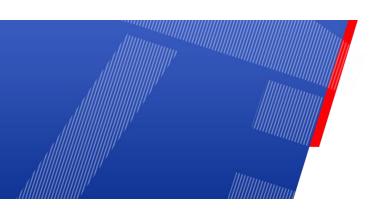




Markov Decision Processes II





Georges SakrESIB



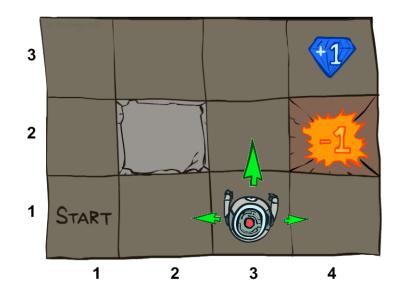






Example: Grid World

- A maze-like problem
 - The agent lives in a grid
 - Walls block the agent's path
- Noisy movement: actions do not always go as planned
 - 80% of the time, the action North takes the agent North
 - 10% of the time, North takes the agent West; 10% East
 - If there is a wall in the direction the agent would have been taken, the agent stays put
- The agent receives rewards each time step
 - Small "living" reward each step (can be negative)
 - Big rewards come at the end (good or bad)
- Goal: maximize sum of (discounted) rewards





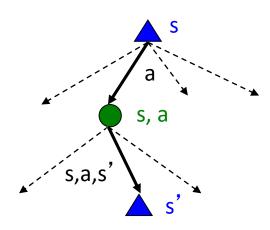


Recap: MDPs

- Markov decision processes:
 - States S
 - Actions A
 - Transitions P(s'|s,a) (or T(s,a,s'))
 - Rewards R(s,a,s') (and discount γ)
 - Start state s₀

• Quantities:

- Policy = map of states to actions
- Utility = sum of discounted rewards
- Values = expected future utility from a state (max node)
- Q-Values = expected future utility from a q-state (chance node)

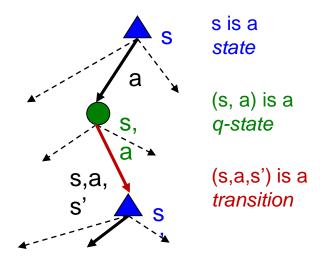






Optimal Quantities

- The value (utility) of a state s:
 - V*(s) = expected utility starting in s and acting optimally
- The value (utility) of a q-state (s,a):
 - Q*(s,a) = expected utility starting out having taken action a from state s and (thereafter) acting optimally
- The optimal policy: $\pi^*(s) = optimal action from state s$

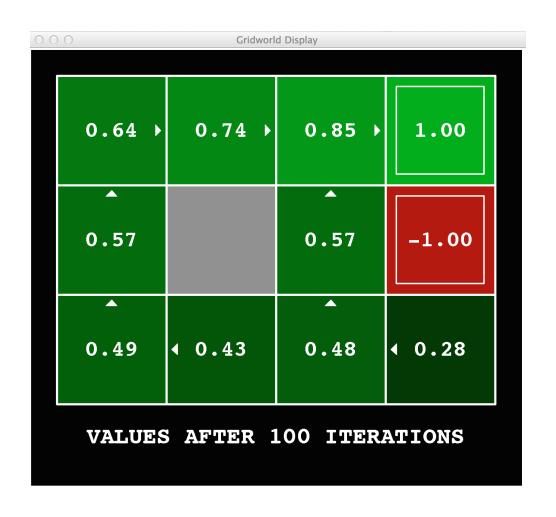


[Demo: gridworld values (L9D1)]





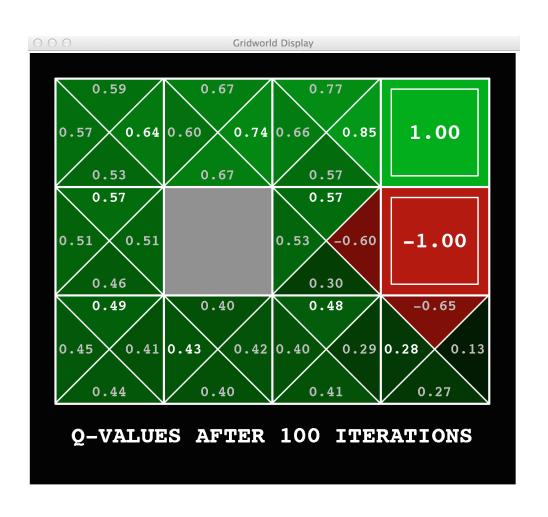
Gridworld Values V*





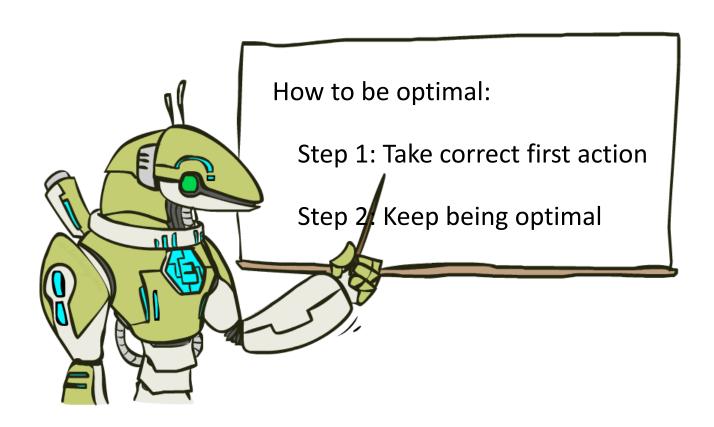


Gridworld: Q*



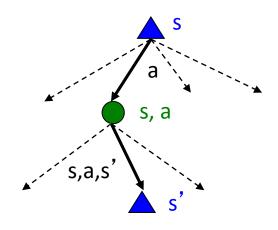








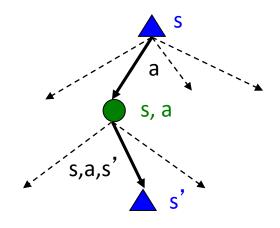








$$V^*(s) = \max_a Q^*(s, a)$$

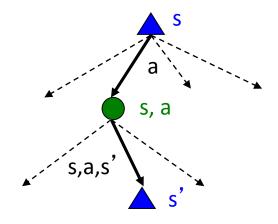






$$V^*(s) = \max_a Q^*(s, a)$$

$$Q^{*}(s, a) = \sum_{s'} T(s, a, s') \left[R(s, a, s') + \gamma V^{*}(s') \right]$$



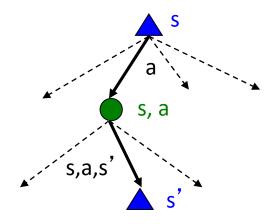




$$V^*(s) = \max_{a} Q^*(s, a)$$

$$Q^*(s, a) = \sum_{s'} T(s, a, s') \left[R(s, a, s') + \gamma V^*(s') \right]$$

$$V^*(s) = \max_{a} \sum_{s'} T(s, a, s') \left[R(s, a, s') + \gamma V^*(s') \right]$$





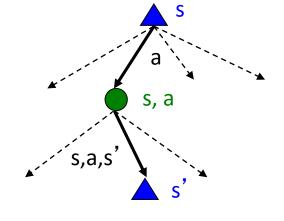


 Definition of "optimal utility" via expectimax recurrence gives a simple one-step lookahead relationship amongst optimal utility values

$$V^*(s) = \max_{a} Q^*(s, a)$$

$$Q^*(s, a) = \sum_{s'} T(s, a, s') \left[R(s, a, s') + \gamma V^*(s') \right]$$

$$V^*(s) = \max_{a} \sum_{s'} T(s, a, s') \left[R(s, a, s') + \gamma V^*(s') \right]$$



 These are the Bellman equations, and they characterize optimal values in a way we'll use over and over

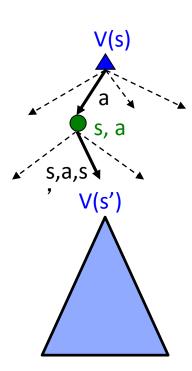








Bellman equations characterize the optimal values:

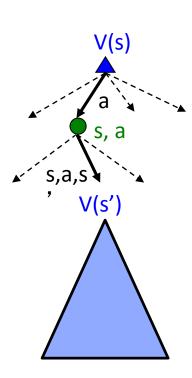






Bellman equations characterize the optimal values:

$$V^*(s) = \max_{a} \sum_{s'} T(s, a, s') \left[R(s, a, s') + \gamma V^*(s') \right]$$



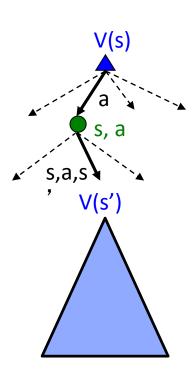




Bellman equations characterize the optimal values:

$$V^*(s) = \max_{a} \sum_{s'} T(s, a, s') \left[R(s, a, s') + \gamma V^*(s') \right]$$

Value iteration computes them:





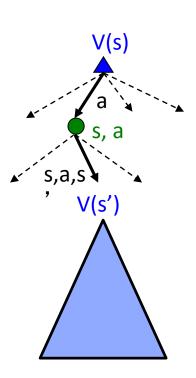


Bellman equations characterize the optimal values:

$$V^*(s) = \max_{a} \sum_{s'} T(s, a, s') \left[R(s, a, s') + \gamma V^*(s') \right]$$

Value iteration computes them:

$$V_{k+1}(s) \leftarrow \max_{a} \sum_{s'} T(s, a, s') \left[R(s, a, s') + \gamma V_k(s') \right]$$







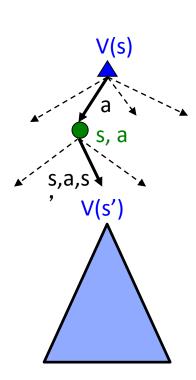
Bellman equations characterize the optimal values:

$$V^*(s) = \max_{a} \sum_{s'} T(s, a, s') \left[R(s, a, s') + \gamma V^*(s') \right]$$

Value iteration computes them:

$$V_{k+1}(s) \leftarrow \max_{a} \sum_{s'} T(s, a, s') \left[R(s, a, s') + \gamma V_k(s') \right]$$

- Value iteration is just a fixed point solution method
 - ... though the V_k vectors are also interpretable as time-limited values







Convergence*

How do we know the V_k vectors are going to converge?





Convergence*

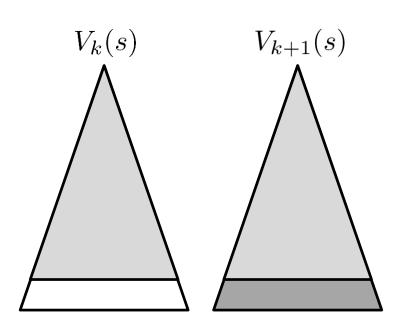
- How do we know the V_k vectors are going to converge?
- Case 1: If the tree has maximum depth M, then V_M holds the actual untruncated values





Convergence*

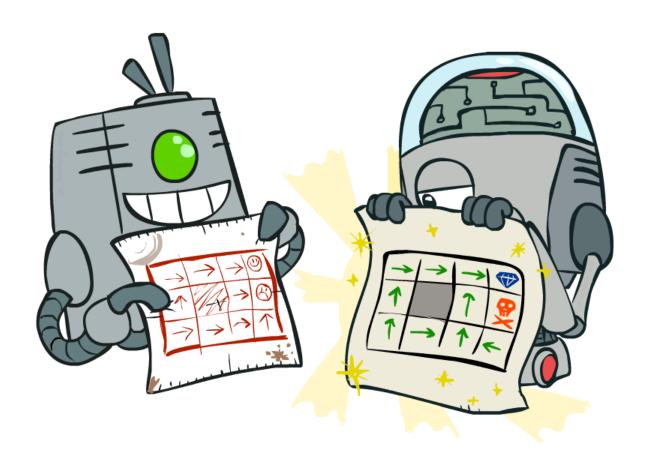
- How do we know the V_k vectors are going to converge?
- Case 1: If the tree has maximum depth M, then V_M holds the actual untruncated values
- Case 2: If the discount is less than 1
 - Sketch: For any state V_k and V_{k+1} can be viewed as depth k+1 expectimax results in nearly identical search trees
 - The difference is that on the bottom layer, V_{k+1} has actual rewards while V_k has zeros
 - That last layer is at best all R_{MAX}
 - It is at worst R_{MIN}
 - But everything is discounted by γ^k that far out
 - So V_k and V_{k+1} are at most γ^k max |R| different
 - So as k increases, the values converge





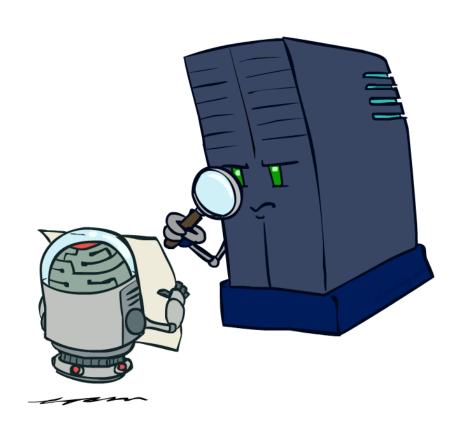


Policy Methods







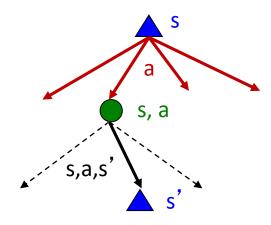






Fixed Policies

Do the optimal action



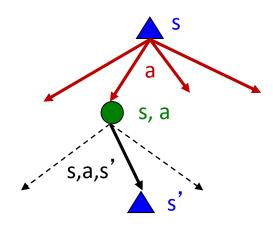
Expectimax trees max over all actions to compute the optimal values



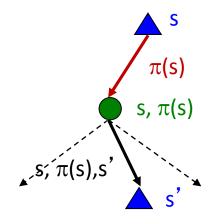


Fixed Policies

Do the optimal action



Do what π says to do



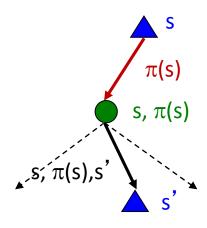
- Expectimax trees max over all actions to compute the optimal values
- If we fixed some policy $\pi(s)$, then the tree would be simpler only one action per state
 - ... though the tree's value would depend on which policy we fixed





Utilities for a Fixed Policy

- Another basic operation: compute the utility of a state s under a fixed (generally non-optimal) policy
- Define the utility of a state s, under a fixed policy π : $V^{\pi}(s)$ = expected total discounted rewards starting in s and following π

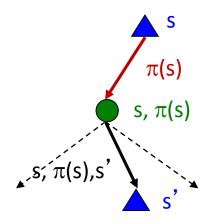






Utilities for a Fixed Policy

- Another basic operation: compute the utility of a state s under a fixed (generally non-optimal) policy
- Define the utility of a state s, under a fixed policy π : $V^{\pi}(s)$ = expected total discounted rewards starting in s and following π
- Recursive relation (one-step look-ahead / Bellman equation):



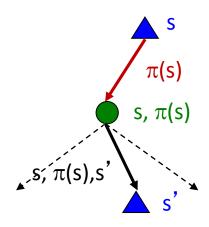




Utilities for a Fixed Policy

- Another basic operation: compute the utility of a state s under a fixed (generally non-optimal) policy
- Define the utility of a state s, under a fixed policy π : $V^{\pi}(s)$ = expected total discounted rewards starting in s and following π
- Recursive relation (one-step look-ahead / Bellman equation):

$$V^{\pi}(s) = \sum_{s'} T(s, \pi(s), s') [R(s, \pi(s), s') + \gamma V^{\pi}(s')]$$











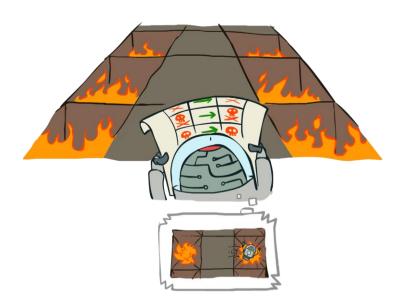
Always Go Right







Always Go Right



Always Go Forward



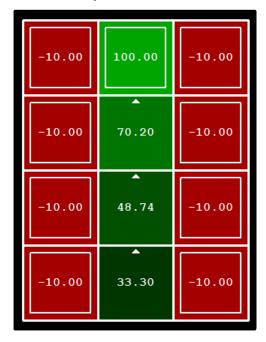




Always Go Right



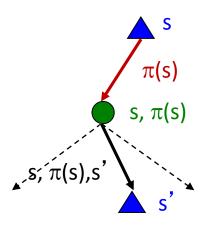
Always Go Forward







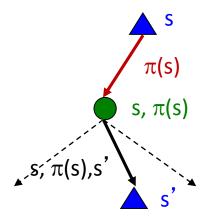
• How do we calculate the V's for a fixed policy π ?







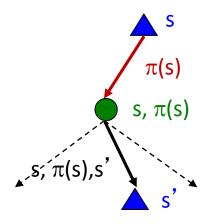
- How do we calculate the V's for a fixed policy π ?
- Idea 1: Turn recursive Bellman equations into updates







- How do we calculate the V's for a fixed policy π ?
- Idea 1: Turn recursive Bellman equations into updates (like value iteration)

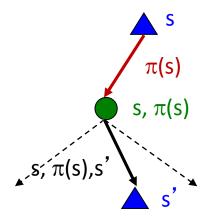






- How do we calculate the V's for a fixed policy π ?
- Idea 1: Turn recursive Bellman equations into updates (like value iteration)

$$V_0^{\pi}(s) = 0$$



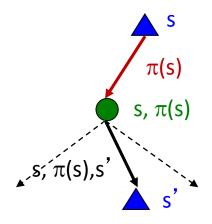




- How do we calculate the V's for a fixed policy π ?
- Idea 1: Turn recursive Bellman equations into updates (like value iteration)

$$V_0^{\pi}(s) = 0$$

$$V_{k+1}^{\pi}(s) \leftarrow \sum_{s'} T(s, \pi(s), s') [R(s, \pi(s), s') + \gamma V_k^{\pi}(s')]$$



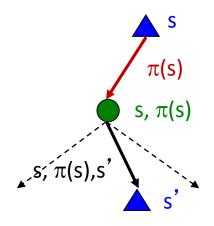




- How do we calculate the V's for a fixed policy π ?
- Idea 1: Turn recursive Bellman equations into updates (like value iteration)

$$V_0^{\pi}(s) = 0$$

$$V_{k+1}^{\pi}(s) \leftarrow \sum_{s'} T(s, \pi(s), s') [R(s, \pi(s), s') + \gamma V_k^{\pi}(s')]$$



Efficiency: O(S²) per iteration

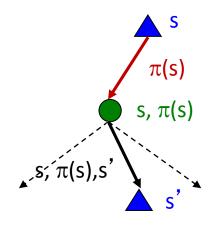




- How do we calculate the V's for a fixed policy π ?
- Idea 1: Turn recursive Bellman equations into updates (like value iteration)

$$V_0^{\pi}(s) = 0$$

$$V_{k+1}^{\pi}(s) \leftarrow \sum_{s'} T(s, \pi(s), s') [R(s, \pi(s), s') + \gamma V_k^{\pi}(s')]$$



- Efficiency: O(S²) per iteration
- Idea 2: Without the maxes, the Bellman equations are just a linear system

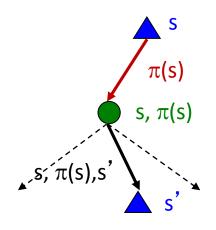




- How do we calculate the V's for a fixed policy π ?
- Idea 1: Turn recursive Bellman equations into updates (like value iteration)

$$V_0^{\pi}(s) = 0$$

$$V_{k+1}^{\pi}(s) \leftarrow \sum_{s'} T(s, \pi(s), s') [R(s, \pi(s), s') + \gamma V_k^{\pi}(s')]$$

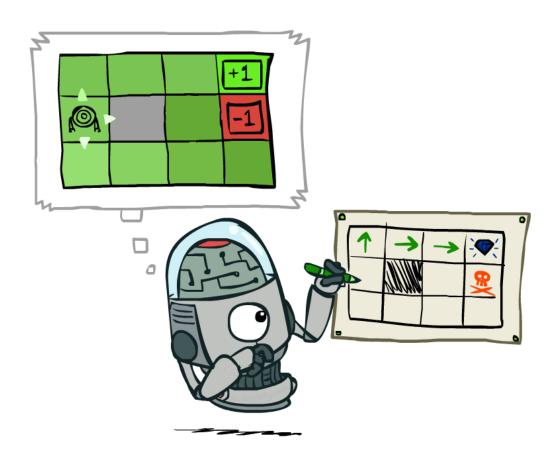


- Efficiency: O(S²) per iteration
- Idea 2: Without the maxes, the Bellman equations are just a linear system
 - Solve with Matlab (or your favorite linear system solver)





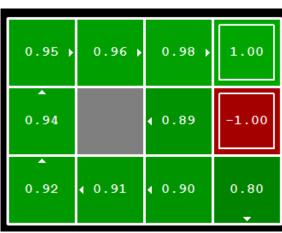
Policy Extraction







Let's imagine we have the optimal values V*(s)







Let's imagine we have the optimal values V*(s)

How should we act?

0.95 ▶	0.96 ▶	0.98 ▶	1.00
0.94		∢ 0.89	-1.00
0.92	∢ 0.91	∢ 0.90	0.80





- Let's imagine we have the optimal values V*(s)
- How should we act?
 - It's not obvious!







- Let's imagine we have the optimal values V*(s)
- How should we act?
 - It's not obvious!
- We need to do a mini-expectimax (one step)

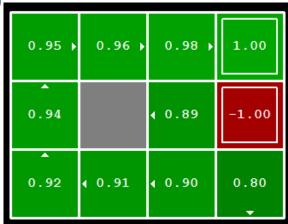


$$\pi^*(s) =$$





- Let's imagine we have the optimal values V*(s)
- How should we act?
 - It's not obvious!
- We need to do a mini-expectimax (one step)

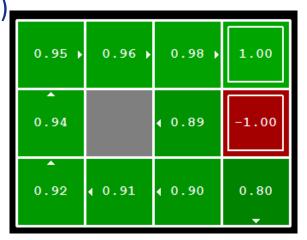


$$\pi^*(s) = \arg\max_{a} \sum_{s'} T(s, a, s') [R(s, a, s') + \gamma V^*(s')]$$





- Let's imagine we have the optimal values V*(s)
- How should we act?
 - It's not obvious!
- We need to do a mini-expectimax (one step)



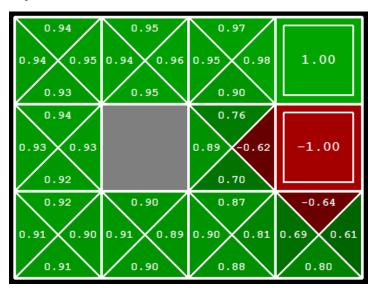
$$\pi^*(s) = \arg\max_{a} \sum_{s'} T(s, a, s') [R(s, a, s') + \gamma V^*(s')]$$

This is called policy extraction, since it gets the policy implied by the values





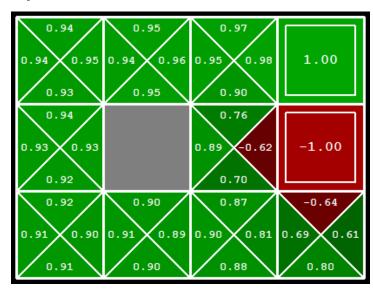
Let's imagine we have the optimal q-values:







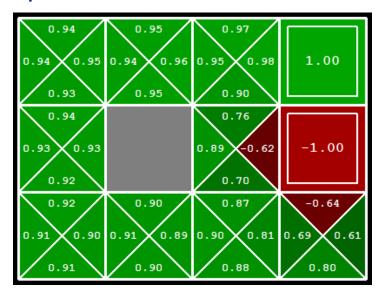
- Let's imagine we have the optimal q-values:
- How should we act?







- Let's imagine we have the optimal q-values:
- How should we act?
 - Completely trivial to decide!

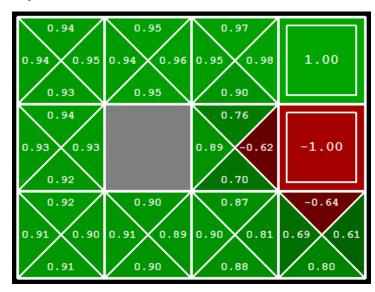






- Let's imagine we have the optimal q-values:
- How should we act?
 - Completely trivial to decide!

$$\pi^*(s) = \arg\max_{a} Q^*(s, a)$$

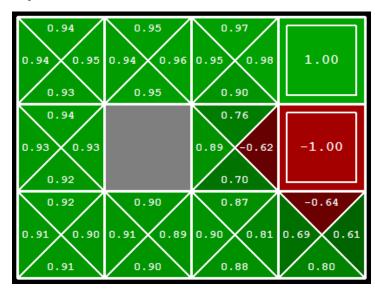






- Let's imagine we have the optimal q-values:
- How should we act?
 - Completely trivial to decide!

$$\pi^*(s) = \underset{a}{\operatorname{arg\,max}} Q^*(s, a)$$

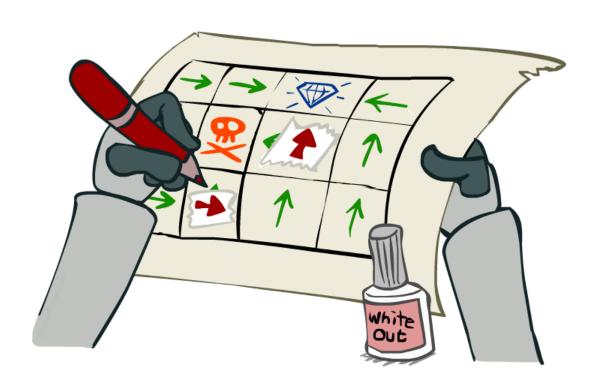


 Important lesson: actions are easier to select from qvalues than values!





Policy Iteration

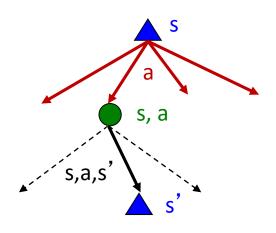






Value iteration repeats the Bellman updates:

$$V_{k+1}(s) \leftarrow \max_{a} \sum_{s'} T(s, a, s') \left[R(s, a, s') + \gamma V_k(s') \right]$$



[Demo: value iteration (L9D2)]

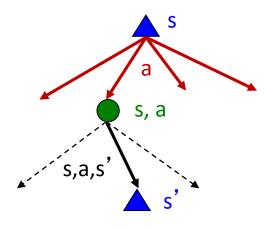




Value iteration repeats the Bellman updates:

$$V_{k+1}(s) \leftarrow \max_{a} \sum_{s'} T(s, a, s') \left[R(s, a, s') + \gamma V_k(s') \right]$$

Problem 1: It's slow – O(S²A) per iteration



[Demo: value iteration (L9D2)]

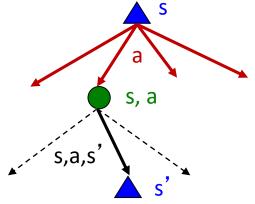




Value iteration repeats the Bellman updates:

$$V_{k+1}(s) \leftarrow \max_{a} \sum_{s'} T(s, a, s') \left[R(s, a, s') + \gamma V_k(s') \right]$$

Problem 1: It's slow – O(S²A) per iteration



Problem 2: The "max" at each state rarely changes

[Demo: value iteration (L9D2)]

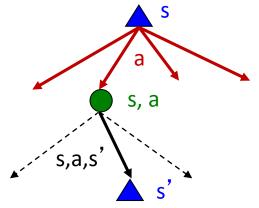




Value iteration repeats the Bellman updates:

$$V_{k+1}(s) \leftarrow \max_{a} \sum_{s'} T(s, a, s') \left[R(s, a, s') + \gamma V_k(s') \right]$$

Problem 1: It's slow – $O(S^2A)$ per iteration



- Problem 2: The "max" at each state rarely changes
- Problem 3: The policy often converges long before the values

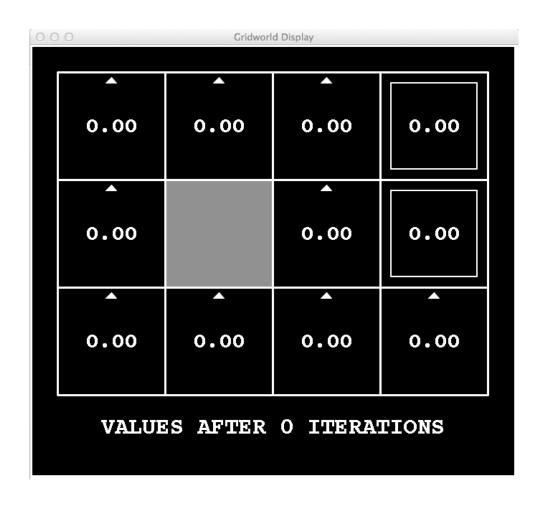
[Demo: value iteration

(L9D2)]



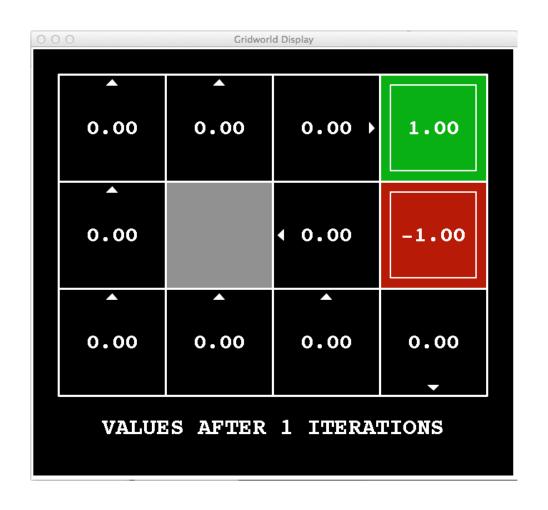








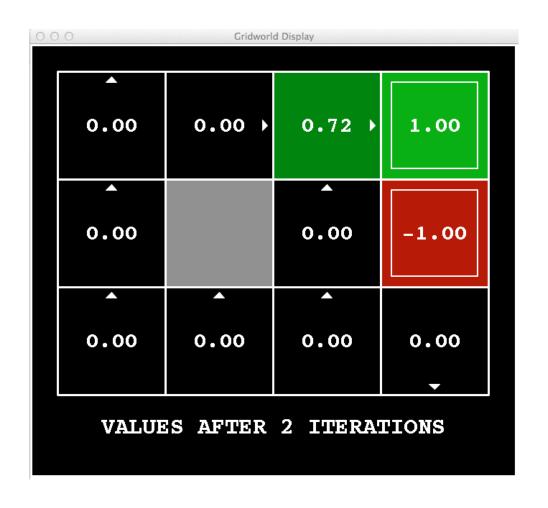








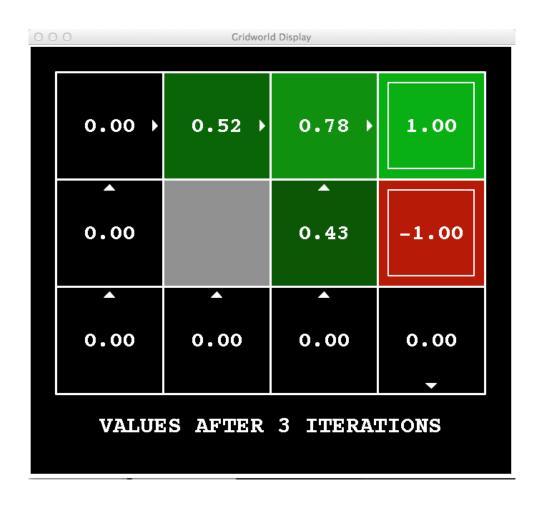
















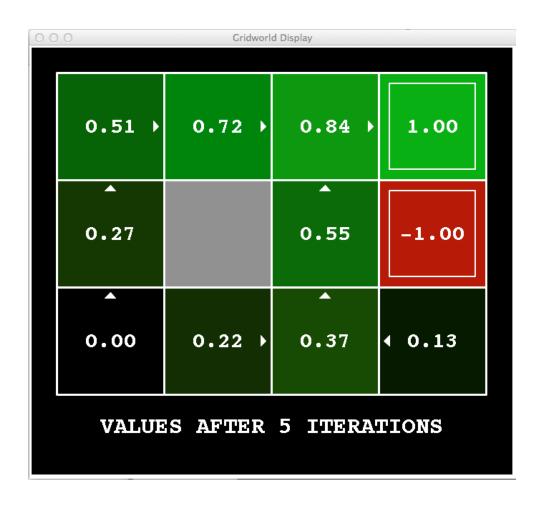








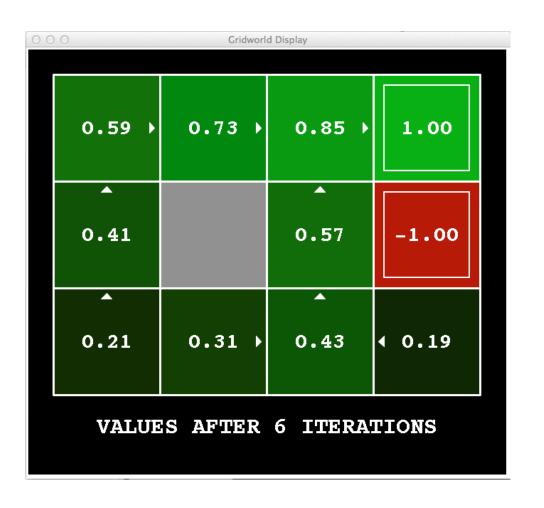








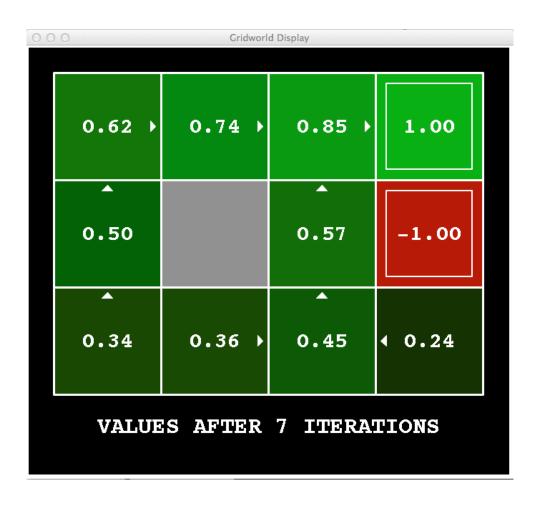








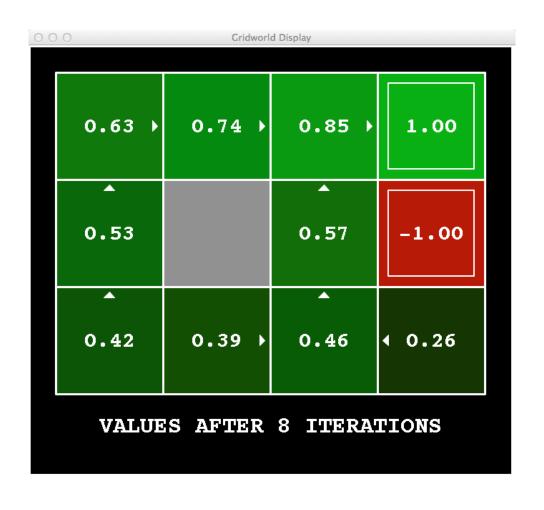








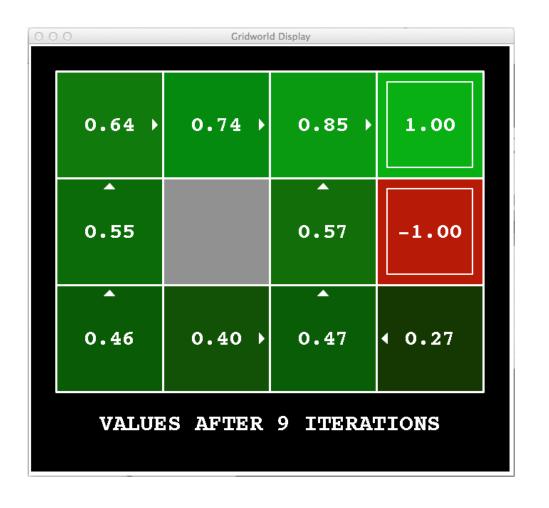






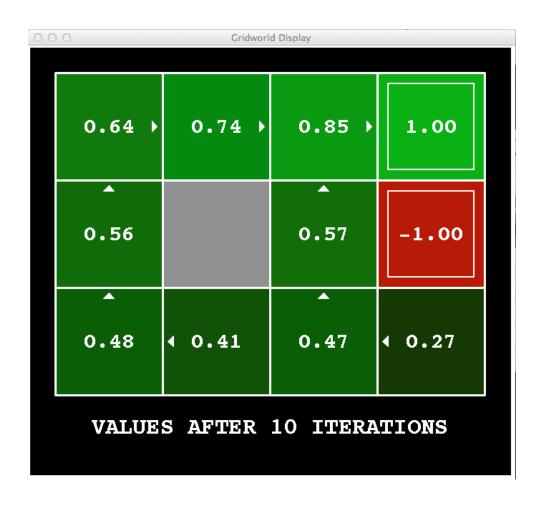






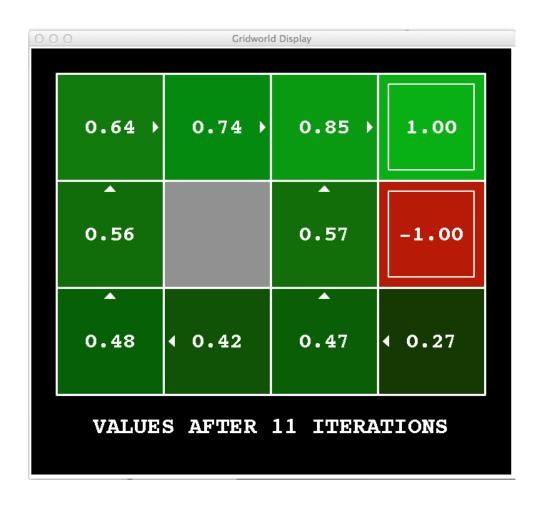






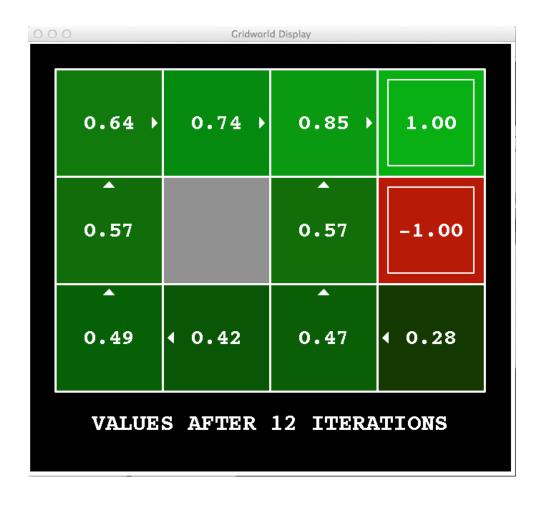






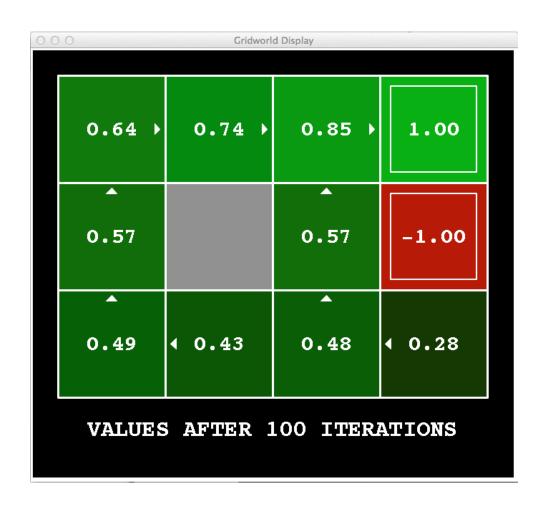
















Policy Iteration

• Alternative approach for optimal values:



- Alternative approach for optimal values:
 - Step 1: Policy evaluation: calculate utilities for some fixed policy (not optimal utilities!) until convergence





- Alternative approach for optimal values:
 - Step 1: Policy evaluation: calculate utilities for some fixed policy (not optimal utilities!) until convergence
 - Step 2: Policy improvement: update policy using one-step lookahead with resulting converged (but not optimal!) utilities as future values



- Alternative approach for optimal values:
 - Step 1: Policy evaluation: calculate utilities for some fixed policy (not optimal utilities!) until convergence
 - Step 2: Policy improvement: update policy using one-step lookahead with resulting converged (but not optimal!) utilities as future values
 - Repeat steps until policy converges



- Alternative approach for optimal values:
 - Step 1: Policy evaluation: calculate utilities for some fixed policy (not optimal utilities!) until convergence
 - Step 2: Policy improvement: update policy using one-step lookahead with resulting converged (but not optimal!) utilities as future values
 - Repeat steps until policy converges
- This is policy iteration
 - It's still optimal!
 - Can converge (much) faster under some conditions





- Evaluation: For fixed current policy π , find values with policy evaluation:
 - Iterate until values converge:





- Evaluation: For fixed current policy π , find values with policy evaluation:
 - Iterate until values converge:

$$V_{k+1}^{\pi_i}(s) \leftarrow \sum_{s'} T(s, \pi_i(s), s') \left[R(s, \pi_i(s), s') + \gamma V_k^{\pi_i}(s') \right]$$





- Evaluation: For fixed current policy π , find values with policy evaluation:
 - Iterate until values converge:

$$V_{k+1}^{\pi_i}(s) \leftarrow \sum_{s'} T(s, \pi_i(s), s') \left[R(s, \pi_i(s), s') + \gamma V_k^{\pi_i}(s') \right]$$

- Improvement: For fixed values, get a better policy using policy extraction
 - One-step look-ahead:





- Evaluation: For fixed current policy π , find values with policy evaluation:
 - Iterate until values converge:

$$V_{k+1}^{\pi_i}(s) \leftarrow \sum_{s'} T(s, \pi_i(s), s') \left[R(s, \pi_i(s), s') + \gamma V_k^{\pi_i}(s') \right]$$

- Improvement: For fixed values, get a better policy using policy extraction
 - One-step look-ahead:

$$\pi_{i+1}(s) = \arg\max_{a} \sum_{s'} T(s, a, s') \left[R(s, a, s') + \gamma V^{\pi_i}(s') \right]$$





 Both value iteration and policy iteration compute the same thing (all optimal values)





- Both value iteration and policy iteration compute the same thing (all optimal values)
- In value iteration:
 - Every iteration updates both the values and (implicitly) the policy
 - We don't track the policy, but taking the max over actions implicitly recomputes it





- Both value iteration and policy iteration compute the same thing (all optimal values)
- In value iteration:
 - Every iteration updates both the values and (implicitly) the policy
 - We don't track the policy, but taking the max over actions implicitly recomputes it
- In policy iteration:
 - We do several passes that update utilities with fixed policy (each pass is fast because we consider only one action, not all of them)
 - After the policy is evaluated, a new policy is chosen (slow like a value iteration pass)
 - The new policy will be better (or we're done)





- Both value iteration and policy iteration compute the same thing (all optimal values)
- In value iteration:
 - Every iteration updates both the values and (implicitly) the policy
 - We don't track the policy, but taking the max over actions implicitly recomputes it
- In policy iteration:
 - We do several passes that update utilities with fixed policy (each pass is fast because we consider only one action, not all of them)
 - After the policy is evaluated, a new policy is chosen (slow like a value iteration pass)
 - The new policy will be better (or we're done)
- Both are dynamic programs for solving MDPs





Summary: MDP Algorithms

- So you want to....
 - Compute optimal values: use value iteration or policy iteration
 - Compute values for a particular policy: use policy evaluation
 - Turn your values into a policy: use policy extraction (one-step lookahead)





Summary: MDP Algorithms

- So you want to....
 - Compute optimal values: use value iteration or policy iteration
 - Compute values for a particular policy: use policy evaluation
 - Turn your values into a policy: use policy extraction (one-step lookahead)
- These all look the same!
 - They basically are they are all variations of Bellman updates
 - They all use one-step lookahead expectimax fragments
 - They differ only in whether we plug in a fixed policy or max over actions





Double Bandits



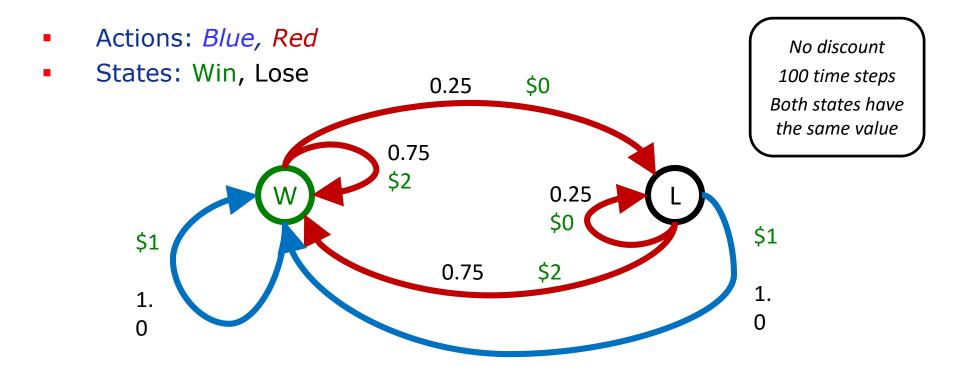








Double-Bandit MDP





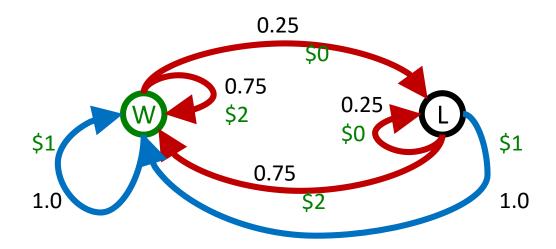


Offline Planning

- Solving MDPs is offline planning
 - You determine all quantities through computation
 - You need to know the details of the MDP
 - You do not actually play the game!

No discount 100 time steps Both states have the same value

	$\overline{}$
	Value
Play Red	150
Play Blue	100



























\$2 \$2









\$2 \$2 \$0









\$2 \$2 \$0 \$2









\$2 \$2 \$0 \$2 \$2









\$2 \$2 \$0 \$2 \$2 \$2









\$2 \$2 \$0 \$2 \$2

\$2 \$2









\$2 \$2 \$0 \$2 \$2

\$2 \$2 \$0









\$2 \$2 \$0 \$2 \$2

\$2 \$2 \$0 \$0









\$2 \$2 \$0 \$2 \$2

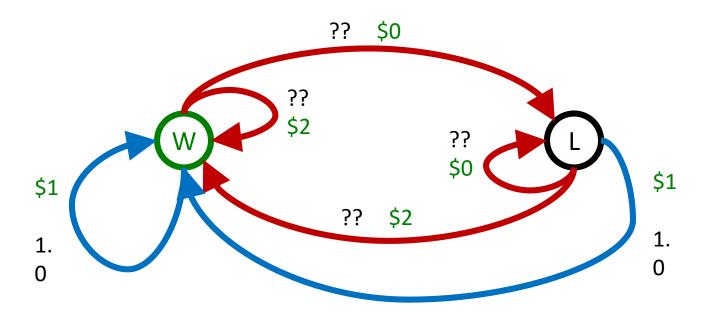
\$2 \$2 \$0 \$0 \$0





Online Planning

Rules changed! Red's win chance is different.



























\$0 \$0









\$0 \$0 \$0









\$0 \$0 \$0 \$2









\$0 \$0 \$0 \$2 \$0









\$0 \$0 \$0 \$2 \$0 \$2









\$0 \$0 \$0 \$2 \$0 \$2 \$0









\$0 \$0 \$0 \$2 \$0

\$2 \$0 \$0









\$0 \$0 \$0 \$2 \$0

\$2 \$0 \$0 \$0









\$0 \$0 \$0 \$2 \$0

\$2 \$0 \$0 \$0 \$0





What Just Happened?







What Just Happened?

- That wasn't planning, it was learning!
 - Specifically, reinforcement learning
 - There was an MDP, but you couldn't solve it with just computation
 - You needed to actually act to figure it out







What Just Happened?

- That wasn't planning, it was learning!
 - Specifically, reinforcement learning
 - There was an MDP, but you couldn't solve it with just computation
 - You needed to actually act to figure it out



- Important ideas in reinforcement learning that came up
 - Exploration: you have to try unknown actions to get information
 - Exploitation: eventually, you have to use what you know
 - Regret: even if you learn intelligently, you make mistakes
 - Sampling: because of chance, you have to try things repeatedly
 - Difficulty: learning can be much harder than solving a known MDP





Next Time: Reinforcement Learning!