Plane intersections

Start with the equation for checking wether a point $\bar{P}$ is on the plane defined by the normal $\bar{n_p}$ and the distance from the origin d. And the equation for a ray.

$(\bar{P} - d\bar{n_p}) \cdot \bar{n_p} = 0$

$\bar{P} = \bar{O_r} + t\bar{d_r}$

Substitute $\bar{P}$ from the ray equation into the plane equation.

$(\bar{O_r} + t\bar{d_r} - d\bar{n_p}) \cdot \bar{n_p} = 0$

Reorder terms and isolate t to the left hand side.

$(\bar{O_r} + t\bar{d_r} - d\bar{n_p}) \cdot \bar{n_p} = 0 \Rightarrow$

$(\bar{O_r} - d\bar{n_p}) \cdot \bar{n_p} + t\bar{d_r} \cdot \bar{n_p} = 0 \Rightarrow$

$t\bar{d_r} \cdot \bar{n_p} = (d\bar{n_p} - \bar{O_r}) \cdot \bar{n_p} \Rightarrow$

$t = \frac{(d\bar{n_p} - \bar{O_r}) \cdot \bar{n_p}}{\bar{d_r} \cdot \bar{n_p}}$

Expand the numerator, and because $\bar{n_p}$ is normal $\bar{n_p}^2 = 1$ is always true, we can substitute in 1 instead

$t = \frac{d\bar{n_p}^2 - \bar{O_r} \cdot \bar{n_p}}{\bar{d_r} \cdot \bar{n_p}} \Rightarrow$

$t = \frac{d - \bar{O_r} \cdot \bar{n_p}}{\bar{d_r} \cdot \bar{n_p}}$

There is no intesection if $\bar{d_r} \cdot \bar{n_p} = 0$

Sphere intersections

Start with the equation for a sphere and the equation for a ray.

$r_s^2 = (\bar{P} - \bar{O}_s)^2$

$\bar{P} = \bar{O}_r + t\bar{d}_r$

Substitute $\bar{P}$ from the ray equation into the sphere equation.

$r_s^2 = (\bar{P} - \bar{O}_s)^2 \Rightarrow$

$r_s^2 = (\bar{O}_r + t\bar{d}_r - \bar{O}_s)^2$

We add $\bar{v}$ and substitute it into our previous equation.

$\bar{v} = \bar{O}_r - \bar{O}_s$

$r_s^2 = (\bar{O}_r + t\bar{d}_r - \bar{O}_s)^2 \Rightarrow$

$r_s^2 = (\bar{v} + t\bar{d}_r)^2$

Expand the right hand side

$r_s^2 = (\bar{v} + t\bar{d}_r)^2 \Rightarrow$

$r_s^2 = \bar{v}^2 + 2\bar{v} \cdot t\bar{d}_r + t^2\bar{d}_r{}^2$

Reorganise the terms and we get our quadratic which we can easily solve.

$r_s^2 = \bar{v}^2 + 2\bar{v} \cdot t\bar{d}_r + t^2\bar{d}_r{}^2 \Rightarrow$

$t^2\bar{d}_r{}^2 + 2\bar{v} \cdot t\bar{d}_r + \bar{v}^2 - r_s^2 = 0$

$a = \bar{d}_r{}^2, b = 2\bar{v} \cdot \bar{d}_r, c = \bar{v}^2 - r_s^2$

$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

There is no intersection if $b^2 - 4ac < 0$