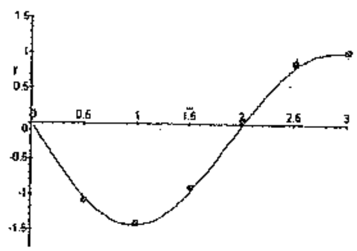
11. Calculate $\widehat{f}(\omega) = \int_1^2 te^{-i\omega t}dt$

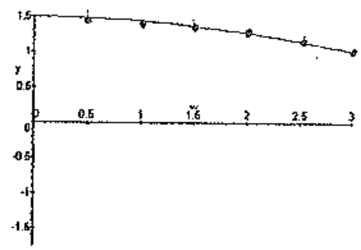
$$= \frac{\cos(2\omega) + 2\omega\sin(2\omega) - \cos(\omega) - \omega\sin(\omega)}{\omega^2} + i\frac{2\omega\cos(2\omega) - \sin(2\omega) - \omega\cos(\omega) + \sin(\omega)}{\omega^2}.$$

Table of Comparative Values

ω	$\widehat{f}(\omega)$	DFT approx.	$\lvert\widehat{f}(\omega)\rvert$	$\lvert$DFT$\rvert$	$\lvert\widehat{f}(\omega) - \text{DFT}\rvert$
$1/2$	$1.057912 - 1.042138i$	$1.044499 - 1.048022i$	1.485001	1.479638	.014647
1	$.020675 - 1.440422i$	$-.005727 - 1.437246i$	1.440571	1.437257	.026592
$3/2$	$-.948272 - .986537i$	$-.974848 - .960163i$	1.368384	1.368299	.037441
2	$-1.270825 - .029045i$	$-1.275097 + .017012i$	1.271157	1.275210	.046255
$5/2$	$-.832959 + .796571i$	$-.802578 + .839322i$	1.152539	1.161289	.052447
3	$-.016632 + 1.016837i$	$.037444 + 1.029887i$	1.016973	1.030567	.055629


Graph of $\text{Re}(\widehat{f}(\omega))$

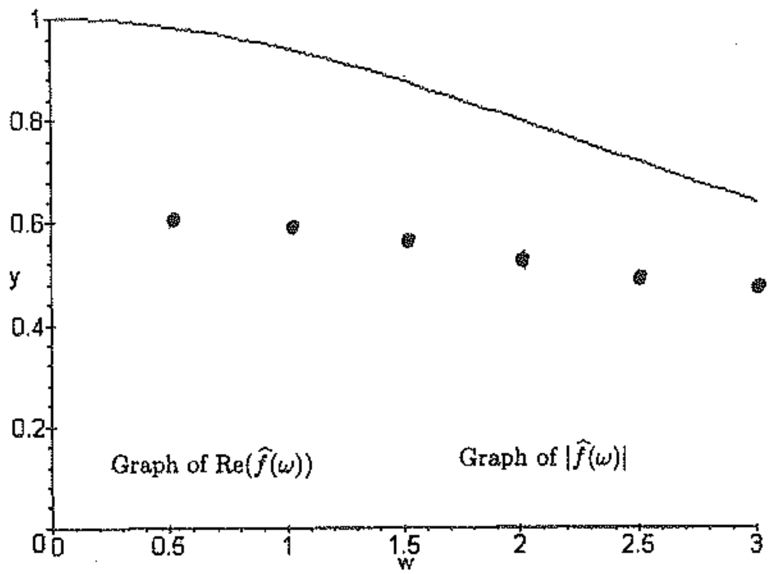

Graph of $\text{Im}(\widehat{f}(\omega))$


Graph of $\lvert\widehat{f}(\omega)\rvert$

12. Calculate $\widehat{f}(\omega) = \int_{-\infty}^{\infty} 2e^{-4|t|}e^{-i\omega t}dt = \dfrac{16}{16+\omega^2}$. Since $\text{Im}(\widehat{f}(\omega)) = 0$, and $|\widehat{f}(\omega)| = \text{Re}(\widehat{f}(\omega))$, only one set of approximations is needed here.

Table of Comparative Values

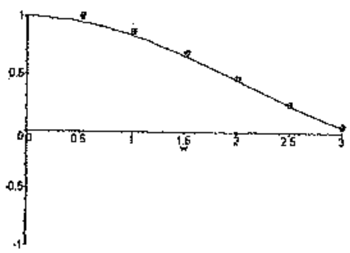
ω	$\widehat{f}(\omega)$	DFT approx.	$\lvert\widehat{f}(\omega)\rvert$	$\lvert$DFT$\rvert$	$\lvert\widehat{f}(\omega) - \text{DFT}\rvert$
1/2	.984615	$.596892 - .060741i$	.984615	.599975	.392452
1	.941176	$.575175 - .116053i$	.941176	.586766	.383960
3/2	.876712	$.542947 - .161992i$	.876712	.566597	.371000
2	.800000	$.504596 - .196810i$	.800000	.541619	.354961
5/2	.719101	$.464153 - .220730i$	.719101	.513965	.337224
3	.640000	$.424611 - .235211i$	.640000	.485406	.318930



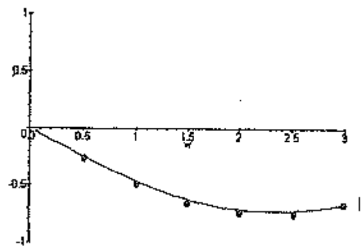
13. Calculate $\widehat{f}(\omega) = \int_0^1 e^{-i\omega t}dt = \dfrac{\sin(\omega)}{\omega} + i\dfrac{\cos(\omega)-1}{\omega}$.

Table of Comparative Values

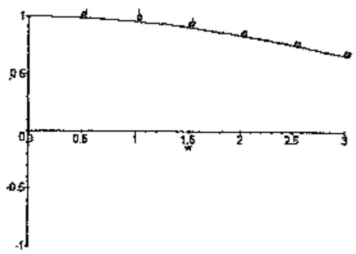
ω	$\widehat{f}(\omega)$	DFT approx.	$\lvert\widehat{f}(\omega)\rvert$	$\lvert$DFT$\rvert$	$\lvert\widehat{f}(\omega) - \text{DFT}\rvert$
1/2	$.958851 - .244835i$	$1.034983 - .259250i$	.989616	1.066958	.077484
1	$.841471 - .459698i$	$.907161 - .484887i$	.958851	1.028619	.070354
3/2	$.664997 - .619509i$	$.716156 - .649086i$	.908852	.966536	.059094
2	$.454649 - .708073i$	$.490757 - .734470i$	.841471	.883340	.044728
5/2	$.239389 - .720457$	$.263625 - .736781i$	.759188	.782524	.029220
3	$.047043 - .663331i$	$.065503 - .665058i$	.664997	.668276	.018543



Graph of $\mathrm{Re}(\widehat{f}(\omega))$



Graph of $\mathrm{Im}(\widehat{f}(\omega))$



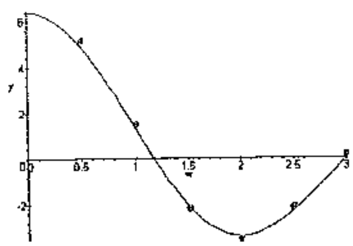
Graph of $\lvert\widehat{f}(\omega)\rvert$

14. Calculate $\widehat{f}(\omega) = \int_0^2 e^t e^{-i\omega t}\,dt$

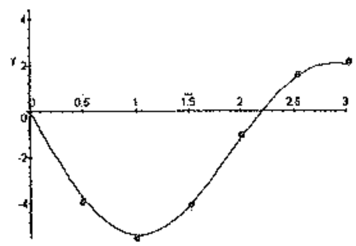
$$= \frac{e^2\cos(2\omega) + e^2\omega\sin(2\omega) - 1}{\omega^2+1} + i\frac{e^2\omega\cos(2\omega) - e^2\sin(2\omega) - \omega}{\omega^2+1}.$$

Table of Comparative Values

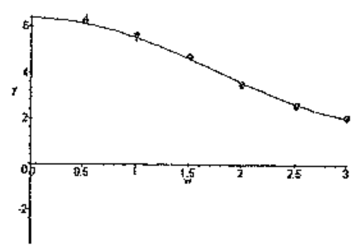
ω	$\widehat{f}(\omega)$	DFT approx.	$\lvert\widehat{f}(\omega)\rvert$	$\lvert\text{DFT}\rvert$	$\lvert\widehat{f}(\omega) - \text{DFT}\rvert$
$1/2$	$4.880930 - 3.77211i$	$4.979178 - 3.852357i$	6.171775	6.295464	.123692
1	$1.321959 - 5.396891i$	$1.334772 - 5.479482i$	5.556438	5.639711	.083579
$3/2$	$-2.077229 - 4.158587i$	$-2.118152 - 4.173814i$	4.648519	4.680522	.043664
2	$-3.402784 - 1.213513i$	$-3.417494 - 1.146478i$	3.612693	3.604674	.068629
$5/2$	$-2.292120 + 1.355246i$	$-2.223975 + 1.446032i$	2.662800	2.652749	.113516
3	$-.009910 + 2.034887$	$.123549 + 2. - 69761i$	2.034911	2.073445	.137940



Graph of $\mathrm{Re}(\widehat{f}(\omega))$



Graph of $\mathrm{Im}(\widehat{f}(\omega))$



Graph of $|\widehat{f}(\omega)|$

In problems 15 through 21, the requested partial sums for the given value of N were computed in the complex series form. Due to possible cancellation of terms when this complex series is put into real form,the graphs of the partial sums may not actually include N non-zero terms. However, the qualitative comparisons of the various partial sums are still quite valid and useful.

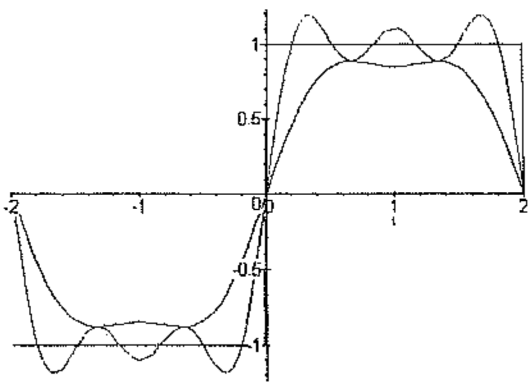
15. The complex Fourier coefficients are calculated by $d_k = \frac{1}{4}\left[\int_{-2}^{0} -e^{-k\pi it/2}dt + \int_0^2 e^{-k\pi it/2}dt\right] = \frac{i}{\pi k}[(-1)^k - 1]$. Note since $d_0 = 0$,this gives the complex form of the Fourier series

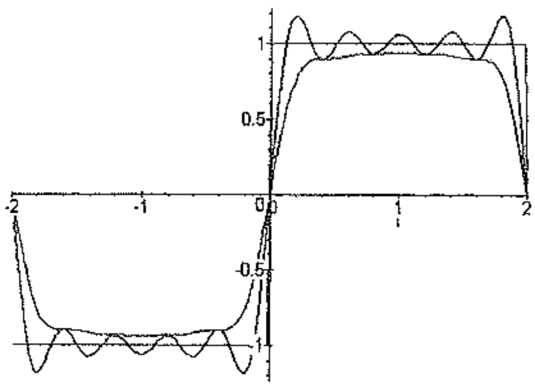
$$\sum_{k=-\infty}^{k=\infty} \frac{i}{\pi k}[(-1)^k - 1] = \sum_{n=1}^{\infty} \frac{2}{n\pi}[1-(-1)^n]\sin\left(\frac{n\pi t}{2}\right) = \frac{4}{\pi}\sum_{p=1}^{\infty}\frac{1}{2p-1}\sin\left(\frac{(2p-1)\pi t}{2}\right).$$

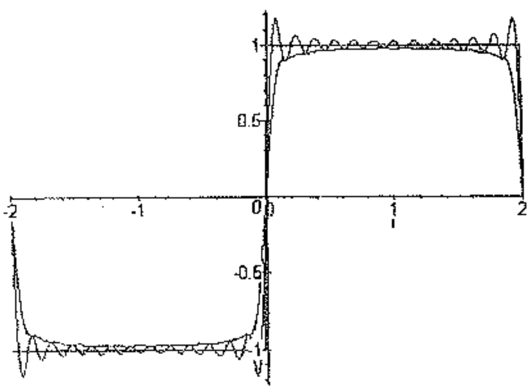
The N^{th} partial sum of the corresponding Cesàro sum is

$$\sigma_N(t) = \frac{4}{\pi}\sum_{n=1}^{N}\left(1-\frac{2p-1}{N}\right)\frac{1}{2p-1}\sin\left(\frac{(2p-1)\pi t}{2}\right).$$

Comparative graphs are shown below for these partial sums for $N = 5, 10, 25$ (see remark preceding this problem).







16. Complex Fourier coefficients are computed by

$$d_0 = \int_{-2}^{1} t^2 dt + \int_{1}^{2} (2+t)dt = 13/8; d_k = \frac{1}{4}\left[\int_{-2}^{1} t^2 e^{-\pi kit/2} dt + \int_{1}^{2}(2+t)e^{-\pi kit/2}dt\right]$$

$$= \frac{1}{k^3\pi^3}\left\{\sin\left(\frac{k\pi}{2}\right)[-4-k\pi^2] + k\pi\cos\left(\frac{k\pi}{2}\right) + 5k\pi(-1)^k\right\}$$

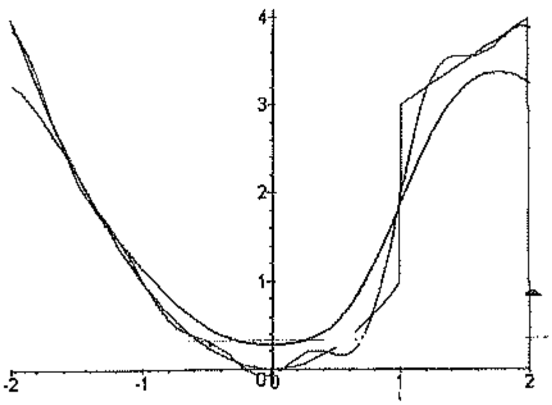
$$+\frac{i}{k^3\pi^3}\left\{-k\pi\sin\left(\frac{k\pi}{2}\right) + \cos\left(\frac{k\pi}{2}\right)[-4-k^2\pi^2] + 4(-1)^k\right\}.$$

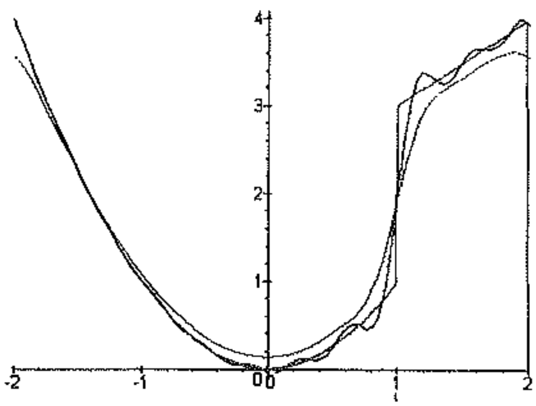
When put into real form, this gives the Fourier series

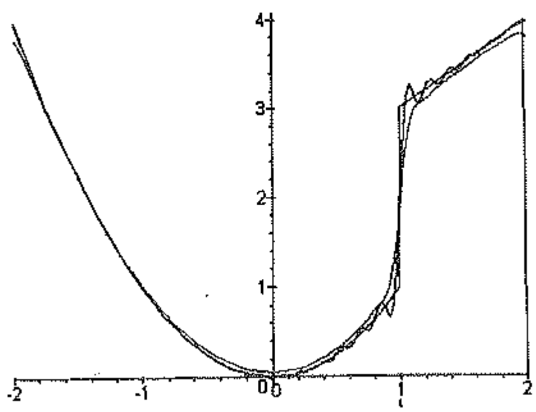
$$\frac{13}{8} + \sum_{n=1}^{\infty} \frac{2}{n^3\pi^3}.\left\{\sin\left(\frac{n\pi}{2}\right)[-4-n\pi^2] + n\pi\cos\left(\frac{n\pi}{2}\right) + 5n\pi(-1)^n\right\}\cos\left(\frac{n\pi t}{2}\right)$$

$$-\frac{2}{k^3\pi^3}\left\{-k\pi\sin\left(\frac{k\pi}{2}\right) + \cos\left(\frac{k\pi}{2}\right)[-4-k^2\pi^2] + 4(-1)^k\right\}\sin\left(\frac{n\pi t}{2}\right).$$

For the Cesàro N^{th} partial sum, simply insert the Cesàro factor $\left(1 - \frac{|n|}{N}\right)$ into the coefficients for the N^{th} partial sum of this series. Comparative graphs are shown below for $N = 5, 10, 25$.







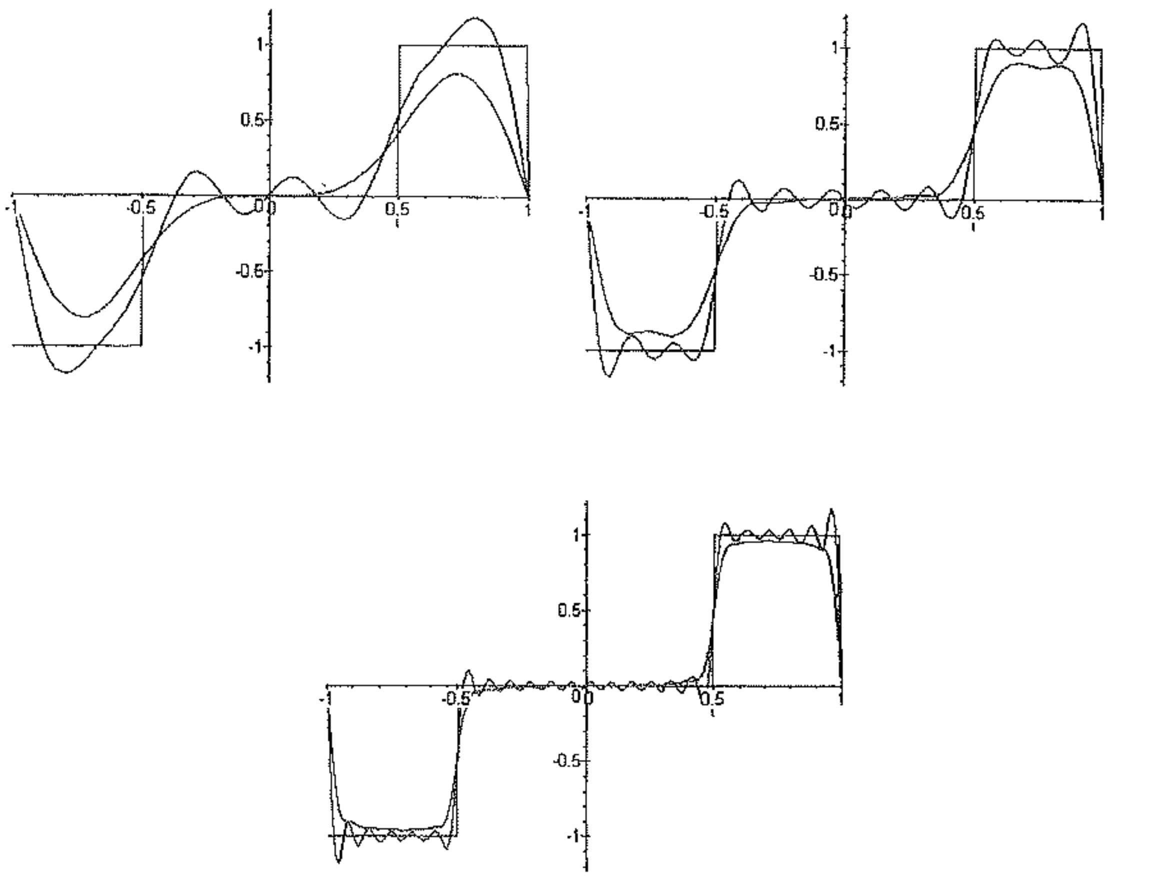
17. Complex coefficients are $d_0 = 0, d_k = \dfrac{i}{k\pi}\left[(-1)^k - \cos\left(\dfrac{k\pi}{2}\right)\right], k \geq 1$. This gives the real form of the Fourier series as

$$\sum_{n=1}^{\infty} \frac{2}{n\pi}\left[\cos\left(\frac{n\pi}{2}\right) - (-1)^n\right]\sin(n\pi t).$$

For the N^{th} Cesàro partial sum we have

$$\sigma_N(t) = \sum_{n=1}^{\infty}\left(1 - \frac{|n|}{N}\right)\frac{2}{n\pi}\left[\cos\left(\frac{n\pi}{2}\right) - (-1)^n\right]\sin(n\pi t).$$

Comparative graphs of the partial sums of the series and the N^{th} Cesàro partial sum are given for $N = 5, 10, 25$.



18. The complex Fourier coefficients are found to be $d_0 = \dfrac{\sin(3) - \sin(1) + e^3 - e^{-1}}{6}$ and

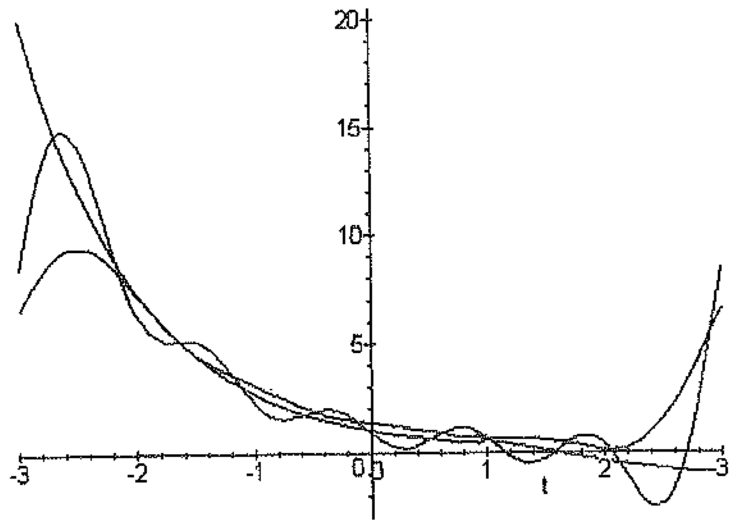
$$d_k = \frac{1}{2(k^2\pi^2+9)}\left\{e^{-1}[k\pi\sin\left(\frac{k\pi}{3}\right) - 3\cos\left(\frac{k\pi}{3}\right)] + 3e^3(-1)^k\right\}$$

$$+\frac{1}{2(k^2\pi^2-9)}\left\{3\cos\left(\frac{k\pi}{3}\right)\sin(1) - k\pi\sin\left(\frac{k\pi}{3}\right)\cos(1) - 3\sin(3)(-1)^k\right\}$$

$$+i\frac{1}{2(k^2\pi^2+9)}\left\{e^{-1}[k\pi\cos\left(\frac{k\pi}{3}\right) + 3\sin\left(\frac{k\pi}{3}\right)] - k\pi e^3(-1)^k\right\}$$

$$-i\frac{1}{2(k^2\pi^2-9)}\left\{3sin\left(\frac{k\pi}{3}\right)\sin(1) + k\pi\cos\left(\frac{k\pi}{3}\right)\cos(1) + k\pi e^3(-1)^k\right\}.$$

The graphs of $S_5(t)$ and $\sigma_5(t)$ are shown below.



19. Calculate complex coefficients to be

$$d_0 = \frac{1}{2}\left[\int_{-1}^{0}(2+t)dt + \int_0^1 7dt\right] = \frac{17}{4}; d_k = \frac{1}{2k^2\pi^2}[1-(-1)^k] + \frac{i}{2k\pi}[6(-1)^k - 5], k \geq 1.$$

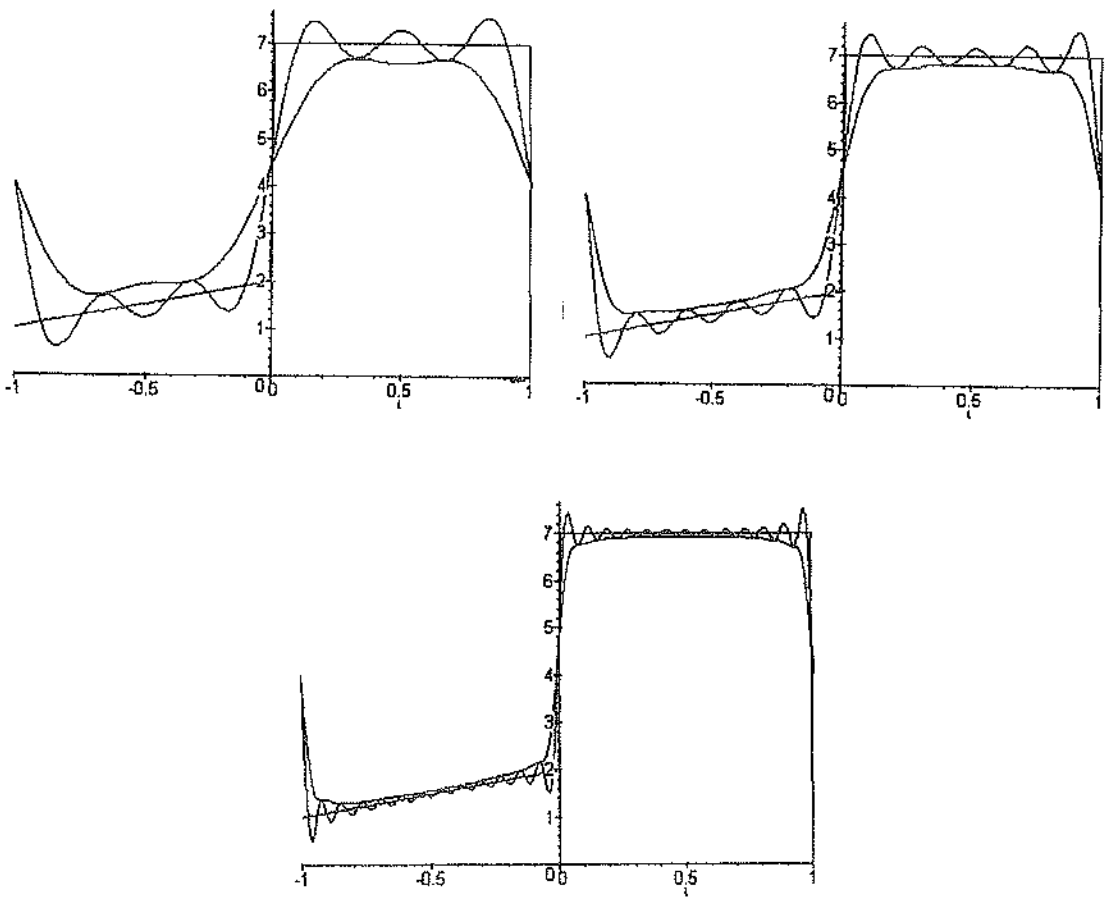
These give the real form of the Fourier series as

$$\frac{17}{4} + \sum_{n=1}^{\infty} \frac{1}{n^2\pi^2}[1-(-1)^n]\cos(n\pi t) + \sum_{n=1}^{\infty} \frac{i}{n\pi}[5-6(-1)^n]\sin(n\pi t).$$

For the N^{th} Cesàro partial sum we have

$$\sigma_N = \frac{17}{4} + \sum_{n=1}^{\infty} \frac{1}{n^2\pi^2}\left(1-\frac{|n|}{N}\right)[1-(-1)^n]\cos(n\pi t) + \sum_{n=1}^{\infty} \frac{i}{n\pi}\left(1-\frac{|n|}{N}\right)[5-6(-1)^n]\sin(n\pi t).$$

Comparative graphs of partial sums for $N = 5, 10, 25$ are shown below.



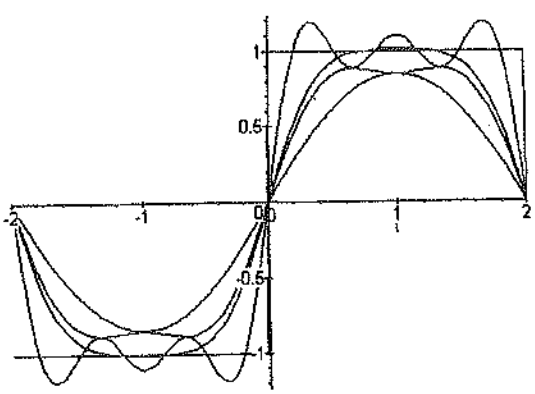
20. The complex coefficients are found to be $d_0 = 0$

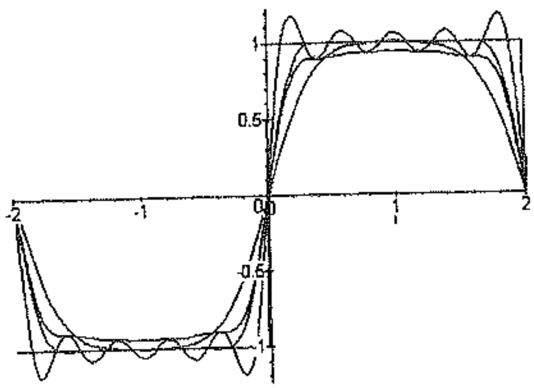
$$d_k = \frac{1}{4}\left[-\int_{-2}^{0} e^{-ik\pi t/2}dt + \int_{0}^{2} e^{-ik\pi t/2}dt\right] = -\frac{i}{k\pi}[1-(-1)^k].$$

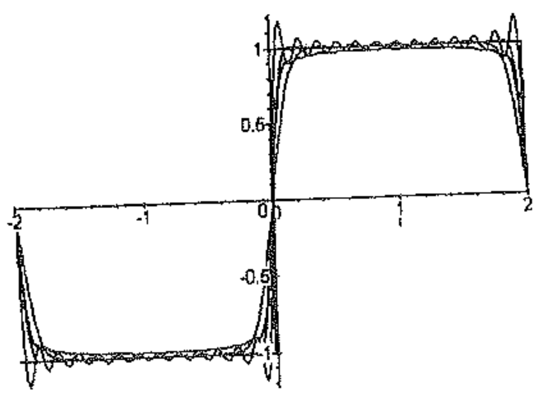
This gives the Fourier series in real form as

$$\sum_{n=1}^{\infty} \frac{2}{n\pi}[1-(-1)^n]\sin\left(\frac{n\pi}{2}\right).$$

To see the effects of each filter on the N^{th} partial sum, insert the factor $\left(1-\frac{n}{N}\right)$ for the Cesàro filtered series, $\left(.54+.46\cos\left(\frac{\pi n}{N}\right)\right)$ for the Hamming filter, and $e^{-a\pi^2 n^2/N^2}$ for the Gauss filtered series ($a = 1$ was used for the graphs). Graphs of these filtered series are shown with the unfiltered series for $N = 5, 10, 25$.



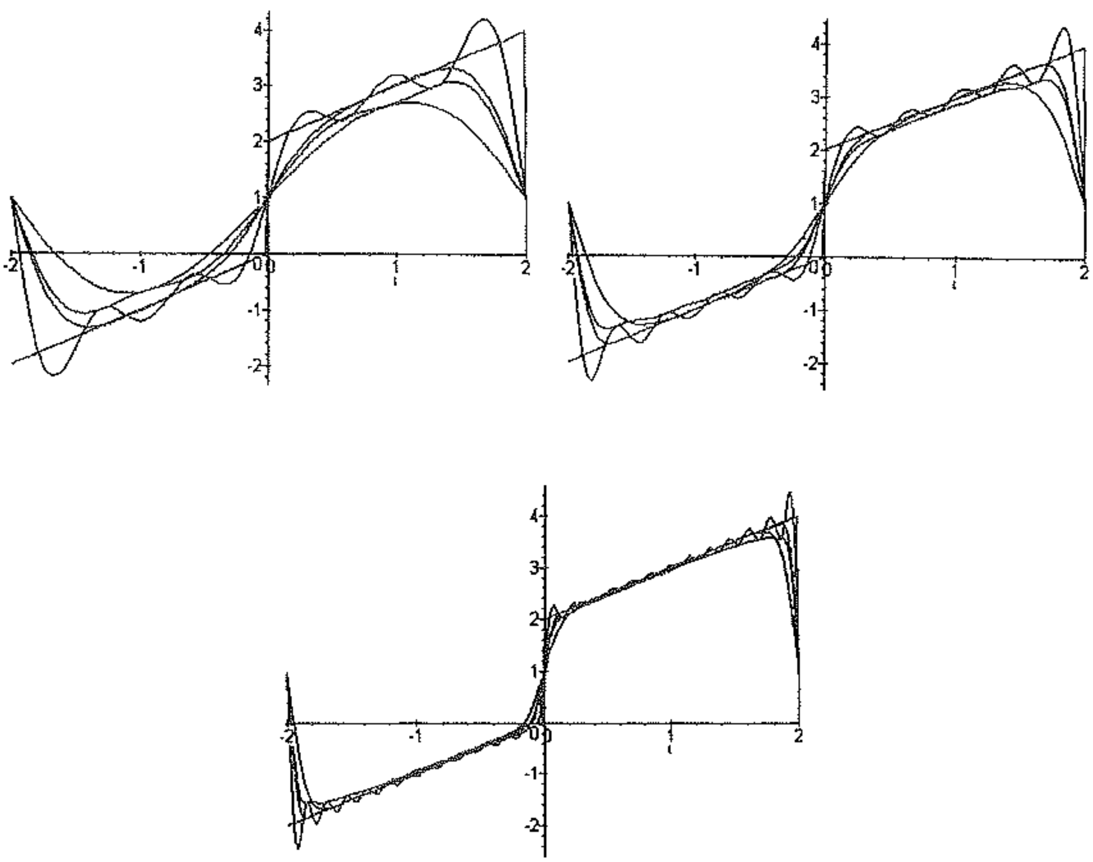




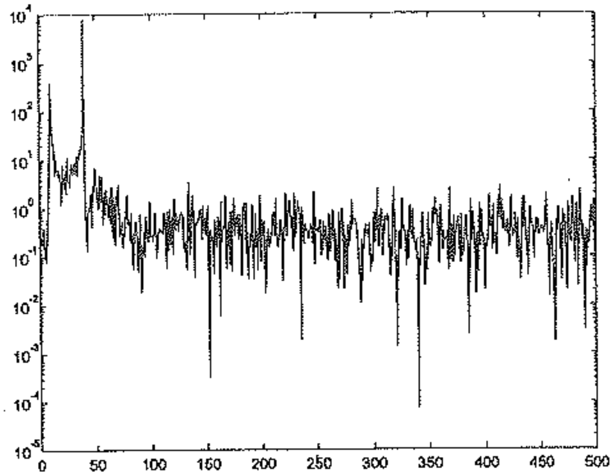
21. The complex Fourier coefficients are found to be $d_0 = 1, d_k = \dfrac{i}{k\pi}[3(-1)^k - 1], k \geq 1$. These give the real form of the Fourier series as

$$1 + \sum_{n-1}^{N} \frac{i}{k\pi}[3(-1)^k - 1] \sin\left(\frac{n\pi t}{2}\right).$$

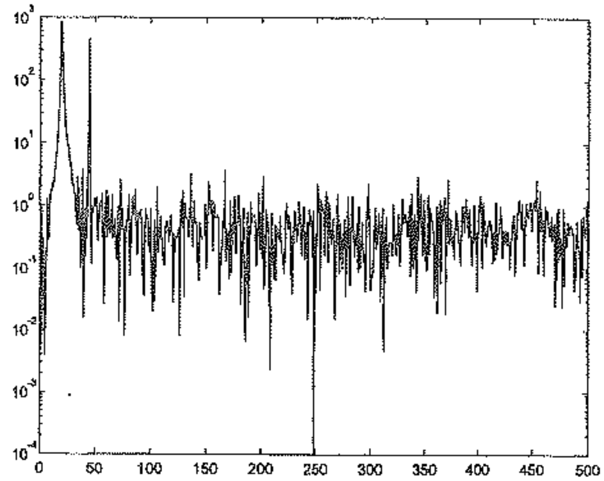
To see the effects of each filter on the N^{th} partial sum, insert the factor $\left(1 - \dfrac{n}{N}\right)$ for the Cesàro filtered series, $\left(.54 + .46\cos\left(\dfrac{\pi n}{N}\right)\right)$ for the Hamming filter, and $e^{-a\pi^2 n^2/N^2}$ for the Gauss filtered series ($a = 1$ was used for the graphs). Graphs of these filtered series are shown with the unfiltered series for $N = 5, 10, 25$.



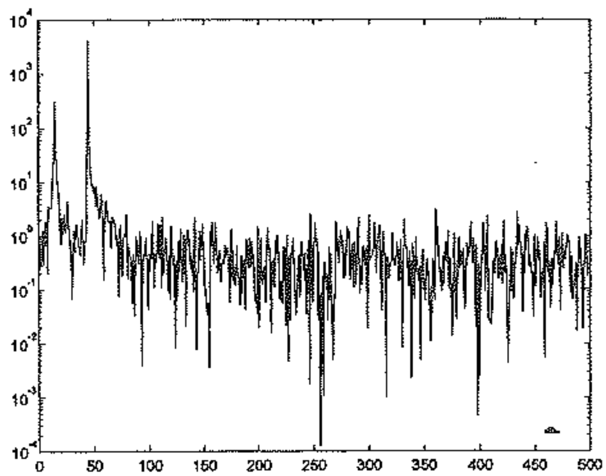
Section 15.9 The Fast Fourier Transform



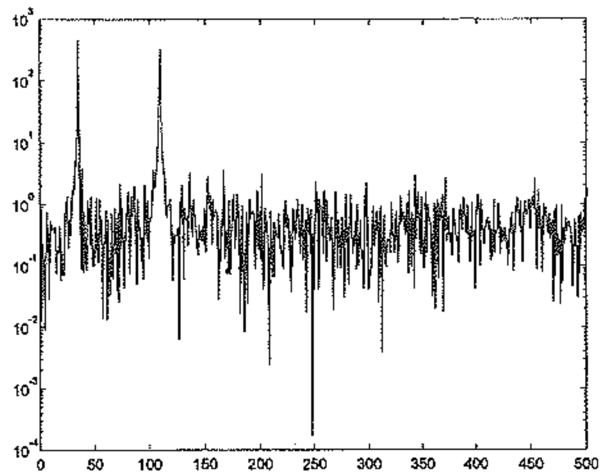
$y = 4\sin(80\pi t) - \sin(20\pi t)$



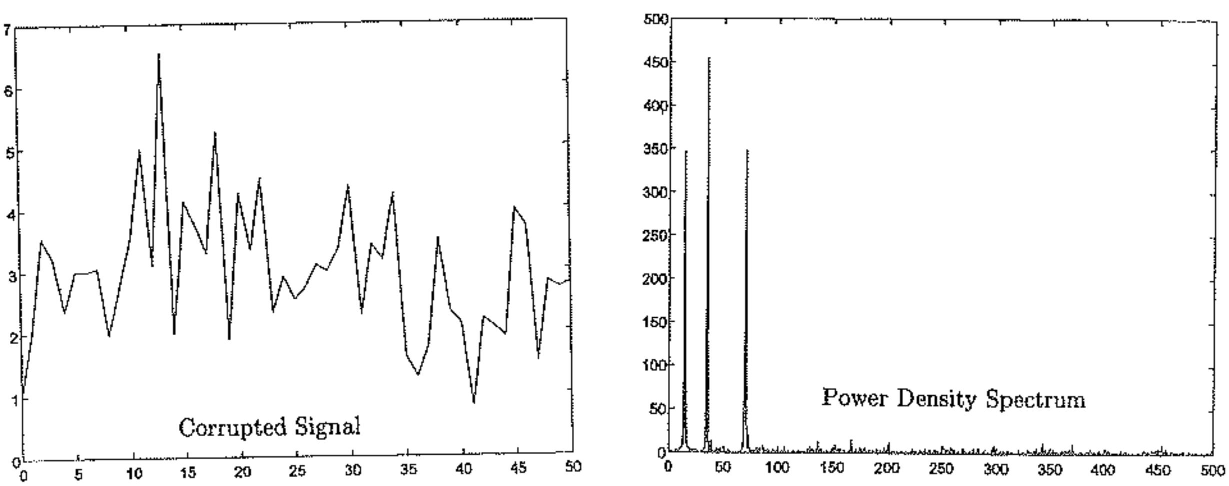
$y = 2\cos(40\pi t) + \sin(90\pi t)$



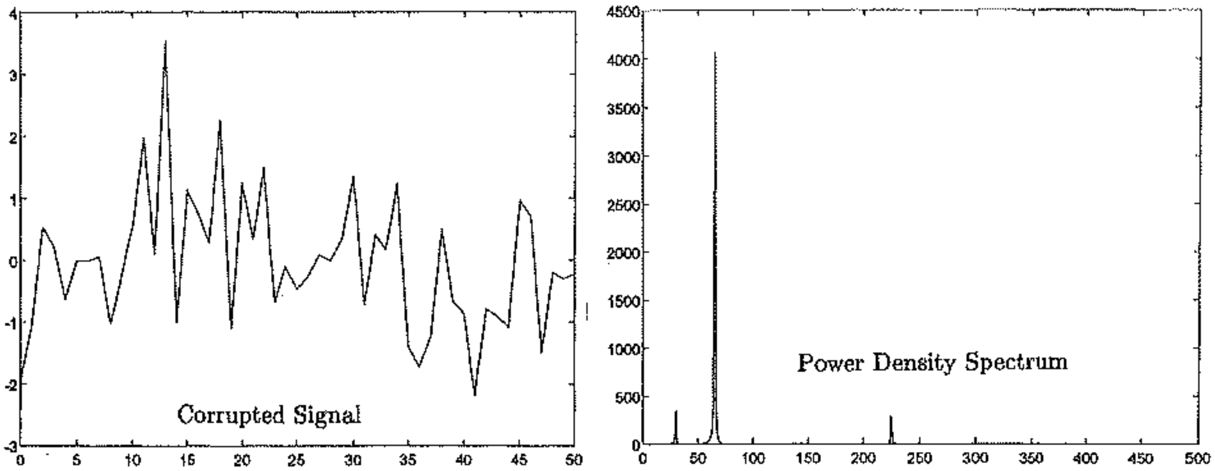
$y = 3\cos(90\pi t) - \sin(30\pi t)$



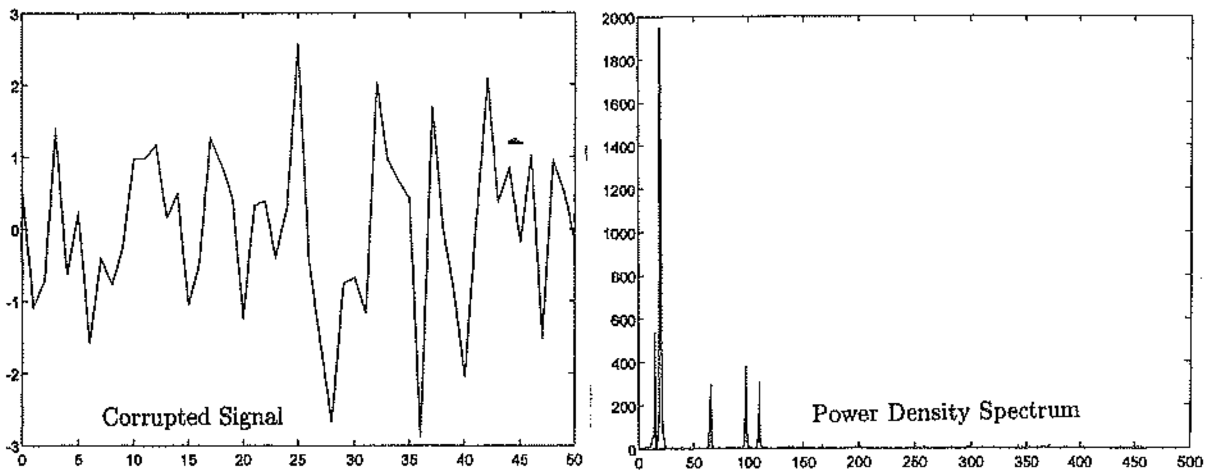
$y = \cos(220\pi t) = \cos(70\pi t)$



$y = \cos(30\pi t) + \cos(70\pi t) + \cos(140\pi t)$



$$y = \sin(60\pi t) + 4\sin(130\pi t) + \sin 2405\pi t)$$



$$y = \sin(30\pi t) + 3\sin(40\pi t) + \sin(130\pi t) + \sin(196\pi t) + \sin(220\pi t)$$

Chapter Sixteen - Special Functions, Orthogonal Expansions and Wavelets

Section 16.1 Legendre Polynomials

1. A routine calculation verifies this, e.g. with $n = 3, \lambda = 3(3+1) = 12, P_3(x) = \frac{1}{2}(5x^3 - 3x)$, and

$$(1-x^2)P_3'' - 2xP_3' + 12P_3 = (1-x^2)(15x) - 2x\left(\frac{15}{2}x^2 - 3\right) + 12\left[\frac{1}{2}(5x^3 - 3x)\right]$$

$= 15x - 15x^3 - 15x^3 + 3x + 30x^3 - 18x = 0.$

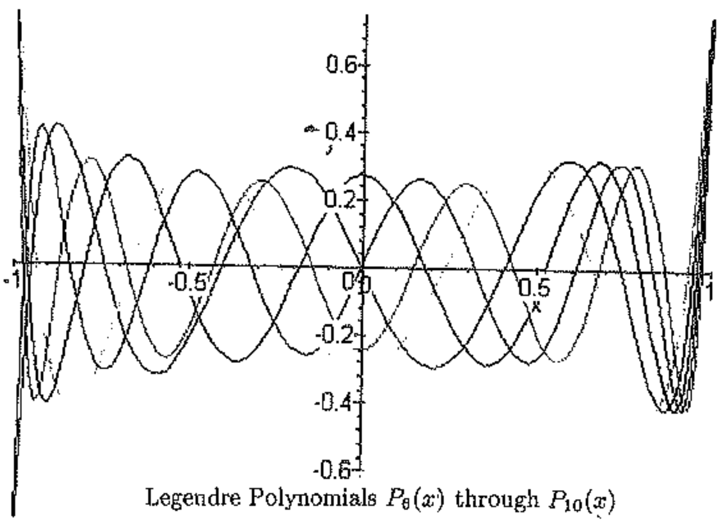
2. $P_6(x) = \frac{11}{6}xP_5(x) - \frac{5}{6}P_4(x) = \frac{1}{16}[231x^6 - 315x^4 + 105x^2 - 5]$;

$P_7(x) = \frac{13}{7}xP_6(x) - \frac{6}{7}P_5(x) = \frac{1}{16}[429x^7 - 693x^5 + 315x^3 - 35x]$;

$P_8(x) = \frac{1}{128}[6435x^8 - 12,012x^6 + 6930x^4 - 1260x^2 + 35]$;

$P_9(x) = \frac{1}{1152}[109,395x^9 - 231,660x^7 + 162,162x^5 - 41,580x^3 + 2835]$;

$P_{10}(x) = \frac{1}{11520}[2,078,505x^{10} - 4,922,775x^8 + 4,054,050x^6 - 1,351,350x^4 + 155925x^2 - 2835]$



Legendre Polynomials $P_6(x)$ through $P_{10}(x)$

3. $n = 1; \frac{1}{2}\frac{d}{dx}[x^2 - 1] = x = P_1(x)$;

$n = 2; \frac{1}{4 \cdot 2!}\frac{d^2}{dx^2}[(x^2-1)^2] = \frac{1}{8}\frac{d^2}{dx^2}[x^4 - 2x^2 + 1]$

$= \frac{1}{2}(3x^2 - 1) = P_2(x);$

$n = 3; \frac{1}{8 \cdot 3!} \frac{d^3}{dx^3}[(x^2 - 1)^3] = \frac{1}{48} \frac{d^3}{dx^3}[x^6 - 3x^4 + 3x^2 - 1]$
$= \frac{1}{2}[5x^3 - 3x] = P_3(x);$

$n = 4; \frac{1}{16 \cdot 4!} \frac{d^4}{dx^4}[(x^2 - 1)^4] = \frac{1}{384} \frac{d^4}{dx^4}[x^8 - 4x^6 + 6x^4 - 4x^2 + 1]$
$= \frac{1}{8}[35x^4 - 30x^2 + 3] = P_4(x);$

$n = 5; \frac{1}{32 \cdot 5!} \frac{d^5}{dx^5}[(x^2 - 1)^5] = \frac{1}{3840} \frac{d^5}{dx^5}[x^{10} - 5x^8 + 10x^6 - 10x^4 - 5x^2 + 1]$
$= \frac{1}{8}[63x^5 - 70x^3 + 15x] = P_5(x).$

4. With $n = 3$ we get,
$\frac{1}{\pi} \int_0^\pi \left(x + \sqrt{x^2 - 1}\cos(\theta)\right)^3 d\theta$
$= \frac{1}{\pi} \int_0^\pi \left(x^3 + 3x^2\sqrt{x^2 - 1}\cos(\theta) + 3x(x^2 - 1)\cos^2(\theta) + (x^2 - 1)\sqrt{x^2 - 1}\cos^3(\theta)\right) d\theta$
$= \frac{1}{\pi}\left[\pi x^3 + 3x(x^2 - 1)\frac{\pi}{2}\right] = \frac{1}{2}(5x^3 - 3x);$
With $n = 4$ we get,
$\frac{1}{\pi} \int_0^\pi \left(x + \sqrt{x^2 - 1}\cos(\theta)\right)^4 d\theta$
$= \frac{1}{\pi} \int_0^\pi \left(x^4 + 4x^3\sqrt{x^2 - 1}\cos(\theta) + 6x^2(x^2 - 1)\cos^2(\theta) + 4x\sqrt{x^2 - 1}\cos^3(\theta) + (x^2 - 1)^2\cos^4(\theta)\right) d\theta$
$= \frac{1}{\pi}\left[\pi x^4 + 6x^2(x^2 - 1)\frac{\pi}{2} + (x^2 - 1)^2\frac{3\pi}{8}\right] = 18(35x^4 - 30x^2 + 3);$
For $n = 5$, expand the integrand by the binomial theorem, integrate and simplify.

5. $P_0(x) = (-1)^0 x^0 = 1; P_1(x) = \frac{(-1)^0 2!}{2} x^1 = x;$
$P_2(x) = (-1)^0 \frac{4!}{2!2!} x^2 + \frac{(-1)^1 2!}{2^2} x^0 = \frac{1}{2}(3x^2 - 1);$
$P_3(x) = \frac{(-1)^0 6!}{2^3 3! 3!} x^3 + \frac{(-1)^1 4!}{2^3 2!} x = \frac{1}{2}(5x^3 - 3x);$
$P_4(x) = \frac{(-1)^0 8!}{2^4 4! 4!} x^4 - \frac{6!}{2^4 3! 2!} x^2 + \frac{4!}{2^4 2! 2!} = \frac{1}{8}(35x^4 - 30x^2 + 3);$
$P_5(x) = \frac{(-1)^0 10!}{2^5 5! 5!} x^5 - \frac{8!}{2^5 4! 3!} x^3 + \frac{6!}{2^5 2! 3!} x = \frac{1}{8}(63x^5 - 70x^3 + 15x).$

6. By Problem 3 and the Rodrigues's formula,

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}[(x^2 - 1)^n]$$

$$= \frac{1}{2^n n!} \frac{d^n}{dx^n}[(x + 1)^n (x - 1)^n]$$

$$= \frac{1}{2^n n!}\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\frac{d^k}{dx^k}[(x+1)^n]\frac{d^{n-k}}{dx^{n-k}}[(x-1)^n],$$

where we used Leibniz formula for the n^{th} derivative of a product.

7. Let $Q_n(x) = P_n(x)z(x)$, and substitute into Legendre's equation to get
$(1-x^2)(P_n''z + 2P_n'z' + P_nz'') - 2x(P_n'z + P_nz') + n(n+1)P_nz$
$= z[(1-x^2)P_n'' - 2xP_n' + n(n+1)P_n] + z''(1-x^2)P_n + z'[2(1-x^2)P_n' - 2xP_n] = 0$. The first term is zero because P_n is a solution of Legendre's equation. Thus $z(x)$ satisfies $\frac{z''}{z'} - \frac{2x}{1-x^2} + \frac{2P_n'}{P_n} = 0$. Integrate this equation to get $\ln|z'| + \ln|1-x^2| + 2\ln|P_n| = c$, or solving for $z'(x) = \frac{K}{(1-x^2)[P_n(x)]^2}$. Integrate again to get $z(x) = K\int\frac{1}{(1-x^2)[P_n(x)]^2}dx$ so $Q_n(x) = P_n(x)\int\frac{1}{P_n^2(x)(1-x^2)}dx$ gives a second linearly independent solution.

8. From Problem 7 we get
$Q_0(x) = \int\frac{1}{1-x^2}dx = \frac{1}{2}\int\left(\frac{1}{1+x}+\frac{1}{1-x}\right)dx$
$= \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right), -1 < x < 1,$
$Q_1(x) = x\int\frac{1}{x^2(1-x^2)}dx = x\int\left[\frac{1}{x^2}+\frac{1}{2}\left(\frac{1}{1+x}+\frac{1}{1-x}\right)\right]dx$
$= -1 + \frac{x}{2}\ln\left(\frac{1+x}{1-x}\right),$
$Q_2(x) = 2(3x^2-1)\int\frac{1}{(3x^2-1)^2(1-x^2)}dx$
$= \frac{1}{4}(3x^2-1)\int\left[\frac{1}{x+1} - \frac{1}{x-1} + \frac{1}{(x+\frac{1}{\sqrt{3}})^2} + \frac{1}{(x-\frac{1}{\sqrt{3}})^2}\right]dx$
$= \frac{1}{4}(3x^2-1)\ln\left(\frac{1+x}{1-x}\right) - \frac{3}{2}x.$

9. (a) From Fig. 16.4 and the law of cosines,

$$R^2 = r^2 + d^2 - 2rd\cos(\theta),\ \text{or}\ R = d\left[1 - \frac{2r}{d}\cos(\theta) + \left(\frac{r}{d}\right)^2\right]^{1/2},$$

thus

$$\varphi(x,y,z) = \frac{1}{d\sqrt{1-\frac{2r}{d}\cos(\theta)+(\frac{r}{d})^2}}.$$

(b) By the binomial series

$$\frac{1}{\sqrt{1-2at+t^2}} = \sum_{n=0}^{\infty}\frac{(-\frac{1}{2})(-\frac{3}{2})(\cdots)(-\frac{1}{2}-n+1)}{n!}(t^2-2at)^n,$$

But

$$(t^2 - 2at)^n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} t^{2k}(-2at)^{n-k},$$

so

$$\frac{1}{\sqrt{1-2at+t^2}} = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(-1)^n 1\cdot 3\cdots(2n-1)}{2^n n!} \frac{n!}{k!(n-k)!}(-2a)^{n-k}t^{n+k}$$

$$= \sum_{n=0}^{\infty} \sum_{\substack{j+k=n\\ k\le j}} \frac{(-1)^j 1\cdot 3\cdots(2j-1)j!(-1)^{j-k}(2)^{j-k}a^{j-k}}{2^j j!k!(j-k)!} t^n = \sum_{n=0}^{\infty} c_n(a)t^n$$

where the coefficient of t^n is $c_n(a) = \sum_{\substack{j+k=n\\ k\le j}} \frac{(-1)^k(2j)!}{2^{k+j}j!k!(j-k)!}a^{j-k}$, which can be written

$$c_n(a) = \sum_{k=1}^{[n/2]} \frac{(-1)^k[2(n-k)]!}{2^n k!(n-k)!(n-2k)!}a^{n-2k}, \text{ and this is } P_n(a).$$

(c) If $r < d$, let $a = \cos(\theta)$ and $t = r/d$ to get

$$\phi(r) = \frac{1}{d\sqrt{1-2at\cos(\theta)+t^2}}$$

$$= \frac{1}{d}\sum_{n=0}^{\infty} P_n(\cos(\theta))\left(\frac{r}{d}\right)^n = \sum_{n=0}^{\infty} \frac{1}{d^{n+1}} P_n(\cos(\theta))r^n.$$

(d) If $r > d$, write $\phi(r) = \phi(r(x,y,z)) = \dfrac{1}{r\sqrt{1-2\frac{d}{r}\cos(\theta)+(\frac{d}{r})^2}}$, and use (c) with $a = \cos(\theta), t = \dfrac{d}{r}$ to get

$$\phi(r) = \frac{1}{r}\sum_{n=0}^{\infty} P_n(\cos(\theta))\left(\frac{d}{r}\right)^n = \frac{1}{r}\sum_{n=0}^{\infty} d^n P_n(\cos(\theta))r^{-n}.$$

10. Using the generating function for the Legendre polynomials we have

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n, \ -1 < t < 1.$$

Substituting $x = 1/2, t = 1/2$ gives $\dfrac{1}{\sqrt{1-\frac{1}{2}+\frac{1}{4}}} = \dfrac{2}{\sqrt{3}} = \sum_{n=0}^{\infty} \dfrac{1}{2^n} P_n\left(\dfrac{1}{2}\right)$. Divide by 2 to get the result.

11. From Problem 5, we have

$$P_{2n+1}(x) = \sum_{k=0}^{n} (-1)^k \frac{(4n+2-2k)!}{2^{2n+1}k!(2n+1-k)!(2n+1-2k)!} x^{2n+1-2k}$$

and $P_{2n}(x) = \sum_{k=0}^{n} (-1)^k \dfrac{(4n-2k)!}{2^{2n}k!(2n-k)!(2n-2k)!} x^{2n-2k}$.

Thus $P_{2n+1}(0) = 0$, and $P_{2n}(0) = \dfrac{(-1)^n(2n)!}{2^{2n}(n!)^2}$

12. Using a bit of matrix notation and the formulas given in the text for $P_0(x), P_1(x), \ldots, P_4(x)$, we can write

$$\begin{pmatrix} P_0(x) \\ P_1(x) \\ P_2(x) \\ P_3(x) \\ P_4(x) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1/2 & 0 & 3/2 & 0 & 0 \\ 0 & -3/2 & 0 & 5/2 & 0 \\ 3/8 & 0 & -30/8 & 0 & 35/8 \end{pmatrix} \begin{pmatrix} 1 \\ x \\ x^2 \\ x^3 \\ x^4 \end{pmatrix}.$$

Inverting the above matrix gives

$$\begin{pmatrix} 1 \\ x \\ x^2 \\ x^3 \\ x^4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1/3 & 0 & 2/3 & 0 & 0 \\ 0 & 3/5 & 0 & 2/5 & 0 \\ 1/5 & 0 & 4/7 & 0 & 8/35 \end{pmatrix} \begin{pmatrix} P_0(x) \\ P_1(x) \\ P_2(x) \\ P_3(x) \\ P_4(x) \end{pmatrix}$$

which easily allows us to express powers of x through x^4 in terms of Legendre polynomials. Then

(a) $1 + 2x - x^2 = \frac{2}{3}P_0(x) + 2P_1(x) - \frac{2}{3}P_2(x)$

(b) $2x + x^2 - 5x^3 = \frac{1}{3}P_0(x) + 2P_1(x) + \frac{11}{3}P_2(x) - 2P_3(x)$

(c) $2 - x^2 + 4x^4 = \frac{37}{15}P_0(x) + \frac{34}{21}P_2(x) + \frac{32}{35}P_4(x)$

13. We have $\sin\left(\frac{\pi x}{2}\right) = \sum_{n=0}^{\infty} a_n P_n(x)$, where $a_n = \frac{2n+1}{2}\int_{-1}^{1} \sin\left(\frac{\pi x}{2}\right) P_n(x)dx, n = 0, 1, 2, \ldots$.

Note $a_{2k} = 0, k = 0, 1, 2, \ldots$, since $P_{2k}(x)$ is even and $\sin\left(\frac{\pi x}{2}\right)$ is odd. Using the formulas given for $P_1(x), P_3(x)$ and $P_5(x)$ in the text we calculate

$$a_1 = \frac{3}{2}\int_{-1}^{1} \sin\left(\frac{\pi x}{2}\right) P_1(x)dx = \frac{12}{\pi^2},$$

$$a_3 = \frac{7}{2}\int_{-1}^{1} \sin\left(\frac{\pi x}{2}\right) P_3(x)dx = \frac{168}{\pi^4}(\pi^2 - 10),$$

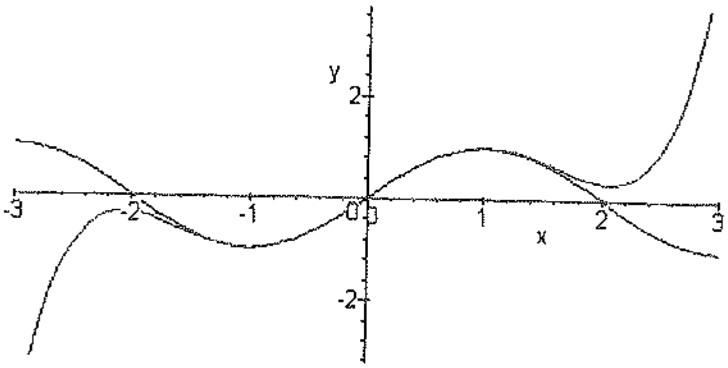
$$a_5 = \frac{7}{2}\int_{-1}^{1} \sin\left(\frac{\pi x}{2}\right) P_5(x)dx = \frac{660}{\pi^6}(\pi^4 - 112\pi^2 + 1008);$$

Thus the first three non-zero terms of the Legendre series give

$$\sin\left(\frac{\pi x}{2}\right) \approx \frac{12}{\pi^2}x + \frac{168}{\pi^4}(\pi^2 - 10)(5x^3 - 3x)$$

$$+\frac{660}{\pi^6}(\pi^4 - 112\pi^2 + 1008)(63x^5 - 70x^3 + 15x) + \ldots$$

This approximation and $\sin\left(\frac{\pi x}{2}\right)$ are graphed below.



14. We have $e^{-x} = \sum_{n=0}^{\infty} a_n P_n(x)$, where $a_n = \frac{2n+1}{2}\int_{-1}^{1} e^{-x} P_n(x), n = 0, 1, 2, \ldots$

Calculate

$$a_0 = \frac{1}{2}\int_{-1}^{1} e^{-x} P(x)dx = \frac{(e - e^{-1})}{2} = \sinh(e),$$

$$a_1 = \frac{3}{2}\int_{-1}^{1} e^{-x} P_1(x)dx = -3e^{-1},$$

$$a_2 = \frac{5}{2}\int_{-1}^{1} e^{-x} P_2(x)dx = \frac{5}{2e}(e^2 - 7),$$

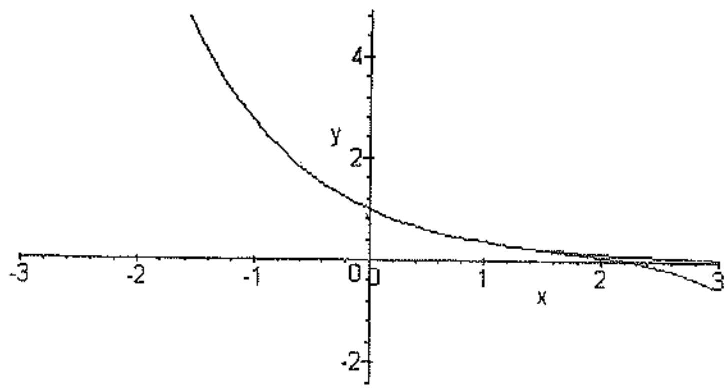
$$a_3 = \frac{7}{2}\int_{-1}^{1} e^{-x} P_3(x)dx = \frac{7}{2e}(5e^2 - 37),$$

$$a_4 = \frac{9}{2}\int_{-1}^{1} e^{-x} P_4(x)dx = \frac{9}{2e}(18e^2 - 133).$$

The first five terms of the Legendre series expansion of e^{-x} give

$$e^{-x} \approx \sinh(e) - \frac{3}{e}x + \frac{5}{4e}(e^2-7)(3x^2-1) + \frac{7}{4e}(5e^2-37)5x^3-3x) + \frac{9}{16e}(18e^2-133)(35x^4-30x^2+3) + \cdots.$$

This approximation and e^{-x} are graphed below.



15. We have $\sin^2(x) = \sum_{n=0}^{\infty} a_n P_n(x)$, where $a_n = \frac{2n+1}{2}\int_{-1}^{1} \sin^2(x) P_n(x)dx, n = 0, 1, 2, \ldots.$

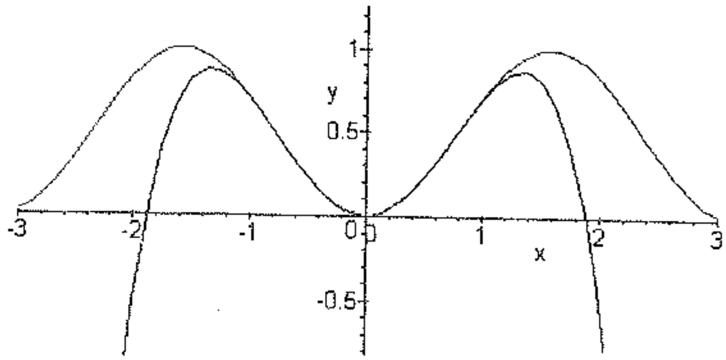
Note $a_{2k+1} = 0, k = 0, 1, 2, \ldots$, since $P_{2k+1}(x)$ is odd and $\sin^2(x)$ is even. Using the formulas given for $P_0(x), P_2(x)$ and $P_4(x)$ in the text we calculate

$$a_0 = \frac{1}{2}\int_{-1}^{1} \sin^2(x)dx = \frac{1}{2}[1 - \cos(1)\sin)1)],$$

$$a_2 = \frac{5}{2}\int_{-1}^{1} \sin^2(x)P_2(x)dx = \frac{5}{8}(3 - \cos(1)\sin(1) - 6\cos^2(1)),$$

$$a_4 = \frac{9}{2}\int_{-1}^{1} \sin^2(x)P_4(x)dx = -\frac{9}{32}[65 - 59\cos(1)\sin(1) - 65\cos^2(1)].$$

Thus the first three non-zero terms of the Legendre series give
$\sin^2(x) \approx \frac{1}{2}[1 - \cos(1)\sin)1)] + \frac{5}{16}(3 - \cos(1)\sin(1) - 6\cos^2(1))(3x^2 - 1)$
$-\frac{9}{256}[65 - 59\cos(1)\sin(1) - 65\cos^2(1)](35x^4 - 30x^2 + 3) + \cdots$
This approximation and $\sin^2(x)$ are graphed below.



16. We have $\cos(x) - \sin(x) = \sum_{n=0}^{\infty} a_n P_n(x)$, where $a_n = \frac{2n+1}{2}\int_{-1}^{1}(\cos(x) - \sin(x))P_n(x)dx, n = 0, 1, 2, \ldots$. Using the formulas given for $P_0(x) through P_4(x)$ we calculate

$$a_0 = \frac{1}{2}\int_{-1}^{1}(\cos(x) - \sin(x))dx = \sin(1),$$

$$a_1 = \frac{3}{2}\int_{-1}^{1}(\cos(x) - \sin(x))P_1(x)dx = 3[\cos(1) - \sin(1)],$$

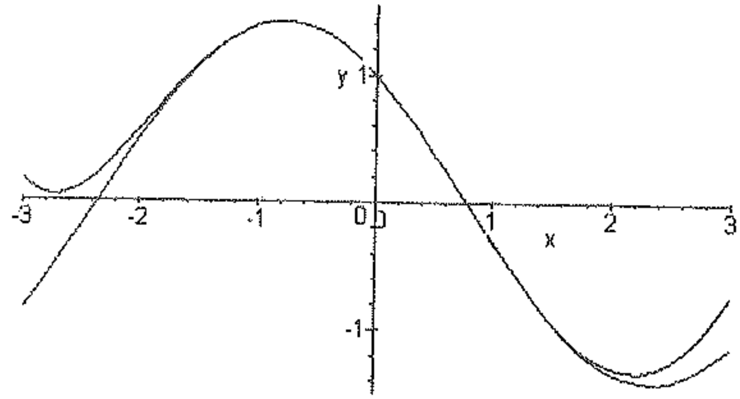
$$a_2 = \frac{5}{2}\int_{-1}^{1}(\cos(x) - \sin(x))P_2(x)dx = 5[3\cos(1) - 2\sin(1)],$$

$$a_3 = \frac{7}{2}\int_{-1}^{1}(\cos(x) - \sin(x))P_3(x)dx = 7[9\sin(1) - 14\cos(1)],$$

$$a_4 = \frac{9}{2}\int_{-1}^{1}(\cos(x) - \sin(x))P_3(x)dx = 9[61\sin(1) - 95\cos(1)].$$

Thus the first five terms of the Legendre series give
$\cos(x) - \sin(x) \approx \sin(1) + 3[\cos(1) - \sin(1)]x + \frac{5}{2}[3\cos(1) - 2\sin(1)](3x^2 - 1)$

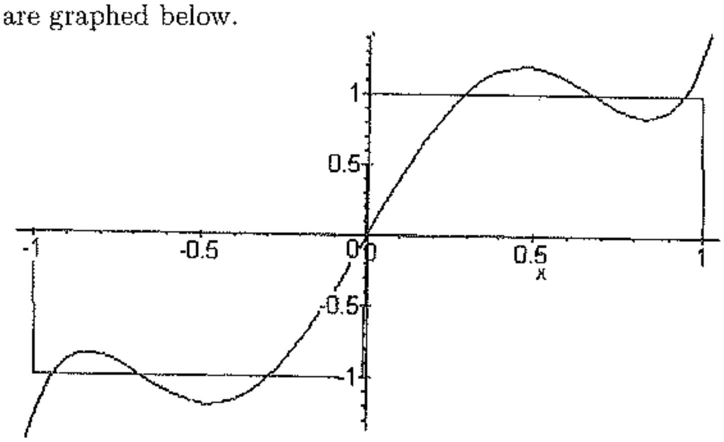
$+\dfrac{7}{2}[9\sin(1)-14\cos(1)](5x^3-3x)+98[61\sin(1)-95\cos(1)](35x^4-30x^2+3)+\cdots$
This approximation and $\cos(x)-\sin(x)$ are graphed below.



17. We have $f(x)=\sum_{n=0}^{\infty}a_nP_n(x)$, where $a_n=\dfrac{2n+1}{2}\left[\int_{-1}^{0}P_n(x)dx+\int_0^1 P_n(x)dx\right], n=0,1,2,\ldots$. Since f is odd and $P_{2n}(x)$ is even, the coefficients $a_{2n}=0, n=0,1,2,\ldots$. Calculate $a_1=\dfrac{3}{2}, a_3=-\dfrac{7}{8}, a_5=\dfrac{11}{16}$. The first few terms of the Legendre series of f gives

$$f(x)\approx\frac{3}{2}x-\frac{7}{16}(5x^3-3x)+\frac{11}{128}(63x^5-70x^2+15x)+\cdots$$

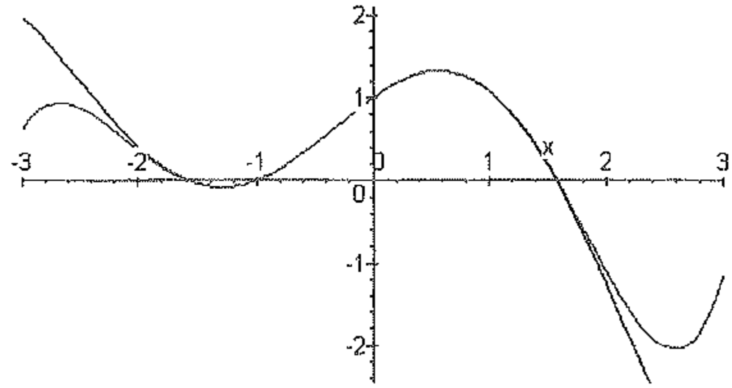
This approximation and f are graphed below.



18. The Legendre series of $f(x)=(x+1)\cos(x)$ has the form $\sum_{n=0}^{\infty}a+nP_n(x)$ where $a_n=\dfrac{2n+1}{2}\int_{-1}^{1}(x+1)\cos(x)P_n(x)dx$. Calculation of the first few coefficients gives values $a_0=\sin(1), a_1=3[2\cos(1)-\sin(1)], a_2=5[3\cos(1)-2\sin(1)], a_3=7[34\sin(1)-53\cos(1)], a_4=9[61\sin(1)-95\cos(1)]$. The first five terms of the Legendre series of $(x+1)\cos(x)$ gives $(x+1)\cos(x)\approx\sin(1)+3[2\cos(1)-\sin(1)]x+\dfrac{5}{2}[3\cos(1)-2\sin(1)](3x^2-1)+\dfrac{7}{2}[34\sin(1)-53\cos(1)](5x^3-3x)$

$+\dfrac{9}{16}[61\sin(1)-95\cos(1)](35x^4-30x^2+3)+\cdots$

This approximation and $(x+1)\cos(x)$ are graphed below.



Section 16.2 Bessel Functions

1. Substitute $y = x^a J_\nu(bx^c)$, $y' = ax^{a-1}J_\nu(bx^c) + x^a bcx^{c-1}J'_\nu(bx^c)$ and $y'' = a(a-1)x^{a-2}J_\nu(bx^c) + \{2ax^{a-1}bcx^{c-1} + x^a bc(c-1)x^{c-2}\}J'_\nu(bx^c) + x^a b^2c^2x^{2c-2}J''_\nu(bx^c)$ into the differential equation and simplify to get $c^2x^{a-2}\{(bx^c)^2J''_\nu(bx^c) + bx^cJ'_\nu(bx^c) + [(bx^c)^2 - \nu^2]J_\nu(bx^c)\} = 0$.

2. Comparing the given equation with the general equation in Problem 1 we identify $1 - 2a = \frac{1}{3}, b^2c^2 = 1, 2c - 2 = 0$, and $a^2 - \nu^2c^2 = 7/144$. Solving in turn for a, c, b, ν we find $a = 1/3, c = 1, b = 1$, and $\nu = 1/4$. The general solution is $y = c_1x^{1/3}J_{1/4}(x) + c_2x^{1/3}J_{-1/4}(x)$.

3. We want $1 - 2a = 1, b^2c^2 = 4, 2c - 2 = 2, a^2 - \nu^2c^2 = -4/9$. Solving, we find $a = 0, c = 2, b = 1, \nu = 1/3$. The general solution is $y = c_1J_{1/3}(x^2) + c_2J_{-1/3}(x^2)$.

4. We want $2a - 1 = 5, b^2c^2 = 64, 2c - 2 = 6, a^2 - \nu^2c^2 = 5$. Solving, we find $a = 3, c = 4, b = 2, \nu = 1/2$. The general solution is $y = c_1x^3J_{1/2}(2x^4) + c_2x^3J_{-1/2}(2x^4)$.

5. We want $1 - 2a = 3, b^2c^2 = 16, 2c - 2 = 2, a^2 - \nu^2c^2 = -5/4$. Solving, we find $a = -1, c = 2, b = 2, \nu = 3/4$. The general solution is $y = c_1x^{-1}J_{3/4}(2x^2) + c_2x^{-1}J_{-3/4}(2x^2)$.

6. We want $2a - 1 = 3, b^2c^2 = 9, 2c - 2 = 4, a^2 - \nu^2c^2 = 0$. Solving, we find $a = 2, c = 3, b = 1, \nu = 2/3$. The general solution is $y = c_1x^2J_{2/3}(x^3) + c_2x^2J_{-2/3}(x^3)$.

7. We want $2a - 1 = 7, b^2c^2 = 36, 2c - 2 = 4, a^2 - \nu^2c^2 = 175/16$. Solving, we find $a = 4, c = 3, b = 2, \nu = 3/4$. The general solution is $y = c_1x^4J_{3/4}(2x^3) + c_2x^4J_{-3/4}(2x^3)$.

8. We want $1 - 2a = 1, b^2c^2 = 0, a^2 - \nu^2c^2 = -1/16$. Solving, we find $a = 0, b = 0$ and this method produces only the trivial solution. The equation is an Euler equation, so put $y = x^r$ to get $r(r-1) + r - 1/16 = 0$ which has roots $r_1 = \frac{1}{4}$ and $r_2 = -\frac{1}{4}$. The general solution is $y = c_1x^{1/4} + c_2x^{-1/4}$.

9. We want $1 - 2a = 5, b^2c^2 = 81, 2c - 2 = 4, a^2 - \nu^2c^2 = 7/4$. Solving, we find $a = -2, c = 3, b = 3, \nu = 1/2$. The general solution is $y = c_1x^{-2}J_{1/2}(3x^3) + c_2x^{-2}J_{-1/2}(3x^3)$.

10. Differentiate $y = b^{-1}u^{-1}u'$ to get $y' = b^{-1}[-u^{-2}(u')^2 + u^{-1}u'']$ and substitute into $\frac{dy}{dx} + by^2 = cx^m$ to get $b^{-1}[-u^{-2}(u')^2 + u^{-1}u''] + b[b^{-1}u^{-1}u']^2 = cx^m$ or upon simplifying $u'' - bcx^mu = 0$. By Problem 1 (using α, β, γ instead of a, b, c) we want $2\alpha - 1 = 0, 2\gamma - 2 =$

$m, \beta^2\gamma^2 = -bc, \alpha^2 - \nu^2\gamma^2 = 0$. Solving we have $\alpha = 1/2, \gamma = \dfrac{m+2}{2}, \beta = \dfrac{2}{m+2}\sqrt{-bc}, \nu = \dfrac{1}{m+2}$. The general solution for u is

$$u(x) = c_1 x^{1/2} J_{1/(m+2)}\left(\frac{2\sqrt{-bc}}{m+2}x^{(m+2)/2}\right) + c_2 x^{1/2} J_{-1/(m+2)}\left(\frac{2\sqrt{-bc}}{m+2}x^{(m+2)/2}\right).$$

To complete the solution, substitute a solution for u into $y = \dfrac{u'}{bu}$, e.g. take $c_1 = 1, c_2 = 0$ and get

$$y(x) = \frac{1}{2bx} + \frac{J'_{1/(m+2)}(\frac{2\sqrt{-bc}}{m+2}x^{(m+2)/2})}{bJ_{1/(m+2)}(\frac{2\sqrt{-bc}}{m+2}x^{(m+2)/2})}.$$

11. With $z = \sqrt{x} = x^{1/2}$, we get $\dfrac{dy}{dx} = \dfrac{dy}{dz}\left(\dfrac{1}{2}x^{-1/2}\right), \dfrac{d^2}{dx^2} = -\dfrac{1}{4}(x)^{-3/2}\dfrac{dy}{dz}$ $+\dfrac{d^2y}{dz^2}\left(\dfrac{1}{4}x^{-1}\right)$. These can be rewritten in terms of z as $\dfrac{dy}{dx} = \dfrac{1}{2}z^{-1}\dfrac{dy}{dz}, \dfrac{d^2y}{dx^2} = \dfrac{1}{4}z^{-2}\dfrac{d^2y}{dz^2} - \dfrac{1}{4}z^{-3}\dfrac{dy}{dz}$.
The transformed equation becomes

$$4z^4\left[\frac{1}{4}z^{-2}\frac{d^2y}{dz^2} - \frac{1}{4}z^{-3}\frac{dy}{dz}\right] + 4z^2\left[\frac{1}{2}z^{-1}\frac{dy}{dz}\right] + (z^2-9)y = 0,$$

or upon simplifying $z^2y'' + zy' + (z^2-9)y = 0$.
This equation has general solution $y(z) = c_1J_3(z) + c_2Y_3(z)$, so $y(x) = c_1J_3(\sqrt{x}) + c_2Y_3(\sqrt{x})$.

12. With $z = x^{3/2}$, the transformed equation is

$$4z^{4/3}\left[\frac{9}{4}z^{2/3}\frac{d^2y}{dz^2} + \frac{3}{4}z^{-1/3}\frac{dy}{dz}\right] + 4z^{2/3}\left[\frac{3}{2}z^{1/3}\frac{dy}{dz}\right] + (9z^2-36)y = 0.$$

This simplifies to $z^2y''+zy'+(z^2-4)y = 0$, which has general solution $y(z) = c_1J_2(z)+c_2Y_2(z)$. Thus $y(x) = c_1J_2(x^{3/2}) + c_2Y(x^{3/2})$.

13. With $z = 2x^{1/3}$, the transformed equation is

$$9\left(\frac{z}{2}\right)^6\left[\frac{4}{9}\left(\frac{z}{2}\right)^{-4}\frac{d^2y}{dz^2} - \frac{4}{9}\left(\frac{z}{2}\right)^{-5}\frac{dy}{dz}\right]$$
$$+9\left(\frac{z}{2}\right)^3\left[\frac{2}{3}\left(\frac{z}{2}\right)^{-2}\frac{dy}{dz}\right] + (4\left(\frac{z}{2}\right)^2 - 16)y = 0.$$

This simplifies to $z^2y'' + zy' + (z^2-16)y = 0$, which has general solution $y(z) = c_1J_4(z) + c_2Y_4(z)$. Thus $y(x) = c_1J_4(2x^{1/3}) + c_2Y_4(2x^{1/3})$

14. With $u = y/x^2$, or $y = x^2u$, the transformed equation is $9x^2[x^2u''+4xu'+2u] - 27x[x^2u' + 2xu] + (9x^2+35)x^2u = 0$. Then, simplifying and dividing by $9x^2$, $x^2u'' + xu' + (x^2-1/2)u = 0$, which has general solution $u(x) = c_1J_{1/3}(x) + c_2Y_{1/3}(x)$. Thus $y(x) = c_1x^2J_{1/3}(x) + c_2x^2y_{1/3}(x)$.

15. With $u = x^{-2/3}y$, or $y = x^{2/3}u$, the transformed equation is

$$36x^2[x^{2/3}u'' + \frac{4}{3}x^{-1/3}u' - \frac{2}{9}x^{-4/3}u] - 12x[x^{2/3}u' + \frac{2}{3}x^{-1/3}u] + (36x^2 + 7)x^{2/3}u = 0.$$

Collect terms and divide by $36x^{2/3}$ to get $x^2u'' + xu' + (x^2 - 1/4)u = 0$, which as general solution $u(x) = c_1J_{1/2}(x) + c_2y_{1/2}(x)$. Thus $y(x) = c_1x^{2/3}J_{1/2}(x) + c_2x^{2/3}Y_{1/2}(x)$

16. With $u = y\sqrt{x}$, or $y = x^{-1/2}u$, the transformed equation is

$$4x^2[x^{-1/2}u'' - x^{-3/2}u' + \frac{3}{4}x^{-5/2}u] + 8x[x^{1/2}u' - \frac{1}{2}x^{-1/2}u] + (4x^2 - 35)x^{-1/2}u = 0.$$

Collect terms and multiply by $\frac{1}{4}\sqrt{x}$ to get $x^2u'' + xu' + (x^2 - 9)u = 0$, which has general solution $u(x) = c_1J_3(x) + c_2Y_3(x)$. Then $y(x) = c_1x^{-1/2}J_3(x) + c_2x^{-1/2}Y_3(x)$.

17. Let $z = \frac{2k}{3}x^{3/2}, y = \sqrt{x}J_{1/3}(z(x))$ and calculate

$$\frac{dy}{dx} = \frac{1}{2}x^{-1/2}J_{1/3}(z) + kxJ'_{1/3}(z),$$

and

$$\frac{d^2y}{dx^2} = -\frac{1}{4}x^{-3/2}J_{1/3}(z) + J'_{1/3}(z)\left[k + \frac{1}{2}k\right] + k^2x^{3/2}J''_{1/3}(z).$$

Substitute into the equation to get

$$\frac{d^2y}{dx^2} + k^2xy = k^2x^{3/2}J''_{1/3} + \frac{3}{2}kJ'_{1/3} + \left(k^2x^{3/2} - \frac{1}{4}x^{-3/2}\right)J_{1/3}$$

$$= \frac{9}{4}x^{-3/2}\left\{z^2J''_{1/3} + zJ'_{1/3} + (z^2 - \frac{1}{9})J_{1/3}\right\} = 0,$$

and observe that this is zero because $J_{1/3}(z)$ is a solution of Bessel's equation with $\nu = 1/3$.

18. Set $2a-1 = 3, 2c-2 = 0, b^2c^2 = 4, a^2 - n^2c^2 = -5$ and solve for $a = 2, c = 1, b = 2, n = 3$ to get general solution $y = c_1x^2J_3(2x) + c_2x^2Y_3(2x)$

19. Set $2a - 1 = 1, 2c - 2 = 0, b^2c^2 = 1$, and $a^2 - n^2c^2 = -3$ and solve for $a = 1, c = 1, b = 1, n = 2$ to get general solution $y = c_1xJ_2(x) + c_2xY_2(x)$.

20. Set $2a - 1 = 5, 2c - 2 = 0, b^2c^2 = 1, a^2 - n^2c^2 = -7$ to get $a = 3, c = 1, b = 1, n = 4$. The general solution is $y = c_1x^3J_4(x) + c_2x^3Y_4(x)$.

21. Set $2a - 1 = 3, 2c - 2 = -1, b^2c^2 = 1/4, a^2 - n^2c^2 = 3$ and solve to get $a = 2, c = 1/2, b = 1, n = 2$ to get general solution $y = c_1x^2J_2(\sqrt{x}) + c_2x^2Y_2(\sqrt{x})$.

22. Set $2a - 1 = 1, 2c - 2 = 2, b^2c^2 = 16, a^2 - n^2c^2 = -15$ to get $a = 1, c = 2, b = 2, n = 2$. The general solution is $y = c_1xJ_2(2x^2) + c_2xY_2(2x^2)$.

23. Using $\nu = -3/2$ and the formula for $J_{3/2}(x)$ from Example 16.4,

$$x^{-3/2}J_{5/2}(x) = -[x^{-3/2}J_{3/2}(x)]' = -\left[x^{-2}\sqrt{\frac{2}{\pi}}\left(\frac{\sin(x)}{x} - \cos(x)\right)\right]'$$

$$= -\sqrt{\frac{2}{\pi}}\left[x^{-3}(\sin(x) - x\cos(x))\right]' = -\sqrt{\frac{2}{\pi}}\left[-3x^{-4}(\sin(x) - x\cos(x)) + x^{-3}(x\sin(x))\right]$$

$$= \sqrt{\frac{2}{\pi}}x^{-2}\left[\left(\frac{3}{x^2} - 1\right)\sin(x) - \frac{3}{x}\cos(x)\right].$$

Now multiply by $x^{3/2}$ to get

$$J_{5/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\left(\frac{3}{x^2} - 1\right)\sin(x) - \frac{3}{x}\cos(x)\right].$$

24. By the second equality of Problem 33 with $\nu = -3/2$, and the formula in the text for $J_{-3/2}(x)$ we get

$$x^{-3/2}J_{-5/2}(x) = \frac{d}{dx}\left[x^{-2}\sqrt{\frac{2}{\pi}}\left(-\sin(x) - \frac{\cos(x)}{x}\right)\right] = \sqrt{\frac{2}{\pi}}x^{-2}\left[\left(\frac{3}{x^2} - 1\right)\cos(x) + \frac{\sin(x)}{x}\right].$$

Multiply by $x^{3/2}$ to get the formula for $J_{-5/2}(x)$.

25. First, $[J_0(s)]' = -J_1(s)$. Integrate from 0 to α to get $\int_0^\alpha J_1(s)ds = -J_0(s)|_0^\alpha = J_0(0) - J_0(\alpha) = 1$, since $J_0(\alpha) = 0$. Let $s = \alpha x$ in the integral to get $\alpha\int_0^1 J_1(\alpha x)dx = 1$.

26. (a) With $u = J_0(\alpha x), u' = \alpha J_0'(\alpha x), u'' = \alpha^2 J_0''(\alpha x)$, so $xu'' + u' + \alpha^2 xu = \alpha[x\alpha J_0''(\alpha x) + J_0'(\alpha x) + \alpha x J_0(\alpha x)] = 0$. Similarly for $v = J_0(\beta x)$.
(b) Then $xvu'' + vu' + \alpha^2 xvu - [xuv'' + uv' + \beta^2 xuv] = 0$, so $(\beta^2 - \alpha^2)xuv = x[vu'' - uv''] + vu' - uv' = [x(vu' - uv')]'$. Integrate this last equation to get $(\beta^2 - \alpha^2)\int xJ_0(\alpha x)J_0(\beta x)dx = x[\alpha J_0'(\alpha x)J_0(\beta x) - \beta J_0'(\beta x)J_0(\alpha x)]$.

27. $I_0(x)$ is a solution of $y'' + \frac{1}{x}y' - y = 0$, so $xI_0'' + I_0' = xI_0$, hence $[xI_0'(x)]' = xI_0(x)$.

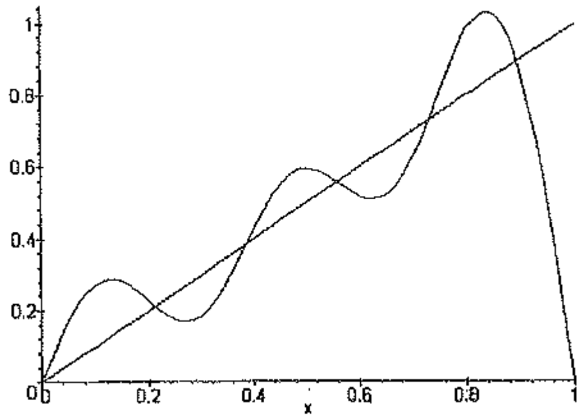
28. For a given function $f(x)$, the Fourier-Bessel coefficients for an expansion of $f(x) = \sum_{n=1}^{\infty} a_n J_1(j_n x)$ in terms of J_1 are given by

$$a_n = \frac{2}{[J_2(j_n)]^2}\int_0^1 \xi f(\xi)J_1(j_n\xi)d\xi, n \geq 1.$$

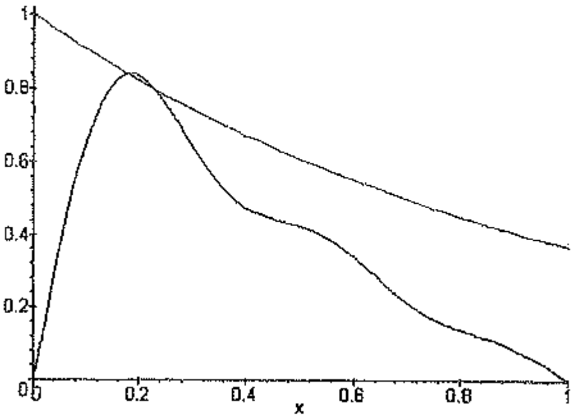
The first five zeros of $J_1(x)$ and the coefficients for the expansion of each of the functions (a) $f(x) = x$; (b) $f(x) = e^{-x}$; (c) $f(x) = xE^{-x}$; (d) $f(x) = x^2e^{-x}$ are given in the table below. These were computed with Maple to nine decimal places and rounded to five decimals in the table. The graphs of the fifth partial sum and each function are displayed below.

n	zero of $J_1(x)$	$f(x) = x$	$f(x) = e^{-x}$	$f(x) = xe^{-x}$	$f(x) = x^2e^{-x}$
1	3.83171	1.29596	.76073	.14553	−.09913
2	7.01559	−.94990	.43775	.02615	−.02355
3	10.17347	.78729	.30547	.00877	−.00835
4	13.32370	−.68744	.23430	.00394	−.00383
5	16.47063	.61806	.18995	.00209	−.00205

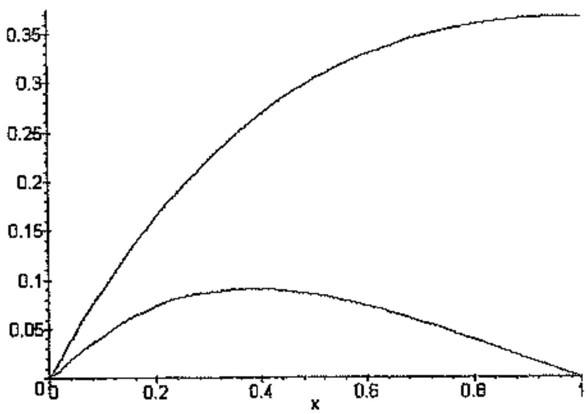
In general, observe that the approximation to $f(x)$ by these Fourier-Bessel partial sums is not very good for these few terms. Part of the problem is also the type functions given here. As a bonus, this same exercise is carried out with $f(x) = \sin(\pi x)$, and the graph of its fifth partial sum shows much better agreement.



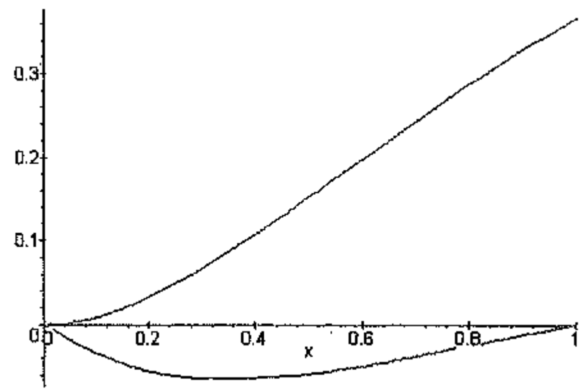
Fourier-Bessel $J_1(x)$ approximation of $f(x) = x$



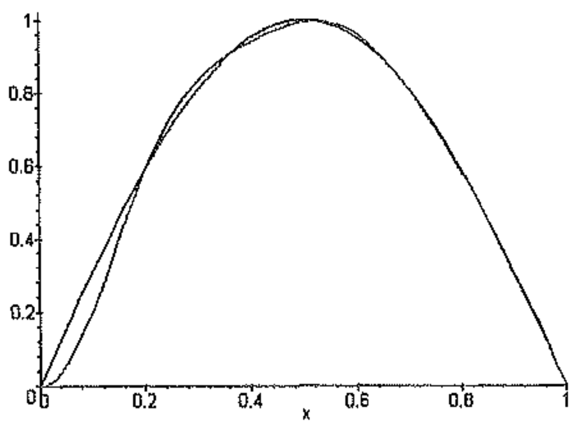
Fourier-Bessel $J_1(x)$ approximation of $f(x) = e^{-x}$



Fourier-Bessel $J_1(x)$ approximation of $f(x) = xe^{-x}$



Fourier-Bessel $J_2(x)$ approximation of $f(x) = x^2e^{-x}$



Bonus Fourier-Bessel $J_1(x)$ approximation of $\sin(\pi x)$

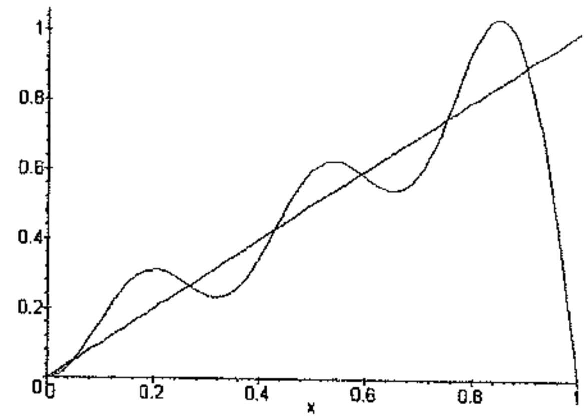
29. For a given function $f(x)$, the Fourier-Bessel coefficients for an expansion of $f(x) = \sum_{n=1}^{\infty} a_n J_2(j_n x)$ in terms of J_2 are given by

$$a_n = \frac{2}{[J_3(j_n)]^2} \int_0^1 \xi f(\xi) J_2(j_n \xi) d\xi, n \geq 1.$$

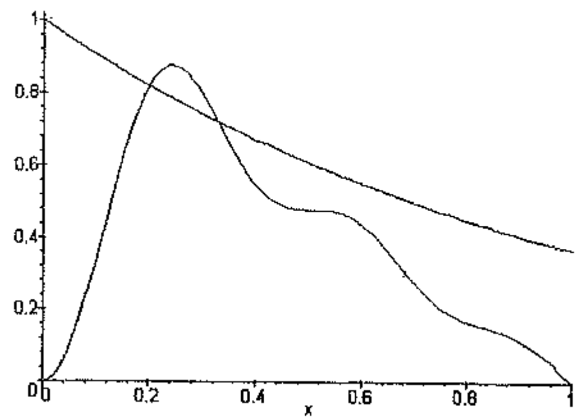
The first five zeros of $J_2(x)$ and the coefficients for the expansion of each of the functions (a) $f(x) = x$; (b) $f(x) = e^{-x}$; (c) $f(x) = xE^{-x}$; (d) $f(x) = x^2 e^{-x}$ are given in the table below. These were computed with Maple to nine decimal places and rounded to five decimals in the table. The graphs of the fifth partial sum and each function are displayed below.

n	zero of $J_2(x)$	$f(x) = x$	$f(x) = e^{-x}$	$f(x) = xe^{-x}$	$f(x) = x^2e^{-x}$
1	5.13562	1.6739	.94224	.34983	.06390
2	8.41724	−.77749	.63155	.13190	.00918
3	11.61984	.82807	.47796	.06949	.00255
4	14.79595	−.62012	.38506	.04293	.00098
5	17.95982	.62816	.32257	.02916	.00045

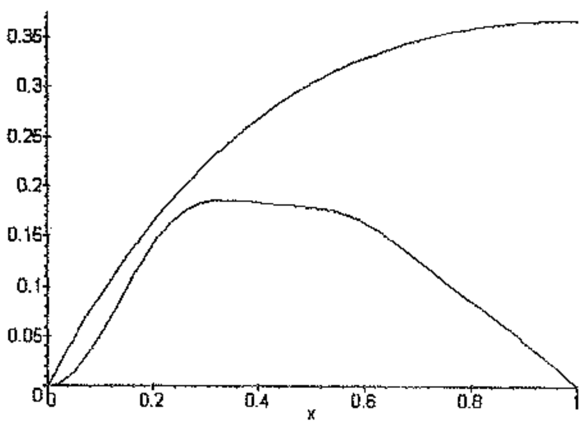
Again, observe the rather poor approximation to these functions by these abbreviated partial sums. More terms of the series could be expected to give better approximations.



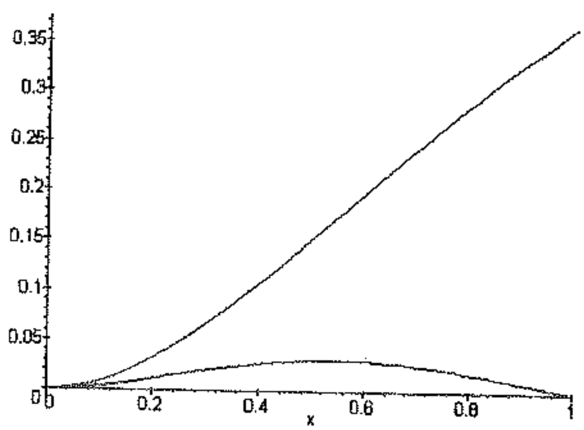
Fourier-Bessel $J_2(x)$ approximation of $f(x) = x$



Fourier-Bessel $J_2(x)$ approximation of $f(x) = e^{-x}$



Fourier-Bessel $J_2(x)$ approximation of $f(x) = xe^{-x}$



Fourier-Bessel $J_2(x)$ approximation of $f(x) = x^2e^{-x}$

Section 16.3 Sturm-Liouville Theory and Eigenfunction Expansions

Problems 1 through 8 are all either regular or periodic Sturm-Liouville problems on finite intervals. In each case we can show that there are no negative eigenvalues for the problem. Summarized below are the eigenvalue, eigenvector sets.

1. Regular on $[0, L]$; eigenvalues are $\lambda_n = \left[\frac{(2n-1)\pi}{2L}\right]^2, n = 1, 2, 3, \ldots$; eigenfunctions $\phi_n(x) = \sin\left(\frac{(2n-1)\pi x}{2L}\right)$.

2. Regular on $[0, \pi]$; eigenvalues are $\lambda_0 = 0, \phi_0(x) = 1,\ \lambda_n = n^2, n = 1, 2, 3, \ldots$; eigenfunctions $\phi_n(x) = \cos(n\pi x)$.

3. Regular on $[0, 4]$; eigenvalues are $\lambda_n = \left[\left(n - \frac{1}{2}\right)\frac{\pi}{4}\right]^2, n = 1, 2, 3, \ldots$; eigenfunctions are $\phi_n(x) = \cos\left[\left(n - \frac{1}{2}\right)\frac{\pi x}{4}\right]$.

4. Periodic on $[0, \pi]$; eigenvalues are $\lambda_0 = 0$, with eigenfunction $\phi_0(x) = 1; \lambda_n = 4n^2, n = 1, 2, 3, \ldots$; with eigenfunction $\phi_n(x) = a_n\cos(2nx) + b_n\sin(2nx)$, not both a_n, b_n zero.

5. Periodic on $[-3\pi, 3\pi]$; eigenvalues are $\lambda_0 = 0$, with eigenfunction $\phi_0(x) = 1; \lambda_n = \frac{n^2}{9}, n = 1, 2, 3, \ldots$, with eigenfunctions $\phi_n(x) = a_n\cos\left(\frac{nx}{3}\right) + b_n\sin\left(\frac{nx}{3}\right)$, not both a_n and b_n zero.

6. Regular on $[0, \pi]$; eigenvalues are the positive solutions of the equation

$$\sin(\sqrt{\lambda}\pi) + 2\sqrt{\lambda}\cos(\sqrt{\lambda}\pi) = 0.$$

There are infinitely many solution of which the first four are approximately
$\lambda_1 = .48705;\ \lambda_2 = 2.54914;\ \lambda_3 = 6.56059;\ \lambda_4 = 12.56423.$
eigenfunctions are $\phi_n(x) = \sin(\sqrt{\lambda_n}x)$.

7. Regular on $[0, 1]$; eigenvalues are positive solutions of the equation $\tan(\sqrt{\lambda}) = 1/(2\sqrt{\lambda})$. There are infinitely many such solutions of which the first four are approximately
$\lambda_1 = .42676;\ \lambda_2 = 10.8393;\ \lambda_3 = 40.4702;\ \lambda_4 = 89.8227.$
eigenfunctions are $\phi_n(x) = 2\sqrt{\lambda_n}\cos(\sqrt{\lambda_n}x) + \sin(\sqrt{\lambda_n}x)$.

8. Regular on $[0, 1]$; to simplify the calculation of eigenvalues and eigenfunctions, let $y(x) = e^{-x}u(x)$. The differential equation becomes $e^{-x}[u'' + \lambda u] = 0$ or $u'' + \lambda u = 0$ since $e^{-x} \neq 0$, and boundary conditions become $u(0) = u(1) = 0$. Thus eigenvalues are $\lambda_n = n^2\pi^2, n = 1, 2, 3, \ldots$ and eigenfunctions are $\phi_n(x) = e^{-x}u_n(x) = e^{-x}\sin(n\pi x)$.

9. Regular on $[0, \pi]$; eigenvalues are $\lambda_n = 1 + n^2$; eigenfunctions are $\phi_n(x) = e^{-x}\sin(nx), n = 1, 2, \ldots$

10. Regular on $[0, 8]$; eigenvalues are $\lambda_n = 8 + n^2\pi^2$; eigenfunctions are $\phi_n(x) = e^{3x}\sin(n\pi x), n = 1, 2, \ldots$

11. Regular on $[1, e^3]$; eigenvalues are $\lambda_n = 1 + \frac{n^2\pi^2}{9}$;
eigenfunctions are $\phi_n(x) = \frac{1}{x}\sin\left[\frac{n\pi}{3}\ln(x)\right], n = 1, 2, \ldots$

12. Regular on $[0, e^4]$; eigenvalues are $\lambda_n = -3 + \frac{n^2\pi^2}{16}$;
eigenfunctions are $\phi_n(x) = x\sin\left[\frac{n\pi}{4}\ln(x)\right], n = 1, 2, \ldots$.

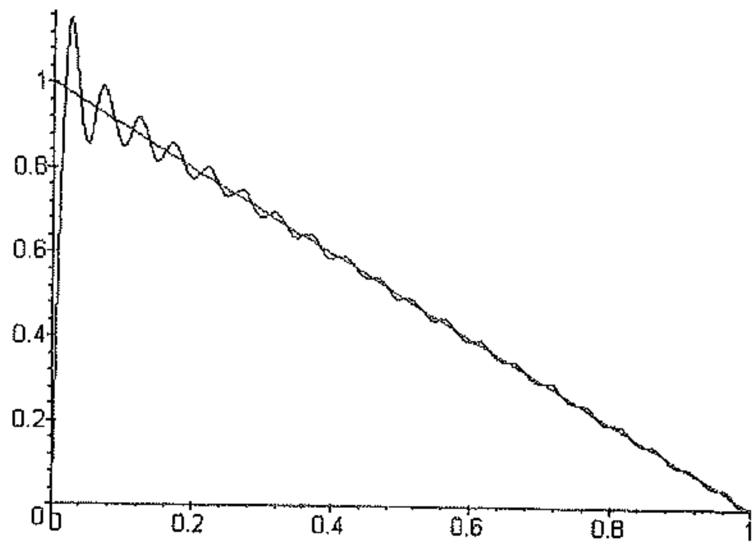
13. The eigenfunctions are easily found to be $\phi_n(x) = \sin\left(\frac{n\pi x}{L}\right), n = 1, 2, 3, \ldots$.
The coefficients of the eigenfunction expansion are

$$c_n = \frac{2}{L}\int_0^L (1-x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{2}{n\pi}[1 + (-1)^n(L-1)], n \geq 1.$$

Thus

$$\sum_{n=1}^{\infty}\frac{2}{n\pi}[1 + (-1)^n(L-1)]\sin\left(\frac{n\pi x}{L}\right) = 1 - x, \text{ for } 0 < x < L.$$

For the graph with $L = 1$, the series simplifies to $\sum_{n=1}^{\infty}\frac{2}{n\pi}\sin(n\pi x)$. The graph of the fortieth partial sum of this series is shown below.



14. The eigenfunctions from Problem 1 (with $L = \pi$) are $\phi_n(x) = \sin\left[(2n-1)\frac{x}{2}\right], n = 1, 2, \ldots$. The coefficients of the eigenfunction expansion are

$$c_n = \frac{2}{\pi}\int_0^{\pi}|x|\sin\left[(2n-1)\frac{x}{2}\right]dx = \frac{8}{\pi}\frac{(-1)^{n+1}}{(2n-1)^2}, n \geq 1.$$

Thus

$$\frac{8}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\sin\left[(2n-1)\frac{x}{2}\right] = |x|, 0 \leq x \leq \pi.$$

15. The eigenfunctions from Problem 3 are $\phi_n(x) = \cos\left(\frac{(2n-1)\pi x}{8}\right), n = 1, 2, 3, \ldots$
The coefficients of the eigenfunction expansion are

$$c_n = \frac{1}{2}\int_0^2 -\cos\left(\frac{(2n-1)\pi x}{8}\right)dx + \frac{1}{2}\int_2^4 \cos\left(\frac{(2n-1)\pi x}{8}\right)dx$$

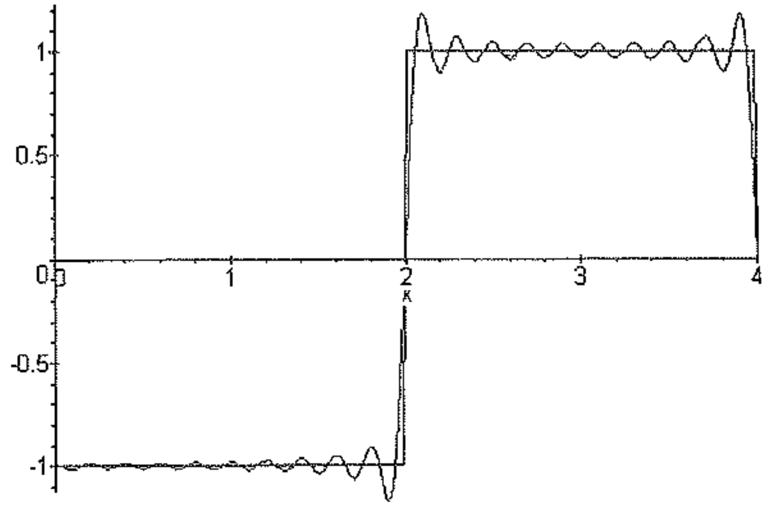
$$= \frac{4}{\pi(2n-1)}\left[(-1)^{n+1} + \sqrt{2}\left(\cos\left(\frac{n\pi}{2}\right) - \sin\left(\frac{n\pi}{2}\right)\right)\right], n \geq 1.$$

Thus

$$\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1} + \sqrt{2}(\cos(\frac{n\pi}{2}) - \sin(\frac{n\pi}{2}))}{2n-1}\cos\left(\frac{(2n-1)\pi x}{8}\right)$$

$$= \begin{cases} -1, & 0 < x < 2 \\ 0, & x = 0, 2, 4 \\ 1, & 2 < x < 4 \end{cases}$$

Comparative graphs of $f(x)$ and S_{40} are shown below. Note the clear illustration of the Gibbs phenomenon at $x = 2$ and $x = 4$



16. The eigenfunctions are $\phi_0(x) = 1$, and $\phi_n(x) = \cos(nx), n \geq 1$. The coefficients of the eigenfunction expansion are $c_0 = \dfrac{1}{\pi}\displaystyle\int_0^{\pi}\sin(2x)dx = 0$,
$c_n = \dfrac{2}{\pi}\displaystyle\int_0^{\pi}\sin(2x)\cos(nx)dx = \dfrac{4}{\pi}\dfrac{[(-1)^n - 1]}{n^2-4}, n \geq 1, n \neq 2;$
$c_2 = \dfrac{2}{\pi}\displaystyle\int_0^{\pi}\sin(2x)\cos(2x)dx = 0;$
Thus the eigenfunction expansion is

$$\sum_{n=0}^{\infty} c_n\phi_n(x) = \frac{4}{\pi}\sum_{\substack{n=1\\ n\neq 2}}^{\infty}\frac{[(-1)^n-1]}{n^2-4}\cos(nx) = -\frac{8}{\pi}\sum_{p=1}^{\infty}\frac{1}{(2p-1)^2-4}\cos((2p-1)x) = \sin(2x), 0 \leq x \leq \pi.$$

17. The eigenfunctions are $\phi_0(x) = 1$, and $\phi_n(x) = a_n\cos\left(\dfrac{nx}{3}\right) + b_n\sin\left(\dfrac{nx}{3}\right), n = 1, 2, 3, \ldots$. The coefficients of the eigenfunction expansion are $c_0 = \dfrac{1}{6\pi}\displaystyle\int_{-3\pi}^{3\pi}x^2dx = 3\pi^2$

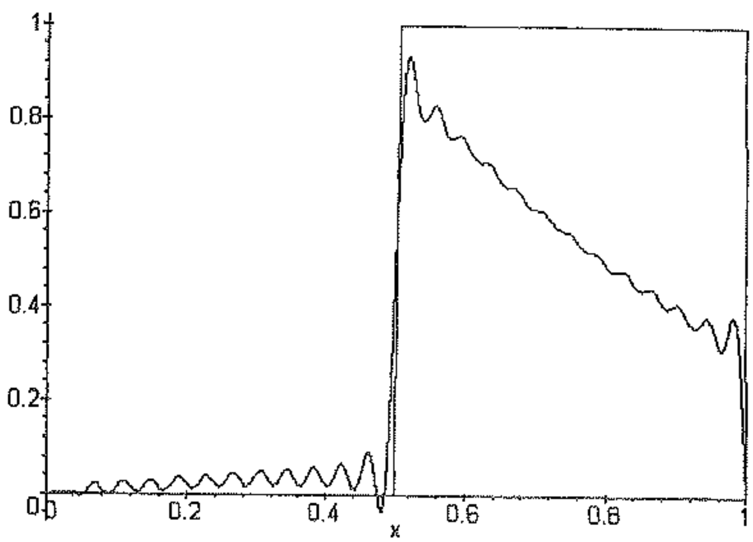
$$a_n = \frac{1}{3\pi}\int_{-3\pi}^{3\pi} x^2 \cos\left(\frac{nx}{3}\right) dx = \frac{36}{n^2}(-1)^n, n \geq 1$$
$$b_n = \frac{1}{3\pi}\int_{-3\pi}^{3\pi} x^2 \sin\left(\frac{nx}{3}\right) dx = 0, n \geq 1.$$
Thus $3\pi^2 + 36\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\left(\frac{nx}{3}\right) = x^2, -3\pi < x < 3\pi.$

18. The eigenfunctions from Problem 8 are $\phi_n(x) = e^{-x}\sin(n\pi x), n = 1, 2, 3, \ldots$
The coefficients of the eigenfunction expansion are

$$c_n = \frac{(f \cdot \phi_n)}{(\phi_n \cdot \phi_n)} = \frac{\int_{1/2}^1 e^{-x}\sin(n\pi x)dx}{\int_0^1 e^{-2x}\sin^2(n\pi x)dx}$$
$$= \frac{4}{n^2\pi^2}\frac{[e^{-1}n\pi(-1)^n - e^{-1/2}\{n\pi\cos(\frac{n\pi}{2}) + \sin(\frac{n\pi}{2})\}]}{(e^{-2}-1)}, n \geq 1.$$

Thus
$$\sum_{n=1}^{\infty}\frac{4}{n^2\pi^2}\frac{[e^{-1}n\pi(-1)^n - e^{-1/2}\{n\pi\cos(\frac{n\pi}{2}) + \sin(\frac{n\pi}{2})\}]}{(e^{-2}-1)}e^{-x}\sin(n\pi x)$$
$$= \begin{cases} 0, & 0 < x < \frac{1}{2} \\ \frac{1}{2}, & x = 0, \frac{1}{2}, 1 \\ 1, & \frac{1}{2} < x < 1 \end{cases}$$

Comparative graphs of $f(x)$ and $S_{50}(x)$ are shown below. Note the generally poor agreement, especially near $x = 1$.



19. Eigenfunctions of the regular Sturm-Liouville problem $y'' + \lambda y = 0, y'(0) = 0, y(4) = 0$ are any non-zero multiple of $\cos\left(\frac{(2n-1)\pi x}{8}\right)$. To normalize these eigenfunctions, calculate $\int_0^4 \cos^2\left(\frac{(2n-1)\pi x}{8}\right) = 2$. Thus normalized eigenfunctions are

$$\phi_n(x) = \frac{1}{\sqrt{2}}\cos\left(\frac{(2n-1)\pi x}{8}\right); n \geq 1.$$

Now calculate

$$(\phi_n \cdot f) = \frac{1}{\sqrt{2}} \int_0^4 x(4-x)\cos\left(\frac{(2n-1)\pi x}{8}\right)dx = -\frac{256}{\sqrt{2}} \frac{\{4(-1)^n + (2n-1_\pi\}}{\pi^3(2n-1)^3}$$

Certainly $f(x) = x(4-x)$ in integrable on $[0,4]$, and $(f \cdot f) = \int_0^4 x^2(4-x)^2 dx = \frac{512}{15}$. Thus Bessel's inequality gives (after a bit of simplification) that

$$\sum_{n=1}^{\infty} \left(\frac{\{4(-1)^n + (2n-1_\pi\}}{\pi^3(2n-1)^3}\right)^2 \leq \frac{512}{15} \frac{2}{(256)^2} = \frac{1}{960}.$$

20. Eigenfunctions of the regular Sturm-Liouville problem $y'' + \lambda y = 0, y(0) = 0, y(\pi) + 2y'(\pi) = 0$ were found to be any non-zero multiple of $\sin(\sqrt{\lambda_n}x)$ where $\sin(\sqrt{\lambda_n}\pi) + 2\sqrt{\lambda_n}\cos(\sqrt{\lambda_n}\pi) = 0$, recall λ_n is the n^{th} positive root of the equation $\tan(\sqrt{\lambda}\pi) = -2\sqrt{\lambda}$. The equation defining λ_n will be used several times in the following calculations to simplify certain results. To normalize these eigenfunctions, calculate

$$\int_0^\pi \sin^2(\sqrt{\lambda}x)dx = \frac{1}{2}\int_0^\pi 1 - \cos(2\sqrt{\lambda_n})x) = \frac{1}{2}\left[\pi - \frac{2\sin(\sqrt{\lambda_n}\pi)\cos(\sqrt{\lambda_n}\pi)}{2\sqrt{\lambda_n}}\right].$$

$$= \frac{1}{2}\left[\pi + 2\cos^2(\sqrt{\lambda_n}\pi)\right]$$

Thus normalized eigenfunctions are

$$\phi_n(x) = \sqrt{\frac{2}{\pi + 2\cos^2(\sqrt{\lambda_n}\pi)}}\sin(\sqrt{\lambda_n}x).$$

Next calculate

$$(\phi_n \cdot f) = \sqrt{\frac{2}{\pi + 2\cos^2(\sqrt{\lambda_n}\pi)}} \int_0^\pi e^{-x}\sin(\sqrt{\lambda_n}x)dx$$

$$= \sqrt{\frac{2}{\pi + 2\cos^2(\sqrt{\lambda_n}\pi)}}\left[\frac{e^{-\pi}}{1+\lambda_n}\left\{-\sin(\sqrt{\lambda_n}\pi) - \sqrt{\lambda_n}\cos(\sqrt{\lambda_n}\pi)\right\} + \frac{\sqrt{\lambda_n}}{1+\lambda_n}\right]$$

$$= \sqrt{\frac{2}{\pi + 2\cos^2(\sqrt{\lambda_n}\pi)}}\frac{\lambda_n}{1+\lambda_n}\left(1 + e^{-\pi}\cos(\sqrt{\lambda_n}\pi)\right).$$

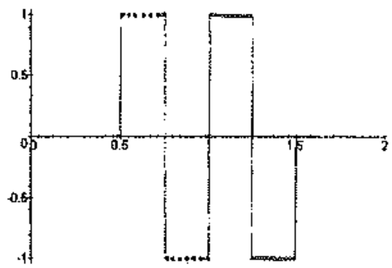
Also $(f \cdot f) = \int_0^\pi e^{-2x}dx = \frac{1}{2}(1 - e^{-2\pi}) = e^{-\pi}\sinh(\pi)$. From Bessel's inequality it follows that

$$\sum_{n=1}^{\infty} \frac{2\lambda_n^2}{(1+\lambda_n)^2(\pi + 2\cos^2(\sqrt{\lambda_n}\pi))} \cdot \left(1 + e^{-\pi}\cos(\sqrt{\lambda_n}\pi)\right)^2 \leq e^{-\pi}\sinh(\pi).$$

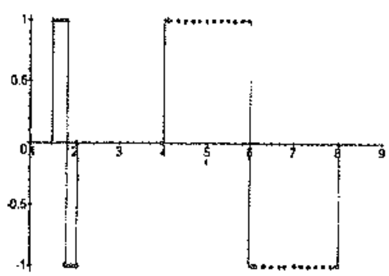
Section 16.4 Wavelets

1. We give the proof of orthogonality of $\sigma_{m,n}$ and $\sigma_{p,q}$ in two parts.
Case 1 Suppose $m \neq p$ so the wavelets have different length intervals of support, and for definiteness suppose that $m > p$. Then $\sigma_{p,q}$ has support on $\left[\frac{q}{2^p}, \frac{q+1}{2^p}\right)$, having value $+1$ on $I_{left} = \left[\frac{q}{2^p}, \frac{q+\frac{1}{2}}{2^p}\right)$ and value -1 on $I_{right} = \left[\frac{q+\frac{1}{2}}{2^p}, \frac{q+1}{2^p}\right)$. Each half of this interval has length $\left|\frac{q+1}{2^p} - \frac{q+\frac{1}{2}}{2^p}\right| = \left|\frac{q+\frac{1}{2}}{2^p} - \frac{q}{2^p}\right| = \frac{1}{2^{p+1}} = \frac{2^{m-p-1}}{2^m}$. Thus on each of I_{left} and I_{right} there are exactly $2^{m-p-1} \geq 1$ spaces of width $\frac{1}{2^m}$. If $\sigma_{m,n}$ starts a cycle on either I_{left} or on I_{right}, then $\sigma_{m,n}$ will complete its cycle while $\sigma_{p,q}$ has a constant value of either $+1$ or -1. Clearly in this case $\int_{-\infty}^{\infty} \sigma_{m,n}(t)\sigma_{p,q}(t)dt = 0$. If $\sigma_{m,n}$ starts its cycle to the right of $\frac{q+1}{2^p}$ then clearly $\sigma_{m,n}$ and $\sigma_{p,q}$ have disjoint support, so $\int_{-\infty}^{\infty} \sigma_{m,n}(t)\sigma_{p,q}(t)dt = 0$. Finally consider $\frac{n}{2^m} < \frac{q}{2^p}$. Then $0 < \frac{q}{2^p} - \frac{n}{2^m} = \frac{q2^{m-p} - n}{2^m}$. The numerator of this last fraction is a positive integer which counts the number of $\frac{1}{2^m}$ length steps between $\frac{n}{2^m}$ and $\frac{q}{2^P}$. Thus the support of $\sigma_{m,n}$ (which covers one such step) is disjoint from the support of $\sigma_{p,q}$ and certainly $\int_{-\infty}^{\infty} \sigma_{m,n}(t)\sigma_{p,q}(t)dt = 0$. This concludes the Case $m > p$.
Case 2 Suppose $m = p$ and $n \neq q$ so the wavelets have same length support. In this case it is easy to see that the support of the wavelets is disjoint, for otherwise we would have $n < q < n + 1$ (or $q < n < q + 1$). Both of these are clearly impossible since n and q are integers. Thus $\int_{-\infty}^{\infty} \sigma_{m,n}(t)\sigma_{m,q}(t)dt = 0,\ n \neq q$.

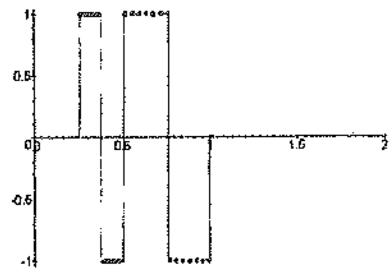
2. The graph to the right shows $\sigma_{1,1}(t)$ (dotted graph) and $\sigma_{1,2}(t)$ (solid line) on the same axes. These functions are clearly non-zero over disjoint intervals, $\sigma_{1,1}(t) \neq 0$ on $[1/2, 1)$ and $\sigma_{1,2}(t) \neq 0$ on $[1, 3/2)$. It follows that $\int_{-\infty}^{\infty} \sigma_{1,1}(t)\sigma_{1,2}(t)dt = 0$.



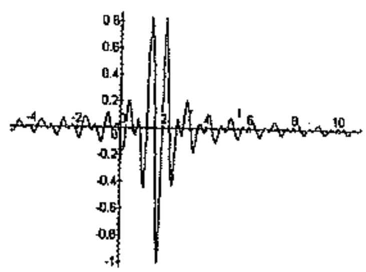
3. The graph to the right shows $\sigma_{1,3}(t)$ (solid line) and $\sigma_{-2,1}(t)$ (dotted graph) on the same axes. These functions are clearly non-zero over disjoint intervals, $\sigma_{1,3}(t) \neq 0$ on $[3/2, 2)$ and $\sigma_{-2,1}(t) \neq 0$ on $[4, 8)$. It follows that $\int_{-\infty}^{\infty} \sigma_{1,3}(t)\sigma_{-2,1}(t)dt = 0$.



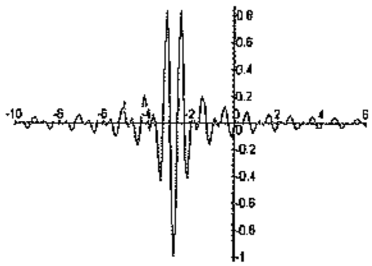
4. The graph shows $\sigma_{2,1}(t)$ (solid line) and $\sigma_{1,1}(t)$ (dotted graph) on the same axes. These functions are clearly non-zero over disjoint intervals, as $\sigma_{2,1}(t) \neq 0$ only on $[1/4, 1/2]$ and $\sigma_{1,1}(t) \neq 0$ only on $[1/2, 1]$. Therefore $\int_{-\infty}^{\infty} \sigma_{2,1}(t)\sigma_{1,1}(t)dt = 0$.



5. Graph of $\psi(2t-3) = \psi(2(t-3/2)$. Note shift of 3/2 units to right and compression by factor of 2.



6. Graph of $\psi(2t+6) = \psi(2(t-(-3))$. Note shift of 3 units to left and compression by a factor of 2.



7. By using the definition of the wavelets $\sigma_{m,n}$ we find that f can be written as

$$f(t) = \begin{cases} 4, & \text{on } [-16,-12) \\ -4, & \text{on } [-12,-8) \\ 6, & \text{on } [2,3) \\ -6, & \text{on } [3,4) \end{cases}$$

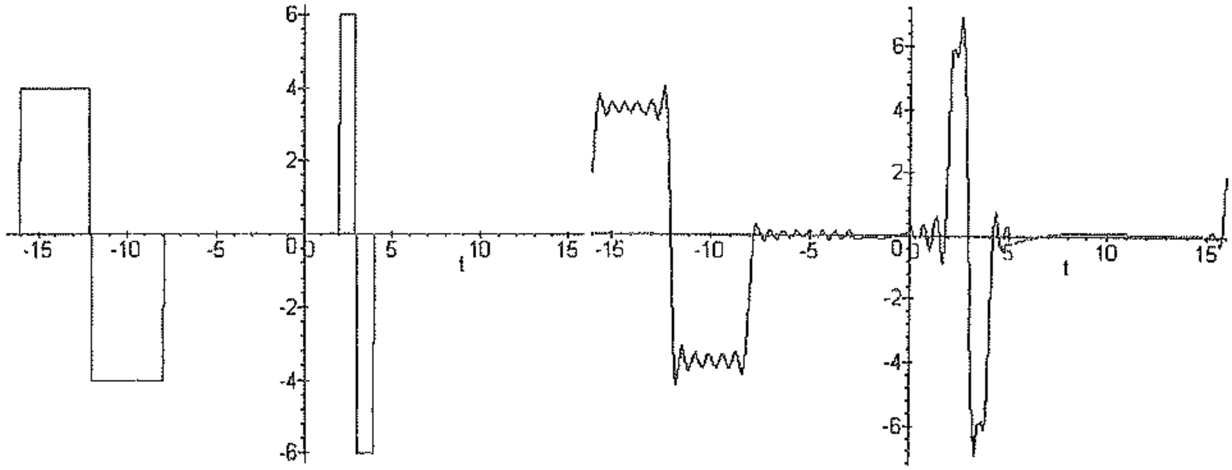
The Fourier coefficients of f are computed to be $a_0 = 0$, and

$$a_n = \frac{1}{n\pi}\left[12\sin\left(\frac{3n\pi}{16}\right) - 6\sin\left(\frac{n\pi}{8}\right) - 6\sin\left(\frac{n\pi}{4}\right) - 8\sin\left(\frac{3n\pi}{4}\right) + 4\sin\left(\frac{n\pi}{2}\right)\right]$$

for $n \geq 1$ and

$$b_n = \frac{1}{n\pi}\left[-12\cos\left(\frac{3n\pi}{16}\right) + 6\cos\left(\frac{n\pi}{8}\right) + 6\cos\left(\frac{n\pi}{4}\right) - 8\cos\left(\frac{3n\pi}{4}\right) + 4\sin\left(\frac{n\pi}{2}\right)\right]$$

for $n \geq 1$. The graphs of f and the fiftieth partial sum of its Fourier series are shown below.



8. By using the definition of the wavelets $\sigma_{m,n}$ we find that f can be written as

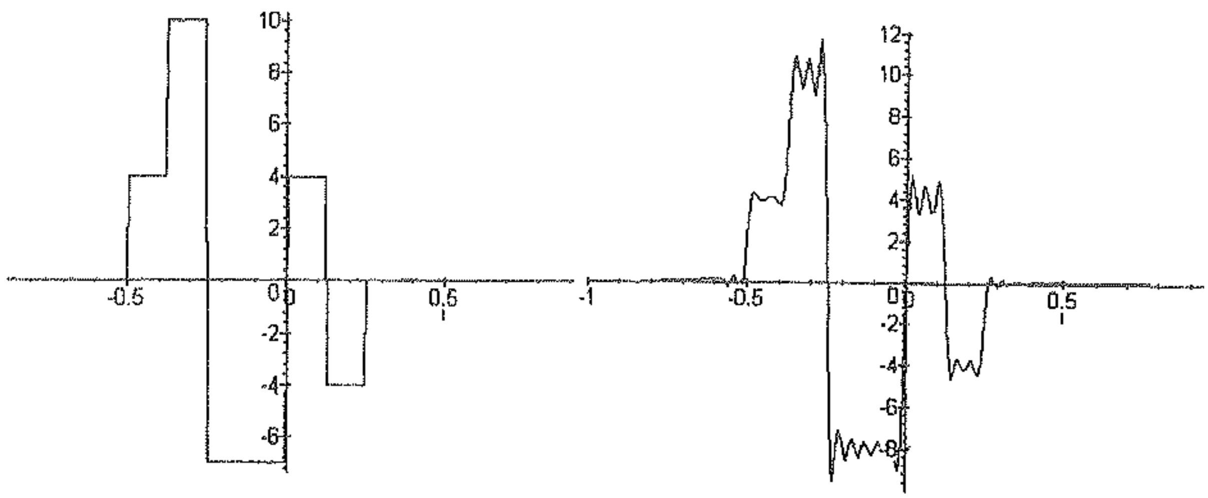
$$f(t) = \begin{cases} 4, & \text{on } [-1/2, -3/8) \\ 10, & \text{on } [-3/8, -1/4) \\ -8, & \text{on } [-1/4, 0) \\ 4, & \text{on } [0, 1/8) \\ -4, & \text{on } [1/8, 1/4) \end{cases}$$

The Fourier coefficients of f are computed to be

$$a_0 = -\frac{1}{8},\ a_n = \frac{1}{n\pi}\left[6\sin\left(\frac{3n\pi}{8}\right) + 4\sin\left(\frac{n\pi}{2}\right) - 22\sin\left(\frac{n\pi}{4}\right) + 8\sin\left(\frac{n\pi}{8}\right)\right], n \geq 1.$$

$$b_n = \frac{1}{n\pi}\left[6\cos\left(\frac{3n\pi}{8}\right) + 4\cos\left(\frac{n\pi}{2}\right) - 14\cos\left(\frac{n\pi}{4}\right) - 8\cos\left(\frac{n\pi}{8}\right) + 12\right], n \geq 1.$$

The graphs of f and the fiftieth partial sum of its Fourier series are shown below.



9. By using the definition of the wavelets $\sigma_{m,n}$ we find that f can be written as

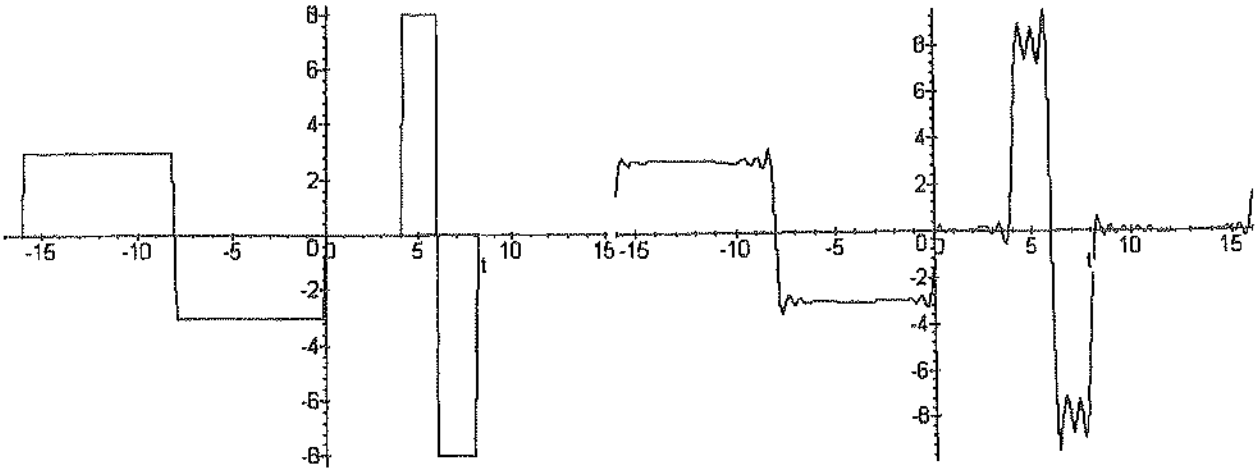
$$f(t) = \begin{cases} 3, & \text{on } [-16,-8) \\ -3, & \text{on } [-8,0) \\ 8, & \text{on } [4,6) \\ -8, & \text{on } [6,8) \end{cases}$$

The Fourier coefficients of f are computed to be

$$a_0 = 0,\ a_n = \frac{1}{n\pi}\left[-14\sin\left(\frac{n\pi}{2}\right) + 16\sin\left(\frac{3n\pi}{8}\right) - 8\sin\left(\frac{n\pi}{4}\right)\right], n \geq 1.$$

$$b_n = \frac{1}{n\pi}\left[2\cos\left(\frac{n\pi}{2}\right) - 16\cos\left(\frac{3n\pi}{8}\right) + 8\cos\left(\frac{n\pi}{4}\right) + 3(-1)^n + 3\right], n \geq 1.$$

The graphs of f and the fiftieth partial sum of its Fourier series are shown below.



10. By using the definition of the wavelets $\sigma_{m,n}$ we find that f can be written as

$$f(t) = \begin{cases} 1, & \text{on } [-8,-6) \\ -1, & \text{on } [-6,-4) \\ 2, & \text{on } [-1,-3/4) \\ -2, & \text{on } [-3/4,-1/2) \\ 4, & \text{on } [3/2,7/4) \\ -4, & \text{on } [7/4,2) \end{cases}$$

The Fourier coefficients of f are computed to be $a_0 = 0$ and

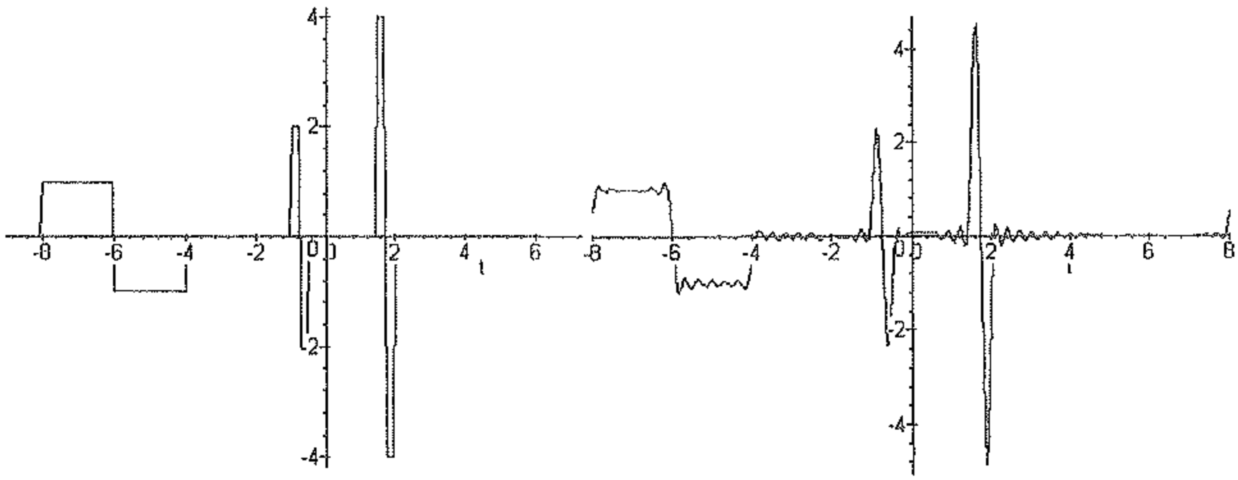
$$a_n = \frac{1}{n\pi}\left[-2\sin\left(\frac{3n\pi}{4}\right) + \sin\left(\frac{n\pi}{2}\right) - 4\sin\left(\frac{3n\pi}{32}\right) + 2\sin\left(\frac{n\pi}{8}\right)\right.$$

$$\left. +2\sin\left(\frac{n\pi}{16}\right) + 8\sin\left(\frac{7n\pi}{32}\right) - 4\sin\left(\frac{3n\pi}{16}\right) - 4\sin\left(\frac{n\pi}{4}\right)\right], n \geq 1.$$

$$b_n = \frac{1}{n\pi}\left[-2\cos\left(\frac{3n\pi}{4}\right) + \cos\left(\frac{n\pi}{2}\right) - 4\cos\left(\frac{3n\pi}{32}\right) + 2\cos\left(\frac{n\pi}{8}\right)\right.$$

$$\left. +2\cos\left(\frac{n\pi}{16}\right) - 8\cos\left(\frac{7n\pi}{32}\right) + 4\cos\left(\frac{3n\pi}{16}\right) + 4\cos\left(\frac{n\pi}{4}\right) + (-1)^n\right], n \geq 1.$$

The graphs of f and the fiftieth partial sum of its Fourier series are shown below.



Chapter Seventeen - The Wave Equation

Section 17.1 The Wave Equation and Initial and Boundary Conditions

1. Compute $y_{tt} = -\frac{n^2\pi^2c^2}{L^2}\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right)$ and $y_{xx} = -\frac{n^2\pi^2}{L^2}\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right)$. We see $y_{tt} = c^2 y_{xx}$ so $y(x,t)$ satisfies the one dimensional wave equation.

2. We compute

$$z_{tt} = -(n^2+m^2)c^2\sin(nx)\cos(my)\cos(\sqrt{n^2+m^2}ct);$$

$$z_{xx} = -n^2\sin(nx)\cos(my)\cos(\sqrt{n^2+m^2}ct);$$

$$z_{yy} = -m^2\sin(nx)\cos(my)\cos(\sqrt{n^2+m^2}ct)$$

so $z_{tt} = c^2(z_{xx}+z_{yy})$.

3. Compute $y_{xx} = \frac{1}{2}[f''(x+ct)+f''(x-ct)]$ and $y_{tt} = \frac{1}{2}[c^2f''(x+ct)+c^2f''(x-ct)]$, so $y_{tt} = c^2y_{xx}$.

4. First compute

$$y_{tt} = -c^2\sin(x)\cos(ct) - c\cos(x)\sin(ct),$$

and then

$$c^2y_{xx} = c^2[-\sin(x)\cos(ct) - \frac{1}{c}\cos(x)\sin(ct)] = -c^2\sin(x)\cos(ct) - c\cos(x)\sin(ct)$$

so $y_{tt} = c^2y_{xx}$ and the differential equation is satisfied. Now check the boundary conditions

$$y(0,t) = y(2\pi,t) = \frac{1}{c}\sin(ct),$$

for all $t > 0$, and the initial conditions,

$$y(x,0) = \sin(x), \frac{\partial y}{\partial t}(x,0) = -c\sin(x)\sin(ct) + \frac{c}{c}\cos(x)\cos(ct)\Big|_{t=0} = \cos(x).$$

5. Let $z(x,y,t)$ denote the vertical displacement of the point of the membrane located at point (x,y) at time $t>0$. Since the membrane occupies the region $0 \le x \le a, 0 \le y \le b$ and is rigidly fastened at all points of the rectangular boundary, we have the boundary value problem

$$\frac{\partial^2 z}{\partial t^2} = c^2\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right),\ 0<x<a, 0<y<b, t>0$$

subject to boundary conditions

$$z(0,y,t) = z(a,y,t) = z(x,0,t) = z(x,b,t) = 0$$

and initial conditions

$$z(x,y,0)=f(x,y),\frac{\partial z}{\partial t}(x,y,0)=g(x,y).$$

6. Let $u(x,t)$ denote the transverse displacement at time t of the point of the string located at point x. Then

$$\frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2}-k\left(\frac{\partial u}{\partial t}\right)^2,\ 0<x<L,t>0$$

with boundary conditions

$$u(0,t)=u(L,t)=0,t>0$$

and initial conditions

$$u(x,0)=f(x),\frac{\partial u}{\partial t}(x,0)=0,0<x<L.$$

Section 17.2 Fourier Series Solutions of the Wave Equation

In Problems 1 through 8, separation of variables for the wave equation with the fixed end conditions at $x=0$ and $x=L$ gives the general solution

$$y(x,t)=\sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{L}\right)\left[a_n\cos\left(\frac{cn\pi t}{L}\right)+\frac{L}{cn\pi}b_n\sin\left(\frac{cn\pi t}{L}\right)\right]$$

where

$$a_n=\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx,n\geq 1$$

and $f(x)$ is the given initial displacement, and

$$b_n=\frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx,n\geq 1$$

and $g(x)$ is the given initial velocity. To complete the solution requires only an identification of the values of c and L, and calculation of the Fourier coefficient of $f(x)$ and $g(x)$.

1. We have $c=c,L=2,f(x)=0,g(x)=2x[1-H(x-1)]$ and by calculating Fourier coefficients,

$$a_n=0,n\geq 1,b_n=\frac{4}{n^2\pi^2}\left[n\pi\cos\left(\frac{n\pi}{2}\right)-2\sin\left(\frac{n\pi}{2}\right)\right],n\geq 1.$$

These coefficients can be simplified as

$$b_{2n-1}=\frac{8(-1)^n}{(2n-1)^2\pi^2},n\geq 1,\ \text{and}\ b_{2n}=\frac{2}{n\pi}(-1)^n,n\geq 1.$$

Then

$$y(x,t)=\frac{16}{c\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^3}\sin\left[(2n-1)\frac{\pi x}{2}\right]\sin\left[(2n-1)\frac{\pi ct}{2}\right]$$

$$+\frac{2}{c\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\sin(n\pi x)\sin(n\pi ct).$$

2. We have $c = 3, L = 4, f(x) = 2\sin(\pi x), g(x) = 0$ so $a_n = 2$ if $n = 2, a_n = 0, n \neq 2, b_n = 0$, and $y(x,t) = 2\sin(\pi x)\cos(3\pi t)$.

3. We have $c = 2, L = 3, f(x) = 0$ and $g(x) = x(3-x)$, so

$$a_n = 0, n \geq 1, b_n = \frac{36}{n^3\pi^3}[1-(-1)^n], n \geq 1,$$

and

$$y(x,t) = \frac{108}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}\sin\left[(2n-1)\frac{\pi x}{3}\right]\cos\left[\frac{2(2n-1)\pi t}{3}\right].$$

4. We have $c = 3, L = \pi, f(x) = \sin x, g(x) = 1$, so

$$a_1 = 1, a_n = 0, \text{ if } n \neq 1;\ b_n = \frac{2}{n\pi}[1-(-1)^n].$$

Then

$$y(x,t) = \sin(x)\cos(3t) + \frac{4}{3\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\sin[(2n-1)x]\sin[3(2n-1)t].$$

5. Here $c = 2\sqrt{2}, L = 2\pi, g(x) = 0$, and $f(x) = 3x[1-H(x-\pi)] + (6\pi - 3x)H(x-\pi)$, for $0 \leq x \leq 2\pi$. Compute

$$a_n = \frac{24}{\pi n^2}\sin\left(\frac{n\pi}{2}\right);\ \text{so } a_{2n} = 0, n \geq 1, a_{2n-1} = \frac{24(-1)^{n+1}}{\pi(2n-1)^2}, n \geq 1, b_n = 0, n \geq 1.$$

Then

$$y(x,t) = \frac{24}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\sin\left[(2n-1)\frac{\pi x}{2}\right]\cos[(2n-1)\sqrt{2}t].$$

6. Here $c = 2, L = 5, f(x) = 0$, and $g(x) = (5-x)[H(x-4) - H(x-5)]$. Then

$$a_n = 0, n \geq 1, b_n = \frac{2}{n^2\pi^2}\left[n\pi\cos\left(\frac{4}{5}n\pi\right) + 5\sin\left(\frac{4}{5}n\pi\right)\right].$$

Then

$$y(x,t) = \frac{5}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^3}\sin\left(\frac{n\pi x}{5}\right)\left\{5\sin\left(\frac{4}{5}n\pi\right) + n\pi\cos\left(\frac{4}{5}n\pi\right)\right\}\sin\left(\frac{2n\pi t}{5}\right).$$

7. Here $c = 3, L = 2, f(x) = x(x-2)$ and $g(x) = 3[H(x-1/2) - H(x-1)]$. Calculate

$$a_n = -\frac{16}{n^3\pi^3}[1-(-1)^n], n \geq 1;\ b_n = \frac{6}{n\pi}\left[\cos\left(\frac{n\pi}{4}\right) - \cos\left(\frac{n\pi}{2}\right)\right], n \geq 1.$$

Then

$$y(x,t) = -\frac{32}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\sin\left[(2n-1)\frac{\pi x}{2}\right]\cos\left[\frac{3(2n-1)\pi t}{2}\right]$$

$$+\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[\cos\left(\frac{n\pi}{4}\right)-\cos\left(\frac{n\pi}{2}\right)\right]\sin\left(\frac{n\pi x}{2}\right)\sin\left(\frac{3n\pi t}{2}\right).$$

8. Here $c=5, L=\pi, f(x)=\sin(2x), g(x)=\pi-x$, so we compute $a_2=1, a_n=0, n\neq 2, b_n=\frac{2}{n}, n\geq 1$. Then

$$y(x,t)=\sin(2x)\cos(10t)+\frac{2}{5}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin(nx)\sin(5nt).$$

9. The equation is not separable as is, so let $y(x,t)=Y(x,t)-h(x)$ and choose h to cancel the $2x$ term in the equation and preserve the homogeneous boundary conditions at $x=0$ and $x=2$, i.e. $h(0)=h(2)=0$. We find $h(x)=\frac{1}{9}(x^3-4x)$ and then Y satisfies the problem $Y_{tt}=3Y_{xx}, Y(0,t)=Y(2,t)=0, Y(x,0)=h(x)=\frac{1}{9}(x^3-4x), \frac{\partial Y}{\partial t}(x,0)=0$. The Fourier coefficients for the solution for $Y(x,t)$ are $a_n=\frac{32(-1)^n}{3n^3\pi^3}, n\geq 1, b_n=0, n\geq 1$. Then

$$y(x,t)=-\frac{1}{9}(x^3-4x)+\frac{32}{3\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\sin\left(\frac{n\pi x}{2}\right)\cos\left(\sqrt{3}\frac{n\pi t}{2}\right).$$

10. Following the hint for Problem 9, let $y(x,t)=Y(x,t)-h(x)$ and choose $h(x)$ to make the differential equation homogeneous (and separable) and preserve the homogeneous boundary conditions at $x=0$ and $x=4$. We find $h(x)=\frac{1}{108}(64x-x^4)$ meets these requirements and $Y(x,t)$ should satisfy $Y_{tt}=9Y_{xx}, Y(0,t)=Y(4,t)=0, Y_t(x,0)=0, Y(x,0)=\frac{1}{108}(64x-x^4)$. The solution for $Y(x,t)$ is easily found by separation of variables and then

$$y(x,t)=\frac{1}{108}(x^4-64x)-\frac{512}{9\pi^5}\sum_{n=1}^{\infty}\frac{[2(1-(-1)^n)+n^2\pi^2(-1)^n]}{n^5}\sin\left(\frac{n\pi x}{4}\right)\cos\left(\frac{3n\pi t}{4}\right).$$

11. Let $y(x,t)=Y(x,t)-h(x)$, with h chosen so $-h''(x)-\cos(x)=0, h(0)=h(2\pi)=0$. Thus $h(x)=\cos(x)-1$, and $Y(x,t)$ satisfies $Y_{tt}=Y_{xx}, Y(0,t)=Y(2\pi,t)=0, Y_t(x,0)=0, Y(x,0)=\cos(x)-1$. Calculation of the Fourier coefficients of $Y(x,0)=\cos(x)-1$ gives $a_2=0, a_n=\frac{8}{\pi n(n^2-4)}[1-(-1)^n], n\neq 2$. Thus

$$a_{2n-1}=\frac{16}{\pi(2n-1)[(2n-1)^2-4]}, n\geq 1,$$

so

$$y(x,t)=1-\cos(x)+\frac{16}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)[(2n-1)^2-4]}\sin\left[\frac{(2n-1)x}{2}\right]\cos\left[\frac{(2n-1)t}{2}\right].$$

12. (a) Substitution of $u(x,t) = X(x)T(t)$ gives $a^2X^{(4)}T + XT'' = 0$ or upon separating variables $X^{(4)} - \lambda X = 0$, and $T'' + a^2\lambda T = 0$.
(b) The boundary conditions on $u(x,t)$ impose conditions $X''(0) = X''(\pi) = X^{(3)}(0) = X^{(3)}(\pi) = 0$. Now consider cases on λ.

$\underline{\lambda = 0}$ gives $X(x) = A + Bx + Cx^2 + Dx^3$, and the boundary conditions in turn require $C = 0, 6D\pi = 0$ so $D = 0, A, B$ arbitrary. Thus $\lambda = 0$ is an eigenvalue with $X_0(x) = A + Bx, T_0(t) = \alpha + \beta t$.

$\underline{\lambda > 0}$ let $\lambda = \alpha^4, \alpha$ real. Then the general solution for $X(x)$ is $X(x) = A\cos(\alpha x) + B\sin(\alpha x) + C\cosh(\alpha x) + D\sinh(\alpha x)$. The boundary conditions in turn give the four equations $-A + C = 0, -A\cos(\alpha\pi) - B\sin(\alpha\pi) + C\cosh(\alpha\pi) + D\sinh(\alpha\pi) = 0, -B + D = 0, A\sin(\alpha\pi) - B\cos(\alpha\pi) + C\sinh(\alpha\pi) + D\cosh(\alpha\pi) = 0$. Clearly we get $A = C$ and $B = D$ with the other two equation then becoming $C[\cosh(\alpha\pi) - \cos(\alpha\pi)] + D[\sinh(\alpha\pi) - \sin(\alpha\pi)] = 0, C[\sinh(\alpha\pi) + \sin(\alpha\pi)] + D[\cosh(\alpha\pi) - \cos(\alpha\pi)] = 0$. The determinant of the coefficients of this two by two system is $\cosh^2(\alpha\pi) - \sinh^2(\alpha\pi) + \cos^2(\alpha\pi) + \sin^2(\alpha\pi) - 2\cosh(\alpha\pi)\cos(\alpha\pi) = 2[1 - \cosh(\alpha\pi)\cos(\alpha\pi)]$. This is zero, hence there will be nontrivial solutions, if $\cos(\alpha\pi)\cosh(\alpha\pi) = 1$. This equation has an infinite number of positive solutions, $\alpha_n, n = 1, 2, \ldots$, each giving an eigenvalue $\lambda_n = \alpha_n^4$, and eigenfunction $X_n(x) = r_n[\cos(\alpha_n x) + \cosh(\alpha_n x)] + \sin(\alpha_n x) + \sinh(\alpha_n x)$, where $r_n = \dfrac{C_n}{D_n} = \dfrac{\sin(\alpha_n\pi) - \sinh(\alpha_n\pi)}{\cosh(\alpha_n\pi) - \cos(\alpha_n\pi)}$. For each α_n we get $T_n(t) = A_n\cos(a\alpha_n^2 t) + B_n\sin(a\alpha_n^2 t)$.

$\underline{\lambda < 0}$, let $\lambda = -4\alpha^4, \alpha > 0$ and real. The roots of the characteristic equation, $r^4 + 4\alpha^4 = 0$ are $\alpha(1+i), \alpha(1-i), \alpha(-1+i), \alpha(-1-i)$, so we find general solution $X(x) = e^{\alpha x}[A\cos(\alpha x) + B\sin(\alpha x)] + e^{-\alpha x}[C\cos(\alpha x) + D\sin(\alpha x)]$. The boundary conditions give the four equations $B - D = 0$ from $X''(0) = 0$; from $X^{(3)}(0) = 0$ we get $A - B - C - D = 0; X''(\pi) = 0$ gives $-Ae^{\alpha\pi}\sin(\alpha\pi) + Be^{\alpha\pi}\cos(\alpha\pi) + Ce^{-\alpha\pi}\sin(\alpha\pi) - De^{-\alpha\pi}\cos(\alpha\pi) = 0$; and $X^{(3)}(\pi) = 0$ gives $-Ae^{\alpha\pi}[\cos(\alpha\pi) + \sin(\alpha\pi)] + Be^{\alpha\pi}[\cos(\alpha\pi) - \sin(\alpha\pi)] + Ce^{-\alpha\pi}[\cos(\alpha\pi) - \sin(\alpha\pi)] - De^{-\alpha\pi}[\cos(\alpha\pi) + \sin(\alpha\pi)] = 0$. The determinant of this four by four system is $\cosh(2\alpha\pi) - \cos(2\alpha\pi)$ which is zero if and only if $\alpha = 0$. Thus there are no nontrivial solutions for $\alpha > 0$, and hence no negative eigenvalues.
(c) We proceed as in part(b) by considering cases on λ. The boundary conditions are $X(0) = X(\pi) = X''(0) = X''(\pi) = 0$.

$\underline{\lambda = 0}$ gives $X(x) = A + Bx + Cx^2 + Dx^3$. The boundary conditions give $X(0) = A = 0, X''(0) = 2C = 0; X''(\pi) = 6D\pi = 0$, and finally $X(\pi) = B\pi = 0$ so we have only the trivial solution and $\lambda = 0$ is not an eigenvalue.

$\underline{\lambda > 0}$, let $\lambda = \alpha^4, \alpha$ real and get $X(x) = A\cos(\alpha x) + B\sin(\alpha x) + C\cosh(\alpha x) + D\sinh(\alpha x)$. $X(0) = 0$ gives $A + C = 0$, and $X''(0) = 0$ gives $-A + C = 0$ so $A = C = 0$. Then $X(\pi) = B\sin(\alpha\pi) + D\sinh(\alpha\pi)$ and $X''(\pi) = -B\sin(\alpha\pi) + D\sinh(\alpha\pi) = 0$. The determinant of coefficients is $\sinh^2(\alpha\pi)\sin^2(\alpha\pi)$ which is zero for $\alpha = n, n = 1, 2, \ldots$. This gives eigenvalues $\lambda_n = n^4$ with $X_n(x) = \sin(nx)$, and $T_n(t) = A_n\cos(an^2t) + B_n\sin(an^2t)$.

$\underline{\text{For } \lambda < 0}$, let $\lambda = -4\alpha^2, \alpha > 0$ and real to get $X(x) = e^{\alpha x}[A\cos(\alpha x) + B\sin(\alpha x)] + e^{-\alpha x}[C\cos(\alpha x) + D\sin(\alpha x)]$ $X(0) = 0$ gives $A + C = 0; X''(0) = 0$ gives $B - D = 0$. Use these results in X and compute $X(\pi) = 0$ and $X''(\pi) = 0$ to get the equations $A\cos(\alpha\pi)\sinh(\alpha\pi) + B\sin(\alpha\pi)\cosh(\alpha\pi) = 0, -A\sin(\alpha\pi)\cosh(\alpha\pi) + B\cos(\alpha\pi)\sinh(\alpha\pi) = 0$. The determinant of coefficients is $\cos^2(\alpha\pi)\sinh^2(\alpha\pi) + \sin^2(\alpha\pi)\cosh^2(\alpha\pi) = \sin^2(\alpha\pi) + \sinh^2(\alpha\pi) \neq 0$ if $\alpha > 0$. Hence there are no nontrivial solutions and no negative eigenvalues.

13. Separation of variables gives the ordinary differential equations $X'' + \lambda X = 0, X(0) = X(L) = 0$; and $T'' + AT' + (B + c^2\lambda)T = 0, T'(0) = 0$. Eigenvalues and eigenfunctions for X are $\lambda = \dfrac{n^2\pi^2}{L^2}$, with $X_n(x) = \sin\left(\dfrac{n\pi x}{L}\right)$. With these values we get characteristic equation $r^2 + Ar + (B + c^2n^2\pi^2/L^2) = 0$, with roots $r = -\dfrac{A}{2} \pm \dfrac{1}{2}\sqrt{A^2 - 4\left(B + \dfrac{c^2n^2\pi^2}{L^2}\right)}$. The given condition $A^2L^2 < 4(BL^2 + c^2\pi^2)$ ensures these roots are complex. Let $r_n^2 = 4(BL^2 + c^2n^2\pi^2) - A^2L^2$, and the roots are $r = -\dfrac{A}{2} \pm \dfrac{ir_n}{2L}$. Thus

$$T_n(t) = e^{-At/2}\left[a_n\cos\left(\frac{r_nt}{2L}\right) + b_n\sin\left(\frac{r_nt}{2L}\right)\right].$$

Then $T_n'(0) = 0$ gives $-\dfrac{Aa_n}{2} + \dfrac{b_nr_n}{2L} = 0$, so $b_n = \dfrac{ALa_n}{r_n}$ and by superposition

$$u(x,t) = e^{-At/2}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{L}\right)a_n\left[\cos\left(\frac{r_nt}{2L}\right) + \frac{AL}{r_n}\sin\left(\frac{r_nt}{2L}\right)\right].$$

To satisfy $u(x,0) = f(x)$ choose $a_n = \dfrac{2}{L}\displaystyle\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx, n \geq 1$.

14. (a)The function $\phi(x)$ should satisfy $\phi''(x) = -\frac{5}{9}x^3, \phi(0) = \phi(4) = 0$. This is easily solved to give $\phi(x) = \frac{1}{36}x(256 - x^4)$. The solution for $w(x,t)$ will have the form

$$w(x,t) = \sum_{n=1}^{\infty} \alpha_n \sin\left(\frac{n\pi x}{4}\right)\cos\left(\frac{3n\pi t}{4}\right)$$

where coefficients

$$\alpha_n = \frac{2}{4}\int_0^4 \left[\cos(\pi x) - \frac{1}{36}x(256 - x^4)\right]\sin\left(\frac{n\pi x}{4}\right)dx$$

$$= \begin{cases} \frac{2n[1-(-1)^n]}{\pi(n^2-16)} + \frac{10240(-1)^n(n^2\pi^2-6)}{9n^5\pi^5} & , n \neq 4 \\ \frac{10(16\pi^2-6)}{9\pi^5} & , n = 4 \end{cases}$$

The solution of the forced motion is given by

$$y(x,t) = \sum_{\substack{n=1 \\ n\neq 4}}^{\infty} \left\{\frac{2n[1-(-1)^n]}{\pi(n^2-16)} + \frac{10240(-1)^n(n^2\pi^2-6)}{9n^5\pi^5}\right\}\sin\left(\frac{n\pi x}{4}\right)\cos\left(\frac{3n\pi t}{4}\right)$$

$$+\frac{10(16\pi^2-6)}{9\pi^5}\sin(\pi x)\cos(3\pi t) + \frac{1}{36}x(256 - x^4)$$

(b) Without the forcing term, the solution has the form

$$y(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{4}\right)\cos\left(\frac{3n\pi t}{4}\right)$$

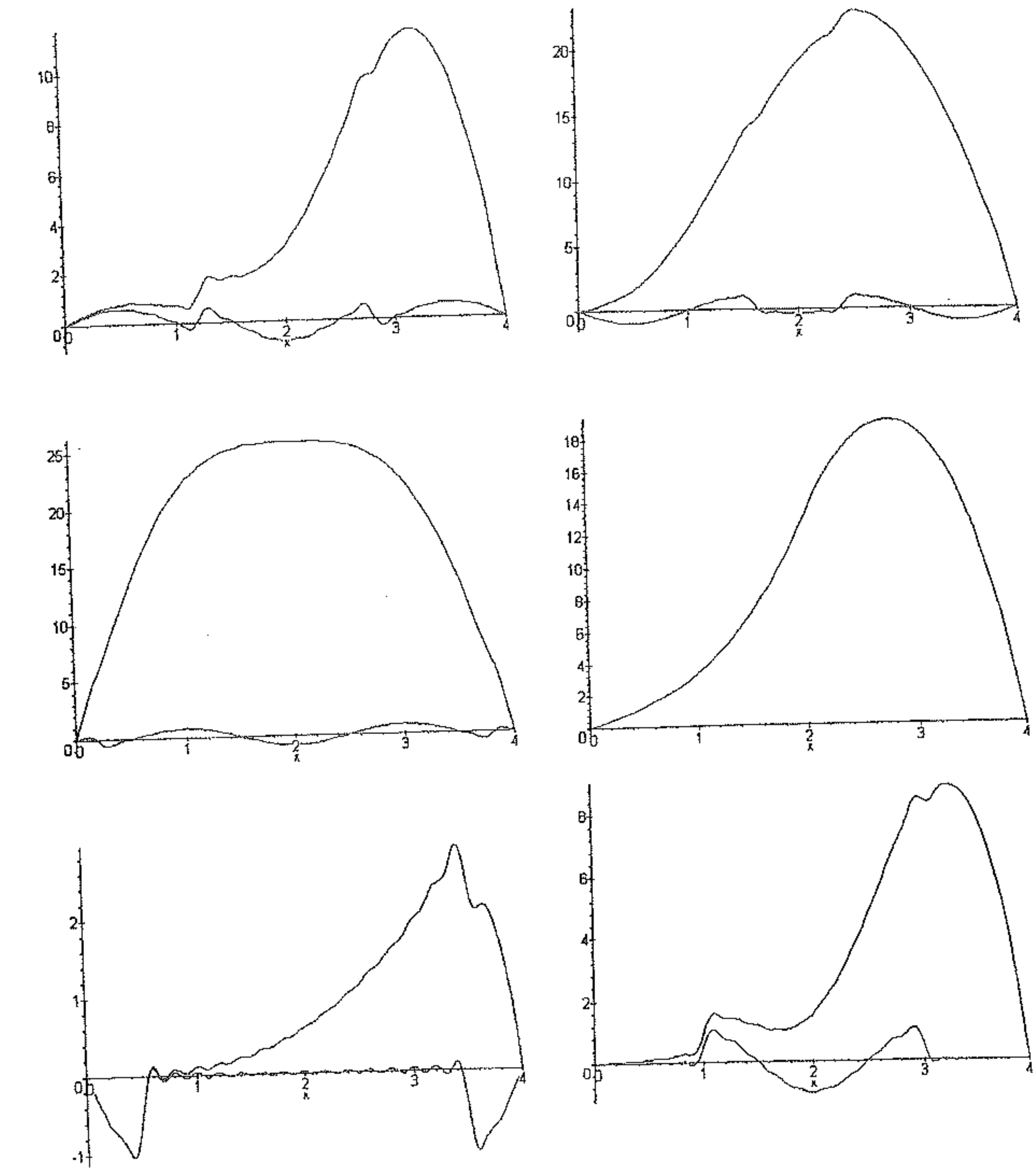
where

$$a_n = \frac{2}{4}\int_0^4 \cos(\pi x)\sin\left(\frac{n\pi x}{4}\right)dx = \begin{cases} \frac{2n[1-(-1)^{n+1}]}{\pi(n^2-16)} & , n \neq 4 \\ 0 & , n = 4 \end{cases}$$

Thus the unforced motion is given by

$$y(x,t) = \sum_{\substack{n=1 \\ n\neq 4}}^{\infty} \frac{2n[1-(-1)^{n+1}]}{\pi(n^2-16)}\sin\left(\frac{n\pi x}{4}\right)\cos\left(\frac{3n\pi t}{4}\right)$$

(c) The graphs of forced motion and unforced motion are shown below for $t = .4, .8, 1.4, 2, 2.5, 3$.



15. (a)The function $\phi(x)$ should satisfy $\phi''(x) = -\frac{\cos(\pi x)}{9}, \phi(0) = \phi(4) = 0$. This is easily solved to give $\phi(x) = \frac{1}{9\pi^2}[\cos(\pi x) - 1]$. The solution for $w(x,t)$ will have the form

$$w(x,t) = \sum_{n=1}^{\infty} \alpha_n \sin\left(\frac{n\pi x}{4}\right) \cos\left(\frac{3n\pi t}{4}\right)$$

where coefficients

$$\alpha_n = \frac{2}{4}\int_0^4 \left\{x(4-x) - \frac{1}{9\pi^2}[\cos(\pi x) - 1]\right\} \sin\left(\frac{n\pi x}{4}\right) dx$$

$$= \begin{cases} \frac{64}{\pi^3}\left(\frac{2}{(2p-1)^3} + \frac{1}{9}\frac{1}{(2p-1)((2p-1)^2-16)}\right) & , n = 2p-1, p \geq 1 \\ 0 & , n = 2p, p \geq 1 \end{cases}$$

The solution for the forced motion is given by

$$y(x,t) = \frac{64}{\pi^3}\sum_{p=1}^{\infty}\left(\frac{2}{(2p-1)^3} - \frac{1}{9}\frac{1}{(2p-1)((2p-1)^2-16)}\right)\sin\left(\frac{(2p-1)\pi x}{4}\right)\cos\left(\frac{3(2p-1)\pi t}{4}\right)$$

$$+\frac{1}{9\pi^2}[\cos(\pi x) - 1]$$

(b)Without the forcing term, the solution has the form

$$y(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{4}\right)\cos\left(\frac{3n\pi t}{4}\right)$$

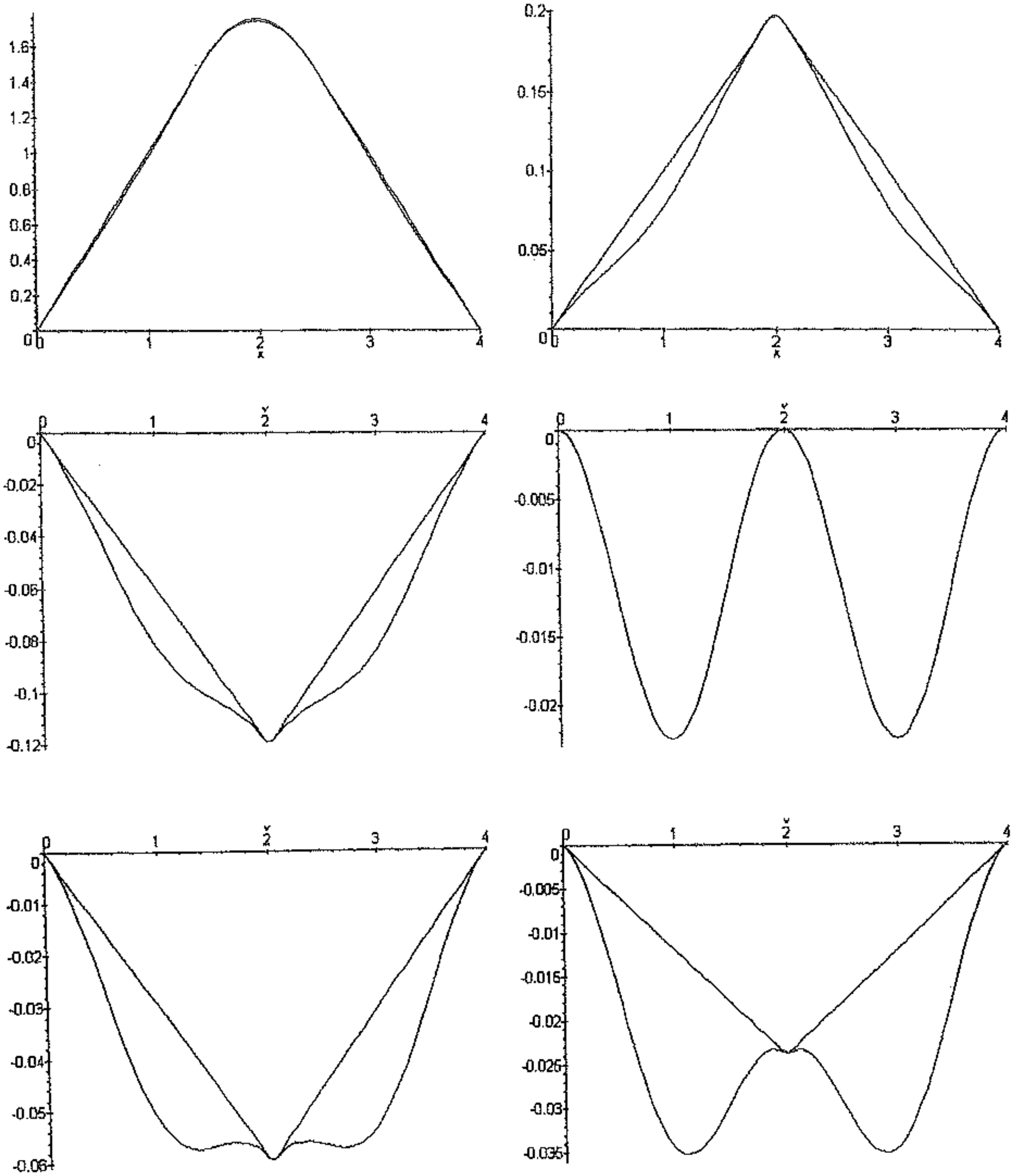
where

$$a_n = \frac{2}{4}\int_0^4 x(4-x)\sin\left(\frac{n\pi x}{4}\right) dx = \frac{64[1-(-1)^n]}{n^3\pi^3}$$

Thus the unforced motion is given by

$$y(x,t) = \sum_{n=1}^{\infty}\frac{64[1-(-1)^n]}{n^3\pi^3}\sin\left(\frac{n\pi x}{4}\right)\cos\left(\frac{3n\pi t}{4}\right)$$

(c) The graphs of forced motion and unforced motion are shown below for $t = .5, .65, 1.99, 1.995, 1.998, 2$.



16. (a)The function $\phi(x)$ should satisfy $\phi'' = e^{-x}/9, \phi(0) = \phi(4) = 0$. This is easily solved to give $\phi(x) = \frac{1}{36}(4e^{-x} + (1-e^{-4})x - 4)$. The solution for $w(x,t)$ will have the form

$$w(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{4}\right) \cos\left(\frac{3n\pi t}{4}\right)$$

where

$$\begin{aligned} a_n &= \frac{2}{4}\int_0^4 \left(\sin(\pi x) - \frac{1}{36}(4e^{-x} + (1-e^{-4})x - 4)\right) \sin\left(\frac{n\pi x}{4}\right) dx \\ &= \begin{cases} \frac{32}{9}\frac{1-e^{-4}(-1)^n}{16+n^2\pi^2} + 1 & , n \neq 4 \\ \frac{32}{9}\frac{1-e^{-4}(-1)^n}{16+n^2\pi^2} & , n = 4 \end{cases} \end{aligned}$$

The solution of the forced motion is given by

$$y(x,t) = \frac{32}{9}\sum_{n=1}^{\infty} \frac{1-e^{-4}(-1)^n}{16+n^2\pi^2} \sin\left(\frac{n\pi x}{4}\right) \cos\left(\frac{3n\pi t}{4}\right)$$

$$+ \sin(\pi x)\cos(3\pi t) + \frac{1}{36}(4e^{-x} + (1-e^{-4})x - 4)$$

(b)Without the forcing term, the solution has the form

$$y(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{4}\right) \cos\left(\frac{3n\pi t}{4}\right)$$

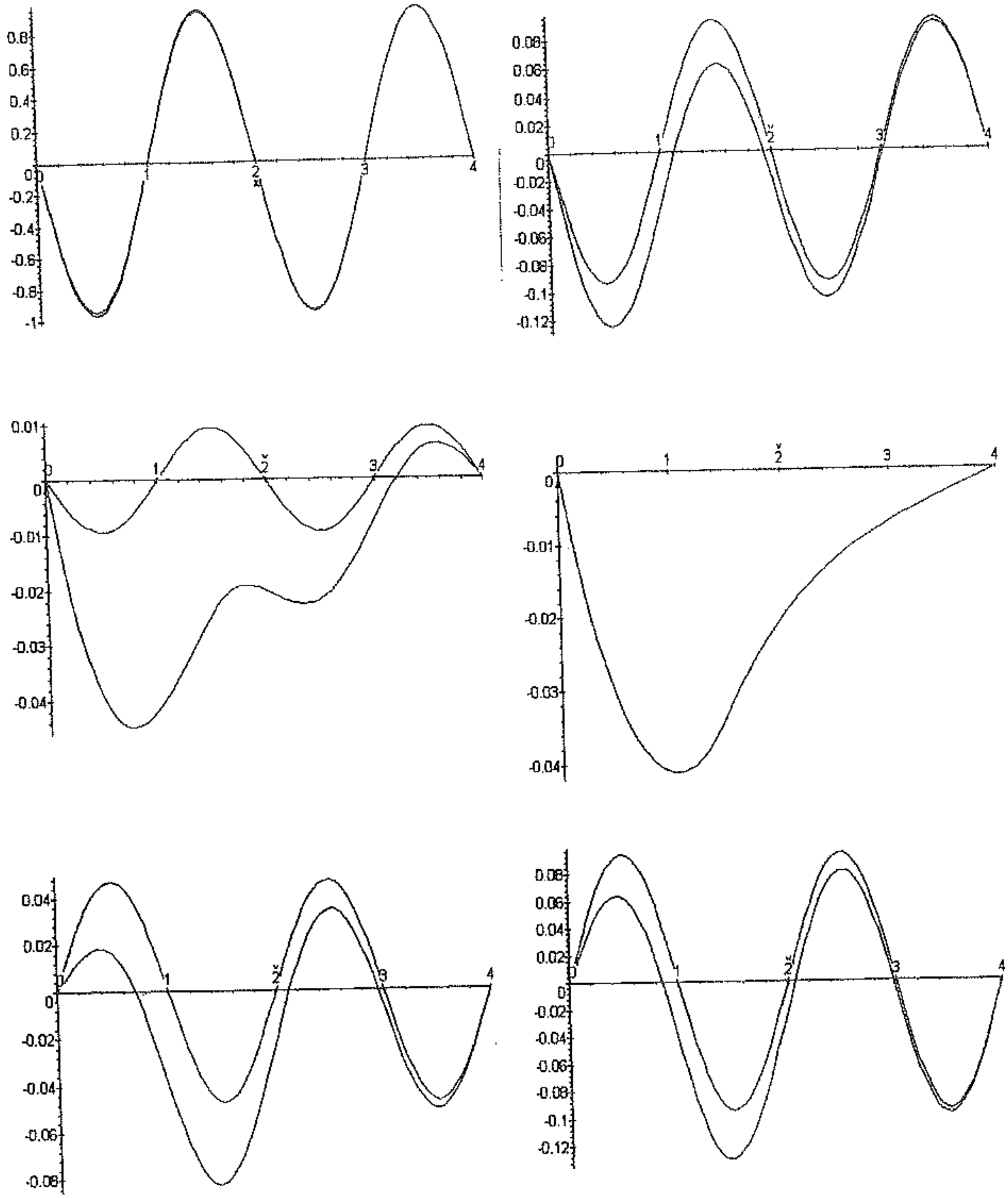
where

$$a_n = \frac{2}{4}\int_0^4 \sin(\pi x) \sin\left(\frac{n\pi x}{4}\right) dx = \begin{cases} 0 & , n \neq 4 \\ 1 & , n = 4 \end{cases}$$

Thus the unforced motion is given by

$$y(x,t) = \sin(\pi x)\cos(3\pi t).$$

(c) The graphs of forced motion and unforced motion are shown below for $t = .3, .49, .499, .5, .505, .51$.



17. $y_{j,1} = 0.25$ for $j = 1, 2, \cdots, 19$

$y_{1,2} = 0.08438$, $y_{j,2} = 0.05$ for $j = 2, 3, \cdots, 19$
$y_{1,3} = 0.13634$, $y_{2,3} = 0.077149$, $y_{j,3} = 0.075$ for $j = 3, 4, \cdots, 19$
$y_{1,4} = 0.17608$, $y_{2,4} = 0.10786$, $y_{3,4} = 0.10013$, $y_{j,4} = 0.1$ for $j = 4, 5, \cdots, 19$
$y_{1,5} = 0.20055$, $y_{2,5} = 0.14235$, $y_{3,5} = 0.12574$, $y_{4,5} = 0.12501$
$y_{j,5} = 0.125$ for $j = 5, 6, \cdots, 19$
19. We give $y_{j,k}$ for $j = 1, 2, \cdots, 9$, first for $k = -1$, then $k = 0, 1, \cdots, 5$.
$y_{j,-1}$: 0.08075, 0.127, 0.14475, 0.14, 0.11875, 0.087, 0.05075, 0.016, −0.01125
$y_{j,0}$: 0.081, 0.128, 0.147, 0.144, 0.125, 0.096, 0.063, 0.032, 0.009
$y_{j,1}$: 0.079125, 0.1735, 0.14788, 0.147, 0.13063,
0.10475, 0.075375, 0.0485, 0.030125
$y_{j,2}$: 0.0057813, 00.2115, 0.77078, 0.14903, 0.13567,
0.11328, 0.087906, 0.065531, 0.050516
$y_{j,3}$: −0.055066, 0.27160, 1.3199, 0.18908, 0.14015,
0.12162, 0.10062, 0.083022, 0.068688
$y_{j,4}$: −0.092055, 0.3768, 1.7328, 0.29675, 0.14653,
0.12981, 0.11355, 0.10072, 0.083463
$y_{j,5}$: −0.093987, 0.53745, 1.9712, 0.48652, 0.16125,
0.13803, 0.12669, 0.11814, 0.0941

Section 17.3 Wave Motion Along Infinite and Semi-Infinite Strings

For Problems 1 through 6, the Fourier integral on $-\infty < x < \infty$ yields a solution of the wave equation with specified initial condition $f(x)$, and initial velocity $g(x)$ in the form

$$y(x,t) = \int_0^\infty \{[a_\omega \cos(\omega x) + b_\omega \sin(\omega x)]\cos(c\omega t) + [c_\omega \cos(\omega x) + d_\omega \sin(\omega x)]\sin(c\omega t)\}d\omega$$

where

$$a_\omega = \frac{1}{\pi}\int_{-\infty}^\infty f(\xi)\cos(\omega\xi)d\xi, \quad b_\omega = \frac{1}{\pi}\int_{-\infty}^\infty f(\xi)\sin(\omega\xi)d\xi;$$

and

$$c_\omega = \frac{1}{\pi c\omega}\int_{-\infty}^\infty g(\xi)\cos(\omega\xi)d\xi, \quad d_\omega = \frac{1}{\pi c\omega}\int_{-\infty}^\infty g(\xi)\sin(\omega\xi)d\xi;$$

1. For the Fourier integral solution, calculate

$$a_\omega = \frac{1}{\pi}\int_{-\infty}^\infty e^{-5|\xi|}\cos(\omega\xi)d\xi = \frac{10}{\pi(25+\omega^2)};$$

$$b_\omega = \frac{1}{\pi}\int_{-\infty}^\infty e^{-5|\xi|}\sin(\omega\xi)d\xi = 0;$$

Clearly $c_\omega = d_\omega = 0$, so the solution is

$$y(x,t) = \frac{10}{\pi}\int_0^\infty \left[\frac{1}{25+\omega^2}\right]\cos(\omega x)\cos(12\omega t)d\omega.$$

Solving with the Fourier transform in x, with $\hat{y} = \mathcal{F}[y(x,t); x \to \omega]$, we get the transformed problem (in t)

$$\hat{y}'' + 144\omega^2\hat{y} = 0; \hat{y}(\omega, 0) = \int_{-\infty}^{\infty} e^{-5|x|} e^{-i\omega\xi} d\xi = \frac{10}{25+\omega^2}; \hat{y}'(\omega, 0) = 0.$$

The solution of the transformed problem is $\hat{y}(\omega,t) = \frac{10}{25+\omega^2}\cos(12\omega t)$. Inverting gives the solution

$$y(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{10}{25+\omega^2}\cos(12\omega t) e^{i\omega x} d\omega.$$

2. For the Fourier integral solution, clearly $c_\omega = d_\omega = 0$, so calculate

$$a_\omega = \frac{1}{\pi}\int_0^8 (8-\xi)\cos(\omega\xi)d\xi = \frac{1-\cos(8\omega)}{\pi\omega^2};$$

$$b_\omega = \frac{1}{\pi}\int_0^8 (8-\xi)\sin(\omega\xi)d\xi = \frac{8\omega - \sin(8\omega)}{\pi\omega^2};$$

The solution is

$$y(x,t) = \int_0^\infty \left\{ \left[\frac{1-\cos(8\omega)}{\pi\omega^2}\right]\cos(\omega x) + \left[\frac{8\omega - \sin(8\omega)}{\pi\omega^2}\right]\sin(\omega x) \right\} \cos(8\omega t) d\omega.$$

Solving with the Fourier transform in x, with $\hat{y} = \mathcal{F}[y(x,t); x \to \omega]$, we get the transformed problem (in t)

$$\hat{y}'' + 16\omega^2\hat{y} = 0; \hat{y}(\omega, 0) = \int_0^8 (8-\xi)e^{-i\omega\xi}d\xi = \frac{1 - 8\omega i - e^{-8\omega i}}{\omega^2}; \hat{y}'(\omega, 0) = 0.$$

The solution of the transformed problem is

$$\hat{y}(\omega, t) = \frac{1 - 8\omega i - e^{-8\omega i}}{\omega^2}.$$

Inverting gives the solution

$$y(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1 - 8\omega i - e^{-8\omega i}}{\omega^2} e^{i\omega x} d\omega.$$

3. For the Fourier integral solution, clearly $a_\omega = b_\omega = 0$, so calculate

$$c_\omega = \frac{1}{4\pi\omega}\int_{-\pi}^{\pi} \sin(\xi)\cos(\omega\xi)d\xi = 0;$$

$$d_\omega = \frac{1}{4\pi\omega}\int_{-\pi}^{\pi} \sin(\xi)\sin(\omega\xi)d\xi = -\frac{\sin(\omega\pi)}{2\pi\omega(\omega^2-1)};$$

The solution is

$$y(x,t) = \int_0^\infty \left[-\frac{\sin(\omega\pi)}{2\pi\omega(\omega^2-1)}\right]\sin(\omega x)\sin(4\omega t)d\omega.$$

Solving with the Fourier transform in x, with $\hat{y} = \mathcal{F}[y(x,t); x \to \omega]$, we get the transformed problem (in t)

$$\hat{y}'' + 16\omega^2\hat{y} = 0; \hat{y}(\omega,0) = 0; \hat{y}'(\omega,0) = \int_{-\pi}^{\pi} \sin(\xi)e^{-i\omega\xi} = \frac{2i\sin(\pi\omega)}{(\omega^2-1)}.$$

The solution of the transformed problem is

$$\hat{y}(\omega,t) = \frac{2i\sin(\pi\omega)}{4\omega(\omega^2-1)}\sin(4\omega t).$$

Inverting gives the solution

$$y(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{i\sin(\pi\omega)}{2\omega(\omega^2-1)}\sin(4\omega t)e^{i\omega x}d\omega.$$

4. For the Fourier integral solution, clearly $c_\omega = d_\omega = 0$, so calculate

$$a_\omega = \frac{1}{\pi}\int_{-2}^{2}(2-|\xi|)\cos(\omega\xi)d\xi = \frac{2}{\pi\omega^2}(1-\cos(2\omega));$$

$$b_\omega = \frac{1}{\pi}\int_{-2}^{2}(2-|\xi|)\sin(\omega\xi)d\xi = 0;$$

The solution is

$$y(x,t) = \int_0^\infty \left[\frac{2}{\pi\omega^2}(1-\cos(2\omega))\right]\cos(\omega x)\cos(\omega t)d\omega.$$

Solving with the Fourier transform in x, with $\hat{y} = \mathcal{F}[y(x,t); x \to \omega]$, we get the transformed problem (in t)

$$\hat{y}'' + \omega^2\hat{y} = 0; \hat{y}(\omega,0) = \int_{-2}^{2}(2-|\xi|)e^{-i\omega\xi}d\xi = \frac{2}{\omega^2}(1-\cos(2\omega)); \hat{y}'(\omega,0) = 0.$$

The solution of the transformed problem is $\hat{y}(\omega,t) = \frac{2}{\omega^2}(1-\cos(2\omega))\cos(\omega t)$. Inverting gives the solution

$$y(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2}{\omega^2}(1-\cos(2\omega))\cos(\omega t)e^{i\omega x}d\omega.$$

5. For the Fourier integral solution, clearly $a_\omega = b_\omega = 0$, so calculate

$$c_\omega = \frac{1}{3\pi\omega}\int_1^\infty e^{-2\xi}\cos(\omega\xi)d\xi = \frac{e^{-2}}{3\pi\omega}\frac{(2\cos(\omega) - \omega\sin(\omega))}{4+\omega^2};$$

$$d_\omega = \frac{1}{3\pi\omega}\int_1^\infty e^{-2\xi}\sin(\omega\xi)d\xi = \frac{e^{-2}}{3\pi\omega}\frac{(2\sin(\omega) + \omega\cos(\omega))}{4+\omega^2};$$

The solution is

$$y(x,t) = \int_0^\infty \left\{\left[\frac{e^{-2}}{3\pi\omega}\frac{(2\cos(\omega) - \omega\sin(\omega))}{4+\omega^2}\right]\cos(\omega x) + \left[\frac{e^{-2}}{3\pi\omega}\frac{(2\sin(\omega) + \omega\cos(\omega))}{4+\omega^2}\right]\sin(\omega x)\right\}\sin(3\omega t)d\omega.$$

Solving with the Fourier transform in x, with $\hat{y} = \mathcal{F}[y(x,t); x \to \omega]$, we get the transformed problem (in t)

$$\hat{y}'' + 9\omega^2\hat{y} = 0; \hat{y}(\omega, 0) = 0; \hat{y}'(\omega, 0) = \mathcal{F}\left[e^{-2x}H(x-1)\right] = \frac{(2-i\omega)e^{-(2+i\omega)}}{4+\omega^2}.$$

The solution of the transformed problem is $\hat{y}(\omega,t) = \frac{(2-i\omega)e^{-(2+i\omega)}}{3\omega(4+\omega^2)}\sin(3\omega t)$. Inverting gives the solution

$$y(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{(2-i\omega)e^{-(2+i\omega)}}{3\omega(4+\omega^2)}\sin(3\omega t)e^{i\omega x}d\omega.$$

6. For the Fourier integral solution, clearly $a_\omega = b_\omega = 0$, so calculate

$$c_\omega = \frac{1}{2\pi\omega}\int_{-2}^{2} \operatorname{sgn}(\xi)\cos(\omega\xi)d\xi = 0;$$

$$d_\omega = \frac{1}{2\pi\omega}\int_{-2}^{2} \operatorname{sgn}(\xi)\sin(\omega\xi)d\xi = \frac{1-\cos(2\omega)}{\pi\omega^2};$$

The solution is

$$y(x,t) = \int_0^\infty \left[\frac{1-\cos(2\omega)}{\pi\omega^2}\right]\sin(\omega x)\sin(2\omega t)d\omega.$$

Solving with the Fourier transform in x, with $\hat{y} = \mathcal{F}[y(x,t); x \to \omega]$, we get the transformed problem (in t)

$$\hat{y}'' + 4\omega^2\hat{y} = 0; \hat{y}(\omega, 0) = 0; \hat{y}'(\omega, 0) = \int_{-2}^{2} \operatorname{sgn}(\xi)e^{-i\omega\xi}d\xi = \frac{2(1-\cos(2\omega))}{\omega}.$$

The solution of the transformed problem is $\hat{y}(\omega,t) = \frac{(1-\cos(2\omega))}{\omega^2}\sin(2\omega t)$. Inverting gives the solution

$$y(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{(1-\cos(2\omega))}{\omega^2}\sin(2\omega t)e^{i\omega x}d\omega.$$

For Problems 7 through 11, separation of variables and the requirement of a solution bounded for all $x > 0$ and $t > 0$ yields a solution of the boundary value problem $\frac{\partial^2 y}{\partial^2} = c^2\frac{\partial^2 y}{\partial x^2}(x > 0, t > 0), y(0,t) = 0 (t > 0), y(x,0) = f(x), \frac{\partial y}{\partial t}(x,0) = g(x)(x > 0)$ having the form $y(x,t) = \int_0^\infty \sin(\omega x)[a_\omega\cos(\alpha\omega t) + b_\omega\sin(\alpha\omega t)]d\omega$. The initial conditions $y(x,0) = f(x) = \int_0^\infty a_\omega\sin(\omega x)d\omega$ and $\frac{\partial y}{\partial t}(x,0) = g(x) = \int_0^\infty c\omega b_\omega\sin(\omega x)dx$ are satisfied by choosing

$$a_\omega = \frac{2}{\pi}\int_0^\infty f(\xi)\sin(\omega\xi)d\xi \text{ and } b_\omega = \frac{2}{\pi c\omega}\int_0^\infty g(\xi)\sin(\omega\xi)d\xi.$$

7. To solve with the Fourier sine integral, calculate

$$a_\omega = \frac{2}{\pi}\int_0^1 \xi(1-\xi)\sin(\omega\xi)d\xi = \frac{2}{\pi}\left[\frac{2}{\omega^3}(1-\cos(\omega)) - \frac{\sin(\omega)}{\omega^2}\right]$$

and

$$b_\omega = 0$$

to get the solution

$$y(x,t) = \frac{2}{\pi}\int_0^\infty \left\{\frac{2(1-\cos(\omega)) - \omega\sin(\omega)}{\omega^3}\right\}\sin(\omega x)\cos(3\omega t)d\omega.$$

Solving with the Fourier Sine transform in x, with $\hat{y}_s(\omega,t) = \mathcal{F}_s[y(x,t); x \to \omega]$, we get the transformed problem (in t)

$$\hat{y}_s'' + 9\omega^2\hat{y} = 0; \hat{y}_s(\omega,0) = \int_0^1 \xi(1-\xi)\sin(\omega\xi)d\xi = \left\{\frac{2(1-\cos(\omega)) - \omega\sin(\omega)}{\omega^3}\right\}; \hat{y}_s'(\omega,0) = 0;$$

The solution of the transformed problem is

$$\hat{y}(\omega,t) = \left\{\frac{2(1-\cos(\omega)) - \omega\sin(\omega)}{\omega^3}\right\}\cos(3\omega t)$$

and by inverting we get the solution

$$y(x,t) = \frac{2}{\pi}\int_0^\infty \left\{\frac{2(1-\cos(\omega)) - \omega\sin(\omega)}{\omega^3}\right\}\sin(\omega x)\cos(3\omega t)d\omega.$$

8. To solve with the Fourier sine integral, calculate $a_\omega = 0$; and

$$b_\omega = \frac{2}{3\pi\omega}\int_4^{11} 2\sin(\omega x)dx = \frac{4(\cos(4\omega) - \cos(11\omega))}{3\pi\omega^2};$$

to get the solution

$$y(x,t) = \frac{4}{3\pi}\int_0^\infty \frac{(\cos(4\omega) - \cos(11\omega))}{\omega^2}\sin(\omega x)\sin(3\omega t)d\omega.$$

Solving with the Fourier Sine transform in x, with $\hat{y}_s = \mathcal{F}_s[y(x,t); x \to \omega]$, we get the transformed problem (in t)

$$\hat{y}_s'' + 9\omega^2\hat{y} = 0; \hat{y}_s(\omega,0) = 0; \hat{y}_s'(\omega,0) = \int_4^{11} 2\sin(\omega\xi)d\xi = \frac{2(\cos(4\omega) - \cos(11\omega))}{\omega};$$

The solution of the transformed problem is

$$\hat{y}(\omega,t) = \frac{2(\cos(4\omega) - \cos(11\omega))}{3\omega^2}\sin(3\omega t)$$

and by inverting we get the solution

$$y(x,t) = \frac{4}{3\pi}\int_0^\infty \frac{(\cos(4\omega) - \cos(11\omega))}{\omega^2}\sin(\omega x)\sin(3\omega t)d\omega.$$

9. To solve with the Fourier sine integral, calculate $a_\omega = 0$; and

$$b_\omega = \frac{2}{2\pi\omega}\int_{\pi/2}^{5\pi/2} \cos(\xi)\sin(\omega\xi)d\xi = \frac{\sin(\omega\pi/2) - \sin(5\omega\pi/2)}{\pi\omega(\omega^2 - 1)}$$

to get the solution

$$y(x,t)=\int_0^\infty \frac{\sin(\omega\pi/2)-\sin(5\omega\pi/2)}{\pi\omega(\omega^2-1)}\sin(\omega x)\sin(2\omega t)d\omega.$$

Solving with the Fourier Sine transform in x, with $\hat{y}_s=\mathcal{F}_s[y(x,t);x\to\omega]$, we get the transformed problem (in t)

$$\hat{y}_s''+4\omega^2\hat{y}=0;\hat{y}_s(\omega,0)=0;\hat{y}_s'(\omega,0)=\int_{\pi/2}^{5\pi/2}\cos(\xi)\sin(\omega\xi)d\xi=\frac{\sin(\omega\pi/2)-\sin(5\omega\pi/2)}{\omega^2-1};$$

The solution of the transformed problem is

$$\hat{y}(\omega,t)=\frac{\sin(\omega\pi/2)-\sin(5\omega\pi/2)}{2\omega(\omega^2-1)}\sin(2\omega t)$$

and by inverting we get the solution

$$y(x,t)=\frac{2}{\pi}\int_0^\infty \frac{\sin(\omega\pi/2)-\sin(5\omega\pi/2)}{2\omega(\omega^2-1)}\sin(\omega x)\sin(2\omega t)d\omega.$$

10. To solve with the Fourier sine integral, clearly $b_\omega=0$, so calculate

$$a_\omega=\frac{2}{\pi}\int_0^\infty -2e^{-\xi}\sin(\omega\xi)d\xi=-\frac{4\omega}{\pi(1+\omega^2)};$$

to get the solution

$$y(x,t)=-\frac{4}{\pi}\int_0^\infty \frac{\omega}{1+\omega^2}\sin(\omega x)\cos(6\omega t)d\omega.$$

Solving with the Fourier Sine transform in x, with $\hat{y}_s=\mathcal{F}_s[y(x,t);x\to\omega]$, we get the transformed problem (in t)

$$\hat{y}_s''+36\omega^2\hat{y}=0;\hat{y}_s(\omega,0)=\int_0^\infty -2e^{-\xi}\sin(\omega\xi)d\xi=-\frac{2\omega}{1+\omega^2};\hat{y}_s'(\omega,0)=0;$$

The solution of the transformed problem is

$$\hat{y}(\omega,t)=-\frac{2\omega}{1+\omega^2}\cos(6\omega t)$$

and by inverting we get the solution

$$y(x,t)=-\frac{4}{\pi}\int_0^\infty \frac{\omega}{1+\omega^2}\sin(\omega x)\cos(6\omega t)d\omega.$$

11. To solve with the Fourier sine integral, calculate $a_\omega=0$; and

$$b_\omega=\frac{2}{14\pi\omega}\int_0^3 \xi^2(3-\xi)\sin(\omega\xi)d\xi=\frac{3}{7\pi\omega^5}(2\sin(3\omega)-4\omega\cos(3\omega)-3\omega^2\sin(3\omega)-2\omega);$$

to get the solution

$$y(x,t)=\int_0^\infty \frac{3}{7\pi\omega^5}(2\sin(3\omega)-4\omega\cos(3\omega)-3\omega^2\sin(3\omega)-2\omega)\sin(\omega x)\sin(14\omega t)d\omega.$$

Solving with the Fourier Sine transform in x, with $\hat{y}_s = \mathcal{F}_s[y(x,t); x \to \omega]$, we get the transformed problem (in t)

$$\hat{y}_s'' + 196\omega^2\hat{y} = 0; \hat{y}_s(\omega,0) = 0; \hat{y}_s'(\omega,0) = \int_0^3 \xi^2(3-\xi)\sin(\omega\xi)d\xi$$

$$= \frac{3}{\omega^4}(2\sin(3\omega) - 4\omega\cos(3\omega) - 3\omega^2\sin(3\omega) - 2\omega);$$

The solution of the transformed problem is

$$\hat{y}(\omega,t) = \frac{3}{14\omega^5}(2\sin(3\omega) - 4\omega\cos(3\omega) - 3\omega^2\sin(3\omega) - 2\omega)\sin(3\omega t)$$

and by inverting we get the solution

$$y(x,t) = \frac{2}{\pi}\int_0^\infty \frac{3}{14\omega^5}(2\sin(3\omega) - 4\omega\cos(3\omega) - 3\omega^2\sin(3\omega) - 2\omega)\sin(\omega x)\sin(3\omega t)d\omega.$$

12. Before transforming, write the boundary condition at $x = 0$ as $y(0,t) = \sin(2\pi t)[1 - H(t-1)] = \sin(2\pi t) - \sin(2\pi(t-1))H(t-1)$. Transform the equation and boundary conditions to get $s^2Y = c^2Y''$, since $Y(x,0) = Y'(x,0) = 0$, and $Y(0,s) = \dfrac{2\pi}{s^2+4\pi^2}[1-e^{-s}]$. The solution is $Y(x,s) = c_1(s)e^{sx/c} + c_2(s)e^{-sx/c}$. For bounded solutions choose $c_1(s) = 0$, then

$$Y(0,s) = \frac{2\pi}{s^2+4\pi^2}[1-e^{-s}] = c_2(s), \text{ and } Y(x,s) = \frac{2\pi}{s^2+4\pi^2}[e^{-sx/c} - e^{-s(1+x/c)}].$$

Invert to get

$$y(x,t) = \sin\left[2\pi\left(t-\frac{x}{c}\right)\right]H\left(t-\frac{x}{c}\right) - \sin\left[2\pi\left(t-\frac{x}{c}-1\right)\right]H\left(t-1-\frac{x}{c}\right),$$

or

$$y(x,t) = \begin{cases} \sin\left[2\pi\left(t-\frac{x}{c}\right)\right], & \text{if } \frac{x}{c} < t < 1+\frac{x}{c} \\ 0, & \text{otherwise} \end{cases}$$

13. Transform the differential equation and the boundary condition using the Laplace transform in the t variable with $Y(x,s) = \mathcal{L}[y(x,t)]$ to get

$$s^2Y(x,s) - sy(x,0) - y_t(x,0) = c^2Y''(x,s), \text{ and } Y(0,s) = \frac{1}{s^2}, y(x,0) = 0, y_t(x,0) = A.$$

The ordinary differential equation for $Y(x,s)$ is $c^2Y''(x,s) - s^2Y(x,s) = -A$, which has general solution

$$Y(x,s) = c_1(s)e^{sx/c} + c_2(s)e^{-sx/c} + \frac{A}{s^2}.$$

The requirement that solutions remain bounded as $s \to \infty$ dictates $c_1(s) = 0$, then $Y(0,s) = \dfrac{1}{s^2} = c_2(s) + \dfrac{A}{s^2}$ gives $c_2(s) = \dfrac{1-A}{s^2}$. Now invert $Y(x,s) = (1-A)\dfrac{e^{-sx/c}}{s^2} + \dfrac{A}{s^2}$ to get solution

$$y(x,t) = (1-A)\left(t-\frac{x}{c}\right)H\left(t-\frac{x}{c}\right) + At = \begin{cases} At, & \text{if } t < x/c \\ t + (A-1)\frac{x}{c}, & \text{if } t > x/c \end{cases}$$

Section 17.4 Characteristics and d'Alembert's Solution

1. With $c = 1$ the characteristics are $x - t = k_1, x + t = k_2$. Solutions of the wave equation from d'Alembert's formula are

$$u(x,t) = x^2 - xt + t^2.$$

2. With $c = 4$ the characteristics are $x - 4t = k_1, x + 4t = k_2$. Solutions of the wave equation from d'Alembert's formula are

$$u(x,t) = x^2 + 16t^2 - 2x + \frac{1}{8}\{\sin(x+4t) - \sin(x-4t)\}$$

$$= x^2 + 16t^2 - 2x + \frac{1}{4}\cos(x)\sin(4t).$$

3. With $c = 7$ the characteristics are $x - 7t = k_1, x + 7t = k_2$. Solutions of the wave equation from d'Alembert's formula are

$$u(x,t) = \frac{1}{2}\{\cos(\pi(x-7t)) + \cos(\pi(x+7t))\} + t - x^2t - \frac{49}{3}t^3 = \frac{1}{2}\cos(\pi x)\cos(7\pi t) + t - x^2t - \frac{49}{3}t^3.$$

4. With $c = 5$ the characteristics are $x - 5t = k_1, x + 5t = k_2$. Solutions of the wave equation from d'Alembert's formula are

$$u(x,t) = \frac{1}{2}\{\sin(2(x-5t)) + \sin(2(x+5t))\} + x^3t + 25xt^3 = \sin(2x)\cos(10t) + x^3t + 25xt^3.$$

5. With $c = 14$ the characteristics are $x - 14t = k_1, x + 14t = k_2$. Solutions of the wave equation from d'Alembert's formula are

$$u(x,t) = \frac{1}{2}\{e^{x-14t} + e^{x+14t}\} + xt = e^x\cosh(14t) + xt.$$

6. With $c = 12$ the characteristics are $x - 12t = k_1, x + 12t = k_2$. Solutions of the wave equation from d'Alembert's formula are

$$u(x,t) = x^2 + 144t^2 - 5x + 3t.$$

7. Characteristics are $x - 4t = k_1, x + 4t = k_2$. Solutions are

$$u(x,t) = \frac{1}{2}[f(x-4t) + f(x+4t)] + \frac{1}{8}\int_{x-4t}^{x+4t} e^{-\xi}d\xi + \frac{1}{8}\int_0^t\int_{x-4t+4\eta}^{x+4t-4\eta}(\xi+\eta)d\xi d\eta$$

$$= x + \frac{1}{4}e^{-x}\sinh(4t) + \frac{1}{2}xt^2 + \frac{1}{6}t^3$$

8. Characteristics are $x - 2t = k_1, x + 2t = k_2$. Solutions are

$$u(x,t) = \frac{1}{2}[f(x-2t) + f(x+2t)] + \frac{1}{4}\int_{x-2t}^{x+2t} 2\xi d\xi + \frac{1}{4}\int_0^t\int_{x-2t+2\eta}^{x+2t-2\eta}(2\xi\eta)d\xi d\eta$$

$$= \frac{1}{2}[\sin(x-2t) + \sin(x+2t)] + 2xt + \frac{1}{3}xt^3$$

9. Characteristics are $x - 8t = k_1, x + 8t = k_2$. Solutions are

$$u(x,t) = \frac{1}{2}[f(x-8t) + f(x+8t)] + \frac{1}{16}\int_{x-8t}^{x+8t} \cos(2\xi)d\xi + \frac{1}{16}\int_0^t \int_{x-8t+8\eta}^{x+8t-8\eta} (\eta\cos(\xi))d\xi d\eta$$
$$= x^2 + 64t^2 - x + \frac{1}{32}[\sin(2(x+8t)) - \sin(2(x-8t))] + \frac{1}{12}xt^4$$

10. Characteristics are $x - 4t = k_1, x + 4t = k_2$. Solutions are

$$u(x,t) = \frac{1}{2}[f(x-4t) + f(x+4t)] + \frac{1}{8}\int_{x-4t}^{x+4t} \xi e^{-\xi} d\xi + \frac{1}{8}\int_0^t \int_{x-4t+4\eta}^{x+4t-4\eta} (\xi\sin(\eta))d\xi d\eta$$
$$= x^2 + 16t^2 + \frac{1}{2}xe^{-x}\sinh(4t) + \frac{1}{4}e^{-x}\sinh(4t) - te^{-x}\cosh(4t) - x\sin(t) + xt$$

11. Characteristics are $x - 3t = k_1, x + 3t = k_2$. Solutions are

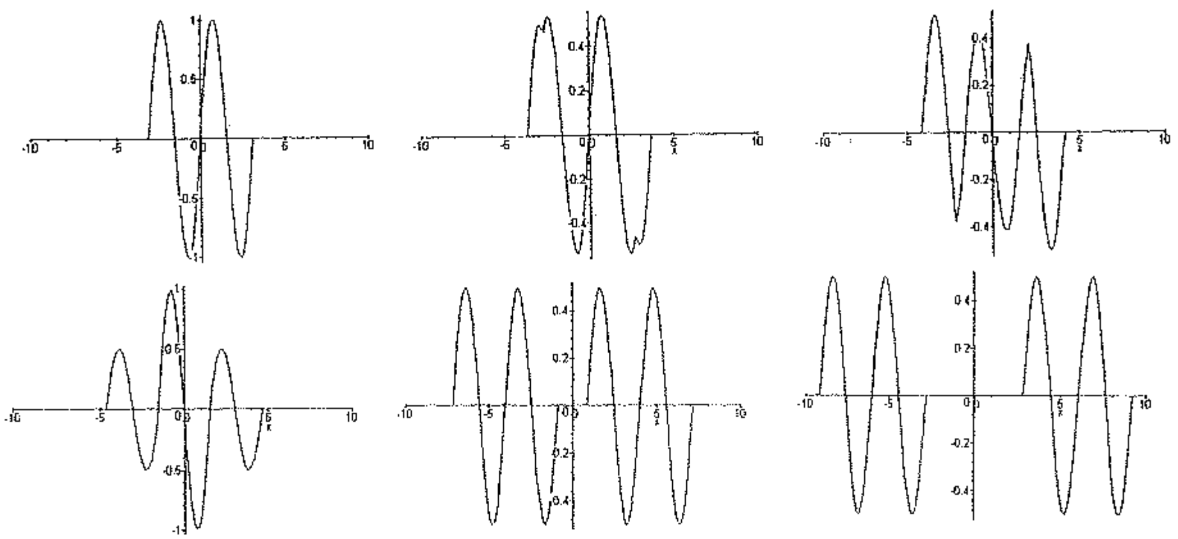
$$u(x,t) = \frac{1}{2}[f(x-3t) + f(x+3t)] + \frac{1}{6}\int_{x-3t}^{x+3t} d\xi + \frac{1}{6}\int_0^t \int_{x-3t+3\eta}^{x+3t-3\eta} (\xi\sin(\eta))d\xi d\eta$$
$$= \frac{1}{2}[\cosh(x-3t) + \cosh(x+3t)] + t + \frac{1}{4}xt^4$$

12. Characteristics are $x - 7t = k_1, x + 7t = k_2$. Solutions are

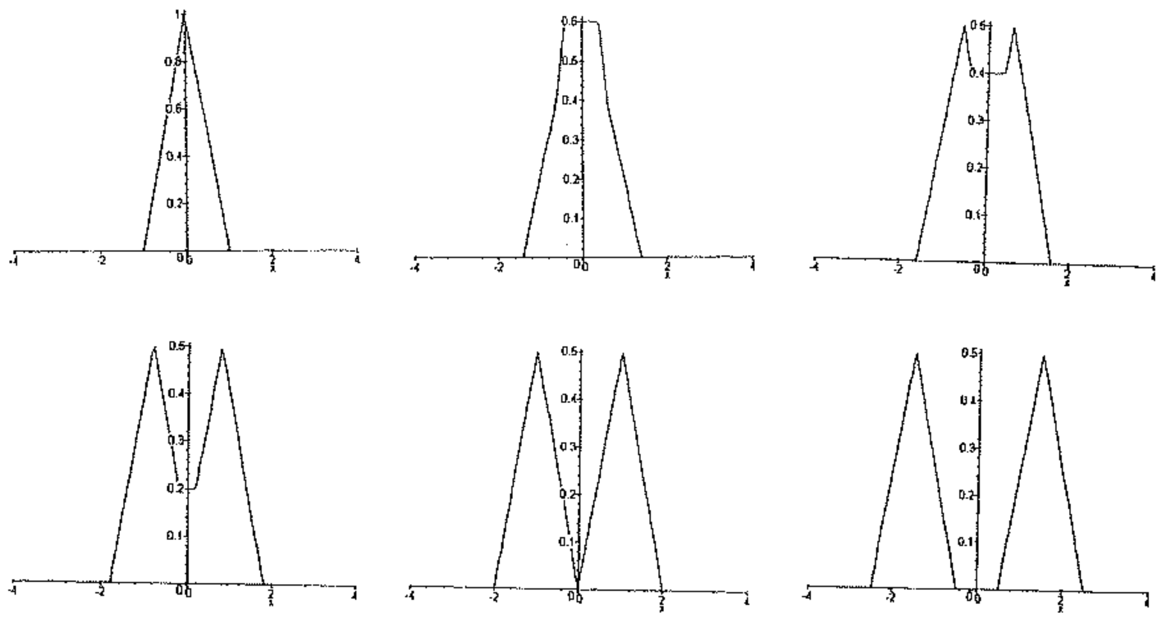
$$u(x,t) = \frac{1}{2}[f(x-7t) + f(x+7t)] + \frac{1}{14}\int_{x-7t}^{x+7t} \sin(\xi)d\xi + \frac{1}{14}\int_0^t \int_{x-7t+7\eta}^{x+7t-7\eta} (\xi - \cos(\eta))d\xi d\eta$$
$$= x - \frac{1}{14}[\cos(x-7t) - \cos(x+7t)] + \frac{1}{2}xt^2 + \cos(t)$$

For Problems 13 through 18, the graph of the initial displacement is shown as $u(x,0) = f(x)$, and displacement curves are shown as an average of a forward wave $f(x-t)$, and a backward wave $f(x+t)$, at the stated value of t.

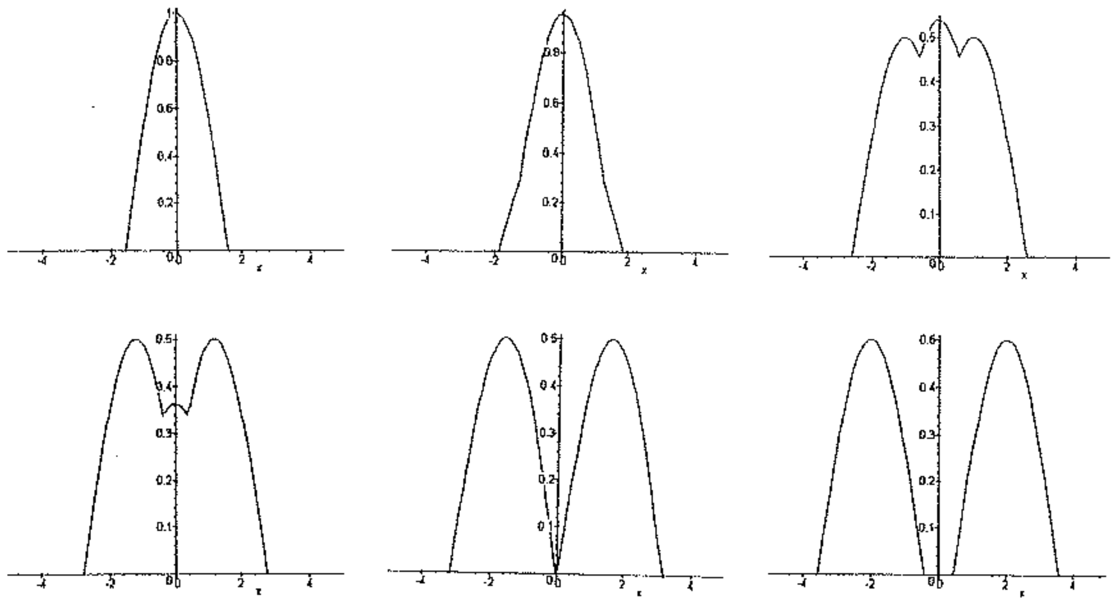
13. Forward and Backward waves at times $t = 0, .5, 1, 1.5, 4, 6$ seconds



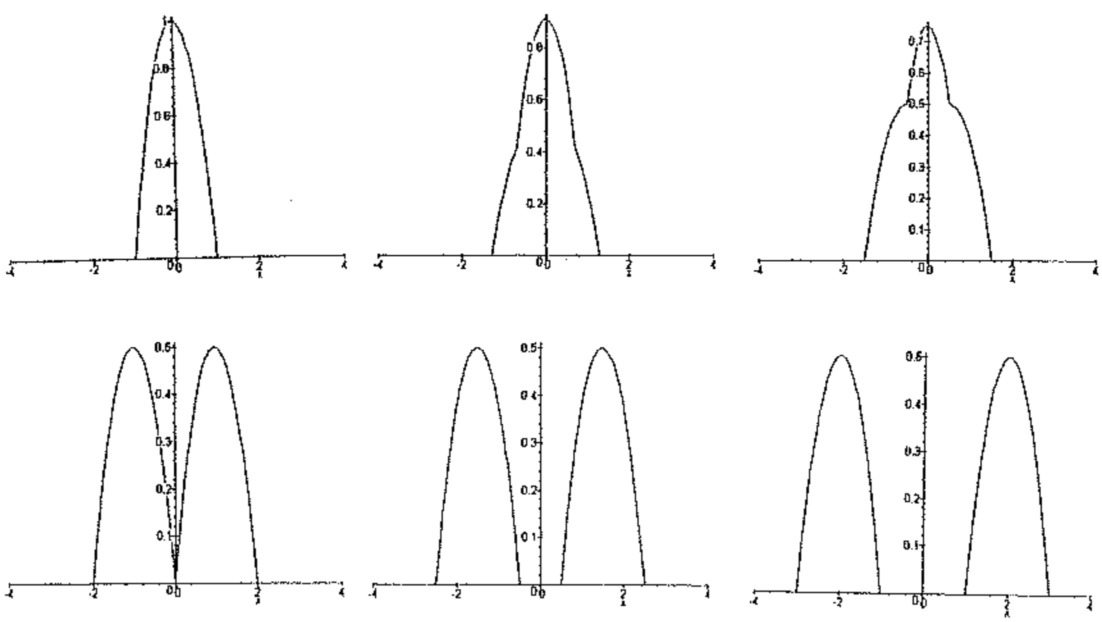
14. Forward and Backward waves at times $t = 0, .4, .6, .8, , 1, 1.5$ seconds



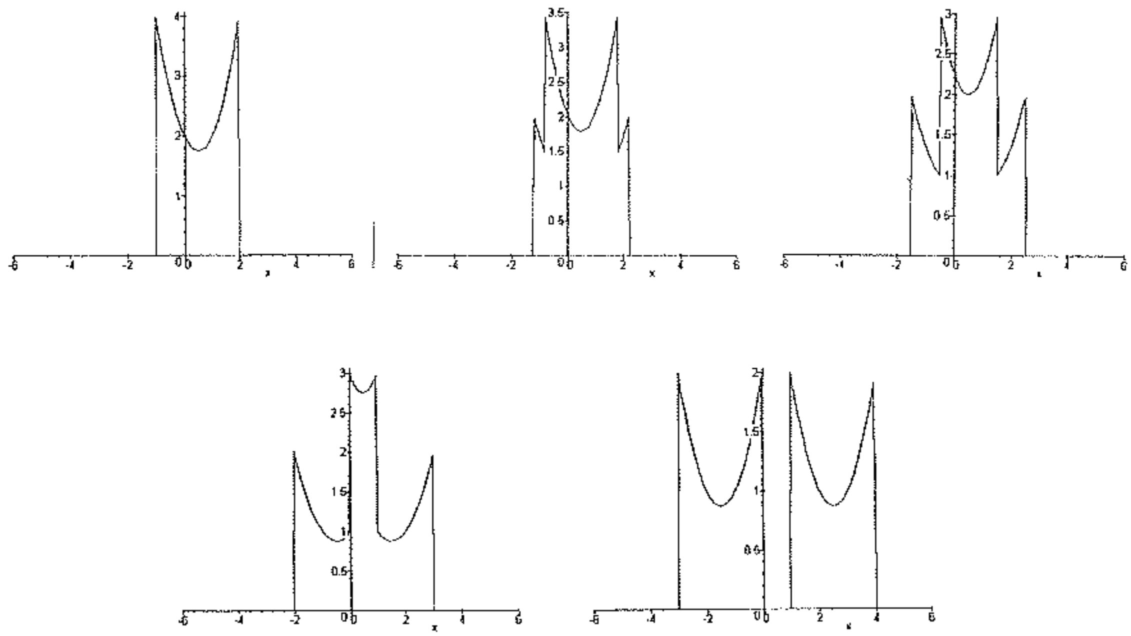
15. Forward and Backward waves at times $t = 0, .3, 1, 1.2, 1.5, 2$ seconds



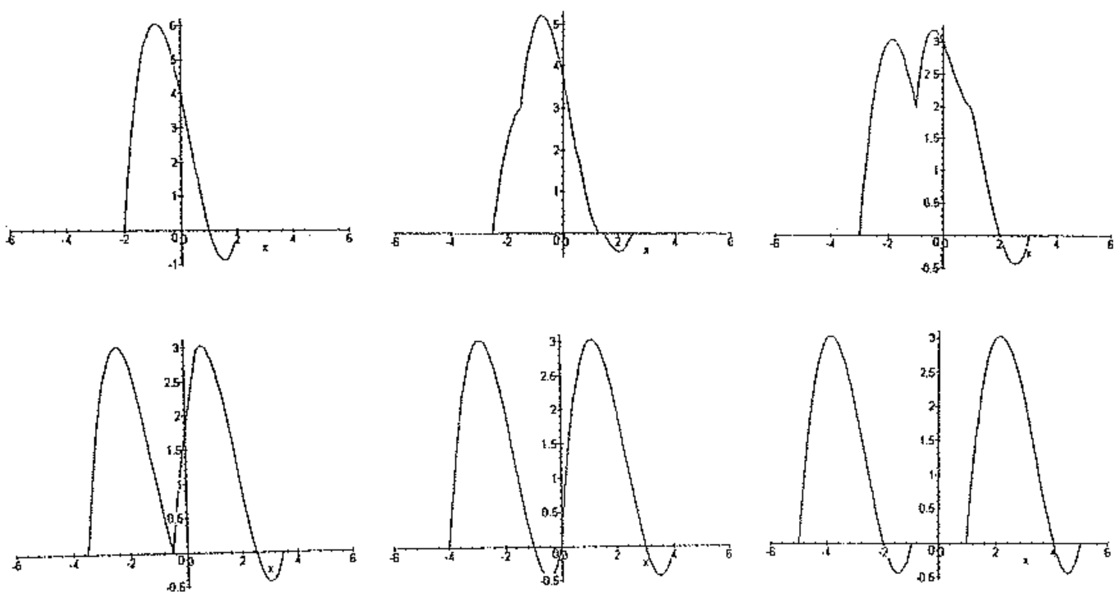
16. Forward and Backward waves at times $t = 0, .3, .5, 1, 1.5, 2$ seconds



17. Forward and Backward waves at times $t = 0, .2, .5, 1, 2$ seconds



18. Forward and Backward waves at times $t = 0, .5, 1, 1.5, 2, 3$ seconds



Section 17.5 Normal Modes of Vibration of a Circular Elastic Membrane

The form of the solution is $z(r,t) = \sum_{n=1}^{\infty} a_n J_0(j_n r)\cos(j_n t)$ where j_n is the n^{th} zero of J_0. For a given initial displacement $f(r)$ the coefficients are given by

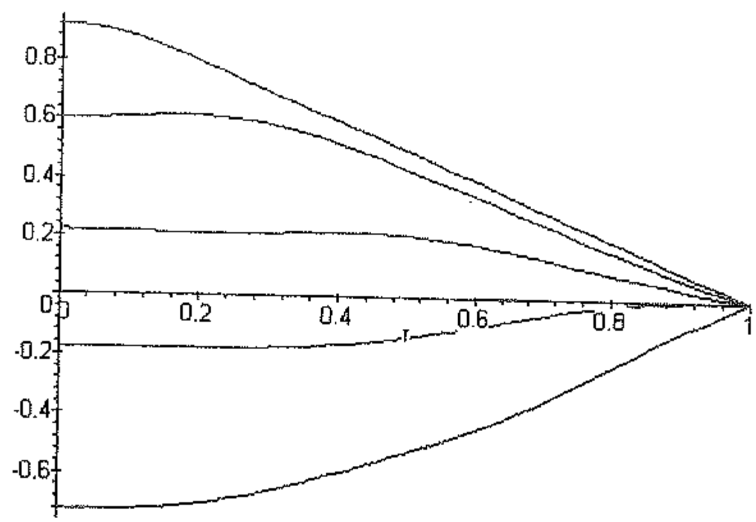
$$a_n = \frac{2}{[J_1(j_n)]^2} \int_0^1 s f(s) J_0(j_n s) ds; n \geq 1.$$

For each of these solutions, the coefficients will be computed using Maple with the results listed below. The graph of the fifth partial sum is displayed for the stated values of t.

1. For $f(r) = 1 - r$ the computed coefficients rounded to five decimal places are:

$a_1 = .78452, a_2 = .06869, a_3 = .05311, a_4 = .01736, a_5 = .01698$

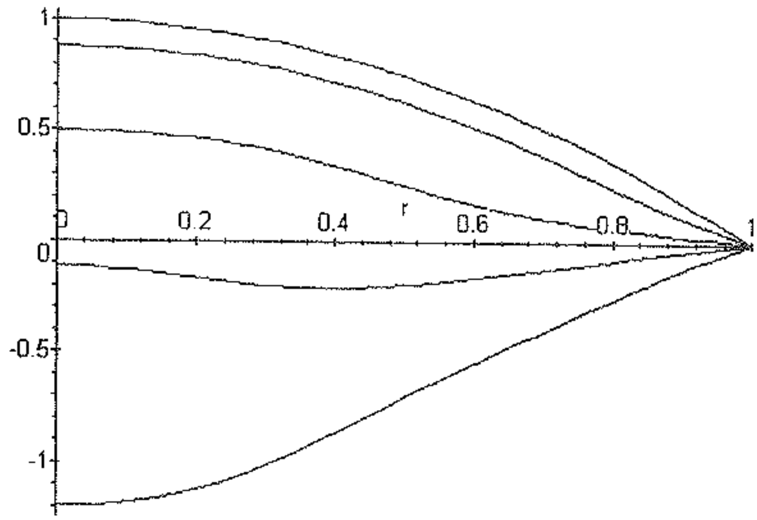
The graph below shows the displacement at times $t = .05, .25, .5, .75, 1.25$



2. For $f(r) = 1 - r^2$ the computed coefficients rounded to five decimal places are:

$a_1 = 1.10802, a_2 = -.13978, a_3 = .04548, a_4 = -.02099, a_5 = .011637$

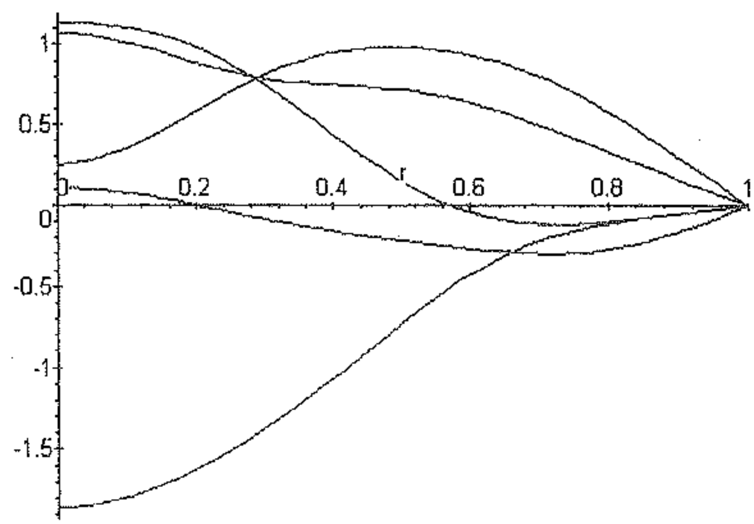
The graph below shows the displacement at times $t = .05, .25, .5, .75, 1.25$



3. For $f(r) = \sin(\pi r)$ the computed coefficients rounded to five decimal places are:

$a_1 = 1.25335, a_2 = -.80469, a_3 = -.11615, a_4 = -.09814, a_5 = -.03740$

The graph below shows the displacement at times $t = .05, .25, .5, .75, 1.25$



Section 17.6 Vibrations of a Circular Elastic Membrane, Revisited

1. Following the solution in the text, and including the general solution for the temporal component of the solution $T_{nk}(t)$, we can show that the general solution for the vibration of a circular membrane fixed at the edge $R = 2$, with prescribed initial displacement $f(r, \theta)$, and prescribed initial velocity $g(r, \theta)$, and wave number $c = 2$ is given by the function

$$z(r,\theta,t) = \sum_{n=0}^{\infty}\sum_{k=1}^{\infty}[a_{nk}\cos(n\theta) + b_{nk}\sin(n\theta)]J_n\left(\frac{j_{nk}}{2}r\right)\cos(j_{nk}t)$$

$$+\sum_{p=0}^{\infty}\sum_{q=1}^{\infty}[c_{pq}\cos(p\theta) + d_{pq}\sin(p\theta)]J_p\left(\frac{j_{pq}}{2}r\right)\sin(j_{pq}t).$$

Evaluation of this solution at $t = 0$ gives the conditions that the coefficients a_{nk}, b_{nk} be chosen so that

$$z(r,\theta,0) = f(r,\theta) = (4-r^2)\frac{1-\cos(2\theta)}{2} = \sum_{n=0}^{\infty}\sum_{k=1}^{\infty}[a_{nk}\cos(n\theta) + b_{nk}\sin(n\theta)]J_n\left(\frac{j_{nk}}{2}r\right).$$

For the initial velocity condition we choose coefficients c_{pq}, d_{pq} so that

$$v_t(r,\theta,0) = g(r,\theta) = \theta = \sum_{p=0}^{\infty}\sum_{q=1}^{\infty}[c_{pq}j_{pq}\cos(p\theta) + d_{pq}j_{pq}\sin(p\theta)]J_p\left(\frac{j_{pq}}{2}r\right).$$

We can exploit the simple nature of the θ dependence in $f(r, \theta)$ (orthogonality in disguise) to conclude, by matching coefficients of the $\cos(n\theta)$ terms for $n = 0$ and $n = 2$, that

$$\frac{(4-r^2)}{2} = \frac{\alpha_0(r)}{2} = \sum_{k=1}^{\infty}a_{0k}J_0\left(\frac{j_{0k}}{2}r\right) \quad\text{and}\quad -\frac{(4-r^2)}{2} = \alpha_2(r) = \sum_{k=1}^{\infty}a_{2k}J_2\left(\frac{j_{2k}}{2}r\right).$$

Furthermore $\alpha_n(r) = 0,\ n \neq 2; \beta_n(r) = 0,\ n \geq 0$; from which it follows that $a_{nk} = 0, n \neq 0, n \neq 2, k \geq 1; b_{nk} = 0, n \geq 0, k \geq 1$.
Finally using the orthogonality of the Bessel function $\left\{J_0\left(\frac{j_{0k}}{2}r\right)\right\}_{k=1}^{\infty}$ and $\left\{J_2\left(\frac{j_{0k}}{2}r\right)\right\}_{k=1}^{\infty}$ we can calculate the coefficients

$$a_{0k} = \frac{2}{[J_1(j_{0k})]^2}\int_0^1 \xi(1-\xi^2)J_0(j_{0k}\xi)d\xi,\ k \geq 1;$$

and

$$a_{2k} = \frac{4}{[J_3(j_{2k})]^2}\int_0^1 \xi(\xi^2-1)J_2(j_{2k}\xi)d\xi,\ k \geq 1.$$

For the coefficients c_{pq}, d_{pq} associated with the velocity series of the solution note that

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\theta\cos(p\theta)d\theta = 0, p \geq 0; \frac{1}{\pi}\int_{-\pi}^{\pi}\theta\sin(p\theta)d\theta = \frac{2}{p}(-1)^{p+1}, p \geq 1.$$

It follows that $c_{pq} = 0, p \geq 0, q \geq 1$; and

$$\sum_{q=1}^{\infty}d_{pq}j_{pq}J_p\left(\frac{j_{pq}}{2}r\right) = \frac{2}{p}(-1)^{p+1} = \beta_p(r), p \geq 1;$$

and the coefficients d_{pq} are given by the formulas

$$d_{pq} = \frac{4(-1)^{p+1}}{pj_{pq}[J_{p+1}(j_{pq})]^2} \int_0^1 \xi J_p(j_{pq}\xi)d\xi, p \geq 1, q \geq 1.$$

With these formulas for the coefficients, the complete solution can be written

$$z(r,\theta,t) = \sum_{k=1}^{\infty} \left(\frac{2}{[J_1(j_{0k})]^2} \int_0^1 \xi(1-\xi^2) J_0(j_{0k}\xi)d\xi \right) J_0\left(\frac{j_{0k}}{2}r\right) \cos(j_{0k}t)$$

$$+\cos(2\theta) \sum_{k=1}^{\infty} \left(\frac{4}{[J_3(j_{2k})]^2} \int_0^1 \xi(\xi^2-1) J_2(j_{2k}\xi)d\xi \right) J_2\left(\frac{j_{2k}}{2}r\right) \cos(j_{2k}t)$$

$$+\sum_{p=1}^{\infty} \sin(p\theta) \sum_{q=1}^{\infty} \left(\frac{4(-1)^{p+1}}{pj_{pq}[J_{p+1}(j_{pq})]^2} \int_0^1 \xi J_p(j_{pq}\xi)d\xi \right) J_p\left(\frac{j_{pq}}{2}r\right) \sin(j_{pq}t).$$

Since the Problem requests five terms independent of θ and four term having a θ dependence, we calculate approximate values of the coefficients $a_{0k}, k = 1,2,3,4,5$; $a_{2k}, k = 1,2,3,4$; $d_{1q}, q = 1,2,3,4$. The following table of values was computed to nine places using Maple and rounded to six places in the table.

Table of Zeros of Bessel Functions and Coefficients

k	j_{0k}	a_{0k}	j_{2k}	a_{2k}	j_{1k}	d_{1k}
1	2.404826	1.108022	5.135622	-2.976777	3.831706	1.155175
2	5.520078	-.139778	8.417244	-1.434294	7.015587	-.147414
3	8.653728	.045476	11.619841	-1.140494	10.173468	.217155
4	11.791534	-.020991	14.795952	-.832713	13.323692	-.068294
5	14.930918	.011636	17.959820	-.718490	16.470630	-.098518

Using the values from the previous table we can write the requested number of terms of the solution to get

$z(r,\theta,t) \approx \{1.108022 J_0(1.202413r)\cos(2.404826t) - .139778 J_0(2.760039r)\cos(5.520078t) + .045476 J_0(4.326864r)\cos(8.653728t) - .020991 J_0(5.895767r)\cos(11.791534t) + .011636 J_0(7.465459r)\cos(14.930918t) + \cdots\}$

$+\cos(2\theta)\{-2.976777 J_2(2.567811r)\cos(5.135622t) - 1.434294 J_2(4.208622r)\cos(8.417244t) - 1.140494 J_2(5.809921r)\cos(11.619841t) - .832713 J_2(7.397976r)\cos(14.795952t)\cdots\}$

$+\sin(\theta)\{1.155175 J_1(1.915853r)\sin(3.831706t) - .147414 J_1(3.507794r)\sin(7.015587t) + .217155 J_1(5.086734r)\sin(10.173468t) - .068294 J_1(6.661846r)\sin(13.323692t) + \cdots\}$

2.Evaluate the solution at $r = 0$ to get $z(0,\theta,t) = \sum_{n=0}^{\infty}\sum_{k=1}^{\infty}[a_{nk}\cos(n\theta) + b_{nk}\sin(n\theta)]$ $J_n(0)\cos\left(\frac{z_{nk}}{R}at\right)$, which we would like to show is zero for all $t \geq 0$. Now $J_n(0) = 0$ for all $n \geq 1$, and $J_0(0) = 1$ so this reduces to showing $z(0,\theta,t) = \sum_{k=1}^{\infty} a_{0k}\cos\left(\frac{z_{0k}}{R}at\right) = 0$ for all

$t \geq 0$. From the formulas for the coefficients we have

$$a_{0k} = \frac{1}{2\pi \int_0^R rJ_0^2(\frac{z_{0k}}{R}r)}dr \int_0^R rJ_0\left(\frac{z_{ok}}{R}r\right)\int_{-\pi}^{\pi} f(r,\theta)d\theta dr.$$

Since $f(r,\theta)$ is odd in θ, the integral $\int_{-\pi}^{\pi} f(r,\theta)d\theta = 0$, hence $a_{0k} = 0$, all $k \geq 1$, and it follows that $z(0,\theta,t) = 0$ for all t, as was to be shown.

Section 17.7 Vibrations of a Rectangular Membrane

1. Assume $z(x,y,t) = X(x)Y(y)T(t)$ to get $\frac{T''}{T} - \frac{X''}{X} = \frac{Y''}{Y} = -\alpha$ and then separate again to get $\frac{T''}{T} + \alpha = \frac{X''}{X} = -\lambda$. This gives the three separated boundary value problems

$$\begin{aligned}&X'' + \lambda X = 0, X(0) = X(2\pi) = 0,\\&Y'' + \alpha Y = 0, Y(0) = Y(2\pi) = 0,\\&T'' + (\alpha+\lambda)T = 0, T'(0) = 0.\end{aligned}$$

The eigenvalues and eigenfunctions are respectively $\lambda = n^2/4, X_n(x) = \sin\left(\frac{nx}{2}\right), \alpha = m^2/4, Y_m(y) = \sin\left(\frac{my}{2}\right), T_{nm}(t) = \cos(t\sqrt{n^2+m^2}/2)$ and the solution is of the form

$$z(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{nm}\sin\left(\frac{nx}{2}\right)\sin\left(\frac{my}{2}\right)\cos(t\sqrt{n^2+m^2}/2).$$

The coefficients are computed from

$$a_{nm} = \frac{1}{\pi^2}\int_0^{2\pi} x^2\sin\left(\frac{nx}{2}\right)dx\int_0^{2\pi}\sin(y)\sin\left(\frac{my}{2}\right)dy$$

$$= \frac{1}{\pi^2}\left(-\frac{8}{n^3}[2(1-(-1)^n)+n^2\pi^2(-1)^n]\right)\pi\delta_{2m} = -\frac{8}{\pi}\frac{1}{n^3}\left[2(1-(-1)^n)+n^2\pi^2(-1)^n\right]\delta_{2m}.$$

The solution is

$$z(x,y,t) = -\frac{8}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^3}[2(1-(-1)^n)+n^2\pi^2(-1)^n]\sin\left(\frac{nx}{2}\right)\sin(y)\cos(t\sqrt{n^2+4}/2).$$

2. Eigenfunctions in x and y are $X_n(t) = \sin(nx)$ and $Y_n(y) = \sin(my)$, and the solution for $T(t)$ is $T(t) = a_{nm}\cos(3t\sqrt{n^2+m^2}) + b_{nm}\sin(3t\sqrt{n^2+m^2})$. Superpose these products with a double sum on n and m and choose coefficients to satisfy $z(x,y,0) = \sin(x)\cos(y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{nm}\sin(nx)\sin(my)$ and $\frac{\partial z}{\partial t}(x,y,0) = xy = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} 3\sqrt{n^2+m^2}b_{nm}\sin(nx)\sin(my)$. Note that $a_{nm} = 0$ unless $n = 1$, and then $a_{1m} = \frac{2}{\pi}\int_0^{\pi}\cos(y)\sin(my)dy$. For $m = 1, a_{11} = 0$,

and for $m \geq 2$ we have $a_{1m} = \frac{2}{\pi} \cdot \frac{m}{m^2-1}[1-(-1)^m]$.
For b_{nm} calculate

$$b_{nm} = \frac{1}{3\sqrt{n^2+m^2}} \frac{2}{\pi} \int_0^{\pi} x \sin(nx) dx \frac{2}{\pi} \int_0^{\pi} y \sin(my) dy$$

$$= \frac{4}{3\pi^2\sqrt{n^2+m^2}} \left[-\frac{\pi(-1)^n}{n}\right] \left[-\frac{\pi(-1)^n}{m}\right].$$

The solution can be written

$$z(x,y,t) = \frac{4}{\pi} \sum_{p=2}^{\infty} \frac{(2p-1)}{[(2p-1)^2-1]} \sin(x) \sin((2p-1)y) \cos\left[3t\sqrt{1+(2p-1)^2}\right]$$

$$+\frac{4}{3} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{(-1)^{n+m}}{nm\sqrt{n^2+m^2}} \sin(nx) \sin(my) \sin[3t\sqrt{n^2+m^2}].$$

3. By separation of variables the eigenfunctions in x and y are easily found to be $X_n(x) = \sin\left(\frac{nx}{2}\right)$ and $Y_m(y) = \sin\left(\frac{my}{2}\right)$. The solution for $T_{nm}(t)$ is $T_{nm}(t) = a_{nm}\cos(t\sqrt{n^2+m^2}) + b_{nm}\sin(t\sqrt{n^2+m^2})$ and the condition $z(x,y,0) = 0$ gives $a_{nm} = 0$. Thus the solution is of the form

$$z(x,y,t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} b_{nm} \sin\left(\frac{nx}{2}\right) \sin\left(\frac{my}{2}\right) \sin(t\sqrt{n^2+m^2}).$$

To satisfy $\frac{\partial z}{\partial t}(x,y,0) = 1$ choose coefficients

$$b_{nm} = \frac{1}{\sqrt{n^2+m^2}} \frac{1}{\pi} \int_0^{2\pi} \sin\left(\frac{nx}{2}\right) dx \cdot \frac{1}{\pi} \int_0^{2\pi} \sin\left(\frac{my}{2}\right) dy$$

$$= \frac{1}{\pi^2\sqrt{n^2+m^2}} \left[\frac{2(1-(-1)^n)}{n}\right] \left[\frac{2(1-(-1)^m)}{m}\right].$$

Then

$$z(x,y,t) = \frac{16}{\pi^2} \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \frac{1}{(2p-1)(2q-1)\sqrt{(2p-1)^2+(2q-1)^2}}.$$

$$\sin\left[(2p-1)\frac{x}{2}\right] \sin\left[(2q-1)\frac{y}{2}\right] \sin\left[t\sqrt{(2p-1)^2+(2q-1)^2}\right].$$

Chapter Eighteen - The Heat Equation

Section 18.1 The Heat Equation and Initial and Boundary Conditions

1. With $u(x,t)$ the temperature at time t of the cross section located at point x we have
$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, t > 0, 0 < x < L,$$
with boundary conditions
$$u(0,t) = 0, \frac{\partial u}{\partial x}(L,t) = 0, t > 0$$
and initial condition
$$u(x,0) = f(x), 0 < x < L.$$

2. With $u(x,t)$ the temperature at time t of the cross section located at point x we have
$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, t > 0, 0 < x < L,$$
with boundary conditions
$$u(0,t) = \alpha(t), u(L,t) = \beta(t), t > 0$$
and initial condition
$$u(x,0) = f(x), 0 < x < L.$$

3. With $u(x,t)$ the temperature at time t of the cross section located at point x we have
$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, t > 0, 0 < x < L,$$
with boundary conditions
$$\frac{\partial u}{\partial x}(0,t) = 0, u(L,t) = \beta(t), t > 0$$
and initial condition
$$u(x,0) = f(x), 0 < x < L.$$

Section 18.2 Fourier Series Solutions of the Heat Equation

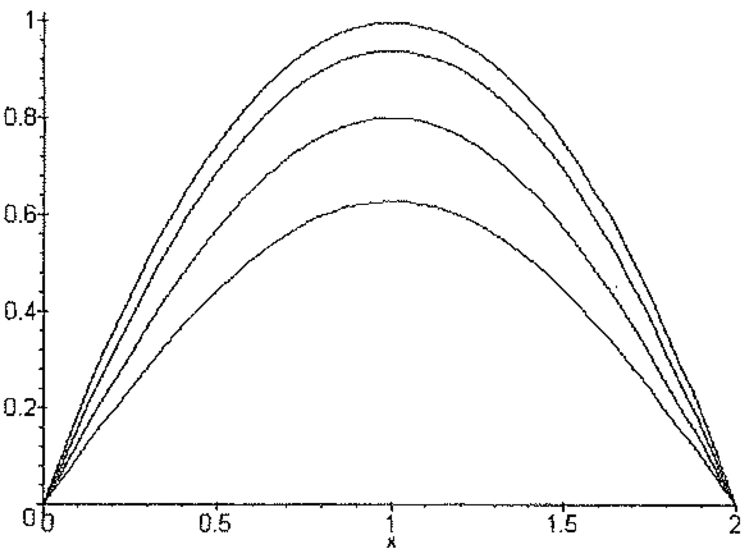
For the first three problems of this section, the separation of variables technique and the boundary conditions $u(0,t) = u(L,t) = 0$ give the same set of eigenvalues and eigenfunctions which are
$$\lambda_n = \frac{n^2\pi^2}{L}, \text{ and } X_n(x) = \sin\left(\frac{n\pi x}{L}\right), n \geq 1.$$
The corresponding time functions are
$$T_n(t) = e^{-kn^2\pi^2 t/L^2}$$
and by superposition the solution has the form $u(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) e^{-kn^2\pi^2 t/L^2}$ where the c_n are to be determined so that $u(x,0) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) = f(x)$ where f is the given initial function.

1. Here $f(x) = x(L-x)$, so c_n are given by

$$c_n = \frac{2}{L}\int_0^L x(L-x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{4L^2}{n^3\pi^3}[1-(-1)^n], n \geq 1.$$

Then

$$u(x,t) = \frac{8L^2}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\sin\left[\frac{(2n-1)\pi x}{L}\right]e^{-k(2n-1)^2\pi^2 t/L^2}.$$

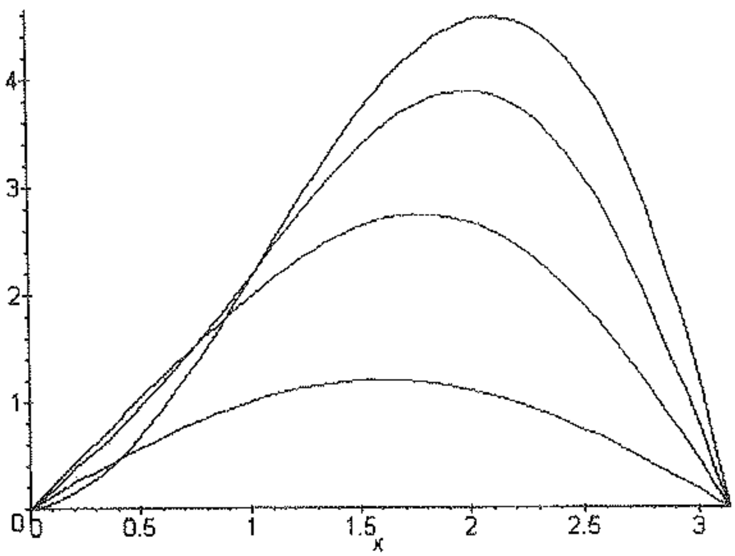


2. We have $k=4$, and $f(x) = x^2(L-x)$, so

$$c_n = \frac{2}{L}\int_0^L x^2(L-x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{4L^3}{\pi^3}\left[\frac{1+2(-1)^n}{n^3}\right], n \geq 1.$$

Then

$$u(x,t) = \frac{4L^3}{\pi^3}\sum_{n=1}^{\infty}\left[\frac{1+2(-1)^n}{n^3}\right]\sin\left(\frac{n\pi x}{L}\right)e^{-4n^2\pi^2 t/L^2}.$$

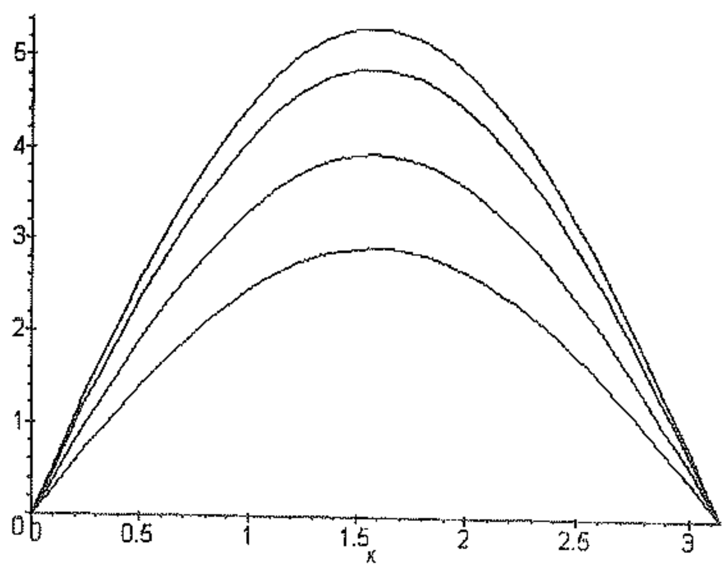


3. We have $k = 3$ and $f(x) = L\left[1 - \cos\left(\frac{2\pi x}{L}\right)\right]$, so

$$c_n = \frac{2}{L}\int_0^L L\left[1 - \cos\left(\frac{2\pi x}{L}\right)\right]\sin\left(\frac{n\pi x}{L}\right)dx = \frac{8L}{n\pi(n^2-4)}[\cos(n\pi) - 1] = \frac{8L[(-1)^n - 1]}{n\pi(n^2-4)}.$$

Then

$$u(x,t) = -\frac{16L}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)[(2n-1)^2-4]}\sin\left[\frac{(2n-1)\pi x}{L}\right]e^{-3(2n-1)^2}\pi^2 t/L^2.$$



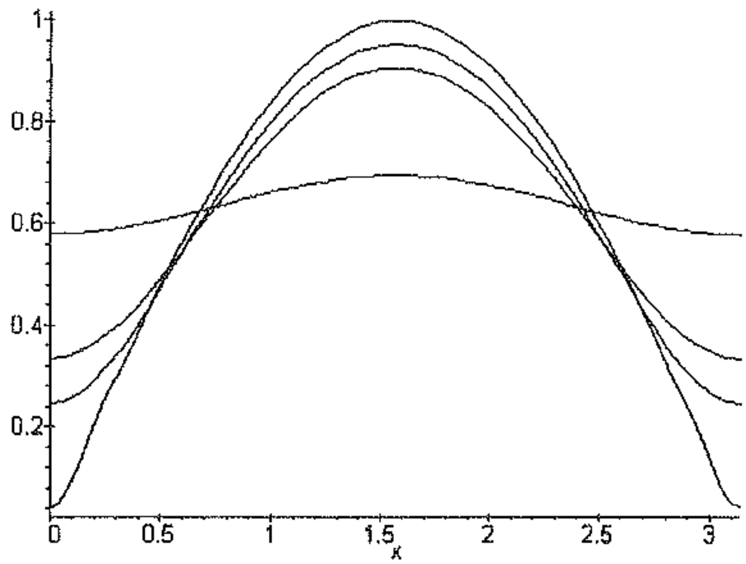
In Problems 4 through 7, separation of variables and the insulated end conditions $\frac{\partial u}{\partial x}(0,t) = 0$, $\frac{\partial u}{\partial x}(L,t) = 0$ give the eigenvalues $\lambda_0 = 1$ with $X_0(x) = 1$ and for $n \geq 1$, $\lambda_n = \frac{n^2\pi^2}{L^2}$ with $X_n(x) = \cos\left(\frac{n\pi x}{L}\right)$. The associated time functions are $T_n(t) = e^{-kn^2\pi^2 t/L^2}$, $n \geq 0$, and $u(x,t)$ has the form $u(x,t) = c_0 + \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi x}{L}\right) e^{-kn^2\pi^2 t/L^2}$ where $c_0 = \frac{1}{L}\int_0^L f(x)dx$, $c_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$.

4. Here $k = 1, L = \pi, f(x) = \sin(x)$ so $c_0 = \frac{1}{\pi}\int_0^{\pi} \sin(x)dx = \frac{2}{\pi}$;

$$c_n = \frac{2}{\pi}\int_0^{\pi} \sin(x)\cos(nx)dx = -\frac{2}{\pi}\frac{\left[1-(-1)^{n+1}\right]}{(n^2-1)}, n \geq 1 = \begin{cases} -\frac{4}{\pi}\left(\frac{1}{n^2-1}\right), & n = 2,4,6,\ldots \\ 0, & n = 1,3,5,\ldots \end{cases}$$

Then

$$u(x,t) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\cos(2nx)e^{-4n^2 t}.$$

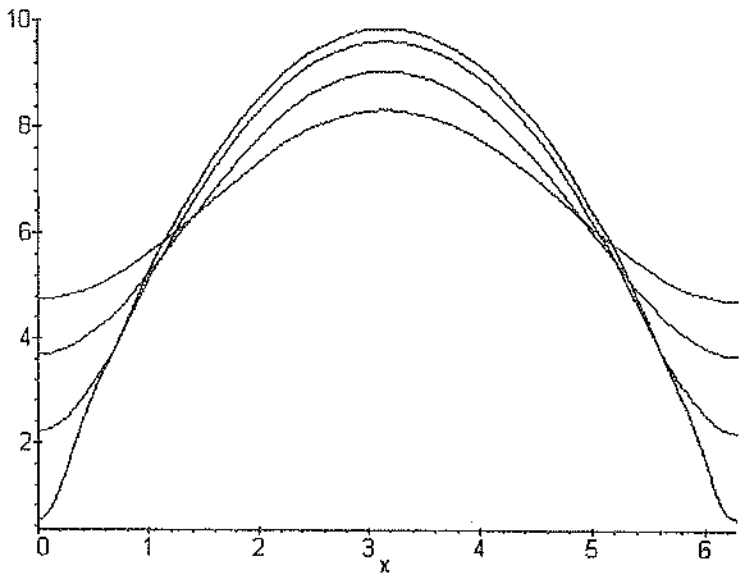


5. Here $k = 4, L = 2\pi, f(x) = x(2\pi - x)$, so $c_0 = \frac{1}{2\pi}\int_0^{2\pi} x(2\pi - x)dx = \frac{2\pi^2}{3}$;

$$c_n = \frac{1}{\pi}\int_0^{2\pi} x(2\pi - x)\cos(nx)dx = -\frac{4}{n^2}, n \geq 1.$$

Then

$$u(x,t) = \frac{2\pi^2}{3} - 4\sum_{n=1}^{\infty}\frac{1}{n^2}\cos(nx)e^{-4n^2 t}.$$

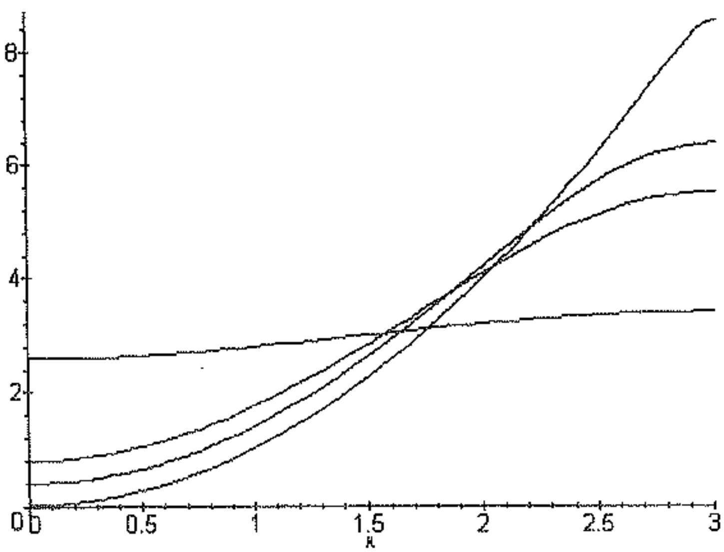


6. Here $k = 4, L = 3, f(x) = x^2$, so $c_0 = \dfrac{1}{3}\displaystyle\int_0^3 x^2 dx = 3$;

$$c_n = \frac{2}{3}\int_0^3 x^2 \cos\left(\frac{n\pi x}{3}\right) dx = \frac{36(-1)^n}{n^2\pi^2},\ n \geq 1.$$

Then

$$u(x,t) = 3 + \frac{36}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} cos\left(\frac{n\pi x}{3}\right) e^{-4n^2\pi^2 t/9}.$$

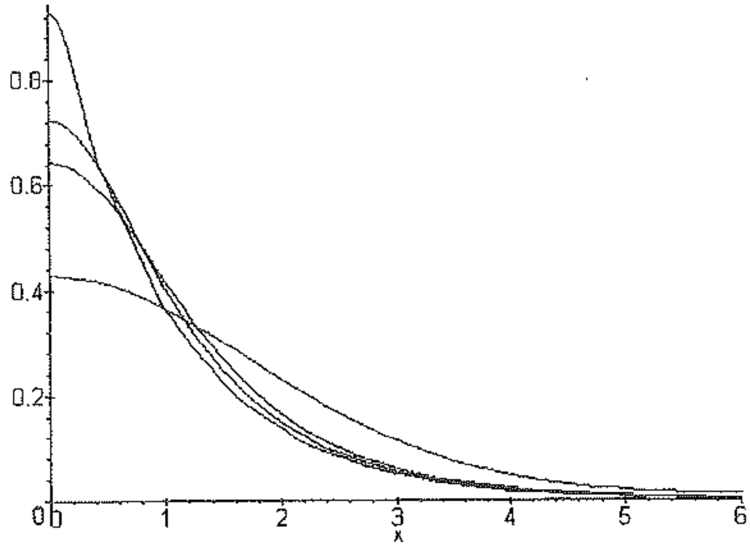


7. Here $k = 2, L = 6, f(x) = e^{-x}$, so $c_0 = \frac{1}{6}\int_0^6 e^{-x}dx = \frac{1}{6}\left(1 - e^{-6}\right)$;

$$C_n = \frac{1}{3}\int_0^6 e^{-x}\cos\left(\frac{n\pi x}{6}\right)dx = \frac{12}{36 + n^2\pi^2}\left[1 - (-1)^n e^{-6}\right], \ n \geq 1.$$

Then

$$u(x,t) = \frac{1}{6}\left(1 - \frac{1}{e^6}\right) + \frac{12}{e^6}\sum_{n=1}^{\infty}\left(\frac{e^6 - (-1)^n}{36 + n^2\pi^2}\right)\cos\left(\frac{n\pi x}{6}\right)e^{-n^2\pi^2 t/18}.$$



8. The boundary value problem is $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) = 0, u(x,0) = B$, so $c_0 = \frac{1}{L}\int_0^L B dx = B, c_n = \frac{2}{L}\int_0^L B\cos\left(\frac{n\pi x}{L}\right)dx = 0, n \geq 1$. Then $u(x,t) = B$.

9. The boundary value problem for temperature is

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, 0 < x < L, t > 0, u(0,t) = 0, \frac{\partial u}{\partial x}(L,t) = 0, t > 0, u(x,0) = B, 0 < x < L.$$

Assuming $u(x,t) = X(x)T(t)$ gives separation of variables into the two ordinary problems

$$X'' + \lambda X = 0, X(0) = 0, X'(L) = 0; \ T' + k\lambda T = 0.$$

The boundary conditions at $x = 0$ and $x = L$ gives the eigenvalues

$$\lambda_n = \frac{(2n-1)^2\pi^2}{4L^2}, n \geq 1 \text{ with associated eigenfunctions } X_n(x) = \sin\left[\frac{(2n-1)\pi x}{2L}\right].$$

The corresponding time functions are $T_n(x) = e^{-k(2n-1)^2\pi^2 t/4L^2}$ and by superposition we get

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\left[\frac{(2n-1)\pi x}{2L}\right]e^{-k(2n-1)^2\pi^2 t/4L^2}.$$

The c_n are given by

$$c_n = \frac{2}{L}\int_0^L B\sin\left[\frac{(2n-1)\pi x}{2L}\right]dx = \frac{4B}{\pi(2n-1)}.$$

Then

$$u(x,t) = \frac{4B}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)}\sin\left[\frac{(2n-1)\pi x}{2L}\right]e^{-k(2n-1)^2\pi^2 t/4L^2}.$$

10. The boundary value problem is

$$\frac{\partial u}{\partial t} = 9\frac{\partial^2 u}{\partial x^2}, 0 < x < 2, t > 0, u(0,t) = 0, \frac{\partial u}{\partial x}(2,t) = 0, t > 0, u(x,0) = x^2.$$

From Problem 7 with $k = 9, L = 2$, the general form of the solution is

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\left[\frac{(2n-1)\pi x}{4}\right]e^{-9\pi^2(2n-1)^2 t/16}$$

with

$$c_n = \int_0^2 x^2 \sin\left[\frac{(2n-1)\pi x}{4}\right]dx = -\frac{64}{\pi^3}\left[\frac{2+(-1)^n(2n-1)\pi}{(2n-1)^3}\right], n \geq 1.$$

Then

$$u(x,t) = -\frac{64}{\pi^3}\sum_{n=1}^{\infty}\left[\frac{2+(-1)^n(2n-1)\pi}{(2n-1)^3}\right]\sin\left[\frac{(2n-1)\pi x}{4}\right]e^{-9\pi^2(2n-1)^2 t/16}.$$

For fixed x, $\lim_{t\to\infty} u(x,t) = 0$, all $0 \leq x \leq 2$.

11. For the suggested transformation , calculate

$$\frac{\partial u}{\partial t} = e^{\alpha x+\beta t}\left[\beta v + \frac{\partial v}{\partial t}\right]; \frac{\partial u}{\partial x} = e^{\alpha x+\beta t}\left[\alpha v + \frac{\partial v}{\partial x}\right]; \text{ and } \frac{\partial^2 u}{\partial x^2} = e^{\alpha x+\beta t}\left[\alpha^2 v + 2\alpha\frac{\partial v}{\partial x} + \frac{\partial^2 v}{\partial x^2}\right].$$

Substitute these into the original problem and cancel the exponential factor $e^{\alpha x+\beta t}$ to get

$$\beta v + \frac{\partial v}{\partial t} = k\left(\alpha^2 v + 2\alpha\frac{\partial v}{\partial x} + \frac{\partial^2 v}{\partial x^2} + A\alpha v + A\frac{\partial v}{\partial x} + Bv\right).$$

Now choose α, β to eliminate terms containing v, or $\dfrac{\partial v}{\partial x}$. Thus choose $2\alpha + A = 0$ and $k(\alpha^2 + A\alpha + B) - \beta = 0$, or $\alpha = -\dfrac{A}{2}$ and $\beta = k\left(B - \dfrac{A^2}{4}\right)$. With these choices $v(x,t)$ should satisfy the standard heat equation, boundary and initial conditions given by

$$\frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2}; v(0,t) = v(L,t) = 0, v(x,0) = e^{-\alpha x}u(x,0).$$

12. Following the method of Problem 11 with $A = 4, B = 2, k = 1$, we find that $\alpha = -2, \beta = -2$ to get the transformation e^{-2x-2t}. Then $u(x,t) = e^{-2x-2t}v(x,t)$ and v satisfies the standard heat equation and boundary conditions

$$v_t = v_{xx}, 0 < x < \pi, t > 0; v(0,t) = v(\pi,t) = 0, v(x,0) = e^{2x}u(x,0) = x(\pi - x)e^{2x}.$$

The form of the solution for $v(x,t)$, found by a standard separation of variables, is $v(x,y) = \sum_{n=1}^{\infty} a_n \sin(nx) e^{-n^2 t}$. From the initial condition we have $v(x,0) = x(\pi - x)e^{2x} = \sum_{n=1}^{\infty} a_n \sin(nx)$, so calculate

$$a_n = \frac{2}{\pi}\int_0^{\pi} x(\pi - x)e^{2x}\sin(nx)dx = \frac{4n}{\pi(4+n^2)^3}\left\{(-1)^n e^{2\pi}(12 + 8\pi - n^2(1+2\pi))\right.$$

$$\left. -(12 + 8\pi - n^2(1-2\pi))\right\}, n \geq 1.$$

The solution for the original problem is given by

$$u(x,t) = e^{-2x-2t}\sum_{n=1}^{\infty}\frac{4n}{\pi(4+n^2)^3}\left\{(-1)^n e^{2\pi}(12 + 8\pi - n^2(1+2\pi))\right.$$

$$\left. -(12 + 8\pi - n^2(1-2\pi))\right\}\sin(nx)e^{-n^2 t}.$$

13. Following the method of Problem 11 with $A = 6, B = 0, k = 1$, we find that $\alpha = -3, \beta = -9$ to get the transformation e^{-3x-9t}. Then $u(x,t) = e^{-3x-9t}v(x,t)$ and v satisfies the standard heat equation and boundary conditions

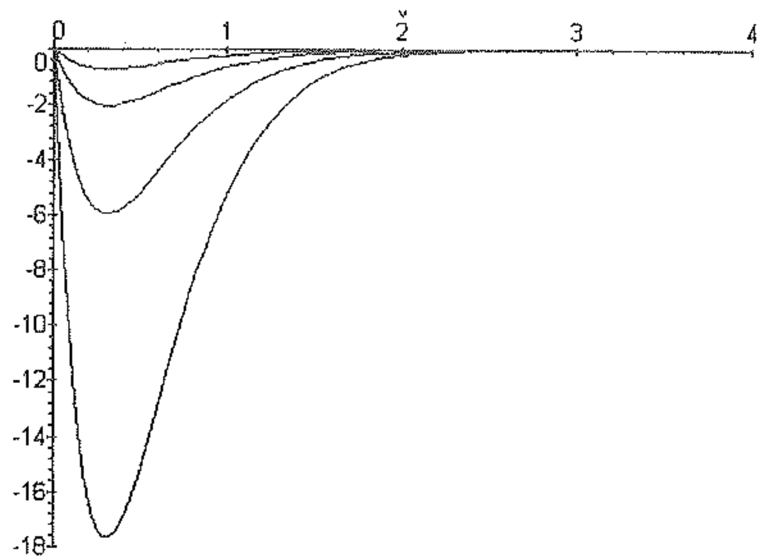
$$v_t = v_{xx}, 0 < x < 4, t > 0; v(0,t) = v(4,t) = 0, v(x,0) = e^{3x}u(x,0) = e^{3x}.$$

The form of the solution for $v(x,t)$, found by a standard separation of variables, is $v(x,y) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{nx\pi}{4}\right) e^{-n^2\pi^2 t/16}$. From the initial condition we have $v(x,0) = e^{3x} = \sum_{n=1}^{\infty} a_n \sin(nx)$, so calculate

$$a_n = \frac{1}{2}\int_0^4 e^{3x}\sin\left(\frac{nx\pi}{4}\right)dx = \frac{2n\pi}{144 + n^2\pi^2}[1 - e^{12}(-1)^n], n \geq 1$$

The solution for the original problem is given by

$$u(x,t) = e^{-3x-9t}\sum_{n=1}^{\infty}\frac{2n\pi}{144 + n^2\pi^2}[1 - e^{12}(-1)^n]\sin\left(\frac{nx\pi}{4}\right)e^{-n^2\pi^2 t/16}.$$



14. Following the method of Problem 11 with $A = -6, B = 0, k = 1$, we find that $\alpha = 3, \beta = -9$ to get the transformation e^{3x-9t}. Then $u(x,t) = e^{3x-9t}v(x,t)$ and v satisfies the standard heat equation and boundary conditions

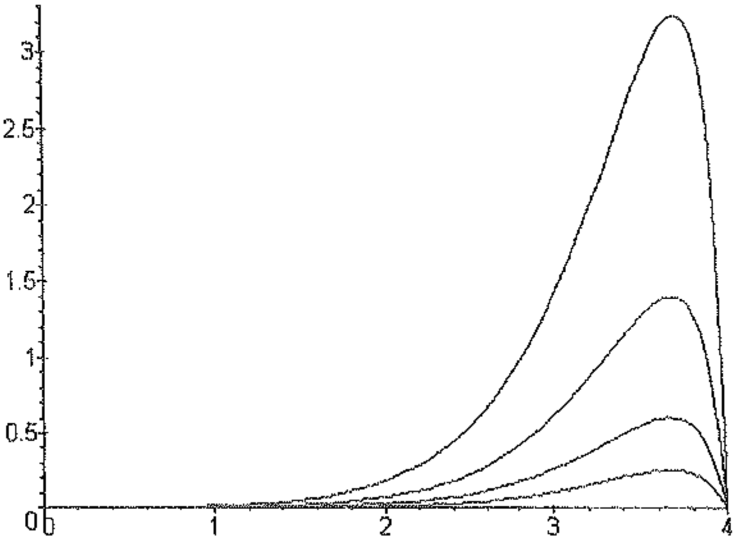
$$v_t = v_{xx}, 0 < x < \pi, t > 0; v(0,t) = v(\pi,t) = 0, v(x,0) = e^{-3x}u(x,0) = x^2(\pi - x)e^{-3x}.$$

The form of the solution for $v(x,t)$, found by a standard separation of variables, is $v(x,y) = \sum_{n=1}^{\infty} a_n \sin(nx)e^{-n^2t}$. From the initial condition we have $v(x,0) = e^{-3x}x^2(\pi - x) = \sum_{n=1}^{\infty} a_n \sin(nx)$, so calculate

$$a_n = \frac{2}{\pi}\int_0^{\pi} e^{-3x}x^2(\pi - x)\sin(nx)dx =$$

The solution for the original problem is given by

$$u(x,t) = e^{3x-9t}\sum_{n=1}^{\infty} \sin(nx)e^{-n^2t}.$$



15. To remove the non-zero boundary condition at each end of the interval $[0,1]$, we introduce a linear function $L(x)$ such that $L(0)=2, L(1)=5$ and define $v(x,t)=u(x,t)-L(x)$. Since $L''(x)=0$, v will satisfy the same partial differential equation as u, and the function $L(x)$ will transfer the nonhomogeneity to the initial conditions. Thus we choose $L(x)=2x+3$, and $v(x,t)$ satisfies the standard heat equation

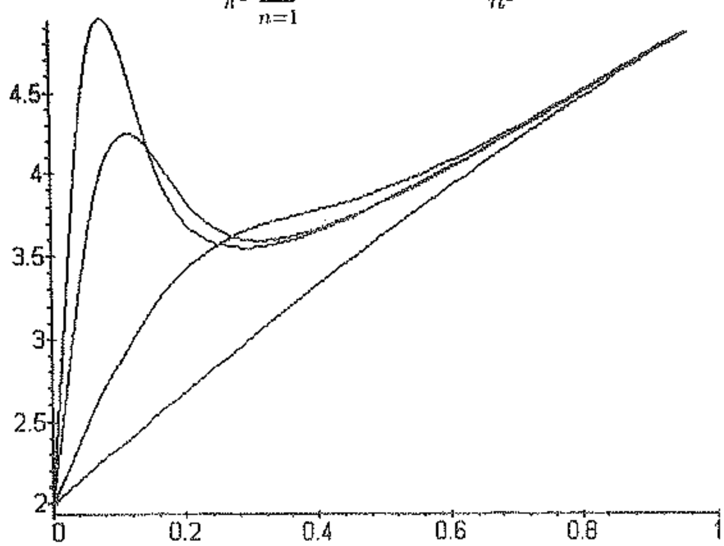
$$v_t = v_{xx}, 0<x<1, t>); v(0,t)=v(1,t)=0, v(x,t)=u(x,0)-L(x)=x^2-3x-2.$$

By a standard separation of variables argument, $v(x,t)=\sum_{n=1}^{\infty} a_n \sin(n\pi x)e^{-n^2t^2}$. The initial condition gives

$$a_n = 2\int_0^1 (x^2-3x-2)\sin(n\pi x)dx = \frac{4}{n^3\pi^3}[(-1)^n(1+2n^2\pi^2)-(1+n^2\pi^2)], n\geq 1.$$

Then

$$u(x,t)=(2+3x)+\frac{4}{\pi^3}\sum_{n=1}^{\infty}\frac{[(-1)^n(1+2n^2\pi^2)-(1+n^2\pi^2)]}{n^3}\sin(n\pi x)e^{-n^2\pi^2 t}.$$



16. To introduce a homogeneous boundary condition at $x = 0$ and preserve the one at $x = L$, let

$$u(x,t) = V(x,t) + T - \frac{Tx}{L}.$$

Then V satisfies

$$V_t = kV_{xx}$$

$$V(0,t) = V(L,t) = 0$$

$$V(x,0) = x(L-x) - T + \frac{Tx}{L} = \frac{1}{L}(Lx - T)(L-x).$$

Separation of variables gives

$$V(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) a_n e^{-kn^2\pi^2 t/L^2}$$

with

$$\begin{aligned} a_n &= \frac{2}{L}\int_0^L \frac{1}{L}(Lx - T)(L-x)\sin\left(\frac{n\pi x}{L}\right) dx \\ &= \frac{2}{n^3\pi^3}\left[2L^2\left(1-(-1)^n\right) - n^2\pi^2 T\right]. \end{aligned}$$

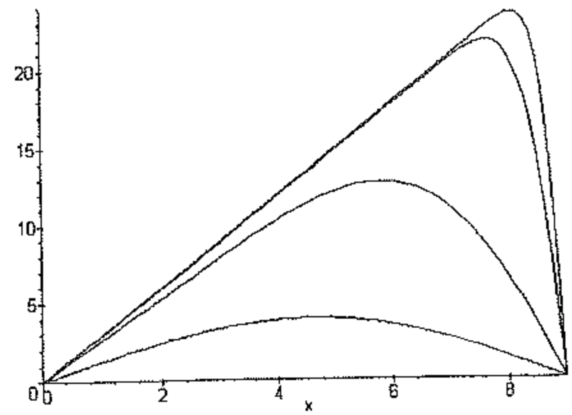
Then

$$u(x,t) = \frac{T}{L}(L-x) + \frac{2}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^3}\sin\left(\frac{n\pi x}{L}\right)\left[2L^2\left(1-(-1)^n\right) - n^2\pi^2 T\right] e^{-kn^2\pi^2 t/L^2}.$$

17. Let $u(x,t) = w(x,t)e^{-\alpha t}$ and choose α to eliminate the $-Aw$ term in the differential equation. Substitution gives $-\alpha w e^{-\alpha t} + w_t e^{-\alpha t} = 4w_{xx}e^{-\alpha t} - Awe^{-\alpha t}$, thus choose $\alpha = A$. Then w satisfies $w_t = 4w_{xx}, w(0,t) = w(9,t) = 0, w(x,0) = 3x$. By separation of variables

$$w(x,t) = \frac{54}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\left(\frac{n\pi x}{9}\right) e^{-4n^2\pi^2 t/81}$$

so $u(x,t) = e^{-At}w(x,t)$. Solutions are shown for $A = \frac{1}{4}, \frac{1}{2}, 1, 3$.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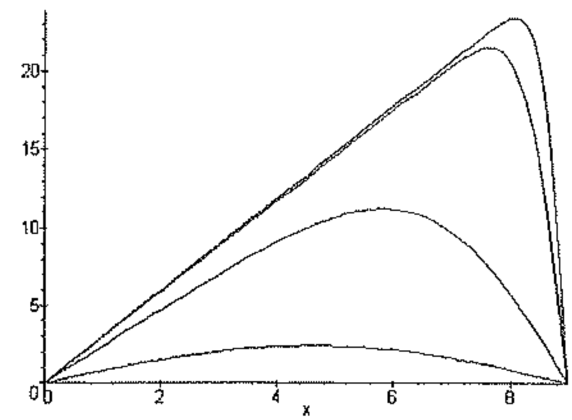
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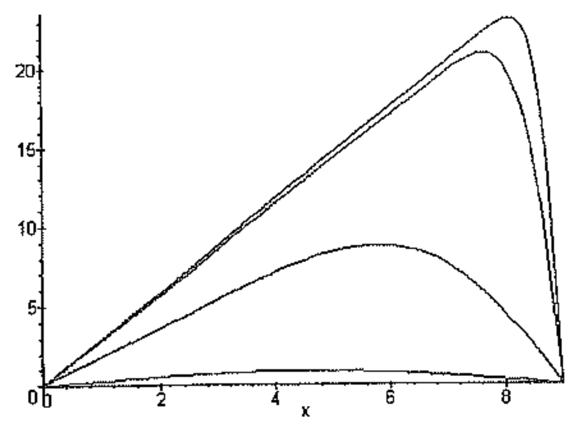
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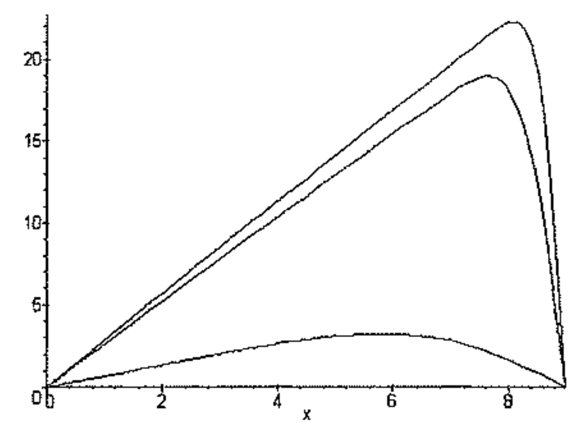
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18. Let $u(x,t) = V(x,t) + T\left(1 - \frac{x}{L}\right)$ then V satisfies $V_t = 9V_{xx}; V(0,t) = V(L,t) = 0; V(0,x) = -T\left(1 - \frac{x}{L}\right)$. By separation of variables

$$V(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) e^{-9n^2\pi^2 t/L^2}$$

with

$$a_n = \frac{2}{L}\int_0^L -T\left(1 - \frac{x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = -\frac{2T}{n\pi}.$$

Then

$$u(x,t) = T\left(1 - \frac{x}{L}\right) - \frac{2T}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi x}{L}\right)e^{-9n^2\pi^2 t/L^2}.$$

For Problems 19 through 23, the heat equation on $[0, L]$, with a source term $F(x,t)$, having end conditions $u(0,t) = u(L,0) = 0$, and initial temperature function $u(x,0) = f(x)$ we find a solution in the form

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t)\sin\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)e^{-kn^2\pi^2 t/l^2}$$

where

$$B_n(t) = \frac{2}{L}\int_0^L F((\xi,t)\sin\left(\frac{N\pi\xi}{L}\right)d\xi, n = 1,2,3,\ldots$$

$$b_n = \frac{2}{L}\int_0^L f(\xi)\sin\left(\frac{n\pi\xi}{L}\right)d\xi, n = 1,2,3,\ldots$$

$T_n(t)$ is the solution of the first order differential equation

$$T_n' t) + k\frac{n^2\pi^2}{L^2}T_n(t) = B_n(t), T_n(0) = b_n.$$

We summarize the calculations for each problem, write the solution and show the graphs of the solution with and without internal source for various times. The second term above in the solution represents the solution without the internal source $F(x,t)$.

19. With $k = 4, L = \pi, f(x) = x(\pi - x), F(x,t) = t$ we have

$$B_n(t) = \frac{2}{\pi}\int_0^\pi t\sin(n\xi)d\xi = \frac{2t}{n\pi}[1-(-1)^n], b_n = \frac{4}{\pi n^3}[1-(-1)^n], n \geq 1$$

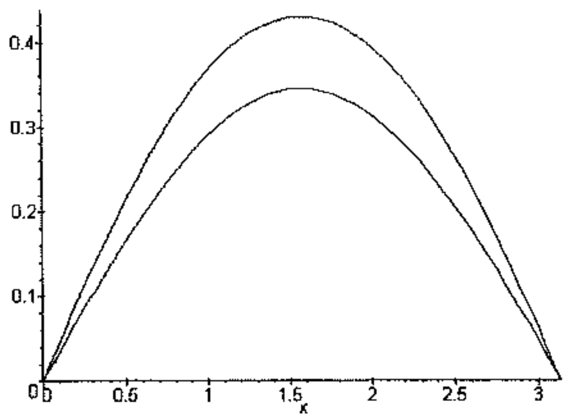
$$T_n(t) = \frac{1}{8\pi n^5}[1-(-1)^n]\left\{-1+4n^2t+e^{-4n^2t}\right\}$$

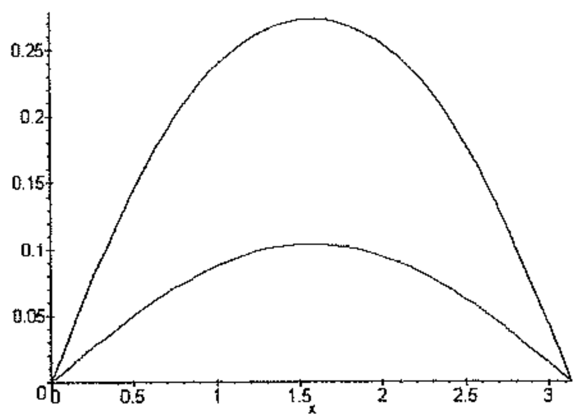
and solution

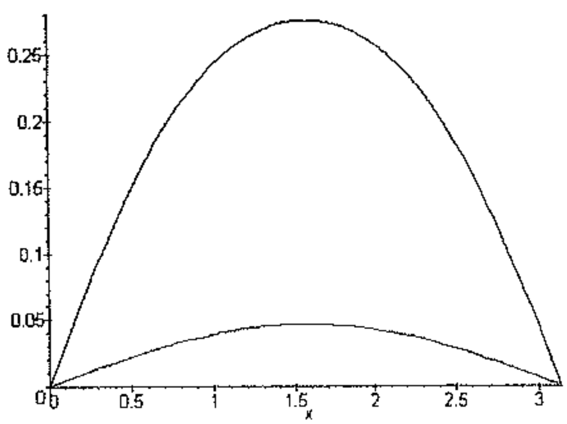
$$u(x,t) = \sum_{n=1}^{\infty}\frac{1}{8\pi n^5}[1-(-1)^n]\left\{-1+4n^2t+e^{-4n^2t}\right\}\sin(nx)$$

$$+\sum_{n=1}^{\infty}\frac{4}{\pi n^3}[1-(-1)^n]\sin(nx)e^{-4n^2t}.$$

Solutions with and without source are shown for $t = .5, .8, 1$ seconds.







20. With $k = 1, L = 4, f(x) = 1 F(x,t) = x\sin(t)$ we have

$$B_n(t) = \frac{1}{2}\int_0^4 \xi\sin(t)\sin\left(\frac{n\pi\xi}{4}\right)d\xi = \frac{8\sin(t)}{n\pi}(-1)^{n+1}, b_n = \frac{1}{2}\int_0^4 \sin\left(\frac{n\pi\xi}{4}\right)d\xi = \frac{2}{n\pi}[1-(-1)^n], n \geq 1.$$

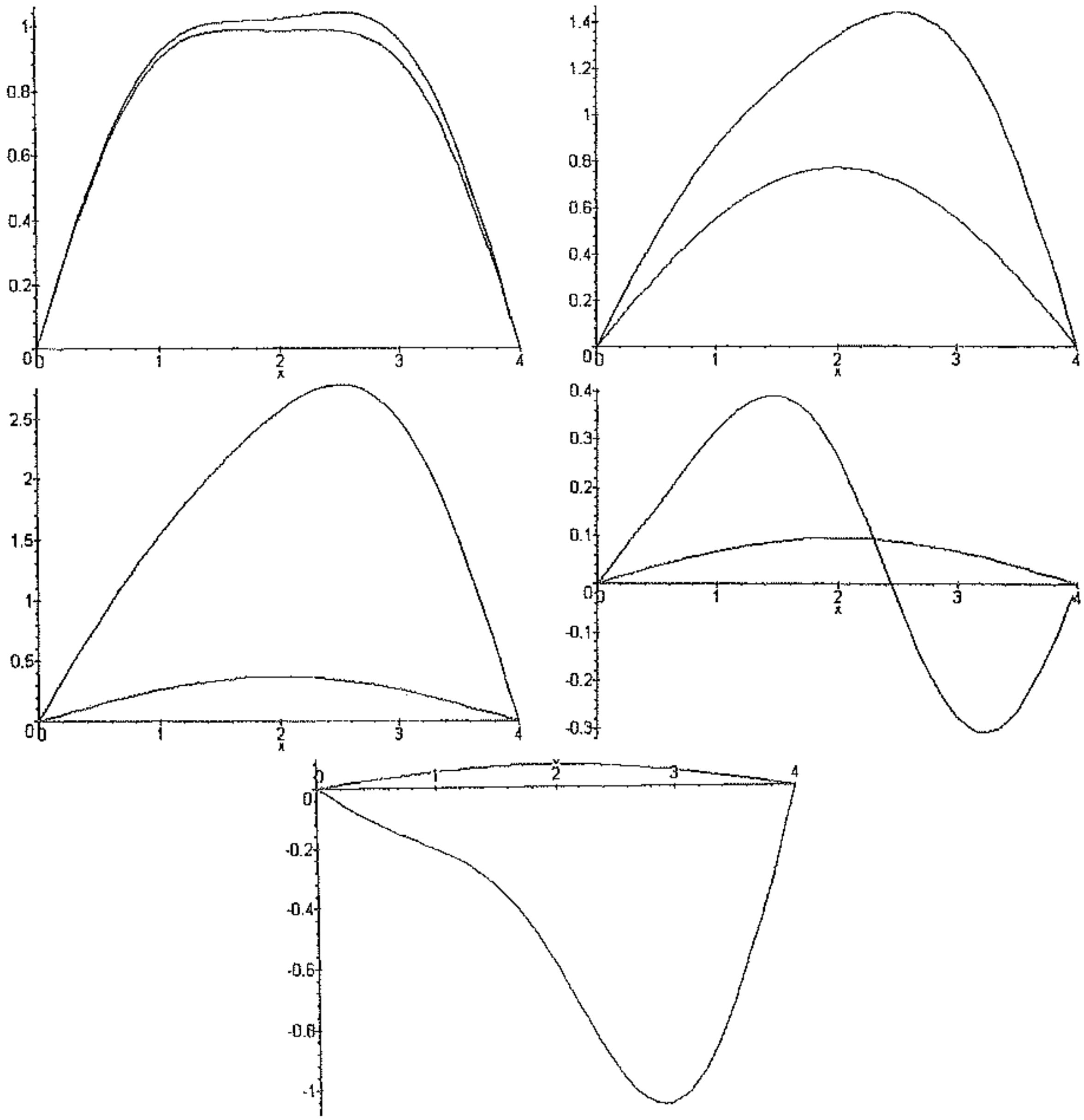
$$T_n(t) = \frac{128(-1)^n}{n\pi(n^4\pi^4+256)}\left\{16\cos(t) - n^2\pi^2\sin(t) - 16e^{-n^2\pi^2t/16}\right\}$$

and solution

$$u(x,t) = \frac{128(-1)^n}{n\pi(n^4\pi^4+256)}\left\{16\cos(t) - n^2\pi^2\sin(t) - 16e^{-n^2\pi^2t/16}\right\}\sin\left(\frac{n\pi x}{4}\right)$$

$$+\sum_{n=1}^{\infty}\frac{2}{n\pi}[1-(-1)^n]\sin\left(\frac{n\pi x}{4}\right)e^{-n^2\pi^2t/16}$$

Solutions with and without source are shown for $t = .2, .8, 2, 4.2, 4.7$ seconds.



21. With $k=1, L=5, f(x)=x^2(5-x), F(x,t)=t\cos(x)$ we have

$$B_n(t)=\frac{2}{5}\int_0^5 t\cos(\xi)\sin\left(\frac{n\pi\xi}{5}\right)d\xi=\frac{2t}{(n^2\pi^2-25)}\left[(-1)^{n+1}(5+n\pi)+n\pi\right], n\geq 1$$

$$b_n=\frac{2}{5}\int_0^5 \xi^2(5-\xi)\sin\left(\frac{n\pi\xi}{5}\right)d\xi=\frac{500}{n^3\pi^3}[(-1)^{n+1}-1], n\geq 1$$

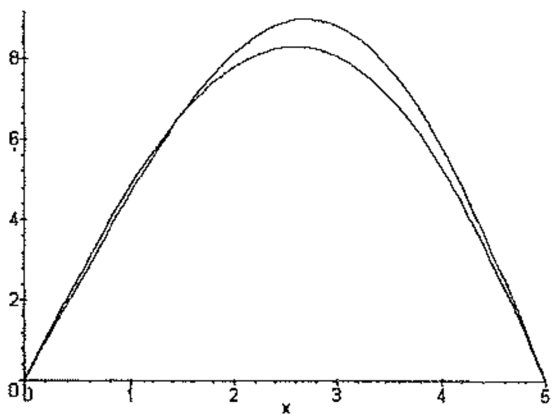
$$T_n(t)=\frac{50[1-\cos(5)(-1)^n]}{n^3\pi^3(n^2\pi^2-25)}\left\{n^2\pi^2t-25+25e^{-n^2\pi^2t/25}\right\}$$

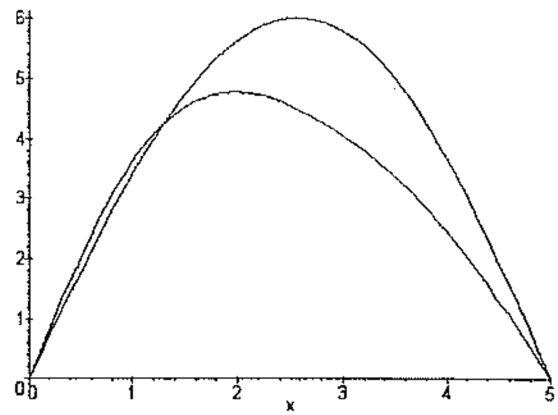
and solution

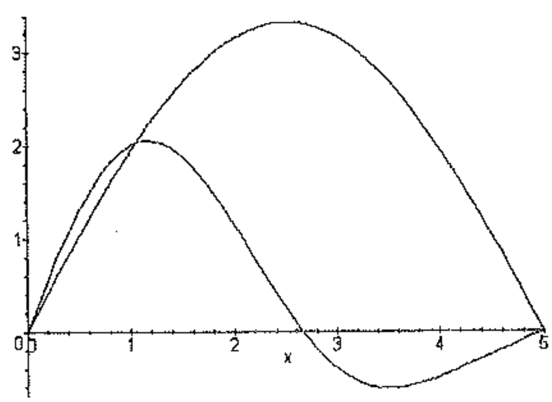
$$u(x,t)=\sum_{n=1}^{\infty}\frac{50[1-\cos(5)(-1)^n]}{n^3\pi^3(n^2\pi^2-25)}\left\{n^2\pi^2t-25+25e^{-n^2\pi^2t/25}\right\}\sin\left(\frac{n\pi x}{5}\right)$$

$$+\sum_{n=1}^{\infty}\frac{500}{n^3\pi^3}[(-1)^{n+1}-1]\sin\left(\frac{n\pi x}{5}\right)e^{-n^2\pi^2t/25}$$

Solutions with and without source are shown for $t=1.5, 2.5, 4$ seconds.







22. With $k = 4, L = 2, f(x) = \sin\left(\frac{\pi x}{2}\right), F(x,t) = K(H(x) - H(x-1))$ we have

$$B_n(t) = \int_0^1 K \sin\left(\frac{n\pi\xi}{2}\right) d\xi = \frac{2K}{n\pi}\left[1 - \cos\left(\frac{n\pi}{2}\right)\right], n \geq 1, b_1 = 1, b_n = 0, n \neq 1$$

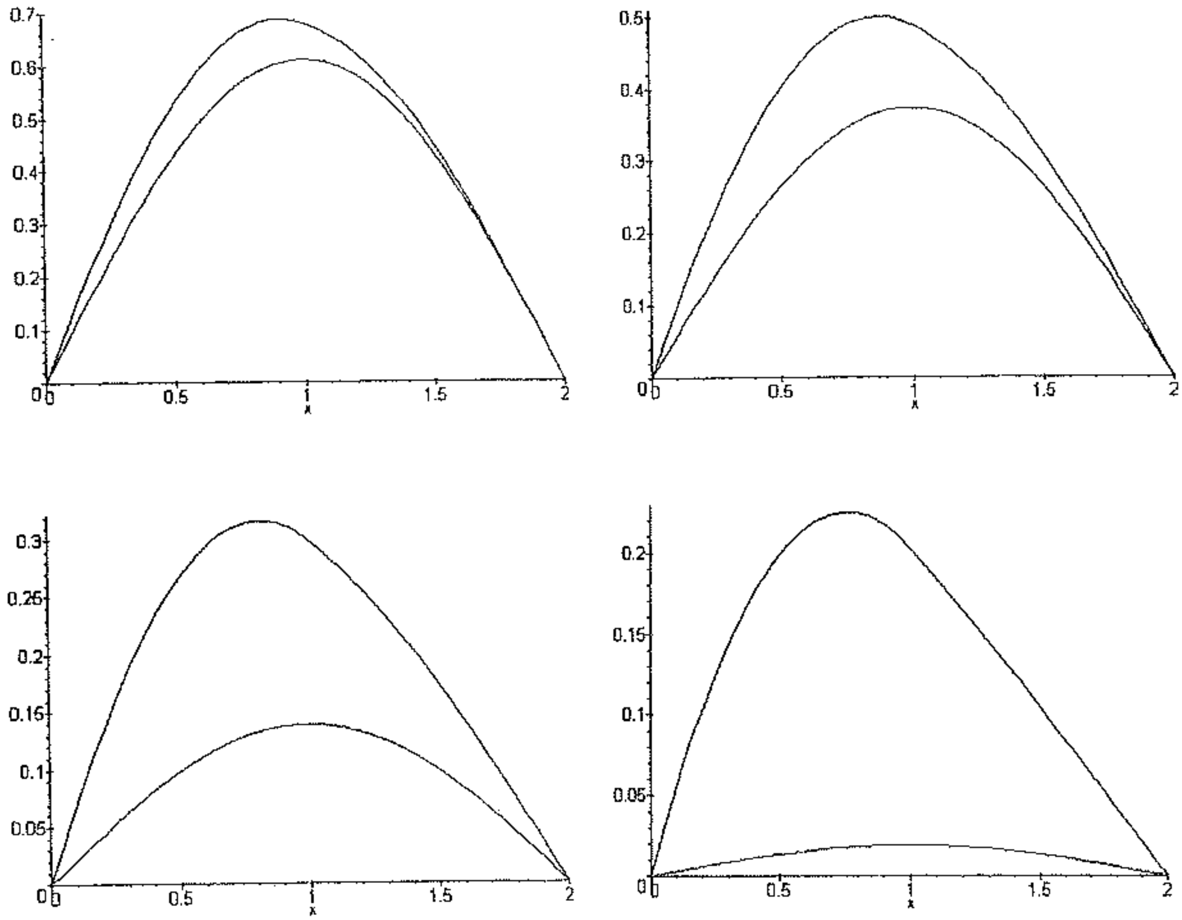
$$T_n(t) = \frac{2K}{n^3\pi^3}\left[1 - \cos\left(\frac{n\pi}{2}\right)\right]\left\{1 - e^{-n^2\pi^2 t}\right\}$$

and solution

$$u(x,t) = \sum_{n=1}^{\infty} \frac{2K}{n^3\pi^3}\left[1 - \cos\left(\frac{n\pi}{2}\right)\right]\left\{1 - e^{-n^2\pi^2 t}\right\} \sin\left(\frac{n\pi x}{5}\right)$$

$$+ \sin\left(\frac{\pi x}{2}\right) e^{-\pi^2 t}$$

Solutions are shown with and without source (using $K = 3$) for $t = .05, .1, .2, .4$



23. With $k = 16, L = 3, f(x) = K, F(x,t) = xt$ we have

$$B_n(t) = \frac{2}{3}\int_0^3 \xi t \sin\left(\frac{n\pi\xi}{3}\right) d\xi = \frac{6t}{n\pi}(-1)^{n+1}, n \geq 1$$

$$b_n = \frac{2}{3}\int_0^3 K \sin\left(\frac{n\pi\xi}{3}\right) d\xi = \frac{2K}{n\pi}[1-(-1)^n], n \geq 1$$

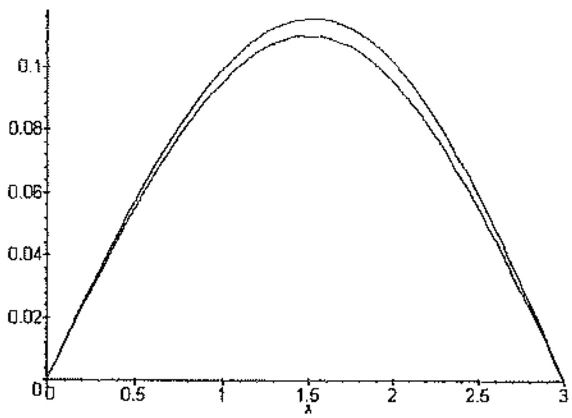
$$T_n(t) = \frac{27(-1)^{n+1}}{128n^5\pi^5}\left\{16n^2\pi^2 - 9 + 9e^{-16n^2\pi^2 t/9}\right\}$$

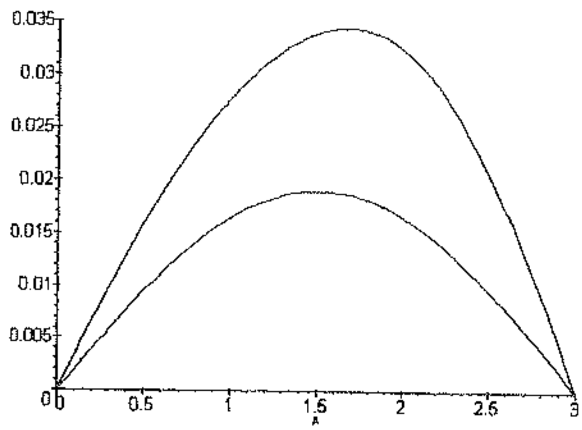
and solution

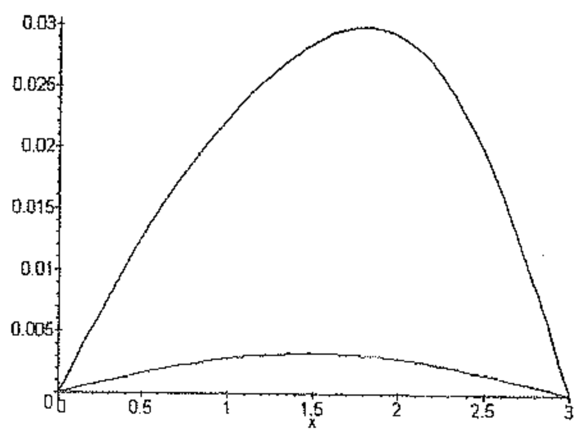
$$u(x,t) = \sum_{n=1}^{\infty} \frac{27(-1)^{n+1}}{128n^5\pi^5}\left\{16n^2\pi^2 - 9 + 9e^{-16n^2\pi^2 t/9}\right\}\sin\left(\frac{n\pi x}{3}\right)$$

$$+\sum_{n=1}^{\infty} \frac{2K}{n\pi}[1-(-1)^n]\sin\left(\frac{n\pi x}{3}\right)e^{-16n^2\pi^2 t/9}$$

Solutions are shown with and without source using $K = 1/2$) for $t = .1, .2, .3$







25. In the following, $j = 1, 2, \cdots, 9$.

$u_{j,0}$: 0.009, 0.032, 0.063, 0.096, 0.125, 0.144
0.147, 0.128, 0.081
$u_{j,1}$: 0.0125, 0.034, 0.0635, 0.095, 0.1225
0.14, 0.1415, 0.121, 0.0725
$u_{j,2}$: 0.01475, 0.064125, 0.089, 0.094, 0.1195
0.136, 0.136, 0.114, 0.0665
$u_{j,3}$: 0.023381, 0.058, 0.084031, 0.099125, 0.11725
0.13188, 0.1305, 0.10763, 0.06175
27. In the following, $j = 1, 2, \cdots, 9$.
$u_{j,0}$: 0.098769, 0.19021, 0.2673, 0.32361, 0.35355
0.35267, 0.31779, 0.24721, 0.14079
$u_{j,1}$: 0.096937, 0.18622, 0.26211, 0.31702, 0.34585
0.34417, 0.30887, 0.23825, 0.13220
$u_{j,2}$: 0.095124, 0.18307, 0.25697, 0.3105, 0.33822
0.33577, 0.30004, 0.22939, 0.12566
$u_{j,3}$: 0.09330, 0.17956, 0.25188, 0.30405, 0.33062
0.32745, 0.29131, 0.22112, 0.12018

Section 18.3 Heat Conduction in Infinite Media

In each of Problems 1 through 4, separation of variables and the requirement of a bounded solution yields a solution of the boundary value problem $\dfrac{\partial u}{\partial t} = k\dfrac{\partial^2 u}{\partial x^2}$, $-\infty < x < \infty, t > 0$, $u(x,0) = f(x)$ of the form $u(x,t) = \int_0^\infty [a_\omega \cos(\omega x) + b_\omega \sin(\omega x)]e^{-\omega^2 kt}d\omega$, where $a_\omega = \dfrac{1}{\pi}\int_{-\infty}^{\infty} f(\xi)\cos(\omega\xi)d\xi$ and $b_\omega = \dfrac{1}{\pi}\int_{-\infty}^{\infty} f(\xi)\sin(\omega\xi)d\xi$. Calculation of these Fourier integral coefficients and substitution yields the integral form of the solution.

To write the solution of the heat equation $u_t = ku_{xx}, u(x,0) = f(x), -\infty < x < \infty$ using Fourier transforms, let $\widehat{u}(\omega,t) = \mathcal{F}[u(x,t); x \to \omega] = \int_{-\infty}^{\infty} u(x,t)e^{-i\omega t}dx$. This gives the transformed problem $\dfrac{d\widehat{u}}{dt} + k\omega^2\widehat{u} = 0, \widehat{u}(\omega,0) = \mathcal{F}[f(x); x \to \omega] = \widehat{f}(\omega)$. The general solution of this transformed problem is easily found to be $\widehat{u}(\omega,t) = \widehat{f}(\omega)e^{-k\omega^2 t}$. To recover the original solution, we invert this transform by the convolution theorem. For this result, we need $\mathcal{F}^{-1}\left[e^{-k\omega^2 t}\right] = \dfrac{1}{2\sqrt{\pi kt}}e^{-x^2/4kt}$. Then the convolution theorem gives

$$u(x,t) = \mathcal{F}^{-1}\left[\widehat{f}(\omega)e^{-k\omega^2 t}\right] = \frac{1}{2\sqrt{\pi kt}}\int_{-\infty}^{\infty} f(\xi)e^{-(x-\xi)^2/4kt}d\xi.$$

1. With $f(x) = e^{-4|x|}$ we get

$$a_\omega = \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-4|\xi|}\cos(\omega\xi)d\xi = \frac{8}{\pi}\frac{1}{16+\omega^2},$$

and

$$b_\omega = \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-4|\xi|}\sin(\omega\xi)d\xi = 0.$$

Substitution into the integral formula gives the solution

$$u(x,t) = \frac{8}{\pi}\int_0^\infty \left[\frac{1}{16+\omega^2}\cos(\omega x)\right] e^{-\omega^2 kt}d\omega.$$

For the solution by Fourier transforms we get

$$u(x,t) = \frac{1}{2\sqrt{\pi kt}}\int_{-\infty}^{\infty} e^{-4|\xi|}e^{-(x-\xi)^2/4kt}d\xi.$$

2. With the given $f(x)$ we get

$$a_\omega = \frac{1}{\pi}\int_{-\pi}^{\pi}\sin(\xi)\cos(\omega\xi)d\xi = 0,$$

and

$$b_\omega = \frac{1}{\pi}\int_{-\pi}^{\pi}\sin(\xi)\sin(\omega\xi)d\xi = \frac{2}{\pi}\frac{\sin(\omega\pi)}{\omega^2-1}.$$

Substitution into the integral formula gives the solution

$$u(x,t) = \frac{2}{\pi}\int_0^\infty \left[\frac{\sin(\omega\pi)}{\omega^2-1}\sin(\omega x)\right] e^{-\omega^2 kt}e^{-\omega^2 kt}d\omega.$$

For the solution by Fourier transforms, we first write $f(x) = \sin(x)[H(x+\pi) - H(x-\pi)]$ to get the solution

$$\begin{aligned}u(x,t) &= \frac{1}{2\sqrt{\pi kt}}\int_{-\infty}^{\infty}\sin(\xi)[H(\xi+\pi) - H(\xi-\pi)]e^{-(x-\xi)^2/4kt}d\xi \\ &= \frac{1}{2\sqrt{\pi kt}}\int_{-\pi}^{\pi}\sin(\xi)e^{-(x-\xi)^2/4kt}d\xi.\end{aligned}$$

3. With the given $f(x)$ we get

$$a_\omega = \frac{1}{\pi}\int_0^4 \xi cos(\omega\xi)d\xi = \frac{1}{\pi}\frac{4\omega\sin(4\omega) + \cos(4\omega) - 1}{\omega^2},$$

and

$$b_\omega = \frac{1}{\pi}\int_0^4 \xi\sin(\omega\xi)d\xi = \frac{1}{\pi}\frac{\sin(4\omega) - 4\omega\cos(4\omega)}{\omega^2}.$$

Substitution into the integral formula gives the solution

$$u(x,t) = \frac{1}{\pi}\int_0^\infty \left[\left(\frac{4\omega\sin(4\omega)+\cos(4\omega)-1}{\omega^2}\right)\cos(\omega x) + \left(\frac{\sin(4\omega)-4\omega\cos(4\omega)}{\omega^2}\right)\sin(\omega x)\right] e^{-\omega^2 kt}d\omega.$$

For the soution by Fourier transforms, write $f(x) = x[H(x) - H(x-4)]$ to get solution

$$u(x,t) = \frac{1}{2\sqrt{\pi kt}}\int_{-\infty}^{\infty}\xi[H(\xi) - H(\xi-4)]e^{-(x-\xi)^2/4kt}d\xi$$

$$= \frac{1}{2\sqrt{\pi kt}} \int_0^4 \xi e^{-(x-\xi)^2/4kt} d\xi.$$

4. With the given $f(x)$ we get

$$a_\omega = \frac{1}{\pi} \int_{-1}^{1} e^{-\xi} \cos(\omega\xi) d\xi = \frac{2}{\pi} \frac{\cos(\omega)\sinh(1) + \omega\sin(\omega)\cosh(1)}{\omega^2+1},$$

and

$$b_\omega = \frac{1}{\pi} \int_{-1}^{1} e^{-\xi} \sin(\omega\xi) d\xi = \frac{2}{\pi} \frac{\omega\cos(\omega)\sinh(1) - \sin(\omega)\cosh(1)}{\omega^2+1}.$$

Substitution into the integral formula gives the solution

$$u(x,t) = \frac{2}{\pi} \int_0^\infty \left[\frac{\cos(\omega)\sinh(1) + \omega\sin(\omega)\cosh(1)}{\omega^2+1} \cos(\omega x) \right.$$

$$\left. + \frac{\omega\cos(\omega)\sinh(1) - \sin(\omega)\cosh(1)}{\omega^2+1} \sin(\omega x) \right] e^{-\omega^2 kt} d\omega.$$

For the Fourier transform solution, we write $f(x) = e^{-x}[H(x+1) - H(x-1)]$, to get solution

$$u(x,t) = \frac{1}{2\sqrt{\pi kt}} \int_{-\infty}^{\infty} e^{-\xi}[H(\xi+1) - H(\xi-1)] e^{-(x-\xi)^2/4kt} d\xi$$

$$= \frac{1}{2\sqrt{\pi kt}} \int_{-1}^{1} e^{-\xi} e^{-(x-\xi)^2/4kt} d\xi.$$

5. For the given $f(x) = e^{-\alpha x}$ we calculate

$$b_\omega = \frac{2}{\pi} \int_0^\infty e^{-\alpha\xi} \sin(\omega\xi) d\xi = \frac{2}{\pi} \frac{\omega}{\omega^2+\alpha^2}.$$

Substitution gives the solution

$$u(x,t) = \frac{2}{\pi} \int_0^\infty \left(\frac{\omega}{\omega^2+\alpha^2} \right) \sin(\omega x) e^{-k\omega^2 t} d\omega.$$

6. For the given $f(x) = xe^{-\alpha x}$ we calculate

$$b_\omega = \frac{2}{\pi} \int_0^\infty \xi e^{-\alpha\xi} \sin(\omega\xi) d\xi = \frac{2}{\pi} \frac{2\alpha\omega}{(\alpha^2+\omega^2)^2}.$$

Substitution gives the solution

$$u(x,t) = \frac{2}{\pi} \int_0^\infty \left(\frac{2\alpha\omega}{(\alpha^2+\omega^2)^2} \right) \sin(\omega x) e^{-k\omega^2 t} d\omega.$$

7. For the given $f(x)$ we calculate

$$b_\omega = \frac{2}{\pi} \int_0^h \sin(\omega\xi) d\xi = \frac{2}{\pi} \frac{1-\cos(\omega h)}{\omega}.$$

Substitution gives the solution

$$u(x,t) = \frac{2}{\pi}\int_0^\infty \left(\frac{1-\cos(\omega h)}{\omega}\right)\sin(\omega x)e^{-k\omega^2 t}d\omega.$$

8. For the given $f(x)$ we calculate

$$b_\omega = \frac{2}{\pi}\int_0^2 \xi\sin(\omega\xi)d\xi = \frac{2}{\pi}\frac{\sin(2\omega)-2\omega\cos(2\omega)}{\omega^2}.$$

Substitution gives the solution

$$u(x,t) = \frac{2}{\pi}\int_0^\infty \left(\frac{\sin(2\omega)-2\omega\cos(2\omega)}{\omega^2}\right)\sin(\omega x)e^{-k\omega^2 t}d\omega.$$

9. Apply a Fourier sine transform on x to get $\frac{dU_s}{dt} + \omega^2 U_s + tU_s = 0$, with $U_s(\omega,0) = \mathcal{F}_s[xe^{-x}] = \frac{2\omega}{(1+\omega^2)^2}$. The solution of this first order linear equation is easily found to be

$$U_s(\omega,t) = \frac{2\omega}{(1+\omega^2)^2}e^{-(\omega^2 t + t^2/2)},$$

and by the inversion formula the solution is

$$u(x,t) = \frac{4}{\pi}\int_0^\infty \frac{\omega}{(1+\omega^2)^2}e^{-(\omega^2 t+t^2/2)}\sin(\omega x)d\omega.$$

10. Apply the Fourier Cosine transform on x to get $\frac{dU_c}{dt}(\omega,t) = -\omega^2 U_c(\omega,t) - U_c(\omega,t) - f(t)$ or equivalently, the first order linear equation $U_c' + (1+\omega^2)U_c = -f(t), U_c(\omega,0) = 0$. The solution of this equation is

$$U_c(\omega,t) = -e^{-(1+\omega^2)t}\int_0^t f(\tau)e^{(1+\omega^2)\tau}d\tau = -f(t) * e^{-(1+\omega^2)t}$$

and by inversion of this transform we get

$$u(x,t) = -\frac{2}{\pi}\int_0^\infty f(t) * e^{-(1+\omega^2)t}\cos(\omega x)d\omega.$$

11. Transform to get $sU(x,s) = kU''(x,s), U(0,s) = \frac{2}{s^3}$. The solution of this second order equation is $U(x,s) = c_1(s)e^{\sqrt{s}x/\sqrt{k}} + c_2(s)e^{-\sqrt{s}x/\sqrt{k}}$. For bounded solutions, take $c_1(s) = 0$, then $U(0,s) = \frac{2}{s^3} = c_2(s)$ gives $U(x,s) = \frac{2}{s^3}e^{-\sqrt{s}x/\sqrt{k}}$. To invert this transform we use the convolution theorem and the results $\mathcal{L}^{-1}\left[\frac{2}{s^2}\right] = 2t$ and $\mathcal{L}^{-1}\left[\frac{1}{s}e^{-k\sqrt{s}}\right] = \operatorname{erfc}\left(\frac{k}{2\sqrt{t}}\right)$ to get solution

$$u(x,t) = \int_0^t 2(t-\tau)\operatorname{erfc}\left(\frac{x}{2\sqrt{k\tau}}\right)d\tau.$$

12. Transform on t to get $sU(x,s) - e^{-x} = kU''(x,s), U(0,s) = 0$. The solution of this equation is $U(x,s) = c_1(s)e^{\sqrt{s}x/\sqrt{k}} + c_2(s)e^{-\sqrt{s}x/\sqrt{k}} + \frac{1}{s-k}e^{-x}$. The requirement of solutions $U(x,s)$ remaining bounded dictates $c_1(s) = 0$, and $U(0,s) = 0 = c_2(s) + \frac{1}{s-k}$, which gives solution

$$U(x,s) = \frac{1}{s-k}e^{-x} - \frac{1}{s-k}e^{-\sqrt{s}x/\sqrt{k}}.$$

Invert this transform to obtain

$$u(x,t) = e^{(kt-x)} - \int_0^t e^{k(t-\tau)}\operatorname{erfc}\left(\frac{x}{2\sqrt{k\tau}}\right) d\tau.$$

Section 18.4 Heat Conduction in an Infinite Cylinder

In Problems 1, 2 and 3, the temperature function in an infinitely long cylinder of radius R with outer surface maintained at temperature zero, $u(R,t) = 0$, with temperature distribution independent of θ, and with initial temperature given by $u(r,0) = f(r)$, has the form $u(r,t) = \sum_{n=1}^{\infty} a_n J_0\left(\frac{j_n r}{R}\right) e^{-j_n^k t/R^2}$ where the coefficients a_n are calculated by

$$a_n = \frac{2}{[J_1(j_n)]^2}\int_0^1 \xi f(R\xi) J_0(j_n\xi)d\xi,$$

where j_n is the n^{th} positive zero of $J_0(z)$.

1. With $R = 1, f(r) = r$, we get coefficients

$$a_n = \frac{2}{[J_1(j_n)]^2}\int_0^1 \xi^2 J_0(j_n\xi)d\xi.$$

These coefficients were computed to 9 decimal places using Maple, then rounded to four decimal places. The first five coefficients are:

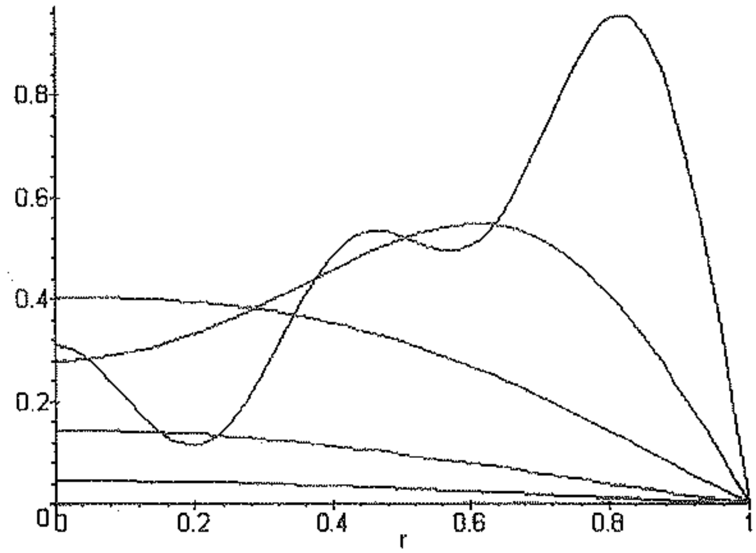
$$a_1 = .8175;\ a_2 = -1.1335;\ a_3 = .7983;\ a_4 = -.7470;\ a_5 = .6315;$$

and the first five terms in the series solution (with $k = 1$) give

$$u(r,t) \approx .8175J_0(2.40483r)e^{-5.7832t} - 1.1335J_0(5.5201r)e^{-30.5588t} + .7983J_0(8.6537r)e^{-74.8791t}$$

$$-.74701J_0(11.7914r)e^{-139.0402t} + .6316J_0(14.9309r)e^{-222.9324t} + \cdots.$$

These approximations are graphed below for $t = .001, .025, .1, .3, .5$ seconds.



2. With $R = 3, f(r) = e^R$, we get coefficients

$$a_n = \frac{2}{[J_1(j_n)]^2} \int_0^1 \xi e^{3\xi} J_0(j_n \xi) d\xi.$$

These coefficients were computed to 9 decimal places using Maple, then rounded to four decimal places. The first five coefficients are:

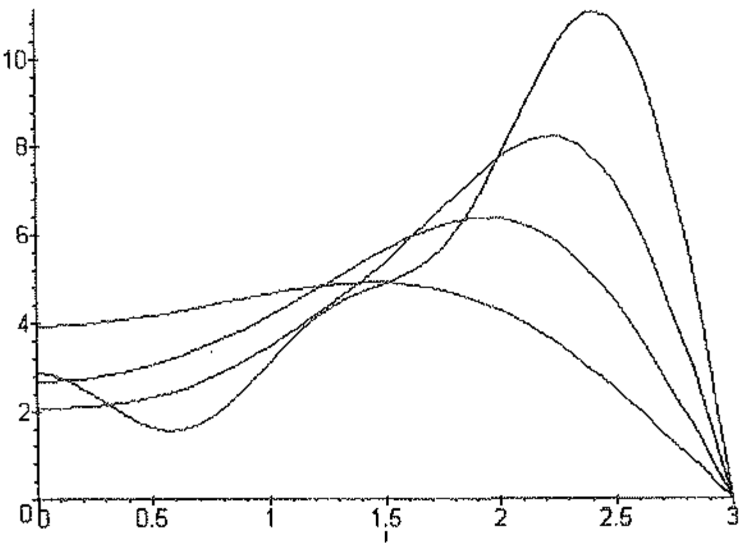
$$a_1 = 9.1181; \; a_2 = -15.3926; \; a_3 = 14.6004; \; a_4 = -13.5432; \; a_5 = 12.3173;$$

and the first five terms in the series solution (with $k = 16$) give

$$u(r,t) \approx 9.1181 J_0(.8016r) e^{-10.2812t} - 15.3926 J_0(1.8400r) e^{-54.1711t} + 14.6004 J_0(2.8846r) e^{-133.1325t}$$

$$-13.5432 J_0(3.9305r) e^{-247.1827t} + 12.3173 J_0(4.9770r) e^{-396.3241t} + \cdots .$$

These approximations are graphed below for $t = .0025, .001, .005, .01, .2$ seconds.



3. With $R = 3, f(r) = 9 - r^2$, we get coefficients

$$4.a_n = \frac{2}{[J_1(j_n)]^2}\int_0^1 \xi(9-(3\xi)^2)J_0(j_n\xi)d\xi.$$

These coefficients were computed to 9 decimal places using Maple, then rounded to four decimal places. The first five coefficients are:

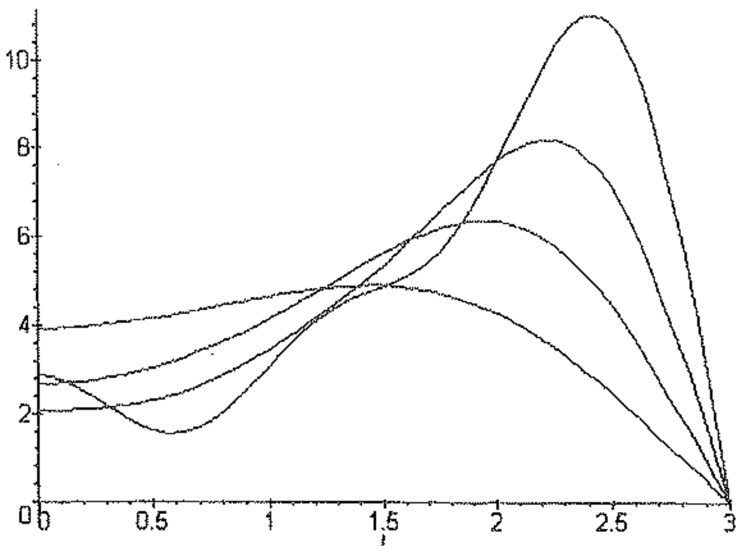
$$a_1 = 9.9722;\ a_2 = -1.2580;\ a_3 = .4093;\ a_4 = -.1889;\ a_5 = .1047;$$

and the first five terms in the series solution (with $k = 1/2$) give

$$u(r,t) \approx 9.9722 J_0(.8016r)e^{-.3213t} - 1.2580 J_0(1.8400r)e^{-1.6929t} + .4093 J_0(2.8846r)e^{-4.1604t}$$

$$-.1889 J_0(3.9305r)e^{-7.7245t} + .1047 J_0(4.9770r)e^{-12.3851t} + \cdots.$$

These approximations are graphed below for $t = .001, .05, .25, .5, 1$ seconds.



4. The boundary value problem to be solved is

$$u_t = k(u_{rr} + \frac{1}{r}u_r), 0 < r < R;\ u_r(R,t) = -Au(R,t); u(r,0) = f(r).$$

Assume that $u(r,t) = F(r)T(t)$ and use separation of variables to get the two problems $F'' + \frac{1}{r}F' + \lambda F = 0, F'(R) + AF(R) = 0$, and $T' + \lambda kT = 0$. Since the solution is to remain bounded in time, there will be no nonpositive eigenvalues. Hence put $\lambda = \omega^2 > 0$ and get $r^2F'' + rF' + r^2k^2F = 0$ which has general solution $F(r) = c_1J_0(\omega r) + c_2Y_0(\omega r)$. Since Y_0 is unbounded at $r = 0$, choose $c_2 = 0$. From the boundary condition at R we get $\omega J_0'(\omega R) + AJ_0(\omega R) = 0$. Since the problem for $F(r)$ is a Sturm Liouville problem on $[0, R]$ it follows that there will be an infinite number of positive solutions, z_n, of the equation $\omega J_0'(\omega R) + AJ_0(\omega R) = 0$ (these solutions provide the eigenvalues) and that the corresponding eigenfunctions will be orthogonal on $[0, R]$. Thus letting $\omega = z_n, n \geq 1$ denote the n^{th} positive solution of $\omega J_0'(\omega R) + AJ_0(\omega R) = 0$ we have $\int_0^R rJ_0(z_nr)J_0(z_mr)dr = 0$ if $n \neq m$. With $\lambda = z_n^2$ we solve for $T(t) = e^{-kz_n^2t}$ and use superposition to obtain $u(r,t) = \sum_{n=1}^{\infty} A_nJ_0(z_nr)e^{-kz_n^2t}$ as a solution of the differential equation and the boundary condition $u_r(R,t) = -Au(R,t)$. To complete the solution we satisfy $u(r,0) = f(r) = \sum_{n=1}^{\infty} A_nJ_0(z_nr)$ by using the orthogonality of $\{J_0(z_nr)\}_{n=1}^{\infty}$ to calculate

$$A_n = \frac{\int_0^R rf(r)J_0(z_nr)dr}{\int_0^R r[J_0(z_nr)]^2dr}, n \geq 1.$$

Chapter Nineteen - The Potential Equation

Section 19.1 Harmonic Functions and the Dirichlet Problem

1. Suppose that both f and g are harmonic on a set D in the plane. Then $f_{xx}+f_{yy}=0$ and $g_{xx}+g_{yy}=0$ throughout D. Now consider $(\alpha f+\beta g)_{xx}+(\alpha f+\beta g)_{yy}=\alpha(f_{xx}+f_{yy})+\beta(g_{xx}+g_{yy})=0+0=0$ throughout D. It follows that linear combinations of harmonic functions are harmonic.

2. (a) Calculate $(x^3-3xy^2)_{xx}+(x^3-3xy^2)_{yy}=6x-6x=0$;
(b) Calculate $(3x^2y-y^3)_{xx}+(3x^2y-y^3)_{yy}=6y-6y=0$;
(c) Calculate $(x^4-6x^2y^2+y^4)_{xx}+(x^4-6x^2y^2+y^4)_{yy}=(12x^2-12y^2)+(-12x^2+12y^2)=0$;
(d) Calculate $(4x^3y-4xy^3)_{xx}+(4x^3y-4xy^3)_{yy}=24xy-24xy=0$;
(e) Calculate $(\sin(x)\cosh(y))_{xx}+(\sin(x)\cosh(y))_{yy}=-\sin(x)\cosh(y)+\sin(x)\cosh(y)=0$;
(f) Calculate $(\cos(x)\sinh(y))_{xx}+(\cos(x)\sinh(y))_{yy}=-\cos(x)\sinh(y)+\cos(x)\sinh(y)=0$;
(g) Calculate $(e^{-x}\cos(y))_{xx}+(e^{-x}\cos(y))_{yy}=e^{-x}\cos(y)+e^{-x}(-\cos(y))=0$;
It follows that all these functions are harmonic throughout the entire plane.

3. Calculate $\dfrac{\partial}{\partial x}\ln(x^2+y^2)=\dfrac{2x}{x^2+y^2}$; $\dfrac{\partial}{\partial y}\ln(x^2+y^2)=\dfrac{2y}{x^2+y^2}$; and then second partials $(\ln(x^2+y^2))_{xx}=\dfrac{2(x^2+y^2)-4x^2}{(x^2+y^2)^2}=\dfrac{2y^2-2x^2}{(x^2+y^2)^2}$; $(\ln(x^2+y^2))_{yy}=\dfrac{2(x^2+y^2)-4y^2}{(x^2+y^2)^2}=\dfrac{2x^2-2y^2}{(x^2+y^2)^2}$. All these partials are continuous at all points except $(x,y)=(0,0)$ and clearly the sum of second partials is zero.

4. The Laplacian in polar coordinates is given by $\nabla_{r\theta}=\dfrac{\partial^2}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2}{\partial\theta^2}$. When applied to $r^n\cos(n\theta)$ and $r^n\sin(n\theta)$ gives

$$\nabla_{r\theta}(r^n\cos(n\theta))=n(n-1)r^{n-2}\cos(n\theta)+\frac{1}{r}nr^{n-1}\cos(n\theta)+\frac{1}{r^2}r^n(-n^2\cos(n\theta))$$

$$=r^{n-2}\cos(n\theta)[n(n-1)+n-n^2]=0.$$

Similarly

$$\nabla_{r\theta}(r^n\sin(n\theta))=n(n-1)r^{n-2}\sin(n\theta)+\frac{1}{r}nr^{n-1}\sin(n\theta)+\frac{1}{r^2}r^n(-n^2\sin(n\theta))$$

$$=r^{n-2}\sin(n\theta)[n(n-1)+n-n^2]=0.$$

5. These functions are not defined at $r=0$, so we exclude the origin. Calculate

$$\nabla_{r\theta}(r^n\cos(n\theta))=-n(-n-1)r^{-n-2}\cos(n\theta)+\frac{1}{r}(-n)r^{-n-1}\cos(n\theta)+\frac{1}{r^2}r^{-n}(-n^2\cos(n\theta))$$

$$=r^{n-2}\cos(n\theta)[n(n+1)-n-n^2]=0.$$

Similarly

$$\nabla_{r\theta}(r^n \sin(n\theta)) = -n(-n-1)r^{-n-2}\sin(n\theta) + \frac{1}{r}(-n)r^{-n-1}\sin(n\theta) + \frac{1}{r^2}r^{-n}(-n^2\sin(n\theta))$$

$$= r^{n-2}\sin(n\theta)[n(n+1) - n - n^2] = 0.$$

Section 19.2 Dirichlet Problem for a Rectangle

1. Let $u(x,y) = X(x)Y(y)$ and separate variables to get the two ordinary differential equation problems $X'' + \lambda X = 0, X(0) = 0, X(1) = 0; Y'' - \lambda Y = 0, Y(\pi) = 0$. The regular Sturm Liouville problem for X is a familiar one with eigenvalues $\lambda_n = n^2\pi^2, n = 1, 2, \ldots$ with corresponding eigenfunctions $X_n(x) = \sin(n\pi x)$. With these values of λ, the problem in Y has solutions which can be expressed in terms of hyperbolic functions as $Y_n(y) = a_n \sinh(n\pi y) + b_n \sinh(n\pi(\pi - y))$. The zero boundary condition along $y = \pi$ gives $Y_n(\pi) = a_n \sinh(n\pi^2) = 0$ so choose $a_n = 0$. By superposition we get $u(x,y) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)\sinh(n\pi(\pi - y))$ which satisfies the partial differential equation and the zero boundary conditions along three edges, $x = 0, x = 1, y = \pi$. On the edge $y = 0$ we get $u(x,0) = \sin(\pi x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)\sinh(n\pi^2)$, which we can satisfy by choosing $b_n \sinh(n\pi^2)$ as the Fourier coefficients of $\sin(\pi x)$. Thus take $b_1 = \dfrac{1}{\sinh(\pi^2)}, b_n = 0, n \neq 1$ to get the solution

$$u(x,y) = \frac{1}{\sinh(\pi^2)}\sin(\pi x)\sinh(\pi(\pi - y)).$$

2. The pair of homogeneous boundary conditions on the edges $y = 0$ and $y = 2$, with separation of variables, gives the problem $Y'' + \lambda Y = 0, Y(0) = 0, Y(2) = 0$; which has eigenvalues and eigenfunctions $\lambda_n = \dfrac{n^2\pi^2}{4}, Y_n(y) = \sin(n\pi y/2), n = 1, 2, \ldots$. We write the corresponding solution for $X_n(x) = a_n \sinh(n\pi x/2) + b_n \sinh(n\pi(3-x)/2)$. Superposition gives the solution $u(x,y) = \sum_{n=1}^{\infty}[a_n \sinh(n\pi x/2) + b_n \sinh(n\pi(3-x)/2)]\sin(n\pi y/2)$. The boundary condition on $x = 3$ gives $u(3,y) = 0 = \sum_{n=1}^{\infty}[a_n \sinh(n\pi 3/2)]\sin(n\pi y/2)$, so choose $a_n = 0,\ n \geq$

On edge $x = 0$ we have $u(0,y) = y(2-y) = \sum_{n=1}^{\infty} b_n \sinh(3n\pi/2)\sin(n\pi y/2)$. To complete the solution, calculate

$$b_n = \frac{1}{\sinh(3n\pi/2)}\int_0^2 y(2-y)\sin(n\pi y/2)dy = \frac{1}{\sinh(3n\pi/2)}\frac{16[1-(-1)^n]}{n^3\pi^3}, n \geq 1.$$

The solution is

$$u(x,y) = \frac{16}{\pi^3}\sum_{n=1}^{\infty}\frac{[1-(-1)^n]}{n^3}\sin\left(\frac{n\pi y}{2}\right)\frac{\sinh\left(\frac{n\pi}{2}(3-x)\right)}{\sinh\left(\frac{3n\pi}{2}\right)}$$

Before solving the remainder of these problems, some general comments are in order. Separation of variables on Laplace's equation in Cartesian coordinates will always give rise to the pair of differential equations $X'' + \lambda X = 0, Y'' - \lambda Y = 0$. The sign of λ will be determined by the boundary conditions in such a way that the trigonometric terms go with the equation on which opposite coordinate faces have homogeneous boundary conditions, *i.e.* $X(0) = 0, X(L) = 0$ tell us to choose $\lambda = \alpha^2 > 0$ with the X equation. The other coordinate equation will then have exponential solutions, which we choose to write in terms of the hyperbolic function 'sinh' with shifted argument (if necessary). This makes the form of the solution which satisfies three homogeneous boundary conditions immediately evident. The inclusion of the final hyperbolic factor in the general form of the solution makes the final determination of Fourier coefficients quite routine. These ideas are used in the remainder of the problems in this section.

3. The homogeneous boundary conditions on the opposite edges $x = 0, x = 1$ and on the edge $y = 0$, gives a solution of the general form

$$u(x,y) = \sum_{n=1}^{\infty} a_n \sin(n\pi x)\frac{\sinh(n\pi y)}{\sinh(4n\pi)}$$

this function clearly satisfies the boundary conditions on the edges $x = 0, x = 1, y = 0$. To complete the solution evaluate $u(x,4) = \sum_{n=1}^{\infty} a_n \sin(n\pi x)$ and calculate

$$a_n = \frac{1}{2}\int_0^4 x\cos\left(\frac{\pi x}{2}\right)\sin(n\pi x)dx = \frac{8}{\pi}\frac{(2n^3+n)}{(4n^2-1)}, n \geq 1.$$

The solution is given by

$$u(x,y) = \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{(2n^3+n)}{(4n^2-1)}\sin(n\pi x)\frac{\sinh(n\pi y)}{\sinh(4n\pi)}$$

4. Since no pair of opposite coordinate edges have homogeneous boundary conditions, we separate this problem into two problems. Let $u(x,y) = v(x,y) + w(x,y)$ where

$$\begin{array}{ll} \nabla^2 v = 0, 0 < x < \pi, 0 < y < \pi & \nabla^2 w = 0, 0 < x < \pi, 0 < y < \pi \\ v(x,0) = v(x,\pi) = 0 & w(0,y) = w(\pi,y) = 0 \\ v(\pi,y) = 0, v(0,y) = \sin(y) & w(x,\pi) = 0, w(x,0) = x(\pi - x) \end{array}$$

The problem for v has the general form

$$v(x,y) = \sum_{n=1}^{\infty} a_n \sin(ny)\frac{\sinh(n(\pi - y))}{\sinh(n\pi)}$$

and this satisfies the three homogeneous boundary conditions. evaluate $v(0,y) = \sin(y) = \sum_{n=1}^{\infty} a_n \sin(ny)$ and we see immediately that $a_1 = 1, a_n = 0, n \geq 2$. Thus $v(x,y) = \sin(y)\dfrac{\sinh(\pi - x)}{\sinh(\pi)}$.

For w we find a solution of the form

$$w(x,y) = \sum_{n=1}^{\infty} b_n \sin(nx) \frac{\sinh(n(\pi - y))}{\sinh(n\pi)}.$$

Evaluate $w(x,0) = x(\pi - x) = \sum_{n=1}^{\infty} b_n \sin(nx)$. Calculate

$$b_n = \frac{2}{\pi} \int_0^{\pi} x(\pi - x) \sin(nx) dx = \frac{4}{n^3\pi}[1 - (-1)^n], n \geq 1.$$

The final solution is

$$u(x,y) = \sin(y) \frac{\sinh(\pi - x)}{\sinh(\pi)} + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{[1-(-1)^n]}{n^3} \sin(nx) \frac{\sinh(n(\pi - y))}{\sinh(n\pi)}$$

5. Since no pair of opposite coordinate edges have homogeneous boundary conditions, we separate this problem into two problems. Let $u(x,y) = v(x,y) + w(x,y)$ where

$$\begin{array}{ll} \nabla^2 v = 0, 0 < x < 2, 0 < y < \pi & \nabla^2 w = 0, 0 < x < 2, 0 < y < \pi \\ v(0,y) = v(\pi, y) = 0 & w(x,0) = w(x,\pi) = 0 \\ v(x,0) = 0, v(x,\pi) = x\sin(\pi x) & w(0,x) = 0, w(0,2) = \sin(y) \end{array}$$

Begin with the problem for w which has the form

$$w(x,y) = \sum_{n=1}^{\infty} b_n \sin(ny) \frac{\sinh(nx)}{\sinh(2n)}.$$

Evaluate $w(2,y) = \sin(y) = \sum_{n=1}^{\infty} b_n \sin(ny)$ and we see immediately that $b_1 = 1, b_n = 0, n \geq 2$.
Thus $w(x,y) = \sin(y)\frac{\sinh(x)}{\sinh(2)}$.
For $v(x,y)$ we find a solution of the form

$$v(x,y) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{2}\right) \frac{\sinh(\frac{n\pi y}{2})}{\sinh(\frac{n\pi^2}{2})}.$$

Evaluate $v(x,\pi) = x\sin(\pi x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{2}\right)$, so calculate

$$a_n = \int_0^2 x\sin(\pi x)\sin\left(\frac{n\pi x}{2}\right) dx = \frac{16n}{\pi^2(n^2-4)^2}[(-1)^n - 1], n \neq 2, a_2 = 1.$$

The complete solution is

$$u(x,y) = \sin(y)\frac{\sinh(x)}{\sinh(2)} + \sin(\pi x)\frac{\sinh(\pi y)}{\sinh(\pi^2)} + \frac{16}{\pi^2} \sum_{\substack{n=1 \\ n \neq 2}}^{\infty} \frac{n}{(n^2-4)^2}[(-1)^n - 1] \sin\left(\frac{n\pi x}{2}\right) \frac{\sinh(\frac{n\pi y}{2})}{\sinh(\frac{n\pi^2}{2})}.$$

6. The homogeneous boundary conditions on opposite edges $y = 0, y = b$ and the homogeneous edge condition at $x = 0$ give a solution of the general form

$$u(x,y) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{(2n-1)\pi}{2}\frac{y}{b}\right) \frac{\sinh\left(\frac{(2n-1)\pi}{2}\frac{x}{b}\right)}{\sinh\left(\frac{(2n-1)\pi}{2}\frac{a}{b}\right)}$$

which satisfies the three homogeneous boundary conditions. Evaluate $u(a,y) = g(y) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{(2n-1)\pi}{2}\frac{y}{b}\right)$. Finally calculate

$$a_n = \frac{2}{b}\int_0^b g(y) \sin\left(\frac{(2n-1)\pi}{2}\frac{y}{b}\right), n \geq 1.$$

7. The homogeneous boundary conditions on opposite edges $x = 0, x = a$ and the homogeneous edge condition at $y = 0$ give a solution of the general form

$$u(x,y) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{(2n-1)\pi}{2}\frac{x}{a}\right) \frac{\sinh\left(\frac{(2n-1)\pi}{2}\frac{y}{a}\right)}{\sinh\left(\frac{(2n-1)\pi}{2}\frac{b}{a}\right)}$$

which satisfies the three homogeneous boundary conditions. Evaluate $u(x,b) = f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{(2n-1)\pi}{2}\frac{x}{a}\right)$. Finally calculate

$$a_n = \frac{2}{a}\int_0^a g(y) \sin\left(\frac{(2n-1)\pi}{2}\frac{x}{a}\right), n \geq 1.$$

8. The homogeneous boundary conditions on edges $x = 0, x = a$ and $y = 0$ give a solution of the general form

$$u(x,y) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{a}\right) \frac{\sinh\left(\frac{n\pi y}{a}\right)}{\sinh\left(\frac{n\pi b}{a}\right)}$$

Evaluate $u(x,b) = x(x-a)^2 = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{a}\right)$, and calculate

$$a_n = \frac{2}{a}\int_0^a x(x-a)^2 \sin\left(\frac{n\pi x}{a}\right) dx = \frac{4}{n^3\pi^3}[1 + 2n\pi(-1)^n], n \geq 1.$$

The solution is

$$u(x,y) = \frac{4}{\pi^3}\sum_{n=1}^{\infty} \frac{[1 + 2n\pi(-1)^n]}{n^3} \sin\left(\frac{n\pi x}{a}\right) \frac{\sinh\left(\frac{n\pi y}{a}\right)}{\sinh\left(\frac{n\pi b}{a}\right)}$$

9. Here again we decompose the problem into two simpler problems with three homogeneous boundary conditions. Let $u(x,y) = v(x,y) + w(x,y)$ where

$$\begin{array}{ll} \nabla^2 v = 0, 0 < x < 4, 0 < y < 1 & \nabla^2 w = 0, 0 < x < 4, 0 < y < 1 \\ v(x,0) = v(x,1) = 0 & w(x,0) = w(x,1) = 0 \\ v(4,y) = 0, v(0,y) = \sin(\pi y) & w(0,y) = 0, w(4,y) = y(1-y) \end{array}$$

The problem for v has solution of the form

$$v(x,y) = \sum_{n=1}^{\infty} \sin(n\pi y)\frac{\sinh(n\pi(4-x))}{\sinh(4n\pi)}$$

Evaluate $v(0,y) = \sin(\pi y) = \sum_{n=1}^{\infty} a_n \sin(n\pi y)$, from which we see immediately that $a_1 = 1, a_n = 0, n \geq 2$. Thus

$$v(x,y) = \sin(\pi y)\frac{\sinh(\pi(4-x))}{\sinh(4\pi)}$$

The solution for w has the form

$$w(x,y) = \sum_{n=1}^{\infty} b_n \sin(n\pi y)\frac{\sinh(\pi x)}{\sinh(4\pi)}$$

Evaluate $w(4,y) = y(1-y) = \sum_{n=1}^{\infty} b_n \sin(n\pi y)$, so calculate

$$b_n = 2\int_0^1 y(1-y)\sin(n\pi y)dy = \frac{8}{\pi}\frac{(2n^3+n)}{(4n^2-1)^2}, n \geq 1.$$

The complete solution has the form

$$u(x,y) = \sin(\pi y)\frac{\sinh(\pi(4-x))}{\sinh(4\pi)} + \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{(2n^3+n)}{(4n^2-1)^2}\sin(n\pi y)\frac{\sinh(\pi x)}{\sinh(4\pi)}$$

Section 19.3 Dirichlet Problem for a Disk

Problems 1 through 8 all have solution of the form $u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n r^n \cos(n\theta) + b_n r^n \sin(n\theta)$, where the coefficients a_n and b_n are determined to be Fourier coefficients of the given boundary function $f(\theta)$.

1. Because of the simple form of $f(\theta) = 1$ we easily match coefficients getting $a_0/2 = 1$, all other coefficients being zero. The solution is $u(r,\theta) = 1$.

2. Here again we can easily match the nonzero coefficient getting $a_4(3)^4 = 8$, all other coefficients are zero. The solution is $u(r,\theta) = 8\left(\frac{r}{3}\right)^4\cos(4\theta)$.

3. Calculate $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}(\xi^2-\xi)d\xi = \frac{2\pi^2}{3}$,
$a_n = \frac{1}{2^n\pi}\int_{-\pi}^{\pi}(\xi^2-\xi)\cos(n\xi)d\xi = \frac{4(-1)^n}{n^2 2^n}, n \geq 1$;
$b_n = \frac{1}{2^n\pi}\int_{-\pi}^{\pi}(\xi^2-\xi)\sin(n\xi)d\xi = \frac{2(-1)^n}{n2^n}, n \geq 1$.

The solution is

$$u(r,\theta)=\frac{\pi^2}{3}+2\sum_{n=1}^{\infty}\left(\frac{r}{2}\right)^n\frac{(-1)^n}{n^2}(2\cos(n\theta)+n\sin(n\theta).$$

4. Calculate $a_n=\dfrac{1}{\pi 5^n}\displaystyle\int_{-\pi}^{\pi}\xi\cos(\xi)\cos(n\xi)d\xi=0,n\geq 1;$
$b_n=\dfrac{1}{\pi 5^n}\displaystyle\int_{-\pi}^{\pi}\xi\cos(\xi)\sin(n\xi)d\xi=\dfrac{2(-1)^n(n^3-n)}{(n^2-1)^2 5^n},n\geq 2,$
$b_1=\dfrac{1}{5\pi}\displaystyle\int_{-\pi}^{\pi}\xi\cos(\xi)\sin(\xi)d\xi=-\dfrac{1}{10}.$

The solution is

$$u(r,\theta)=-\frac{r}{10}\sin(\theta)+2\sum_{n=2}^{\infty}\frac{(-1)^n(n^3-n)}{(n^2-1)^2}\left(\frac{r}{5}\right)^n\sin(n\theta)$$

5. Calculate $a_0=\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi}e^{-\xi}d\xi=2\sinh(\pi)/\pi,$
$a_n=\dfrac{1}{4^n\pi}\displaystyle\int_{-\pi}^{\pi}e^{-\xi}\cos(n\xi)d\xi=\dfrac{2\cosh(\pi)}{\pi}\dfrac{(-1)^n}{4^n(n^2+1)},n\geq 1;$
$b_n=\dfrac{1}{4^n\pi}\displaystyle\int_{-\pi}^{\pi}e^{-\xi}\sin(n\xi)d\xi=\dfrac{2\sinh(\pi)}{\pi}\dfrac{n(-1)^n}{4^n(n^2+1)},n\geq 1.$

The solution is

$$u(r,\theta)=\frac{\sinh(\pi)}{\pi}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}\left(\frac{r}{4}\right)^n[\cosh(\pi)\cos(n\theta)+n\sinh(\pi)\sin(n\theta)]$$

6. Write $\sin^2(\theta)=\frac{1}{2}(1-\cos(2\theta))$ and we can easily identify nonzero coefficients as $a_0=1,a_2=-\frac{1}{2}$ (all others being zero) to get the solution $u(r,\theta)=\dfrac{1}{2}-\dfrac{r}{2}\cos(2\theta).$

7. Calculate $a_0=\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi}d\xi=2,$
$a_n=\dfrac{1}{\pi 8^n}\displaystyle\int_{-\pi}^{\pi}(1-\xi^3)\cos(n\xi)d\xi=0,n\geq 1;$
$b_n=\dfrac{1}{\pi 8^n}\displaystyle\int_{-\pi}^{\pi}(1-\xi^3)\sin(n\xi)d\xi=\dfrac{4(-1)^{n+1}}{\pi n^4 8^n}(6n-n^3\pi^3),n\geq 1.$

The solution is

$$u(r,\theta)=1+\frac{4}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(6n-n^3\pi^3)}{n^4}\left(\frac{r}{8}\right)^n\sin(n\theta)$$

8. Calculate $a_0=\dfrac{1}{pi}\displaystyle\int_{-\pi}^{\pi}\xi e^{2\xi}d\xi=\dfrac{2\cosh(2\pi)-\sinh(2\pi)}{2\pi},$
$a_n=\dfrac{1}{\pi 4^n}\displaystyle\int_{-\pi}^{\pi}\xi e^{2\xi}\cos(n\xi)d\xi=\dfrac{2(-1)^n}{\pi(n^2+4)^2 4^n}[\cosh(2\pi)(8\pi+2\pi n^2)+\sinh(2\pi)(n^2+4)],n\geq 1;$

$$b_n = \frac{1}{\pi 4^n}\int_{-\pi}^{\pi}\xi e^{2\xi}\sin(n\xi)d\xi = \frac{2(-1)^n}{\pi(n^2+4)^2 4^n}[\cosh(2\pi)(4n\pi + n^3\pi^3) - \sinh(2\pi)(4n)], n \geq 1.$$

The solution is

$$u(r,\theta) = \frac{2\cosh(2\pi) - \sinh(2\pi)}{4\pi}$$

$$+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{(n^2+4)^2}\left(\frac{r}{4}\right)^n[\cosh(2\pi)(8\pi + 2\pi n^2) + \sinh(2\pi)(n^2+4)]\cos(n\theta)$$

$$+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{(n^2+4)^2}\left(\frac{r}{4}\right)^n[\cosh(2\pi)(4n\pi + n^3\pi^3) - \sinh(2\pi)(4n)]\sin(n\theta)$$

In problems 9 through 12, we can actually identify the Fourier coefficients in the general solution $u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n r^n\cos(n\theta) + b_n r^n\sin(n\theta)$ by simply matching like terms, since the boundary function $f(\theta)$ is of such a simple form in each problem. For more complicated $f(\theta)$ we would use the orthogonality properties and compute the Fourier coefficients.

9. The Dirichlet problem in polar coordinates is $\nabla^2 u(r,\theta) = 0, 0 \leq r < 4, -\pi \leq \theta \leq \pi, u(4,\theta) = 16\cos^2(\theta) = 8(1+\cos(2\theta))$. Identify $\frac{a_0}{2} = 8, a_2(4)^2 = 8$, all other coefficients being zero. We get solution

$$u(r,\theta) = 8 + 8\left(\frac{r}{4}\right)^2\cos(2\theta).$$

Converting back to rectangular coordinates using $x = r\cos(\theta), y\ r\sin(\theta)$ and a trig identity for $\cos(2\theta) = 2\cos^2(\theta) - 1$ gives $u(x,y) = 8 + \frac{1}{2}(x^2 - y^2)$.

10. The Dirichlet problem in polar coordinates is $\nabla^2 u(r,\theta) = 0, 0 \leq r < 3, -\pi \leq \theta \leq \pi, u(3,\theta) = 3(\cos(\theta) - \sin(\theta))$. Identify $3 = 3a_1, 3 = 3b_1$, all other coefficients being zero. We get solution

$$u(r,\theta) = r(\cos(\theta) - \sin(\theta)).$$

In terms of rectangular coordinates this is easily seen to be $u(x,y) = x - y$.

11. The Dirichlet problem in polar coordinates is $\nabla^2 u(r,\theta) = 0, 0 \leq r < 2, -\pi \leq \theta \leq \pi, u(2,\theta) = 4(\cos^2(\theta) - \sin^2(\theta)) = 4\cos(2\theta)$. Identify $4 = a_2(2)^2$, all other coefficients being zero to get solution

$$u(r,\theta) = r^2\cos(2\theta).$$

In rectangular coordinates we easily get $u(x,y) = x^2 - y^2$.

12. The Dirichlet problem in polar coordinates is $\nabla^2 u(r,\theta) = 0, 0 \leq r < 5, -\pi \leq \theta \leq \pi, u(5,\theta) = 25\sin(\theta)\cos(\theta) = \frac{25}{2}\sin(2\theta)$. Identify $\frac{25}{2} = a_2(5)^2$, all other coefficients being zero to get solution

$$u(r,\theta) = \frac{1}{2}r^2\sin(2\theta).$$

To convert to rectangular coordinates, write $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$ to get $u(x,y) = xy$.

Section 19.4 Poisson's Integral Formula for the Disk

1. From Poisson's integral formula (19.2) with $R=1, f(\theta)=\theta$, we get the integral formula

$$u(r,\theta)=\frac{1-r^2}{2\pi}\int_{-\pi}^{\pi}\frac{\xi}{1+r^2-2r\cos(\xi-\theta)}d\xi.$$

the requested numerical values are $u(1/2,\pi)=.1256637\times 10^{-11}; u(3/4,\pi/3)=.882613, u(0.2,\pi/4)=.2465422;$

2. From Poisson's integral formula (19.2) with $R=4, f(\theta)=\sin(4\theta)$, we get the integral formula

$$u(r,\theta)=\frac{16-r^2}{2\pi}\int_{-\pi}^{\pi}\frac{\sin(4\xi)}{16+r^2-8r\cos(\xi-\theta)}d\xi.$$

the requested numerical values are $u(1,\pi/6)=.0033829; u(3,7\pi/2)=.30997\times 10^{-12}, u(1,\pi/4)=.4105\times 10^{-12}, u(2.5,\pi/12)=.132145$

3. From Poisson's integral formula (19.2) with $R=15, f(\theta)=\theta^3-\theta$, we get the integral formula

$$u(r,\theta)=\frac{225-r^2}{2\pi}\int_{-\pi}^{\pi}\frac{\xi^3-\xi}{225+r^2-30r\cos(\xi-\theta)}d\xi.$$

the requested numerical values are $u(4,\pi)=.837758\times 10^{-12}, u(12,3\pi/2)=-2.571176, u(8,\pi/4)=.59705, u(7,0)=-.628319\times 10^{-11}$

4. From Poisson's integral formula (19.2) with $R=6, f(\theta)=e^{-\theta}$, we get the integral formula

$$u(r,\theta)=\frac{36-r^2}{2\pi}\int_{-\pi}^{\pi}\frac{e^{-\xi}}{36+r^2-12r\cos(\xi-\theta)}d\xi.$$

the requested numerical values are $u(5.5,3\pi/5)=.409013, u(4,2\pi/7)=1.174463, u(1,\pi)=4.333381, u(4,\pi/4)=1.209883$

5. If f has period P then $\int_a^{a+P}f(x)dx=\int_0^P f(x)dx$ for every value of a. If the function $f(\xi)$ which appears in Poisson's integral formula has period 2π, we can apply this result to write Poisson's integral formula given in equation (18.2) as

$$u(r,\theta)=\frac{1}{2\pi}\int_0^{2\pi}\frac{R^2-r^2}{R^2+r^2-2Rr\cos(\xi-\theta)}f(\xi)d\xi.$$

This idea was used to get the formula given in the statement of this problem since $\sin(n\theta)$ is period with period 2π. Following the suggestion in the Problem we find (after a little simplifying)

$$u(R/2,\pi/2)=\frac{R^n}{2^n}\sin\left(\frac{n\pi}{2}\right)=\frac{R^n}{2\pi}\int_0^{2\pi}\frac{3\sin(n\xi)}{5-4\cos(\xi-\frac{\pi}{2})}d\xi.$$

But $\cos(\xi-\frac{\pi}{2})=\sin(\xi)$, and a bit more simplifying gives

$$\int_0^{2\pi}\frac{\sin(n\xi)}{5-4\sin(\xi)}d\xi=\frac{\pi}{3(2^{n-1})}\sin\left(\frac{n\pi}{2}\right).$$

6. From Problem 5 we get

$$u(R/2,\pi)=\frac{R^n}{2^n}\sin(n\pi)=\frac{R^n}{2\pi}\int_0^{2\pi}\frac{3/4\sin(n\xi)}{5/4-\cos(\pi-\xi)}d\xi,$$

which simplifies to give

$$\int_0^{2\pi}\frac{\sin(n\xi)}{5+4\cos(\xi)}d\xi=0.$$

7. Using known solution $u(r,\theta)=r^n\cos(n\theta)$ in Poisson's integral formula gives the result

$$u(r,\theta)=r^n\cos(n\theta)=\frac{R^n}{2\pi}\int_0^{2\pi}\frac{(R^2-r^2)\cos(n\xi)}{R^2+r^2-2Rr\cos(\xi-\theta)}d\xi.$$

Evaluate $u(R/2,\pi/2)=\frac{R^n}{2^n}\cos\left(\frac{n\pi}{2}\right)=\frac{R^n}{2\pi}\int_0^{2\pi}\frac{\frac{3}{4}\cos(n\xi)}{\frac{5}{4}-\cos(\xi-\frac{\pi}{2})}d\xi$, which simplifies nicely to give

$$\int_0^{2\pi}\frac{\cos(n\xi)}{5-4\sin(\xi)}d\xi=\frac{\pi}{3(2^{n-1})}\cos\left(\frac{n\pi}{2}\right).$$

Similarly we find by evaluating $(u(R/2,\pi)$ and simplifying that

$$\int_0^{2\pi}\frac{\cos(n\xi)}{5+4\cos(\xi)}d\xi=\frac{\pi(-1)^n}{3(2^{n-1})}.$$

8. With $u(r,\theta)=1$ as the solution of the Dirichlet problem, the boundary function $f(\theta)=1$, from which we get the integration formula

$$\int_0^{2\pi}\frac{R^2-r^2}{R^2+r^2-2Rr\cos(\xi-\theta)}d\xi=2\pi.$$

Section 19.5 Dirichlet Problems in Unbounded Regions

1. Use the integral formula for the solution on the upper half plane to get

$$u(x,y)=\frac{y}{\pi}\left[\int_{-4}^{0}\frac{-1}{y^2+(\xi-x)^2}d\xi+\int_0^4\frac{1}{y^2+(\xi-x)^2}d\xi\right]$$

$$=\frac{y}{\pi}\int_0^4\left[\frac{-1}{y^2+(\xi+x)^2}+\frac{1}{y^2+(\xi-x)^2}\right]d\xi$$

$$=\frac{1}{\pi}\left[2\mathrm{Tan}^{-1}\left(\frac{x}{y}\right)-\mathrm{Tan}^{-1}\left(\frac{4+x}{y}\right)+\mathrm{Tan}^{-1}\left(\frac{4-x}{y}\right)\right],\ \text{for } -\infty<x<\infty,\ y>0$$

2. The integral formula for the upper half plane gives solution

$$u(x,y)=\frac{y}{\pi}\int_{-\infty}^{\infty}\frac{e^{-|x|}}{y^2+(\xi-x)^2}d\xi,\ -\infty<x<\infty,y>0$$

3. From the formula developed in the text for the right quarter plane, we can write the integral solution of this problem as

$$u(x,y) = \frac{2}{\pi}\int_0^\infty \left[\frac{1}{y^2+(\xi-x)^2} - \frac{1}{y^2+(\xi+x)^2}\right] e^{-\xi}\cos(\xi)d\xi.$$

4. First separate variables by assuming a product function $u(x,y) = X(x)Y(y)$ to get the two problems $X'' - \omega^2 X = 0, Y'' + \omega^2 Y = 0$. To determine values of ω^2 we exploit the one homogeneous boundary condition $u(x,0) = X(x)Y(0) = 0$ and the condition that $X(x)$ remains bounded as $x \to \infty$ to obtain product solutions of the form $X(\omega)Y(\omega) = B(\omega)\sin(\omega y)e^{\omega x}$ for each $\omega > 0$. By superposition over all ω values, we find $u(x,y) = \int_0^\infty B(\omega)\sin(\omega y)e^{-\omega x}d\omega$. This expression satisfies Laplace's equation, remains bounded as $x \to \infty$, and $u(x,0) = 0$. It remains to satisfy $u(0,y) = g(y) = \int_0^\infty B(\omega)\sin(\omega y)d\omega$. But this is exactly the sine integral representation of $g(y)$, so take $B(\omega) = \frac{2}{\pi}\int_0^\infty g(\xi)\sin(\omega\xi)d\xi$ to get the integral solution

$$u(x,y) = \frac{2}{\pi}\int_0^\infty \left(\int_0^\infty g(\xi)\sin(\omega\xi)d\xi\right)\sin(\omega y)e^{-\omega x}d\omega.$$

As a second method of solution we use a Fourier sine transform on y. Let $\tilde{u}_s(x,\omega) = \mathcal{F}_s[u(x,y), y \to \omega]$. This gives the transformed problem $\tilde{u}_s'' - \omega^2\tilde{u}_s = 0, \tilde{u}_s$ remains bounded as $x \to \infty, \tilde{u}_s(0,\omega) = \tilde{g}_s(\omega)$. The solution of this problem is easily found to be $\tilde{u}_s(x,\omega) = \tilde{g}(\omega)e^{-\omega x}$. Invert this transform to get

$$u(x,y) = \frac{2}{\pi}\int_0^\infty \tilde{g}_s(\omega)\sin(\omega y)e^{-\omega x}d\omega.$$

To make this solution look like the one obtained by separation of variables, replace $\tilde{g}_s(\omega)$ by the definition of its Fourier sine transform. This gives

$$u(x,y) = \frac{2}{\pi}\int_0^\infty \left(\int_0^\infty g(\xi)\sin(\xi\omega)d\xi\right)\sin(\omega y)e^{-\omega x}d\omega.$$

5. This problem can be solved by splitting it into two problems each having a single homogeneous boundary condition on one edge. One of the problems will be precisely Problem 4, solved above. For the other, interchange x and y and switch the names of f and g, finally add these two solutions to get $u(x,y) =$

$$\frac{2}{\pi}\int_0^\infty \left(\int_0^\infty f(\xi)\sin(\omega\xi)d\xi\right)\sin(\omega x)e^{-\omega y}d\omega + \frac{2}{\pi}\int_0^\infty \left(\int_0^\infty g(\xi)\sin(\omega\xi)d\xi\right)\sin(\omega y)e^{-\omega x}d\omega.$$

6. If $u(x,y)$ is harmonic in the upper half plane and satisfies $u(x,0) = f(x)$ along the real axis, the for $y < 0$ define $v(x,y) = u(x,-y)$. It is easily checked that $\nabla^2 v = \nabla^2 u; v(x,0) = u(x,0) = f(x)$. Thus the formula for $v(x,y)$ can be written

$$v(x,y) = \frac{-y}{\pi}\int_{-infty}^\infty \frac{f(\xi)}{y^2+(\xi-x)^2}d\xi.$$

7. Since we have zero boundary conditions on $x = 0$ and $x = \pi$, we use a finite Fourier sine transform in x. Let $\widetilde{u}_S(n, y) = \mathfrak{F}_S[u(x, y)](n)$ and transform Laplace's equation and the boundary condition along the edge $y = 0$ to get

$$\frac{\partial^2 \widetilde{u}_S(n, y)}{\partial y^2} - n^2 \widetilde{u}_S(n, y) = 0, \widetilde{u}_S(n, 0) = \begin{cases} B\pi/2 \text{ for } n = 1 \\ 0 \text{ for } n \neq 1. \end{cases}$$

The general solution of the differential equation is $\widetilde{u}_S(n, y) = c_n e^{-ny} + d_n e^{ny}$. Since the solution must remain bounded as $y \to \infty$, choose $d_n = 0$. The transformed boundary condition then gives $c_1 = B\pi/2$, $c_n = 0$ for $n \neq 1$. Since these are the Fourier sine series coefficients for the solution, we have by inverting that $u(x, y) = Be^{-y} \sin(x)$.

8. For the given boundary value problem we use the Fourier transform on x since the boundary function $u(x, 0) = e^{-ax} H(x)$ has a Fourier transform. Let $\hat{u}(\omega, y) = \mathcal{F}[u(x, y), x \to \omega]$. Transforming the problem gives $\hat{u}'' - \omega^2 \hat{u} = 0, \hat{u}(\omega, 0) = 1/(a + i\omega) = \frac{a-i\omega}{a^2+\omega^2}$. The solution is $\hat{u} = A(\omega)e^{-\omega y} + B(\omega)e^{\omega y}$, where $-\infty < \omega < \infty$ and $\hat{u}(\omega, y)$ must be bounded. Thus write $\hat{u}(\omega, y) = C(\omega)e^{-|\omega|y}$ and choose $C(\omega) = \frac{a-i\omega}{a^2+\omega^2}$. Next invert the transform to get

$$u(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{-|\omega|y}}{a^2 + \omega^2} e^{i\omega x} d\omega.$$

To simplify this inverse Fourier transform, break the integral into two parts over $(-\infty, 0]$ and $[0, \infty)$, make the change of variable $\omega = -\eta$ in the first integral, rename $\omega = \eta$ in the second, simplify and recombine to get solution

$$u(x, y) = \frac{1}{\pi} \int_0^{\infty} \frac{e^{-\eta}}{a^2 + \eta^2} (a \cos(\eta x) + \eta \sin(\eta x)) d\eta.$$

9. Since the boundary data is prescribed along $x = 0$ and $x = \pi$ we use a finite Fourier sine transform on x. Let $\tilde{u}_s(n, y) = \mathcal{F}_s[u(x, y), x \to n]$ and transform Laplace's equation and the boundary condition along the edge $y = 0$ to get

$$\frac{\partial^2 \tilde{u}_s(n, y)}{\partial y^2} - n^2 \tilde{u}_s = 2n(-1)^n, \ \tilde{u}_s(n, 0) = -\frac{4}{n}[1 - (-1)^n].$$

The general solution of this equation subject to the prescribed initial conditions and the condition that it remains bounded as $y \to \infty$ is $\tilde{u}_s(n, y) = \left(\frac{6}{n}(-1)^n - \frac{4}{n}\right) e^{-ny} - \frac{2}{n}(-1)^n$. By inverting this transform we get the solution

$$u(x, y) = \frac{2}{\pi} \sum_{n=1}^{\infty} \left[\left(\frac{6}{n}(-1)^n - \frac{4}{n} \right) e^{-ny} - \frac{2}{n}(-1)^n \right] \sin nx.$$

10. Since the boundary data is prescribed as function values along $x = 0$ and $x = \pi$ we use a finite Fourier sine transform on x. Let $\tilde{u}_s(n, y) = \mathcal{F}_s[u(x, y), x \to n]$ and transform Laplace's equation and the boundary conditions along the edges $y = 0$ and $y = 2$ to get

$$\frac{\partial^2 \tilde{u}_s(n, y)}{\partial y^2} - n^2 \tilde{u}_s = 4n(-1)^n, \ \tilde{u}_s'(n, 0) = 0, \ \tilde{u}_s(n, 2) = 0.$$

The general solution of this equation can be written $\tilde{u}_s(n,y) = c_c\cosh(ny) + d_n\sinh(n(2-y)) - \frac{4}{n}(-1)^n$. Fitting the mixed conditions at $y=0$ and $y=2$ gives $\tilde{u}_s(n,y) = \frac{4}{n}(-1)^n\left[\frac{\cosh(ny)}{\cosh(2n)} - 1\right]$. By inverting we get the series solution

$$u(x,y) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{4}{n}(-1)^n\left[\frac{\cosh(ny)}{\cosh(2n)} - 1\right]\sin(nx).$$

11. Because of the given homogeneous boundary condition along $y=0$, and the particular function specified along the edge $x=0$ we try a Fourier sine transform in y. Let $\tilde{u}_s(x,\omega) = \mathcal{F}_s[u(x,y), y\to\omega]$ to get the transformed problem $\tilde{u}_s'' - \omega^2\tilde{u}_s = 0,\ \tilde{u}(0,\omega) = 1/(1+\omega^2),\ \tilde{u}(x,\omega)$ remains bounded as $x\to\infty$. The solution of the transformed problem is easily found to be $\tilde{u}(x,\omega) = e^{-\omega x}/(1+\omega^2)$. Invert this transform to get the solution

$$u(x,y) = \frac{2}{\pi}\int_0^{\infty}\frac{e^{-\omega x}}{1+\omega^2}\sin(\omega y)d\omega.$$

12. The solution for the right half plane can be obtained from the integral formula for the upper half plane by interchanging x and y to get

$$u(x,y) = \frac{x}{\pi}\int_{-1}^{1}\frac{1}{x^2+(\eta-y)^2}d\eta$$

$$= \frac{1}{\pi}\left[\mathrm{Tan}^{-1}\left(\frac{1-y}{x}\right) + \mathrm{Tan}^{-1}\left(\frac{1+y}{x}\right)\right],\ \text{for } -\infty < y < \infty, x > 0$$

13. The solution for the upper half plane is given by the integral formula as

$$u(x,y) = \frac{y}{\pi}\int_4^8\frac{A}{y^2+(\xi-x)^2}d\xi = \frac{A}{\pi}\left[\mathrm{Tan}^{-1}\left(\frac{8-x}{y}\right) - \mathrm{Tan}^{-1}\left(\frac{4-x}{y}\right)\right].$$

14. With zero function values given on the left edge, we use a Fourier sine transform on x. Let $\tilde{u}_s(\omega,y) = \mathcal{F}_s[u(x,y), x\to\omega]$ to get the transformed problem $\tilde{u}_s'' - \omega^2\tilde{u}_s = 0, \tilde{u}_s(\omega,0) = 0, \tilde{u}_s(\omega,1) = \tilde{f}_s(\omega,y)$. We easily find the solution $\tilde{u} - s(\omega,y) = \tilde{f}_s(\omega)\sinh(\omega y)/\sinh(\omega)$. Invert this transform to get the general form of the solution

$$u(x,y) = \frac{2}{\pi}\int_0^{\infty}\tilde{f}_s(\omega)\frac{\sinh(\omega y)}{\sinh(\omega)}\sin(\omega x)d\omega.$$

Section 19.6 A Dirichlet Problem for a Cube

1. Using two separation of variables steps, and exploiting the fact that we have zero boundary conditions on opposite pairs of faces $x(0) = 0, x(1) = 0$; $y(0) = 0, y(1) = 0$; and using the fact that $z(0) = 0$, we find fundamental product solutions of the form
$x_n(x) = \sin(n\pi x), y_r(y) = \sin(r\pi y), z_{mr}(z) = \sinh(\pi\sqrt{n^2+r^2}z)$.

Superposing these solutions over all positive integers $n \geq 1, r \geq 1$ we have the basic form of a solution given by

$$u(x,y,z) = \sum_{n=1}^{\infty}\sum_{r=1}^{\infty} a_n b_r \sin(n\pi x)\sin(r\pi y)\sinh(\pi\sqrt{n^2+r^2}z);$$

and this function satisfies Laplace's equation and the five zero boundary conditions. For the remaining boundary condition on $z = 1$ we have

$$u(x,y,1) = \sum_{n=1}^{\infty}\sum_{r=1}^{\infty} a_n b_r \sin(n\pi x)\sin(r\pi y)\sinh(\pi\sqrt{n^2+r^2}).$$

We satisfy this final boundary condition by choosing the constants $a_n b_r \sinh(\sqrt{n^2+r^2})$ as the double Fourier coefficients of xy. In this case, since the boundary function on $z = 1$ can be factored as $f(x)g(y)$, this calculation is a bit easier. Calculate

$$a_n b_r \sinh(\pi\sqrt{n^2+r^2}) = 2\int_0^1 x\sin(n\pi x)dx \cdot 2\int_0^1 y\sin(r\pi y)dy$$

$$= \frac{2(-1)^{n+1}}{n\pi}\frac{2(-1)^{r+1}}{r\pi}.$$

The solution is given by

$$u(x,y,z) = \frac{4}{\pi^2}\sum_{n=1}^{\infty}\sum_{r=1}^{\infty}\frac{(-1)^{n+1}}{n}\frac{(-1)^{r+1}}{r}\frac{1}{\sinh(\pi\sqrt{n^2+r^2})}\sin(n\pi x)\sin(r\pi y)\sinh(\pi\sqrt{n^2+r^2}z)$$

2. Two separations of variables and use of the homogeneous boundary conditions on $0 \leq y \leq 2\pi$ and $0 \leq z \leq 1$ and at $x = 0$ gives a solution of the form

$$u(x,y,z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{nm}\sin\left(\frac{ny}{2}\right)\sin(m\pi z)\sinh(\sqrt{n^2+4m^2\pi^2}x/2).$$

Since $u(2\pi, y, z) = 2$, the orthogonality gives coefficients

$$a_{nm} = \frac{1}{\sinh(\sqrt{n^2+4m^2\pi^2}\pi)}\cdot\frac{1}{\pi}\int_0^{2\pi} 2\sin\left(\frac{ny}{2}\right)dy \cdot 2\int_0^1 \sin(m\pi z)dz$$

$$= \frac{4}{\pi\sinh(\sqrt{n^2+4m^2\pi^2}\pi)}[1-(-1)^n][1-(-1)^m].$$

Then

$$u(x,y,z) = \frac{16}{\pi}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\sinh(\sqrt{(2n-1)^2+4\pi^2(2m-1)^2}x/2)}{\sinh(\sqrt{(2n-1)^2+4\pi^2(2m-1)^2}\pi)}\sin\left[(2n-1)\frac{y}{2}\right]\sin[(2m-1)\pi z].$$

3. We write the solution of this problem as a sum of solutions of the two simpler problems $\nabla^2 w = 0$,

$w(0, y, z) = w(1, y, z). = w(x, 0, z) = w(x, 2\pi, z) = w(x, y, 0) = 0$, and $w(x, y, \pi) = 1$ and $\nabla^2 v = 0, v(0, y, z) = v(1, y, z) = v(x, y, 0) = v(x, y, \pi) = v(x, 0, z) = 0$, and $v(x, 2\pi, z) = 1$, each of which has homogeneous boundary conditions on opposite ends of two coordinate intervals. The eigenfunctions of each problem are easily found and we get

$$w(x, y, z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{nm} \sin(n\pi x) \sin\left(\frac{my}{2}\right) \sinh(\sqrt{4n^2\pi^2 + m^2}z/2)$$

and

$$v(x, y, z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} b_{nm} \sin(n\pi x) \sin(mz) \sinh(\sqrt{n^2\pi^2 + m^2}y).$$

To satisfy $w(x, y, \pi) = 1$ calculate

$$a_{nm} = \frac{1}{\sinh(\sqrt{4n^2\pi^2 + m^2}\pi/2)} \cdot 2\int_0^1 \sin(n\pi x)dx \cdot \frac{1}{\pi}\int_0^{2\pi} \sin\left(\frac{my}{2}\right) dy$$

$$= \frac{2}{\pi \sinh(\sqrt{4n^2\pi^2 + m^2}\pi/2)} \left[\frac{1 - (-1)^n}{n\pi}\right] \left[\frac{2}{m}(1 - (-1)^m)\right].$$

To satisfy $v(x, 2\pi, z) = 2$ calculate

$$b_{nm} = \frac{1}{\sinh(\sqrt{n^2\pi^2 + m^2}2\pi)} \cdot 2\int_0^1 2\sin(n\pi x)dx \cdot \frac{2}{\pi}\int_0^{\pi} \sin(mz)dz$$

$$= \frac{8}{\pi \sinh(\sqrt{n^2\pi^2 + m^2}2\pi)} \left[\frac{1 - (-1)^n}{n\pi}\right] \left[\frac{1 - (-1)^n}{m}\right].$$

The solution for $u(x, y, z)$ can be written $u(x, y, z) =$

$$\frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{\sin[(2n-1)\pi x]}{2n-1}\sum_{m=1}^{\infty}\frac{1}{2m-1}\left\{\frac{\sin[(2m-1)y/2]\sinh[\sqrt{4\pi^2(2n-1)^2 + (2m-1)^2}z/2]}{\sinh(\sqrt{4\pi^2(2n-1)^2 + (2m-1)^2}\pi/2)}\right.$$

$$\left. + \frac{2\sin[(2m-1)z]\sinh[\sqrt{\pi^2(2n-1)^2 + (2m-1)^2}y]}{\sinh(\sqrt{\pi^2(2n-1)^2 + (2m-1)^2}2\pi)}\right\}.$$

4. Since only one opposite set of faces has zero boundary conditions, we decompose this problem into two simpler problems and then add the solutions. Let $u(x, y, z) = v(x, y, z) + w(x, y, z)$ where u and w are the respective solutions of

$$\begin{array}{ll}
\nabla^2 v = 0, 0 < x < 1, 0 < y < 2, 0 < z < \pi & \nabla^2 w = 0, 0 < x < 1, 0 < y < 2, 0 < z < \pi \\
v(0, y, z) = 0, v(1, y, z) = \sin(\pi y)\sin(z) & w(0, y, z) = w(1, y, z) = 0 \\
v(x, 0, z) = v(x, 2, z) = 0 & w(x, 0, z) = w(x, 2, z) = 0 \\
v(x, y, 0) = 0; v(x, y, \pi) = 0 & w(x, y, 0) = x^2(1 - x)y(2 - y), w(x, y, \pi) = 0
\end{array}$$

Each of these problems can be solved by separation of variables and application of the zero boundary conditions on opposite faces to identify solutions of the form

$$v(x,y,z)=\sum_{n=1}^{\infty}\sum_{r=1}^{\infty}a_{nr}\sinh\left(\sqrt{\left(\frac{n\pi}{2}\right)^2+r^2}x\right)\sin\left(\frac{n\pi y}{2}\right)\sin(rz).$$

and

$$w(x,y,z)=\sum_{k=1}^{\infty}\sum_{p=1}^{\infty}b_{kp}\sin(k\pi x)\sin\left(\frac{p\pi y}{2}\right)\sinh\left(\sqrt{(k\pi)^2+\left(\frac{p\pi}{2}\right)^2}(\pi-z)\right).$$

The coefficient problem for $v(x,y,z)$ is nearly trivial. We see by inspection that $a_{21}\sqrt{\pi^2+1}=1$, all other $a_{nr}=0$. This is due to the boundary function at $x=1$ being simply a finite product of eigenfunctions. The coefficients b_{kp} require a bit more work, but are given by

$$b_{kp}\sinh\left(\frac{\pi}{2}\sqrt{p^2+4k^2}\right)=2\int_0^1 x^2(1-x)\sin(k\pi x)dx\cdot\int_0^2 y(2-y)\sin\left(\frac{p\pi y}{2}\right)dy$$

$$=-4\frac{[1+2(-1)^n]}{k^3\pi^3}\cdot 16\frac{[1-(-1)^p]}{p^3\pi^3}=-\frac{64}{\pi^6}[1+2(-1)^n][1-(-1)^p]$$

Reassembling these coefficients, adding the two solutions for v and w gives the final solution of the original problem as $u(x,y,z)=$

$$\frac{1}{\sqrt{1+\pi^2}}\sinh\left(\sqrt{1+\pi^2}x\right)\sin(\pi y)\sin(z)$$

$$-\frac{64}{\pi^6}\sum_{k=1}^{\infty}\sum_{p=1}^{\infty}\frac{[1+2(-1)^n][1-(-1)^p]}{\sinh\left(\frac{\pi}{2}\sqrt{p^2+4k^2}\right)}\sin(k\pi x)\sin\left(\frac{p\pi y}{2}\right)\sinh\left(\frac{\pi}{2}\sqrt{p^2+4k^2}(\pi-z)\right)$$

Section 19.7 The Steady State Heat Equation for a Solid Sphere

From the development in the text for steady state temperature in a sphere, the solution is given by

$$u(\rho,\phi)=\sum_{n=0}^{\infty}\frac{2n+1}{2}\left(\int_{-1}^{1}f(\arccos(x))P_n(x)dx\right)\left(\frac{\rho}{R}\right)^n P_n(\cos(\phi)).$$

In Problems 1 through 4 we approximate the required integrals and write six terms of the solution with approximate numerical coefficients. Those integrals not listed in the tables are all zero, as can easily be seen by considering even-odd properties of the function $f(\arccos(x))$ and the even-odd properties of the Legendre polynomials $P_n(x)$.

1. For $f(\phi) = A\phi^2$ the integrals to be approximated are $I_n = \int_{-1}^{1}(\arccos(x))P_n(x)dx, n = 0, 1, \ldots, 5$. We insert the factor of A after the approximations are made. Approximate integral values are:

$$\begin{aligned}
I_0 &= \int_{-1}^{1}(\arccos(x))^2 dx && \approx 5.86960441\\
I_1 &= \int_{-1}^{1}(\arccos(x))^2 x dx && \approx -2.46740110\\
I_2 &= \int_{-1}^{1}(\arccos(x))^2 \frac{(3x^2-1)}{2} dx && \approx .44444444\\
I_3 &= \int_{-1}^{1}(\arccos(x))^2 \frac{(5x^3-3x)}{2} dx && \approx -.154212688\\
I_4 &= \int_{-1}^{1}(\arccos(x))^2 \frac{(35x^4-30x^2+3)}{8} dx && \approx .07111111\\
I_5 &= \int_{-1}^{1}(\arccos(x))^2 \frac{(63x^5-70x^3+15x)}{8} dx && \approx -.03855314
\end{aligned}$$

It follows that the first six terms of the series give the approximation

$$u(\rho, \phi) \approx A \Bigg[5.86960441 - \frac{3}{2}(2.46740)\frac{\rho}{R}\cos(\phi) + \frac{5}{4}(.44444)\left(\frac{\rho}{R}\right)^2 (3\cos^2(\phi) - 1)$$

$$-\frac{7}{4}(.154212)\left(\frac{\rho}{R}\right)^3 (5\cos^3(\phi) - 3\cos(\phi)) + \frac{9}{16}(.071111)\left(\frac{\rho}{R}\right)^4 (35\cos^4(\phi) - 30\cos^3(\phi) + 3)$$

$$-\frac{11}{16}(.03855314)\left(\frac{\rho}{R}\right)^5 (63\cos^5(\phi) - 70\cos^3(\phi) + 15\cos(\phi)) + \cdots$$

2. For $f(\phi) = \sin(\phi)$ use the identity $\sin(\arccos(x)) = \sqrt{1-x^2}$. Since this function is even and $P_n(x)$ is odd for n odd, the values of $I_1, I_3, I_5 = 0$. Approximate integral values are:

$$\begin{aligned}
I_0 &= \int_{-1}^{1}\sqrt{1-x^2}dx && \approx 1.570796327\\
I_2 &= \int_{-1}^{1}\sqrt{1-x^2}\frac{(3x^2-1)}{2}dx && \approx -.196349541\\
I_4 &= \int_{-1}^{1}\sqrt{1-x^2}\frac{(35x^4-30x^2+3)}{8}dx && \approx -.024543693
\end{aligned}$$

These values give the approximation

$$U(\rho.\phi) \approx 1.570796327 - \frac{5}{2}(.196349541)\left(\frac{\rho}{R}\right)^2 (3\cos^2(\phi) - 1)$$

$$-\frac{9}{2}(.024543693)\left(\frac{\rho}{R}\right)^4 (35\cos^4(\phi) - 30\cos^3(\phi) + 3) + \cdots$$

3. For $f(\phi) = \phi^3$, the integrals to be approximated are given in the table below. Approximate integral values are:

$$
\begin{aligned}
I_0 &= \int_{-1}^{1} (\arccos(x))^3 dx & &\approx 12.15672076\\
I_1 &= \int_{-1}^{1} (\arccos(x))^3 x dx & &\approx -6.57347193\\
I_2 &= \int_{-1}^{1} (\arccos(x))^3 \frac{(3x^2-1)}{2} dx & &\approx 2.09439510\\
I_3 &= \int_{-1}^{1} (\arccos(x))^3 \frac{(5x^3-3x)}{2} dx & &\approx -.686958537\\
I_4 &= \int_{-1}^{1} (\arccos(x))^3 \frac{(35x^4-30x^2+3)}{8} dx & &\approx .3351032164\\
I_5 &= \int_{-1}^{1} (\arccos(x))^3 \frac{(63x^5-70x^3+15x)}{8} dx & &\approx -.177875557
\end{aligned}
$$

It follows that the first six terms of the series give the approximation

$$u(\rho,\phi) \approx 5.86960441 - \frac{3}{2}(6.573472)\frac{\rho}{R}\cos(\phi) + \frac{5}{4}(2.094395)\left(\frac{\rho}{R}\right)^2 (3\cos^2(\phi) - 1)$$

$$-\frac{7}{4}(.6869585)\left(\frac{\rho}{R}\right)^3 (5\cos^3(\phi) - 3\cos(\phi)) + \frac{9}{16}(.33510322)\left(\frac{\rho}{R}\right)^4 (35\cos^4(\phi) - 30\cos^3(\phi) + 3)$$

$$-\frac{11}{16}(.17787555)\left(\frac{\rho}{R}\right)^5 (63\cos^5(\phi) - 70\cos^3(\phi) + 15\cos(\phi)) + \cdots$$

4. For $f(\phi) = 2 - \phi^2$, we use the known orthogonality of $\{P_n(x)\}_{n=0}^{\infty}$ to simplify the calculations here. Recall $\int_{-1}^{1} 2P_n(x)dx = 0, n \geq 1$, by orthogonality of $P_0(x)$ and $P_n(x), n \geq 1$. So for $n \geq 1$, $\int_{-1}^{1}(2 - (\arccos(x))^2)P_n(x)dx = -\int_{-1}^{1}(\arccos(x))^2 P_n(x)dx$ and these values were computed for Problem 1. By direct evaluation and the value I_0 from problem 1, $\int_{-1}^{1}(2 - (\arccos(x))^2)dx = -1.86960441$

It follows that the first six terms of the series give the approximation

$$u(\rho,\phi) \approx -1.86960441 + \frac{3}{2}(2.46740)\frac{\rho}{R}\cos(\phi) - \frac{5}{4}(.44444)\left(\frac{\rho}{R}\right)^2 (3\cos^2(\phi) - 1)$$

$$+\frac{7}{4}(.154212)\left(\frac{\rho}{R}\right)^3 (5\cos^3(\phi) - 3\cos(\phi)) - \frac{9}{16}(.071111)\left(\frac{\rho}{R}\right)^4 (35\cos^4(\phi) - 30\cos^3(\phi) + 3)$$

$$+\frac{11}{16}(.03855314)\left(\frac{\rho}{R}\right)^5 (63\cos^5(\phi) - 70\cos^3(\phi) + 15\cos(\phi)) + \cdots$$

5. The boundary value problem to be solved is

$$
\begin{aligned}
&u_{\rho\rho} + \frac{2}{\rho}u_\rho + \frac{1}{\rho^2}u_{\phi\phi} + \frac{\cot(\phi)}{\rho^2}u_\phi = 0, R_1 \leq \rho \leq R_2\\
&u(R_1,\phi) = T_1, u(R_2,\phi) = 0.
\end{aligned}
$$

Assume $u(\rho,\phi) = F(\rho)\Phi(\phi)$ and separate variables to get the two problems $F'' + \dfrac{2}{\rho}F' - \dfrac{\lambda}{\rho^2}F = 0, R_1 < \rho < R_2$ and $\Phi'' + \cot(\phi)\Phi' + \lambda\Phi = 0, -\pi/2 < \phi < \pi/2$. As before, the solution for $\Phi(\phi)$ which remains bounded on $-\pi/2 < \phi < \pi/2$ is $\Phi_n(\phi) = P_n(\cos(\phi))$, the n^{th} Legendre polynomial corresponding to $\lambda_n = n(n+1)$. The corresponding radial function $F(\rho)$ is easily found, by solving the Euler equation $\rho^2F''+2\rho F'-n(n+1)F = 0$, to be $F(\rho) = a_n\rho^n + b_n\rho^{-n-1}$. By superposition we have $u(\rho,\phi) = \sum_{n=0}^{\infty}[a_n\rho^n + b_n\rho^{-n-1}]P_n(\cos(\phi))$.
The condition specified on $\rho = R_1$ requires

$$u(R_1,\phi) = T_1 = \sum_{n=0}^{\infty}[a_nR_1^n + b_nR_1^{-n-1}]P_n(\cos(\phi)),$$

while the condition on $\rho = R_2$ requires

$$u(R_2,\phi) = 0 = \sum_{n=0}^{\infty}[a_nR_2^n + b_nR_2^{-n-1}]P_n(\cos(\phi)).$$

From the orthogonality of $\{P_n(x)\}_{n=0}^{\infty}$ on $-1 \le x \le 1$, we conclude that

$$a_0 + \frac{b_0}{R_1} = T_1, a_0 + \frac{b_0}{R_2} = 0, \text{ and } a_nR_1^n + \frac{b_n}{R_1^{n+1}} = 0, a_nR_2^n + \frac{b_n}{R_2^{n+1}} = 0, n \ge 1.$$

Solve for

$$a_0 = \frac{T_1R_1}{R_1 - R_2}, b_0 = -\frac{T_1R_1R_2}{R_1 - R_2}, a_n = b_n = 0, n \ge 1.$$

The solution is

$$u(\rho,\phi) = \frac{T_1R_1}{R_2 - R_1}\left[\frac{R_2}{\rho} - 1\right].$$

6. Since the solution for Problem 5 was obtained in closed form, no numerical approximations of the solution are necessary.

7. The boundary value problem is

$$\begin{aligned} &u_{\rho\rho} + \frac{2}{\rho}u_\rho + \frac{1}{\rho^2}u_{\phi\phi} + \frac{\cot(\phi)}{\rho^2}u_\phi = 0 \\ &u(\rho,\pi/2) = 0, u(R,\phi) = A \end{aligned}$$

Assume $u(\rho,\phi) = F(\rho)\Phi(\phi)$ and separate variables to get the two problems $F'' + \dfrac{2}{\rho}F' - \dfrac{\lambda F}{p^2} = 0; \Phi'' + (\cot)\phi\Phi' + \lambda\Phi = 0$, and $\Phi(\pi/2) = 0$. Making the change of variable $x = \cos(\phi)$ we identify the equation for Φ as Legendre's equation, hence the eigenvalues are $\lambda_n = n(n+1)$ and $\Phi_n(\phi) = P_n(\cos(\phi))$. The boundary condition $\Phi(\pi/2) = 0$ becomes $P_n(0) = 0$. Since $P_n(0) \neq 0$ if n is even, we choose to retain only the odd Legendre polynomials. With $\lambda = n(n+1)$ the equation for $F(\rho)$ has solution $F(\rho) = a_n\rho^n + b_n\rho^{-n-1}$. To have bounded solutions at $\rho = 0$, the center of the sphere, choose $b_n = 0$. Superposition then gives the solution $u(\rho,\phi) = \sum_{n=1}^{\infty}a_{2n-1}\rho^{2n-1}P_{2n-1}(\cos(\phi))$. To complete the solution, choose coefficients

a_{2n-1} so that $u(R,\phi) = A = \sum_{n=1}^{\infty} a_{2n-1}R^{2n-1}P_{2n-1}(\cos(\phi))$. By the orthogonality of the Legendre polynomials, $a_{2n-1} = \frac{A}{R^{2n-1}}\frac{\int_0^1 P_{2n-1}(x)dx}{\int_0^1 [P_{2n-1}(x)]^2 dx}, n \geq 1$. From Theorem 16.7, Section 16.1, and the fact that $P_{2n-1}(x)$ is an odd function, we can show

$$\int_0^1 [P_{2n-1}(x)]^2 dx = \frac{1}{2}\int_{-1}^1 [P_{2n-1}(x)]^2 dx = \frac{1}{4n-1}.$$

Previously it was shown that $nP_n(x) - xP_n'(x) + P_{n-1}'(x) = 0, n \geq 1$. Rewrite this result with n replaced by $2n-1$ and integrate the result over $[0,1]$ to get

$$(2n-1)\int_0^1 P_{2n-1}(x)dx - \int_0^1 xP_{2n-1}'(x)dx + \int_0^1 P_{2n-2}'(x)dx = 0$$

Integrate the second integral by parts and simplify the result to get

$$2n\int_0^1 P_{2n-1}(x)dx - P_{2n-1}(1) + P_{2n-2}(1) - p_{2n-2}(0) = 0$$

In section 16.1, it was shown using the generating function that $P_n(1) = 1$, and we get the result

$$P_{2n-2}(0) = (-1)^{n-1}\frac{(2n-2)!}{2^{2n-2}[(n-1)!]^2}, n \geq 1.$$

Substitute these results in the relationship above for $\int_0^1 P_{2n-1}(x)dx$ and simplify the result to finally obtain

$$\int_0^1 P_{2n-1}(x)dx = \frac{(-1)^{n-1}(2n)!}{(2n-1)2^{2n}(n!)^2}$$

Finally use these results in the coefficients a_{2n-1} and the solution can be written

$$u(\rho,\phi) = A\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(4n-1)(2n)!}{(2n-1)2^{2n}(n!)^2}\left(\frac{\rho}{R}\right)^{2n-1}P_{2n-1}(\cos(\phi)).$$

8. The boundary value problem for $u(\rho,\phi)$ can be expressed as $u_{\rho\rho}+\frac{2}{\rho}u_\rho+\frac{1}{\rho^2}u_{\phi\phi}+\frac{\cot(\phi)}{\rho^2}u_\phi = 0, u(R,\phi) = A, \frac{\partial u}{\partial\phi}(\rho,\pi/2) = 0$. Since the base of the hemisphere is insulated, no heat can escape, and since the surface $\rho = R$ is maintained at A, the steady state temperature will clearly be a constant so $u(\rho,\phi) = A$ for $0 < \rho < R, 0 < \phi < \pi/2$. This can easily be verified by substitution.

9. The solution is identical to problem 7 to the point of choosing coefficients of the Legendre-Fourier series. Hence $u(\rho,\phi) = \sum_{n=1}^{\infty} a_{2n-1}\rho^{2n-1}P_{2n-1}(\cos(\phi))$. Now choose coefficients so that

$$u(R,\phi) = f(\phi) = \sum_{n=1}^{\infty} a_{2n-1}R^{2n-1}P_{2n-1}(\cos(\phi)).$$

Thus

$$a_{2n-1} = \frac{4n-1}{R^{2n-1}}\int_0^1 f(\cos^{-1}(x))P_{2n-1}(x)dx.$$

Section 19.8 The Neumann Problem

1. Check the necessary condition for a solution and find $\int_0^1 4\cos(\pi x)dx = 0$, so a solution is possible. From the zero boundary conditions on the opposite edges $x = 0$ and $x = 1$ we find by separation of variables that it should have the form

$$u(x,y) = c_0 + \sum_{n=1}^{\infty}[c_n \cosh(n\pi y) + d_n \cosh(n\pi(1-y))]\cos(n\pi x).$$

The boundary condition at $y = 1$ becomes

$$\frac{\partial u}{\partial y}(x,1) = 0 = \sum_{n=1}^{\infty} n\pi c_n \sinh(n\pi)\cos(n\pi x)$$

so $c_n = 0, n \geq 1$. On the edge $y = 0$ we have

$$\frac{\partial u}{\partial y}(x,0) = 4\cos(n\pi x) = \sum_{n=1}^{\infty} -n\pi d_n \sinh(n\pi)\cos(n\pi x);$$

so $-d_1\pi\sinh(\pi) = 4, d_n = 0, n \geq 2$. A solution is

$$u(x,y) = c_0 - \frac{4}{\pi\sinh(\pi)}\cosh(\pi(1-y))\cos(\pi x).$$

2. Check the necessary condition for a solution and find $\int_0^\pi \left(y - \frac{y}{2}\right)dy = \frac{\pi^2}{2} - \frac{\pi^2}{2} = 0$, so a solution is possible. From the zero boundary conditions on opposite edges $y = 0$ and $y = \pi$ by separation of variables we see that it should have the form

$$u(x,y) = c_0 + \sum_{n=1}^{\infty}[c_n \cosh(nx) + d_n \cosh(n(1-x))]\cos(ny).$$

The boundary conditions on edges $x = 0$ and $x = 1$ then give the equations

$$\frac{\partial u}{\partial x}(0,y) = y - \frac{\pi}{2} = \sum_{n=1}^{\infty} -nd_n \sinh(n)\cos(ny)$$

and

$$\frac{\partial u}{\partial x}(1,y) = \cos(y) = \sum_{n=1}^{\infty} nc_n \sinh(n)\cos(ny).$$

Thus calculate

$$c_n = \frac{1}{n\sinh(n)}\cdot\frac{2}{\pi}\int_0^\pi \cos(y)\cos(ny)dy = \begin{cases} 0 & ,n \neq 1 \\ \frac{1}{\sinh(1)} & ,n = 1 \end{cases}$$

$$d_n = \frac{-1}{n\sinh(n)}\cdot\frac{2}{\pi}\int_0^\pi \left(y - \frac{\pi}{2}\right)\cos(ny)dy = \frac{2[1-(-1)^n]}{\pi n^3\sinh(n)}, n \geq 1.$$

$$d_n = \frac{-1}{n\sinh(n)} \cdot \frac{2}{\pi}\int_0^\pi \left(y - \frac{\pi}{2}\right)\cos(ny)dy = \frac{2[1-(-1)^n]}{\pi n^3 \sinh(n)}, n \geq 1.$$

A solution is

$$u(x,y) = c_0 + \frac{\cosh(x)}{\sinh(1)}\cos(y) + \sum_{n=1}^{\infty} \frac{2[(-1)^n - 1]}{\pi n^3 \sinh(n)} \cosh(n(1-x))\cos(ny).$$

3. Check the necessary conditions for a solution and find $\int_0^\pi \cos(3x)dx = 0$, so a solution is possible. From the zero boundary conditions an edges $x = 0$ and $x = \pi$ and separation of variables, we find the solution should have the form

$$u(x,y) = c_0 + \sum_{n=1}^{\infty} [c_n \cosh(ny) + d_n \cosh(n(\pi - y))]\cos(nx).$$

The boundary condition at $y = 0$ gives

$$\frac{\partial u}{\partial y}(x,0) = \cos(3x) = \sum_{n=1}^{\infty} -nd_n \sinh(n\pi)\cos(nx);$$

so $d_3 = \dfrac{-1}{3\sinh(3\pi)}, d_n = 0, n \neq 3$. The boundary condition at $y = \pi$ gives

$$\frac{\partial u}{\partial y}(x,\pi) = 6x - 3\pi = \sum_{n=1}^{\infty} nc_n \sinh(n\pi)\cos(nx).$$

Calculate

$$c_n = \frac{1}{n\sinh(n\pi)} \cdot \frac{2}{\pi}\int_0^\pi (6x - 3\pi)\cos(nx)dx = \frac{1}{n\sinh(n\pi)} \cdot \frac{12}{n^2\pi}[(-1)^n - 1], n \geq 1.$$

The solution is

$$u(x,y) = -\frac{\cosh(3(\pi - y))}{3\sinh(3\pi)}\cos(3x) + \sum_{n=1}^{\infty} \frac{12[(-1)^n - 1]}{n^3 \pi \sinh(n\pi)} \cosh(ny)\cos(nx) + c_0.$$

4. Assume $u(x,y) = X(x)Y(y)$ and separate variables to get $X'' - \lambda X = 0, X'(0) = X'(\pi) = 0$ and $Y'' + \lambda Y = 0$. The solutions for X are $X_0 = 1, X_n(x) = \cos(nx), n \geq 1$. The for each $n \geq 1$ we have $Y_n(y) = c_n \cosh(ny) + d_n \cosh(n(\pi - y))$ and by superposition we find a solution of the form

$$u(x,y) = c_0 + \sum_{n=1}^{\infty} [c_n \cosh(ny) + d_n \cosh(n(\pi - y))]\cos(nx).$$

At the edge $y = \pi$ we have

$$u(x,\pi) = 0 = c_0 + \sum_{n=1}^{\infty} [c_n \cosh(n\pi) + d_n]\cos(nx)$$

and along the edge $y = 0$ we find

$$u(x,0) = f(x) = c_0 + \sum_{n=1}^{\infty}[c_n + d_n \cosh(n\pi)]/cos(nx).$$

Thus the coefficients should be chosen (if possible) to satisfy the system of equations $c_0 = 0$, and for each $n \geq 1$

$$\begin{aligned} c_n \cosh(n\pi) + d_n &= 0 \\ c_n + d_n \cosh(n\pi) &= \tfrac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)dx \end{aligned}$$

For each $n \geq 1$ we have

$$\begin{vmatrix} \cosh(n\pi) & 1 \\ 1 & \cosh(n\pi) \end{vmatrix} = \cosh^2(n\pi) - 1 = \sinh^2(n\pi) \neq 0.$$

Thus the system of equations for $\{c_n\}, \{d_n\}$ will have a unique solution, and the solution $u(x,y)$ will be uniquely and completely determined.

5. Using a separation of variables with $u(x,y) = X(x)Y(y)$ we find $X'' - \lambda X = 0, Y'' + Y = 0, Y(0) = Y(1) = 0$. Solutions are $Y_n(y) = \sin(n\pi y), n \geq 1, X_n(x) = c_n \cosh(n\pi x) + d_n \cosh(n\pi(x-1))$, and a solution will have the form

$$u(x,y) = \sum_{n=1}^{\infty}[c_n \cosh(n\pi x) + d_n \cosh(n\pi(1-x))]\sin(n\pi y).$$

The boundary conditions at $x = 1$ and $x = 0$ give respectively, the equations

$$\frac{\partial u}{\partial x}(1,y) = 0 = \sum_{n=1}^{\infty} n\pi c_n \sinh(n\pi)\sin(n\pi y);$$

and

$$\frac{\partial u}{\partial x}(0,y) = 3y^2 - 2y = \sum_{n=1}^{\infty} -n\pi \sinh(n\pi)\sin(n\pi y).$$

It follows that $c_n = 0, n \geq 1$, and

$$d_n = \frac{-2}{n\pi \sinh(n\pi)}\int_0^1 (3y^2 - 2y)\sin(n\pi y)dy = \frac{2}{n^4\pi^4 \sinh(n\pi)}\{n^2\pi^2(-1)^n + 6[1-(-1)^n]\}, n \geq 1.$$

This gives the solution

$$u(x,y) = \sum_{n=1}^{\infty} \frac{2}{n^4\pi^4 \sinh(n\pi)}\{n^2\pi^2(-1)^n + 6[1-(-1)^n]\}\cosh(n\pi(1-x))\sin(n\pi y).$$

6. Since $\displaystyle\int_{-\pi}^{\pi} \sin(3\theta)d\theta = 0$, a solution is possible and will have the form

$$u(r,\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n r^n \cos(n\theta) + b_n r^n \sin(n\theta).$$

The boundary condition on $r = R$ gives

$$\frac{\partial u}{\partial r}(R,\theta) = \sin(3\theta) = \sum_{n=1}^{\infty}[na_nR^{n-1}\cos(n\theta) + nb_nR^{n-1}\sin(n\theta)].$$

By inspection (orthogonality in disguise in a simple situation) we see $a_n = 0, n \geq 1; 3b_3R^2 = 1, b_n = 0, n \neq 3$. These coefficients give the solution

$$u(r,\theta) = \frac{1}{2}a_0 + \frac{R}{3}\left(\frac{r}{R}\right)^3 \sin(3\theta).$$

7. Since $\int_{-\pi}^{\pi} \cos(2\theta)d\theta = 0$, a solution is possible and will have the form

$$u(r,\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n r^n \cos(n\theta) + b_n r^n \sin(n\theta)$$

The boundary condition on $r = R$ gives

$$\frac{\partial u}{\partial r}(R,\theta) = \cos(2\theta) = \sum_{n=1}^{\infty}[na_nR^{n-1}\cos(n\theta) + nb_nR^{n-1}\sin(n\theta)].$$

Again in this simple case we see the coefficients $b_n = 0, n \geq 1; 2a_2R = 1; a_n = 0, n \neq 2$; to get the solution

$$u(r,\theta) = \frac{1}{2}a_0 + \frac{R}{2}\left(\frac{r}{R}\right)^2 \cos(2\theta).$$

8. First check the necessary condition for a solution and find $\int_{-\infty}^{\infty} xe^{-|x|}dx = 0$, so a solution is possible. Using the formula developed in the text in Section 18.8.3, we can write the solution for $u(x, y)$ to be

$$u(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \ln(y^2 + 9\xi - y)^2)\xi e^{-|\xi|}d\xi + c.$$

9. Since $\int_{-\infty}^{\infty} e^{-|x|}\sin(x)dx = 0$, the necessary condition for a solution is satisfied and we can write the solution in the form

$$u(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \ln(y^2 + 9\xi - y)^2)e^{-|\xi|}\sin(\xi)d\xi + c.$$

10. The solution of the Dirichlet problem for the lower half plane was found in Section 18.5, Problem 6. Using that result and the technique given in the text for solving a Neumann problem in the upper half plane, we can write the solution for the lower half plane as

$$u(x,y) = -\frac{1}{2\pi}\int_{infty}^{\infty} \ln(y^2 + (\xi - x)^2)f(\xi)d\xi + c.$$

Another argument for this sign change is that the sign of the outer normal derivative changes along the real axis when going from the upper half plane to the lower half plane.

11. To solve this problem, apply a Fourier cosine transform Let $U(\omega, y) = \mathcal{F}_c[u(x,y); x \to \omega]$, and apply the transform to Laplace's equation on the upper right quarter plane to get

$$-\omega^2 U - \frac{\partial u}{\partial x}(0,y) + U_{yy} = 0.$$

Since $u_x(0,y) = 0$ we get $U_{yy} - \omega^2 U = 0$. This equation has solution $U(\omega,y) = a_\omega e^{-\omega y} + b_\omega e^{\omega y}$. To have bounded solutions for $y > 0$, take $b_\omega = 0$. Now invert the cosine transform to get

$$u(x,y) = \int_0^\infty a_\omega \cos(\omega x) e^{-\omega y} d\omega.$$

From this, calculate

$$\frac{\partial u}{\partial y}(x,0) = \int_0^\infty -\omega a_\omega \cos(\omega x) d\omega = f(x);$$

to get

$$a_\omega = -\frac{2}{\pi\omega} \int_0^\infty f(\xi) \cos(\omega\xi) d\xi.$$

12. Because of the boundary condition at $x = 0$, we use a Fourier sine transform on x. This gives the transformed problem $-\omega^2 U(\omega,y) + \omega u(0,y) + U(\omega,y)_{yy} = 0$. Putting $u(0,y) = 0$ and solving for bounded solutions gives $U(\omega,y) = a_\omega e^{-\omega y}$. Then calculate $U_y(\omega,0) = -\omega a_\omega = \widetilde{f}_s(\omega)$. Solve this for $a_\omega = -\frac{2}{\pi\omega}\int_0^\infty f(\xi)\sin(\omega\xi)d\xi$. The solution is given by

$$u(x,y) = \int_0^\infty a_\omega e^{-\omega y} \sin(\omega x) d\omega.$$

Chapter Twenty - Geometry and Arithmetic of Complex Numbers

Section 20.1 Complex Numbers

1. $(3-4i)(6+2i) = (18+8)+i(-24+6) = 26-18i$

2. $i(6-2i)+|1-i| = 6i+2+\sqrt{1+1} = 2+\sqrt{2}+6i$

3. $\dfrac{2+i}{4-7i} = \dfrac{(2+i)(4+7i)}{(4-7i)(4+7i)} = \dfrac{1}{65}(8-7+18i) = \dfrac{1}{65}(1+18i)$

4. $\dfrac{(2+i)-(3-4i)}{(5-i)(3+i)} = \dfrac{(-1+5i)}{(16+2i)}\dfrac{(16-2i)}{(16-2i)} = \dfrac{-6+82i}{260} = \dfrac{-3+41i}{130}$

5. $(17-6i)\overline{(-4-12i)} = (17-6i)(-4+12i) = 4+228i$

6. $\left|\dfrac{3i}{-4+8i}\right| = \dfrac{|3i|}{|-4+8i|} = \dfrac{3}{\sqrt{80}} = \dfrac{3}{4\sqrt{5}}$

7. $i^3-4i^2+2 = -i+4+2 = 6-i$

8. $(3+i)^3 = 27+3(3)^2i+3(3)(i)^2+i^3 = 27+27i-9-i = 18+26i$

9. $\left(\dfrac{-6+2i}{1-8i}\right)^2 = \left[\dfrac{(-6+2i)(1+8i)}{(1-8i)(1+8i)}\right]^2 = \dfrac{(-22-46i)^2}{(65)^2} = \dfrac{-1632+2024i}{4225}$

10. $(-3-8i)(2i)(4-i) = (-3-8i)(2+8i) = 58-40i$

11. Certainly, $i^1 = i, i^2 = -1, i^3 = i\cdot i^2 = -i, i^4 = (i^2)^2 = (-1)^2 = 1$.
Now $(i)^{4n+k} = i^{4n}\cdot i^k = (i^4)^n i^k = i^k$ and the result follows.

12. With $z = a+bi, z^2 = (a^2-b^2)+2abi$, so $Re(z^2) = a^2-b^2$ and $Im(z^2) = 2ab$.

13. We have $z^2-iz+1 = a^2-b^2+b+1+(2ab-a)i$, so $Re(z^2-iz+1) = a^2-b^2+b+1$, and $Im(z^2-iz+1) = 2ab-a$.

14. Let $z = a+bi$, then $\overline{z} = a-bi, z^2 = a^2-b^2+2abi$, and $(\overline{z})^2 = a^2-b^2-2abi$. Now $z^2 = (\overline{z})^2$ if and only if $2ab = -2ab$. But this is true if and only if $ab = 0$. Since a and b are real, we conclude $ab = 0$ if and only if either $a = 0$, in which case z is pure imaginary, or $b = 0$, in which case z is real.

15. Let z, w, u be three points in the complex plane which form the vertices of a triangle (labeled so we traverse the triangle in a clockwise direction through z, w, u in order). The respective sides of this triangle are vectors represented by the complex numbers $(w-z), (u-w)$ and $(z-u)$, hence the triangle will be equilateral if and only if $|w-z| = |u-w| = |z-u|$, and each side vector can be rotated by $\theta = 2\pi/3$ clockwise to align with the next side. Thus $(u-w) = (w-z)e^{-2\pi i/3}$ and $(z-u) = (u-w)e^{-2\pi i/3}$. Dividing these equations gives $\dfrac{u-w}{z-u} = \dfrac{w-z}{u-w}$; and cross multiplying gives $w^2-2wu+u^2 = zw+uz-uw-z^2$; finally rearranging we get $z^2+w^2+u^2 = zw+zu+wu$.

16. Let $z = a+bi$, then $iz = -b+ai$. We have $Re(iz) = -b = -Im(z)$, and $Im(iz) = a = Re(z)$.

17. $\arg(3i) = \dfrac{\pi}{2}+2k\pi, k$ any integer.

18. $\arg(-2+2i) = \dfrac{3\pi}{4}+2\pi k, k$ any integer.

19. $\arg(-3+2i) = \text{Tan}^{-1}\left(-\frac{2}{3}\right) + \pi + 2k\pi, k$ any integer, where $\text{Tan}^{-1}\left(-\frac{2}{3}\right)$ is the principal value of $\text{Tan}^{-1}(\theta)$ and satisfies $-\frac{\pi}{2} < \text{Tan}^{-1}(\theta) < \frac{\pi}{2}$. Since the complex number $-3+2i$ corresponds to a point in quadrant II, we need to add π to $\text{Tan}^{-1}\left(-\frac{2}{3}\right)$.

20. $\arg(8+i) = \text{Tan}^{-1}\left(\frac{1}{8}\right) + 2k\pi, k$ any integer.

21. $\arg(-4) = \pi + 2k\pi, k$ any integer.

22. $\arg(3-4i) = \text{Tan}^{-1}\left(-\frac{4}{3}\right) + 2k\pi, k$ any integer.

23. For $z = -2+2i$, we have $|z| = \sqrt{4+4} = 2\sqrt{2}$, and $\theta = \arg(-2+2i) = \text{Tan}^{-1}(-1)+\pi = \frac{3\pi}{4}$ is an argument of $(-2+2i)$. So $-2+2i = 2\sqrt{2}e^{3\pi i/4}$ in polar form.

24. We have $|-7i| = 7$, and an argument is $\theta = -\frac{\pi}{2}$, so polar form is $7e^{-i\pi/2}$.

25. We have $|5-2i| = \sqrt{29}, \theta = \text{Tan}^{-1}\left(-\frac{2}{5}\right)$ is one argument, so polar form is $\sqrt{29}e^{i\text{Tan}^{-1}(-2/5)}$

26. We have $|-4-i| = \sqrt{17}, \theta = \text{Tan}^{-1}\left(\frac{1}{4}\right) + \pi$ is one argument, so polar form is $\sqrt{17}e^{i(\text{Tan}^{-1}(\frac{1}{4})+\pi)}$

27. We have $|8+i| = \sqrt{65}$, and by Problem 20, one argument is $\theta = \text{Tan}^{-1}\left(\frac{1}{8}\right)$, so polar form is $\sqrt{65}e^{i\text{Tan}^{-1}(\frac{1}{8})}$

28. We have $|-12+3i| = \sqrt{153}, \theta = \text{Tan}^{-1}\left(-\frac{1}{4}\right) + \pi$ is an argument, so polar form is $\sqrt{153}e^{i(\text{Tan}^{-1}(-\frac{1}{4})+\pi)}$

29. Suppose $|z| = 1$, then $z\overline{z} = 1$ and $|\overline{z}| = 1$. We have $\left|\frac{z-w}{1-\overline{z}w}\right| = \left|\frac{z-w}{z\overline{z}-\overline{z}w}\right| = \frac{|z-w|}{|\overline{z}||z-w|} = \frac{1}{|\overline{z}|} = 1$. If $|w| = 1$, then $\overline{w}w = 1$. We have $\left|\frac{z-w}{1-\overline{z}w}\right| = \left|\frac{z-w}{\overline{w}w-\overline{z}w}\right| = \frac{1}{|w|}\frac{|z-w|}{|\overline{w}-\overline{z}|} = 1$, since $|z-w| = |\overline{w-z}| = |\overline{w}-\overline{z}|$.

30. $|z+w|^2||z-w|^2 = (z+w)(\overline{z+w})+(z-w)(\overline{z-w}) = z\overline{z}+w\overline{z}+z\overline{w}+w\overline{w}+z\overline{z}-w\overline{z}-z\overline{w}+w\overline{w} = |z|^2+|w|^2+|z|^2+|w|^2 = 2(|z|^2+|w|^2)$.

Section 20.2 Loci and Sets of Points in the Complex Plane

1. $|z-8+4i| = 9$ is a circle of radius 9 centered at $z_0 = 8-4i$.

2. $|z| = |z-i|$ is the set of points equidistant from $z = 0$ and $z = i$, or equivalently from $(0,0)$ and $(0,1)$ which is the line $y = \frac{1}{2}$ or $z = x + \frac{1}{2}i, -\infty < x < \infty$.

3. With $z = x+iy$, this set is described by $x^2+y^2+y = 16$ or $x^2 + \left(y+\frac{1}{2}\right)^2 = \frac{65}{4}$; a circle of radius $\frac{\sqrt{65}}{2}$ centered at $\left(0,-\frac{1}{2}\right)$ or $z = -\frac{1}{2}i$.

4. $|z-i|+|z|=9$ describes an ellipse with foci at $(0,1)$ and $(0,0)$. Specifically, with $z=x+iy$ we get $\dfrac{x^2}{20}+\dfrac{4(y-1/2)^2}{81}=1$.

5. With $z=x+iy$, this set is described by $\sqrt{x^2+y^2}+x=0$ or $y=0, x\leq 0$, the negative real axis.

6. Putting $z=x+iy$, we get $(x+iy)+(x-iy)^2=4$ or $x+x^2-y^2=4$ and $y-2xy=0$ by equating real and imaginary parts. With $y=0$ (from $y(1-2x)=0$) we get $x=\dfrac{-1\pm\sqrt{17}}{2}$; taking $x=\dfrac{1}{2}$, we get $y^2=\dfrac{-13}{4}$ so there are no such points. The set $z+\overline{z}^2=4$ describes two points $z=\dfrac{-1\pm\sqrt{17}}{2}$.

7. Putting $z=x+iy$ we obtain $y-1=x+1$ or the line $y=x+2$

8. Putting $z=x+iy$, we obtain $\sqrt{x^2+y^2}=y-1$, or equvialently $x^2+2y-1=0, y\geq 1$. The first equation describes a parabola, vertex at $(0,-1/2)$ and opening down, hence there are no points on this parabola having $y\geq 1$. The statement $|z|=Im(z-i)$ is an empty set. We can see this directly by observing $\sqrt{x^2+y^2}\geq\sqrt{y^2}=|y|\geq y>y-1$. Hence no (x,y) could satisfy $\sqrt{x^2+y^2}=y-1$.

9. $|z+1+6i|=|z-3+i|$ is the set of all points equidistant from $(-1,-6)$ and $(3,-1)$. But this is the straight line through $(1,-7/2)$ having slope $-4/5$ or $10y+8x+27=0$. An alternate solution is to start with $(x+1)^2+(y+6)^2=(x-3)^2+(y+1)^2$, expand and simplify to get $8x+10y+27=0$

10. $|z-4i|\leq|z+1|$ is the set of all points as close or closer to $(0,4)$ as they are to $(-1,0)$. Thus we need $x^2+(y-4)^2\leq(x+1)^2+y^2$ or simplifying, $0\leq 2x+8y-15$, a half plane.

11. $|z+2+i|>|z-1|$ is the set of all points further from $(-2,-1)$ than from $(1,0)$. Thus $(x+2)^2+(y+1)^2>(x-1)^2+y^2$, which simplifies to $3x+y+2>0$, the equation of a half plane.

12. S is the set of all points outside the circle of radius 2 centered at the origin. S is open. The set of limit points of S is the set of all points in S together with the set of points on the circle $|z|=2$. Boundary points of S are all $|z|=2$, and this circle is also the boundary of S. $S^c=\{|z|\geq 2\}$. S is not compact since it is not closed.

13. K is the set of all points $z=x+iy$ which satisfy $(x-1)^2+y^2\leq x^2+(y+4)^2$ or when simplified, the half plane $0\leq 2x+8y+15$. The set K is closed, so $K^c=K$. Every point of K is a limit point of K. The set of boundary points of $K=$ boundary of $K=\{z|z=x+iy$ and $2x+8y+15=0\}$, a line. K is not compact since it is not bounded.

14. T is the closed annular region between concentric cirlces of radii 2 and $2\sqrt{2}$ centered at $z=-i$, so $T^c=T$. Every limit point of T is a point of T and conversely. The set of boundary points of $T=$ boundary of $T=\{z||z+i|=2$ or $|z+i|=2\sqrt{2}\}$. T is closed and bounded, hence compact.

15. M is an open half plane consisting of all $z=x+iy$ with $y<7$. The limit points of M are all $z=x+iy$ with $y\leq 7$. The set of boundary points of $M=$ boundary of $M=\{z|z=x+iy, y=7\}$, a horizontal line. M is not compact, being neither closed nor bounded.

16. R is a set which is neither open nor closed. The set of limit points of R = the set of boundary points of R = boundary of $R = \{z = x+iy | x = 0, y = \frac{1}{n}\} \cup \{z = x+iy | y = 0, x = \frac{1}{m}\}$, m, n positive integers. Finally $R^c = R \cup \{$ boundary of $R\}$. R is bounded but not closed, hence not compact.

17. U is the vertical strip of $z = x+iy$, where $1 < x \leq 3$. This set is neither open nor closed. The set of limit points of U is the vertical strip of $z = x+iy$, where $1 \leq x \leq 3$. The boundary of U = the set of boundary points of U = the two vertical lines $x = 1, x = 3$. Finally $U^c = \{z | 1 \leq Re(z) \leq 3\}$. U is not bounded, hence not compact.

18. V is the rectangular set $= \{z = x+iy | 2 < x \leq 3, -1 < y < 1\}$. This set is neither open nor closed. The set of limit points of $V = \{z = x+iy | 2 \leq x \leq 3, -1 \leq y \leq 1\}$. The boundary of V = the set of boundary points of $V = \{z = x+iy | x = 2, -1 \leq y \leq 1\} \cup \{z = x+iy | x = 3, -1 \leq y \leq 1\} \cup \{z = x+iy | 2 \leq x \leq 3, y = -1\} \cup \{z = x+iy | 2 \leq x \leq 3, y = 1\}$. Finally $V^c = \{z = x+iy | 2 \leq x \leq 3, -1 \leq y \leq 1\}$. V is not closed, so it is not compact.

19. Putting $z = x+iy$, we find $z \epsilon W$ iff $x > y^2$. This is the open region inside the parabola $x = y^2$ which opens to the right along the x axis and has vertex at the origin. The set of limit points of W are all points of W together with the parabola $x = y^2$. The boundary of W is the parabola $x = y^2$, and these are all the boundary points of W. Finally $W^c = \{z = x+iy | x \geq y^2\}$. W is not closed, so it is not compact.

20. (a) We give the argument in two parts. First suppose $w \notin S$. Let $\epsilon = \min_{1 \leq i \leq n} |w - z_i|/2$. Such a minimum exists because S is a finite set. Then the open disk about w of radius ϵ contains no point of S, hence w is not a limit point of S. For $w \in S$, say $w = z_k$ let $\epsilon = \min_{1 \leq i \leq n, i \neq k} |z_k - z_i|/2$. Then the open disk about w of radius ϵ contains no point of S except z_k, hence w in not a limit point of S. It follows that S has no limit points.

(a) (b) Clearly every open disk about z_k contains $z_k \in S$ and at least one point not in S, in particular $w = z_k + \epsilon$, where ϵ is defined in the second part of (a) above.

(c) S is closed because every $z_k \in S$ is a boundary point of S.

21. $\lim_{n\to\infty} \left(1 + \frac{2in}{n+1}\right) = \lim_{n\to\infty} \left(1 + \frac{2i}{1+\frac{1}{n}}\right) = 1 + 2i$

22. The sequence $\{i^{2n}\}$ diverges since these are subsequences which converge to different limits. If $n = 2k$, then $i^{2n} = i^{4k} = 1$, and $\{i^{4k}\} \to 1$, but if $n = 2k+1, i^{2n} = i^{4k+1} = i$ and $\{i^{4k+1}\} \to i$.

23. $\lim_{n\to\infty} \left[\frac{1+2n^2}{n^2} - \left(\frac{n-1}{n}\right) i\right] = \lim_{n\to\infty} \left[\frac{1}{n^2} + 2 - \left(1 - \frac{1}{n}\right) i\right] = 2 - i$

24. The sequence $\{e^{n\pi i/3}\}$ diverges. Consider the subsequences obtained if $n = 6k, e^{6k\pi i/3} = e^{2\pi ik} = 1$, so $\{e^{n\pi i/3}\}_{n=6k} \to 1$, whereas if $n = 6k+3, e^{(6k+3)\pi i/2} = e^{(2k\pi+\pi)i} = -1$ so $\{e^{n\pi i/3}\}_{n=6k+3} \to -1$.

25. This is a constant sequence, since $-i^{4n} = -1$ for all $n \geq 1$, so $\lim_{n\to\infty} (-i^{4n}) = -1$.

26. This sequence diverges, since $\sin(n)$ does not have a limit as $n \to \infty$.

27. $\lim_{n\to\infty}\left(\dfrac{1+3n^2 i}{2n^2-n}\right)=\lim_{n\to\infty}\left(\dfrac{\frac{1}{n^2}+3i}{2-\frac{1}{n}}\right)=\dfrac{3}{2}i$

28. Taking $n=6k$ we get $\{e^{6k\pi i/3}\}=\{e^{2k\pi i}\}=1,1,1,\ldots$ which converges to 1. Taking $n=6K+3$ we get $\{e^{(6k+3)\pi i/3}\}=\{e^{(2k+1)\pi i}\}=-1,-1,-1,\ldots$ which converges to -1.

29. For $n=2k$ we have $\{i^{4k}\}=1,1,1,\ldots$ which converges to 1. For $n=2k+1$ we have $\{i^{4k+2}\}=-1,-1,-1,\ldots$ which converges to -1.

Chapter Twenty One - Complex Functions

Section 21.1 Limits, Continuity and Derivatives

1. With $z = x+iy$, we find $f(z) = z-i = x+iy-i = x+i(y-1)$; Thus $u(x,y) = x, v(x,y) = y-1$. For the Cauchy-Riemann equation, compute $\frac{\partial u}{\partial x} = 1, \frac{\partial v}{\partial y} = 1$, and $\frac{\partial u}{\partial y} = 0, \frac{\partial v}{\partial x} = 0$; Since u, v, u_x, v_y, u_y, v_x are all continuous and the Cauchy-Riemann equations hold everywhere, f is differentiable everywhere.

2. $f(x+iy) = (x+iy)^2 - i(x+iy) = (x^2 - y^2 + y) + i(2xy - x)$; Thus $u(x,y) = x^2 - y^2 + y, v(x,y) = 2xy - x$. For the Cauchy-Riemann equations, compute $\frac{\partial u}{\partial x} = 2x, \frac{\partial v}{\partial y} = 2x, \frac{\partial u}{\partial y} = -2y+1, \frac{\partial v}{\partial x} = 2y - 1$; so Cauchy-Riemann equations hold everywhere. Since u, v and all partials are continuous, f is differentiable everywhere.

3. $f(x+iy) = |x+iy| = \sqrt{x^2+y^2}$; so $u(x,y) = \sqrt{x^2+y^2}, v(x,y) = 0$. For the Cauchy-Riemann equations, compute

$$\frac{\partial u}{\partial x} = \frac{x}{\sqrt{x^2+y^2}}, \frac{\partial v}{\partial y} = 0, \frac{\partial u}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}, \frac{\partial v}{\partial x} = 0.$$

We see that the Cauchy-Riemann hold nowhere, and $f(z) = |z|$ is differentiable nowhere.

4. $f(x+iy) = \frac{2x+1+2yi}{x+iy} = \frac{[(2x+1)+2yi][x-iy]}{x^2+y^2} = \frac{2x^2+x+2y^2+i(-y)}{x^2+y^2}$; Thus $u(x,y) = \frac{2x^2+x+2y^2}{x^2+y^2}, v(x,y) = \frac{-y}{x^2+y^2}$. For the Cauchy-Riemann equations, compute

$$\frac{\partial u}{\partial x} = \frac{(x^2+y^2)(4x+1) - (2x^2+x+2y^2)(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2},$$
$$\frac{\partial v}{\partial y} = \frac{-(x^2+y^2)+y(2y)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2},$$
$$\frac{\partial u}{\partial y} = \frac{4y(x^2+y^2) - 2y(2x^2+x+2y^2)}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2},$$
$$\frac{\partial v}{\partial x} = \frac{2xy}{(x^2+y^2)^2}.$$

Since u, v, and all partial are continuous at all $z \neq 0$, and the Cauchy-Riemann equations hold for all $z \neq 0$, f is differentiable for all $z \neq 0$.

5. $f(x+iy) = i|x+iy|^2 = i(x^2+y^2)$, so $u(x,y) = 0, v(x,y) = x^2+y^2$. For the Cauchy-Riemann equations, compute $\frac{\partial u}{\partial x} = 0, \frac{\partial v}{\partial y} = 2y, \frac{\partial u}{\partial y} = 0, \frac{\partial v}{\partial x} = 2x$. The Cauchy-Riemann equations hold only for $x = y = 0$, i.e. $z = 0$ but f is not differentiable at $z = 0$.

6. $f(x+iy) = x+iy+Im(x+iy) = (x+y)+iy$, so $u(x,y) = x+y, v(x,y) = y$. For the Cauchy-Riemann equations, compute $\frac{\partial u}{\partial x} = 1, \frac{\partial v}{\partial y} = 1, \frac{\partial u}{\partial y} = 1, \frac{\partial v}{\partial x} = 0$. The Cauchy-Riemann equations hold nowhere, so f is not differentiable at any point.

7. $f(x+iy) = \frac{x+iy}{Re(x+iy)} = \frac{x+iy}{x} = 1+i\left(\frac{y}{x}\right)$; so $u(x,y) = 1, v(x,y) = \frac{y}{x}$. For the Cauchy-Riemann equations, compute $\frac{\partial u}{\partial x} = 0, \frac{\partial v}{\partial y} = \frac{1}{x}, \frac{\partial u}{\partial y} = 0, \frac{\partial v}{\partial x} = -\frac{y}{x^2}$. The Cauchy-Riemann

equations hold nowhere, so f is not differentiable at any point.

8. $f(x+iy) = (x+iy)^3 - 8(x+iy) + 2 = x^3 - 3xy^2 - 8x + 2 + i(3x^2y - y^3 - 8y)$, so $u(x,y) = x^3 - 3xy^2 - 8x + 2, v(x,y) = 3x^2y - y^3 - 8y$. For the Cauchy-Riemann equations, compute $\dfrac{\partial u}{\partial x} = 3x^2 - 3y^2 - 8, \dfrac{\partial v}{\partial y} = 3x^2 - 3y^2 - 8, \dfrac{\partial u}{\partial y} = -6xy, \dfrac{\partial v}{\partial x} = 6xy$. The Cauchy-Riemann equations hold everywhere, u, v, and all partial are continuous everywhere, so f is differentiable everywhere.

9. $f(x+iy) = (x-iy)^2 = x^2 - y^2 - 2xyi$, so $u(x,y) = x^2 - y^2, v(x,y) = -2xy$. For the Cauchy-Riemann equations, compute $\dfrac{\partial u}{\partial x} = 2x, \dfrac{\partial v}{\partial y} = -2x, \dfrac{\partial u}{\partial y} = -2y, \dfrac{\partial v}{\partial x} = -2y$. The Cauchy-Riemann equations hold only at $x = y = 0$ or $z = 0$, but f is not differentiable at $z = 0$.

10. $f(x+iy) = i(x+iy) + |x+iy| = -y + \sqrt{x^2+y^2} + ix$; so $u(x,y) = -y + \sqrt{x^2+y^2}, v(x,y) = x$. For the Cauchy-Riemann equations, compute $\dfrac{\partial u}{\partial x} = \dfrac{x}{\sqrt{x^2+y^2}}, \dfrac{\partial v}{\partial y} = 0, \dfrac{\partial u}{\partial y} = -1 + \dfrac{y}{\sqrt{x^2+y^2}}, \dfrac{\partial v}{\partial x} = 1$. The Cauchy-Riemann equations hold nowhere, so f is not differentiable at any point.

11. $f(x+iy) = -4(x+iy) + \dfrac{1}{x+iy} = -4x + \dfrac{x}{x^2+y^2} + i\left(-4y - \dfrac{y}{x^2+y^2}\right)$; so

$$u(x,y) = -4x + \frac{x}{x^2+y^2}, v(x,y) = -4y - \frac{y}{x^2+y^2}.$$

For the Cauchy-Riemann equations, compute

$$\frac{\partial u}{\partial x} = -4 + \frac{y^2 - x^2}{(x^2+y^2)^2}, \frac{\partial v}{\partial y} = -4 + \frac{y^2 - x^2}{x^2+y^2},$$

$$\frac{\partial u}{\partial y} = \frac{-2xy}{(x^2+y^2)^2}, \frac{\partial v}{\partial x} = \frac{2xy}{(x^2+y^2)^2}.$$

Since u, v, and all partials are continuous at all points except $z = 0$, and the Cauchy-Riemann equations hold for all $z \neq 0$, f is differentiable for all $z \neq 0$.

12. $f(x+iy) = \dfrac{x + i(y-1)}{x + i(y+1)} = \dfrac{x^2 + y^2 - 1 - 2xi}{x^2 + (y+1)^2}$, so

$$u(x,y) = \frac{x^2 + y^2 - 1}{x^2 + (y+1)^2}, v(x,y) = \frac{-2x}{x^2 + (y+1)^2}.$$

For the Cauchy-Riemann equations, compute

$$\frac{\partial u}{\partial x} = \frac{2x(x^2 + (y+1)^2) - 2x(x^2 + y^2 - 1)}{[x^2 + (y+1)^2]^2} = \frac{4x(y+1)}{[x^2 + (y+1)^2]^2},$$

$$\frac{\partial v}{\partial y} = \frac{4x(y+1)}{[x^2 + (y+1)^2]^2},$$

$$\frac{\partial u}{\partial y} = \frac{2y(x^2+(y+1)^2) - 2(y+1)(x^2+y^2-1)}{[x^2+(y+1)^2]^2} = \frac{2(y+1)^2 - 2x^2}{[x^2+(y+1)^2]^2},$$

$$\frac{\partial v}{\partial x} = \frac{-2(x^2+(y+1)^2)+4x^2}{[x^2+(y+1)^2]^2} = \frac{2x^2-2(y+1)^2}{[x^2+(y+1)^2]^2}.$$

Since u, v, and all partials are continuous at all points except $x = 0, y = -1$, or $z = -i$, and the Cauchy-Riemann equations holds at all such points, f is differentiable at all points except at $z = -i$.

Section 21.2 Power Series

In Problems 1 through 8 we apply the ratio test to determine the open disk and radius of convergence.

1. Consider the ratio $\left|\frac{c_{n+1}(z+3i)^{n+1}}{C_n(z+3i)^n}\right| = \frac{|z+3i|}{2}\left(\frac{n+2}{n+1}\right) \to \frac{|z+3i|}{2}$ as $n \to \infty$. For convergence we have $\frac{|z+3i|}{2} < 1$ or $|z+3i| < 2$. This is an open disk of radius $R = 2$ centered at $z = -3i$.

2. Consider the ratio $\left|\frac{c_{n+1}(z-i)^{2n+2}}{c_n(z-i)^{2n}}\right| = |z-i|^2\frac{(2n+1)^2}{(2n+3)^2} \to |z-i|^2$ as $n \to \infty$. The series converges on the open disk $|z-i| < 1$ which has radius $R = 1$ and center $z = i$.

3. Consider the ratio

$$\left|\frac{c_{n+1}(z-1+3i)^{n+1}}{c_n(z-i+i)^n}\right| = |z-(1-3i)|\frac{(n+1)^{n+1}}{(n+2)^{n+1}}\frac{(n+1)^n}{n^n}$$

$$= |z-(1-3i)|\left(1-\frac{1}{n+2}\right)^{n+1}\left(1+\frac{1}{n}\right)^n.$$

Since $\lim_{n\to\infty}\left(1+\frac{k}{n}\right)^n = e^k$, we have $\left|\frac{c_{n+1}(z-1+3i)^{n+1}}{c_n(z-1+3i)^n}\right| \to |z-(1-3i)|$ as $n \to \infty$. The series converges on the open disk $|z-(1-3i)| < 1$ of radius $R = 1$, center at $z = 1-3i$.

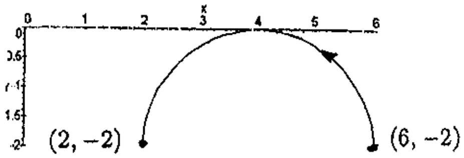
4. Consider the ratio $\left|\frac{c_{n+1}(z+3-4i)^{n+1}}{c_n(z+3-4i)^n}\right| = |z-(-3+4i)|\left|\frac{2i}{5+i}\right| \to |z-(-3+4i)|\frac{2}{\sqrt{26}}$ as $n \to \infty$. The series converges on the open disk $|z-(-3+4i)| < \frac{\sqrt{26}}{2}$ of radius $R = \frac{\sqrt{26}}{2}$, center at $z = -3+4i$.

5. Consider the ratio $\left|\frac{c_{n+1}(z+8i)^{n+1}}{c_n(z+8i)^n}\right| = \frac{|z+8i|}{2}|i|$. This series is geometric and converges on the open disk $|z+8i| < 2$ of radius $R = 2$, center at $z = -8i$.

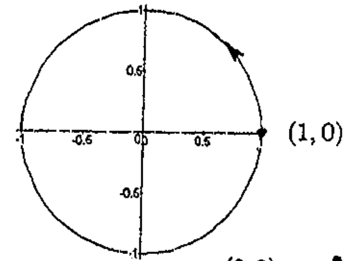
Chapter Twenty Two - Complex Integration

Section 22.1 Curves in the Plane

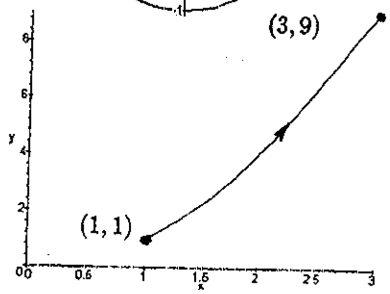
1. Identify $z_0 = 4 - 2i, r = 2$ and $\Gamma(t) = z_0 + re^{it}$ represents the upper half of a circle of radius 2 centered at $(4, -2)$ drawn from initial point $(6, -2)$ counterclockwise to the terminal point $(2, -2)$ The curve is simple but not closed; and has tangent $\Gamma'(t) = 2ie^{it} = -2\sin(t) + 2i\cos(t) = -2\sin(t)\mathbf{i} + 2\cos(t)\mathbf{j}$ at each point



2. $\Gamma(t) = ie^{2it} = -2\sin(t) + 2i\cos(t)$ describes two revolutions around the circle of radius $r = 1$ drawn counterclockwise from initial point $(1, 0)$ to terminal point $(1, 0)$. The curve is closed but not simple; and has tangent $\Gamma'(t) = -2e^{2it} = -2\cos(2t) - 2i\sin(2t) = -2\cos(2t)\mathbf{i} - 2\sin(2t)\mathbf{j}$ at each point.

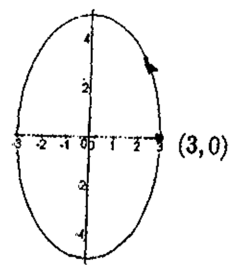


3. Put $x = t, y = t^2$ and we see $\Gamma(t)$ describes a portion of the parabola $y = x^2$ drawn from initial point $(1, 1)$ to terminal point $(3, 9)$. The curve is simple, not closed; and has tangent $\Gamma' = 1 + 2it = \mathbf{i} + 2t\mathbf{j}$ at every point

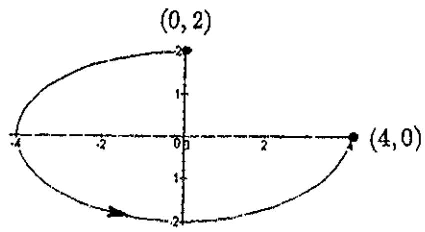


4. With $x = 3\cos(t), y = 5\sin)t)$ we see $\left(\frac{x}{3}\right)^2 + \left(\frac{y}{5}\right)^2 = 1$, so $\Gamma(t)$ traces once around an ellipse in a counterclockwise direction from initial point $(3, 0)$ to terminal point $(3, 0)$ The curve is simple and closed; with tangent $\Gamma'(t) = -3\sin(t) + 5\cos(t)i = -3\sin(t)\mathbf{i} + 5\cos(t)\mathbf{j}$ at every point.

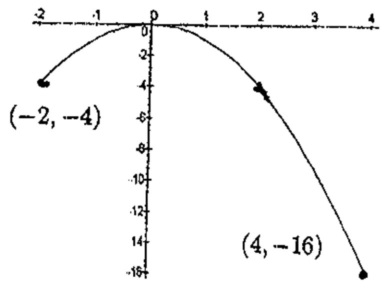
5. $\Theta(t)$ describes the same ellipse as in 4 above, except $\Theta(t)$ traces two times around the ellipse. the curve is closed, but not simple; a tangent is $\Theta'(t) = -3\sin(t) + 5\cos(t)i = -3\sin(t)\mathbf{i} + 5\cos(t)\mathbf{j}$ at every point



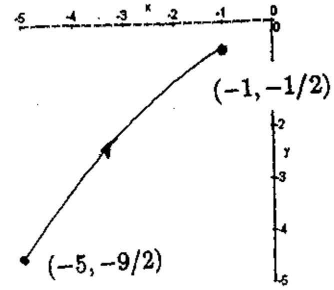
6. put $x = 4\sin(t), y = -2\cos(t)$ to get $\left(\frac{x}{4}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$, so $\Lambda(t)$ traces in a counterclockwise direction around a portion of this ellipse from initial point $(0, 2)$ to terminal point $(4, 0)$. The curve is simple, but not closed; with tangent $\Lambda'(t) = 4\cos(t) + 2\sin(t)i = 4\cos(t)\mathbf{i} + 2\sin(t)\mathbf{j}$ at every point



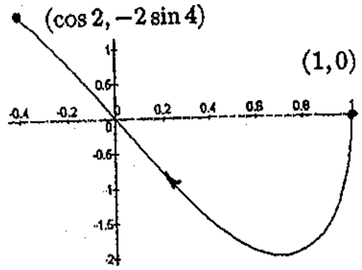
7. $\Psi(t)$ traces a portion of the parabola $y = -x^2$ from initial point $(-2, -4)$ to terminal point $(4, -16)$. The curve is simple, but not closed; a tangent in $\Psi'(t) = 1 - 2it = \mathbf{i} - 2t\mathbf{j}$ at every point



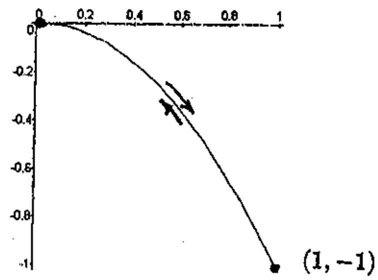
8. Eliminate the parameter from $x = 2t + 1, y = -\frac{1}{2}t^2$ to get $y = -\frac{1}{2}\left(\frac{x-1}{2}\right)^2$. So $\Phi(t)$ traces a portion of this parabola from initial point $(-5, -9/2)$ to terminal point $(-1, -1/2)$. The curve is simple, not closed; and has tangent $\Phi'(t) = 2 - ti = 2\mathbf{i} - t\mathbf{j}$ at every point.



9. The curve traced by $\Gamma(t)$ is a portion of a figure eight curve from initial point $(1, 0)$ to terminal point $(\cos 2, -2\sin 4)$. The curve is simple, not closed; with tangent $\Gamma'(t) = -\sin(t) - 4\cos(2t)i = -\sin(t)\mathbf{i} - 4\cos(2t)\mathbf{j}$ at every point



10 Eliminating the parameter t shows $y = -x^2$ so the curve is a portion of a parabola. Since $x = t^2 \geq 0$ and $y = -t^4 \leq 0$ we get only the portion lying in quadrant four Thus $\Delta(t)$ traces from initial point $(1, -1)$ to the origin $(0, 0)$ and then back to the terminal point $(1, -1)$. The curve is closed, but not simple; with tangent $\Delta'(t) = 2t - 4t^3 i = 2t\mathbf{i} - 4t^3\mathbf{j}$ at every point except $(0, 0$.



Section 22.2 The Integral of a Complex Function

1. $\int_\Gamma 1dz = \int_1^3 (2t - i)dt = 8 - 2i$

2. $\int_\Gamma (z^2 - iz)dz = \left(\frac{z^3}{3} - \frac{iz^2}{2}\right)\Big|_2^{2i} = \frac{1}{3}(-8 + 4i)$

3. For z on $\Gamma, z = 1 + (1+i)t, 0 \le t \le 1$, so $\int_\Gamma Re(z)dz = \int_0^1 (1+t)(1+i)dt = \frac{3}{2}(1+i)$

4. For z on $\Gamma, z = 4e^{it}, \pi/2 \le t \le 3\pi/2$, so $\int_\Gamma \frac{1}{z}dz = \int_{\pi/2}^{3\pi/2} \frac{1}{4e^{it}}(4ie^{it})dt = i\pi$

5. $\int_\Gamma (z-1)dz = \frac{(z-1)^2}{2}\Big|_{2i}^{1-4i} = \frac{1}{2}[(-4i)^2 - (2i-1)^2] = -\frac{13}{2} + 2i$

6. $\int_\Gamma iz^2dz = \frac{i}{3}(z^3|_{1+2i}^{3+i}) = \frac{1}{3}(-28 + 29i)$

7. $\int_\Gamma \sin(2z)dz = -\frac{1}{2}\cos(2z)\Big|_{-i}^{-4i} = -\frac{1}{2}[\cosh(8) - \cosh(2)]$

8. $\int_\Gamma (1 + z^2)dz = \left(z + \frac{1}{3}z^3\right)\Big|_{-3i}^{3i} = -12i$

9. $\int_\Gamma -i\cos(z)dz = -i\sin(z)|_0^{2+i} = -i\sin(2+i) = -i[\sin(2)\cosh(1) + i\cos(2)\sinh(1)]$
$= -\cos(2)\sinh(1) - i\sin(2)\cosh(1)$

10. Parametrize Γ by $z = -4 + t(i+4), 0 \le t \le 1$ so on $\Gamma, |z|^2 = 16(t-1)^2 + t^2$.
Then $\int_\Gamma |z|^2dz = \int_0^1 [16(t-1)^2 + t^2](i+4)dt = \frac{(i+4)}{3}\left[16(t-1)^3 + t^3\right]\Big|_0^1 = \frac{17}{3}(4+i)$.

11. Γ as given runs from $z_1 = 0$ to $z_2 = 2 - 4i$, so $\int_\Gamma (z-i)^3dz = \frac{1}{4}(z-i)^4\Big|_0^{(2-4i)}$
$= \frac{1}{4}[(2-5i)^4 - (-i)^4] = 10 + 210i$

12. $\int_\Gamma e^{iz}dz = -ie^{iz}\Big|_{-2}^{-4-i} = -i[e^{1-4i} - e^{-2i}] = -e\sin(4) + \sin(2) + i[\cos(2) - e\cos(4)]$

13 Parametrize Γ by $z = t(-4+3i), 0 \le t \le 1$, so $\int_\Gamma i\bar{z}dz = \int_0^1 i(-4t-3ti)(-4+3i)dt$
$= (-4+3i)\left[\frac{3}{2} - 2i\right] = \frac{25}{2}i$

14 Parametrize Γ by $z = e^{it}, 0 \le t \le 2\pi$, so $\int_\Gamma Im(z)dz = \int_0^{2\pi} \sin(t)ie^{it}dt$
$= \int_0^{2\pi} [-\sin^2(t) + i\sin(t)\cos(t)]dt = -\pi$

15 Parametrize Γ by $z = -i + t(1+i), 0 \le t \le 1$, so $\int_\Gamma |z|^2 dz$
$= \int_0^1 [t^2 + (t-1)^2](1+i)dt = \frac{2}{3}(1+i)$

16 To obtain a bound on $\left|\int_\Gamma \cos(z^2)dz\right|$ we begin with $\left|\int_\Gamma \cos(z^2)dz\right| \le \int_\Gamma |\cos(z^2)|dz \le M$ (length of Γ), where M is an upper bound on $|\cos(z^2)|$ With $z = x+iy$ we have $z^2 = x^2 - y^2 + 2xyi$, so $|\cos(z^2)| = |\cos(x^2 - y^2 + 2xyi)| = |\cos(x^2-y^2)\cosh(2xy) - i\sin(x^2 - y^2)\sinh(2xy)| \le \cosh(2xy) + \sinh(2xy) = e^{2xy}$ For points on $\Gamma, x = 4\cos(\theta), y = 4\sin(\theta)$ so $e^{2xy} = e^{16\sin(2\theta)} \le e^{16} = M$. Also length of $\Gamma = 8\pi$, so $\left|\int_\Gamma \cos(z^2)dz\right| \le 8\pi e^{16}$

17 We begin with $\left|\int_\Gamma \frac{1}{1+z}dz\right| \le M$ (length of Γ) where $M \ge \left|\frac{1}{1+z}\right| = \frac{1}{|1+z|}$ for all z on Γ. The point on Γ closest to $z = -1$ is $2+i$, hence for z on $\Gamma, |z+1| = |z-(-1)| \ge |2+i+1| = \sqrt{10}$, and $\frac{1}{|z+1|} \le \frac{1}{\sqrt{10}} = M$ The length of Γ is $|(4+2i)-(2+i)| = \sqrt{5}$, so $\left|\int_\Gamma \frac{1}{z+1}dz\right| \le \frac{\sqrt{5}}{\sqrt{10}} = \frac{1}{\sqrt{2}}$

Section 22.3 Cauchy's Theorem

1. Since $\sin(3z)$ is differentiable everywhere, $\int_\Gamma \sin(3z)dz = 0$ by Cauchy's Theorem

2. Let Γ be parametrized by $z = i+3e^{it}, 0 \le t \le 2\pi$, so $\int_\Gamma \frac{2z}{z-i}dz = \int_0^{2\pi} \left(\frac{2i+6e^{it}}{3e^{it}}\right) i3e^{it}dt = \int_0^{2\pi} (-2+6ie^{it})dt = -4\pi$

3. Let Γ be parametrized by $z = 2i + 2e^{it}, 0 \le t \le 2\pi$, so $\int_\Gamma \frac{1}{(z-2i)^3}dz = \int_0^{2\pi} \frac{2ie^{it}}{(2e^{it})^3}dt = \frac{i}{4}\int_0^{2\pi} e^{-2it}dt = 0$

4. Since $z^2\sin(z)$ is differentiable everywhere, $\int_\Gamma z^2\sin(z)dz = 0$ by Cauchy's Theorem.

5 Describe Γ by $z = e^{it}, 0 \le t \le 2\pi$, so $\int_\Gamma \bar{z}dz = \int_0^{2\pi} e^{-it}ie^{it}dt = 2\pi i$

6 Describe Γ by $z = 5e^{it}, 0 \le t \le 2\pi$, so $\int_\Gamma \frac{1}{\bar{z}} = \int_0^{2\pi} \frac{1}{5e^{-it}}(5ie^{it})dt = 0$

7 Since ze^z is differentiable everywhere, $\int_\Gamma ze^z dz = 0$ by Cauchy's Theorem

8. Since $z^2 - 4z + i$ is differentiable everywhere, $\int_\Gamma (z^2 - 4z + i)dz = 0$ by Cauchy's Theorem

9. Describe Γ by $z = 7e^{it}, 0 \leq t \leq 2\pi$, so $\int_\Gamma |z|^2 dz = \int_0^{2\pi} (49)7ie^{-it}dt = 0$

10. $\sin\left(\frac{1}{z}\right)$ is differentiable at all points except $z = 0$, but $z = 0$ is outside $|z - 1 + 2i| = 1$, so $\int_\Gamma \sin\left(\frac{1}{z}\right) dz = 0$, by Cauchy's Theorem

11. Describe Γ by $z = 2e^{it}, 0 \leq t \leq 2\pi$, so $\int_\Gamma Re(z)dz = \int_0^{2\pi} 2\cos(t)(2ie^{-it})dt$
$= \int_0^{2\pi} [4i\cos^2(t) - 4\cos(t)\sin(t)]dt = 4\pi i$

12. We have $\int_\Gamma (z^2 + Im(z))dz = \int_\Gamma z^2 dz + \int_\Gamma Im(z)dz$ The first integral is zero by Cauchy's Theorem. For the integral $\int_\Gamma Im(z)dz$ we parametrize the four sides of Γ by $S_1 : z = -2it, 0 \leq t \leq 1, S_2 : z = 2t - 2i, 0 \leq t \leq 1; S_3 : z = 2 - 2i(1-t); 0 \leq t \leq 1; S_4 : z = 2(1-t), 0 \leq t \leq 1$
Then $\int_\Gamma Im(z)dz = \int_0^1 (-2t)(-2i)dt + \int_0^1 (-2)(2)dt + \int_0^1 -2(1-t)(2i)dt + \int_0^1 0dt$
$= 2i - 4 - 2i = -4$, so $\int_\Gamma (z^2 + Im(z))dz = -4$

Section 22.4 Consequences of Cauchy's Theorem

1. $\int_\Gamma \frac{z^4}{z - 2i}dz = 2\pi i(2i)^4 = 32\pi i$

2. $\int_\Gamma \frac{\sin(z^2)}{z-5}dz = 2\pi i[\sin(5^2)] = 2\pi\sin(25)i$

3. $\int_\Gamma \frac{z^2 - 5z + i}{z - 1 + 2i}dz = 2\pi i[(1-2i)^3 - 5(1-2i) + i] = 2\pi i[-8 + 7i]$

4. $\int_\Gamma \frac{2z}{(z-2)^2}dz = 2\pi i\frac{d}{dz}(2z^3)\Big|_{z=2} = 48\pi i$

5. $\int_\Gamma \frac{ie^z}{(z-2+i)^2}dz = 2\pi i\frac{d}{dz}(ie^z)\Big|_{z=2-i} = -2\pi e^{2-i} = -2\pi e^2[\cos(1) - \sin(1)i]$

6. $\int_\Gamma \frac{\cos(z-i)}{(z+2i)^3}dz = \frac{2\pi i}{2}\frac{d^2}{dz^2}(\cos(z-i))\Big|_{z=-2i} = -\pi i\cos(-3i) = -\pi i\cosh(3)$

7. $\int_\Gamma \frac{z\sin(3z)}{(z+4)^3}dz = \frac{2\pi i}{2}\frac{d^2}{dz^2}(z\sin(3z))\Big|_{z=-4} = \pi i[6\cos(12) - 36\sin(12)]$

8 Parametrize Γ by $z = 1 - (i+1)t, 0 \leq t \leq 1$, so for z on Γ,

$$2i\overline{z}|z| = 2i((1-t) + it)\sqrt{1 - 2t + 2t^2},$$

and

$$\int_\Gamma 2i\bar{z}|z|dz = \int_0^1 2i[(1-t)+it]\sqrt{1-2t+2t^2}(-1-i)dt$$

$$= 2\int_0^1 \sqrt{1-2t+2t^2}dt + 2i\int_0^1 (2t-1)\sqrt{1-2t+2t^2}dt$$

$$= 1 + \frac{\sqrt{2}}{4}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)$$

9. $\displaystyle\int_\Gamma -\frac{(2+i)\sin(z^4)}{(z+4)^2}dz = -(2\pi i)(2+i)\,\frac{d}{dz}(\sin(z^4))\Big|_{z=-4} = 2\pi(1-2i)\left[4z^3\cos(z^4)\right]\Big|_{z=-4}$
$= -512\pi(1-2i)\cos(256)$

10. $\displaystyle\int_\Gamma (z-i)^2dz = \frac{1}{3}(z-i)^3\Big|_i^{-i} = \frac{8}{3}i$

11. Parametrize Γ by $z = (3+i)+t(-1-6i), 0 \le t \le 1$, so for z on Γ, $Re(z+4) = 7-t$, and $\displaystyle\int_\Gamma Re(z+4)dz = \int_0^1 (7-t)(-1-6i)dt = (-1-6i)\left(\frac{13}{2}\right) = -\frac{13}{2} - 39i$

12. $\displaystyle\int_\Gamma \frac{3z^2\cosh(z)}{(2+2i)^2} = 2\pi i\,\frac{d}{dz}\left[3z^2\cosh(z)\right]\Big|_{z=-2i} = 2\pi i\left[6z\cosh(z) + 3z^2\sinh(z)\right]\Big|_{z=-2i}$

$$= 2\pi i\left[-12i\cosh(2i) + 12\sinh(2i)\right] = 24\pi[\cos(2) - \sin(2)].$$

13. Consider $\displaystyle\oint_\Gamma \frac{e^z}{z}dz$ with $\Gamma : z = e^{i\theta}, 0 \le \theta \le 2\pi$. By Cauchy's integral formula $\displaystyle\oint_\Gamma \frac{e^z}{z}dz = 2\pi i$.
By parametrizing Γ and direct evaluation we have

$$\oint_\Gamma \frac{e^z}{z}dz = \int_0^{2\pi} \frac{e^{(\cos\theta + i\sin\theta)}}{e^{i\theta}} ie^{i\theta}d\theta = i\int_0^{2\pi} e^{\cos(\theta)}\cos(\sin(\theta))d\theta - \int_0^{2\pi} e^{\cos(\theta)}\sin(\sin(\theta))d\theta.$$

By equating Imaginary parts of $\displaystyle\oint_\Gamma \frac{e^z}{z}dz$ we conclude,

$$\int_0^{2\pi} e^{\cos(\theta)}\cos(\sin(\theta))d\theta = 2\pi.$$

14. First observe $f(z) = \dfrac{z-4i}{z^3+4z} = \dfrac{z-4i}{z(z-2i)(z+2i)}$. Let Γ_1, Γ_2 and Γ_3 be non-intersecting circles enclosing $z_1 = 0, z_2 = 2i$, and $z_3 = -2i$ respectively. By the Deformation Theorem, $\displaystyle\oint_\Gamma f(z)dz = \oint_{\Gamma_1} f(z)dz + \oint_{\Gamma_2} f(z)dz + \oint_{\Gamma_3} f(z)dz$, and each of these three integrals can be evaluated by Cauchy's integral theorem. We have,

$$\oint_{\Gamma_1} \frac{z-4i}{z(z-2i)(z+2i)}dz = 2\pi i\left[\frac{-4i}{(-2i)(2i)}\right] = 2\pi i(-i),$$

$$\oint_{\Gamma_2} \frac{z-4i}{z(z-2i)(z+2i)} dz = 2\pi i \left[\frac{-2i}{(2i)(4i)}\right] = 2\pi i \left(\frac{i}{4}\right),$$

$$\oint_{\Gamma_3} \frac{z-4i}{z(z-2i)(z+2i)} dz = 2\pi i \left[\frac{-6i}{(-2i)(-4i)}\right] = 2\pi i \left(\frac{3i}{4}\right).$$

So,

$$\oint_{\Gamma} \frac{z-4i}{z^3+4z} dz = 2\pi i \left[-i + \frac{i}{4} + \frac{3i}{4}\right] = 0$$

Chapter Twenty Three - Series Representations of Functions

Section 23.1 Power Series Representations

1. $\cos(2z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(2z)^{2n}; |z| < \infty$

2. $e^{-z} = e^{-(z+3i-3i)} = e^{3i}\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}(z+3i)^n; |z+3i| < \infty$

3. $\frac{1}{1-z} = \frac{1}{1-4i-(z-4i)} = \frac{1}{1-4i}\left(\frac{1}{1-\frac{z-4i}{1-4i}}\right) = \frac{1}{(1-4i)}\sum_{n=0}^{\infty}\left(\frac{1}{1-4i}\right)^n (z-4i)^n; |z-4i| < \sqrt{17}$

4. $\sin(z^2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(z^2)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n}; |z| < \infty$

5. $\frac{1}{(1-z)^2} = \frac{d}{dz}\left(\frac{1}{1-z}\right) = \frac{d}{dz}\left(\sum_{n=0}^{\infty} z^n\right) = \sum_{n=1}^{\infty} nz^{n-1}; |z| < 1$

6. $\frac{1}{2+z} = \frac{1}{(3-8i)+z-(1-8i)} = \frac{1}{3-8i}\left(\frac{1}{1+\frac{z-(1-8i)}{3-8i}}\right)$

$$= \frac{1}{3-8i}\sum_{n=0}^{\infty}\frac{(-1)^n}{(3-8i)^n}(z-1+8i)^n; |z-1+8i| < \sqrt{65}$$

7. $z^2-3z+i = i-3(z-2+i+2-i)+(z-2+i+2-i)^2 = -3+(1-2i)(z-2+i)+(z-2+i)^2, |z| < \infty$

8. First use partial fractions to write

$$\frac{1}{z^2+2} = -\frac{i}{2\sqrt{2}}\left[\frac{1}{z-\sqrt{2}i} - \frac{1}{z+\sqrt{2}i}\right] = \frac{i}{2\sqrt{2}}\left[\frac{1}{\sqrt{2}i-z} + \frac{1}{\sqrt{2}i+z}\right]$$

$$= \frac{i}{2\sqrt{2}}\left[\frac{1}{i(\sqrt{2}-1)-(z-i)} + \frac{1}{i(\sqrt{2}+1)+z-i}\right]$$

So $1 + \frac{1}{2+z^2}$

$$= 1 + \frac{1}{4-2\sqrt{2}}\sum_{n=0}^{\infty}\left(\frac{-i}{\sqrt{2}-1}\right)^n (z-i)^n + \frac{1}{4+2\sqrt{2}}\sum_{n=0}^{\infty}\left(\frac{i}{\sqrt{2}+1}\right)^n (z-i)^n$$

$$= 1 + \sum_{n=0}^{\infty}\left[\frac{1}{4-2\sqrt{2}}\left(\frac{-i}{\sqrt{2}-1}\right)^n + \frac{1}{4+2\sqrt{2}}\left(\frac{i}{\sqrt{2}+1}\right)^n\right](z-i)^n; |z-i| < \sqrt{2}-1$$

9. Let $f(z) = (z-9)^2$, then $f'(z) = 2(z-9), f''(z) = 2, f^{(n)}(z) = 0$, all $n \geq 3$.
Then $f(1+i) = (-8+i)^2 = 63-16i, f'(1+i) = 2(-8+i) = -16+2i, f''(1+i) = 2$.
Therefore $(z-9)^2 = (63-16i) + (-16+2i)(z-1-i) + (z-1-i)^2; |z| < \infty$

10. $e^z - i\sin(z) = \sum_{n=0}^{\infty}\frac{1}{n!}z^n - i\sum_{n=0}^{\infty}\frac{(-1)^n z^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}\frac{1}{(2n)!}z^{2n} + \sum_{n=0}^{\infty}\frac{[1+i(-1)^{n+1}]}{(2n+1)!}z^{2n+1}; |z| < \infty$

11. $\sin(z+i) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}(z+i)^{2n+1}; |z+i| < \infty$

12. $\frac{3}{z-4i} = \frac{3}{-4i-5+(z+5)} = -\frac{3}{5+4i}\sum_{n=0}^{\infty}\left(\frac{1}{5+4i}\right)^n (z+5)^n; |z-5| < \sqrt{41}$

13. We are given $f(0) = 1, f'(0) = i$ and $f''(z) = 2f(z) + 1$. Differentiating repeatedly we get $f^{(n)}(z) = 2f^{(n-2)}(z)$, all $n \geq 3$. A simple induction argument gives
$f(0) = 1, f^{(2n)}(0) = 2^n + 2^{n-1}$, all $n \geq 1$,
$f'(0) = i, f^{(2n+1)}(0) = 2^n i$, all $n \geq 1$. Thus $f(z) = 1+iz+\sum_{n=1}^{\infty}\frac{2^n+2^{n-1}}{(2n)!}z^{2n}+\sum_{n=1}^{\infty}\frac{2^n i}{(2n+1)!}z^{2n+1}$
The ratio test establishes that both series have $R = +\infty$, thus the series converges for all $|z| < \infty$

14. (a) With $f(z) = \sin^2(z)$, we get $f'(z) = 2\sin(z)\cos(z) = \sin(2z), f''(z) = 2\cos(2z)$, $f'''(z) = -4\sin(2z), f^{(4)}(z) = -8\cos(2z), f^{(5)}(z) = 16\sin(2z), f^{(6)}(z) = 32\cos(2z)$,
Evaluating at $z = 0$ and using $c_n = \frac{f^{(n)}(0)}{n!}$ gives $\sin^2(z) = z^2 - \frac{1}{3}z^4 + \frac{2}{45}z^6 + \dots$
(b) By multiplying Maclaurin series we get $\sin^2(z) = \left(z - \frac{z^3}{6} + \frac{z^5}{120} - \cdots\right)\left(z - \frac{z^3}{6} + \frac{z^5}{120} - \cdots\right) =$
$z^2 - z^4\left(\frac{1}{6}+\frac{1}{6}\right) + z^6\left(\frac{1}{120}+\frac{1}{36}+\frac{1}{120}\right) \quad = z^2 - \frac{1}{3}z^4 + \frac{2}{45}z^6 + \dots$
This can also be done by using the identity $\sin^2(z) = \frac{1-\cos(2z)}{2}$ and known series to get

$$\sin^2(z) = \frac{1}{2}\left[1 - \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}(2z)^{2n}\right] = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n)!}2^{2n-1}z^{2n}$$

(c) $(\sin(z))^2 = \left(\frac{e^{iz}-e^{-iz}}{2i}\right)^2 = -\frac{1}{4}(e^{2iz}-2+e^{-2iz}) = \frac{1}{2} - \frac{1}{4}\sum_{k=0}^{\infty}\frac{(2iz)^k}{k!} - \frac{1}{4}\sum_{k=0}^{\infty}\frac{(-1)^k(2iz)^k}{k!} =$
$\frac{1}{2} - \frac{1}{4}\sum_{n=0}^{\infty}\left[\frac{(2iz)^{2n}}{(2n)!} + \frac{(2iz)^{2n}}{(2n)!}\right] = \frac{1}{2}\left[1 - \sum_{n=0}^{\infty}\frac{(-1)^n(2z)^{2n}}{(2n)!}\right] = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n)!}2^{2n-1}z^{2n}$

15. We begin by considering the integral

$$\frac{1}{2\pi i}\oint_\gamma \frac{z^n}{n!w^{n+1}}e^{zw}dw = \frac{1}{2\pi i}\oint_\gamma \frac{z^n}{n!w^{n+1}}\sum_{k=0}^{\infty}\frac{(zw)^k}{k!}dw = \frac{1}{2\pi i}\oint_\gamma \sum_{k=0}^{\infty}\frac{z^{n+k}w^{k-n-1}}{n!k!}dw,$$

where $\gamma : |w| = 1$.
Now put $w = e^{i\theta}, 0 \leq \theta \leq 2\pi$ to get

$$\frac{1}{2\pi i}\int_0^{2\pi}\sum_{k=0}^{\infty}\frac{z^{n+k}e^{i(k-n-1)\theta}}{n!k!}ie^{i\theta}d\theta = \sum_{k=0}^{\infty}\frac{1}{2\pi}\int_0^{2\pi}\frac{z^{n+k}}{n!k!}e^{i(k-n)\theta}d\theta$$

Each of these integrals is zero except when $k = n$, in which case the integral equals 2π. Thus

$\frac{1}{2\pi i}\oint_\gamma \frac{z^n}{n!w^{n+1}}e^{zw}dw = \frac{(z^n)^2}{(n!)^2}$. Finally we have for the series

$$\sum_{n=0}^{\infty}\frac{1}{(n!)^2}z^{2n} = \sum_{n=0}^{\infty}\frac{1}{2\pi i}\oint_\gamma \frac{z^n}{n!w^{n+1}}e^{zw}dw = \frac{1}{2\pi}\int_0^{2\pi}\sum_{n=0}^{\infty}\frac{z^n}{n!e^{i(n+1)\theta}}e^{ze^{i\theta}}e^{i\theta}d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\left[\sum_{n=0}^{\infty}\frac{(ze^{-i\theta})^n}{n!}\right]e^{ze^{i\theta}}d\theta = \frac{1}{2\pi}\int_0^{2\pi}e^{ze^{-i\theta}}e^{ze^{i\theta}}d\theta = \frac{1}{2\pi}\int_0^{2\pi}e^{z(e^{-i\theta}+e^{i\theta})d\theta}$$

$$= \frac{1}{2\pi}\int_0^{2\pi}e^{2z\cos(\theta)}d\theta$$

16. From Theorem 22.16, we have $|\cos(z)| = |\cos(x)\cosh(y) - \sin(x)\sinh)y)| = \cos^2(x)\cosh^2(y) + \sin^2(x)\sinh^2(y) = (1 - sin^2(x))\cosh^2(y) + \sin^2(x)(\cosh^2(y) - 1) = \cosh^2(y) + \sin^2(x)$. On the given square this attains its maximum value at $y = \pi, x = \pi/2$ of $\cosh^2(\pi) + 1$.

17. From Theorem 22.12 (7), we have $|e^z| = |e^x(\cos(y) + i\sin(y))| = e^{Re(z)} = e^x$. Since e^x is increasing it will attain its maximum value of e at all points on the right hand edge of the square

18. From Theorem 22.16, we have $|\sin(z)| = |\sin(x)\cosh(y) + i\cos(x)\sinh(y)| = \sin^2(x)\cosh^(y) + \cos^2(x)\sinh^(y) = (1 - \cos^2(x))\cosh^(y) + \cos^2(x)(\cosh^2(y) - 1) = \cosh(y) - \cos^2(y)$. On the given square this will attain its maximum value at $y = 1, x = \pi/2$ or $3\pi/2$ of $\cosh(1)$.

Section 23.2 The Laurent Expansion

1. By partial fractions,

$$\frac{2z}{1+z^2} = \left[\frac{1}{z-i} + \frac{1}{z+i}\right] = \frac{1}{z-i} + \frac{1}{2i + (z-i)} = \frac{1}{z-i} + \frac{1}{2i}\left[\frac{1}{1+\frac{z-i}{2i}}\right]$$

$$= \frac{1}{z-i} + \frac{1}{2i}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2i)^n}(z-i)^n; 0 < |z-i| < 2$$

2. $\frac{\sin(z)}{z^2} = \frac{1}{z^2}\sum_{n=0}^{\infty}\frac{(-1)^n z^{2n+1}}{(2n+1)!} = \frac{1}{z} + \sum_{n=1}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{2n-1}, 0 < |z| < \infty$

3. $\frac{1-\cos(2z)}{z^2} = \frac{1}{z^2}\left[1 - \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}(2z)^{2n}\right] = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}4^n}{(2n)!}z^{2n-2}; 0 < |z| < \infty$

4. $z^2\cos\left(\frac{i}{z}\right) = z^2\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left(\frac{i}{z}\right)^{2n} = \sum_{n=0}^{\infty}\frac{1}{(2n)!}z^{2-2n}; 0 < |z| < \infty$

5. $\frac{z^2}{1-z} = \frac{[(z-1)+1]^2}{1-z} = -\frac{[1 + 2(z-1) + (z-1)^2]}{z-1} = -\frac{1}{z-1} - 2 - (z-1); 0 < |z-1| < \infty$

6. $\dfrac{z^2+1}{2z-1} = \dfrac{1}{2}\left[\dfrac{z^2+1}{z-1/2}\right] = \dfrac{1}{2}\left[\dfrac{1+[(z-\frac{1}{2})+\frac{1}{2}]^2}{(z-\frac{1}{2})}\right] = \dfrac{5}{8}\left(\dfrac{1}{z-1/2}\right) + \dfrac{1}{2} + \dfrac{1}{2}\left(z-\dfrac{1}{2}\right); 0 <$
$\left|z-\dfrac{1}{2}\right|$

7. $\dfrac{e^{z^2}}{z^2} = \dfrac{1}{z^2}\sum_{n=0}^{\infty}\dfrac{1}{n!}z^{2n} = \dfrac{1}{z^2} + \sum_{n=1}^{\infty}\dfrac{1}{n!}z^{2n-2}; 0 < |z| < \infty$

8. $\dfrac{\sin(4z)}{z} = \dfrac{1}{z}\sum_{n=0}^{\infty}\dfrac{(-1)^n}{(2n+1)!}(4z)^{2n+1} = 4 + \sum_{n=1}^{\infty}\dfrac{(-1)^n 4^{2n+1}}{(2n+1)!}z^{2n}; 0 < |z| < \infty$

9. $\dfrac{z+i}{z-i} = \dfrac{2i+(z-i)}{z-i} = 1 + \dfrac{2i}{z-i}; 0 < |z-i|$

10. $\sinh\left(\dfrac{1}{z^3}\right) = \sum_{n=0}^{\infty}\dfrac{1}{(2n+1)!}\left(\dfrac{1}{z^3}\right)^{2n+1} = \sum_{n=0}^{\infty}\dfrac{1}{(2n+1)!}z^{-6n-3}; 0 < |z| < \infty$

Chapter Twenty Four - Singularities and the Residue Theorem

Section 24.1 Singularities

1. $\dfrac{\cos(z)}{z^2}$ has a singularity at $z = 0$. Since $\cos(0) = 1$, by Theorem 24.2, $z = 0$ is a pole of order 2

2. $z = i$ is a simple pole, $z = -i$ is a pole of order 2

3. $(z+2i)e^{\frac{1}{z}} = (z+2i)\sum_{n=0}^{\infty}\frac{1}{n!}\frac{1}{z^n} = 2i\sum_{n=0}^{\infty}\frac{z^{-n}}{n!} + \sum_{n=0}^{\infty}\frac{1}{n!}z^{1-n}$, thus $z = 0$ is an essential singularity.

4. $\lim_{z\to\pi}\dfrac{\sin(z)}{(z-\pi)} = \lim_{z\to\pi}\cos(z) = -1$, so $\dfrac{\sin(z)}{z-\pi}$ has a removable singularity at $z = \pi$

5. $\dfrac{\cos(2z)}{(z-1)^2(z^2+1)} = \dfrac{\cos(2z)}{(z-1)^2(z-i)(z+1)}$ By Theorem 24.2, there are simple poles at $z = i, z = -i$, and a pole of order 2 at $z = 1$.

6. $z = -1$ is a pole of order 2

7. $\dfrac{z-i}{z^2+1} = \dfrac{(z-i)}{(z-i)(z+i)}$ Since $\lim_{z\to i}\dfrac{z-1}{z^2+1} = \dfrac{1}{2i} \neq 0$, $z = i$ is a removable singularity; $z = -i$ is a simple pole by Theorem 24.2.

8. The only zero of $\sinh(z)$ is $z = 0$ which is also a zero of $\sin(z)$ Thus $z = 0$ is a removable singularity, in fact $\lim_{z\to 0}\dfrac{\sin(z)}{\sinh(z)} = 1$.

9. $\dfrac{z}{z^4-1} = \dfrac{z}{(z^2-1)(z^2+1)} = \dfrac{z}{(z-1)(z+1)(z-i)(z+i)}$, so there are simple poles at each of $z = 1, -1, i, -i$.

10. $\tan(z) = \dfrac{\sin(z)}{\cos(z)}, d(z) = \cos(z)$ has a zero at $z_n = \dfrac{\pi}{2}(2n+1), n$ any integer, and since $d'(z_n) = -\sin(z_n) = (-1)^{n+1} \neq 0$ each of these is a simple zero. Thus $\tan(z)$ has a simple pole at each $z_n = \dfrac{\pi}{2}(2n+1)$, n any integer.

11. $\dfrac{1}{\cos(z)}$ has a simple pole at each $z_n = \dfrac{\pi}{2}(2n+1), n$ any integer (see Problem 10 solution).

12. $e^{1/z(z+1)}$ has essential singularities at each of $z = 0$ and $z = -1$.

13. Since g has a pole of order m at $z_0, g(z) = \dfrac{h(z)}{(z-z_0)^m}$ on same annulus $0 < |z - z_0| < R$ where h is differentiable at $z_0, h(z_0) \neq 0$. Then on some annulus $0 < |z - z_0| < r (r \leq R)$ we have $(fg)(z) = \dfrac{f(z)h(z)}{(z-z_0)^m}$, and $f(z_0)h(z_0) \neq 0$. Thus fg has a pole of order m at z_0.

14. Since h has a zero of order 2 at $z_0, h(z) = (z-z_0)^2\phi(z)$ where ϕ is differentiable at z_0 and $\phi(z_0) \neq 0$ Then $\dfrac{g(z)}{h(z)} = \dfrac{g(z)}{(z-z_0)^2\phi(z)}$, on same annulus $0 < |z - z_0| < r$, and $\lim_{z\to z_0}\dfrac{(z-z_0)^2 g(z)}{h(z)} = \lim_{z\to z_0}\dfrac{g(z)}{\phi(z)} = \dfrac{g(z_0)}{\phi(z_0)} \neq 0$, so $\dfrac{g(z)}{h(z)}$ has a pole of order 2 at z_0

15. Since h has a zero of order 3 at $z_0, h(z) = (z - z_0)^3\phi(z)$ where ϕ is differentiable at z_0 and $\phi(z_0) \neq 0$ Then $\dfrac{g(z)}{h(z)} = \dfrac{g(z)}{(z-z_0)^3\phi(z)}$ on same annulus $0 < |z - z_0| < r$,
and $\lim_{z\to z_0}(z-z_0)^3\dfrac{g(z)}{h(z)} = \lim_{z\to z_0}\dfrac{g(z)}{\phi(z)} = \dfrac{g(z_0)}{\phi(z_0)} \neq 0$, so $\dfrac{g(z)}{h(z)}$ has a pole of order 3 at z_0.

Section 24.2 The Residue Theorem

1. The circle Γ encircles $z = 1$ at which $f(z) = \dfrac{1+z^2}{(z-1)^2(z+2i)}$ has a pole of order 2, and $z = -2i$ which is a simple pole. Then

$$Res(f(z), 1) = \lim_{z\to 1}\frac{d}{dz}\left(\frac{1+z^2}{z+2i}\right) = \frac{4i}{-3+4i},$$

and

$$Res(f(z), -2i) = \lim_{z\to -2i}\frac{1+z^2}{(z-1)^2} = \frac{-3}{-3+4i},$$

hence

$$\oint_\Gamma \frac{1+z^2}{(z-1)^2(z+2i)}dz = 2\pi i\left[\frac{-3}{-3+4i} + \frac{4i}{-3+4i}\right] = 2\pi i.$$

2. The circle Γ encircles $z = i$ at which point $f(z) = \dfrac{2z}{(z-i)^2}$ has a pole of order 2. By Theorem 24.7, $Res(f(z), i) = \lim_{z\to i}\dfrac{d}{dz}(2z) = 2$, hence $\displaystyle\int_\Gamma \frac{2z}{(2-i)^2}dz = 2\pi i(2) = 4\pi i$.

3. The circle Γ does not encircle $z = 0$ which is the only singularity of $f(z) = \dfrac{e^z}{z}$, so $\displaystyle\oint_\Gamma \frac{e^z}{z}dz = 0$

4. The square Γ encloses both $z = 2i$ and $z = -2i$ which are each simple poles of $f(z) = \dfrac{\cos(z)}{z^2+4}$. Then

$$Res(f(z), 2i) = \lim_{z\to 2i}\frac{\cos(z)}{z+2i} = \frac{\cosh(2)}{4i};\ Res(f(z), -2i) = \lim_{z\to -2i}\frac{\cos(z)}{z-2i} = \frac{\cosh(2)}{-4i},$$

so

$$\oint_\Gamma \frac{\cos(z)}{z^2+4}dz = 2\pi i\left[\frac{\cosh(2)}{4i} - \frac{\cosh(2)}{4i}\right] = 0$$

5 The square Γ encloses both $z = \sqrt{6}i$ and $z = -\sqrt{6}i$ which are each simple poles of $f(z) = \dfrac{z+i}{z^2+6}$. We have

$$Res(f(z), \sqrt{6}i) = \lim_{z\to\sqrt{6}i}\left(\frac{z+i}{z+\sqrt{6}i}\right) = \frac{\sqrt{6}+1}{2\sqrt{6}};$$

$$Res(f(z), -\sqrt{6}i) = \lim_{z\to -\sqrt{6}i}\left(\frac{z+i}{z-\sqrt{6}i}\right) = \frac{\sqrt{6}-1}{2\sqrt{6}}$$

and

$$\oint_\Gamma \frac{z+i}{z^2+6}dz = 2\pi i\left(\frac{\sqrt{6}+1}{2\sqrt{6}} + \frac{\sqrt{6}-1}{2\sqrt{6}}\right) = 2\pi i$$

6. The circle Γ encloses $z = -1/2$ which is a simple pole of $f(z) = \dfrac{z-i}{2z+1}$, with

$$Res(f(z), -1/2) = \lim_{z\to -1/2} \frac{(z-i)}{2} = \left(-\frac{1}{4} - \frac{i}{2}\right).$$

Hence

$$\oint_\Gamma \frac{z-i}{2z+1}dz = 2\pi i\left(-\frac{1}{4} - \frac{i}{2}\right) = \pi - \frac{\pi}{2}i.$$

7. The circle Γ encloses the origin $z = 0$ which is a simple pole (by Theorem 24 3) of $f(z) = \dfrac{z}{\sin^2(z)}$ having $Res(f(z), 0) = \lim_{z\to 0}(\dfrac{z^2}{\sinh^2(z)}) = \lim_{z\to 0}\left[\dfrac{1}{(\frac{\sinh(z)}{z})}\right]^2 = \left[\dfrac{1}{\lim_{z\to 0}(1 + \frac{z^2}{6} + \cdots)}\right]^2 = 1$

Thus $\oint_\Gamma \dfrac{z}{\sinh^2(z)}dz = 2\pi i$

8 The circle Γ encloses the origin $z = 0$ which is a simple pole of $f(z) = \dfrac{\cos(z)}{ze^z}$ having $Res(f(z), 0) = \lim_{z\to 0}\dfrac{\cos(z)}{e^z} = 1$ Thus $\oint_\Gamma \dfrac{\cos(z)}{ze^z}dz = 2\pi i$

9 The circle Γ encloses only the simple pole at $z = -3i$, and

$$Res(f(z), -3i) = \lim_{z\to -3i}\left(\frac{iz}{(z-3i)(z-i)}\right) = -\frac{1}{8}$$

Hence

$$\oint_\Gamma \frac{iz}{(z^2+9)(z-i)}dz = 2\pi i\left(-\frac{1}{8}\right) = -\frac{\pi i}{4}$$

10. Γ encloses $z = 0$ which is an essential singularity of $f(z) = e^{2/z^2}$. We have for $0 < |z|, e^{2/z^2} = 1 + \dfrac{2}{z^2} + \dfrac{2}{z^4} + \cdots$. The residue at $z = 0$ is the coefficient of $\dfrac{1}{z}$ in this series, hence $Res(f(z), 0) = 0$ and $\oint_\Gamma e^{2/z^2}dz = 0$.

11 $f(z) = \dfrac{8z - 4i + 1}{z + 4i}$ is differentiable on and inside $\Gamma : |z + i| = 2$, so $\oint_\Gamma \dfrac{8z-4i+1}{z+4i}dz = 0$

12. Γ encloses the point $z = 1 - 2i$ which is a simple pole of $f(z) = \dfrac{z^2}{z-1+2i}$ having

$$Res(f(z), 1-2i) = \lim_{z\to 1-2i}(z^2) = -3 - 4i$$

Thus

$$\oint_\Gamma \frac{z^2}{z-1+2i}dz = 2\pi i(-3-4i) = 2\pi(4-3i)$$

13 $f(z) = \coth(z) = \dfrac{\cosh(z)}{\sinh(z)} = \dfrac{i\cosh(z)}{\sin(iz)}$ has simple poles at $z = n\pi i, n$ any integer, but only $z = 0$ lies inside $|z - i| = 2$ We have $Res(f(z), 0) = \dfrac{\cosh(0)}{\cosh(0)} = 1$, by Theorem 24.6, Corollary 24.1 Thus $\oint_\Gamma \coth(z)dz = 2\pi i$

14. The circle $|z - 2| = 2$ encloses only the simple pole of $f(z) = \dfrac{(1-z)^2}{z^3 - 8}$ at $z = 2$ The residue is $Res(f(z), 2) = \lim_{z\to 2}\left(\dfrac{(1-z)^2}{z^2+2z+4}\right) = \dfrac{1}{12}$, hence $\oint_\Gamma \dfrac{(1-z)^2}{z^3-8}dz = 2\pi i\left(\dfrac{1}{12}\right) = \dfrac{\pi i}{6}$

15. $z = 0$ and $z = 4i$ are both simple poles of $f(z) = \dfrac{e^{2z}}{z(z-4i)}$.
We find

$$Res(f(z), 0) = \lim_{z\to 0}\left(\frac{e^{2z}}{z-4i}\right) = -\frac{1}{4i};\ Res(f(z), 4i) = \lim_{z\to 4i}\left(\frac{e^{2z}}{z}\right) = \frac{e^{8i}}{4i}$$

Hence

$$\oint_\Gamma \frac{e^{2z}}{z(z-4i)}dz = 2\pi i\left[\frac{e^{8i}}{4i} - \frac{1}{4i}\right] = \frac{\pi}{2}[\cos(8) - 1 + i\sin(8)]$$

16. $f(z) = \dfrac{z^2}{(z-1)^2}$ has a pole of order 2 at $z = 1$, and $Res(f(z), 1) = \lim_{z\to 1}\dfrac{d}{dz}(z^2) = 2$, so $\oint_\Gamma \dfrac{z^2}{(z-1)^2}dz = 2\pi i(2) = 4\pi i$

17 By Theorem 24.7, the residue at z_0 is $Res\left(\dfrac{g(z)}{h(z)}, z_0\right) = \lim_{z\to z_0}\dfrac{d}{dz}\left((z-z_0)^2\dfrac{g(z)}{h(z)}\right)$. Replace $h(z) = (z-z_0)^2\phi(z)$ to get $Res\left(\dfrac{g(z)}{h(z)}, z_0\right) = \lim_{z\to z_0}\dfrac{d}{dz}\left(\dfrac{g(z)}{\phi(z)}\right) = \dfrac{\phi(z_0)g'(z_0) - \phi'(z_0)g(z_0)}{[\phi(z_0)]^2}$.
From $h(z) = (z-z_0)^2\phi(z)$, we find $h''(z) = 2\phi(z) + 4(z-z_0)\phi'(z) + (z-z_0)^2\phi''(z)$ and $h^{(3)}(z) = 6\phi'(z) + 6(z-z_0)\phi''(z) + (z-z_0)^2\phi^{(3)}(z)$, so $\phi(z_0) = \dfrac{h''(z_0)}{2}$ and $\phi'(z_0) = \dfrac{h^{(3)}(z_0)}{6}$.
Substitution of these values gives $Res\left(\dfrac{g(z)}{h(z)}, z_0\right) = \dfrac{2g'(z_0)}{h''(z_0)} - \dfrac{2}{3}\dfrac{g(z_0)h^{(3)}(z_0)}{[h''(z_0)]^2}$

18. To determine the residue at z_0 we use Laurent series representations
We have on $0 < |z - z_0| < r$

$$\frac{g(z)}{h(z)} = \frac{\sum_{n=0}^{\infty}\frac{g^{(n)}(z_0)}{n!}(z-z_0)^n}{\sum_{n=3}^{\infty}\frac{h^{(n)}(z)}{n!}(z-z_0)^n} = \frac{1}{(z-z_0)^3}\frac{g(z_0) + g'(z_0)(z-z_0) + \frac{g''(z_0)}{2}(z-z_0)^2 + \cdots}{\frac{h^{(3)}(z_0)}{6} + \frac{h^{(4)}(z_0)}{24}(z-z_0) + \frac{h^{(4)}(z_0)}{120}(z-z_0)^2 + }$$

and we need the coefficient of the $\dfrac{1}{z-z_0}$ term in the expression on the right. To simplify the book keeping, we will do the calculation at $z = 0$. Write

$$\frac{g(0) + g'(0)z + \frac{g''(0)}{2}z^2 + \frac{g^{(3)}(0)}{6}z^3 + }{\frac{h^{(3)}(0)}{6} + \frac{h^4(0)}{24}z + \frac{h^{(5)}(0)}{120}z^2 \cdots} = a_0 + a_1 z + a_2 z^2 + \cdots$$

Now cross multiply this relationship to get

$$(a_0+a_1z+a_2z^2+\cdots)(\frac{h^{(3)}(0)}{6}+\frac{h^{(4)}(0)}{24}z+\frac{h^{(5)}(0)}{120}z^2+\cdots) = g(0)+g'(0)z+\frac{g''(0)}{2}z^2+\frac{g^{(3)}(0)}{6}z^3+\cdots$$

Multiply the series on the left and equate coefficients of like powers to get

$$a_0\frac{h^{(3)}(0)}{6} = g(0); a_1\frac{h^{(3)}(0)}{6} + a_0\frac{h^{(4)}(0)}{24} = g'(0); a_2\frac{h^{(3)}(0)}{6} + a_1\frac{h^{(4)}(0)}{24} + a_0\frac{h^{(5)}(0)}{120} = g''(0).$$

Solve these equation successively for a_0, a_1 and finally

$$a_2 = 3\frac{g''(0)}{h^{(3)}(0)} - \frac{3}{10}\frac{g(0)h^{(5)}(0)}{(h^{(3)}(0))^2} + \frac{3}{8}\frac{h^{(4)}(0)}{(h^{(3)}(0))^3}\left[g(0)h^{(4)}(0) - 4g'(0)h^{(3)}(0)\right]$$

which is the given formula for the residue (See also Problem 19 below)

19 Using the notation introduced in the exercise for G_j and H_j we have $\frac{g(z)}{h(z)} =$

$$\frac{G_0 + G_1(z-z_0) + G_2(z-z_0)^2 + \cdots + G_{k-1}(z-z_0)^{k-1} + G_k(z-z_0)^k + \cdots}{H_k(z-z_0)^k + H_{k+1}(z-z_0)^{k+1} + H_{k+2}(z-z_0)^{k+2} + \cdots H_{2k+1}(z-z_0)^{2k+1} + H_{2k}(z-z_0)^{2k} + \cdots} =$$

$\frac{b_{-k}}{(z-z_0)^k} + \frac{b_{-(k-1)}}{(z-z_0)^{k-1}} + \cdots + \frac{b_{-1}}{(z-z_0)} + c_0 + c_1(z-z_0) + \cdots$ We know $G_0 \neq 0, H_k \neq 0$ so $\frac{g(z)}{h(z)}$ has a pole of order k and the residue is b_{-1} Cross multiply this last fractional identity to get

$G_0 + G_1(z-z_0) + G_2(z-z_0)^2 + \cdots + G_{k-1}(z-z_0)^{k-1} + \cdots$
$= H_kb_{-k} + [H_kb_{-(k-1)} + H_{k+1}b_{-k}](z-z_0) + [H_{k+2}b_{-k} + H_{k+1}b_{-(k-1)} + H_kb_{-(k-2)}](z-z_0)^2 +$
$[H_{k+3}b_{-k} + H_{k+2}b_{-(k-1)} + H_{k+1}b_{-(k-2)} + H_kb_{-(k-3)}](z-z_0)^3 + \cdots + [H_{2k-1}b_{-k} + H_{2k-2}b_{-(k-1)} +$
$\cdots + H_kb_{-(k-1)}](z-z_0)^{k-1} + \cdots$

Now equate like coefficients of powers of $(z-z_0)$ to obtain the system of equations

$$\begin{pmatrix} H_k & 0 & 0 & \cdots & 0 & 0 \\ H_{k+1} & H_k & 0 & \cdots & 0 & 0 \\ H_{k+2} & H_{k+1} & H_k & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ H_{2k-1} & H_{2k-2} & H_{2k-3} & \cdots & H_{k+1} & H_k \end{pmatrix} \begin{pmatrix} b_{-k} \\ b_{-(k-1)} \\ b_{-(k-2)} \\ \vdots \\ b_{-1} \end{pmatrix} = \begin{pmatrix} G_0 \\ G_1 \\ G_2 \\ \vdots \\ G_{k-1} \end{pmatrix}$$

Solving these equations for b_{-1} by Cramer's rule gives the stated result for the residue at z_0

Section 24.3 Some Applications of the Residue Theorem

1. (a) $f(z) = \frac{z}{1+z^2}$ has simple poles at $z = \pm i$, with $Res(f(z), \pm i) = \left.\frac{z}{2z}\right|_{z=\pm i} = \frac{1}{2}$, so by the Residue Theoem $\oint_{|z|=2} \frac{z}{1+z^2}dz = 2\pi i\left(\frac{1}{2}+\frac{1}{2}\right) = 2\pi i$.
(b) Apply the Argument Principle with $f(z) = z^2+1$ which has 2 zeros and no poles inside $|z| = 2$ Hence $\frac{1}{2\pi i}\oint_{|z|=2}\frac{2z}{z^2+1}dz = 2 - 0 = 2$, so $\oint_{|z|=2}\frac{z}{z^2+1}dz = 2\pi i$

2. (a) $f(z) = \tan z$ has simple poles at $z = \pm\frac{\pi}{2}$, with $Res\left(\tan(z), \pm\frac{\pi}{2}\right) = \left.\frac{\sin(z)}{-\sin(z)}\right|_{z=\pm\frac{\pi}{2}} =$ -1. By the Residue Theoem $\oint_{|z|=\pi} \tan(z)dz = 2\pi i(-1-1) = -4\pi i$.
(b) Apply the Argument Principle with $f(z) = \cos(z)$ which has 2 zeros at $\pm\frac{\pi}{2}$, and no poles inside $|z| = \pi$, so $Z = 2, P = 0$, and $\frac{1}{2\pi i}\oint_{|z|=\pi} \frac{-\sin(z)}{\cos(z)}dz = 2$, so $\oint_{|z|=\pi} \tan(z)dz = -4\pi i$.

3. (a) $f(z) = \frac{z+1}{z^2+2z+4}$ has simple poles at $z = -1 \pm \sqrt{3}i$, and both are outside of $|z| = 1$, so by the Residue Theorem, $\oint_{|z|=1} \frac{z+1}{z^2+2z+4}dz = 0$.
(b) By the Argument Principle, since $f(z) = z^2 + 2z + 4$ has no zeros and no poles inside $|z| = 1, Z = 0, P = 0$ and $\frac{1}{2\pi i}\oint_{|z|=1} \frac{2z+2}{z^2+2z+4}dz = 0$, hence $\oint_{|z|=1} \frac{z+1}{z^2+2z+4}dz = 0$

4. With $z_1, \ldots, z_n$ distinct and $f(z) = \frac{p'(z)}{p(z)}$ we see that f has a simple pole at each $z_j, j =$ $1, 2, \ldots, n$, and $Res(f(z), z_j) = \frac{p'(z)}{p'(z)}|_{z=z_j} = 1$ for each $j = 1, 2, \ldots, n$. Thus by the Residue Theorem, $\oint_\Gamma \frac{p'(z)}{p(z)}dz = 2\pi i(1 + 1 + \cdots + 1) = 2n\pi i$.
(b) By the Argument Principle, since p has n zeros inside Γ, and no poles, $Z = n, P = 0$ and $\oint_\Gamma \frac{p'(z)}{p(z)}dz = 2\pi i(Z - P) = 2n\pi i$.

5. $F(z) = \frac{z}{z^2+9}$ has simple poles at $z = \pm 3i$. We compute $Res\left(\frac{ze^{zt}}{z^2+9}, 3i\right) = \frac{3ie^{3it}}{2(3i)} = \frac{e^{3it}}{2}$, and $Res\left(\frac{ze^{zt}}{z^2+9}, -3i\right) = \frac{-3ie^{3it}}{2(3i)} = \frac{e^{-3it}}{2}$. An inverse Laplace transform of $F(z)$ is therefore $f(t) = \frac{e^{3it} + e^{-3it}}{2} = \cos(3t)$.

6. $F(z) = \frac{1}{(z+3)} =$ has a pole of order 2 at $z = -3$. We compute $Res\left(\frac{e^{zt}}{(z+3)^2}, -3\right) =$ $\lim_{z\to-3}\frac{d}{dz}(e^{zt}) = \lim_{z\to-3}[te^{zt}] = te^{-3t} = f(t)$

7. $F(z) = \frac{1}{(z-2)^2(z+4)}$ has a simple pole at $z = -4$, and a pole of order 2 at $z = 2$. We compute

$$Res(e^{zt}F(z), -4) = \frac{1}{36}e^{-4t};$$

$$Res(e^{zt}F(z), 2) = \lim_{z\to2}\frac{d}{dz}(e^{zt}(z+4)^{-1}) = \lim_{z\to2}[te^{zt}(z+4)^{-1} - e^{zt}(z+4)^{-2}] = \frac{t}{6}e^{2t} - \frac{e^{2t}}{36}$$

An inverse Laplace transform of $F(z)$ is therefore $\frac{1}{36}e^{-4t} + \frac{t}{6}e^{2t} - \frac{e^{2t}}{36}$

8. $F(z) = \frac{1}{(z^2+9)(z-2)^2}$ has simple poles at $z = \pm 3i$, and a pole of order 2 at $z = 2$. We

compute

$$Res(e^{zt}F(z), 3i) = \frac{e^{3it}}{6i(3i-2)^2},$$

$$Res(e^{zt}F(z), -3i) = \frac{e^{-3it}}{-6i(-3i-2)^2},$$

$$Res(e^{zt}F(z), 2) = \lim_{z\to 2}\frac{d}{dz}(e^{zt}(z^2+9)^{-1}) = \frac{t}{13}e^{2t} - \frac{4}{169}e^{2t}.$$

An inverse Laplace transform of $F(z)$ is therefore

$$\frac{t}{13}e^{2t} - \frac{4}{169}e^{2t} + 2Re\left(\frac{e^{3it}}{6i(3i-2)^2}\right).$$

We have

$$\frac{e^{3it}}{6i(3i-2)^2} = \frac{[\cos(3t) + i\sin(3t)](72+30i)}{6084}$$

which has Real part

$$\frac{72\cos(3t) - 30\sin(3t)}{6084}$$

The inverse transform of $F(z)$ is therefore

$$\frac{t}{13}e^{2t} - \frac{4}{169}e^{2t} + \frac{12\cos(3t) - 5\sin(3t)}{507}$$

9 $F(z) = \dfrac{1}{(z+5)^3}$ has a pole of order 3 at $z = -5$, so $Res(e^{zt}F(z), -5) = \lim_{z\to -5}\dfrac{d^2}{dz^2}(e^{zt}) = \dfrac{1}{2}[t^2e^{-5t}]$, which is the desired inverse Laplace transform.

10 With $z = e^{i\theta}$, we have $\sin(\theta) = \dfrac{1}{2i}\left(z - \dfrac{1}{z}\right), d\theta = \dfrac{dz}{iz}$, and the real integral becomes $\displaystyle\int_0^{2\pi}\frac{1}{6+\sin(\theta)}d\theta = \oint_{|z|=1}\frac{1}{6+\frac{1}{2i}(z-\frac{1}{z})}\frac{dz}{iz} = 2\oint_{|z|=1}\frac{1}{z^2+12iz+1}dz$. Let $f(z) = \dfrac{1}{z^2+12iz+1}$, which has simple poles at $z_1 = -6i + \sqrt{35}i$ and $z_2 = -6i - \sqrt{35}i$. Only z_1 lies inside $|z| = 1$, and we compute $Res(f(z), z_1) = \dfrac{1}{2z_1 + 12i} = \dfrac{1}{2\sqrt{35}i}$.

Thus we have

$$\int_0^{2\pi}\frac{1}{6+\sin(\theta)}d\theta = 2\oint_{|z|=1}f(z)dz = 2(2\pi i)\left(\frac{1}{2\sqrt{35}i}\right) = \frac{2\pi}{\sqrt{35}}$$

11 With $z = e^{i\theta}$, we have $\cos(\theta) = \dfrac{1}{2}\left(z + \dfrac{1}{z}\right), d\theta = \dfrac{dz}{iz}$, and the real integral becomes $\displaystyle\int_0^{2\pi}\frac{1}{2-\cos(\theta)}d\theta = \oint_{|z|=1}\frac{1}{2-\frac{1}{2}(z+\frac{1}{z})}\frac{dz}{iz} = -\frac{2}{i}\oint_{|z|=1}\frac{1}{z^2-4z+1}dz$. Let $f(z) = \dfrac{1}{z^2-4z+1}$,

which has simple poles at $z_1 = 2-\sqrt{3}$ and $z_2 = 2+\sqrt{3}$. Only z_1 lies inside $|z|=1$, and we compute $Res(f(z), z_1) = \dfrac{1}{2z_1 - 4} = -\dfrac{1}{2\sqrt{3}}$.
Thus we have

$$\int_0^{2\pi} \frac{1}{2-\cos(\theta)} d\theta = -\frac{2}{i}\oint_{|z|=1} f(z)dz = -\frac{2}{i}(2\pi i)\left(-\frac{1}{2\sqrt{3}}\right) = \frac{2\pi}{\sqrt{3}}$$

12 $f(z) = \dfrac{1}{z^4+1}$ has simple poles in the upper half plane at the two fourth roots of -1 with positive imaginary part, $z_1 = \dfrac{1}{\sqrt{2}}(1+i), z_2 = \dfrac{1}{\sqrt{2}}(-1+i)$.
Compute

$$Res(f(z), z_1) = \frac{1}{4z_1^3} = -\frac{1}{4}z_1; Res(f(z), z_2) = -\frac{1}{4}z_2$$

Then

$$\int_{-\infty}^{\infty} \frac{1}{x^4+1} dx = 2\pi i\left[-\frac{1}{4}(z_1+z_2)\right] = \frac{\pi}{\sqrt{2}}$$

13 First note $\displaystyle\int_0^{\infty} \frac{1}{x^6+1}dx = \frac{1}{2}\int_{-\infty}^{\infty}\frac{1}{x^6+1}dx$. $f(z) = \dfrac{1}{z^6+1}$ has simple poles in the upper half plane at $z_1 = i, z_2 = \dfrac{1}{2}(\sqrt{3}+i), z_3 = \dfrac{1}{2}(-\sqrt{3}+i)$. At each pole compute

$$Res\left(\frac{1}{z^6+1}, z_k\right) = \frac{1}{6z_k^5} = \frac{z_k}{-6}$$

Then

$$\int_0^{\infty} \frac{1}{x^6+1}dx = \frac{1}{2}(2\pi i)\left[-\frac{1}{6}(z_1+z_2+z_3)\right] = \frac{\pi}{3}$$

14 $f(z) = \dfrac{1}{z^2-2z+6}$ has a simple pole in the upper half plane at $z_1 = 1+\sqrt{5}i$ with $Res(f(z), z_1) = \dfrac{1}{2z_1-2} = \dfrac{1}{2\sqrt{5}i}$
Then

$$\int_{-\infty}^{\infty} \frac{1}{x^2-2x+6}dx = 2\pi i\left(\frac{1}{2\sqrt{5}i}\right) = \frac{\pi}{\sqrt{5}}$$

15 Take $f(z) = \dfrac{ze^{2iz}}{z^4+16}$ which has simple poles in the upper half plane at $z_1 = \sqrt{2}(1+i)$ and $z_2 = \sqrt{2}(-1+i)$ Compute $Res(f(z), z_k) = \left.\dfrac{ze^{2iz}}{4z^3}\right|_{z_k} = \dfrac{e^{2iz_k}}{4z_k^2}$ So

$$Res(f(z), z_1) = \frac{e^{2\sqrt{2}(-1+i)}}{8(2i)}; \; Res(f(z), z_2) = \frac{e^{2\sqrt{2}(-1-i)}}{8(-2i)}$$

Then

$$\int_{-\infty}^{\infty} \frac{x\sin(2x)}{x^4+16}dx = Im\left[2\pi i\left(\frac{e^{-2\sqrt{2}}}{8}\right)\left(\frac{e^{2\sqrt{2}i} - e^{-2\sqrt{2}i}}{2i}\right)\right] = \frac{\pi e^{-2\sqrt{2}}}{4}\sin(2\sqrt{2})$$

16. This can be done by complex variables methods, but by a simple change of variable, $\theta = 2\pi - \phi$, we can show

$$\int_{\pi}^{2\pi} \frac{\sin(\theta)}{2+\sin^2(\theta)} d\theta = -\int_0^{\pi} \frac{\sin(\phi)}{2+\sin^2(\theta)} d\phi,$$

so

$$\int_0^{2\pi} \frac{\sin(\theta)}{2+\sin^2(\theta)} d\theta = 0.$$

17. $f(z) = \dfrac{1}{z(z+4)(z^2+16)}$ has simple poles on the real axis at $z = 0$ and $z = -4$, and a simple pole in the upper half plane at $z = 4i$.
Compute

$$Res(f(z),0) = \frac{1}{64};\ Res(f(z),-4) = -\frac{1}{128};\ Res(f(z),4i) = \frac{-1+i}{256}$$

Then

$$\int_{-\infty}^{\infty} \frac{1}{x(x+4)(x^2+16)} dx = \pi i\left[\frac{1}{64} - \frac{1}{128}\right] + 2\pi i\left[\frac{-1+i}{256}\right] = -\frac{\pi}{128}$$

18. $f(z) = \dfrac{e^{iz}}{z^2-4z+5}$ has a simple pole in the upper half plane at $z = 2+i$. Compute $Res(f(z), 2+i) = \dfrac{e^{-1+2i}}{2i}$. Then $\displaystyle\int_{-\infty}^{\infty} \frac{\sin(x)}{x^2-4x+5} dx = Im\left[(2\pi i)\left(\frac{e^{-1+2i}}{2i}\right)\right] = \pi e^{-1}\sin(2)$.

19. First use the identity $\cos^2(x) = \dfrac{1+\cos(2x)}{2}$, and evaluate the equivalent integral $\displaystyle\frac{1}{2}\int_{-\infty}^{\infty} \frac{1+\cos(2x)}{(x^2+4)^2}$. Take $f(z) = \dfrac{1+e^{2iz}}{(z^2+4)^2}$ which has a pole of order two at $z = 2i$ in the upper half plane.
Compute

$$Res(f(z),2i) = \lim_{z\to 2i} \frac{d}{dz}\left[\frac{1+e^{2iz}}{(z+2i)^2}\right] = \frac{1+5e^{-4}}{32i}$$

Then

$$\int_{-\infty}^{\infty} \frac{\cos^2(x)}{(x^2+4)^2} dx = \frac{1}{2} Re\left[2\pi i\left(\frac{1+5e^{-4}}{32i}\right)\right] = \frac{\pi}{32}(1+5e^{-4}).$$

20. With $z = e^{i\theta}, \cos(\theta) = \dfrac{1}{2}\left(z+\dfrac{1}{z}\right), \sin(\theta) = \dfrac{1}{2i}(z-\dfrac{1}{2}), d\theta = \dfrac{dz}{iz}$, so

$$\int_0^{2\pi} \frac{\sin(\theta)+\cos(\theta)}{2-\cos(\theta)} d\theta = \int_0^{2\pi} \frac{\frac{1}{2i}(z-\frac{1}{z}) + \frac{1}{2}(z+\frac{2}{z})}{2-\frac{1}{2}(z-\frac{1}{z})} \frac{dz}{iz} = \oint_{|z|=1} \frac{(1+i)z^2-1+i}{z(z^2-4z+1)} dz.$$

The zeros of the denominator are $z = 0, z = 2-\sqrt{3}$, and $z = 2+\sqrt{3}$, but only the first two lie inside $|z| = 1$. Calculate

$$Res(f(z),0) = \lim_{z\to 0}\left(\frac{(1+i)z^2-1+i}{z^2-4z+1}\right) = -1+i;$$

$$Res(f(z), 2-\sqrt{3}) = \left(\frac{(1+i)z^2 - 1 + i}{3z^2 - 8z + 1}\right)|_{z=2-\sqrt{3}} = 1 + \frac{4-2\sqrt{3}}{3-2\sqrt{3}}i.$$

So

$$\int_0^{2\pi} \frac{\sin(\theta)+\cos(\theta)}{2-\cos(\theta)} d\theta = 2\pi i \left[-1 + i + 1 + \frac{4-2\sqrt{3}}{3-2\sqrt{3}} i\right] = \frac{2\pi}{3}[2\sqrt{3} - 3].$$

21. $f(z) = \dfrac{1}{(z-4)(z^5+1)}$ has real simple poles at $z = 4$ and $z = -1$, and simple poles in the upper half plane at $z_1 = e^{i\pi/5}$ and $z_2 = e^{3\pi i/5}$. At the real poles we compute

$$Res(f(z), 4) = \frac{1}{4^5+1} = \frac{1}{1025}; \ Res(f(z), -1) = \frac{1}{(-5)(5)} = -\frac{1}{25}.$$

At the complex poles z_1 and z_2 we have

$$Res(f(z), z_k) = \left.\frac{1}{(z-4)(5z^4)}\right|_{z_k} = \frac{z_k}{5(4-z_k)}.$$

With a bit of algebra we can show

$$Res(f(z), z_1) = \frac{1}{5}\left[\frac{4e^{i\pi/5} - 1}{17 - 8\cos(\pi/5)}\right]; \ Res(f(z), z_2) = \frac{1}{5}\left[\frac{4e^{3\pi i/5} - 1}{17 - 8\cos(3\pi/5)}\right].$$

Then

$$I = \int_{-\infty}^{\infty} \frac{1}{(x-4)(x^5+1)} dx = \pi i \left[\frac{1}{1025} - \frac{1}{25}\right] + \frac{2\pi i}{5}\left[\frac{4e^{i\pi/5} - 1}{17 - 8\cos(\pi/5)} + \frac{4e^{3i\pi/5} - 1}{17 - 8\cos(3\pi/5)}\right]$$

Since the integral is real, the imaginary side of the right hand side must be zero, and the real part is the value of the integral. This allows us to write

$$I = -\frac{8\pi}{5} Im\left\{\frac{e^{i\pi/5}}{17 - 8\cos(\pi/5)} + \frac{e^{3\pi i/5}}{17 - 8\cos(3\pi/5)}\right\} = -\frac{8\pi}{5}\left\{\frac{\sin(\pi/5)}{17 - 8\cos(\pi/5)} + \frac{\sin(3\pi/5)}{17 - 8\cos(3\pi/5)}\right\}$$

22. We identify $\alpha = 3/4 < 1, q(z) = z^4 + 1$ which has simple roots at the four fourth roots of -1. Then $\displaystyle\int_0^{\infty} \frac{x^{3/4}}{x^4+1} dx = \frac{2\pi i}{1+i}\sum_{k=1}^{4} Res(f(z), z_k) = \pi(1+i)\sum_{k=1}^{4} Res(f(z), z_k)$. Computing residues at $z_k = e^{i(\pi/4 + k\pi/2)}, k = 0, 1, 2, 3$. gives $Res(f(z), z_k) = \dfrac{z^{3/4}}{4z^3}|_{z_k} = -\dfrac{1}{4} z_k^{7/4}$. With $z_1 = e^{i\pi/4}, z_2 = e^{3\pi i/4}, z_3 = e^{5\pi i/4}, z_4 = e^{7\pi i/4}$, we find

$$\int_0^{\infty} \frac{x^{3/4}}{x^4+1} dx = -\frac{\pi\sqrt{2}}{4} e^{i\pi/4}[e^{7\pi i/16} - e^{5\pi i/16} + e^{3\pi i/16} - e^{\pi i/16}]$$

$$= \frac{\pi\sqrt{2}}{4}\left\{\sin\left(\frac{3\pi}{16}\right) - \sin\left(\frac{\pi}{16}\right)\right\} = \pi\sqrt{2}\sin\left(\frac{\pi}{16}\right)\cos\left(\frac{\pi}{8}\right)$$

23. Take $f(z) = \dfrac{e^{i\alpha z}}{z^2+1}$, which has only the one simple pole at $z = i$ in the upper half plane. Compute $Res(f(z), i) = \dfrac{e^{-\alpha}}{2i}$, then $\displaystyle\int_{-\infty}^{\infty} \frac{\cos(\alpha x)}{x^2+1}dx = Re\left[2\pi i\left(\frac{e^{-\alpha}}{2i}\right)\right] = \pi e^{-\alpha}$

24. Take $f(z) = \dfrac{e^{i\alpha z}}{(z^2+\beta^2)^2}$, which has only $z = \beta i$, a pole of order two, in the upper half plane.

Compute

$$Res\left(\frac{e^{i\alpha z}}{(z^2+\beta^2)^2}, \beta i\right) = \lim_{z\to\beta i}\frac{d}{dz}\left[\frac{e^{i\alpha z}}{(z+\beta i)^2}\right]$$

$$= \lim_{z\to\beta i}\left[\frac{(z+\beta i)^2 i\alpha e^{i\alpha z} - 2e^{i\alpha z}(z+\beta i)}{(z+\beta i)^4}\right] = \frac{e^{-\alpha\beta}}{4i\beta^3}(1+\alpha\beta)$$

Then

$$\int_{-\infty}^{\infty}\frac{\cos(\alpha x)}{(x^2+\beta^2)^2}dx = Re[(2\pi i)\frac{e^{-\alpha\beta}}{4i\beta^3}(1+\alpha\beta)] = \frac{\pi}{2\beta^3}e^{-\alpha\beta}(1+\alpha\beta).$$

25. $$\int_0^{2\pi}\frac{1}{\alpha^2\cos^2(\theta)+\beta^2\sin^2(\theta)}d\theta = \oint_{|z|=1}\frac{1}{\frac{\alpha^2}{4}(z+\frac{1}{z})^2 - \frac{\beta^2}{4}(z-\frac{1}{z})^2}\frac{dz}{iz}$$
$$= \frac{4}{i}\int_{|z|=1}\frac{z}{(\alpha^2-\beta^2)z^4+2(\alpha^2+\beta^2)z^2+(\alpha^2-\beta^2)}dz.$$

The poles of the integrand lying inside $|z| = 1$ will be solutions of

$$(\alpha^2-\beta^2)z^4+2(\alpha^2+\beta^2)+(\alpha^2-\beta^2) = 0, \text{ which satisfy } |z| < 1.$$

With a bit of algebra we find $z^2 = \dfrac{\beta-\alpha}{\beta+\alpha}$ or $z^2 = \dfrac{\beta+\alpha}{\beta-\alpha}$. Since α and β are positive, $\dfrac{\beta-\alpha}{\beta+\alpha} < 1$ and $\dfrac{\beta+\alpha}{\beta-\alpha} > 1$. Let z_1 and z_2 be the values for which $z^2 = \dfrac{\beta-\alpha}{\beta+\alpha}$ and compute

$$Res(f(z), z_k) = \frac{z}{4(\alpha^2-\beta^2)z^3+4(\alpha^2+\beta^2)z}\bigg|_{z_k} = \frac{1}{4(\alpha^2-\beta^2)(\frac{\beta-\alpha}{\beta+\alpha})+4(\alpha^2+\beta^2)} = \frac{1}{8\alpha\beta}$$

So

$$\int_0^{2\pi}\frac{1}{\alpha^2\cos^2(\theta)+\beta^2\sin^2(\theta)}d\theta = \frac{4}{i}(2\pi i)\left(\frac{1}{8\alpha\beta}+\frac{1}{8\alpha\beta}\right) = \frac{2\pi}{\alpha\beta}$$

26. Let $g(\theta) = \dfrac{1}{\alpha+\sin^2(\theta)}$ and write $\displaystyle\int_0^{2\pi} g(\theta)d\theta = \int_0^{\pi/2} g(\theta)d\theta + \int_{\pi/2}^{\pi}(\theta)g(\theta)d\theta + \int_{\pi}^{3\pi/2} g(\theta)d\theta + \int_{3\pi/2}^{2\pi} g(\theta)d\theta$. In the second, third and fourth integrals let $\theta = \pi - u, \theta = \pi + u, \theta = 2\pi - u$ respectively to show that

$$\int_0^{\pi/2}\frac{1}{\alpha+\sin^2(\theta)}d\theta = \frac{1}{4}\int_0^{2\pi} g(\theta)d\theta = \frac{1}{4}\oint_{|z|=1}\frac{1}{\alpha-\frac{1}{4}(z-\frac{1}{z})^2}\frac{dz}{iz} = -\frac{1}{i}\oint_{|z|=1}\frac{z}{z^4-(2+4\alpha)z^2+1}dz.$$

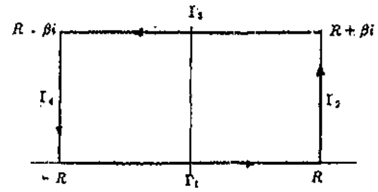
$f(z) = \dfrac{z}{z^4 - (2+4\alpha)z^2 + 1}$ has simple poles at z_1 and z_2 for which $z_k^2 = (1+2\alpha) - 2\sqrt{\alpha^2+\alpha}$, and

$$Res(f(z), z_k) = \left.\frac{z}{4z^3 - (4+8\alpha)z}\right|_{z_k} = \frac{1}{-8\sqrt{\alpha^2+\alpha}}$$

at both z_1 and z_2 So

$$\int_0^{\pi/2} \frac{1}{\alpha + \sin^2(\theta)} d\theta = -\frac{1}{i}(2\pi i)\left[\frac{2}{-8\sqrt{\alpha^2+\alpha}}\right] = \frac{\pi}{2\sqrt{\alpha^2+\alpha}}$$

27. Let Γ be the suggested path with sides labeled $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ as shown.



By Cauchy's Theorem, $0 = \oint_\Gamma e^{-z^2} dz = \int_{\Gamma_1} e^{-z^2} dz + \int_{\Gamma_2} e^{-z^2} dz + \int_{\Gamma_3} e^{-z^2} dz + \int_{\Gamma_4} e^{-z^2} dz$

On $\Gamma_1, z = x$ so $\int_{\Gamma_1} e^{-z^2} dz = \int_{-R}^{R} e^{-x^2} dx$

On $\Gamma_3, z = x+i\beta$ so $\int_{\Gamma_2} e^{-z^2} dz = \int_R^{-R} e^{-(x^2+2x\beta i-\beta^2)} dx = -e^{\beta^2} \int_{-R}^{R} e^{-x^2}[\cos(2\beta x) - i\sin(2\beta x)]dx$

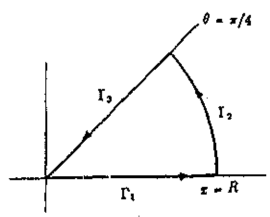
On $\Gamma_2, z = R+it$ so $\int_{\Gamma_2} e^{-z^2} dz = \int_0^\beta e^{-(R^2+2Rti-t^2)} i dt = ie^{-R^2} \int_0^\beta e^{t^2}[\cos(2Rt) - i\sin(2Rt)]dt$.

On $\Gamma_4, z = -R+it$ so $\int_{\Gamma_4} e^{-z^2} dz = \int_\beta^0 e^{-(R^2-2itR-t^2)} i dt = ie^{-R^2} \int_0^\beta e^{t^2}[-\cos(2Rt) - i\sin(2Rt)]dt$.

We now add these integrals and then let $R \to +\infty$.

This gives $\int_{-R}^{R} e^{-x^2} dx - e^{\beta^2} \int_{-R}^{R} e^{-x^2} \cos(2\beta x) dx + 2e^{-R^2} \int_0^\beta e^{t^2} \sin(2Rt) dt = 0$. Note that the term containing $\int_{-R}^{R} e^{-x^2} \sin(2\beta x) dx$ is 0 since the integrand is an odd function Also the integral $\int_0^\beta e^{t^2} \sin(2Rt) dt$ remains bounded for all R. Finally letting $R \to \infty$ and using $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$ we get $\int_{-\infty}^{\infty} e^{-x^2} \cos(2\beta x) dx = \sqrt{\pi} e^{-\beta^2}$ Then $\int_0^\infty e^{-x^2} \cos(2\beta x) dx = \dfrac{\sqrt{\pi}}{2} e^{-\beta^2}$

28 Let Γ be the path shown.



By Cauchy's Theorem, $\int_\Gamma e^{iz^2} dz = 0 = \int_{\Gamma_1} e^{iz^2} dz + \int_{\Gamma_2} e^{iz^2} dz + \int_{\Gamma_3} e^{iz^2} dz$

On $\Gamma_1, z = x$, $\int_{\Gamma_1} e^{iz^2} dz = \int_0^R e^{ix^2} dx = \int_0^R [\cos(x^2) + i\sin(x^2)]dx$.

On $\Gamma_2, z = Re^{i\theta}$, $\int_{\Gamma_2} e^{iz^2} dz = \int_0^{\pi/4} e^{iR^2 e^{i2\theta}} iRe^{i\theta} d\theta$.

On $\Gamma_3, z = re^{i\pi/4}$, $\int_{\Gamma_3} e^{iz^2} dz = \int_R^0 e^{-r^2} e^{i\pi/4} dr$

Now estimate the integral on Γ_2 as follows:

First make the change of variable $u = 2\theta$, so

$$\int_{\Gamma_2} e^{iz^2} dz = \frac{1}{2}\int_0^{\pi/2} e^{(iR^2\cos(u) - R^2\sin(u))} iRe^{iu/2} du$$

Then

$$\left|\int_{\Gamma_2} e^{iz^2} dz\right| \le \frac{R}{2}\int_0^{\pi/2} |e^{iR^2\cos(u)}||e^{iu/2}|e^{-R^2\sin(u)}du = \frac{R}{2}\int_0^{\pi/2} e^{-R^2\sin(u)}du \le \frac{R}{2}\left(\frac{\pi}{2R^2}\right) = \frac{\pi}{4R}$$

Add the integrals on $\Gamma_1, \Gamma_2, \Gamma_3$, let $R \to \infty$, and use the above bound to get

$$\int_0^\infty [\cos(x^2) + i\sin(x^2)]dx - e^{i\pi/4}\int_0^\infty e^{-r^2} dr = 0.$$

Finally use $\displaystyle\int_0^\infty e^{-r^2} dr = \frac{\sqrt{\pi}}{2}$, and equate real and imaginary parts of

$$\int_0^\infty \cos(x^2)dx + i\int_0^\infty \sin(x^2)dx = \frac{\pi}{2\sqrt{2}}(1+i)$$

29. First show by a change of variable that

$$\int_0^\infty \frac{x\sin(\alpha x)}{x^4+\beta^4}dx = \frac{1}{2}\int_{-\infty}^\infty \frac{x\sin(\alpha x)}{x^4+\beta^4}dx.$$

Take $f(z) = \dfrac{ze^{i\alpha z}}{z^4+\beta^4}$ which has simple poles in the upper half plane at $z_1 = \beta e^{i\pi/4}$ and $z_2 = \beta e^{3\pi i/4}$.

Calculate

$$Res(f(z), z_k) = \left.\frac{ze^{i\alpha z}}{4z^3}\right|_{z_k} = \frac{e^{i\alpha z_k}}{4z_k^2};$$

$$Res(f(z), z_1) = \frac{e^{i\alpha\beta e^{i\pi/4}}}{4\beta^2 i};\ Res(f(z), z_2) = \frac{e^{i\alpha\beta e^{3i\pi/4}}}{-4\beta^2 i}.$$

Then

$$\int_0^\infty \frac{x\sin(\alpha x)}{x^4+\beta^4}dx = \frac{1}{2}Im\left[\frac{2\pi i}{4\beta^2}\left(e^{i\alpha\beta e^{i\pi/4}} - e^{i\alpha\beta e^{3i\pi/4}}\right)\frac{1}{i}\right] = \frac{\pi e^{-\alpha\beta/\sqrt{2}}}{2\beta^2}\sin\left(\frac{\alpha\beta}{\sqrt{2}}\right).$$

30. $\displaystyle\int_0^{2\pi} \frac{1}{(\alpha+\beta\cos(\theta))^2}d\theta = \int_0^{\pi} \frac{1}{(\alpha+\beta\cos(\theta))^2}d\theta + \int_\pi^{2\pi} \frac{1}{(\alpha+\beta\cos(\theta))^2}d\theta$

In the second integral, let $\theta = 2\pi - u$ to show

$$\int_0^{\pi} \frac{1}{(\alpha+\beta\cos(\theta))^2}d\theta = \frac{1}{2}\int_0^{2\pi} \frac{1}{(\alpha+\beta\cos(\theta))^2}d\theta$$

$$= \frac{1}{2}\int_{|z|=1} \frac{1}{[\alpha+\frac{\beta}{2}(z+\frac{1}{z})]^2}\frac{dz}{iz} = \frac{2}{i}\int_{|z|=1} \frac{z}{[\beta z^2 + 2\alpha z + \beta]^2}$$

$f(z) = \dfrac{z}{[\beta z^2 + 2\alpha z + \beta]^2}$ has poles or order two at the zeros of $\beta z^2 + 2\alpha z + \beta$, which are $z_1 = \dfrac{-\alpha + \sqrt{\alpha^2 - \beta^2}}{\beta}$ and $z_2 = \dfrac{-\alpha - \sqrt{\alpha^2 - \beta^2}}{\beta}$, the second one lying outside $|z| = 1$ Compute the residue at z_1 by

$$Res(f(z), z_1) = \lim_{z \to z_1} \frac{d}{dz}\left(\frac{z}{\beta^2 (z - z_2)^2}\right) = \frac{1}{\beta^2} \lim_{z \to z_1} \left[\frac{(z - z_2)^2 - 2z(z - z_2)}{(z - z_2)^4}\right]$$

$$= \frac{1}{\beta^2}\left(\frac{\alpha\beta^2}{4(\alpha^2 - \beta^2)^{3/2}}\right) = \frac{\alpha}{4(\alpha^2 - \beta^2)^{3/2}}$$

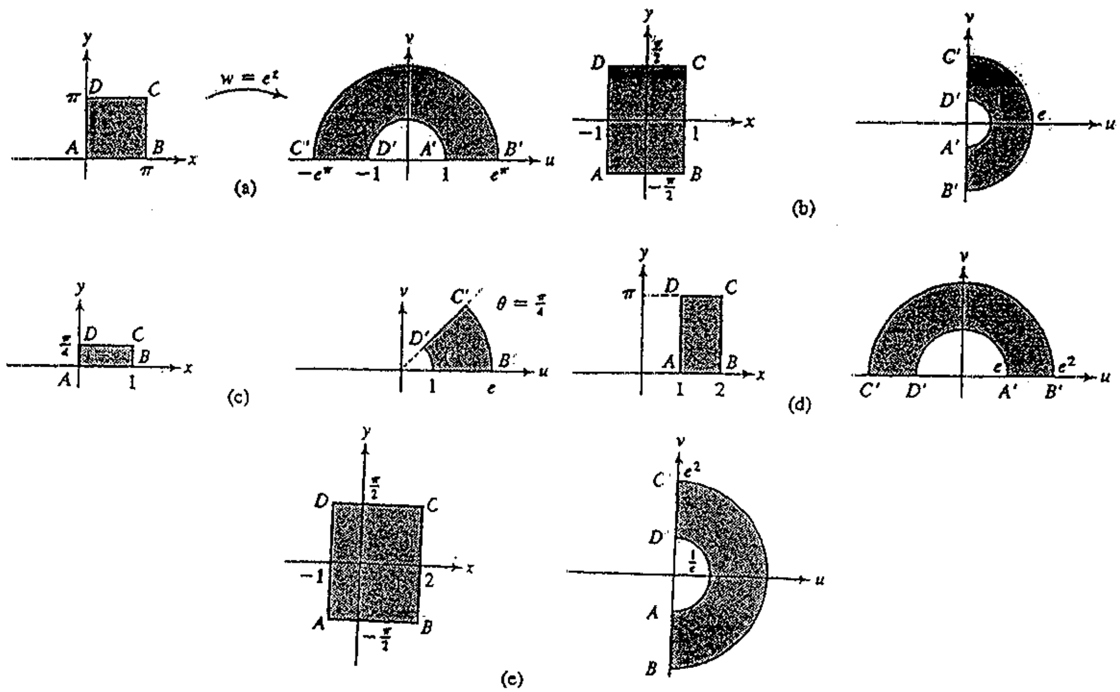
So

$$\int_0^{\pi} \frac{1}{(\alpha + \beta\cos(\theta))^2} d\theta = \frac{2}{i}(2\pi i)\frac{\alpha}{4(\alpha^2 - \beta^2)^{3/2}} = \frac{\pi\alpha}{(\alpha^2 - \beta^2)^{3/2}}$$

Chapter Twenty Five - Conformal Mappings

Section 25.1 Functions as Mappings

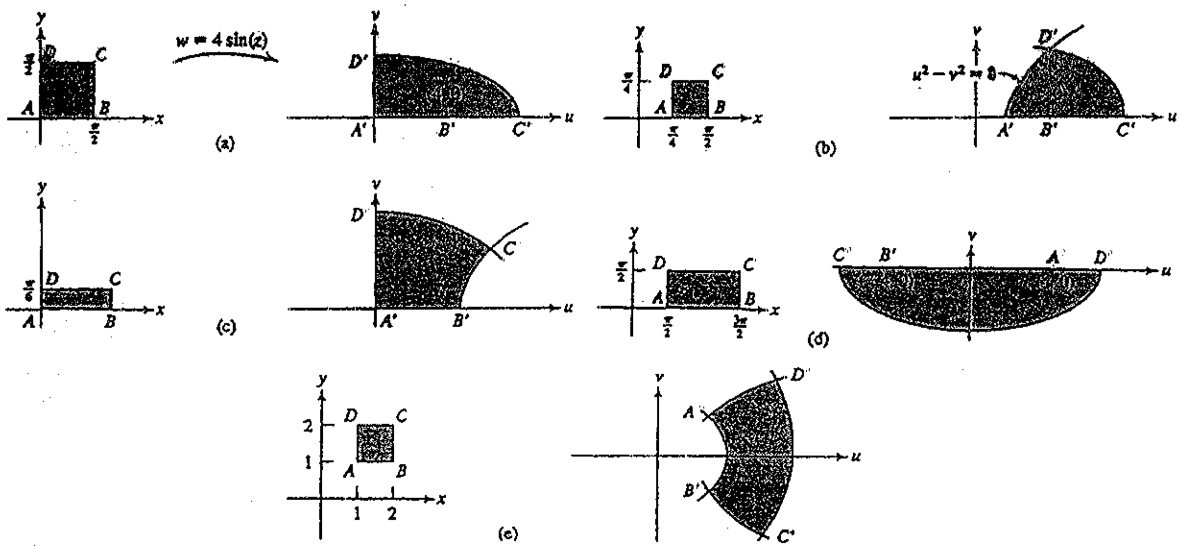
1. With $z = x + iy$ and $w = u + iv$, the mapping $w = e^z$ gives $u + iv = e^x(\cos(y) + i\sin(y))$. From this we see that vertical lines, $x = x_0$ in the z-plane map onto circles $|w| = e^{x_0}$ in the w-plane and horizontal lines, $y = y_0$ in the z-plane map onto rays $\arg(w) = y_0$ in the w-plane. Specific examples for (a) - (e) are illustrated below.



2. With $z = x + iy$ and $w = u + iv$, the mapping $w = \cos(z)$ gives $u + iv = \cos(x)\cosh(y) - i\sin(x)\sinh(y)$. The vertical lines $x = (2k+1)\pi/2$, k any integer, map onto $u = 0, v = (-1)^{k+1}\sinh(y)$ which is the vertical axis in the w-plane. Vertical lines $x = k\pi$, k any integer, map onto $u = (-1)^k\cosh(y), v = 0$ which is a portion of the real axis $|u| \geq 1$ in the w-plane. For $x \neq n\pi/2$ but $x = x_0$ we get hyperbolas $\left(\dfrac{u}{\cos(x_0)}\right)^2 - \left(\dfrac{v}{\sin(x_0)}\right)^2 = 1$ which all have foci at $(\pm 1, 0)$ and intercepts $(\pm\cos(x_0), 0)$. The horizontal line $y = 0$ maps onto $u = \cos x$ or $-1 \leq u \leq 1, v = 0$. Other horizontal lines $y = y_0 \neq 0$ map onto ellipses $\left(\dfrac{u}{\cos(y_0)}\right)^2 + \left(\dfrac{v}{\sin(y_0)}\right)^2 = 1$ which all have foci at $(\pm 1, 0)$. For specific examples of (a) - (e), see the solution of Problem 16(a) below.

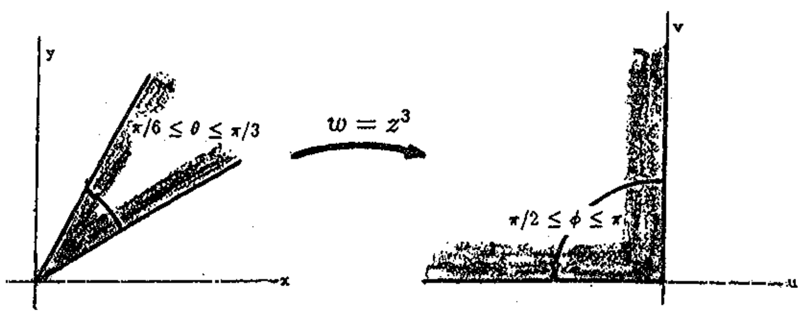
3. With $z = x + iy$ and $w = u + iv$, the mapping $w = 4\sin(z)$ gives $u + iv = 4\sin(x + iy) = 4\sin(x)\cosh(y) + 4i\cos(x)\sinh(y)$. The vertical lines $x = k\pi, k$ any integer, map onto $u = 0, v = 4(-1)^k\sinh(y)$ which is the vertical axis in the w plane. Vertical lines $x = (2k+1)\pi/2, k$ any integer, map onto $u = 4(-1)^{k+1}\cosh(y), v = 0$ which is a portion of the real axis $|u| \geq 4$ in the w plane. For $x \neq n\pi/2$ but $x = x_0$ we get hyperbolas $\left(\dfrac{u}{4\sin(x_0)}\right)^2 - \left(\dfrac{v}{4\cos(x_0)}\right)^2 = 1$

in the w plane. For $x \neq n\pi/2$ but $x = x_0$ we get hyperbolas $\left(\frac{u}{4\sin(x_0)}\right)^2 - \left(\frac{v}{4\cos(x_0)}\right)^2 = 1$ which all have foci at $(\pm 4, 0)$ and intercepts at $(\pm 4\sin(x_0), 0)$. The horizontal line $y = 0$ maps onto $u = 4\sin(x), v = 0$ or $-4 \leq u \leq 4$. Other horizontal lines are $y = y_0 \neq 0$ and map onto ellipses $\left(\frac{u}{4\cosh(y_0)}\right)^2 + \left(\frac{v}{4\sinh(y_0)}\right)^2 = 1$ which all have foci $(\pm 4, 0)$. Specific examples of (a) - (e) are illustrated below



4 Let $z = re^{i\theta}, \pi/4 \leq \theta \leq 5\pi/4$ Then $w = z^2 = r^2 e^{i2\theta} = r^2 e^{i\phi}, \pi/2 \leq \phi \leq 5\pi/2$, the entire w plane.

5 Let $z = re^{i\theta}, \pi/6 \leq \theta \leq \pi/3$ Then $w = z^3 = r^3 e^{3i\theta} = r^3 e^{i\phi}, \pi/2 \leq \phi \leq \pi$, the second quadrant of the w plane.



6. Let $z = re^{i\theta}$, and $w = u + iv$ to get $u + iv = \frac{1}{2}\left(re^{i\theta} + \frac{1}{r}e^{-i\theta}\right)$, so $u = \frac{1}{2}\left(r + \frac{1}{r}\right)\cos\theta$, $v = \frac{1}{2}\left(r - \frac{1}{r}\right)\sin\theta$ Then $\left[\frac{u}{\frac{1}{2}(r+\frac{1}{r})}\right]^2 + \left[\frac{v}{\frac{1}{2}(r-\frac{1}{r})}\right]^2 = 1$ which is an ellipse. Since $\left(r + \frac{1}{r}\right) > \left(r - \frac{1}{r}\right)$ the foci in the w plane are at $(\pm c, 0)$ where $c^2 = \frac{1}{4}\left[\left(r + \frac{1}{r}\right)^2 - \left(r - \frac{1}{r}\right)^2\right] = 1$.

7. We get $u = \frac{1}{2}\left(r + \frac{1}{r}\right)\cos(k), v = \frac{1}{2}\left(r - \frac{1}{r}\right)\sin(k)$, or

$$\frac{u}{\cos(k)} = \frac{1}{2}\left(r + \frac{1}{r}\right); \ \frac{v}{\sin(k)} = \frac{1}{2}\left(r - \frac{1}{r}\right).$$

Add these last two equations to get $\frac{u}{\cos(k)} + \frac{v}{\sin(k)} = r$, subtract to get $\frac{u}{\cos(k)} - \frac{v}{\sin(k)} = \frac{1}{r}$, now multiply these two equations to get

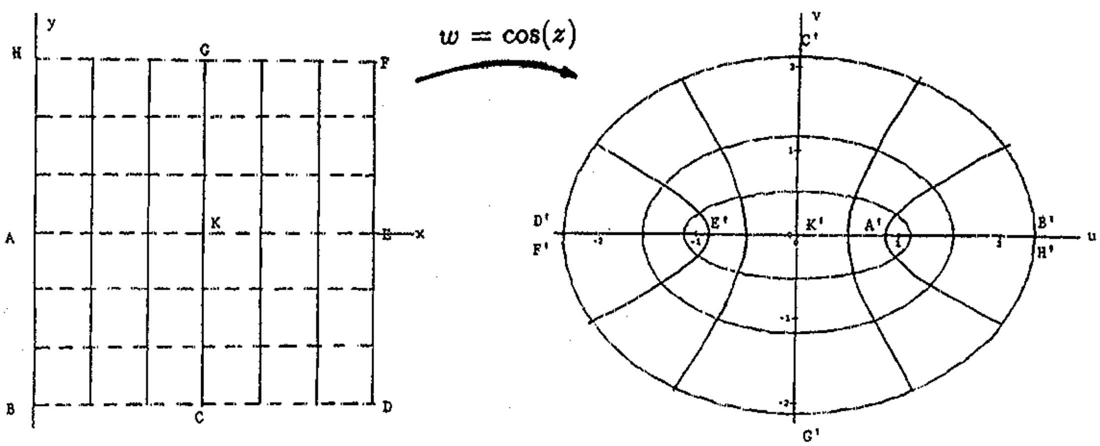
$$\frac{u^2}{(\cos(k))^2} - \frac{v^2}{(\sin(k))^2} = 1,$$

an hyperbola with foci at $(\pm c, 0)$ where $c^2 = (\cos(k))^2 + (\sin(k))^2 = 1$. The cases $k = n\pi$ give $u = \frac{1}{2}\left(r + \frac{1}{r}\right)(-1)^n, v = 0$ which gives $u \geq 1$ for $n = 2p$ and $u \leq -1$ for $n = 2p + 1$, while $k = \left(n + \frac{1}{2}\right)\pi$ gives $u = 0, -\infty < v < \infty$ which is the imaginary w axis.

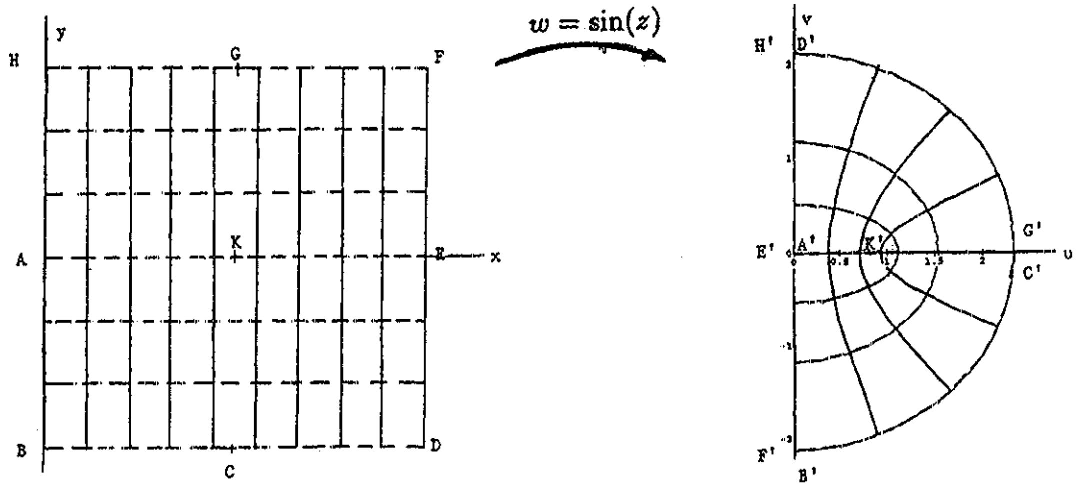
8. Every circle or line in the $z = x + iy$ plane can be written $A(x^2 + y^2) + Bx + Cy + D = 0$ or $A|z|^2 + \frac{B}{2}(z + \overline{z}) + \frac{C}{2}\left(\frac{z - \overline{z}}{i}\right) + D = 0$. Then $w = \frac{1}{z}$(or $z = \frac{1}{w}$) gives $A\left|\frac{1}{w}\right|^2 + \frac{B}{2}\left(\frac{1}{w} + \frac{1}{\overline{w}}\right) + \frac{C}{2i}\left(\frac{1}{w} - \frac{1}{\overline{w}}\right) + D = 0$. Write $|w|^2 = w\overline{w}$ and multiply by $w\overline{w}$ to get $A + \frac{B}{2}(\overline{w} + w) + \frac{C}{2}\left(\frac{\overline{w} - w}{i}\right) + D|w|^2 = 0$, the equation of a line or circle in the w plane.
For examples, take $D = 0$ and the circle $A(x^2 + y^2) + Bx + Cy = 0$ maps to the line $A + Bu - Cv = 0$. With $A = 0$, the line $Bx + Cy + D = 0$ maps to the circle $D(u^2 + v^2) - Cv + Bu = 0$ passing through $w = 0$

9. Let $z = x + iy, -\infty < x < \infty,\ 0 \leq y \leq 2\pi$. Then $w = e^z = e^x(\cos(y) + i\sin(y)) = u + iv$. The image is the entire w plane except for $w = 0$

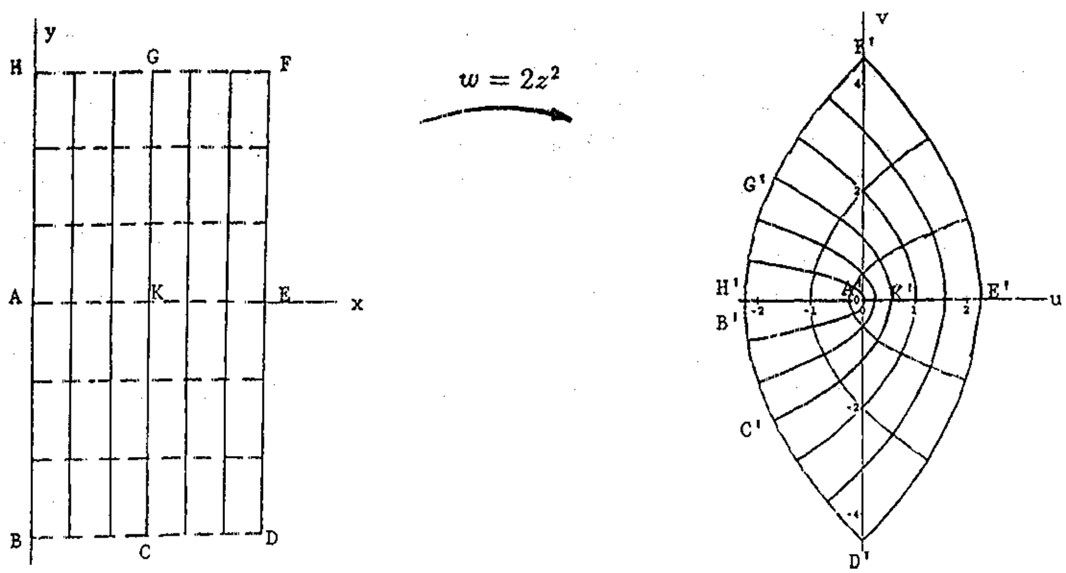
10. (a) Write $w = u+iv = \cos(x+iy) = \cos(x)\cosh(y) - i\sin(x)\sinh(y)$, so $u = \cos(x)\cosh(y)$ and $v = -\sin(x)\sinh(y)$. On the left and right edges of D we have $x = 0$ and $x = \pi$ respectively, so $v = 0$ along both these edges. The edge portions AB and AH each map onto $1 \le u \le \cosh(\alpha)$, while edge portions ED and EF each map onto $-\cosh(\alpha) \le u \le -1$. The vertical line $x = \pi/2$ maps onto $u = 0, -\sinh(\alpha) \le v \le \sinh(\alpha)$ as shown. Other vertical lines have the form $x = x_0$ and map onto hyperbolas $\left(\frac{u}{\cos(x_0)}\right)^2 - \left(\frac{v}{\sin(x_0)}\right)^2 = 1$ which all have foci at $(\pm 1, 0)$. The horizontal line $y = 0$ in D maps onto $-1 \le u \le 1, v = 0$. Other horizontal lines have the form $y = y_0$ and map onto the ellipses $\left(\frac{u}{\cosh(y_0)}\right)^2 - \left(\frac{v}{\sinh(y_0)}\right)^2 = 1$ which also all have foci at $(\pm 1, 0)$.



(b) Write $w = u+iv = \sin(x+iy) = \sin(x)\cos(y) + i\cos(x)\sinh(y)$, so $u = \sin(x)\cosh(y)$ and $v = \cos(x)\sinh(y)$. On the left and right edges of D we have $x = 0$ and $x = \pi$ respectively, so $u = 0$ along both these edges. The edge portions AB and EF eaach map onto $0 \ge v \ge -\sinh(\alpha)$, while edge portions AH and ED each map onto $0 \le v \le \sinh(\alpha)$. The vertical line $x = \pi/2$ maps onto $v = 0, u = \cosh(y)$. Other vertical lines have the form $x = x_0$ and map onto $\left(\frac{u}{\sin(x_0)}\right)^2 - \left(\frac{v}{\cos(x_0)}\right)^2 = 1$ which all have foci at $(\pm 1, 0)$. The horizontal line $y = 0$ in D maps onto $v = 0, u = \sin(x)$ or $0 \le u \le 1$. Other horizontal lines have the form $y = y_0$ and map onto ellipses $\left(\frac{u}{\cosh(y_0)}\right)^2 + \left(\frac{v}{\sinh(y_0)}\right)^2 = 1$ which have foci at $(\pm 1, 0)$. Finally note that since $\sin(\pi/2 - z) = \sin(\pi/2 + z)$, every point w in the image of D has two preimages in D



(c) Write $w = u + iv = 2(x + iy)^2 = 2(x^2 - y^2) + 4xyi$, so $u = 2(x^2 - y^2)$ and $v = 4xy$. The vertical line $x = 0$ maps onto $u = -2y^2, v = 0$ which is the negative u axis. Other vertical lines of the form $x = x_0 \neq 0$ map onto the parabolas $u = 2x_0^2 - \dfrac{v^2}{8x_0^2}$ having intercepts at $(2x_0^2, 0)$ and opening left. The horizontal line $y = 0$ maps onto $u = 2y^2 \geq 0, v = 0$ which is the positive u axis. Other horizontal lines of the form $y = y_0 \neq 0$ map onto the parabolas $u = \dfrac{v^2}{8y_0^2} - 2y_0^2$ having intercept at $(-2y_0^2, 0)$ and opening right. The plot below is a plot of the mapping $w = 2z^2$ defined on $z = \{x + iy, 0 \leq x \leq 1.5, -1.5 \leq y \leq 1.5\}$.



Section 25.2 Conformal Mappings

1 With $z_1 = 1, z_2 = 2, z_3 = 3$ and $w_1 = 1, w_2 = -i, w_3 = 1+i$, Theorem 25.3 gives $(1-w)(1+2i)(-1)(3-z) = (1-z)(1)(1+i)(1+i-w)$
Solve for

$$w = \frac{(1+4i)z - (3+8i)}{(2+3i)z - (4+7i)}$$

2. $w = \dfrac{(1+i)z - (2+2i)}{(3-i)z - 2}$

3. Since $w_3 = \infty$, use Theorem 25.4 with $z_1 = 1, z_2 = 2i, z_3 = 4, w_1 = 1+i, w_2 = 3-i$ to get $(1+i-w)(1-2i)(4-z) = (1-z)(-2+2i)(4-2i)$. Solve for $w = \dfrac{(33+i)z - (48+16i)}{5(z-4)}$

4 $w = \dfrac{(4-2i)z - (6+12i)}{(5+i)(z+1)} = \dfrac{(9-7i)z - (21+27i)}{13(z+1)}$

5 $w = \dfrac{(3+22i)z + (4-75i)}{(2+3i)z - (21-4i)}$

6 Solve for $z = \dfrac{2i}{w}$. Then $Re(z) = \dfrac{z+\overline{z}}{2} = -4$ becomes $z + \overline{z} = -8$ or $\dfrac{2i}{w} - \dfrac{2i}{\overline{w}} = -8$ Clear of fractions to get $8w\overline{w} - 2i(w - \overline{w}) = 0$ With $w = u+iv$ we get $2(u^2+v^2)+v = 0$. Complete the square to get $u^2 + \left(v + \dfrac{1}{4}\right)^2 = \dfrac{1}{4}$, a circle of radius $\dfrac{1}{2}$ centered at $\left(0, -\dfrac{1}{4}\right)$.

7. Solve for $z = \dfrac{w+4}{2i}$ Then $Re(z) = \dfrac{z+\overline{z}}{2} = 5$ becomes $\dfrac{w+4}{2i} - \dfrac{\overline{w}+4}{2i} = 10$. Putting $w = u+iv$ we get $\dfrac{w-\overline{w}}{2i} = v = 10$, a horizontal line.

8 Solve for $z = \dfrac{-i}{iw-1} = \dfrac{-1}{w+i}$ Substitute into the given line to get

$$\frac{1}{2}\left(\frac{-1}{w+i} - \frac{1}{\overline{w}-i}\right) + \frac{1}{2i}\left(\frac{-1}{w+i} + \frac{1}{\overline{w}-i}\right) = 4$$

Clear of fractions to write as

$$8i(w+i)(\overline{w}-i) = -i(w+\overline{w}) + (2i + w) = \overline{w}).$$

Put $w = u+iv$ to get $4(u^2+v^2) + 7v + u + 3 = 0$, or

$$\left(u + \frac{1}{8}\right)^2 + \left(v + \frac{7}{8}\right)^2 = \frac{1}{32},$$

a circle centered at $(-\frac{1}{8}, -\frac{7}{8})$ of radius $\sqrt{2}/8$

9. Solve for $z = \dfrac{-w-1+i}{2w-1}$, so $|z| = 4$ becomes $\left|\dfrac{-w-1+i}{2w-1}\right| = 4$ or $|w+1-i| = 4|2w-1|$ Put $w = u+iv$ and get $(u+1)^2 + (v-1)^2 = 16(2u-1)^2 + 64v^2$ Expand and simplify as $63u^2 - 66u + 63v^2 + 2v = -14$. Complete the square to get

$$\left(u - \frac{11}{21}\right)^2 + \left(v + \frac{1}{63}\right)^2 = \frac{208}{3969}.$$

10. Solve for $z = \dfrac{w+i}{3}$ and substitute into $|z-4| = 3$ to get $\left|\dfrac{w+i}{3} - 4\right| = 3$ or $|w-(12-i)| = 9$, a circle of radius 9, centered at $w_0 = 12 - i$ or $(12, -1)$ in the w plane.

11. Solve for $z = \dfrac{iw+5}{2-w}$. Then compute

$$z + \overline{z} = 2Re(z) = 2Re\left(\frac{(5-v)+iu}{(2-u)-iv}\right) = \frac{2[(5-v)(2-u)-uv]}{(u-2)^2+v^2} = \frac{20-4v-10u}{(u-2)^2+v^2},$$

and

$$\frac{1}{2i}(z-\overline{z}) = Im(z) = \left[\frac{(2-u)u+v(5-v)}{(u-2)^2+v^2}\right].$$

Substitute into the given line and clear of fractions to get

$$20 - 4v - 10u - 3[2u - u^2 + 5v - v^2] - 5[u^2 - 4u + 4 + v^2] = 0,$$

or equivalently, $2u^2 + 2v^2 - 4u + 19v = 0$ Complete the square and get

$$(u-1)^2 + \left(v + \frac{19}{4}\right)^2 = \frac{377}{16},$$

a circle centered at $\left(1, -\dfrac{19}{4}\right)$ of radius $\dfrac{1}{4}\sqrt{377}$

12. Solve for $z = \dfrac{-2}{w-3i-1}$ and substitute into $|z-i| = 1$ to get $\left|\dfrac{-2}{w-3i-1} - i\right| = 1$ or equivalently $|w-3i-1| = |-2-iw-3+i| = |-i(w-5i-1)|$. Putting $w = u+iv$ and squaring gives

$$(u-1)^2 + (v-3)^2 = (u-1)^2 + (v-5)^2$$

Expand and simplify to get $4v = 16$ or $v = 4$

13. $w = \overline{z}$ is not conformal because it reverse orientation and hence does not preserve angles. Specifically, take $c_1 : |z| = 1, c_2 : z = t, 0 \le t < \infty$. These curves intersect at $(1,0)$ with a $\pi/2$ angle between c_1 and c_2. Let γ_1 and γ_2 be the images of c_1 and c_2 under $w = \overline{z}$ Then $\gamma_1 : w = e^{-i\theta}, 0 \le \theta \le 2\pi$ is the unit circle with clockwise orientation and $\gamma_2 : w = t, 0 \le t < \infty$. γ_1 and γ_2 intersect at $(1,0)$ but the angle between tangents is $-\pi/2$.

14. Let $w = T_1(z) = \dfrac{az+b}{cz+d}$, and $\xi = T_2(w) = \dfrac{\alpha w+\beta}{\gamma w+\delta}$.
Then

$$\xi = (T_2 \circ T_1)(z) = T_2(T_1(z)) = \frac{\alpha(\frac{az+b}{cz+d}) + \beta}{\gamma(\frac{az+b}{cz+d}) + \delta} = \frac{(\alpha a + \beta c)z + \alpha b + \beta d}{(\gamma a + \delta c)z + \gamma b + \delta d}$$

is a linear fractional transformation
The condition

$$\begin{vmatrix} \alpha a + \beta c & \alpha b + \beta d \\ \gamma a + \delta c & \gamma b + \delta d \end{vmatrix} \neq 0$$

is met because this is the determinant of the product

$$\left|\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right| = \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix}\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0.$$

15. Let $w = T(z) = \dfrac{az+b}{cz+d}$. Solve this equation for $z = \dfrac{-dw+b}{cw-a}$ We claim $z = T^*(w)$ is the inverse of T. Compute

$$(T^* \circ T)(z) = T^*(T(z)) = \frac{-d(\frac{az+b}{cz+d})+b}{c(\frac{az+b}{cz+d})-a} = \frac{z(cb-ad)}{(cb-ad)} = z.$$

Similarly $(T \circ T^*)(w) = w$.

16. By Theorem 2, every linear fractional transformation maps lines/circles onto lines/circles Thus the circle $|z| = 1$ is mapped by any linear fractional transformation to a circle or a line, but the boundary of the ellipse $u^2/4 + v^2 = 1/16$ is neither a line nor a circle. Since no linear fractional transformation maps $|z| = 1$ onto the ellipse, no linear fractional transformation can map $|z| < 1$ onto the interior of the ellipse

17. If $f(z) = \dfrac{az+b}{cz+d}$ is neither a translation nor the identity mapping then $c \neq 0$. z is a fixed point of f if and only if $z = \dfrac{az+b}{cz+d}$, or equivalently $cz^2 + (d-a)z - b = 0$ Since $c \neq 0$, this quadratic equation has either one or two solutions. Note that a translation has no fixed points while the identity has only fixed points.

18. By Problem 17, any linear fractional transformation having three fixed points must be either a translation or the identity But a translation has no fixed points (or one if we admit $z = \infty$) Thus f must be the identity mapping

19. $w = w_2 + i$, a translation; $w_2 = -4w_1$, a magnification; $w_1 = \dfrac{1}{z}$, an inversion.

20. By division we get $w = \dfrac{1}{2} - \dfrac{8+i}{4z+2i} = \dfrac{1}{2} - \dfrac{8+i}{4}\left(\dfrac{1}{z+i/2}\right)$ so $w = w_3 + \dfrac{1}{2}$, a translation; $w_3 = -\dfrac{8+i}{4}w_2$, a rotation/magnification; $w_2 = \dfrac{1}{w_1}$, an inversion, $w_1 = z + \dfrac{i}{2}$, a translation

21 $w = w_1 - (2-7i)$, a translation; $w_1 = iz$, a rotation.

22. $w = 1 - \dfrac{4+i}{z+3+i}$ by division, so $w = w_3 + 1$, a translation; $w_3 = -(4+i)w_2$, a rotation/magnification; $w_2 = \dfrac{1}{w_1}$, an inversion; $w_1 = z + 3 + i$, a translation.

Section 25.3 Construction of Conformal Mappings Between Domains

In Problems 1 through 6, there are many correct mappings having the stated property. We give one such solution with some explanation of the strategy which leads to it.

1. Both domains here are circles, but of different radii and centers. Thus we can map $|z| < 3$ onto $|w - 1 + i| < 6$ by a using scaling factor of 2, followed by a translation to match centers Thus take $w = 2z + 1 - i$.

2. We can achieve this mapping from $|z| < 3$ to $|w-1+i| > 6$ in three steps, one of which will be an inversion. First invert in $|z| = 3$, by $w_1 = \dfrac{1}{z}$, our new circle now has radius $|w_1| = \dfrac{1}{3}$

Next expand by a factor of 18 so the radii match. Thus $w_2 = 18w_1 = \frac{18}{z}$. Finally translate centers to match by $w = w_2 + 1 - i$. Then

$$w = \frac{18}{z} + 1 - i = \frac{(1-i)z + 18}{z}$$

will be such a mapping.

3. Since we are mapping the interior of a z disk to the exterior of a w disk we need an inversion. First translate by $w_1 = z + 2i$, so our w_1 disk has center at the origin. Then invert by $w_2 = \frac{1}{w_1}, = \frac{1}{z+2i}$. Next scale by a factor of 2 to match radii of boundaries, by $w_3 = 2w_2 = \frac{2}{z+2i}$. Finally translate the center to $w = 3$ by

$$w = w_3 + 3 = \frac{2}{z+i} + 3 = \frac{3z + 2 + 6i}{z + 2i}.$$

4. The mapping of the half plane $Re(z) > 1$ onto the half plane $Im(w) > -1$ can be accomplished by first rotating by $\pi/2$ counterclockwise with $w_1 = iz$, then shifting down 2 units by $w = w_1 - 2i$, so $w = iz - 2i = i(z-2)$ is such a mapping. Note that the algebra of $w = i(z-2)$ reveals another solution, i.e. shift to the left two units, then rotate by $\pi/2$ counterclockwise.

5. We know that we can map the line $Re(z) = 0$ onto the circle $|w| = 4$ by a linear fractional transformation, but we need to check where $Re(z) < 0$ goes, i.e. inside or outside. We accomplish this in two steps. First construct a linear fractional transformation which maps $z_1 = 0$ to $\eta_1 = 1, z_2 = i$ to $\eta_2 = i$ in such a way that $z = -1$ goes to $\eta = 0$. Thus we want $\eta = \frac{a(z+1)}{cz+d}$. From $\eta(0) = 1$ we get $1 = \frac{a}{d}$ so $a = d$. From $\eta(i)$ we get $\frac{a(i+1)}{ci+a} = i$ so $c = -a$. Then $\eta = \frac{a(z+1)}{-a(z-1)} = \frac{z+1}{1-z}$ maps $Re(z) < 0$ onto $|\eta| < 1$. Note $\eta(\infty) = -1$. To complete the solution take $w = 4\eta = 4\left(\frac{z+1}{1-z}\right)$. Note that the mapping given preserves boundary orientation and hence is conformal.

6. Construct the mapping in several steps. First map $Im(z) > -4$ to $Im(w_1) > 0$ by $w_1 = z + 4i$. Next map $Im(w_1)$ to $|w_2| < 1$. Many mappings will do this, but one is $w_2 = \frac{w_1 - i}{w_1 + i} = \frac{z+3i}{z+5i}$. Next do an inversion onto $|w_3| > 1$ by $w_3 = \frac{1}{w_2} = \frac{z+5i}{z+3i}$. Finally, scale the radius by 2, translate $w_3 = 0$ to $w = i$ by $w = 2w_3 + i$. A completed mapping is given by

$$w = 2\left(\frac{z+5i}{z+3i}\right) + i = \frac{(2+i)z - (3-10i)}{z+3i}$$

7. Let $z = re^{i\theta}, r > 0, 0 < \theta < \pi$ and take $w = z^{1/3} = \sqrt[3]{r}e^{i\theta/3} = \rho e^{i\phi}$ where $\rho > 0, 0 < \phi < \pi/3$. Since $\frac{dw}{dz} = \frac{1}{3}z^{-2/3} \neq 0$, the mapping is conformal.

8. Consider $z = x + iy = re^{i\theta}$, with $y > 0$. Then Arg $(z) = \theta$ is unique and $w =$ Log $(z) = \ln(r) + i\theta$. Since $r > 0$, the values of $\ln(r)$ vary from $-\infty$ to ∞ as x varies from $-\infty$ to ∞. Clearly $Im(w) = \theta\epsilon(0, \pi)$. Thus Log (z) is in the strip $0 < Im(w) < \pi$. To show $w =$ Log (z)

is onto, for any $w = u + iv$ in $0 < Im(w) < \pi$, take $z = e^w$. Then $Im(z) = e^u > 0$ and Log $(z) = \ln(e^u) + i$ Arg $(e^w) = u + iv = w$ Finally $\frac{d\text{Log}}{dz}(z) = \frac{1}{z} \neq 0$ so the mapping is conformal.

9. To show

$$w = f(z) = 2i\int_0^x (\xi+1)^{-1/2}(\xi-1)^{-1/2}\xi^{-1/2}d\xi$$

maps the upper half plane onto the given rectangle we evaluate $f(-1)$, $f(0)$, $f(1)$ and $f(\infty)$ and show that these are the vertices of the given rectangle

First $f(0) = 0$ is clear.

Now

$$f(1) = 2i\int_0^1 (\xi^2-1)^{-1/2}\xi^{-1/2}d\xi = 2i\int_0^1 \frac{(1-\xi^2)^{-1/2}}{\sqrt{-1}}\xi^{-1/2}d\xi = 2\int_0^1 (1-\xi^2)^{-1/2}\xi^{-1/2}d\xi.$$

Let $\xi = u^{1/2}$ to get

$$f(1) = \int_0^1 (1-u)^{-1/2}u^{-3/4}du$$

This last integral is a Beta function defined by

$$B(m,n) = \int_0^1 u^{m-1}(1-u)^{n-1}du = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}, \text{ so } f(1) = \frac{\Gamma(1/4)\Gamma(1/2)}{\Gamma(3/4)} = c.$$

Next calculate $f(-1) = 2i\int_0^{-1} (\xi^2-1)^{-1/2}\xi^{-1/2}d\xi$, and let $\xi = -u$ to get

$$f(-1) = 2i\int_0^1 (1-u^2)^{-1/2}u^{-1/2}du.$$

This integral can also be expressed in terms of Beta functions as

$$B(1/4, 1/2) = \frac{\Gamma(1/4)\Gamma(1/2)}{\Gamma(3/4)}, \text{ so } f(-1) = \frac{i\Gamma(1/2)\Gamma(1/4)}{\Gamma(3/4)} = ic.$$

Finally calculate

$$f(\infty) = 2i\int_0^\infty (\xi+1)^{-1/2}(\xi-1)^{-1/2}\xi^{-1/2}d\xi$$

$$= 2i\int_0^1 (\xi+1)^{-1/2}(\xi-1)^{-1/2}\xi^{-1/2}d\xi + 2i\int_1^\infty (\xi+1)^{-1/2}(\xi-1)^{-1/2}\xi^{-1/2}d\xi.$$

This first integral is $\frac{\Gamma(1/2)\Gamma(1/4)}{\Gamma(3/4)}$ as above; in the second let $\xi = 1/u$, to get

$$f(\infty) = c + 2i\int_1^0 \left(\frac{1+u}{u}\right)^{-1/2}\left(\frac{1-u}{u}\right)^{-1/2}u^{1/2}\left(-\frac{du}{u^2}\right)$$

$$= c + 2i\int_0^1 (1-u^2)^{-1/2}u^{-1/2}du = c(1+i).$$

This completes the proof.

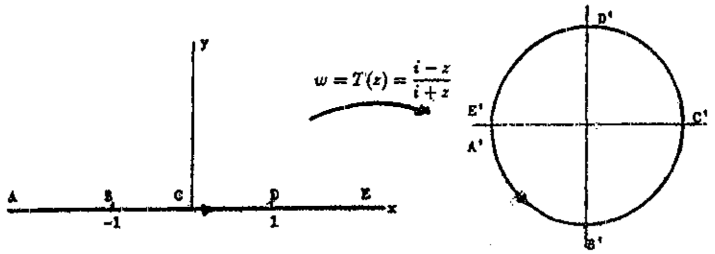
10. For any three pairs of points $(z_1, w_1), (z_2, w_2)$, and (z_3, w_3), there is a unique linear fractional transformation T with $T(z_j) = w_j$, and it is defined by $\frac{z_1 - z}{z_1 - z_2}\frac{z_3 - z_2}{z_3 - z}\frac{w_1 - w}{w_1 - w_2}\frac{w_3 - w_2}{w_3 - w}$. Thus $[z_1, z_2, z_3, z] = [T(z_1), T(z_2), T(z_3), T(z)]$ for all z, with $w = T(z)$. In particular, with $z = z_4$, we get $[z_1, z_2, z_3, z_4] = [T(z_1), T(z_2), T(z_3), T(z_4)]$

11. For $w = f(z) = 1 - \frac{z_3 - z_4}{z_3 - z_2}\frac{z - z_2}{z - z_4}$ calculate $w_2 = f(z_2) = 1, w_3 = f(z_3) = 0, w_4 = f(z_4) = \infty$. Thus f is the unique linear fractional transformation defined in Problem 10, so $[z_1, z_2, z_3, z_4] = f(z_1)$.

12. In Problem 10, w_2, w_3, w_4 all lie on a line, the real axis. Since circles/lines map to circles/lines, it follows that $[z_1, z_2, z_3, z_4]$ is real if and only if z_1, z_2, z_3, z_4 all lie on the same line or circle.

Section 25.4 Harmonic Functions and the Dirichlet Problem

1. We begin with a conformal mapping of the upper half plane $Im(z) > 0$ to the unit disk. One such mapping is $w = T(z) = \dfrac{i-z}{i+z}$ which maps as shown



The solution of the Dirichlet problem is then $u(x, y) = Re[f(z)]$ where

$$f(z) = \frac{1}{2\pi i}\int_C g(\xi)\left(\frac{T(\xi)+T(z)}{T(\xi)-T(z)}\right)\frac{T'(\xi)}{T(\xi)}d\xi.$$

On C we have $\xi = t, -\infty < t < \infty$ which preserves the proper orientation as shown above We easily find $T'(z) = \dfrac{-2i}{(i+z)^2}$ and now calculate

$$\frac{T(\xi)+T(z)}{T(\xi)-T(z)}\frac{T'(\xi)}{T(\xi)} = \frac{\frac{i-t}{i+t}+\frac{i-z}{i+z}}{\frac{i-t}{i+t}-\frac{i-z}{i+z}}\left(\frac{i+t}{i-t}\right)\left(\frac{-2i}{(i+t)^2}\right)$$

With some effort this simplifies to $\dfrac{-2(1+tz)}{(z-t)(1+t^2)}$, and by putting $z = x + iy$ we simplify further to get

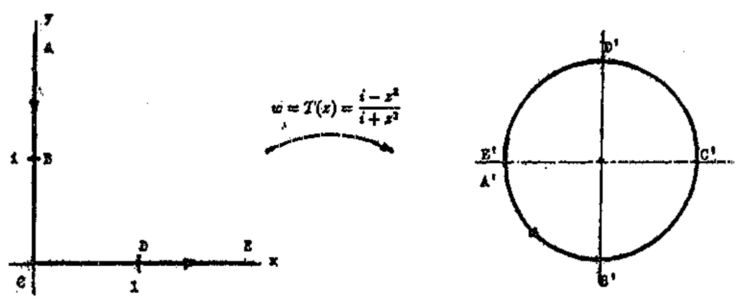
$$\frac{-2[\{(1+tx)(x-t)-ty^2\}-iy(1+t^2)]}{(x-t)^2+y^2}\frac{1}{1+t^2}$$

Substitute this expression in the integral above and extract the real part, recalling that $g(t)$ is real valued to get

$$u(x, y) = \frac{y}{\pi}\int_{-\infty}^{\infty}\frac{g(t)}{(x-t)^2+y^2}dt$$

as the integral solution for the upper half plane $y > 0$

2. A conformal map of the first quadrant of the z plane to the unit disk is given by the composite mapping $w = T(z) = \dfrac{i-z^2}{i+z^2}$ which maps as shown



Then a solution of the Dirichlet problem is given by

$$u(x,y) = Re\left[\frac{1}{2\pi i}\int_C g(\xi)\left(\frac{T(\xi)+T(z)}{T(\xi)-T(z)}\right)\frac{T'(\xi)}{T(\xi)}d\xi\right]$$

Compute $\dfrac{T'(z)}{T(z)} = \dfrac{2zi}{1+z^4}$ Along AC we have $z = it$ for $\infty > t > 0$ and

$$\frac{T(\xi)+T(z)}{T(\xi)-T(z)} = \frac{\frac{i+t^2}{-t^2} + \frac{i-z^2}{i+z^2}}{\frac{i+t^2}{i-t^2} - \frac{i-z^2}{i+z^2}} = \frac{t^2z^2-1}{i(t^2+z^2)}$$

Along $CE, \xi = t, 0 < t < \infty$ and

$$\frac{T(\xi)+T(z)}{T(\xi)-T(z)} = \frac{t^2z^2+1}{i(t^2-z^2)}$$

Substitute these values into the integral formula above to get

$$u(x,y) = Re\left[\frac{1}{2\pi i}\int_\infty^0 g(it)\frac{t^2z^2-1}{i(t^2+z^2)}\left(\frac{-2t}{1+t^4}\right)idt + \frac{1}{2\pi i}\int_0^\infty g(t)\frac{t^2z^2+1}{i(t^2-z^2)}\left(\frac{2ti}{1+t^4}\right)dt\right]$$

To compute the real part of these integral expressions, recall $g(it) = g(0,t)$ and $g(t) = g(t,0)$ are both real valued. Then we need $Im\left(\dfrac{t^2z^2-1}{t^2+z^2}\right)$ and $Im\left(\dfrac{t^2z^2+1}{t^2-z^2}\right)$. Put $z = x+iy$ to get

$$Im\left(\frac{t^2(x+iy)^2-1}{t^2+(x+iy)^2}\right) = Im\left[\frac{t^2(x^2-y^2)-1+2xyt^2i}{t^2+x^2-y^2+2xyi}\right] = \frac{2xy(t^4+1)}{(t^2+x^2-y^2)^2+4x^2y^2},$$

and

$$Im\left(\frac{t^2z^2+1}{t^2-z^2}\right) = \frac{2xy(t^4+1)}{t^2-x^2+y^2+4xy}$$

Substitute these in the integrals above, simplify and get the final form of the solution of the Dirichlet problem as

$$u(x,y) = \frac{2xy}{\pi}\int_0^\infty \frac{tg(0,t)}{(t^2+x^2-y^2)^2+4x^2y^2}dt + \frac{2xy}{\pi}\int_0^\infty \frac{tg(t,0)}{(t^2-x^2+y^2)+4x^2y^2}dt.$$

3. A conformal mapping of $|z-z_0| < R$ to $|w| < 1$ is $w = T(z) = \dfrac{z-z_0}{R}$. With this mapping the solution of the Dirichlet problem can be expressed as

$$u(x,y) = Re\left[\frac{1}{2\pi i}\int_c g(\xi)\left(\frac{T(\xi)+T(z)}{T(\xi)-T(z)}\right)\frac{T'(\xi)}{T(\xi)}d\xi\right]$$

Now on C, the boundary of $|z-z_0| < R$ we have $\xi = z_0 + Re^{it}, 0 \le t \le 2\pi$ and we calculate

$$\frac{T(\xi)+T(z)}{T(\xi)-T(z)}\frac{T'(\xi)}{T(\xi)}d\xi = \frac{Re^{it}+(z-z_0)}{Re^{it}-(z-z_0)}\frac{ie^{it}}{e^{it}}dt.$$

Since $g(\xi) = g(z_0 + Re^{it})$ is real valued, we can write the solution as

$$u(x,y) = \frac{1}{2\pi}\int_0^{2\pi} g(x_0 + R\cos(t), y_0 + R\sin(t))K(x,y,t)dt$$

where

$$K(x,y,t) = Re\left[\frac{(R\cos(t) + x - x_0) + i(R\sin(t) + y - y_0)}{(R\cos(t) - x + x_0) + i(R\sin(t) - y + y_0)}\right]$$

$$= \left\{\frac{R^2 - (x-x_0)^2 - (y-y_0)^2}{R^2 + (x-x_0)^2 + (y-y_0)^2 - 2R(x-x_0)\cos(t) - 2R(y-y_0)\sin(t)}\right\}$$

4. From Example 1 of the text, the integral solution for the right half plane is given by

$$u(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{g(it)x}{x^2 + (t-y)^2}dt,$$

and this was derived by parametrizing the imaginary axis by $\xi = it, \infty > t > -\infty$. Since the boundary function is nonzero only for $-1 \le y \le 1$ we get solution

$$u(x,y) = \frac{x}{\pi}\int_{-1}^{1} \frac{g(0,t)}{x^2 + (t-y)^2}dt = \frac{x}{\pi}\int_{-1}^{1} \frac{1}{x^2 + (t-y)^2}dt$$

In this case, we can evaluate the integral to get

$$u(x,y) = \frac{x}{\pi}\left[\mathrm{Tan}^{-1}\left(\frac{1-y}{x}\right) + \mathrm{Tan}^{-1}\left(\frac{1+y}{x}\right)\right], x > 0, -\infty < y < \infty.$$

5. Use Poisson's integral formula from the text to get

$$u(r\cos(\theta), r\sin(\theta)) = \frac{1}{2\pi}\int_0^{2\pi} \frac{r(\cos(\phi) - \sin(\phi))(1-r^2)}{1 + r^2 - 2r\cos(\phi - \theta)}d\phi$$

6. By Poisson's integral formula we have

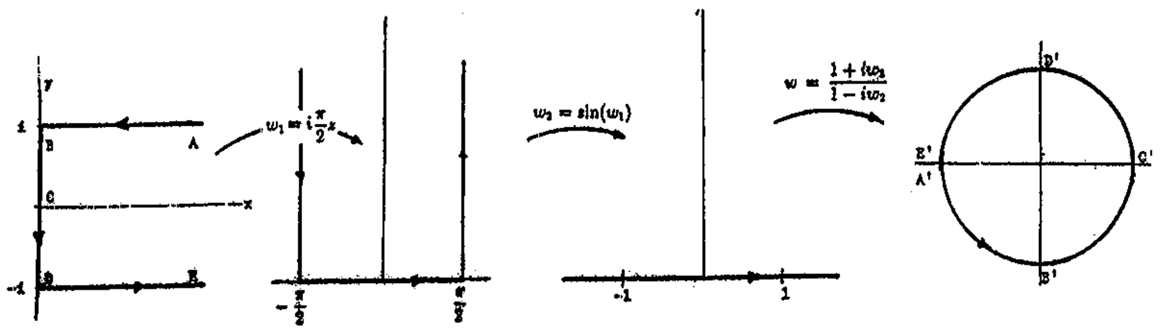
$$u(r\cos\theta, r\sin(\theta)) = \frac{1}{2\pi}\int_0^{\pi/4} \frac{(1-r^2)}{1 + r^2 - 2r\cos(\phi - \theta)}d\phi$$

since $g(e^{i\phi}) = 0$ if $\pi/4 < \phi < 2\pi$

7. We first construct a mapping from the given strip S onto the unit circle. Begin with $w_1 = i\pi z/2$ which rotates the strip 90° counterclockwise and expands it to $-\pi/2 \le Re(w_1) \le \pi/2$. Next apply $w_2 = \sin(w_1)$ which by example 25.6 of Section 25.1 maps the w_1 strip onto the upper half plane. Finally the linear fractional transformation which maps $-1 \to -i, 0 \to 1, 1 \to i$ by $w = \dfrac{i - w_2}{i + w_2}$ maps the upper half plane of the w_2-plane to the unit disk in the w-plane. Note $w(i) = 0$ and the order along the boundary is preserved. The completed mapping is

$$w = T(z) = \frac{i - \sin(\pi iz/2)}{i + \sin(\pi iz/2)} = \frac{1 - \sinh(\pi z/2)}{1 + \sinh(\pi z/2)}$$

which maps the strip $-1 < Im(z) < 1, Re(z) > 0$ onto the unit disk.



The solution of the Dirichlet problem can be expressed as

$$u(x,y) = Re\left[\frac{1}{2\pi i}\int_C g(\xi)\left(\frac{T(\xi)+T(z)}{T(\xi)-T(z)}\right)\frac{T'(\xi)}{T(\xi)}d\xi\right]$$

Since $g(\xi) = 0$ along the upper and lower edges of the strip S, the solution simplifies to

$$u(x,y) = Re\left[\frac{1}{2\pi i}\int_{BD} g(\xi)\left(\frac{T(\xi)+T(z)}{T(\xi)-T(z)}\right)\frac{T'(\xi)}{T(\xi)}d\xi\right]$$

Along $BD, \xi = it, 1 \geq t \geq -1, g(\xi) = g(it) = g(0,t) = 1-|t|$ and $T(\xi) = T(it) = \frac{i+\sin(\pi t/2)}{i-\sin(\pi t/2)}$ To simplify the solution, we calculate $\frac{T(\xi)+T(z)}{T(\xi)-T(z)}\frac{T'(\xi)}{T(\xi)}d\xi$ so we can extract the real part of the integral solution. We find

$$\frac{T'(it)}{T(it)}d(it) = \frac{\pi\cos(\pi t/2)}{1+\sin^2(\pi t/2)}dt, \text{ and } \frac{T(\xi)+T(z)}{T(\xi)-T(z)} = \frac{|T(\xi)|^2 + 2iIm(T(z)\overline{T(\xi)}) + |T(z)|^2}{|T(\xi)|^2 - 2Re(T(z)\overline{T(\xi)}) + |T(z)|^2}$$

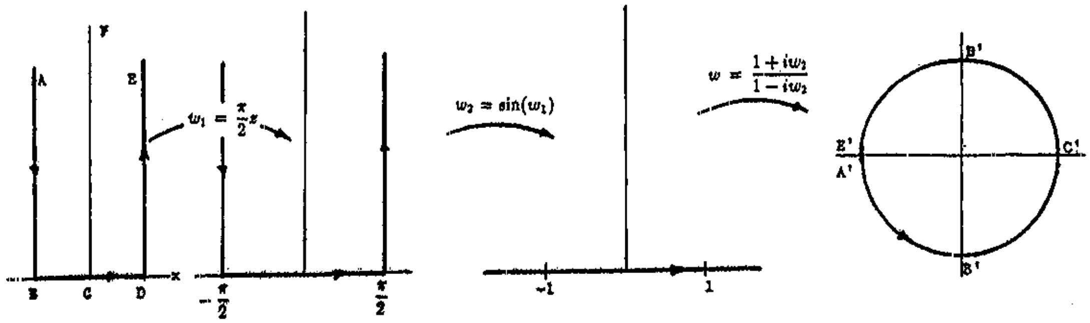
Now recall $|T(\xi)|^2 = 1$ since T map the boundary of the strip on $|w| = 1$, and by looking ahead a bit we see the solution can be written as

$$u(x,y) = \int_1^{-1}\frac{(1-|t|)\cos(\pi t/2)}{1+\sin^2(\pi t/2)}\frac{Im(T(z)\overline{T(it)})}{1-2Re(T(z)\overline{T(it)})+|T(z)|^2}dt$$

8. We first find a mapping of S onto $|w| < 1$ This mapping can be constructed in several steps using examples presented in the text and linear fractional transformations First let $w_1 = \frac{\pi}{2}z$ which expands the strip to $-\pi/2 <$ Im $(w_1) < \pi/2$, Re $(w_1) > 0$. Next, from Example 25 6 of Section 25 1, the mapping $w_2 = \sin(w_1)$ maps the expanded strip to the upper half plane. Next construct a linear fractional transformation mapping $-1 \to -i, 0 \to 1, 1 \to i$ by $w = \frac{1+iw_2}{1-iw_2}$ Since the points $w = -i, 1, i$ all lie on $|w| = 1$, this last mapping maps the real axis to $|w| = 1$ Also $w(i) = 0$, so $Re(w_2) > 0$ maps to $|w| < 1$, interior to the circle $|w| = 1$. Finally the composite mapping

$$w = T(z) = \frac{1+i\sin(\frac{\pi z}{2})}{1-i\sin(\frac{\pi z}{2})} = \frac{1+\sinh(\frac{\pi i z}{2})}{1-\sinh(\frac{\pi i z}{2})}$$

maps S conformally onto $|w| < 1$. These mappings are illustrated below.



Note that they preserve orientation, and the intervals on which g has its various definitions are easily identified. The solution of the Dirichlet problem can be expressed as $u(x,y) = Re[f_1(z) + f_2(z) + f_3(z)]$ where

$$f_1(z) = \frac{1}{2\pi i}\int_{AB} g(\xi)\left(\frac{T(\xi)+T(z)}{T(\xi)-T(z)}\right)\frac{T'(\xi)}{T(\xi)}d\xi;$$

$$f_2(z) = \frac{1}{2\pi i}\int_{BD} g(\xi)\left(\frac{T(\xi)+T(z)}{T(\xi)-T(z)}\right)\frac{T'(\xi)}{T(\xi)}d\xi;$$

$$f_3(z) = \frac{1}{2\pi i}\int_{DE} g(\xi)\left(\frac{T(\xi)+T(z)}{T(\xi)-T(z)}\right)\frac{T'(\xi)}{T(\xi)}d\xi.$$

To simplify the solution, calculate $\frac{T(\xi)+T(z)}{T(\xi)-T(z)}\frac{T'(\xi)}{T(\xi)}d\xi$ along each part of the path of integration. Calculate

$$\frac{T'(z)}{T(z)} = \frac{i\pi\cos(\pi z/2)}{1+\sin^2(\pi z/2)},$$

and

$$\frac{T(\xi)+T(z)}{T(\xi)-T(z)} = \frac{|T(\xi)|^2 + 2iIm(T(z)\overline{T(\xi)}) + |T(z)|^2}{|T(\xi)|^2 - 2Re(T(z)\overline{T(\xi)}) + |T(z)|^2} = \frac{1 + 2iIm(T(z)\overline{T(\xi)}) + |T(z)|^2}{1 - 2Re(T(z)\overline{T(\xi)}) + |T(z)|^2}$$

since $|T(\xi)| = 1$. Along $AB, \xi = -1+it, \infty > t \geq 0; g(\xi) = g(-1,t) = e^{-t}$ and

$$\frac{T'(\xi)}{T(\xi)} = \frac{i\pi\sinh(\pi t/2)}{1+\cosh^2(\pi t/2)}dt$$

With these calculations, the solution can be expressed as

$$u(x,y) = -\frac{1}{2}\int_{\infty}^{0}\frac{e^{-t}\sinh(\pi t/2)}{1+\cosh^2(\pi t/2)}\ \frac{1+|T(z)|^2}{1-2Re(T(z)\overline{T}(-1+it)) + |T(z)|^2}dt$$

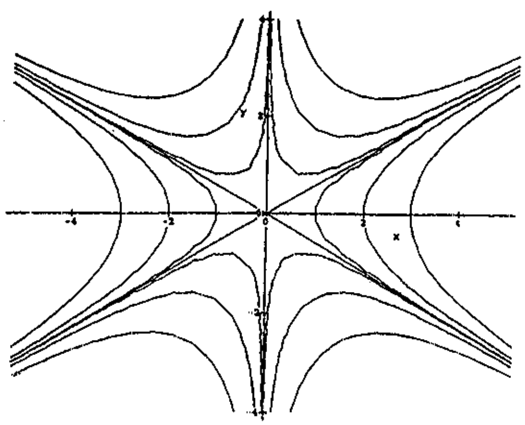
$$+\frac{1}{2}\int_{-1}^{1}\frac{\cos(\pi t/2)}{1+\sin^2(\pi t/2)}\;\frac{1+|T(z)|^2}{1-2Re(T(z)\overline{T(t)})+|T(z)|^2}dt$$

$$+\frac{1}{2}\int_{0}^{\infty}\frac{e^{-t}\sinh(\pi t/2)}{1+\cosh^2(\pi t/2)}\;\frac{1+|T(z)|^2}{1-2Re(T(z)\overline{T(1+it)})+|T(z)|^2}dt$$

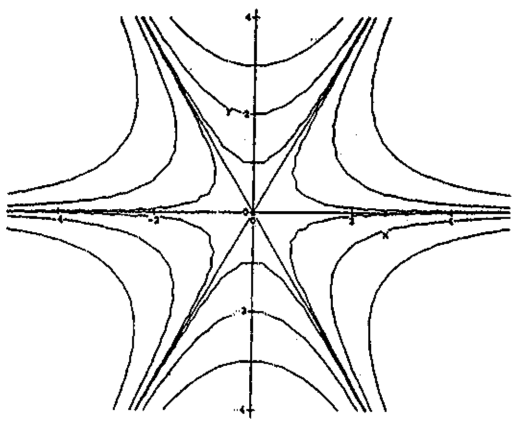
Section 25.5 Complex Function Models in Plane Fluid Flow

1. Write $a = Ke^{i\theta}$, put $z = x+iy$ and compute $f(z) = Ke^{i\theta}[x+iy] = K[x\cos(\theta) - y\sin(\theta)] + iK[x\sin(\theta) + y\cos(\theta)]$. With $f(z) = \phi(x,y) + i\psi(x,y)$ we identify equipotential curves as $\phi(x,y) = K[x\cos(\theta) - y\sin(\theta)] =$ constant. These are lines $y = \cot(\theta)x + b$, of slope $\cot(\theta)$. Streamlines are $\psi(x,y) = K[x\sin(\theta) + y\cos(\theta)] =$ constant, or $y = -\tan(\theta)x + c$. Note that streamlines are perpendicular to equipotential curves since $-\cot(\theta)\tan(\theta) = -1$. To find the velocity, compute $\mathbf{V}(x,y) = \overline{f'(z)} = \overline{a} = Ke^{-i\theta}$, a constant velocity. Since $f'(z) \neq 0$, there are no stagnation points, hence no sinks or sources.

2. To determine equipotential curves and streamlines, put $z = x+iy$ and compute $f(z) = z^3 = (x+iy)^3 = (x^3 - 3xy^2) + i(3x^2y - y^3) = \phi(x,y) + i\psi(x,y)$. We identify equipotential curves as $\phi(x,y) = x^3 - 3xy^2 =$ constant and streamlines as $\psi(x,y) = 3x^2y - y^3 =$ constant. To analyze and sketch these curves consider equipotential curves, $\phi(x,y) = x(x^2-3y^2) = c_1$. For $c_1 = 0$ we get $x = 0$ (the y axis), or $y = \pm\frac{1}{\sqrt{3}}x$ (lines passing through the origin). These lines divide the plane into six wedge shaped regions and on each region both x and (x^2-3y^2) are of one sign. Specifically, on the region $R_1 = \{(x,y)|x>0, -\frac{1}{\sqrt{3}}x < y < \frac{1}{\sqrt{3}}x\}$ we have $x(x^2-3y^2) > 0$, so for $c_1 > 0$ there will be points in R_1 satisfying $x(x^2-3y^2) = c_1$. As we move from region to region (counterclockwise around the origin) the sign of $x(x^2-3y^2)$ alternates in adjacent regions. Some of these equipotential curves and streamlines are sketched below for $c_1 = 1, 8, 27$, and $c_1 = -1, -8, 27$.



Equipotential curves for flow $f(z) = z^3$



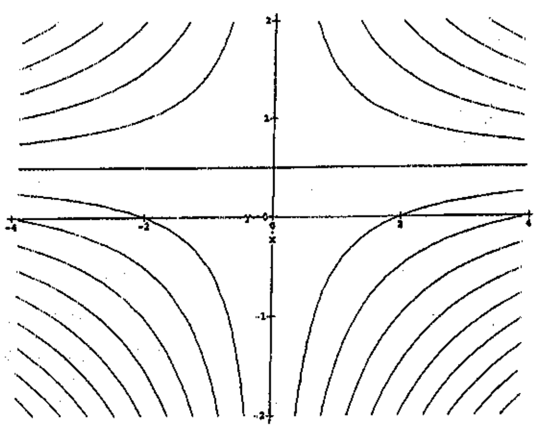
Streamlines for flow $f(z) = z^3$

Since the streamlines have the equation $3x^2y - y^3 = c_2$, we see that these curves can be obtained from the equipotential curves by replacing x by $(-y)$ and y by x This amounts to a 90° clockwise rotation of the above graph of equipotential curves

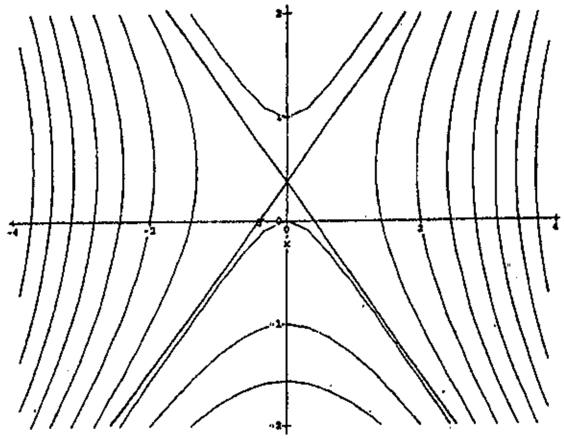
For the velocity of this flow compute $\mathbf{V}(x, y) = \overline{f'(z)} = \overline{3z^2} = 3(x^2 - y^2) - 6xyi$ and identify $u(x, y) = 3(x^2 - y^2)$ and $v(x, y) = -6xy$. Since $f'(0) = 0$, the origin is a stagnation point of the flow By Theorem 2, the flow is both irrotational and solenoidal Calculate $\oint_{|z|=r} -v(x, y)dx + u(x, y)dy = \oint_{|z|=r} 6xydx + 3(x^2 - y^2)dy = 0$ (by Green's Theorem) and we see the origin is neither a source nor a sink. Finally we have on $|z| = r$, that $|\mathbf{V}(z)| = |\overline{3z^2}| = 3r^2$ so the velocity is larger further from the origin In summary, we can envision the potential $f(z) = z^3$ as describing fluid motion along the streamlines $3x^2y - y^3 = c_2$, with the straight lines $y = \pm\sqrt{3}x$ and $y = 0$ acting as barriers of the flow(sides of a container) As particles near the origin, they slow down, and then speed up as they recede from the origin.

3. For $f(z) = \cos(z)$ with $z = x + iy$ compute $f(z) = \cos(x + iy) = \cos(x)\cosh(y) - i\sin(x)\sinh(y) = \phi(x, y) + i\psi(x, y)$. Equipotential curves are $\phi(x, y) = \cos(x)\cosh(y) = c_1$ and streamlines are $\psi(x, y) = -\sin(x)\sinh(y) = c_2$.

4 For $f(z) = z + iz^2$ with $z = x + iy$ compute $f(z) = (x + iy) + i(x + iy)^2 = (x - 2xy) + i(y + x^2 - y^2) = \phi(x, y) + i\psi(x, y)$ Equipotential curves are $\phi(x, y) = x - 2xy = c_1$. If $c_1 = 0$ we get $x = 0$ or $y = 1/2$ For $c_1 \neq 0$ we get the translated hyperbolas



Streamlines are $\psi(x, y) = y + x^2 - y^2 = c_2$ These can be written $x^2 - \left(y - \frac{1}{2}\right)^2 = c_2 - \frac{1}{4}$. For $c_2 = \frac{1}{4}$ we get the lines $y = \frac{1}{2} \pm x$, and for $c_2 \neq \frac{1}{4}$ we have hyperbolas.



For the velocity compute $\mathbf{V}(x) = \overline{f'(z)} = \overline{1+2iz} = (1-2y) - 2xi$, so $u(x,y) = 1-2y$ and $v(x,y) = -2x$. Since $u(x,y) = 1-2y$ and $-v(x,y) = 2x$ satisfy the Cauchy Riemann equations, the flow is both irrotational and solenoidal. Since $f'(i/2) = 0$ we see that $z = i/2$ (or $(x,y) = (0,1/2)$) is a stagnation point. On circles $|z - 1/2| = r$ we have $|\overline{f'(z)}| = 2\sqrt{x^2 + (y-1/2)^2} = 2r$, so the velocity is smaller nearer the point $(0,1/2)$. We can envision the flow as fluid confined to one of the regions between lines $y = 1/2 \pm x$ with fluid motion along the hyperbolic streamlines $x^2 - (y-1/2)^2 = c_2 - 1/4$.

5. Since $f(z) = K\log(z - z_0) = K\ln|z - z_0| + iK\arg(z - z_0) = \phi(x,y) + i\psi(x,y)$ we see that equipotential curves are $\phi(x,y) = K\ln|z - z_0| =$ constant, which are concentric circles centered at z_0; while streamlines are $\psi(x,y) = K\arg(z - z_0) =$ constant, which are half lines emanating from z_0. Now calculate the velocity,

$$\overline{f'(z)} = \overline{\frac{K}{z - z_0}} = \frac{K}{|z - z_0|^2}[(x - x_0) + i(y - y_0)] = u(x,y) + v(x,y)i.$$

For any circle $\gamma : |z - z_0| = r$ we have

$$\oint_\gamma -v(x,y)dx + u(x,y)dy = \int_0^{2\pi}\left[-\frac{K}{r^2}(r\sin(t))(-r\sin(t)) + \frac{K}{r^2}(r\cos(t))(r\cos(t))\right]dt = 2\pi K.$$

Thus $z = z_0$ is a source if $K > 0$ and a sink if $K < 0$

6. Write $f(z) = K\ \mathrm{Log}\left(\frac{z-a}{z-b}\right) = \frac{K}{2}\ln\left(\frac{|z|^2 + |a|^2 - 2Re(a\overline{z})}{|z|^2 + |b|^2 - 2Re(b\overline{z})}\right) + iK\ \mathrm{Arg}\left(\frac{z-a}{z-b}\right) = \phi(x,y) + i\psi(x,y)$. Equipotential curves have equations $\frac{|z|^2 + |a|^2 - 2Re(a\overline{z})}{|z|^2 + |b|^2 - 2Re(b\overline{z})} = c_1$, and streamlines have equations $Arg\left(\frac{z-a}{z-b}\right) = c_2$. To analyze these families of curves in detail let $a = a_1 + a_2 i, b = b_1 + b_2 i, z = x + iy$. Then equipotential curves can be expressed as $(x^2+y^2)(1-c_1) - 2\{a_1 x + a_2 y - c_1(b_2 x + b_2 y)\} + a_1^2 + a_2^2 - c_1(b_1^2 + b_2^2) = 0$. For $c_1 = 1$, we get the line $(a_1 - b_1)x + (a_2 - b_2)y = \frac{1}{2}[(a_1^2 + a_2^2) - (b_1^2 + b_2^2)]$. For $c_1 \neq 1$, we get the equations $\left(x - \frac{a_1 - c_1 b_1}{1 - c_1}\right)^2 + \left(y - \frac{a_2 - c_1 b_2}{1 - c_1}\right)^2 = r_e^2$, where $r_e^2 = \frac{c_1}{(1-c_1)^2}[(a_1 - b_1)^2 + (a_2 - b_2)^2]$.

For $c_1 > 0$, these equations describe circles having centers $\left(\dfrac{a_1 - c_1 b_1}{1 - c_1}, \dfrac{a_1 - c_1 b_2}{1 - c_1}\right)$ and radii $r_e = \dfrac{\sqrt{c_1}}{|1 - c_1|}\sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2}$.
Note that the centers of these circles all lie on the line $(a_2 - b_2)x - (a_1 - b_1)y + a_1 b_2 - a_2 b_1 = 0$, which is the line connecting $a = a_1 + a_2 i$ and $b = b_1 + b_2 i$ in the complex plane. This line containing the centers of all the equipotential curves (for $c_1 \neq 1$) is perpendicular to the equipotential curve obtained for $c_1 = 1$, and these two lines intersect at $\left[\frac{1}{2}(a_1 + b_1), \frac{1}{2}(a_2 + b_2)\right]$ which is the midpoint of the line segment joining a and b in the complex plane

For the streamlines, write

$$\frac{z-a}{z-b} = \frac{(z-a)(\bar{z} - \bar{b})}{|z-b|^2} = \frac{|z|^2 - (a\bar{z} + \bar{b}z) + a\bar{b}}{|z-b|^2}$$

$$= \frac{x^2 + y^2 - [(a_1 + b_1)x + (a_2 + b_2)y] + a_1 b_1 + a_2 b_2}{(x - b_1)^2 + (x - b_2)^2} + i\frac{a_2 b_1 - a_1 b_2 - x(a_2 - b_2) + y(a_1 - b_1)}{(x - b_1)^2 + (x - b_2)^2}$$

Then a curve $\text{Arg}\left(\dfrac{z-a}{z-b}\right) = c_2$ has the form

$$\frac{a_2 b_1 - a_1 b_2 - x(a_2 - b_2) + y(a_1 - b_1)}{x^2 + y^2 - [(a_1 + b_1)x + (a_2 + b_2)y] + a_1 b_1 + a_2 b_2} = c_2$$

For $c_2 = 0$, we get the line $(a_2 - b_2)x - (a_1 - b_1)y + a_1 b_2 - a_2 b_1 = 0$, which connects $a = a_1 + a_2 i$ and $b = b_1 + b_2 i$ in the complex plane For $c_2 \neq 0$, we get the equations

$$\left(x + \frac{a_2 - b_2 - c_2(a_1 + b_1)}{2c_2}\right)^2 + \left(y - \frac{a_1 - b_1 + c_2(a_2 + b_2)}{2c_2}\right)^2 = r_s^2,$$

where

$$r_s^2 = \frac{1 + c_2^2}{4c_2^2}\left[(a_1 - b_1)^2 + (a_2 - b_2)^2\right].$$

These equations describe circles having centers

$$\left(-\frac{a_2 - b_2 - c_2(a_1 + b_1)}{2c_2}, \frac{a_1 - b_1 + c_2(a_2 + b_2)}{2c_2}\right)$$

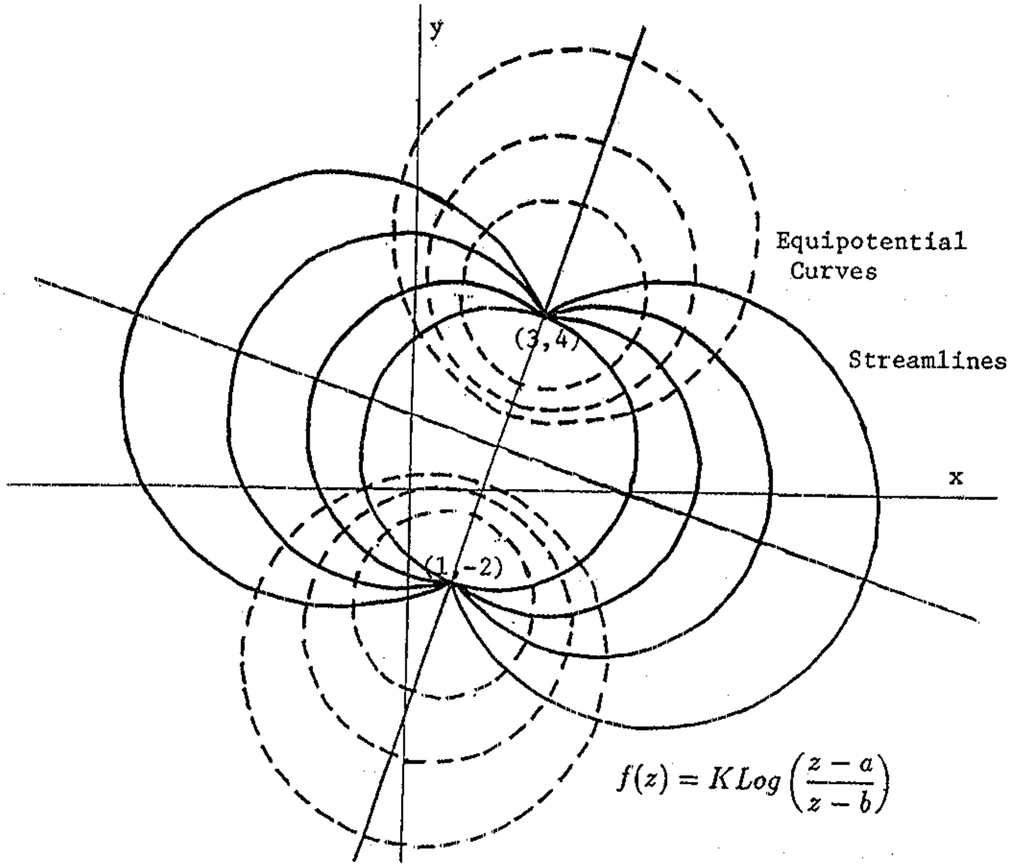
and radii

$$r_s = \frac{\sqrt{1 + c_2^2}}{2|c_2|}\sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2}$$

The centers of all these circles lie on the line

$$(a_1 - b_1)x + (a_2 - b_2)y - \frac{1}{2}[(a_1^2 - b_1^2) + (a_2^2 - b_2^2)] = 0,$$

which is the perpendicular bisector of the line segment through a and b, and each circle passes through both a and b. Some streamlines and equipotential curves for the case $a = 1 - 2i$ and $b = 3 + 4i$ are sketched below.



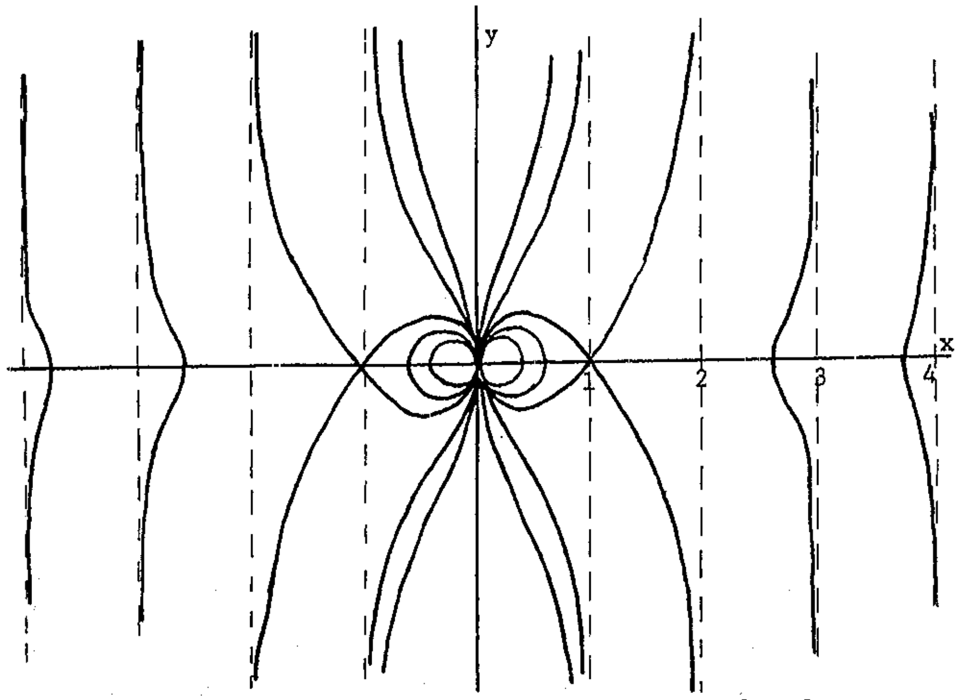
7. Write $f(z) = k\left[x + iy + \frac{1}{x+iy}\right] = \frac{kx(x^2+y^2+1)}{x^2+y^2} + i\frac{ky(x^2+y^2-1)}{x^2+y^2}$, so equipotential curves are

$$\phi(x,y) = \frac{kx(x^2+y^2+1)}{x^2+y^2} = c_1,$$

and streamlines are

$$\psi(x,y) = \frac{ky(x^2+y^2-1)}{x^2+y^2} = c_2.$$

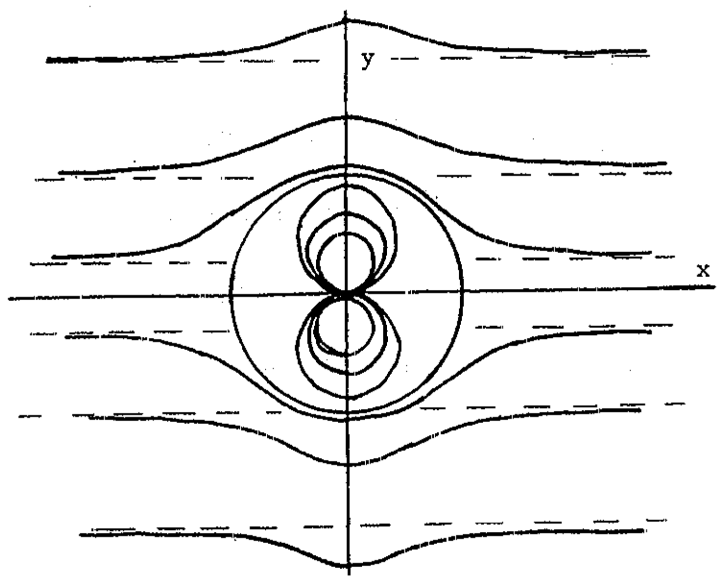
An analysis of the equipotential curves shows that for $c_1 = 0$ we get $x = 0$, the y axis. For $c_1 \neq 0$, let $c_1 = kb$ and solve for $y^2 = -\frac{x(x^2-bx+1)}{x-b}$. From this form, we see that the equipotential curves have vertical asymptotes at $x = b$. For $|b| < 2$, these curves intersect the x axis exactly once at $x = 0$. For $|b| = 2$, they intersect the x axis at $x = 0$ and at $x = \operatorname{sgn}(b)$. For $|b| > 2$, they intersect the x axis at $x = 0$ and at $x = \frac{1}{2}[b \pm \sqrt{b^2-4}]$. Some equipotential curves are sketched below.



For streamlines, we get when $c_2 = 0$, that $y = 0$ (the x axis) or $x^2 + y^2 = 1$, the unit circle. For $c_2 \neq 0$, let $c_2 = 2kc$ and solve for

$$x^2 = -\frac{y(y^2 - 2cy - 1)}{y - 2c} = -\frac{y[y - (c + \sqrt{1 + c^2})][y - (c - \sqrt{1 + c^2})]}{y - 2c}$$

For each $c \neq 0$, these streamlines have horizontal asymptotes at $y = 2c$ and intersect the y axis at $y = 0$. If $c > 0$, there is also an intercept at $y = c - \sqrt{1 + c^2} < 0$. For $c < 0$, there is an intercept at $y = c + \sqrt{1 + c^2} > 0$. Some streamlines are sketched below.



The function f models flow around the unit circle.

8 From the solution of Problem 6, write

$$f(z) = \frac{m-ik}{2\pi}\left[\frac{|z|^2+|a|^2-2Re(a\overline{z})}{|z|^2+|b|^2-2Re(b\overline{z})} + iArg\left(\frac{z-a}{z-b}\right)\right] = \phi(x,y)+i\psi(x,y),$$

so equipotential curves are

$$\phi(x,y) = \frac{m}{2\pi}\left[\frac{|z|^2+|a|^2-2Re(a\overline{z})}{|z|^2+|b|^2-2Re(b\overline{z})}\right] + \frac{k}{2\pi}Arg\left(\frac{z-a}{z-b}\right) = c_1,$$

and streamlines are

$$\psi(x,y) = \frac{m}{2\pi}Arg\left(\frac{z-a}{z-b}\right) - \frac{k}{2\pi}\left[\frac{|z|^2+|a|^2-2Re(a\overline{z})}{|z|^2+|b|^2-2Re(b\overline{z})}\right] = c_2.$$

Calculate $f'(z) = \frac{m-ik}{2\pi}\left[\frac{a-b}{(z-a)(z-b)}\right]$. Since the velocity of this flow is $\overline{f'(z)} = u(x,y)+iv(x,y)$, we have $u(x,y)-iv(x,y) = f'(z)$, and $f'(z)dz = (u-iv)(dx+idy) = (udx+vdy)+i(-vdx+udy)$. Thus $\oint_\Gamma f'(z)dz = \oint_\Gamma(udx+vdy)+i\oint_\Gamma(-vdx+udy)$ and the integral on the left can be computed easily by the residue theorem. We find

$$\oint_\Gamma f'(z)dz = 2\pi i\left(\frac{m-ik}{2\pi}\right)Res\left(\frac{a-b}{(z-a)(z-b)}\right)$$

at poles enclosed by Γ. Since $\frac{a-b}{(z-a)(z-b)}$ has simple poles at both $z=a$ and $z=b$, we find

$$Res\left(\frac{a-b}{(z-a)(z-b)};z=a\right) = 1; \text{ and } Res\left(\frac{a-b}{(z-a)(z-b)};z=b\right) = -1.$$

Thus if Γ encloses only $z=a$,

$$\oint_\Gamma(udx+vdy)+i\oint_\Gamma(-vdx+udy) = k+im,$$

so $z=a$ is a source of strength m and a vortex of strength k. If Γ encloses only $z=b$,

$$\oint_\Gamma(udx+vdy)+i\oint_\Gamma(-vdx+udy) = -k-im,$$

so $z=b$ is a sink of strength m and a vortex of strength $-k$

9 Put $z=x+iy$ to get

$$f(z) = k\left(x+iy+\frac{1}{x+iy}\right)+\frac{ib}{2\pi}\,\text{Log}(x+iy)$$

$$= \frac{kx(x^2+y^2+1)}{x^2+y^2} - \frac{b}{2\pi}Arg(x+iy) + i\left\{\frac{ky(x^2+y^2-1)}{x^2+y^2}+\frac{b}{4\pi}\ln(x^2+y^2)\right\}$$

Equipotential curves are

$$\phi(x,y) = \frac{kx(x^2+y^2+1)}{x^2+y^2} - \frac{b}{2\pi}Arg(x+iy) = c_1,$$

and streamlines are

$$\psi(x,y) = \frac{ky(x^2+y^2-1)}{x^2+y^2} + \frac{b}{4\pi}\ln(x^2+y^2) = c_2$$

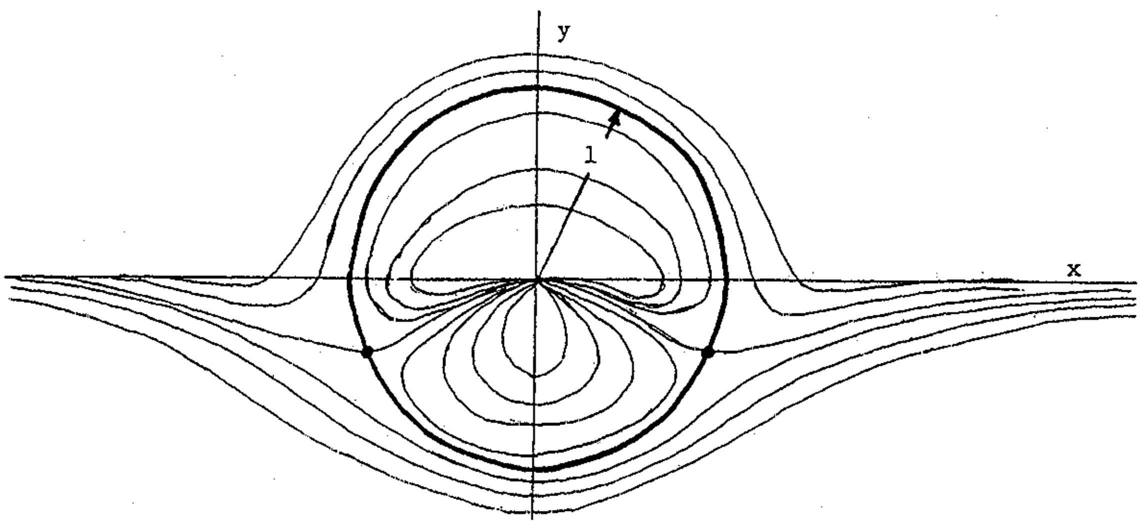
Compute

$$f'(z) = k\left(1-\frac{1}{z^2}\right) + \frac{ib}{2\pi z} = \frac{1}{z^2}\left[kz^2 + \frac{ib}{2\pi}z - k\right],$$

so stagnation points occur where $f'(x) = 0$, which gives

$$z = -\frac{ib}{4\pi k} \pm \sqrt{1 - \frac{b^2}{16\pi^2k^2}}$$

Those points lie on the unit circle symmetrically across the y axis. In polar coordinates, the streamlines can be expressed as $k(r^2-1)\sin(\theta) + \frac{b}{2\pi}r\ln(r) - c_2r = 0$ For $c_2 = 0$, we get $r = 1$, or $\sin(\theta) = \frac{b}{2\pi k}\left(\frac{r\ln(r)}{1-r^2}\right)$. For $c_2 \neq 0$, we get $\sin(\theta) = \frac{b}{2\pi k}\frac{r[\ln(r)-c_2]}{1-r^2}$ Some streamlines are sketched below.



10. Write

$$f(z) = iKa\sqrt{3}\left[\text{Log}\left(z - \frac{ia\sqrt{3}}{2}\right) - \text{Log}\left(z + \frac{ia\sqrt{3}}{2}\right)\right]$$

and compute

$$f'(z) = iKa\sqrt{3}\frac{ia\sqrt{3}}{(z-\frac{ia\sqrt{3}}{2})(z+\frac{ia\sqrt{3}}{2})} = \frac{-3Ka^2\left((\overline{z})^2+\frac{3a^2}{4}\right)}{(z^2+\frac{3a^2}{4})\left((\overline{z})^2+\frac{3a^2}{4}\right)}$$

Parametrize the boundary of $4x^2+4(y-a)^2 = a^2$ by $x = \frac{a}{2}\cos(\theta), y = a+\frac{a}{2}\sin(\theta)$, with $0 \le \theta \le 2\pi$ to get

$$f'(z(\theta)) = \frac{6K[\sin(\theta)+i\cos(\theta)]}{2+\sin(\theta)}$$

Then for $\overline{f'(z)} = u(x(\theta),y(\theta)) + iv(x(\theta),y(\theta))$ we identify

$$u(x(\theta),y(\theta)) = \frac{6K\sin(\theta)}{2+\sin(\theta)};\ v(x(\theta),y(\theta)) = -\frac{6K\cos(\theta)}{2+\sin(\theta)}$$

Now compute the circulation, $\oint_C udx + vdy =$

$$\int_0^{2\pi}\left\{\frac{6K\sin(\theta)}{2+\sin(\theta)}\left[-\frac{a}{2}\sin(\theta)\right] - \frac{6K\cos(\theta)}{2+\sin(\theta)}\left[\frac{a}{2}\cos(\theta)\right]\right\}d\theta = -3Ka\int_0^{2\pi}\frac{1}{2+\sin(\theta)}d\theta < 0,$$

since the integrand is positive. Since the circulation of this flow about $(0,a)$ is not zero, the flow is not irrotational.
Next compute the flux, $\oint -vdx + udy =$

$$\int_0^{2\pi}\left\{\frac{6K\cos(\theta)}{2+\sin(\theta)}\left[-\frac{a}{2}\sin(\theta)\right] + \frac{6K\sin(\theta)}{2+\sin(\theta)}\left[\frac{a}{2}\cos(\theta)\right]\right\}d\theta = 0,$$

so the flow is solenoidal.

11. From Problem 10, find

$$[f'(z)]^2 = \frac{9a^4K^2}{(z-\frac{ia\sqrt{3}}{2})^2(z+\frac{ia\sqrt{3}}{2})^2}$$

By Blasius's theorem, the thrust of the fluid outside the barrier $4x^2+4(y-a)^2 = a^2$ is the vector $A\mathbf{i}+B\mathbf{j}$ where

$$A - Bi = \frac{1}{2}i\rho\oint_\gamma [f'(z)]^2dz = \frac{i\rho}{2}\oint_\gamma \frac{9a^4K^2}{(z-\frac{ia\sqrt{3}}{2})^2(z+\frac{ia\sqrt{3}}{2})^2}dz =$$

$$-\pi\rho Res\left([f'(z)]^2; z = \frac{ia\sqrt{3}}{2}\right) = -\pi\rho(9a^4K^2)\frac{d}{dz}\left[(z+\frac{ia\sqrt{3}}{2})^{-2}\right]\Bigg|_{z=ia\sqrt{3}/2}$$

$$= -9\pi a^4K^2\rho[-2(ia\sqrt{3})^{-3}] = -\frac{18\pi a^4K^2\rho}{3\sqrt{3}a^3}i$$

Thus the vertical component of thrust is $B = 2\sqrt{3}\pi a\rho K^2$.

Chapter Twenty Six - Counting and Probability

Section 26.1 The Multiplication Principle

1 (a) A typical outcome can be represented as a string $a_1 \cdots a_{12}$ of symbols, in which, for $j = 1, 2, \cdots, 7$, each a_j is an H or T (head or tail for each of the coin flips), and each a_j for $j = 8, 9, \cdots, 12$ is one of the integers 1, 2, 3, 4, 5, or 6, showing the number of dots on the face of the die that came up on that toss

(b) The number of outcomes is $2^7 6^5$, or 995,328 This is because each coin flip has 2 outcomes, and there are 7 coin flips for 2^7 outcomes, and each die has 6 outcomes, and there are five dice tossed.

2. The number of seconds is the number of possible choices There are 40 ways to pick a chest, and 6 drawers in each chest, for 40(6), or 240 outcomes

3. The number of outcomes is $6^{10}(15)$, since there are 6 possible outcomes of each of the 10 dice, and there are 15 integers from 7 to 21 inclusive.

4. The man can choose any one of seven pairs of shoes, one of twelve pairs of trousers, one of fifteen shirts, one of ten ties, one of seven jackets/sweaters, one of two coats and one of four hats. Here we aggregate the jackets and sweaters into a collection of seven objects, of which one is chosen, because the man takes a sweater or jacket, but not both. The number of outfits is 7(12)(15)(10)(7)(2)(4), of 705,600.

If an outfit can include a jacket and/or sweater, than we must allow for a jacket and sweater ($(4)(3) = 12$ ways to take one of each), or a jacket and no sweater (3 ways to pick one jacket and no sweater), or a sweater and no jacket (4 ways to do this), for a total of $12 + 3 + 4 = 19$ ways to form the sweater/jacket part of the outfit Similarly, we must allow for a coat and no hat (2 ways), or a hat and no coat (4 ways), or a hat and coat ($2(4) = 8$ ways), for a total of $2 + 4 + 8 = 14$ ways to handle the hat and/or coat. Now the number of outfits that can be formed is 7(12)(15)(10)(19)(14), or 3,351,600

5. The number of codes is $4^3 5^3$, or 8,000 This is because any of four numbers can be selected for each of the first three entries in the code, and any of five letters for the last three

6. The number of tables is 6(15)(4), or 360 tables.

7. There are 7(15)(8)(17) sets, or 14,280 sets.

8. An outcome is a string *abcd* of four numbers, each of which can be any integers from 1 through 50 inclusive. There are 50^4, or 6,250,000 outcomes.

9. The number of committees is 7(11)(6)(14), or 6,468

10. An outcome is a string $a_1 \cdots a_{20}$, where each a_j is a string $b_{j1}b_{j2} \cdots b_{j6}$ and each b_{ji} is an H (if that flip came up heads) or a T (for tails) Each string a_j can occur 2^6 ways (two possible outcomes on each of six flips). Since there are twenty repetitions of these six flips, the total number of outcomes is $(2^6)^{20}$, or 2^{120} This is

$$1,329,227,995,784,915,872,903,807,060,280,344,576.$$

Section 26.2 Permutations

1. There are 9!, or 362,880, arrangements

2. There are 26 lower case letters This means that there are 26 choices for the first letter of the code, 25 for the second (because the code consists of distinct letters), 24 for the third, and so on down to the seventeenth letter The number of codes is therefore $26(25)(24)\cdots(10)$, or 156,000.

3. There are 9! such ID numbers.

4. The first plan allows for 5!, or 120 passwords. In the second plan, there are five symbols available for each of the places in the five-symbol password, for a total of $5^5 = 3,125$ passwords.

5. (a) There are $7! = 5,040$ arrangements.

(b) Now the first letter must be a, so there are six letters left to place in any order, for a total of $6! = 720$ arrangements

(c) With a first and g fifth, there are five symbols left to put in any order, and this can be done in $5! = 120$ ways.

6. There are $n!$ different ID numbers that can be formed. To accommodate 20,000 people, we need to choose n so that $n! \geq 20,000$. After a little experimentation, we find that $8! = 40,320$, more than enough. But $7! = 5,040$, too small, so the smallest number of symbols we can get by with is $n = 8$

If we need to accommodate 1,000,000 people, we need $n! \geq 1,000,000$. Again, try some values of n and we find that $10! = 3,628,800$, more than enough. But $9! = 362,880$, too small, so the smallest integer we can use for 1,000,000 people is $n = 10$

7. Regardless of pattern, the point is that six objects are being chosen in some order, so there are $6! = 720$ patterns.

8. There are $15! = 1,307,674,368,000$ arrangements. The fact that the pattern is a triangle is irrelevant.

9. There are $12! = 479,001,600$ outcomes.

10. There are ten letters from a to l inclusive. If three of the places are fixed, there are seven letters available to put in any order to fill out the string, and the number of ways to do this is 7!.

11. There are six even numbers between 1 and 12 inclusive, so there are 6! ways to choose them.

12.

n	$n!$	$(\frac{n}{e})^n \sqrt{2\pi n}$	% Error
1	1	0.92214	7.79
2	2	1.919	4.05
3	6	5.8362	2.73
4	24	23.506	2.06
5	120	118.02	1.65
6	720	710.08	1.38
7	5040	4980.4	1.18
8	40320	39902	1.04
9	362880	$3.5940(10)^5$	0.92
10	3628800	$3.5987(10)^6$	0.83
11	39916800	$3.9616(10)^7$	0.75
12	479001600	$4.7569(10)^8$	0.69
13	6227020800	$6.1872(10)^9$	0.64
14	87178291200	$8.6661(10)^{10}$	0.59
15	1307674368000	$1.3004(10)^{12}$	0.55
16	20922789888000	$2.0814(10)^{13}$	0.52
17	355687428096000	$3.5395(10)^{14}$	0.49
18	6402373705728000	$6.3728(10)^{15}$	0.46
19	121645100408832000	$1.2111(10)^{17}$	0.44
20	2432902008176640000	$2.4228(10)^{18}$	0.41

Section 26.3 Choosing r Objects From n Objects

1 Since the order of the draw is important in determining the prize, there are ${}_{25}P_7$ outcomes. This is

$${}_{25}P_7 = \frac{25!}{18!} = 25(24)(23) \cdots (19) = 2,422,728,000.$$

If each outcome has the same reward, order no longer is important and the number of outcomes is ${}_{25}C_7$, which is

$$\frac{25!}{18!7!} = \frac{1}{7!}(2422728000) = 480,700.$$

2 The number of ballots is ${}_{16}P_5 = 524,160$

3 Since it is best to win the draw early, order counts and the number of possibilities is ${}_{22}P_6 = 53,721,360$

4 (a) The number of choices is ${}_{20}P_3 = 6,840$

(b) If the list begins with 4, there are only two numbers to choose from the remaining nineteen. There are ${}_{19}P_2 = 342$ ways to do this, with order. This is 5% of the 6,840 choices. This percentage does not depend on which particular number is put into the first slot

(c) 5% of the choices end with 9 (or any of the other numbers)

(d) If the first number is 3 and the last is 15, then there are 18 choices for the middle number, hence 18 arrangements beginning with 3 and ending with 15

5 Without order, the number of hands is ${}_{52}C_{10} = 15,820,024,220$. Taking order into account, the number of ten card hands is ${}_{52}P_{10} = 57,407,703,889,536,000$.

6 Disregarding order, the number of nine man lineups is ${}_{17}C_9 = 24,310$ If order makes a difference, the number is ${}_{17}P_9 = 8,821,612,800$

7 The number of choices is ${}_{20}C_4 = 4,845$

8. The number is ${}_{40}C_{12} = 5,586,853,480$

9 With order, the number of hands is ${}_{52}P_5 = 311,875,200$. Without order, the number is ${}_{52}C_5 = 2,598,960$.

10 With order, the number is ${}_{17}P_7 = 98,017,920$ Disregarding order, the number is ${}_{17}C_7 = 19,448$

11 Compute

$$\binom{n}{n-k} = \frac{n!}{(n-k)!(n-(n-k))!}$$

$$= \frac{n!}{k!(n-k)!} = \binom{n}{k}$$

12 (a) The tree diagram is shown below

(b) The outcomes with the second and third coins heads all have the appearance $aHHbc$, in which a, b and c can each be either an H or a T. From the tree diagram, it is easy to list all such outcomes by following along branches and looking for paths in which the second and third vertices are both H These outcomes are

$$HHHHH, HHHHT, HHHTH, HHHTT,$$

$$THHHH, THHHT, THHTH, THHTT$$

Note that it is reasonable that there should be eight such outcomes, since we can put either H or T in the a, b and c slots, for $2^3 = 8$ possibilities

(c) These outcomes all have the form $HabTc$, so again we expect 8 such outcomes. They are

$$HHHTH, HHHTT, HHTTH, HHTTT,$$

$$HTHTH, HTHTT, HTTTH, HTTTT.$$

(d) This question can have two answers, depending on interpretation

If we require exactly two heads and exactly two tails, there are no such outcomes, since there are five coin flips.

The other alternative is that there are at least two heads and at least two tails (since, if there are at least two tails, then there are two tails, plus perhaps a third, fourth or fifth tail). These outcomes are (reading from the top of the tree diagram)

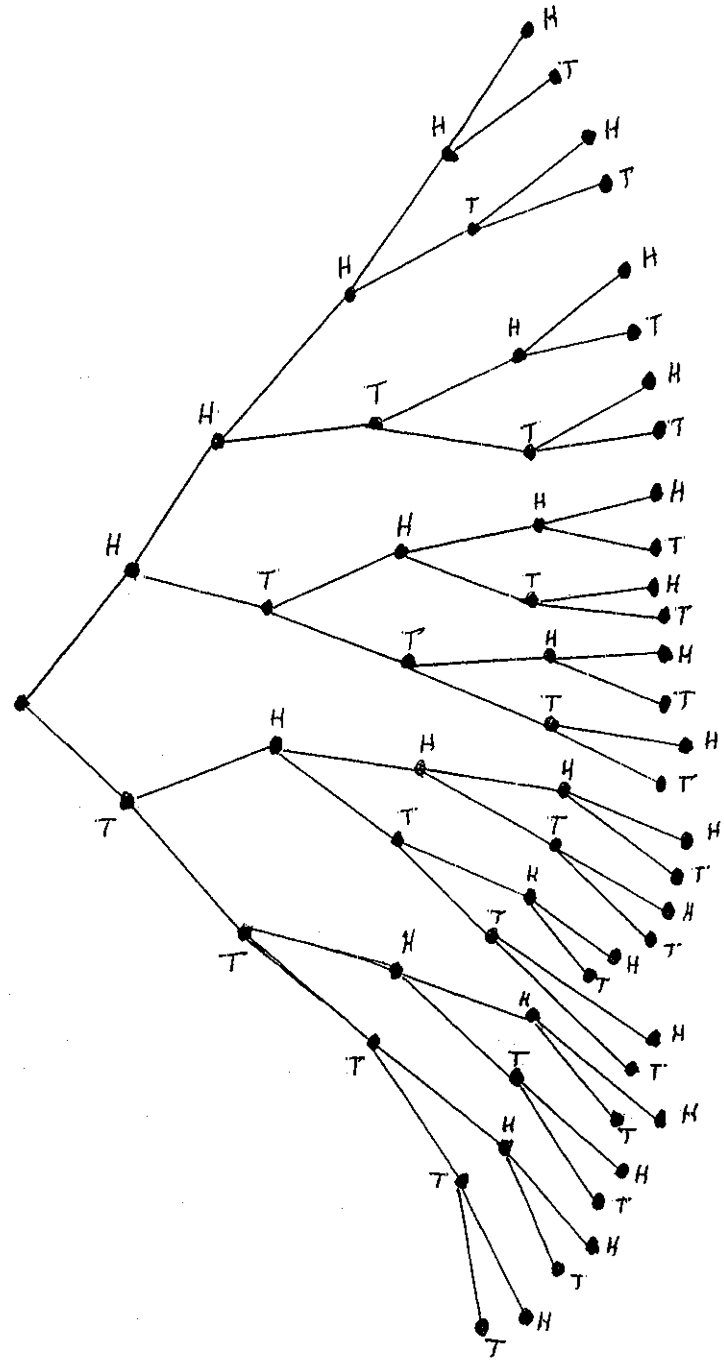
$$HHHTT, HHTHT, HHTTH, HHTTT, HTHHT, HTHTH,$$

$$HTHTT, HTTHH, HTTHT, HTTTH, THHHT, THHTH,$$

$$THHTH, THTHH, THTHT, THTTH, TTHHH,$$

$$TTHHT, TTHTH, TTTHH.$$

Notice that ten begin with H and ten with T. We would be suspicious of a miscount if this symmetry were violated.



Tree Diagram for Problem 12

13. (a) The tree diagram is shown below. Because there are 216 outcomes, some ingenuity must be invoked in the drawing.

(b) The following outcomes (triple of numbers on the faces of the dice) have even sums The list was compiled by starting at vertex one (first die came up 1) and moving counterclockwise through the other verticces in the tree diagram:

$$112, 114, 116, 121, 123, 125, 132, 134, 136,$$

$$141, 143, 145, 152, 154, 156, 161, 163, 165,$$

$$211, 213, 215, 222, 224, 226, 231, 233, 235,$$

$$242, 244, 246, 251, 253, 255, 262, 264, 266,$$

$$312, 314, 316, 321, 323, 325, 332, 334, 336,$$

$$341, 343, 345, 352, 354, 356, 361, 363, 365,$$

$$411, 413, 415, 422, 424, 426, 431, 433, 435,$$

$$442, 444, 446, 451, 453, 455, 462, 464, 466,$$

$$512, 514, 516, 521, 523, 525, 532, 534, 536,$$

$$541, 543, 545, 552, 554, 556, 561, 563, 565,$$

$$611, 613, 615, 622, 624, 626, 631, 633, 635,$$

$$642, 644, 646, 651, 653, 655, 662, 664, 666$$

This list has 108 outcomes This makes sense. In three rolls of the dice, there are $6^3 = 216$ outcomes, and the sum should be even exactly half of these.

(c) We want all outcomes abc in which a and c are odd. There are three odd numbers that can go in the first and third slots, and any of the six numbers can go in the second, so there are $3(3)(6) = 54$ such outcomes This will serve as a partial check when we think we have the list all of them. It is efficient to list the outcomes with first and third die odd as triples xby, in which b can be any of 1, 2, 3, 4, 5 or 6. The triples are

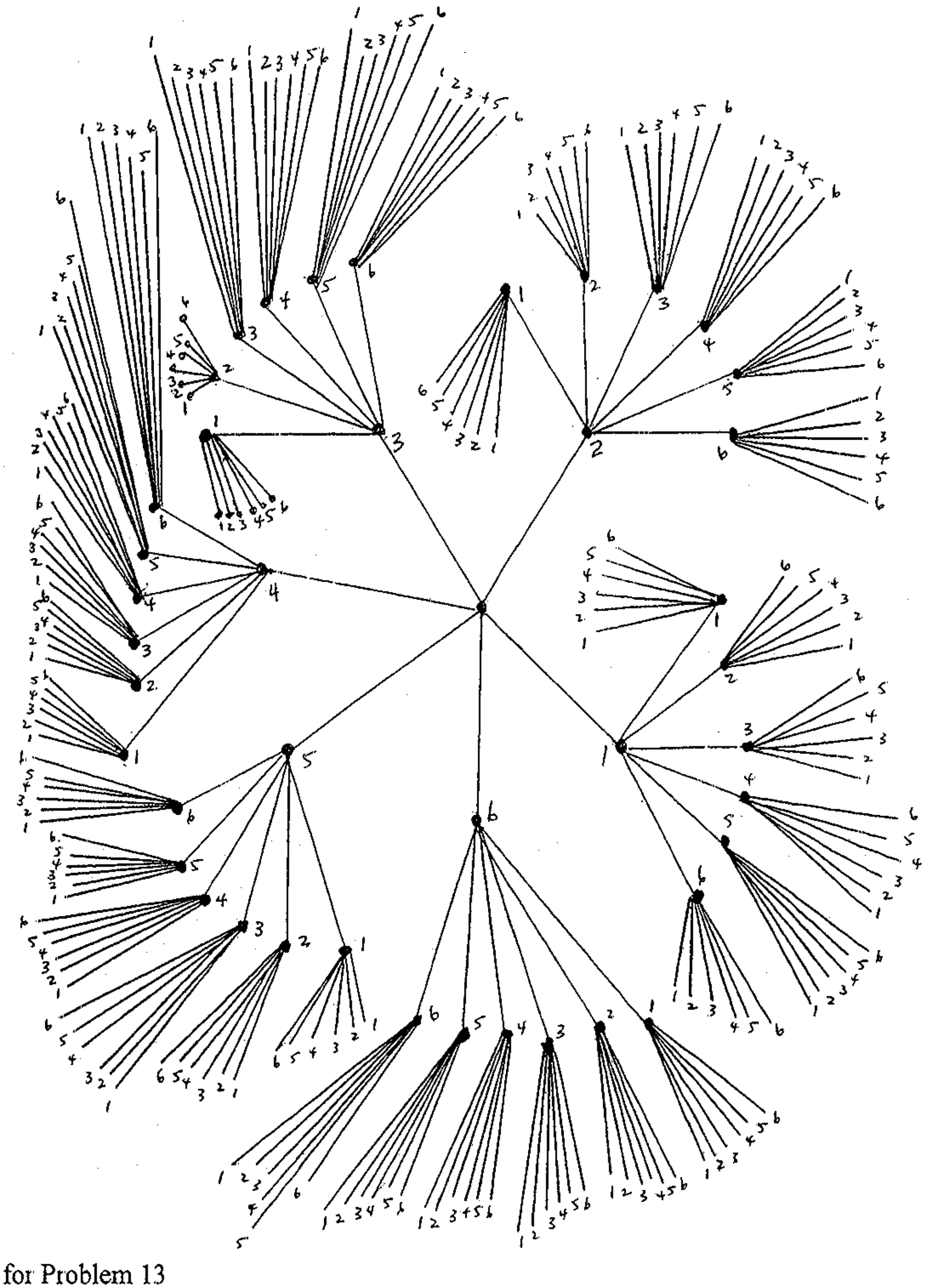
$$1b1, 1b3, 1b5, 3b1, 3b3, 3b5, 5b1, 5b3, 5b5$$

There are 54 such triples.

(d) There are $2(6)(3) = 36$ such outcomes. They are all of the form xby, in which x must be 2 or 5, b can be any of the die's numbers, and c must be 3, 4 or 6. We can list them efficiently as

$$2b3, 2b4, 2b6, 5b3, 5b4, 5b6,$$

in which b can be any of 1, 2, 3, 4, 5 or 6.



Tree Diagram for Problem 13

14. A tree diagram is shown below.

(b) We want the strings $Hxyz$, where x can be either H or T, and $y+z$ is odd Those beginning HH can have the dice come up in the pairs:

$$12, 14, 16, 32, 34, 36, 52, 54, 56,$$

$$21, 23, 25, 41, 43, 45, 61, 63, 65.$$

Those beginning with HH can have the same dice pairs. There are 36 outcomes with first coin toss heads and the sum of the dice odd.

(c) The outcomes in which the first die comes up 1 or 5 can be listed:

$$HH1x, HH5x, HT1x, HT5x,$$

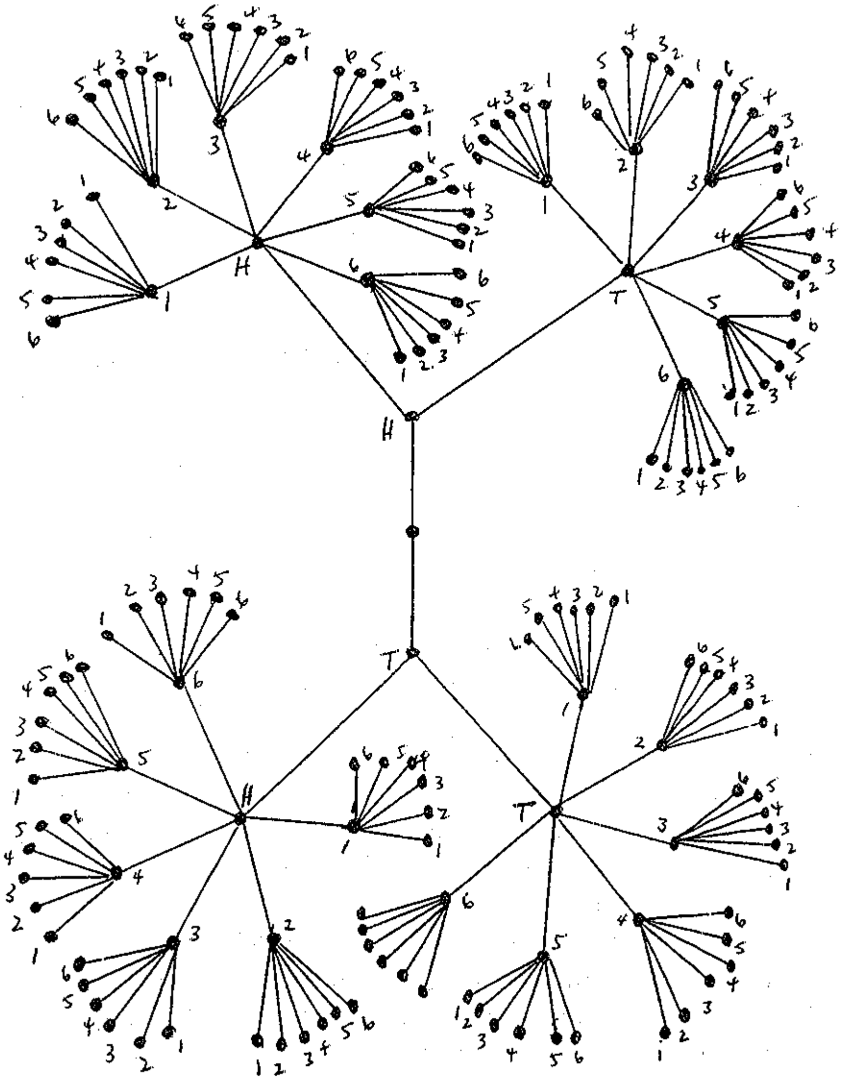
$$TH1x, TH5x, TT1x, TT5x,$$

in which x can be 1, 2, 3, 4, 5 or 6 There are 48 such outcomes.

(d) We want both coins to come up tails, and the second die to be 1, 3, or 5. There are eighteen such outcomes, and each has one of the forms:

$$TTx1, TTx3 \text{ or } TTx5,$$

where x can be 1, 2, 3, 4, 5 or 6



Tree Diagram for Problem 14

Section 26.4 Events and Sample Spaces

1. If we roll four dice, a typical outcome has the form $xyzk$, where each letter can be any of the numbers 1, 2, 3, 4, 5 or 6 There are 6^4 outcomes

Event A consists of all outcomes in which the dice total 9. The four dice total 9 if they come up any of the following ways (in any order):

$$6,1,1,1;5,2,1,1;4,2,2,1;4,3,1,1 \text{ or } 3,3,2,1$$

There are 4 ways to roll a six and three ones There are twelve ways of rolling a five, a two and two ones, as well as of rolling a four, three and two ones, or a two, one and two threes. Therefore A has 40 outcomes in it.

B consists of all outcomes in which each roll is either a 1 or 2. There are sixteen outcomes in B. They are:

$$1111, 2222, 1112, 1211, 1121, 2111,$$

$$1122, 1212, 1221, 2211, 2121, 2112,$$

$$2221, 2212, 2122, 1222.$$

2. If seven cards are drawn from fifty two cards, without regard to order, then the number of possible hands is ${}_{52}C_7 = 133,784,560$.

Event W consists of all outcomes in which each card is a 2 or 3 Then the hand must be four $2's$ and three $3's$, or four $3's$ and three $2's$. Each kind of hand can happen four ways, so W has 8 outcomes in it.

K consists of hands having only kings and/or aces. We are therefore choosing 7 cards from 8 cards, without regard to order. There are 8 ways to do this (by excluding one card). Then K has 8 outcomes in it.

M consists of hands with exactly one ace. We are therefore choosing 6 cards from 48 cards (deck with the aces removed), without regard to order, and there are ${}_{48}C_6$ ways to do this. Then we can pick any of the four aces, for a total of $4({}_{48}C_6) = 49,086,048$ outcomes in M.

3. Pick four distinct letters from the English alphabet, without regard to order. There are ${}_{26}C_4 = 14,950$ ways to do this.

A consists of outcomes having an a and a z. We are therefore left to pick 2 letters from 24, and there are ${}_{24}C_2 = 276$ ways to do this, hence 276 outcomes in A

W consists of outcomes using only letters from a through g inclusive Now we are picking four letters from seven letters, without regard to order, so the number of outcomes in W is ${}_7C_4 = 35$.

4. There are ${}_{26}P_5$ ways to choose five letters from the English alphabet, taking order into account This is $7,893,600$ ways

I consists of outcomes beginning with zk There are therefore three letters left to choose from 24 letters, and the number of ways to do this is ${}_{24}P_3 = 12,144$

W consists of all outcomes with third letter e, and so has ${}_{25}P_4 = 303,600$ outcomes in it.

P consists of all outcomes having $chjk$ as the first four letters This leaves 22 choices for the fifth letter, hence 22 outcomes in P

5 Flip three coins and roll three dice The number of outcomes is $2^3 6^3 = 1728$

C consists of all outcomes having three heads and dice totalling 15. There is only one way to get three heads. For dice totalling 15, we could roll $5,5,5$ (one way to do this), or $6,4,5$ (six ways to do this), or $6,6,3$ (three ways to do this), for a total of 10 outcomes in C

D consists of all outcomes with at least two heads and dice totalling at least 15 (so there can be two or three heads, and dice totalling 15, 16, 17 or 18). There are four ways to roll exactly two heads, or three heads There are 10 ways for the dice to total 15. For a total of 16, we can roll 6, 6, 4 (three ways to do this) or 6, 5, 5 (three ways). For a total of 17, we can roll 6, 6, 5 (three ways) And for a total of 18 we must roll 6, 6, 6 (one way). By the multiplication principle, there are therefore $(4)(10+3+3+3+1)=80$ outcomes in D.

E consists of all outcomes with exactly one head (therefore two tails), and showing only a 5 or 6 on each die. There are 3 ways of rolling exactly one head For all the dice to have only fives or sixes, the possibilities are:

$$5,5,5 \text{ (one way)}$$

$$6,6,6 \text{ (one way)}$$

$$5,5,6;5,6,5;6,5,5 \text{ (three possibilities)}$$

$$6,6,5;6,5,6;5,6,6 \text{ (three possibilities)}.$$

There are therefore eight ways the dice can come up with only fives and/or sixes. By the multiplication principle, the number of outcomes in E is $(3)(8)=24$

6. There are 11 ways to pick an integer from 0 through 10 inclusive. Assume we pick the letters without regard to order. Then there are ${}_{26}C_3 = 2,600$ ways to do this. The total number of outcomes is $(11)(2,600)=28,600$.

M is the event that the number chosen is even There are six even numbers in the given range, so M consists of $(6)(2600)=15,600$ outcomes This is more than half the total number of outcomes because there are only five odd numbers in the given range

K is the event that the number is 3, 5 or 7, and one of the letters is w. Since one letter is w, we are picking two letters from the remaining twenty five, and there are ${}_{25}C_2 = 300$ ways to do this The number of outcomes in K is $(3)(300)=900$

If we pick the letters taking order into account, replace ${}_bC_a$ with ${}_bP_a$ in the relevant places

7 If five dice are rolled, there are $6^5 = 7,776$ outcomes

U is the event that the dice total at least 28. This means that the dice can total 28, 29 or 30 The dice can come up:

$$6,6,6,6,4 \text{ (five ways to do this)},$$

$$6,6,6,6,5 \text{ (five ways)}$$

$$6,6,6,6,6 \text{ (one way)}$$

or

$$6,6,6,5,5 \text{ (there are } {}_5C_2 = 10 \text{ ways to do this.}$$

There are therefore $5+5+1+10=21$ outcomes in U.

K is the event that the dice total no more than 7. They can therefore total 5, 6 or 7 The possibilities are:

$$1,1,1,1,1 \text{ (one way)}$$

$$1,1,1,1,2 \text{ (five ways)}$$

$$1,1,1,1,3 \text{ (five ways)}$$

$$1,1,1,2,2 \text{ } ({}_5C_2 = 10 \text{ ways})$$

The number of outcomes in K is $1+5+5+10=21$.

By symmetry, we should not be surprised that the number of ways of rolling 7 or less is the same as the number of ways of rolling 28 or more, since 7 is two more than the minimum possible, and 28 is two less than the maximum possible

8. Twenty three coins are flipped. There are $2^{23} = 8,388,608$ outcomes.

R is the event that at least twenty heads come up. This means that the number of heads can be exactly 20, 21, 22 or 23. There are ${}_{23}C_{20}$ ways to get exactly twenty heads (pick twenty of the twenty three flips and designate them heads, make the others tails). In total, the number of ways at least twenty heads can come up is

$${}_{23}C_{20} + {}_{23}C_{21} + {}_{23}C_{22} + {}_{23}C_{23} = 2047$$

This is the number of outcomes in R.

Q is the event that the last nineteen flips all came up heads. The first four flips are still random, so there are $2^4 = 16$ outcomes in Q

9. If sixteen cards are drawn at random from a standard deck, without regard to order, the number of outcomes is ${}_{52}C_{16} = 10,363,194,502,115$

Y is the event that every card drawn was a face card or ace. There are four jacks, queens and kings, for twelve face cards, and four aces. There is only one way to draw (without regard to order) sixteen cards and have them all face cards or aces. Y has one outcome in it.

M is the event that four jacks, four kings and three aces were drawn. This fixes eleven of the sixteen cards. There is only one to pick four jacks, and only one way to pick the four kings (disregarding order). There are four ways to pick three aces. The other five cards can be any cards drawn from the remaining forty cards in the deck, and there are ${}_{40}C_5 = 658,008$ ways to do this. The number of outcomes in M is $(4)(658,008) = 2,632,032$

10. We are really just picking four integers from $1, 2, \ldots, 10$. Suppose the order of the choice does not matter. Then the number of outcomes is ${}_{10}C_4 = 210$. (If we want to take order into account, the number is ${}_{10}P_4 = 5,040$.

Y is the event that the numbers on the sides chosen totals exactly 14. This can happen if the sides chosen are

$$1,2,3,8; 1,2,4,7; 1,2,5,6;$$

$$1,3,4,6 \text{ or } 2,3,4,5$$

The number of outcomes in Y is 5.

P is the event that sides 2, 3 and 7 were picked. This leaves seven ways to pick the fourth side, so there are 7 outcomes in P.

Section 26.5 The Probability of An Event

1 The number of outcomes is $2^5 = 32$.

(a) There are ${}_5C_2 = 10$ ways of getting exactly two heads. Therefore

$$\Pr(\text{exactly two heads}) = \frac{10}{32} = \frac{5}{16}$$

(b) We get at least two heads if we get two, three, four or five heads. There are ten ways of getting exactly two heads. The number of ways of getting exactly three heads is also ten (this is the same as the number of ways of getting exactly two tails). There are five ways of getting exactly four heads, and one way of getting five heads. Therefore

$$\Pr(\text{at least two heads}) = \frac{10+10+5+1}{32} = \frac{26}{32} = \frac{13}{16}$$

2. If four dice are rolled, there are $6^4 = 1,296$ outcomes.

(a) The number of ways of getting exactly two $4's$ is $25({}_4C_2) = 150$. This is because we can get exactly two $4's$ by picking any two of the dice (choose two out of four) and imagining they came up 4, while each of the other two can have any of five outcomes $(1, 2, 3, 5, 6)$. The probability of getting exactly two $4's$ is therefore

$$\Pr(\text{exactly two } 4's) = \frac{150}{1296} = \frac{23}{216}.$$

(b) The number of ways of getting exactly three $4's$ is $5({}_4C_1) = 5(4) = 20$, so

$$\Pr(\text{exactly three } 4's) = \frac{20}{1296} = \frac{5}{324}.$$

(c) We can get at least two $4's$ if we get exactly two, exactly three or four heads. We know that there are 150 ways of getting exactly two $4's$, and 20 ways of getting exactly three $4's$. Clearly there is one way of getting four $4's$. Then

$$\Pr(\text{at least two } 4's) = \frac{150+20+1}{1296} = \frac{19}{144}.$$

(d) For the dice to total 22, the numbers showing on the four dice could be

$$6, 6, 6, 4 \text{ (happens four ways)}$$

or

$$6, 6, 5, 5 \text{ (6 ways)}.$$

Then

$$\Pr(\text{dice total } 22) = \frac{10}{1296} = \frac{5}{648}.$$

3 The number of ways of drawng two cards without regard to order is ${}_{52}C_2 = 1,326$.

(a) Since we get two kings if we pick two of the four kings, there are ${}_4C_2 = 6$ ways of getting two kings Then

$$\Pr(\text{two kings}) = \frac{6}{1326} = \frac{1}{221}.$$

(b) The aces and face cards number 16 of the 52 cards If none of the two cards drawn is a face card or ace, then these cards are drawn from the 36 remaining cards. Without order, there are ${}_{36}C_2 = 630$. Then

$$\Pr(\text{no ace or face card}) = \frac{630}{1326} = \frac{105}{221}.$$

4 In choosing, with order, four letters from the English alphabet, the number of outcomes is ${}_{26}P_4 = 14,950$

(a) If the first letter is q, we are picking, with order, 3 letters from the remaining 25 letters There are ${}_{25}P_3 = 2,300$ ways to do this. Then

$$\Pr(\text{first letter is } q) = \frac{2300}{14950} = \frac{2}{13}.$$

(b) Suppose a and b are two of the four letters. How many ways can this happen? Imagine a string of four chosen letters. Pick any two of these, in order, to put a and b in. There are ${}_4P_2 = 12$ ways to do this. For the other two places, we can put (counting order) any two of the remaining 24 letters. There are ${}_{24}P_2$ ways to pick these other two letters The number of ordered strings of four letters with a and b as two letters is $({}_4P_2)({}_{24}P_2) = 564$. Therefore

$$\Pr(a \text{ and } b \text{ were chosen}) = \frac{560}{14950} = \frac{7}{187}$$

(c) The probability of $abdz$ is $1/14950$, since this string can occur in exactly one way

5. The number of ways of picking (without order) three of the eight bowling balls is ${}_8C_3 = 56$

(a) For none of the balls to be defective, they had to come from the six nondefective balls. There are ${}_6C_3 = 20$ ways to pick these three balls, so

$$\Pr(\text{no defective balls were chosen}) = \frac{20}{56} = \frac{5}{14}$$

(b) There are two ways to take one defective ball Then two balls must be chosen from the remaining six, which can be done in ${}_6C_2 = 15$ ways. There are therefore $2(15) = 30$ ways to select exactly one defective ball in choosing three of eight Then

$$\Pr(\text{exactly one defective ball}) = \frac{30}{56} = \frac{15}{28}.$$

(c) There are ${}_3C_2 = 3$ ways of choosing two of the defective balls out of the three. The third ball is then one of the remaining six, and there are 6 ways of picking it, so there are $3(6) = 18$ ways of ending up with both defective balls. Then

$$\Pr(\text{both defective balls were chosen}) = \frac{18}{56} = \frac{9}{28}.$$

6 There are $4^7 = 16,384$ outcomes.

(a) There is exactly one way each die can come up 3, so

$$\Pr(\text{all faces come up } 3) = \frac{1}{16384}.$$

(b) There are ${}_7C_5 = 21$ ways of picking five of the tosses and imagining they came up 1, and the other two tosses came up 4. Therefore

$$\Pr(\text{five } 1's, \text{ two } 4's) = \frac{21}{16384}$$

(c) The only ways to roll a total of 26 are to roll (in any order):

$$4, 4, 4, 4, 4, 4, 2 \text{ (seven ways to do this)}$$

or

$$4, 4, 4, 4, 4, 3, 3 \text{ (21 ways to do this)}$$

Therefore

$$\Pr(\text{rolling a total of } 26) = \frac{7+21}{16384} = \frac{7}{4096}.$$

(d) The sum is at least 26 if it is 26, 27 or 28. We already know that there are 28 ways to roll 26. To roll a 27, we must roll (in any order)

$$4, 4, 4, 4, 4, 4, 3.$$

There are seven ways to do this. To roll a 28, all the dice must come up 4. There are therefore $28 + 7 + 1 = 36$ ways to roll at least a 26, so

$$\Pr(\text{roll at least a } 26) = \frac{36}{16384} = \frac{9}{4096}.$$

7. There are ${}_{20}P_5 = 1,860,480$ outcomes.
(a) There is only one way to make this draw, so

$$\Pr(\text{drawing } 1, 2, 3, 4, 5) = \frac{1}{1860480}.$$

(b) We want the probability that, in five ordered drawings from a collection of balls numbered $1, \cdots, 20$, the ball numbered 3 was selected. We therefore need the number of ordered choices of five of the twenty numbers, that include the number 3. We can think of this as choosing, in order, 4 of the nineteen numbers from 1 through 20 (with 3 removed), and then inserting a 3 in any of the positions from first through fifth number of the sequence. This will result in all ordered sequences of length five from the twenty numbers, and containing the number 3. There are therefore $5({}_{19}P_4) = 465,120$ such sequences. Therefore

$$\Pr(\text{selecting a } 3) = \frac{465120}{1860480} = \frac{1}{4}.$$

(c) We need to count all the sequences of the type we are considering, but containing at least one even number. This is a difficult count, since we must include the sequences having one, two, three, four or five even numbers. It is easier (see below) to count the sequences that have no even number, hence are formed from just the odd numbers in $1, \cdots, 20$. If this number is y, then the number x of sequences having an even number is $x = 1,860,480 - y$, since a sequence must either contain an even number or not.

But y is easy to obtain, since we are using just the ten odd numbers, and the number of ordered five term sequences of these is ${}_{10}P_5 = 30,240$. Therefore

$$x = 1860480 - 30240 = 1830240.$$

Then

$$\Pr(\text{sequence has an even number}) = \frac{1830240}{1860480}$$

This is about 0.984, so it is "very likely" that any sequence of the type we are forming will have an even number.

If for some reason we wanted to directly count the number of ways an even ball could be selected, think of this as the number of ways exactly one even ball could be selected, plus the number of ways exactly two even balls could be selected, and so on up to the number of ways exactly five even balls could be selected. Adding these numbers, we get that the number of ways an even ball can be selected is

$$\begin{gathered} {}_5C_1({}_{10}P_1)({}_{10}P_4) + {}_5C_2({}_{10}P_2)({}_{10}P_3) + {}_5C_3({}_{10}P_3)({}_{10}P_2) \\ + {}_5C_4({}_{10}P_4)({}_{10}P_1) + {}_5C_5({}_{10}P_5)({}_{10}P_0) \end{gathered}$$

It is routine to verify that this gives the same result for x that we derived previously

8. An outcome of this experiment is a string abc, in which each letter represents (in some way) a chosen drawer out of the nine available drawers. The number of such outcomes, without regard to order, is ${}_9C_3 = 84$.

(a) A person can get at least $\$1,000$ by picking exactly one drawer with the thousand dollar bill and two without ($2({}_7C_2)$ ways to do this), or both drawers with thousand dollar bills and one of the other drawers (7 ways to do this). The number of ways of getting at least a thousand dollars is therefore $42 + 7$, or 49. Then

$$\Pr(\text{getting at least } \$1,000) = \frac{49}{84} = \frac{7}{12}$$

(b) The probability that a person can end up with less than one dollar is 0, since this cannot happen

(c) The payoff is \$1.50 exactly when the person choose three drawers, each containing \$0.50. The number of ways this can happen is ${}_7C_3 = 35$, so

$$\Pr(\$1.50 \text{ payoff}) = \frac{35}{84} = \frac{5}{12}$$

Notice that the sum of the probabilities in (a) and (c) is 1. This is because, in choosing three drawers, the person must either choose all drawers having the fifty cents, or must choose at least one of the drawers having the thousand dollars.

9. The number of hands (disregarding order) is ${}_{52}C_5 = 2,598,960$

(a) The number of hands containing exactly one jack and exactly one king is $(4)(4)({}_{44}C_3) = 211,904$. This is because there are four ways of getting a jack, four ways of getting a king, and then ${}_{44}C_3$ ways of drawing (without order) three cards from the remaining 44 cards. Then

$$\Pr(\text{exactly one king, exactly one jack}) = \frac{211904}{2598960} = \frac{1892}{23205}$$

This is about 0.082, so this event is not very likely

(b) The hand can contain at least two aces if it contains two, three, or four aces. Thus the number of such hands is

$$({}_4C_2)({}_{48}C_3) + ({}_4C_3)({}_{48}C_2) + ({}_4C_4)({}_{48}C_1) = 108,336.$$

Then

$$\Pr(\text{at least two aces}) = \frac{108336}{2598960},$$

which is about 0.042. As we might expect, it is very unlikely to draw at least two aces

10. Twenty integers are chosen, without regard to order, from the 101 numbers $0, 1, \cdots, 100$ There are ${}_{101}C_{20} = 668,324,943,343,021,950,370$ ways to do this.

(a) There are 21 numbers in the $80 - 100$ range, inclusive. The number of ways of choosing twenty of these is ${}_{21}C_{20} = 21$. Therefore

$$\Pr(\text{all numbers chosen are larger than } 79) = \frac{21}{668\,324\,943\,343\,021\,950\,370},$$

about 3.1422×10^{-20} In the practical world we would say that this event is essentially impossible

(b) We interpret the problem to mean that exactly one of the numbers is 5 There are ${}_{100}C_{19} = 132,341,572,939,212,267,400$ ways to choose nineteen of the numbers different from 5, and of course one way to choose 5 for the twentieth number Then

$$\Pr(\text{one number chosen is } 5) = \frac{132341572939212267400}{668324943343021950370},$$

about 0.198.

Section 26.6 Complementary Events

1. Roll seven dice We want the probability that at least two came up 4

At least two means two, three, four, five, six or seven dice come up 4. We can count the number of ways this can happen But it may be easier to look at the complementary event, which is that either no 4, or exactly one 4, comes up If no 4 comes up, then each roll can have five outcomes, for a total of 5^7 possible outcomes. If exactly one 4 comes up, then the other dice have five possible numbers showing, for $7(5^6)$ possible outcomes (the die that comes up 4 may be any of the seven dice). This means that the event that at most one 4 comes up has $5^7 + 7(5^6) = 187,500$ outcomes. The experiment of rolling seven dice has $6^7 = 279,936$ outcomes. Therefore

$$\Pr(\text{at most one } 4) = \frac{187500}{279936},$$

about 0.669 Then

$$\Pr(\text{at least one } 4) = 1 - \frac{187500}{279936},$$

about 0.330

2. Fourteen coins are flipped. There are $2^{14} = 16,384$ outcomes of fourteen coin flips We want the probability that at least three came up heads

The complementary event is that fewer than three heads came up This means that either no heads, one head or two heads came up No heads can occur in exactly one way (all coins tails). One head can occur in fourteen ways (any one of the coins can be a head, the others all tails). Exactly two heads can occur in ${}_{14}C_2 = 91$ ways (pick two flips for the heads, imagine all the other coins came up tails). Then

$$\Pr(\text{fewer than three heads}) = \frac{1 + 14 + 91}{16384} = \frac{106}{16384}$$

Then

$$\Pr(\text{at least three heads}) = 1 - \frac{106}{16384},$$

about 0.994 It is very likely to get three or more heads in fourteen flips.

3 Draw five cards from a standard deck. Without regard to order, there are ${}_{52}C_5 = 2,598,960$ hands Consider the event E that at least one card is a face card, an ace, or is numbered 4 or higher The complementary event is that the hand has no face card and no ace, and every card numbers 2 or 3 This leaves just two's and three's to form the hand With these eight cards, the number of possible hands is ${}_8C_5 = 56$. Then

$$\Pr(E^C) = \frac{56}{2598960}.$$

Then

$$\Pr(E) = 1 - \frac{56}{2598960},$$

about 0.999.

4 Two coins are flipped and five dice rolled. The number of outcomes is $2^2 6^5 = 31,104$. We want the probability of the event E that two heads and at least one 4 come up

E^C is the event that two heads did not come up, or at least one 4 did not come up. This can happen if two heads do not come up, and the dice can be anything ($3(6^5)$ ways, or if there are two heads, and the dice show no $4's$ (this happens in 5^5 ways) Therefore

$$\Pr(E^C) = \frac{3(6^5) + 5^5}{31104} = \frac{26453}{31104},$$

about 0.85. Then

$$\Pr(E) = 1 - \frac{26453}{31104},$$

about 0.15.

5. Four numbers are chosen (without regard to order) from the fifty five numbers $1, 2, \cdots, 55$. We want the probability of the event E that at least one number is greater than 4?

E^C is the event that all four numbers are less than or equal to 4, hence must be chosen from the numbers $1, 2, 3, 4$. Since we are picking four numbers, this can only be done in one way The total number of ways of picking the four numbers is ${}_{55}C_4 = 341,055$. Then

$$\Pr(E^C) = \frac{1}{341055}.$$

Then

$$\Pr(E^C) = 1 - \frac{1}{341055}$$

6. We roll six dice The number of outcomes is $6^6 = 46,656$. We want the probability of the event E that they total at least 11, which means they can total anything in $11, 12, \cdots, 36$.

E^C is the event that they total less than 11. With six dice, this means that they total $6, 7, 8, 9$ or 10 It would appear easier to enumerate E^C than E

There is one way the six dice can total 6 (they all come up 1).

There are six ways they can total 7, namely five dice come up 1 and one comes up 2.

To total 8, we can roll $1, 1, 1, 1, 2, 2$ (${}_6C_2 = 15$ ways) or $1, 1, 1, 1, 1, 3$ (six ways).

To total 9, we can roll $2, 2, 2, 1, 1, 1$ (${}_6C_3 = 20$ ways), or $4, 1, 1, 1, 1, 1$ (six ways), or $3, 2, 1, 1, 1, 1$ ($2({}_6C_2) = 30$ ways).

To roll 10, we can get 2, 2, 2, 2, 1, 1 (${}_6C_2 = 15$ ways), or 4, 2, 1, 1, 1, 1 (30 ways), or 5, 1, 1, 1, 1, 1 (six ways), or 3, 3, 1, 1, 1, 1 (15 ways), or 3, 2, 2, 1, 1, 1 ($6({}_5C_2) = 60$ ways).

Finally, to roll 11, we can roll 2, 2, 2, 2, 2, 1 (six ways), or 3, 2, 2, 2, 1, 1 (60 ways), or 3, 3, 2, 1, 1, 1 (60 ways), or 4, 3, 1, 1, 1, 1 (30 ways), or 5, 2, 1, 1, 1, 1 (30 ways), or 6, 1, 1, 1, 1, 1 (six ways).

Then E^C has 402 outcomes in it, so

$$\Pr(E^C) = \frac{402}{46656},$$

about 0 0086. Then

$$\Pr(E) = 1 - \frac{402}{46656},$$

about 0.991 38

7. We have $N \geq 2$ people in a room, and we want to compute the probability that at least two have the same birthday Suppose p is the probability that at least two people have the same birthday This will be some function of N, which we would like to find Since at least two people could mean $2, 3, 4, \cdots N$, involving a lot of possibilities if N is fairly large, it would appear to be easier to consider the complementary probability $q = 1 - p$ that no two people have the same birthday. Thus concentrate on computing q.

Assume that the year has 365 days The experiment here is to list each person's birthday (as a numbered day of the year from 1 to 365), resulting in a list of N numbers The total number of such sequences that is possible is

$$(365)(365)(365) \cdots (365) = (365)^N,$$

since to begin each person could have any one of the 365 days as a birthday Now count the number of sequences of birthdays in which no two people have the same birthday The first person now has 365 possible birthdays, but the second person only 364, and the third person 363, and so on, if all birthdays are to be different. Thus the number of sequences with no duplicated birthdays is

$$(365)(364) \cdots (365 - N + 1)$$

This means that

$$q = \frac{(365)(364) \cdots (365 - N + 1)}{(365)^N}.$$

We can compute this for any N we want, and then, from q, compute $p = 1 - q$.

If you compute p for $N = 1, 2, 3, \cdots$, you find a perhaps surprising result With $N = 5$, we get $p \approx 0.024$; with $N = 15$, $p \approx 0\ 25$, and with $N = 25$, $p \approx 0\ 57$. With only twenty five people, there is a better than even bet that at least two will have the same birthday! For $N \geq 60$, the probability exceeds 0 99, making it highly likely in a group of sixty or more people that at least two have the same birthday This appears to violate the common intuition that it should take many more people before there is such high likelihood of at least two birthdays coinciding

Section 26.7 Conditional Probability

1. (a) This experiment has four outcomes. Let E be the event that the first coin comes up heads. Then $E = \{HH, HT\}$, containing two outcomes, so

$$\Pr(E) = \frac{1}{2}$$

(b) Let U be the event that at least one coin came up heads. Then $U = \{HH, HT, TH\}$ and $E \cap U = \{HH, HT\}$, with two outcomes. Then

$$\Pr(E \mid U) = \frac{\text{number of outcomes in } E \cap U}{\text{number of outcomes in } U} = \frac{2}{3}$$

We could alternatively compute

$$\Pr(E \mid U) = \frac{\Pr(E \cap U)}{\Pr(U)} = \frac{2/4}{3/4} = \frac{2}{3}.$$

2. (a) Let E be the event that exactly three of the four coins come up heads. Then

$$\Pr(E) = \frac{4}{2^4} = \frac{1}{4}.$$

(b) Here we must be clear on what we mean by " .. if we know that one came up tails". Consider two cases.

Suppose first we interpret this to mean that we know that exactly one came up tails. Let this be the event U. Now we know with certainty that exactly three coins came up heads, so

$$\Pr(E \mid U) = 1.$$

On the other hand, suppose we interpret the condition to mean that we know at least one coin came up tails. Now let U be this event. Then U contains all outcomes except $HHHH$. Further, $E \cap U = E$, so

$$\Pr(E \mid U) = \frac{\Pr(E \cap U)}{\Pr(U)} = \frac{\Pr(E)}{\Pr(U)} = \frac{1/4}{15/16} = \frac{1}{4}\frac{16}{15} = \frac{4}{15}.$$

3. Flip four coins. There are $2^4 = 16$ outcomes

(a) Let E be the event that at least three come up tails. Then E has five outcomes ($E = \{TTTH, TTHT, THTT, HTTT, TTTT\}$0, so

$$\Pr(E) = \frac{5}{16}.$$

(b) We want the probability that at least three came up tails, having seen two come up tails. This means we know that there were at least two tails. Let U be the event that at least two tails came up. Then $U = \{HHTT, HTHT, THHT, THTH, TTHH, HTTH, TTTH, TTHT, THTT, HTTT, TTTT\}$. Then $\Pr(U) = 11/16$. Further, $E \cap U = E$, so

$$\Pr(E \mid U) = \frac{\Pr(E \cap U)}{\Pr(U)} = \frac{\Pr(E)}{\Pr(U)} = \frac{5/16}{11/16} = \frac{5}{11}$$

4. If we deal six cards from a standard deck, without regard to order, then the number of outcomes is ${}_{52}C_6 = 20{,}358{,}520$

(a) We want the probability that exactly two face cards were dealt. In this case two cards were taken from the 12 face cards, and the remaining cards from the 40 other cards. Thus the number of ways of getting exactly two face cards (event E) is $({}_{12}C_2)({}_{40}C_4) = 6{,}031{,}740$. Then

$$\Pr(E) = \frac{6031740}{20358520},$$

about 0.296.

(b) We want the probability that exactly two face cards were dealt, knowing that a king was dealt. Let U be the event that a king was dealt. This can occur if one, two, three or four kings were dealt, along with other cards Count:

$$\text{hands with one king: } 4({}_{48}C_5)$$

$$\text{hands with two kings: } 6({}_{48}C_4)$$

$$\text{hands with three kings: } 4({}_{48}C_3)$$

$$\text{hands with four kings: } {}_{48}C_2$$

Then the number of outcomes in U is $8,087,008$.

Now we need to consider $E \cap U$, which consists of all hands with exactly two face cards, and knowing that a king was dealt In this case, one of the two face cards is a king, and the other face card can be another king, a jack or an ace There are $4(11)({}_{40}C_4) = 4,021,160$ outcomes in this event. Then

$$\Pr(E \mid U) = \frac{\Pr(E \cap U)}{\Pr(U)} = \frac{\text{number of outcomes in } E \cap U}{\text{number of outcomes in } U}$$
$$= \frac{4021160}{8087008},$$

about 0.497.

5. Roll dice four times First we want the probability that the dice total exactly 19.

The number of outcomes is $6^4 = 1,296$ Now we want the number of outcomes in the event E that the dice total 19 In order to come up 19, the dice can show:

$$6, 6, 6, 1 \text{ (four ways)},$$

$$6, 6, 5, 2 \text{ (twelve ways)},$$

$$6, 6, 4, 3 \text{ (twelve ways)}$$

$$6, 5, 5, 3 \text{ (twelve ways)},$$

$$6, 5, 4, 4 \text{ (twelve ways)},$$

$$5, 5, 5, 4 \text{ (four ways)}.$$

Then E has 56 outcomes in it, and

$$\Pr(E) = \frac{56}{1296},$$

about 0 043

Now suppose we want the probability that the dice total exactly 19, if we know that one die came up 1 Consistent with this, we could have one, two, three or four dice come up 1. We need to count the number of outcomes in this event U that (at least) one die came up 1. But U consists of all outcomes except those in which no die came up 1 There are $5^4 = 625$ such outcomes Therefore U has

$$6^4 - 5^4 = 1296 - 625 = 671$$

outcomes Further $E \cap U$ consists of outcomes totalling 19, and having at least one 1, and there are four such outcomes. Therefore

$$\Pr(E \mid U) = \frac{4}{671},$$

about 0 059.

6 Roll two dice. We want the probability that they sum to at least 9, knowing that one die came up even?

Two dice can sum to at least 9 if they sum to 9, 10, 11 or 12. This can happen in the following ways:

sum to 9: 6, 3 or 5, 4 (four ways)

sum to 10: 5, 5 or 6, 4 (three ways)

sum to 11: 6, 5 (two ways)

sum to 12: 6, 6 (one way).

Let E be the event that the dice sum to at least 9, and U the event that one die came up even We interpret this to mean that one or both dice can come up even. Then $E \cap U$ consists of eight outcomes. We must count the outcomes in U These consist of:

exactly one even: eighteen such outcomes

or

both even: nine outcomes

Then

$$\Pr(E \mid U) = \frac{\text{number of outcomes in } E \cap U}{\text{number of outcomes in } U} = \frac{8}{17},$$

about 0.47.

7. Deal a five card hand, without regard to order. We want the probability that the hand has four aces, if we know that a four of spades was dealt.

Let E be the event that the hand has four aces, and U the event that a four of spades was dealt. Now $E \cap U$ consists of all outcomes in which there are four aces and a four of spades.

Disregarding order, there is one five card hand having four aces and a four of spades, hence one outcome in $E \cap U$ We need to count the outcomes in U If one card is a four of spaces, the other four cards can be any four of the remaining fifty one cards. There are therefore ${}_{51}C_4 = 249,900$ unordered hands having the four of spades Then

$$\Pr(E \mid U) = \frac{1}{249,900},$$

about $4(10^{-6})$.

8. Let E be the event that seven coin flips will produce at least five heads, and U the event that four flips came up heads. Then $E \cap U$ is the even that five, six or seven heads come up, and U is the event that four, five, six or seven heads come up

The number of outcomes in $E \cap U$ is $({}_7C_5 + {}_7C_6 + {}_7C_7 = 29$ and the number of outcomes in U is $29 + {}_7C_4 = 64$ Then

$$\Pr(E \mid U) = \frac{29}{64},$$

about 0.453.

9. Imagine tossing four dice We want the probability that all four came up an odd number. This probability is

$$\Pr(\text{all odd inn four tosses}) = \frac{3^4}{6^4} = \frac{1}{2^4}.$$

Now let E be the event that all four dice come up odd, and let U be the event that one die came up 1 and another came up 5 We want $\Pr(E \mid U)$

The number of outcomes in U is $2({}_4C_2)(6)(6) = 432$. Next, $E \cap U$ consists of all outcomes in which it is known that one die comes up 1 and one 5, and all four come up odd This means we roll 1, 5, odd, odd, and there are $2({}_4C_2)(3)(3) = 108$ Then

$$\Pr(E \mid U) = \frac{108}{432} = \frac{1}{4}.$$

Finally, suppose we know that the second die came up 6. Now the probability that all four came up odd is zero

10. Suppose $A = \{a_1, a_2, \cdots, a_n\}$ and $B = \{b_1, b_2, \cdots, b_r\}$ Consider two cases.

Case 1: $A \cap B = \emptyset$. Now none of the $a's$ can equal any of the $b's$, and $A \cup B = \{a_1, \cdots, a_n, b_1, \cdots, b_r\}$. If N is the number of outcomes of the experiment, then

$$\Pr(A \cup B) = \frac{n+r}{N} = \frac{n}{N} + \frac{r}{N} = \Pr(A) + \Pr(B),$$

and this gives the requested equation in this case, because now $\Pr(A \cap B) = \Pr(\emptyset) = 0$.

Case 2: Suppose $A \cap B \neq \emptyset$ This means that some of the $a's$ coincide with some of the $b's$. To be specific, suppose $a_1 = b_1, \cdots, a_s = b_s$ If $s = r = n$, then $A = B = A \cup B = A \cap B$, so

$$\Pr(A) + \Pr(B) - \Pr(A \cap B) = \Pr(A) + \Pr(A) - \Pr(A) = \Pr(A) = \Pr(A \cup B)$$

Thus suppose $s < n$ or $s < r$ (or perhaps both) Say, to be specific, that $s < n$. Then $A \cap B = \{a_1, \cdots, a_s\}$ has s outcomes in it, and $A \cup B$ has $r + (n - s)$ outcomes in it. Then

$$\Pr(A \cup B) = \frac{r+n-s}{N} = \frac{r}{N} + \frac{n}{N} - \frac{s}{N}$$
$$= \Pr(B) + \Pr(A) - \Pr(A \cap B)$$

It is easy to show that $\Pr(A) + \Pr(B) \neq \Pr(A \cup B)$ if A and B have outcomes in common For example, let the experiment be flipping two coins and let A be the event that at least one head results, and let B be the event that at least one tail results. Then $A \cup B$ is the entire sample space, so $\Pr(A \cup B) = 1$. But $\Pr(A) = \Pr(B) = \frac{3}{4}$, so $\Pr(A) + \Pr(B) = \frac{3}{2}$

Section 26.8 Independent Events

1. Flip four coins E is the event that exactly one coin comes up heads, and U is the event that at least three coins come up tails. Now $E \cap U$ consists of the outcomes in which one coin comes up heads and the others come up tails. This can happen in four ways, so $\Pr(E \cap U) = 4/2^4 = 1/4$ But $\Pr(E) = 1/4$ and $\Pr(U) = 5/16$ Since $\Pr(E \cap U) \neq \Pr(E)\Pr(U)$, these events are not independent

This is intuitively apparent, since the knowledge that exactly one coin comes up heads influences the probability that at least three come up tails (by removing one possible way this can happen)

2. Draw two cards from a deck E is the event that both cards are aces U is the event that one card is a diamond and the other a spade

We will assume that the order of the draw does not count for anything Now

$$\Pr(E) = \frac{{}_4C_2}{{}_{52}C_2} = \frac{6}{1326} = \frac{1}{221}$$

and

$$\Pr(U) = \frac{(13)(13)}{({}_{52}C_2)} = \frac{169}{1326} = \frac{13}{102}$$

Finally, $E \cap U$ consists of all outcomes in which the ace of diamonds and the ace of spades are drawn This can happen one way, so

$$\Pr(E \cap U) = \frac{1}{1326} \neq \Pr(E)\Pr(U),$$

so E and U are not independent

3. Two dice are rolled. E is the event that they total more than eleven (hence 11 or 12). U is the event that at least one die comes up even. Now the dice can total eleven in two ways, and they can total twelve in one way, so

$$\Pr(E) = \frac{3}{6^2} = \frac{1}{12}$$

And U consists of all outcomes with exactly one even (eighteen ways this can happen), together with all outcomes with both even (nine ways), so

$$\Pr(U) = \frac{27}{6^2} = \frac{3}{4}$$

Finally, $E \cap U$ consists of all outcomes with at least one even, and totalling eleven or twelve. This can happen in two ways (roll a six and a five, in either order), So

$$\Pr(E \cap U) = \frac{2}{36} = \frac{1}{18}$$

Since $\Pr(E \cap U) \neq \Pr(E)\Pr(U)$, these events are not independent.

4. E is the event that at least three of the six children are girls, and U is the event that at least two are girls, out of six children. In the absence of other information, we assume that the probability of a boy is 1/2, and the same for a girl The number of outcomes in E is

$${}_6C_3 +{}_6 C_4 +{}_6 C_5 +{}_6 C_6 = 42,$$

so

$$\Pr(E) = \frac{42}{2^6} = \frac{21}{32}$$

The number of outcomes in U is

$$42 +{}_6 C_2 = 57$$

so

$$\Pr(U) = \frac{57}{2^6} = \frac{57}{64}$$

Now $E \cap U = E$, so $\Pr(E \cap U) = 21/32 \neq \Pr(E)\Pr(U)$, so E and U are not independent

5 Flip two coins and then roll two dice. E is the event that at least one head comes up. U is the event that at least one die comes up 6. And $E \cap U$ is the event that at least one coin comes up heads and at least one die comes up 6.

The number of outcomes in E is $(3)(36) = 108$. Then

$$\Pr(E) = \frac{108}{2^2 6^2} = \frac{3}{4}$$

$$\frac{108}{4(36)}$$

The number of outcomes in U is $(4)(11) = 44$, so

$$\Pr(U) = \frac{4(11)}{4(36)} = \frac{11}{36}$$

Finally, $E \cup U$ consists of all outcomes having at least one head and at least one 6. There are $(3)(11) = 33$, so

$$\Pr(E \cap U) = \frac{33}{4(36)} = \frac{11}{48}.$$

Now

$$\Pr(E)\Pr(U) = \frac{3}{4}\left(\frac{11}{36}\right) = \frac{11}{48} = \Pr(E \cap U),$$

so E and U are independent.

6. Pick two cards from a standard deck, with replacement. E is the event that the first card drawn was a king. U is the event that the second card drawn was an ace. $E \cap U$ is the event that the first card drawn was a king and the second card, an ace

E can happen in $4(52)$ ways, so

$$\Pr(E) = \frac{4(52)}{(52)(52)} = \frac{1}{13}$$

U can happen in $52(4)$ ways, so

$$\Pr(U) = \frac{1}{13}.$$

$E \cap U$ can happen in $4(4) = 16$ ways, so

$$\Pr(E \cap U) = \frac{16}{52(52)} = \frac{1}{169} = \Pr(E)\Pr(U).$$

Since $\Pr(E)\Pr(U) \neq \Pr(E \cap U)$, these events are independent.

7 Deal two cards without replacement. E is the event that the first card drawn was a jack of diamonds. U is the event that the second card was a club or spade. $E \cap U$ is the event that the first card drawn was a jack of diamonds, and the second card a club or spade

E can happen in 51 ways, since the deal is without replacement, so

$$\Pr(E) = \frac{51}{52(51)} = \frac{1}{52}.$$

We need to count the outcomes in U. If the first card drawn was a club or spade, there are 26 choices for this first card, and then 25 for the second, for 26(25) outcomes. If the first card drawn was not a club or spade, then there are 26 choices for the first card, and 26 for the second as well, for 26^2 outcomes. Then U has $26(25+26) = 26(51)$ outcomes in it, and

$$\Pr(U) = \frac{26(51)}{52(51)} = \frac{1}{2}.$$

Finally, $E \cap U$ has 26 outcomes in it, so

$$\Pr(E \cap U) = \frac{26}{52(51)} = \frac{1}{102}$$

Now compute

$$\Pr(E)\Pr(U) = \frac{1}{52}\left(\frac{1}{2}\right) = \frac{1}{104}.$$

Then E and U are not independent

8. Flip four coins. E is the event that the first coin is a head. U is the event that the last coin is a tail. Then $E \cap U$ is the event that the first coin is a head and the last coin a tail.

E can happen in 2^3 ways, so

$$\Pr(E) = \frac{2^3}{2^4} = \frac{1}{2}.$$

U can also happen in 8 ways, so

$$\Pr(U) = \frac{1}{2}.$$

These make sense, since on any flip there is a 1/2 probability of getting a head or tail.

$E \cap U$ can happen in 4 ways, so

$$\Pr(E \cap U) = \frac{4}{16} = \frac{1}{4}.$$

Since $\Pr(E)\Pr(U) = \Pr(E \cap U)$, these events are independent

9. The dishonest coin is flipped four times. We want the probability of getting at least two heads in four flips. There are $2^4 = 16$ outcomes, and the following have at least two heads:

$$HHHH, HHHT, HHTH, HTHH, THHH,$$

$$HHTT, HTHT, HTTH, TTHH, THTH, THHT.$$

The probability of a head is 0.4. Therefore the probability of getting a tail is 0.6. Then the probability of getting at least two heads, from the list of outcomes, is

$$\Pr(\text{at least two heads})$$

$$= (0.4)^4 + 3(0.4)^3(0.6) + 6(0.4)^2(0.6)^2 = 0.4864.$$

The probability of getting exactly two heads is

$$\Pr(\text{exactly two heads}) = 6(0.4)^2(0.6)^2 = 0.3456.$$

10. The probability of rolling a 5 in three tosses is zero, since three of the numbers 1, 4 and 6 cannot sum to 5.

11 Let r be the probability of drawing a red, g that of drawing a green, and b the probability of drawing a blue marble. If we reach into the jar and pull out a marble, then it is certain that it is red, blue or green Therefore

$$r + b + g = 1.$$

But $b = 2g = 3r$, so

$$r + 3r + \frac{3}{2}r = \frac{11}{2}r = 1$$

Then

$$r = \frac{2}{11}, g = \frac{3}{2}\frac{2}{11} = \frac{3}{11}, \text{ and } b = \frac{6}{11}$$

Now we know the probability of picking any marble out of the jar. Imagine picking three marbles We want the probability of picking exactly two red ones. The possible draws with exactly two red marbles are:

$$rrb, rbr, brr, rrg, rgr, grr.$$

Then

$$\begin{aligned} &\Pr(\text{exactly two red marbles in three draws}) \\ &= 3\left(\frac{2}{11}\right)^2 \frac{6}{11} + 3\left(\frac{2}{11}\right)^2 \frac{3}{11} = \frac{108}{1331}, \end{aligned}$$

about 0.081.

Section 26.9 Tree Diagrams in Computing Probabilities

1. The tree diagram shows the outcomes, and we get the probabilities by multiplying numbers on edges of paths. The outcomes are payoffs of \$0, \$5, \$10, \$50, or \$1,200. From the edges of the tree, we read:

$$\Pr(0) = \frac{1}{4}\left(\frac{1}{2}\right) = \frac{1}{8},$$

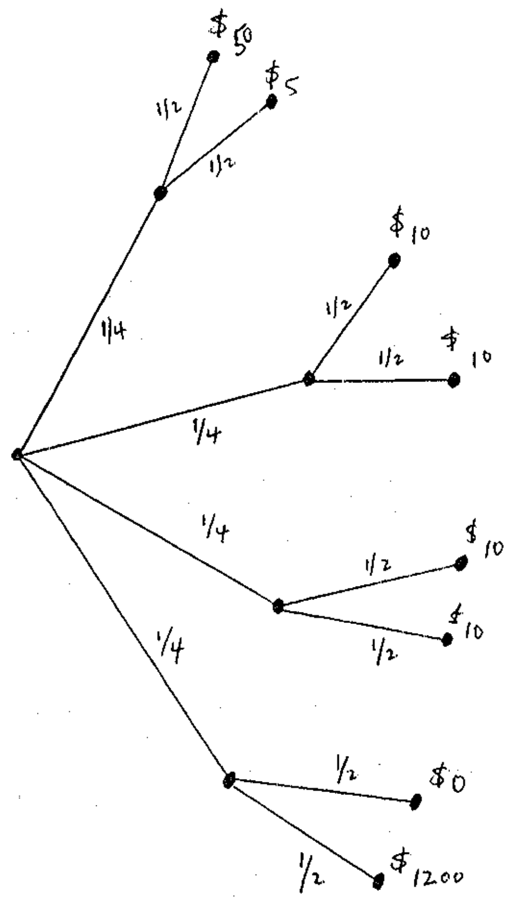
$$\Pr(5) = \frac{1}{4}\left(\frac{1}{2}\right) = \frac{1}{8},$$

$$\Pr(10) = 4\frac{1}{4}\left(\frac{1}{2}\right) = \frac{1}{2},$$

$$\Pr(50) = \frac{1}{4}\left(\frac{1}{2}\right) = \frac{1}{8},$$

and

$$\Pr(1200) = \frac{1}{4}\left(\frac{1}{2}\right) = \frac{1}{8}.$$



Section 26 9 – Tree Diagram for Problem 1

2. From the tree diagram, we read

$$\Pr(\text{receive nothing}) = \frac{1}{3}\left(\frac{1}{2}\right) + 2\left(\frac{1}{3}\right)^2 = \frac{7}{18},$$

$$\Pr(\text{diamonds}) = \frac{1}{3}\left(\frac{1}{2}\right)\left(\frac{1}{4}\right) = \frac{1}{24},$$

$$\Pr(\text{cash}) = \frac{1}{3}\left(\frac{1}{2}\right)\left(\frac{1}{4}\right) = \frac{1}{24},$$

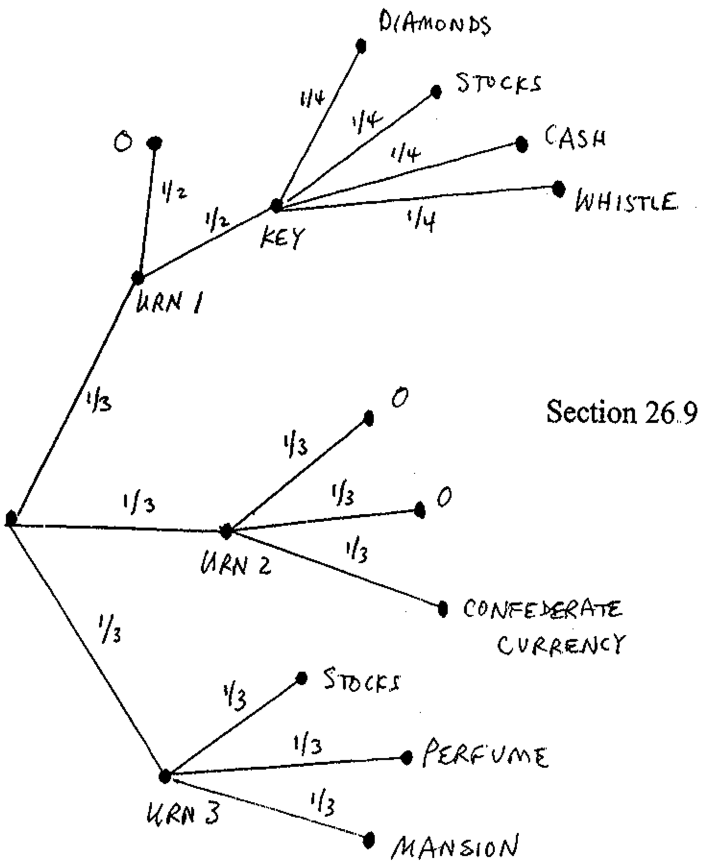
$$\Pr(\text{whistle}) = \frac{1}{3}\left(\frac{1}{2}\right)\left(\frac{1}{4}\right) = \frac{1}{24},$$

$$\Pr(\text{Confederate currency}) = \left(\frac{1}{3}\right)^2 = \frac{1}{9},$$

$$\Pr(\text{perfume}) = \left(\frac{1}{3}\right)^2 = \frac{1}{9},$$

$$\Pr(\text{mansion}) = \left(\frac{1}{3}\right)^2 = \frac{1}{9},$$

$$\Pr(\text{some stocks}) = \frac{1}{3}\left(\frac{1}{2}\right)\left(\frac{1}{4}\right) + \left(\frac{1}{3}\right)^2 = \frac{11}{72}$$

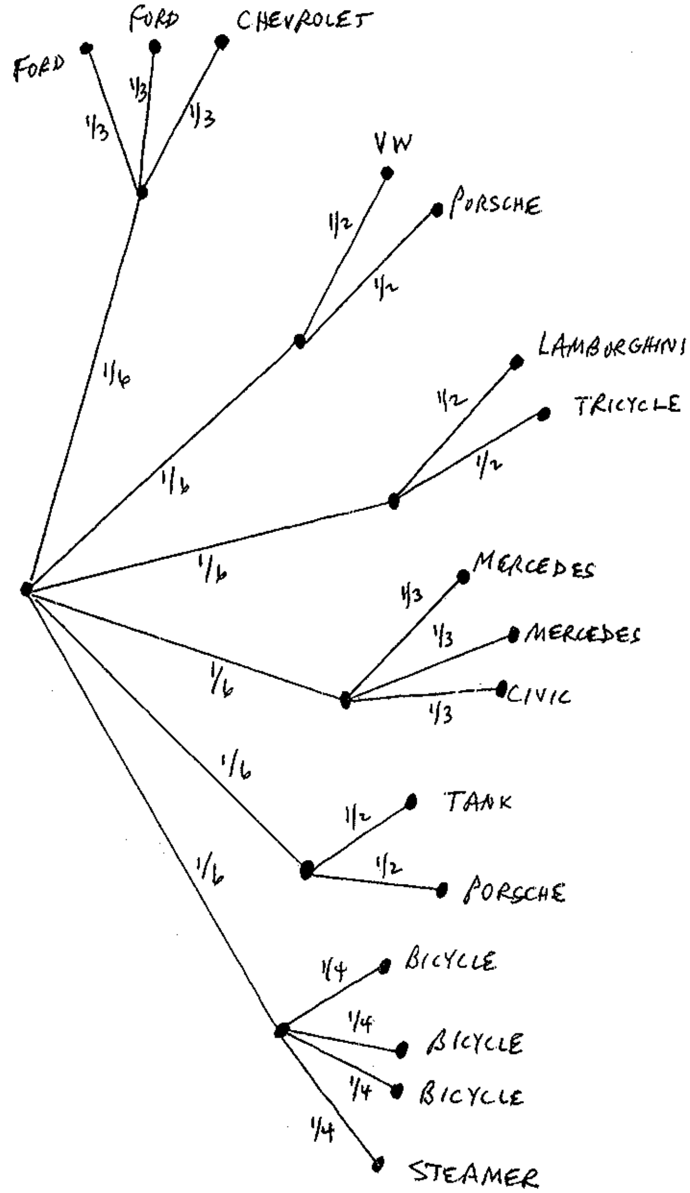


Section 26.9 – Tree Diagram for Problem 2

3. From the tree diagram,

$$\Pr(\text{Ford}) = 2\left(\frac{1}{6}\right)\left(\frac{1}{3}\right) = \frac{1}{9},$$

$$\Pr(\text{Chevrolet}) = \frac{1}{6}\left(\frac{1}{3}\right) = \frac{1}{18},$$
$$\Pr(\text{VW}) = \frac{1}{6}\left(\frac{1}{2}\right) = \frac{1}{12},$$
$$\Pr(\text{Porsche}) = 2\frac{1}{6}\left(\frac{1}{2}\right) = \frac{1}{6},$$
$$\Pr(\text{Lamborghini}) = \frac{1}{6}\left(\frac{1}{2}\right) = \frac{1}{12},$$
$$\Pr(\text{tricycle}) = \frac{1}{6}\left(\frac{1}{2}\right) = \frac{1}{12},$$
$$\Pr(\text{Mercedes}) = 2\left(\frac{1}{6}\right)\left(\frac{1}{3}\right) = \frac{1}{9},$$
$$\Pr(\text{Honda}) = \frac{1}{6}\left(\frac{1}{3}\right) = \frac{1}{18},$$
$$\Pr(\text{tank}) = \frac{1}{6}\left(\frac{1}{2}\right) = \frac{1}{12},$$
$$\Pr(\text{bicycle}) = 3\left(\frac{1}{6}\right)\left(\frac{1}{4}\right) = \frac{1}{8},$$
$$\Pr(\text{Stanley Steamer}) = \frac{1}{6}\left(\frac{1}{4}\right) = \frac{1}{24}$$

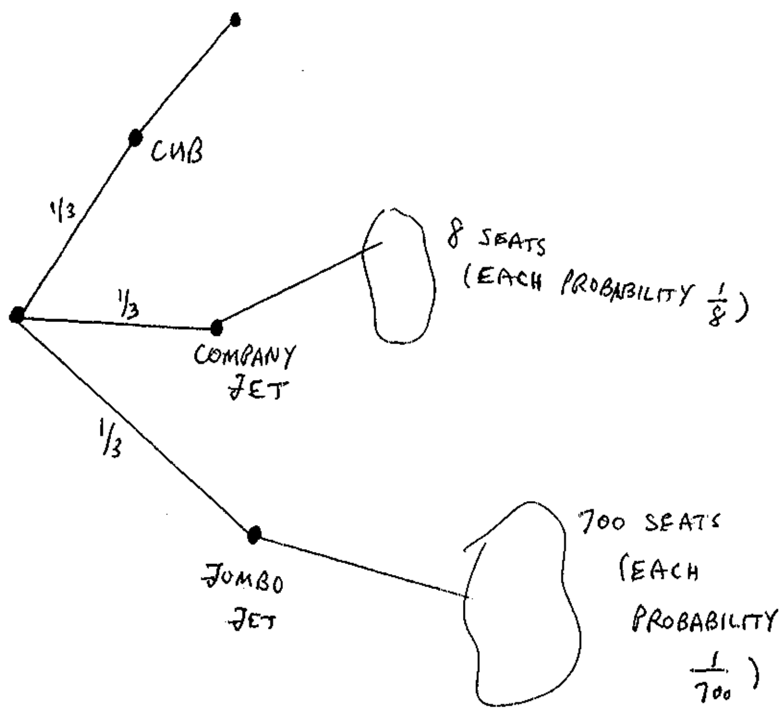


Section 26.9 - Tree Diagram for Problem 3

4. We don't need a tree diagram for this, but we can use one to visualize the probabilities. One such diagram is drawn below. Of course, we do not want to put 700 dots for the seats in the jumbo jet, so these are just indicated, along with the fact that half of these seats have odd numbers

Based on the diagram, the probability of picking an odd numbered seat on the jumbo jet is

$$\Pr(\text{odd numbered jumbo jet seat}) = 350\left(\frac{1}{3}\right)\left(\frac{1}{700}\right) = \frac{1}{6}$$



Section 26.9 – Tree Diagram for Problem 4

5 From the tree diagram, read:

$$\Pr(0) = 5\left(\frac{1}{5}\right)\left(\frac{1}{4}\right) = \frac{1}{4},$$

$$\Pr(20) = 2\left(\frac{1}{5}\right)\left(\frac{1}{4}\right) = \frac{1}{10},$$

$$\Pr(\text{nickel}) = \frac{1}{5}\left(\frac{1}{4}\right) = \frac{1}{20},$$

$$\Pr(\text{chair}) = \frac{1}{5}\left(\frac{1}{4}\right) = \frac{1}{20},$$

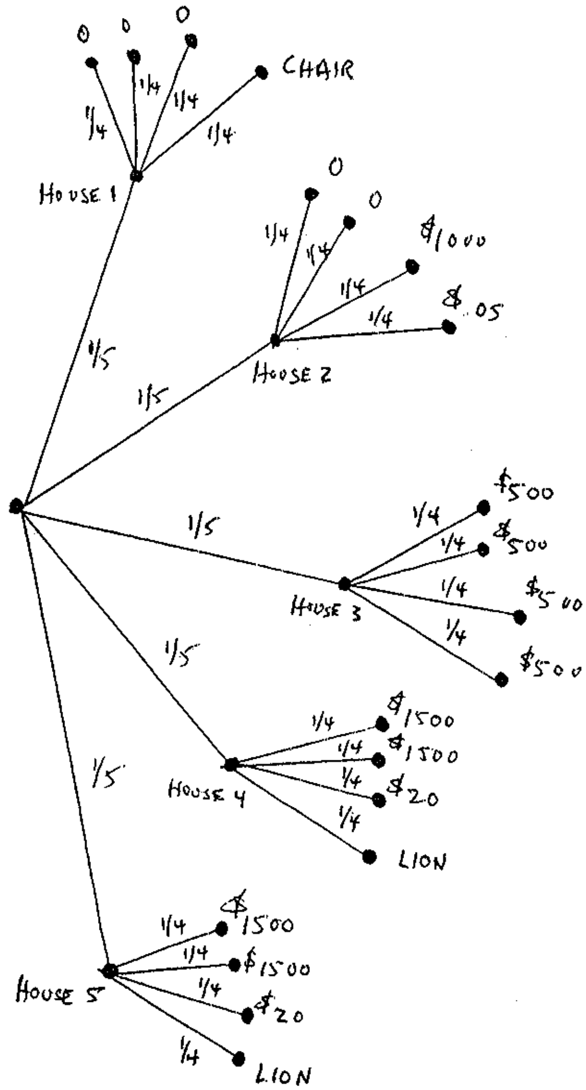
$$\Pr(500) = 4\left(\frac{1}{5}\right)\left(\frac{1}{4}\right) = \frac{1}{5},$$

$$\Pr(1000) = \frac{1}{5}\left(\frac{1}{4}\right) = \frac{1}{20},$$

$$\Pr(1500) = 4\left(\frac{1}{5}\right)\left(\frac{1}{4}\right) = \frac{1}{5},$$

and

$$\Pr(\text{lion}) = 2\left(\frac{1}{5}\right)\left(\frac{1}{4}\right) = \frac{1}{10}$$



Section 26 9 – Tree Diagram for Problem 5

Section 26.10 Bayes' Theorem

1. The company hires people in four different groupings:

years experience	% of time products defective	% as a decimal
$\geq$ 10 years exp.	1	0.01
$5 - 10$ years exp	3	0.03
$1 - 5$ years exp.	5.7	0.057
<1 year exp.	11.3	0.113

The following table displays these groupings by work experience, and for each, the percentage this group constitutes of the total work force (written as a decimal), and a reiteration of the percentage (as a decimal) of products they produce that are defective. By writing these percentages as decimals, they can be thought of as probabilities. For example, if 45% of the workforce is in one category, then the probability that a worker is in that category is 0.45.

group	% of workforce (E_j)	% of time products defective
$\geq$ 10 years exp.	0.45	0.01
$5 - 10$ years exp	0.37	0.03
$1 - 5$ years exp.	0.07	0.057
<1 year exp	0.11	0.113

We want for each category or grouping, the probability that a worker is in this grouping, knowing that an item is defective. Let U be the event that an item is defective, and E_j is the event that a worker is in grouping j (column two of the last table). We will use

$$\Pr(E_j \mid U)$$

$$= \frac{\Pr(E_j)\Pr(U \mid E_j)}{\Pr(E_1)\Pr(U \mid E_1) + \Pr(E_2)\Pr(U \mid E_2) + \Pr(E_3)\Pr(U \mid E_3) + \Pr(E_4)\Pr(U \mid E_4)}$$

for $j = 1, 2, 3, 4$, with the numbers $\Pr(U \mid E_j)$ read from column three of the last table. Now compute

$$\Pr(E_1 \mid U) =$$

$$\frac{(0.45)(0.01)}{(0.45)(0.01) + (0.37)(0.03) + (0.07)(0.057) + (0.11)(0.113)}$$

$$= 0.141.$$

This is the probability that a worker was in the ten or more years experience group, knowing that a product was defective.

For $j = 2, 3, 4$, the denominator in the Bayes formula is the same, and we need only vary the numerator (which is one term of the denominator). We get

$$\Pr(E_2 \mid U) =$$

$$\frac{(0.37)(0.03)}{(0.45)(0.01) + (0.37)(0.03) + (0.07)(0.057) + (0.11)(0.113)}$$

$$= 0.347,$$

$$\Pr(E_3 \mid U) = \frac{(0.07)(0.057)}{(0.45)(0.01) + (0.37)(0.03) + (0.07)(0.057) + (0.11)(0.113)} = 0.125,$$

and

$$\Pr(E_4 \mid U) = \frac{(0.11)(0.0113)}{(0.45)(0.01) + (0.37)(0.03) + (0.07)(0.057) + (0.11)(0.113)} = 0.039$$

2 The following table displays the information given:

city	% of keyboards manu	% defective
Los Angeles	0.37	0.02
Detroit	0.63	0.053

Let U be the probability that a keyboard is defective (column three) Let E_1, E_2 be the probability that a keyboard is from a particular city (column 2, reading down the column). We want the probability that a keyboard is from a particular city, knowing that it is defective Compute

Pr(manufactured in Los Angeles | the keyboard is defective

$$\Pr(E_1 \mid U) = \frac{(0.37)(0.02)}{(0.37)(0.02) + (0.63)(0.053)} = 0.181$$

And

$$\Pr(E_2 \mid U) = \frac{(0.63)(0.053)}{(0.37)(0.02) + (0.63)(0.053)} = 0.819.$$

3. Display the data as follows:

	survive < 1 yr.	survive ≥ 2 yrs.	total group pop
adult men	619	257	876
adult women	471	320	791
boys	155	104	155
girls	190	152	190

(a) Suppose a patient is selected. We want the probability that the patient is an adult man, if we are told that he or she survived at least two more years. Compute

Pr(adult man | survived ≥ 2 years)

Now the total number of patients is $876 + 791 + 155 + 190 = 2012$ The probability of choosing an adult man is therefore (to the best of our information) $\frac{876}{2012}$ Arguing similarly for the other probabilities in the general Bayes formula, we get

Pr(adult man | survived ≥ 2 years) =

$$\frac{\left(\frac{190}{2012}\right)\left(\frac{38}{190}\right)}{\left(\frac{876}{2012}\right)\left(\frac{257}{876}\right)+\left(\frac{791}{2012}\right)\left(\frac{320}{791}\right)+\left(\frac{155}{2012}\right)\left(\frac{104}{155}\right)+\left(\frac{190}{2012}\right)\left(\frac{152}{190}\right)}$$
$$= 0.309$$

(b) Compute

$$\Pr(\text{choose a girl} \mid \text{person survived } < 1 \text{ year}) =$$
$$= \frac{\left(\frac{190}{2012}\right)\left(\frac{38}{190}\right)}{\left(\frac{190}{2012}\right)\left(\frac{38}{190}\right)+\left(\frac{155}{2012}\right)\left(\frac{51}{155}\right)+\left(\frac{791}{2012}\right)\left(\frac{471}{791}\right)+\left(\frac{876}{2012}\right)\left(\frac{619}{876}\right)}$$
$$= 0.032.$$

4. Write the information in decimal form:

	% of total	% fatal
Lower Amazon	0.62	0.14
Sub-Saharan Africa	0.25	0.06
Newark, New Jersey	0.13	0.005

For each of these three locations, we want the probability that the herb came from that location, knowing that it was fatal. Let E_j be the event that the herb came from location j, ordered down column one of the table. Then

$$\Pr(\text{came from Lower Amazon} \mid \text{fatal}) = \Pr(E_1 \mid U)$$
$$= \frac{\Pr(E_1)\Pr(U \mid E_1)}{\Pr(E_1)\Pr(U \mid E_1)+\Pr(E_2)\Pr(U \mid E_2)+\Pr(E_3)\Pr(U \mid E_3)}$$
$$= \frac{(0.62)(0.14)}{(0.62)(0.14)+(0.25)(0.06)+(0.13)(0.005)} = 0.847,$$

and, similarly,

$$\Pr(E_2 \mid U)$$
$$= \frac{(0.25)(0.06)}{(0.62)(0.14)+(0.25)(0.06)+(0.13)(0.005)} = 0.146$$

and

$$\Pr(E_3 \mid U)$$
$$= \frac{(0.13)(0.005)}{(0.62)(0.14)+(0.25)(0.06)+(0.13)(0.005)} = 0.0063$$

5. Let E_j be the event that the gun is from city j in the table. Let U be the event that the gun explodes. The denominator in the general Bayes formula is

$$(.15)(.02)+(.07)(.01)+(.21)(.06)+(.04)(.02)+(.09)(.03)+(.44)(.09)$$
$$= 0.0594.$$

Then

$$\Pr(E_1 \mid U) = \frac{(.15)(.02)}{.0594} = 0.0505,$$

$$\Pr(E_2 \mid U) = \frac{(.07)(.01)}{0594} = 0\ 0118,$$

$$\Pr(E_3 \mid U) = \frac{(.21)(.06)}{.0594} = 0\ 212,$$

$$\Pr(E_4 \mid U) = \frac{(.04)(.02)}{0594} = 0\ 0135,$$

$$\Pr(E_5 \mid U) = \frac{(.09)(.03)}{.0594} = 0.0455,$$

$$\Pr(E_6 \mid U) = \frac{(.44)(.09)}{.0594} = 0.666.$$

Section 26.11 Expected Value

1. First compute the probability that at least three heads come up in seven flips. It is easier to compute the complementary probability that fewer than three heads come up, which means that no heads, one head or two heads come up. The number of ways this can happen is ${}_7C_2 + {}_7C_1 + {}_7C_0$, which equals 29 Since there are $2^7 = 128$ outcomes, then

$$\Pr(\text{fewer than three heads}) = \frac{29}{128}$$

so

$$\Pr(\text{at least three heads}) = 1 - \frac{29}{128} = \frac{99}{128}$$

The player wins five dollars if there are at least three heads Therefore the player's expected value is

$$\frac{99}{128}(5) - \frac{29}{128}(9),$$

which is about \$1 83. On average, the player expects to win this amount each time the game is played.

2 Toss five dice The player wins if at least three come up 5 or 6 First we need the probability of this event To get at least three dice to come up 5 or 6, we can have exactly three, exactly four or exactly five. For exactly three, imagine picking 3 of the five dice (${}_5C_3 = 10$ ways), to come up 5 or 6 (8 ways to arrange three dice in fives and sixes) The other two dice can have any other number, for four possibilities each There are therefore $(10)(8)(4)(4) = 1280$ ways of getting three fives or sixes.

Now for four fives or sixes. Now pick any four of the dice (${}_5C_4 = 5$ ways to do this). There are 16 ways to pick four of the five dice to be fives or sixes The remaining die can have any of the other four numbers. Therefore there are $16(5)(4) = 320$ ways of getting four fives or sixes

Finally, for five fives and sixes, each die is a five or six. The number of ways of doing this is ${}_5C_5 +_5 C_4 +_5 C_3 +_5 C_2 +_5 C_1 +_5 C_0$, or 32

The number of ways of getting at least three dice come up five or six is $1280 + 320 + 32$, or 1632 Therefore

$$\Pr(\text{at least three come up 5 or 6}) = \frac{1632}{6^5} = \frac{17}{81},$$

about 0.21

The expected value to the player is

$$\frac{17}{81}(50) - \left(1 - \frac{17}{81}\right)(9),$$

about \$3.38.

3. Deal six cards, without regard to order. There are ${}_{52}C_6 = 20,358,520$ hands The number of hands with no aces is ${}_{48}C_6 = 12,271,512$ Then

$$\Pr(\text{no ace}) = \frac{12271512}{20358520},$$

about 0.603. Then

$$\Pr(\text{at least one ace}) = 1 - \frac{12271512}{20358520},$$

about 0.397. For the player, the expect value is about

$$(.397)(45) - (.603)(30),$$

or -0.225. The player can expect on average to lose between 22 and 23 cents per game.

4. Flip nine coins. There are $2^9 = 512$ outcomes. We need the probability that seven or more come up tails. This happens if exactly seven, or exactly eight, or exactly nine coins come up tails. This can happen in ${}_9C_7 +{}_9 C_8 +{}_9 C_9$ ways, or 46 ways. Then

$$\Pr(\text{at least seven tails}) = \frac{46}{512},$$

about 0 09. The expected value for the player is

$$\frac{46}{512}(50) - \left(1 - \frac{46}{512}\right)(3),$$

about 1 7617 On average, the player expects to win about \$1 76 per game.

5. There are ${}_{20}C_3 = 1140$ ways to do this, disregarding the order of the draw We need the probability that at least two are numbered even. This happens if exactly two or exactly three have even numbers. For exactly two even, we can pick two of the ten even numbers (${}_{10}C_2$ ways), and with each choice, any of the ten odd, for $10({}_{10}C_2)$ ways of picking exactly two even. We can pick exactly three even in ${}_{10}C_3$ ways. Then the number of ways of picking two or three even numbered marbles is $(10)({}_{10}C_2) +{}_{10} C_3$, or 570 Then

$$\Pr(\text{two or three even numbers chosen}) = \frac{570}{1140} = \frac{1}{2},$$

as we might expect (any draw should yield no odd or one odd, or two odd or three odd, and each of these complementary events has the same probability) The expected value to the player is

$$3\left(\frac{1}{2}\right) - 7\left(\frac{1}{2}\right),$$

or -2. On average, the player expects to lose \$2 per game

6 If 16 dice are rolled, there are $6^{16} = 2,821,109,907,456$ outcomes. We need Pr(at least five sixes) At least five sixes means exactly five, six, seven, … , sixteen sixes It is easier to look at the

complementary event that fewer than five sixes come up. This occurs if exactly no sixes, or exactly one, two, three, or four sixes come up

There are 5^{16} ways to have no sixes (each die comes up $1, 2, 3, 4, 5$) There are $({}_{16}C_2)5^{14}$ ways to get exactly two sixes (pick two dice and imagine them sixes, the other fourteen dice have no sixes). Similarly, there are $({}_{16}C_3)5^{13}$ ways to get exactly three sixes, and $({}_{16}C_4)5^{12}$ ways to get exactly four sixes. Then

$$\Pr(< 5 \text{ sixes}) = \frac{5^{16} + ({}_{16}C_2)5^{14} + ({}_{16}C_3)5^{13} + ({}_{16}C_4)5^{12}}{6^{16}}$$
$$= \frac{5^{16} + 120(5^{14}) + 560(5^{13}) + 1820(5^{12})}{6^{16}},$$

about 0.714 This means that

$$\Pr(\geq 5 \text{ sixes}) = 1 - \Pr(< 5 \text{ sixes}),$$

is about equal to $1 - 0.714$, or 0.286. The expected value for the player is

$$(0.286)(70)1 - (0.714)(4)$$

which is approximately equal to 17.164. The player can expect to average about \$17.16 per game.

7. We need the probability that, in choosing four of the hats (without regard to order), we draw at least three red hats (meaning exactly two or exactly three hats) If we draw all three red hats, then we must draw one blue hat, and there are four ways to do this If we draw exactly two red hats, there are 3 ways to do this with three red hats available, and then there are ${}_4C_2$ ways of picking the other two hats from the four blue hats This means there are $3(6) = 18$ ways of picking exactly two red hats Then there are 22 ways of drawing at least two red hats, and

$$\Pr(\geq 2 \text{ red hats}) = \frac{22}{35}$$

The payoff for the player is

$$\left(\frac{22}{35}\right)(10) - \left(1 - \frac{22}{35}\right)(5),$$

which is about 4.42. The player can expect on average to win \$4.42 per game.

8. Flip six coins and roll four dice. We need the probability that four heads or a total of at least 24 on the dice come up.

We interpret "four" to mean at least four, not exactly four Thus we need

$$\Pr(\geq 4 \text{ heads or } \geq 24 \text{ total on the dice}).$$

There are $2^6 6^4$ outcomes of six coin flips and four dice rolls.

Next, there are 22 ways to flip at least four heads (${}_6C_4 = 15$ ways of getting exactly four, 6 ways of getting exactly five heads, and 1 way of getting all six heads). With each of these, the dice can come out any of 6^4 ways, for $(22)(6^4)$ possible ways to flip at least four heads. There is only one way for four dice to total 24, so there are 2^6 outcomes of six coin flips and four dice rolls, with the dice totalling 24 Then

$$\Pr(\geq 4 \text{ heads of } \geq 24 \text{ total on the dice})$$

$$= \frac{(22)(6^4) + 2^6}{2^6 6^4},$$

about 0.345. The expected value of the player is therefore approximately

$$(0\ 345)(25) - (1 - 0\ 345)(3),$$

which is about 6.66 The player can expect on average to win \$6.66 each game.

Chapter Twenty Seven - Statistics

Section 27.1 Measures of Center and Variation

1. (a) The mean is

$$\overline{x} = \frac{-5-2-1.5+0+0+1+1+1+2+7+9+11.6}{12}$$
$$= 2.0083$$

(to four decimal places). For s, first compute

$$\sum_{j=1}^{12}(x_j - \overline{x})^2 = 254.41$$

Then

$$s = \sqrt{\frac{254.41}{11}} = 4.809$$

For comparison,

$$\frac{1}{4}(\text{largest number} - \text{smallest number}) = \frac{1}{4}(11.6-(-5)) = 4.15$$

(b)

$$\overline{x} = \frac{7+1-4+2+2+1+4+1+1+2+6+3}{11} = 2.1667$$

After a little arithmetic we get

$$s = \sqrt{\frac{85.667}{11}} = 2.7907$$

We also find that

$$\frac{1}{4}(\text{largest number} - \text{smallest number}) = \frac{1}{4}(7-(-4)) = 2.75$$

(c)

$$\overline{x} = \frac{5-2-10+5+2+2+5+3+5}{9} = 1.6667$$

and

$$s = \sqrt{\frac{196}{8}} = 4.9497$$

For comparison,

$$\frac{1}{4}(\text{largest number } - \text{ smallest number}) = \frac{1}{4}(5-(-10)) = 3.75$$

2. (a) The mean is

$$\overline{x} = \frac{-4-6+2.5+3+8+5-3-4+8+3+2.2}{11} = 1.3364.$$

For the median, write the list in nondescending order:

$$-6, -4, -4, -3, 2.2, 2.5, 3, 3, 5, 8, 8.$$

With an odd number of numbers in the list, the median is the sixth number, 2.5

The standard deviation is

$$s = \sqrt{\frac{239.44}{10}} = 4.8933$$

(b) The mean is

$$\overline{x} = \frac{1+1+1-1+2+3-1+4+2}{9} = 1.3333$$

Arranged in nondescending order, the sequence is

$$-1, -1, 1, 1, 1, 2, 2, 3, 4$$

The median is 1 (the fifth number from the left).

The standard deviation is

$$s = \sqrt{\frac{22}{8}} = 1.6583.$$

(c) The mean is

$$\overline{x} = \frac{3-4+2+1.5-4-4+2+1+7}{9} = 0.5$$

In nondecreasing order, the data list is

$$-4, -4, -4, 1, 1.5, 2, 2, 3, 7.$$

The median is 1.5

The standard deviation is

$$s = \sqrt{\frac{115}{8}} = 3.7914$$

(d) The mean is

$$\overline{x} = \frac{9.3+9.5+9.7+10+8.4+8.7+8.8+8.8+4.1}{9} = 8.5889.$$

Rewrite the data list as

$$4.1, 8.4, 8.7, 8.8, 8.8, 9.3, 9.5, 9.7, 10$$

The median is 8.8, the fifth number from the left.

The standard deviation is

$$s = \sqrt{\frac{24.849}{8}} = 1.7624.$$

(e) The mean is

$$\overline{x} = \frac{-16 - 14 - 10 + 0 + 0 + 1 + 1 + 3 + 5 + 7}{10} = -2.3$$

The data is already listed in nondescending order. The median is the average of the fifth and sixth entries, or $(0 + 1)/2 = 0.5$

The standard deviation is

$$s = \sqrt{\frac{584.1}{9}} = 8.0561.$$

3. The mean is

$$\overline{x} = \frac{4(-3) + 2(-1) + 6(0) + 4(1) + 12(3) + 3(4)}{31} = 1.2258$$

The median is the sisteenth number from the left, or 1. The standard deviation is

$$s = \sqrt{\frac{151.42}{30}} = 2.2466.$$

4. The mean is

$$\overline{x} = \frac{4(-12) + 2(-9.7) + 6(-8) + 4(-7.6) + 12(-5.1) + 3(4)}{31} = -6.2903.$$

The data is already listed in nondescending order. The number of entries is 31, an odd number. The sixteenth number from the left is the median. This is -7.6 (the right most -7.6 in the string of four such numbers).

The standard deviation is

$$s = \sqrt{\frac{512.73}{30}} = 4.1341$$

Section 27.2 Random Variables and Probability Distributions

1. If we roll two dice, there are 36 possible outcomes. The sum of the numbers than can come up on the two dice are as follows:

2	3	4	5	6	7
3	4	5	6	7	8
4	5	6	7	8	9
5	6	7	8	9	10
6	7	8	9	10	11
7	8	9	10	11	12

For example, if o is the outcome that one die comes up 2 and the other 3, then the sum of the dice is 5. This number occurs four times in the outcomes of rolling two dice, so $X(5) = 4$

Below is a frequency table for these sums, giving the number of times each occurs:

sum	2	3	4	5	6	7
frequency	1	2	3	4	5	6

sum	8	9	10	11	12
frequency	5	4	3	2	1

Based on this table, we assign probabilities to the random variable as follows:

$$P(2) = \frac{1}{36}$$

(that is, the probability that the random variable has value 2 is 1/36, because this number comes up only once in the thirty six outcomes);

$$P(3) = P(11) = \frac{2}{36}$$

(the probability that the random variable has value 3 or 11 is 2/36, because each of these sums of the two dice occurs twice in the thirty six outcomes),

$$\begin{aligned} P(4) = P(10) &= \frac{3}{36}, \\ P(5) = P(9) &= \frac{4}{36}, \\ P(6) = P(8) &= \frac{5}{36}, \\ P(7) &= \frac{6}{36}, \\ P(12) &= \frac{1}{36} \end{aligned}$$

Notice that $\sum_x P(x) = 1$, as required.

The mean of X is

$$\begin{aligned} \mu &= \sum_x xP(x) \\ &= 2\left(\frac{1}{36}\right) + 3\left(\frac{2}{36}\right) + 4\left(\frac{3}{36}\right) + 5\left(\frac{4}{36}\right) + 6\left(\frac{5}{36}\right) + 7\left(\frac{6}{36}\right) \\ &\quad +8\left(\frac{5}{36}\right) + 9\left(\frac{4}{36}\right) + 10\left(\frac{3}{36}\right) + 11\left(\frac{2}{36}\right) + 12\left(\frac{1}{36}\right) \\ &= 7 \end{aligned}$$

This is interpreted to mean that, on average, we expect to come up with a seven if we roll two dice This is a reasonable expectation in view of the fact that there are more ways to roll 7 than any other sum.

The standard deviation is

$$\sigma = \sqrt{\sum_x (x-7)^2 P(x)}$$

Compute

$$\sum_x (x-7)^2 P(x)$$

$$= (2-7)^2\left(\frac{1}{36}\right) + (3-7)^2\left(\frac{2}{36}\right) + (4-7)^2\left(\frac{3}{36}\right) + (5-7)^2\left(\frac{4}{36}\right)$$

$$+(6-7)^2\left(\frac{5}{36}\right) + (7-7)^2\left(\frac{6}{36}\right) + (8-7)^2\left(\frac{5}{36}\right) + (9-7)^2\left(\frac{4}{36}\right)$$

$$+(10-7)^2\left(\frac{3}{36}\right) + (11-7)^2\left(\frac{2}{36}\right) + (12-7)^2\left(\frac{1}{36}\right)$$

$$= 5.8333.$$

Then

$$\sigma = \sqrt{5.8333} = 2.4152.$$

2 Flip four coins, with sixteen possible outcomes. If o is an outcome, $X(o)$ can have only two possible values, namely 1 if two, three or four tails are in o, and 3 otherwise (one tail or no tails). There are five outcomes with one tail or no tail, and eleven outcomes with two or more tails, so

$$P(1) = \frac{11}{16} \text{ and } P(3) = \frac{5}{16}.$$

The mean is

$$\mu = \sum_x xP(x) = 1\left(\frac{11}{16}\right) + 3\left(\frac{5}{16}\right) = \frac{26}{16} = 1.625.$$

For the standard deviation of X, first compute

$$\sum_x (x-\mu)^2 P(x)$$

$$= (1-1.625)^2\left(\frac{11}{16}\right) + (3-1.625)^2\left(\frac{5}{16}\right) = 0.85938$$

Then

$$\sigma = \sqrt{0.85938} = 0.92703$$

3. We have

$$X(1) = 0,$$

$$X(2) = X(3) = X(5) = X(7) = X(11) = X(13) = X(17) = X(19) = 1,$$

$$X(4) = X(6) = X(9) = X(10) = X(14) = X(15) = 2,$$

$$X(8) = X(12) = X(18) = X(20) = 3,$$

$$X(16) = 4$$

The values assumed by X are $0, 1, 2, 3, 4$. From the list of values, we get

$$P(0) = \frac{1}{20}, P(1) = \frac{8}{20}, P(2) = \frac{6}{20},$$

$$P(3) = \frac{4}{20}, P(4) = \frac{1}{20}.$$

These are the probabilities of the values of the random variable X.

The mean of X is

$$\mu = \sum_x xP(x)$$

$$= 0\left(\frac{1}{20}\right) + 1\left(\frac{8}{20}\right) + 2\left(\frac{6}{20}\right) + 3\left(\frac{4}{20}\right) + 4\left(\frac{1}{20}\right) = 1.8$$

The determine the standard deviation of X, first compute

$$\sum_x (x-\mu)^2 P(x)$$

$$= (0-1.8)^2\left(\frac{1}{20}\right)^2 + (1-1.8)^2\left(\frac{8}{20}\right)^2 + (2-1.8)^2\left(\frac{6}{20}\right)^2$$

$$= (3-1.8)^2\left(\frac{4}{20}\right)^2 + (4-1.8)^2\left(\frac{1}{20}\right)^2$$

$$= 0.1838$$

Then

$$\sigma = \sqrt{0.1838} = 0.42872$$

4. The outcomes of two rolls of the dice are

1, 1	1, 2	1, 3	1, 4	1, 5	1, 6
2, 1	2, 2	2, 3	2, 4	2, 5	2, 6
3, 1	3, 2	3, 3	3, 4	3, 5	3, 6
4, 1	4, 2	4, 3	4, 4	4, 5	4, 6
5, 1	5, 2	5, 3	5, 4	5, 5	5, 6
6, 1	6, 2	6, 3	6, 4	6, 5	6, 6

Then

$$X(1,1) = 1, X(2,2) = 2, X(3,3) = 3,$$

$$X(4,4) = 4, X(5,5) = 5, X(6,6) = 6,$$

$$X(1,2) = X(2,1) = X(2,4) = X(4,2) = X(3,6) = X(6,3) = 2,$$

$$X(1,3) = X(3,1) = X(2,6) = X(6,2) = 3,$$

$$X(1,4) = X(4,1) = 4, X(1,5) = X(5,1) = 5, X(1,6) = X(6,1) = 6,$$

$$X(2,3) = X(3,2) = X(4,6) = X(6,4) = 3/2,$$

$$X(2,5) = X(5,2) = 5/2,$$

$$X(3,4) = X(4,3) = 4/3,$$

$$X(5,3) = X(3,5) = 5/3,$$

$$X(4,5) = X(5,4) = 5/4,$$

$$X(5,6) = X(6,5) = 6/5$$

The values X assumes are $1, 2, 3, 4, 5, 6, 3/2, 5/2, 4/3, 5/3, 5/4, 6/5$. We find that

$$P(1) = \frac{1}{36}, P(2) = \frac{7}{36}, P(3) = \frac{5}{36},$$

$$P(4) = \frac{3}{36}, P(5) = \frac{3}{36}, P(6) = \frac{3}{36},$$

$$P(3/2) = \frac{4}{36}, P(5/2) = P(4/3) = P(5/3) = P(5/4) = P(6/5) = \frac{2}{36}.$$

The mean is

$$\begin{aligned}\mu &= \sum_x xP(x)\\ &= 1\left(\frac{1}{36}\right) + 2\left(\frac{7}{36}\right) + 3\left(\frac{5}{36}\right) + 4\left(\frac{3}{36}\right) + 5\left(\frac{3}{36}\right) + 6\left(\frac{3}{36}\right)\\ &+\frac{3}{2}\left(\frac{4}{36}\right) + \frac{5}{2}\left(\frac{2}{36}\right) + \frac{4}{3}\left(\frac{2}{36}\right) + \frac{5}{3}\left(\frac{2}{36}\right) + \frac{5}{4}\left(\frac{2}{36}\right) + \frac{6}{5}\left(\frac{2}{36}\right)\\ &= 2.6916.\end{aligned}$$

To compute the standard deviation, first calculate

$$\begin{aligned}&\sum_x (x-\mu)^2 P(x)\\ &= (1-2.6916)^2\left(\frac{1}{36}\right) + (2-2.6916)^2\left(\frac{7}{36}\right) + (3-2.6916)^2\left(\frac{5}{36}\right)\\ &+(4-2.6916)^2\left(\frac{3}{36}\right) + (5-2.6916)^2\left(\frac{3}{36}\right) + (6-2.6916)^2\left(\frac{3}{36}\right)\\ &+\left(\frac{3}{2}-2.6916\right)^2\left(\frac{4}{36}\right) + \left(\frac{5}{2}-2.6916\right)^2\left(\frac{2}{36}\right) + \left(\frac{4}{3}-2.6916\right)^2\left(\frac{2}{36}\right)\\ &+\left(\frac{5}{3}-2.6916\right)^2\left(\frac{2}{36}\right) + \left(\frac{5}{4}-2.6916\right)^2\left(\frac{2}{36}\right) + \left(\frac{6}{5}-2.6916\right)^2\left(\frac{2}{36}\right)\\ &= 2.2442.\end{aligned}$$

Then

$$\sigma = \sqrt{2.2442} = 1.4981.$$

5. Draw two cards from a deck. There are ${}_{52}C_2 = 1,326$ ways to do this (disregarding order).

If o is an outcome in which both cards are numbered, then $X(o) =$ sum of the numbers on the cards. If exactly one of the cards is a face card or ace, then $X(o) = 11$, and if both cards are from the face cards and aces, then $X(o) = 12$. Therefore the values of $X(o)$ for all the possible outcomes are $4, 5, 6, \cdots$,By a routine but tedious counting of the ways the numbered cards can total $4, 5, \cdots, 20$, we find that

$$P(4) = \frac{6}{1326}, P(5) = \frac{16}{1326}, P(6) = \frac{22}{1326}, P(7) = \frac{32}{1326},$$

$$P(8) = \frac{38}{1326}, P(9) = \frac{48}{1326}, P(10) = \frac{54}{1326}, P(11) = \frac{640}{1326},$$

$$P(12) = \frac{190}{1326}, P(13) = \frac{64}{1326}, P(14) = \frac{54}{1326}, P(15) = \frac{48}{1326},$$

$$P(16) = \frac{38}{1326}, P(17) = \frac{32}{1326}, P(18) = \frac{22}{1326}, P(19) = \frac{16}{1326}$$

and

$$P(20) = \frac{6}{1326}$$

Using these, compute

$$\mu = \sum_x xP(x) = 11\,566,$$

and

$$\sigma = \sqrt{(x-\mu)^2 P(x)} = \sqrt{5\,4289} = 2\,33$$

6 X takes on the values 1, 2 and π. In particular,

$$X(1) = X(5) = X(7) = X(11) = X(13) = X(17)$$

$$X(19) = X(23) = X(25) = X(29) = \pi,$$

$$X(2) = X(4) = X(6) = \cdots = X(\text{even integer}) = 1$$

$$X(3) = X(9) = X(15) = X(21) = X(27) = 2$$

Then

$$P(1) = \frac{15}{30} = \frac{1}{2}, P(2) = \frac{5}{30} = \frac{1}{6}, \text{ and } P(\pi) = \frac{10}{30} = \frac{1}{3}$$

Then

$$\mu = 1\left(\frac{1}{2}\right) + 2\left(\frac{1}{6}\right) + \pi\left(\frac{1}{3}\right) = \frac{5}{6} + \frac{\pi}{3},$$

approximately 1.8805.

For the standard deviation, first compute

$$\sum_x (x-\mu)^2 P(x)$$

$$= (1 - 1.8805)^2\left(\frac{1}{2}\right) + (2 - 1\,8805)^2\left(\frac{1}{6}\right) + (\pi - 1\,8805)^2\left(\frac{1}{3}\right),$$

which is approximately 0 920 14 Then

$$\sigma = \sqrt{0.92014} = 0\,95924$$

Section 27.3 The Binomial and Poisson Distributions

1 (a)

$$P(2) = \binom{8}{2}(0.43)^2(1-0.43)^6 = 0.17756$$

(b)

$$P(3) = \binom{4}{3}(0.7)^3(1-0.7) = 0.4116$$

(c)

$$P(3) = \binom{6}{3}(0.5)^3(1-0.5)^3 = 0.3125$$

(d)

$$\begin{aligned} &P(2)+P(3)+P(4)+P(5) \\ &= \binom{10}{2}(0.6)^2(0.4)^8 + \binom{10}{3}(0.6)^3(0.4)^7 \\ &+ \binom{10}{4}(0.6)^4(0.4)^6 + \binom{10}{5}(0.6)^5(0.4)^5 \\ &= 0.36521 \end{aligned}$$

(e)

$$\begin{aligned} P(7)+P(8) &= \binom{8}{7}(0.4)^7(0.6) + \binom{8}{8}(0.4)^8 \\ &= 0.0085197 \end{aligned}$$

(f)

$$\begin{aligned} &P(2)+P(3)+P(4) \\ &= \binom{10}{2}(0.58)^2(0.42)^8 + \binom{10}{3}(0.58)^3(0.42)^7 + \binom{10}{4}(0.58)^4(0.42)^6 \\ &= 0.19908 \end{aligned}$$

(g)

$$\begin{aligned} P(3)+P(7) &= \binom{10}{3}(0.35)^3(0.65)^7 + \binom{10}{7}(0.35)^7(0.65)^3 \\ &= 0.27342 \end{aligned}$$

(h)

$$\begin{aligned} &P(1)+P(3)+P(5) \\ &= \binom{7}{1}(0.24)(0.76)^6 + \binom{7}{3}(0.24)^3(0.76)^4 + \binom{7}{5}(0.24)^5(0.76)^2 \\ &= 0.49481 \end{aligned}$$

2 (a) The probability of getting exactly one question right is

$$P(1) = \binom{6}{1}(0.25)(0.75)^5 = 0.35596.$$

(b) The probability of getting exactly two questions right is

$$P(2) = \binom{6}{2}(0.25)^2(0.75)^4 = 0.29663$$

(c) The probability of getting four or five right is

$$P(4) + P(5) = \binom{6}{4}(0.25)^4(0.75)^2 + \binom{6}{5}(0.25)^5(0.75)$$
$$= 0.037354$$

(d) The probability of getting all six right is

$$P(6) = \binom{6}{6}(0.25)^6 = 0.00024414$$

3 Now the number of choices for each question is three, instead of four, in the setting of Problem 2. This changes the probability of guessing the right answer on any given question from 1/4 to 1/3. Now we have the following results.

(a) The probability of getting exactly one question right is

$$P(1) = \binom{6}{1}(1/3)(2/3)^5 = 0.26337$$

(b) The probability of getting exactly two questions right is

$$P(2) = \binom{6}{2}(1/3)^2(2/3)^4 = 0.32922$$

(c) The probability of getting four or five right is

$$P(4) + P(5) = \binom{6}{4}(1/3)^4(2/3)^2 + \binom{6}{5}(1/3)^5(2/3) = 0.098765.$$

(d) The probability of getting all six right is

$$P(6) = \binom{6}{6}(1/3)^6 = 0.0013717.$$

4 Now there are ten questions, each with four choices. The probability of guessing the right answer on any question is 0.25

(a) The probability of getting exactly one question right is

$$P(1) = \binom{10}{1}(0.25)(0.75)^9 = 0.18771.$$

(b) The probability of getting exactly two questions right is

$$P(2) = \binom{10}{2}(0.25)^2(0.75)^8 = 0.28157$$

(c) The probability of getting four or five right is

$$P(4) + P(5)$$

$$= \binom{10}{4}(0.25)^4(0.75)^6 + \binom{10}{5}(0.25)^5(0.75)^5 = 0.20440$$

(d) The probability of getting all ten right is

$$P(10) = \binom{10}{10}(0.25)^{10} = 9.5367(10^{-7})$$

(e) The probability of getting five, six or seven questions right is

$$P(5) + P(6) + P(7)$$

$$= \binom{10}{5}(0.25)^5(0.75)^5 + \binom{10}{6}(0.25)^6(0.75)^4 + \binom{10}{7}(0.25)^7(0.75)^3$$

$$= 7.7711(10^{-2}).$$

5. Here, $N = 100$, $p = 0.032$ (probability of producing a defective starter), and we want the probality that two or fewer starters are defective This number is

$$P(\leq 2) = P(0) + P(1) + P(2)$$

$$= \binom{100}{0}(0.032)^0(1-0.032)^{100} + \binom{100}{1}(0.032)(1-0.032)^{99}$$

$$+ \binom{100}{2}(0.032)^2(1-0.032)^{98} = 0.37583.$$

6. Here $N = 1000$ and $p = 0.07$. We want

$$P(92) = \binom{1000}{92}(0.07)^{92}(1-0.07)^{908} = 0.0014477.$$

8. (a)

$$P(6) = \frac{(0.9)^6 e^{-0.9}}{6!} = 3.0009(10^{-4})$$

(b)

$$P(10) = \frac{(0.85)^{10} e^{-0.85}}{10!} = 2.3189(10^{-8})$$

(c)

$$P(4) = \frac{(0.92)^4 e^{-0.92}}{4!} = 0.011896$$

(d)

$$P(8) = \frac{(0.87)^8 e^{-0\ 87}}{8!} = 3\ 4103(10^{-6})$$

(e) We want

$$\begin{aligned} &P(1) + P(2) + P(3) + P(4) + P(5) \\ &= \frac{(0.94)^1 e^{-0\ 94}}{1!} + \frac{(0.94)^2 e^{-0\ 94}}{2!} + \frac{(0.94)^3 e^{-0\ 94}}{3!} \\ &+ \frac{(0.94)^4 e^{-0\ 94}}{4!} + \frac{(0.94)^5 e^{-0\ 94}}{5!} \\ &= 0\ 60894 \end{aligned}$$

(f)

$$\begin{aligned} &P(3) + P(4) + P(5) \\ &= \frac{(0.64)^3 e^{-0\ 64}}{3!} + \frac{(0.64)^4 e^{-0\ 64}}{4!} + \frac{(0.64)^5 e^{-0\ 64}}{5!} \\ &= 0\ 027196 \end{aligned}$$

(g)

$$\begin{aligned} P(3) + P(8) &= \frac{(0.75)^3 e^{-0\ 75}}{3!} + \frac{(0.75)^8 e^{-0\ 75}}{8!} \\ &= 0\ 033214 \end{aligned}$$

(h)

$$\begin{aligned} &P(1) + P(3) + P(10) \\ &= \frac{(0.97)^1 e^{-0\ 97}}{1!} + \frac{(0.97)^3 e^{-0\ 97}}{3!} + \frac{(0.97)^{10} e^{-0\ 97}}{10!} \\ &= 0\ 42537 \end{aligned}$$

9. Let $\mu = 295/320 = 0.922$ (to three places)

(a)

$$P(3) = \frac{(0.922)^3 e^{-0.922}}{3!} = 0.052$$

(b)

$$\begin{aligned} &P(2) + P(3) + P(4) + P(5) \\ &= \frac{(0.922)^2 e^{-0\ 922}}{2!} + \frac{(0.922)^3 e^{-0\ 922}}{3!} \\ &+ \frac{(0.922)^4 e^{-0\ 922}}{4!} + \frac{(0.922)^5 e^{-0\ 922}}{5!} \\ &= 0.23519 \end{aligned}$$

10 Take

$$\mu = \frac{997850}{1000000} = 0\ 998$$

We want

$$P(3) = \frac{(0.998)^3 e^{-0\ 998}}{3!} = 0.061068$$

11. Take

$$\mu = \frac{476}{500} = 0.952$$

The probability that any region was hit exactly three times is

$$P(3) = \frac{(0.952)^3 e^{-0.952}}{3!} = 0.055502$$

The probability that any region was hit from two to six times, inclusive, is

$$\begin{aligned} &P(2) + P(3) + P(4) + P(5) + P(6) \\ &= \frac{(0.952)^2 e^{-0.952}}{2!} + \frac{(0.952)^3 e^{-0.952}}{3!} + \frac{(0.952)^4 e^{-0.952}}{4!} \\ &\quad + \frac{(0.952)^5 e^{-0.952}}{5!} + \frac{(0.952)^6 e^{-0.952}}{6!} \\ &= 0.24653 \end{aligned}$$

12. The probability of getting a 15 is $1/24$, or 0.041667. Think of the interval as the 100 games, and μ is the ratio of occurrences of 15 per interval (100 games). This is, on average, $(100)(0.041667)$, or 4.1667.

(a)

$$P(7) = \frac{(4.1667)^7 e^{-4.1667}}{7!} = 0.067072$$

(b)

$$P(3) = \frac{(4.1667)^3 e^{-4.1667}}{3!} = 0.18692.$$

(c)

$$P(0) = \frac{(4.1667)^0 e^{-4.1667}}{0!} = 0.015503.$$

Section 27.4 Normally Distributed Data and the Bell Curve

1. Imagine rolling a die 90 times. We are interested in the outcome that a 6 comes up (by this we mean, a 6 on at least one die).

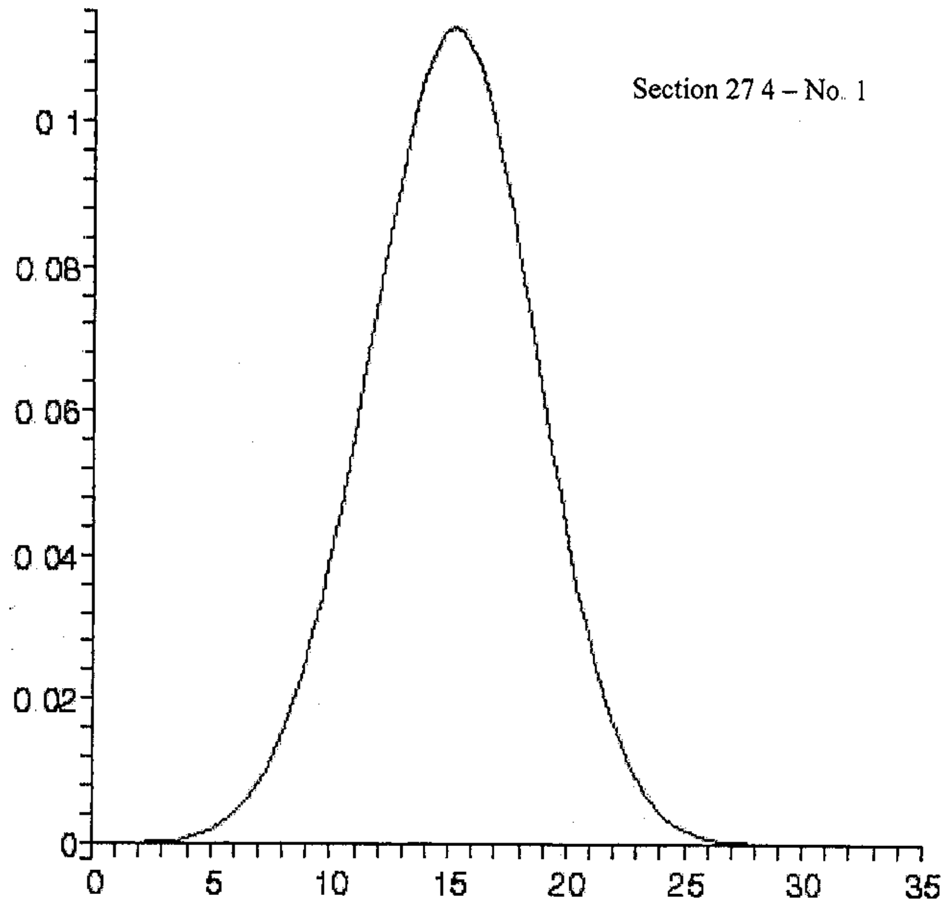
(a) Here $N = 90$, the number of rolls, and $p = 1/6$, the probability of a six on any toss. The mean is $\mu = Np = (90)(1/6) = 15$, and the standard deviation is

$$\sigma = \sqrt{Np(1-p)} = \sqrt{90\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)} = \sqrt{\frac{75}{6}} = 5\sqrt{\frac{3}{6}} = 5\sqrt{\frac{1}{2}},$$

about 3.5355.

(b) We have $2\sigma^2 = 25$. The bell curve for this experiment is the graph of the exponential function

$$\frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2} = \frac{1}{\sqrt{2\pi}(5)\sqrt{\frac{1}{2}}} e^{-(x-15)^2/25} = \frac{1}{5\sqrt{\pi}} e^{-(x-15)^2/25}$$



(c) The probability that a six comes up between 30 and 60 times is

$$P(30 \le x \le 60) = \frac{1}{5\sqrt{\pi}} \int_{30-1/2}^{60+1/2} e^{-(x-15)^2/25} dx = (2.0549)(10^{-5}).$$

As we might expect, this event is very unlikely

(d) The probability that a six comes up between 2 and 80 times is

$$P(2 \le x \le 80) = \frac{1}{5\sqrt{\pi}} \int_{3/2}^{161/2} e^{-(x-15)^2/25} dx = 0.99993.$$

It is extremely like that we will get somewhere between two and eighty sixes

(e) The probability that a six comes up at least thirty five times is

$$P(x \ge 35) = \frac{1}{5\sqrt{\pi}} \int_{69/2}^{\infty} e^{-(x-15)^2/25} dx = 1.7396(10^{-8})$$

(f) The probability that we get fewer than twenty sixes (that is, nineteen or less in number) is

$$P(x \leq 19) = \frac{1}{5\sqrt{\pi}} \int_{-\infty}^{39/2} e^{-(x-15)^2/25} dx = 0\ 89845$$

(g) The probability that exactly forty five heads came up is

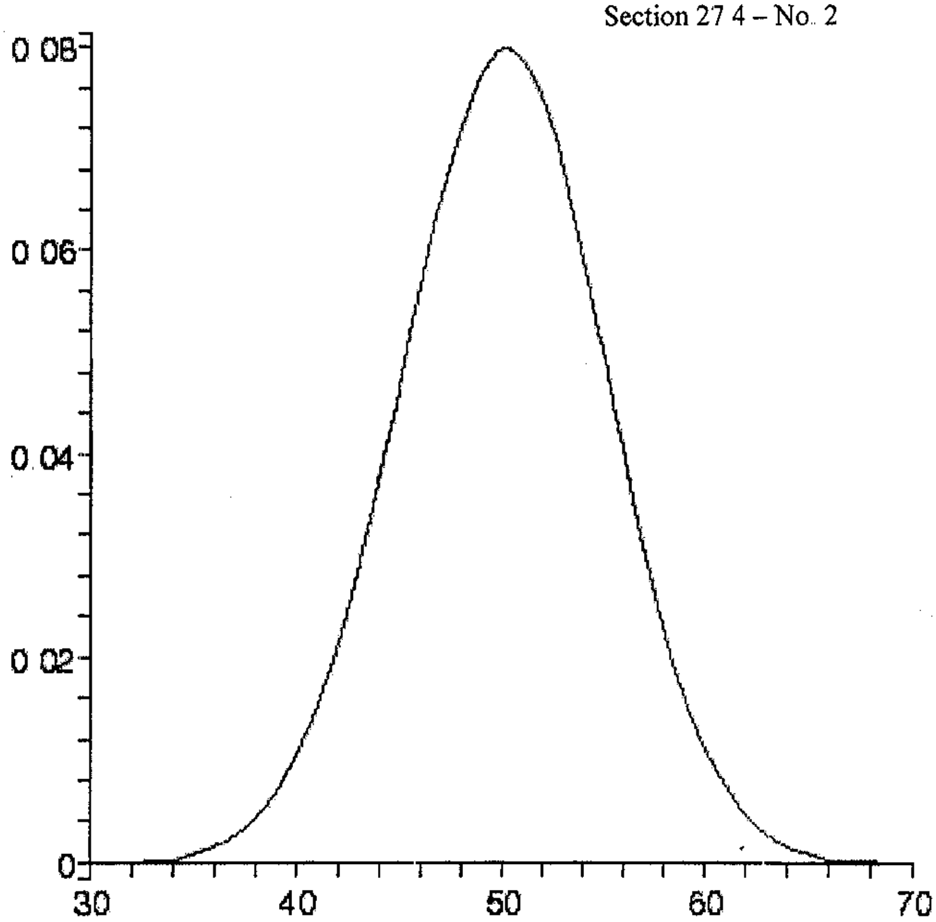
$$P(x = 45) = \frac{1}{5\sqrt{\pi}} \int_{89/2}^{91/2} e^{-(x-15)^2/25} dx = 3.2795(10^{-17}).$$

2. (a) Assuming an honest coin, the mean is $\mu = 100(1/2) = 50$ and the standard deviation is $\sigma = \sqrt{100(1/2)(1/2)} = 5$.

(b) The bell curve for this experiment is the graph of the exponential function

$$\frac{1}{5\sqrt{2\pi}} e^{-(x-50)^2/50}$$

Section 27 4 – No. 2



(c)

$$P(20 \le x \le 40)$$

$$= \frac{1}{5\sqrt{2\pi}} \int_{39/2}^{81/2} e^{-(x-50)^2/50} dx = .028717$$

(d)

$$P(10 \le x \le 50) = 0.53983.$$

(e)

$$P(x \ge 30) = \frac{1}{5\sqrt{2\pi}} \int_{59/2}^{\infty} e^{-(x-50)^2/50} dx = 0.99998.$$

(f) Fewer than sixty means fifty nine or less occurrences. This probability is

$$P(x \le 59) = \frac{1}{5\sqrt{2\pi}} \int_{-\infty}^{119/2} e^{-(x-50)^2/50} dx = 0.97128.$$

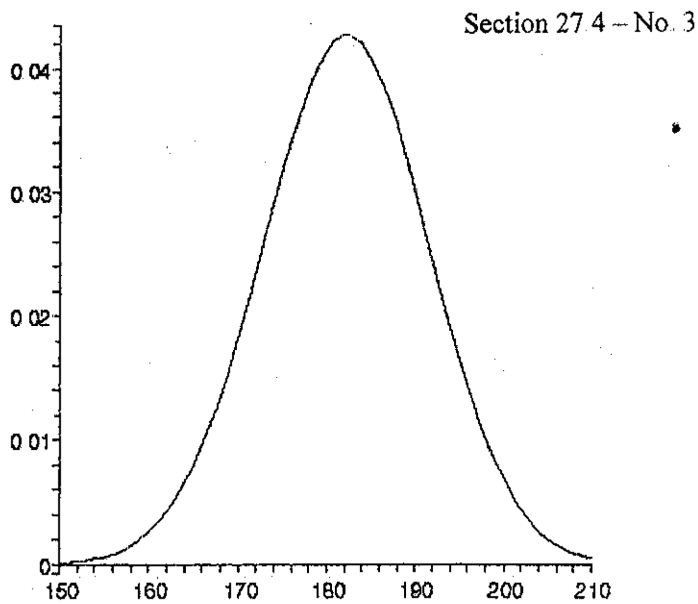
(g)

$$P(x = 55) = \frac{1}{5\sqrt{2\pi}} \int_{109/2}^{111/2} e^{-(x-50)^2/50} dx = 0.048394.$$

3. (a) Take $p = 0.52$, so $q = 1 - p = 0.48$. The mean is $\mu = (350)(0.52) = 182$. The standard deviation is $\sigma = \sqrt{(350)(0.52)(0.48)} = 9.3467$

(b) Compute $2\sigma^2 = 174.72$. The bell curve is the graph of the exponential function

$$\frac{1}{(9.3467)\sqrt{2\pi}} e^{-(x-182)^2/174.72}$$



(c)

$$P(x \geq 220) \doteq \frac{1}{(9.3467)\sqrt{2\pi}} \int_{439/2}^{\infty} e^{-(x-182)^2/174.72} dx$$
$$= 3.0087(10^{-5})$$

(d)

$$P(x \geq 150) = \frac{1}{(9.3467)\sqrt{2\pi}} \int_{299/2}^{\infty} e^{-(x-182)^2/174.72} dx$$
$$= 0.99974.$$

(e)

$$P(120 \leq x \leq 250) = \frac{1}{(9.3467)\sqrt{2\pi}} \int_{239/2}^{501/2} e^{-(x-182)^2/174.72} dx$$

This integral comes out to be 1 within the approximation scheme used to evaluate it. This means that this probability is, for all practical purposes, 1.

(f)

$$P(x = 180) = \frac{1}{(9.3467)\sqrt{2\pi}} \int_{359/2}^{361/2} e^{-(x-182)^2/174.72} dx$$
$$= 0.041698.$$

4. The mean is $\mu = (1000)(0.58) = 580$ and $\sigma = \sqrt{1000(0.58)(0.42)} = 15.608$. Compute $2\sigma^2 = 487.2$. The bell curve for this experiment is the graph of the exponential function

$$\frac{1}{(15.608)\sqrt{2\pi}} e^{-(x-580)^2/487.2}$$

(a)

$$P(x \geq 400) = 0.99998.$$

(b)

$$P(400 \leq x \leq 600) = \frac{1}{(15.608)\sqrt{2\pi}} \int_{799/2}^{1201/2} e^{-(x-580)^2/487.2} dx$$
$$= 0.99547.$$

(c)

$$P(x \leq 450) = \frac{1}{(15.608)\sqrt{2\pi}} \int_{-\infty}^{901/2} e^{-(x-580)^2/487.2} dx$$

$$= 5.3300(10^{-17})$$

(d)

$$P(x = 520) = \frac{1}{(15\,608)\sqrt{2\pi}} \int_{1039/2}^{1041/2} e^{-(x-580)^2/487.2} dx$$
$$= 1.5831(10^{-5}).$$

5. Take $\mu = 750$ and $\sigma = 65$ Then $2\sigma^2 = 8450$, and the bell curve is the graph of the function

$$\frac{1}{(65)\sqrt{2\pi}} e^{-(x-750)^2/8450}$$

The probability that a typical passenger rides between two hundred fifty and six hundred miles per month is

$$P(250 \leq x \leq 600) = \frac{1}{(65)\sqrt{2\pi}} \int_{499/2}^{1201/2} e^{-(x-750)^2/8450} dx$$
$$= 0.010724$$

By contrast,

$$P(600 \leq x \leq 900) = \frac{1}{(65)\sqrt{2\pi}} \int_{1199/2}^{1801/2} e^{-(x-750)^2/8450} dx$$
$$= 0.97941.$$

It is very likely that the average passenger rides between six hundred and nine hundred miles per month.

6 Here $\mu = 940$ and $\sigma = 76$ Then $2\sigma^2 = 11552$ a d we use the exponential function

$$\frac{1}{(76)\sqrt{2\pi}} e^{-(x-940)^2/11552}$$

Then

$$P(750 \leq x \leq 1000) = \frac{1}{(76)\sqrt{2\pi}} \int_{1499/2}^{2001/2} e^{-(x-940)^2/11552} dx$$
$$= 0.7809$$

And

$$P(300 \leq x \leq 500) = \frac{1}{(76)\sqrt{2\pi}} \int_{599/2}^{1001/2} e^{-(x-940)^2/11552} dx$$
$$= 3\,6713(10^{-9}).$$

7. Multiply these percentages by 100 to convert 247 to 24 7 and 021 to 2 1 Now $\mu = 24\,7$ and $\sigma = 2\,1$, so $2\sigma^2 = 8.82$ and

$$P(24.5 \leq x \leq 27\,0) = \frac{1}{(2.1)\sqrt{2\pi}} \int_{24}^{55/2} e^{-(x-24\,7)^2/8\,82} dx$$
$$= 0.53935$$

This is the probability that a batter hit between 245 and 270

Next,

$$P(x \geq 30) = 0.01135$$

The reason this does not appear consistent with what we actually see in the American League is the unrealistically small standard deviation assumed here.

8 Here $N = 2000$ and $p = 0.42$ (probability of a head on a single coin flip). Then $\mu = Np = 2000(0.42) = 840$ and $\sigma = \sqrt{2000(0.42)(0.58)} = 22.073$.

(a) We want to go one standard deviation to the left and right of the mean. Compute

$$840 - 22.073 = 817.93$$

so take $a = 817$, since numbers of heads must be an integer. And

$$840 + 22.073 = 862.07$$

so take $b = 863$.

(b) Now we want to take points two standard deviations to the left and right of the mean. Compute

$$840 - 2(22.073) = 795.85$$

and take $a = 795$. Taking two standard deviations to the right of the mean, compute

$$840 + 2(22.073) = 884.15$$

so take $b = 885$.

(c) For a .99 probability of seeing between a and b heads, choose a as three standard deviations to the left of the mean, which is

$$840 - 3(22.073) = 773.78$$

Take $a = 773$. Similarly, b is three standard deviations to the right of the mean. Compute

$$840 + 3(22.073) = 906.22.$$

Take $b = 907$.

9. The probability of a die coming up even is 0.5. Now $\mu = 550(0.5) = 275$ and $\sigma = 11.726$.

(a) For a probability of 0.68 of seeing between a and b even rolls, go one standard deviation to the left and right of the mean. First,

$$275 - 11.726 = 263.27$$

so take $a = 263$. And

$$275 + 11.726 = 286.73$$

so take $b = 287$.

(b) For a probability of 0.95 of seeing between a and b even tosses, so two standard deviations to each side of the mean. Compute

$$275 - 2(11.726) = 251.55$$

so choose $a = 251$. And

$$275 + 2(11.726) = 298.45$$

so let $b = 299$

(c) Now go three standard deviations to the left and right of the mean. First,

$$275 - 3(11.726) = 239.82$$

so take $a = 239$. And

$$275 + 3(11.726) = 310.18,$$

so take $b = 311$.

Section 27.5 Sampling Distributions and the Central Limit Theorem

1. (a) The table below gives the samples means corresponding to the samples in Table 27.8:

$\overline{x}$	$\overline{x}$	$\overline{x}$	$\overline{x}$	$\overline{x}$
6.2	5.8	6.8	6.2	4.2
6.2	5	5.6	7	5.6
5.2	6	6.8	5.4	6
7	5.2	5.6	5.4	5.6
5.6	6.4	6	6.4	6.4
6.4	5.8	7	6	6.4

The mean of the sample means is 179.2/30, or 5.97. The mean of the entire population (the 150 individual numbers of all the samples) is 896/150, or 5.97.

(b) We find that the standard deviation of the population is

$$\sigma = \frac{\sqrt{25871}}{75},$$

approximately 2.14459. The standard deviation of the sample means is

$$\sigma_{means} = 0.662885.$$

Since the sample size is $n = 5$, compute $\sigma/\sqrt{5} = 0.95909$, differing from σ_{means} by about 0.296. The approximation in this example is not as good as we might want because the sample size of five is very small.

2. The sample means are given in the following table, with entries corresponding to the samples given in Table 27.9:

$\overline{x}$	$\overline{x}$	$\overline{x}$	$\overline{x}$	$\overline{x}$
4.857	5.286	4.714	3.423	4.857
5.0	2.857	4.142	2.857	3.857
4.143	4.143	5.286	4.143	3.857
4.429	3.714	2.714	4.143	5.286
4.286	4.0	1.714	3.286	4.571
2.857	3.571	4.429	4.714	3.143

The mean of the sample means is 119 90/30 = 3.9966. The mean of the population is 851/210 = 4.053 The difference between these two is due to rounding in computing the sample means.

(b) We find that, for the entire population, the standard deviation is

$$\sigma = \frac{2\sqrt{31001}}{105},$$

while the standard deviation of the sample means is

$$\sigma_{means} = 0.8565047426.$$

With samples of size 7, compute

$$\frac{\sigma}{\sqrt{7}} = \frac{2\sqrt{31001}}{105}\left(\frac{1}{\sqrt{7}}\right) = 1\ 2676$$

Again, the approximation would improve if we took larger samples.

3. Take $\mu = 670$ as the average score, and $\sigma = 105$ Now let

$$\mu_{means} = 670, \sigma_{means} = \frac{105}{\sqrt{30}}.$$

Then $2(\sigma_{means})^2 = 735$. Using these numbers, define the exponential function

$$y = \frac{\sqrt{30}}{\sqrt{2\pi}(105)} e^{-(x-670)^2/735}.$$

(a) The probability that all thirty students score in the 700 – 750 range is

$$P(700 \leq x \leq 750) = \frac{\sqrt{30}}{\sqrt{2\pi}(105)} \int_{700}^{750} e^{-(x-670)^2/735} dx$$
$$= 0.0588$$

This is not very likely.

(b) The probability that all thirty recruits score at or above average is

$$P(x \geq 670) = \frac{\sqrt{30}}{\sqrt{2\pi}(105)} \int_{670}^{900} e^{-(x-670)^2/735} dx = 0.5.$$

$$\frac{\sqrt{30}}{\sqrt{2\pi}(105)} \int_{670}^{900} e^{-(x-670)^2/735} dx$$

(c) Now we want a sample size of $n = 7$. Recompute $\sigma_{means} = \sigma/\sqrt{7} = 105/\sqrt{7}$. Then $2\sigma_{mean}^2 = 3150$, so the probability tht seven of the candidates score 800 or higher is

$$P(800 \leq x \leq 900) = \frac{\sqrt{7}}{(105)\sqrt{2\pi}} \int_{800}^{900} e^{-(x-670)^2/3150} dx$$
$$= 0.00052701,$$

an extremely unlikely event in view of the fact that the average is 670 and seven is nearly one fourth of the recruiting class

4. Since six recruits are chosen, choose $n = 6$ This assumes that recruits do not share pants. Take $\mu = 31$ and $\sigma = 3.2$. Then $\mu_{means} = 31$ and

$$\sigma_{means} = \frac{\sigma}{\sqrt{6}} = \frac{3.2}{\sqrt{6}}.$$

Then $2\sigma^2_{mean} = \frac{1}{3}(3.2)^2 = 3.4133$

(a) The probability that the mean length of the sample is at least twenty nine inches is approximately

$$P(x \geq 29) = \frac{\sqrt{6}}{(3.2)\sqrt{2\pi}} \int_{29}^{\infty} e^{-(x-31)^2/3.4133} dx = 0.9371.$$

Notice that three sample standard deviations to the left of the mean is nearly 27

(b)

$$P(29 \leq x \leq 33) = \frac{\sqrt{6}}{(3.2)\sqrt{2\pi}} \int_{29}^{33} e^{-(x-31)^2/3.4133} dx = 0.87421.$$

(c)

$$P(x \leq 32) = \frac{\sqrt{6}}{(3.2)\sqrt{2\pi}} \int_{-\infty}^{32} e^{-(x-31)^2/3.4133} dx = 0.778$$

5 The braces can handle an average of $\mu = 5000$ pounds per square inch, with a standard deviation of $\sigma = 800$ To sample twenty braces, choose $n = 20$ Then $\mu_{means} = 5000$ and

$$\sigma_{means} = \frac{800}{\sqrt{20}}.$$

Compute $2\sigma^2_{mean} = \frac{1}{10}(640000) = 64000$.

(a) The probability that the mean pressure tolerance is at least 4,800 pounds per square inch is

$$P(x \geq 4800) = \frac{\sqrt{20}}{800\sqrt{2\pi}} \int_{4800}^{\infty} e^{-(x-5000)^2/64000} dx$$
$$= 0.86822$$

(b)

$$P(4700 \leq x \leq 5100) = \frac{\sqrt{20}}{800\sqrt{2\pi}} \int_{4700}^{5100} e^{-(x-5000)^2/64000} dx$$
$$= 0.66516$$

(c)

$$P(x \geq 4500) = \frac{\sqrt{20}}{800\sqrt{2\pi}} \int_{4500}^{\infty} e^{-(x-5000)^2/64000} dx$$
$$= 0.99741$$

6. Now $\mu = 17$ and $\sigma = 1.8$. Sample size is $n = 30$ Then $\mu_{means} = 17$ and

$$\sigma_{means} = \frac{1.8}{\sqrt{30}}.$$

(a) Compute

$$2\sigma^2_{mean} = \frac{2}{30}(1.8)^2 = 0.216.$$

Then

$$P(16.8 \le x \le 17.2) = \frac{\sqrt{30}}{(1.8)\sqrt{2\pi}} \int_{16.8}^{17.2} e^{-(x-17)^2/0.216} dx$$
$$= 0.4572.$$

(b)

$$P(x \ge 16.7) = \frac{\sqrt{30}}{(1.8)\sqrt{2\pi}} \int_{16.7}^{\infty} e^{-(x-17)^2/0.216} dx$$
$$= 0.81934.$$

(c)

$$P(x \ge 17.2) = \frac{\sqrt{30}}{(1.8)\sqrt{2\pi}} \int_{17.2}^{\infty} e^{-(x-17)^2/0.216} dx$$
$$= 0.2714.$$

Section 27.6 Confidence Intervals and Estimating Population Proportion

1. Here we have $n = 1200$ and the best estimate for $\widetilde{p}$ is $\widetilde{p} = 0.02$. Then $\widetilde{q} = 0.98$.
(a) For a 99% confidence level, $z_{\alpha/2} = 2.575$. Compute

$$\epsilon = (2.575)\sqrt{\frac{(0.02)(0.98)}{1200}} = 0.0104.$$

The confidence interval is

$$0.02 - 0.01 < p < 0.02 + 0.01$$

or

$$0.01 < p < 0.03.$$

(b) With a 95% confidence level, use $z_{\alpha/2} = 1.96$ and compute

$$\epsilon = (1.96)\sqrt{\frac{(0.02)(0.98)}{1200}} = 0.008.$$

Now the confidence interval is

$$0.02 - 0.008 < p < 0.02 + 0.008,$$

or

$$0.012 < p < 0.028.$$

(c) For a 90% confidence level, use $z_{\alpha/2} = 1.645$ to get

$$\epsilon = (1.645)\sqrt{\frac{(0.02)(0.98)}{1200}} = 0.0065.$$

Now the confidence interval is

$$0.02 - 0.0065 < p < 0.02 + 0.0065,$$

or

$$0.0135 < p < 0.0265$$

(d) Replicate the solutions in parts (a), (b) and (c), with 1200 replaced by 800. We get the following.

(a) For a 99% confidence level, $z_{\alpha/2} = 2.575$. Compute

$$\epsilon = (2.575)\sqrt{\frac{(0.02)(0.98)}{800}} = 0.0127.$$

The confidence interval is

$$0.02 - 0.0127 < p < 0.02 + 0.0127$$

or

$$0.0073 < p < 0.0327$$

(b) With a 95% confidence level, use $z_{\alpha/2} = 1.96$ and compute

$$\epsilon = (1.96)\sqrt{\frac{(0.02)(0.98)}{800}} = 0.0097.$$

Now the confidence interval is

$$0.02 - 0.0097 < p < 0.02 + 0.097,$$

or

$$0.0103 < p < 0.0297.$$

(c) For a 90% confidence level, use $z_{\alpha/2} = 1.645$ to get

$$\epsilon = (1.645)\sqrt{\frac{(0.02)(0.98)}{800}} = 0.0081.$$

Now the confidence interval is

$$0.02 - 0.0081 < p < 0.02 + 0.0081,$$

or

$$0.0119 < p < 0.0281$$

As is apparent from equation (27.6), decreasing the sample size increases the width of the respective confidence intervals.

2. Now $n = 750$ and $\widetilde{p} = 0.22$.

(a) For a 99% confidence interval, compute

$$\epsilon = (2.575)\sqrt{\frac{(0.22)(0.78)}{750}} = 0.039.$$

The interval is

$$0.22 - 0.039 < p < 0.22 < 0.039$$

or

$$0.181 < p < 0.259$$

(b) For a 95% interval, compute

$$\epsilon = (1.96)\sqrt{\frac{(0.22)(0.78)}{800}} = 0.029$$

The interval is

$$0.191 < p < 0.249.$$

3. Take $n = 200$ and $\widetilde{p} = 87/200 = 0.435$

(a) For a 95% confidence interval, compute

$$\epsilon = (1.96)\sqrt{\frac{(0.435)(0.565)}{200}} = 0.069.$$

The interval is

$$0.366 < p < 0.504$$

(b) For an 85% confidence interval, compute

$$\epsilon = (1.44)\sqrt{\frac{(0.435)(0.565)}{200}} = 0.05.$$

The interval is

$$0.435 - 0.05 < p < 0.435 + 0.05$$

or

$$0.385 < p < 0.485$$

4. Now $n = 100$ and $\widetilde{p} = 0.07.$

(a) For a 99% confidence interval, compute

$$\epsilon = (2.575)\sqrt{\frac{(0.07)(0.93)}{100}} = 0.066$$

The interval is

$$0.004 < p < 0.136.$$

(b) For a 90% confidence interval, we need

$$\epsilon = (1.645)\sqrt{\frac{(0.07)(0.93)}{100}} = 0.042.$$

The interval is

$$0.028 < p < 0.112$$

5. (a) For a 95% confidence level, choose $z_{\alpha/2} = 1.96$ We want $\epsilon = 0.02$, so

$$n = \frac{(z_{\alpha/2})^2 \widetilde{p}\widetilde{q}}{\epsilon^2} = \frac{(1.96)^2}{(0.02)^2}\widetilde{p}\widetilde{q}.$$

We have no information with which to estimate $\widetilde{p}$, so take $\widetilde{p}\widetilde{q} = 0.25$ to get

$$n = \frac{(1.96)^2}{(0.02)^2}(0.25) = 2401$$

Actually this computation yields 192.08, but the number of people must be an integer, so we took $n = 192$. One could also be more conservative and use $n = 193$

(b) If the executive is willing to allow a five percent maximum error, now choose $\epsilon = 0.05$ and compute

$$n = \left(\frac{1.96}{0.05}\right)^2 (0.25) = 384.$$

Actually, this computation yields $n = 384.16$, but the number of people must be an integer.

(c) Now suppose we use the previous year's figure to estimate $\widetilde{p} = 0.37$. Let $\epsilon = 0.02$. For a 95% confidence level, the number of people surveyed should be

$$n = \left(\frac{1.95}{0.02}\right)^2 (0.37)(0.63) = 2215.9.$$

Actually, survey 2216 people.

(d) If we use more current information to estimate $\widetilde{p} = 0.32$, then the calculation in (c) is adjusted to

$$n = \left(\frac{1.95}{0.02}\right)^2 (0.32)(0.68) = 2068.6.$$

Now survey 2069 people. With a lower sample proportion, we do not need to interview as many people.

6. (a) Let $\epsilon = 0.03$. For a 99% confidence level in the survey's results, use

$$n = \left(\frac{2.575}{0.03}\right)^2 (0.25) = 1841.8,$$

so she should survey 1842 people

(b) If a 95% confidence level is good enough, now set

$$n = \left(\frac{1.95}{0.03}\right)^2 (0.25) = 1056.3,$$

so survey 1056 people. A lower confidence level requires fewer people in the survey.

(c) Now take $\epsilon = 0.02$ and compute

$$n = \left(\frac{2.575}{0.02}\right)^2 (0.12)(0.88) = 1750.5,$$

so survey 1751 people.

7. (a) An estimate of the number of people is given by

$$n = \left(\frac{1.44}{0.07}\right)^2 (0.25) = 105.8,$$

so survey 109 people.

(b) Now adjust the estimate to

$$n = \left(\frac{1.44}{0.12}\right)^2 (0.25) = 36,$$

so survey 36 people A much higher error tolerance allows a survey involving fewer people, as we might expect

(c) Now estimate

$$n = \left(\frac{1.44}{0.07}\right)^2 (0.06)(0.94) = 23.868,$$

so we can now get by surveying only 24 people.

Section 27.7 Estimating Population Mean and the Student t - Distribution

1. (a) For a 99% confidence level, compute

$$\epsilon = z_{\alpha/2}\frac{\sigma}{\sqrt{n}} = (2.575)\frac{1.04}{\sqrt{350}} = 0.14315.$$

Then

$$71.857 = 72 - 0.143 = \overline{x} - \epsilon < \mu < \overline{x} + \epsilon = 72 + 0.143 = 72.143.$$

(b) Estimate

$$n = \left(\frac{\sigma z_{\alpha/2}}{\epsilon}\right)^2 = \left(\frac{(1.04)(2.575)}{0.2}\right)^2 = 179.29,$$

so choose $n = 180$.

2. (a) For a 95% confidence level, estimate the maximum error in the population mean as

$$\epsilon = (1.96)\frac{0.7}{\sqrt{50}} = 0.194.$$

Then

$$119 - 0.194 < \mu < 119 + 0.194$$

or

$$118.81 < \mu < 119.194.$$

(b) Estimate

$$n = \left(\frac{0.7(2.575)}{0.2}\right)^2 = 81.225,$$

so 82 is a good estimate for the sample size

3. (a) Compute

$$\epsilon = (1.96)\frac{2.4}{10} = 0.47.$$

Then estimate

$$106 - 0.47 < \mu < 106 + 0.47$$

or

$$105.53 < \mu < 106.47.$$

(b) Estimate

$$n = \left(\frac{(2.4)(2.575)}{0.1}\right)^2 = 3819.2$$

so 3820 is a good sample size to use

4. Compute

$$\epsilon_t = t_{\alpha/2}\frac{s}{\sqrt{n}} = (2.009)\frac{0.05}{\sqrt{50}} = 0.0142$$

Here the given critical value $t_{\alpha/2}$ is for a 95% confidence interval. The confidence interval is

$$0.8 - 0.0142 < \mu < 0.8 + 0.0142$$

or

$$0.7858 < \mu < 0.8142.$$

5. Estimate

$$\epsilon_t = (1.992)\frac{0.7}{\sqrt{75}} = 0.161$$

The confidence interval is

$$3 - 0.161 < \mu < 3 + 0.161$$

or

$$2.839 < \mu < 3.161.$$

6. Estimate

$$\epsilon_t = (1.971)\frac{0.01}{\sqrt{200}} = 0.0014$$

The confidence interval is

$$7 - 0.0014 < \mu < 7 + 0.0014$$

or

$$6.9986 < \mu < 7.0014$$

Section 27.8 Correlation and Regression

1. Compute

$$\sum_{i=1}^{n} x_i = 52.38, \sum_{i=1}^{n} y_i = 73.05,$$

$$\sum_{i=1}^{n} x_i^2 = 420.88, \sum_{i=1}^{n} y_i^2 = 631.9,$$

$$\left(\sum_{i=1}^{n} x_i\right)^2 = 2743.7, \left(\sum_{i=1}^{n} y_i\right)^2 = 5336.3,$$

$$\overline{x} = 5.238, \overline{y} = 7.305, \text{ and } \sum_{i=1}^{n} x_i y_i = 502.50$$

Then

$$c = \frac{10(502.50) - (52.38)(73.05)}{\sqrt{10(420.88) - 2743.7}\sqrt{10(631.9) - 5336.3}} = 0.99895$$

There is a significant linear correlation between the $x_i's$ and the $y_i's$. A graph of the data is shown below

Next, compute

$$b = \frac{10(502.50) - (52.38)(73.05)}{10(420.88) - 2743.7} = 0.81813$$

and

$$a = \overline{y} - b\overline{x} = 3.0196$$

The regression line has equation

$$y = 3.0196 + 0.81813x$$

Next compute

$$S_r = \sqrt{\frac{631.9 - (3.0196)(73.05) - (0.81813)(502.50)}{8}} = 0.1612.$$

With $n = 10$ and a 95% confidence level, $t_{\alpha/2} = 2.306$ (from a table of $t-$ distribution critical points).

If $x = 0.4$, the regression line returns a value of

$$y = 3.0196 + 0.81813(0.4) = 3.3469$$

For this value of x, compute

$$\epsilon_r = (2.306)(0.1612)\sqrt{1 + \frac{1}{10} + \frac{10(0.4 - 5.238)^2}{10(420.88) - 2743.7}} = 0.41722$$

The confidence interval for this value of y (for $x = 0\,4$) is

$$3\,3469 - 0.41722 < y < 3\,3469 + 0.41722,$$

or

$$2\,9297 < y < 3\,7641.$$

If $x = 6.2$, the predicted value of y is

$$y = 3\,0196 + 0\,81813(6\,2) = 8\,092.$$

Now

$$\epsilon_r = (2.306)(0.1612)\sqrt{1 + \frac{1}{10} + \frac{10(6.2 - 5.238)^2}{10(420\,88) - 2743\,7}}$$
$$= 0.39099$$

The confidence interval is

$$7.701 < y < 8\,4830.$$

If $x = 15.1$, the predicted value of y is

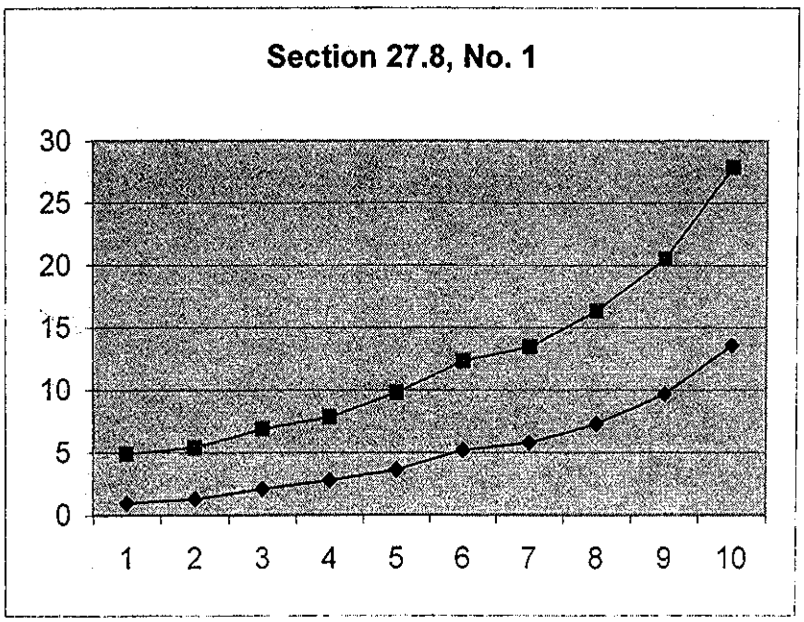
$$y = 3\,0196 + 0.81813(15\,1) = 15.373.$$

Now compute

$$\epsilon_r = (2.306)(0.1612)\sqrt{1 + \frac{1}{10} + \frac{10(15.373 - 5.238)^2}{10(420.88) - 2743.7}}$$
$$= 0.49888$$

The confidence interval is

$$14\,874 < y < 15.872.$$



2. Compute

$$\sum x_i = 47.3, \left(\sum x_i\right)^2 = 2237\ 3, \overline{x} = 4\ 73,$$

$$\sum y_i = -103\ 99, \left(\sum y_i\right)^2 = 10813.92, \overline{y} = -10\ 399,$$

$$\sum {x_i}^2 = 441\ 43, \sum y_i^2 = 2947.8151$$

and

$$\sum x_i y_i = -1128.866.$$

Then

$$c = \frac{10(-1128.866) - (47.3)(-103.99)}{\sqrt{10(441\ 43) - (47\ 3)^2}\sqrt{10(2947.8151) - (10813.92)}}$$
$$= -0\ 99931,$$

so there is a significant linear correlation between the $x_i's$ and the $y_i's$. A graph of the data is shown below.

Now compute

$$b = \frac{10(-1128.866) - (47.3)(-103.99)}{10(441.43) - (2237\ 3)}$$

$$= -2\ 9262$$

and

$$a = -10.399 - (-2.9262)(4\ 73) = 3.4419$$

The regression line is

$$y = 3.4419 - 2.9262x$$

We will need

$$S_r = \sqrt{\frac{2947.8151 - 3.4419(-103.99) + 2.9262(-1128.866)}{8}}$$

$$= 0\ 55347.$$

The predicted value going with $x = 5\ 2$ is

$$y = 3\ 4419 - 2.9262(5\ 2) = -11\ 774$$

Compute

$$\epsilon_r = (2\ 306)(0\ 55347)\sqrt{1 + \frac{1}{10} + \frac{10(5.2 - 4.73)^2}{10(441\ 43) - (2237.3)}}$$
$$= 1\ 3392.$$

The confidence interval for this predicted value of y is

$$-13.113 < y < -10.435.$$

If $x = 8.6$, the regression line predicts that

$$y = 3.4419 - 2.9262(8.6) = -21.723.$$

Now

$$S_r = (2.306)(0.55347)\sqrt{1 + \frac{1}{10} + \frac{10(8.6 - 4.73)^2}{10(441.43) - (2237.3)}}$$
$$= 1.3798.$$

so the confidence interval is

$$-23.103 < y < -20.343$$

If $x = 17.8$, we get

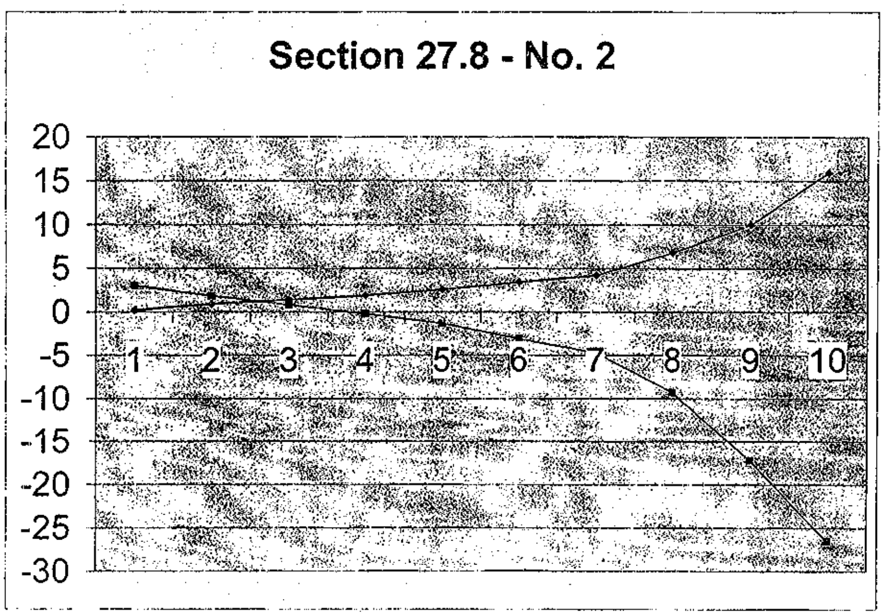
$$y = 3.4419 - 2.9262(17.8) = -48.644.$$

Now

$$\epsilon_r = (2.306)(0.55347)\sqrt{1 + \frac{1}{10} + \frac{10(17.8 - 4.73)^2}{10(441.43) - (2237.3)}}$$
$$= 1.7522,$$

so the confidence interval is

$$-50.396 < y < -46.892.$$



3 Compute

$$\sum x_i = 147.11, \sum x_i^2 = 3200.2, \overline{x} = 14.711,$$
$$\sum y_i = 12.25, \sum y_i^2 = 676.79, \overline{y} = 1.225,$$
$$\left(\sum x_i\right)^2 = 21641, \left(\sum y_i\right)^2 = 150.06,$$

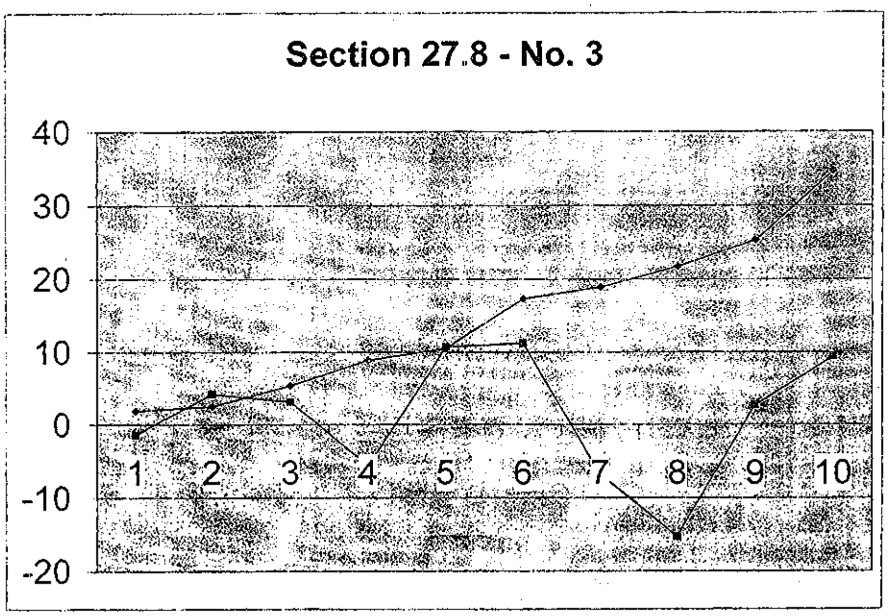
and

$$\sum x_i y_i = 215.02.$$

Then

$$c = \frac{10(215.02) - (147.11)(12.25)}{\sqrt{10(3200.2) - 21641}\sqrt{10(676.79) - 150.06}}$$
$$= 0.042039$$

There is no significant linear correlation between the x_i's and the y_i's. A graph of the data is shown below.



4. Compute

$$\sum x_i = 200.4, \sum x_i^2 = 5701.38, \overline{x} = 20.04$$

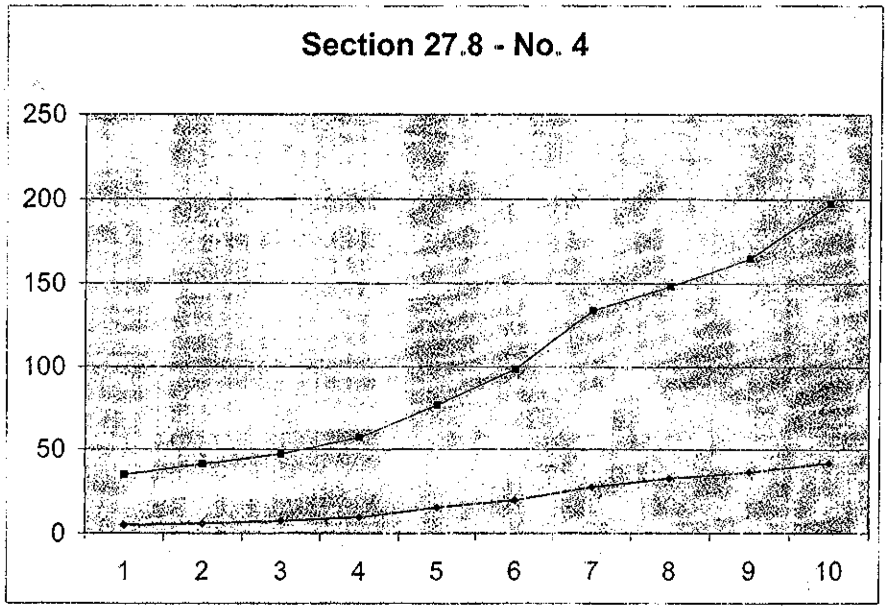
$$\sum y_i = 1000.3, \sum y_i^2 = 130147.33, \overline{y} = 100.03$$

$$\sum x_i y_i = 27156.78, \left(\sum x_i\right)^2 = 40160.16, \left(\sum y_i\right)^2 = 1000600$$

Now

$$c = \frac{10(27156.78) - (20.04)(1000.3)}{\sqrt{10(5701.38) - 40160.16}\sqrt{10(130147.33) - 1000600}}$$
$$= 3.5321$$

There is no significant linear correlation between the two data sets. A graph of the data is given below



5. We find that

$$\sum x_i = 127.0, \sum x_i^2 = 2904.4, \overline{x} = 12.7,$$

$$\sum y_i = 97.42, \sum y_i^2 = 1945.0, \overline{y} = 9.742,$$

$$\left(\sum x_i\right)^2 = 16129, \left(\sum y_i\right)^2 = 9490.7,$$

$$\sum x_i y_i = 2369.6$$

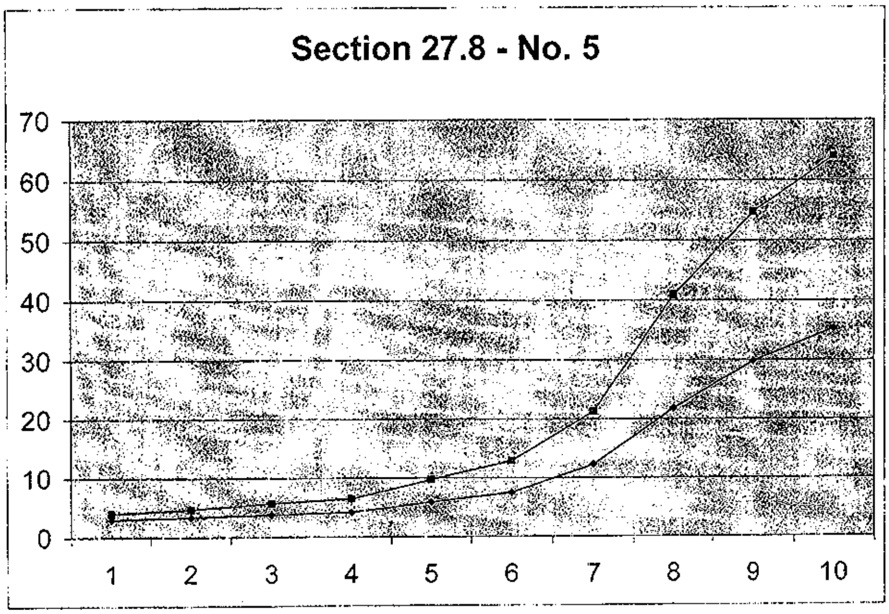
Compute

$$c = \frac{10(2369.6) - (127)(97.42)}{\sqrt{10(2904.4) - 16129}\sqrt{10(1945) - 9490.7}}$$
$$= 0.99845,$$

indicating a significant linear correlation between the data sets. A graph of the data is shown below.

Compute $b = 0.87678$ and $a = -1.3931$ to get the regression line

$$y = -1.3931 + 0.87678x$$



Next compute

$$S_r = 0.66229$$

If $x = 0.6$, the regression line predicts that $y = -0.86703$. We get $\epsilon_r = 1.6823$, so the confidence interval is $-2.5493 < y < 0.81527$.

If $x = 3$, we get $y = 1.2372$. Now $\epsilon_r = 1.6540$ and the confidence interval is $-0.4168 < y < 2.8912$.

If $x = 39$, we get $y = 32.801$. Now $\epsilon_r = 1.9532$ and the confidence interval is $30.848 < y < 34.754$.

Finally, if $x = 42.1$, we get $y = 35.519$ and $\epsilon_r = 2.0314$ for a confidence interval of $33.488 < y < 37.55$

6 We find that

$$\sum x_i = 51.8, \sum x_i^2 = 279.82, \overline{x} = 5.18$$

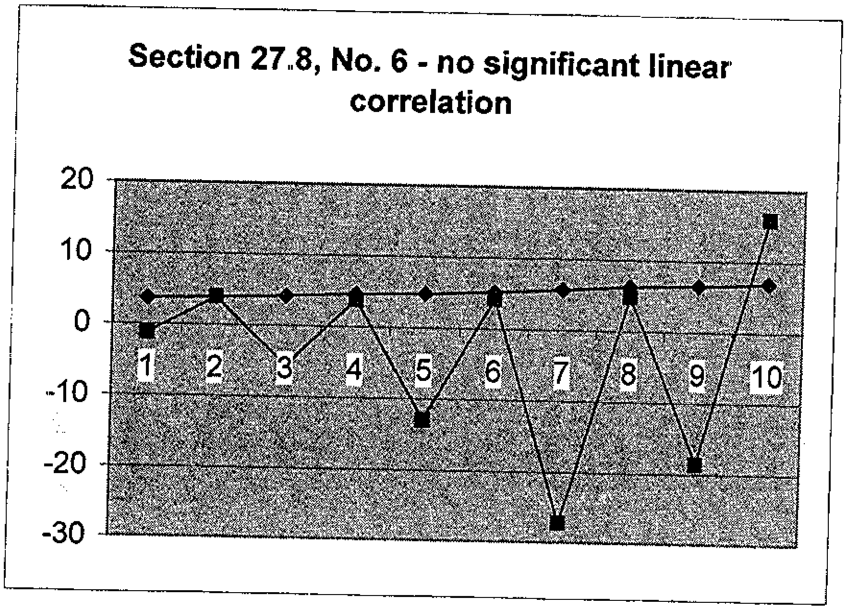
$$\sum y_i = -32.9, \sum y_i^2 = 1608.15, \overline{y} = -3.29,$$

$$\sum x_i y_i = -171.34, \left(\sum x_i\right)^2 = 2683.24, \left(\sum y_i\right)^2 = 1082.41.$$

Compute

$$c = \frac{10(-171.34) - (51.8)(-32.9)}{\sqrt{10(279.82) - 2683.24}\sqrt{10(1608.15) - 1082.41}} = 0.00699$$

There is no significant linear correlation between the $x_i's$ and the $y_i's$. A graph of the data is shown below.



7. Note now that there are $n = 15$ pairs of points. We find that

$$\sum x_i = 277.5, \sum x_i^2 = 7315.9, \overline{x} = 18.5$$

$$\sum y_i = 398.65, \sum y_i^2 = 16115, \overline{y} = 26.577,$$

$$\sum x_i y_i = 10843, \left(\sum x_i\right)^2 = 77006, \left(\sum y_i\right)^2 = 158920.$$

Compute $c = 0.99919$, so there is a significant linear correlation between the $x_i's$ and the $y_i's$. A graph of the data is shown below.

Compute $b = 1.5892$ and $a = -2.8232$, so the regression line has equation

$$y = -2.8232 + 1.5892x$$

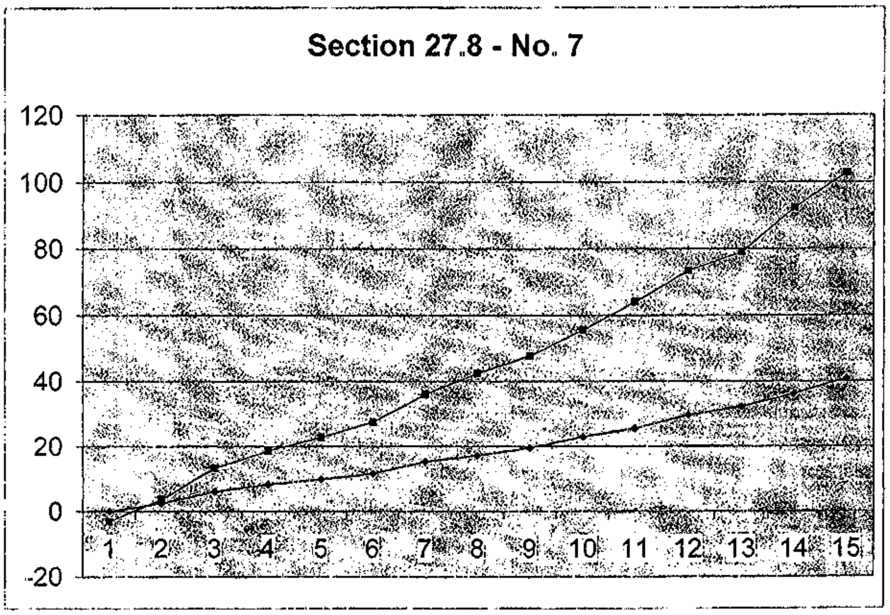
Next, compute $S_\tau = 0.82149$.

If $x = -4$, then the regression line gives $y = -9.18$. With $n = 15$, a table of $t-$ distribution critical values gives $t_{\alpha/2} = 2.160$, with which we compute $\epsilon_\tau = 2.1588$. The confidence interval is $-11.339 < y < -7.0212$

If $x = 11.1$, we get $y = 14.817$, and $\epsilon_\tau = 1.9794$ for a confidence interval of $12.838 < y < 16.796$.

If $x = 30.1$, we get $y = 45.012$ and $\epsilon_\tau = 2.0122$ for a confidence interval of $43 < y < 47.024$.

If $x = 66.7$, $y = 103.18$ and $\epsilon_\tau = 2.7656$ for a confidence interval of $100.41 < y < 105.95$



8. There are $n = 15$ data pairs. Compute

$$\sum x_i = 127.5, \sum x_i^2 = 2095.17, \overline{x} = 8.5$$

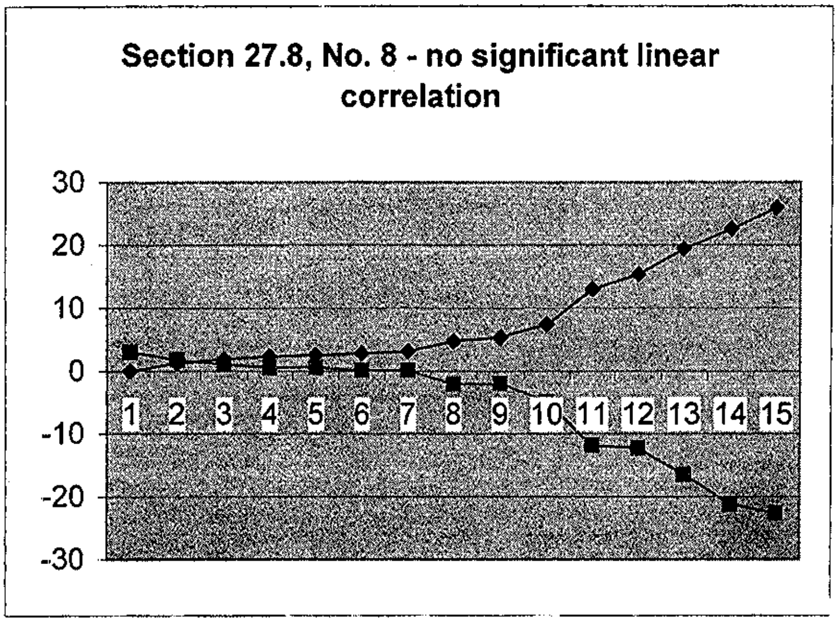
$$\sum y_i = -213.83 =, \sum y_i^2 = 7214.7485, \overline{y} = -14.2553$$

$$\sum x_i y_i = -3868.95, \left(\sum x_i\right)^2 = 16256.25, \left(\sum y_i\right)^2 = 45723.2689$$

Then

$$c = \frac{15(-3868.95) - (127.5)(-213.83)}{\sqrt{15(7315.9) - 16256.25}\sqrt{15(7214.7485) - 45723.2689}}$$
$$-0.40257$$

There is no significant linear correlation between the $x^{i\prime}s$ and the $y_i^\prime s$. A graph of the data is shown below.



9. As a preliminary to carrying out the statistical analysis, compute

$$\sum x_i = 674, \sum x_i^2 = 46506, \overline{x} = 67.4$$

$$\sum y_i = 533, \sum y_i^2 = 29073, \overline{y} = 53.3$$

$$\sum x_i y_i = 36755, \left(\sum x_i\right)^2 = 454276, \left(\sum y_i\right)^2 = 284089.$$

(a) Compute

$$c \doteq \frac{10(36755) - (674)(533)}{\sqrt{10(46506) - 454276}\sqrt{10(29073) - 284089}} = 0.98172.$$

Therefore there is a significant linear correlation between games played by this player, and games won by the team

(b) We want the regression line Compute

$$b = \frac{10(36755) - (674)(533)}{10(46506) - 454276} = 0.7704$$

and

$$a = 53.3 - (0.7704)(67.4) = 1.375.$$

The equation of the regression line is

$$y = 1.375 + 0.7704x$$

(c) Compute

$$c^2 = (0.98172)^2 = 0.96377.$$

We conclude that about 96.4% of the teams wins are explained by this player's presence on the court.

(d) Suppose this player is in 85 games Compute from the regression line that

$$y(85) = 1\,375 + 0.7704(85) = 66.859$$

We estimate that the team will will 67 of these games. For the confidence interval of this prediction, compute

$$S_r = \sqrt{\frac{29073 - 1.375(533) - 0.7704(36755)}{8}}$$

$$= 1.7347$$

Then

$$\epsilon_r = 2\,306(1\,7347)\sqrt{1 + \frac{1}{10} + \frac{10(85 - 67.4)^2}{10(46506) - 454276}}$$
$$= 4.7115.$$

The confidence interval is

$$67 - 4.7115 < y < 67 + 4.7115,$$

or, since the number of games must be an integer,

$$62 < y < 72$$

10 (a) First compute the numbers we need:

$$\sum x_i = 72600, \overline{x} = 7260, \sum x_i^2 = 586680000,$$

$$\sum y_i = 2255, \overline{y} = 225\,5, \sum y_i^2 = 563855,$$

$$\sum x_i y_i = 18180500, \left(\sum x_i\right)^2 = 5270760000, \left(\sum y_i\right)^2 = 5085025.$$

Now compute

$$c = \frac{10(18180500) - (72600)(2255)}{\sqrt{10(586680000) - 5270760000}\sqrt{10(563855) - (5085025)}}$$
$$= 0.99605$$

There is therefore a significant linear correlation between the number of patients and those exhibiting the side effect.

(b) We want the equation of the regression line Compute

$$b = \frac{10(18180500) - (72600)(2255)}{10(586680000) - 5270760000}$$

$$= 0\,030354$$

and

$$a = 225\,5 - (0\,030354)(7260) = 5\,1300$$

The regression line has equation

$$y = 5\,1300 + 0\,030354x$$

(c) From the regression line, we estimate that the number of patients, out of 17,000, having this side effect is

$$y = 5.1300 + 0.030354(17000) = 521.15,$$

or, since the number of patients is an integer, 522.

(d) Suppose now we consider a population of 17,500 patients. From the regression line, the number with the side effect is estimated at

$$5.1300 + 0.030354(17500),$$

or, in terms of an integer value, 537 For a 95% confidence interval, first compute

$$S_r = \sqrt{\frac{563855 - 5.13(2255) - 0.030345(18180500)}{8}}$$
$$= 8.6572.$$

Then

$$\epsilon_r = (2.306)(8.6572)\sqrt{1 + \frac{1}{10} + \frac{10(17500 - 7260)^2}{10(586680000) - 5270760000}}$$
$$= 33.757.$$

The confidence interval is

$$537 - 33.757 < y < 537 + 33.757$$

or

$$503 < y < 571.$$

11. (a) First compute

$$\sum x_i = 151, \overline{x} = 15.1, \sum x_i^2 = 2453$$
$$\sum y_i = 58, \overline{y} = 5.8, \sum y_i^2 = 358$$
$$\sum x_i y_i = 934, \left(\sum x_i\right)^2 = 22801, \left(\sum y_i\right)^2 = 3364.$$

Next,

$$c = \frac{10(934) - (151)(58)}{\sqrt{10(2453) - 22801}\sqrt{10(358) - 3364}} = 0.95235$$

There is a significant linear correlation between the number of visits and the number of sets.

(b) Compute

$$b = \frac{10(934) - (151)(58)}{10(2453) - 22801} = 0.33661$$

and

$$a = 5.8 - (0.33661)(15.1) = 0.71719.$$

The equation of the regression line is

$$y = 0.71719 + 0.33661x$$

(c) Since

$$c^2 = (0.95235)^2 = 0.90697,$$

it is estimated that this person is responsible for about 90% of the major jewelry thefts in the city each year.

(d) Compute $y(25) = 0.71719 + 0.33661(25) = 9.1324$ The regression line suggests that, with twenty five visits, this person will be involved in 9 major thefts. For the confidence interval, first compute

$$S_r = \sqrt{\frac{358 - (0.71719)(58) - (0.33661)(934)}{8}} = 0.50155.$$

Then

$$\begin{aligned}\epsilon_r &= (2.306)(0.50155)\sqrt{1 + \frac{1}{10} + \frac{10(25 - 15.1)^2}{10(2453) - 22801}} \\ &= 1.4943.\end{aligned}$$

The confidence interval is

$$9 - 1.4943 < y < 9 + 1.4943$$

or, in terms of integers for number of thefts,

$$7 < y < 11.$$

12. Compute

$$\sum x_i = 4950, \sum x_i^2 = 2490950, \overline{x} = 495$$

$$\sum y_i = 43246, \sum y_i^2 = 190193552, \overline{y} = 4324.6$$

$$\sum x_i y_i = 21765805, \left(\sum x_i\right)^2 = 24502500, \left(\sum y_i\right)^2 = 1870216516.$$

(a) Compute

$$\begin{aligned}c &= \frac{10(21765805) - (4950)(43246)}{\sqrt{10(2490950) - (24502500)}\sqrt{10(190193552) - (1870216516)}} \\ &= 0.99926.\end{aligned}$$

Therefore there is a significant linear correlation between the additive and children who are cavity free.

(b) Compute

$$\begin{aligned}b &= \frac{10(21765805) - (4950)(43246)}{10(2490950) - 24502500} \\ &= 8.8215\end{aligned}$$

and

$$a = 4324.6 - (8.8215)(495) = -42.043.$$

The regression line has equation

$$y = -42.043 + 8.8215x.$$

(c) Compute

$$c^2 = (.99926)^2 = 0.99852,$$

which estimates that 99.8% of the children who were cavity free have this circumstance explained by the chemical additive to the water supply

(d) First, $y(650) = -42.043 + 8.8215(650) = 5691.9$. The estimate is that adding 650 pounds of the chemical will result in 5692 children who are cavity free.

To determine a 95% confidence interval for this estimate, first compute

$$S_r = \sqrt{\frac{190193552 - (-42.043)(43246) - (8.8215)(21765805)}{8}}$$
$$= 24.225.$$

Then

$$\epsilon_r = (2.306)(24.225)\sqrt{1 + \frac{1}{10} + \frac{10(650 - 495)^2}{10(2490950) - 24502500}}$$
$$= 72.628.$$

The 95% confidence interval is

$$5691.9 - 72.628 < y < 5691.9 + 72.628,$$

or

$$5619 < y < 5765.$$